Special Values of Anticyclotomic L-functions

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Abstract

This thesis consists of four chapters and deals with two different problems which are both related to the broad topic of special values of anticyclotomic L-functions.

In Chapter 3, we generalize some results of Vatsal on studying the special values of Rankin-Selberg L-functions in an anticyclotomic $\mathbb{Z}_p$-extension. Let g be a cuspidal Hilbert modular form of parallel weight $(2, ..., 2)$ and level $\mathcal{N}$ over a totally real field F, and let K/F be a totally imaginary quadratic extension of relative discriminant $\mathcal{D}$. We study the l-adic valuation of the special values $L(g, \chi, \frac{1}{2})$ as χ varies over the ring class characters of K of $\mathcal{P}$-power conductor, for some fixed prime ideal $\mathcal{P}$. We prove our results under the only assumption that the prime to $\mathcal{P}$ part of $\mathcal{N}$ is relatively prime to $\mathcal{D}$.

In Chapter 4, we compute a basis for the two-dimensional subspace $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ of half-integral weight modular forms associated, via the Shimura correspondence, to a newform $F \in S_{k-1}(\Gamma_0(N))$, which satisfies $L(F, \frac{1}{2}) \neq 0$. Here, we let k be a positive integer such that $k \equiv 3 \mod 4$ and N be a positive square-free integer. This is accomplished by using a result of Waldspurger, which allows one to produce a basis for the forms that correspond to a given F via local considerations, once a form in the Kohnen space has been determined. The squares of the Fourier coefficients of these forms are known to be essentially proportional to the central critical values of the L-function of F twisted by some quadratic characters.

Preface

A version of Chapter 4 has been published under the title Ternary Quadratic Forms and Half-Integral Weight Modular Forms in LMS Journal of Computation and Mathematics [10].

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Chapter 1

Introduction and Organization

1.1 Introduction

Let π be an irreducible cuspidal automorphic representation of GL_2 over a totally real field F, and let K/F be a totally imaginary quadratic extension. Attached to π and a Hecke character χ of K is the Rankin-Selberg L-function $L(\pi, \chi, s)$. If w is the central character of π and $\chi \cdot w = 1$ on $\mathbb{A}_F^* \subset \mathbb{A}_K^*$, then $L(\pi, \chi, s)$ is entire and satisfies the functional equation

$$L(\pi, \chi, s) = \epsilon(\pi, \chi, s) L(\pi, \chi, 1 - s),$$

where $\epsilon(\pi, \chi, \frac{1}{2}) = \epsilon(\pi, \chi) \in \{\pm 1\}$. If $\epsilon(\pi, \chi) = -1$, the special value $L(\pi, \chi, \frac{1}{2})$ vanishes trivially. Hence, we assume throughout that $\epsilon(\pi, \chi) = 1$ in which case we say that the pair (π, χ) is even. Obtaining a formula for the special value $L(\pi, \chi, \frac{1}{2})$ in the even case has been the subject of extensive research for over 25 years now. There are several results in the literature relating $L(\pi, \chi, \frac{1}{2})$ to a finite sum $\mathrm{a}(x, \chi)$ which is essentially the height of a twisted CM point on the Shimura curve associated to some carefully chosen totally definite quaternion algebra B. In 1985, Waldspurger proved a fundamental theorem (Théorème 2 in [28]) which states that under very mild conditions on π and χ, $L(\pi, \chi, \frac{1}{2}) \neq 0$ if and only if $\mathrm{a}(x, \chi) \neq 0$. However, this result doesn't give a precise formula for the special value $L(\pi, \chi, \frac{1}{2})$ in terms of $\mathrm{a}(x, \chi)$. Most authors refer to such a formula as a Gross-Zagier formula, and it is expected (but not known yet) that it exists in full generality. The general scheme for obtaining a Gross-Zagier formula involves the construction of a discrete set of CM points and a function ψ on this set induced by some automorphic form θ in the representation space of π', the unique cuspidal automorphic representation on B associated to π via the Jacquet-Langlands correspondence. In this framework, the Gross-Zagier sum $\mathrm{a}(x, \chi)$ is given by

$$\mathrm{a}(x, \chi) = \int_{\mathrm{Gal}_K^{\mathrm{ab}}} \chi(\sigma) \psi(\sigma.x),$$

where x is a CM point of conductor equal to that of χ, and $\mathrm{Gal}_K^{\mathrm{ab}}$ is the Galois group of the maximal abelian extension of K acting continuously on the set of CM points. In fact, one can easily check that $\mathrm{a}(x,\chi)$ is a finite sum.

In 1987, Gross established the main identity which expresses the special value $L(\pi,\chi,\frac{1}{2})$ in terms of the height pairing on the CM points (see [9]). In Gross' result, π corresponds to a weight 2 newform of prime conductor and trivial central character. Subsequently, Zhang obtained a generalization of this result to the case where π corresponds to a Hilbert newform g of parallel weight $(2,...,2)$ and trivial central character. More specifically, Zhang proved in [32] that if the level $\mathcal{N}$ of g, the conductor $\mathcal{C}$ of χ, and the discriminant $\mathcal{D}$ of K/F are pairwise coprime, then

$$L(\pi,\chi,\frac{1}{2}) = \frac{C}{\sqrt{D}}\,\langle g,g\rangle\,|\mathrm{a}(x,\chi)|^2,$$

where $\langle g,g\rangle$ is the Petersson inner product, D is the absolute norm of $\mathcal{C}^2\mathcal{D}$ and C is a non-zero constant independent of χ.

Results as such were the point of departure in the work of Vatsal ([24], [25]) and Cornut-Vatsal ([7]) on the non-vanishing of $L(\pi,\chi,\frac{1}{2})$ in anticyclotomic towers. We will now give a brief account of the main results in [7], [24] and [25] which will be discussed more elaborately in Chapter 3 of this dissertation.

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N, and let $K/\mathbb{Q}$ be an imaginary quadratic field extension of discriminant D such that N and D are relatively prime. Denote by K_∞ the anticyclotomic $\mathbb{Z}_p$-extension of K where p is a given prime number with $p \nmid ND$. In 2002, Vatsal succeeded in settling a conjecture of Mazur pertaining to the size of the Mordell-Weil group $E(K_\infty)$. In fact, Mazur's conjecture predicts that the group $E(K_\infty)$ is finitely generated, and Vatsal proved in [24] that this is true, at least when E is ordinary at p, or when the class number of K is prime to p.

In more concrete terms, Vatsal considered the modular form g associated to E and the family of Rankin -Selberg L-functions $L(g,\chi,s)$ as χ varies over ring class characters of K of p-power conductor. Under certain conditions on g and χ, the result of Vatsal asserts that the special values $L(g,\chi,\frac{1}{2})$ are non-vanishing for all but finitely many χ, provided that p is an ordinary prime for g or p does not divide the class number of K. One consequence of this result is the non-triviality of certain Euler systems as formulated by Bertolini-Darmon in [3] which in its turn implies that the desired statement about the Mordell-Weil group is indeed true.

In 2004, Cornut and Vatsal generalized in [7] the above mentioned work of Vatsal to totally real fields. Numerous technical complications arise due to the fact that a more general number field F is considered. However, the basic arguments are ultimately the same, as the authors invoke deep theorems of Ratner [19] on uniform distribution of unipotent orbits on p-adic Lie groups to deduce the desired result.

In 2003, Vatsal extended the results and methods of [24] to study the variation of the λ-adic absolute value of $L^{\mathrm{al}}(g, \chi, \frac{1}{2})$ as a function of χ, where λ is a fixed prime of $\bar{\mathbb{Q}}$ with residue characteristic l. In Chapter 3 of this dissertation, we generalize this work to totally real fields while removing most of the restrictions on N, p, D and l (Theorem 3.4.14). We use the improved formalism developed in [7] to achieve this purpose.

We now give an overview of our work in Chapter 3. Let π be an irreducible automorphic representation of GL_2 over a totally real field corresponding to a cuspidal Hilbert modular newform g of level $\mathcal{N}$, trivial character and parallel weight $(2, ..., 2)$. The Hecke eigenvalues of g are denoted by a_v ($T_v g = a_v g$). Let $\mathcal{P}$ be a prime ideal of F such that $\mathcal{P}$ lies over an odd rational prime p. Let l be a rational prime, and denote by E_l the l-adically complete discrete valuation ring containing the Hecke eigenvalues of g. To simplify the exposition of the introduction, we assume that $l \neq p$ although we should mention that the case $l = p$ does not give rise to significant complications. Let χ be a ring class character of K of conductor $\mathcal{P}^n$ such that $\chi = 1$ when restricted to $\mathbb{A}_F^* \subset \mathbb{A}_K^*$. In addition to some mild restrictions, we assume that the prime to $\mathcal{P}$ part of $\mathcal{N}$ is relatively prime to the discriminant $\mathcal{D}$ of K/F. We also impose sufficient conditions to make the sign in the functional equation of $L(\pi, \chi, s)$ equal $+1$ for all but finitely many characters χ of the type considered above.

Let $G(n)$ be the Galois group of the ring class field of conductor $\mathcal{P}^n$ over K. We have the decomposition $G(n) = G_0 \times H(n)$ where G_0 is known as the tame subgroup of $G(n)$ and $H(n)$ as the wild subgroup of $G(n)$. By class field theory, we can view χ as a character of $G(n)$. Hence, χ can be written as the product of a tamely ramified character χ_0 of G_0 and a wild character χ_1 of $H(n)$. It can be shown that $G(n)$ acts simply transitively on the set of CM points of conductor $\mathcal{P}^n$ on the Shimura curve associated to some carefully chosen totally definite quaternion algebra B. Given a CM point x of conductor $\mathcal{P}^n$ and a ring class character χ of the same conductor, we define the Gross-Zagier sum

$$\mathrm{a}(x, \chi) = \frac{1}{|G(n)|} \sum_{\sigma \in G(n)} \chi(\sigma)\psi(\sigma.x),$$

where ψ is the E_l-valued function on the set of Heegner points, associated to g via the Jacquet-Langlands correspondence. In the light of the existing Gross-Zagier formulae, our job is reduced to studying the l-adic valuation of this sum. More precisely, our goal is to prove an analogue of Proposition 4.1 and Corollary 4.2 in [25] for a Hilbert modular form g over a totally real field F, while removing the assumptions on l and the class number of K. Before we state the results we obtained in this direction, it is perhaps more enlightening to shed some light on Vatsal's result in which π corresponds to a weight 2 newform g for $\Gamma_0(N)$ such that N, p, and D are pairwise relatively prime.

Theorem 1.1.1. *Let the tame character χ_0 be given such that its order is prime to p. Then, under some restrictions on l and the Hecke field of g, we have*

$$ord_\lambda(\mathrm{a}(x,\chi)) < \mu \tag{1.1}$$

for all $n \gg 0$ and $\chi = \chi_0\chi_1$, where μ is the smallest integer such that $a_q \not\equiv 1+q \mod \lambda^\mu$ for some $q \nmid pND$.

Note that the restriction on the Hecke field of g, which will be made clear in Section 3.4, was overlooked in [25]. More importantly, we remark that (1.1) is mistakenly given as an equality in [25]. The source of this mistake is an error made in the proof of Proposition 5.3 part (2). We mention here that in order to make the necessary corrections, we need to modify the definition of the constant μ from the one given in [25]. We say a bit more about this issue in Chapter 3.

Recall that E_l is an l-adically complete discrete valuation ring containing the Hecke eigenvalues of g. We may assume without loss of generality that E_l contains the values of χ_0 and the *p-th* roots of unity. We consider the trace of $\mathrm{a}(x,\chi)$ taken from $E_l(\chi_1)$ to E_l:

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \sum_{\sigma\in\mathrm{Gal}(E_l(\chi_1)/E_l)} \sigma(\mathrm{a}(x,\chi)).$$

This trace expression is different than the average expression

$$\mathrm{b}(x,\chi_0) = \sum_{\chi_1\in\widehat{H(n)}} \mathrm{a}(x,\chi_0\chi_1)$$

considered in the work of Vatsal and Cornut-Vatsal. In particular, given any $\chi = \chi_0\chi_1$, the non-vanishing of $\mathrm{Tr}(\mathrm{a}(x,\chi))$ implies that of $\mathrm{a}(x,\chi)$. After a series of reductions, we prove the following proposition in two level raising steps (Proposition 3.4.9 and Proposition 3.4.12).

Proposition 1.1.2. *The trace expression simplifies to*

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|} \sum_{\sigma\in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}),$$

where G_1 and G_2 are certain subgroups of $G(n)$, $\psi_{m,\mathcal{D}}$ is a function of higher level induced by ψ, and $x_{m,\mathcal{D}}$ is a CM point of higher level and conductor $\mathcal{P}^n$.

Hence, the problem is reduced to studying the λ-adic valuation of

$$\sum_{\sigma\in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}).$$

Finally, we state the following theorem which is the main result in Chapter 3 (Theorem 3.4.14).

Theorem 1.1.3. *Let χ_0 be any character of G_0. For any CM point x of conductor $\mathcal{P}^n$ with $n \gg 0$, there exists some $y \in G(n).x$ such that*

$$ord_\lambda\left(\sum_{\sigma\in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.y)\right) < \mu,$$

where μ is precisely given in Definition 3.4.8.

Consider an irreducible cuspidal automorphic representation of GL_2 over $\mathbb{Q}$ corresponding to a newform F. An interesting situation occurs if we let K vary over the imaginary quadratic extensions of $\mathbb{Q}$ and let χ be the trivial character of K. In this case we have the decomposition

$$L(\pi,\chi,\frac{1}{2}) = L(\pi,\frac{1}{2})L(\pi\otimes\eta,\frac{1}{2}),$$

where η is the quadratic character of $\mathbb{Q}$ associated with the extension K/F. By a striking result of Waldspurger [29], we know that the special values $L(\pi\otimes\eta,\frac{1}{2})$ as K vary over the imaginary quadratic extensions of $\mathbb{Q}$ are related to the Fourier coefficients of some half-integral weight modular form corresponding to F via the Shimura correspondence. Hence, the Gross-Zagier formula yields a very interesting interpretation of the height pairing $\mathrm{a}(x,\chi)$ in terms of the Fourier coefficients of some half-integral weight modular form. This observation motivates to some extent our work in Chapter 4 of this dissertation. The contents of this chapter have been published in the LMS Journal of Computation and Mathematics [10].

Let k be a positive integer such that $k \equiv 3 \mod 4$, and let N be a positive square-free integer. In Chapter 4, we compute a basis for the two-dimensional subspace $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ of half-integral weight modular forms associated, via the Shimura correspondence, to a newform $F \in S_{k-1}(\Gamma_0(N))$, which satisfies $L(F, \frac{1}{2}) \neq 0$. Waldspurger's Theorem asserts that there exists a basis for $S_{\frac{k}{2}}(4N, F)$ such that for every positive integer n the Fourier coefficient $a_n(f_i)$ of a basis element f_i is the product of two factors:

- a product of local terms $c_i(n, F)$ each of which is completely determined by the local components of F according to explicit formulae given in [29];
- a global factor $A_F(n)$ whose square is the central critical value of the L-function of the newform F twisted by a quadratic character depending on n.

In view of this remarkable result, our task is reduced to computing the global factors $A_F(n)$. Toward this end, we use a construction drawn from the articles [9] and [4] to compute a non-zero modular form g as a linear combination of ternary theta series $g([I_i])$ generated by the norm form N of some carefully chosen quaternion algebra B. This modular form belongs to the Kohnen subspace of $S_{\frac{3}{2}}(\Gamma_0(4N), F)$. The determination of the theta series $g([I_i])$ up to a precision T amounts to computing the number of times $N(x)$ represents $1, 2, 3, ..., T$ as x varies in some ternary lattice R_i. Therefore, it takes time roughly proportional to $T^{\frac{3}{2}}$. We use the Brandt module package in Sage to compute g as a linear combination of $g([I_i])$.

Now we show how we use the form g to compute yet another half-integral weight modular form h which also maps to F via the Shimura correspondence. These two forms make up a basis for the space $S_{\frac{k}{2}}(\Gamma_0(N), F)$. Let the Fourier expansion of the given newform be $F = \sum_{n \geq 1} b_n q^n$ and let $g = \sum_{n \geq 1} a_n q^n$ be the Fourier expansion of the half-integral weight modular form obtained above. For every prime number p not dividing N, we put $\lambda_p = b_p p^{1-\frac{k}{2}}$, $\alpha_p + \alpha'_p = \lambda_p$, and $\alpha_p \alpha'_p = 1$. For a positive square-free integer t, we denote by Δ_{-t} the fundamental discriminant corresponding to $-t$. We obtain the following theorems which are the main results in Chapter 4 (Theorem 4.4.5 and Theorem 4.4.7).

Theorem 1.1.4. *For a positive square-free integer t satisfying* $\left(\frac{\Delta_{-t}}{p}\right) =$

$-p^{\frac{1}{2}}\lambda'_p$ *for all* p *such that* $p|N$ *and* $p \nmid \Delta_{-t}$*, we have*

$$A_F(t) = r2^{\frac{\nu_t}{2}}|\Delta_{-t}|^{\frac{2-k}{4}}a_{|\Delta_{-t}|},$$

where $2^{\frac{\nu_t}{2}} = \prod_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{-\frac{1}{2}}$*, and* r *is a non-zero complex constant depending only on* F*.*

In what follows, the terms $C_1(n)$ and $C_2(n)$ refer to certain products of the local factors $c_i(n, F)$ determined in Waldspurger's paper. We assume that $\alpha_2 \neq \alpha'_2$ to simplify exposition.

Theorem 1.1.5. *Let* $F = \sum_{n\geq 1} b_n q^n$ *be a newform in* $S_{k-1}^{\text{new}}(\Gamma_0(N))$ *with odd square-free level* N *such that* $k \equiv 3 \mod 4$ *and* $L(F, 1) \neq 0$*.*
Let $g = \sum_{n\geq 1} a_n q^n \in S_{\frac{k}{2}}(\Gamma_0(4N), F)$ *be the form obtained above. Put*

$$f_1 = \sum_{n\geq 1} a_n(f_1)q^n \quad \textit{and} \quad f_2 = \sum_{n\geq 1} a_n(f_2)q^n,$$

where if $\left(\frac{\Delta_{-n^{\text{sf}}}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p$ *for all* p *such that* $p|N$ *and* $p \nmid \Delta_{-n^{\text{sf}}}$*, we have*

$$a_n(f_1) = 2^{\frac{\nu_n}{2}} a_{|\Delta_{-n^{\text{sf}}}|}|\Delta_{-n^{\text{sf}}}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} C_1(n)$$

$$a_n(f_2) = 2^{\frac{\nu_n}{2}} a_{|\Delta_{-n^{\text{sf}}}|}|\Delta_{-n^{\text{sf}}}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} C_2(n),$$

and otherwise we have $a_n(f_1) = a_n(f_2) = 0$*. Then* f_1 *and* f_2 *form a basis for* $S_{\frac{k}{2}}(\Gamma_0(4N), F)$

In order to use Theorem 1.1.5 as an effective tool for computing a basis for $S_{\frac{3}{2}}(\Gamma_0(4N), F)$, we make the following important observation. Let $h = 2^{\frac{1}{2}}(f_1 + f_2)$ and write $h = \sum_{n\geq 1} a_n(h)q^n$. It turns out that the Fourier coefficients of h can be expressed in terms of the Fourier coefficients of g by means of simple formulae. Take, for example, a positive square-free integer n such that $\left(\frac{n}{p}\right) = -p^{\frac{1}{2}}\lambda'_p$ whenever $p|N$ and $p \nmid n$ (since, for square-free n, $a_n(h) = 0$ otherwise). We have

$$a_n(h) = \begin{cases} a_n(g)(b_2 - 2(2, n)_2) & \text{if } n \equiv 3 \mod 4; \\ 2a_{4n}(g) & \text{otherwise.} \end{cases}$$

We obtain a similar expression of $a_n(h)$ in terms of $a_n(g)$ for when n is a positive integer $n = n^{\mathrm{sf}}(y)^2$, which is not divisible by any square prime to $4N$. A recursive formula is then used to compute $a_n(h)$ for an arbitrary positive integer n.

We implemented these formulae in Sage. The result is a function that outputs the Fourier expansion of h up to a desired precision. Thus, we get a basis for $S_{\frac{3}{2}}(\Gamma_0(4N), F)$ consisting of the modular forms g and h.

1.2 Organization

The thesis is organized as follows. Chapter 1 is an introduction to the thesis in which we discuss the motivation and related research to our work, and we give a brief account of our main results.

In Chapter 2, we introduce the basic definitions and all the preliminary material that will be used throughout the thesis. In the first section, we recall some arithmetic aspects of Hilbert modular forms. In Section 2.2 we give an overview of some basics from the theory of Quaternion Algebras. We then explore the connection between automorphic forms on totally definite quaternion algebras and adelic Hilbert modular forms; this connection is made possible by means of the Jacquet-Langlands correspondence. In Section 2.3, we recall the definition and some of the basic properties of Rankin-Selberg L-functions.

In Chapter 3, we generalize the results of Vatsal on studying the l-adic valuation of the special values of Rankin-Selberg L-functions in an anticyclotomic $\mathbb{Z}_p$-extension. First we introduce some notation and fix a set of hypotheses which we require throughout the chapter. In the second section, we recall the construction of CM points associated to a totally definite quaternion algebra. In Section 3.3, we recall the fundamental result of Cornut-Vatsal [7] on uniform distribution of CM points. In Section 3.4, we explore the Gross-Zagier sum $\mathrm{a}(x, \chi)$ which is closely related to the special value $L(\pi, \chi, \frac{1}{2})$. Then we prove some propositions and lemmas related to this sum leading to the main result Theorem 3.4.14.

In Chapter 4, we compute a basis for the subspace $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ of half-integral weight modular forms associated, via the Shimura correspondence, to a newform F of weight $k-1$ and odd square-free level N. First, we recall the definition and some basic properties of the Shimura correspondence. In Section 4.2, we present a construction of a half-integral weight modular form g which belongs to the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ for a given newform $F \in S_{k-1}^{\mathrm{new}}(\Gamma_0(N))$. The squares of the Fourier coefficients

of this form are essentially proportional to the central critical values of the L-function of F twisted with some quadratic characters. In Section 4.3, we digress briefly to discuss the representation-theoretic interpretation of the Shimura correspondence as a theta correspondence. In Section 4.4, we state the full result of Waldspurger. Next, we show that the space $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ is in fact two-dimensional and has a distinguished basis $\{f_1, f_2\}$ such that either $f_1 - f_2$ or f_2 belongs to the Kohnen subspace. We use this to express the global factors $A_F(n)$ in terms of the coefficients of g (Theorem 4.4.5). Finally, we explicitly determine the Fourier coefficients of the two modular forms that generate $S_{\frac{k}{2}}(\Gamma_0(4N), F)$, thus, arriving at our main contribution given in Theorem 4.4.7. The last section contains examples to illustrate the calculations carried out in Section 4.4.

