Uniqueness of Lagrangian mean curvature flow and minimal immersions with free boundary

by

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Abstract

In this thesis we investigate some problems on the uniqueness of mean curvature flow and the existence of minimal surfaces, by geometric and analytic methods. A summary of the main results is as follows.

(*i*) The special Lagrangian submanifolds form a very important class of minimal submanifolds, which can be constructed via the method of mean curvature flow. In the graphical setting, the potential function for the Lagrangian mean curvature flow satisfies a fully nonlinear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{j=1}^{n} \arctan \lambda_j \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

where the λ_j's are the eigenvalues of the Hessian D^2u.

We prove a uniqueness result for unbounded solutions of (1) without any growth condition, via the method of viscosity solutions ([4], [13]): *for any continuous u_0 in $\mathbb{R}^n$, there is a unique continuous viscosity solution to (1) in $\mathbb{R}^n \times [0, \infty)$.*

(*ii*) Let N be a complete, homogeneously regular Riemannian manifold of $\dim N \geq 3$ and let M be a compact submanifold of N. Let Σ be a compact Riemann surface with boundary. A branched immersion $u : (\Sigma, \partial\Sigma) \to (N, M)$ is a *minimal surface with free boundary in M* if $u(\Sigma)$ has zero mean curvature and $u(\Sigma)$ is orthogonal to M along $u(\partial\Sigma) \subseteq M$.

We study the free boundary problem for minimal immersions of compact bordered Riemann surfaces and prove that

- if Σ is not a disk, then there exists a free boundary minimal immersion of Σ minimizing area in any given conjugacy class of a map in $C^0(\Sigma, \partial\Sigma; N, M)$ that is incompressible;
- the kernel of $i_* : \pi_1(M) \to \pi_1(N)$ admits a generating set such that each member is freely homotopic to the boundary of an area minimizing disk that solves the free boundary problem.

(*iii*) Under certain nonnegativity assumptions on the curvature of a 3-manifold N and convexity assumptions on the boundary $M = \partial N$, we investigate controlling topology for free boundary minimal surfaces of low index:

- We derive bounds on the genus, number of boundary components;
- We prove a rigidity result;
- We give area estimates in term of the scalar curvature of N.

Preface

This thesis is based on two research papers:

[1] J. Chen, C. Pang, Uniqueness of unbounded solutions of Lagrangian mean curvature flow, *C. R. Math. Acad. Sci. Paris* **347** (2009), 1031–1034.

[2] J. Chen, A. Fraser, and C. Pang, Minimal immersions of compact bordered Riemann surfaces with free boundary, to appear in *Trans. Amer. Math. Soc.*, arXiv:1209.1165.

All authors contributed equally, and are listed alphabetically, as is the convention in mathematics.

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Dedication

Through the Ph.D. process at UBC and at all times, I was accompanied by my Aunt's unconditional warmth, love and devotion. To the memory of Hibiscus mutabilis.

Chapter 1

Introduction

The field of minimal surfaces has its origin in the mid eighteenth century with the publication of Lagrange's famous memoir "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies" and Euler's paper on the minimizing properties of the catenoid. The study of minimal surfaces has remained a vibrant area of research since Lagrange's memoir.

A generalization of the concept of a minimal surface to higher dimensions is a minimal submanifold. Let $F_0 : M^n \hookrightarrow \bar{M}^{n+k}$ be an isometrically immersed submanifold. A minimal submanifold is a critical point of the volume functional. Let $F : (-\epsilon, \epsilon) \times M \to \bar{M}$ be a family of immersions such that $F(0, \cdot) = F_0$. Denote by $X = (F_* \frac{\partial}{\partial t})(0, \cdot)$ the variational vector field. The first variation formula says that M is minimal if

$$\frac{d}{dt} \operatorname{Vol}(F_t(M))\Big|_{t=0} = -\int_M \left\langle X, \vec{\boldsymbol{H}} \right\rangle d\sigma = 0$$

where $\vec{\boldsymbol{H}}$ is mean curvature vector. Thus a submanifold is minimal if its mean curvature vector vanishes. In an orthonormal basis $\{e_i\}$ for the tangent space at one point, $\vec{\boldsymbol{H}}$ is given by $\vec{\boldsymbol{H}} = \sum A(e_i, e_i)$, where A is the second fundamental form.

The field of minimal submanifolds has been a central area of research and is closely related to other branches of mathematics and mathematical physics such as topology, partial differential equations and mathematical relativity. It has been an active research field which has given rise to a wide variety of problems, which led to the discovery of many profound phenomena in mathematics. However, the understanding of minimal submanifolds is still far from being complete even in dimension two, i.e. in the case of minimal surfaces; and there are many open problems even in one of the simplest cases of embedded minimal surfaces in the Euclidean 3-sphere. For example, one of the many conjectures concerning embedded minimal surfaces in the 3-sphere, the so-called Lawson conjecture, was only recently solved by S. Brendle.

A special class of minimal submanifolds arises from calibrated geometry (cf. [25]). Let X be a Riemannian manifold. A calibration on X is a closed p-form φ that satisfies

$$\varphi(\sigma) \leq \text{ Vol}(\sigma) \tag{1}$$

for all oriented tangent p-planes σ on X, and X is called a calibrated manifold. The oriented p-planes at which equality is achieved are called φ-planes. A φ-submanifold is a compact oriented p-dimensional submanifold M of X such that the tangent space at each point is a φ-plane. This implies

$$\varphi(M) = \text{ Vol}(M). \tag{2}$$

It is a fundamental fact of calibrated geometry that φ-submanifolds are homologically volume minimizing (since for any M' with $\partial M' = \partial M$ which is homological to M, by (1), (2) and since φ is a closed form, we have $\text{Vol}(M) = \int_M \varphi = \int_{M'} \varphi \leq \text{Vol}(M')$). In particular, φ-submanifolds have vanishing mean curvature vectors.

Many interesting examples of calibrated manifolds exist, such as Kähler, Calabi-Yau, G_2, or spin(7) manifolds. One of the calibrated geometries of primary importance is associated to the form

$$\varphi = \mathbf{Re}(dz_1 \wedge \ldots \wedge dz_n)$$

in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. The φ-submanifolds consist of Lagrangian submanifolds of $\mathbb{C}^n$ which are stationary, and are therefore called special Lagrangian submanifolds.

Since all Lagrangian planes in $\mathbb{C}^n$ are equivalent under $\boldsymbol{SU}_n$ to the real plane $\mathbb{R}^n = (\mathbb{R}^n, 0) \subset \mathbb{C}^n$, we may consider a special Lagrangian submanifold M to be given locally as a graph $(x, F(x))$ over a domain in $\mathbb{R}^n$. Then the graph is Lagrangian if and only if the Jacobian matrix $(\partial F^i/\partial x_j)$ is symmetric. In particular, if the domain is simple-connected (for example an entire graph) the Lagrangian condition implies that locally $F = \nabla f$ for some potential function. In order for the graph to be special Lagrangian it must be Lagrangian and satisfy one other condition. Let Hess $f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ denote the Hessian of f. Then it is special Lagrangian if and only if f satisfies a fully nonlinear elliptic equation

$$\mathbf{Im}\{\det_{\mathbb{C}}(I + i\text{Hess } f)\} = 0.$$

In particular, this equation reads $\Delta f = \det(\text{Hess } f)$ in dimension three. That is the Laplacian of f equals the Monge-Ampére of f.

The existence of minimal submanifolds in $\mathbb{R}^n$ or general Riemannian manifolds is a fundamental topic in the theory of minimal submanifolds. One approach that has been proposed for constructing minimal submanifolds is via the parabolic method, i.e. the mean curvature flow. Namely, given an embedding $F_0 : M \to N$, the mean curvature flow is a family of maps $F : M \times [0,T) \to N$ that satisfy

$$\begin{cases} \frac{\partial F}{\partial t} = \vec{\boldsymbol{H}} \\ F(x,0) = F_0(x). \end{cases}$$

A stationary solution, if one exists, will be a minimal submanifold.

In the Lagrangian setting, the special Lagrangian mean curvature flow equation for graphs takes the form

$$\begin{cases} \frac{\partial f}{\partial t} = \frac{1}{\sqrt{-1}} \log \frac{\det(I_n + \sqrt{-1} D^2 f)}{\sqrt{\det(I_n + (D^2 f)^2}} \\ f(x,0) = f_0(x) \end{cases} \tag{3}$$

which is a fully nonlinear parabolic equation. Suppose a solution exists. Then a family of diffeomorphisms $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ can be constructed accordingly such that $F(x,t) = (\phi_t(x), \nabla f(\phi_t(x), t))$ evolves by the mean curvature flow equation

$$\begin{cases} \frac{\partial F}{\partial t} = \vec{\boldsymbol{H}} \\ F(x,0) = \nabla f_0(x). \end{cases}$$

A stationary solution will be minimal. The long time existence for solutions to (3) with $C^{1,1}$ initial data and a certain bound on the Hessian has been established by Chau-Chen-He [6] and Chau-Chen-Yuan [8]. The solutions become smooth immediately. For less smooth initial data, to study the problem one has to use other weaker solutions.

Since the differential operator is in the Hessian form, a natural way to study the problem is by viscosity solutions. In Chapter 2, we prove the existence of a viscosity solution to (3) for any continuous initial data. On the other hand, uniqueness results can also find many applications. We also investigate the uniqueness problem for (3) and prove the following result (a similar result holds for the Cauchy-Dirichlet problem).

Theorem 1.0.1. *Let u and v be an upper semicontinuous and a lower semicontinuous viscosity subsolution and supersolution to (3) in $\mathbb{R}^n \times [0,T)$ respectively. If $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}^n$, then $u \le v$ in $\mathbb{R}^n \times [0,T)$. In particular, for any continuous function u_0 in $\mathbb{R}^n$, there is a unique continuous viscosity solution to (3) in $\mathbb{R}^n \times [0,\infty)$.*

From a purely analytic point of view, when the domain is noncompact, it is often necessary to impose certain growth conditions on the solutions at infinity in order to obtain uniqueness for nonlinear parabolic equations. In our proof, no growth condition is assumed. Instead, a certain boundedness of the differential operator is crucial in applying a comparison theorem of Barles-Biton-Ley ([4]) for viscosity solutions of a class of fully nonlinear parabolic equations.

This result has been used by Chau-Chen-He ([7]) in studying the rigidity of self-similar solutions of the Lagrangian mean curvature flow (see also [9]).

There is another way of proving the existence of minimal submanifolds, which can be formulated into a variational problem of certain functionals in various settings. In this approach, the Euler-Lagrange equation of the functional plays an important role, which is an elliptic differential equation, and therefore we may call this approach an elliptic method.

The boundary value problem for minimal surfaces began with the study of Plateau's problem, which is the question of finding a surface of least area spanning a fixed simple Jordan curve. A general solution was obtained by Douglas and simultaneously by Radó. For minimal surfaces in Riemannian manifolds, the problem was solved by Morrey [45], and was later supplemented by results of Lemaire [36], and Jost [29].

A general existence result for minimal immersions of closed Riemann surfaces of genus ≥ 1 in compact manifolds was proved independently by Schoen and Yau [50] (using the energy $E(f) = \int_\Sigma |\nabla f|^2 d\mu_g$), and Sacks and Uhlenbeck [48, 49] (using perturbed energy functionals $E_\alpha(f) = \int_\Sigma (1+|\nabla f|^2)^\alpha d\mu_g$). Their theorems show that for any continuous map $f : \Sigma \to N$ such that $f_* : \pi_1(\Sigma) \to \pi_1(N)$ is injective, there exists a branched minimal immersion of Σ minimizing area among all maps which induce the same action on the fundamental group. Sacks and Uhlenbeck [48] also established an existence theory for minimal 2-spheres. In particular, they proved that if the universal covering space of N is not contractible, then there exists a non-trivial branched minimal immersion of the 2-sphere into N.

Another natural boundary value problem is the free boundary problem, where the boundary of a surface is allowed to vary in some supporting submanifold. Let M be a closed submanifold of a Riemannian manifold N. A branched immersion $u : (\Sigma, \partial\Sigma) \to (N, M)$ of a surface Σ with nonempty boundary $\partial\Sigma$ is a *minimal surface with free boundary in* M if $u(\Sigma)$ has zero mean curvature and $u(\Sigma)$ is orthogonal to M along $u(\partial\Sigma) \subseteq M$. Existence results for free boundary minimal surfaces in various settings have been much studied.

In Chapter 3 we study the free boundary problem for minimal immersions of compact bordered Riemann surfaces. The purpose is twofold. In the first part, we prove a general existence theorem for compact bordered Riemann surfaces of any topological type in complete Riemannian manifolds, assuming certain incompressibility conditions. In the second part, we investigate controlling the topology of free boundary minimal surfaces of low index in 3-manifolds, under certain nonnegativity assumptions on the curvature and convexity assumptions on the boundary of the 3-manifold. Our existence result is:

Theorem 1.0.2. *Let N be a complete, homogeneously regular Riemannian manifold of $dim N \geq 3$ and let M be a compact submanifold of N. Then,*

(*i*) *if Σ is a compact, connected orientable surface of genus g and with $k \geq 1$ boundary components that is not a disk, and $f : \Sigma \to N$ is a continuous map with $f(\partial\Sigma) \subset M$ such that*

$$f_* : \pi_1(\Sigma) \times \pi_1(\Sigma, \partial\Sigma) \to \pi_1(N) \times \pi_1(N, M)$$

is injective, then there exists a branched minimal immersion $(\Sigma, \partial\Sigma) \to (N, M)$ solving the free boundary problem, and minimizing area among all maps $(\Sigma, \partial\Sigma) \to (N, M)$ that induce the same action as f on the fundamental groups;

(*ii*) *there exists a generating set $\{\gamma_j\}$ for $\ker i_*$, where*

$$i_* : \pi_1(M) \to \pi_1(N)$$

is the homomorphism induced by the inclusion, such that each γ_j is freely homotopic to the boundary of an area minimizing disk that solves the free boundary problem.

The disk case, part (*ii*) of the above theorem, was already proved by Ye [57] by a different method. Existence results for disk-type solutions in various settings have been studied by Meeks and Yau [44], Jost [30–32], Struwe [54], Kuwert [35], Fraser [22], among others. Embedded free boundary solutions of prescribed topological type in 3-manifolds with mean convex boundary were produced by Jost [31] assuming a Douglas type condition. Recently M. Li [37] proved existence of embedded solutions of controlled topological type in 3-manifolds with no convexity assumption on the boundary.

We take the Sacks-Uhlenbeck approach of working with the perturbed energy. The analytic foundation is already established in the interior [48] and in the boundary [22] settings. Following the ideas of Schoen-Yau [50] and Sacks-Uhlenbeck [49] for closed surfaces, for each conformal structure on the bordered surface we produce an energy-minimizing map which induces the same action on the fundamental group as a given continuous map $f : (\Sigma, \partial\Sigma) \to (N, M)$, and then we minimize over all conformal structures to produce a branched minimal surface. The key point is to understand the limiting of the conformal structures in the boundary setting. The incompressibility assumptions on the fundamental groups prevent degeneration in the limiting of the conformal structures, in the form of pinching of an interior or boundary circle or of a curve connecting two boundary circles of the surface.

Remark. In the special case that M is 1-dimensional, the theorem asserts a solution to the Plateau problem, i.e. an area-minimizing immersion of Σ with fixed boundary as M in N.

Besides preventing degeneration of a sequence of conformal structures on Σ, the incompressibility assumptions are also used to rule out triviality of the energy-minimizing maps for each fixed conformal structure.

When the ambient manifold has positive curvature, there are some strong restrictions on the topology of stable or index one minimal surfaces. In the second part of the paper, we investigate the relationship between the topology of free boundary minimal surfaces and the geometry of the ambient manifold, such as nonnegativity of the curvature and convexity of the boundary, by means of the second variation formula. One of the reasons we are interested in understanding the topology of minimal surfaces in a Riemannian manifold is because it provides information about the ambient manifold (e.g. see [56]).

For closed minimal surfaces in 3-manifolds of positive Ricci curvature, it is conjectured that there should exist a bound on the genus of any minimal surface in terms of its Morse index, and it is known that any index 1 surface must have genus at most three ([47]). For minimal surfaces with free boundary in 3-manifolds with nonnegative Ricci curvature and weakly convex boundary, we obtain a bound on the genus and number of boundary components of any index 1 free boundary minimal surface. Also, there is a bound on the area of stable or index 1 free boundary solutions in terms of the topology of the surface and a positive lower bound on the ambient scalar curvature.

Now we state the theorem.

Theorem 1.0.3. *Let N be a 3-dimensional Riemannian manifold with smooth boundary ∂N. Suppose Σ is a compact orientable two-sided surface of genus g and with $k \geq 1$ boundary components, solving the free boundary problem $(\Sigma, \partial\Sigma) \to (N, \partial N)$.*

(i) *Suppose $Ric(N) \geq 0$ and ∂N is weakly convex. If Σ has index 1, then:*

   a) *$g + k \leq 3$ if g is even;*

   b) *$g + k \leq 4$ if g is odd.*

(ii) *Suppose the scalar curvature $R(N) \geq 0$ and ∂N is weakly mean convex. If Σ is stable, then Σ is either a disk or a totally geodesic and flat cylinder.*
If Σ has index 1, then:

   a) *$k \leq 5$ if g is even;*

   b) *$k \leq 7$ if g is odd.*

(iii) *Suppose N has scalar curvature $R \geq R_0 > 0$ and ∂N is weakly mean convex.*

   a) *if Σ is stable, then it is a disk and $Area(\Sigma) \leq \frac{2\pi}{R_0}$;*

   b) *if Σ has index 1, then $Area(\Sigma) \leq \frac{2\pi(7-(-1)^g-k)}{R_0}$.*

M. Li [38] proved an area bound and rigidity result for free boundary minimal surfaces in strictly convex domains in $\mathbb{R}^3$. These area estimates are free boundary analogs of the area estimates for closed stable and index 1 minimal surfaces in 3-manifolds of positive scalar curvature of Marques and Neves [42]. The rigidity result for stable surfaces in part (*ii*) can be viewed as the free boundary analog of results of Schoen and Yau for compact ambient manifolds ([50] Theorem 5.1), and Fischer-Colbrie and Schoen for complete ambient manifolds ([21] Theorem 3).

Chapter 2

Uniqueness of unbounded solutions of Lagrangian mean curvature flow for graphs

2.1 Lagrangian mean curvature flow

The special Lagrangian graphs equation was derived in Harvey-Lawson [25]. Suppose $F : \mathbb{R}^n \to \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ is a C^1-mapping. Then the graph of F is Lagrangian if and only if $F = \nabla f$ for some potential function f in $C^2(\mathbb{R}^n)$. In addition, the graph $(x, \nabla f)$ is a special Lagrangian submanifold in $\mathbb{C}^n$ if and only if f satisfies the following differential equation

$$\mathbf{Im}\left\{\det{}_{\mathbb{C}}\big(I_n + iD^2 f\big)\right\} = 0.$$

Since this condition is equivalent to

$$\det\big(I_n + iD^2 f\big) = \overline{\det\big(I_n + iD^2 f\big)} = \det\big(I_n - iD^2 f\big),$$

we have

$$i \log \frac{\det\big(I_n + iD^2 f\big)}{\det\big(I_n - iD^2 f\big)} = 0 \pmod{2\pi}.$$

The parabolic version of this equation, i.e. the Lagrangian mean curvature flow equation, thus takes the form

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{1}{2i} \log \frac{\det\big(I_n + iD^2 f\big)}{\det\big(I_n - iD^2 f\big)} \\ &= \frac{1}{i} \log \frac{\det(I_n + iD^2 f)}{\sqrt{\det(I_n + (D^2 f)^2}}\end{aligned}$$

where f is a function $\mathbb{R}^n \times [0, t) \to \mathbb{R}^{2n}$. The factor $\frac{1}{i}$ is introduced for the right hand side to take real values. Indeed denote by λ_j's the eigenvalues of $D^2 f$. We have

$$\det(I_n + iD^2 f) = \prod(1 + i\lambda_j) = \prod(1 + \lambda_j^2)^{\frac{1}{2}} \cdot e^{i \arctan \lambda_j}$$

and then

$$\begin{aligned}\frac{1}{2i} \log \frac{\det\big(I_n + iD^2 f\big)}{\det\big(I_n - iD^2 f\big)} &= \frac{1}{2i} \log \frac{e^{i \sum \arctan \lambda_j}}{e^{-i \sum \arctan \lambda_j}} \\ &= \sum_{j=1}^{n} \arctan \lambda_j \pmod{2\pi}.\end{aligned}$$

Hence we can write the equation also as

$$\frac{\partial f}{\partial t} = \sum_{j=1}^{n} \arctan \lambda_j \pmod{\pi}.$$

Accordingly the initial value problem can be formulated as the following

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{j=1}^{n} \arctan \lambda_j \\ u(x,0) = u_0(x) \end{cases} \tag{2.1.1}$$

for $u_0 : \mathbb{R}^n \to \mathbb{R}$ a given function and $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$. For each t, $(x, \nabla u(x))$ defines a Lagrangian submanifold in $\mathbb{R}^{2n}$. In particular, for a smooth stationary solution to (2.1.1), the graph of its gradient is a special Lagrangian submanifold in $\mathbb{R}^{2n}$, which can be seen from

$$\frac{\det\left(I_n + iD^2 f\right)}{\det\left(I_n - iD^2 f\right)} = 1$$

(see also [53], [52]). Suppose u is a regular solution to (2.1.1). Then a family of diffeomorphisms $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ can be constructed such that $F(x,t) = (\phi_t(x), \nabla u(\phi_t(x), t))$ solves the mean curvature flow equation

$$\begin{cases} \frac{\partial F}{\partial t} = \vec{\boldsymbol{H}} \\ F(x,0) = \nabla f_0(x), \end{cases}$$

and so we call a solution to 2.1.1 the Lagrangian mean curvature flow.

The smooth longtime existence of solutions to (2.1.1) has been established in Chau-Chen-He [6] and Chau-Chen-Yuan [8] with $C^{1,1}$ initial u_0 and assuming a certain bound on the Lipschitz norm of Du_0. In this chapter, we study uniqueness problems for the Lagrangian mean curvature flow. First, we consider (2.1.1) on unbounded domain $\mathbb{R}^n$. The result is

Theorem 2.1.1. *Let u and v be an upper semicontinuous and a lower semicontinuous viscosity subsolution and supersolution to (2.1.1) in $\mathbb{R}^n \times [0,T)$ respectively. If $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}^n$, then $u \le v$ in $\mathbb{R}^n \times [0,T)$. In particular, for any continuous function u_0 in $\mathbb{R}^n$, there is a unique continuous viscosity solution to (2.1.1) in $\mathbb{R}^n \times [0,\infty)$.*

Then we also consider the Cauchy-Dirichlet problem for the Lagrangian mean curvature flow. The key ingredient in the proof is a comparison theorem of Barles-Biton-Ley for a class of parabolic equations.

