Essays on Competing Auctions

by

Cristián Andrés Troncoso-Valverde

B.Sc., Universidad de Talca, 1999
M.A., Concordia University, 2001

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Economics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

May 2013

Abstract

This dissertation studies two elements of auction design that are important to understand environments where multiple auctioneers compete against each other: heterogeneity in bidders' preferences, and endogenous information structures.

The first research chapter studies a model of competing auctions in which bidders have heterogeneous preferences. I provide a novel characterization of the set of participation rules and show that contrary to results in models with homogeneous goods, bidders' selection of trading partners is non random. I also show that changes in reserve prices affect not only the distribution of valuations of participants but also the probability with which every bidder visits the auctions. This introduces a novel trade–off between screening and traffic effect not present in models with homogeneous goods.

The second research chapter examines a model of competing auctions in which sellers can release information that allows bidders to learn their valuations before choosing trading partners. I provide a set of sufficient conditions for the existence of a unique equilibrium in which both sellers supply information. These conditions involve restrictions on the prior distribution of bidders valuations. The existence of this equilibrium is independent of the number of bidders, which differs considerably from results in models with a single auctioneer where releasing information is optimal for the auctioneer only if the number of bidders is sufficiently large.

The last chapter reexamines the problem of information provision in competing auctions in a framework where sellers can also post reserve prices. The inclusion of reserve prices makes the existence of an equilibrium in which both sellers do not supply information less likely because sellers can use reserve prices to appropriate of some of the surplus generated by information provision. I show the existence of a threshold number of bidders such that the information provision game admits a unique equilibrium in which both sellers release information provided that the actual number of bidders is above this threshold.

Preface

This dissertation is original and it is the result of independent work carried out by the author, Cristián Troncoso-Valverde. No part of this dissertation has been published or submitted for publication.

Table of Contents

Appendices

List of Figures

Acknowledgements

I am deeply grateful to my supervisor, Michael Peters, for his invaluable support, patience, and encouragement during the development of this dissertation. I am also grateful to Li Hao for his insightful comments, constructive suggestions, and enlightening advice.

I would like to express my sincere appreciation to my fellow classmates Marcos Agurto, Jonathan Goyette, Jian Mardukhi, Takuro Miyamoto, and Germán Pupato, for enhancing my experience as a Ph.D. student at UBC. In particular, I thank Marcos and Germán for helping me get through some difficult times.

Many thanks to my friends Pablo Morán, Stefania Contreras, and Leopoldo Gutierrez. Pablo was always ready to listen to my ideas and to offer honest advise and valuable suggestions, and Stefania and Leo did not hesitate to open the door of their home to me. Thank you for your generosity, kindness and true friendship. I would also like to thank my colleagues at Universidad Diego Portales for their support during the last year of my Ph.D. studies.

A huge thank to my parents and siblings and also to my extended family. They did never hesitate to help in any way they could to make our stay in Vancouver more enjoyable, and our return to Chile less stressful. But more importantly, they were always there to offer their support and to provide comfort in those many times of distress. For them, my most sincere and honest appreciation and my eternal gratitude. Without you all, this accomplishment would have never seen the light of day.

Last, but by no means least, I want to thank my wife, María Cristina, for her infinite and unconditional love; thank you for holding my hand and walking with me all these years.

Dedication

To my wife María Cristina, for always believing in me.

To my daughter María Ignacia and my son Emilio, for being the source of my inspiration.

To my parents María and Rubén, and my siblings Gabriela and Iván, for always standing by me.

Chapter 1

Introduction

In the literature on competing mechanism, multiple sellers compete against each other giving buyers the chance to choose among multiple alternatives. This competitive consideration is important because it offers an answer to the striking theoretical prediction –due to Cremer and McLean (1988), that arises in environments with a single seller. In such environments, if bidders' valuations are even slightly correlated then the seller can design a mechanism –an auction with a system of fees, that enables him to extract all the surplus. As pointed out by Peters (2010), the fact that no one has yet come up with a 'real life' example in which these fees are actually used, makes Cremer and McLean's result implausible. Competition among sellers provides one possible resolution to this issue.

Instead of considering a general space of mechanism, the literature on competing auctions assumes that the set of mechanism available to sellers are auctions. From a theoretical point of view, this restriction appears to be motivated by the infinite regress problem described by McAfee (1993) and later formalized by Epstein and Peters (1999). In a nutshell, the infinite regress problem arises because the description of feasible mechanisms in competing mechanism games may depend on the other principals' mechanisms and so on. Thus, by focusing on auctions this literature has not only avoided this infinite regress problem but it has also shown how competition boosts the appearance of simple equilibrium mechanisms, that in this case correspond to auctions with equilibrium reserve prices converging to sellers' production costs (Peters and Severinov, 1997; Hernando-Veciana, 2005; Virag, 2010).

An assumption common to all models of competing auctions developed so far is the perfect substitutability of the goods offered by the sellers. However, there are many cases in which items can be better described as imperfect substitutes. Every day, online platforms such as eBay, host thousands of independent auctions where sellers offer items of similar but not identical characteristics. Bapna, Dellarocas, and Rice (2010) mention the case of 'iPod Nano 4GB', where a search on any random day of 2009 was likely to reveal more than one thousand listings with none of these listings offering exactly identical items. In this case, assuming that goods are perfect substitutes may be misleading because buyers may assign different valuations to the items even if all of them are units of the same type of good. The second chapter of this dissertation investigates this issue: it studies an otherwise standard model of competing auctions where bidders place different valuations on each of the items up for sale. Evidently, the fact that bidders place different values to different items makes any bidder's type a collection of random variables, which introduces important technical challenges when modeling bidders' participation decisions. I overcome these challenges by extending the idea of cutoff values –used in (monopolistic) auction models with costly participation, to cutoff functions. The use of cutoff functions provides a clear and novel characterization of bidders' participation rules that serves to highlight the role played by heterogeneity as a coordinating device for bidders' participation decisions. I show that cutoff functions can be used to completely characterize the set of (symmetric) participation rules used by bidders in equilibrium and prove that all these rules must share this coordination property. This suggests one of the main differences between models with homogeneous and heterogeneous goods: while in the former there is a coordination failure due to bidders' randomization in the selection of trading partners, heterogeneity in bidders' preferences rules out any randomization and hence, it eliminates coordination failures as a source of friction in the market. I also show that changes in reserve prices affect the participation decision of every buyer regardless of her vector of valuations. With homogeneous items a decrease in seller j's reserve price has two consequences on the visiting decisions of bidders: (i) those bidders with valuations just below seller j's reserve price begin to visit this seller with probability one; (ii) some bidders who were mixing among some subset of sellers find profitable to bid for sure at seller j's auction. However, when items are heterogeneous a change in a reserve price affects not only the decision of those bidders whose valuations are close to seller j's reserve price but also the participation decision of bidders with relatively high valuations who where indifferent before the change in the reserve price took place. This adds a novel trade–off between screening and traffic effect not present in models with homogeneous goods.

Another assumption common to all competing auction models in the literature is the exogenous nature of information structures. We can think of many examples in which this assumption may not hold. For instance, sellers participating in online auctions usually have the option of posting pictures or adding descriptions of the objects that they wish to sell. A casual look at eBay reveals that while some sellers provide very detailed descriptions others simply post basic information and omit any detail that might be of relevance for some potential buyer. Chapters three and four of this dissertation are devoted to the analysis of this issue.

In the third chapter I study a model in which two sellers running second-price auctions without reserve prices compete by choosing information structures from a binary set. There are two sellers with unit supply and multiple bidders with unit demand. Sellers can release information about their items prior to commencing with the auction. I assume that the set of information structures is binary in the sense that sellers can choose to either let every bidder perfectly learn her valuation or not. Choosing not to release information is interpreted as choosing not to add more information to the public pool of information contained in the prior distribution of valuations. Despite this binary structure, the fact that bidders select trading partners after observing the information structures chosen by sellers, allows to capture in a relatively simple way the relationship between information and competition by making participation depend on bidder's private information. I prove that full information provision can be supported as the unique equilibrium of the game provided that the distribution of bidders' valuations is increasing and convex. Intuitively, when a seller unilaterally decides to release information (while his competitor is expected not to do so), the deviating seller is able to screen bidders and attract only those who value his item the most. However, if only a few number of bidders have valuations above the mean, there is a chance that releasing information hurts the deviating seller by decreasing expected traffic. Although this negative traffic effect disappears as the number of potential bidders grows large, it leaves open the possibility of having equilibria where both sellers optimally choose not to provide information. It turns out that this kind of issues can be avoided if the prior distribution function of bidders' valuations is increasing and convex. Finally, I provide a characterization of information in terms of its strategic value for sellers.

In the fourth chapter, I revisit the problem of endogenous information structures in competing auctions in a framework where sellers can also post reserve prices. I maintain the assumption of a binary set of information structures and I assume that sellers choose reserve prices from a finite subset of the $[v_0, 1]$ interval. The addition of reserve prices as a second strategic variable induces continuation equilibria that are characterized by cutoff functions for which no closed–form solution is available (except for the case in which reserve prices are equal), making the issue of equilibrium existence a difficult task. However, the finiteness of the set of reserve prices allows the use of standard game theoretical tools to claim existence of an equilibrium. I show that equilibria in which both sellers do not supply information are less likely to exist –compared to the existence of such equilibria in a game where sellers cannot post reserve prices, because reserve prices allow sellers to appropriate of some of the surplus generated by information provision. I also provide a sufficient condition in terms of a critical number of bidders such that the unique equilibrium of the game is one in which both sellers supply information provided that the actual number of bidders is above this threshold.

1.1 Related Literature

As mentioned above, this dissertation belongs to the strand of literature dealing with competing auctions, which is directly related to the literature on competing mechanism design initiated by McAfee (1993). McAfee studied a (dynamic) model where several sellers compete by choosing direct mechanisms followed by bidders who decide on which seller to visit. Using an equilibrium concept called Competitive Subform Consistent Equilibrium (CSCE), McAfee shows that equilibrium mechanisms resemble second-price auctions with zero reserve prices and that buyers randomize over the sellers they visit. McAfee arrived at this conclusion under two simplifying assumptions: (i) the set of available mechanism is equal to the set of direct mechanisms; (ii) in equilibrium, all agents believe that the expected profits of any buyer (whether this buyer participate or not in any given mechanism) is invariant to sellers' unilateral deviations. He used the first assumption in order to avoid the infinite regress problem when defining the strategy space of sellers[1], while the second one served him to reduce the strategic interactions occurring in the market. Peters and Severinov (1997) relaxed McAfee's second assumption at the cost of a more restrictive version of the first. Instead of considering the full set of direct mechanisms, these authors model competition in auctions (which reduces to competition in nonnegative reserve prices) but take full account of the strategic interactions of their model. They arrived to a similar conclusion to McAfee's (i.e., equilibrium reserve prices are equal to sellers' production costs) provided that the number of agents in the market is large (infinite). However, Peters and Severinov did not prove existence of an equilibrium when the number of agents is assumed finite. Instead, they showed that if an equilibrium for each finite market size version of the model exists, then reserve prices must converge to zero as the buyer-to-seller ratio becomes sufficiently large. Burguet and Sakovics (1999) addressed the question of equilibrium existence in the finite version of Peters and Severinov's game and concluded that convergence does indeed need the large market assumption. In order to make their point, Burguet and Sakovics considered a competing auction model with only two sellers and proved that the equilibrium probability of posting reserve prices equal to zero is nil. Virag (2010) extended Burguet and Sakovics's analysis to any arbitrary (finite) number of sellers and showed that the key element to sustain converge of equilibrium reserve price is a sufficiently large number of sellers[2]due to the negligible effect on the utility levels of the bidders when the number of sellers grows large.

An important element of this dissertation is the idea of cutoff functions to model bidders' participation decision. This idea is borrowed from the literature that studies endogenous participation in auctions, where it is customary to make bidders' participation endogenous by adding an entry costs that must be paid before bidders can submit a bid[3]. The first paper to formally introduce the concept of

[1]Epstein and Peters (1999) have shown that such infinite regress converges to a universal type space in which the standard revelation principle can be applied. The main drawback of their approach is its analytical complexity which severely limits its applicability to practical contracting situations.

[2]Hernando-Veciana (2005) also considers the question of convergence in competing auction models with finitely many reserve prices. He shows that an equilibrium in which each seller posts a reserve prices equal to his production costs exists if we let let the number of agents becomes sufficiently large.

[3]This is the approach followed by McAfee and McMillan (1987), Levin and Smith (1994),

cutoff function is Green and Laffont (1984). These authors model participation in auctions assuming that bidders possess two privately known pieces of information: their willingness to pay and their costs of participation. The authors characterize bidders' participation decision through a relationship such that for each value of the bidder's willingness to pay there is a critical value of the bidder's participation cost above which she does not participate. They also show that the participation constraint is binding at many points in the parameter space, instead of just one as it would be the case in models with deterministic (or commonly known) participation costs. Lu (2006) generalizes Green and Laffont's approach and shows that any equilibrium entry strategy can be characterized through a set of continuous and monotonic shutdown curves that separate bidders' types into participating and non-participating groups. I take this approach even further and show existence of cutoff functions in a model where the cost of entry can be considered as endogenous.

This dissertation is also related to the literature that studies endogenous information structures in auctions. The celebrated linkage principle of Milgrom and Weber (1982) establishes that in auctions where valuations are affiliated, the seller must provide bidders with as much information as possible about the value of the object. Intuitively, by supplying all available information the seller reduces bidders' private information increasing the expected price of the object. This conclusion, however, relies heavily on the assumption of affiliation of valuations. In simple terms, affiliated valuations implies that all bidders react symmetrically to the information revealed by the seller. Hence, by providing more information the seller is able to increase the bids of all bidders, and in particular the bids of those who lose the auction and determine the price finally paid by the winner. Bergemann and Pesendorfer (2007) is perhaps the first paper to consider the problem of optimal information structures in independent private value auctions. In their model, the auctioneer can control the accuracy with which each bidder learns her valuation of the object along with the decision of whom to sell the object. They show that optimal information structures can be represented by finite partitions, and that this partitions are asymmetric across bidders. The focus of the analysis in this dissertation differs from that of Bergemann and Pesendorfer in that we restrict

Samuelson (1985), Vagstad (2007), Menezes and Monteiro (2000), and Tan and Yilankaya (2006) among others.

sellers to treat bidders symmetrically with respect to information provision. Thus, the approach I follow here is closer to the one followed by Ganuza (2004). He studies a model in which an auctioneer must decide how much information to reveal prior to commencing with the auction, and where the seller must treat bidders symmetrically with respect to information provision. Ganuza's main conclusion is that the auctioneer releases less information than what would be efficient but that this inefficiency disappears as the number of bidders becomes large. Ganuza and Penalva (2004) generalizes Ganuza's approach (by considering more general information structures) and arrive to a similar conclusion: the seller provides less than the efficient level of information and such inefficiency vanishes as the number of buyers grows large. In an even more general setup, Board (2009) shows that releasing information is never optimal for a seller running a second-price auction when the number of bidders is restricted to two. In the second research chapter of this dissertation I show that this may no longer be true if we have two competing auctioneers.

One paper that combines endogenous participation with information provision is Vagstad (2007). In this paper, potential bidders must pay an entry cost before learning their valuations but the auctioneer can release information that allows bidders to perfectly learn their valuations. In this model, early information induces screening of high-valuation bidders, an effect that is also present in the models of Chapters 3 and 4 of this dissertation. However, Vastag considers only one seller and hence, the value of bidders' outside option cannot be directly affected by information provided by the auctioneer. He finds that the auctioneer has too weak an incentive to produce early information because releasing information can drastically reduce entry, reducing profits relative to what the seller could obtain by not releasing information. In contrast, I show that this need not be the case when competition among sellers is explicitly taken into consideration. The difference arises because competition makes information affect the value of bidders' outside option changing the incentives that each seller faces when deciding whether or not to provide information to the bidders.

Forand (2009) is, up to my knowledge, the only work that examines the problem of endogenous information in competitive environments. In his model, two sellers direct the search of two bidders through commitments to provide information. He shows that relative to monopoly, competition always improves informational efficiency, a result similar to ours. However, Forand models bidders' participation decision in ex-ante rather than ex-post terms, which precludes the study of issues such as the screening role played by information in competitive environments. Moreover, his model considers competition through commitments to providing information rather than actual provision of information. Although Forand's model is more general than the models considered here –he allows sellers to choose from the set of direct mechanisms while I only consider auctions, the analysis in this dissertation does not restrict to the two-bidder case and it allows valuations to be drawn from a continuous rather than a discrete probability distribution.

Finally, this dissertation relates to, but it is separated from, the strand of literature that studies the incentives to provide information in environments where sellers compete through prices (as opposed to competing in auctions). Damiano and Li (2007) are among the few to study this issue in a two-sellers model with one buyer and binary information structures. They show that information is used by sellers to soften price competition as it makes items more differentiated in terms of their quality. Ivanov (2009) extends this idea by considering a model with an arbitrary number of sellers and a continuum of types but a single buyer. He shows that there exists a critical number of sellers such that the unique symmetric equilibrium has all sellers providing information and charging prices equal to their marginal costs.

Chapter 2

Competing Auctions with Heterogeneous Goods

2.1 Introduction

The literature on competing auctions has focused on the analysis of models where sellers offer their items to a pool of bidders who consider the items to be perfect substitutes. However, there are multiple cases in which bidders may consider these items to be different objects even if they are units of the same good. For example, a casual look at online platforms reveals that sellers supply different amounts of information about their objects. Having different amounts of information may result in situations where bidders regard the items as close but imperfect substitutes because the assessments of the objects may differ depending on how much information about the item each bidder has. The model I develop in this chapter offers a theoretical framework that can be used to analyze issues like this.

I consider a model with two risk–neutral sellers with unit supply who post reserve prices, and n risk–neutral buyers ($n \geq 2$) with unit demands who place different valuations to the items up for sale. Evidently, the fact that bidders have different valuations implies that bidders' types are collections of random variables, which introduces important technical challenges when modeling bidders' participation decisions. One of the contributions of this chapter is to provide a complete characterization of the participation rules used by bidders in any symmetric equilibrium. These participation rules are conceptually similar to cutoff strategies used in (monopolistic) auction models with costly participation (Green and Laffont, 1984; Samuelson, 1985; Vagstad, 2007). Our first important result establishes that no matter what participation strategy bidders may use, there always is a best response to this strategy that can be described in terms of a nondecreasing and continuous function ρ with the property that a bidder with valuations (v_1, v_2) visits seller 1 if and only if $v_2 \leq \rho(v_1)$, and visits seller 2 with probability one if and only if $v_2 > \rho(v_1)$. This ensures that whenever the game possesses a continuation equilibrium then it must also posses an equilibrium in which bidders use pure strategies. This last finding is important because it points toward one of the main differences between models with homogeneous and heterogeneous goods: while the former involves a coordination failure due to buyers' randomization in the selection of trading partners, heterogeneity in bidders' preferences mitigates coordination failures as a source of friction in the market.

Another interesting result of this chapter is the way in which a change in some reserve price affects the demand faced by a seller. With homogeneous goods, a decrease in seller j's reserve price has two consequences on the visiting decisions of bidders: (i) those bidders with valuations just below seller j's reserve price begin to visit seller j with probability one; and (ii) some bidders who were mixing among sellers find profitable to bid for sure at seller j's auction. The interesting observation is the fact that this change in seller j's reserve price does not affect the probability with which bidders with high valuations visit the seller. This is no longer true when goods are assumed to be heterogeneous. Since equilibrium participation rules can be characterized by a continuous and increasing function, we are able to show that unilateral changes in reserve prices affect not only the types who participate but also the probability with which each bidder chooses to visit the auction. Thus, when a seller unilaterally decreases his reserve price he affects the participation decisions of bidders with high valuations who were just indifferent before the change in the reserve price took place. This introduces a novel trade–off between traffic and screening effects not present in models with homogeneous goods.

The rest of the chapter is organized as follows. The model is outlined in section 2.2. Section 2.3 characterizes the participation rules used by bidders in equilibrium, and section 2.4 characterizes the equilibrium set of the sellers' game. The chapter ends with some concluding remarks in section 2.5.

2.2 The Model

Consider an economy in which trade takes place using second-price sealed bid auctions. The economy is populated by two risk-neutral sellers (seller 1 and seller 2) with unit supply, and n risk neutral bidders, $n \geq 2$, with unit demands[4]. Sellers are indexed by $j \in \{1,2\}$, and bidders are indexed by $i = \{1,\cdots,n\}$. Bidders attach different valuations to each of the items offered by the sellers. Buyer i's true vector of valuations is denoted by a random vector $V_i = (V_{i1}, V_{i2})$, where V_{ij} represents bidder i's valuation of item j. The random vector V_i is assumed to be a collection of independently and identically distributed random variables, each following a cumulative distribution function F with continuously differentiable, bounded and positive density $f > 0$, and support $[0,1]$. A bidder i whose realization of V_{ij} is v_{ij} and who trades with seller j at price p_j gets a surplus $v_{ij} - p_j$, while seller j gets a surplus p_j. In case the object is left unsold, the seller derives a value v_0 from his item whereas the bidder derives a payoff equal to zero. The value of v_0 is common knowledge among players, nonnegative and less than one. In what follows, if X_l is a set and $l \in \{1,\cdots,L\}$ then $X = \Pi_{l=1}^{L} X_l$ and $X_{-l} = \Pi_{k \neq l} X_k$; thus $X = X_l \times X_{-l}$. Furthermore, $x \in X$ then $x = (x_i, x_{-i})$ with $x_{-i} \in X_{-i}$, $x_{-i} = (x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_L)$.

The game we study is similar in almost all respects to the standard competing auction model with homogeneous goods (Peters and Severinov, 1997; Burguet and Sakovics, 1999; Virag, 2010) with the exception that bidders have heterogeneous preferences. The game begins when Nature draws a vector of independent and identically distributed valuations from the common prior distribution F for each bidder $i = 1,\cdots,n$, and privately communicates it to bidder i alone. After Nature has moved, sellers simultaneously announce nonnegative reserve prices, which become common knowledge right after announced. After observing these reserve prices, each bidder independently and simultaneously decides on whether to participate in some auction or to leave the market. If a bidder decides to participate in some auction, she is restricted to choose one and only one seller as her trading partner. If a bidder chooses not to participate, then this bidder obtains an exogenous

[4] Hereafter, masculine pronouns will be used to refer to sellers and feminine ones will be used to refer to bidders.

given payoff equal to zero. After bidders have selected their trading partners, the bidding process takes place: every bidder visiting seller j learns the actual number of participants in j's auction and submits a bid to this seller, who then collects all bids and awards the good to the highest bidder (in case of a tie, the good is randomly assigned among the highest bidders), and asks for a price equal to the second-highest bid. Then, the game ends and any unrealized payoff is realized.

2.3 Bidder's Participation Game

A strategy for a bidder is a rule that specifies a participation and a bidding decision as a function of bidder's information in stage two of the game. As it is customary in the literature of competing auctions (Peters and Severinov, 1997; Virag, 2010), we will assume that conditional on participating every bidder bids her estimate truthfully (v_{i1} or v_{i2} depending on which auction bidder i has chosen to bid in). The main advantage of this assumption is the reduction of bidder's strategies to rules that specify the probabilities with which they visit each seller. Furthermore, it is straightforward to check that truthful bidding constitutes a Bayesian equilibrium of any bidding continuation game. Thus, a strategy for bidder i is a mapping $\pi_i : [0,1]^2 \times [v_0,1]^2 \rightarrow [0,1]^2$ with $\pi_i(v_i,r) = \{\pi_{i1}(v_i,r), \pi_{i2}(v_i,r)\}$, $\pi_{ij} \geq 0$, and $\pi_{i1} + \pi_{i2} \leq 1$, such that $\pi_{ij}(v_i,r)$ delivers the probability with which bidder i bids in auction j as a function of her vector of valuations $v_i = (v_{i1}, v_{i2})$ and the vector of reserve prices (r_1, r_2). We restrict attention to equilibria in which every bidder uses a *symmetric participation rule*. A participation rule is symmetric if for a given vector of reserve prices, two bidders with the same vector of valuations visit seller j with the same probability, $\pi_{ij}(\cdot) = \pi_{kj}(\cdot) \equiv \pi_j(\cdot)$, $i \neq k$. We also adhere to the convention to treat the decision not to bid in any auction as equivalent to the decision to submit a non serious bid in auction 1. Thus, if $\pi(v,r)$ stands for the probability that a bidder with valuations $v = (v_1, v_2)$ visits auction 1 then $\pi_2(v,r) = 1 - \pi(v,r)$ is the corresponding probability that this bidder visits seller 2. Finally, we let S be the strategy space for bidder i, i.e., the set of all (measurable) mappings π.

Our assumption of truthful bidding suggests that payoffs should be functions of the valuation of that particular item and the vector of reserve prices. Intuitively, once a bidder has chosen an auction all that matters is the valuation of that particular item because this should determine how much to bid. However, that bids depend on a single valuation is neither sufficient nor necessary to have payoffs depend on a single valuation because participation strategies are functions of the whole vector of valuations. Nonetheless, we will treat bidders' payoffs as if they were functions of a single valuation, which will considerably simplify exposition.

Let $Q_1(v_1;\pi,r_1)$ represent the (reduced form) probability that a bidder with valuation v_1 trades with seller 1 when the participation rule used by other bidders is π, and reserve prices are $(r_1,r_2)\in[v_0,1]^2$. Then, the (reduced-form) payoff that this bidder expects in auction 1 can be written as the difference of two terms: her probability of trading with seller 1 times his valuation of this item, minus the expected price she pays. If $\mathscr{U}_1(v_1;\pi,r_1)$ denotes this payoff, then:

$$\mathscr{U}_1(v_1;\pi,r_1) = v_1 Q_1(v_1;\pi,r_1) - P_1(v_1;\pi,r_1)$$

with $\mathscr{U}_2(v_2;\pi,r_2)$ written likewise[5]. It is fairly clear that the payoff of any type whose valuation falls below the respective reserve price should be equal to zero. Similarly, any type whose valuation v_1 (resp. v_2) is above r_1 (resp. r_2) should expect a positive payoff because there always is a positive chance to trade with seller 1 (resp. seller 2) (the probability of having everybody else's valuations below v_1 (resp. v_2) is strictly positive if $v_1 > r_1$). Notice that this event is independent of the participation decisions of other bidders and hence, $Q_1(v_1;\pi,r_1) > 0$ (resp. $Q_2(v_2;\pi,r_2) > 0$) for every type such that $v_1 > r_1$ (resp. $v_2 > r_2$). This observation together with standard incentive compatibility arguments (Riley and Samuelson, 1981) allows us to establish monotonicity with respect to v_j of the payoff functions $\mathscr{U}_j(\cdot;\pi,r_j)$.

Lemma 1. *$\mathscr{U}_j$ is nondecreasing and continuous with respect to $v_j\in[0,1]$. Moreover,*

$$\mathscr{U}_j(v_j;\pi,r_j) = \max\left\{0; \int_{r_j}^{v_j} Q_j(\xi;\pi,r_j)d\xi\right\} \tag{2.1}$$

[5] Of course, all these functions may also depend on the whole vector of reserve prices through its effect on π. However, we suppress this dependence to simplify notation whenever there is no risk of confusion.

$j = 1, 2$.

Proof. In the appendix. □

Lemma 1 is important because it helps us derive the following useful property of the set of best responses of any bidder to any participation rule π used by the remaining ones.

Proposition 2. *Take any bidder and let ω be any best response to the symmetric participation rule π used by other bidders. Then, there exists a $\omega' \in S$ and a nondecreasing and continuous function $\rho : [0,1] \to \mathbb{R}$ with the property that $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$, and $\omega'(v,r) = 0$ if and only if $v_2 > \rho(v_1)$, and such that ω' is also a best response to π.*

We outline the proof for the case in which both reserve prices are strictly below one, relegating the other cases (together with the proofs of monotonicity and continuity of ρ) to the appendix. Let $\max\{r_1; r_2\} < 1$ and suppose that every bidder other than bidder 1 uses the participation rule π to choose trading partners. A necessary and sufficient condition for the participation rule ω' to be bidder 1's best response to π is that for every type $(v_1, v_2) \in [0,1]^2$, and every $(r_1, r_2) \in [v_0, 1]^2$,

$$\omega'(v,r) = \begin{cases} 0 & \text{if } \mathscr{U}_1(v_1;\pi,r_1) < \mathscr{U}_2(v_2;\pi,r_2) \\ 1 & \text{if } \mathscr{U}_1(v_1;\pi,r_1) > \mathscr{U}_2(v_2;\pi,r_2) \\ \in [0,1] & \text{if } \mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(v_2;\pi,r_2) \end{cases} \tag{2.2}$$

where $\mathscr{U}_1(\cdot;\pi,r_1)$ is bidder 1's payoff when she bids in auction 1, and $\mathscr{U}_2(\cdot;\pi,r_2)$ is her payoff when she bids in auction 2. Since $\mathscr{U}_1$ and $\mathscr{U}_2$ are both continuous functions (lemma 1) defined on the closed interval $[0,1]$, we can use the intermediate value theorem to claim the existence of a pair of numbers $(v_1^*, v_2^*) \in [0,1]^2$ such that $u_1 = \mathscr{U}_1(v_1^*;\pi,r_1)$ and $u_2 = \mathscr{U}_1(v_2^*;\pi,r_2)$ for every number u_1 between $\mathscr{U}_1(0;\pi,r_1)$ and $\mathscr{U}_1(1;\pi,r_1)$, and every number u_2 between $\mathscr{U}_2(0;\pi,r_2)$ and $\mathscr{U}_2(1;\pi,r_2)$ respectively. First, suppose that $\mathscr{U}_1(1;\pi,r_1) \leq \mathscr{U}_2(1;\pi,r_2)$ then we can assign to every $v_1 \in [0,1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(\rho(v_1);\pi,r_2)$. This mapping ρ has the property that $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$ because lemma 1 ensures that $\mathscr{U}_2(v_2;\pi,r_2) = \mathscr{U}_2(\rho(v_1);\pi,r_1) = \mathscr{U}_1(v_1;\pi,r_1) = 0$

whenever $v_1 \le r_1$ (and hence, we can assign the same number $\rho(v_1)$ to every such v_1), and $\mathscr{U}_2(v_2;\pi,r_2) < \mathscr{U}_2(\rho(v_1);\pi,r_1) = \mathscr{U}_1(v_1;\pi,r_1)$ whenever $v_1 > r_1$. Second, if $\mathscr{U}_1(1;\pi,r_1) > \mathscr{U}_2(1;\pi,r_2)$ then there are values of v_1 for which bidder 1 strictly prefers to visit seller 1. Let $\bar{v}_1$ be implicitly defined by $\mathscr{U}_1(\bar{v}_1;\pi,r_1) = \mathscr{U}_2(1;\pi,r_2)$. Clearly, $\bar{v}_1 > r_1$ because $\mathscr{U}_1$ and $\mathscr{U}_2$ are increasing functions and hence, $\mathscr{U}_1(1;\pi,r_2) > \mathscr{U}_2(r_2;\pi,r_2) = 0$. Using a similar argument to the one employed in the previous case we can assign to every $v_1 \in [0,\bar{v}_1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(\rho(v_1);\pi,r_2)$. For values of v_1 outside $[0,\bar{v}_1]$, we let $\rho(v_1)$ take the value of one such that $\mathscr{U}_1(v_1;\pi,r_1) > \mathscr{U}_2(\rho(v_1);\pi,r_2)$ holds for every $v_1 > \bar{v}_1$. Then, $\omega'(v,r) = 1$ if and only if $v_2 \le \rho(v_1)$. Figure 2.3 provides a graphical interpretation of the ρ function and the best response ω'.

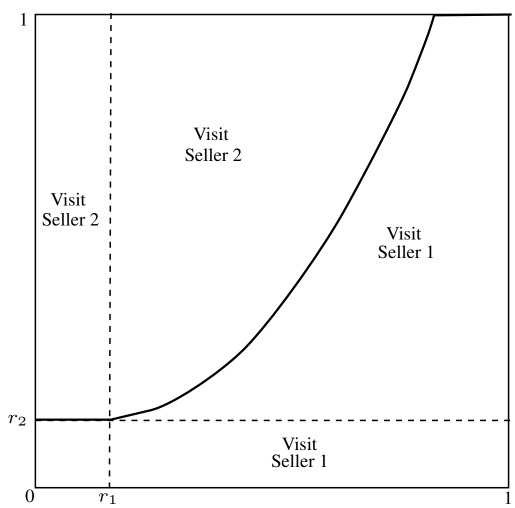

Figure 2.1: ρ function

An interesting implication of proposition 2 is the fact that no matter what participation rule bidders may use, every best response to it can be characterized by a pure strategy that is defined in terms of the associated function ρ. This suggests that for every continuation equilibria (if one exists at all) we can find another one in which bidders use pure strategies.

Corollary 3. *Take any pair of reserve prices $(r_1,r_2) \in [v_0,1]^2$ and consider the continuation game in which bidders simultaneously select trading partners. If this* *continuation game possesses an equilibrium then it must also posses an equilibrium in which bidders use pure strategies.*

A second interesting implication of proposition 2 is the existence of a function ρ that is associated to the best response ω'. The existence of this function allows us to write the payoff that a bidder with valuations v_1 and v_2 expects in auction 1 and 2 in terms of ρ as follows:

$$\mathscr{U}_1(v_1;\rho,r) = \max\left\{0; \int_{r_1}^{v_1}\left[1-\int_{t_1}^{1} F(\rho(\hat{t}_1))f(\hat{t}_1)d\hat{t}_1\right]^{n-1} dt_1\right\} \tag{2.3}$$

$$\mathscr{U}_2(v_2;\rho,r_2) = \max\left\{0; \int_{r_2}^{v_2}\left[F(t_2)F(\rho^{-1}(t_2)) + \int_{\rho^{-1}(t_2)}^{1} F(\rho(\hat{t}_2))f(\hat{t}_2)d\hat{t}_2\right]^{n-1} dt_2\right\} \tag{2.4}$$

where $\rho^{-1}(t_2)$ is defined as follows:

$$\rho^{-1}(t_2) = \begin{cases} 0 & \text{if } t_2 < \rho(0) \\ \max\{t_1 \in [0,1] : t_2 \geq \rho(t_1)\} & \text{if } t_2 \geq \rho(0) \end{cases} \tag{2.5}$$

In the construction of the payoff functions we have implicitly used a nice insight due to McAfee (1993) regarding the probability with which any given bidder trades with each seller: any bidder with valuation v_j who plans to bid in seller j's auction wins the item when (i) no other bidder visits seller j; or (ii) any other participant has a valuation lower than bidder i's valuation of this item.