Chapter 2

Background

2.1 Hilbert Modular Forms

We begin by recalling some arithmetic aspects of Hilbert modular forms which are primary objects of interest for this dissertation. The exposition in this section follows that of Shimura in [21]. Let F be a totally real number field of degree d over $\mathbb{Q}$, and let $J_F = \{\tau_1, ..., \tau_d\}$ be the set of all real embeddings $F \hookrightarrow \mathbb{R}$. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$ and $z = (z_1, ..., z_d) \in \mathbb{C}^d$, we write $z^{\mathbf{k}} = \prod_{i=1}^d z_i^{k_i}$. An element $a \in F$ is said to be totally positive ($a \gg 0$) if $\tau(a) > 0$ for all $\tau \in J_F$. We denote by O_F the ring of integers of F and by $\mathcal{D}$ its different over $\mathbb{Q}$. For each prime ideal $\mathcal{P}$ of F, we denote by $F_{\mathcal{P}}$ and $O_{F,\mathcal{P}}$ the completions of F and O_F, respectively, at $\mathcal{P}$. Moreover, given a fractional ideal $\mathcal{I}$ of F, we define the localization $\mathcal{I}_{\mathcal{P}}$ of $\mathcal{I}$ at $\mathcal{P}$ as a submodule of $F_{\mathcal{P}}$, and we put $\widehat{\mathcal{I}} = \prod_{\mathcal{P}} \mathcal{I}_{\mathcal{P}}$. We let $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$ be the ring of adeles of F and $\widehat{F} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$ be the ring of finite adeles. The archimedean component of $\mathbb{A}_F$ is $F_\infty = F \otimes \mathbb{R}$, and F_∞^+ is the identity component in F_∞^*. If $z \in \mathbb{A}_F$, we let z_∞ be its archimedean component. For $t \in \mathbb{A}_F^*$, we denote by tO_F the fractional ideal in F associated to t unless otherwise specified. Here $\mathbb{A}_F^*$ is the group of ideles.

2.1.1 Classical Hilbert Modular Forms

We have an embedding of $\mathrm{GL}_2(F)$ into $\prod_{i=1}^d \mathrm{GL}_2(\mathbb{R})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} \tau(a) & \tau(b) \\ \tau(c) & \tau(d) \end{pmatrix} \right)_{\tau \in J_F}.$$

We often identify $\mathrm{GL}_2(F)$ with its image in $\prod_{i=1}^d \mathrm{GL}_2(\mathbb{R})$ under this embedding. For any subring R in F, we define $\mathrm{GL}_2^+(R)$ to be the subgroup of

elements with totally positive determinant. More precisely, we have

$$\mathrm{GL}_2^+(R) = \{\gamma \in \mathrm{GL}_2(R) : (\tau_1(\gamma), ..., \tau_d(\gamma)) \in \prod_{i=1}^{d} \mathrm{GL}_2^+(\mathbb{R})\},$$

where

$$\mathrm{GL}_2^+(\mathbb{R}) = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) : \det(\gamma) > 0\}.$$

Let $\mathbb{H}^d$ be the d-fold Cartesian product of the Poincaré upper-half plane. The well-known action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$ by fractional linear transformations extends naturally to an action of $\prod_{i=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}^d$: For every $z = (z_1, ..., z_d) \in \mathbb{H}^d$ and $\gamma = (\gamma_1, ..., \gamma_d) \in \prod_{i=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}^d$ with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, we let γ act on z by

$$\gamma.z = (\gamma_1.z_1, ..., \gamma_d.z_d),$$
$$\gamma_i.z_i = \frac{a_i(z_i) + b_i}{c_i(z_i) + d_i}.$$

Moreover, for a given $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$, the group $\prod_{i=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$ acts on all complex-valued functions on $\mathbb{H}^d$ as follows. Given $f : \mathbb{H}^d \to \mathbb{C}$ and $\gamma \in \prod_{i=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$ as above, we put

$$(f|_{\mathbf{k}}\gamma)(z) = \left(\prod_{i=1}^{d} (c_i z_i + d_i)^{-k_i}\right) f(\gamma.z)$$

and

$$(f||_{\mathbf{k}}\gamma)(z) := \left(\prod_{i=1}^{d} \det(\gamma_i)^{\frac{k_i}{2}}\right) (f|_{\mathbf{k}}\gamma)(z).$$

A subgroup Γ of $\mathrm{GL}_2^+(F)$ is called a congruence subgroup if it contains the subgroup

$$\Gamma_N = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_F) : \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \in N.M_2(O_F)\right\}$$

for some positive integer N, and if $\Gamma/\Gamma \cap F$ is commensurable with $\mathrm{PSL}_2(O_K)$.

Two examples of congruence subgroups that appear frequently in literature are

$$\Gamma_0(\mathcal{I},\mathcal{N}) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} O_F & \mathcal{I}^{-1} \\ \mathcal{I}\mathcal{N} & O_F \end{pmatrix} : \det(\gamma) \in O_F^{*+}\right\} \tag{2.1}$$

and

$$\Gamma_1(\mathcal{I},\mathcal{N}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{I},\mathcal{N}) : d \equiv 1 \mod \mathcal{N}\right\}, \tag{2.2}$$

where $\mathcal{I}$ is a fractional ideal of F and $\mathcal{N}$ is an ideal of O_F.

Definition 2.1.1. *Let* $\mathbf{k}$ *be a tuple in* $\mathbb{Z}^d$ *and let* Γ *be a congruence subgroup. A holomorphic function* $f : \mathbb{H}^d \to \mathbb{C}$ *is a classical Hilbert modular form of weight* $\mathbf{k}$ *and level* Γ *if*

1. $f||_{\mathbf{k}}\gamma = f$ *for all* $\gamma \in \Gamma$
2. f *is holomorphic at the cusps* $\Gamma\backslash\mathbb{P}^1(F)$

The space of all classical Hilbert modular forms of weight $\mathbf{k}$ *and level* Γ *is denoted by* $\mathcal{M}_{\mathbf{k}}(\Gamma)$.

The condition of holomorphicity at the cusps is unnecessary for $[F : \mathbb{Q}] > 1$ since it is readily met by any holomorphic function $f : \mathbb{H}^d \to \mathbb{C}$ which satisfies the transformation formula $f||_{\mathbf{k}}\gamma = f$ for all γ in some congruence subgroup Γ. This result is known as Koecher's principle.

Every classical Hilbert modular form f admits a Fourier expansion of the form

$$f(z) = \sum_{\xi} c(\xi) e_F(\xi z),$$

where $z = (z_1, ..., z_d) \in \mathbb{H}^d$, $e_F(\xi z) = e^{2\pi i \mathrm{Tr}(\xi z)}$ and $\mathrm{Tr}(\xi z) = \sum_{i=1}^d \tau_i(\xi) z_i$. Here ξ runs over 0 and all totally positive elements ($\xi \gg 0$) of some lattice in F, and $c(\xi)$ are complex numbers.

Definition 2.1.2. *A classical Hilbert modular form of weight* $\mathbf{k}$ *and level* Γ *is a cusp form if the constant term in the Fourier expansion of* $f|_{\mathbf{k}}\gamma$ *is 0 for all* $\gamma \in \mathrm{GL}_2^+(F)$. *The space of all classical Hilbert cusp forms of weight* $\mathbf{k}$ *and level* Γ *is denoted by* $\mathcal{S}_{\mathbf{k}}(\Gamma)$.

The spaces $\mathcal{M}_{\mathbf{k}}(\Gamma)$ and $\mathcal{S}_{\mathbf{k}}(\Gamma)$ are finite dimensional $\mathbb{C}$-vector spaces. In what follows we list some basic properties of these spaces. The reader is referred to Shimura [21] for proofs and more elaborate discussions on such results.

Proposition 2.1.3. *If $\mathcal{M}_{\mathbf{k}}(\Gamma) \neq 0$, then either all k_i are positive or $\mathbf{k} = (0, ..., 0)$ in which case $\mathcal{M}_{\mathbf{0}}(\Gamma) = \mathbb{C}$ and $\mathcal{S}_{\mathbf{0}}(\Gamma) = (0)$.*

Proposition 2.1.4. *We have $\mathcal{M}_{\mathbf{k}}(\Gamma) = \mathcal{S}_{\mathbf{k}}(\Gamma)$ unless $k_1 = ... = k_d$.*

Finally we introduce an inner product on $\mathcal{M}_{\mathbf{k}}(\Gamma)$. Let f and g be classical Hilbert modular forms in $\mathcal{M}_{\mathbf{k}}(\Gamma)$ such that fg is a cusp form. We put

$$\langle f, g\rangle = \mu(\Gamma\backslash\mathbb{H}^d)^{-1} \int_{\Gamma\backslash\mathbb{H}^d} \overline{f(z)} g(z) y^{\mathbf{k}} d\mu(z),$$

where $d\mu(z) = \prod_{\nu=1}^{d} y_\nu^{-2} dx_\nu dy_\nu$ with $z_\nu = x_\nu + iy_\nu$ and $\mu(\Gamma\backslash\mathbb{H}^d)$ is the measure of a fundamental domain for $\Gamma\backslash\mathbb{H}^d$ with respect to $d\mu(z)$.

2.1.2 Adelic Hilbert Modular forms

We denote by $K_\infty^+ = \prod_{\nu=1}^{d} \mathbb{R}^* SO_2(\mathbb{R})$ the stabilizer of $\mathbf{j} = (i, ..., i)$ in $\prod_{\nu=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.1.5. *Let $\mathbf{k}$ be a tuple in $\mathbb{Z}^d$ and let K be a an open compact subgroup of $\mathrm{GL}_2(\widehat{F})$. A function $f : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ is an adelic Hilbert modular form of weight $\mathbf{k}$ with respect to K if it is left $\mathrm{GL}_2(F)$-invariant, right K-invariant and for all $g \in \mathrm{GL}_2(\widehat{F})$ the function*

$$f_g : \gamma = (\gamma_1, ..., \gamma_d) \in G_\infty^+ \mapsto \left(\prod_{\nu=1}^{d} \det(\gamma_\nu)^{-\frac{k_\nu}{2}} (c_\nu i + d_\nu)^{k_\nu} \right) f(g\gamma)$$

factors through a holomorphic function of $G_\infty^+/K_\infty^+ \simeq \mathbb{H}^d$. Here G_∞^+ is $\prod_{\nu=1}^{d} \mathrm{GL}_2^+(\mathbb{R})$.

An adelic Hilbert modular form of weight $\mathbf{k}$ with respect to K is called a cusp form if

$$\int_{F\backslash\mathbb{A}_F} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0, \quad \textit{for all } g \in \mathrm{GL}_2(\mathbb{A}_F).$$

For an integral ideal $\mathcal{N}$ of O_F, we define the following open compact subgroup of $\mathrm{GL}_2(\widehat{F})$: $K_0(\mathcal{N}) = \prod_{\mathcal{P}} K_0(\mathcal{N})_{\mathcal{P}}$ where the product is taken over all prime ideal $\mathcal{P}$ of F and

$$K_0(\mathcal{N})_{\mathcal{P}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_{F,\mathcal{P}}) : c \in \mathcal{N}_{\mathcal{P}} \right\}. \tag{2.3}$$

This is the standard Iwahori congruence subgroup of $\mathrm{GL}_2(\widehat{F})$ of level $\mathcal{N}$. We also define

$$K_1(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(\mathcal{N}) : d \equiv 1 \mod \mathcal{N} \right\}. \tag{2.4}$$

An adelic Hilbert modular form of weight $\mathbf{k}$ with respect to $K_0(\mathcal{N})$ is said to have level $\mathcal{N}$. The space of all such forms is denoted by $M_{\mathbf{k}}(\mathcal{N})$. The space of all adelic Hilbert cusp forms of weight $\mathbf{k}$ and level $\mathcal{N}$ is denoted by $S_{\mathbf{k}}(\mathcal{N})$.

Since the image of $K_0(\mathcal{N})$ in $\widehat{F}^*$ under the determinant map is $\prod_{\mathcal{P}} O^*_{F,\mathcal{P}}$, the strong approximation theorem for GL_2 yields a bijection between

$$\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/G^+_\infty K_0(\mathcal{N})$$

and $F^*\backslash \mathbb{A}^*_F/F^+_\infty \widehat{O_F}^*$, the narrow ideal class group of F. This allows us to relate the adelic and the classical ideal theoretic formulations of Hilbert modular forms as follows. Choose elements $g_1, ..., g_h$ in $\mathrm{GL}_2(\widehat{F})$ such that $\{\det(g_1), ..., \det(g_h)\}$ forms a set of representatives for $F^*\backslash \mathbb{A}^*_F/F^+_\infty \widehat{O_F}^*$. We put $t_i = \det(h_i) \in \widehat{F}^*$ for all $1 \leq i \leq h$, and we let $\mathcal{I}_i$ be the fractional ideal of F associated to t_i so that $\{\mathcal{I}_1, ..., \mathcal{I}_h\}$ forms a set of representatives for the narrow ideal class group of F.

The map $f \mapsto f_g$ for $g \in \mathrm{GL}_2(\widehat{F})$, which was introduced in Definition 2.1.5 induces the following isomorphisms:

$$M_k(\mathcal{N}) \simeq \bigoplus_{1 \leq i \leq h} \mathcal{M}_k(\Gamma_{g_i}) \tag{2.5}$$

$$S_k(\mathcal{N}) \simeq \bigoplus_{1 \leq i \leq h} \mathcal{S}_k(\Gamma_{g_i}) \tag{2.6}$$

$$f \mapsto (f_{g_1}, ..., f_{g_h}), \tag{2.7}$$

where $\Gamma_{g_i} = \mathrm{GL}^+_2(F) \cap g_i^{-1} K_0(\mathcal{N}) g_i G^+_\infty$. With the notation of 2.1, we have $\Gamma_{g_i} = \Gamma_0(\mathcal{I}_i, \mathcal{N})$.

In what follows, we define the Fourier coefficients of an adelic Hilbert modular form f of weight k and level $\mathcal{N}$. Let $(f_1, ..., f_h)$ be the h-tuple of classical Hilbert modular forms associated to f via the isomorphism 2.5 . For $1 \leq i \leq h$, we know that $f_i \in \mathcal{M}_k(\Gamma_0(\mathcal{I}_i, \mathcal{N}))$ for some fractional ideal $\mathcal{I}_i$ of F. Since $\Gamma_0(\mathcal{I}_i, \mathcal{N})$ contains all elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with

$b \in \mathcal{I}_i^{-1}$, f_i has a Fourier expansion

$$f_i(z) = a_i(0) + \sum_{\substack{\xi \in \mathcal{I}_i \mathcal{D}^{-1} \\ \xi \gg 0}} a_i(\xi) e_F(\xi z).$$

Notice that $a_i(\xi)\xi^{-\frac{k}{2}}$ for $\xi \neq 0$ depends only on the ideal ξO_F. This follows from the simple observation that $\Gamma_0(\mathcal{I}_i, \mathcal{N})$ contains all elements of the form $\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$ with $\epsilon \in O_F^*$ so that $a_i(\epsilon\xi) = \epsilon^{\frac{k}{2}} a_i(\xi)$.

Every non-zero integral ideal $\mathcal{M}$ of F can be written as $\mathcal{M} = \xi \mathcal{I}_i^{-1} \mathcal{D}$ for a unique $i \in \{1, ...h\}$ and a unique totally positive $\xi \in \mathcal{I}_i \mathcal{D}^{-1}$. Define :

$$c(\mathcal{M}, f) = \begin{cases} a_i(\xi)\xi^{-\frac{k}{2}} & \text{if } \mathcal{M} \text{ is integral and } \mathcal{M} = \xi \mathcal{I}_i^{-1} \mathcal{D} \\ 0 & \text{otherwise} \end{cases}$$

Then f has the following adelic Fourier expansion at infinity: for $y \in \mathbb{A}_F^*$, $y_\infty \gg 0$ and $x \in \mathbb{A}_F$, we have

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \xi \in F} c(\xi y O_F, f)(\xi y_\infty)^{\frac{k}{2}} e_F(\xi \mathbf{j} y_\infty) \chi_F(\xi x)$$
$$+ \begin{cases} c_0(y O_F)(\mathrm{N}(y O_F)^{-1} |y_\infty|)^{\frac{k_1}{2}} & \text{if } k_1 = ... = k_d \\ 0 & \text{otherwise,} \end{cases}$$

where χ_F is the standard additive character of $\mathbb{A}_F/F$ such that $\chi_F(x_\infty) = e_F(x_\infty)$ for $x_\infty \in F_\infty$, and c_0 is a function on the narrow ideal class group of F defined by

$$c_0(\eta \mathcal{I}_i^{-1}) = a_i(0) \mathrm{N}(\mathcal{I}_i)^{-\frac{k_1}{2}} \quad \text{for } 0 \ll \eta \in F.$$

One of the problems that is encountered when studying classical Hilbert modular forms is the absence of an action of Hecke operators under which the space of all such forms (of given weight and level) is stable. However, this difficulty is easily overcome by working with the larger space of adelic Hilbert modular forms which unlike its classical counterpart, is invariant under the action of Hecke operators defined naturally as follows. The space $M_k(\mathcal{N})$ admits a left action of the Hecke algebra of $K_0(\mathcal{N})$-biinvariant compactly supported functions on $\mathrm{GL}_2(\widehat{F})$. More precisely, for every $g \in \mathrm{GL}_2(\widehat{F})$, the Hecke operator corresponding to the characteristic function of $K_0(\mathcal{N}) g K_0(\mathcal{N})$ is denoted by $[K_0(\mathcal{N}) g K_0(\mathcal{N})]$ and maps f to $\sum_i f(\cdot g_i)$, where $K_0(\mathcal{N}) g K_0(\mathcal{N}) = \bigcup_j g_j K_0(\mathcal{N})$ is a disjoint union.

We now introduce the standard Hecke operators $T_{\mathcal{P}}$, $U_{\mathcal{P}}$ and $[\mathcal{P}]_k$. For a prime ideal $\mathcal{P}$ of F, we let $\pi_{\mathcal{P}}$ be a uniformizer of $F_{\mathcal{P}}$. We define $T_{\mathcal{P}}$ by putting

$$T_{\mathcal{P}} = [K_0(\mathcal{N}) \begin{pmatrix} \pi_{\mathcal{P}} & 0 \\ 0 & 1 \end{pmatrix} K_0(\mathcal{N})].$$

We write $U_{\mathcal{P}}$ for $T_{\mathcal{P}}$ if $\mathcal{P} \mid \mathcal{N}$. We also define the diamond operator $[\mathcal{P}]_k$ by putting

$$([\mathcal{P}]_k f)(x)) = \begin{cases} f(\pi_{\mathcal{P}} x) & \text{if } \mathcal{P} \nmid \mathcal{N} \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathrm{GL}_2(\widehat{F})$. This definition can be extended to integral ideals of F by multiplicativity. Notice that the action of the diamond operators on the space $M_{\mathbf{k}}(\mathcal{N})$ is trivial in our case because we limit our discussion to Hilbert Modular forms with the trivial character modulo $\mathcal{N}$. For a finite order Hecke character ψ of F whose conductor divides $\mathcal{N}$, the space of adelic Hilbert modular forms of weight $\mathbf{k}$, level $\mathcal{N}$ and character ψ is given by:

$$M_k(\mathcal{N}, \psi) = \{f \in M_{\mathbf{k}}(K_1(\mathcal{N})) : f(zg) = \psi(z) f(g) \text{ for all } z \in \mathbb{A}_F^*\},$$

where the open compact subgroup $K_1(\mathcal{N})$ is as defined in 2.4.

The subalgebra generated by the standard Hecke operators $T_{\mathcal{P}}$, $U_{\mathcal{P}}$ and $[\mathcal{P}]_k$ for all prime ideals $\mathcal{P}$ is commutative. It is often referred to as the standard Hecke algebra.

Let f and g be adelic Hilbert cusp forms in $S_{\mathbf{k}}(\mathcal{N})$, and let $(f_1, ..., f_h)$ and $(g_1, ..., g_h)$ be the associated h-tuples respectively. The Petersson inner product

$$(f, g) = \sum_{i=1}^{h} \langle f_i, g_i \rangle$$

endows $S_{\mathbf{k}}(\mathcal{N})$ with a structure of a Hermitian space with respect to which the operators $T_{\mathcal{P}}$ are normal.

The standard Hecke operators satisfy various identities which are all elegantly accounted for in the following formal Euler product:

$$\sum_{\mathcal{M}} T_{\mathcal{M}} \mathrm{N}(\mathcal{M})^{-s} = \prod_{\mathcal{P}} (1 - T_{\mathcal{P}} \mathrm{N}(\mathcal{P})^{-s} + [\mathcal{P}]_k \mathrm{N}(\mathcal{P})^{1-2s})^{-1},$$

where $\mathcal{M}$ runs over all integral ideals of F.

Following the notation of Shimura [21], we let k_0 be the largest of $k_1, ..., k_d$ and put

$$T'_{\mathcal{M}} = \mathrm{N}(\mathcal{M})^{\frac{k_0 - 2}{2}} T_{\mathcal{M}},$$

$$C(\mathcal{A}, f) = \mathrm{N}(\mathcal{A})^{\frac{k_0}{2}} c(\mathcal{A}, f).$$

Notice that if $\mathcal{A} = \xi \mathcal{I}_i^{-1} \mathcal{D}$ for some totally positive element $\xi \in F$, we get

$$C(\mathcal{A}, f) = \mathrm{N}(\mathcal{I}_i \mathcal{D}^{-1})^{-\frac{k_0}{2}} a_i(\xi) \xi^{\frac{\mathbf{k_0} - \mathbf{k}}{2}},$$

where $\mathbf{k_0}$ here is understood to be the d-tuple $(k_0, ..., k_0)$. Then the Fourier coefficients of $T_{\mathcal{M}} f$ satisfy the following identity:

$$C(\mathcal{A}, T'_{\mathcal{M}} f) = \sum_{\mathcal{A} + \mathcal{M} \subset \mathcal{C}} \mathrm{N}(\mathcal{C})^{k_0 - 1} C(\mathcal{C}^{-2} \mathcal{A} \mathcal{M}, f).$$

We also have

$$C(\mathcal{A}, U_{\mathcal{P}} f) = C(\mathcal{A}\mathcal{P}, f),$$

where $\mathcal{P}$ is a prime ideal dividing the level $\mathcal{N}$.

A nonzero Hilbert modular form $f \in M_{\mathbf{k}}(\mathcal{N})$ that is an eigenform for all standard Hecke operators is called an eigenform. If $T'_{\mathcal{M}} f = \lambda_{\mathcal{M}} f$, then $C(\mathcal{M}, f) = \lambda_{\mathcal{M}} C(O_F, f)$. An eigenform f is said to be normalized if $C(O_F, f) = 1$.

Proposition 2.1.6. *[21] The eigenvalues of $T_{\mathcal{M}}$ are algebraic numbers. Moreover, if $k_1 = ... = k_d$, the eigenvalues of $T'_{\mathcal{M}}$ are algebraic integers.*

A Hilbert modular form in $M_{\mathbf{k}}(\mathcal{N})$ is said to be primitive if it is orthogonal with respect to the Petersson inner product to all forms coming from lower levels. The interested reader may consult Miyake [16] for more details on the construction of the subspace of primitive forms. A normalized primitive eigenform in $M_{\mathbf{k}}(\mathcal{N})$ is called a newform. The Strong Multiplicity One Theorem for GL_2 ([5] Theorem 3.3.6) implies that a primitive form that is an eigenform for all Hecke operators $T_{\mathcal{P}}$ ($\mathcal{P} \nmid \mathcal{N}$) is necessarily an eigenform for the Hecke operators $U_{\mathcal{P}}$ ($\mathcal{P} | \mathcal{N}$) as well.