2.2 A comparison theorem for viscosity solutions

2.2.1 The notion of viscosity solutions

The theory of viscosity solutions provides a framework for proving comparison and uniqueness theorems, and existence theorems for fully nonlinear partial differential equations of second order, which have found enormous important applications (cf. Crandall-Iishi-Lions [13]).

Suppose an elliptic equation is in the form $F(x, u, Du, D^2u) = 0$, where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \boldsymbol{S}_n \to \mathbb{R}$ and $\boldsymbol{S}_n$ is the space of symmetric $n \times n$ matrices, and F satisfies a monotonicity condition

$$F(x, r, p, X) \le F(x, s, p, Y), \quad \text{wherever } r \le s \text{ and } Y \le X$$

where $\boldsymbol{S}_n$ is equipped with its usual order, i.e. we say $X \ge Y$ if and only if $(Xe, e) \ge (Ye, e)$, $\forall e \in \mathbb{R}^n$.

Let $\boldsymbol{USC}$ ($\boldsymbol{LSC}$) denote the set of upper (lower) semicontinuous functions on $\Omega \subset \mathbb{R}^n$. For $u : \Omega \to \mathbb{R}$ and $x \in \Omega$, let $J^{2,+}_{\Omega} u(x)$, the second-order superject of u at x, be the set of $(p, X) \in \mathbb{R}^n \times \boldsymbol{S}_n$ such that

$$u(x') \le u(x) + p \cdot (x' - x) + X(x' - x) \cdot (x' - x)/2 + o\big(|x' - x|^2\big)$$

for any $x' \to x$.

Definition 2.2.1 ([13])**.** *A viscosity subsolution (supersolution) of $F = 0$ on Ω is a function $u \in$ $\boldsymbol{USC}$ ($\boldsymbol{LSC}$) such that*

$$F\big(x, u(x), p, X\big) \le (\ge)\ 0, \quad \forall x \in \Omega,\ (p, X) \in J^{2,+}_{\Omega} u(x)\ \Big(-J^{2,-}_{\Omega} u(x)\Big).$$

A viscosity solution of $F = 0$ is both a viscosity subsolution and a viscosity supersolution.

With certain conditions on the differential operator F, a maximal principle and a comparison result for viscosity solutions can be established ([13]).

Furthermore, the definition of viscosity solutions and the comparison result can be extended to parabolic equations

$$u_t + F\big(t, x, u, Du, D^2u\big) = 0$$

where u is a function $u(t, x)$, $Du = D_x u(t, x)$, $D^2u = D^2_x u(t, x)$ and $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \boldsymbol{S}_n \to \mathbb{R}$ ([13]).

2.2.2 A comparison theorem of of Barles-Biton-Ley

Barles, Biton and Ley obtained a general comparison result (Theorem 2.1 in [4]) for the viscosity solutions of a class of fully nonlinear parabolic equations, as well as an existence result (Theorem 3.1 in [4]).

We now describe the assumptions in the comparison and existence results in [4]. For any $X \in \boldsymbol{S}_n$, there exists an orthogonal matrix P such that $X = P\Lambda P^T$, where Λ is the diagonal matrix with diagonal entries consisting of eigenvalues of X. Let Λ^+ be the diagonal matrix obtained by replacing the negative eigenvalues in Λ with 0's. Then we define $X^+ = P\Lambda^+ P^T$.

Suppose F is a continuous function from $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times S_n$ to $\mathbb{R}$. Consider the following assumptions on F ([4]):

(H1) For any $R > 0$, there exists a function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0^+) = 0$ and

$$F(y,t,\eta(x-y),Y) - F(x,t,\eta(x-y),X) \leq m_R(\eta\,|x-y|^2 + |x-y|)$$

for all $x, y \in \bar{B}_R(0)$ and $t \in [0,T]$, whenever $X, Y \in \boldsymbol{S}_n$ and $\eta > 0$ satisfy

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

(H2) There exist $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x,t,p,X) - F(x,t,q,Y) \leq K_1\,|p-q|\,(1+|x|) + K_2\left(\mathrm{tr}\,(Y-X)^+\right)^\alpha$$

for every $(x,t,p,q,X,Y) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times S_n \times S_n$.

The operator F is degenerate elliptic if (H2) holds.

Theorem 2.2.2. *(Barles-Biton-Ley) Let u and v be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution respectively of*

$$\begin{aligned} \frac{\partial u}{\partial t} + F(x,t,Du,D^2u) &= 0 \quad \text{in } \mathbb{R}^n \times [0,T) \\ u(\cdot,0) &= u_0 \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Assume that (H1) and (H2) hold for F. Then

(i) If $u(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}^n$, then $u \leq v$ in $\mathbb{R}^n \times [0,T)$.

(ii) If $u_0 \in C(\mathbb{R}^n)$, there is a unique continuous viscosity solution in $\mathbb{R}^n \times [0,\infty)$.

In particular, they showed that (2.1.1) admits a unique longtime continuous viscosity solution for any continuous function u_0 in $\mathbb{R}$ when $n = 1$.

In the next section, we observe, via elementary methods, that the hypotheses in the general theorems in [4] are valid for the geometric evolution equation (2.1.1) in general dimensions.

2.3 Hypotheses (H1)

We now present the proof of Theorem 2.1.1. We define $F : \boldsymbol{S}_n \to \mathbb{R}$ by

$$F(X) = -i \log \frac{\det(I + iX)}{\det(I + X^2)^{\frac{1}{2}}} = -\frac{i}{2} \log \frac{\det(I + iX)}{\det(I - iX)}. \tag{2.3.1}$$

That F takes real values follows easily from

$$\overline{F(X)} = \frac{i}{2} \log \frac{\det(I - iX)}{\det(I + iX)} = F(X).$$

Note that $F(D^2u)$ is equal to $\sum \arctan \lambda_j$ modulo 2π (see Section 2.1). Therefore the flow (2.1.1) is equivalent to

$$u_t + (-F(D^2u)) = 0.$$

Since $F(x, t, p, X) = F(X)$ is independent of x, the right hand side of the inequality for F in (H1) must be zero, namely $m_R = 0$. By multiplying an arbitrary vector $(\xi, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and its transpose to the second matrix inequality in (H1), we see that $X \leq Y$. Therefore, in order to establish (H1) it suffices to show:

(H1′) For any $X, Y \in S_n$, if $X \geq Y$ then $F(X) \geq F(Y)$.

2.3.1 First proof

For any $X, Y \in S_n$ and $t \in [0, 1]$, define

$$f_{XY}(t) = F(tX + (1 - t)Y).$$

We will show that $f_{XY}(t)$ is nondecreasing in $t \in [0, 1]$ and then (H1′) will follow as $f_{XY}(0) = F(Y)$ and $f_{XY}(1) = F(X)$. Set

$$A = I + i(tX + (1 - t)Y)$$

and

$$B = I - i(tX + (1 - t)Y).$$

Then

$$f_{XY}(t) = -\frac{i}{2}(\log\det A - \log\det B).$$

It follows that $AB = BA$ and

$$\left(A^{-1} + B^{-1}\right) \cdot \frac{AB}{2} = \frac{A+B}{2} = I.$$

Note that both A and B are invertible matrices for all $t \in [0,1]$. Hence, by using the formula $\partial_t \ln\det G = \mathrm{tr}(G^{-1}\partial_t G)$ for $G(t) \in GL(n,\mathbb{R})$, we have

$$\begin{aligned} f'_{XY}(t) &= -\frac{i}{2}\mathrm{tr}\left(A^{-1}\cdot\partial_t A - B^{-1}\cdot\partial_t B\right) \\ &= -\frac{i}{2}\mathrm{tr}\left((A^{-1}+B^{-1})\cdot i(X-Y)\right) \\ &= -\frac{i}{2}\mathrm{tr}\left((A+B)(AB)^{-1}\cdot i(X-Y)\right) \\ &= \mathrm{tr}\left((I+(tX+(1-t)Y)^2)^{-1}\cdot(X-Y)\right). \end{aligned} \tag{2.3.2}$$

Since $tX + (1-t)Y$ is real symmetric, the matrix

$$C = I + (tX + (1-t)Y)^2$$

is positive definite, hence so is C^{-1}. There exists a matrix $Q \in GL(n,\mathbb{R})$ such that $C = QQ^T$. By the assumption $X \geq Y$, we have

$$\begin{aligned} \mathrm{tr}\left(C^{-1}(X-Y)\right) &= \mathrm{tr}\left(Q\cdot Q^T(X-Y)\right) \\ &= \mathrm{tr}\left(Q^T(X-Y)\cdot Q\right) \\ &\geq 0 \end{aligned}$$

since $Q^T(X-Y)Q$ is positive semidefinite. Therefore, we have shown that (H1) is valid for F defined in (2.3.1).

2.3.2 Second proof

We notice that (H1′), for the operator $F(X) = \sum \arctan\lambda_j(X)$, also follows from the basic fact: Suppose $X, Y \in S_n$ and that the eigenvalues λ_j of X and μ_j of Y are in descending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If $X \geq Y$, then $\lambda_j \geq \mu_j$ for $j = 1, \cdots, n$ (see p.182 in [27]). Here we include a proof.

Proof. Let e_i be an eigenvector to λ_i such that $\{e_i\}$ form an orthogonal basis. Denote $V_k = \mathrm{span}\{e_1, \cdots, e_k\}$ and $V_k^\perp = \mathrm{span}\{e_{k+1}, \cdots, e_n\}$ its orthogonal complement. Also let h_i be an eigenvector to μ_i and denote $W_k = \text{span}\{h_1, \cdots, h_k\}$ and $W_k^\perp = \text{span}\{h_{k+1}, \cdots, h_n\}$. By the variational characterization and the descending order of eigenvalues, we have

$$\lambda_k = \max_{e \in V_{k-1}^\perp} \frac{(Xe, e)}{\|e\|^2}, \quad \mu_{k-1} = \min_{h \in W_{k-1}} \frac{(Yh, h)}{\|h\|^2}.$$

Now we look at the intersection $V_{k-1}^\perp \cap W_{k-1}$:

i) $V_{k-1}^\perp \cap W_{k-1}$ contains an unit vector $e = h$: then $\lambda_k \geq (Xe, e) \geq (Yh, h) \geq \mu_{k-1} \geq \mu_k$, where the first inequality is by the variational formula and $e \in V_{k-1}^\perp$, the second is by $X \geq Y$, and the third is by the variational formula and $h \in W_{k-1}$.

ii) $V_{k-1}^\perp \cap W_{k-1} = \{0\}$: then $V = V_{k-1}^\perp \oplus W_{k-1}$ (the sum may not be orthogonal). This implies that for any $h^\perp \in W_{k-1}^\perp$, there exist unit vectors $h \in W_{k-1}$ and $e^\perp \in V_{k-1}^\perp$ such that $h^\perp - ah = be^\perp$. Since h and $h^\perp$ are orthogonal (with respect to the Euclidean form), we get

$$(h^\perp, h^\perp) + a^2 = b^2.$$

On the other hand, since W_{k-1} and $W_{k-1}^\perp$ are orthogonal also with respect to the bilinear form Y, we get

$$b^2(Ye^\perp, e^\perp) = a^2(Yh, h) + (Yh^\perp, h^\perp).$$

From these two equalities we have

$$\begin{aligned} \frac{(Yh^\perp, h^\perp)}{(h^\perp, h^\perp)} &= \frac{b^2(Ye^\perp, e^\perp) - a^2(Yh, h)}{b^2 - a^2} \\ &\leq \frac{b^2(Xe^\perp, e^\perp) - a^2\mu_{k-1}}{b^2 - a^2} \\ &\leq \frac{b^2\lambda_k - a^2\mu_{k-1}}{b^2 - a^2} \end{aligned}$$

where the first inequality is by $X \geq Y$ and the variational formula, and the second inequality is by the variational formula. Taking the sup over $W_{k-1}^\perp$, we have $(b^2 - a^2)\mu_k \leq b^2\lambda_k - a^2\mu_{k-1}$. From this we derive $\lambda_k \geq \mu_k$.

We have proved in either case, $\lambda_j \geq \mu_j$ for $j = 1, \cdots, n$. □

2.4 Hypotheses (H2)

As $F(x,t,p,X)$ is independent of p, (H2) reads: there exist constants $K>0$ and $0<\alpha<1$ such that $F(X)-F(Y)\le K\left(\operatorname{tr}(X-Y)^+\right)^\alpha$ for all $X,Y\in \boldsymbol{S}_n$.

For any $X,Y\in \boldsymbol{S}_n$, integrating (2.3.2) leads to

$$F(X)-F(Y)=\int_0^1 \operatorname{tr}\left(C^{-1}(X-Y)\right)\,dt. \tag{2.4.1}$$

For $X-Y\in \boldsymbol{S}_n$ there exists an orthogonal matrix P such that $X-Y=P\Lambda P^T$ where the diagonal matrix Λ has diagonal entries $\lambda_1,\cdots,\lambda_n$. Let $\lambda_j^+=\max\{\lambda_j,0\}$. Since $0<C^{-1}\le I$, we have $0<P^TC^{-1}P\le I$. If c_{jj} denote the diagonal entries of $P^TC^{-1}P$ for $j=1,\cdots,n$, then $c_{jj}=\langle P^TC^{-1}Pe_j,e_j\rangle$ where $\{e_1,\cdots,e_n\}$ is the standard basis for $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. It follows that $0<c_{jj}\le 1$ for $j=1,\cdots,n$. Then

$$\begin{aligned}\operatorname{tr}\left(C^{-1}(X-Y)\right) &= \operatorname{tr}\left(P^TC^{-1}P\cdot P^T(X-Y)P\right)\\ &= \operatorname{tr}\left(P^TC^{-1}P\cdot\Lambda\right)\\ &= \sum c_{jj}\lambda_j\\ &\le \sum \lambda_j^+\\ &= \operatorname{tr}(X-Y)^+.\end{aligned}$$

Substituting the above inequality into (2.4.1) implies: for any $X,Y\in \boldsymbol{S}_n$ we have

$$F(X)-F(Y)\le \operatorname{tr}(X-Y)^+.$$

Because $\arctan x$ is in $(-\pi/2,\pi/2)$, we have $F(X)-F(Y)<n\pi$. For any constant α with $0<\alpha<1$,

(i) if $\operatorname{tr}(X-Y)^+\le 1$, then

$$F(X)-F(Y)\le \operatorname{tr}(X-Y)^+\le n\pi\,\left[\operatorname{tr}(X-Y)^+\right]^\alpha$$

(ii) if $\operatorname{tr}(X-Y)^+>1$, then

$$F(X)-F(Y)\le n\pi\le n\pi\,\left[\operatorname{tr}(X-Y)^+\right]^\alpha.$$

Therefore, (H2) holds for $K_2=n\pi$ and any constants $K_1>0$ and α with $0<\alpha<1$.

Now Theorem 2.1.1 follows immediately from Theorem 2.2.2.

We also mention the uniqueness of viscosity solutions of the Cauchy-Dirichlet problem for (2.1.1). Since the operator $F(X) = \sum \arctan \lambda_j(X)$ satisfies (H1′), which is exactly the fundamental monotonicity condition for $-F$ in [13], $-F$ is proper in the sense of [13] (p.2 [13]). As (H1) holds, Theorem 8.2 in [13] is valid for (2.1.1):

Theorem 2.4.1. *The continuous viscosity solution to the following Cauchy-Dirichlet problem is unique:*

$$
\begin{aligned}
&u_t = \sum_{j=1}^{n} \arctan \lambda_j, \quad \text{in } (0,T) \times \Omega \\
&u(t,x) = 0, \qquad \text{for } 0 \le t < T \text{ and } x \in \partial\Omega \\
&u(0,x) = \psi(x), \quad \text{for } x \in \overline{\Omega}
\end{aligned}
$$

where λ_j, $j = 1, \cdots, n$, are the eigenvalues of D^2u, $\Omega \subset \mathbb{R}^n$ is open and bounded and $T > 0$ and $\psi \in C(\overline{\Omega})$. If u is an upper semicontinuous viscosity solution and v is a lower semicontinuous viscosity solution of the Cauchy-Dirichlet problem, then $u \le v$ on $[0,T) \times \Omega$.

Note that the initial boundary conditions for the subsolution and supersolution are: $u(x,t) \le 0 \le v(x,t)$ for $t \in [0,T)$ and $x \in \partial\Omega$, and $u(x,0) \le \psi(x) \le v(v,0)$ for $x \in \overline{\Omega}$.

Chapter 3

Minimal immersions of compact bordered Riemann surfaces with free boundary

3.1 Introduction

In this chapter, we study the existence theory and second variation theory for free boundary minimal surfaces. Suppose N is a complete Riemannian manifold of $\dim(N) \geq 3$, M is a compact submanifold of N, and Σ is a compact Riemann surface with nonempty boundary $\partial\Sigma$. A *minimal surface of N with free boundary in M* is an immersion $u : (\Sigma, \partial\Sigma) \to (N, M)$ (possibly with branched points) such that $u(\Sigma)$ has zero mean curvature and $u(\Sigma)$ is orthogonal to M along $u(\partial\Sigma) \subseteq M$.

We begin by reviewing some basic facts about the induced homomorphism on the fundamental group and recall the Euler-Lagrange equations for the α-energy. Then we cite the homogeneously regular condition to be used in our setting, and give a detailed proof of the ϵ-regularity theorem first proved by Sacks-Uhlenbeck for closed surfaces ([48]) and then extended to the boundary situation by Fraser ([22]). Using these main estimates we obtain global convergence of critical maps of the α-energy, and the results are compiled in Subsection 3.2.4. In particular, we prove the existence of a harmonic map in any conjugacy class. Furthermore, under stronger conditions on the topology of N and M, we prove the existence of a harmonic map in any homotopy class. There we give two proofs, the first of which is based on a construction by Sacks-Uhlenbeck ([48]), and the second is based on a topological fact.

Then we recall the notion of the Riemann moduli space and the Teichmüller space, and a structure theorem on the space of conformal structures for compact bordered Riemann surfaces. We show how to reduce the minimal area problem to a convergence problem for conformal structures in the moduli space. Then using the incompressible assumptions, we prove a general existence theorem for free boundary minimal immersions of compact bordered Riemann surfaces which are not a disk. Two different proofs are presented in Section 3.4.

Next we consider minimizing disks. We prove the kernel of $i_* : \pi_1(M) \to \pi_1(N)$ admits a generating set such that each member is freely homotopic to the boundary of an area minimizing disk that solves the free boundary problem $(D, \partial D) \to (N, M)$.

In the second part, we investigate controlling the topology of free boundary minimal surfaces of low index in 3-manifolds, under certain nonnegativity assumptions on the curvature and convexity assumptions on the boundary of the 3-manifold. We derive bounds on the genus and number of boundary components, and area estimates. We also prove a rigidity result for stable minimal surfaces.

3.2 Existence of minimizing harmonic maps in a conjugacy class

Throughout this chapter the terms "fundamental group" and "relative fundamental group" will implicitly refer to some base point $*$. Since the isomorphism class of the fundamental group of a path-connected space is not affected by the choice of base-point, without ambiguity, we will write $\pi_1(\,\cdot\,)$ and $\pi_1(\cdot\,,\cdot)$ while omitting the base point.

Any continuous map f gives rise to a homomorphism $f_* : \pi_1(\,\cdot\,,*) \to \pi_1(\,\cdot\,, f(*))$, which we will call the induced map of f. We shall say two maps f and g induce the same action on the fundamental group, if there exists a path λ from $f(*)$ to $g(*)$, such that $f_* = \lambda_*^{-1} \circ g_* \circ \lambda_*$, or equivalently, if f_* and g_* represent the same homomorphism after identifying the fundamental groups with different base-points through an isomorphism

$$I_\lambda : \pi_1(\,\cdot\,, f(*)) \to \pi_1(\,\cdot\,, g(*)), \quad \sigma \mapsto \lambda \cdot \sigma \cdot \lambda^{-1}.$$

In this case, we will briefly say f_* is conjugate to g_* and will write $f_* \sim g_*$ for short.

Finally we recall that the relative fundamental group $\pi_n(X, A, *)$ for a triple $\{*\} \subset A \subset X$, is a group only for $n \geq 2$. When $n = 1$, it is the set of homotopy classes of paths from the base point $*$ to a varying point in A.

Let N, M and Σ be as defined in Theorem 1.1. Given a continuous map $f : \Sigma \to N$ with $f(\partial\Sigma) \subseteq M$, denote by f_* the induced homomorphism as indicated in each of the following situations:

1) Σ is not a disk, $\quad f_* : \pi_1(\Sigma) \times \pi_1(\Sigma, \partial\Sigma) \to \pi_1(N) \times \pi_1(N, M)$
2) Σ is a disk D, $\quad f_* : \pi_1(\partial D) \to \pi_1(M)$.

We will use the terminology "the conjugacy class of f_*" to denote the set of maps for which the induced homomorphisms on the above fundamental groups are conjugate to f_*.

3.2.1 Existence of minimizers for E_α

Suppose a conformal structure on Σ is fixed, a Riemannian metric compatible with this conformal structure is given, and this metric defines an area element $d\mu$. Let $N \hookrightarrow \mathbb{R}^K$ be a C^∞ isometric embedding for sufficiently large K. Set

$$W^{1,p}(\Sigma, N) = \{u \in W^{1,p}(\Sigma, \mathbb{R}^K) \mid u(x) \in N \text{ a.e. } x \in \Sigma\}.$$

For $\alpha > 1$, we define the α-energy

$$E_\alpha(u) = \int_\Sigma \left(1 + |\nabla u|^2\right)^\alpha d\mu$$

on the admissible space

$$W_\alpha = \{u \in W^{1,2\alpha}(\Sigma, N) \mid u(\partial\Sigma) \subseteq M,\ u_* \sim f_*\}.$$

Note that by the Sobolev embedding theorem, each u in $W^{1,2\alpha}(\Sigma, N)$ is continuous.