Take any bidder (say bidder 1) and suppose that every other bidder is using a function ρ to select trading partners[6]. Intuitively, for values of v_1 not too high (and reserve prices below one), bidder 1's best response function should deliver a value $\rho'(v_1)$ such that the type $(v_1, \rho'(v_1))$ is indifferent about which seller to visit. This value $\rho(v_1)$ can, in principle, be obtained by equating the expected payoffs given

[6]Strictly speaking, the function ρ is not a strategy but the function used to describe one. However, once we know the function ρ we can define the strategy ω' associated to it as done in proposition 2.

in Eq. (2.3) and (2.4). Thus, given v_1 the number $\rho(v_1)$ that satisfies this equality will have the property that bidder 1 wants to visit seller 1 if and only if $v_2 \leq \rho'(v_1)$ else she visits seller 2 with probability one. The only problem with this approach is the possibility that there is no value of v_2 such that payoffs are equal since Eq. (2.3) and (2.4) depend on the particular ρ function being used by other bidders. In this case, there will be types of bidder 1 who strictly prefer to visit seller 1 and hence, bidder 1's best response should deliver a value of one for any such type.

Formally, bidder 1's best response is a mapping T taking elements from the set of nondecreasing and continuous functions defined on $[0,1]$, and delivering another function ρ' that represents bidder 1's best response to the function ρ used by other bidders. Let $\mathscr{R}$ be the set of continuous and non decreasing functions mapping elements from $[0,1]$ into $\mathbb{R}$. Bidder 1's best response mapping T on $\mathscr{R}$ can be defined by:

$$T\rho(v_1) = \max\{v_2 \in [0,1] : \mathscr{U}_2(v_2;\rho,r_2) \leq \mathscr{U}_1(v_1;\rho,r_1)\} \tag{2.6}$$

where $\mathscr{U}_1(v_1;\rho,r_1)$ and $\mathscr{U}_2(v_2;\rho,r_2)$ are given by Eq. (2.3) and Eq. (2.4) respectively. Figure 2.3 gives a graphical representation of the procedure used to obtain bidder 1's best response function.

We should point out that any fixed point ρ^* of T can be used to construct a pure strategy ω^* that constitutes a symmetric continuation equilibrium. To see how this works, let ρ^* be a fixed point of T and consider the following symmetric pure strategy: $\omega^*(v,r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and $\omega^*(v,r) = 0$ if and only if $v_2 > \rho^*(v_1)$. Suppose that every bidder but bidder 1 conforms to this strategy. We can compute bidder 1's payoffs as done in Eq. (2.3) and Eq. (2.4) above. Since ρ^* is a fixed point of T it satisfies $T\rho^* = \rho^*$ and $\mathscr{U}_2(\rho^*(v_1);\rho^*,r_2) \leq \mathscr{U}_1(v_1;\rho^*,r_1)$ for all $v_1 \in [0,1]$. Take any type (v_1,v_2) of bidder 1. First, suppose that $\mathscr{U}_2(\rho^*(v_1);\rho^*,r_2) < \mathscr{U}_1(v_1;\rho^*,r_1)$. Then $\rho^*(v_1)$ must equal one as otherwise there would be some $\tilde{v}_2$ such that $\rho^*(v_1) < \tilde{v}_2 < 1$ and $\mathscr{U}_2(\rho^*(v_1);\rho^*,r_2) < \mathscr{U}_2(\tilde{v}_2;\rho^*,r_2) \leq \mathscr{U}_1(v_1;\rho^*,r_1)$, contradicting the fact that $\rho^*(v_1) = T\rho^*(v_1)$ is the highest such number. It follows that $v_2 \leq \rho^*(v_1)$ and $\mathscr{U}_2(v_2;\rho^*,r_2) \leq \mathscr{U}_2(\rho^*(v_1);\rho^*,r_2)$ (because $\mathscr{U}_2$ is increasing in v_2 from lemma A.1) and bidder 1 should visit seller 1 for sure. Second, suppose that $\mathscr{U}_2(\rho^*(v_1);\rho^*,r_2) = \mathscr{U}_1(v_1;\rho^*,r_1)$. If $v_1 \leq r_1$

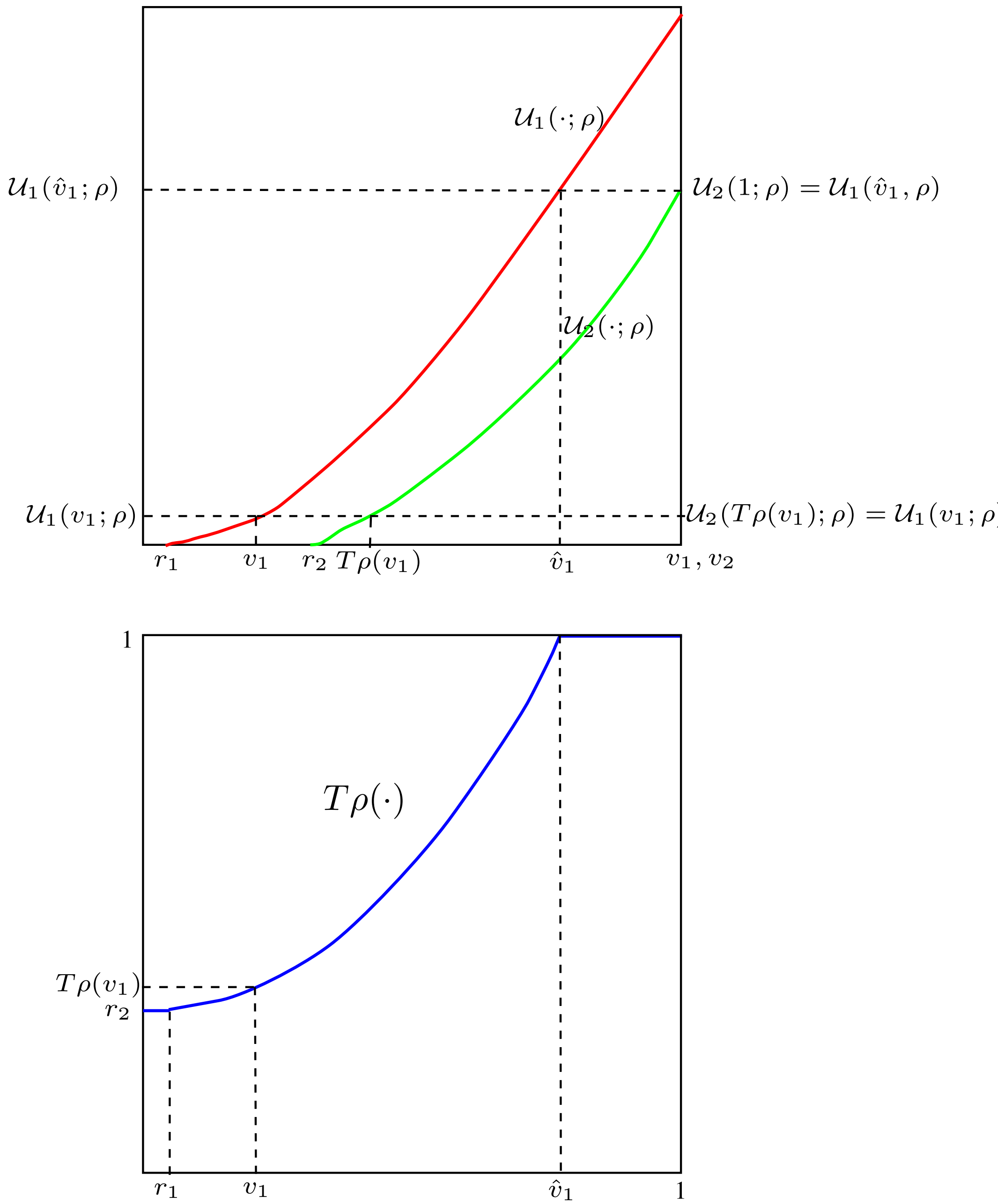

Figure 2.2: Bidder 1's Best Response Mapping

then $\rho^*(v_1) = r_2$ and it is (weakly) better for bidder 1 to visit seller 1 whenever $v_2 \leq \rho^*(v_1)$ and seller 2 for sure whenever $v_2 > \rho^*(v_1)$. If $v_1 > r_1$ then $\rho^*(v_1) > r_2$ and bidder 1 should visit seller 1 if and only if $v_2 \leq \rho^*(v_1)$ and seller 2 for sure otherwise. Overall, this means that the pure strategy ω^* must be bidder 1's best response to ω^* and thus, ω^* is a symmetric (Bayesian) equilibrium of the bidders' participation game.

The converse of the previous statement is also true. That is, the function associated to any equilibrium strategy must necessarily be a fixed point of T. To see why, suppose that a symmetric continuation equilibrium exists and let π^* be the strategy used by bidders in this equilibrium. From proposition 2.6, the set of best responses to π^* must contain a strategy ω^* that is characterized by a nondecreasing and continuous function ρ^* such that $\omega^*(v,r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and $\omega^*(v,r) = 0$ if and only if $v_2 > \rho^*(v_1)$. Since π^* is a symmetric equilibrium, the strategy ω^* must be a best response to itself. If this were not the case, we could construct a strategy ω' different from ω^* that yields a strictly higher payoff than strategy ω^* for some type of bidder 1. However, ω^* is a best response to π^* and hence, these two strategies must give the same payoff to every type of bidder 1. This means that the strategy ω' must yield a strictly higher payoff to this type of bidder 1 than the payoff associated to strategy π^*, contradicting the fact that π^* is a symmetric continuation equilibrium strategy. Since $\omega^*(v,r) = 1$ if and only if $v_2 \leq \rho^*(v_1)$ and ω^* is a best response to itself, the function ρ^* must necessarily satisfy $T\rho^*(v_1) = \rho^*(v_1)$ for all $v_1 \in [0,1]$, which implies that ρ^* is indeed a fixed point of T.

The previous discussion allows us to redirect questions about existence and uniqueness of a continuation equilibrium to questions about existence and uniqueness of a fixed point of the best response operator T. The next theorem establishes existence and uniqueness of such fixed point.

Theorem 4. *Let $v_0 > 0$ and consider the bidders' participation game following any history in which reserve prices are $(r_1, r_2) \in [v_0, 1]^2$. Then, there exists a unique continuous and nondecreasing function $\rho^* : [0,1] \to [0,1]$ such that $T\rho^* = \rho^*$. The* *function ρ^* is defined by:*

$$\rho^*(v_1) = \begin{cases} \min\{1, r_2\} & \text{if } \max\{r_1, r_2\} = 1 \\ \varphi^*(v_1) & \text{if } \max\{r_1, r_2\} < 1 \end{cases}$$

where,

$$\varphi^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ \min\{z(v_1); 1\} & \text{if } v_1 \geq r_1 \end{cases}$$

and the function z solves the following equation:

$$\frac{d}{dt}z(t) = \left(\frac{1 - \int_t^1 F(z(\tau))f(\tau)d\tau}{F(z(t))F(t) + \int_t^1 F(z(\tau))f(\tau)d\tau} \right)^{n-1} \qquad t \in [r_1, 1]$$

with initial condition $z(r_1) = r_2$.

Proof. In the appendix. □

The proof of the theorem makes extensive use of some properties of the best response operator that must hold true no matter what function other bidders use to select trading partners. For these properties to hold, it is not necessary that v_0 is strictly positive (they also hold if $v_0 = 0$) but we need a strictly positive v_0 in the course of proving existence and uniqueness of the function z for any arbitrary pair of reserve prices[7].

As an example of the properties of T that we exploit, take the case in which $r_1 = 1$. From Eq. (2.6), bidder 1's best response to any function ρ used by other bidders must be constant and equal to $\min\{1; r_2\}$. To see why, observe that the payoff that bidder 1 expects if she attends to auction 1 is nonpositive no matter what ρ or v_1 is. If $r_2 = 1$ then her expected payoff at auction 2 is also nonpositive and hence, bidder 1 should visit seller 1 with probability one (where she submits a non-serious bid). If $r_2 < 1$ then bidder 1 will select seller 2 with probability one whenever her valuation of item 2 is above r_2 regardless of the function ρ used by bidders other than bidder 1. Thus, $T\rho(v_1) \equiv r_2 = \min\{1; r_2\}$.

[7] Proposition 6 below shows how to extend Theorem 4 to continuation games in which $r_1 = r_2$ and $v_0 = 0$.

Perhaps, the most interesting property arises in cases where both reserve prices are strictly below one. As our previous discussion suggests, we can –at least in principle, find bidder 1's best response to ρ by equating the expected payoffs that bidder 1 would obtain when the other bidders use the function ρ to select trading partners. This idea works fine so long as the value of v_1 given ρ is not too high as to make impossible to find a value of v_2 such that payoffs are equal. Nonetheless, one would suspect that payoff should be equal at least within some subinterval of $[0,1]$. Part (ii) of the next lemma shows that this is indeed the case and it also summarizes some other useful properties of the best response operator T.

Lemma 5. *Suppose that $v_0 \in [0,1)$ and let $r_1 \in [v_0,1]$ and $r_2 \in [v_0,1]$ be any two reserve prices announced by sellers 1 and 2 respectively. Then, for any $\rho \in \mathscr{R}$:*

1. *If $\max\{r_1,r_2\} = 1$, then $T\rho(v_1) = \min\{1;r_2\}$ for all $v_1 \in [0,1]$;*
2. *If $\max\{r_1,r_2\} < 1$, then:*
   - *(i) $T\rho(v_1) = r_2$ for all $v_1 \leq r_1$;*
   - *(ii) there exists some $\bar{v}_1$ (that may depend on ρ) satisfying $r_1 < \bar{v}_1 \leq 1$ such that $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(T\rho(v_1);\rho,r_2)$ for all $v_1 \in [r_1,\bar{v}_1]$.*
   - *(iii) If $\bar{v}_1 < 1$ then $T\rho(v_1) = 1$ for all $v_1 \geq \bar{v}_1$.*

Proof. In the appendix. □

As already mentioned, part (ii) of lemma 5 is perhaps the most useful property of the best response operator that we use to show existence and uniqueness of a fixed point for T. To understand why this is so, recall our discussion about the proof of Proposition 2 for the case in which both reserve prices are strictly below one. The idea was to assign to every $v_1 \in [0,1]$ a number $v_2^* \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(v_2^*;\pi,r_2)$. Using the payoff functions given by Eq. (2.3) and (2.4), we can use a similar argument to show existence of a number $T\rho(v_1)$ such that $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(T\rho(v_1);\rho,r_2)$ regardless of whether $\mathscr{U}_1(1;\rho,r_1) \leq \mathscr{U}_2(1;\rho,r_2)$ or $\mathscr{U}_1(1;\rho,r_1) > \mathscr{U}_2(1;\rho,r_2)$. In the former case, we can assign a number between r_2 and one to every $v_1 \in [r_1,1]$ because the highest payoff that any bidder expects if bidding in auction 1 is never greater than the highest payoff that she expects in auction 2. In the latter case, we can repeat the above process but this time within some non–empty interval of the form $[r_1, \bar{v}_1]$, $r_1 < \bar{v}_1 \leq 1$. The value $\bar{v}_1$ may depend on the particular function ρ used by other bidders but it is not difficult to show that it must always lie strictly above r_1.

The above discussion suggests to use part (ii) of lemma 5 to construct a necessary condition for the best response operator in the form of a integro-differential equation that must hold everywhere with respect to $v_1 \in [r_1, \bar{v}_1]$,

$$\frac{dT\rho(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^{1} F(\rho(t)) f(t) dt}{F(T\rho(v_1)) F(\rho^{-1}(T\rho(v_1)) + \int_{\rho^{-1}(T\rho(v_1))}^{1} F(\rho(t)) f(t) dt} \right)^{n-1} \tag{2.7}$$

plus an initial condition $T\rho(r_1) = r_2$ that follows from part (i) of the lemma. The numerator and denominator of the right–hand–side of this last expression are the probabilities of trading with seller 1 and seller 2 respectively, for a type of bidder whose valuations are $(v_1, T\rho(v_1))$, when every of the remaining $(n-1)$ bidders use the function ρ to select trading partners. Since any fixed point of T must satisfy $T\rho^* = \rho^*$ for all $v_1 \in [0,1]$, the above equation gives a condition that we can exploit to find a fixed point of T. There are two technical difficulties with this approach. First, the interval within which the above equation holds true is endogenous. Second, Eq. (2.7) is an integro–differential equation and hence, it is not possible to directly apply any of the standard tools from the theory of differential equations to this problem. To overcome these difficulties we construct an auxiliary problem where we establish existence and uniqueness of a pair of functions that solve a system of two differential equations related to Eq. (2.7) that holds for all $v_1 \in [r_1, 1]$. In order for this auxiliary problem to possess a unique solution it is sufficient that v_0 be strictly positive. We then use the solution to this auxiliary problem to construct a unique function ρ^* and show that this function must be the unique fixed point of T.

A class of continuation games that will arise in chapters 3 and 4 of this dissertation and that is not covered by theorem 4 is the class of continuation games following histories in which $r_1 = r_2 = 0$. As the proof of theorem 4 shows in more detail, a $v_0 > 0$ is sufficient to make the denominator of the left-hand side of Eq. (2.7) well defined under any possible combination of reserve prices that sellers may choose. However, a positive value of v_0 is stronger than needed in cases where both reserve prices are equal. Intuitively, if $r_1 = r_2$ then sellers can be considered ex-ante identical so long as valuations are equal. Thus, we may guess that a bidder with valuations (v_1, v_2) should prefer to bid in auction 1 (resp. auction 2) whenever her valuation of item 1 (resp. item 2) is above her valuation of item 2 (resp. item 1) if this bidder expects everybody else to use this same participation strategy. This gives us $\rho^*(v) = v$, $v \in [0,1]$, as a candidate for a fixed point of T even if $r_1 = r_2 = 0$. The next result formalizes this intuition.

Proposition 6. *Consider any continuation game following a history in which $r_1 = r_2$, with $r_1 \in [0,1]$ and $r_2 \in [0,1]$. Then, the participation strategy:*

$$\pi(s,r) \quad = \quad \begin{cases} 1 & \text{if } v_1 \geq \rho^*(v_1) \\ 0 & \text{if } v_1 < \rho^*(v_1) \end{cases}$$

constitutes the unique symmetric equilibrium of this continuation game. The function $\rho^* : [0,1] \to [0,1]$ is defined by:

$$\rho^*(v_1) = \begin{cases} \min\{1; r_2\} & \text{if } \max\{r_1; r_2\} = 1 \\ \varphi^*(v_1) & \text{if } \max\{r_1; r_2\} < 1 \end{cases}$$

where:

$$\varphi^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ v_1 & \text{if } v_1 \geq r_1 \end{cases}$$

We outline the proof of the function $\rho^*(v_1) = v_1$ being a fixed point of T relegating the proof of uniqueness to the appendix. From lemma 5, the best response operator $T\rho^*$ must satisfies $T\rho^*(0) = 0$ and $\mathscr{U}_2(T\rho^*(v_1); \rho^*, r) = \mathscr{U}_1(v_1; \rho^*, r)$ for $v_1 \in [0, \bar{v}]$ where $\bar{v}$ is implicitly defined by $\mathscr{U}_1(\bar{v}; \rho^*, r) = \mathscr{U}_2(1; \rho^*, r)$. As previously mentioned, these two properties of the best response operator hold true even if v_0 is equal to zero. Using Eq. (2.3) and (2.4) we obtain the following payoff functions when bidders other than bidder 1 use this function ρ^* to select trading partners:

$$\mathscr{U}_1(v_1; \rho^*, r) = \int_0^{v_1} \left(\frac{1}{2} + \frac{F^2(t)}{2} \right)^{n-1} dt$$

and,

$$\mathscr{U}_1(v_2;\rho^*,r) = \int_0^{v_2} \left(\frac{1}{2} + \frac{F^2(t)}{2}\right)^{n-1} dt$$

Hence, $\bar{v} = 1$ and $\mathscr{U}_1(v_1;\rho^*,r) = \mathscr{U}_2(v_2;\rho^*,r)$ if and only if $v_1 = v_2$. It follows that:

$$\begin{aligned} T\rho^*(v_1) &= \max\{v_2 : \mathscr{U}_2(v_2;\rho^*,r) \leq \mathscr{U}_1(v_1;\rho^*,r)\} \\ &= v_1 \\ &= \rho^*(v_1) \end{aligned}$$

for every $v_1 \in [0,1]$, which shows that $\rho^*(v_1) = v_1$, $v_1 \in [0,1]$, is a fixed point of T when $r_1 = r_2$ and $v_0 \geq 0$.

2.3.1 Heterogeneity as a Coordination Device

We end this section with some discussion about the role played by heterogeneity in the selection of trading partners, and how this selection rule is affected by changes in reserve prices. Observe that apart from providing a complete and novel characterization of the set of (symmetric) participation rules, Theorem 4 also rules out randomization in the selection of trading partners. Indeed, theorem 4 shows that the set of types who wish to randomize between sellers (those lying on the cutoff function) must have zero measure. Perhaps more importantly, this lack of randomization is a property that must hold true in every symmetric continuation equilibrium of the game. This differentiates our model from current models in the literature where bidders always randomize in their choices of trading partners[8], which in turn introduces frictions in the market due to a coordination failure. Contrary to this, the introduction of heterogeneity eliminates this market friction by coordinating the visiting decisions of bidders. In this sense, heterogeneity acts as a device that rules out market frictions due to coordination problems in the selection of trading partners.

A second issue closely related to the previous one is the way in which changes

[8] These models also admit continuation equilibria in which bidders choose trading partners using pure strategies. However, such continuation equilibria require some sort of sunspot that allows bidders to coordinate on their visiting decisions.

in some reserve price affect bidders' visiting decisions. The literature on competing auctions (Peters and Severinov, 1997; Burguet and Sakovics, 1999; Virag, 2010) has shown that in the case of homogeneous goods, a change in some seller's reserve price changes the set of types that visits an auction but it does not change the probability with which each bidder participates. Thus, a higher reserve price has the effect of shutting down the participation of those bidders whose valuations are close to the reserve price, making less likely that bidders with high valuations face an opponent. As Peters (2010) points out, this means that sellers who compete in auctions do not compete directly for the high valuation bidders since only low valuation types alter their behavior in response to changes in reserve prices. This is no longer true when items are assumed to be heterogeneous. As shown by theorems 3 and 4, bidders use functions to select trading partners and hence, changes in some reserve price will have an effect on the participation decisions of the whole set of types. In particular, some bidders with high valuations will also respond by shifting from the high-reserve to the low–reserve price auction. This additional shift in the number of bidders who now find profitable to attend to the low–reserve price auction will reduce expected traffic, adding a new channel through which reserve prices affect sellers profits.

Proposition 7. *Take any two distinct pair of reserve prices* (r_1, r_2) *and* $(\hat{r}_1, r_2)$*, with* $r_1 < \hat{r}_1$*. Let* $\rho(\cdot;(r_1, r_2))$ *and* $\rho(\cdot;(\hat{r}_1, r_2))$ *be the equilibrium functions used by bidders to select trading partners when reserve prices are* (r_1, r_2) *and* $(\hat{r}_1, r_2)$ *respectively. Then,* $\rho(v_1;(r_1, r_2)) \geq \rho(v_1;(\hat{r}_1, r_2))$ *for every* $v_1 \in [0,1]$*. Furthermore, if* $r_2 < 1$ *then there exists a nonempty interval* $\Omega \subseteq [0,1]$ *such that* $\rho(v_1;(r_1, r_2)) > \rho(v_1;(\hat{r}_1, r_2))$ *for all* $v_1 \in \Omega$*.*

Proof. In the appendix. □

2.4 The Sellers' Game

The existence of a function ρ^* that can be associated to any equilibrium in the bidders' continuation game makes it possible to describe sellers' reduced form payoffs only in terms of ρ^*. To stress the fact that this function depends on the reserve prices announced by the sellers, we will explicitly write $\rho^*(v_1;(r_1, r_2))$ to indicate the function used by bidders to select trading partners in the continuation game following a history in which reserve prices are r_1 and r_2 respectively.

A strategy for seller j is his choice of reserve price $r_j \in [v_0, 1]$. Suppose that sellers announce some pair of reserve prices $r_1 \in [v_0, 1]$ and $r_2 \in [v_0, 1]$ respectively, and that bidders' choice of trading partners is described by the continuation equilibrium strategy ω^* characterized by the function ρ^* . From the perspective of seller 1, if he announces a reserve price equal to one then every bidder whose valuation of item 2 is above r_2 will visit seller 2 for sure and any bidder whose valuation is below r_2 will come to his auction where she will submit a non-serious bid. This implies that the expected payoff of seller 1 when he posts a reserve price of one must be equal to v_0, the value that seller 1 attaches to his own item in case he does not sell it. In all other cases (i.e., in all cases where seller 1 posts a reserve price strictly lower than one), seller 1's expected payoff can be calculated as the sum of three terms: (i) v_0 times the probability that he receives no visitors[9]; (ii) the payoff that seller 1 expects when a single bidder visits his auction; and (iii) the payoff that he expects when two or more bidders visit his auction. Thus, all that we need to know to compute each of these three terms is the probability with which any given bidder visits his auction, and the expected value of the second highest type of those bidders who chooses to visit the auction, whenever two or more bidders visit.

Let $G_1(v_1; \rho^*, r)$ denote the probability that a bidder with valuation v_1 trades with seller 1 when the vector of reserve prices is $r = (r_1, r_2) \in [v_0, 1]^2$ and the function to select trading partners is ρ^*. In what follows and provided that there is no risk of confusion, we will sometimes write $G_1(v_1)$ instead of $G_1(v_1; \rho^*, r)$. From theorem 4, this probability can be computed as follows:

$$G_1(v_1) = \left[1 - \int_{v_1}^{1} F(\rho^*(t_1; r))dF(t_1)\right]$$

As the item offered by seller 1 will be left unsold when every bidder either goes to seller 2's or comes to seller 1 as a non–serious bidder, seller 1's payoff in case

[9] Since bidders who do not want to participate in any auction are treated as visitors to auction 1 where they submit non–serious bids, the probability that seller 1 has no visitors is equivalent to the probability that seller 1 only receives non-serious bidders.

that he does not sell his item is:

$$R_1^0(r_1, r_2; \rho^*) = v_0 G_1^n(r_1)$$

Similarly, seller 1's payoff in case he receives a single serious bid is:

$$R_1^1(r_1, r_2; \rho^*) = n r_1 G_1^{n-1}(r_1)(1 - G_1(r_1))$$

and the expected payoff when two or more bidders bids in auction 1 is:

$$R_1^{2^+}(r_1, r_2; \rho^*) = n(n-1) \int_{r_1}^{1} t_1 \left[1 - G_1(t_1)\right] \left[G_1(t_1)\right]^{n-2} dG_1(t_1)$$

Consequently, seller 1's payoff function can be written as follows:

$$R_1(r_1, r_2; \rho^*) =$$

$$\begin{cases} v_0 & \text{if } r_1 = 1 \\ R_1^0(r_1, r_2; \rho^*) + R_1^1(r_1, r_2; \rho^*) + R_1^{2^+}(r_1, r_2; \rho^*) & \text{if } v_0 \leq r_1 < 1 \end{cases}$$

Seller 2's payoff function can be derived likewise. Let $G_2(t_2; \rho^*, r)$ (or simply $G_2(t_2)$ if there is no risk of confusion) be the probability that a bidder with valuation t_2 trades with seller 2:

$$\begin{aligned} G_2(t_2) &= \left[F(t_2) F(\rho^{*-1}(t_2; r)) + \int_{\rho^{*-1}(t_2; r)}^{1} F(\rho^*(\tau; r)) f(\tau) d\tau\right] \\ &= \left[1 - \int_{t_2}^{1} F(\rho^{*-1}(\tau; r)) f(\tau) d\tau\right] \end{aligned}$$

where:

$$\rho^{*-1}(t_2; r) = \begin{cases} 0 & \text{if } t_2 < r_2 \\ \max\{s \in [0,1] : t_2 \geq \rho^*(s; r)\} & \text{if } t_2 \geq r_2 \end{cases}$$

Then, seller 2's payoff function is:

$$R_2(r_1, r_2; \rho^*) =$$

$$\begin{cases} v_0 & \text{if } r_2 = 1 \\ R_2^0(r_1,r_2;\rho^*) + R_2^1(r_1,r_2;\rho^*) + R_2^{2^+}(r_1,r_2;\rho^*) & \text{if } v_0 \le r_2 < 1 \end{cases}$$

where:

$$\begin{aligned} R_2^0(r_1,r_2;\rho^*) &= v_0 G_2^n(r_2) \\ R_2^1(r_1,r_2;\rho^*) &= n r_2 G_2^{n-1}(r_2)(1-G_2(r_2)) \\ R_2^{2^+}(r_1,r_2;\rho^*) &= n(n-1)\int_{r_2}^1 t_2\,[1-G_2(t_2)]\,[G_2(t_2)]^{n-2}\,dG_2(t_2) \end{aligned}$$

We now argue that $R_1(r_1,r_2;\rho^*)$ is a continuous function of r_1 on $[v_0,1]$. Note that this claim requires a careful proof because ρ^* is the solution of a integro–differential equation and thus, it is not immediate that standard tools from the theory of differential equations (such as Grownwall's inequality) can be applied. The proof of proposition 8 contains a formal proof of the continuity of $R_1(r_1,r_2;\rho^*)$ and $R_2(r_1,r_2;\rho^*)$ with respect to r_1 and r_2 respectively.

When seller 1 slightly changes r_1 he induces a direct effect on R_1 produced by the direct change in r_1, and an indirect effect triggered by the change induced by r_1 on the function used by bidders to select trading partners. Since the 'shape' of this function also depends on what reserve price seller 2 has decided to post, we consider two cases. First, suppose that seller 2 sets a reserve prices of one. From theorem 4 bidders select trading partners using the function $\rho^*(v_1) \equiv 1$ independent of what reserve price seller 1 sets. Then, $G_1(t_1;\rho^*,r) = F(t_1)$ for every $t \in [r_1,1]$ and every $r_1 \in [v_0,1]$. Furthermore, as $F(r_1)$ tends to one with $r_1 \to 1$, $R_1^0(r_1,1;\rho^*) + R_1^1(r_1,1;\rho^*) + R_1^{2^+}(r_1,1;\rho^*)$ must tend to $v_0 = R_1(1,1;\rho^*)$ and hence, $R_1(r_1,1;\rho^*)$ must be continuous in r_1 on $[v_0,1]$ when $r_2 = 1$.

Second, consider the case in which $r_2 < 1$. Let r_1 approach one from the left, i.e., for any $\varepsilon > 0$ let $\lambda = \varepsilon F^2(v_0)$ and consider any r_1 satisfying $0 < 1 - r_1 < \lambda$. Since $r_1 < 1$, lemma 27 implies the existence of a continuous and increasing function z^* that satisfies:

$$z^*(v_1;r) = r_2 + \int_{r_1}^{v_1} \frac{dz^*(t;r)}{dt}dt$$

for $v_1 \in [r_1,1]$, and $z^*(r_1) = r_2$, such that $\rho^*(v_1;r) = \min\{z^*(v_1;r);1\}$ when $v_1 \ge r_1$,

$r = (r_1, r_2)$. Since $z^*(\cdot; r)$ is defined for all $v_1 \in [r_1, 1]$,

$$\begin{aligned}
\sup_{v_1 \in [r_1,1]} |z^*(v_1, r) - r_2| &= \sup_{v_1 \in [r_1,1]} \left| \int_{r_1}^{v_1} \frac{dz^*(t;r)}{dt} dt \right| \\
&\leq \sup_{v_1 \in [r_1,1]} \int_{r_1}^{v_1} \left| \frac{dz^*(t;r)}{dt} \right| dt \\
&\leq (1 - r_1) \frac{1}{F^2(v_0)} \\
&< \frac{\lambda}{F^2(v_0)} \\
&= \varepsilon
\end{aligned}$$

and $z^*(v_1; r)$ approaches the constant function r_2 uniformly as r_1 approaches one from the left. Thus, $\min\{z^*(v_1), 1\}$ becomes arbitrarily close to r_2 as $r_1 \to 1$ and hence, $\rho^* \to r_2$ as $r_1 \to 1$. Consequently,

$$\begin{aligned}
R_1^0(r_1, 1; \rho^*) &\longrightarrow v_0 \\
R_1^1(r_1, 1; \rho^*) &\longrightarrow 0 \\
R_1^{2^+}(r_1, 1; \rho^*) &\longrightarrow 0
\end{aligned}$$

as $r_1 \to 1$, which means that $\lim_{r_1^- \to 1} R_1(r_1, r_2; \rho^*) = v_0 = R_1(1, r_2; \rho^*)$ and $R_1(r_1, r_2; \rho^*)$ is (left) continuous at $r_1 = 1$ whenever $r_2 < 1$.

Finally, let $r_2 < 1$ and take any $r_1 \in [v_0, 1)$. From part (1) of lemma 34 (in the appendix), for every $\varepsilon > 0$ we can find some $\lambda > 0$ such that $|r_1 - r_1'| < \lambda$ implies $\sup_{v_1 \in [0,1]} |\rho(v_1, r) - \rho(v_1, r')| < \varepsilon$. Since F is continuous and the integral sign preserves continuity, $G_1(t_1)$ must vary continuously with r_1 on $[v_0, 1)$ because it does not directly depend on r_1 and it depends continuously on ρ. We conclude that $R_1(r_1, r_2; \rho^*)$ is a continuous function of r_1 on $[v_0, 1)$ and hence, on the close interval $[v_0, 1]$. Similarly, we can show that $R_2(r_1, r_2; \rho^*)$ is a continuous function of r_2 on $[v_0, 1]$. Therefore, by Glicksberg's theorem (Theorem 3 in Dasgupta and Maskin, 1986) the whole game must admit an equilibrium in which bidders use symmetric strategies.

Proposition 8. *The competing auction game with heterogeneous goods admits a*

Perfect Bayesian equilibrium in which bidders follow symmetric strategies.

Proof. In the appendix. □

2.5 Concluding Remarks

This chapter develops a model of competing auctions in which bidders' preferences are assumed to be heterogeneous. I provide a complete and novel characterization of the participation rules used by bidders in any symmetric equilibrium in terms of a nondecreasing and continuous function ρ with the property that a bidder with valuations (v_1, v_2) visits seller 1 if and only if $v_2 \leq \rho(v_1)$, and visits seller 2 with probability one if and only if $v_2 > \rho(v_1)$. This characterization ensures that if the game possesses a continuation equilibrium then it must also posses an equilibrium in which bidders use pure strategies. Thus, heterogeneity in bidders' preferences rules out any randomization and it eliminates coordination failures as a source of friction in the market.