Proposition 2.1.7. *[21] Suppose that f is a newform of level $\mathcal{N}$ and weight $\mathbf{k}$ such that $k_1 \equiv ... \equiv k_d \mod 2$, and let $\mathbb{Q}(f)$ be the field generated over $\mathbb{Q}$ by $C(\mathcal{M}, f)$ for all $\mathcal{M}$. Then $\mathbb{Q}(f)$ is either totally real or a totally imaginary quadratic extension of a totally real algebraic number field.*

Another important result states that there is a bijection between the newforms f in $S_{\mathbf{k}}(\mathcal{N})$ and cuspidal automorphic representations π of GL_2, of conductor $\mathcal{N}$ and trivial central character, and such that π_∞ belongs to the holomorphic discrete series of weight $\mathbf{k}$.

Finally we mention that one can attach to $f \in M_{\mathbf{k}}(\mathcal{N})$ a Dirichlet series

$$L(f,s) = \sum_{\mathcal{M}} C(\mathcal{M}, f)\mathrm{N}(\mathcal{M})^{-s},$$

where $\mathcal{M}$ runs over all integral ideals of F. This converges for sufficiently large $\mathrm{Re}(s)$, and can be continued to a meromorphic function on the whole s-plane. It is entire if f is a cusp form.

2.2 Automorphic Forms on Totally Definite Quaternion Algebras

In this section we recall some basics from the theory of Quaternion Algebras which are used extensively in Chapters 3 and 4 of this dissertation. We then explore the connection between automorphic forms on totally definite quaternion algebra and adelic Hilbert modular forms; this connection is made possible by means of the Jacquet-Langlands correspondence.

2.2.1 Quaternion Algebras

The references for the material covered in this subsection are [1], [26] and [27].

Definition 2.2.1. *Let F be any field. A quaternion algebra B over F is a central simple F-algebra that has dimension 4 over F.*

Over a field F with characteristic different from 2, every quaternion algebra B has an F-basis $\{1, i, j, k\}$ satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$ and $ij = k$ for some $a, b \in F^*$. In this case we denote the quaternion algebra B by $\left(\frac{a,b}{F}\right)$. Every quaternion F-algebra $B = \left(\frac{a,b}{F}\right)$ admits a standard involution of the first kind $\omega \mapsto \overline{\omega}$ called conjugation. If $\omega = x + yi + zj + tij$, with $x, y, z, t \in F$, then $\overline{w} = x - yi - zj - tij$. The reduced trace and reduced norm are defined by $\mathrm{trd}(\omega) = \omega + \overline{\omega}$ and $\mathrm{nrd}(\omega) = \omega\overline{\omega}$, respectively.

A quaternion algebra over F is either a division F-algebra or a matrix F-algebra. If F is algebraically closed, then B is necessarily a matrix algebra. If F is a local field ($\neq \mathbb{C}$), there exists a unique division quaternion F-algebra up to isomorphism.

Let F be a number field and for each place v of F, let F_v be the corresponding local field.

Definition 2.2.2. *Let B be a quaternion algebra over F. For each place v of F, $B_v = F_v \otimes B$ is a quaternion algebra over F_v. If B_v is a division algebra, we say that B_v is ramified at v; otherwise, we say that B is split or unramified at v ($B_v \simeq M_2(F_v)$). A quaternion algebra that ramifies at all the infinite places of F is said to be totally definite.*

Note that if F is not a totally real number field, every quaternion algebra over F is not totally definite.

Definition 2.2.3. *The reduced discriminant D_B of a quaternion B over F is the square-free product of the prime ideals of O_F that ramify in B.*

It is well-known that two quaternion F-algebra are isomorphic if and only if they are ramified at the same places. The following theorem classifies quaternion algebras over F.

Theorem 2.2.4. *A quaternion algebra B over F is ramified at a finite even number of places. Moreover, given an even number of noncomplex places of F, there exists a quaternion algebra B over F that ramifies exactly at these places; B is unique up to isomorphism.*

Definition 2.2.5. *A field K containing F is splitting field for B if $K \otimes_F B$ is split, i.e. $K \otimes_F B \simeq M_2(K)$.*

Given a quaternion algebra B over F, there always exists a field K that splits B. In fact, if $\overline{F}$ is an algebraic closure of F, then $\overline{F}$ splits H. The following standard result provides necessary and sufficient conditions under which a quadratic field extension of F splits B.

Proposition 2.2.6. *Let B be a quaternion algebra over F and K a quadratic field over F. Then the following conditions are equivalent:*

(a) K splits B.

(b) There exists an F-embedding $K \hookrightarrow B$.

(c) Every place v in F that ramifies in B is not totally split in K.

(d) K is F-isomorphic to a maximal subfield of B containing F.

In what follows, we recall the theory of integral structures in quaternion algebras. Note that this discussion applies to quaternion algebras over any field F that arises as the field of fractions of a Dedekind ring. For our purposes F will always be a number field or the completion of a number field at a finite place, and B will always be a quaternion algebra over F.

Definition 2.2.7. *An element $\omega \in B$ is said to be integral over F if its reduced characteristic polynomial $x^2 - \mathrm{trd}(\omega)x + \mathrm{nrd}(\omega)$ has coefficients in O_F.*

This is consistent with the notion of integrality from commutative algebra. However, unlike the commutative case, the set of integral elements in a quaternion algebra need not form a ring.

Definition 2.2.8. *An O_F-lattice Λ of B is a finitely generated O_F-submodule contained in B. An O_F-ideal I of B is an O_F-lattice such that $F \otimes_{O_F} I \simeq B$. The reduced norm $\mathrm{nrd}(I)$ of an O_F-ideal I is the fractional ideal in F generated by the reduced norms of its elements.*

Definition 2.2.9. *An O_F-order $R \subset B$ is an O_F-ideal of B which is also a subring of B.*

Corollary 2.2.10. *If $R \subset B$ is an O_F-order, then every $\omega \in R$ is integral over F.*

To simplify notation, we refer to O_F-ideals and O_F-orders of B as ideals and orders of B. An important construction of orders comes from associating left and right orders with ideals. Namely, if $I \subset B$ is an ideal, we define the left and right orders to be the sets

$$R_l(I) = \{\omega \in B : \omega I \subset I\} \quad \text{and} \quad R_r(I) = \{\omega \in B : I\omega \subset I\},$$

respectively. These sets are orders but not necessarily equal. If $R_l(I) = R_r(I) = R$, we say that I is a two-sided or bilateral fractional R-ideal. In general, we say that I is a left fractional $R_l(I)$-ideal, and a right fractional $R_r(I)$-ideal. Moreover, the ideal I is integral if and only if it is contained in $R_l(I) \cap R_r(I)$. Equivalently, the ideal I is integral if and only if it is a left ideal of $R_l(I)$ and a right ideal of $R_r(I)$ in the usual sense.

Definition 2.2.11. *Let I be an ideal in B. We say I is left invertible if there exists a left fractional $R_r(I)$-ideal $I^{-1} \subset B$ such that $I^{-1}I = R_r(I)$.*

The notion of right invertibility is defined analogously. However, right and left invertibility are equivalent in the presence of a standard involution, and so we simply say I is invertible. Notice that if I is an invertible two-sided fractional R-ideal, then its right inverse is equal to its left inverse is equal to $\{\omega \in B : I\omega I \subset I\}$.

Proposition 2.2.12. *An ideal $I \subset B$ is invertible if and only if I is locally principal.*

Just as in the commutative setting, the discriminant of an order is realized as the reduced norm of the different ideal. More precisely, we have the following definition.

Definition 2.2.13. *The different $D(R)$ of an order $R \subset B$ is the bilateral integral ideal of R defined as $D(R) = (R^{\#})^{-1}$, where $R^{\#}$ is the dual of R with respect to the trace form:*

$$R^{\#} = \{\omega \in B : \mathrm{trd}(\omega R) \subset O_F\}.$$

The reduced discriminant of R, denoted by $\Delta(R)$, is the reduced norm of $D(R)$.

We now discuss ideal classes associated to quaternion orders. Let $R \subset B$ be an order, and let I and J be invertible right fractional R-ideals. We say that I and J are equivalent on the left if $I = bJ$ for some $b \in B^*$. The set $\mathrm{Cl}(R)$ of left-ideal classes of R is defined as the set of invertible right fractional R-ideals modulo equivalence on the left. One can define the set of right-ideal classes of R in a similar fashion. However, this doesn't yield anything new because conjugation $I \mapsto \overline{I} = \{\overline{\omega} : \omega \in I\}$ induces a bijection between the sets of right and left fractional R-ideals.

The following well-known result provides an adelic interpretation of ideal classes in a quaternion algebra, which is the non-commutative analogue of the class field theoretic realization of ideal classes in a number field. It follows almost immediately from Proposition 2.2.12 above.

Proposition 2.2.14. *The set $\mathrm{Cl}(R)$ is in bijection with $B^*\backslash\widehat{B}^*/\widehat{R}^*$, where $\widehat{B}$ is the ring of finite adeles of B and $\widehat{R} = R \otimes \widehat{\mathbb{Z}}$.*

Let S be a finite set of places of F containing the archimedean places and let $O_{F,S}$ be the ring of S-integers in F. Then $O_{F,S}$ is a Dedekind domain with field of fractions F. Let R be an $O_{F,S}$-order of B. In what follows we study the set $\mathrm{Cl}(R)$ by examining the effect of the reduced norm map on the double coset space $B^*\backslash\widehat{B}^*/\widehat{R}^*$. It is easy to verify that $\mathrm{nrd}(\widehat{B}^*) = \widehat{F}^*$. A less obvious result known as Eichler's theorem on norms asserts that $\mathrm{nrd}(B^*) = F^*_{(+)}$, where $F^*_{(+)}$ is the set of all $x \in F$ such that $v(x) > 0$ for all ramified real archimedean places v of F. Hence, the reduced norm map induces a surjective map:

$$c : B^*\backslash\widehat{B}^*/\widehat{R}^* \to F^*_{(+)}\backslash\widehat{F}^*/\mathrm{nrd}(\widehat{R}^*). \tag{2.8}$$

Theorem 2.2.15. *If S contains a place v such that B is unramfied at v, then the map 2.8 is a bijection for all $O_{F,S}$-orders $R \subset B$.*

This rather remarkable result follows immediately from Eichler's strong approximation theorem for quaternion algebras.

Theorem 2.2.16. *Let B_1^* and $\widehat{B_1}^*$ be the kernels of the reduced norm maps* $\mathrm{nrd} : B^* \to F_{(+)}^*$ *and* $\mathrm{nrd} : \widehat{B}^* \to \widehat{F}^*$ *respectively. If S contains a place v such that B is unramfied at v, then B_1^* is dense in $\widehat{B_1}^*$.*

For the rest of this subsection we study two types of quaternion orders which will be used extensively in the following chapters.

Definition 2.2.17. *Let B be quaternion algebra over a number field or the completion of a number field at a finite place. An order $R \subset B$ is maximal if it is not properly contained in another order. An order $R \subset B$ is Eichler if it can be written as the intersection of two maximal orders.*

Needless to say, the above definition is valid for any quaternion algebra over a field F which arises as the field of fractions of a Dedekind ring.

Let B be a quaternion algebra over a number field F. First we study Eichler orders in the local setting. To this end, we fix a finite place v of F and consider the quaternion algebra $B_v = F_v \otimes B$. Let π_v be a uniformizer of O_{F_v}, the ring of integers of F_v. Recall that B_v is either a division algebra or a matrix algebra. It suffices to study orders in these cases since one can easily check that the property of being an Eichler order is preserved by isomorphisms.

Proposition 2.2.18. *If B_v is a division algebra, then B_v contains a unique maximal order $R_v = \{\omega \in B_v : \mathrm{nrd}(\omega) \in O_{F_v}\}$. Hence, R_v is the unique Eichler order in B_v.*

Lemma 2.2.19. *If B_v is the matrix algebra $M_2(F_v)$, then the maximal orders in B_v are the $\mathrm{GL}_2(F_v)$-conjugate orders of $M_2(O_{F_v})$.*

Proposition 2.2.20. *Let $R_v \subset M_2(F_v)$ be an order. The following conditions are equivalent:*

(a) R_v is an Eichler order

(b) There exists a unique pair $\{R_1, R_2\}$ of maximal orders of $M_2(F_v)$ such that $R_v = R_1 \cap R_2$.

(c) There exists a unique non-negative integer n such that the order R_v is conjugate to the order $\begin{pmatrix} O_{F_v} & O_{F_v} \\ \pi_v^n O_{F_v} & O_{F_v} \end{pmatrix}$, which is known as the standard Eichler order of level $\pi_v^n O_{F_v}$.

The ideal $N_{R_v} = \pi_v^n O_{F_v}$, determined in statement (c), is called the level of the local Eichler order $R_v \subset M_2(F_v)$.

Definition 2.2.21. *Let $R_v \subset B_v$ be an Eichler order. The level of R_v is the ideal*

$$N_{R_v} = \begin{cases} O_{F_v} & \text{if } B_v \text{ is a division algebra,} \\ N_{\phi(R_v)} & \text{where } \phi : B_v \to M_2(F_v) \text{ is an isomorphism.} \end{cases}$$

Let us now study Eichler orders in the global setting. Recall that B is a quaternion algebra over a number field F with ring of integers O_F. For an order R of B and a place v of F, put $R_v = O_{F_v} \otimes R$. If v is a finite place, $R_v \subset B_v$ is a local O_{F_v}-order; if v is an infinite place, consider $O_{F_v} = F_v$ and $R_v = B_v$. One can easily check that $R = B \cap \prod_v R_v$.

Proposition 2.2.22. *Let R be an order of B. Then, R is a maximal (resp. Eichler) order if and only if R_v is a maximal (resp. Eichler) order for every finite place v of F.*

Hence, we arrive at the following definition of the level of a global Eichler order.

Definition 2.2.23. *The level N_R of a global Eichler order R is the unique integral ideal $\mathcal{N} \subset O_F$ such that $\mathcal{N}_v$ is the level of each local order R_v at each finite place v of F. Thus, $\mathcal{N} = \prod_v N_{R_v}$.*

Note that maximal orders are characterized by their discriminants. In fact, an order $R \subset B$ is maximal if and only if $\Delta(R) = D_B$. This is not the case for Eichler orders. Nevertheless, we have the following result.

Proposition 2.2.24. *If $R \subset B$ is an Eichler order, then N_R is coprime to D_B and $\Delta(R) = N_R D_B$.*

2.2.2 Jacquet Langlands Correspondence

We follow the exposition given in Cornut-Vatsal [7] Section 5. Let B be a totally definite quaternion algebra over a number field F, and let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ be the algebraic group over $\mathbb{Q}$ whose set of points on a commutative $\mathbb{Q}$-algebra A is given by $G(A) = (B \otimes A)^*$. Notice that G is a reductive group with center $Z = \mathrm{Res}_{F/\mathbb{Q}}(F^*)$, and the reduced norm map $\mathrm{nrd} : B \to F$ induces a morphism $\mathrm{nrd} : G \to Z$.

Definition 2.2.25. *An automorphic form of weight 2 on G is a smooth (=locally constant) function $\theta : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to \mathbb{C}$.*

Adapting the notation from [7], we denote the space of all automorphic forms of weight 2 on G by S_2. We let $G(\mathbb{A}_f)$ act on S_2 by right translation:

$$(g.\theta)(x) = \theta(xg), \quad g \in G(\mathbb{A}_f) \text{ and } x \in G(\mathbb{Q})\backslash G(\mathbb{A}_f).$$

This admissible left action gives rise to a semi-simple representation of $G(\mathbb{A}_f)$ known as the right regular representation, and the space S_2 is the algebraic direct sum of its irreducible subrepresentations.

Definition 2.2.26. *An irreducible representation π' of $G(\mathbb{A}_f)$ is automorphic if it occurs in S_2. We denote by $S_2(\pi')$ the corresponding subspace of S_2, so that $S_2 \simeq \oplus_{\pi'} S_2(\pi')$.*

An automorphic representation π' of $G(\mathbb{A}_f)$ is either of dimension one or infinite dimension. If π' is 1-dimesional, it corresponds to a smooth character χ of $G(\mathbb{A}_f)$ which is trivial on $G(\mathbb{Q})$ and factors through nrd : $G(\mathbb{A}_f) \to Z(\mathbb{A}_f)$. A function $\theta \in S_2$ is said to be Eisenstein if it belongs to the subspace spanned by these finite dimensional subrepresentations of S_2. Equivalently, θ is Eisenstein if and only if it factors through the reduced norm. If π' is infinite dimensional, we say that π' is cuspidal. A function $\theta \in S_2$ is said to be a cusp form if it belongs to the $G(\mathbb{A}_f)$-invariant subspace of S_2 spanned by its irreducible cuspidal subrepresentations.

We say that a function $\theta \in S_2$ has level H if it is fixed by some open compact subgroup H of $G(\mathbb{A}_f)$. Hence, the subspace S_2^H of all such functions may be identified with the finite dimensional space of all complex-valued functions on the finite double quotient

$$M_H = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/H.$$

For each place v of F at which H is maximal and B is unramified, there is a natural Hecke action on S_2^H given by:

$$T_v\theta(x) = \sum_{i \in I_v} \theta(x_{v,i}), \quad x = [g] \in M_H \text{ and } x_{v,i} = [g\eta_{v,i}] \in M_H.$$

Here, $H_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} H_v = \coprod_{i \in I_v} \eta_{v,i} H_v$ with ϖ_v a local uniformizer in F_v.

Let f be a cuspidal Hilbert newform of parallel weight 2 and level $\mathcal{N}$ such that the automorphic representation of GL_2 associated with f is square-integrable (modulo the center) at the places of F where B is ramified. The Jacquet-Langlands correspondence which will be stated shortly implies that there exists a function $\theta \in S_2^H$, for some open compact subgroup $H \subset G(\mathbb{A}_f)$,

such that θ is an eigenfunction for all Hecke operators T_v with the same eigenvalues as f. Here we will only mention that the subgroup H corresponds to an order $R \subset B$ with $\Delta(R) = \mathcal{N}$.

Theorem 2.2.27. *There is a natural bijection between the set of infinite dimensional automorphic representations π' of G and the the set of cuspidal representations π of GL_2 which satisfy π_v is square-integrable (modulo the center) for all places v of F at which B is ramified.*

2.3 Rankin-Selberg L-functions

In this section we recall some of the basic properties of Rankin-Selberg L-functions with emphasis on the special type of such L- functions which we study in Chapter 3. We follow the exposition of Cornut-Vatsal in [7] Section 1.

Let F be a number field and $\mathbb{A}_F$ its adele ring. Let π_1 and π_2 be automorphic cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central characters and conductors $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. The Rankin-Selberg L-function attached to the pair $\{\pi_1, \pi_2\}$ is first defined as a product of Euler factors over all places of F:

$$L(\pi_1, \pi_2, s) = \prod_v L_v(\pi_{1,v}, \pi_{2,v}, s),$$

where the factors are of degree smaller than or equal to four.

For every finite place v of F not dividing $\mathcal{N}_1\mathcal{N}_2$, let $\alpha_1(v)$ and $\alpha_2(v)$ be the Satake parameters of $\pi_{1,v}$, and let $\beta_{1,v}$ and $\beta_{2,v}$ be the Satake parameters of $\pi_{2,v}$. Then

$$L_v(\pi_{1,v}, \pi_{2,v}, s) = \prod_{i=1}^{2}\prod_{j=1}^{2}(1 - \alpha_{i,v}\beta_{j,v}q_v^{-s})^{-1},$$

where q_v is the cardinality of the residue field at v. There is a similar but more complicated definition of the Euler factors at the remaining places of F. The reader is referred to [12] and [11] for an elaborate discussion on this topic.

Let π be an automorphic cuspidal representation of GL_2 over a totally real number field F, and let K be a totally imaginary quadratic extension of F. Given a quasi-character χ of $\mathbb{A}_K^*/K^*$, we denote by $L(\pi, \chi, s)$ the Rankin-Selberg L-function attached to the pair $\{\pi, \pi(\chi)\}$, where $\pi(\chi)$ is the automorphic representation of GL_2 associated with χ. It is well-known that

$L(\pi, \chi, s)$ has a meromorphic extension to $\mathbb{C}$ with functional equation

$$L(\pi, \chi, s) = \epsilon(\pi, \chi, s) L(\tilde{\pi}, \chi^{-1}, 1-s),$$

where $\tilde{\pi}$ is the contragredient of π and $\epsilon(\pi, \chi, s)$ is a certain ϵ-factor. If w is the central character of π and $\chi \cdot w = 1$ on $\mathbb{A}_F^* \subset \mathbb{A}_K^*$, then $L(\pi, \chi, s)$ is entire and $L(\pi, \chi, s) = L(\tilde{\pi}, \chi^{-1}, s)$. Hence, the functional equation becomes

$$L(\pi, \chi, s) = \epsilon(\pi, \chi, s) L(\pi, \chi, 1-s),$$

and the parity of the order of vanishing of $L(\pi, \chi, s)$ at $s = \frac{1}{2}$ is determined by the value of

$$\epsilon(\pi, \chi) = \epsilon(\pi, \chi, \frac{1}{2}) \in \{\pm 1\}.$$

We say that the pair (π, χ) is even if $\epsilon(\pi, \chi) = 1$. Otherwise, we say that pair (π, χ) is odd.

Definition 2.3.1. *A character χ is said to be a ring class character if there exists some O_F-ideal $\mathcal{C}$ such that χ factors through the finite group*

$$\mathbb{A}_K^* / (K^* K_\infty^* \widehat{O}_{\mathcal{C}}^*),$$

where $K_\infty = K \otimes \mathbb{R}$ and $O_{\mathcal{C}} = O_F + \mathcal{C} O_K$. The conductor $c(\chi)$ of χ is the largest such ideal $\mathcal{C}$.

For our purposes, we view the automorphic cuspidal representation π as being fixed of parallel weight $(2, 2, .., 2)$ and level $\mathcal{N}$. We let χ vary through the collection of ring class characters of $\mathcal{P}$-power conductor, where $\mathcal{P}$ is a fixed maximal ideal in O_F. We also assume that the prime to $\mathcal{P}$ part $\mathcal{N}'$ of $\mathcal{N}$ is relatively prime to the discriminant $\mathcal{D}$ of K/F. Under these conditions, Cornut and Vatsal obtained the following result in [7]:

Proposition 2.3.2. *For all but finitely many ring class characters of P-power conductor, we have $\epsilon(\pi, \chi) = (-1)^{|S|}$, where S is the union of all archimedean places of F together with the those finite places of F which do not divide P, are inert in K and divide $\mathcal{N}$ to an odd power. In other words, we have $\epsilon(\pi, \chi) = (-1)^{[F:\mathbb{Q}]} \eta(\mathcal{N}')$, where η is the quadratic Hecke character of F attached to K/F.*

For the rest of the current section, we assume that $(-1)^{[F:\mathbb{Q}]} \eta(\mathcal{N}') = 1$ in which case we say that the tuple (π, K, P) is definite. Let B be the quaternion algebra over F such that $Ram(B) = S$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ be the algebraic group over $\mathbb{Q}$ associated to B^*, and $Z = \mathrm{Res}_{F/\mathbb{Q}}(F^*)$ be

its center. Since every place in F that ramifies in B is inert in K, there exists an F-embedding $K \hookrightarrow B$. After fixing such an embedding, the group $T = \mathrm{Res}_{F/\mathbb{Q}}(K^*)$ can be viewed as a maximal sub-torus of G defined over $\mathbb{Q}$. Let π' be the automorphic representation of G that corresponds to $\pi = JL(\pi')$ via the Jacquet-Langlands correspondence.