Proposition 3.2.1. *E_α attains the infimum at some $u_\alpha \in W_\alpha$, $\forall \alpha > 1$.*

Proof. Let $I_\alpha = \inf_{W_\alpha} E_\alpha$. Let $\{u_k^\alpha\}$ be an E_α-minimizing sequence of maps; that is $E_\alpha(u_k^\alpha) \to I_\alpha$. From the Sobolev embedding

$$W^{1,2\alpha}(\Sigma, N) \hookrightarrow C^{0,\frac{\alpha-1}{\alpha}}(\Sigma, N),$$

the sequence $\{u_k^\alpha\}$ is equicontinuous, so the Arzelà-Ascoli theorem yields a subsequence, which we still denote by $\{u_k^\alpha\}$, that converges uniformly to a map u_α in $C^{0,\beta}(\Sigma, N)$ for any $\beta \in \left[0, \frac{\alpha-1}{\alpha}\right)$, and $u_\alpha(\partial\Sigma) \subseteq M$. Furthermore, when k is sufficiently large, u_k^α is homotopic to u_α, and hence $(u_\alpha)_* \sim (u_k^\alpha)_* \sim f_*$. On the other hand, from the weak compactness of the unit ball in $W^{1,2\alpha}(\Sigma, N)$, a subsequence of $\{u_k^\alpha\}$ converges weakly to some u'_α in $W^{1,2\alpha}(\Sigma, N)$. It follows that the two limits from the strong convergence and the weak convergence agree, that is $u_\alpha = u'_\alpha \in W^{1,2\alpha}(\Sigma, N)$. Thus u_α is in W_α. Now from the lower semi-continuity of the α-energy, we have $E_\alpha(u_\alpha) = I_\alpha$. □

3.2.2 Euler-Lagrange equations for E_α

Now we investigate the convergence of $\{u_\alpha\}$ for a sequence $\alpha \to 1$. The key to obtaining a limiting harmonic map is to get estimates for a sequence of critical maps of $E_\alpha$that are independent of α. For this, the coefficients in the Euler-Lagrange equations for E_α for a sequence $\alpha \to 1$ have to be uniformly controlled. When N is compact, these coefficients can be given in terms of the second fundamental form of some isometric embedding $N \hookrightarrow \mathbb{R}^K$ (cf. page 154, Colding-Minicozzi [12]). In the situation where the ambient manifold is noncompact, we shall employ an intrinsic version of the Euler-Lagrange equation for E_α given in terms of intrinsic geometric quantities of the ambient manifold.

We now give the intrinsic version of the Euler-Lagrange equations (cf. Fraser [22]). Let (x_1, x_2) be the standard Euclidean coordinates on a disk $D \subset \Sigma$. Given $p \in M$, we choose Fermi coordinates $f^1, \cdots, f^n$ on an open neighbourhood V of p in N that satisfy the following:

(*i*) $V \cap M$ is the zero set of $f^{m+1}, \cdots, f^n$

(*ii*) the metric g on N in these coordinates satisfies $g_{ab}(p) = 0$ for any $p \in M$ when $a = 1, \cdots, m$ and $b = m+1, \cdots, n$ where $g_{ab} = g(\frac{\partial}{\partial f^a}, \frac{\partial}{\partial f^b})$).

Let Γ^a_{bc} denote the Christoffel symbols of N in a local coordinates. The Euler-Lagrange equation is given by

$$\Delta u^a + \sum_{i=1}^{2} \sum_{b,c=1}^{n} \Gamma^a_{bc} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^i} + \frac{\alpha - 1}{1 + |du|^2} \sum_{i=1}^{2} \frac{\partial}{\partial x^i}(|du|^2) \frac{\partial u^a}{\partial x^i} = 0$$

at interior points in Σ. If a disk is centred at a boundary point in $\partial\Sigma \subset M$, using Fermi coordinates, one can write the boundary conditions as

$$(B) \begin{cases} \frac{\partial u^a}{\partial r}(x) = 0 & \text{for } a = 1, \cdots, m \\ u^a(x) = 0 & \text{for } a = m+1, \cdots, n. \end{cases}$$

Proposition 3.2.2. $u_\alpha \in C^\infty(\Sigma, N)$.

Proof. The result follows from Sacks and Uhlenbeck [48] for interior regularity, and from Fraser [22] for boundary regularity. □

Now we cover Σ by small disks of radius R in the interior, and half disks of radius R along the boundary, where locally we choose Fermi coordinates. When we scale these disks and half disks to unit size the energy integral becomes $E_\alpha(u) = R^{2(1-\alpha)} \int_D (R^2 + |\nabla u|^2)^\alpha d\mu$ where D is the unit disk. The Euler-Lagrange equations appear in the following forms after this conformal dilation:

$$\Delta u^a + \sum_{i=1}^{2} \sum_{b,c=1}^{n} \Gamma^a_{bc} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^i} + \frac{\alpha - 1}{R + |du|^2} \sum_{i=1}^{2} \frac{\partial}{\partial x^i}(|du|^2) \frac{\partial u^a}{\partial x^i} = 0 \qquad (3.2.1)$$

or

$$\begin{aligned} \Delta u^a \quad &+ \quad \sum_{i=1}^{2} \sum_{b,c=1}^{n} \Gamma^a_{bc} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^i} + \frac{\alpha - 1}{R + |du|^2} \sum_{i,j=1}^{2} \sum_{b,c=1}^{n} \Bigg(\frac{\partial (g_{bc} \circ u)}{\partial x^i} \frac{\partial u^b}{\partial x^j} \frac{\partial u^c}{\partial x^j} \\ &+ \quad 2 g_{bc}(u) \frac{\partial^2 u^b}{\partial x^i \partial x^j} \frac{\partial u^c}{\partial x^j} \Bigg) \frac{\partial u^a}{\partial x^i} = 0 \end{aligned}$$

with the boundary condition (B) on half disks in Fermi coordinates.

Since N is allowed to be noncompact, we impose suitable conditions on N. A complete Riemannian manifold N is *homogeneously regular* if its injectivity radius is bounded from below and its sectional curvature is bounded (see [43] p. 623, [45]).

Definition 3.2.3. *A Riemannian manifold N is said to be homogeneously regular if there exist positive numbers λ, Λ and K, independent of p_0, such that each point p_0 of N lies in an open set B_{p_0} in N which can be mapped onto the unit ball $B_1(0)$ by a bi-Lipschitz map such that p_0 corresponds to the origin and the following hold*

$$1)\quad \lambda|x|^2 \leq \langle x, x\rangle_{g(p)} \leq \Lambda|x|^2, \quad \textit{for all} \;\; p \in B_1(0),\; x \in T_pB_1(0);$$

$$2)\quad \sup_{B_1(0)} \left|\frac{\partial g_{ij}}{\partial x^k}\right|, \left|\frac{\partial^2 g_{ij}}{\partial x^k \partial x^n}\right| \leq K, \quad \textit{for } i,\; j,\; k,\; n = 1, 2, \cdots, n,$$

where $g = g_{ij}dx^i dx^j$ is the metric on $B_1(0)$ induced from that on B_{p_0}.

With this condition, and the assumption that the boundary of the surface lies in a compact submanifold of N, we can derive the main estimate for critical maps of the α-energy at interior points of Σ in a similar manner as in the case of closed surfaces in compact manifolds (Proposition 3.2 of [48]). In fact, the smaller the disk, the nearer to Euclidean is the metric on the expanded disk. Thus a priori estimates are uniform in $\alpha \geq 1$ and $0 < R \leq 1$. In the boundary situation, let $D_r^+ = \Sigma \cap D_r$ for a point $x \in \partial\Sigma$, where D_r the disk of radius r about x. If the total energy of u is small enough, its derivative is bounded on D_r^+ by some constant C depending on r and the geometry of N in $u(D_r^+)$, by Lemma 1.6 in [22]. Since N is homogeneously regular, $C = C(r, K, \lambda, \Lambda)$. By choosing r sufficiently small, we can assume $u(\partial\Sigma \cap D_r)$ is contained in a coordinate chart in N with Fermi coordinates defined as above. Therefore the Euler-Lagrange equations (3.2.1) with the boundary condition (B) are valid and we can derive the main estimate at boundary points (Proposition 1.7, Fraser [22]).

3.2.3 Main estimates

We state the main interior and boundary estimates (cf. Sacks-Uhlenbeck [48], Fraser [22]).

Theorem 3.2.4. *There exist $\epsilon > 0$ and $\alpha_0 > 1$ such that: if $u_\alpha : \Sigma \to N$ with $u_\alpha(\partial\Sigma) \subset M$ is a critical map of E_α on W_α and $E(u|_D) < \epsilon$, then there is an estimate uniform in $1 \le \alpha \le \alpha_0$,*

$$\|\nabla u\|_{D',1,p} \le K(p, D', D, N)\, \|\nabla u\|_{D,0,2}$$

for any $D' \subset D$, $1 < p < \infty$.

Proof. First we prove interior estimate, i.e. $\Sigma \cap D = \emptyset$. We denote

$$\sum_{i=1}^{2} \sum_{b,c=1}^{n} \Gamma^a_{bc} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^i} = A(\nabla u, \nabla u)$$

$$\sum_{i,j=1}^{2} \sum_{b,c=1}^{n} \frac{\partial (g_{bc} \circ u)}{\partial x^i} \frac{\partial u^b}{\partial x^j} \frac{\partial u^c}{\partial x^j} \frac{\partial u^a}{\partial x^i} = \dot{G}(\nabla u, \nabla u, \nabla u)$$

and

$$\sum_{i,j=1}^{2} \sum_{b,c=1}^{n} 2 g_{bc}(u) \frac{\partial^2 u^b}{\partial x^i \partial x^j} \frac{\partial u^c}{\partial x^j} \frac{\partial u^a}{\partial x^i} = G(\nabla^2 u(\nabla u), \nabla u) = G(\nabla^2 u, \nabla u, \nabla u)$$

where A, $\dot{G}$, and G are linear in each variable respectively, and they are given in terms of the metric and its derivatives. Then we can write the Euler-Lagrange equation (3.2.1) as

$$\Delta u + A(\nabla u, \nabla u) + \frac{\alpha - 1}{R + |\nabla u|^2} \Big(\dot{G}(\nabla u, \nabla u, \nabla u) + G(\nabla^2 u, \nabla u, \nabla u) \Big) = 0 \tag{3.2.2}$$

Denote by (D_1, D_2, φ) two disks $D_2 \subsetneq D_1$ and a smooth function φ on Σ, such that $\mathrm{supp}(\varphi) \subset D_1$ and $\varphi = 1$ on D_2.

We compute

$$\begin{aligned} \varphi \nabla u &= \nabla(u\varphi) - u \nabla \varphi \\ \varphi \Delta u &= \Delta(u\varphi) - u \Delta \varphi - 2 \langle \nabla u, \nabla \varphi \rangle \\ \varphi \nabla^2 u &= \nabla^2(u\varphi) - u \nabla^2 \varphi - 2 \nabla u \otimes \nabla \varphi. \end{aligned}$$

Multiplying (3.2.2) by φ, and inserting the above equalities, we get

$$\begin{aligned} & \Delta(u\varphi) - u\Delta\varphi - 2\langle \nabla u, \nabla\varphi\rangle + A\big(\nabla(u\varphi) - u\nabla\varphi, \nabla u\big) \\ + \quad & \frac{\alpha-1}{R+|\nabla u|^2}\Big(\dot{G}\big(\nabla(u\varphi) - u\nabla\varphi, \nabla u, \nabla u\big) \\ + \quad & G\big(\nabla^2(u\varphi) - u\nabla^2\varphi - 2\nabla u\otimes\nabla\varphi, \nabla u, \nabla u\big)\Big) = 0. \end{aligned}$$

Since $|\nabla u|^2 < R + |\nabla u|^2$, we have

$$\begin{aligned} & \Delta(u\varphi) + A(\nabla u\varphi, \nabla u) \\ & + \frac{\alpha-1}{R+|\nabla u|^2}\Big(\dot{G}\big(\nabla u\varphi, \nabla u, \nabla u\big) + G\big(\nabla^2(u\varphi), \nabla u, \nabla u\big)\Big) \\ = \quad & u\Delta\varphi + 2\langle \nabla u, \nabla\varphi\rangle + A\big(u\nabla\varphi, \nabla u\big) \\ & + \frac{\alpha-1}{R+|\nabla u|^2}\Big(\dot{G}\big(u\nabla\varphi, \nabla u, \nabla u\big) + G\big(u\nabla^2\varphi + 2\nabla u\otimes\nabla\varphi, \nabla u, \nabla u\big)\Big) \\ \leq \quad & K(\varphi, A, \dot{G}, G)\Big(|u| + |\nabla u| + |u||\nabla u|\Big) \end{aligned}$$

where K is a constant depending on $A, \dot{G}, G$ and φ. From this we derive

$$\|\Delta(u\varphi)\|_{0,p} \leq K\Big((\alpha-1)\|u\varphi\|_{2,p} + \big\||\nabla(u\varphi)||\nabla u|\big\|_{0,p} + \|u\|_{1,p} + \big\||u||\nabla u|\big\|_{0,p}\Big).$$

By the L_p estimates for second-order elliptic equations, (see for example Theorem 8.2 in Agmon [3], or Theorem 9.13 in Gilbarg-Trudinger [23]), the operator $\Delta^{-1} : W^{0,p}(D) \to (W^{2,p}(D) \cap W_0^{1,p}(D))$ is bounded, that is

$$\|h\|_{2,p} \leq C\big(\,\|\Delta h\|_{0,p} + \|h\|_{0,p}\,\big)$$

for any $h \in W^{2,p}(D) \cap W_0^{1,p}(D)$, where C is a constant depending on p, the elliptic constant λ for the Laplacian on D and the constant K in the theorem in [3]. We note that K (Definition 5.1 in [3]) involves bounds on the metric and its first and second order derivatives. Therefore C depends on p, the metric of N and its derivatives up to order two on the domain D (see also Gilbarg-Trudinger [23]). Therefore when $\alpha - 1 > 0$ is sufficiently small, we get an estimate

$$\|u\varphi\|_{2,p} \leq K\Big(\big\||\nabla(u\varphi)||\nabla u|\big\|_{0,p} + \|u\|_{1,p} + \big\||u||\nabla u|\big\|_{0,p}\Big) \qquad (3.2.3)$$

where K is a constant depending on the metric of N and its derivatives up to order two on the domain D, φ and p.

Since we can assume the total energy of u is small enough, its derivative is bounded on D_r by some constant C depending on N and r, by Lemma 1.6 in [22]. But by assumption N is homogeneously regular. Therefore the derivative of u is bounded on Σ and we get

$$\|u\varphi\|_{2,p} \le K\big(\big\||\nabla(u\varphi)||\nabla u|\big\|_{0,p} + \|u\|_{1,p}\big).$$

From the Holder inequality, we have

$$\big\||\nabla(u\varphi)||\nabla u|\big\|_{0,p} \le \|\nabla(u\varphi)\|_{0,\lambda p}\,\|\nabla u\|_{0,\mu p}$$

for any $\lambda, \mu > 0$ such that $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. Thus

$$\|u\varphi\|_{2,p} \le K\big(\,\|\nabla(u\varphi)\|_{0,\lambda p}\,\|\nabla u\|_{0,\mu p} + \|u\|_{1,p}\big). \tag{3.2.4}$$

Now letting

$$p = 2-\delta,\ \ \lambda = \frac{2}{\delta} \text{ and } \mu = \frac{2}{2-\delta}, \quad \text{where } 0 < \delta < 2$$

and using that $\|u\varphi\|_{1,2(2-\delta)/\delta}$ is uniformly bounded by $\|u\varphi\|_{2,2-\delta}$, we get

$$\|u\varphi\|_{1,\frac{2(2-\delta)}{\delta}} \le K\big(\,\|\nabla(u\varphi)\|_{0,\frac{2(2-\delta)}{\delta}}\,\|\nabla u\|_{0,2} + \|u\|_{1,2-\delta}\big).$$

Then for u such that $\|\nabla u\|_{0,2}\,K < 1$, we get

$$\|u\varphi\|_{1,\frac{2(2-\delta)}{\delta}} \le \frac{K}{1-\|\nabla u\|_{0,2}\,K}\,\|u\|_{1,2-\delta}\,.$$

Or equivalently, for any $p > 0$ and u of small energy, we have an estimate

$$\|u\varphi\|_{1,p} \le K\,\|u\|_{1,\frac{2p}{p+2}} \tag{3.2.5}$$

where K depends on p, φ and the energy of u.

Now let D be the unit disk on which the rescaled map u is defined, and $D' \subsetneq D'' \subsetneq D$ be two smaller disks. Applying the above argument to (D'', D', φ) we get from (3.2.5)

$$\|u\varphi\|_{D'',1,p} \le K_1\,\|u\|_{D'',1,\frac{2p}{p+2}}$$

and applying the above argument to (D, D'', ϕ) we get

$$\|u\|_{D'',1,p} = \|u\phi\|_{D'',1,p} \le \|u\phi\|_{1,p} \le K_2\,\|u\|_{1,\frac{2p}{p+2}}$$

where we suppress the subscript of domain for D. This implies that

$$\begin{aligned} \|u\|_{D'',1,\mu p} &\leq K_1 \|u\|_{1,\frac{2\mu p}{\mu p+2}} && (3.2.6)\\ \|u\varphi\|_{D'',1,\lambda p} &\leq K_2 \|u\|_{D'',1,\frac{2\lambda p}{\lambda p+2}}\,. && (3.2.7)\end{aligned}$$

From these two inequalities and (3.2.4) for (D'', D', φ), we get

$$\begin{aligned} \|u\|_{D',2,p} &\leq \|u\varphi\|_{D'',2,p}\\ &\leq K\Big(\|\nabla(u\varphi)\|_{D'',0,\lambda p} \|\nabla u\|_{D'',0,\mu p} + \|u\|_{D'',1,p}\Big)\\ &\leq K\Big(\|u\|_{D'',1,\frac{2\lambda p}{\lambda p+2}} \|u\|_{1,\frac{2\mu p}{\mu p+2}} + \|u\|_{1,\frac{2p}{p+2}}\Big)\\ &\leq K\Big(\|u\|_{1,\frac{2\lambda p}{\lambda p+2}} \|u\|_{1,\frac{2\mu p}{\mu p+2}} + \|u\|_{1,\frac{2p}{p+2}}\Big)\end{aligned}$$

where in the third inequality we have used (3.2.6) and (3.2.7) for the three norms.

The Holder inequality implies

$$\|f\|_{\frac{2\lambda p}{\lambda p+2}} \|g\|_{\frac{2\mu p}{\mu p+2}} \leq \|f\|_2^{\frac{1}{2}} \|g\|_2^{\frac{1}{2}} \|1\|_p$$

for $\frac{1}{\lambda} + \frac{1}{\mu} = 1$, and

$$\|f\|_{\frac{2p}{p+2}} \leq \|f\|_2 \|1\|_p\,.$$

Therefore we get

$$\begin{aligned} \|u\|_{D',2,p} &\leq K\Big(\|u\|_{1,\frac{2\lambda p}{\lambda p+2}} \|u\|_{1,\frac{2\mu p}{\mu p+2}} + \|u\|_{1,\frac{2p}{p+2}}\Big)\\ &\leq K\Big(\big(\|u\|_{\frac{2\lambda p}{\lambda p+2}} + \|\nabla u\|_{\frac{2\lambda p}{\lambda p+2}}\big)\big(\|u\|_{\frac{2\mu p}{\mu p+2}} + \|\nabla u\|_{\frac{2\mu p}{\mu p+2}}\big)\\ &\quad + \|u\|_{\frac{2p}{p+2}} + \|\nabla u\|_{\frac{2p}{p+2}}\Big)\\ &\leq K \|u\|_{1,2}\end{aligned}$$

for any critical map of E_α with $\alpha - 1 > 0$ and the energy of u sufficiently small.

We can also write this estimate in an intrinsic form

$$\|\nabla u\|_{D',1,p} \leq K \|\nabla u\|_{0,2}$$

by considering an isometric embedding $N \hookrightarrow \mathbb{R}^K$, choosing the origin such that $\int_D u = 0$ and the Poincaré inequality.

In the boundary situation, we consider two disks $D_{r'}$ and D_r around a boundary point with $r' < r$, and a smooth function φ with support in D_r and is 1 on $D_{r'}$, satisfying the boundary condition $\frac{\partial \varphi}{\partial r} = 0$ on $\partial\Sigma \cap D_r$. If the total energy of u is small enough, its derivative is bounded on $\Sigma \cap D_r$ by Lemma 1.6 in [22]. Then by choosing r sufficiently small, we can assume $u(\partial\Sigma \cap D_r)$ is contained in a coordinate chart in N with Fermi coordinates defined as above. Therefore the estimate (3.2.2) is available. Then we have (3.2.3) and by a similar argument, we have

$$\|\nabla u\|_{D_{r'},1,p} \leq K \|\nabla u\|_{D_r,0,2} .$$

□

3.2.4 Convergence of critical maps for E_α as $\alpha \to 1$

Next we consider the convergence of a sequence of critical maps of E_α as $\alpha \to 1$. Notice that for a sequence of minimizing maps u_α of E_α we have a uniform energy bound. Let f_0 be a smooth map in the homotopy class of f, which exists since $C^\infty(\Sigma, N)$ is dense in $C(\Sigma, N)$. Then $f_0 \in W_\alpha$ for all $\alpha > 0$. Since u_α minimizes E_α on W_α, we have

$$\int_\Sigma \left(1 + |\nabla u_\alpha|^2\right)^\alpha d\mu \leq \int_\Sigma \left(1 + |\nabla f_0|^2\right)^\alpha d\mu.$$

Then for $\alpha \in (1, 2)$, we get that the energy of u_α is uniformly bounded as

$$\int_\Sigma |\nabla u_\alpha|^2 \, d\mu \leq \int_\Sigma \left(1 + |\nabla u_\alpha|^2\right)^\alpha d\mu \leq \int_\Sigma \left(1 + |\nabla f_0|^2\right)^2 d\mu.$$

Lemma 3.2.5. *Let u_α be a sequence of critical maps of E_α with $E(u_\alpha) \leq B$. Then a subsequence $u_\alpha \to u$ strongly in $L^2(\Sigma, \mathbb{R}^K)$, weakly in $W^{1,2}(\Sigma, \mathbb{R}^K)$, and $E(u) \leq \lim_{\alpha \to 1} E(u_\alpha)$.*

We then have the following convergence result for critical maps of small energy.