Another interesting result is the effect that a change in some reserve price causes on the participation decisions of bidders. Unilateral changes in reserve prices affect not only the types who visit each auction but also the probability with which they visit. In particular, a change in a reserve price affects the participation decisions of bidders with high valuations who were indifferent before the change in the reserve price took place, which adds a novel trade–off between traffic and screening effects not present in models with homogeneous goods.

We can consider an extension of the current model to a competing auction model with more than two sellers. As the existence of a function that characterizes bidders' participation decision does not appear to hinge on the existence of just two sellers, an outstanding conjecture is that such characterization would still be available if the number of sellers is augmented. Moreover, with an arbitrary number of sellers other issues in the literature such as convergence of reserve prices could be analyzed. Explicitly verifying this conjecture is the direction for future work.

Chapter 3

Competing through Information in Auctions

3.1 Introduction

There are many situations in which sellers have the ability to control the amount of information available to potential buyers. Online auctions are a good example of this since sellers participating in these sites have the option of posting pictures and adding descriptions of the objects they wish to sell. A casual look at these sites reveals that while some sellers provide very detailed descriptions others simply post very basic information omitting details that may be of interest to some bidders. In this chapter I develop a simple model aimed at investigating the incentives of competing auctioneers who can use information to attract potential buyers.

The model I develop captures in a simple way the relationship between competition and information provision. There are only two sellers with unit supply who compete for the unit demands of $n \geq 2$ potential bidders. Sellers can release information before bidders decide which auction they want to attend to. I assume that information structures are binary: sellers can only choose between letting bidders learn their valuations or leaving them uninformed. When a seller chooses to release information, bidders privately learn their true valuation before selecting trading partners, which makes participation decisions depend on bidder's private information.

The main contribution of this chapter is to provide conditions under which there exists a unique equilibrium where both sellers supply information. These conditions translates into restrictions on the distribution of bidders valuations. When sufficient probability mass is allocated to the right of the average valuation there can not exist an equilibrium in which sellers do not provide information. Intuitively, more mass allocated to the right of the average valuation makes more likely that bidders draw valuations above this value, making price above average valuation a more likely event. However, as price is computed conditional on participation it is plausible that releasing information hurts the deviating seller by decreasing expected traffic. I show that if the distribution of valuations is monotone increasing and convex then traffic is also favored by information provision and hence, the unique equilibrium of the game is one in which both sellers release all available information.

The rest of the chapter is organized as follows. Section 3.2 outlines the model. Section 3.3 characterizes bidders' participation rules used to select trading partners. Section 3.4 describes the sellers' game induced by the continuation equilibrium found in section 3.3 and presents the main results of the chapter. Section 3.7 concludes with some final comments and conclusions.

3.2 The Model

Consider an economy in which trade takes place using second-price sealed bid auctions. The economy is populated by two risk-neutral sellers (seller 1 and seller 2) with unit supply, and $n \geq 2$ risk neutral bidders with unit demands indexed by $i \in \{1, \cdots, n\}$. Sellers attach no value to either item. Before any interaction with sellers takes place, each bidder privately observes the realization $(s_1, s_2) \in [0,1]^2$ of a pair of identically distributed random variables (S_1, S_2) with common distribution functions F and support $[0,1]$. We assume that F is at least twice continuously differentiable and has a strictly positive and bounded density function $f > 0$.

Each bidder is unsure about how exactly these signals translate into valuations. Sellers can help reduce this uncertainty by supplying information about the characteristics of their respective items. Before the auction takes place, each seller independently and simultaneously decides on whether to reveal or not information to potential bidders. That is, a given seller can choose between granting full access to information about his[10] product or limiting any access to it. In the former case,

[10]Similar to chapter 2, we use male pronouns to refer to sellers and female pronouns to refer to bidders.

each bidder –using her signals, is able to perfectly infer the value that she attaches to this item whereas in the latter case (i.e., if a seller chooses not to supply information) she remains uncertain about it. Let $p_j = 0$ if seller j chooses not to inform bidders and $p_j = 1$ if he chooses to do so. Then, bidder i's posterior valuation of item j when her signals is s_j and seller j's choice is p_j becomes:

$$\omega_{ij}(s_{ij}, p_j) = \begin{cases} s_{ij} & \text{if } p_j = 1 \\ \mu & \text{if } p_j = 0 \end{cases}$$

In words, when seller j chooses to supply information each bidder i privately learns that her valuation is equal to her signal s_{ij}. Alternatively, when seller j chooses not to provide information then bidder i considers her signal as an indistinguishable and independent draw from the distribution F and therefore, she forms an estimate $\mu = \int_0^1 sdF(s)$ about her valuation of item j. We can interpret μ as the best estimate bidders can make based solely on public information contained in the distribution function F. A bidder i with valuation v_{ij} who trades with seller j at price p gets a surplus $v_{ij} - p_j$, while seller j gets a surplus p_j. In case there is no trade both bidder and seller receive an exogenous payoff of zero.

The timing of events is as follows. At the beginning of the game Nature privately communicates to each bidder i a pair of signals $(s_{i,1}, s_{i,2}) \in [0,1]^2$. Without observing any of these signals each seller j independently and simultaneously decides whether to inform or not bidders by choosing some $p_j \in \{0,1\}$. After sellers have chosen some pair (p_1, p_2), each bidder i simultaneously and independently decides whether to participate in some auction and which auction she wishes to do so. We restrict each bidder to choose one and only one seller as her trading partner. This is a commonly used assumption in the literature of competing auctions, and it is made in order to keep tractability of the model. After bidders have assigned themselves into the different auctions, each seller collects the bids and awards the good using a second price sealed-bid auction without reserve price[11]. Finally, payoff are realized and the game ends.

[11]Normalizing reserve prices to be equal to zero is with loss of generality and it is made largely due to its convenience and because it allows us to more easily compare our results with those in the existing literature.

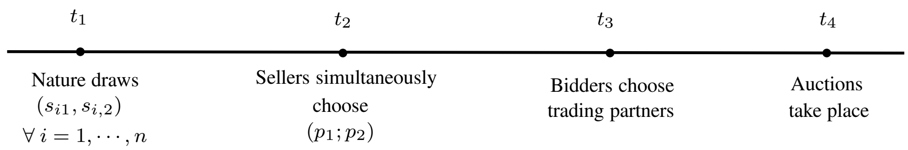

Figure 3.1: Timing of the Game

3.3 Bidders' Participation Game

By the time bidder i must submit a bid (say in auction j), she already knows whether her signal reflects her true valuation of that item or not. Therefore, conditional on participation, bidding s_{ij} when $p_j = 1$ and μ when $p_j = 0$ for every $i = 1, \ldots, n$ is a weakly dominant strategy for every bidder. Similar to what we did in chapter 2, we *assume* that bidding is truthful. Consequently, a strategy for bidder i is a mapping $\pi_i : \{0,1\}^2 \times [0,1]^2 \longrightarrow [0,1]$, where $\pi_i(p,s)$ specifies the probability with which bidder i visits seller 1 as a function of the possible choices of sellers' information structures $p \in \{0,1\}^2$, and bidder i's signals $s_i \in [0,1]^2$. We also adhere to the convention to treat the decision not to bid in any auction as equivalent to the decision to submit a non serious bid in auction 1. Thus, if $\pi(p,s)$ stands for the probability that a bidder who has observed the pair of signals $s = (s_1, s_2)$, and sellers' choices of information structures $p = (p_1, p_2)$, then $1 - \pi(p,s)$ is the probability that this bidder visits seller 2.

We focus on the existence of (Perfect Bayesian) equilibria in which bidders use *symmetric participation rules.* A participation rule is symmetric if two bidders with the same vector of estimates visit seller 1 (resp. seller 2) with the same probability, $\pi_i(\cdot) = \pi_k(\cdot) \equiv \pi(\cdot)$, $i \neq k \in \{1,2\}$.

Given the binary nature of information structures, the analysis of bidders' participation decisions can be decomposed into three types of continuation games: (i) both sellers announce uninformative structures $(p_1 = p_2 = 0)$; (ii) only one seller announces an uninformative structure while the other announces a perfectly informative one $(p_j = 1; p_{-j} = 0)$; and (iii) both sellers announce perfectly informative structures $(p_1 = p_2 = 1)$.

We begin with case (i). Suppose that bidders use the participation rule $\pi(s,p)$ to select trading partners. Since sellers choose $p_1 = p_2 = 0$ then the best estimate that each bidder i can form about v_{ij} must equal the average valuation μ. Therefore, bidder 1 makes a positive payoff if only if she is the only bidder participating in the auction (in the event that two or more bidders visit the price will equal μ). The probability that any given bidder chooses to visit seller 1 is:

$$q = \int_0^1 \int_0^1 \pi((s_1, s_2); (0,0)) dF(s_1) dF(s_2)$$

which gives us the expected payoffs of any participant in auction 1 and auction 2 as:

$$\begin{aligned} \mathscr{U}_j(q) &= \max\left\{0; \mu(1-q)^{n-1}\right\} \\ \mathscr{U}_2(q) &= \max\left\{0; \mu q^{n-1}\right\} \end{aligned}$$

It is not difficult to check that the value of q can be pinned down by solving the following equation:

$$\mu(1-q)^{n-1} = \mu q^{n-1} \tag{3.1}$$

from where we obtain $q = 1/2$.

Case (ii) is slightly more involved. For simplicity, consider the case in which seller 1 chooses a perfectly informative structure and seller 2 chooses an uninformative one (the case in which seller 2 chooses $p_2 = 1$ and seller 1 chooses $p_1 = 0$ can be dealt with similarly). In this framework, signals related to item 1 can be used to obtain a perfect estimate of bidder's true valuation of item 1 since $v_{i1} = s_{i1}$ for all $i = 1, \ldots, n$. Alternatively, signals related to item 2 are pure i.i.d noise coming from the distribution F and hence, $v_{i2} = \mu \ \forall i$. To obtain the payoff that a bidder expects to receive in each auction, we use the reduced-form approach employed in chapter 2 and write these payoffs as the difference between bidder's valuation times the probability that she trades with the seller minus the price she expects to pay. Let $Q_1(s_1)$ be the reduced form probability of trading with seller 1 when bidder's valuation is s_1. Following McAfee (1993), $Q_1(s_1)$ can be written as:

$$Q_1(s_1) = 1 - \int_{s_1}^1 \int_0^1 \pi((t_1, t_2); (0,0)) dF(t_2) dF(t_1)$$

where $\pi(s,p)$ corresponds to the probability with which any bidder with signals (s_1,s_2) visits seller 1. Therefore, bidder 1's expected payoff in auction 1 when $p_1=1$, $p_2=0$ can be written as:

$$\mathscr{U}_1(s_1)=\max\left\{0;\mathscr{U}_1(0)+\int_0^{s_1}Q_1(t)dt\right\}$$

and bidder 1's expected payoff if she visits seller 2 is:

$$\mathscr{U}_2 \quad = \max \quad \left\{0;\mu\int_0^1\int_0^1\pi((s_1,s_2);(0,0))dF(s_2)dF(s_1)\right\}$$

since this bidder wins item 2 if and only if she is alone at this auction because all bidders who come to auction 2 do so with the same valuation μ. As the type with signal $s_1=0$ expects a payoff equal to zero in auction 1, $\mathscr{U}_1(0)=0$ and hence,

$$\mathscr{U}_1(s_1)=\int_0^{s_1}Q_1(t)dt$$

It is fairly clear that so long as bidder 1's signal is such that $\mathscr{U}_1(s_1)$ is greater than $\mathscr{U}_2$ then this type of bidder 1 should bid in auction 1 whereas the opposite should hold true when $\mathscr{U}_1(s_1)<\mathscr{U}_2$. We can use the same logic used in the derivation of the continuation equilibrium of the model in chapter 2 to argue that we can always find a best response strategy characterized by some cutoff *value* s^* such that bidders with valuations of item 1 above s^* goes to seller 1 with probability one, while bidders with valuation below s^* visit seller 2 with probability one. Thus, our problem reduces to finding a value of s^*such that the associated strategy just described is a best response to itself.

Suppose that any bidder other than bidder 1 uses a cutoff strategy with cutoff point s^*. A type of bidder 1 with signal $s_1=s^*$ expects a positive payoff in auction 1 only if she is the unique participant in this auction. As the probability that any other bidder does not visit this same seller is equal to $F(s^*)^{n-1}$, the expected payoff of type s^* of bidder 1 in auction 1 must be:

$$\mathscr{U}_1(s^*)=s^*F(s^*)^{n-1}$$

Alternatively, if this type bids in auction 2, her expected payoff will be equal to:

$$\mathscr{U}_2 \quad = \quad \mu(1-F(s^*))^{n-1}$$

Therefore, bidder 1 with valuation s^* of item 1 is indifferent between auction 1 and auction 2 whenever her expected payoffs are equal, i.e., whenever the following condition holds true;

$$s^*F(s^*)^{n-1} = \mu(1-F(s^*))^{n-1} \tag{3.2}$$

Lemma 9. *There exists a unique value $s^* \in (0,1)$ that solves Eq.* (3.2)*.*

To prove the lemma let $\varphi(s) := sF(s)^{n-1} - \mu(1-F(s))^{n-1}$. It is immediate that φ is a continuous function of $s \in [0,1]$. Moreover, $\varphi(0) = -\mu < 0$ and $\varphi(1) = 1 > 0$ because $F(0) = 0$ and $F(1) = 1$. Therefore, there must exist some $s^* \in (0,1)$ such that $\varphi(s^*) = 0$. Uniqueness of s^* follows from $\frac{d}{dz}\varphi(s) > 0$ for all $s \in (0,1)$.

It is straightforward to check that bidder 1 will find optimal to visit seller 1 (resp. seller 2) whenever her valuation of item 1 is greater (resp. lower) than s^* because $\mathscr{U}_1(s_1) \geq \mathscr{U}_1(s^*)$ whenever $s_1 \geq s^*$ and $\mathscr{U}_1(s) < \mathscr{U}_1(s^*) = \mathscr{U}_2$ whenever $s_1 < s^*$[12].

The only remaining case is that in which both sellers choose to supply information. Since seller 1 and seller 2 choose $p_1 = p_2 = 1$, bidders are perfectly informed about their true valuations of each item before they decide on which auction they want to submit their bids. Thus, the problem of selecting trading partners is identical to the problem faced by bidders participating in a competing auction model with heterogeneous goods with reserve prices normalized to zero. The next lemma (whose proof is identical to the proof of proposition 6 in chapter 2) ensures existence and uniqueness of a continuation equilibrium in this case.

Lemma 10. *Consider the continuation game following a history in which sellers choose $p_1 = p_2 = 1$. Then, in the unique symmetric continuation equilibrium bidders use a strategy characterized by a nondecreasing and continuous function*

[12]This follows from $\mathscr{U}_1$ being a continuous and increasing function of s_1 and Lemma 2 in Myerson (1981).

$\rho^* : [0,1] \to [0,1]$ *such that bidder with valuations* (v_1, v_2) *visits seller 1 with probability one if and only if* $v_1 \geq \rho^*(v_2)$*, and visits seller 2 with probability one if and only if* $v_1 < \rho^*(v_2)$*. The function* ρ^* *is such that* $\rho^*(v_1) = v_2$ *for all* $v_1 \in [0,1]$.

3.4 Equilibria without Provision of Information

The continuation equilibrium described in the previous section determines a normal form game between the sellers in which the action space and payoff for each auctioneer is $\{0,1\}$ and:

$$R_j(p_j, p_j) = \sum_{k=2}^{n} \binom{n}{k} q_j^k (1-q_j)^{n-k} T_{jk}(p_j, p_{-j}) \tag{3.3}$$

$j = 1,2$, where $p_j \in \{0,1\}$ and $p_{-j} \in \{0,1\}$ represents the choice of information structures made by seller j and seller $-j$ respectively, $q_j := q_j(p)$, $p = (p_j, p_{-j})$, is the probability that a given bidder visits seller j, and $T_{jk}(p_j, p_{-j})$ is the price seller j expects to receive when he is matched with exactly k bidders given sellers' choices of information structures. Of course, the specific form of this payoff function depends on sellers' choices of information structures since these choices affect the expected traffic (through their effect on q_j) and the expected price.

From the previous section, we know that any continuation game following a history in which $p_1 = p_2 = 0$ will have bidders choosing trading partners with probability a half, just as Eq. (3.1) indicates. Moreover, as both sellers choose uninformative structures, posterior valuations of both items are equal to μ. Hence, Eq. (3.3) becomes:

$$\begin{aligned} R_j(0,0) &= \sum_{k=2}^{n} \binom{n}{k} q_j^k (1-q_j)^{n-k} \mu \\ &= \left[1 - (1+n)\left(\frac{1}{2}\right)^n\right] \mu \end{aligned} \tag{3.4}$$

The payoff when seller 1 chooses to supply information while seller 2 does not can be derived likewise. From lemma 9, in any continuation game following a history in which $p_1 = 1$ and $p_2 = 0$ bidders choose trading partners based on whether their privately known valuation of item 1 is above or below certain common threshold value defined by Eq. (3.2). This gives the probability with which a bidder visits seller 1 as $1 - F(s^*)$, and the probability with which a bidder visits seller 2 as $F(s^*)$. Furthermore, seller 2's expected price $T_{2k}(1,0)$ must equal μ for all $k = 2, \ldots, n$ because seller 2 does not supply information, whereas $T_{1k}(1,0)$ is equal to the expected value of the second-order statistic of the distribution function of types when exactly k bidders choose to visit seller 1. Let $H_1(s; s^*, k)$ be the distribution of the second order statistic conditional on being exactly k visitors, $k \geq 2$, at seller 1's auction. As only bidders with valuations above s^* visit this auction, the distribution of types conditional on visiting is given by the truncation of F from below with truncation point s^*. Therefore, Eq. (3.3) when $p_1 = 1$ and $p_2 = 0$ becomes:

$$R_1(1,0) = \sum_{k=2}^{n} \binom{n}{k} [1 - F(s^*)]^k [F(s^*)]^{n-k} \left(\int_{s^*}^{1} s dH_1(s; s^*, k) \right) \tag{3.5}$$

Clearly, the most important element in Eq. (3.5) is the cutoff point s^*. Intuitively, the location of this cutoff value depend on the shape of the distribution function F. More precisely, this value should vary with the probability mass allocated around the average valuation μ. The reason for this is as follows. When more mass is allocated to the right of μ, bidders are more likely to draw valuations that are greater than μ. Since relatively high valuations make more attractive to visit auction 1, the valuation of the indifferent type should be higher to counterbalance this extra traffic. This has a nice implication for seller 1's payoff. Since a cutoff higher than μ means that conditional on visiting bidders do so with valuations that are never less than μ, the price expected by seller 1 can never be lower than μ. The next lemma makes precise this intuition by showing the exact relationship between the value of s^* and the shape of F.

Lemma 11. *Assume that $p_1 = 1$ and $p_2 = 0$, and let s^* be the unique solution to Eq.* (3.2)*. Let m be the median of F, i.e., the value that satisfies $F(m) = 1/2$. Then, s^* satisfies $\mu \leq s^* \leq m$ if and only if $F(\mu) \leq \frac{1}{2}$, and $m < s^* < \mu$ if and only if $F(\mu) > \frac{1}{2}$.*

Proof. In the appendix. □

The importance of this lemma is to provide sufficient conditions under which there cannot be an equilibrium without provision of information.

Proposition 12. *Let m and μ be the median and mean of F respectively. If $m \geq \mu$ then there is no equilibrium in which both sellers choose uninformative structures.*

To prove this proposition, first notice that if $\mu \leq m$ then $F(\mu) \leq 1/2$ because F is an increasing function of s. Therefore, Lemma 11 allows us to conclude that the cutoff value must satisfy $\mu \leq s^* \leq m$, which in turn implies $F(\mu) \leq F(s^*) \leq F(m) = 1/2$. This gives us a probability of visiting seller 1, $1 - F(s^*)$, that is weakly greater than $1/2$. On the other hand, if seller 1 were to choose $p_1 = 0$ while seller 2 kept his choice at $p_2 = 0$, seller 1 would be visited by any bidder with probability $1/2$. This means that having $F(\mu) \leq 1/2$ is enough to guarantee that traffic is never lowered with seller 1's choice $p_1 = 1$. Furthermore, since $F(\mu) \leq 1/2$ implies that $\mu \leq s^*$(which follows from lemma 11), conditional on visiting any given bidder comes with a valuation that is never lower than the expected price that seller 1 would receive if he chose $p_1 = 0$. Thus so long as two bidders visit the auction, the price seller 1 expects if he chooses $p_1 = 1$ is always above the price he would receive by setting $p_1 = 0$. Thus, as traffic cannot be decreased and the expected price is increased by seller 1's choice of $p_1 = 1$ when seller 2 chooses $p_2 = 0$ and F satisfies $F(\mu) \leq 1/2$, we conclude that $R_1(1,0) > R_1(0,0)$ holds for all $n \geq 2$.

Things are more complicated when F satisfies $F(\mu) > 1/2$. First, as $F(\mu) > 1/2$ makes the cutoff value lie below μ but above m, traffic hurts seller 1's profits when he chooses $p_1 = 1$ and seller 2 chooses $p_2 = 0$. However, it may still be possible for the expected price to be sufficiently high to overcome the reduction in expected traffic. For example, the distribution function $F(v) = v^a$, with $a = 1/4$ satisfies the condition $F(\mu) > 1/2$ (it gives $F(\mu) = 0.67$) but it can be easily shown that $R_1(1,0) > R_1(0,0)$ still holds even if $n = 2$. Second, the existence of an equilibrium without provision of information requires the market to be small since it is possible to show that for a sufficiently large n, $R_1(1,0) > R_1(0,0)$ holds even if $F(\mu) > 1/2$ because the expected price tends to one as n grows large. The proof of this claim is similar to the one found in the literature of auction theory where seller's revenue increases with the number of bidders. The difference between the standard case and ours is the use of a cutoff value that also depends on n. If, for instance, the cutoff increased as n grew large then we would have to compare second-order statistics coming from distributions with increasing truncation points making seller 1's payoff not monotonic with respect to n [13]. However, the cutoff value (implicitly) defined by Eq. (3.2) cannot increase with n when $F(\mu) > 1/2$.

Lemma 13. *Suppose that $F(\mu) > \frac{1}{2}$. Let s_n^* denote the unique solution to Eq.* (3.2) *when the number of bidders is equal to n. Then, $s_n^* > s_{n+1}^*$.*

Proof. In the appendix. □

According to lemma 13, increasing the number of bidders when $F(\mu) > 1/2$ can never hurt traffic if seller 1 supplies information in response to not provision by his competitor. Moreover, price cannot decrease either because it is the equal to the expected value of a second-order statistic (see Ganuza and Penalva (2006) or Shaked and Shantikumar (1994) for a proof of this claim). Thus, the existence of an equilibrium in which sellers do not supply information information should only happen in markets where the number of bidders is small.

Proposition 14. *Consider the information provision game played by sellers followed by continuation game in which bidders select trading partners using the equilibrium participation rules described in the previous section (and subsequently bid their valuations truthfully). Let s^*be the unique solution to Eq.* (3.2). *Then, $F(s^*) > 1/2$ is necessary and $F(s^*) > 3/4$ is sufficient for the existence of some n^* such that for all $n \geq n^*$ both sellers do not provide information (i.e., $p_1 = p_2 = 0$) in equilibrium.*

Proof. In the appendix. □

3.5 Equilibria with Provision of Information

Consider any continuation game following a history in which both sellers decide to supply information. According to lemma 10, bidders choose trading partners using

[13] Menezes and Monteiro (2000) study a model in which participation is endogenous but the cost of participating is given exogenously. In this setup, seller's payoff may increase or decrease as n grows large because the truncation point increases with the number of potential bidders. Thus, when the number of potential bidders increase from n to $n+1$ the price changes from a second-order statistics of $n+1$ draws from a fixed distribution truncated at some point x to the second-order statistics of n draws from the same distribution truncated at some point $y < x$.

the function $\rho^*(s_1) = s_1$, $s_1 \in [0,1]$. As before, let $T_{1k}(1,1)$ be seller 1's expected price when this seller is matched with exactly k bidders, $k = 2,\ldots,n$. Then, seller 1's expected profit when he and his competitor announce informative structures is:

$$R_1(1,1) \quad = \quad \sum_{k=2}^{n} \binom{n}{k} q^n (1-q)^{n-k} T_{1k}(1,1) \tag{3.6}$$

where $q = 1/2$ is the probability that any given bidder visits seller 1. Alternatively, if seller 1 unilaterally deviates to $p_1 = 0$ his payoff would be:

$$R_1(0,1) \quad = \quad \sum_{k=2}^{n} \binom{n}{k} [F(s^*)]^k [1-F(s^*)]^{n-k} \mu \tag{3.7}$$

because any given bidder visits seller 1 whenever her valuation of item 2 falls *below* the cutoff value s^* given by Eq. (3.2).

Although the payoff seller 1 expects when he deviates to $p_1 = 0$ depends on the value of the cutoff s^*, the comparison between this payoff and the one that seller 1 expects if he chooses $p_1 = 1$ involves more than simple statements about the location of this value. When seller 1 deviates from $p_1 = 1$ to $p_1 = 0$, he expects changes in his payoff due to changes in expected traffic and in expected price. From Eq. (3.7), it is clear that the effect of this deviation on expected traffic depends on the value of the cutoff s^*. However, Eq. (3.6) shows that the change on the expected price depends on the underlying distribution of bidders' valuations generated by the participation rule given in lemma 10. This complicates the comparison of both payoffs at least without some additional structure on the problem. In what follows, we will perform such comparison under some conditions on one of the two primitives of the model that determines traffic and price: (1) the number of bidders in the market; (2) the distribution of bidders' valuations. In the first case, the idea is to exploit the relationship between the expected value of a second order statistics and n whereas in the second one, we will specify some (family of) distributions that ensures existence (and uniqueness) of an equilibrium in which $p_1 = p_2 = 1$ regardless of the number of bidders.

3.5.1 Number of Bidders

Let $G_1(s_1)$ be the probability that a bidder with valuation s_1 trades with seller 1 when $p_1 = 1$ and $p_2 = 1$. As previously discussed, $G_1(s_1)$ must be equal to the sum of the probability of being alone plus the probability of having every other participant with a valuation below s_1. Hence,

$$\begin{aligned} G_1(s_1) &= \left(1 - q + \int_0^{s_1} F(t) f(t) dt\right) \\ &= \left(\frac{1}{2} + \frac{F^2(s_1)}{2}\right) \end{aligned}$$

where $q := \int_0^1 F(\rho^*(s)) f(s) ds = \frac{1}{2}$ denotes the probability any given bidder comes to auction 1. Following Virag (2010), we can write seller 1's payoff when $p_1 = p_2 = 1$ as follows:

$$R_1(1,1) = n(n-1) \int_0^1 s G_1^{n-2}(s)(1 - G(s)) dG(s)$$

Define $H_1(s)$ by:

$$H_1(s) = G_1^n(s) + n G_1^{n-1}(s)(1 - G_1(s))$$

Then, $dH_1(s) = n(n-1) G_1^{n-2}(s)(1 - G(s))$ and hence,

$$R_1(1,1) = \int_0^1 s dH_1(s)$$

which it can be further rewritten (using integration by parts) as:

$$R_1(1,1) = 1 - \int_0^1 H_1(t) dt$$

Observe that $nG_1^{n-1}(s)$ tends to zero at a faster rate than $F^{n-1}(s)$ because $G_1(s) \geq F(s)$ for all $s \in [0,1]$. This means that there must exist some $n_0 \geq 2$ such that $nG_1^{n-1}(s) - F^{n-1}(s) < 0$ for all $n > n_0$ and all $s \in [0,1]$. Hence,

$$\begin{aligned} H_1(s) - F(s) &= G_1^n(t) + nG_1^{n-1}(s)(1 - G_1(s)) - F(s) \\ &\leq nG_1^{n-1}(s) - (n-1)G_1^n(s) - F^{n-1}(s) \\ &< 0 \end{aligned}$$

for every $s \in [0,1]$ provided that $n > n_0$. Thus,

$$\begin{aligned} R_1(1,1) - R_1(0,1) &> 1 - \int_0^1 H_1(s)ds - \mu \\ &= 1 - \int_0^1 H_1(s)ds - \left(1 - \int_0^1 F(s)ds\right) \\ &= \int_0^1 \{F(s) - H_1(s)\}\, ds \\ &> 0 \end{aligned}$$

if $n > n_0$ because $R_1(0,1)$ is strictly lower than μ and $\mu = 1 - \int_0^1 F(s)ds$. We conclude that for every $n > n_0$, $(p_1 = 1; p_2 = 1)$ must be an equilibrium of the game. We can extend the analysis to show the existence of some $n^* \geq 2$ such that $(p_1 = 1; p_2 = 1)$ is the unique equilibrium of the game. To do so, first observe that if F satisfies $F(\mu) \leq 1/2$, then lemma 11 and proposition 12 ensures that $R_1(1,0) > R_1(0,0)$ for all $n \geq 2$. Therefore, if $n > n_0$ and $F(\mu) \leq 1/2$, $(p_1 = 1; p_2 = 1)$ must be the unique equilibrium of the game. Second, if F satisfies $F(\mu) > 1/2$ then $m \leq s^* \leq \mu$ (lemma 11). Let $\tilde{G}_1(s; s^*)$ be the distribution of valuations at seller 1 conditional on participating in this auction and define $\tilde{H}_1(s; v^*)$ similarly to $H_1(s)$. Then,

$$\sum_{k=2}^{n} \binom{n}{k} q^n (1-q)^{n-k} T_{1k}(1,0) = \int_{s^*}^1 s dH_1(s; s^*)$$

where $q := 1 - F(v^*)$. Moreover,

$$\int_{s^*}^1 s d\tilde{H}_1(s; s^*) = 1 - s^* \tilde{H}_1(s^*) - \int_{s^*}^1 \tilde{H}_1(s; s^*) dt$$

after integration by parts. Let n_1 be such that for all $n > n_1$:

$$F(s) - F^n(s) - nF^{n-1}(s)(1 - F(s)) \geq 0$$

for all $s \in [0,1]$. Then,

$$\begin{aligned} R_1(1,0) - \mu &= 1 - s^*\tilde{H}_1(s^*) - \int_{s^*}^{1} \tilde{H}_1(s;s^*)ds - 1 + \int_0^1 F(s)ds \\ &= \int_0^{s^*} F(s)ds - s^*\tilde{H}_1(s^*,s^*) + \int_{s^*}^{1} \left\{F(s) - \tilde{H}_1(s;s^*)\right\} ds \\ &= \int_0^{s^*} \left\{F(s) - \tilde{H}_1(s^*,s^*)\right\} ds + \int_{s^*}^{1} \left\{F(s) - \tilde{H}_1(s;s^*)\right\} ds \\ &\geq 0 \end{aligned}$$

and $R_1(1,0) > R_1(0,0)$ if $n > n_1$ because the cutoff value decreases as n grows large when $F(\mu) > 1/2$ (lemma 13) and $V_1(0,0)$ is strictly lower than μ. Therefore, by setting $n^* = \max\{n_0, n_1\}$ we have shown the existence of a threshold number of bidders such that $p_1 = p_2 = 1$ is the unique equilibrium of the game provided that $n > n^*$.

Proposition 15. *Consider the information provision game in which sellers choose information structures from the set $\{0,1\}$ and bidders select trading partners using the participation rules described in the previous section. Then, there exists a critical number of bidders n^* such that for all $n > n^*$ the unique equilibrium of the game has both sellers choosing $p_1 = p_2 = 1$.*

3.5.2 Distribution of Valuations

As before, let s^* be the cutoff value used by bidders to select trading partners when sellers choose $p_1 = 0$ and $p_2 = 1$ respectively. As it is seller 1 who is setting $p_1 = 0$, any bidder will visit his auction provided that her valuation of item 2 is below s^* giving $F(s^*)$ as the probability that this bidder visits seller 1. Thus, if F satisfies $F(s^*) \leq \frac{1}{2}$ then traffic is reduced if seller 1 chooses $p_1 = 0$ when seller 2 is choosing $p_2 = 1$. Furthermore, since the distribution of the second order statistics implied by the probability distribution $F^2(s)$ increases with k in the first-order stochastic sense, $T_{1k}(1,1) > \mu$ for all $k = 3,\ldots,n$ provided that we can ensure $T_{12}(1,1) > \mu$. Let $H_{12}(s)$ be the distribution of the second order statistics when seller 1 is matched with exactly two bidders. Then,

$$H_{12}(s) = F^4(s) + 2F^2(s)(1 - F^2(s))$$

because the distribution of valuations conditional on participating when bidders use the function $\rho(s_1) = s_1$ is $F^2(s_1)$. Thus, the difference between the expected price when $(p_1 = 1; p_2 = 1)$ and $(p_1 = 0; p_2 = 1)$ is:

$$\begin{aligned} T_{12}(1,1) - \mu &= \int_0^1 s dH_{12}(s) - \int_0^1 s dF(s) \\ &= 1 - \int_0^1 H_{12}(s) ds - \left(1 - \int_0^1 F(s) ds\right) \\ &= \int_0^1 \left\{F(s) + F^4(s) - 2F^2(s)\right\} ds \end{aligned}$$

Suppose that F is convex, that is, suppose that in addition to $F' := f > 0$, F satisfies $F'' := f' \geq 0$. Let $u := F(s)$, $\alpha(u) := F^{-1}(u)$, and $\frac{d}{du}\alpha(u) := \alpha'(u) = \frac{1}{f(u)}$, where F^{-1} is the (right) inverse of F. Integrating by parts the difference between $T_{12}(1,1)$ and μ yields:

$$T_{12}(1,1) - \mu \quad = \quad \alpha'(s)\beta(s)\big|_0^1 - \int_0^1 \beta(u)\alpha''(u) du$$

where $\beta(s) = \int_0^s \{u + u^4 - 2u^2\} du \geq 0$, $s \in [0,1]$, and $\alpha''(u) = \frac{d^2}{du^2}\alpha(u) = -\frac{f'(u)}{f(u)}$. Since $\alpha'(u) > 0$ because $f > 0$, the whole expression is positive provided that $f' \geq 0$, which is precisely what the convexity of F ensures. Thus, convexity and monotonicity of F are enough to guarantee that the expected price when sellers set $p_1 = p_2 = 1$ is not lower than the price when they set $p_1 = 0$ and $p_2 = 1$. Moreover, F convex implies that $F(\mu) \leq 1/2$ and hence, $R_1(1,1) \geq R_1(0,1)$ for all $n \geq 2$. Finally, proposition 12 ensures that this equilibrium must be unique because $F(\mu) \leq 1/2$.