In 1985, Waldspurger proved a fundamental theorem (Théorème 2 in [31]) yielding a criterion for the non-vanishing of $L(\pi, \chi, \frac{1}{2})$ in a very general setting. Consider the linear form $l_\chi : S_2(\pi') \to \mathbb{C}$ defined by the period integral:

$$l_\chi(\phi) = \int_{Z(\mathbb{A})T(\mathbb{Q})\backslash T(\mathbb{A})} \chi(t)\phi(t)\, dt$$

Theorem 2.3.3. *For a ring class character χ of P-power conductor that satisfies $\chi.w = 1$ on $\mathbb{A}_F^*$, we have*

$$L(\pi, \chi, \frac{1}{2}) \neq 0 \;\Leftrightarrow\; \exists \phi \in \; S_2(\pi') : l_\chi(\phi) \neq 0$$

However, the result of Waldspurger doesn't give a precise formula for the special value $L(\pi, \chi, \frac{1}{2})$. Several authors have subsequently taken up the task of specifying a test vector ϕ on which to evaluate the linear functional l_χ and finding a Gross-Zagier formula for $L(\pi, \chi, \frac{1}{2})$ in terms of $l_\chi(\phi)$. In [32], Zhang has obtained a formula of this type under the assumption that the central character of π is trivial and that $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{D}$ are pairwise co-prime. Another significant improvement in this direction has been made in [15] by Martin and Whitehouse who established a Gross-Zagier formula under the only assumption that $\mathcal{P}$ does not divide $\mathcal{N}$.

Chapter 3

Special Values of Anticyclotomic L-functions modulo λ

In this chapter we generalize some results of Vatsal on studying the special values of Rankin-Selberg L-functions in an anticyclotomic $\mathbb{Z}_p$-extension. Let g be a cuspidal Hilbert modular form of parallel weight $(2, ..., 2)$ and level $\mathcal{N}$ over a totally real field F, and let K/F be a totally imaginary quadratic extension of relative discriminant $\mathcal{D}$. We study the l-adic valuation of the special values $L(g, \chi, \frac{1}{2})$ as χ varies over the ring class characters of K of $\mathcal{P}$-power conductor, for some fixed prime ideal $\mathcal{P}$. We prove our results under the only assumption that the prime to $\mathcal{P}$ part of $\mathcal{N}$ is relatively prime to $\mathcal{D}$.

3.1 Preliminaries and Notations

Let us first fix some notation. We write $\mathbb{A}$ (resp. $\mathbb{A}_f$) for the ring of adeles (resp. finite adeles) of $\mathbb{Q}$. Let F be a totally real number field, and let K be a totally imaginary quadratic extension of F the discriminant of which we denote by D. The ring of adeles of F is $\mathbb{A}_F = \mathbb{A} \otimes_{\mathbb{Q}} F$, and the ring of finite adeles of F is $\widehat{F} = \mathbb{A}_f \otimes_{\mathbb{Q}} F$. Similarly, we write $\mathbb{A}_K$ (resp. $\widehat{K}$) for the ring of adeles (resp. finite adeles) of K.

We consider a cuspidal Hilbert modular newform g of level $\mathcal{N}$, trivial character and parallel weight $(2, ..., 2)$. We denote by a_v the Hecke eigenvalues of g ($T_v g = a_v g$) and by π the automorphic irreducible representation of GL_2 over F corresponding to g. Let $\mathcal{P}$ be a prime ideal of F such that $\mathcal{P}$ lies over an odd rational prime p, and let $\pi_{\mathcal{P}}$ be a uniformizer of $F_{\mathcal{P}}$. By abuse of notation, we also denote the maximal ideal in $O_{F,\mathcal{P}}$ by $\mathcal{P}$. Let χ be a finite-order Hecke character of K of conductor $\mathcal{P}^n$.

The data of the previous paragraph is to remain fixed and the following hypotheses are assumed throughout this chapter:

1. The representations π and $\pi \otimes \eta$ are distinct, where η is the quadratic

character associated to the extension K/F. We say that the pair (π, K) is non-exceptional.

2. The prime to $\mathcal{P}$ part $\mathcal{N}'$ of $\mathcal{N}$ is relatively prime to the discriminant D of K/F

3. The character χ is a ring class character of $\mathcal{P}$-power conductor, and it is trivial when restricted to $\mathbb{A}_F^* \subset \mathbb{A}_K^*$.

4. Let S be the set of all the Archimedean places of F, together with those finite places of F which do not divide $\mathcal{P}$, are inert in K, and divide $\mathcal{N}$ to an odd power. We require S to have an even cardinality.

It follows from the last condition that the sign in the functional equation of $L(\pi, \chi, s)$ is $+1$ for all but finitely many characters χ that satisfy condition (3) (see Lemma 1.1 in [7]).

Let l be any rational prime. Fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_l}$, and denote by E the subalgebra of $\overline{\mathbb{Q}_l}$ generated by the images of the Hecke eigenvalues of g. Write E_l for the integral closure of E in its field of fractions and λ for the maximal ideal in E_l.

3.2 CM Points and Galois Action

Let B be the totally definite quaternion algebra over F such that $\mathrm{Ram}(B) = S$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$ be the algebraic group over $\mathbb{Q}$ associated to B^*. Thus, the center of G is $Z = \mathrm{Res}_{F/\mathbb{Q}}(F^*)$. Since every place in F that ramifies in B is inert in K, there exists an F-embedding $K \hookrightarrow B$. After fixing such an embedding, the group $T = \mathrm{Res}_{F/\mathbb{Q}}(K^*)$ can be viewed as a maximal sub-torus of G defined over $\mathbb{Q}$.

In what follows, we sketch the construction of an O_F-order R of reduced discriminant $\mathcal{N}$ in B following [7] and [33]. Let $\mathcal{N}'$ be the prime to $\mathcal{P}$ part of $\mathcal{N}$, and write $\mathcal{N} = \mathcal{P}^\delta \mathcal{N}'$. Let R_0 be an Eichler order of level $\mathcal{P}^\delta$ in B. We choose R_0 such that the O_F-order $O = O_K \cap R_0$ has a $\mathcal{P}$-power conductor. For example, if $\mathcal{P}$ does not divide $\mathcal{N}$, we require that R_0 optimally contains O_K. Denote by $\mathcal{N}_B$ the discriminant of B/F, and let $\mathcal{M}_K$ be an ideal in O_K which has relative norm $\mathcal{N}'/\mathcal{N}_B$. We may find such an ideal $\mathcal{M}_K$ as follows. For each prime $\mathfrak{P}$ dividing $\mathcal{N}'$, let $\mathfrak{P}_K$ be a prime of O_K dividing $\mathfrak{P}$. We put

$$\mathcal{M}_K = \prod_{\mathfrak{P}|\mathcal{N}_B} \mathfrak{P}_K^{[ord_{\mathfrak{P}}(\mathcal{N})/2]} \cdot \prod_{\mathfrak{P}|\frac{\mathcal{N}'}{\mathcal{N}_B}} \mathfrak{P}_K^{ord_{\mathfrak{P}}(\mathcal{N})}.$$

Finally, we obtain R by the following formula:

$$R = O + (O \cap \mathcal{M}_K).R_0.$$

In particular, $R_{\mathcal{P}} = R_{0,\mathcal{P}}$ is an Eichler order of level $\mathcal{P}^\delta$. Define an open compact subgroup H of $G(\mathbb{A}_f)$ by $H = \widehat{R}^*$. The subgroup H is sometimes referred to as the level structure. This gives rise to the finite sets

$$M_H = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/H,$$

and

$$N_H = Z(\mathbb{Q})^+\backslash Z(\mathbb{A}_f)/\mathrm{nrd}(H).$$

It also gives rise to the set of CM points

$$CM_H = T(\mathbb{Q})\backslash G(\mathbb{A}_f)/H.$$

Notice that any function on M_H induces a function on CM_H via the obvious reduction map $\mathrm{red} : CM_H \to M_H$.

The action of $T(\mathbb{A}_f)$ on CM_H by left multiplication in $G(\mathbb{A}_f)$ factors through the reciprocity map $\mathrm{rec}_K : T(\mathbb{A}_f) \to \mathrm{Gal}_K^{\mathrm{ab}}$. This induces an action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on CM_H. Hence, for $x = [g] \in CM_H$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$, we have $\sigma.x = [\beta g]$ where $\beta \in T(\mathbb{A}_f)$ is such that $\mathrm{rec}_K(\beta) = \sigma$.

Moreover, the reduced norm map on $G(\mathbb{A}_f)$ induces the map $\mathrm{c} : M_H \to N_H$. Hence, the action of $\mathrm{Gal}_F^{\mathrm{ab}}$ on N_H induces an action of $\mathrm{Gal}_K^{\mathrm{ab}}$ on N_H. For $x = [z] \in N_H$ and $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$, we have $\sigma.x = [\mathrm{nrd}(\beta)z]$ where $\beta \in T(\mathbb{A}_f)$ is such that $\mathrm{rec}_K(\beta) = \sigma$.

We now introduce the notion of a CM point with a $\mathcal{P}$-power conductor.

Definition 3.2.1. *We say that $x = [g]$ is a CM point of conductor $\mathcal{P}^n$ and write $x \in CM_H(\mathcal{P}^n)$ if $T(\mathbb{A}_f) \cap gHg^{-1} = \widehat{O_{\mathcal{P}^n}}^*$, where $O_{\mathcal{P}^n} \subset O_K$ is the O_F-order of conductor $\mathcal{P}^n$.*

Choose $\alpha_{\mathcal{P}} \in O_{K,\mathcal{P}}$ such that $\{1, \alpha_{\mathcal{P}}\}$ is an $O_{F,\mathcal{P}}$-basis of $O_{K,\mathcal{P}}$. Since $O_{\mathcal{P}^n,\mathcal{P}} = O_{F,\mathcal{P}} + \mathcal{P}^n O_{K,\mathcal{P}}$, the set $\{1, \pi_{\mathcal{P}}^n \alpha_{\mathcal{P}}\}$ is an $O_{F,\mathcal{P}}$-basis of $O_{\mathcal{P}^n,\mathcal{P}}$. We fix the embedding $K_{\mathcal{P}} \hookrightarrow M_2(F_{\mathcal{P}})$ defined by

$$a + b\alpha_{\mathcal{P}} \mapsto \begin{pmatrix} a + b\mathrm{Tr}\alpha_{\mathcal{P}} & b\mathrm{N}\alpha_{\mathcal{P}} \\ -b & a \end{pmatrix},$$

where Tr and N denote the trace and norm maps.

Lemma 3.2.2. *Consider $g_{\mathcal{P}} \in B_{\mathcal{P}} \simeq M_2(F_{\mathcal{P}})$ specified as:*

$$g_{\mathcal{P}} = \begin{pmatrix} \pi_{\mathcal{P}}^n \mathrm{N}\alpha_{\mathcal{P}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for n large enough, the order $g_{\mathcal{P}} R_{\mathcal{P}} g_{\mathcal{P}}^{-1}$ in $B_{\mathcal{P}}$ optimally contains the $O_{F,\mathcal{P}}$-order in $K_{\mathcal{P}}$ of conductor $\mathcal{P}^n$.

Proof. Let $\tau = a + b\alpha_{\mathcal{P}}$ be any element in $K_{\mathcal{P}}$. We have

$$g_{\mathcal{P}}^{-1} \tau g_{\mathcal{P}} = \begin{pmatrix} a + b\mathrm{Tr}\alpha_{\mathcal{P}} & b\pi_{\mathcal{P}}^{-n} \\ -b\pi_{\mathcal{P}}^n \mathrm{N}\alpha_{\mathcal{P}} & a \end{pmatrix}.$$

Recall that, by construction, $R_{\mathcal{P}}$ is an Eichler order (of level $\mathcal{P}^\delta$) in $B_{\mathcal{P}} \simeq M_2(F_{\mathcal{P}})$. Without loss of generality, we identify $R_{\mathcal{P}}$ with the order

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O_{F,\mathcal{P}}) : c \equiv 0 \mod \pi_{\mathcal{P}}^\delta \right\}.$$

Then, for all $n \gg 0$, we have $g_{\mathcal{P}}^{-1} \tau g_{\mathcal{P}} \in R_P$ if and only if $\tau \in O_{\mathcal{P}^n,\mathcal{P}}$. In other words, the order $g_{\mathcal{P}} R_{\mathcal{P}} g_{\mathcal{P}}^{-1}$ in $B_{\mathcal{P}}$ optimally contains the $O_{F,\mathcal{P}}$-order in $K_{\mathcal{P}}$ of conductor $\mathcal{P}^n$. □

In the sequel, we shall fix a choice of CM point $x = [g] \in CM_H(\mathcal{P}^n)$ such that its $\mathcal{P}$-*th* component is as specified in the previous lemma.

For later reference, notice that $R_{\mathcal{P}}^*$ is isomorphic to the Iwahori subgroup

$$U_0(O_{F,\mathcal{P}}, \mathcal{P}^\delta) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_{F,\mathcal{P}}) : c \equiv 0 \mod \pi_{\mathcal{P}}^\delta \right\}.$$

3.3 Uniform Distribution of CM Points

The CM points are uniformly distributed on the components of the Shimura curve associated to B. This was the most crucial idea behind Vatsal's proof of Mazur's conjecture for weight two modular forms over $\mathbb{Q}$. In this section, we recall a crucial result on the uniform distribution of CM points due to Cornut-Vatsal, which we use to prove our main theorem. To describe this result, we need to introduce some more notation.

Let $K[\mathcal{P}^n]$ be the ring class field over K of conductor $\mathcal{P}^n$. In other words, $K[\mathcal{P}^n]$ is the abelian extension of K associated by class field theory to the subgroup $K^* K_\infty^* \widehat{O}_{\mathcal{P}^n}^*$ of $\mathbb{A}_K^*$. Let $G(n)$ denote the Galois group of this extension. We have

$$G(n) = \mathrm{Gal}(K[\mathcal{P}^n]/K) \simeq \mathbb{A}_K^* / (K^* K_\infty^* \widehat{O}_{\mathcal{P}^n}^*)$$

via the reciprocity map of K.

Set $K[\mathcal{P}^\infty] = \cup_{n\geq 0} K[\mathcal{P}^n]$, so that $G(\infty) = \mathrm{Gal}(K[\mathcal{P}^\infty]/K)$. The torsion subgroup of $G(\infty)$ is denoted by G_0. It is finite and $G(\infty)/G_0$ is a free $\mathbb{Z}_p$-module of rank $[F_{\mathcal{P}} : \mathbb{Q}_p]$. The reciprocity map of K maps $\mathbb{A}_F^* \subset \mathbb{A}_K^*$ onto the subgroup $G_2 \simeq Pic(O_F)$ of G_0.

Let $G(\infty)'$ be the subgroup of $G(\infty)$ generated by the Frobeniuses of the primes of K which are not above $\mathcal{P}$. Write $G_1 = G_0 \cap G(\infty)'$. Let $\mathcal{D}'$ be the square-free product of the primes $\mathcal{Q} \neq \mathcal{P}$ of F which ramify in K. Then G_1/G_2 is an $\mathbb{F}_2$-vector space with basis

$$\{\sigma_{\mathcal{Q}} \mod G_2 : \mathcal{Q}|\mathcal{D}'\},$$

where $\sigma_{\mathcal{Q}} = \mathrm{Frob}_{\mathfrak{Q}}$ and $\mathfrak{Q}$ is the prime of K above $\mathcal{Q}$.

Loosely speaking, the uniform distribution result in [24] states the following. Let p_1 and p_2 be arbitrary double cosets in M_H, and let σ be an arbitrary nontrivial element of G_0 with $\sigma \notin G_1$. Then there exists a CM point $x \in CM(\mathcal{P}^n)$ such that $\mathrm{red}(x) = p_1$ and $\mathrm{red}(\sigma.x) = p_2$ whenever n is sufficiently large.

In what follows, we describe the result of Cornut and Vatsal which extends and refines Vatsal's theorem alluded to in the previous paragraph.

Let $\mathcal{R}$ be a set of representatives for G_0/G_1 containing 1. We have the following maps:

$$\mathrm{RED} : CM_H(\mathcal{P}^\infty) \to M_H^{\mathcal{R}}, \quad x \mapsto (\mathrm{red}(\tau.x))_{\tau\in\mathcal{R}}$$

$$\mathrm{C} : M_H^{\mathcal{R}} \to N_H^{\mathcal{R}}, \qquad (a_\tau)_{\tau\in\mathcal{R}} \mapsto (c(a_\tau))_{\tau\in\mathcal{R}}$$

and the composite map

$$\mathrm{C} \circ \mathrm{RED} : CM_H(\mathcal{P}^\infty) \to N_H^{\mathcal{R}},$$

which is $G(\infty)$-equivariant.

The following is the key theorem of Cornut-Vatsal as stated in [7]. However, the reader is referred to [6] for a proof of this result.

Theorem 3.3.1 (See [7])**.** *For all but finitely many $x \in CM_H(\mathcal{P}^\infty)$,*

$$\mathrm{RED}(G(\infty).x) = \mathrm{C}^{-1}(G(\infty).C \circ \mathrm{RED}(x))$$

3.4 Toward Computing $ord_\lambda(L^{\mathrm{al}}(\pi,\chi,\frac{1}{2}))$

Let π' be the unique cuspidal automorphic representation on B that is associated to π by the Jacquet-Langlands correspondence ($\pi = \mathrm{JL}(\pi')$). We

associate to g a unique function $\theta \in S_2(\pi')$, where $S_2(\pi')$ is the representation space of π'. One can view θ as

$$\theta : M_H \to E_g,$$

where E_g is the Hecke field of g. This yields the function $\psi = \theta \circ \mathrm{red}$ on CM_H. The space of functions on M_H is endowed by an action of Hecke operators T_v. This action agrees with the classical Hecke action on the space of Hilbert modular forms in the sense that θ has the same eigenvalues as g for all T_v, $v \nmid \mathcal{N}$. In particular, we may view θ as taking values in E_l:

$$\theta : M_H \to E_l.$$

Without loss of generality, we may also assume that $\theta([g])$ is a λ-adic unit for some $[g] \in M_H$.

There are several results in the literature which relate the special value $L(\pi,\chi,\frac{1}{2})$ to $|\mathrm{a}(x,\chi)|^2$, for some CM point $x \in CM_H(\mathcal{P}^n)$, with

$$\mathrm{a}(x,\chi) = \frac{1}{|G(n)|}\sum_{\sigma\in G(n)} \chi(\sigma)\psi(\sigma.x).$$

In 1985, Waldspurger proved a fundamental theorem (Théorème 2 in [31]) yielding a criterion for the non-vanishing of $L(\pi,\chi,\frac{1}{2})$ in a very general setting. Roughly speaking, the result of Waldspurger states that, under very mild conditions on π and χ, $L(\pi,\chi,\frac{1}{2}) \neq 0$ if and only if $|\mathrm{a}(x,\chi)|^2 \neq 0$. However, this result doesn't give a precise formula for the special value $L(\pi,\chi,\frac{1}{2})$ in terms of $|\mathrm{a}(x,\chi)|^2$. Most authors refer to such a formula as a Gross-Zagier formula, and it is expected (but not known yet) that it exists in full generality. Nevertheless, Zhang has proven a formula of this type in [32] under the assumption that the central character of π is trivial and that $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{D}$ are pairwise co-prime. Another significant improvement in this direction has been made in [15] by Martin and Whitehouse who established a Gross-Zagier formula under the only assumption that $\mathcal{P}$ does not divide $\mathcal{N}$.

Results as such were the point of departure in the work of Vatsal [24] and Cornut-Vatsal [7] on the non-vanishing of $L(\pi,\chi,\frac{1}{2})$. However, in order to study this special value modulo a given prime $\lambda \in \bar{\mathbb{Q}}$, Vatsal employed in [25] a construction of Hida to define a canonical period Ω_g^{can} such that

$$L^{\mathrm{al}}(\pi,\chi,\frac{1}{2}) = \frac{L(\pi,\chi,\frac{1}{2})}{\Omega_g^{\mathrm{can}}}$$

is a λ-adic integer. Then Vatsal incorporated this period into the Gross-Zagier formula obtained by Zhang to deduce in one simple step an exact statement about the λ-adic valuation of $L(\pi,\chi,\frac{1}{2})$ from that of $\mathrm{a}(x,\chi)$. Unfortunately, such a construction is not yet known in the degree of generality required in this work. Hence, we will only concern ourselves with studying the value of $\mathrm{a}(x,\chi)$ modulo a given prime ideal $\lambda \subset \bar{\mathbb{Q}}$ as χ varies over the ring class characters of K of $\mathcal{P}$-power conductor.

Adapting the notation from [7], we identify ring class characters of $\mathcal{P}$-power conductor with finite-order characters of $G(\infty)$. Hence, given a character χ_0 of G_0 such that $\chi_0 = 1$ on G_2, denote by $P(n,\chi_0)$ the set of primitive characters of $G(n)$ (do not factor through $G(n-1)$) and induce χ_0 on G_0. In [24] and [7], the authors proved that for each character χ_0 of G_0 and all but finitely many n, there exists a character $\chi \in P(n,\chi_0)$ such that $L(\pi,\chi,\frac{1}{2}) \neq 0$. Moreover, Vatsal showed in [24] that if χ_0 has order prime to p and the Hecke field of g is linearly disjoint from the field generated over $\mathbb{Q}$ by the *p-th* roots of unity, then $L(\pi,\chi,\frac{1}{2}) \neq 0$ for all $\chi \in P(n,\chi_0)$ with n sufficiently large. We remark that this statement differs slightly from the statement given in [24] since the condition on the Hecke field of g was overlooked there. In [25], Vatsal extended the results and methods of [24] to study the algebraic part of the special value $L(\pi,\chi,\frac{1}{2})$ modulo a given prime $\lambda \in \bar{\mathbb{Q}}$ of characteristic l. Vatsal proved that for all $n \gg 0$, there exists $\chi \in P(n,\chi_0)$ such that

$$ord_\lambda\left(\frac{(L(\pi,\chi,\frac{1}{2})}{\Omega_g^{\mathrm{can}}\mathrm{C}_{\mathrm{csp}}}\right) < ord_\lambda(\mathrm{C}_{\mathrm{Eis}}^2), \tag{3.1}$$

where $\mathrm{C}_{\mathrm{Eis}}$ is a constant that measures the congruence between g and the space of Eisenstein Series, and $\mathrm{C}_{\mathrm{csp}}$ is a constant that measures the congruence between g and some cusp forms of lower levels. Moreover, if χ_0 has order prime to p, the Hecke field of g is linearly disjoint from the field generated over $\mathbb{Q}$ by the *p-th* roots of unity, and l satisfies certain conditions (see next paragraph), then (3.1) is true for all $\chi \in P(n,\chi_0)$ with n sufficiently large. We remark that (3.1) is mistakenly given as an equality in [25], the source of the mistake being an error made in the proof of Proposition 5.3 part (2). We provide a correct version of this result in Proposition 3.4.12 below. We mention here that establishing the correct statement involves modifying the choice of the constant $\mathrm{C}_{\mathrm{Eis}}$. In fact, the statement becomes correct if we take $\mathrm{C}_{\mathrm{Eis}} = \tilde{\lambda}^\mu$ where $\tilde{\lambda}$ is a λ-adic uniformizer in E_l and μ is the constant given in Definition 3.4.8 below.