Lemma 3.2.6 ([48], [22])**.** *Let $u_\alpha : \Sigma \to N$ with $u_\alpha(\partial\Sigma) \subset M$ be critical maps of E_α on W_α for a sequence $\alpha \to 1$, that converge weakly in $L^2(\Sigma, \mathbb{R}^K)$. Then there exists $\epsilon > 0$ such that if $E(u_\alpha|_D) < \epsilon$, then $\{u_\alpha\} \to u$ in $C^1(\overline{D}_{\frac{r}{2}}, N)$ and $u : \overline{D}_{\frac{r}{2}} \to N$ is a smooth harmonic map such that $u(D_{\frac{r}{2}})$ meets M orthogonally along $u(\partial\Sigma \cap D_{\frac{r}{2}})$.*

We can now deduce global convergence of critical maps of the α-energy away from a finite number of points.

Theorem 3.2.7 ([48], [22])**.** *Let $u_\alpha : \Sigma \to N$ with $u_\alpha(\partial\Sigma) \subset M$ be critical maps of E_α for a sequence $\alpha \to 1$, that converge weakly in $L^2(\Sigma, \mathbb{R}^K)$, with $E(u_\alpha) < B$. Then there exists a finite set of points $\{z_1, \cdots, z_l\}$ of Σ such that $u_\alpha \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$ and $u : (\Sigma, \partial\Sigma) \to (N, M)$ is a smooth harmonic map satisfying the free boundary condition. Furthermore, when Σ is not a disk, if each u_α induces the same action on the fundamental group as f, then so does u.*

Proof. The convergence part is Theorem 4.4 in [48] and Theorem 1.15 in [22]. We need only verify that the induced map of u_α on the fundamental group is preserved in the limiting process, regardless of a finite set of points where bubbling may occur.

Choose, as generators of $\pi_1(\Sigma)$, $k+2g$ loops through a base point $*$ in Σ such that all of these curves $\{\gamma_j\}$ stay away from the points $\{z_1, \cdots, z_l\}$ where the C^1 convergence fails. Since $u_\alpha \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$, $u_\alpha(\cup_j \gamma_j)$ is homotopic to $u(\cup_j \gamma_j)$ for α sufficiently close to 1. It follows that there exists a path connecting $u_\alpha(*)$ and $u(*)$ such that $u_\alpha(\gamma_j)$ can be deformed to $u(\gamma_j)$ along the same path for each j. Therefore by definition, u induces the same action as u_α, and thus f on $\pi_1(\Sigma)$. On the other hand, $\pi_1(\Sigma, \partial\Sigma)$ is the set of free homotopy classes of paths from a fixed point $* \in \partial\Sigma$ to a varying point on $\partial\Sigma$, and for each class a representative can be chosen away from the points $\{z_1, \cdots, z_l\}$. Therefore u induces the same action as u_α for α sufficiently close to 1, and thus f on $\pi_1(\Sigma, \partial\Sigma)$. □

Geometrically the condition of preserving the action on π_1 is equivalent to the condition that the free homotopy class of Σ_1 is preserved, where Σ_1 is the union of a set of base-pointed generators for π_1. Since the 1-skeleton Σ_1 can deform around any nontrivial S^2 in the ambient manifold without affecting the homotopy class of Σ_1, the action on π_1 is not affected by $\pi_2(N)$ (whereas the homotopy class of $\Sigma_2 = \Sigma$ is changed).

Therefore in the special case that minimizing sequences $\{u_k^\alpha\}$ for E_α, and minimizing sequences $\{u_\alpha\}$ for E (but each need not be minimizing for E_α) are considered in the admissible spaces W_f^α and W_f with a fixed conjugacy class, we have the following result of C^1-convergence.

Corollary 3.2.8. *Let u_α be a sequence of minimizing maps of E_α in W_f^α for a sequence $\alpha \to 1$, that converges in $L^2(\Sigma, \mathbb{R}^K)$. Then a subsequence $u_\alpha \to u$ in $C^1(\Sigma, N)$ where u is a smooth harmonic map in W_f.*

Proof. Let $f_0 \in W_f$ be a smooth map. Since u_α is minimizing, from the above we have $E(u_\alpha) \leq E_2(f_0)$ provided $\alpha \leq 2$. Thus by Theorem 3.2.7 there exists a subsequence $u_\alpha \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$. Choose small disks or half disks $D_i(\rho)$ of radius ρ around each z_i. Denote $A = \cup D_i(\rho)$ and $B = \Sigma - A$. We can get modified maps $\tilde{u}_\alpha$ that agree with u_α on B and such that $\tilde{u}_\alpha \to u$ in $C^1(\Sigma, N)$ (for the details see the proof of Theorem 3.2.9).

Choose a set of generators for $\pi_1(\Sigma)$ and any nontrivial element of $\pi_1(\Sigma, \partial\Sigma)$ away from these disks. Then $(\tilde{u}_\alpha)_* \sim (u_\alpha)_*$. Since u_α is minimizing in W_f^α, it follows that

$$E(u_\alpha|_\Sigma) \leq E(\tilde{u}_\alpha|_\Sigma).$$

Since $\tilde{u}_\alpha = u_\alpha$ on B, this implies

$$E(u_\alpha|_A) \leq E(\tilde{u}_\alpha|_A).$$

On the other hand, by the C^1-convergence we have

$$E(\tilde{u}_\alpha|_A) \to E(u|_A) \le |A| \max |\nabla u|^2.$$

Hence for $\alpha - 1$ sufficiently small

$$E(u_\alpha|_A) \le (1+\sigma)|A| \max |\nabla u|^2 \le (1+\sigma)\pi\rho^2 l \|u\|^2_{1,\infty}$$

where $\sigma > 0$ is a constant. Now we can choose ρ sufficiently small such that $E(u_\alpha|_A) < \epsilon$, where ϵ is as in Lemma 3.2.6. Then by Lemma 3.2.6, we get $u_\alpha \to u$ in $C^1(\Sigma, N)$. □

Remark. In comparison, blowup points for a sequence of minimizing maps in a fixed homotopy class can occur by the existence of nontrivial elements in $\pi_2(N)$ or $\pi_2(N, M)$. Therefore when Σ is a disk D, to show the action on π_1 is preserved in the limit, the argument above cannot be applied as a blowup point may be on the generator of $\pi_1(\partial D)$, for a sequence of minimizing maps in a fixed homotopy class of the boundary curve. See Section 3.5.

3.2.5 Existence of a harmonic map in any homotopy class

The convergence may fail at a finite number of interior or boundary points, where bubbling occurs. Thus the homotopy class of $\{u_\alpha\}$ can be altered in the limiting process. However, under stronger conditions on the topology of N and M, we obtain the existence of a harmonic map in any homotopy class. This can be interpreted as the free boundary analog of Theorem 5.1 in [48].

Theorem 3.2.9. *Let N, M and Σ be as in Theorem 1.0.2. If in addition $\pi_2(N) = 0$ and $\pi_2(N, M) = 0$, then there exists a minimizing harmonic map satisfying the free boundary condition in every free homotopy class of maps in $C^0((\Sigma, \partial\Sigma), (N, M))$.*

I. Analytic proof

First we give a proof by modifying each u_α in its homotopy class using the topological condition, to get a sequence of modified maps that converges in $C^1(\Sigma)$. Then we show the C^1-convergence of $\{u_\alpha\}$ by comparing the two sequences and the minimizing property of u_α.

Proof. Let u_α be a minimizing map of E_α in a fixed homotopy class of f for a sequence $\alpha \to 1$. Then $\{u_\alpha\}$ has uniformly bounded energy by the note after Proposition 3.2.1. By Theorem 3.2.7, there exists a finite set of points

$\{z_1, \cdots, z_l\}$ such that a subsequence $u_\alpha \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$ and $u : (\Sigma, \partial\Sigma) \to (N, M)$ is a smooth harmonic map satisfying the free boundary condition. We claim that under the topological assumptions of the theorem, $u_\alpha \to u$ in $C^1(\Sigma, N)$.

At each point z_i where the C^1 convergence fails, center a small disk D_ρ in Σ about z_i of radius ρ, where ρ is small enough so that $z_j \notin \bar{D}_\rho$ for $j \neq i$. First, assume z_i is an interior point, and choose ρ small enough so that $D_\rho \cap \partial\Sigma = \emptyset$. Let $\eta(r)$ be a smooth function that is 1 for $r \geq 1$ and 0 for $r \leq \frac{1}{2}$, and as in [48] Theorem 5.1, define a modified map $\widehat{u}_\alpha$ by

$$\widehat{u}_\alpha(z) = \exp_{u(z)} \Big(\eta\big(|z|/\rho\big) \exp^{-1}_{u(z)} \left(u_\alpha(z)\right)\Big), \tag{3.2.8}$$

where exp is the exponential map on N. Then $\widehat{u}_\alpha$ agrees with u_α outside D_ρ and with u on $D_{\rho/2}$, and $\widehat{u}_\alpha \to u$ in $C^1(D_\rho, N)$. By assumption, $\pi_2(N) = 0$ and so u_α and $\widehat{u}_\alpha$ are homotopic.

If z_i is a boundary point, let $A \subset \partial\Sigma$ be the segments of the intersection of $\partial\Sigma$ with the annulus $\{\frac{\rho}{2} < |z| < \rho\}$. For the map defined in (3.2.8), $\widehat{u}_\alpha(A)$ may not lie in M, however we can modify the map so that it does satisfy the boundary condition. Since $u_\alpha \to u$ in C^1 on $\bar{D}_\rho - D_{\rho/2}$ and $u(\partial\Sigma) \subset M$, we may choose a neighborhood Ω_α of A in $\bar{D}_\rho - D_{\rho/2}$ that lies in a tubular neighborhood of $\partial\Sigma$ and so that the nearest point projection from $\partial\Omega_\alpha \cap \mathrm{int}(\Sigma)$ to $\partial\Sigma$ is one-to-one, such that $\widehat{u}_\alpha(\Omega_\alpha)$ lies in a tubular neighborhood of M in N, with $|\Omega_\alpha| \to 0$ as $\alpha \to 1$. On Ω_α, we redefine $\widehat{u}_\alpha$ to map each geodesic segment between a point z of $\partial\Omega_\alpha - A$ and its nearest point in $\partial\Sigma$ proportionally to the geodesic segment in N between $\widehat{u}_\alpha(z)$ and its nearest point in M. This modified $\widehat{u}_\alpha$ is piecewise smooth since u, u_α, the exponential map, and the nearest point projection maps are smooth. Moreover, since $u_\alpha \to u$ in C^1 on $\bar{D}_\rho - D_{\rho/2}$, $|\nabla \widehat{u}_\alpha|$ is bounded independent of α on Ω_α, and since $|\Omega_\alpha| \to 0$ we have $\lim_{\alpha\to1} E_\alpha(\widehat{u}_\alpha|_{\Omega_\alpha}) = 0$. Finally, since $\widehat{u}_\alpha = u_\alpha$ on the half circle $\{|z| = \rho\}$, the assumption $\pi_2(N, M) = 0$ implies that there exists a homotopy between the disk $u_\alpha(D_\rho) \cup \widehat{u}_\alpha(D_\rho)$ and a disk in M, relative to the boundary $u_\alpha(D_\rho \cap \partial\Sigma) \cup \widehat{u}_\alpha(D_\rho \cap \partial\Sigma)$. Hence the two disks bound a 3-dimensional disk in N and there exists a homotopy between u_α and $\widehat{u}_\alpha$ relative to the half circle $\{|z| = \rho\}$, mapping the boundary $D_\rho \cap \partial\Sigma$ into M.

Therefore in either case, whether z_i is an interior or boundary point of Σ, we have defined a map $\widehat{u}_\alpha$ homotopic to u_α such that

$$\lim_{\alpha\to1} \tilde{E}_\alpha(\widehat{u}_\alpha|_{D_\rho}) = E(u|_{D_\rho}),$$

where $\tilde{E}_\alpha(u) = \int \big((1+|\nabla u|^2)^\alpha - 1\big)\, d\mu$. Since u_α is minimizing for E_α in its homotopy class, $E_\alpha(u_\alpha|_{D_\rho}) \le E_\alpha(\widehat{u}_\alpha|_{D_\rho})$. Therefore,

$$\begin{aligned}\limsup_{\alpha\to 1} E(u_\alpha|_{D_\rho}) \le \limsup_{\alpha\to 1} \tilde{E}_\alpha(u_\alpha|_{D_\rho}) \le \lim_{\alpha\to 1} \tilde{E}_\alpha(\widehat{u}_\alpha|_{D_\rho}) = E(u|_{D_\rho}) \\ \le \pi\rho^2 \|u\|_{1,\infty}^2.\end{aligned}$$

Choose ρ sufficiently small so that $\pi\rho^2\|u\|_{1,\infty}^2 < \epsilon/2$, where ϵ is as in Lemma 3.2.6. Then for α sufficiently close to 1, we have $E(u_\alpha|_{D_\rho}) < \epsilon$ and by Lemma 3.2.6, $u_\alpha \to u$ in C^1 on D_ρ. Hence $u_\alpha \to u$ in $C^1(\Sigma, N)$ and u is in the same free homotopy class as f. □

II. Topological proof

Lemma 3.2.10. *Suppose N is a manifold, M is a submanifold of N, and Σ is a compact orientable surface. Let $f_0, f_1 : \Sigma \to N$ be two maps which induce the same action on the fundamental group $\pi_1(\Sigma) \to \pi_1(N)$.*

(*i*) *If $\pi_2(N) = 0$, then f_0 and f_1 are homotopic.*

(*ii*) *Suppose $\partial\Sigma \neq \emptyset$, and f_0, f_1 map $\partial\Sigma$ to M. If furthermore a homotopy $\{f_t\}_{t\in[0,1]}$ exists such that each boundary component C_s contains a point q_s, for which the curve $\{f_t(q_s)\}_{t\in[0,1]}$ is null-homotopic in $\pi_1(N, M)$, and we assume $\pi_2(N, M) = 0$, then a homotopy between f_0 and f_1 can be chosen such that $f_t(\partial\Sigma) \subset M, \forall t \in [0, 1]$.*

Proof. We note that the Lemma is trivial when Σ is the sphere or the disk. Thus we consider compact orientable surfaces which are not a sphere or a disk.

(i). We deal with $(g \ge 1, k = 0)$, $(g = 0, k \ge 2)$ and $(g \ge 1, k \ge 1)$ respectively. The fundamental polygon for a compact surface with $g \ge 1$ and $k = 0$ is a $4g$-sided domain, with the edges labelled as

$$\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_{2g-1}, \sigma_{2g}, \sigma_{2g-1}^{-1}, \sigma_{2g}^{-1}$$

where $\{\sigma_i\}_{1\le i\le 2g}$ is a set of generators for $\pi_1(\Sigma)$. Then the condition that f_0 and f_1 induce the same action (see the definition in section 3.2) implies that $f_0(\bigcup_i \sigma_i)$ is homotopic to $f_1(\bigcup_i \sigma_i)$, i.e. there exists a homotopy

$$\phi : \bigcup \sigma_i \times I \to N, \quad \phi(x, 0) = f_0|_{\cup\sigma_i}(x),\ \phi(x, 1) = f_1|_{\cup\sigma_i}(x).$$

Then we can extend this map to a polyhedron

$$\phi : \bigcup \sigma_i \cup (\Sigma, 0) \cup (\Sigma, 1) \to N$$

by letting $\phi(x, 0) = f_0(x)$, $\phi(x, 1) = f_1(x)$ for $x \in \Sigma$. Moreover, by the fact that a polyhedron is homeomorphic to S^2 and the assumption $\pi_2(N) = 0$, we can extend ϕ to the solid polyhedron

$$\tilde{\phi} : \Sigma \times I \to N, \quad \tilde{\phi}(x, 0) = f_0(x),\ \tilde{\phi}(x, 1) = f_1(x).$$

It follows that f_0 and f_1 are homotopic.

Surfaces of $g = 0$ and $k \geq 2$ can be realized as a k-sided polygon with the edges labelled as $\sigma_1, \cdots, \sigma_k$, where $\{\sigma_i\}_{1 \leq i \leq k}$ is a set of generators for $\pi_1(\Sigma)$ satisfying the relation $\prod_i \sigma_i = 1$ (so it is a free group with $k-1$ generators). By a similar argument, we can show f_0 and f_1 are homotopic.

Suppose $g \geq 1$ and $k \geq 1$ (Σ has non-empty boundary). Then Σ can be realized as a $4g$-sided polygon with k non-intersecting disks $\{D_j\}$ in the interior of this polygon removed. We denote by S_j the loops formed by attaching some segments from the boundary of these disks to the vertex of the polygon, i.e. the base point. Then the edges σ_i and the loops S_j form a set of generators for $\pi_1(\Sigma)$. Since f_0 and f_1 induce the same action, there exists a homotopy between f_0 and f_1 on $A = \bigcup_i \sigma_i \cup \bigcup_j S_j$

$$\phi : \Big(\bigcup_{i=1}^{g} \sigma_i \cup \bigcup_{j=1}^{k} S_j\Big) \times I \to N, \quad \phi(x, 0) = f_0|_A(x),\ \phi(x, 1) = f_1|_A(x).$$

Then by a similar argument, f_0 and f_1 are homotopic, as the curve A can be considered as the boundary of a disk-type surface.

(ii). We consider the boundary problem $(\Sigma, \partial\Sigma) \to (N, M)$ assuming $\partial\Sigma \neq \emptyset$. The hypothesis in the theorem asserts the existence of a homotopy

$$\varphi : \Sigma \times I \to N, \quad \varphi(x, 0) = f_0(x),\ \varphi(x, 1) = f_1(x). \tag{3.2.9}$$

By assumption, each boundary circle C_s contains a point $q_s \in C_s$ such that $\varphi(\{q_s\} \times I)$ is homotopic to a segment $\lambda_s \subset M$ connecting $f_0(q_s)$ and $f_1(q_s)$, by a homotopy relative to the end points

$$\varphi'_s : I \times I \to N, \quad \varphi'(t, 0) = \varphi(q_s, t),\ \varphi'(t, 1) = \lambda_s(t)$$

where the first factor I refers to the domain for the curve, and the second factor I refers to a parameter of the deformation. Next we shall modify the homotopy φ in (3.2.9) such that $\{q_s\} \times I$ is mapped to $\lambda_s \subset M$.

We denote

the half disk D^+	$\{(x,y,z) \in B : x = 0, z \geq 0\}$
the diameter S	$\{(x,y,z) \in B : x = 0, z = 0\}$
the half circle C	$\{(x,y,z) \in S^2 : x = 0, z \geq 0\}$
the half ball B^+	$\{(x,y,z) \in B : z \geq 0\}$
the base disk D_0	$\{(x,y,z) \in B : z = 0\}$

where B is the unit ball in $\mathbb{R}^3$. Then $S = D^+ \cap D_0$ and $C = D^+ \cap S^2$. Now we extend φ to $(\Sigma \times I) \cup B^+$ in steps.

a). Since φ'_s is relative to the end points, the domain for this homotopy can be considered as D^+ such that S represents the domain for the curve $\varphi(\{q_s\} \times I)$ and C represents the domain for the curve $\lambda_s \subset M$. Identifying $S \subset D_0$ with $\{q_s\} \times I \subset \partial\Sigma \times I$, we may consider D_0 as a subset of $\partial\Sigma \times I$. Since φ is defined on $\Sigma \times I$, we have that $\varphi|_{D_0}$ and $\varphi'_s|_{D^+}$ agree on $S = D_0 \cap D^+$ and this defines a map $\varphi|_{D_0} \cup \varphi'_s|_{D^+}$, or $\varphi|_{\Sigma\times I} \cup \varphi'_s|_{D^+}$.

b). We take B^+ as the union of two balls

$$B_1^+ = \{(x,y,z) \in B^+ : x \leq 0\}, \quad B_2^+ = \{(x,y,z) \in B^+ : x \geq 0\}$$

and let

$$S_1 = D^+ \cup \big(B_1^+ \cap D_0\big), \quad S_2 = D^+ \cup \big(B_2^+ \cap D_0\big)$$

be two half spheres of B_1^+ and B_2^+ respectively. Then $S_1 \cup S_2 = D_0 \cup D^+$. Moreover, there exist retraction maps

$$\Pi_1 : B_1^+ \to S_1, \quad \Pi_2 : B_2^+ \to S_2$$

such that the pull-back of the map $\varphi|_{D_0} \cup \varphi'_s|_{D^+}$ on S_i by Π_i defines a map

$$\varphi_s : B^+ \to N$$

which agrees with φ on $D_0 = (\Sigma \times I) \cap B^+$. Therefore the union of the two maps $\varphi|_{\Sigma\times I} \cup \varphi_s|_{B^+}$ defines a map.

c). We still denote this union map $\varphi|_{\Sigma\times I} \cup \varphi_s|_{B^+}$ by φ. Since there is a retraction from B^+ to D_0, the adding of B^+ doesn't affect the topological type of $\Sigma \times I$, and so the domain for φ can be considered as $\Sigma \times I$. More importantly, since D_0 is in $\partial\Sigma \times I$, φ_0 is again a homotopy between f_0 and f_1, as $\varphi(\cdot, t)$ is kept for $t = 0, 1$. But then $\{q_s\} \times I$ will be mapped to $\lambda_s \subset M$.