Proposition 16. *Suppose that the distribution of bidders' valuations F is increasing and convex, i.e., F satisfies $F' = f > 0$ and $F'' = f' \geq 0$. Then, the information* *provision game has a unique (symmetric) equilibrium in which both sellers release all available information to bidders.*

It is almost immediate that the need of having F convex can be relaxed if instead F satisfies: (i) $F(\mu) \leq 1/2$; (ii) $\int_0^1 \left\{F(s) + F^4(s) - 2F^2(s)\right\} ds \geq 0$ since (i) and lemma 11 ensure that traffic is never reduced and (ii) ensures that the difference between the expected price and μ is positive if seller 1 chooses $p_1 = 1$ and seller 2 chooses $p_2 = 0$, making unilateral deviations unprofitable.

3.6 The Strategic Value of Information

An interesting question that we may ask in the competitive framework developed in this chapter is the strategic value attached by sellers to the supply of information. In the terminology of Bulow, Geanakoplos, and Klemperer (1985), providing information is an strategic complement (resp. substitute) if the sellers' choices of providing information mutually reinforce (resp. offset) each other. Based on our previous discussion, we may conjecture that the incentives to provide information are weakened by the competitor's decision to supply information because the seller who decides to unilaterally supply information (given that the other seller is expected to supply information as well) cannot induce a lower truncated distribution of posterior valuations. However, supply of information also affects expected traffic, which we have seen it depends on location parameters of F. The next result provides one case in which incentives are indeed weaker in environments where both sellers are expected to provide information. These environments correspond to those in which traffic is unaffected by information provision[14], which happens when the distribution of bidders' valuations satisfies $F(\mu) = 1/2$. Observe that the condition $F(\mu) = 1/2$ holds whenever F is symmetric around its mean. However, requiring F to satisfy $F(\mu) = 1/2$ is weaker than requiring F to be symmetric because not all distribution functions satisfying $F(\mu) = 1/2$ are symmetric.

Proposition 17. *Suppose that F satisfies $F(\mu) = 1/2$. Let ΔR_j be the difference between the change in profits when seller j supplies information against a competitor*

[14]The cases in which information does have an effect on traffic are much more involved because of the complication in disentangling the traffic from the price effect.

who does not, and the change in profits when seller j supplies information against a competitor who also does so. Then, $\Delta R_j \leq 0$ for all $n \geq 2$, i.e., information provision behaves as an strategic substitute for all $n \geq 2$.

Proof. In the appendix. □

Intuitively, when seller 1 decides to supply information and he expects seller 2 not to do so, he can shut down low-valuation types by inducing a continuation equilibrium in which only bidders with valuations above certain cutoff value visit his auction. This is beneficial for the seller because he can capture more surplus without having to give away too much rents (in the form of informational rents) to bidders. In contrasts, supplying information in environments where seller 2 is also expected to supply information as well, does not have the effect of precluding the visit of low-valuation customers. Of course, the key for proposition 17 to hold is the fact that information only acts on sellers' profits through its effect on the expected price. To avoid having to deal with the traffic effect, proposition 17 requires F to satisfy $F(\mu) = 1/2$, which eliminates the effect of information provision on expected traffic.

3.7 Concluding Remarks

This chapter develops a simple model aimed at investigating sellers' incentives to provide information in competing auctions. The model considers two sellers who can choose to inform or not bidders before they choose trading partners. I show that supplying information by both sellers is the unique equilibrium of the game provided that the distribution function of bidders' valuations is increasing and convex. Distributions with this property ensure that both traffic and price are favored by the supply of information and make unilateral deviations (to not provision of information) unprofitable. I also provide a characterization of information in terms of its strategic value for sellers where I show that the value of information to a seller decreases with the decision of his competitor to provide information. In this sense, information behaves as an strategic substitute at least when traffic is unchanged by the sellers' decisions to provide information.

Future work includes extending the binary set of information structures into a finite or even a continuous one. The binary case is special in the sense that sellers cannot fine tune the amount of information that they supply to bidders. Therefore, sellers may not have sufficient incentives to supply information when they expect all other sellers to do so because informational rents are related to the degree of heterogeneity in bidders' ex-post distribution of valuations. This is one reason that explains the need to restrict the distribution of bidder's valuations to those satisfying the condition mentioned in the previous paragraph. The price that we may have to pay if we generalize the model in this direction is the additional layer of complexity in the characterization of the equilibrium in bidders' participation games. One conjecture is that characterizing continuation games using cutoff functions remains available. However, a formal verification of this conjecture is left for future work.

Chapter 4

Provision of Information in Competing Auctions

4.1 Introduction

This chapter studies a competing auction model in which two sellers can control the the amount of information available to potential buyers and they can also post reserve prices. As mentioned in the introduction of the previous chapter, there are many situations in which sellers can voluntary add or omit details of the items that may be of interest for some of the potential buyers. However, this provision of information is usually made in environments where sellers control another important variable: reserve price. In this chapter I explore the issue of information provision in competing auctions in environments where sellers can also post reserve prices.

The model I develop shares features from the models of the two previous chapters of this dissertation. There are only two sellers with unit supply who compete for the unit demands of $n \geq 2$ potential bidders. Apart from choosing information structures from a binary set, sellers can also post nonnegative reserve price before bidders decide where they want to submit a bid. Thus, I maintain the assumption of the set of information structures being binary, which means that sellers can only choose between letting bidders learn their valuations perfectly or leave them completely uninformed. I also assume that sellers can choose reserve prices from a finite subset $\mathscr{R} \subseteq [v_0, 1]$. Of course, the finiteness of the set of reserve prices is with loss of generality but it ensures the existence of an equilibrium. The issue of equilibrium existence in models where sellers control information and reserve prices is complicated because payoff functions induced by continuation equilibria in which both sellers choose to supply information depend on a cutoff function for which no closed-form solution is available. Moreover, reserve prices can be used as a tool to extract (part of the) surplus generated when information is supplied to bidders, which introduces many more deviations that we have to account for when characterizing the equilibrium set. However, having a finite set of reserve prices allows me to rely on standard game theoretical tools to claim existence of an equilibrium for the game. I then show that the existence of an equilibrium in which both sellers do not supply information is less likely than the existence of such equilibrium in games where sellers can only decide on information provision. The reason is the use of reserve prices to extract part of the surplus originated by the seller's decision to supply information. I also provide a set of sufficient conditions in terms of the number of bidders such that the unique equilibrium of the game is one in which both sellers supply information.

The rest of the chapter is organized as follows. The next section outlines the model whereas section 4.3 characterizes the equilibrium set of the game. Section 4.4 ends the chapter with some conclusions and final remarks.

4.2 The Model

Consider an economy in which trade takes place using second-price sealed bid auctions. The economy is populated by two risk-neutral sellers (seller 1 and seller 2) with unit supply, and $n \geq 2$ risk neutral buyers with unit demands indexed by $i \in \{1, \cdots, n\}$. Sellers attach to either item a nonnegative value v_0, which is common knowledge among players. Before any interaction with the sellers takes place, each bidder privately observes the realization $(s_1, s_2) \in [0,1]^2$ of a pair of identically distributed random variables following a distribution function F with support $[0,1]$. The function F is at least twice continuously differentiable with bounded density function $f > 0$. We also assume that sellers' value v_0 is strictly lower than the average valuation of bidders, $0 < v_0 < \mu = \int_0^1 sf(v)dv$.

Each bidder is unsure about how exactly these signals translate into valuations. Sellers can help reduce this uncertainty by supplying information about their respective items. Before the auction takes place, each seller independently and simultaneously decides on the minimum acceptable bid (reserve prices) and the amount of information he wants to reveal to potential buyers. Sellers can post reserve prices from the finite subset $\mathscr{R} \subset [v_0, 1]$, with $v_0 \in \mathscr{R}$, and they can choose between informing and not informing bidders about the characteristics of their goods. We let $r_j \in \mathscr{R}$ and $p_j \in \{0,1\}$ be seller j's reserve price and choice of information structure, with $p_j = 1$ if seller j provides information, and $p_j = 0$ if he does not. The game begins when Nature privately communicates to each bidder i a pair of signals $(s_{i1}, s_{i2}) \in [0,1]^2$. Then, each seller independently and simultaneously chooses a reserve price and an information structure from the sets $\mathscr{R}$ and $\{0,1\}$ respectively. These choices become common knowledge right after announced. Having observed sellers' choices each bidder i independently and simultaneously decides whether to participate in some auction and which auction to do so. We restrict bidders to choose one and only one seller as their respective trading partner. After bidders have assigned themselves into the different auctions, each seller collects the bids and awards the good using a second price sealed-bid auction with reserve price equal to that announced in stage two of the game. Finally, payoff are realized and the game ends.

4.3 Analysis

Similar to chapter 3 we assume truthful bidding. As by the time bidder i must submit a bid (say in auction j), she has already observed whether $v_{ij} = s_{ij}$ (which happens if $p_j = 1$), bidding s_{ij} when $p_j = 1$ and μ when $p_j = 0$ for every $i = 1, \ldots, n$ constitutes a Bayesian-Nash equilibrium of the bidding game. Thus, a strategy for bidder i in the present setting is a mapping $\pi_i : [0,1]^2 \times \{0,1\}^2 \times \mathscr{R}^2 \longrightarrow [0,1]$, where $\pi_i(s,p,r)$ specifies a probability with which bidder i visits seller j as a function of bidder's signals $s = (s_1, s_2)$, sellers' choices of information structures $p = (p_1, p_2)$, and reserve prices $r = (r_1, r_2)$. Furthermore, we adhere to the convention to treat any the decision not to bid in any auction as equivalent to the decision to submit a non serious bid in auction 1. Thus, if $\pi(s,p,r)$ stands for the probability that a bidder who has observed the pair of signals s, and the sellers' choices of information structures and reserve prices p and r, then $1 - \pi(s,p,r)$ is the probability that this bidder visits seller 2. Our equilibrium concept is symmetric perfect

Bayesian equilibrium[15].

4.3.1 Bidders' Participation Game

Given the similarities between the model in this chapter and those in the preceding two chapters, it is not surprising that the analysis of the bidders' participation game resembles previous ones. We consider three types of continuation games that may arise depending on whether both sellers supply (resp. do not supply) information or only one of them decides to do so. First, consider continuation games following histories in which $p_1 = p_2 = 0$. If sellers' posted prices are (r_1, r_2) then the expected payoffs of any participant in auction 1 and auction 2 respectively are:

$$\begin{aligned} \mathscr{U}_j &= \max\left\{0; (\mu - r_1)(1-q)^{n-1}\right\} \\ \mathscr{U}_2 &= \max\left\{0; (\mu - r_2)q^{n-1}\right\} \end{aligned}$$

where:

$$q = \int_0^1 \int_0^1 \pi((s_1, s_2); (0,0), (r_1, r_2)) dF(s_1) dF(s_2)$$

because $p_1 = p_2 = 0$ implies $v_1 = v_2 = \mu$ for all bidders. If $r_1 > \mu$ then $q = 0$ if whereas if $r_2 > \mu$ then $q = 1$ (recall that bidders who do not want to participate are treated as non serious bidders in auction 1). When both reserve prices are below μ, the value of q can be pinned down by solving the following equation:

$$(\mu - r_1)(1-q)^{n-1} = (\mu - r_2)q^{n-1} \tag{4.1}$$

As expected, q collapses to $1/2$ whenever sellers are ex-ante identical, i.e., whenever reserve prices satisfy $r_1 = r_2$, it decreases with higher values of r_1, and it increases with higher values of r_2.

Next, consider any continuation game following a history in which seller 1 chooses a perfectly informative structure and seller 2 chooses an uninformative one. Let $Q_j(s_j, r)$ be the reduced form probability of trading with seller j when

[15] That is, equilibria in which bidders use *symmetric participation rules.* A participation rule is symmetric if two bidders with the same vector of estimates visit seller 1 (resp. seller 2) with the same probability, $\pi_i(\cdot) = \pi_k(\cdot) \equiv \pi(\cdot)$, $i \neq k \in \{1,2\}$.

bidder's signal is s_j and reserve prices are r. Then Q_1 and Q_2 can be written as:

$$Q_1(s_1,r) = \left(1-\int_{s_1}^{1}\int_0^1 \pi((s_1,s_2),(1,0),r)dF(s_2)dF(t)\right)^{n-1}$$

$$Q_2(r) = \left(\int_0^1\int_0^1 \pi((s_1,s_2),(0,1),r)dF(s_1)dF(s_2)\right)^{n-1}$$

because all bidders who come to auction 2 do so with the same valuation μ. Then, bidder's payoffs when $p_1 = 1$, $p_2 = 0$, and reserve prices are $(r_1,r_2) \in [v_0,1]^2$, can be written as:

$$\mathscr{U}_1(v,r,p) = \max\left\{0; \mathscr{U}_1(0,r) + \int_0^v Q_1(t,r)dt\right\}$$
$$\mathscr{U}_2(\mu,r,p) = \max\{0; (\mu - r_2)Q_2(r)\}$$

Suppose that any bidder other than bidder 1 uses a cutoff strategy with cutoff point s^* such that bidders with valuation of item 1 above s^* visit seller 1 for sure, and bidders with valuations of item 1 below s^* visit seller 2 for sure. If $r_2 > \mu$ then a strategy with a cutoff equal to zero will be a best response to itself (with those bidders with valuations below r_1 bidding non seriously in auction 1). If $r_1 < 1$ and $r_2 < \mu$, a type of bidder 1 with a valuation of item 1 exactly equal to s^* expects a positive payoff in auction 1 if and only if she is the only participant in this auction. As the probability that any other bidder does not visit this same seller is equal to $F(s^*)^{n-1}$, the expected payoff of type s^* of bidder 1 in auction 1 is:

$$\mathscr{U}_1(s^*,r,p) = (s^* - r_1)F(s^*)^{n-1}$$

and this type's payoff if she bids in auction 2:

$$\mathscr{U}_2(\mu,r,p) = (\mu - r_2)(1-F(s^*))^{n-1}$$

Therefore, bidder 1 with valuation s^* of item 1 is indifferent between auction 1 and auction 2 whenever her expected payoffs are equal, i.e., whenever the following condition holds true:

$$(s^* - r_1)F(s^*)^{n-1} = (\mu - r_2)(1 - F(s^*))^{n-1} \tag{4.2}$$

It is straightforward to check that Eq. (4.2) has a unique solution and that this solution satisfies $r_1 < s^* < 1$. Therefore, bidder 1's best response to

$$\pi^*(s,p,r) = \begin{cases} 1 & \text{if } (s_1 \geq s^*; s_2 \in [0,1]) \\ 0 & \text{if } (s_1 < s^*; s_2 \in [0,1]) \end{cases}$$

is to also use π^*. It is not difficult to check that the value of s^* increases with higher values of r_1 and decreases with higher values of r_2. That is, the probability that any given bidder visits seller 1 (which is equal to $1 - F(s^*)$) decreases with higher values of r_1 and increases with higher values of the competitor's reserve price.

Finally, consider any continuation game following a history in which both sellers choose perfectly informative structures. Since sellers choose $p_1 = p_2 = 1$, bidders perfectly learn their true valuations before choosing trading partners. Therefore, the participation decision problem becomes identical to the problem faced by bidders participating in a competing auction model with heterogeneous goods. The next lemma (whose proof is identical to the proof of proposition 2 in chapter 2) ensures existence and uniqueness of a continuation equilibrium in this case.

Lemma 18. *Consider the continuation game following a history in which sellers choose perfectly informative structures (i.e., $p_1 = p_2 = 1$) and a vector of non-negative reserve prices $r \in [v_0, 1]$. Then, in the unique symmetric continuation equilibrium bidders use a strategy characterized by a nondecreasing and continuous function $\rho^* : [0,1] \to [0,1]$ such that bidder with valuations (v_1, v_2) visits seller 1 with probability one if and only if $v_1 \geq \rho(v_2)$, and visits seller 2 with probability one if and only if $v_1 < \rho(v_2)$.*

4.3.2 The Sellers' Game

The continuation equilibrium described in the previous section determines a normal form game between sellers. The action space for each seller is $\{0,1\} \times \mathscr{R}$ and the payoff function of seller j, $j = 1,2$, is:

$$R_j^{(p_1,p_2)}(r_1,r_2) =$$

$$v_0(1-q_j)^n + nr_jq_j^n(1-q_j)^{n-1} + \sum_{k=2}^{n}\binom{n}{k}q_j^k(1-q_j)^{n-k}T_{jk}^{(p_1,p_2)}(r_1,r_2) \qquad (4.3)$$

where $p_j \in \{0,1\}$ and $p_{-j} \in \{0,1\}$ represent the choice of information structure made by seller j and seller $-j$ respectively, $(r_j, r_{-j}) \in \mathscr{R}^2$ is the vector of reserve prices, $q_j := q_j(p,r)$, $p = (p_j, p_{-j})$, is the probability that a given bidder visits seller j, and $T_{jk}^{(p_1,p_2)}(r_1,r_2)$ is the price seller j expects to receive when he is matched with exactly k bidders given sellers' choices of information provision and reserve prices, $k = 2,3,\ldots,n$. The first term in Eq. (4.3) captures the profit that seller j expects if nobody visits his auction[16]; the second represents his expected payoff in case only one bidder visits, and the third one his payoff in the event that two or more bidders choose seller j as their trading partner. Of course, the specific form of this payoff function depends on the vectors of information levels and reserve prices chosen by sellers. However, as strategy spaces are finite existence of an equilibrium for the sellers' game follows from standard existence theorems (e.g. proposition 8.D.2 in Mas-Colell, Whinston, and Green (1995)).

Lemma 19. *The game admits an equilibrium in which bidders use symmetric strategies.*

Suppose that an equilibrium in which sellers do not provide information exists. In this putative equilibrium, reserve prices below μ will affect traffic and expected price in the event that exactly one bidder visits and expected traffic in cases where two or more bidders visit the auction [17]. Let $r_1 \in \mathscr{R}$ and $r_2 \in \mathscr{R}$ be any two reserve prices posted by seller 1 and seller 2 respectively. If $r_1 > \mu$ then seller 1's profit is equal to v_0 regardless of what reserve price seller 2 has chosen to post. If $r_1 < \mu$ and $r_2 \geq \mu$ then every bidder will visit seller 1's auction with probability one and

[16] In case that $j = 1$, this term represents seller 1's expected payoff when only non serious bidders visit is auction.

[17] To see this observe that when both sellers choose uninformative structures, the price conditional on receiving two or more visits is equal to the average valuation, which is independent of the actual reserve prices set by sellers.

seller 1's expected profit will be equal to μ. Finally, if $r_1 < \mu$ and $r_2 < \mu$, seller 1's expected profit is the sum of three terms: the payoff in case no bidder visits; (ii) the payoff in case exactly one bidder visits; and (iii) the payoff in case two or more bidders visit:

$$R_1^{(0,0)}(r_1, r_2) =$$

$$\begin{cases} v_0 & \text{if } r_1 \geq \mu \text{ and } r_2 \in [v_0, 1] \\ \mu & \text{if } r_1 < \mu \text{ and } r_2 \geq \mu \\ v_0(1-q)^n + nr_1 q(1-q)^{n-1} + \sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{n-k} \mu & \text{if } r_1 < \mu \text{ and } r_2 < \mu \end{cases}$$

where q is the solution to Eq. (4.1) and represents the probability that a given bidder visits auction 1 when reserve prices are (r_1, r_2).

Our first result shows that it is possible to find a symmetric Nash equilibrium in the game where sellers are assumed to choose $p_1 = p_2 = 0$, provided that the set $\mathscr{R}$ is sufficiently rich in the sense that it includes sufficiently many reserve prices. First, use Eq. (4.1) to write $R_1^{(0,0)}$ directly in terms of q and r_2. Since q solves $(\mu - r_1)(1-q)^{n-1} = (\mu - r_2)q^{n-1}$,

$$nr_1 q(1-q)^{n-1} = n\mu q(1-q)^{n-1} - n(\mu - r_2)q^n$$

and hence,

$$R_1^{(0,0)}(q, r_2) =$$

$$\begin{cases} v_0 & \text{if } r_1 \geq \mu \text{ and } r_2 \in [v_0, 1] \\ \mu & \text{if } r_1 < \mu \text{ and } r_2 \geq \mu \\ v_0(1-q)^n + \mu - [\mu(1-q)^n + n(\mu - r_2)q^n] & \text{if } r_1 < \mu \text{ and } r_2 < \mu \end{cases}$$

It is almost immediate that conditional on choosing $p_1 = p_2 = 0$, posting a reserve price greater than μ cannot be optimal for either seller because any of them could do strictly better by lowering his price and receiving the visit of some bidder. Therefore, if r^* is an optimal symmetric reserve price when $p_1 = p_2 = 0$ it must be the case that $r^* < \mu$.

Momentarily suppose that instead of choosing from a finite set of reserve prices, sellers could choose reserve prices from the continuous interval $[v_0, 1]$. Then, seller

1's problem becomes one in which this seller chooses some $q \in [0,1]$ to solve the following problem:

$$\max_{q \in (0,1)} \tilde{R}_1^{(0,0)}(q, r_2) = v_0(1-q)^n + \mu - [\mu(1-q)^n + n(\mu - r_2)q^n]$$

The first order condition for this problem is:

$$\begin{aligned} \frac{\partial}{\partial q}\tilde{R}_1^{(0,0)}(q, r_2) &= nq^{n-1}\left[-v_0\left(\frac{1-q}{q}\right)^{n-1} - n(\mu - r_2) + \mu\left(\frac{1-q}{q}\right)^{n-1}\right] \\ &= 0 \end{aligned}$$

In any symmetric equilibrium, $r_1^* = r_2^*$ and hence, this first order condition becomes:

$$\Psi(r_1^*) = \left(\frac{1}{2}\right)^{n-1} n\left[-v_0 - n(\mu - r_1^*) + \mu\right] = 0$$

because Eq. (4.1) implies that $q = 1/2$ whenever $r_1 = r_2$. From this last condition we obtain $r_1^* = r_2^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$ as the unique candidate for an equilibrium. Moreover, $\Psi(v_0) < 0$, $\Psi(\mu) > 0$ (because $n \geq 2$ and $v_0 < \mu$), and $\Psi' > 0\ \forall r_1 \in [v_0, \mu]$ from where we conclude that this candidate must be unique. Finally, differentiating $\tilde{R}_1^{(0,0)}(q, r_2)$ twice with respect to q gives:

$$\begin{aligned} \frac{\partial^2}{\partial q^2}\tilde{R}_1^{(0,0)}(q, r_2) &= n(n-1)q^{n-2}\left[(v_0 - \mu)\left(\frac{1-q}{q}\right)^{n-1} - n(\mu - r_2)\right] \\ &\leq 0 \end{aligned}$$

because $v_0 < \mu$ and $r_2 \leq \mu$. Hence, $\tilde{R}_1^{(0,0)}(q, r_2)$ is concave with respect to q for any $r_2 \in [0, \mu]$, and $r_1^* = r_2^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$ is the unique symmetric reserve price that satisfies:

$$\tilde{R}_1^{(0,0)}(r^*, r^*) \geq \tilde{R}_1^{(0,0)}(\hat{r}, r^*) \qquad \hat{r} \in [v_0, 1]$$

and similarly for seller 2. Since this condition holds for all $\hat{r}$ in the interval $[v_0, 1]$ it must also hold for any finite subset of $[v_0, 1]$ that includes r^*. Thus, as we have assumed that $r^* \in \mathscr{R}$, then it must be true that r^* satisfies:

$$R_j^{(0,0)}(r^*, r^*) \geq R_j^{(0,0)}(\hat{r}_j, r^*)$$

for all $\hat{r} \in \mathscr{R}$ and $j \in \{1,2\}$.

Lemma 20. *Suppose that both sellers choose not to provide information (i.e. they choose* $p_1 = p_2 = 0$*). Suppose further that for each* n *the reserve price* $r^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$ *belongs to* $\mathscr{R}$*. Then,* r^* *is the unique symmetric reserve price such that* $R_j^{(0,0)}(r^*, r^*) \geq R_j^{(0,0)}(\hat{r}_j, r^*)$ *holds for every* $\hat{r}_j \in \mathscr{R}$ *and* $j \in \{1,2\}$*.*

Two comments about this lemma are in order here. First, the requirement that r^* belong to $\mathscr{R}$ may be somewhat relaxed if we assume that the set $\mathscr{R}$ includes sufficiently many reserve prices in the sense that the distance between any two adjacent elements of $\mathscr{R}$ is small. This is a consequence of the mean value theorem applied to the continuous counterpart of $R_1^{(0,0)}$ since

$$\left| R_j^{(0,0)}(r_j^k, r^*) - R_j^{(0,0)}(r_j^{k+1}, r^*) \right| \leq h \sup_{r \in [v_0, 1]} \left| R_j^{'(0,0)}(r, r^*) \right|$$

where $h = \left| r_j^k - r_j^{k+1} \right|$ is the distance between adjacent reserve prices. Thus, we can approximate the value of $R_j^{(0,0)}$ at (r^*, r^*) reasonably well if h is small (though not infinitesimal) using the continuous counterpart of $R_j^{(0,0)}$. Second, notice that the symmetric reserve price r^* increases as the number of bidders grows large even though competition is (mostly) guided by traffic and not by price.

Next, consider seller 1's profit when he chooses $p_1 = 1$ and $r_1 \in \mathscr{R}$, whereas seller 2 chooses $p_2 = 0$ and some $r_2 \in \mathscr{R}$. If $1 \in \mathscr{R}$ and $r_1 = 1$ then the continuation game will only have non–serious bidders visiting seller 1 whereas if seller 1 picks any reserve price other than one and seller 2 chooses $r_2 \geq \mu$, then bidders will visit auction 1 with probability one. Finally, if seller 1 chooses a reserve prices strictly below one and seller 2 chooses a reserve price strictly below μ, the probability with which any bidder visits seller 1 is $q := 1 - F(s^*)$[18], where s^* is the value of the cutoff given implicitly by Eq. (4.2). Hence, seller 1's payoff when $p_1 = 1$ and $p_2 = 0$ is equal to:

$$R_1^{(1,0)}(r_1, r_2) =$$

[18] If $r_1 = 1$ and $r_2 \geq \mu$ then every bidders visits seller 1 for sure where they submit non serious bids.

$$\begin{cases} v_0 & \text{if } r_1 \geq 1,\ r_2 \in \mathscr{R} \\ T_{1n}^{(1,0)}(r_1, r_2) & \text{if } r_1 < 1,\ r_2 \geq \mu \\ v_0(1-q)^n + nr_1q(1-q)^{n-1} + \sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{k-n} T_{1k}^{(1,0)}(r_1, r_2) & \text{if } r_1 < 1,\ r_2 < \mu \end{cases}$$

where $T_{1k}^{(1,0)}(r_1, r_2)$ is the price seller 1 expects to receive when he is matched with exactly k bidders, $k = 2, \ldots, n$. Recall that this price is equal to the expected value of the second order statistic induced by the truncated distribution F with lower truncation point s^*.

Suppose that there exists a (symmetric) equilibrium in which both sellers choose $p_1 = p_2 = 0$. From lemma 20, the unique symmetric reserve that sellers can post in this putative equilibrium is $r^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$. Thus, seller 1's expected payoff in this putative equilibrium becomes:

$$R_1^{(0,0)}(r_1^*, r_2^*) = \mu - (\mu - v_0)\left(\frac{1}{2}\right)^{n-1} \tag{4.4}$$

because $q = 1/2$ whenever $r_1 = r_2$. Let m and μ be the median and mean of F respectively and suppose that $m \geq \mu$ holds. We will show that whenever the previous condition holds, there exists a profitable unilateral deviation for seller 1 in which he chooses $p_1 = 1$ and $\tilde{r}_1 = m - \frac{(\mu - v_0)}{n} > v_0$. Again, we will assume that $\tilde{r}_1$ belongs to $\mathscr{R}$ even though the conclusions should remain true if h is small enough. As seller 2 is supposed to choose $p_2 = 0$ and to post a reserve price equal to $r^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$, the continuation equilibrium following seller 1 deviation has any bidder choosing auction 1 whenever her valuation of item 1 is above the cutoff value s^*:

$$\begin{aligned} (s^* - \tilde{r}_1)F^{n-1}(s^*) &= (\mu - r^*)(1 - F(s^*))^{n-1} \\ &= \frac{(\mu - v_0)}{n}(1 - F(s^*))^{n-1} \end{aligned}$$

where the last line follows after replacing the value $r^* = \left(\frac{n-1}{n}\right)\mu + \frac{v_0}{n}$ into the first line. We already know (lemma 4.2) that the solution to this equation must be unique. Let $s^* = m$. Then,

$$
\begin{aligned}
(s^* - \tilde{r}_1)F^{n-1}(s^*) &= (m - \tilde{r}_1)F^{n-1}(m) = \\
&= \left(m - m + \frac{(\mu - v_0)}{n}\right)\left(\frac{1}{2}\right)^{n-1} \\
&= \frac{(\mu - v_0)}{n}\left(\frac{1}{2}\right)^{m-1} \\
&= (\mu - r^*)(1 - F(s^*))^{n-1}
\end{aligned}
$$

and then, $s^* = m$ must be value of the cutoff used by bidders in this continuation equilibrium. Since $m \geq \mu$ by assumption, any bidder who visits seller 1 must bid a valuation that is at least equal to m. Thus, the price that seller 1 expects when he deviates to $p_1 = 1$ and $\tilde{r}_1$ can never be less than m. Consequently,

$$
\begin{aligned}
\sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{n-k} T_{1k}^{(1,0)}(r_1, r_2) &> \left(1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1}\right) m \\
&= \left(1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n\right) m
\end{aligned}
$$

and hence,

$$
\begin{aligned}
R_1^{(1,0)}(\tilde{r}_1, r^*) &> v_0\left(\frac{1}{2}\right)^n + n\left(m - \frac{(\mu - v_0)}{n}\right)\left(\frac{1}{2}\right)^n + \left(1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n\right) m \\
&= \mu - (v_0 - \mu)\left(\frac{1}{2}\right)^{n-1} \\
&= R_1^{(1,0)}(r^*, r^*)
\end{aligned}
$$

It is possible to relax a little bit the condition that $m \geq \mu$. As discussed in chapter 3, having $\mu > m$ allows for the possibility that reduced traffic overcomes any potential gain due to a higher expected price. However, in chapter 3 we abstracted from reserve prices in order to isolate the effects of information on sellers' profits whereas in the current chapter sellers can use reserve prices to affect the location of this cutoff. For instance, suppose that $n = 2$ and μ, m and v_0 satisfies $0 < v_0 \leq 2m - \mu$. This condition is likely to be satisfied when the mean of F is not too far apart from its median, and sellers' valuation v_0 is not too high. Then, if

$v_0 \leq \frac{n}{n-1}m - \frac{1}{n-1}\mu$ and $\tilde{r}_1 = m - \frac{(\mu - v_0)}{n} \in \mathscr{R}$, we can construct a profitable deviation to $p_1 = 1$ and $\tilde{r}_1 = m - \frac{(\mu - m)}{n}$ for seller 1.

Proposition 21. *Suppose that for each* $n \geq 2$*,* m*,* μ *and* v_0 *satisfies* $\frac{n}{n-1}m - \frac{1}{n-1}\mu \geq v_0$*, and* $m - \frac{(\mu - v_0)}{n} \in \mathscr{R}$*. Then, there cannot be an equilibrium in which both sellers choose* $p_1 = p_2 = 0$ *and post symmetric reserve prices.*

The previous two propositions provide conditions that suggest the impossibility of having a symmetric equilibrium in which sellers do not supply information. However, both statements depend on the set of reserve prices to be rich enough to include certain elements that guarantee the existence of profitable unilateral deviations. This is restrictive and we would like a more general statement that does not depend on the shape of the set $\mathscr{R}$ and that it accounts for the possibility that sellers post reserve prices using some general (not necessarily symmetric) strategy σ_j. In what follows, we will address this issue by imposing some restrictions on the number of bidders in the market.

Suppose that there exists an equilibrium in which both sellers choose $p_1 = p_2 = 0$ and post reserve prices according to some strategy (σ_1^0, σ_2^0). We have already argued that seller 1's payoff $R_1^{(0,0)}(\cdot,\cdot)$ is bounded above by μ because the best possible scenario is one in which $r_2 \geq \mu$ and $r_1 < \mu$ with seller 1's expected profit equal to μ[19]. If seller 1 chooses $p_1 = 1$ and posts some reserve price $r_1 \in \mathscr{R}$ (while seller 2 still chooses $p_2 = 0$ and posts reserve prices according to σ_2^0), his profits would be:

$$R_1^{(1,0)}(r_1, \sigma_2) = \sum_{r_2 \in \mathscr{R}} \sigma_2^0(r_2)\upsilon_1^{(1,0)}(r_1, r_2)$$

where:

$$\upsilon_1^{(1,0)}(r_1, r_2) = v_0(1-q)^n + nr_1q(1-q)^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \tilde{q}^k (1-\tilde{q})^{n-k} T_{1k}^{(1,0)}(r_1, r_2)$$

In this expression, $T_{1k}^{(1,0)}(r_1, r_2)$ is the expected price seller 1 receives when

[19]If both reserve prices are below μ, then:

$$\begin{aligned} R_1^{(0,0)}(r_1, r_2) &= \mu - \left((\mu - v_0)(1-q)^n + n(\mu - r_1)q(1-q)^{n-1}\right) \\ &< \mu \end{aligned}$$

he is matched with exactly k bidders, $k = 2, \ldots, n$, and $q := 1 - F(s_n)$ where s_n satisfies:

$$(s_n - r_1)F^{n-1}(s_n) = (\mu - r_2)(1 - F(s_n))^{n-1}$$

Let

$$H_1^{(1,0)}(t) = F^n(t) + nF^{n-1}(t)(1 - F(t))$$

and rewrite the last term in $\upsilon_1^{(1,0)}(r_1, r_2)$ as:

$$\int_{s_n}^{1} t dH_1^{(1,0)}(t) = 1 - s_n H_1^{(1,0)}(s_n) - \int_{s_n}^{1} H_1^{(1,0)}(t) dt$$

Then,

$$\begin{aligned}
\upsilon_1^{(1,0)}(r_1, r_2) &= v_0(1-q)^n + nr_1 q(1-q)^{n-1} + \int_{s_n}^{1} t dH_1^{(1,0)}(t) \\
&\geq v_0(1-q)^n + nv_0 q(1-q)^{n-1} + \int_{s_n}^{1} t dH_1^{(1,0)}(t) \\
&= v_0 H_1^{(1,0)}(s_n) + \left(1 - s_n H_1^{(1,0)}(s_n) - \int_{s_n}^{1} H_1^{(1,0)}(t) dt\right) \\
&= 1 - (s_n - v_0) H_1^{(1,0)}(s_n) - \int_{s_n}^{1} H_1^{(1,0)}(t) dt
\end{aligned}$$

Lemma 22. *Let $\eta(s_n) = 1 - (s_n - v_0)H_1^{(1,0)}(s_n) - \int_{s_n}^{1} H_1^{(1,0)}(t)dt$. Then, $s_n > s_n'$ implies $\eta(s_n) \leq \eta(s_n')$.*

Proof. In the appendix. □

Let $\tilde{s}_n$ be the solution to Eq. (4.2) when seller 1 and seller 2 post reserve prices equal to v_0. Since higher values of r_2 decreases the value of s_n, $s_n < \tilde{s}_n$ and hence, $\eta(s_n) > \eta(\tilde{s}_n)$ for every $r_2 > v_0$ and every $n \geq 2$ if r_1 is fixed at v_0. Define $\bar{s}_n = m$ (where m is the median of F) whenever F satisfies $F(\mu) \leq 1/2$, and $\bar{s}_n = \tilde{s}_n$ if F satisfies $F(\mu) > 1/2$. In the appendix we check that $\tilde{s}_n$ is bounded above by m whenever $F(\mu) \leq 1/2$ no matter what n is, and that $\tilde{s}_n$ decreases with n whenever $F(\mu) > 1/2$. Thus, $s_n \leq \bar{s}_n$ for all n and hence, $\upsilon_1^{(1,0)}(r_1, r_2) > \eta(\bar{s}_n)$.