Given $x = [g] \in CM_H(\mathcal{P}^n)$ and a ring class character χ of conductor $\mathcal{P}^n$,

we define the Gross-Zagier sum

$$\mathrm{a}(x,\chi) = \frac{1}{|G(n)|}\sum_{\sigma\in G(n)} \chi(\sigma)\psi(\sigma.x).$$

In order to prove that a family of values is non-vanishing, it is a standard technique to compute their average. Hence, in order to show that $\mathrm{a}(x,\chi)\neq 0$ is non-vanishing for some $\chi\in P(n,\chi_0)$, it suffices to show that

$$\mathrm{b}(x,\chi_0) = \sum_{\chi\in P(n,\chi_0)} \mathrm{a}(x,\chi)\neq 0.$$

If the order of χ_0 is prime to p, then all the characters in $P(n,\chi_0)$ are in fact conjugates under the action of $\mathrm{Aut}(\mathbb{C})$. If, in addition, the Hecke field of g is linearly disjoint from the field generated over $\mathbb{Q}$ by the p-th roots of unity, it follows that the sums $\mathrm{a}(x,\chi)$ are also conjugates for all $\chi\in P(n,\chi_0)$. Hence, the non-vanishing of $\mathrm{a}(x,\chi)$ for some $\chi\in P(n,\chi_0)$ forces the non-vanishing of $\mathrm{a}(x,\chi)$ for all $\chi\in P(n,\chi_0)$. Moreover, Vatsal noticed in [25] that if l splits completely in the field $\mathbb{Q}(\chi_0)$ generated by the values of χ_0, and if it is inert in the field $\mathbb{Q}(\mu_{p^\infty})$ generated by all p-power roots of unity, then all the characters in $P(n,\chi_0)$ are conjugates under the action of a decomposition group D_λ. Thus, the sums $\mathrm{a}(x,\chi)$ have the same λ-adic valuation for all $\chi\in P(n,\chi_0)$.

Our goal is to prove an analogue of Theorem 1.2 in [25] (see also Proposition 4.1 and Corollary 4.2) for a Hilbert modular form g over a totally real field F, while removing the above mentioned assumptions on l and the order of χ_0. More precisely, given a character χ_0 of G_0, we prove that

$$ord_\lambda(\mathrm{a}(x,\chi)) < \mu$$

for all $\chi\in P(n,\chi_0)$ with $n\gg 0$, where μ is a constant to be specified at a later stage (see Definition 3.4.8).

We know by Lemma 2.8 in [7] that we can identify G_0 with its image $G_0(n)$ in $G(n)$ whenever n is sufficiently large. We denote the quotient group $G(n)/G_0(n)$ by $H(n)$. Suppose that χ_0 is a fixed character of G_0 and let $\chi\in P(n,\chi_0)$. One can express χ as $\chi=\chi_0'\chi_1$, where χ_0' is some character of $G(n)$ inducing χ_0 on $G_0(n)\simeq G_0$, and χ_1 is some character of $H(n)$.

Recall that E_l is an l-adically complete discrete valuation ring containing the Fourier coefficients of g. Enlarge E_l if necessary to contain the values of χ_0, and let $E_l(\chi_1)$ be the field obtained by adjoining to E_l the values of χ_1. We assume without loss of generality that the order of χ_0 is divisible by p.

Consider the trace of $\mathrm{a}(x,\chi)$ taken from $E_l(\chi_1)$ to E_l:

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \sum_{\sigma\in\mathrm{Gal}(E_l(\chi_1)/E_l)} \sigma(\mathrm{a}(x,\chi)).$$

This trace expression is different than the average expression

$$\mathrm{b}(x,\chi) = \sum_{\chi\in P(n,\chi)} \mathrm{a}(x,\chi)$$

considered in the work of Vatsal and Cornut-Vatsal. Nevertheless, the same approach is followed to study both expressions. Evidently, given any $\chi \in P(n,\chi_0)$, the non-vanishing of $\mathrm{Tr}(\mathrm{a}(x,\chi))$ would then imply the non-vanishing of $\mathrm{a}(x,\chi)$.

Notice that

$$\begin{aligned}\mathrm{Tr}(\mathrm{a}(x,\chi)) &= \frac{1}{|G(n)|}\mathrm{Tr}_{E_l(\chi_1)/E_l}\sum_{\sigma\in G_0(n)}\sum_{\tau\in H(n)}\chi_0'(\sigma)\chi_1(\tau)\psi(\sigma\tau.x)\\ &= \frac{1}{|G(n)|}\sum_{\sigma\in G_0}\chi_0(\sigma)\sum_{\tau\in H(n)}\psi(\sigma\tau.x)\mathrm{Tr}_{E_l(\chi_1)/E_l}\chi_1(\tau)\\ &= \frac{[E_l(\chi_1):E_l]}{|G(n)|}\sum_{\sigma\in G_0}\chi_0(\sigma)\sum_{\substack{\tau\in H(n)\\ \chi_1(\tau)\in E_l}}\psi(\sigma\tau.x)\chi_1(\tau).\end{aligned}$$

Let m be the highest power of p dividing the order of χ_0, and put

$$Z(n,m) = \{\tau\in H(n) : \mathrm{order}(\tau) \mid p^m\}.$$

Since $Z(n,m) = \{\tau\in H(n) : \chi_1(\tau)\in E_l\}$, we get

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{[E_l(\chi_1):E_l]}{|G(n)|}\sum_{\sigma\in G_0}\chi_0(\sigma)\sum_{\tau\in Z(n,m)}\chi_1(\tau)\psi(\sigma\tau.x).$$

Lemma 3.4.1. *For $n \gg 0$, $Z(n,m) \simeq O_F/\mathcal{P}^m$.*

Proof. It follows from the definition of $Z(n,m)$ that $Z(n,m) = \ker(H(n) \twoheadrightarrow H(n-m))$. We also have an isomorphism between $Z(n,m)$ and $\ker(G(n) \twoheadrightarrow G(n-m))$ induced by the natural quotient map $G(n) \twoheadrightarrow H(n)$. The reciprocity map induces an isomorphism between $\ker(G(n) \twoheadrightarrow G(n-m))$ and

$$K^*\widehat{O}^*_{\mathcal{P}^{n-m}}/K^*\widehat{O}^*_{\mathcal{P}^n} \simeq O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^{n-m}}O^*_{\mathcal{P}^n,\mathcal{P}}.$$

Notice that $O^*_{\mathcal{P}^{n-m}} = O^*_F$ is contained in $O^*_{\mathcal{P}^n,\mathcal{P}}$ for sufficiently large n, so that

$$\ker(G(n) \twoheadrightarrow G(n-m)) \simeq O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}}.$$

On the other hand,

$$O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}} = \{1 + a\alpha_{\mathcal{P}}\pi_{\mathcal{P}}^{n-m} \mod O^*_{\mathcal{P}^n,\mathcal{P}} : a \in O_{F,\mathcal{P}}/\mathcal{P}^m\},$$

where $\alpha_{\mathcal{P}} \in O_{K,\mathcal{P}}$ is such that $\{1, \alpha_{\mathcal{P}}\}$ is an $O_{F,\mathcal{P}}$-basis of $O_{K,\mathcal{P}}$. This yields the desired isomorphism. □

Define a function θ_m on $G(\mathbb{A}_f)$ by:

$$\theta_m(g) = \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g.(1,1,...,\underbrace{\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}}_{\mathcal{P}^{\mathrm{th}}\text{place}},...,1,1)),$$

where $\tau_a \in Z(n,m)$ is the element associated to $a \in O_{F,\mathcal{P}}/\mathcal{P}^m$.

To simplify notation, we identify $(1,1,..,b_{\mathcal{P}},..,1,1) \in G(\mathbb{A}_f)$ with $b_{\mathcal{P}} \in \mathrm{GL}_2(F_{\mathcal{P}}) \simeq B^*_{\mathcal{P}}$. For example, we write

$$\theta_m(g) = \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g.\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}).$$

Lemma 3.4.2. *The function θ_m has level $H_m = \widehat{R_m}^*$ for some O_F-order R_m which agrees with R outside $\mathcal{P}$. At $\mathcal{P}$, the subgroup $R^*_{m,\mathcal{P}}$ is isomorphic to the Iwahori subgroup*

$$U_1(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)}) : a \equiv 1 \mod \pi_{\mathcal{P}}^{\max(2m,\delta)} \right\},$$

where δ is such that $\mathcal{N} = \mathcal{P}^\delta\mathcal{N}'$ and $\mathcal{N}'$ is prime to $\mathcal{P}$.

Proof. Let γ be any element in $U_0(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)})$. The Iwahori factorization of γ yields $\gamma = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. We view γ as an element in $G(\mathbb{A}_f)$. For $g \in G(\mathbb{A}_f)$, we have

$$\begin{aligned}
\theta_m(g\gamma) &= \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g\gamma\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}) \\
&= \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ad_1^{-1}d_2})\theta(g\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-la\pi_{\mathcal{P}}^{-m} & -la^2\pi_{\mathcal{P}}^{-2m} \\ l & 1+la\pi_{\mathcal{P}}^{-m} \end{pmatrix}) \\
&= \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ad_1^{-1}d_2})\theta(g\begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix})
\end{aligned}$$

If we further assume that $\gamma \in U_1(O_{F,\mathcal{P}}, \mathcal{P}^{\max(2m,\delta)})$, then $d_1 \equiv d_2 \equiv 1 \mod \pi_{\mathcal{P}}^{\max(2m,\delta)}$. Hence, $\theta_m(g\gamma) = \theta_m(g)$. □

We now make the following important observation. Since $\mathrm{a}(\gamma.x,\chi) = \chi^{-1}(\gamma)\mathrm{a}(x,\chi)$ for any $\gamma \in G(n)$, and $G(n)$ acts simply and transitively on $CM_H(\mathcal{P}^n)$, it suffices to study the λ-adic valuation of $\mathrm{a}(y,\chi)$ for any $y \in CM_H(\mathcal{P}^n)$. Henceforth, we shall take for x the CM point obtained in Lemma 3.2.2 and fix the choice of representative g determined in the lemma as well. We denote by x_m the class of g in CM_{H_m}.

Lemma 3.4.3. *Let ψ_m denote the function induced by θ_m on CM_{H_m}. We have*

$$\sum_{\tau \in Z(n,m)} \chi_1(\tau)\psi(\tau.x) = \psi_m(x_m).$$

Proof. By Lemma 3.4.1, we view $\tau \in Z(m,n)$ as the class of $1+a\alpha_{\mathcal{P}}\pi_{\mathcal{P}}^{n-m}$ in $O^*_{\mathcal{P}^{n-m},\mathcal{P}}/O^*_{\mathcal{P}^n,\mathcal{P}}$ for some $a \in O_{F,\mathcal{P}}/\mathcal{P}^m$. We then identify τ with its image in $\mathrm{GL}_2(O_{F,\mathcal{P}})$:

$$\tau \mapsto \begin{pmatrix} 1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{n-m}\mathrm{N}\alpha_{\mathcal{P}} \\ -a\pi_{\mathcal{P}}^{n-m} & 1 \end{pmatrix}.$$

Write $x = [g]$, where $g \in G(\mathbb{A}_f)$ is as defined in Lemma 3.2.2. We have

$$\begin{aligned}
\sum_{\tau \in Z(n,m)} \chi_1(\tau)\psi(\tau.x) &= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\psi(\tau_a.x) \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(\begin{pmatrix} 1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{n-m}\mathrm{N}\alpha_{\mathcal{P}} \\ -a\pi_{\mathcal{P}}^{n-m} & 1 \end{pmatrix} g) \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g \begin{pmatrix} 1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{-m} \\ -a\pi_{\mathcal{P}}^{2n-m}\mathrm{N}\alpha_{\mathcal{P}} & 1 \end{pmatrix}) \\
&= \sum_{a \in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_a)\theta(g \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}) \\
&= \psi_m(x_m).
\end{aligned}$$

The fourth line follows from the fact that

$$\begin{pmatrix} 1+a\pi_{\mathcal{P}}^{n-m}\mathrm{Tr}\alpha_{\mathcal{P}} & a\pi_{\mathcal{P}}^{-m} \\ -a\pi_{\mathcal{P}}^{2n-m}\mathrm{N}\alpha_{\mathcal{P}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix},$$

where $d_1, d_2 \in O_{F,\mathcal{P}}$, and $l \in \mathcal{P}^\delta$ for $n \gg 0$. □

Since $\chi_0 = 1$ on G_2, we have:

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|} \sum_{\sigma\in G_0/G_2} \chi_0(\sigma)\psi_m(\sigma.x_m).$$

We can reduce the above sum into something even simpler by means of another level raising step.

Proposition 3.4.4. *There exists an O_F-order $R_{m,\mathcal{D}}$, a nonzero function $\theta_{m,\mathcal{D}}$ of level $H_{m,\mathcal{D}} = \widehat{R_{m,\mathcal{D}}}^*$ on $G(\mathbb{A}_f)$, and for each $n \geq 0$, a Galois equivariant map $x \mapsto x_{m,\mathcal{D}}$ from $CM_H(\mathcal{P}^n)$ to $CM_{H_{m,\mathcal{D}}}(\mathcal{P}^n)$ such that*

$$\psi_{m,\mathcal{D}}(x_{m,\mathcal{D}}) = \sum_{\tau\in G_1/G_2} \chi_0(\tau)\psi_m(\tau.x_m),$$

where $\psi_{m,\mathcal{D}} = \theta_{m,\mathcal{D}} \circ \mathrm{red}$.

Proof. The reader is referred to the proof of Lemma 5.9 in [7] □

Hence, the trace expression simplifies to

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|} \sum_{\sigma\in G_0/G_1} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}).$$

We now study the λ-adic valuation of the sum

$$\sum_{\sigma\in\mathcal{R}} \chi_0(\sigma)\psi_{m,\mathcal{D}}(\sigma.x_{m,\mathcal{D}}),$$

where $\mathcal{R}$ is a set of representatives for G_0/G_1 containing 1.

Definition 3.4.5. *Let k be any ring. A k-valued function ϕ on M_H is said to be Eisenstein if it factors through N_H via the map* c*, where as ϕ is said to be exceptional if there exists $z \in N_H$ such that ϕ is constant on $c^{-1}(\sigma.z)$ for all $\sigma \in \mathrm{Gal}_K^{\mathrm{ab}}$.*

Choose an ideal C in O_F such that $\mathrm{nrd}(H)$ contains all elements of $\widehat{O_F}^*$ congruent to 1 modulo C. Such an integral ideal exists because $\mathrm{nrd}(H)$ is open in $\widehat{F}^*$. For a finite prime v of F, let q_v be the cardinality of the residue class field at v. Denote by S the set of all finite places of F that do not divide $\mathcal{N}$ and correspond to a principal prime ideal aO_F with $a \equiv 1 \mod C$ and a is totally positive.

Lemma 3.4.6. *If ϕ is Eisenstein modulo λ^r for some positive integer r, then $a_v \equiv q_v + 1 \mod \lambda^r$ for all $v \in S$.*

Proof. Let v be a finite place in F corresponding to a principal prime ideal $Q = aO_F$ with $a \equiv 1 \mod C$ and a is totally positive. Choose $x \in M_H$ such that $\phi(x)$ is a λ-adic unit. By definition, we know that

$$T_v\phi(x) = \sum_{i\in I_v} \phi(x\eta_{v,i}).$$

Here $H_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} H_v = \coprod_{i\in I_v} \eta_{v,i}H_v$. Notice that $c(x\eta_{v,i}) = c(x\eta_v)$, where $\eta_v = (1,..,1,\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},1,...,1)$.
Since ϕ is Eisenstein modulo λ^r, we get

$$T_v\phi(x) \equiv (1+q_v)\phi(x\eta_v) \mod \lambda^r.$$

Choose $d \in G(\mathbb{Q})$ such that $\mathrm{nrd}(d) = a$. Notice that

$$\mathrm{nrd}(\eta_v^{-1}d) = (a,...,a,1,a,...,a) \equiv 1 \mod C.$$

We thus obtain an element $h \in H$ such that $\mathrm{nrd}(h) = \mathrm{nrd}(\eta_v^{-1}d)$. Hence,

$$\phi(x\eta_v) \equiv \phi(x\eta_v d^{-1}) \equiv \phi(xh^{-1}) \equiv \phi(x) \mod \lambda^r.$$

On the other hand, we know that $T_v\phi(x) = a_v\phi(x)$. Putting all of this together gives $a_v\phi(x) \equiv (1+q_v)\phi(x) \mod \lambda^r$, which implies that $a_v \equiv 1+q_v \mod \lambda^r$ since $\phi(x)$ is a λ-adic unit. □

Lemma 3.4.7. *If ϕ is exceptional but non-Eisenstein modulo λ^r for some positive integer r, then a_v is a λ-adic non-unit for all finite places v of F that are inert in K and do not divide $\mathcal{N}$.*

Proof. The argument given here is drawn from [7]. Recall that we have an action of the group $\mathrm{Gal}_K^{\mathrm{ab}}$ on N_H, and one can show that there are at most two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbit in N_H. If there were only one $\mathrm{Gal}_K^{\mathrm{ab}}$-orbit in N_H, then any exceptional function on M_H would also be Eisenstein. Since ϕ is exceptional and non-Eisenstein modulo λ^r, we know there must be exactly two $\mathrm{Gal}_K^{\mathrm{ab}}$-orbits in N_H, which we denote by X and Y with ϕ being constant modulo λ^r on $\mathrm{c}^{-1}(z)$ for all $z \in X$. Since ϕ is non-Eisenstein modulo λ^r, there exist $y \in Y$ and some $x_1, x_2 \in \mathrm{c}^{-1}(y)$ such that $\phi(x_1) \not\equiv \phi(x_2) \mod \lambda^r$.

Let v be a finite place of F that is inert in K and does not divide $\mathcal{N}$. For any $x \in M_H$, we know that

$$T_v\phi(x) = a_v\phi(x) = \sum_{i\in I_v} \phi(x\eta_{v,i}).$$

We also know that if $x \in \mathrm{c}^{-1}(y)$ then $x\eta_{v,i} \in \mathrm{c}^{-1}(\mathrm{Frob}_v.y)$. Since v is inert in K, we get $\mathrm{Frob}_v.y \in X$, so that ϕ is constant modulo λ^r on $\mathrm{c}^{-1}(\mathrm{Frob}_v.y)$ with $\phi(v,y)$ being the common value. Hence,

$$a_v\phi(x_1) \equiv (1+q_v)\phi(v,y) \equiv a_v\phi(x_2) \mod \lambda^r.$$

It follows that a_v is a λ-adic non-unit, since otherwise $\phi(x_1)$ and $\phi(x_2)$ would be congruent modulo λ^r. □

We shall assume henceforth that g satisfies the condition: a_v is a λ-adic unit for some v inert in K, $v \nmid \mathcal{N}$.

Definition 3.4.8. *Let μ be the smallest integer such that $a_v \not\equiv 1+q_v$ mod λ^μ for some $v \in S, v \nmid \mathcal{D}$.*

It follows immediately from the definition of μ that the function θ is non-exceptional modulo λ^μ.

We let f be the inertia degree of $\mathcal{P}$ over p. If λ lies above p ($l = p$), we let e be the corresponding ramification index. In this case, we denote by k the ring $E_l/\lambda^{emf+\mu}E_l$. If λ does not lie above p ($l \neq p$), we denote by k the ring $E_l/\lambda^\mu E_l$. We shall view θ, θ_m and $\theta_{m,\mathcal{D}}$ as k-valued functions.

Proposition 3.4.9. *The function $\theta_m : M_{H_m} \to k$ is a non-zero eigenfunction for all Hecke operators T_v ($v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}'$) with $T_v\theta_m = a_v\theta_m$.*

Proof. It is clear that θ_m is an eigenfunction for all Hecke operators T_v ($v \nmid \mathcal{P}\mathcal{N}'\mathcal{D}'$) with $T_v\theta_m = a_v\theta_m$. If $\theta_m = 0$ as a k-valued function, then

$$\begin{aligned}
0 &= \sum_{u\in(O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.\theta_m \\
&= \sum_{u\in(O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \chi_1(\tau_{ua}) \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}.\theta \\
&= \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}^m} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-m} \\ 0 & 1 \end{pmatrix}.\theta \sum_{u\in(O_{F,\mathcal{P}}/\mathcal{P}^m)^*} \chi_1(\tau_{ua})
\end{aligned}$$

Let $q = p^f$ denote the cardinality of the residue class field $O_F/\mathcal{P}$. By means of Lemma 3.4.11 below, we get

$$\begin{aligned}
0 &= q^{m-1}(q-1)\theta - q^{m-1} \sum_{a\in(O_{F,\mathcal{P}}/\mathcal{P})^*} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix}.\theta \\
&= q^m\theta - q^{m-1} \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix}.\theta \\
&= q^{m-1}\left(q\theta - \sum_{a\in O_{F,\mathcal{P}}/\mathcal{P}} \begin{pmatrix} 1 & a\pi_{\mathcal{P}}^{-1} \\ 0 & 1 \end{pmatrix}.\theta\right) \\
&= q^{m-1}\theta^+.
\end{aligned}$$

This yields a contradiction since $q^{m-1}\theta^+$ is non-zero by Lemma 4.12 in [7]; the proof of this lemma uses the fact that θ is non-Eisenstein modulo λ^μ. The reader is referred to [7] for a description of the function θ^+ and its properties (see, for example, Section 1.6, Theorem 5.10 and the Appendix). □

Corollary 3.4.10. *θ_m is non-exceptional as a k-valued function.*

Lemma 3.4.11. *For $a \in O_F/\mathcal{P}^m$, we have*

$$\sum_{u\in(O_F/\mathcal{P}^m)^*} \chi_1(\tau_{ua}) = \begin{cases} q^{m-1}(q-1) & a \in \mathcal{P}^m \\ -q^{m-1} & a \in \mathcal{P}^{m-1}/\mathcal{P}^m \text{ and } a \notin \mathcal{P}^m \\ 0 & \text{otherwise} \end{cases}$$

Proof. The statement of the lemma follows trivially for $a \equiv 0 \mod \mathcal{P}^m$. For the remaining cases, we write

$$\sum_{u\in(O_F/\mathcal{P}^m)^*} \chi_1(\tau_{ua}) = \sum_{u\in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) - \sum_{u\in\mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua})$$

Notice that if $a \in \mathcal{P}^{m-1}/\mathcal{P}^m$, we have

$$\sum_{u\in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) = 0 \text{ and } \sum_{u\in\mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua}) = q^{m-1}.$$

Otherwise, we get

$$\sum_{u\in O_F/\mathcal{P}^m} \chi_1(\tau_{ua}) = \sum_{u\in\mathcal{P}/\mathcal{P}^m} \chi_1(\tau_{ua}) = 0$$

. □

Proposition 3.4.12. *The function $\theta_{m,\mathcal{D}} : M_{H_{m,\mathcal{D}}} \to k$ is a non-zero eigenfunction for all Hecke operators T_v away from $\mathcal{PN}'\mathcal{D}'$ with $T_v\theta_{m,\mathcal{D}} = a_v\theta_{m,\mathcal{D}}$.*

Proof. By definition (see [7] p. 57),

$$\theta_{m,\mathcal{D}} = \sum_{d|\mathcal{D}'} \chi_0(\sigma_d)(\alpha_d.\theta_m),$$

where $\alpha_d = \prod_{Q|d} \alpha_Q$, and α_Q is any element in $R_Q \sim M_2(O_{F,Q})$. Notice that $\theta_{m,\mathcal{D}}$ is left-invariant under $H_{m,\mathcal{D}} = \widehat{R}_{m,\mathcal{D}}$ where $R_{m,\mathcal{D}}$ is the unique O_F-order which agrees with R_m outside $\mathcal{D}'$ and equals $R_Q \cap \alpha_Q R_Q \alpha_Q^{-1}$ at $Q \mid \mathcal{D}'$.