Now the loop

$$f_0(C_s) \cdot \lambda_s \cdot f_1(C_s) \cdot \lambda_s^{-1} \subset M$$

forms the boundary of a disk

$$\varphi : C_s \times I \to N.$$

Recall that the condition $\pi_2(N, M) = 0$ implies any disk D_1 with boundary in M is homotopic to a point in M by a homotopy from $D_1 \times I$ to N. Identifying $D_1 \times \{1\}$ to a point, we get a map from a 3-dimensional cone to N with base disk D_1 such that $\partial D \times \{t\}$ is mapped to M for any $t \in [0, 1]$. Thus the union of the boundary circles is a disk $D_2 \subset M$, which has the same boundary $\partial D \times \{0\}$ as D_1. Since the two disks bound a ball, there exists a homotopy between them relative to the boundary. Using this fact and that the disk $\varphi(C_s \times I)$ lies entirely in M, we get a relative homotopy

$$\tilde{\varphi}_s : \{C_s \times I\} \times I \to N$$

satisfying the initial condition

$$\tilde{\varphi}_s(p, 0) = \varphi|_{C_s \times I}(p), \qquad p \in C_s \times I \tag{3.2.10}$$

$$\tilde{\varphi}_s(p, 1) \in M, \qquad p \in C_s \times I \tag{3.2.11}$$

and the boundary condition

$$\tilde{\varphi}_s\big((x, 0), t\big) = f_0(x), \qquad x \in C_s,\ t \in I \tag{3.2.12}$$

$$\tilde{\varphi}_s\big((x, 1), t\big) = f_1(x), \qquad x \in C_s,\ t \in I \tag{3.2.13}$$

as the boundary loop is $(C_s \times \{0\}) \cup (C_s \times \{1\}) \cup (\{q_s\} \times I)$. Now by (3.2.10), we can consider the union of $\varphi|_{\Sigma \times I}$ and $\tilde{\varphi}_s|_{C_s \times I}$, $s = 1, \cdots, k$, as a homotopy defined on $\Sigma \times I$, since there exists a retraction from $\{C_s \times I\} \times I$ to $C_s \times I$. By (3.2.12) and (3.2.13) this is a homotopy $\{f_t\}_{0 \le t \le 1}$ between f_0 and f_1, and from (3.2.11) satisfying $f_t(\partial M) \subset M$ for all $t \in [0, 1]$. $\square$

Proof of Theorem 3.2.9. Let u_α be as in the proof in Subsection 3.2.5 such that u_α is a minimizing map for E_α in a given homotopy class f and $u_\alpha \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_q\}, N)$. Then u_α and u induce the same action by Theorem 3.2.7 and satisfy the hypotheses in Lemma 3.2.10 for $\alpha - 1$ sufficiently small, where the initial homotopy as in (ii) can be defined by a linear homotopy in the tangent bundle through the exponential map (where the condition $\pi_2(N) = 0$ is used). Then we can choose a point p in every boundary circle of $\partial\Sigma$ such that the geodesic connecting $u_\alpha(p)$ and $u(p)$ lies in a tubular neighborhood of M for $\alpha - 1$ sufficiently small. Then by Lemma 3.2.10 there exists a homotopy between u and $u_\alpha$in the space $C^0((\Sigma, \partial\Sigma); (N, M))$, and u is in the prescribed homotopy class f.

Remark 1). The requirement that the homotopy in (ii) take boundary value in M is because we minimize the energy in the space $C((\Sigma, \partial\Sigma); (N, M))$.

Remark 2). From the exact homotopy sequence:

$$\cdots \to \pi_2(M) \to \pi_2(N) \xrightarrow{\pi} \pi_2(N, M) \xrightarrow{\partial} \pi_1(M) \xrightarrow{i} \pi_1(N) \to \cdots$$

we can deduce that if $\pi_2(N, M) = 0$, then $i : \pi_1(M) \to \pi_1(N)$ is injective. In particular, there is no minimizing disk with free boundary in M.

3.2.6 Convergence of harmonic maps for varying conformal structures

We will need the following convergence result for harmonic maps with respect to varying conformal structures on Σ.

Theorem 3.2.11. *Let $u_i : (\Sigma, \partial\Sigma) \to (N, M)$ be a harmonic map satisfying the free boundary condition for a conformal structure c_i on Σ, where Σ is not of disk type. Suppose c_i converges to a conformal structure c in the C^∞-topology and $E(u_i, c_i) \leq B$. Then there exist a subsequence $\{u_i\}$ and a finite set of points $\{z_1, \cdots, z_l\}$ such that $u_i \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$, where $u : (\Sigma, \partial\Sigma) \to (N, M)$ is a smooth harmonic map satisfying the free boundary condition, and $E(u, c) \leq \liminf_{i\to\infty} E(u_i, c_i)$. Furthermore, if each u_i induces the same action on the fundamental group as f, then so does u.*

Proof. The convergence for varying conformal structures on a closed surface lying in a bounded set is shown in [49] Theorem 2.3, and the argument carries through for bordered surfaces. The argument that the induced map of u_i on the fundamental group is preserved in the limiting process is as in the proof of Theorem 3.2.7 above. □

As in Corollary 3.2.8, we also have a stronger result for a sequence of minimizing maps.

Corollary 3.2.12. *Let c_i be a sequence of conformal structures and u_i be a minimizing map for c_i in W_f. Suppose c_i converges to a conformal structure c. Then a subsequence $u_i \to u$ in $C^1(\Sigma, N)$ where u is a smooth harmonic map.*

Proof. Let $f_0 \in W_f$ be a smooth map. Since u_i is minimizing for c_i, $c_i \to c$ and $E(h, \cdot)$ is continuous, we get $E(u_i, c_i) \leq E(f_0, c_i) \to E(f_0, c)$, and so $E(u_i, c_i)$ is uniformly bounded. Since $c_i \to c$, a subsequence $\{u_i\}$ converges weakly in $L^2(\Sigma, \mathbb{R}^K)$ with respect to the conformal structure c. From Theorem 3.2.11, $u_i \to u$ in $C^1(\Sigma - \{z_1, \cdots, z_l\}, N)$. Choose small disks or half disks $D_i(\rho)$ of radius ρ around each z_i. Denote $A = \cup D_i(\rho)$ and $B = \Sigma - A$. We can define modified maps $\tilde{u}_i$ that agree with u_i on B and such that $\tilde{u}_i \to u$ in $C^1(\Sigma, N)$ (see the proof of Theorem 3.2.9).

We can choose a set of generators for $\pi_1(\Sigma)$ and any nontrivial element in $\pi_1(\Sigma, \partial\Sigma)$ away from these disks. Then $(\tilde{u}_i)_* \sim (u_i)_*$. Since u_i is minimizing in W_f, it follows that

$$E(u_i|_\Sigma, c_i) \leq E(\tilde{u}_i|_\Sigma, c_i).$$

Since $\tilde{u}_i = u_i$ on B, this implies

$$E(u_i|_A, c_i) \leq E(\tilde{u}_i|_A, c_i).$$

Let $\phi_i : (A, c_i) \to (A, c)$ be a conformal map with respect to the pull-back conformal structure $\phi_i^*(c) = c_i$ on each disk or half disk. Since $c_i \to c$, we have $\phi_i \to id$. Then $\tilde{u}_i \circ \phi_i^{-1} \to u$ in $C^1(A, N)$ and so

$$E(\tilde{u}_i \circ \phi_i^{-1}|_A, c) \to E(u|_A, c) \leq |A| \max |\nabla u|^2.$$

By conformal invariance, we have

$$E(u_i \circ \phi_i^{-1}|_A, c) = E(u_i|_A, \phi_i^*(c)) = E(u_i|_A, c_i)$$

and

$$E(\tilde{u}_i \circ \phi_i^{-1}|_A, c) = E(\tilde{u}_i|_A, \phi_i^* c) = E(\tilde{u}_i|_A, c_i).$$

Combining the above, we have for sufficiently large i

$$\begin{aligned} E(u_i \circ \phi_i^{-1}|_A, c) &= E(u_i|_A, c_i) \\ &\leq E(\tilde{u}_i|_A, c_i) \\ &= E(\tilde{u}_i \circ \phi_i^{-1}|_A, c) \\ &\leq (1+\sigma)|A| \max |\nabla u|^2 \\ &\leq (1+\sigma)\pi\rho^2 l \|u\|_{1,\infty}^2 \end{aligned}$$

where $\sigma > 0$ is a constant. Now we can choose ρ sufficiently small such that $E(u_\alpha|_A) < \epsilon$, where ϵ is as in Lemma 3.2.6. Then since $\phi_i \to id$, for sufficiently large i, we have $E(u_i|_A) < \epsilon$ and by Lemma 3.2.6, u_i converges to u in $C^1(A, N)$. □

3.3 A variational problem for conformal structures on compact Riemann surfaces

3.3.1 The notion of Riemann moduli space and Teichmüller space

A conformal structure on a smooth surface is a collection of local charts $\{f_\alpha : U_\alpha \to \mathbb{C}\}$ such that the transition maps are holomorphic. Let M_0 be an orientable surface. A conformal structure on M_0 can also be considered as an element (M, f), that is

$$M_0 \xrightarrow{f} M$$

where f is a diffeomorphism and M is a Riemann surface with a conformal structure c. In this sense the two factors (M, c) and f determine a conformal structure on M_0 (to make it a Riemann surface). But if we ignore the diffeomorphism f, we get an element (M, c). The space of all such elements is called the Riemann moduli space for M_0.

With the pull-back conformal structure, the diffeomorphism

$$(M_0, f^*c) \xrightarrow{f} (M, c)$$

is holomorphic. Suppose $M_0 \xrightarrow{f} M$ and $M_0 \xrightarrow{f'} M'$ represent the same conformal structure, that is we have

$$\begin{array}{ccc} (M_0, f^*c) & \xrightarrow{f} & (M, c) \\ {\scriptstyle id}\downarrow & & \\ (M_0, (f')^*c') & \xrightarrow{f'} & (M', c') \end{array}$$

Then $M \xrightarrow{f' \circ f^{-1}} M'$ is biholomorphic. Therefore (M, c) and (M', c') represent the same element if and only if there exists a biholomorphic map between them.

Suppose two conformal structures on M_0 are given by equivalent elements (M, c) and (M', c'). In the above sense, we don't distinguish (M, c) and (M', c') and will write (M, c). Then we will have a commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M \\ {\scriptstyle h}\uparrow & & \downarrow{\scriptstyle id} \\ M_0 & \xrightarrow{f \circ h} & M \end{array}$$

where h is a diffeomorphism of M_0. Writing $c' = f^*c$, then $h : (M_0, h^*c') \to (M_0, c')$ is holomorphic. Therefore we can also say two conformal structures c and c' on M_0 are equivalent if and only if there exists a diffeomorphism h of M_0 such that $h : (M_0, c') \to (M_0, c)$ is holomorphic. Now we define the Riemann moduli space of a Riemann surface.

Let Σ be an orientable surface. Let $\mathcal{D}(\Sigma)$ denote the topological group of diffeomorphisms of Σ onto itself with the C^∞-topology of uniform convergence on compact sets of all differentials. Let $\mathcal{M}(\Sigma)$ denote the space of conformal structures on Σ. There is a natural action

$$\mathcal{M}(\Sigma) \times \mathcal{D}(\Sigma) \to \mathcal{M}(\Sigma)$$

by pulling back conformal structures. The Riemann moduli space of Σ is defined as the quotient

$$\mathcal{R}(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}(\Sigma)$$

consisting of equivalent conformal structures with respect to this action. With the natural topology, the Riemann moduli space is not a manifold for any Riemann surface of Euler characteristic $\chi(\Sigma) \leq 0$.

The Teichmüller space is the universal covering space of any conceivable moduli space of a Riemann surface, and has the structure of a manifold. Precisely, let $\mathcal{D}_0(\Sigma)$ denote the subgroup of $\mathcal{D}(\Sigma)$ consisting of orientation-preserving diffeomorphisms which are homotopic to the identity. Then the Teichmüller space is defined as $\mathcal{T}(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma)$. The Teichmüller space and the Riemann moduli space are related by

$$\mathcal{R} = \mathcal{T}/\mathrm{Mod}$$

where $\mathrm{Mod} = \mathcal{D}/\mathcal{D}_0$ is the Teichmüller modular group.

An equivalent definition of the Teichmüller space can be formulated by reducing the homotopy condition to the conjugacy condition. A marking on a Riemann surface Σ is a generating set of simple closed curves for $\pi_1(\Sigma)$. We denote by Σ_1 the union of these generators. Two markings are equivalent if and only if the 1-skeletons Σ_1 for the markings are freely homotopic. We say two marked Riemann surface (Σ, c, Σ_1) and (Σ, c', Σ_1') are equivalent if and only if there exists a holomorphic diffeomorphism $f : (\Sigma, c) \to (\Sigma, c')$ such that $f(\Sigma_1)$ and Σ_1' are equivalent. Then one can show that the equivalence classes of marked surfaces form a space which is diffeomorphic to the Teichmüller space of Σ defined as above.

For bordered surfaces, there are two kinds of Teichmüller spaces. We denote the following subgroups of $\mathcal{D}(\Sigma)$:

$\mathcal{D}_0(\Sigma)$:	orientation-preserving diffeomorphisms that are homotopic to the identity and map each boundary curve onto itself;
$\mathcal{D}_0(\Sigma;\partial\Sigma)$:	orientation-preserving diffeomorphisms that are homotopic to the identity and keep the boundary fixed;
$\mathcal{D}_0(\Sigma;U)$:	orientation-preserving diffeomorphisms that are homotopic to the identity and keep a subset $U \subsetneq \partial\Sigma$ fixed.

The unreduced Teichmüller space is defined as

$$\mathcal{T}(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma;\partial\Sigma)$$

and the reduced Teichmüller space is defined as

$$\mathcal{T}^*(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma;U)\ (= \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma) \text{ for } U = \emptyset).$$

For example, when Σ is the annulus, $\mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma;x_0)$ is $(0,\infty)$ where x_0 is any point on $\partial\Sigma$, and $\mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma;\partial\Sigma)$ is infinite-dimensional. The reduced Teichmüller space and the Riemann moduli space are related by $\mathcal{R}(\Sigma) = \mathcal{T}^*(\Sigma)/\mathrm{Mod}(\Sigma)$ where $\mathrm{Mod}(\Sigma)$ is the reduced Teichmüller modular group.

Definition 3.3.1. *Specifically for a compact bordered Riemann surface Σ of $\chi(\Sigma) \le 0$, we define*

$$\begin{array}{lll} 1) & \mathcal{T}^*(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma), & \textit{if } \chi(\Sigma) < 0, \\ 2) & \mathcal{T}^*(\Sigma) = \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma;x_0), & \textit{if it is the annulus.} \end{array}$$

The study of Teichmüller spaces for bordered surfaces can be reduced to one without boundary. The construction is due to Schottky. Namely, a Riemann surface 2Σ without boundary can be formed as the union of a bordered Riemann surface Σ and an *exact duplicate* of it, with a conformal structure $2c$ induced by that on Σ. This conformal structure is *symmetric* in the sense that there is an antiholomorphic diffeomorphism S of 2Σ such that $S^2 = id$ and it leaves $\partial\Sigma$ fixed.

Let $\mathcal{M}^S$ be the space of conformal structures on 2Σ, for which S is antiholomorphic and $\mathcal{D}_0^S$ be the subgroup of $\mathcal{D}_0$ consisting of diffeomorphisms which commute with S. Then $\mathcal{T}^* \cong \mathcal{M}^S/\mathcal{D}_0^S$ (see e.g. [55]).

3.3.2 The structure of $\mathcal{M}$ for compact bordered surfaces

We have the following theorem concerning the structure of $\mathcal{M}(\Sigma)$ (cf. [16]).

Theorem 3.3.2 (Earle-Schatz)**.** *Suppose Σ is a compact bordered Riemann surface of Euler characteristic $\chi(\Sigma) \leq 0$. Then*

(i) *$\mathcal{M}(\Sigma)$ is a contractible Fréchet manifold,*

(ii) *$\mathcal{D}_0(\Sigma)$ or $\mathcal{D}_0(\Sigma; x_0)$ acts freely, continuously, and properly on $\mathcal{M}(\Sigma)$,*

(iii) *$\mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma) = \mathcal{T}^*(\Sigma)$ is a principal $\mathcal{D}_0(\Sigma)$ fibre-bundle. $\mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma)/\mathcal{D}_0(\Sigma; x_0)$ is a principal $\mathcal{D}_0(\Sigma; x_0)$ fibre-bundle.*

In both cases, it follows that $\mathcal{M}(\Sigma)$ is the universal $\mathcal{D}_0(\Sigma)$ (or $\mathcal{D}_0(\Sigma; x_0)$) bundle. On the other hand, it is known that $\mathcal{T}^*(\Sigma)$ is homeomorphic to $\mathbb{R}^{6g-6+3k}$ for $\chi(\Sigma) < 0$, and $\mathbb{R}$ for the annulus. Then from the theory of fibre bundles, $\mathcal{M}(\Sigma)$ is homeomorphic to $\mathcal{T}^*(\Sigma) \times \mathcal{D}_0(\Sigma)$ (or $\mathcal{T}^*(\Sigma) \times \mathcal{D}_0(\Sigma; x_0)$) and $\mathcal{D}_0(\Sigma)$ (or $\mathcal{D}_0(\Sigma; x_0)$) is contractible.

Corollary 3.3.3. *Suppose Σ is a compact bordered Riemann surface. Then $\mathcal{M}(\Sigma)$ has a trivial bundle structure $\mathcal{M}(\Sigma) \cong \mathcal{T}^*(\Sigma) \times \mathcal{D}_0(\Sigma)$ if $\chi(\Sigma) < 0$, and $\mathcal{M}(\Sigma) \cong \mathcal{T}^*(\Sigma) \times \mathcal{D}_0(\Sigma; x_0)$ if Σ is the annulus.*

Remark 3.3.4. Parallel theorems for closed Riemann surfaces of $g > 1$ were proved in Earle-Eells [15]. Later Fischer-Tromba ([20]) showed the homeomorphisms in these theorems were indeed diffeomorphisms. The differential geometric version of a Teichmüller theory for bordered Riemann surfaces of $\chi(\Sigma) < 0$ was proved in Tomi-Tromba [55] using the Schottky construction (see also section 4.3, [14]).

The structure theory of conformal structures implies the following result on the metrics (see [48] for the case of closed Riemann surfaces).

Corollary 3.3.5. *Suppose Σ is a compact bordered Riemann surface of $\chi(\Sigma) < 0$. Let $g(t)$ be a variation of a metric $g = g(0)$ on Σ. Then every such variation arises from a composition of*

(i) *the pull-back of metric by a C^∞ family of diffeomorphisms in $\mathcal{D}_0$,*

(ii) *a smooth curve in the Teichmüller space,*

(iii) *a family of conformal changes in the metric.*

Proof. Let $\mathcal{P}$ be the space of C^∞ positive functions and $\mathcal{G}$ be the space of smooth metrics on Σ. Any two metrics g and g' are conformal if $g' = \lambda g$ for some $\lambda \in \mathcal{P}$ and a representative metric in a conformal class $\{\lambda g\}$ can be chosen to have Gauss curvature as -1. These metrics form a space $\mathcal{M}_{-1} = \mathcal{G}/\mathcal{P}$. From [55], $\mathcal{M}_{-1}$ is diffeomorphic to $\mathcal{M}$. Therefore a variation of metrics decomposes as a family of conformal changes and a smooth curve in $\mathcal{M}$. But by the above results, the latter decomposes as a smooth curve in $\mathcal{T}^*$ and the pull-back by a C^∞ family of diffeomorphisms in $\mathcal{D}_0$. □

3.3.3 Variational problems on the space of conformal structures

In some variational problems, one often needs to produce critical points or minimizers for certain functionals defined on the space of conformal structures $\mathcal{M}$ (possibly with some function space). This can be done by proving the convergence of a minimizing sequence of conformal structures for the functional. A natural idea is to work with the total space $\mathcal{M}$. However, this space is infinite-dimensional and the compactification problem is rather difficult. Thus we often have to look at smaller spaces which allow for compactification, so that we can get a convergent minimizing sequence. For compact Riemann surfaces, there are two candidates for this role, the Riemann moduli space $\mathcal{R}$ and the (reduced) Teichmüller space $\mathcal{T}^*$, as both spaces allow for compactification. Then we have two different situations.

i) Subspace. The first approach is to define the functional on $\mathcal{R}$ or $\mathcal{T}^*$ by considering the space as a "subspace" of $\mathcal{M}$. Let π denote the projection

$$\pi : \mathcal{M} \to \mathcal{M}/\mathcal{D} = \mathcal{R}, \quad \pi : \mathcal{M} \to \mathcal{M}/\mathcal{D}_0 = \mathcal{T}^*.$$

Suppose we can "embed" $\mathcal{R}$ or $\mathcal{T}^*$ into $\mathcal{M}$ such that an equivalent class $[x]$ is mapped to a conformal structure c_x. A natural condition will be that $c_x \in [x]$. Although this can be realized by choosing an arbitrary c_x for each $[c]$, we require that the map be continous (and smooth), since we should also consider the topology (and geometry), such as the compactification (and differentiable features of the manifold and the functional). Hence the proposition that $\mathcal{R}$ or $\mathcal{T}^*$ can be embedded as a subspace of $\mathcal{M}$ is equivalent to the existence of a *continuous* (*smooth*) section

$$i : \mathcal{R} \to \mathcal{M}, \quad i : \mathcal{T}^* \to \mathcal{M}$$

such that $\pi \circ i = id$. Here we also have to differ the cases of $\mathcal{R}$ and $\mathcal{T}^*$.

From Theorem 3.3.2, we know that $\mathcal{D}_0$ acts freely, continuously and properly on $\mathcal{M}$, and hence $\mathcal{M} \to \mathcal{M}/\mathcal{D}_0$ defines a principal $\mathcal{D}_0$ fibre-bundle. In comparison, the action of $\mathcal{D}$ is only effective * and $\mathcal{M} \to \mathcal{M}/\mathcal{D}$ doesn't define a principal bundle.

*A group action $G \times X \to X$ is free if no element in G has a fixed point in X, i.e. $\forall (g, x) \in G \times X$, $g \cdot x \neq x$; and it is effective if $\forall g \in G$, $g \neq id$.

For the Teichmüller space, from the theory of fibre-bundles, we know that the existence of a continuous (smooth) section $i : \mathcal{T}^* \to \mathcal{M}$ such that $\pi \circ i = id$ is equivalent to the fact that the principal bundle $\mathcal{M}$ is homeomorphic (diffeomorphic) to the trivial bundle $\mathcal{T}^* \times \mathcal{D}_0$ (Corollary 8.3, [28]). On the other hand, this is true by Theorem 3.3.2 (and Remark 3.3.4). Thus $\mathcal{T}^*$ can be realized as a subspace (smooth submanifold) of $\mathcal{M}$ in a natural way so that it inherits the topology (geometry) of the total space and is homeomorphic (diffeomorphic) to $\mathbb{R}^{6g-6}$.

Therefore any functional on the space of conformal structures (and possibly some function space) can be defined on the Teichmüller space as a subspace of the former space.

ii) Quotient space. For any functional H on $\mathcal{M}$ (and possibly some function spaces X), to define it on $\mathcal{R}$ or $\mathcal{T}^*$ as a quotient space of $\mathcal{M}$, the functional should be invariant with respect to the action of $\mathcal{D}$ or $\mathcal{D}_0$. Namely, we should have $H(x, f^*c) = H(x, c)$ for any $x \in X$ and $f \in \mathcal{D}$ (or $\mathcal{D}_0$), which is a very strong condition. However, if we only care about obtaining a critical point or a minimizer for H on $X \times \mathcal{M}$, we can instead look at a modified functional

$$\bar{H}(c) = \inf_X H(\cdot, c), \quad c \in \mathcal{M}$$

with the invariance condition that $\bar{H}(f^*c) = \bar{H}(c)$ for any $f \in \mathcal{D}$ (or $\mathcal{D}_0$). If this is satisfied, then $\bar{H}$ is well-defined on $\mathcal{R}$ (or $\mathcal{T}^*$).