Define n_0 to be the value of n such that $F(v_0) - H_1^{(1,1)}(\bar{s}_n) > 0$ if $n > n_0$, and n_1 to be the value of n such that $F(t) - H_1^{(1,1)}(t) > 0$ for all $t \in [\bar{s}_n, 1]$ whenever

$n > n_1$. Then,

$$\begin{aligned} \eta(\bar{s}_n) - \mu &= (\bar{s}_n - v_0)(F(v_0) - H_1^{(1,1)}(\bar{s}_n)) + \int_{\bar{s}_n}^{1} \left\{ F(t) - H_1^{(1,0)}(t) \right\} dt \\ &> 0 \end{aligned}$$

if $n > \max\{n_0, n_1\}$ because lemma 35 ensures that the value $\bar{s}_n$ does not increase with n. Thus, it is profitable for seller 1 to unilaterally deviate and choose $p_1 = 1$ and $\tilde{r}_1 = v_0$ whenever the number of bidders is above the critical value n^*.

Proposition 23. *Consider the sellers' game in which sellers choose binary information structures and post reserve prices from some finite subset $\mathscr{R} \subset [v_0, 1]$ with $v_0 \in \mathscr{R}$. Then, there exists a critical number of bidders n^* such that if $n > n^*$ then this game has no equilibrium in which both sellers choose uninformative structures.*

We now consider the question of existence of an equilibrium in which sellers supply information. This task is complicated because payoff functions depend on a cutoff function for which no closed-form solution is available. Using the results from chapter 2, we can write seller 1's and seller 2's expected payoffs when they both announce $p_1 = p_2 = 1$ and $r_1 \in \mathscr{R}$, $r_2 \in \mathscr{R}$, as:

$$\upsilon_1^{(1,1)}(r_1, r_2; \rho^*) = \tag{4.5}$$

$$\begin{cases} v_0 & \text{if } r_1 = 1 \\ R_1^0(r_1, r_2; \rho^*) + R_1^1(r_1, r_2; \rho^*) + R_1^{2^+}(r_1, r_2; \rho^*) & \text{if } v_0 \le r_1 < 1 \end{cases}$$

where:

$$\begin{aligned} R_1^0(r_1, r_2; \rho^*) &= v_0 G_1^n(r_1) \\ R_1^1(r_1, r_2; \rho^*) &= n r_1 G_1^{n-1}(r_1)(1 - G_1(r_1)) \\ R_1^{2^+}(r_1, r_2; \rho^*) &= n(n-1) \int_{r_1}^{1} t_1 \left[1 - G_1(t_1)\right] \left[G_1(t_1)\right]^{n-2} dG_1(t_1) \end{aligned}$$

and:

$$G_1(t_1) = \left[1 - \int_{t_1}^{1} F(\rho^*(\tau;r))dF(\tau)\right]$$

Likewise,

$$\upsilon_2^{(1,1)}(r_1,r_2;\rho^*) =$$

$$\begin{cases} v_0 & \text{if } r_2 = 1 \\ \tilde{R}_2^0(r_1,r_2;\rho^*) + \tilde{R}_2^1(r_1,r_2;\rho^*) + \tilde{R}_2^{2^+}(r_1,r_2;\rho^*) & \text{if } v_0 \leq r_2 < 1 \end{cases}$$

where:

$$\begin{aligned} \tilde{R}_2^0(r_1,r_2;\rho^*) &= v_0 G_2^n(r_2) \\ \tilde{R}_2^1(r_1,r_2;\rho^*) &= n r_2 G_2^{n-1}(r_2)(1-G_2(r_2)) \\ \tilde{R}_2^{2^+}(r_1,r_2;\rho^*) &= n(n-1)\int_{r_2}^{1} t_2\left[1-G_2(t_2)\right]\left[G_2(t_2)\right]^{n-2} dG_2(t_2) \end{aligned}$$

and:

$$G_2(t_2) = \left[F(t_2)F(\rho^{*-1}(t_2;r)) + \int_{\rho^{*-1}(t_2;r)}^{1} F(\rho^*(\tau;r))f(\tau)d\tau\right]$$

with:

$$\rho^{*-1}(t_2;r) = \begin{cases} 0 & \text{if } t_2 < r_2 \\ \max\{s \in [0,1] : t_2 \geq \rho^*(s;r)\} & \text{if } t_2 \geq r_2 \end{cases}$$

As mentioned before, payoff functions depend on the cutoff function ρ^* for which there is no closed-form solution except in continuation games following histories in which $r_1 = r_2$ (proposition 6 in chapter 2). This makes a direct analysis of equilibria in which sellers post reserve prices using pure strategies very difficult. Therefore, we consider equilibria in which sellers use mixed strategies to post reserve prices.

Proposition 8 in the appendix provides an existence result for games in which bidders have heterogeneous tastes. In the present context, this result implies the existence of a pair of distribution of reserve prices (σ_1^1, σ_2^1) used by sellers to post reserve prices whenever they both announce $p_1 = p_2 = 1$, with the property that no seller has incentives to unilaterally deviate and charge a reserve price outside the support of this distribution.

Lemma 24. *Suppose that both sellers announce $p_1 = p_2 = 1$. Then, there exists a pair of distribution of reserve prices (σ_1^1, σ_2^1) such that seller j does not have incentives to unilaterally deviate and charge a reserve price outside the support of σ_j^1, $j = 1,2$.*

Although no much else can be said about the pair (σ_1^1, σ_2^1), it is not difficult to show that a reserve price of one cannot belong to the support of σ_j^1 because seller j could do better by switching probability mass to some reserve price strictly below one. Lemma C.4 in the appendix confirms this intuition.

Let $R_1^{(1,1)}(\sigma_1, \sigma_2)$ be seller 1's payoff when $p_1 = p_2 = 1$ and sellers post reserve prices according to (σ_1, σ_2):

$$R_1^{(1,1)}(\sigma_1, \sigma_2) = \sum_{r_1 \in \mathscr{R}} \sum_{r_2 \in \mathscr{R}} \sigma_1(r_1)\sigma_2(r_2)\upsilon_1^{(1,1)}(r_1, r_2; \rho^*)$$

Since $r_1 = 1$ receives no positive weight in equilibrium (lemma C.4), $\upsilon_1^{(1,1)}(r_1, r_2; \rho^*)$ can be simply written as:

$$\upsilon_1^{(1,1)}(r_1, r_2; \rho^*) =$$

$$v_0 G_1^n(r_1) + n r_1 G_1^{n-1}(r_1)(1 - G_1(r_1)) + n(n-1)\int_{r_1}^{1} t_1 \left[1 - G_1(t_1)\right]\left[G_1(t_1)\right]^{n-2} dG_1(t_1)$$

where the function ρ^* determines the distribution of valuations conditional on participating, G_1. Let $H_1^{(1,1)}(t)$ be given by:

$$H_1^{(1,1)}(t) = G_1^n(t) + nG_1^{n-1}(t)(1 - G_1(t))$$

Then, $\upsilon_1^{(1,1)}(r_1, r_2)$ becomes:

$$\upsilon_1^{(1,1)}(r_{1,} r_2) = 1 - (r_1 - v_0)G_1^n(r_1) - \int_{r_1}^{1} H_1^{(1,1)}(t)dt \tag{4.6}$$

Suppose that instead of choosing $p_1 = 1$ and posting reserve prices according to σ_1, seller 1 unilaterally deviates and chooses $p_1 = 0$ and some arbitrary $\tilde{r}_1 \in \mathscr{R}$. This deviation induces a continuation equilibrium in which bidders no longer use a common function to select trading partners but a common cutoff value to determine which seller to visit. Thus, bidders whose valuations of item 2 are above the threshold value s^* will visit seller 2 for sure whereas bidders whose valuations of item 2 are below s^* will visit seller 1 with probability one. From our previous discussions, we know that if $\tilde{r}_1 \geq \mu$ then only non-serious bidders will visit seller 1 because they cannot make a positive payoff in this auction. Therefore, seller 1 will retain the object giving him a payoff of v_0 and thus, $R_1^{(0,1)}(\tilde{r}_1, \sigma_2) = v_0$. If $\tilde{r}_1 < \mu$ then bidders with valuations of item 2 below s^* will visit seller 1 for sure, where the value of s^* satisfies:

$$(s^* - r_2)F(s^*)^{n-1} = (\mu - \tilde{r}_1)(1 - F(s^*))^{n-1} \tag{4.7}$$

Thus, seller 1's payoff when he deviates to $p_1 = 0$ given some seller 2's reserve price $r_2 \in \mathscr{R}$ is:

$$\upsilon_1^{(0,1)}(\tilde{r}_1, r_2) = v_0(1-\tilde{q})^n + n\tilde{r}_1\tilde{q}(1-\tilde{q})^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \tilde{q}^k (1-\tilde{q})^{n-k}\mu$$

where $\tilde{q} := F(s^*)$. Simplifying the second term in the right–hand side of this expression gives us:

$$\upsilon_1^{(0,1)}(\tilde{r}_1, r_2) = \mu - (\mu - v_0)(1-\tilde{q})^n - n(\mu - \tilde{r}_1)\tilde{q}(1-\tilde{q})^{n-1}$$

from where it follows that:

$$\begin{aligned} R_1^{(0,1)}(\tilde{r}_1, \sigma_2) &= \sum_{r_2 \in \mathscr{R}} \sigma_2(r_2)\upsilon_1^{(0,1)}(\tilde{r}_1, r_2) \\ &\leq \mu \end{aligned}$$

Thus, a sufficient condition for the existence of an equilibrium with supply of information is that seller's payoffs when they both announce $p_1 = p_2 = 1$ and post reserve prices according to (σ_1^1, σ_2^1) is greater than or equal to μ.

Define $\overline{G}(t)$ by:

$$\overline{G}(t) := (1 - F(v_0)(1 - F(t)))$$

Then,

$$\begin{aligned} G_1(t) &= \left(1 - \int_t^1 F(\rho(t) f(t) dt\right) \\ &\leq \left(1 - F(r_2) \int_t^1 f(t) dt\right) \\ &\leq (1 - F(v_0)(1 - F(t))) \end{aligned}$$

because $F(\rho(t)) \geq F(\rho(r_1)) \geq F(v_0)$ for all $t \in [0,1]$ and all $(r_1, r_2) \in \mathscr{R}^2$. Define n_2 as the lowest value of n such that:

$$\int_{v_0}^1 \left\{F(t) - \overline{G}^n(t) - n\overline{G}^{n-1}(t)(1 - \overline{G}(t))\right\} dt \quad \geq \quad 0$$

Similarly, defined n_3 as the lowest value of n such that:

$$(\bar{r} - v_0)\left(\overline{G}^n(\bar{r}) - F(v_0)\right) \leq 0$$

where $\bar{r}$ is the highest reserve price in $\mathscr{R}$ that is strictly lower than one. Notice that the value of n_3 grows with $\bar{r}$ because values of $\bar{r}$ close to 1 implies values of $\overline{G}$ close to one. Hence, if $n > \max\{n_2, n_3\}$,

$$\begin{aligned} \upsilon_1^{(1,1)}(r_1, r_2) - \mu &= 1 - (r_1 - v_0) G_1^n(r_1) - \int_{r_1}^1 H_1^{(1,1)}(t) dt - \left(1 - \int_0^1 F(t)\right) \\ &\geq \int_0^{v_0} F(t) dt + (r_1 - v_0)(F(v_0) - G_1^n(r_1)) + \int_{r_1}^1 \left\{F(t) - H_1^{(1,1)}(t)\right\} dt \\ &> \int_0^{v_0} F(t) dt + (r_1 - v_0)\left(F(v_0) - \overline{G}_1^n(r_1)\right) + \int_{r_1}^1 \left\{F(t) - \overline{H}_1^{(1,1)}(t)\right\} dt \\ &> 0 \end{aligned}$$

for all $(r_1, r_2) \in \mathscr{R}$. Therefore,

$$\begin{aligned} R_1^{(1,1)}(\sigma_1,\sigma_2) &= \sum_{r_1\in\mathscr{R}}\sum_{r_2\in\mathscr{R}} \sigma_1(r_1)\sigma_2(r_2)\upsilon_1^{(1,1)}(r_1,r_2) \\ &> \mu \end{aligned}$$

provided that $n > \max\{n_2, n_3\}$, and supplying information is an equilibrium of the game.

Proposition 25. *Consider the information provision game in which each seller chooses a binary information structure and posts a reserve price from the set $\mathscr{R}$. Then, there exists a threshold number of bidders n^* such that for all $n > n^*$ the game has an equilibrium in which both sellers supply information.*

4.4 Concluding Remarks

This chapter develops a model in which auctioneers can compete in two dimensions: they can post reserve prices and they can also supply information to bidders. As expected, the addition of reserve price as a second strategic variable makes the characterization of the set of equilibria much harder. Firstly, reserve prices give sellers a tool to extract (part of the) surplus generated when information is supplied to bidders. Secondly, releasing information induces continuation equilibria that are characterized by a function for which no closed–form solution is available (except for the case in which reserve prices are equal). This makes the analysis of equilibria in which sellers use pure strategies very difficult. I show that for a sufficiently rich set of reserve prices the conditions that ensures the non existence of equilibria without provision of information are weaker than similar conditions derived in a model where sellers cannot post reserve prices. I also show that for a sufficiently large number of bidders the unique equilibrium of the game is one in which both sellers supply information.

A natural direction for future work is the extension of the finite set of reserve prices into a continuous set. As mentioned in the main text, the finiteness of the set of reserve prices ensures the existence of an equilibrium regardless of the number of bidders in the market. This is important because existence of equilibria when reserve prices belong to a continuous set requires some additional work on the properties of the cutoff functions used by bidders to select trading partners. Although existence of an equilibrium with a continuum of reserve prices when sellers are assumed to choose informative structures is ensured by proposition 8 in chapter 2, nothing guarantees that such existence result remains valid if in addition to reserve prices sellers can also choose the information structure that will prevail in his auction. Showing that this is indeed the case constitutes the main task for future work in this direction.

Chapter 5

Conclusions

This dissertation studies two elements of competing auction design that are important to understand environments where multiple auctioneers compete for the attention of customers: heterogeneity in bidders' preferences and endogenous information structures. The second chapter of this dissertation analyzes a model of competing auctions in which bidders have heterogeneous preferences whereas the third and fourth chapters are devoted to the analysis of information provision in competing auctions.

This dissertation shows that relaxing the assumption of items being perfect substitutes results in an equilibrium considerably different from the equilibria found in previous research. In particular, bidders' participation decisions are no longer random, which suggests that heterogeneity mitigates coordination failures as a source of friction in the market. Furthermore, a change in a reserve price affects the participation decisions of every bidder regardless of her valuation. This means that bidders with high valuations also modify their participation decisions in response to a change in a reserve price, which adds a novel trade–off between traffic and screening effects not present in models with homogeneous goods.

This dissertation also addresses the issue of endogenous information structures in competing auctions. The main finding is the existence of a class of games that admits a unique equilibrium in which both sellers supply information. This result is at odds with previous findings where a single auctioneer finds optimal to release information only if the number of bidders is sufficiently large. When reserve prices are introduced, there exists a threshold number of bidders that guarantees the existence of a unique equilibrium in which both sellers supply information.

Future work includes extending the current model to a competing auction model with more than two sellers. As the existence of a function that characterizes bidders' participation decision does not appear to hinge on the existence of just two sellers, an outstanding conjecture is that such characterization would still be available if the number of sellers is augmented.

Another direction for future work is the extension of the binary set of information structures into a finite or even a continuous one. The binary case considered in this thesis is special in the sense that it does not allow sellers to fine tune the amount of information (and hence, the amount of informational rents) that they give to bidders. One may conjecture that a richer set of information structures would make harder to support equilibria with supply of information because seller can now exploit a larger number of deviations. In order to verify this conjecture, one first needs to generalize the characterization of bidders' participation game to account for the possibility of endogenous ex-post distribution of valuations. The verification of this conjecture is left for future work.

Bibliography

BAPNA, R., C. DELLAROCAS, AND S. RICE (2010): "Vertically differentiated simultaneous Vickrey auctions: Theory and experimental evidence," *Management Science*, 56(7), 1074–1092.

BERGEMANN, D., AND M. PESENDORFER (2007): "Information structures in optimal auctions," *Journal of Economic Theory*, 137(1), 580–609.

BOARD, S. (2009): "Revealing information in auctions: the allocation effect," *Economic Theory*, 38(1), 125–135.

BULOW, J., J. GEANAKOPLOS, AND P. KLEMPERER (1985): "Multimarket oligopoly: Strategic substitutes and complements," *The Journal of Political Economy*, 93(3), 488–511.

BURGUET, R., AND J. SAKOVICS (1999): "Imperfect Competition in Auction Designs," *International Economic Review*, 40(1), 231–247.

CREMER, J., AND R. MCLEAN (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1247.

DAMIANO, E., AND H. LI (2007): "Information Provision and Price Competition," Mimeo, University of Toronto.

DASGUPTA, P., AND E. MASKIN (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

EPSTEIN, L., AND M. PETERS (1999): "A Revelation Principle for Competing Mechanisms," *Journal of Economic Theory*, 88(1), 19–161.

FORAND, J. G. (2009): “Competing Through Information Provision,” Manuscript. Department of Economics, University of Toronto.

GANUZA, J., AND J. PENALVA (2004): “Optimal Information Transmission in Private Value Auctions,” Discussion paper, Mimeo, Universidad Carlos III, Madrid.

GANUZA, J.-J. (2004): “Ignorance Promotes Competition: An Auction Model of Endogenous Private Valuations,” *RAND Journal of Economics*, 35(3), 583–598.

GANUZA, J. J., AND J. S. PENALVA (2006): “On Information and Competition in Private Value Auctions,” Economics Working Papers 937, Department of Economics and Business, Universitat Pompeu Fabra.

GREEN, J., AND J.-J. LAFFONT (1984): “Participation constraints in the vickrey auction,” *Economics Letters*, 16(1-2), 31–36.

HERNANDO-VECIANA, A. (2005): “Competition among auctioneers in large markets,” *Journal of Economic Theory*, 121(1), 107–127.

IVANOV, M. (2009): “Information Revelation in Competitive Markets,” Mimeo, McMaster University.

KREIDER, D. L., R. KULLER, AND D. OSTBERG (1968): *Elementary Differential Equations*. Addison-Wesley Publishing Company.

LEVIN, D., AND J. SMITH (1994): “Equilibrium in Auctions with Entry,” *The American Economic Review*, pp. 585–599.

LU, J. (2006): “Endogenous entry and auctions design with private participation costs,” MPRA Paper 934, University Library of Munich, Germany.

MAS-COLELL, A., M. WHINSTON, AND J. GREEN (1995): *Microeconomic Theory*. Oxford.

MCAFEE, R., AND J. MCMILLAN (1987): “Auctions with entry,” *Economics Letters*, 23(4), 343–347.

MCAFEE, R. P. (1993): “Mechanism Design by Competing Sellers,” *Econometrica*, 61(6), 1281–1312.

MENEZES, F., AND P. MONTEIRO (2000): "Auctions with endogenous participation," *Review of Economic Design*, 5(1), 71–89.

MILGROM, P., AND R. WEBER (1982): "A theory of auctions and competitive bidding," *Econometrica*, pp. 1089–1122.

MYERSON, R. (1981): "Optimal auction design," *Mathematics of operations research*, 6(1), 58–73.

PETERS, M. (2010): "Competing Mechanisms," Manuscript, The University of British Columbia.

PETERS, M., AND S. SEVERINOV (1997): "Competition Among Sellers who offer Auctions Instead of Prices," *Journal of Economic Theory*, 75, 141–179.

RILEY, J. G., AND W. F. SAMUELSON (1981): "Optimal Auctions," *American Economic Review*, 71(3), 381–92.

SAMUELSON, W. F. (1985): "Competitive bidding with entry costs," *Economics Letters*, 17(1-2), 53–57.

SHAKED, M., AND J. SHANTIKUMAR (1994): *Stochastic Orders and their Applications*. Academic Press.

TAN, G., AND O. YILANKAYA (2006): "Equilibria in Second Price Auctions with Participation Costs," *Journal of Economic Theory*, 130(1), 205–219.

VAGSTAD, S. (2007): "Should auctioneers supply early information for prospective bidders?," *International Journal of Industrial Organization*, 25(3), 597–614.

VIRAG, G. (2010): "Competing auctions: finite markets and convergence," *Theoretical Economics*, 5(2), 241–274.

Appendix A

Appendix for Chapter 2

A.1 Proof of Lemma 1

Pick any $\pi \in S$. If $r_j = 1$, $\mathscr{U}_j(v_j;\pi,r_j) = 0$ for all $v_j \in [0,1]$ and hence, $\mathscr{U}_j$ is continuous and nondecreasing. Let $r_j < 1$. If $\max\{v_j, \hat{v}_j\} < r_j$, where v_j and $\hat{v}_j$ are two valuations of item j, then $\mathscr{U}_j(v_j;\pi,r_j) = \mathscr{U}_j(\hat{v}_j;\pi,r_j) = 0$. If v_j and $\hat{v}_j$ satisfy $v_j \leq r_j < \hat{v}_j$ then $\mathscr{U}_j(v_j;\pi,r_j) = 0 < \mathscr{U}_j(\hat{v}_j;\pi,r_j)$ and $Q_j(\hat{v}_j;\pi,r_j) > 0$ because $F(\hat{v}_j) > F(r_j) \geq 0$. Thus, $\hat{v}_j > v_j$ implies $\mathscr{U}_j(\hat{v}_j;\pi,r_j) > \mathscr{U}_j(v_j;\pi,r_j)$, and $\mathscr{U}_j$ is monotonic. Finally, let v_j and $\hat{v}_j$ satisfy $\min\{v_j, \hat{v}_j\} > r_j$. Incentive compatibility conditions imply that:

$$\mathscr{U}_j(v_j;\pi,r) - \mathscr{U}_j(\hat{v}_j;\pi,r) \geq Q_j(\hat{v}_j;\pi,r_j)(v_j - \hat{v}_j) \tag{A.1}$$

The right-hand side of this expression is strictly positive so long as $v_j > \hat{v}_j$ because $Q_j(\hat{v}_j;\pi,r_j) > 0$. Therefore, $\mathscr{U}_j$ is strictly increasing whenever $v_j > r_j$. Furthermore, incentive compatibility also implies that $\frac{d\mathscr{U}_1(v_1;\pi,r_1)}{dv_1} = Q_1(v_1;\pi,r_1)$ (Myerson, 1981). Since $Q_j(\cdot;\pi,r_j)$ is monotonic, it is Riemann integrable. Therefore, for $v_j > r_j$,

$$\begin{aligned} \mathscr{U}_j(v_j;\pi,r_1) &= \mathscr{U}_j(r_j;\pi,r_j) + \int_{r_j}^{v_j} Q_j(\xi;\pi,r_j)d\xi \\ &= \int_{r_j}^{v_j} Q_j(\xi;\pi,r_j)d\xi \end{aligned}$$

because $\mathscr{U}_j(r_j;\pi,r_j) = 0$. Continuity of this function stems from the fact that its derivative is Lebesgue integrable for all $v_j \in [r_j, 1]$, $r_j \in [v_0, 1)$.

A.2 Proof of Proposition 2

Take any bidder i and any type $(v_1, v_2) \in [0,1]^2$. A necessary and sufficient condition for ω' to be a best response to π is that for every $(v_1, v_2) \in [0,1]^2$, and every $(r_1, r_2) \in [v_0, 1]^2$,

$$\omega'(v,r) = \begin{cases} 0 & \text{if } \mathscr{U}_1(v_1;\pi,r_1) < \mathscr{U}_2(v_2;\pi,r_2) \\ 1 & \text{if } \mathscr{U}_1(v_1;\pi,r_1) > \mathscr{U}_2(v_2;\pi,r_2) \\ \in [0,1] & \text{if } \mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(v_2;\pi,r_2) \end{cases} \tag{A.2}$$

If $r_1 \le r_2 = 1$ then $\mathscr{U}_2(v_2;\pi,r_2) = 0 \le \mathscr{U}_1(v_1;\pi,r_1)$ from Eq. (2.1) regardless of π. Let $\rho(v_1) \equiv 1$. This function is continuous and nondecreasing and $\omega'(v,r) = 1$ if and only if $v_2 \le \rho(v_1)$ because $\mathscr{U}_2(v_2;\pi,r_2) = \mathscr{U}_2(\rho(v_1);\pi,r_2) = 0 \le \mathscr{U}_1(v_1;\pi,r_1)$ for all $(v_1,v_2) \in [0,1]^2$ and non participants are treated as non serious bidders in auction 1. Next, consider the case in which $r_2 < r_1 = 1$. Then, $\mathscr{U}_1(v_1;\pi,r_1) = 0$ for all $v_1 \in [0,1]$ no matter what π is. Since $r_2 < 1$, $\mathscr{U}_2(v_2;\pi,r_2) = 0$ for all $v_2 \in [0,r_2]$ and $\mathscr{U}_2(v_2;\pi,r_2) > 0$ for all $v_2 \in (r_2,1]$. Let $\rho(v_1) = r_2$ for all $v_1 \in [0,1]$. Then, $\omega'(v,r) = 1$ if and only if $v_2 \le \rho(v_1)$ because $\mathscr{U}_2(v_2;\pi,r_2) = \mathscr{U}_2(\rho(v_1);\pi,r_1) = 0$ when $v_2 \le r_2 = \rho(v_1)$ and $\mathscr{U}_1(v_1;\pi,r_1) = 0$ for all $v_1 \in [0,1]$. Clearly, $\rho(v_1) \equiv r_2$ is a continuous and nondecreasing function of v_1.

Let $\max\{r_1,r_2\} < 1$. Since $\mathscr{U}_j(\cdot;\pi,r_j)$ is continuous and v_j belongs to a compact set, $\mathscr{U}_j(\cdot;\pi,r_j)$ is bounded, $j = 1,2$. Let $I_1 = [0,\bar{u}_1]$ and $I_2 = [0,\bar{u}_2]$ be the (compact) image of $\mathscr{U}_1(\cdot;\pi,r_1)$ and $\mathscr{U}_2(\cdot;\pi,r_1)$ on $[0,1]$ respectively. Clearly $I_1 \cap I_2 \neq \emptyset$ because $\mathscr{U}_j(r_j;\pi,r_j) = 0 \in I_j$, $j = 1,2$. Furthermore, from the intermediate value theorem there must exist a number $v_2^* \in [0,1]$ such that for every $u_2 \in I_2$, $\mathscr{U}_2(v_2^*;\pi,r_2) = u_2$, and there must exist some $v_1 \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = u_1$ for every $u_1 \in I_1$. There are two cases of interest.

(i) $I_1 \subseteq I_2$. Then, $u_1 \in I_2$ and hence, we can assign to every $v_1 \in [0,1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(\rho(v_1);\pi,r_2)$. This function ρ has the property that $\omega'(v,r) = 1$ if and only if $v_2 \le \rho(v_1)$ because lemma 1 ensures that $\mathscr{U}_2(v_2;\pi,r_2) = \mathscr{U}_2(\rho(v_1);\pi,r_1) = \mathscr{U}_1(v_1;\pi,r_1) = 0$ whenever $v_1 \le r_1$ and we can assign the same number $\rho(v_1)$ to every such v_1, and $\mathscr{U}_2(v_2;\pi,r_2) < \mathscr{U}_2(\rho(v_1);\pi,r_1) = \mathscr{U}_1(v_1;\pi,r_1)$ whenever $v_1 > r_1$.

(ii) $I_2 \subset I_1$. Then, there are values of $\mathscr{U}_1(\cdot;\pi,r_1)$ that falls outside the range of $\mathscr{U}_2(\cdot;\pi,r_2)$. This means that for sufficiently high values of v_1, the bidder will strictly prefer seller 1 over seller 2. Let $\bar{v}_1$ be implicitly defined by $\mathscr{U}_1(\bar{v}_1;\pi,r_1) = \mathscr{U}_2(1;\pi,r_2)$. Then, a similar argument to the one employed in case (i) above allows to assign to every $v_1 \in [0,\bar{v}_1]$ a number $\rho(v_1) \in [0,1]$ such that $\mathscr{U}_1(v_1;\pi,r_1) = \mathscr{U}_2(\rho(v_1);\pi,r_2)$. For values of v_1 outside $[0,\bar{v}_1]$, we let $\rho(v_1)$ take the value of one such that $\mathscr{U}_1(v_1;\pi,r_1) > \mathscr{U}_2(\rho(v_1);\pi,r_2)$ for every $v_1 > \bar{v}_1$. Then, $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$.

The above two cases show existence of a function ρ such that $\omega'(v,r) = 1$ if and only if $v_2 \leq \rho(v_1)$ and $\omega'(v,r) = 0$ if and only if $v_2 > \rho(v_1)$. Next, we show that this function ρ is nondecreasing. Cases where some reserve price is equal to one are straightforward because the function ρ is constant. Hence, consider the cases where $\max\{r_1;r_2\} < 1$. Take any v_1 and v_1' in $[0,1]$ such that $v_1' < v_1$. From cases (i) and (ii) above, for each v_1 the number $\rho(v_1)$ satisfies $\mathscr{U}_1(v_1;\pi,r_1) \geq \mathscr{U}_2(\rho(v_1);\pi,r_2)$. If $\mathscr{U}_1(v_1';\pi,r_1) > \mathscr{U}_2(\rho(v_1');\pi,r_2)$ then $\rho(v_1') = 1$ and since $v_1' < v_1$, $\mathscr{U}_1(v_1';\pi,r_1) < \mathscr{U}_1(v_1;\pi,r_1)$ and hence, $\rho(v_1') = \rho(v_1) = 1$ from case (ii) above. Thus, let $\mathscr{U}_1(v_1';\pi,r_1) = \mathscr{U}_2(\rho(v_1');\pi,r_2)$ and suppose that $\rho(v_1') > \rho(v_1)$. Take any $\tilde{v}_2$ such that $\rho(v_1) < \tilde{v}_2 < \rho(v_1')$. Then, $\omega'(\tilde{v},r) = 0$, $\tilde{v} = (v_1,\tilde{v}_2)$, because $\rho(v_1) < \tilde{v}_2$ and hence, $\mathscr{U}_2(\tilde{v}_2;\pi,r_2) > \mathscr{U}_1(v_1;\pi,r_1)$ from (A.2). Since lemma 1 ensures that $\mathscr{U}_1(v_1;\pi,r_1) \geq \mathscr{U}_1(v_1';\pi,r_1)$ because $v_1 > v_1'$, we must have $\mathscr{U}_2(\tilde{v}_2;\pi,r_2) > \mathscr{U}_1(v_1';\pi,r_1) = \mathscr{U}_2(\rho(v_1');\pi,r_2)$ which is possible only if $\tilde{v}_2 > \rho(v_1')$ because $\mathscr{U}_2$ is nondecreasing in v_2, a contradiction. Therefore $v_1' < v_1$ must imply $\rho(v_1') \leq \rho(v_1)$ and the function ρ must be nondecreasing.

Finally, we verify that ρ is continuous. Again, cases where $\max\{r_1,r_2\} = 1$ imply ρ constant. Hence, assume that $\max\{r_1;r_2\} < 1$ and suppose that ρ is not continuous. Since ρ is nondecreasing, it must have a countable number of points at which it is discontinuous. Let $\tilde{v}_1$ be any such point. Then, $\lim_{v_1^+ \to \tilde{v}_1} \rho(v_1) < \lim_{v_1^- \to \tilde{v}_1} \rho(v_1) = \rho(\tilde{v}_1)$ because ρ is nondecreasing. Take any $\tilde{v}_2$ such that $\lim_{v_1^+ \to \tilde{v}_1} \rho(v_1) < \tilde{v}_2 < \rho(\tilde{v}_1)$. As $\tilde{v}_2 < \rho(\tilde{v}_1)$ then $\omega'(\tilde{v},r) = 1$ and type $(\tilde{v}_1,\tilde{v}_2)$ should visit seller 1 for sure. Continuity of $\mathscr{U}_1$ with respect to v_1 ensures the existence of some v_1 such that $\mathscr{U}_1(v_1;\pi,r_1)$ is arbitrarily close to $\mathscr{U}_1(\tilde{v}_1;\pi,r_1)$ implying that type $(v_1,\tilde{v}_2)$ should also strictly prefer seller 1 over seller 2. However, $\rho(v_1) < \tilde{v}_2$ because ρ is discontinuous at $\tilde{v}_1$ and $\omega'((v_1,\tilde{v}_2),r) = 0$ a contradiction.

A.3 Proof of Lemma 5

To prove part (1), we only need to show that it holds for reserve prices such that $r_1 < 1$ and $r_2 = 1$ since the other cases are covered in the main text. As $r_2 = 1$, $\mathscr{U}_2(v_2;\rho,r_2) = 0$ no matter what v_2 or ρ is. Therefore, bidder 1 should visit seller 1 with probability one because she expects a payoff of zero if her valuation of item 1 is below r_1 whereas her expected payoff when $v_1 \in (r_1, 1]$ is strictly positive. Thus, $T\rho(v_1) = 1 = \min\{1, r_2\}$ for all $v_1 \in [0,1]$ and all $\rho \in \mathscr{R}$ as claimed.