If we fix a prime divisor Q of $\mathcal{D}'$, it is easy to see that $\theta_{m,\mathcal{D}}$ can be rewritten as

$$\theta_{m,\mathcal{D}} = \sum_{d|\frac{\mathcal{D}'}{Q}} \chi_0(\sigma_d)(\alpha_d.\theta_m) + \chi_0(\sigma_Q)\alpha_Q. \sum_{d|\frac{\mathcal{D}'}{Q}} \chi_0(\sigma_d)(\alpha_d.\theta_m).$$

Let ϑ_1 and ϑ_2 be k-valued functions on M_{H_m} satisfying $T_v\vartheta_i = a_v\vartheta_i$ for all $v \nmid \mathcal{PN}'\mathcal{D}'$. We claim that any nontrivial linear combination $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is non-zero in k. If $a\vartheta_1 + b\alpha_Q.\vartheta_2 = 0$ for some scalars a and b, then $a\vartheta_1 = -b\alpha_q.\vartheta_2$ is fixed under the group spanned by R_Q^* and $\alpha_Q R_Q^* \alpha_Q^{-1}$ which contains the kernel of the reduced norm map $B_{\mathcal{P}} \to F_{\mathcal{P}}$. It follows from the strong approximation theorem ([26] p. 81) that ϑ_1 factors through the norm map as a k-valued function, which is a contradiction to the fact that $a_v \not\equiv q_v + 1 \mod \lambda^\mu$ for some $v \in S$ (Lemma 3.4.6). Hence, $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is non-zero. Not only this, but $a\vartheta_1 + b\alpha_Q.\vartheta_2$ is also an eigenfunction for all T_v ($v \nmid \mathcal{PN}'\mathcal{D}'$) with the same eigenvalues as ϑ_1 and ϑ_2.

In light of the above observation, we proceed by induction on the number of prime ideal divisors of $\mathcal{D}'$ to prove that $\theta_{m,\mathcal{D}}$ is non-zero and satisfies $T_v\theta_{m,\mathcal{D}} = a_v\theta_{m,\mathcal{D}}$ for all $v \nmid \mathcal{PN}'\mathcal{D}'$. This reduces the problem to the case of θ_m which satisfies the required hypothesis by Proposition 3.4.9. □

Corollary 3.4.13. *$\theta_{m,\mathcal{D}}$ is non-exceptional as a k-valued function.*

Finally, we state and prove the main result in this chapter. This result gives an upper bound for the λ-adic valuation of the sum

$$\sum_{\tau\in\mathcal{R}} \chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.x_{m,\mathcal{D}}),$$

which we recall is related to the Gross-Zagier sum $\mathrm{a}(x,\chi)$ by the formula

$$\mathrm{Tr}(\mathrm{a}(x,\chi)) = \frac{|G_2|[E_l(\chi_1):E_l]}{|G(n)|}\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.x_{m,\mathcal{D}}).$$

Theorem 3.4.14. *Let χ_0 be any character of G_0. For any $x \in CM_{H_{m,\mathcal{D}}}(\mathcal{P}^n)$ with $n \gg 0$, there exists some $y \in G(\infty).x$ such that*

$$\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y) \not\equiv 0 \quad (\textit{in } k).$$

Proof. We follow the proof of Corollary 5.7 in [7]. Since $\theta_{m,\mathcal{D}}$ is non-exceptional as a k-valued function, there exists $\sigma \in G(\infty)$ such that $\theta_{m,\mathcal{D}}$ is non-constant as a k-valued function on $\mathrm{c}^{-1}(\mathrm{c}\circ\mathrm{red}(\sigma.x))$. Choose $p_1, p_2 \in \mathrm{c}^{-1}(\mathrm{c}\circ\mathrm{red}(\sigma.x))$ such that $\theta_{m,\mathcal{D}}(p_1) \not\equiv \theta_{m,\mathcal{D}}(p_2)$ (in k). If n is sufficiently large, Theorem 3.3.1 guarantees the existence of $y_1, y_2 \in G(\infty).x$ such that

$$\mathrm{red}(y_1) = p_1, \quad \mathrm{red}(y_2) = p_2$$

and

$$\mathrm{red}(\tau.y_1) = \mathrm{red}(\tau.x) = \mathrm{red}(\tau.y_2) \quad \text{for all } \tau \neq 1 \text{ in } \mathcal{R}.$$

We thus obtain

$$\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_1) - \sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_2) = \theta_{m,\mathcal{D}}(p_1) - \theta_{m,\mathcal{D}}(p_2) \not\equiv 0 \text{ (in } k).$$

Therefore, at least one of the sums

$$\sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_1) \quad \text{or} \quad \sum_{\tau\in\mathcal{R}}\chi_0(\tau)\psi_{m,\mathcal{D}}(\tau.y_2)$$

is non-zero in k. □

Finally, we remark that one can easily obtain a lower bound on the l-adic valuation of the Gross-Zagier sum $\mathrm{a}(x,\chi)$. In fact, let ν be the largest

integer such that θ is Eisenstein modulo λ^ν. Then

$$\begin{aligned}\sum_{\sigma\in G(n)}\chi(\sigma)\theta\circ\mathrm{red}(\sigma.x) &= \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{red}(\sigma.x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{c}\circ\mathrm{red}(\sigma.x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\sigma.\mathrm{c}\circ\mathrm{red}(x))\\ &\equiv \sum_{\sigma\in G(n)}\chi(\sigma)\theta(\mathrm{nrd}(\beta)\mathrm{c}\circ\mathrm{red}(x))\\ &\equiv 0 \mod \lambda^\nu,\end{aligned}$$

where the last line follows from the orthogonality property of group characters. Hence,

$$ord_\lambda\left(\sum_{\sigma\in G(n)}\chi(\sigma)\psi(\sigma.x)\right)\geq\nu.$$

Assume for simplicity that $l\neq p$, then it is obvious that this simple observation combined with Theorem 3.4.14 would give an exact value for the l-adic valuation of $\mathrm{a}(x,\chi)$ if we have $\nu+1=\mu$. However, it is not clear to us whether the statement $\nu+1=\mu$ is true or not. This is a very interesting question, but we choose not to discuss it in this work. We remark only that the answer seems to be connected to multiplicity-one-type results for the component group of a Shimura curve at Eisenstein primes.

Chapter 4

Fourier Coefficients of Half-Integral Weight Modular Forms

Let k be a positive integer such that $k \equiv 3 \mod 4$, and let N be a positive square-free integer. In this chapter, we compute a basis for the two-dimensional subspace $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ of half-integral weight modular forms associated, via the Shimura correspondence, to a newform $F \in S_{k-1}(\Gamma_0(N))$, which satisfies $L(F, \frac{1}{2}) \neq 0$. This is accomplished by using a result of Waldspurger, which allows one to produce a basis for the forms that correspond to a given F via local considerations, once a form in the Kohnen space has been determined.[1]

4.1 The Shimura Correspondence

Let k be an odd positive integer, and M be a positive integer divisible by 4. A modular form of half-integral weight $\frac{k}{2}$ for $\Gamma_0(M)$ is a holomorphic function f on the upper half-plane which is also holomorphic at the cusps and transforms like the k-th power of the theta series

$$\theta(z) = 1 + 2\sum_{n\geq 1} e^{2\pi i n^2 z}$$

under fractional linear transformations of $\Gamma_0(M)$. If f vanishes at all the cusps, we say that it is a cusp form and write $f \in S'_{\frac{k}{2}}(\Gamma_0(M))$. We also denote by $S'_{\frac{k}{2}}(M, \boldsymbol{\chi})$, the space of cusp forms with central character $\boldsymbol{\chi}$ of conductor dividing M. The Kohnen subspace in $S'_{\frac{k}{2}}(M, \boldsymbol{\chi})$ consists of

[1] The contents of this chapter have been published in the LMS Journal of Computation and Mathematics [10]

$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ with the Fourier coefficients a_n satisfying

$$a_n = 0, \text{ when } \boldsymbol{\chi}_{(2)}(-1)(-1)^{\frac{k-1}{2}} n \equiv 2, 3 \mod 4,$$

where $\boldsymbol{\chi}_{(2)}$ is the 2-primary component of $\boldsymbol{\chi}$.

One can define an action of Hecke operators on the space of half-integral weight modular forms. The different Hecke operators commute ($T_{m^2}T_{n^2} = T_{m^2n^2}$, $\gcd(m,n) = 1$), and the operator $T_{p^{2v}}$ is a polynomial in T_{p^2}. However, one can have non-trivial operators T_m only for square m or for $(m, M) \neq 1$. Hence, for a half-integral weight modular form which is an eigen-form for all Hecke operators, one can only relate its coefficients whose indices differ by a perfect square ([13], Proposition 14). For a more detailed exposition on half-integral weight modular forms the reader is referred to [13].

In 1973, Shimura proved a fundamental result which gives a correspondence between modular forms of half-integral weight and modular forms of even integral weight. Let $S_{\frac{3}{2}}(M, \boldsymbol{\chi})$ denote the orthogonal complement (with respect to the Petersson inner product) of the subspace of $S'_{\frac{3}{2}}(M, \boldsymbol{\chi})$ spanned by the Shimura theta series

$$\theta_{\psi,m}(z) = \sum_{n=-\infty}^{\infty} \psi(n) n e^{2\pi i n^2 m z}$$

for all positive integers m and odd primitive Dirichlet characters ψ (see [23]). For an odd integer $k \geq 5$, we put for notational convenience $S_{\frac{k}{2}}(M, \boldsymbol{\chi}) = S'_{\frac{k}{2}}(M, \boldsymbol{\chi})$. Adapting the notation from [29], we let

$$S_{k-1}^{\text{new}}(\boldsymbol{\chi}^2) = \bigcup_{N>0} S_{k-1}^{\text{new}}(N, \boldsymbol{\chi}^2),$$

where $S_{k-1}^{\text{new}}(N, \boldsymbol{\chi}^2)$ is the (finite) subset of newforms in $S_{k-1}(N, \boldsymbol{\chi}^2)$. If $F \in S_{k-1}^{\text{new}}(\boldsymbol{\chi}^2)$ is such that $T_p F = b_p F$ for all p, one defines the Shimura lift of F to be the subspace

$$S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F) = \{f \in S_{\frac{k}{2}}(M, \boldsymbol{\chi}) : T_{p^2} f = b_p f \text{ for almost all } p \not| M\}.$$

Shimura showed that if $f \in S_{\frac{k}{2}}(M, \boldsymbol{\chi})$ is an eigenform for almost all Hecke operators, then there exists a unique $F \in S_{k-1}^{\text{new}}(\boldsymbol{\chi}^2)$ such that $f \in S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$. This assignment is at the heart of the Shimura correspondence. The following is a simplified version of Shimura's original theorem.

Theorem 4.1.1. *(See [20]) Let $f \in S_{\frac{k}{2}}(M, \boldsymbol{\chi})$ be a common eigenfunction for all T_{p^2} with λ_p being the corresponding eigenvalue. Define the sequence of complex numbers $\{b_n\}$ by the formal identity*

$$\sum_{n=1}^{\infty} b_n n^{-s} = \prod_p \frac{1}{1 - \lambda_p p^{-s} + \boldsymbol{\chi}(p^2) p^{k-2-2s}}$$

Then $F(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ belongs to $S_{k-1}(N', \boldsymbol{\chi}^2)$ for some integer N' which is divisible by the conductor of χ^2.

Thus, each eigenform of weight $\frac{k}{2}$ is associated to a form of integral weight $k-1$. However, it is not clear (and in general very far from true) that this correspondence is bijective on the level of eigenforms. Indeed, the principal point for us in this chapter is that the number of eigenforms of weight $\frac{k}{2}$ of a given level and character associated to a fixed F is a rather subtle invariant. To clarify the situation, we introduce the following setup. Let $F \in S_{k-1}^{\text{new}}(\boldsymbol{\chi}^2)$ be a newform of even weight $k-1$, level N and character $\boldsymbol{\chi}^2$ defined modulo N. Let M be a positive integer divisible by 4, and suppose that $\boldsymbol{\chi}$ is defined modulo M. (Here N may or may not divide M.) Consider the space $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ defined in a previous paragraph. We are led to the following questions:

1. What is the dimension of the space $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$?

2. If it is non-zero, can we compute a basis for this space?

It turns out that the answer to these questions is extremely delicate, and relies in a fundamental way on the representation theory of the metaplectic covers of SL_2 and GL_2. The answer is rendered complicated by two main factors: firstly, that a certain global theta correspondence is trivial when a certain L-function vanishes, and secondly, that there are severe local complications in the representation theory of the metaplectic group. The first causes a natural construction of forms in $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ to vanish, while the second shows that there is no theory of newforms on the half-integral side, and that the space one is trying to construct may have high dimension.

The construction of forms in $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ if non-empty for some M and $\boldsymbol{\chi}$ was taken up by Shintani [22], and solved in general by Flicker [8]. It is not at all clear, *a* priori, which characters $\boldsymbol{\chi}$ and which integers M one has to take in order to obtain a non-zero space.

On the other hand, the questions of finding the dimension and a basis for $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ were taken up by Waldspurger in [29]. In fact, Waldspurger

proved that, under quite general conditions on F, N and $\boldsymbol{\chi}$, there exists a basis for $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ such that for every positive integer n the Fourier coefficient $a_n(f_i)$ of a basis element f_i is the product of two factors: a product of local terms $c_i(n, F)$ each of which is completely determined by the local components of F according to explicit formulae given in [29], and a global factor $A_F(n)$ whose square is the central critical value of the L-function of the newform F twisted by a quadratic character depending on n.

A more detailed discussion of the representation-theoretic subtleties of this circle of questions is given in Section 3 below. The main point of our work is to answer these questions in the simplest case, when $\boldsymbol{\chi}$ is trivial, and $M = 4N$, with N odd and square-free. More precisely, we will compute a basis for $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ for an odd square-free integer N and $k \equiv 3 \mod 4$ provided that $L(F, \frac{1}{2}) \neq 0$ (the L-function is normalized so that the functional equation is with respect to $s \to 1 - s$ rather than $s \to k - 1 - s$). In the light of Waldspurger's work, our task is reduced to computing the global factors $A_F(n)$.

The organization of the chapter is as follows. In Section 2, we present a construction of a half-integral weight modular form g which belongs to the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ for a given newform $F \in S_{k-1}^{\text{new}}(\Gamma_0(N))$. The squares of the Fourier coefficients of this form are essentially proportional to the central critical values of the L-function of F twisted by some quadratic characters. In Section 3 we digress briefly to discuss the representation-theoretic interpretation of the Shimura correspondence as a theta correspondence. In Section 4, we state the full result of Waldspurger. Next, we show that the space $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ is in fact two-dimensional and has a distinguished basis $\{f_1, f_2\}$ such that either $f_1 - f_2$ or f_2 belongs to the Kohnen subspace. We use this to express the global factors $A_F(n)$ in terms of the coefficients of g (Theorem 4.4.5). Finally, we explicitly determine the Fourier coefficients of the two modular forms that generate $S_{\frac{k}{2}}(\Gamma_0(4N), F)$, thus, arriving at our main contribution given in Theorem 4.4.7. The last section contains examples to illustrate the calculations carried out in Section 4.

In conclusion, we remark that the problem of finding a basis for $S_{\frac{k}{2}}(M, \boldsymbol{\chi}, F)$ (in a more general setting than the one assumed in our present work) can undoubtedly be solved by representation-theoretic techniques, and by generalizing the framework sketched in Section 3 of the present chapter. We hope to take this up in a future work.

4.2 A Modular Form in $S_{\frac{k}{2}}(\Gamma_0(4N), F)$

We first recall the definition of the L-function associated to a newform $F \in S_{k-1}^{\text{new}}(\Gamma_0(N))$. Following Waldspurger in [29], we set

$$L(F,s) = 2(2\pi)^{1-s-k/2} N^{\frac{s-1+k/2}{2}} \Gamma(s-1+k/2) \sum_{n=1}^{\infty} b_n(F) n^{1-s-k/2} \qquad \operatorname{Re} s > \frac{3}{2}.$$

Then $L(F,s)$ extends to an entire function which satisfies a functional equation with respect to $s \to 1-s$. Moreover, since we assume at the outset that F is a newform with the trivial character, F is also an eigenform for the Atkin-Lehner Operator W_N, so that $W_N F = w_N F$ for some $w_N \in \mathbb{C}$. Hence, the functional equation for $L(F,s)$ takes on the form:

$$L(F,s) = i^{k-1} w_N L(F, 1-s).$$

It is known that W_N is an involution when $k-1$ is even, so $w_N = \pm 1$. In particular, if $k \equiv 3 \mod 4$ and $L(F, \frac{1}{2}) \neq 0$, then the root number in the functional equation is $i^{k-1} w_N = -w_N = 1$. Otherwise, $L(F, \frac{1}{2})$ would vanish trivially.

Let F be a newform in $S_2^{\text{new}}(\Gamma_0(N))$ with an odd square-free level N such that $L(F, \frac{1}{2}) \neq 0$. For each prime divisor p of N we denote by w_p the eigenvalue of the Atkin-Lehner involution W_p. Since $w_N = \prod w_p$ and $w_N = -1$ by the above discussion, the set $S = \{p|N : w_p = -1\}$ has an odd cardinality. Notice that if we consider a newform of higher even weight $k-1$, then $|S|$ is odd if we also assume that $k \equiv 3 \mod 4$.

Consider the definite quaternion algebra B over $\mathbb{Q}$ ramified at S and ∞. Let O be an order in B such that O_q is maximal in B_q for all $q \in S$ and is of index p in a maximal order for the remaining primes p dividing N. Such an order is called an Eichler order of square-free level N in B. The relation between orders of level N and modular forms on $\Gamma_0(N)$ will be clear in what follows.

The Brandt module which we shall denote by $X(\mathbb{R})$ is the free abelian group on the left ideal classes of a quaternion order of level N along with a natural Hecke action determined by the Brandt matrices. Pizer proved in [18] that there exists a Hecke algebra isomorphism between the Brandt module and a subspace of modular forms containing all the newforms of level N ([18], Corollary 2.29 and Remark 2.30). In the representation theoretic language of Jacquet-Langlands, Pizer's result can be interpreted as giving a correspondence between automorphic forms on the adelization $B(\mathbb{A})$ and automorphic forms on $\mathrm{GL}(2, \mathbb{A})$.

Recall that a left O-ideal I is a lattice in B such that $I_p = O_p a_p$ (for some $a_p \in B_p^*$) for every prime p. The set of all such ideals is denoted by C. Two left O-ideals I and J are equivalent if there exists $a \in B^*$ such that $I = Ja$. This gives rise to the set $\bar{C} = \{[I_1], [I_2], ..., [I_H]\}$ of left O-ideal classes which has a finite cardinality H. We also define the norm of an ideal I to be the positive rational number which generates the fractional ideal of $\mathbb{Q}$ generated by $\{N(x) : x \in I\}$. Here $N(x)$ denotes the reduced norm of the quaternion element x.

Define the right order of a left O-ideal I to be the set $O_r(I) = \{a \in B : Ia \subseteq I\}$; this is also an order of level N in B.

Denote by X the free abelian group with basis $\bar{C}$, the set of left O-ideal classes. We shall now illustrate the well-known construction of Hecke operators acting on the $\mathbb{R}$-vector space $X(\mathbb{R}) = X \otimes_{\mathbb{Z}} \mathbb{R}$. We define a height pairing $(\ ,\ )$ on X with integer values by setting $([I_i], [I_j]) = 0$ if $[I_i] \neq [I_j]$ and $([I_i], [I_i]) = \frac{1}{2}\#O_r^*(I_i)$, then extending bi-additively. This height pairing induces an inner product on $X(\mathbb{R})$. For each $n \geq 1$, the Hecke operator $t_n : X(\mathbb{R}) \to X(\mathbb{R})$ is defined by :

$$t_n([I_i]) = \sum_{j=1}^{H} (B(n))_{ij}[I_j].$$

The entries of the Brandt matrix $B(n)$ are calculated using

$$(B(n))_{ij} = \frac{1}{e_j} \times \#\{x \in {I_j}^{-1} I_i : \frac{N(x)}{N({I_j}^{-1} I_i)} = n\},$$

where

$$e_j = \#\{x \in O_r(I_j) : N(x) = 1\} = \#O_r^*(I_j).$$

In other words, $e_j(B(n))_{ij}$ is the nth coefficient in the Fourier expansion of the theta series

$$\theta_{ij}(z) = \sum_{x \in {I_j}^{-1} I_i} q^{\frac{N(x)}{N({I_j}^{-1} I_i)}} \qquad (q = exp(2\pi i z)).$$

In [18], Pizer described an explicit algorithm to compute these matrices. The main procedure computes the number of times $Q(x) = \frac{N(x)}{N(I)}$ represents $1, 2, 3, .., T$ for some given T as x varies over the lattice I in our quaternion algebra. The graph of $Q(x)$ with $x \in \mathbb{R}^4$ is a 4-dimensional paraboloid which has a unique minimum point, and as we move away from this point in any

direction, the values given by $Q(x)$ will always increase. That was the very simple idea behind Pizer's method.

The Hecke operators t_n generate a commutative ring $\mathbb{T}$ of self adjoint operators ([18], Proposition 2.22). The spectral theorem implies that $X(\mathbb{R})$ has an orthogonal basis of eigenvectors for $\mathbb{T}$.