Then the strategy is to first obtain a convergent minimizing sequence for $\bar{H}$ in $\mathcal{R}$ (or $\mathcal{T}^*$) in the compactified space, and furthermore by additional assumptions prove that the limiting $[c]$ is not on the boundary of the compactified space and is in $\mathcal{R}$ (or $\mathcal{T}^*$). Then some representative conformal structure c for $[c]$ and a minimizer for $H(\cdot, c)$ will probably produce a minimizer for H on $X \times \mathcal{M}$.

For example, consider the energy functional $E : W^{1,2}(\Sigma, N) \times \mathcal{M}(\Sigma) \to \mathbb{R}$ for a compact orientable surface Σ

$$E(u, c) = \int |\nabla_g u|_g^2 \, d\mu_g$$

where g is a Riemannian metric* compatible with c and $d\mu$ is the area element induced by g.

*A conformal structure c on a surface induces an almost complex structure J, given by $d\phi^{-1} \circ i \circ d\phi$ where $\{\phi\}$ is a chart for c and i is the multiplication by i on the complex plane. Then in dim-2 J determines uniquely a class of symmetric positive $(2,0)$-tensors (metrics). By the conformal invariance of E, we can choose any such a metric.

The condition $\bar{H}(f^*c) = \bar{H}(c)$ is now

$$\inf_{u\in X} E(u,c) = \inf_{u\in X} E(u, f^*c). \tag{3.3.1}$$

Since $E(u, f^*c) = E(u \circ f^{-1}, c)$ by the conformal invariance of E, this is

$$\inf_{u\in X} E(u,c) = \inf_{u\in X} E(u\circ f^{-1}, c) = \inf_{u\in X_{f^{-1}}} E(u,c)$$

where $X_f = \{x \circ f : x \in X\}$ for a diffeomorphism $f \in \mathcal{D}$ (or $\mathcal{D}_0$). We give two examples of X.

a). $X = W^{1,2}(\Sigma, N)$, where N is a Riemannian manifold. Then $X_f = X$, $\forall f \in \mathcal{D}$ and the condition (3.3.1) is satisfied. Thus E is well-defined on $W^{1,2}(\Sigma, N) \times \mathcal{M}/\mathcal{D}_1$ for any subgroup $\mathcal{D}_1$ of $\mathcal{D}$. In particular, E can be defined on both $W^{1,2}(\Sigma, N) \times \mathcal{R}$ and $W^{1,2}(\Sigma, N) \times \mathcal{T}^*$.

b). X is the function space with a fixed conjugacy class of a given continuous map h

$$X = \{u \in W^{1,2}(\Sigma, N) \mid u(\partial\Sigma) \subseteq M,\ u_* \sim h_*\}$$

where $M \subset N$ is a compact submanifold. Then we have

$$\begin{aligned} X_f &= \{u\circ f \in W^{1,2}(\Sigma, N) \mid u\circ f(\partial\Sigma) \subseteq M,\ (u\circ f)_* \sim h_*\} \\ &= \{u \in W^{1,2}(f(\Sigma), N) \mid u(f(\partial\Sigma)) \subseteq M,\ u_* \sim h_*\circ f_*^{-1}\} \\ &= X \quad \forall f \in \mathcal{D}_0 \end{aligned}$$

(recall that $\mathcal{D}_0$ consists of diffeomorphisms of Σ that map each boundary curve onto itself and are homotopic to the identity). Thus the condition (3.3.1) is satisfied and E can be defined on $W \times \mathcal{T}^*$. In comparison, E cannot be defined on $W \times \mathcal{R}$ as the boundary and the homotopy conditions are not satisfied for all $f \in \mathcal{D}$.

Now the spaces $\mathcal{R}$ and $\mathcal{T}^*$ as quotient spaces of $\mathcal{M}$ (as the definitions for them), both allow for compactification. Thus whenever the condition (3.3.1) is satisfied for some functional H, we can define H on $\mathcal{R}$ and $\mathcal{T}^*$ in a natural way. Note that in the subspace approach, the discussion doesn't depend on the functional.

3.3.4 The minimal area problem, first approach: the functional $\bar{E}$ on the space $\mathcal{M}$

Recall the Euler characteristic of a surface of genus g with k boundary components is $\chi(\Sigma) = 2-2g-k$. On the disk D, there is only one conformal structure, and any smooth harmonic map $u : D \to N$ with $u(\partial D) \subset M$ and meeting M orthogonally along $u(\partial D)$ is conformal, and hence a branched minimal immersion. When Σ is not a disk, that is when $\chi(\Sigma) \leq 0$, in order to produce a branched minimal immersion, we must vary the conformal structure on Σ.

Let $\mathcal{M}(\Sigma)$ denote the space of conformal structures on Σ with the C^∞-topology as in the above. Given a conformal structure $c \in \mathcal{M}(\Sigma)$, let g be a Riemannian metric compatible with c and $d\mu$ be the area element induced by g. We then consider

$$E : W^{1,2}(\Sigma, N) \times \mathcal{M}(\Sigma) \to \mathbb{R}$$

where

$$E(u, c) = \int_\Sigma |\nabla_g u|_g^2 \, d\mu.$$

By virtue of the conformal invariance of the energy functional, this is independent of the choice of the metric g and is well-defined. We note that $E(\cdot, c)$ is lower semi-continuous in u, and $E(u, \cdot)$ is continuous in c.

For each conformal structure c we have produced, by Proposition 3.2.1 and Theorem 3.2.7, when Σ is not a disk, a smooth harmonic map u_c in the admissible space

$$W_f = \{u \in W^{1,2}(\Sigma, N) \mid u(\partial\Sigma) \subseteq M, \ u_* \sim f_*\}$$

such that $E(u_c, c) = \inf_{u \in W_f} E(u, c)$ (note that for $f \in W^{1,2}(\Sigma, N)$, f_* can be defined as in section 1, Schoen-Yau [50]). Now we define a functional on the space of conformal structures $\bar{E} : \mathcal{M}(\Sigma) \to \mathbb{R}$ by

$$\bar{E}(c) = \inf_{u \in W_f} E(u, c) = E(u_c, c). \tag{3.3.2}$$

Lemma 3.3.6. *$\bar{E}$ is continuous on $\mathcal{M}(\Sigma)$.*

Proof. Let $c_i \to c$ be a sequence of conformal structures. Suppose $\bar{E}(c_i) = E(u_i, c_i)$ and $\bar{E}(c) = E(u, c)$ for some $u_i, u \in W_f$. Let $K = \liminf E(u_i, c_i)$, and let $\{(u_{i_k}, c_{i_k})\}$ be a subsequence such that $E(u_{i_k}, c_{i_k}) \to K$. By Theorem 3.2.11 there exists a further subsequence, which we continue to denote by

$\{u_{i_k}\}$, and $u_0 \in W_f$ such that $E(u_0, c) \le \lim_{k\to\infty} E(u_{i_k}, c_{i_k})$. Then,

$$\begin{aligned} E(u,c) &= \lim_i E(u, c_i) \\ &\ge \limsup_i E(u_i, c_i) \qquad \text{(since } u_i \text{ is minimizing for } c_i\text{)} \\ &\ge \liminf_i E(u_i, c_i) \\ &= \lim_k E(u_{i_k}, c_{i_k}) \\ &\ge E(u_0, c) \\ &\ge E(u,c) \qquad \text{(since } u \text{ is minimizing for } c\text{)} \end{aligned}$$

It follows that $\bar{E}(c) = E(u,c) = \lim_i E(u_i, c_i) = \lim_i \bar{E}(c_i)$. □

Suppose $\inf_{\mathcal{M}(\Sigma)} \bar{E}$ is attained at $c \in \mathcal{M}(\Sigma)$. Let u be a minimizing harmonic map for c. Then for any pair $(u', c') \in W_f \times \mathcal{M}(\Sigma)$, we have

$$E(u,c) = \inf_{W_f}(\cdot, c) = \bar{E}(c) \le \bar{E}(c') = \inf_{W_f}(\cdot, c') \le E(u', c').$$

The relationship between such a minimizing pair and the minimal immersion problem is illustrated in the following (cf. Theorem 1.8 [48] and Corollary 1.5 [49]).

Theorem 3.3.7. *If (u,c) is a critical point of E on $W_f \times \mathcal{M}(\Sigma)$, then u is a branched minimal immersion.*

Proof. Any variation of the metric arises from a composition of a conformal change in the metric, and a curve in $\mathcal{M}(\Sigma)$. Hence by the conformal invariance of E, the fact that c is a critical point of $E(u, \cdot)$ on $\mathcal{M}(\Sigma)$ implies that E is critical with respect to any variation of the initial metric induced by c. The computation of Sacks and Uhlenbeck ([48], p.6) shows that u is weakly conformal in the interior of Σ. Then by Gulliver-Osserman-Royden [24], u is a branched minimal immersion. □

Corollary 3.3.8. *If (u,c) is a minimizer of E on $W_f \times \mathcal{M}(\Sigma)$ with respect to all smooth variations of c preserving the action on the fundamental group, then u minimizes area among all branched immersions having the same action.*

Let $\{c_i\}$ be an $\bar{E}$-minimizing sequence, i.e. $\bar{E}(c_i) \to \inf \bar{E}$. If a subsequence converges to a conformal structure c, then $\bar{E}(c) = \inf \bar{E}$ by Lemma 3.3.6. In fact it suffices to have a weaker condition that $\{c_i\}$ converges in the level of the moduli space.

Lemma 3.3.9. *Let $\{c_i\}$ be an $\bar{E}$-minimizing sequence. If there exist diffeomorphisms $\phi_i \in \mathcal{D}(\Sigma)$ and $c \in \mathcal{M}(\Sigma)$, such that $\phi_i^* c_i \to c$ in the C^∞-topology, then there exists a minimizing conformal structure for $\bar{E}$.*

Proof. Let u_i be a minimizing harmonic map for $\phi_i^* c_i$ which induces the same action as $f \circ \phi_i$. Then $u_i \circ \phi_i^{-1}$ induces the same action as f and we have

$$\bar{E}(c_i) \leq E(u_i \circ \phi_i^{-1}, c_i) = E(u_i, \phi_i^* c_i) \leq E(u_{c_i} \circ \phi_i, \phi_i^* c_i) = E(u_{c_i}, c_i) = \bar{E}(c_i)$$

where u_{c_i} denotes a minimizing harmonic map for c_i which induces the same action as f. Therefore,

$$E(u_i, \phi_i^* c_i) = \bar{E}(c_i). \tag{3.3.3}$$

By Theorem 3.2.11 and Corollary 3.2.12, there exists a subsequence u_{i_k} such that $u_{i_k} \to u$ in $C^1(\Sigma)$, and since $\phi_i^* c_i \to c$, we have

$$E(u, c) \leq \liminf E(u_{i_k}, \phi_{i_k}^* c_{i_k}). \tag{3.3.4}$$

For sufficiently large k, $u \circ \phi_{i_k}^{-1}$ induces the same action as $u_{i_k} \circ \phi_{i_k}^{-1}$ on the fundamental group by the proof of Theorem 3.2.11, hence is in the admissible space W_f, since $u_i \circ \phi_i^{-1}$ induces the same action as f. Then we have

$$\begin{aligned}
\bar{E}((\phi_{i_k}^{-1})^* c) &\leq E(u \circ \phi_{i_k}^{-1}, (\phi_{i_k}^{-1})^* c) \\
&= E(u, c) \\
&\leq \liminf E(u_{i_k}, \phi_{i_k}^* c_{i_k}) \\
&= \liminf \bar{E}(c_{i_k}) \\
&= \inf_{\mathcal{M}(\Sigma)} \bar{E}
\end{aligned}$$

where the first equality is by conformal invariance of the energy, the second inequality is by (3.3.4), the second equality is by (3.3.3), and the third equality follows since $\{c_{i_k}\}$ is a minimizing sequence of $\bar{E}$. Therefore

$$\bar{E}((\phi_{i_k}^{-1})^* c) = \inf_{\mathcal{M}(\Sigma)} \bar{E}.$$

This proves the existence of a minimizing conformal structure. In fact, for all large k, $(\phi_{i_k}^{-1})^* c$ are $\bar{E}$-minimizers. $\square$

Therefore by Theorem 3.3.7, Corollary 3.3.8 and Lemma 3.3.9, the minimal area problem is reduced to the convergence problem in the moduli space $\mathcal{R}(\Sigma)$.

3.3.5 The minimal area problem, second approach: the functional $\bar{E}$ on the space $\mathcal{T}^*$

In the first approach to the minimal area problem, we utilize the functional $\bar{E}$ defined on the total space of conformal structures $\mathcal{M}$, and only consider the moduli space later for the convergence problem. However, there is another approach of defining the functional initially on a compactifiable space.

From the discussion in subsection 3.3.3, we know that the admissible space W_f is invariant with respect to the action of the group $\mathcal{D}_0$, and using the conformal invariance of E, we can define $\bar{E}$ on $\mathcal{T}^*$ by

$$\bar{E}([c]) = \inf_{W_f} E(\cdot, c), \quad \forall [c] \in \mathcal{T}^*$$

where c is any representative conformal structure for $[c]$. Now let $[c_i]$ be a $\bar{E}$-minimizing sequence, that is $\bar{E}([c_i]) \to \inf_{\mathcal{T}^*} \bar{E}$ as $i \to \infty$. In the compactification space of $\mathcal{T}^*$, there exists some point $[c_0]$ such that a subsequence $[c_i] \to [c_0]$. If we can prove $[c_0]$ is not on the boundary of the compactification space, it is in $\mathcal{T}^*$. Let c_0 be any representative conformal structure for $[c_0]$. We have $\bar{E}(c_0) = \inf_{\mathcal{T}^*} \bar{E}$. Then we can choose a minimizing harmonic map u with respect to c_0, and we can show that (u, c_0) is a minimizing pair for E on the space $W_f \times \mathcal{M}$.

Moreover, from the above discussion we know that the minimizing conformal structures for $\bar{E}$ on $\mathcal{M}$ (see the definition (3.3.2)) are exactly the fibres (action by $\mathcal{D}_0$) over the minimizers for $\bar{E}$ on $\mathcal{T}^*$.

This allows one to define the functional $\bar{E}$ on $\mathcal{T}^*$ as a quotient space by $\mathcal{D}_0$. But we can also adopt a different setup. We know E is well-defined on $W_f \times \mathcal{T}^*$ by using the section $i : \mathcal{T}^* \to \mathcal{M}$ which exists by Theorem 3.3.2 and Remark 3.3.4 (see the discussion in subsection 3.3.3). Then we can define

$$\bar{E}(c) = \inf_{W_f} E(\cdot, c), \quad \forall c \in \mathcal{T}^* \hookrightarrow \mathcal{M}$$

by considering $\mathcal{T}^*$ as a subspace of $\mathcal{M}$. In the compactification space of $\mathcal{T}^*$, using a similar argument as above, we obtain a minimizing sequence $c_i \to c_0$ in $\mathcal{T}^*$. But this is $i(c_i) = c_i \to c_0 = i(c_0)$ in $\mathcal{M}$. Therefore by Lemma 3.3.6, $\bar{E}(c_0) = \lim \bar{E}(c_i)$ and c_0 is a minimizing conformal structure for $\bar{E}$ on $\mathcal{T}^*$. Now we use the conformal invariance of E to get that c_0 is a minimizing conformal structure for $\bar{E}$ on $\mathcal{M}$. This argument relies crucially on the structure theory of $\mathcal{M}$.

We now prove part (i) of Theorem 1.0.2.

3.4 Minimal surfaces of non-disk type

Throughout we let $\{c_i\}$ be an $\bar{E}$-minimizing sequence of conformal structures, and u_i will denote a minimizing map for c_i, with $E(u_i, c_i) < B$. Now we prove the existence of a minimizing conformal structure for $\bar{E}$ for non-disk surfaces. The first proof is based on the theory of moduli space for *closed* Riemann surfaces via the Schottky construction.

3.4.1 First proof

I. Σ is not a cylinder.

Assume that Σ is a surface with $\chi(\Sigma) < 0$. For each conformal structure c_i in the minimizing sequence, consider the doubled conformal surface. Applying the compactification theorem of the moduli space of conformal structures for the closed doubled conformal surfaces (Lemma 4 of Abikoff [1]), there is a subsequence of $\{c_i\}$ (which we continue to denote by $\{c_i\}$) and there are diffeomorphisms ϕ_i of Σ such that either $\phi_i^* c_i \to c$ in C^∞ or, $(\Sigma, \phi^* c_i)$ converges to a Riemann surface with nodes Σ_∞ corresponding to pinching a set of homotopically nontrivial simple closed curves in the doubled surface to nodes w_m, $m = 1, \ldots n$. In the first case, by Lemma 3.3.9 we are done. In the second case, we have curves γ_m in Σ which are pinched, each of which is either a closed curve (possibly a boundary component) or a curve between two boundary components of Σ (corresponding to a closed curve in the doubled surface that crosses $\partial\Sigma$, and must be reflection invariant across $\partial\Sigma$). We may then argue as in [49] Theorem 4.3. We may choose a nested sequence $\{D_j^m\}$ of closed neighborhoods of γ_m such that D_j^m converges to the node w_m of Σ_∞, and for each fixed j, the change in the conformal structure on Σ as $(\Sigma, \phi_i^* c_i) \to \Sigma_\infty$ is restricted to the interior of $\cup_{m=1}^n D_j^m$ ([5]). Let $\Sigma_j = \Sigma - \cup_{m=1}^n D_j^m$. By Theorem 3.2.11 and Corollary 3.2.12, there is a subsequence $\{u_i^{(1)}\}$ of $\{u_i\}$ that convergences in $C^1(\Sigma_1, N)$ to a smooth harmonic map defined on Σ_1. Given $\{u_i^{(j-1)}\}$, by Theorem 3.2.11 and Corollary 3.2.12, a subsequence $\{u_i^{(j)}\}$ of $\{u_i^{(j-1)}\}$ converges in $C^1(\Sigma_j, N)$ to a smooth harmonic map. Consider the diagonal sequence $\{u_i^{(i)}\}$ which converges to a harmonic map u in $C^1(\Sigma'_\infty, N)$, where Σ'_∞ is the punctured Riemann surface $\Sigma - \{w_1, \ldots, w_n\}$. Since $E(u_i, c_i) < B$ for all i, $E(u) < B$, and by Theorem 1.6 of [48] and Theorem 1.10 of [22], u can be extended to a smooth harmonic map $u : \tilde{\Sigma}_\infty \to N$ satisfying the free boundary condition, where $\tilde{\Sigma}_\infty = \Sigma'_\infty \cup \{q_1, \ldots, q_s, (q_{s+1}, q'_{s+1}), \ldots, (q_n, q'_n)\}$ is the bordered Riemann surface obtained by adding a point q_m at the punctures of Σ'_∞ corresponding to the nodes $w_m \in \Sigma_\infty$ resulting from the pinching of components of $\partial\Sigma$, and adding a pair of points (q_m, q'_m) at the two punctures of Σ'_∞ (which may be boundary points) corresponding to each node $w_m \in \Sigma_\infty$ resulting from the pinching of a closed curve inside Σ or a closed curve in the doubled surface that crosses $\partial\Sigma$. Now let γ be a curve homotopic to γ_m, for any fixed m between 1 and n, chosen to lie in D_j^m for j sufficiently large. Since $\gamma \subset \tilde{\Sigma}_\infty$ is homotopically trivial (either as a closed curve or a relative curve between boundary components) it follows that $u(\gamma)$ is homotopically trivial. But $\lim_{i\to\infty} u_i^{(i)}(\gamma) = u(\gamma)$, so $u_i^{(i)}(\gamma)$ is homotopically trivial for i sufficiently large. Since γ is homotopically nontrivial in Σ, this contradicts our assumption that the induced map on the fundamental groups is injective. Therefore the second case cannot occur.

II. Σ is a cylinder.

A cylinder with a conformal structure can be represented by a parallelogram spanned by the vectors $(1,0)$ and ξ in $\mathbb{R}^2$ with sides corresponding to one of the two generators identified. Two cylinders given by ξ_1, ξ_2, with the same corresponding sides identified, represent conformally equivalent cylinders if $\xi_2 = \tau\xi_1$ for some $\tau \in \mathrm{PSL}(2,\mathbb{Z})$.

Given our minimizing sequence of conformal structures ξ_i, and associated minimizing harmonic maps u_i, there exist elements $\tau_i \in \mathrm{PSL}(2,\mathbb{Z})$ such that $\tau_i\xi_i$ lies in the fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$. If $\mathrm{Im}(\tau_i\xi_i) \le b < \infty$ for all i, then a subsequence of $\{\tau_i\xi_i\}$ converges to η, and by Lemma 3.3.9 we are done.

Otherwise, suppose that $\kappa_i = \mathrm{Im}(\tau_i\xi_i) \to \infty$. Let $\eta_i = \tau_i\xi_i$. Then $v_i = u_i \circ \tau_i^{-1} : (\Sigma, \eta_i) \to N$ is harmonic and $E(v_i, \eta_i) = E(u_i, \xi_i) \le B$. We consider the following two cases:

a) If the sides corresponding to η_i are identified, then on any cylinder $S^1 \times [0,\kappa)$ we can find a subsequence of $\{v_i\}$ which converges in $C^1(S^1 \times [0,\kappa), N)$ to a harmonic map $v : S^1 \times [0,\kappa) \to N$ with $E(v) < B$. Since κ was arbitrary, using a diagonal sequence argument as above, we obtain a harmonic map $v : S^1 \times [0,\infty) \to N$ with $E(v) < B$. But $S^1 \times [0,\infty)$ is conformally $\bar{D} - \{p\}$ for some $p \in D$, and by Theorem 1.6 in [48], v extends to a smooth harmonic map $v : D \to N$ providing a homotopy of $v_i(S^1 \times \{q\}) \simeq v(S^1 \times \{q\})$ to a point for suitable q and i sufficiently large. This implies that the generator $\tau_i^{-1}(S^1 \times \{q\})$ of (Σ, ξ_i) is mapped by u_i, and hence also by f, to a loop homotopic to zero, contradicting the assumption that $f_* : \pi_1(\Sigma) \to \pi_1(N)$ is injective.

b) If the sides corresponding to $(0,1)$ are identified, then on any strip

$[0,1] \times (-\kappa, \kappa)$ we can find a subsequence $\{v_i\}$ which converges in $C^1([0,1] \times (-\kappa, \kappa), N)$ to a harmonic map $v : [0,1] \times (-\kappa, \kappa) \to N)$ with $E(v) < B$. Since κ was arbitrary, we can obtain a harmonic map $v : [0,1] \times \mathbb{R} \to N$ with $E(v) < B$. But $[0,1] \times \mathbb{R}$ is conformally $\bar{D} - \{p_1, p_2\}$ for some $p_1, p_2 \in \partial D$, and by Theorem 1.10 in [22], v extends to a smooth harmonic map $v : \bar{D} \to N$ providing a homotopy of $v_i([0,1] \times \{q\}) \simeq v([0,1] \times \{q\})$ to a point for suitable q and i sufficiently large. This contradicts the assumption that $f_* : \pi_1(\Sigma, \partial\Sigma) \to \pi_1(N, M)$ is injective.