Next, we prove (i) of part (2). Take any type (v_1, v_2) such that $v_1 \leq r_1$. Then, $\mathscr{U}(v_1;\rho,r_1) = 0$ for all $\rho \in \mathscr{R}$ and hence,

$$\begin{aligned} T\rho(v_1) &= \max\{v_2 \in [0,1] : \mathscr{U}_2(v_2;\rho,r_2) \leq \mathscr{U}_1(v_1;\rho,r_1)\} \\ &= r_2 \end{aligned}$$

for all $v_1 \leq r_1$ and all $\rho \in \mathscr{R}$. To prove (ii) and (iii), let $I_1 = [0, \bar{u}_1]$ and $I_2 = [0, \bar{u}_2]$ be the (compact) image of $\mathscr{U}_1(\cdot;\rho,r_1)$ and $\mathscr{U}_2(\cdot;\rho,r_2)$ on $[r_1,1]$ and $[0,1]$ respectively. It is almost immediate that $I_1 \cap I_2 \neq \emptyset$ because $\mathscr{U}_1(r_1;\rho,r_1) = 0 = \mathscr{U}_2(r_2;\rho,r_2)$, regardless of ρ. Consider the following two cases.

(i) $I_1 \subseteq I_2$. Then, $\mathscr{U}_1(\cdot;\rho,r_1) \in I_2$ for every $v_1 \in [r_1,1]$. From the intermediate value theorem we can assign to every $v_1 \in [r_1,1]$ a number $v_2^* \in [0,1]$ such that $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(v_2^*;\rho,r_2)$. Moreover, the fact that $\mathscr{U}_2$ is increasing in $v_2 > r_2$ and $\mathscr{U}_2(v_2;\rho,r_2) = 0$ for all $v_2 \in [0,r_2]$, $\rho \in \mathscr{R}$, implies that this number must be unique. Since $T\rho(v_1)$ delivers the maximum number such that $\mathscr{U}_2(v_2;\rho) \leq \mathscr{U}_1(v_1;\rho)$ holds, $T\rho(v_1) = v_2^*$, and $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(T\rho(v_1);\rho,r_1)$ for all $v_1 \in [r_1,1]$.

(ii) $I_2 \subset I_1$. Then, there are values of v_1 such that $\mathscr{U}_1(\cdot;\rho,r_1)$ falls outside the range of $\mathscr{U}_2(\cdot;\rho,r_2)$. From Eq. (2.4), $\mathscr{U}_2(1;\rho,r_2) \geq \mathscr{U}_2(v_2;\rho,r_2) > 0$ for any $v_2 \in (r_2,1)$ because $r_2 < 1$ and lemma A.1. Moreover, lemma A.1 ensures that $\mathscr{U}_1$ is a continuous function of v_1 for any $\rho \in \mathscr{R}$. Therefore, there must be a number $\bar{v} \leq 1$ such that $\bar{v} = \max\{v_1 \in [0,1] : \mathscr{U}_1(v_1;\rho,r_1) \leq \mathscr{U}_2(1,\rho,r_2)\}$. Furthermore, the number $\bar{v}$ must be strictly greater than r_1 because $\bar{v} = r_1$ would imply the existence of some other number $\tilde{v} > \bar{v}$ such that $\mathscr{U}_1(\tilde{v};\rho,r_1) = \mathscr{U}_2(1,\rho,r_2)$, contradicting the fact that $\bar{v}$ is the maximum such number. Using a similar argument to the one employed in case (i) above allows to assign to every $v_1 \in [0, \bar{v}_1]$ a number $v_2^* \in [0,1]$ such that $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(v_2^*;\rho,r_2)$, and since $T\rho(v_1)$ delivers the maximum number such that $\mathscr{U}_2(v_2;\rho) \leq \mathscr{U}_1(v_1;\rho)$, $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(T\rho(v_1);\rho,r_1)$ for all $v_1 \in [r_1, \bar{v}]$. For values of v_1 greater than $\bar{v}$, $\mathscr{U}_1(v_1;\rho,r_1) > \mathscr{U}_2(1;\rho,r_2)$ and hence, $T\rho(v_1) = 1$ for all $v_1 \geq \bar{v}$. This completes the proof of (ii) and (iii) of the lemma.

A.4 Proof of Theorem 4

Suppose that $\max\{r_1,r_2\} = 1$. Thus, at least one reserve price is exactly equal to one. From part (1) in lemma 5 the best response operator must satisfy $T\rho(v_1) = \min\{1,r_2\}$ for every $v_1 \in [0,1]$ and every $\rho \in \mathscr{R}$. In particular, this must hold true when we take $\rho^*(v_1) \equiv \min\{1,r_2\}$. Therefore, $T\rho^*(v_1) \equiv \min\{1,r_2\} \equiv \rho^*(v_1)$, which implies that ρ^* must be the unique fixed point of T.

Next, suppose that both reserve prices are strictly less than one, i.e., $\max\{r_1,r_2\} < 1$. Then, part (2) of lemma 5 ensures the existence of some nonempty interval $[r_1, \bar{v}_1]$ such that $\mathscr{U}_1(v_1;\rho,r_1) = \mathscr{U}_2(T\rho(v_1);\rho,r_2)$, $v_1 \in [r_1,\bar{v}_1]$. Since this equation holds for every $v_1 \in [r_1,\bar{v}_1]$, both $\mathscr{U}_1$ and $\mathscr{U}_2$ are continuously differentiable functions, and $T\rho(r_1) = r_2$ from part (i) of lemma 5, the function $T\rho$ must be strictly increasing with respect to v_1 within the interval $[r_1,\bar{v}_1]$ and differentiable everywhere with respect to v_1 in $(r_1,\bar{v}_1)$:

$$\frac{dT\rho(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^{1} F(\rho(t)) f(t) dt}{F(T\rho(v_1)) F(\rho^{-1}(T\rho(v_1)) + \int_{\rho^{-1}(T\rho(v_1))}^{1} F(\rho(t)) f(t) dt} \right)^{n-1}$$

where the numerator (resp. denominator) is the slope of $\mathscr{U}_1$ (resp. $\mathscr{U}_2$), i.e., the probability of trading with seller 1 (resp. seller 2) when bidder 1's valuations are $(v_1, T\rho(v_1))$ and the remaining $(n-1)$ bidders use the function ρ. Moreover, if ρ^* is a fixed point of T, then $T\rho^* = \rho^*$ and the above equation becomes:

$$\frac{d\rho^*(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^{1} F(\rho^*(t)) f(t) dt}{F(\rho^*(v_1)) F(\rho^{*-1}(\rho^*(v_1)) + \int_{\rho^{*-1}(\rho^*(v_1))}^{1} F(\rho^*(t)) f(t) dt} \right)^{n-1}$$

where,

$$\rho^{*-1}(v_2) = \begin{cases} 0 & \text{if } v_2 < \rho(0) \\ \max\{t_1 \in [0,1] : v_2 \geq \rho^*(t_1)\} & \text{if } v_2 \geq \rho(0) \end{cases} \tag{A.3}$$

Lemma 26. *Suppose that there exists a continuous and increasing function* $z : [r_1, 1] \to \mathbb{R}$ *that satisfies:*

$$\begin{aligned} \frac{dz(v_1)}{dt} &= \left(\frac{1 - \int_{v_1}^{1} F(z(t_1)) f(t_1) dt_1}{F(z(v_1))F(v_1) + \int_{v_1}^{1} F(z(\tau)) f(\tau) d\tau} \right)^{n-1} \\ z(r_1) &= r_2 \end{aligned} \tag{A.4}$$

where F *is an absolutely continuous distribution function with strictly positive and bounded density* f*, and support* $[0,1]$ *(and hence,* $F(s) = 0$ *for* $s < 0$ *and* $F(s) = 1$ *for* $s > 1$*), and* $r_1 \in (v_0, 1)$*,* $r_2 \in (v_0, 1)$*,* $v_0 > 0$*. Define* ρ^* *as follows:*

$$\rho^*(v_1) = \begin{cases} r_2 & \text{if } v_1 < r_1 \\ \min\{z(v_1), 1\} & \text{if } v_1 \geq r_1 \end{cases}$$

Then, ρ^* *is a fixed point of* T*.*

Proof. From part (i) in lemma 5, $T\rho^*(v_1) = r_2 = \rho^*(v_1)$ whenever $v_1 < r_1$. Hence, let $v_1 \geq r_1$. Suppose that bidders other than bidder 1 uses the function ρ^* defined in the lemma. From (McAfee, 1993), the probability that bidder 1 trades with seller j when her valuation is v_j is equal to the probability that no other bidder visits seller j plus the probability that any other participant has a valuation lower than v_j. Then,

$$H_1(t_1; \rho^*, r_1) = \left(1 - \int_{t_1}^{1} F(\min\{z(\tau), 1\}) f(\tau) d\tau \right)^{n-1}$$

for all $t_1 \geq r_1$. Since $F(s) = 1$ for all $s \geq 1$, $F(\min\{z(v_1), 1\}) = F(z(v_1))$ for all

$v_1 \in [r_1, 1]$ and hence,

$$\begin{aligned} H_1(t_1;\rho^*,r_1) &= \left(1 - \int_{t_1}^{1} F(\min\{z(\tau),1\}) f(\tau) d\tau\right)^{n-1} \\ &= \left(1 - \int_{t_1}^{1} F(z(\tau)) f(\tau) d\tau\right)^{n-1} \\ &= H_1(t_1;z,r_1) \end{aligned}$$

for all $t_1 \in [r_1, 1]$. Therefore,

$$\begin{aligned} \mathscr{U}_1(v_1;\rho^*,r_1) &= \int_{r_1}^{v_1} \left(1 - \int_{t_1}^{1} F(z(\tau)) f(\tau) d\tau\right)^{n-1} dt_1 \\ &= \mathscr{U}_1(v_1;z,r_1) \end{aligned}$$

Define $\bar{v}_1 = \sup\{v_1 : z(v_1) \leq 1\}$. Since z and ρ^* coincides on the interval $[r_1, \bar{v}_1]$ and z is a strictly increasing function of v_1, ρ^* is invertible on $[r_1, \bar{v}_1]$ and $z(\bar{v}_1) = \rho^*(\bar{v}_1) = 1$. Therefore,

$$\begin{aligned} H_2(t_2;z,r) &= \left(F(t_2)F(z^{-1}(t_2)) + \int_{z^{-1}(t_2)}^{1} F(z(\tau)) f(\tau) d\tau\right)^{n-1} \\ &= \left(F(t_2)F(\rho^{*-1}(t_2)) + \int_{\rho^{*-1}(t_2)}^{1} F(\rho(\tau)) f(\tau) d\tau\right)^{n-1} \\ &= H_2(t_2;\rho^*,r) \end{aligned}$$

for all $t_2 \in [r_2, 1)$. Furthermore,

$$\begin{aligned} H_2(1;z,r) &= \left(F(1)F(z^{-1}(1)) + \int_{z^{-1}(1)}^{1} F(z(\tau)) f(\tau) d\tau\right)^{n-1} \\ &= \left(F(\bar{v}_1) + \int_{\bar{v}_1}^{1} F(1) f(\tau) d\tau\right)^{n-1} \\ &= (F(\bar{v}_1) + 1 - F(\bar{v}_1))^{n-1} \\ &= 1 \\ &= H_2(1;\rho^*,r) \end{aligned}$$

where the second line follows from $z(t_1) \geq 1$ whenever $t_1 \geq \bar{v}_1$ and hence,

$$\int_{z^{-1}(1)}^{1} F(z(\tau))f(\tau)d\tau = 1 - F(\bar{v}_1)$$

It follows that $H_2(t_2;\rho^*,r) = H_2(t_2;z,r)$ for all $t_2 \in [r_2,1]$. Since z satisfies Eq. (A.4),

$$H_2(z(v_1);z,r)\frac{dz(v_1)}{dv_1} = H_1(v_1;z,r)$$

holds for every $v_1 \in [r_1,1]$. Integrating both sides of the above equation with respect to v_1 yields:

$$\int_{r_1}^{v_1} H_2(z(t_1);z,r)\frac{dz(t_1)}{dt_1}dt_1 = \int_{r_1}^{v_1} H_1(t_1;z,r)dt_1$$

Let $u(t_1) = \int_{r_2}^{z(t_1)} H_2(t_2,z,r)dt_2$ such that $du = H_2(z(t_1),z,r_2)\dot{z}(t)dt$. Then,

$$\begin{aligned}
\int_{r_1}^{v_1} H_2(z(t_1);z,r)\frac{dz(t_1)}{dt_1}dt_1 &= \int_{r_1}^{v_1} du \\
&= u(v_1) - u(r_1) \\
&= \int_{r_2}^{z(v_1)} H_2(t_2;z,r_2)dt_2
\end{aligned}$$

because $z(r_1) = r_2$. Therefore, z must also satisfy:

$$\begin{aligned}
\int_{r_2}^{z(v_1)} H_2(t_2;z,r_2)dt_2 &= \int_{r_1}^{v_1} H(t_1;z,r)dt_1 \\
&= \mathscr{U}_1(v_1;z,r_1)
\end{aligned}$$

for all $v_1 \geq r_1$ such that $z(v_1) \leq 1$. Since $H_1(\cdot;z,r) = H_1(\cdot;\rho^*,r)$ and $H_2(\cdot;z,r) = H_2(\cdot;\rho^*,r)$,

$$\begin{aligned}
\int_{r_2}^{z(v_1)} H_2(t_2;z,r_2)dt_2 &= \int_{r_2}^{z(v_1)} H_2(t_2;\rho^*,r_2)dt_2 \\
&= \mathscr{U}_1(v_1;\rho^*,r_1)
\end{aligned}$$

From part (ii) of lemma 5, $\mathscr{U}_2(T\rho^*(v_1);\rho^*,r_2) = \mathscr{U}_1(v_1;\rho^*,r_1)$ for all $v_1 \in$

$[r_1, \bar{v}_1]$. Then,

$$\begin{aligned} \mathscr{U}_2(T\rho^*(v_1);\rho^*,r_2) &= \mathscr{U}_1(v_1;\rho^*,r_1) \\ &= \int_{r_2}^{z(v_1)} H_2(t_2;\rho^*,r_2)dt_2 \end{aligned}$$

for every $v_1 \in [r_1, \bar{v}_1]$. Since $v_1 \geq r_1$, and $H_2(v_2;\rho^*,r_2) > 0$, $\mathscr{U}_2$ is increasing in v_2 on $[r_2, 1]$. Therefore, the value $v_2^* \in [0,1]$ satisfying $\mathscr{U}_1(v_1;\rho^*,r_1) = \mathscr{U}_2(v_2^*;\rho^*,r_2)$ must be unique. It follows that $T\rho^*(v_1) = z(v_1) \leq 1$ for all $v_1 \in [r_1, \bar{v}_1]$. because $H_2(\cdot;\rho^*,r_2)$ and $H_2(\cdot;z,r_2)$ are equal everywhere on $[r_2, 1]$. Suppose that $\bar{v}_1 = 1$. Then, $T\rho^*(v_1) = z(v_1) = \min\{z(v_1);1\} = \rho^*(v_1)$ for all $v_1 \in [r_1, 1]$. Next, suppose that $\bar{v}_1 < 1$. Then $T\rho^*(\bar{v}_1) = z(\bar{v}_1) = 1$ because $T\rho^*(v_1) = 1$ for all $v_1 \in [\bar{v}_1, 1]$ from part (ii) in lemma 5. Since z satisfies Eq. (A.4), it is strictly increasing with respect to $v_1 \in [r_1, 1]$. Therefore, $z(v_1) > z(\bar{v}_1) = 1$ for all $v_1 \in [\bar{v}_1, 1]$ and thus, $\min\{z(v_1);1\} = 1$ whenever $v_1 \in [\bar{v}_1, 1]$. Hence, $T\rho^*(v_1) = 1 = \min\{z(v_1);1\} = \rho^*(v_1)$ if $\bar{v}_1 < 1$. We conclude that $T\rho^*(v_1) = \rho^*(v_1)$ for all $v_1 \in [0,1]$ and ρ^* as defined in the lemma is a fixed point of T. □

The rest of the proof is intended to show existence of a function z.

Proposition 27. *Let F be an absolutely continuous distribution function with support $[0,1]$ (hence, it satisfies $F(s) = 0$ for all $s < 0$, $F(s) = 1$ for all $s > 1$), and strictly positive and bounded density f, and let r_1 and r_2 be scalars satisfying $r_1 \in (v_0, 1)$, $r_2 \in (v_0, 1)$, with $v_0 \in (0,1)$. Then, there exists a unique continuous and increasing function $z^* : [r_1, 1] \to \mathbb{R}$ that satisfies:*

$$\frac{dz^*(v_1)}{dv_1} = \left(\frac{1 - \int_{v_1}^1 F(z^*(t))f(t)dt}{F(z^*(v_1))F(v_1) + \int_{v_1}^1 F(z^*(t))f(t)dt} \right)^{n-1} \tag{A.5}$$

$$z^*(r_1) = r_2 \tag{A.6}$$

Proof. In order to prove the proposition we have to show existence of a continuous function defined on the closed interval $[r_1, 1]$ that satisfies the integro–differential equation (A.5), and the initial condition (A.6). Our plan is to use standard tools from the theory of differential equations to show existence and uniqueness of a pair of continuous functions that solves the following initial value problem:

$$
\begin{aligned}
\frac{d\phi(t)}{dt} &= \left(\frac{1-\phi_2(t)}{F(\phi_1(t))F(t)+\phi_2(t)}\right)^{n-1} \\
\frac{d\phi_2(t)}{dt} &= -F(\phi_1(t))f(t) \\
\phi_1(r_1) &= r_2 \\
\phi_2(r_1) &= \theta
\end{aligned}
$$

where $\theta \in (0,1)$ is a constant parameter. Second, we will use this family of solutions indexed by θ with typical element $(\phi_1^\theta;\phi_2^\theta)$, to show existence of a unique root θ^* to the equation:

$$
\phi_2^\theta(1) = 0 \tag{A.7}
$$

that will allow us to uniquely express $\phi_2^{\theta^*}$in terms of $\phi_1^{\theta^*}$:

$$
\begin{aligned}
\phi_2^{\theta^*}(t) &= \phi_2^{\theta^*}(1) + \int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt \\
&= \int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt
\end{aligned}
$$

Third, after replacing $\phi_2^{\theta^*}$ into the above initial value problem we will obtain a unique continuous and increasing function $\phi_1^{\theta^*}$ that satisfies:

$$
\begin{aligned}
\frac{d\phi_1^{\theta^*}(t)}{dt} &= \left(\frac{1-\int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt}{F(\phi_1^{\theta^*}(t))F(t)+\int_t^1 F(\phi_1^{\theta^*}(t))f(t)dt}\right)^{n-1} \\
\phi_1^{\theta^*}(r_1) &= r_2
\end{aligned}
$$

Lemma 28. *There exists a unique pair of continuous functions defined for all $t \in [r_1, 1]$ that solves the following initial value problem:*

$$
\begin{aligned}
\frac{d\phi_1(t)}{dt} &= \left(\frac{1-\phi_2(t)}{F(\phi_1(t))F(t)+\phi_2(t)}\right)^{n-1} \\
\frac{d\phi_2(t)}{dt} &= -F(\phi_1(t))f(t) \\
\phi_1(r_1) &= r_2 \\
\phi_2(r_1) &= \theta
\end{aligned}
$$

with $\theta \in (0,1)$.

Proof. Consider the domain:

$$D = \left\{(t,y_1,y_2) \in \mathbb{R}^3 : v_0 \leq t \leq 1; v_0 \leq y_1 < \infty; 0 \leq y_2 < \infty\right\}$$

and the mapping $h : D \to \mathbb{R}^2$:

$$
\begin{aligned}
h(t,\mathbf{y}) &= [h_1(t,\mathbf{y}); h_2(t,\mathbf{y})] \\
h_1(t,\mathbf{y}(t)) &= \left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1} \\
h_2(t,\mathbf{y}(t)) &= -F(y_1)f(t)
\end{aligned}
$$

where $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. Notice that the denominator of h_1 is positive on D because y_1 and t both bounded away from zero and F increasing ensure that $F(y_1)F(t) + y_2 \geq F^2(v_0) > 0$ for all $(t,\mathbf{y}) \in D$.

Claim 29. The mapping $h(t,\mathbf{y})$ is uniformly continuous with respect to t, bounded, and Lipschitz continuous in $\mathbf{y}$ on D.

Proof. As F is a continuous function, $F(s) > 0$ for $s > 0$, $y_1 \geq v_0 > 0$, and $t \geq r_1 > 0$, $h_1(t,\mathbf{y})$ and $h_2(t,\mathbf{y})$ are continuous functions with respect to t. Moreover, t belongs to the compact interval $[r_1,1]$ and by the Heine–Cantor theorem, both $h_1(t,\mathbf{y})$ and $h_2(t,\mathbf{y})$ must be uniformly continuous functions of t.

In what follows, if $x \in \mathbb{R}$ then $|x|$ denotes Euclidean norm in $\mathbb{R}$ whereas if $x \in \mathbb{R}^n$ $|x|$ denotes the 1-norm, i.e., $|x| := |x|_1 = \sum_{i=1}^{2} |x_1|$. We now show that there exists a constant $B > 0$ such that $|h(t,\mathbf{y})| \leq B$ for all $(t,\mathbf{y}) \in D$. First, let $\bar{f}$ be a bound for f. As $F(s) = 1$ for all $s \geq 1$, then $|-F(y_1)f(t)| \leq \bar{f}$ for all $(t,\mathbf{y}) \in D$. Second, for every $(t,\mathbf{y}) \in D$:

$$\begin{aligned} \frac{1-y_2}{F(y_1)F(t)+y_2} &\leq \frac{1-y_2}{F^2(v_0)} \\ &\leq \frac{1}{F^2(v_0)} \end{aligned}$$

because $F(y_1)F(t)+y_2 \geq F^2(c)$ and $y_2 \geq 0$. Hence, $(1/F^2(v_0))^{n-1}$ is an upper bound for h_1. Similarly,

$$\frac{1-y_2}{F(y_1)F(t)+y_2} \geq \frac{1-y_2}{1+y_2}$$

since $F(s) \leq 1$ for all $s \in \mathbb{R}$. Let y_2 tend to ∞. It is not difficult to check that the right–hand side of this expression tends to -1. Therefore,

$$\begin{aligned} \frac{1-y_2}{F(y_1)F(t)+y_2} &\geq \frac{1-y_2}{1+y_2} \\ &> -1 \end{aligned}$$

and $h_1(t,\mathbf{y}) \leq (-1)^{n-1}$ for all $(t,\mathbf{y}) \in D$. Since $F(v_0) < 1$, $-\left(\frac{1}{F^2(v_0)}\right)^{n-1} < (-1)^{n-1}$ for any $n \geq 2$ finite. It follows that $|h_1(t,\mathbf{y})| \leq \left(\frac{1}{F^2(v_0)}\right)^{n-1}$ and $B = \max\left\{\left(\frac{1}{F^2(v_0)}\right)^{n-1}; \bar{f}\right\}$ is a bound for $h(t,\mathbf{y})$, $(t,\mathbf{y}) \in D$.

Finally, we show that $h(t,\mathbf{y})$ is Lipschitz continuous with respect to $\mathbf{y}$ on D. To demonstrate this, we need to produce a positive constant $M > 0$, independent of $(t,\mathbf{y}) \in D$, satisfying:

$$|h(t;\mathbf{y}_1) - h(t;\mathbf{y}_2)| \leq M\,|\mathbf{y}_1 - \mathbf{y}_2|$$

for every $(t,\mathbf{y}_1) \in D$ and $(t,\mathbf{y}_2) \in D$. Simple computations yield:

$$\frac{\partial h_1(t,\mathbf{y})}{\partial y_1} = -(n-1)\left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1}\left(\frac{(1-y_2)f(y_1)F(t)}{(F(y_1)F(t)+y_2)^2}\right) \quad (A.8)$$

$$\frac{\partial h_1(t,\mathbf{y})}{\partial y_2} = -(n-1)\left(\frac{1-y_2}{F(y_1)F(t)+y_2}\right)^{n-1}\left(\frac{F(y_1)F(t)+1}{(F(y_1)F(t)+y_2)^2}\right) \quad (A.9)$$

$$\frac{\partial h_2(t,\mathbf{y})}{\partial y_1} = -f(y_1)f(t) \quad (A.10)$$

It is almost immediate that $|-f(y_1)f(t)| \leq \bar{f}^2$ and hence, $\frac{\partial h_2(t,\mathbf{y})}{\partial y_1}$ is bounded by $\bar{f}^2$. Next,

$$\left|\frac{1-y_2}{F(y_1)F(t)+y_2}\right| \leq \frac{1}{F^2(v_0)}$$

and,

$$\begin{aligned}\left|\frac{(1-y_2)f(y_1)F(t)}{(F(y_1)F(t)+y_2)^2}\right| &= \left|\frac{(1-y_2)}{F(y_1)F(t)+y_2}\right|\left|\frac{f(y_1)F(t)}{F(y_1)F(t)+y_2}\right| \\ &\leq \left(\frac{1}{F^2(v_0)}\right)\left(\frac{\bar{f}}{F^2(v_0)}\right)\end{aligned}$$

and,

$$\left|\frac{F(y_1)F(t)+1}{(F(y_1)F(t)+y_2)^2}\right| \leq \frac{2}{F^2(v_0)}$$

for every $(t,\mathbf{y}) \in D$. Therefore,

$$\begin{aligned}\left|\frac{\partial h_1(t,\mathbf{y})}{\partial y_1}\right| &\leq (n-1)\left(\frac{1}{F^2(v_0)}\right)^{n-1}\left(\frac{\bar{f}}{F^2(v_0)}\right) \\ &= m_1 \\ \left|\frac{\partial h_1(t,\mathbf{y})}{\partial y_2}\right| &\leq (n-1)\left(\frac{1}{F^2(v_0)}\right)^{n-1}\left(\frac{2}{F^2(v_0)}\right) \\ &= m_2 \\ \left|\frac{\partial h_2(t,\mathbf{y})}{\partial y_1}\right| &\leq \bar{f}^2 \\ &= m_3\end{aligned}$$

and all these three derivatives are continuous and bounded functions in D, with bounds independent of $(t,\mathbf{y}) \in D$. Set $M = \max\{m_1, m_2, m_3\} > 0$. Then, standard arguments imply that:

$$|h(t,\mathbf{y}_1) - h(t,\mathbf{y}_2)| \leq M\,|\mathbf{y}_1(t) - \mathbf{y}_2(t)|$$

holds true for every $(t,\mathbf{y}_1) \in D$ and $(t,\mathbf{y}_2) \in D$. □

Consider the space $\mathscr{C}$ of continuous vector–valued functions $\phi = (\phi_1, \phi_2)$, $\phi_i : [r_1, 1] \to \mathbb{R}$, $i = 1, 2$, equipped with the sup norm, $\|\phi\| = \sup\{|\phi(t)|\,; t \in [r_1, 1]\}$. Let $\mathscr{D} = \{\phi \in \mathscr{C} : \|\phi - \phi_0\| \leq B; \phi_0 = (r_2, \theta)\} \subset \mathscr{C}$, where $B = \max\left\{\left(\frac{1}{F^2(v_0)}\right)^{n-1}; \bar{f}\right\}$ be the subset of continuous and increasing functions whose graph belong to $\mathscr{D}$. Then, $\mathscr{D}$ is a complete metric space because it is a closed subset of a complete metric space. Define the operator K by:

$$\begin{aligned} K\phi(t) &= \phi_0 + \int_{r_1}^{t} h(\tau, \phi(\tau))d\tau \\ \phi_0 &= (r_2, \theta) \end{aligned}$$

Claim 30. K maps $\mathscr{D}$ into itself.

Proof. First, from claim 2 the mapping $h(t,\phi(t))$ is continuous in t on D. Since the integral sign preserves continuity, $K\phi(t)$ must also be continuous in $t \in [r_1, 1]$ when $\phi \in \mathscr{D}$. Second,

$$\begin{aligned} |K\phi(t) - \phi_0(t)| &\leq \int_{r_1}^{t} |h(\tau, \phi(\tau))|\,d\tau \\ &\leq (t - r_1)B \\ &< B \end{aligned}$$

and B is an upper bound of $|K\phi(t) - \phi_0(t)|$, $t \in [r_1, 1]$. Hence,

$$\begin{aligned} \|K\phi - \phi_0\| &= \sup\{|K\phi(t) - \phi_0(t)|\,; t \in [r_1, 1]\} \\ &\leq B \end{aligned}$$

and $K\mathbf{Y} \in \mathscr{D}$. Therefore, for any given $\phi \in \mathscr{D}$ the operator K delivers a continuous function that satisfies $\|K\phi - \phi_0\| \leq B$ and hence, $K\phi \in \mathscr{D}$. □

Claim 31. Let $\mathscr{L}(t) = M\int_{r_1}^{t} d\tau$. For every $t \in [r_1, 1]$ the operator K satisfies:

$$|K^m\phi(t) - K^m\rho(t)| \leq \frac{\mathscr{L}(t)^m}{m!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \tag{A.11}$$

where $m \in \mathbb{N}_0$, $K^m\phi(t) = K[K^{m-1}\phi](t)$, $K^0\phi(t) = \phi$, and $\phi \in \mathscr{D}$, $\rho \in \mathscr{D}$.

Proof. Set $m = 1$. Then,

$$\begin{aligned} |K\phi(t) - K\rho(t)| &\leq \int_{r_1}^{t} |h(\tau,\phi(\tau)) - h(\tau,\phi(\tau))|\, d\tau \\ &\leq \int_{r_1}^{t} M\, |\phi(\tau) - \rho(\tau)|\, d\tau \\ &\leq \mathscr{L}(t) \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

where the second inequality follows from claim 2. Next, suppose that inequality (A.11) holds for some $m > 1$. Then,

$$\begin{aligned} \left|K^{m+1}\phi(t) - K^{m+1}\rho(t)\right| &= |K[K^m\phi](t) - K[K^m\rho](t)| \\ &\leq \int_{r_1}^{t} |h(\tau, K^m\phi(\tau)) - h(\tau, K^m\rho)|\, d\tau \\ &\leq \int_{r_1}^{t} M\, |K^m\phi(\tau) - K^m\rho(\tau)|\, d\tau \\ &\leq \int_{r_1}^{t} M \frac{\mathscr{L}(\tau)^m}{m!} \sup_{s\in[r_1,\tau]} |\phi(s) - \rho(s)|\, d\tau \\ &= \int_{r_1}^{t} \mathscr{L}'(\tau) \frac{\mathscr{L}(\tau)^m}{m!} \sup_{s\in[r_1,\tau]} |\phi(s) - \rho(s)|\, d\tau \\ &= \frac{\mathscr{L}(t)^{m+1}}{(m+1)!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

where the third line follows from claim 2, the fourth line follows from the induction hypothesis, and the sixth line follows from integration by substitution. This shows that (A.11) also holds for $m+1$ and hence, it must hold for any $m \in \mathbb{N}_0$. $\square$

Let $\theta_m = \frac{\mathscr{L}(1)^m}{m!}$. Observe that $\sum_{m=1}^{\infty} \theta_m < \infty$ and hence, this sum converges. Moreover,

$$\begin{aligned} |K^m\phi(t) - K^m\rho(t)| &\leq \frac{\mathscr{L}(t)^m}{m!} \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \\ &\leq \theta_m \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)| \end{aligned}$$

and $\theta_m \sup_{t\in[r_1,1]} |\phi(t) - \rho(t)|$ is a bound for $|K^m\phi(t) - K^m\rho(t)|$. Therefore,

$$\|K^m\phi(t) - K^m\rho(t)\| \leq \theta_m \|\phi(t) - \rho(t)\|$$

and since $\theta_m \to 0$ as $m \to \infty$, there is some m^* such that $K^{m^*}\phi$ is a contraction. Therefore, by theorem 9-9 in (Kreider, Kuller, and Ostberg, 1968) there exists a unique fixed point ϕ^θ of K. Since ϕ^θ is a fixed point of K its graph must belong to $\mathscr{D}$, which implies that ϕ^θ is defined for all $t \in [r_1, 1]$. $\square$

From lemma 2 there exists a unique vector–valued function $\phi^\theta = (\phi_1^\theta, \phi_2^\theta)$ that satisfies:

$$\frac{d\phi_1^\theta(t)}{dt} = \left(\frac{1-\phi_2^\theta(t)}{F(\phi_1^\theta(t))F(t) + \phi_2^\theta(t)}\right)^{n-1} \tag{A.12}$$

$$\frac{d\phi_2^\theta(t)}{dt} = -F(\phi_1^\theta(t))f(t) \tag{A.13}$$

$$\phi_1^\theta(r_1) = r_2 \tag{A.14}$$

$$\phi_2^\theta(r_1) = \theta \tag{A.15}$$

Consider the relation:

$$\phi_2^\theta(1) = 0$$

We want to show that there exists a unique $\theta^* \in (0,1)$ that makes the above relation hold true. We begin by showing existence of such root. Integrate Eq.

(A.13) between r_1 and t to obtain:

$$\phi_2^\theta(t) = \theta - \int_{r_1}^{t} F(\phi_1^\theta(\tau)) f(\tau) d\tau$$

Since $\phi_2^\theta(r_1) = \theta < 1$, and $\phi_2^\theta(t) < \phi_2^\theta(r_1)$ because of Eq. (A.13), $\phi_2^\theta(t) < 1$ for all $t \in [r_1, 1]$. Hence, $1 - \phi_2^\theta(t) > 0$ and ϕ_1^θ is increasing with respect to $t \in [r_1, 1]$. Moreover, F is increasing and $\phi_1^\theta(t) \geq v_0$; then $F(\phi_1^\theta(t)) > F(v_0)$, $t \in (r_1, 1]$. Let $\tilde{\theta}$ be any value of θ living in the open interval $(0, F(v_0)(1 - F(r_1))$. Then,

$$\begin{aligned}
\phi_2^{\tilde{\theta}}(1) &= \tilde{\theta} - \int_{r_1}^{1} F(\phi_1^{\tilde{\theta}}(\tau)) f(\tau) d\tau \\
&< \tilde{\theta} - \int_{r_1}^{1} F(v_0) f(\tau) d\tau \\
&= \tilde{\theta} - F(v_0)(1 - F(r_1)) \\
&< F(v_0)(1 - F(r_1) - 1 + F(r_1)) \\
&= 0
\end{aligned}$$

and $\phi_2^\theta(1)$ must be negative for values of θ close to zero. Likewise, let $\hat{\theta}$ live in the open interval $(1 - F(r_1), 1)$. Then,

$$\begin{aligned}
\phi_2^{\hat{\theta}}(1) &= \hat{\theta} - \int_{r_1}^{1} F(\phi_1^{\hat{\theta}}(\tau)) f(\tau) d\tau \\
&\geq \hat{\theta} - \int_{r_1}^{1} f(\tau) d\tau \\
&= \hat{\theta} - (1 - F(r_1)) \\
&> (1 - F(r_1)) - (1 - F(r_1)) \\
&= 0
\end{aligned}$$

since $F(s) = 1$ for all $s \geq 1$ and hence, $F(\phi_1^\theta(t)) \leq 1$ for all $t \in [0, 1]$ and $\theta \in (0, 1)$. Therefore, $\phi_2^\theta(1)$ must be negative for values of θ close to zero and positive for values of θ close to one, implying the existence of some $\theta^* \in (0, 1)$ such that $\phi_2^{\theta^*}(1) = 0$. Uniqueness follows from the next claim.