As outlined in Section 1, we shall now construct a non-zero modular form g in the Kohnen subspace of $S_{\frac{3}{2}}(\Gamma_0(4N), F)$ as a linear combination of theta series generated by the norm form of B evaluated on some ternary lattices. Needless to say, the ideas here are all well-known and largely drawn from articles [9] and [4]. Our main result is in Section 4 where we use the form g to compute yet another half-integral weight modular form h which also maps to F via the Shimura correspondence. These two forms make up a basis for the space $S_{\frac{k}{2}}(\Gamma_0(N), F)$.

For every left O-ideal I_i in C, let O_i be its right order, and let R_i be the subgroup of trace zero elements in the suborder $\mathbb{Z} + 2O_i$. For every left O-ideal class $[I_i]$, we associate the ternary theta series

$$g([I_i]) = \frac{1}{2} \sum_{x \in R_i} q^{N(x)} = \frac{1}{2} \sum_{D \geq 0} a_D([I_i]) q^D, \tag{4.1}$$

then extend this association by linearity to $X(\mathbb{R})$. Since, the ternary quadratic form $N(x)$ for $x \in R_i$ is a positive definite integral quadratic form with level $4N$ and a square discriminant, these modular forms have weight $\frac{3}{2}$, level $4N$ and the trivial character.

The determination of the theta series $g([I_i])$ up to a precision T amounts to computing the number of times $N(x)$ represents $1, 2, 3, ..., T$ as x varies over all trace zero elements in $\mathbb{Z} + 2O_i$. Therefore, it takes time roughly proportional to $T^{\frac{3}{2}}$. We compute these theta series by implementing a method similar to that of Pizer in [18] (see Section 5 for examples).

The action of the (half-integral) Hecke operators T_{p^2} on the theta series $g(I)$ for $I \in X(\mathbb{R})$ is compatible with the action of the Hecke operators t_n on I. More precisely, one can show that $T_{p^2}(g(I)) = g(t_p(I))$ for all $p \nmid 4N$ and all $I \in X(\mathbb{R})$ (see Proposition 1.7 in [17]).

Theorem 4.2.1. *([4] Theorem 3.2) If I_F is a non-zero element in the F-isotypical component of $X(\mathbb{R})$, then $g = g(I_F)$ is in the Kohnen subspace of $S_{\frac{3}{2}}(\Gamma_0(4N), F)$, and $g(I_F)$ is non-zero if and only if $L(F, \frac{1}{2}) \neq 0$.*

We compute the eigenvector I_F by using the Brandt module package in Sage. Hence, if $I_F = \sum_{i=1}^{H} e_i[I_i]$, then g is computed as the linear combination

$$\sum_{i=1}^{H} e_i g([I_i]).$$

The above procedure can be generalized to compute modular forms of higher weights by using what are known as generalized theta series. Given a newform F in $S_{k-1}(\Gamma_0(N))$ with an odd square-free level N such that $k \equiv 3 \mod 4$ and $L(F, \frac{1}{2}) \neq 0$, the work of Böcherer and Schulze-Pillot guarantees the existence of a non-zero modular form $g = g_F$ in the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$. The form g is obtained as a linear combination of generalized theta series attached to the ternary quadratic form $N(x)$ and some homogeneous harmonic polynomials of degree $\frac{k-3}{2}$ in three variables. The reader is referred to [4] for a thorough discussion of this construction.

4.3 Theta Lifts

In the preceding treatment, we have only considered half-integral weight modular forms with the trivial character. A naturally arising problem is to compute a half-integral weight modular form g with a quadratic character $\boldsymbol{\chi}$, which belongs to the Kohnen subspace and maps to F via the Shimura correspondence. Unfortunately, the above classical construction does not generalize in any obvious or simple way to yield such a form. In theory, however, we are guaranteed the existence of g (under some restrictions on $\boldsymbol{\chi}$) by a beautiful result of Baruch and Mao (Theorem 10.1 in [2]) which, in parts, states the following.

Consider a newform F of even weight $k-1$, odd square-free level N and of trivial character. Let S_N be the set of primes dividing N, and let S be a subset of S_N. Write $N' = \prod_{p \in S} p$, and let $\boldsymbol{\chi} = \prod_{p|2N} \boldsymbol{\chi}_{(p)}$ be any even Dirichlet character defined modulo $4NN'$ such that $\boldsymbol{\chi}_{(p)} \equiv 1$ when $p|(N/N')$ and $\boldsymbol{\chi}_{(p)}(-1) = -1$ when $p|N'$. Set $\chi = \boldsymbol{\chi} \cdot \chi_{-1}^{s}$ where χ_{-1} is the Dirichlet character modulo 4 defined by $\chi_{-1}(n) = \left(\frac{-1}{n}\right)$ and s is the size of S. In particular, χ is unramified at the prime 2. There exists a unique (up to scalar multiple) cusp form g_S that is in the Kohnen subspace of $S_{\frac{k}{2}}(4NN', \boldsymbol{\chi}, F_\chi)$. The squares of the Fourier coefficients of g_S are related to the central values of the quadratic twists of the L-function of F.

In [2], the construction of g_S is of adelic nature because the authors worked in the setting of automorphic representations. A careful translation from the adelic language to the modular form language is needed to write down an explicit expression of g_S as a linear combination of classical theta series. We hope to pursue this work in the future.

Before we proceed into the next section, some remarks on the representation theoretic version of the Shimura correspondence seem to be in order. The theta correspondence for the pairs $(\widetilde{\mathrm{SL}_2}, PB^*)$ and $(\widetilde{\mathrm{SL}_2}, \mathrm{PGL}_2)$ lies at the heart of the above classical constructions. Following Waldspurger's treatment in [28] and [29], the half-integral weight modular form g can be realized as a theta lift from $\mathrm{PGL}_2(\mathbb{A})$ to $\widetilde{\mathrm{SL}_2(\mathbb{A})}$, the two-fold metaplectic cover of $\mathrm{SL}_2(\mathbb{A})$. Moreover, one can obtain g as a theta lift from $PB^*(\mathbb{A})$ to $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ (see [30]). To clarify things in the reader's mind, let us discuss these ideas briefly.

Let $\tilde{A}'_{\frac{k}{2}}(M, \chi_0)$ denote the space of cuspidal automorphic forms $\tilde{\phi} = \otimes_v \tilde{\phi}_v$ on $S_{\mathbb{Q}} \backslash \widetilde{\mathrm{SL}_2(\mathbb{A})}$ with character $\chi_0 = \boldsymbol{\chi} \cdot \chi_{-1}^{\frac{k-1}{2}}$ and level M. This space is determined by a set of local conditions satisfied by the vectors $\tilde{\phi}_v$ (see [29] pp. 381–388 for notation and other details). It is known that there is a natural bijection between $S'_{\frac{k}{2}}(M, \boldsymbol{\chi})$ and $\tilde{A}'_{\frac{k}{2}}(M, \chi_0)$. Thus, half-integral weight modular forms may be viewed as automorphic forms on $S_{\mathbb{Q}} \backslash \widetilde{\mathrm{SL}_2(\mathbb{A})}$ whenever convenient. In this framework, the Shimura correspondence has the following formulation. Suppose that $g \mapsto F$ under the Shimura map. Let $\tilde{\pi}$ be the automorphic representation of $S_{\mathbb{Q}} \backslash \widetilde{\mathrm{SL}_2(\mathbb{A})}$ associated to g, and let π be the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ associated to F. Then, $S_\psi(\tilde{\pi}) = \pi \otimes \chi_0^{-1}$, where ψ is the usual additive character on $\mathbb{A}/\mathbb{Q}$. The space $S_\psi(\tilde{\pi})$ is defined in [28] as $S_\psi(\tilde{\pi}) = \Theta(\tilde{\pi}, \psi^\nu) \otimes \chi_\nu$ for any choice of $\nu \in \mathbb{Q}^*$ such that $\Theta(\tilde{\pi}, \psi^\nu) \neq 0$.

Denote by π' the automorphic representation of PB^* associated to π via the Jacquet-Langlands correspondence. In the previous section, the form g is obtained as a special vector in the representation space of $\Theta(\pi', \psi)$ using the theta correspondence for the pair $(\widetilde{\mathrm{SL}_2}, PB^*)$. Notice that g coincides (up to scalar multiple) with the form g_S for $S = \phi$. This follows directly from the theorem described in the second paragraph of the present section since $N' = 1$ if $\boldsymbol{\chi}$ is the trivial character.

We shall now shed some light on the construction of the form g_S for any S as carried out in [2]. Using the theta correspondence of the pair $(\widetilde{\mathrm{SL}_2}, \mathrm{PGL}_2)$, the form g_S is obtained as a special vector in the representation space $V_{\tilde{\pi}^D}$ of

$$\Theta(\pi \otimes \chi_D, \psi^{|D|}) = \tilde{\pi}^D = \otimes_v \tilde{\pi}_v^D$$

for some fundamental discriminant D determined by S.

Let h be a non-zero modular form in $S_{\frac{k}{2}}(4NN', \boldsymbol{\chi}, F_\chi)$, and denote by

$\tilde{\pi}_h$ the automorphic representation of $S_{\mathbb{Q}}\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ generated by h. One gets that $\tilde{\pi}_h = \tilde{\pi}_{g_S} = \tilde{\pi}^D$. Moreover, the space $V'_{\tilde{\pi}^D} = \tilde{A}_{\frac{k}{2}}(4NN', \chi_0) \cap V_{\tilde{\pi}^D}$ is two-dimensional. In the next section, we verify this fact for $S = \phi$. Roughly speaking, this owes to the fact that the space of vectors in $\tilde{\pi}_2^D$ that satisfy the 2-adic condition forced on a form $\tilde{\phi} \in \tilde{A}_{\frac{k}{2}}(4NN', \chi_0)$ is two-dimensional. This is the space $V'_{\tilde{\pi}^D,2}$ in the factorization $V'_{\tilde{\pi}^D} = \otimes_v V'_{\tilde{\pi}^D,v}$. Moreover, if we let $\tilde{A}^+_{\frac{k}{2}}(4NN', \chi_0)$ be the image of the Kohnen subspace under the natural bijection $S_{\frac{k}{2}}(4NN', \boldsymbol{\chi}) \to \tilde{A}_{\frac{k}{2}}(4NN', \chi_0)$, we get that $\tilde{A}^+_{\frac{k}{2}}(4NN', \chi_0) \cap V_{\tilde{\pi}^D}$ is one-dimensional, and g_S is a non-zero vector in this space. In fact, $\tilde{A}^+_{\frac{k}{2}}(4NN', \chi_0)$ turns out to be the space of forms $\tilde{\phi} = \otimes_v \tilde{\phi}_v$ in $\tilde{A}_{\frac{k}{2}}(4NN', \chi_0)$ with $\tilde{\phi}_2$ being a carefully chosen vector in $V'_{\tilde{\pi}^D,2}$ (see Section 9.4 in [2]). It is exactly this choice of $\tilde{\phi}_2$ that forces g_S to lie in the Kohnen subspace.

4.4 On The Result Of Waldspurger

First, we state Waldspurger's Theorem in its most general form. Let $F \in S^{\mathrm{new}}_{k-1}(\boldsymbol{\chi}^2)$, and let π be the irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to F. The newform F is required to satisfy the following condition: when $\pi_p = \pi(\mu_{1,p}, \mu_{2,p})$ belongs to the principal series, then both characters $\mu_{1,p}$ and $\mu_{2,p}$ are even. Notice that $\mu_{1,p}(-1) = \mu_{2,p}(-1) = 1$ whenever π_p is unramified principal series. In fact, Flicker proved that this condition is satisfied if and only if $S_{\frac{k}{2}}(L, \boldsymbol{\chi}, F) \neq 0$ for some positive integer L. We also require that one of the following conditions is satisfied:

1. The level of F is divisible by 16
2. The conductor of the character χ_0 is divisible by 16, where $\chi_0(n) = \boldsymbol{\chi}(n)\left(\frac{-1}{n}\right)^{\frac{k-1}{2}}$
3. π_2 is not supercuspidal

For every positive integer e and prime number p, Waldspurger used the local components of π and χ_0 to define an integer n_p and a set $U_p(e, F)$ of finitely many functions $c_p : \mathbb{Q}_p^* \to \mathbb{C}$. According to these definitions, given an integer $E \geq 1$, for all but finitely many p the set $U_p(v_p(E), F)$ is a singleton and $n_p = 0$. In fact, we are interested in the finite set $U(E, F) = \prod_{p\ prime} U_p(v_p(E), F)$.

Let $\mathbb{N}^{\mathrm{sf}}$ be the set of positive square-free integers, and let $A : \mathbb{N}^{\mathrm{sf}} \to \mathbb{C}$ be any function. Denote square-free part of a positive integer n by n^{sf}. Given an element $c_E = (c_{E,p}) \in U(E, F)$, we define the function $f(c_E, A)$ on $\mathbb{H}$ by:

$$f(c_E, A) = \sum_{n \geq 1} a_n(c_E, A) q^n,$$

where

$$a_n(c_E, A) = A(n^{\mathrm{sf}}) n^{\frac{k-2}{4}} \prod_p c_{E,p}(n).$$

The integer $\prod_p p^{n_p}$ is denoted by $N(F)$, and the complex vector space generated by the set $\{f(c_E, A_F)\}_{c_E \in U(E,F)}$ is denoted by $U(E, F, A_F)$.

Given a Dirichlet character ν, let $L(\nu, s)$ be the associated L-function. Recall that, for sufficiently large $\operatorname{Re} s$, $L(\nu, s)$ is defined as

$$L(\nu, s) = \pi^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) \sum_{n=1}^{\infty} \hat{\nu}(n) n^{-s},$$

where δ is such that $\nu(-1) = (-1)^\delta$ and $\hat{\nu}$ is the primitive Dirichlet character associated to ν. It is known that $L(\nu, s)$ has a meromorphic continuation to all s. Define $\epsilon(\nu, s)$ as the ϵ-factor that appears in the functional equation $L(\nu^{-1}, 1-s) = \epsilon(\nu, s) L(\nu, s)$.

For any $t \in \mathbb{Z}^*$, denote by χ_t the quadratic Dirichlet character associated with the extension $\mathbb{Q}(\sqrt{t})/\mathbb{Q}$. Notice that $\chi_t(n) = (\frac{\Delta_t}{n})$, where Δ_t is the discriminant of $\mathbb{Q}(\sqrt{t})$. In particular, $\chi_t(n) = 1$ for all $n \in \mathbb{Z}^*$ if t is a perfect square.

Theorem 4.4.1. (*See [29]*) *Given a newform F satisfying the above hypotheses, there exists a function $A_F : \mathbb{N}^{\mathrm{sf}} \to \mathbb{C}$, such that:*

1. $(A_F(t))^2 = L(F \otimes \chi_0^{-1}\chi_t, \frac{1}{2}) \epsilon(\chi_0^{-1}\chi_t, \frac{1}{2})$
2. $S_{\frac{k}{2}}(4N, F, \chi) = \bigoplus U(E, F, A_F)$; *the direct sum being taken over all integers E such that $N(F)|E|4N$*

Our goal is thus reduced to computing the individual spaces $U(E, F, A_F)$. This amounts to computing the half-integral weight modular forms $f(c_E, A_F)$ for all $c_E \in U(E, F)$. Recall that given $c_E = (c_{E,p}) \in U(E, F)$, the function $f(c_E, A_F)$ is written as $\sum_{n \geq 1} a_n(c_E, A_F) q^n$, and the Fourier coefficients

$a_n(c_E, A_F)$ are defined by the formula

$$a_n(c_E, A_F) = A_F(n^{\mathrm{sf}}) n^{\frac{k-2}{4}} \prod_p c_{E,p}(n).$$

The local factor $\prod_p c_{E,p}(n)$ is completely determined by the local properties of F and χ_0. In fact, the various local objects involved in this calculation are given explicitly in [29]. However, the determination of the global factor $A(n^{\mathrm{sf}})$ requires more effort because we are only given its square value. One way to resolve this complication is to use the Fourier coefficients of the half-integral weight modular form g that was computed in Section 2.

Let us now return to our usual setting. Recall that we are given a newform $F = \sum_{n\geq 1} b_n q^n$ in $S_{k-1}(\Gamma_0(N))$ with odd square-free level N such that $k \equiv 3 \mod 4$ and $L(F, \frac{1}{2}) \neq 0$. Let $g = \sum_{n\geq 1} a_n q^n \in S_{\frac{k}{2}}(\Gamma_0(4N), F)$ be the form obtained in Section 2.

For every prime number p not dividing N, we put $\lambda_p = b_p p^{1-\frac{k}{2}}$, $\alpha_p + \alpha'_p = \lambda_p$, and $\alpha_p \alpha'_p = 1$. We also set $\lambda'_p = b_p p^{1-\frac{k}{2}}$ for every prime divisor p of N. We shall now apply the theorem of Waldspurger to compute a basis for the space $S_{\frac{k}{2}}(\Gamma_0(4N), F)$. In order to compute the local factors $\prod_p c_p$ of the Fourier coefficients of the desired basis elements, we use the explicit formulae given in [29] Section 8.

Since the newform F has a square-free level and the trivial character modulo N, we get

$$n_p = \begin{cases} 2 & \text{if } p = 2; \\ 1 & \text{if } p|N; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $N(F) = \prod p^{n_p} = 4N$. We also get

$$S_{\frac{k}{2}}(\Gamma_0(4N), F) = \bigoplus_{N(F)|E|4N} U(E, F, A_F) = U(4N, F, A_F),$$

where $U(E, F, A_F)$ is the vector space generated by the modular forms $\{f(c_E, A_F)\}_{c_E \in U(E,F)}$, and $U(E, F)$ is the finite set $\prod_p U_p(v_p(E), F)$. Recall that for a given prime number p and a positive integer E, the set

$U_p(v_p(E), F)$ consists of finitely many local functions c_p. Using the fact that $E = 4N$ and N is odd and square-free, we get

$$U_2(2, F) = \begin{cases} \{c_2'[\alpha_2], c_2'[\alpha_2']\} & \text{if } \alpha_2 \neq \alpha_2'; \\ \{c_2'[\alpha_2], c_2''[\alpha_2]\} & \text{otherwise,} \end{cases}$$

where the local functions c_2' and c_2'' are defined below.

If p is an odd prime we get,

$$U_p(v_p(4N), F) = \begin{cases} \{c_p^0[\lambda_p]\} & \text{if } p \nmid N; \\ \{c_p^s[\lambda_p']\} & \text{otherwise.} \end{cases}$$

Before we proceed any further, we need to recall the definition of the Hilbert symbol. Given two non-zero elements a and b in a local field K, the Hilbert symbol is defined by

$$(a, b) = \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a non-zero solution in } K^3; \\ -1 & \text{otherwise.} \end{cases}$$

In particular, if $a, b \in \mathbb{Z}$, write $a = 2^\alpha s$ and $b = 2^\beta t$ such that s and t are odd integers. The Hilbert symbol over the 2-adics is

$$(a, b)_2 = (-1)^{\epsilon(s)\epsilon(t) + \alpha\omega(t) + \beta\omega(s)},$$

where $\epsilon(x) = \frac{x-1}{2}$ and $\omega(x) = \frac{x^2-1}{8}$.

In order to evaluate the local functions c_p at an integer n, we need to express n as up^{2h} ($v_p(u) \in \{0, 1\}$), then use the following formulae:

$$c_2'[\delta](n) = \begin{cases} \delta^h(\delta - 2^{-\frac{1}{2}}(2, u)_2) & \text{if } v_2(u) = 0 \text{ and } (u, -1)_2 = -1; \\ \delta^h, & \text{otherwise.} \end{cases}$$

$$c_2''[\delta](n) = \begin{cases} \delta(c_2''[\delta](\frac{n}{4}) + c_2'[\delta](\frac{n}{4})) & \text{if } v_2(n) \geq 2 \\ \delta & \text{if } v_2(n) = 0 \text{ and } (n, -1)_2 = -1; \\ 0 & \text{otherwise.} \end{cases}$$

$$c_p^s[\delta](n) = \begin{cases} \delta^h & \text{if } v_p(u) = 1; \\ 2^{\frac{1}{2}}\delta^h & \text{if } v_p(u) = 0 \text{ and } \left(\frac{-u}{p}\right) = -p^{\frac{1}{2}}\delta; \\ 0, & \text{otherwise.} \end{cases}$$

$$c_p^0[\delta](n) = \begin{cases} 1 & \text{if } v_p(u) = 0 \text{ and } h = 0; \\ b_h - b_{h-1}\left(\frac{-u}{p}\right)p^{-\frac{1}{2}} & \text{if } v_p(u) = 0 \text{ and } h \geq 1; \\ b_h & \text{otherwise,} \end{cases} \tag{4.2}$$

where

$$b_h = \frac{1}{2^h}\left(\sum_{i=0}^{\lfloor \frac{h}{2} \rfloor} \binom{h+1}{2i+1} \delta^{h-2i}(\delta^2 - 4)^i\right).$$

If $\alpha_2 \neq \alpha_2'$, the set $U(4N, F)$ consists of

$$\mathbf{c_1} = c_2'[\alpha_2] \prod_{p|N} c_p^s[\lambda_p'] \prod_{p\nmid 2N} c_p^0[\lambda_p] \quad \text{and} \quad \mathbf{c_2} = c_2'[\alpha_2'] \prod_{p|N} c_p^s[\lambda_p'] \prod_{p\nmid 2N} c_p^0[\lambda_p].$$

If $\alpha_2 = \alpha_2'$, the set $U(4N, F)$ consists of

$$\mathbf{c_1} = c_2'[\alpha_2] \prod_{p|N} c_p^s[\lambda_p'] \prod_{p\nmid 2N} c_p^0[\lambda_p] \quad \text{and} \quad \mathbf{c_2'} = c_2''[\alpha_2] \prod_{p|N} c_p^s[\lambda_p'] \prod_{p\nmid 2N} c_p^0[\lambda_p].$$

Theorem 4.4.1 gives a basis for $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ that consists of the functions

$$f(\mathbf{c_1}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_1}, A_F)q^n \text{ and } f(\mathbf{c_2}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_2}, A_F)q^n \text{ if } \alpha_2 \neq \alpha_2'$$

and

$$f(\mathbf{c_1}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_1}, A_F)q^n \text{ and } f(\mathbf{c_2'}, A_F) = \sum_{n\geq 1} a_n(\mathbf{c_2'}, A_F)q^n \text{ if } \alpha_2 = \alpha_2'.$$

Recall that

$$a_n(\mathbf{c_1}, A_F) = A_F(n^{\mathrm{sf}})n^{\frac{k-2}{4}} c_2'[\alpha_2](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n)$$

$$a_n(\mathbf{c_2}, A_F) = A_F(n^{\mathrm{sf}})n^{\frac{k-2}{4}} c_2'[\alpha_2'](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n)$$

and

$$a_n(\mathbf{c_2'}, A_F) = A_F(n^{\mathrm{sf}})n^{\frac{k-2}{4}} c_2''[\alpha_2](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n).$$

Lemma 4.4.2. *For every positive integer n such that $(-1)^{\frac{k-1}{2}} n = -n \equiv 2, 3 \mod 4$, we have $a_n(\mathbf{c_1}, A_F) = a_n(\mathbf{c_2}, A_F)$ and $a_n(\mathbf{c_2'}, A_F) = 0$.*