3.4.2 Second proof

In the second proof, we look at the moduli space for a bordered surface that is not a cylinder, in the setup of hyperbolic geometry, and the argument is more geometric.

I. Σ is not a cylinder.

Recall that for each conformal structure c_i on Σ, we can form the doubled surface $(2\Sigma, 2c_i)$ (cf. Abikoff [2]). We will use this construction later.

By the existence of a hyperbolic metric on any compact Riemann surface with $\chi < 0$, we can assume a hyperbolic metric g_i on Σ, induced from c_i. Then the following facts hold:

i) Every closed geodesic or geodesic connecting two boundary curves, with respect to this hyperbolic metric, has a collar (a component of a tubular neighborhood minus the geodesic) of area $\geq k_1 > 0$;

ii) Any curve that is homotopically nontrivial in $\pi_1(N)$ or $\pi_1(N, M)$ has length $\geq k_2 > 0$;

iii) $E(u_i, c_i) \leq B$.

Specifically the first fact is from Keen's theorem (page 98 [2], also [33]) and by considering a geodesic connecting two boundary curves as exactly half of a closed geodesic in the doubled surface, the second is implied by the homogeneously regular condition on N, and the third is since $E(u_i, c_i) \to \inf \tilde{E}$. These facts mark the ingredients of a computation by Schoen and Yau (Lemma 3.1, [50]) on the length of closed geodesics in Σ, which also applies to geodesics connecting two boundary curves. To be precise, since

$$f_* : \pi_1(\Sigma) \times \pi_1(\Sigma, \partial\Sigma) \to \pi_1(N) \times \pi_1(N, M)$$

is injective, any curve in a collar as in *i)* that is homotopically nontrivial in $\pi_1(\Sigma)$ or $\pi_1(\Sigma, \partial\Sigma)$, is mapped to a curve in N that is homotopically nontrivial in $\pi_1(N)$ or $\pi_1(N, M)$ respectively. Hence by *ii*), the length of this curve in N is bounded from below. Then it follows from *i*) and *iii*) (cf. Lemma 3.1, [50]) that there is a positive lower bound on the length of any closed geodesic in Σ or geodesic connecting two boundary curves of Σ. In particular, this holds for all boundary curves of Σ, which are geodesics with respect to the hyperbolic metric.

Note that any closed geodesic in the doubled surface $(2\Sigma, 2c_i)$ of (Σ, c_i) arises from a closed geodesic in (Σ, c_i) or the doubling of a geodesic connecting two boundary curves of (Σ, c_i). It follows that there is a positive lower bound on the length of any closed geodesic in the doubled surface. Then by Mumford's compactness theorem ([46], and page 271, [14] for a differential geometric description and a boundary discussion), there exist a subsequence $\{g_i\}$ and diffeomorphisms ϕ_i of Σ onto itself, such that $\phi_i^* g_i$ converges in C^∞ to a hyperbolic metric g on Σ. Let c be the conformal structure associated with g. Since the space of hyperbolic metrics and the space of conformal structures are diffeomorphic for compact Riemann surfaces with $\chi(\Sigma) < 0$, the sequence $\{\phi_i^* c_i\}$ converges to c with the C^∞-topology.

II. Σ is a cylinder.

It is known that the moduli space of a cylinder is $(0, 1)$ and the conformal structures can be realized as $\{A_r\}_{0<r<1}$, where $A_r = \{z | \ r \le |z| \le 1\}$ is an annulus. Suppose c_i is a conformal structure A_{r_i} (in the moduli space). Then $\{r_i\}$ has a limit in $[0, 1]$, which is either 0, or 1, or $r \in (0, 1)$.

a). If a subsequence $r_i \to r \in (0, 1)$, then $c_i \to A_r$ in the moduli space. Hence by Lemma 3.3.9, there exists a minimizing conformal structure.

b). If a subsequence $r_i \to 0$, then by Theorem 3.2.7 and Corollary 3.2.8, there exists a subsequence $\{u_{1,i}\}$ that converges to a harmonic map in A_{r_1}. Given $\{u_{k-1,i}\}$, a subsequence $\{u_{k,i}\}$ converges in A_{r_2}. Then a diagonal subsequence $\{u_{i,i}\}$ converges to a smooth harmonic map u in $C^1(D - \{0\})$. But by the removable singularity theorem of Sacks and Uhlenbeck ([48]), since $E(u) \le B$, u extends to D. It follows that $C = \{|z| = 1\}$ will be mapped to a homotopically trivial curve in $\pi_1(N)$, and for sufficiently large i, $u_i(C)$ is homotopically trivial in $\pi_1(N)$.

On the other hand, the induced map of u_i is injective on $\pi_1(\Sigma) \times \pi_1(\Sigma, \partial\Sigma, *)$, which implies that C can't be mapped to a curve which is homotopically trivial in $\pi_1(N)$. Thus we get a contradiction.

c). If a subsequence $r_i \to 1$, then we get a sequence of maps $u_i : A_{r_i} \to N$ with $E(u_i, A_{r_i}) \le B$. Denote by C_θ the segment $\arg z = \theta$ in A_{r_i}, and l_θ the length of $u_i(C_\theta)$ in N. Suppose $N \hookrightarrow \mathbb{R}^k$ is an isometric embedding. Then,

$$\begin{aligned}
\int_0^{2\pi} (l_\theta)^2 d\theta &= \int_0^{2\pi} \Big(\int_{C_\theta} \Big|\frac{\partial u_i}{\partial r}\Big| dr \Big)^2 d\theta \\
&\leq \int_0^{2\pi} \Big((1-r_i) \int_{C_\theta} \Big|\frac{\partial u_i}{\partial r}\Big|^2 dr \Big) d\theta \\
&= (1-r_i) \int_0^{2\pi} \Big(\int_{C_\theta} \Big|\frac{\partial u_i}{\partial r}\Big|^2 dr \Big) d\theta \\
&\leq \frac{1-r_i}{r_i} \int_0^{2\pi} \int_{C_\theta} \Big(\Big|\frac{\partial u_i}{\partial \theta}\Big|^2 \frac{1}{r^2} + \Big|\frac{\partial u_i}{\partial r}\Big|^2 \Big) r dr d\theta \\
&= \frac{1-r_i}{r_i} \cdot E(u_i).
\end{aligned}$$

By the assumption $E(u_i) \leq B$, the measure of the set of θ for which $l_\theta > \lambda > 0$ goes to zero as $r_i \to 1$, for any $\lambda > 0$. Hence for sufficiently large i, there exists θ such that $u_i(C_\theta)$ lies entirely in a geodesic ball centered at one end point, and thus can be deformed to a path in M relative to the two end points. Therefore $u_i(C_\theta)$ is homotopically trivial in $\pi_1(N, M)$.

Recall that $\pi_1(\Sigma, \partial\Sigma, *)$ is the set of free homotopy classes of paths in Σ, from the base point $* \in \partial\Sigma$ to a varying point in $\partial\Sigma$. When Σ is the cylinder, it consists of two elements, i.e. the trivial map $*$ and any segment connecting the two boundary curves, such as C_θ. Then the injectivity on $\pi_1(\Sigma, \partial\Sigma, *)$ implies that C_θ can't be mapped to a path which is homotopically trivial in $\pi_1(N, M, *)$. Thus we get a contradiction.

Hence $\{c_i\}$ has a limit in the moduli space.

We have proved part (i) of Theorem 1.0.2.

3.5 Minimizing disks

Now we prove part (ii) of Theorem 1.1, that there exists a set of free homotopy classes $\{\Gamma_j\}$ of closed curves in M such that the elements $\{\gamma \in \Gamma_j\}$ form a generating set for $\ker i_*$, where $i_* : \pi_1(M) \to \pi_1(N)$ is the homomorphism induced by the inclusion, and each Γ_j can be represented by the boundary of an area minimizing disk that solves the free boundary problem $(D, \partial D) \to (N, M)$. We will need the following lemma.

3.5.1 An integral identity for critical maps of E_α

Lemma 3.5.1. *Let $u : (D, \partial D) \to (N, M)$ be a critical map of E_α on $W^{1,2\alpha}(D, \partial D; N, M)$. Then u satisfies*

$$\int_D \Big(-\big(1+|\nabla u|^2\big)^\alpha + \alpha\big(1+|\nabla u|^2\big)^{\alpha-1}|\nabla u|^2\Big) z\, dxdy = 0 \tag{3.5.1}$$

where $z = x + iy$ is the complex coordinate on the disk D.

Proof. Writing $u = u(z, \bar z)$, we have $|\nabla u|^2 = |u_x|^2 + |u_y|^2 = 4u_z \cdot u_{\bar z}$ and

$$E_\alpha(u) = \int_D \big(1 + 4u_z \cdot u_{\bar z}\big)^\alpha \, \frac{i}{2}\, dz d\bar z.$$

Given a complex number β, let

$$\varphi_\beta(z) = \frac{z-\beta}{1-\bar\beta z}.$$

Let $\beta(t)$ be a differentiable curve in $\mathbb{C}$ with

$$|\beta(t)| < 1, \;\; \beta(0) = 0.$$

Then $\varphi_t = \varphi_{\beta(t)}$ is a family of automorphisms of the unit disk, which map the boundary to the boundary. Now we define a variation of u by

$$D \xrightarrow{\varphi_t} D \xrightarrow{u} N$$

$$z \mapsto w = \varphi_t(z) \mapsto u(w, \bar w)$$

where

$$z = \varphi_t^{-1}(w) = \frac{w+\beta}{1+\bar\beta w}.$$

Then we have

$$\frac{\partial w}{\partial z} = \frac{1-|\beta|^2}{(1-\bar\beta z)^2}, \quad \frac{\partial z}{\partial w} = \frac{1-|\beta|^2}{(1+\bar\beta w)^2},$$

and using $\beta(0) = 0$, we compute

$$\frac{\partial}{\partial t}\Big(\frac{\partial w}{\partial z}\Big)\Big|_{t=0} = 2\bar\beta'(0) z, \quad \frac{\partial}{\partial t}\Big(\frac{\partial z}{\partial w}\Big)\Big|_{t=0} = -2\bar\beta'(0) w. \tag{3.5.2}$$

We have:

$$E_\alpha(u\circ\varphi_t) = \int_D \left(1 + \frac{\partial w}{\partial z}\frac{\partial \bar w}{\partial \bar z}|\nabla u|^2\right)^\alpha \frac{\partial z}{\partial w}\frac{\partial \bar z}{\partial \bar w}\, \frac{i}{2}\, dw d\bar w.$$

We compute

$$\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} = \frac{(1+\bar{\beta}w)^2(1+\beta\bar{w})^2}{(1-|\beta|^2)^2}, \qquad \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\right)\Big|_{t=0} = 2\bar{\beta}'(0)w + 2\beta'(0)\bar{w}.$$

Using this and (3.5.2), we have

$$\begin{aligned}
& \frac{d}{dt}E_\alpha(u\circ\varphi_t)\Big|_{t=0} \\
= & \int_D \big(1+|\nabla u|^2\big)^\alpha\big(-2\bar{\beta}'(0)w - 2\beta'(0)\bar{w}\big)\,\frac{i}{2}\,dwd\bar{w} \\
& + \int_D \alpha|\nabla u|^2\big(1+|\nabla u|^2\big)^{\alpha-1}\big(2\bar{\beta}'(0)w + 2\beta'(0)\bar{w}\big)\,\frac{i}{2}\,dwd\bar{w} \\
= & \beta'(0)\int_D 2\Big(-\big(1+|\nabla u|^2\big)^\alpha + \alpha\big(1+|\nabla u|^2\big)^{\alpha-1}|\nabla u|^2\Big)\bar{z}\,dxdy \\
& + \bar{\beta}'(0)\int_D 2\Big(-\big(1+|\nabla u|^2\big)^\alpha + \alpha\big(1+|\nabla u|^2\big)^{\alpha-1}|\nabla u|^2\Big)z\,dxdy.
\end{aligned}$$

Since $\beta'(0)$ is arbitrary, we get (3.5.1). □

3.5.2 Nontrivial limit of critical maps of E_α

Now we use the integral identity in Lemma 3.5.1 to show that the limiting map of a sequence of nontrivial critical maps for E_α, in C^1 on the disk minus a boundary point, is not a constant map. This result will be crucial in proving that any free homotopy class in the kernel of $i:\pi_1(M)\to\pi_1(N)$ can be decomposed as free homotopy classes represented by minimizing disks.

Corollary 3.5.2. *Let u_α be a sequence of critical maps of E_α for a sequence $\alpha\to 1$. If $u_\alpha\to u$ in $C^1(\bar{D}-\{p\},N)$ where $p\in\partial D$, and each u_α is nontrivial, then u is not a constant map.*

Proof. From (3.5.1), taking the imaginary part, and using the fact that $\int_D y\,dxdy = 0$, we have

$$\int_D \Big(-\big(1+|\nabla u_\alpha|^2\big)^\alpha + 1 + \alpha\big(1+|\nabla u_\alpha|^2\big)^{\alpha-1}|\nabla u_\alpha|^2\Big)\,y\,dxdy = 0.$$

Note that the integrand is similar to that in the variation formula for the sphere derived in Sacks-Uhlenbeck (Lemma 5.3, page 20, [48]). Thus by the same argument, we have for $1\le\alpha\le 2$,

$$\frac{\alpha}{2}|\nabla u_\alpha|^4 \le \frac{-\big(1+|\nabla u_\alpha|^2\big)^\alpha + 1 + \alpha\big(1+|\nabla u_\alpha|^2\big)^{\alpha-1}|\nabla u_\alpha|^2}{(\alpha-1)\big(1+|\nabla u_\alpha|^2\big)^{\alpha-2}} \le |\nabla u_\alpha|^4.$$

Without loss of generality, we can assume p is the point $(0,1) \in \partial D \subset \mathbb{R}^2$. Dividing D into the upper half disk D^+ and the lower half disk D^-, we have

$$\frac{\alpha}{2} \int\limits_{D^+} \left(1 + |\nabla u_\alpha|^2\right)^{\alpha-2} |\nabla u_\alpha|^4 \, y \, dxdy \leq - \int\limits_{D^-} \left(1 + |\nabla u_\alpha|^2\right)^{\alpha-2} |\nabla u_\alpha|^4 \, y \, dxdy.$$

Assume u is a constant map. Then u_α cannot converge to u in $C^1(D, N)$ (Theorem 1.8 in [22]). Therefore p is a blowup point: that is (by Lemma 1.16 and p. 957 in [22]),

$$b_\alpha = \max_{z \in D} |\nabla u_\alpha(z)| = |\nabla u_\alpha(z_\alpha)| \to \infty$$

where $\lim_{\alpha \to 1} z_\alpha = p$. Consider the rescaled maps $\tilde{u}_\alpha(z) = u_\alpha(z_\alpha + b_\alpha^{-1} z)$. As $\alpha \to 1$, the domains of $\tilde{u}_\alpha$ exhaust either the whole plane or a half plane, and (a subsequence) $\{\tilde{u}_\alpha\}$ converges in C^1 on compact subsets to a nontrivial harmonic map $\tilde{u}$. If $D_R(0)$ denotes the disk of radius R centered at the origin in the plane or the half plane, then we have

$$\begin{aligned}
\frac{1}{2}E(\tilde{u}|_{D_R(0)}) &\le \lim_{\alpha\to 1} \frac{1}{2} \int_{D_R(0)} |\nabla \tilde{u}_\alpha|^2 \, dxdy \\
&= \lim_{\alpha\to 1} \frac{1}{2} \int_{D_{R/b_\alpha}(z_\alpha)} |\nabla u_\alpha|^2 \, dxdy \\
&= \lim_{\alpha\to 1} \frac{1}{2} \int_{D_{R/b_\alpha}(z_\alpha)} \left(|\nabla u_\alpha|^2 - \frac{|\nabla u_\alpha|^2}{1+|\nabla u_\alpha|^2} \right) dxdy \\
&= \lim_{\alpha\to 1} \frac{1}{2} \int_{D_{R/b_\alpha}(z_\alpha)} \frac{|\nabla u_\alpha|^2}{1+|\nabla u_\alpha|^2} \cdot |\nabla u_\alpha|^2 \, dxdy \\
&\le \lim_{\alpha\to 1} \frac{\alpha}{2} \int_{D_{R/b_\alpha}(z_\alpha)} \left(1+|\nabla u_\alpha|^2\right)^{\alpha-1} \frac{|\nabla u_\alpha|^2}{1+|\nabla u_\alpha|^2} \cdot |\nabla u_\alpha|^2 \, y \, dxdy \\
&= \lim_{\alpha\to 1} \frac{\alpha}{2} \int_{D_{R/b_\alpha}(z_\alpha)} \left(1+|\nabla u_\alpha|^2\right)^{\alpha-2} |\nabla u_\alpha|^4 \, y \, dxdy \\
&\le \lim_{\alpha\to 1} \frac{\alpha}{2} \int_{D^+} \left(1+|\nabla u_\alpha|^2\right)^{\alpha-2} |\nabla u_\alpha|^4 \, y \, dxdy \\
&\le \lim_{\alpha\to 1} - \int_{D^-} \left(1+|\nabla u_\alpha|^2\right)^{\alpha-2} |\nabla u_\alpha|^4 \, y \, dxdy \\
&= 0
\end{aligned}$$

where the first equality is by the conformal invariance of the energy functional, the second equality follows since $b_\alpha \to \infty$, and the last equality follows since u is a constant map and $u_\alpha \to u$ in C^1 on D^-, so $|\nabla u_\alpha|^2 \to 0$ uniformly. This contradicts the fact that $\tilde{u}$ is nontrivial. Therefore u is not a constant map. □

3.5.3 Decomposition of the kernel of $i_* : \pi_1(M) \to \pi_1(N)$ by minimizing disks

Now we come back to the specific setting of minimizing disks. Given a basepoint $x_0 \in M$, let

$$i_* : \pi_1(M, x_0) \to \pi_1(N, x_0)$$

be the homomorphism induced by the inclusion i of M in N. Recall that two elements γ and γ' in $\pi_1(M, x_0)$ determine the same free homotopy class of closed curves in M if and only if they belong to the same orbit $\pi_1(M, x_0)\gamma = \pi_1(M, x_0)\gamma'$ under the usual action of $\pi_1(M, x_0)$ on $\pi_1(M, x_0)$. That is, the set of free homotopy classes of closed curves in M is in one-to-one correspondence with the set of orbits $\pi_1(M, x_0)\gamma \subset \pi_1(M, x_0)$ (for further details see [48] p.19). Given an element γ in $\ker i_*$, let Γ be its associated free homotopy class. Let

$$W_\Gamma = \{u \in W^{1,\infty}(D, \partial D; N, M) : [u(\partial D)] = \Gamma\},$$

where we use the notation $[u(\partial D)]$ for the free homotopy class of $u(\partial D)$, and define

$$\mathcal{E}(\Gamma) = \min \{E(u) : u \in W_\Gamma\} = \lim_{\alpha \to 1} \min \{\tilde{E}_\alpha(u) : u \in W_\Gamma\}$$

where $\tilde{E}_\alpha(u) = \int((1 + |\nabla u|^2)^\alpha - 1)d\mu$. Note that $\mathcal{E}(\Gamma) = 0$ if and only if Γ is trivial, and $\mathcal{E}(\Gamma) > \epsilon_0$ otherwise ([22] Theorem 1.8).

Lemma 3.5.3. *Let $\gamma \in \ker i_*$ and let $\Gamma = \pi_1(M, x_0)\gamma$ be its associated free homotopy class. Then either Γ can be represented by the boundary of an area minimizing disk solving the free boundary problem, or for any $\delta > 0$ there exist nontrivial free homotopy classes $\Gamma_1 = \pi_1(M, x_0)\gamma_1$, $\Gamma_2 = \pi_1(M, x_0)\gamma_1$ where $\gamma_1, \gamma_2 \in \ker i_*$, such that*

$$\pi_1(M, x_0)\gamma \subset \pi_1(M, x_0)\gamma_1 + \pi_1(M, x_0)\gamma_2, \quad \mathcal{E}(\Gamma_1) + \mathcal{E}(\Gamma_2) < \mathcal{E}(\Gamma) + \delta.$$

Proof. Since $\gamma \in \ker i_*$ there exists $f : (D, \partial D) \to (N, M)$ such that $[f(\partial D)] = \Gamma$. As in the proof of Proposition 3.2.1, there exists a minimizing map u_α of E_α on W_Γ. By Theorem 3.2.7 there exists a sequence $\alpha \to 1$ such that $u_\alpha \to u$ in C^1 on D minus a finite set of points, and $u : (D, \partial D) \to (N, M)$ is a (possibly trivial) harmonic map satisfying the free boundary condition. If the set of points where the convergence fails is empty, then u is nontrivial, and hence is an area minimizing disk solving the free boundary problem with $[u(\partial D)] = \Gamma$. Otherwise there exists a point p at which u_α fails to converge to u in C^1. Note that p cannot be an interior point. If p is an interior point, then as in the proof of Theorem 3.2.9 we can define a modified map $\widehat{u}_\alpha$ by (3.2.8) with $\widehat{u}_\alpha|_{\partial D} = u_\alpha|_{\partial D}$, so $\widehat{u}_\alpha \in W_\Gamma$ and by the same argument as in the the proof of Theorem 3.2.9 we have $u_\alpha \to u$ in $C^1(D_\rho(p), N)$. Therefore, $p \in \partial D$.