Claim 32. ϕ_1^θ is decreasing and ϕ_2^θ is increasing in $\theta \in (0,1)$ for every $t \in [r_1, 1]$.

Proof. The proof of the claim is by contradiction. Standard considerations in the theory of differential equations (e.g. Theorem 9-12 in (Kreider, Kuller, and Ostberg, 1968)) ensures that $\phi^\theta = (\phi_1^\theta, \phi_2^\theta)$ is continuously differentiable with respect to $\theta \in (0,1)$. Furthermore, $\delta^\theta = (\delta_1^\theta, \delta_2^\theta)$, $\delta_i^\theta = \frac{d\phi_i^\theta(t)}{d\theta}$, must solve the following initial value problem:

$$\frac{d\delta_1^\theta(t)}{dt} = -(n-1)\left(\frac{1-\phi_2^\theta(t)}{F(\phi_1^\theta(t))F(t)+\phi_2^\theta(t)}\right)^{n-2} \times \left(\frac{\delta_1^\theta(t)(1-\phi_2^\theta(t))f(\phi_1^\theta(t))F(t)+\delta_2^\theta(t)(F(\phi_1^\theta(t))F(t)+1)}{(F(\phi_1^\theta(t))F(t)+\phi_2^\theta(t))^2}\right) \tag{A.16}$$

$$\frac{d\delta_2^\theta(t)}{dt} = -f(\phi_1^\theta(t))f(t)\delta_1^\theta(t) \tag{A.17}$$

$$\delta_1^\theta(r_1) = 0 \tag{A.18}$$

$$\delta_2^\theta(r_1) = 1 \tag{A.19}$$

Suppose that there exists some $t \in [r_1, 1]$ such that $\delta_1^\theta(t) > 0$. Since $\delta_1^\theta(r_1) = 0$ and $\left.\frac{d\delta_1^\theta(t)}{dt}\right|_{t=r_1} < 0$, there is some $\varepsilon > 0$ such that $\delta_1^\theta(r_1) = 0$ and $\delta_1^\theta(t) < 0$ for all $t \in [r_1, \varepsilon]$. Thus, if $\delta_1^\theta(t) > 0$ at some t^*, it must be the case that $t^* > r_1$, $\delta_1^\theta(t) < 0$ for $t \in (r_1, t^*)$, and $\delta_1^\theta(t^*) = 0$. This requires the slope of δ_1^θ at t^* to be positive because $\delta_1^\theta(r_1) = 0$ and δ_1^θ must cross the x-axis at t^* from below. Evaluating $\frac{d\delta_1^\theta(t)}{dt}$ at $t = t^*$ yields:

$$\frac{d\delta_1^\theta(t^*)}{dt} = -(n-1)\left(\frac{1-\phi_2^\theta(t^*)}{F(\phi_1^\theta(t^*))F(t^*)+\phi_2^\theta(t^*)}\right)^{n-2}\left(\frac{\delta_2^\theta(t^*)(F(\phi_1^\theta(t^*))F(t^*)+1)}{(F(\phi_1^\theta(t^*))F(t^*)+\phi_2^\theta(t^*))^2}\right) \tag{A.20}$$

I claim that this expression is strictly negative. Integration of Eq. (A.13) between r_1 and t yields:

$$\phi_2^\theta(t) = \theta - \int_{r_1}^{t} F(\phi_1^\theta(\tau))f(\tau)d\tau$$

and hence,

$$\left(\frac{1-\phi_2^\theta(t^*)}{F(\phi_1^\theta(t^*))F(t^*)+\phi_2^\theta(t^*)}\right)^{n-2} = \left(\frac{1-\theta+\int_{r_1}^{t^*}F(\phi_1^\theta(\tau))f(\tau)d\tau}{F(\phi_1^\theta(t^*))F(t^*)+\theta+\int_{r_1}^{t^*}F(\phi_1^\theta(\tau))f(\tau)d\tau}\right)^{n-2} > 0$$

because $\int_{r_1}^1 F(\phi_1^\theta(\tau))f(\tau)d\tau > F(v_0)(1-F(r_1)) > 0$ and $1-\theta>0$. Second, integration of Eq. (A.17) between r_1 and t gives:

$$\delta_2^\theta(t) = 1-\int_{r_1}^t f(\phi_1^\theta(\tau))\delta_1^\theta(\tau)f(\tau)d\tau$$

since $\delta_2^\theta(r_1)=1$. As $\delta_1^\theta(t)<0$ for all $t\in(r_1,t^*)$ and $f>0$, $\delta_2^\theta(t^*)>0$. Therefore, both terms within brackets in Eq. (A.20) are positive, from where it follows that $\frac{d\delta_1^\theta(t^*)}{dt}<0$. This creates the contradiction needed to complete the proof of the claim. □

All the above implies the existence of a unique θ^* such that $\phi_2^{\theta^*}(1)=0$. Let $\phi^{\theta^*}=(\phi_1^{\theta^*},\phi_2^{\theta^*})$ be the unique solution to our initial value problem when θ takes the value θ^*. Integrating Eq. (A.13) between t and one yields:

$$\begin{aligned}\phi_2^{\theta^*}(t) &= \phi_2^{\theta^*}(1)+\int_t^1 F(\phi_1^{\theta^*}(\tau)f(\tau)d\tau\\ &= \int_t^1 F(\phi_1^{\theta^*}(\tau)f(\tau)d\tau\end{aligned}$$

because $\phi_2^{\theta^*}(1)=0$. Therefore, if we let $\phi_1^{\theta^*}(t):=z^*(t)$, the function z^* must be the unique increasing and continuous function that satisfies:

$$\begin{aligned}\frac{dz^*(t)}{dt} &= \left(\frac{1-\int_t^1 F(z^*(t))f(t)dt}{F(z^*(t))F(t)+\int_t^1 F(z^*(t))f(t)dt}\right)^{n-1}\\ z^*(r_1) &= r_2\end{aligned}$$

which is the desired result. □

A.5 Proof of Proposition 6

It is immediate that ρ^* satisfies the conditions imposed by theorem 4 in any continuation game that follows a pair of reserve prices where $\max\{r_1; r_2\} = 1$. Hence, let $r_1 = r_2 < 1$ and suppose that $v_0 > 0$. From proposition 26, there exists a unique continuous and increasing function $z : [r_1, 1] \to \mathbb{R}$ satisfying:

$$\frac{d}{dt}z(t) = \left(\frac{1 - \int_t^1 F(z(\tau))f(\tau)d\tau}{F(z(t))F(t) + \int_t^1 F(z(\tau))f(\tau)d\tau}\right)^{n-1}$$

for $t \in [r_1, 1]$ and $z(r_1) = r_2$. Let $\tilde{z}(v_1) = v_1$, $v_1 \in [r_1, 1]$. Then,

$$\begin{aligned} 1 - \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau &= 1 - \int_t^1 F(\tau)f(\tau)d\tau \\ &= 1 - \left(\frac{1}{2} - \frac{F^2(t)}{2}\right) \\ &= \frac{1}{2} + \frac{F^2(t)}{2} \end{aligned}$$

Likewise,

$$\begin{aligned} F(\tilde{z}(t))F(t) + \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau &= F^2(t) + \frac{1}{2} - \frac{F^2(t)}{2} \\ &= \frac{1}{2} + \frac{F^2(t)}{2} \end{aligned}$$

Thus,

$$\left(\frac{1 - \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau}{F(\tilde{z}(t))F(t) + \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau}\right)^{n-1} = 1$$

and since $\frac{d}{dt}\tilde{z}(t) = 1$,

$$\begin{aligned} \frac{d}{dt}\tilde{z}(t) &= 1 \\ &= \left(\frac{1 - \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau}{F(\tilde{z}(t))F(t) + \int_t^1 F(\tilde{z}(\tau))f(\tau)d\tau}\right)^{n-1} \end{aligned}$$

which shows that $\tilde{z}$ must be the unique function satisfying the conditions described in this proposition. Furthermore, as $\tilde{z}(t) \leq 1$ for all $t \in [r_1, 1]$ and $\tilde{z}(r_1) = r_1 = r_2$ because of the symmetry of reserve prices, the mapping ρ^* defined by:

$$\rho^*(v_1) = \begin{cases} \min\{1; r_2\} & \text{if } \max\{r_1; r_2\} = 1 \\ \varphi^*(v_1) & \text{if } \max\{r_1; r_2\} < 1 \end{cases}$$

must satisfy all the conditions in Theorem 4 and thus, π defined above must be the unique equilibrium of any continuation game in which $r_1 = r_2 < 1$ and $v_0 > 0$.

Now consider the case in which $r_1 = r_2 = 0$. From the main text, $\rho^*(v_1) = v_1$, $v_1 \in [0,1]$, is a fixed point of the best response operator given by Eq. 2.6 when bidders use the function ρ^*. To show uniqueness, suppose that there exists some function $\hat{\rho}(v_1)$ and some nonempty interval $\Omega \subseteq [0,1]$ such that $\rho^*(v_1) \neq \hat{\rho}(v_1)$ for all $v_1 \in \Omega$, and such that $T\hat{\rho} = \hat{\rho}$. Since $\hat{\rho}$ is also a fixed point of T, then $T\hat{\rho}(0) = \hat{\rho}(0) = 0$ because of part (i) of lemma 5. Moreover, from part (ii) of this same lemma the mapping $T\hat{\rho}$ must satisfy:

$$\begin{aligned} \frac{dT\hat{\rho}(v_1)}{dv_1} &= \frac{d\hat{\rho}(v_1)}{dv_1} \\ &= \left(\frac{1 - \int_{v_1}^{1} F(\hat{\rho}(t)) f(t) dt}{F(\hat{\rho}(v_1)) F(\hat{\rho}^{-1}(\hat{\rho}(v_1))) + \int_{\hat{\rho}^{-1}(\hat{\rho}(v_1))}^{1} F(\hat{\rho}(t)) f(t) dt} \right)^{n-1} \end{aligned}$$

for every v_1 within some interval $[0, \bar{v}]$. If $\Omega = [0, \bar{v}]$ then either $\rho^*(v_1) > \hat{\rho}(v_1)$ or $\rho^*(v_1) < \hat{\rho}(v_1)$ for every $v_1 \in (0, \bar{v})$. Suppose that $\rho^*(v_1) > \hat{\rho}(v_1)$. Then, $\frac{d\hat{\rho}(v_1)}{dv_1} < 1$ and hence,

$$\begin{aligned} \hat{\rho}(v_1) &= \int_0^{v_1} \frac{d\hat{\rho}(t)}{dt} dt \\ &< \int_0^{v_1} dt \\ &= \rho^*(v_1) \end{aligned}$$

for every $v_1 \in (0, \bar{v}_1)$, a contradiction. The same argument can be used to generate a contradiction in case that $\Omega = [0, \bar{v}]$ and $\rho^*(v_1) < \hat{\rho}(v_1)$. Hence, suppose that $\Omega \neq [0, \bar{v}_1]$. Since $\rho^*(0) = \hat{\rho}(0) = 0$ and $\rho^* \neq \hat{\rho}$ within the interval Ω, the function

$\hat{\rho}$ must cross the 45 degree line at least once within $[0,1]$. Let $v^s \in (0,1)$ be the highest point at which $\hat{\rho}$ crosses the 45 degree line. Then either $\rho^*(v_1) > \hat{\rho}(v_1)$ or $\rho^*(v_1) < \hat{\rho}(v_1)$ for every $v_1 \in (v^s, \bar{v})$ because v^s is the highest point (strictly less than one) at which $\hat{\rho}$ crosses ρ^*. However, as $\rho^*(v_1) > \hat{\rho}(v_1)$ or $\rho^*(v_1) < \hat{\rho}(v_1)$ for every $v_1 \in (v^s, \bar{v})$ we can find some $\tilde{v}$ strictly within this interval at which $\frac{d\hat{\rho}(\tilde{v}_1)}{dv_1} \neq 1$ holds and we can create a contradiction similar to that in the case where $\Omega = [0, \bar{v}]$. We conclude that $\rho^*(v_1) = \hat{\rho}(v_1)$ for all $v_1 \in [0,1]$ from where it follows that ρ^* must be the unique fixed point of T when $r_1 = r_2 = 0$.

A.6 Proof of Proposition 7

We prove proposition 7 as a special case of the following result.

Proposition 33. *For any pair of reserve prices* $(r_1, r_2) \in [v_0, 1]^2$ *let* $\rho(v_1;(r_1,r_2))$ *be the function used by bidders to select trading partners when reserve prices are* (r_1, r_2)*. Then,*

1. *For any fixed* $r_2 \in [v_0, 1]$*, if* $r_1 < \hat{r}_1$ *then* $\rho(v_1;(r_1,r_2)) \geq \rho(v_1;(\hat{r}_1,r_2))$ *for every* $v_1 \in [0,1]$*. Furthermore, if* $r_2 < 1$ *then* $\rho(v_1;(r_1,r_2)) > \rho(v_1;(\hat{r}_1,r_2))$ *for all* $v_1 \in (r_1, v^s_{\hat{r}})$ *where* $v^s_{\hat{r}} = \max\{v_1 : \rho(v_1;(\hat{r}_1,r_2)) \leq 1\}$.

2. *For any fixed* $r_1 \in [v_0, 1]$*, if* $r_2 < \hat{r}_2$ *then* $\rho(v_1;(r_1,r_2)) \leq \rho(v_1;(r_1,\hat{r}_2))$ *for every* $v_1 \in [0,1]$*. Furthermore, if* $r_1 < 1$ *then* $\rho(v_1;(r_1,r_2)) < \rho(v_1;(r_1,\hat{r}_2))$ *for all* $v_1 \in (r_1, v^s_r)$ *where* $v^s_r = \max\{v_1 : \rho(v_1;(r_1,r_2)) \leq 1\}$.

Proof of Part 1. Since r_2 is fixed, write $\rho(v_1;r_1) := \rho(v_1;(r_1,r_2))$. From theorem 8 if $r_2 = 1$ then $\rho(v_1;r_1) \equiv 1$ no matter what r_1 is, and $\rho(v_1;r_1) \geq \rho(v_1;\hat{r}_1)$ must hold with strict equality. Hence, suppose that $r_2 < 1$. If $\hat{r}_1 = 1$ then $\rho(v_1;\hat{r}_1) = r_2$. Since $r_1 < \hat{r}_1$ then $\rho(v_1;r_1) = r_2$ if $v_1 < r_1$ and $\rho(v_1;r_1) = \min\{z(v_1;r_1), 1\}$ if $v_1 \geq r_1$, where the continuous and increasing function $z : [r_1, 1] \to \mathbb{R}$ satisfies:

$$\frac{d}{dt}z(t;r_1) = \left(\frac{1 - \int_t^1 F(z(\tau,r_1))f(\tau)d\tau}{F(z(t,r_1))F(t) + \int_t^1 F(z(\tau,r_1))f(\tau)d\tau} \right)^{n-1} \qquad t \in [r_1, 1]$$

with initial condition $z(r_1,r_1) = r_2$. As $0 \le F(s) \le 1$ for all s and $z(r_1,r_1) = r_2 < 1$, $\frac{dz(t,r_1)}{dt} > 0$ for all $t \in (r_1, 1]$. Hence, $z(v_1,r_1) > r_2$ and $\rho(v_1,r_1) > r_2 = \rho(v_1;\hat{r}_1)$ for all $v_1 \in (r_1, 1]$ and hence, for all $t \in (r_1, v_r^s)$.

Next, let $\hat{r}_1 < 1$. Since $r_1 < \hat{r}_1$, $[0,r_1]$ is a proper subset of $[0,\hat{r}_1]$. From part (i) of lemma 5, $\rho(v_1;r_1) = \rho(v_1;\hat{r}_1) = r_2$ for $v_1 \in [0,r_1]$, and from part (ii) of this same lemma $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1) = r_2$ for some neighborhood around r_1 where $\rho(\cdot;r_1)$ is an increasing function of v_1. If $v_r^s \le \hat{r}_1$ then $\rho(v_1;r_1) = \rho(v_1;\hat{r}_1)$ for $v_1 \in [0,r_1]$, $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1)$ for all $v_1 \in (r_1, v_1^s)$, (with $\rho(v_r^s;r_1) = z(v_r^s;r_1) = 1$), and $\rho(v_1;r_1) \ge \rho(v_1;\hat{r}_1)$ for $v_1 \in [v_1^s, 1]$, and part (1) of the proposition is completed.

Let $r_2 < 1$, $\hat{r}_1 < 1$, and $v_1^s > \hat{r}_1$. Since $\rho(\cdot,r_1)$ coincides with $z(\cdot;r_1)$ within the nonempty interval (r_1, v_r^s), there must exist a neighborhood around r_1 such that $\rho(v_1;r_1) > r_2 = \rho(v_1;\hat{r}_1)$. Therefore, if $\rho(t;r_1) < \rho(t;\hat{r}_1)$ for some $t \in (r_1, v_r^s)$ then it must be the case that the former function intersects the later one within this interval. Otherwise, $\rho(v_1;r_1) = \rho(v_1;\hat{r}_1) = r_2$ for $v_1 \in [0,r_1]$, $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1)$ for $v_1 \in (r_1, v_1^s)$ and $\rho^*(v_1) \ge \hat{\rho}(v_1)$ for $v_1 \in [v_1^s, 1]$. There are two cases of interest.

Case 1. Suppose that contrary to the claim, $\rho(\cdot;r_1)$ and $\rho(\cdot;\hat{r}_1)$ intersect once within the interval $(\hat{r}_1, v_1^s)$. Then, there exists some $\tilde{v}_1 \in (\hat{r}_1, v_1^s)$ such that $\rho(\tilde{v}_1;r_1) = \rho(\tilde{v}_1;\hat{r}_1)$. Since $\tilde{v}_1 < v_1^s$ and $\rho(v_1^s;r_1) = 1$ by construction, $\rho(\tilde{v}_1;r_1) = \rho(\tilde{v}_1;\hat{r}_1) < 1$, and by part (ii) of lemma 5 the slopes of $\rho(\tilde{v}_1;r_1)$ and $\rho(\tilde{v}_1;\hat{r}_1)$ can be estimated as the ratio of the probabilities of trading with seller 1 and seller 2. Observe that:

$$\left[1 - \int_{\tilde{v}_1}^{1} F(\rho(\tau,r_1))f(\tau)d\tau\right] > \left[1 - \int_{\tilde{v}_1}^{1} F(\rho(\tau;\hat{r}_1))f(\tau)d\tau\right] \tag{A.21}$$

and,

$$\left[F(\rho(\tilde{v}_1;r_1))F(\tilde{v}_1) + \int_{\tilde{v}_1}^{1} F(\rho(\tau;r_1))f(\tau)d\tau\right] < \left[F(\rho(\tilde{v}_1;\hat{r}_1))F(\tilde{v}_1) + \int_{\tilde{v}_1}^{1} F(\rho(\tau;\hat{r}_1))f(\tau)d\tau\right] \tag{A.22}$$

because $\rho(v_1;r_1) < \rho(v_1;\hat{r}_1)$ holds for for all v_1 such that $\tilde{v}_1 < v_1 < v_r^s \le 1$ (and $\rho(v_1;r_1) \ge \rho(v_1;\hat{r}_1)$ for $v_1 \in [v_r^s, 1]$). Nonetheless, $\rho(v_1;r_1)$ intersects $\rho(v_1;\hat{r}_1)$

only once and $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1)$ around some neighborhood of r_1, the function $\rho(\cdot;\hat{r}_1)$ must cut $\rho(\cdot;r_1)$ from below. This means that the slope of $\rho(\cdot;\hat{r}_1)$ must be greater than the slope of $\rho(v_1;r_1)$ at $\tilde{v}_1$. However, inequalities (A.21) and (A.22) imply that the slope of $\rho(\tilde{v}_1;\hat{r}_1)$ is strictly lower than the slope of $\rho(\tilde{v}_1;\hat{r}_1)$, a contradiction.

Case 2. Next, suppose that $\rho(\cdot,r_1)$ and $\rho(\cdot;\hat{r}_1)$ intersect more than once within the interval (r_1,v_1^s). Let $\tilde{v}_1 = \sup\{v_1 \in (r_1,v_1^s) : \rho(v_1;r_1) = \rho(v_1;\hat{r}_1)\}$, i.e., let $\tilde{v}_1$ be the highest value at which $\rho(\cdot;r_1)$ and $\rho(\cdot;\hat{r}_1)$ intersect. Since $\tilde{v}_1 < v_r^s$ and $\rho(v_1^s;r_1) = 1$, $\rho(\tilde{v}_1;r_1) = \rho(\tilde{v}_1;\hat{r}_1) < 1$. Moreover, as these two function do not intersect to the right of $\tilde{v}_1$, then either $\rho(v_1;r_1) < \rho(v_1;\hat{r}_1)$ or $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1)$ for all $v_1 \in (\tilde{v}_1,v_1^s)$. If $\rho(v_1;r_1) < \rho(v_1;\hat{r}_1)$ for all $v_1 \in (\tilde{v}_1,v_r^s)$ then a similar contradiction as the one created in the case above can be constructed. Hence, suppose that $\rho(v_1;r_1) > \rho(v_1;\hat{r}_1)$ for all $v_1 \in (\tilde{v}_1,v_r^s)$. Then,

$$\left[1-\int_{v_1}^{1} F(\rho(\tau;r_1))f(\tau)d\tau\right] < \left[1-\int_{v_1}^{1} F(\rho(\tau;\hat{r}_1))f(\tau)d\tau\right] \tag{A.23}$$

and,

$$\left[F(\rho(v_1;r_1))F(v_1)+\int_{v_1}^{1} F(\rho(\tau;r_1))f(\tau)d\tau\right] > \left[F(\rho(v_1;\hat{r}_1))F(v_1)+\int_{v_1}^{1} F(\rho(\tau;\hat{r}_1))f(\tau)d\tau\right] \tag{A.24}$$

for every $v_1 \in (\tilde{v}_1, v_1^s)$. Take any $v_1' \in (\tilde{v}_1, v_1^s)$. Then,

$$\begin{aligned}
\rho(v_1'; r_1) &= r_2 + \int_{r_1}^{\tilde{v}_1} \frac{d\rho(t; r_1)}{dt} dt + \int_{\tilde{v}_1}^{v_1'} \frac{d\rho(t; r_1)}{dt} dt \\
&= \rho(\tilde{v}_1; r_1) + \int_{\tilde{v}_1}^{v_1'} \frac{d\rho(t; r_1)}{dt} dt \\
&= \rho(\tilde{v}_1; \hat{r}_1) + \int_{\tilde{v}_1}^{v_1'} \frac{d\rho(t; r_1)}{dt} dt \\
&< \rho(\tilde{v}_1; \hat{r}_1) + \int_{\tilde{v}_1}^{v_1'} \frac{d\rho(t; \hat{r}_1)}{dt} dt \\
&= \rho(v_1'; \hat{r}_1)
\end{aligned}$$

where $\rho(\tilde{v}_1; r_1) = \rho(\tilde{v}_1; \hat{r}_1)$ holds by assumption, and $\frac{d}{dt}\rho(t; r_1) < \frac{d}{dt}\rho(t; \hat{r}_1)$ holds because of inequalities (A.23) and (A.24). Thus, $\rho(v_1; r_1) < \rho(v_1; \hat{r}_1)$ even though $v_1' \in (\tilde{v}_1, v_1^s)$, a contradiction.

Proof of Part 2. Similar to the proof of part (1), write $\rho(v_1; r_2) := \rho(v_1; (r_1, r_2))$. If $r_1 = 1$ then $\rho(v_1; r_2) = r_2$ no matter what r_2 is (if $r_2 = 1$ then $\rho(v_1; r_2) = r_2 = 1$) from theorem (8). Therefore, $\rho(v_1; r_2) < \rho(v_1; \hat{r}_2)$ for all $v_1 \in [0,1]$ whenever $r_2 < \hat{r}_2$. Hence, let $r_1 < 1$. If $\hat{r}_2 = 1$ then $\rho(v_1; r_2) < \rho(v_1; \hat{r}_2) = 1$ for all $v_1 \in [0,1)$ and $\rho(1; r_2) = \rho(1; \hat{r}_2) = 1$ because $r_1 < 1$ and $r_2 < 1$ implies that bidders use a nondecreasing function $\rho(v_1; r_2)$ satisfying $\rho(v_1; r_2) = r_2$ if $v_1 < r_1$ and $\rho(v_1; r_2) = \min\{z(v_1; r_2); 1\}$. Therefore, let $r_1 < 1$ and $\hat{r}_2 < 1$. It is sufficient to show that $\rho(v_1; r_2) < \rho(v_1; \hat{r}_2)$ for all $v_1 \in (r_1, v_{\hat{r}}^s)$ where $v_{\hat{r}}^s = \max\{v_1 : \rho(v_1; \hat{r}_2) \leq 1\}$, because $\rho(v_1; r_2) = r_2 < \rho(v_1; \hat{r}_2) = \hat{r}_2$ for all $v_1 \in [0, r_1]$ and $\rho(v_1; r_1) \leq \rho(v_{\hat{r}}^s, \hat{r}_2)$ provided that $\rho(v_1; r_2) < \rho(v_1; \hat{r}_2)$ for all $v_1 \in (r_1, v_{\hat{r}}^s)$ holds true.

Suppose that there exists some $t \in (r_1, v_{\hat{r}}^s)$ and some neighborhood $N_\lambda(t) \subset (r_1, v_{\hat{r}}^s)$ around t such that $\rho(v_1; r_2) \geq \rho(v_1; \hat{r}_2)$ for all $v_1 \in N_\lambda(t)$. Since $\rho(r_1; r_2) < \rho(r_1; \hat{r}_2)$ and ρ is continuous in $v_1 \in [0,1]$, there must exist some $\tilde{t} > r_1$ such that $\rho(v_1, r_1) < \rho(v_1; \hat{r}_1)$ for all $v_1 \in (r_1, \tilde{t})$. Thus, if $\rho(t; r_1) \geq \rho(t; \hat{r}_1)$ for some $t \in (r_1, v_r^s)$ then it must be the case that $\rho(\cdot; r_2)$ intersects $\rho(\cdot; \hat{r}_2)$ on $(r_1, v_{\hat{r}}^s)$. Let t be the highest value at which these two functions intersect. If $\rho(v_1; r_1) < \rho(v_1; \hat{r}_2)$ for all $v_1 \in (r_1, t)$, $\rho(t; r_2) = \rho(t; \hat{r}_2)$ and $\rho(v_1; r_2) > \rho(v_1; \hat{r}_2)$, $\rho(\cdot; r_2)$ intersects $\rho(\cdot; \hat{r}_2)$ from below. As the slope of these functions at t can be computed using the ratio of probabilities of trading, we have:

$$\left[1-\int_t^1 F(\rho(\tau;r_2))f(\tau)d\tau\right] < \left[1-\int_t^1 F(\rho(\tau;\hat{r}_2))f(\tau)d\tau\right] \tag{A.25}$$

and,

$$\left[F(\rho(t;r_2))F(t)+\int_t^1 F(\rho(\tau;r_2))f(\tau)d\tau\right] > \\ \left[F(\rho(t;\hat{r}_2))F(t)+\int_t^1 F(\rho(\tau;\hat{r}_2))f(\tau)d\tau\right] \tag{A.26}$$

from where it follows that $\rho(\tilde{t};r_2) < \rho(\tilde{t};\hat{r}_2)$ for some $\tilde{t} > t$ sufficiently close to $\tilde{t}$, a contradiction. A similar contradiction can be obtained in case that $\rho(t;r_2) = \rho(t;\hat{r}_2)$ and $\rho(v_1;r_2) < \rho(v_1;\hat{r}_2)$ for $v_1 \in (t, v^s_{\hat{r}})$, $\rho(\cdot;r_2)$. We conclude that $\rho(v_1;r_2) = r_2 < \rho(v_1;\hat{r}_2) = \hat{r}_2$ for all $v_1 \in [0, v^s_{\hat{r}})$, and $\rho(v_1;r_2) \leq \rho(v_1;\hat{r}_2)$ for all $v_1 \in [v^s_{\hat{r}}, 1]$ as claimed.

A.7 Proof of Proposition 8

Before proving the proposition, we state and prove the following lemma.

Lemma 34. *For any pair of reserve prices $(r_1,r_2) \in [v_0,1]^2$ let $\rho(v_1;(r_1,r_2))$ be the functions used by bidders to select trading partners when reserve prices are (r_1,r_2). Then,*

1. *For any fixed $r_2 \in [v_0,1]$, $\rho(v_1;(r_1,r_2))$ is a continuous function of r_1 on $[v_0,1)$.*
2. *For any fixed $r_1 \in [v_0,1]$, $\rho(v_1;(r_1,r_2))$ is a continuous function of r_2 on $[v_0,1]$.*

.

Proof of Part 1. Since r_2 is fixed, write $\rho(v_1;r_1) := \rho(v_1;(r_1,r_2))$. If $r_2 = 1$ then $\rho(v_1;r_1) \equiv 1$ for all $r_1 \in [v_0,1]$ and hence, ρ is (uniformly) continuous. If $r_2 < 1$

then:

$$\rho(v_1,r_1)=\begin{cases} r_2 & \text{if } v_1<r_1 \\ \min\{z(v_1,r_1);1\} & \text{if } v_1\geq r_1 \end{cases}$$

because $r_1<1$, with the function z satisfying:

$$\frac{d}{dt}z(t,r_1)=\left(\frac{1-\int_t^1 F(z(\tau,r_1))f(\tau)d\tau}{F(z(t,r_1))F(t)+\int_t^1 F(z(\tau,r_1))f(\tau)d\tau}\right)^{n-1} \qquad t\in[r_1,1]$$

and $z(r_1)=r_2$. Take any $\varepsilon>0$ and let $\lambda=\varepsilon F^2(v_0)$. For any $\bar{r}_1\in[v_0,1)$ define $B_\lambda^+(\bar{r}_1)=\{r_1\in[v_0,1):0<r_1-\bar{r}_1<\lambda\}$ and let $\phi(t)=\rho(t;\bar{r}_1)-\rho(t;r_1)$, $t\in[0,1]$. Since $r_1>\bar{r}_1$ for all $r_1\in B_\lambda^+(\bar{r}_1)$, $\phi(t)\geq 0$ from proposition 7. Define $v_{\bar{r}}^s=\max\{v_1: z(v_1,\bar{r}_1)\leq 1\}$.

Case 1. Suppose that $v_{\bar{r}}^s\leq r_1$. Then:

$$\phi(t)=\begin{cases} 0 & \text{if } t<\bar{r}_1 \\ z(t;\bar{r}_1)-r_2 & \text{if } \bar{r}_1\leq t<v_{\bar{r}}^s \\ 1-r_2 & \text{if } v_{\bar{r}}^s<t\leq r_1 \\ 1-\min\{z(t;r_1);1\} & \text{if } r_1<t\leq 1 \end{cases}$$

The function ϕ reaches its maximum at $t=v_{\bar{r}}^s$ because $\bar{r}_1<v_{\bar{r}}^s$ and hence, (i) $\phi(t)=r_2-r_2=0$ if $t\leq\bar{r}_1$; (ii) $\phi(v_{\bar{r}}^s)=1-r_2\geq z(t;\bar{r}_1)-r_2$ if $t\in[\bar{r}_1,v_{\bar{r}}^s]$; and (iii) $\phi(v_{\bar{r}}^s)=1-r_2\geq 1-\min\{z(t,r_1);1\}$ if $t\in[r_1,1]$. Since the slope of $z(\cdot;\bar{r}_1)$ can never be greater than $\frac{1}{F^2(v_0)}$ (which follows from the proof of proposition 27),

$$\begin{aligned}
\sup_{t\in[0,1]}|\phi(v)| &= \sup_{t\in[0,1]}|\rho(t;\bar{r}_1)-\rho(t;r_1)| \\
&= \phi(v_{\bar{r}}^s) \\
&\leq \frac{1}{F^2(v_0)}|\bar{r}_1-v_{\bar{r}}^s| \\
&\leq \frac{1}{F^2(v_0)}|\bar{r}_1-r_1| \\
&< \varepsilon
\end{aligned}$$

because $v_{\bar{r}}^s<r_1$ and hence, $|\bar{r}_1-r_1|=\bar{r}_1-r_1<\lambda=\varepsilon F^2(v_0)$. A similar conclusion can be obtained when r_1 approaches $\bar{r}_1$ from the left (i.e., when we consider any $r_1 \in B^-_\lambda(\bar{r}_1) = \{r_1 \in [v_0, 1) : 0 < \bar{r}_1 - r_1 < \lambda\}$) by exchanging the roles of $\rho(\cdot; r_1)$ and $\rho(\cdot; \bar{r}_1)$). We then conclude that ρ must be continuous in r_1 on $[v_0, 1)$.

Case 2. Suppose that $v^s_{\bar{r}} > r_1$. Then,

$$\phi(t) = \begin{cases} 0 & \text{if } t < \bar{r}_1 \\ z(t; \bar{r}_1) - r_2 & \text{if } \bar{r}_1 \leq t < r_1 \\ z(t; \bar{r}_1) - z(t, r_1) & \text{if } r_1 \leq t \leq v^s_{\bar{r}} \\ 1 - z(t; r_1) & \text{if } v^s_{\bar{r}} < t \leq v^s_r \\ 0 & \text{if } v^s_r < t \leq 1 \end{cases}$$

From part (1) of proposition (7), $\rho(t; \bar{r}_1) > \rho(t, r_1)$ for all $t \in (r_1, v^s_{\bar{r}})$ and $\rho(t; \bar{r}_1) \geq \rho(t; r_1)$ for $t \in [v^s_{\bar{r}}, 1]$. Since $\rho(\cdot, \bar{r})$ coincides with $z(\cdot; \bar{r}_1)$ on $[\bar{r}, v^s_{\bar{r}}]$, $\rho(t, \bar{r}_1)$ must be increasing in t on $[\bar{r}_1, r_1]$, and $\frac{d\rho(v_1; r_1)}{dv_1} > \frac{d\rho(v_1; \bar{r}_1)}{dv_1}$ on $(r_1, v^s_{\bar{r}})$. Hence, the function ϕ must achieve its maximum at $t \in [r_1, v^s_{\bar{r}}]$. As $\phi'(t) = \frac{dz(t; \bar{r}_1)}{dt} - \frac{dz(t; r_1)}{dt} < 0$ on $[r_1, v^s_{\bar{r}}]$, $\phi(r_1) \geq \phi(t)$ for all $t \in [r_1, v^s_{\bar{r}}]$ and hence, $\phi(r_1) \geq \phi(t)$ for all $t \in [0, 1]$. Therefore,

$$\sup_{t \in [0,1]} |\rho(t, \bar{r}_1) - \rho(t; r_1)| \leq |z(r_1, \bar{r}_1) - z(r_1; r_1)|$$

From the proof of proposition 27, the slope of $z(\cdot; \bar{r}_1)$ can never be greater than $\frac{1}{F^2(v_0)}$. Consequently,

$$|z(r_1; \bar{r}_1) - z(r_1; r_1)| \leq \frac{1}{F^2(v_0)} |r_1 - \bar{r}_1|$$

and thus,

$$\begin{aligned} \sup_{v_1 \in [0,1]} |\rho(v_1; \bar{r}_1) - \rho(v_1; r_1)| &\leq \frac{1}{F^2(v_0)} |r_1 - \bar{r}_1| \\ &< \frac{1}{F^2(v_0)} |r_1 - \bar{r}_1| \\ &< \varepsilon \end{aligned}$$

because $|r_1 - \bar{r}_1| < \lambda = \varepsilon F^2(v_0)$. It is straightforward to check that a similar conclusion holds when we consider any $r_1 \in B_\lambda^-(\bar{r}_1) = \{r_1 \in [v_0, 1) : 0 < \bar{r}_1 - r_1 < \lambda\}$. Thus, ρ must be continuous in r_1 on $[v_0, 1)$ if $r_2 < 1$.

Proof of Part 2. Similar to the case above, write $\rho(v_1; r_2) := \rho(v_1; (r_1, r_2))$. Take any $\varepsilon > 0$ and let $\lambda = \varepsilon$. For any $\bar{r}_2 \in [v_0, 1)$ define $B_\lambda^+(\bar{r}_2) = \{r_2 \in [v_0, 1) : 0 < r_2 - \bar{r}_2 < \lambda\}$ and let $\varphi(t) = \rho(t; r_2) - \rho(t; \bar{r}_2), t \in [0, 1]$. Since $r_2 > \bar{r}_2$ for all $r_2 \in B_\lambda^+(\bar{r}_2)$, $\varphi(t) \geq 0$ from part (2) of proposition (7). Define $v_r^s = \max\{v_1 : z(v_1, r) \leq 1\}$. Then,

$$\varphi(t) = \begin{cases} r_2 - \bar{r}_2 & \text{if } t < r_1 \\ z(t; r_2) - z(t; \bar{r}_2) & \text{if } r_1 \leq t < v_r^s \\ 1 - \min\{z(t; r_2)\} & \text{if } v_r^s < t \leq 1 \end{cases}$$

Since $\rho(\cdot; r_2)$ and $\rho(\cdot; \bar{r}_2)$ coincide with $z(\cdot; r_2)$ and $z(\cdot; \bar{r}_2)$ respectively on $[r_1, v_r^s]$, $\rho(v_1; r_2) > \rho(v_1; \bar{r}_2)$ if $v_1 \in (r_1, v_r^s)$, and $\rho(v_1; r_2) \geq \rho(v_1; \bar{r}_2)$ if $v_1 \in [v_r^s, 1]$, $\varphi'(t) < 0$ for $t \in (r_1, v_r^s)$ and hence, $\varphi(r_1) \geq \varphi(t)$ for all $t \in [0, 1]$. Therefore,

$$\begin{aligned} \sup_{t \in [0,1]} |\varphi(v)| &= \sup_{t \in [0,1]} |\rho(t; r_2) - \rho(t; \bar{r}_2)| \\ &\leq \varphi(r_1) \\ &= |r_2 - \bar{r}_2| \\ &< \varepsilon \end{aligned}$$

Since a similar conclusion can be obtained when r_2 approaches $\bar{r}_2$ from the left. We conclude that ρ must be continuous in r_2 on $[v_0, 1)$ when $r_1 < 1$.

□

Continuity of $R_1(r_1, r_2; \rho)$ follows from part (1) of lemma 34 and the discussion in the main text. We write $\rho(v_1, (r_1, r_2)) := \rho(v_1; r_2)$ whenever r_1 is assumed fixed and there is no risk of confusion. From the main text, the function R_2 is given by:

$$R_2(r_1, r_2; \rho) =$$

$$\begin{cases} v_0 & \text{if } r_2 = 1 \\ R_2^0(r_1,r_2;\rho) + R_2^1(r_1,r_2;\rho) + R_2^{2^+}(r_1,r_2;\rho) & \text{if } v_0 \le r_2 < 1 \end{cases}$$

where:

$$\begin{aligned} R_2^0(r_1,r_2;\rho) &= v_0 G_2^n(r_2;r_2) \\ R_2^1(r_1,r_2;\rho) &= n r_2 G_2^{n-1}(r_2;r_2)(1-G_2(r_2;r_2)) \\ R_2^{2^+}(r_1,r_2;\rho) &= n(n-1)\int_{r_2}^1 t_2\,[1-G_2(t_2;r_2)]\,[G_2(t_2;r_2)]^{n-2}\,dG_2(t_2;r_2) \end{aligned}$$

and $G_2(t_2;r_2) := G_2(t_2;\rho^*,r_2)$ is given by:

$$\begin{aligned} G_2(t_2;r_2) &= \left[F(t_2)F(\rho^{-1}(t_2,r_2)) + \int_{\rho^{-1}(t_2,r_2)}^1 F(\rho(\tau,r_2))f(\tau)d\tau\right] \\ &= \left[1 - \int_{t_2}^1 F(\rho^{-1}(\tau,r_2))f(\tau)d\tau\right] \end{aligned}$$

with:

$$\rho^{-1}(t_2,r) = \begin{cases} 0 & \text{if } t_2 < r_2 \\ \max\{s \in [0,1] : t_2 \ge \rho(s,r_2)\} & \text{if } t_2 \ge r_2 \end{cases}$$

First, let $r_1 = 1$. Then, $\rho(v_1,r_2) \equiv r_2$ for all $v_1 \in [0,1]$. For any $\varepsilon > 0$ let $\lambda = \frac{\varepsilon}{\bar{f}}$ where $|f(s)| \le \bar{f}$ for all $s \in \mathbb{R}$, and define $B_\lambda^-(1) = \{r_2 : [v_0,1] : 0 < 1 - r_2 < \lambda\}$. Since $\rho(v_1;r_2) \equiv r_2 < 1$ for all $r_2 \in B_\lambda(1)$, $\rho^{-1}(v_2,r_2) = 0$ if $v_2 < r_2$ and $\rho^{-1}(v_2,r_2) = 1$ if $v_2 \ge r_2$. Hence, $G_2(t_2;r_2)$ becomes equal to $F(t_2)$, $t_2 \in [r_2,1]$ so long as $r_2 < 1$. Thus,

$$\begin{aligned} |F(r_2) - F(1)| &\le \bar{f}\,|r_2 - 1| \\ &< \bar{f}\frac{\varepsilon}{\bar{f}} \\ &= \varepsilon \end{aligned}$$

and $R_2^0(1,r_2;\rho)$ and $R_2^1(1,r_2;\rho)$ are continuous functions of $r_2 \in [v_0,1]$. Furthermore, $\int_{r_2}^1 t_2(1-F(t_2))F^{n-2}(t_2)dF(t_2)$ must tend to zero as $r_2 \to 1$. Therefore, $R_2(1,r_2;\rho) = R_2^0(1,r_2;\rho) + R_2^1(1,r_2;\rho) + R_2^{2^+}(1,r_2;\rho)$ tends to $R_2(1,1;\rho) = v_0$ and $R_2(1,r_2;\rho)$ is a continuous function of r_2 when $r_1 = 1$.

Second, consider the case where $r_1 < 1$. For any $r_2' \in [v_0, 1)$ and $\varepsilon > 0$ let $\lambda = \frac{c_0 \varepsilon}{\bar{f}^2}$ where $c_0 = \left(\frac{F(v_0)}{2+F(v_0)}\right)^{n-1}$ and define $B_\lambda(r_2') = \{r_2 : |r_2 - r_2'| < \lambda\}$. To simplify notation, let $\underline{r}_2 = \min\{r_2; r_2'\}$ and $\bar{r}_2 = \max\{r_2; r_2'\}$, $r_2 \in B_\lambda(r_2')$. Let $v_{\bar{r}}^s = \max\{t : z(t;\bar{r}_2) \leq 1\}$. Then, for any $t \in (r_1, v_{\bar{r}}^s)$,

$$\begin{aligned} \frac{d}{dt} z(t;\underline{r}_2) &= \left(\frac{1 - \int_t^1 F(\rho(\tau,\underline{r}_2)) f(\tau) d\tau}{F(t)F(z(t;\underline{r}_2)) + \int_t^1 F(z(\tau;\underline{r}_2)) f(\tau) d\tau}\right)^{n-1} \\ &\geq \left(\frac{F(v_0)}{2+F(v_0)}\right)^{n-1} \\ &= c_0 \end{aligned}$$

and the slope of $z(\cdot;\underline{r}_2)$ can never be lower than c_0. Hence,

$$\left|z(t;\underline{r}_2) - z(t';\underline{r}_2)\right| \geq c_0 \left|t - t'\right|$$

for all t and t' on (r_1, v_r^s). From proposition 27, z is continuous and increasing in $(r_1, 1)$ and hence, there must exist a continuous and increasing function $z^{-1} : t \longmapsto z^{-1}(t,r) \in [r_1, 1]$ such that $z^{-1}(z(t)) = z(z^{-1}(t)) = t$. As $t_2 = z(z^{-1}(t_2, \underline{r}_2); r_2) = z(z^{-1}(t_2, \bar{r}_2); \bar{r}_2)$ whenever $t_2 \in [\bar{r}_2, 1]$ we have:

$$\begin{aligned} \left|z^{-1}(t_2,\underline{r}_2) - z^{-1}(t_2,\bar{r}_2)\right| &\leq \frac{1}{c_0}\left|z(z^{-1}(t_2,\underline{r}_2);\underline{r}_2) - z(z^{-1}(t_2;\bar{r}_2);\underline{r}_2)\right| \\ &= \frac{1}{c_0}\left|z(z^{-1}(t_2,\bar{r}_2);\bar{r}_2) - z(z^{-1}(t_2;\bar{r}_2);\underline{r}_2)\right| \\ &\leq \frac{1}{c_0} \sup_{t_1 \in [r_1, v_r^s]} |z(t_1;\bar{r}_2) - z(t_1;\underline{r}_2)| \end{aligned}$$

From part (2) of lemma 34, $\sup_{t_1 \in [r_1, v_r^s]} |z(t_1;\bar{r}_2) - z(t_1;\underline{r}_2)| \leq |\bar{r}_2 - \underline{r}_2|$ and thus,

$$\begin{aligned} \left|z^{-1}(t_2,\underline{r}_2) - z^{-1}(t_2,\bar{r}_2)\right| &\leq \frac{1}{c_0} \sup_{t_1 \in [r_1, v_r^s]} |z(t_1;\bar{r}_2) - z(t_1;r_2)| \\ &= \frac{1}{c_0} \sup_{t \in [0,1]} |\rho(t;\bar{r}_2) - \rho(t;\underline{r}_2)| \\ &\leq \frac{1}{c_0} |\bar{r}_2 - \underline{r}_2| \end{aligned}$$

and as ρ^{-1} and z^{-1} coincide on $[r_2, 1]$,

$$\begin{aligned}\left|\rho^{-1}(\tau;\underline{r}_2)-\rho^{-1}(\tau;\bar{r}_2)\right| &= \left|z^{-1}(\tau;\underline{r}_2)-z^{-1}(\tau;\bar{r}_2)\right| \\ &\leq \frac{1}{c_0}\left|\bar{r}_2-\underline{r}_2\right|\end{aligned}$$

Take any $t_2 < \underline{r}_2$. Then, $\rho^{-1}(t_2;r_2') = \rho^{-1}(t_2;r_2) = 0$ and,

$$\begin{aligned}\left|G_2(t_2)-G_{r_2'}(t_2)\right| &= \left|\int_{t_2}^{1}\left\{F(\rho^{-1}(\tau;r_2'))-F(\rho^{-1}(\tau;r_2))\right\}f(\tau)d\tau\right| \\ &\leq \bar{f}\int_{t_2}^{1}\left|F(\rho^{-1}(\tau;r_2'))-F(\rho^{-1}(\tau;r_2))\right|d\tau \\ &= \int_{\underline{r}_2}^{\bar{r}_2}\left|F(\rho^{-1}(\tau;\underline{r}_2))\right|d\tau+\int_{\bar{r}_2}^{1}\left|F(\rho^{-1}(\tau;\underline{r}_2))-F(\rho^{-1}(\tau;\bar{r}_2))\right|d\tau \\ &\leq \bar{f}(\bar{r}_2-\underline{r}_2)+\bar{f}^2\int_{\bar{r}_2}^{1}\left|\rho^{-1}(\tau;\underline{r}_2)-\rho^{-1}(\tau;\bar{r}_2)\right|d\tau \\ &\leq \bar{f}\,(\bar{r}_2-\underline{r}_2)+\bar{f}^2\frac{1}{c_0}\left|\bar{r}_2-\underline{r}_2\right| \\ &< \left(\frac{c_0\bar{f}+\bar{f}^2}{c_0}\right)\lambda \\ &= \left(\frac{c_0\bar{f}+\bar{f}^2}{c_0}\right)\left(\frac{c_0}{\bar{f}^2}\right)\varepsilon \\ &< \varepsilon\end{aligned}$$

Similarly, if $t_2 \in [\underline{r}_2, \bar{r}_2]$,

$$\begin{aligned}\left|G_2(t_2;r_2)-G_2(t_2;r_2')\right| &\leq \int_{t_2}^{\bar{r}_2}\left|F(\rho^{-1}(\tau;\underline{r}_2))\right|d\tau+\int_{\bar{r}_2}^{1}\left|F(\rho^{-1}(\tau;\underline{r}_2))-F(\rho^{-1}(\tau;\bar{r}_2))\right|d\tau \\ &\leq \bar{f}\,(t_2-\underline{r}_2)+\bar{f}^2\frac{1}{c_0}\left|\bar{r}_2-\underline{r}_2\right| \\ &< \left(\frac{c_0\bar{f}+\bar{f}^2}{c_0}\right)\lambda \\ &< \varepsilon\end{aligned}$$

because $t_2 \in [\underline{r}_2, \bar{r}_2]$ and hence, $|t_2 - \bar{r}_2| \leq |\underline{r}_2 - \bar{r}_2| < \lambda$. Finally, if $t_2 \in [\bar{r}_2, 1]$,

$$\begin{aligned} \left|G_2(t_2;r_2) - G_2(t_2;r_2')\right| &\leq \int_{t_2}^{1} \left|F(\rho^{-1}(\tau;\underline{r}_2)) - F(\rho^{-1}(\tau;\bar{r}_2))\right| d\tau \\ &\leq \bar{f}^2 \frac{1}{c_0} |\bar{r}_2 - \underline{r}_2| \\ &< \left(\frac{\bar{f}^2}{c_0}\right)\lambda \\ &= \varepsilon \end{aligned}$$

from where it follows that $R_2^{2^+}(r_1, r_2; \rho)$ must be continuous in r_2 on $[v_0, 1)$.

Finally, let $r_1 < 1$ and consider what happens when we approach $r_2 = 1$ from the left. As before, let $B_\lambda^-(1) = \{r_2 : 0 < 1 - r_2 < \lambda\}$. Choose any $r_2 \in B_\lambda^-(1)$. Then, for every $t \in [0, r_1]$ $|\rho(t;r_2) - 1| = |r_2 - 1| < \lambda$. Similarly, for any $t \in (r_1, 1]$ either $|\rho(t;r_2) - 1| = |z(t;r_2) - 1| \leq |r_2 - 1| < \lambda$ or $|\rho(t;r_2) - 1| = 0$. The first inequality follows from the fact that if t is such that ρ and z coincide then $1 \geq z(t;r_2) > r_2$ since z is increasing and hence, $|z(t;r_2) - 1| \leq |r_2 - 1|$ whereas the second inequality follows because $\rho(t;r_2) = 1$ whenever ρ and z do not coincide within $(r_1, 1]$. Therefore, $\rho(v_1, r_2)$ must approach one as $r_2 \to 1$ and thus, $G_2(r_2;r_2) \to 1$ and,

$$\begin{aligned} R_2^0(r_1, 1; \rho^*) &\longrightarrow v_0 \\ R_2^1(r_1, 1; \rho^*) &\longrightarrow 0 \\ R_2^{2^+}(r_1, 1; \rho^*) &\longrightarrow 0 \end{aligned}$$

as $r_2 \to 1$. Therefore, $\lim_{r_2^- \to 1} R_2(r_1, r_2; \rho) = v_0 = R_2(r_1, 1; \rho)$ and $R_2(r_1, r_2; \rho)$ is (left) continuous at $r_2 = 1$ when $r_1 < 1$.

Appendix B

Appendix for Chapter 3

B.1 Proof of Lemma 11

It suffices to prove the statement for the case $F(\mu) \leq \frac{1}{2}$ as the case in which $F(\mu) > 1/2$ follows immediately. Suppose that $F(\mu) \leq \frac{1}{2}$. Let $\psi(v) := vF(v)^{n-1} - \mu(1 - F(v))^{n-1}$. Then,

$$\begin{aligned}\psi(\mu) &= \mu F(\mu)^{n-1} - \mu(1-F(\mu))^{n-1} \\ &= \mu\left(F(\mu)^{n-1} - (1-F(\mu))^{n-1}\right) \\ &\leq 0 \\ &= \psi(v^*)\end{aligned}$$

because $F(\mu) \leq 1/2$ implies that $F(\mu)^{n-1} - (1-F(\mu))^{n-1} \leq 0$. Similarly,

$$\begin{aligned}\psi(m) &= mF(m)^{n-1} - \mu(1-F(m))^{n-1} \\ &= (m-\mu)\left(\frac{1}{2}\right)^{n-1} \\ &\geq 0 \\ &= \psi(v^*)\end{aligned}$$

because $F(\mu) \leq F(m) = \frac{1}{2}$ and $f > 0$ (so F is strictly increasing on $[0,1]$) imply $\mu \leq m$. Therefore, $v^* \geq \mu$ and $v^* \leq m$ since the function ψ is also strictly increasing with respect to v. To show necessity, suppose that v^* satisfies $\mu \leq v^* \leq m$. Since $\psi(v)$ is increasing in its argument, we must have $\psi(\mu) \leq \psi(v^*) \leq \psi(m)$ from where it follows that $\psi(\mu) \leq 0$. This in turn implies that the ratio $\frac{F(\mu)}{1-F(\mu)}$ must be less than or equal to one, which is equivalently to $F(\mu) \leq \frac{1}{2}$.

B.2 Proof of Lemma 13

Rewrite Eq. (3.2) as follows:

$$v_n^* \varphi(v_n)^{n-1} = \mu$$

where $\varphi(v_n^*) = \left(\frac{F(v_n^*)}{1-F(v_n^*)}\right)$. Suppose that $v_n^* \leq v_{n+1}^*$. From Lemma 11, $F(\mu) > 1/2$ implies that $F(v_n^*) > 1/2$ for all n and hence, $\varphi(v_n) > 1$ for all n. Furthermore, since $\varphi(s)$ is increasing in s on $[0,1]$, we must have $\varphi(v_n^*)^{n-1} < \varphi(v_n^*)^n \leq \varphi(v_{n+1}^*)^n$. Therefore,

$$\mu = v_n^* \varphi(v_n^*)^{n-1} < v_{n+1}^* \varphi(v_{n+1}^*)^n = \mu$$

a contradiction.

B.3 Proof of Proposition 14

Necessity follows from the contrapositive of Proposition 12. To show sufficiency, suppose that $n = 2$ and that $R_1^{(1,0)} \geq R_1^{(0,0)}$ holds. Let $G^{(1,0)}(v) = \frac{F(x)-F(v^*)}{1-F(v^*)}$ be the distribution of valuations of bidders conditional on visiting seller 1, with v^* the cutoff value employed by bidders to select trading partners. Then, $H^{(1,0)}(x) = [G^{(1,0)}(x)]^2 + 2G^{(1,0)}(x)(1 - G^{(1,0)}(x))$ is the probability distribution of the second highest valuation of bidders who attends to auction 1. It is straightforward to check that $G^{(1,0)}(x)$ first order stochastically dominates $H^{(1,0)}(x)$ and hence, that $\int_{v^*}^1 v dH^{(1,0)}(v) \leq \int_{v^*}^1 v dG^{(1,0)}(v)$ holds (e.g.Ganuza and Penalva (2006)). Therefore,

$$\begin{aligned}
R_1^{(1,0)} - R_1^{(0,0)} &= (1-F(v^*))^2 \left(\int_{v^*}^1 v dH^{(1,0)}(v) \right) - \left(\frac{1}{4}\right)\mu \\
&\leq (1-F(v^*))^2 \left(\int_{v^*}^1 v dG^{(1,0)}(v) \right) - \left(\frac{1}{4}\right)\mu \\
&= (1-F(v^*))^2 \left(\frac{\int_{v^*}^1 v f(v) dv}{1-F(v^*)} \right) - \left(\frac{1}{4}\right)\mu \\
&\leq (1-F(v^*)) \left(\int_0^1 v f(v) dv \right) - \left(\frac{1}{4}\right)\mu \\
&= \left(\frac{3}{4} - F(v^*)\right)\mu
\end{aligned}$$

Hence, if $R_1^{(1,0)} \geq R_1^{(0,0)}$ then $F(v^*) \leq \frac{3}{4}$. Using the contrapositive of this statement we conclude that $F(v^*) > \frac{3}{4}$ is sufficient for the existence of some n ($n = 2$) such that there is an equilibrium in which sellers do not provide information.

B.4 Proof of Proposition 17

Let $j = 1$ and suppose that seller 2 announces $p_2 = 0$. As before, let $R_1^{(a,b)}$ be seller 1's profits when he choose $p_1 = a$ and seller 2 chooses $p_2 = b$, $a, b \in \{0,1\}$. Define ΔR_1^{ni} by:

$$\Delta R_1^{ni} = R_1^{(1,0)} - R_1^{(0,0)}$$

and ΔR_1^i by:

$$\Delta R_1^i = R_1^{(1,1)} - R_1^{(0,1)}$$

We want to establish the sign of $\Delta R_j := \Delta R_1^i - \Delta R_1^{ni}$. Simple algebraic manipulation yields:

$$\begin{aligned}
\Delta R_1 &:= \Delta R_1^i - \Delta R_1^{ni} \\
&= [R_1^{(1,1)} - R_1^{(0,1)}] - [R_1^{(1,0)} - R_1^{(0,0)}] \\
&= [R_1^{(1,1)} - R_1^{(1,0)}] - [R_1^{(0,1)} - R_1^{(0,0)}]
\end{aligned}$$

Let $H_k^{(1,1)}(v)$ be distribution of the second order statistics when the underlying distribution is equal to $F^2(v)$ and seller 1 is matched with exactly k bidders, $k \geq 2$. Similarly, let $H_k^{(1,0)}(v)$ be distribution of the second order statistics when the underlying distribution is $\frac{F(v)-F(v^*)}{1-F(v^*)}$, with v^* the cutoff value given by Eq. (3.2). Let $u = F(v)$. Hence,

$$H_k^{(1,1)}(u) = u^{2k} + ku^{2(k-1)}(1-u^2)$$

and

$$\begin{aligned} H_k^{(1,0)}(u) &= \left(\frac{u-u^*}{1-u^*}\right)^k + k\left(\frac{u-u^*}{1-u^*}\right)^{k-1}\left[1-\left(\frac{u-u^*}{1-u^*}\right)\right] \\ &= (2u-1)^k + k(2u-1)^{k-1}(1-(2u-1)) \end{aligned}$$

because lemma 11 ensures that $v^* = m$ and hence, $u^* = F(v^*) = \frac{1}{2}$ whenever $F(\mu) = 1/2$. Define $\Phi(u,k)$ as follows:

$$\Phi(u,k) := \begin{cases} -H_k^{(1,1)}(u) & \text{if } u < 1/2 \\ H_k^{(1,0)}(u) - H_k^{(1,1)}(u) & \text{if } u \geq 1/2 \end{cases}$$

which, after replacing for the expressions of $H_k^{(1,1)}$ and $H_k^{(1,0)}$ becomes:

$$\Phi(u,k) := \begin{cases} (k-1)u^{2k} - ku^{2k-2} & \text{if } u < 1/2 \\ k\left[(2u-1)^{k-1} - u^{2k-2}\right] - (k-1)\left[(2u-1)^k - u^{2k}\right] & \text{if } u \geq 1/2 \end{cases}$$

Some tedious but otherwise straightforward algebra shows that $\Phi(u,k) \leq 0$ for all $u \in [0,1]$ and all $k \geq 2$. Let $w(u) = F^{-1}(u)$ be the quantile function associated with F. Then, the respective prices seller 1 expects when he announces $p_1 \in \{0,1\}$ and seller 2 announces $p_2 = 0$, and there are exactly k visitors in his auction, $k \geq 2$, are:

$$\begin{aligned} T_{1k}^{(1,1)} &= \int_0^1 w(u)dH_k^{(1,1)} \\ T_{1k}^{(1,0)} &= \int_{1/2}^1 w(u)dH_k^{(1,0)} \end{aligned}$$

Using $\Phi(u,k)$ and integration by parts we obtain:

$$\begin{aligned} T_{1k}^{(1,1)} - T_{1k}^{(1,0)} &= \int_0^1 w'(u)\Phi(u,k)du \\ &\leq 0 \end{aligned}$$

because $\Phi(u,k) \leq 0$ for all $u \in [0,1]$ and all $k \geq 2$, and $w'(u) \geq 0$ for all $u \in [0,1]$ by the implicit function theorem. Therefore, seller 1 expects a higher price when he faces a seller 2 who does not provide information. As we have $F(\mu) = \frac{1}{2}$ and thus $F(v^*) = \frac{1}{2}$, the expected traffic is unaffected by the choice of seller 1. We conclude that $R_1^{(1,1)} - R_1^{(1,0)} \leq 0$. Finally, we must have $R_1^{(0,1)} - R_1^{(0,0)} = 0$ because lemma 11 ensures that $F(v^*) = 1/2$ and hence, the price and the expected traffic are the same regardless of seller 1's choice of information structure. Hence, it follows that $\Delta R_1 \leq 0$ for all $n \geq 2$ and thus, information provision behaves as an strategic substitute when F satisfies $F(\mu) = \frac{1}{2}$.

Appendix C

Appendix for Chapter 4

C.1 Proof of Proposition 21

Consider seller 1's profits when two or more bidders visit his auction:

$$\sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{n-k} T_{1k}^{(1,0)}(r_1, r_2)$$

Following Virag (2010), we can write this expression as follows:

$$\begin{aligned} \sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{n-k} T_{1k}^{(1,0)}(r_1, r_2) &= n(n-1) \int_{s^*}^{1} sF^{n-2}(s)(1-F(s))f(s)ds \\ &= n(n-1) \int_{m}^{1} sF^{n-2}(s)(1-F(s))f(s)ds \end{aligned}$$

or, letting $H(s) = F^n(s) + nF^{n-1}(s)(1-F(s))$, $s \in [m, 1]$,

$$\sum_{k=2}^{n} \binom{n}{k} q^k (1-q)^{n-k} T_{1k}^{(1,0)}(r_1, r_2) = \int_{m}^{1} sdH(s)$$

because the probability of trading with a bidder with valuation $s \geq s^*$ is $F(s)$. Since $H(s)$ is increasing in s on $[m, 1]$, $\int_m^1 sdH(s) > m(1-H(m)) = \left(1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n\right) m$.

Therefore,

$$\begin{aligned} R_1^{(1,0)}(r) &= v_0\left(\frac{1}{2}\right)^n + n\left(m - \frac{(\mu - v_0)}{n}\right)\left(\frac{1}{2}\right)^n + \int_m^1 s dH(s) \\ &> v_0\left(\frac{1}{2}\right)^n + n\left(m - \frac{(\mu - v_0)}{n}\right)\left(\frac{1}{2}\right)^n + \left(1 - \left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n\right)m \\ &= \mu - (v_0 - \mu)\left(\frac{1}{2}\right)^{n-1} \\ &= R_1^{(1,0)}(r_1^*, r_2^*) \end{aligned}$$

and deviating to $p_1 = 1$ and $\tilde{r}_1 = m - \frac{(\mu - m)}{n}$ is profitable for seller 1.

C.2 Lemma 35

Lemma 35. *Let $\bar{s}_n = m$ if $F(\mu) \leq 1/2$ and $\bar{s}_n = \tilde{s}_n$ if $F(\mu) > 1/2$, where $\tilde{s}_n$ is the value of the cutoff when both sellers post reserve prices equal to v_0. Then, $\bar{s}_n \geq \bar{s}_{n+1}$ for all $n \geq 2$.*

Proof. If $F(\mu) \leq 1/2$ then $\bar{s}_n = m$ for all n and the lemma is trivially satisfied. Hence, suppose that $F(\mu) > 1/2$. From Eq. (4.2), we can rearrange terms in the expression defining $\tilde{s}_n$ to obtain:

$$\tilde{s}_n = v_0 + (\mu - v_0)\psi^{n-1}(\tilde{s}_n) \tag{C.1}$$

where $\psi(s) := \left(\frac{1-F(s)}{F(s)}\right)$ is a decreasing function of $s \in [0,1]$. Moreover, $\tilde{s}_n > m$ because:

$$\begin{aligned} (m - v_0)\left(\frac{1}{2}\right)^{n-1} - (\mu - v_0)\left(\frac{1}{2}\right)^{n-1} &= (m - \mu)\left(\frac{1}{2}\right)^{n-1} \\ &< 0 \end{aligned}$$

since $m < \mu$ if $F(\mu) > 1/2$. As $\phi(\tilde{s}_n) := (\tilde{s}_n - v_0)F^{n-1}(\tilde{s}_n) - (\mu - v_0)(1 - F(\tilde{s}_n))^{n-1} = 0$ and ϕ is increasing in s, $m < \tilde{s}_n$ as claimed. Therefore, $F(\tilde{s}_n) > 1/2$ and $\psi(\tilde{s}_n) < 1$

for all n. Suppose that $\tilde{s}_n < \tilde{s}_{n+1}$ holds. Then,

$$\begin{aligned}
\tilde{s}_n - \tilde{s}_{n+1} &= (\mu - v_0)\psi(\tilde{s}_n)^{n-1} - (\mu - v_0)\psi(\tilde{s}_{n+1})^n \\
&> (\mu - v_0)\left[\psi(\tilde{s}_{n+1})^{n-1} - \psi(\tilde{s}_{n+1})^n\right] \\
&\geq (\mu - v_0)\psi(\tilde{s}_{n+1})^{n-1}\left[1 - \psi(\tilde{s}_{n+1})\right] \\
&\geq 0
\end{aligned}$$

where the second line follows because $\psi(\tilde{s}_n) > \psi(\tilde{s}_{n+1})$ due to the fact that ψ is decreasing with respect to its argument and we have assumed that $\tilde{s}_n < \tilde{s}_{n+1}$, and the third line follows because $\psi(s_{n+1}) < 1$. Clearly this is a contradiction and hence, $\bar{s}_n = \tilde{s}_n \geq \tilde{s}_{n+1} = \bar{s}_{n+1}$ as claimed. □

C.3 Proof of lemma 22

Since $s_n > s'_n$,

$$\int_{s'_n}^{1} H_1^{(1,0)}(t)dt = \int_{s'_n}^{s_n} H_1^{(1,0)}(t)dt + \int_{s_n}^{1} H_1^{(1,0)}(t)dt$$

and hence,

$$\begin{aligned}
\int_{s'_n}^{1} H_1^{(1,0)}(t)dt - \int_{s_n}^{1} H_1^{(1,0)}(t)dt &= \int_{s'_n}^{s_n} H_1^{(1,0)}(t)dt \\
&< (s_n - s'_n)H_1^{(1,0)}(s_n)
\end{aligned}$$

because $H_1^{(1,0)}(t)$ is increasing in t. Therefore,

$$\begin{aligned}
\eta(s_n) - \eta(s'_n) &= (s'_n - v_0)H_1^{(1,0)}(s'_n) - (s_n - v_0)H_1^{(1,0)}(s_n) + \int_{s'_n}^{s_n} H_1^{(1,0)}(t)dt \\
&< (s'_n - v_0)H_1^{(1,0)}(s'_n) - (s_n - v_0)H_1^{(1,0)}(s_n) + (s_n - s'_n)H_1^{(1,0)}(s_n) \\
&= (s'_n - v_0)\left(H_1^{(1,0)}(s'_n) - H_1^{(1,0)}(s_n)\right) \\
&< 0
\end{aligned}$$

because $s'_n > v_0$ and $H_1^{(1,0)}(s'_n) < H_1^{(1,0)}(s_n)$ because $s'_n < s_n$.

C.4 Lemma 36

Lemma 36. *Let* (σ_1, σ_2) *be a distribution of reserve prices used by sellers when they both set* $p_1 = p_2 = 1$. *Then,* $1 \notin \text{supp } \sigma_j,\ j = 1,2$.

Proof. Suppose that $1 \in \text{supp } \sigma_1$. Set $\hat{r}_1 = v_0$ (or equal to the closest value to v_0) and compute the difference between the payoff that seller 1 would obtain when $r_1 = v_0$ and his payoff when $r_1 = 1$, given seller 2's mixed strategy σ_2^1:

$$\sum_{r_2 \in \mathscr{R}} \sigma_2^1(r_2) \left\{ \upsilon_1^{(1,1)}(v_0, r_2) - \upsilon_1^{(1,1)}(1, r_2) \right\}$$

Then, for any $r_2 \in \mathscr{R}$,

$$\begin{aligned} \upsilon_1^{(1,1)}(v_0, r_2) - \upsilon_1^{(1,1)}(1, r_2) &= 1 - \int_{v_0}^{1} H_1^{(1,1)}(t)dt - v_0 \\ &> 1 - (1 - v_0) - v_0 \\ &= 0 \end{aligned}$$

because $H_1^{(1,1)}(t)) < 1$ for all $t_1 \in [v_0, 1)$. Therefore, seller 1 could increase his payoff by switching probability mass from $r_1 = 1$ to $r_1 = v_0$ and hence, $1 \notin \text{supp } \sigma_1$ as claimed. □