Now observe that given $\rho > 0$, we can find a neighborhood B of p in $\bar{D}$, with $|B| < \rho$, that contains no other points where the convergence fails, and such that there is a conformal diffeomorphism $h : D - \bar{B} \to B$ leaving $\partial B \cap D$ fixed. The existence of B and h can be seen in the following way. Let $\varphi : D \to H$ be a conformal map from the open disk D to the upper half plane H such that p is mapped to the origin and two nearby points q and $q' \in \partial D$ on either side of p are mapped to 1 and -1. We may choose q and q' sufficiently close to p so that $B := \varphi^{-1}(D^+)$, where $D^+ = D \cap \bar{H}$, has area less than ρ and contains no other points where the convergence fails. Let $S : \bar{D}^+ \to H - D^+$ be the conformal map $S(z) = 1/\bar{z}$, which is the identity map on the half circle. Then we may take $h = \varphi^{-1} \circ S \circ \varphi$.

Using the construction from Theorem 3.2.9, we can define a map $\widehat{u}_\alpha$ that agrees with u_α outside B and with u on neighborhood of p in B, and so that $\lim_{\alpha \to 1} \tilde{E}_\alpha(\widehat{u}_\alpha|_B) = E(u|_B)$. Now define

$$u_\alpha^1 = \begin{cases} u_\alpha & \text{on } D - B \\ \widehat{u}_\alpha & \text{on } B \end{cases}$$

$$u_\alpha^2 = \begin{cases} \widehat{u}_\alpha \circ h & \text{on } D - B \\ u_\alpha & \text{on } B. \end{cases}$$

Let Γ_1 and Γ_2 be the free homotopy classes of $u_\alpha^1(\partial D)$ and $u_\alpha^2(\partial D)$ respectively. Then $\Gamma \subset \Gamma_1 + \Gamma_2$. By the conformality of h, we have

$$\lim_{\alpha \to 1} \tilde{E}_\alpha(u_\alpha^1) = \lim_{\alpha \to 1} \tilde{E}_\alpha(u_\alpha|_{D-B}) + E(u|_B)$$
$$\lim_{\alpha \to 1} \tilde{E}_\alpha(u_\alpha^2) = \lim_{\alpha \to 1} \tilde{E}_\alpha(u_\alpha|_B) + E(u|_B).$$

Choose ρ sufficiently small so that $E(u|_B) \leq \|u\|_{1,\infty}^2 |B| < \|u\|_{1,\infty}^2 \rho < \delta/6$. Then if α is sufficiently close to 1, we have

$$\tilde{E}_\alpha(u_\alpha^1) \leq \tilde{E}_\alpha(u_\alpha|_{D-B}) + \frac{\delta}{3}$$
$$\tilde{E}_\alpha(u_\alpha^2) \leq \tilde{E}_\alpha(u_\alpha|_B) + \frac{\delta}{3},$$

and

$$\mathcal{E}(\Gamma_1) + \mathcal{E}(\Gamma_2) \leq \tilde{E}_\alpha(u_\alpha^1) + \tilde{E}_\alpha(u_\alpha^2) \leq \tilde{E}_\alpha(u_\alpha) + \frac{2\delta}{3} < \mathcal{E}(\Gamma) + \delta, \tag{3.5.3}$$

where the last inequality follows since $\{u_\alpha\}$ is a minimizing sequence for $\mathcal{E}(\Gamma)$. We may assume $\delta < \frac{1}{2}\min\{\epsilon, \epsilon_0\}$. By Lemma 3.2.6, $\tilde{E}_\alpha(u_\alpha^2) \geq \tilde{E}_\alpha(u_\alpha|_B) \geq E(u_\alpha|_B) \geq \epsilon$ for α close to 1, and so

$$\mathcal{E}(\Gamma_1) \leq \tilde{E}_\alpha(u_\alpha^1) \leq \mathcal{E}(\Gamma) + \delta - \epsilon < \mathcal{E}(\Gamma).$$

Therefore $\Gamma_1 \neq \Gamma$ and Γ_2 is nontrivial. It remains to show that Γ_1 is nontrivial. For α sufficiently close to 1, we have

$$\tilde{E}_\alpha(u_\alpha^1) \geq \tilde{E}_\alpha(u_\alpha|_{D-B}) \geq E(u|_{D-B}) - \frac{\delta}{6} > E(u) - \frac{\delta}{3}.$$

If u is nontrivial then $E(u) \geq \epsilon_0$ ([22] Theorem 1.8), and so

$$\tilde{E}_\alpha(u_\alpha^1) > E(u) - \frac{\delta}{3} \geq \epsilon_0 - \frac{\delta}{3} > \delta.$$

If u is trivial, by Corollary 3.5.2 there must be a second point $p' \neq p$ where the convergence $u_\alpha \to u$ fails, and $p' \in \partial D - B$. Then by Lemma 3.2.6, $\tilde{E}_\alpha(u_\alpha^1) \geq \tilde{E}_\alpha(u_\alpha|_{D-B}) \geq E(u_\alpha|_{D-B}) \geq \epsilon$ for α close to 1. In either case, we have $\tilde{E}_\alpha(u_\alpha^1) > \delta$, and then by equation (3.5.3),

$$\mathcal{E}(\Gamma_2) \leq \tilde{E}_\alpha(u_\alpha^2) < \mathcal{E}(\Gamma) + \delta - \tilde{E}(u_\alpha^1) < \mathcal{E}(\Gamma) + \delta - \delta = \mathcal{E}(\Gamma).$$

Therefore, $\Gamma_2 \neq \Gamma$ and Γ_1 is nontrivial. □

Theorem 3.5.4. *There exists a set of free homotopy classes $\{\Gamma_j\}$ of closed curves in M such that the elements $\{\gamma \in \Gamma_j\}$ form a generating set for* $\ker i_*$ *acted on by $\pi_1(M, x_0)$, and each Γ_j can be represented by the boundary of an area minimizing disk that solves the free boundary problem.*

Proof. Let $\{\Gamma_j\}$ be the free homotopy classes that can be represented by the boundary of an area minimizing disk that solves the free boundary problem. Let $P \subset \ker i_*$ be the subgroup generated by $\gamma \in \Gamma_j$. Suppose P is a proper subgroup. Let $I = \inf \mathcal{E}(\Gamma)$ over all free homotopy classes Γ with elements $\gamma \in \Gamma$, $\gamma \notin P$. Then there exists Γ such that $\mathcal{E}(\Gamma) < I + \epsilon_0/2$.

By assumption, Γ cannot be represented by the boundary of an area minimizing disk that solves the free boundary problem, and so by Lemma 3.5.3 there exist nontrivial Γ_1 and Γ_2 with $\pi_1(M, x_0)\gamma \subset \pi_1(M, x_0)\gamma_1 + \pi_1(M, x_0)\gamma_2$ and $\mathcal{E}(\Gamma_1) + \mathcal{E}(\Gamma_2) < \mathcal{E}(\Gamma) + \epsilon_0/2$. Since Γ_1 and Γ_2 are nontrivial, $\mathcal{E}(\Gamma_j) \geq \epsilon_0$ for $j = 1, 2$. This implies $\mathcal{E}(\Gamma_j) < \mathcal{E}(\Gamma) - \epsilon_0/2 < I$. Therefore, by assumption the sets $\pi_1(M, x_0)\gamma_j$ are both in P, and so

$$\pi_1(M, x_0)\gamma \subset \pi_1(M, x_0)\gamma_1 + \pi_1(M, x_0)\gamma_2 \subset P,$$

a contradiction. Therefore $P = \ker i_*$, and so the elements $\{\gamma \in \Gamma_j\}$ form a generating set for $\ker i_*$ acted on by $\pi_1(M, x_0)$, such that each Γ_j can be represented by the boundary of an area minimizing disk that solves the free boundary problem. □

3.6 Topology of minimal surfaces of low index

Let N be a compact 3-manifold with smooth boundary ∂N. Suppose Σ is a compact orientable two-sided minimal surface in N with boundary $\partial\Sigma$ in ∂N solving the free boundary problem $(\Sigma, \partial\Sigma) \to (N, \partial N)$. We will investigate controlling the genus and the number of boundary components of Σ for stable and index 1 minimal surfaces, under certain curvature and boundary assumptions on N.

Let A denote the second fundamental form, and ν denote the unit normal vector field of Σ in N. Let η denote the outward unit conormal of Σ and T the unit tangent vector along $\partial\Sigma$. The index form is the quadratic form

$$I(f,f) = \int_\Sigma \left(|\nabla f|^2 - \left(Ric(\nu) + |A|^2\right) f^2\right) d\mu + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \, f^2 \, ds$$

for any normal variational vector field $f\nu$. The index of Σ is defined as the number of negative eigenvalues of the associated bilinear form.

3.6.1 The first eigenfunction of the index form

A function $f \in W^{1,2}(\Sigma, \mathbb{R})$ is an eigenfunction of the index form with eigenvalue λ if $I(f,g) = \lambda\langle f, g\rangle_{L^2}$ for all $g \in W^{1,2}(\Sigma, \mathbb{R})$, that is f is a weak solution to the following equation

$$\int_\Sigma \left(\nabla f \cdot \nabla g - (Ric(\nu) + |A|^2) f g\right) \, d\mu + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \, f g \, ds = \lambda \int_\Sigma f g \, d\mu.$$

Since all coefficients are smooth up to the boundary, it follows that any eigenfunction of the index form is smooth up to the boundary. Precisely, the interior regularity is asserted in Corollary 7.11, Gilbarg-Trudinger [23], and the boundary regularity is stated in Theorem 5.4 and Section 8.2, Lions-Magenes [41]. Then integrating by parts gives

$$-\int_\Sigma \left(\Delta f + (Ric(\nu) + |A|^2) f + \lambda f\right) g \, d\mu + \int_{\partial\Sigma} \left(\frac{\partial f}{\partial \eta} + \langle \nabla_\nu \nu, \eta\rangle \, f\right) g \, ds = 0.$$

Equivalently f solves the following Robin-type boundary value problem:

$$\begin{cases} \Delta f + \left(Ric(\nu) + |A|^2\right) f = -\lambda f & \text{in } \Sigma \\ \frac{\partial f}{\partial \eta} + \langle \nabla_\nu \nu, \eta\rangle \, f = 0 & \text{on } \partial\Sigma. \end{cases}$$

Standard arguments for self-adjoint operators (Section 8.12, [23]) asserts that the eigenvalues are $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ and the eigenvectors $\{f_i\}$ form an orthogonal basis for $W^{1,2}(\Sigma)$.

Let h be an eigenfunction of λ_1. By the variational characterization of eigenfunctions, we have

$$\lambda_1 = \min_{W^{1,2}(\Sigma)} \frac{I(f,f)}{(f,f)} = \frac{I(h,h)}{(h,h)}.$$

But $|h|$ will satisfy the same equality and hence is also an eigenfunction of λ_1. It follows that $|h|$ is a nonnegative solution to the above boundary value problem. Then the Harnack inequality (Corollary 8.21, [23]) implies that $|h|$ is indeed positive in the interior of Σ. Moreover, by Hopf's Lemma and using the boundary condition, we get that $|h|$ is positive on $\partial\Sigma$ (Lemma 3.4 , [23]). Therefore h has a fixed sign on Σ.

3.6.2 A stability inequality for index-1 minimal surfaces

Suppose the index of a minimal surface is 1. Then any eigenfunction which is not proportional to h has nonnegative eigenvalue. Since their spanning space in $W^{1,2}(\Sigma)$ is the orthogonal complement of h, this implies $I(f,f) \geq 0$, for all f orthogonal to h: $(f,h) = \int_\Sigma fh\, d\mu = 0$.

From the above discussion, let $h > 0$ be a first eigenfunction. We want to use a specific function f orthogonal to h and containing information about the topology of Σ in the second variation formula (index form) above. Using arguments as in [26], [40] page 274 we have the following:

Lemma 3.6.1. *There exists a conformal map $f : \Sigma \to S^2$ such that $\int_\Sigma fh\, d\mu$ $= 0$ and f has degree $\leq [\frac{g+3}{2}]$.*

Proof. By gluing a disk on each boundary component of Σ, we may view Σ as a domain in a compact surface $\bar{\Sigma}$ of genus g. There exists a conformal map from this closed surface to the sphere

$$\psi : \bar{\Sigma} \to S^2$$

of degree $\leq [\frac{g+3}{2}]$ (see [19]). Let G be the group of conformal diffeomorphisms of S^2. We claim there exists $\varphi \in G$ such that

$$\int_\Sigma (\varphi \circ \psi)\, h\, d\mu = 0.$$

To see this, recall that the conformal transformation group G contains a subgroup which is homeomorphic to B^3. That is, given $a \in B^3$ that is not the origin, let $\theta(a) = a/|a| \in S^2$, and let $\varphi(t)$ be the one parameter family of conformal transformations of the ball B^3 that are dilations on the sphere fixing the opposite poles $\theta(a)$ and $-\theta(a)$. In the group $\varphi(t)$ there is a unique conformal automorphism φ_a that maps the origin to a. Define $H : B^3 \to B^3$ by

$$H(a) = \frac{1}{\int_\Sigma h\, d\mu} \int_\Sigma (\varphi_a \circ \psi) h \, d\mu.$$

As a approaches the boundary ∂B^3,

$$\varphi_a(S^2 \setminus \{-a\}) \to a$$

and so

$$\int_\Sigma (\varphi_a \circ \psi) \, d\mu \to a \int_\Sigma h \, d\mu.$$

Therefore, H extends continuously to a map $H : \overline{B^3} \to \overline{B^3}$ which is the identity on ∂B^3. By a standard argument in topology, H must be surjective. Therefore there exists $a \in B^3$ such that $H(a) = 0$, as claimed. □

Now let f_i be the component functions of f that are orthogonal to h. We have

$$I(f_i, f_i) = \int_\Sigma \Big(|\nabla f_i|^2 - \big(Ric(\nu) + |A|^2\big) f_i^2 \Big) \, d\mu + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta \rangle \, f_i^2 \, ds \geq 0.$$

Summing over i, and using $\sum_{i=1}^3 |f_i|^2 = 1$, we get

$$\int_\Sigma \Big(|\nabla f|^2 - \big(Ric(\nu) + |A|^2\big) \Big) \, d\mu + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta \rangle \, ds \geq 0.$$

Since $f : \bar{\Sigma} \to S^2$ is conformal,

$$\int_\Sigma |\nabla f|^2 \, d\mu < \int_{\overline{\Sigma}} |\nabla f|^2 \, d\mu = 2\text{Area}(f(\overline{\Sigma})) = 2\text{Area}(S^2) d(f) \leq 8\pi \Big[\frac{g+3}{2} \Big].$$

Therefore

$$\int_\Sigma (Ric(\nu) + |A|^2) \, d\mu < 8\pi \Big[\frac{g+3}{2} \Big] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta \rangle \, ds.$$

3.6.3 Controlling topology, area estimates, and rigidity for minimal surfaces of low index

Now we prove Theorem 1.0.3.

Proof of theorem 1.0.3. Let Σ be a solution to the free boundary problem $(\Sigma, \partial\Sigma) \to (N, \partial N)$. Choose a local orthonormal frame $\{e_1, e_2, e_3\}$ along Σ such that $e_1 = T$ is the positively oriented unit tangent vector and $e_2 = \eta$ is the outward unit conormal along $\partial\Sigma$, and $e_3 = \nu$ is the globally defined unit normal to Σ. For $1 \leq i < j \leq 3$, let R_{ijij} denote the sectional curvature of N for the section $e_i \wedge e_j$. Let $R = R_{1212} + R_{1313} + R_{2323}$ be the scalar curvature of N, and let

$$R_{33} = R_{1313} + R_{2323} \tag{3.6.1}$$

be the Ricci curvature for $e_3 = \nu$. Let K denote the Gauss curvature of Σ. From the Gauss equation and the fact that Σ is minimal we have

$$K = R_{1212} - \frac{1}{2}|A|^2. \tag{3.6.2}$$

First we assume that Σ has index 1. From the above, we have

$$\int_\Sigma (R_{33} + |A|^2)\, d\mu < 8\pi \left[\frac{g+3}{2}\right] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta \rangle \; ds. \tag{3.6.3}$$

Now we prove the three parts of the theorem.

Part (i) $Ric(N) \geq 0$ and ∂N is weakly convex.

From (3.6.1) and (3.6.2) we have

$$R_{33} + 2K = R_{11} + R_{22} - |A|^2. \tag{3.6.4}$$

Inserting (3.6.4) into (3.6.3), we get

$$\int_\Sigma (R_{11} + R_{22} - 2K)\, d\mu < 8\pi \left[\frac{g+3}{2}\right] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta \rangle \; ds. \tag{3.6.5}$$

By the Gauss-Bonnet theorem,

$$\int_\Sigma K \, d\mu + \int_{\partial\Sigma} k_g \, ds = 2\pi\chi(\Sigma) = 2\pi(2 - 2g - k),$$

where k_g is the geodesic curvature of $\partial\Sigma$ in Σ. Recall that $k_g = -\langle \nabla_T T, \eta \rangle$, so

$$\int_\Sigma K \, d\mu - \int_{\partial\Sigma} \langle \nabla_T T, \eta \rangle \; ds = 2\pi(2 - 2g - k). \tag{3.6.6}$$

Inserting (3.6.6) into (3.6.5), and using the assumption that $Ric(N) \geq 0$, we get

$$4\pi(2g + k - 2) < 8\pi\Big[\frac{g+3}{2}\Big] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds + 2\int_{\partial\Sigma} \langle \nabla_T T, \eta\rangle \; ds.$$

Since η is orthogonal to ∂N and ∂N is weakly convex we get

$$g + \frac{k}{2} - 1 < \Big[\frac{g+3}{2}\Big].$$

Since

$$\Big[\frac{g+3}{2}\Big] = \frac{g + 3 - \frac{1+(-1)^g}{2}}{2},$$

it follows

$$g + k + \frac{1+(-1)^g}{2} < 5.$$

From this we obtain *i*) $g + k \leq 3$ if g is even, *ii*) $g + k \leq 4$ if g is odd.

Part (ii) $R \geq 0$ and ∂N is weakly mean convex.

Adding (3.6.1) and (3.6.2), we have

$$R_{33} + K = R - \frac{1}{2}|A|^2. \tag{3.6.7}$$

Inserting (3.6.7) into (3.6.3), we obtain

$$\int_\Sigma (R - K + \frac{1}{2}|A|^2)\, d\mu < 8\pi\Big[\frac{g+3}{2}\Big] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds. \tag{3.6.8}$$

Then using the nonnegative scalar curvature assumption and (3.6.6), we get

$$-2\pi(2 - 2g - k) < 8\pi\Big[\frac{g+3}{2}\Big] + \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds + \int_{\partial\Sigma} \langle \nabla_T T, \eta\rangle \; ds.$$

Since ∂N is weakly mean convex we obtain

$$g + \frac{k}{2} - 1 < 2\Big[\frac{g+3}{2}\Big].$$

Then

$$g + \frac{k}{2} - 1 < g + 3 - \frac{1+(-1)^g}{2}.$$

From this we obtain *i*) $k \leq 5$ if g is even, *ii*) $k \leq 7$ if g is odd.

We now assume that Σ is stable; that is, $I(f, f) \geq 0$ for all $f \in W^{1,2}(\Sigma)$. Taking f to be a constant function, we obtain

$$\int_\Sigma (R_{33} + |A|^2)\, d\mu \leq \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds.$$

Using (3.6.7) we get

$$\int_\Sigma (R - K + \frac{1}{2}|A|^2)\, d\mu \leq \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds.$$

By (3.6.6) we get

$$\int_\Sigma \big(R + \frac{1}{2}|A|^2\big)\, d\mu - 2\pi(2 - 2g - k) \leq \int_{\partial\Sigma} \langle \nabla_\nu \nu, \eta\rangle \; ds + \int_{\partial\Sigma} \langle \nabla_T T, \eta\rangle \; ds. \tag{3.6.9}$$

Then since $R \geq 0$ and ∂N is weakly mean convex, we get

$$g + \frac{k}{2} - 1 \leq 0.$$

Therefore the only possibilities for (g, k) are $(0, 1)$ or $(0, 2)$ and Σ must be a disk or a cylinder.

If Σ is a cylinder, from the above we must have $R = 0$ and $|A|^2 = 0$ on Σ, and $I(1, 1) = 0$. Therefore $f = 1$ satisfies the Jacobi equation

$$\Delta f + (Ric(\nu) + |A|^2)f = 0.$$

This implies $Ric(\nu) = 0$, and then from (3.6.7), $K = 0$. Hence Σ is a totally geodesic flat cylinder. We remark that this result can be viewed as the free boundary analogue to results of Schoen and Yau for compact minimal surfaces in compact ambient manifolds (Theorem 5.1, [50]), and Fischer-Colbrie and Schoen for complete minimal surfaces in complete ambient manifolds (Theorem 3, [21]).

Part (iii) We now derive the area estimates.

If Σ has index 1, from (3.6.8) and (3.6.6) we get

$$R_0 \cdot \text{Area}(\Sigma) \leq 8\pi\Big(\Big[\frac{g+3}{2}\Big] - \frac{1}{2}\big(g + \frac{k}{2} - 1\big)\Big),$$

and so

$$\text{Area}(\Sigma) \leq \frac{2\pi(7 - (-1)^g - k)}{R_0}.$$

If Σ is stable, from (3.6.9) we get

$$R_0 \cdot \text{Area}(\Sigma) \leq 4\pi\Big(-\big(g + \frac{k}{2} - 1\big)\Big).$$

Then (g, k) has to be $(0, 1)$ and Σ is a disk. Therefore,

$$\text{Area}(\Sigma) \leq \frac{2\pi}{R_0}.$$

This completes the proof of Theorem 1.0.3. □

Remark. Let N be a compact orientable 3-manifold with boundary $\partial N \neq \emptyset$. Theorem 1.0.2 and 1.0.3 imply that if there exists a continuous map from a compact bordered Riemann surface with $g = 0$ and $k \geq 3$ or $g \geq 1$ and $k \geq 1$, satisfying the incompressible assumption in Theorem 1.0.2, then N admits no metric of nonnegative scalar curvature, for which ∂N is weakly mean convex.

In particular, this assumption is satisfied when $\pi_1(N)$ contains a subgroup abstractly isomorphic to the fundamental group of a compact Riemann surface with $g \geq 1$ and $k \geq 1$, or $g = 0$ and $k \geq 3$, and $\pi_1(N, \partial N) = 0$. Precisely, the first condition asserts the existence of a compact Riemann surface of the same topological type in N, which is incompressible. The second condition can be used to deform each boundary circle of this surface to some loop in ∂N by adding a cylinder to the surface. Thus we obtain a compact Riemann surface in N with boundary in ∂N satisfying the incompressible condition.

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