Proof. Write n as $u2^{2h}$ with $v_2(u) \in \{0,1\}$. If $n \equiv 1 \mod 4$, we get $c_2'[\alpha_2](n) = (\alpha_2)^0 = 1 = (\alpha_2')^0 = c_2'[\alpha_2'](n)$ and $c_2''[\alpha_2](n) = 0$ since $h = 0$, $v_2(u) = 0$ and $(u,-1)_2 = 1$. If $n \equiv 2 \mod 4$, we also get $c_2'[\alpha_2](n) = (\alpha_2)^0 = 1 = (\alpha_2')^0 = c_2'[\alpha_2'](n)$ and $c_2''[\alpha_2](n) = 0$ since $h = 0$ and $v_2(u) = 1$. Thus, $a_n(\mathbf{c_1}, A_F) = a_n(\mathbf{c_2}, A_F)$ and $a_n(\mathbf{c_2'}, A_F) = 0$ for all $n \equiv 1, 2 \mod 4$. □

Lemma 4.4.3. *If $\alpha_2 \neq \alpha_2'$, the function $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F)$ belongs to the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$. Otherwise, the function $f(\mathbf{c_2'}, A_F)$ belongs to the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$*

Proof. The coefficients $a_n(\mathbf{c_1}, A_F) - a_n(\mathbf{c_2}, A_F)$ and $a_n(\mathbf{c_2'}, A_F)$ are zero for all n such that $(-1)^{\frac{k-1}{2}} n \equiv 2, 3 \mod 4$. □

Proposition 4.4.4. *If $\alpha_2 \neq \alpha_2'$, the half-integral weight modular form g is a non-zero scalar multiple of $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F)$. Otherwise, g is a non-zero scalar multiple of $f(\mathbf{c_2'}, A_F)$.*

Proof. We know by the multiplicity one result of Kohnen (Theorem 2 in [14]) that the Kohnen subspace of $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ is one-dimensional. We also know from Section 1 that g belongs to the Kohnen subspace. This forces g to be equal to a non-zero scalar multiple of $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F)$ when $\alpha_2 \neq \alpha_2'$ and a non-zero scalar multiple of $f(\mathbf{c_2'}, A_F)$ when $\alpha_2 = \alpha_2'$. □

Let t be a positive square-free integer and denote by Δ_{-t} the fundamental discriminant corresponding to $-t$. Recall that $g = \sum_{n \geq 1} a_n q^n$. Let r_1 and r_2 be the non-zero complex constants such that $f(\mathbf{c_1}, A_F) - f(\mathbf{c_2}, A_F) = r_1 g$ if $\alpha_2 \neq \alpha_2'$ and $f(\mathbf{c_2'}, A_F) = r_2 g$ if $\alpha_2 = \alpha_2'$. We know by the previous proposition that

$$a_{|\Delta_{-t}|}(\mathbf{c_1}, A_F) - a_{|\Delta_{-t}|}(\mathbf{c_2}, A_F) = r_1 a_{|\Delta_{-t}|} \quad \text{if } \alpha_2 \neq \alpha_2'$$

and

$$a_{|\Delta_{-t}|}(\mathbf{c_2'}, A_F) = r_2 a_{|\Delta_{-t}|} \quad \text{if } \alpha_2 = \alpha_2'.$$

We also know that

$$\begin{aligned} r_1 a_{|\Delta_{-t}|} &= a_{|\Delta_{-t}|}(\mathbf{c_1}, A_F) - a_{|\Delta_{-t}|}(\mathbf{c_2}, A_F) \\ &= A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}} \left(c_2'[\alpha_2](|\Delta_{-t}|) - c_2'[\alpha_2'](|\Delta_{-t}|)\right) \prod_{p|N} c_p^s[\lambda_p'](|\Delta_{-t}|) \end{aligned}$$

and

$$r_2 a_{|\Delta_{-t}|} = a_{|\Delta_{-t}|}(\mathbf{c'_2}, A_F) = A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}} c''_2[\alpha_2](|\Delta_{-t}|) \prod_{p|N} c^s_p[\lambda'_p](|\Delta_{-t}|).$$

Notice that the local factors for $p \nmid 2N$ do not appear in the above expressions. In fact, $c^0_p[\lambda_p](|\Delta_{-t}|) = 1$ for all such p since $|\Delta_{-t}|$ is not divisible by the square of any odd prime.

Hence, if $\alpha_2 \neq \alpha'_2$, we get

$$A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}} \left(c'_2[\alpha_2](|\Delta_{-t}|) - c'_2[\alpha'_2](|\Delta_{-t}|)\right) \prod_{p|N} c^s_p[\lambda'_p](|\Delta_{-t}|) = r_1 a_{|\Delta_{-t}|}.$$

Otherwise, we get

$$A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}} c''_2[\alpha_2](|\Delta_{-t}|) \prod_{p|N} c^s_p[\lambda'_p](|\Delta_{-t}|) = r_2 a_{|\Delta_{-t}|}.$$

Since Δ_{-t} is a fundamental discriminant, it is not divisible by any square of any odd prime, and it satisfies $\Delta_{-t} \equiv 1 \mod 4$ or $\Delta_{-t} \equiv 8, 12 \mod 16$. Thus,

$$c'_2[\alpha_2](|\Delta_{-t}|) - c'_2[\alpha'_2](|\Delta_{-t}|) = \alpha_2 - \alpha'_2$$

and

$$c''_2[\alpha_2](|\Delta_{-t}|) = \alpha_2.$$

We still need to investigate the value of $\prod_{p|N} c^s_p[\lambda'_p](|\Delta_{-t}|)$. Since $|\Delta_{-t}|$ is not divisible by the square of any odd prime $c^s_p[\lambda'_p](|\Delta_{-t}|) = 1$ for all p such that $p|N$ and $p|\Delta_{-t}$. If $p|N$ and $p \nmid \Delta_{-t}$ then

$$c^s_p[\lambda'_p](|\Delta_{-t}|) = \begin{cases} 2^{\frac{1}{2}} & \text{if } \left(\frac{\Delta_{-t}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\prod_{p|N} c^s_p[\lambda'_p](|\Delta_{-t}|) = \begin{cases} \prod\limits_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{\frac{1}{2}} & \text{if } \left(\frac{\Delta_{-t}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p \ \forall p,\ p|N \text{ and } p \nmid \Delta_{-t}; \\ 0 & \text{otherwise.} \end{cases}$$

Putting this together gives

$$A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}}(\alpha_2 - \alpha'_2) \prod_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{\frac{1}{2}} = r_1 a_{|\Delta_{-t}|} \quad \text{if } \alpha_2 \neq \alpha'_2$$

and

$$A_F(t)|\Delta_{-t}|^{\frac{k-2}{4}}\alpha_2 \prod_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{\frac{1}{2}} = r_2 a_{|\Delta_{-t}|} \quad \text{if } \alpha_2 = \alpha_2'$$

whenever $\left(\frac{\Delta_{-t}}{p}\right) = -p^{\frac{1}{2}}\lambda_p'$ for all p such that $p|N$ and $p \nmid \Delta_{-t}$. Hence we obtain the following theorem which is the same as Theorem 1.1.4 stated in Chapter 1.

Theorem 4.4.5. *For a positive square-free integer t satisfying $\left(\frac{\Delta_{-t}}{p}\right) = -p^{\frac{1}{2}}\lambda_p'$ for all p such that $p|N$ and $p \nmid \Delta_{-t}$, we have*

$$A_F(t) = r2^{\frac{\nu_t}{2}}|\Delta_{-t}|^{\frac{2-k}{4}} a_{|\Delta_{-t}|},$$

where $2^{\frac{\nu_t}{2}} = \prod_{\substack{p|N \\ p\nmid\Delta_{-t}}} 2^{-\frac{1}{2}}$, and r is a non-zero complex constant depending only on F.

Remark 4.4.6. *Let n be a positive integer such that its square-free part which we denote by n^{sf} satisfies $\left(\frac{\Delta_{-n^{\mathrm{sf}}}}{p}\right) = p^{\frac{1}{2}}\lambda_p'$ for some prime p such that $p|N$ and $p \nmid \Delta_{-n^{\mathrm{sf}}}$. For this particular p, write $n = up^{2h}$ ($v_p(u) \in \{0,1\}$). Since $p \nmid \Delta_{-n^{\mathrm{sf}}}$, we get $v_p(u) = 0$. Moreover, $\left(\frac{-u}{p}\right) = \left(\frac{\Delta_{-n^{\mathrm{sf}}}}{p}\right) = p^{\frac{1}{2}}\lambda_p'$. This forces $c_p^s[\lambda_p']$ to vanish at n. Therefore, $a_n(\mathbf{c_1}, A_F)$, $a_n(\mathbf{c_2}, A_F)$, $a_n(\mathbf{c_2'}, A_F)$ vanish for all such n regardless of the value of $A_F(n^{\mathrm{sf}})$.*

Before stating our main theorem, we shall verify that $f(\mathbf{c_1}, A_F)$ and $f(\mathbf{c_2}, A_F)$ are not scalar multiples of each other even though they share the same Hecke eigenvalues at the primes $p \nmid N$. We assume that $\alpha_2 \neq \alpha_2'$ purely for simplicity of exposition. Let n be a positive integer such that $a_n(\mathbf{c_1}, A_F) \neq 0$. In particular, $A_F(n^{\mathrm{sf}}) \neq 0$ and $c_p^s[\lambda_p'](n) \neq 0$. We set $r = |\Delta_{-n^{\mathrm{sf}}}|$ and $w = 4|\Delta_{-n^{\mathrm{sf}}}|$ to simplify notation. Hence,

$$\prod_{p|N} c_p^s[\lambda_p'](r) = \prod_{p|N} c_p^s[\lambda_p'](w) = \prod_{\substack{p|N \\ p\nmid r}} 2^{\frac{1}{2}}. \tag{4.3}$$

Moreover, for $\delta \in \{\alpha_2, \alpha_2'\}$ we get

$$c_2'[\delta](r) = \begin{cases} \delta - 2^{-\frac{1}{2}}(2,r)_2 & \text{if } r \equiv 3 \mod 4; \\ \delta & \text{otherwise,} \end{cases} \tag{4.4}$$

and

$$c_2'[\delta](w) = \begin{cases} \delta(\delta - 2^{-\frac{1}{2}}(2,r)_2) & \text{if } r \equiv 3 \mod 4; \\ \delta^2 & \text{otherwise.} \end{cases} \tag{4.5}$$

Notice that the coefficients of q^r and q^w in $f(\mathbf{c_1}, A_F)$ and $f(\mathbf{c_2}, A_F)$ are non-zero. The following table shows that these coefficients are not in a constant ratio to each other. In fact, $\frac{a_w(\mathbf{c_1},A_F)}{a_w(\mathbf{c_2},A_F)} = \frac{\alpha_2}{\alpha_2'}\left(\frac{a_r(\mathbf{c_1},A_F)}{a_r(\mathbf{c_2},A_F)}\right)$.

Table 4.1: Ratios of Fourier Coefficients

	$r \equiv 3 \mod 4$	$r \equiv 4, 8 \mod 16$
$\frac{a_r(\mathbf{c_1},A_F)}{a_r(\mathbf{c_2},A_F)}$	$\frac{\alpha_2 - 2^{-\frac{1}{2}}(2,r)_2}{\alpha_2' - 2^{-\frac{1}{2}}(2,r)_2}$	$\frac{\alpha_2}{\alpha_2'}$
$\frac{a_w(\mathbf{c_1},A_F)}{a_w(\mathbf{c_2},A_F)}$	$\frac{\alpha_2(\alpha_2 - 2^{-\frac{1}{2}}(2,r)_2)}{\alpha_2'(\alpha_2' - 2^{-\frac{1}{2}}(2,r)_2)}$	$(\frac{\alpha_2}{\alpha_2'})^2$

In what follows, the terms $K_1(n)$, $K_2(n)$ and $K_2'(n)$ refer to the formulae

$$K_1(n) = 2^{\frac{\nu_n}{2}} a_{|\Delta_{-n^{\mathrm{sf}}}|} |\Delta_{-n^{\mathrm{sf}}}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} c_2'[\alpha_2](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n) \tag{4.6}$$

$$K_2(n) = 2^{\frac{\nu_n}{2}} a_{|\Delta_{-n^{\mathrm{sf}}}|} |\Delta_{-n^{\mathrm{sf}}}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} c_2'[\alpha_2'](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n) \tag{4.7}$$

and

$$K_2'(n) = 2^{\frac{\nu_n}{2}} a_{|\Delta_{-n^{\mathrm{sf}}}|} |\Delta_{-n^{\mathrm{sf}}}|^{\frac{2-k}{4}} n^{\frac{k-2}{4}} c_2''[\alpha_2](n) \prod_{p|N} c_p^s[\lambda_p'](n) \prod_{p\nmid 2N} c_p^0[\lambda_p](n). \tag{4.8}$$

The following Theorem is the main result of the current chapter and is the same as Theorem 1.1.5 stated in Chapter 1.

Theorem 4.4.7. *Let $F = \sum_{n\geq 1} b_n q^n$ be a newform in $S_{k-1}^{\mathrm{new}}(\Gamma_0(N))$ with odd square-free level N such that $k \equiv 3 \mod 4$ and $L(F, \frac{1}{2}) \neq 0$.*
Let $g = \sum_{n\geq 1} a_n q^n \in S_{\frac{k}{2}}(\Gamma_0(4N), F)$ be the form obtained in Section 2. Put $f_1 = \sum_{n\geq 1} a_n(f_1) q^n$, $f_2 = \sum_{n\geq 1} a_n(f_2) q^n$ and $f_2' = \sum_{n\geq 1} a_n(f_2') q^n$ with

$$a_n(f_1) = \begin{cases} K_1(n) & \text{if } \left(\frac{\Delta_{-n^{\mathrm{sf}}}}{p}\right) = -p^{\frac{1}{2}}\lambda_p' \ \forall p,\ p|N \text{ and } p \nmid \Delta_{-n^{\mathrm{sf}}}; \\ 0 & \text{otherwise,} \end{cases}$$

$$a_n(f_2) = \begin{cases} K_2(n) & \textit{if } \left(\frac{\Delta_{-n^{\mathrm{sf}}}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p \ \forall p,\ p|N \textit{ and } p \nmid \Delta_{-n^{\mathrm{sf}}}; \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$a_n(f'_2) = \begin{cases} K'_2(n) & \textit{if } \left(\frac{\Delta_{-n^{\mathrm{sf}}}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p \ \forall p,\ p|N \textit{ and } p \nmid \Delta_{-n^{\mathrm{sf}}}; \\ 0 & \textit{otherwise.} \end{cases}$$

Then $S_{\frac{k}{2}}(\Gamma_0(4N), F)$ is generated by f_1 and f_2 if $\alpha_2 \neq \alpha'_2$, and it is generated by f_1 and f'_2 if $\alpha_2 = \alpha'_2$.

In order to use Theorem 4.4.7 as an effective tool for computing a basis for $S_{\frac{3}{2}}(\Gamma_0(4N), F)$, we need to make the following observations. Once again, we assume that $\alpha_2 \neq \alpha'_2$ to simplify the exposition and avoid unrewarding details.

Let $h = 2^{\frac{1}{2}}(f_1 + f_2)$ and write $h = \sum_{n\geq 1} a_n(h)q^n$. First, we compute the Fourier coefficients $a_n(h)$ for positive square-free integers n such that $\left(\frac{n}{p}\right) = -p^{\frac{1}{2}}\lambda'_p$ whenever $p|N$ and $p \nmid n$ (since, for square-free n, $a_n(h) = 0$ otherwise). A straightforward calculation using Theorem 4.4.7, (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), and (4.8) shows that

$$a_n(h) = \begin{cases} a_n(g)(b_2 - 2(2,n)_2) & \text{if } n \equiv 3 \mod 4; \\ 2a_{4n}(g) & \text{otherwise.} \end{cases}$$

Let us now consider a positive integer n, which is not divisible by any square prime to $4N$, and write $n = n^{\mathrm{sf}}y^2$. We also assume that $\left(\frac{n^{\mathrm{sf}}}{p}\right) = -p^{\frac{1}{2}}\lambda'_p$ whenever $p|N$ and $p \nmid n^{\mathrm{sf}}$. To simplify notation, we set $s = v_2(y)$. Another simple calculation shows that if $n^{\mathrm{sf}} \equiv 3 \mod 4$, then

$$a_n(h) = 2^{\frac{s+1}{2}} a_{n^{\mathrm{sf}}}(g)(\alpha_2^{s+1} + \alpha_2'^{s+1} - 2^{-\frac{1}{2}}(2, n^{\mathrm{sf}})_2(\alpha_2^s + \alpha_2'^s)) \prod_{p|N} b_p^{v_p(y)}.$$

Otherwise, we get

$$a_n(h) = 2^{\frac{s}{2}-1} a_{n^{\mathrm{sf}}}(h)(\alpha_2^s + \alpha_2'^s) \prod_{p|N} b_p^{v_p(y)}.$$

Finally, we use Proposition 14 on page 210 in [13] to compute $a_n(h)$ for an arbitrary positive integer n. Suppose that n_0 is a positive integer, which

is not divisible by the square of any prime $p \nmid 4N$. Then this proposition provides us with the following recursive formulae which hold for all $p \nmid 4N$ and n_1 prime to p:

$$a_{n_0 n_1^2 p^2}(h) = a_{n_0 n_1^2}(h)\left(b_p - \left(\frac{-n_0}{p}\right)\right)$$

and

$$a_{n_0 n_1^2 p^{2(v+1)}}(h) = a_{n_0 n_1^2 p^{2v}}(h)\left(b_p - \left(\frac{-n_0}{p}\right)\right) - p a_{n_0 n_1^2 p^{2(v-1)}}(h) \quad v = 1, 2, ...$$

We implemented all of the above formulae in Sage. The result is a function that outputs the Fourier expansion of h up to a desired precision. Thus, we get a basis for $S_{\frac{3}{2}}(\Gamma_0(4N), F)$ consisting of the modular forms g and h.

4.5 Examples

We are interested in computing a basis for the space $S_{\frac{3}{2}}(\Gamma_0(60), F)$, where F is the newform in $S_2^{\text{new}}(\Gamma_0(15))$ corresponding to the elliptic curve $y^2 + xy + y = x^3 + x^2 - 10x - 10$. We know that

$$F = \sum_{n \geq 1} b_n q^n = q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + O(q^{12}).$$

Since $W_3(F) = F$ and $W_5(F) = -F$, we consider the definite quaternion algebra B ramified at 5. Let O be an order in B of level $N = 15$. O has two left ideal classes $[I_1]$ and $[I_2]$ which give rise to two orders $O_1 = O_r(I_1)$ and $O_2 = O_r(I_2)$. The subgroups R_1 and R_2 of trace zero elements in $\mathbb{Z} + 2O_1$ and $\mathbb{Z} + 2O_2$ have bases $\{i + \frac{1}{3}j + \frac{4}{3}k, \frac{2}{3}j + \frac{8}{3}k, 3k\}$ and $\{\frac{4}{3}j - \frac{2}{3}k, i + \frac{2}{3}j - \frac{1}{3}k, j - 2k\}$ respectively. The two theta series generated by the norm form of the quaternion algebra evaluated on R_1 and R_2 are:

$$\begin{aligned} g(I_1) = \sum_{x \in R_1} q^{N(x)} = \frac{1}{2} &+ q^3 + q^{12} + 3q^{20} + 6q^{23} + q^{27} + 6q^{32} + 6q^{47} + q^{48} \\ &+ 3q^{60} + 6q^{63} + 6q^{68} + 6q^{72} + q^{75} + 3q^{80} + 6q^{83} + 6q^{87} \\ &+ 6q^{92} + 6q^{95} + O(q^{100}) \end{aligned}$$

and

$$g(I_2) = \sum_{x \in R_2} q^{N(x)} = \frac{1}{2} + 2q^8 + q^{12} + q^{15} + q^{20} + 4q^{23} + 2q^{27} + 4q^{32} + 2q^{35} + 8q^{47} + 3q^{48} + q^{60} + 6q^{63} + 6q^{68} + 4q^{72} + 5q^{80} + 4q^{83} + 4q^{87} + 10q^{92} + 6q^{95} + O(q^{100}).$$

The vector $I_F = I_1 - I_2$ is in the F-isotypical component of $X(\mathbb{R})$. Recall that I_F is unique up to a scalar multiple and satisfies $t_p(I_F) = b_p I_F$ for all $p \nmid N$, where the action of the Hecke operators $\{t_p\}_{p \nmid N}$ is determined by the Brandt matrices $\{B_p\}_{p \nmid N}$. Hence, one can find the eigenvector I_F using the Brandt module package in Sage by computing a few Brandt matrices and knowing a few Fourier coefficients of F.

Hence, the modular form

$$g = g(I_1) - g(I_2) = q^3 - 2q^8 - q^{15} + 2q^{20} + 2q^{23} - q^{27} + 2q^{32} - 2q^{35} - 2q^{47} - 2q^{48} + 2q^{60} + 2q^{72} + q^{75} - 2q^{80} + 2q^{83} + 2q^{87} - 4q^{92} + O(q^{100})$$

lies in the space $S_{\frac{3}{2}}(\Gamma_0(60), F)$.

We shall now apply the theorem of Waldspurger to compute a basis for this space. In order to determine the local objects necessary to calculate the local factor $\prod c_p(n)$, we use the explicit formulae given in [29]. As a result of these calculations, we get $N(F) = 2^2 \times 3 \times 5$ and $S_{\frac{3}{2}}(\Gamma_0(60), F) = U(60, F, A_F)$. We also get:

$$U_3(1, F) = \{c_3^s[\lambda_3]\} \quad (\lambda_3 = 3^{-\frac{1}{2}} b_3),$$

$$U_5(1, F) = \{c_5^s[\lambda_5]\} \quad (\lambda_5 = 5^{-\frac{1}{2}} b_5),$$

$$U_2(2, F) = \{c_2'[\alpha_2], c_2'[\alpha_2']\} \quad (\alpha_2 + \alpha_2' = 2^{-\frac{1}{2}} b_2,\ \alpha_2 \alpha_2' = 1),$$

$$U_p(0, F) = \{c_p^0[\lambda_p]\},\ \text{for all } p \neq 2, 3, 5 \quad (\lambda_p = p^{-\frac{1}{2}} b_p).$$

Since the set $U(60, F)$ constitutes of two elements $\mathbf{c_1}$ and $\mathbf{c_2}$, the space $U(60, F, A_F)$ is generated by the two functions $f(\mathbf{c_1}, A_F)$ and $f(\mathbf{c_2}, A_F)$ whose Fourier coefficients are calculated as follows:

$$a_n(\mathbf{c_1}, A_F) = A_F(n^{\mathrm{sf}}) n^{\frac{1}{4}} \underbrace{c_3^s[\lambda_3](n) c_5^s[\lambda_5](n) c_2'[\alpha_2](n) \prod_{p \neq 2,3,5} c_p^0[\lambda_p](n)}_{C_1(n)}$$

$$a_n(\mathbf{c_2}, A_F) = A_F(n^{\mathrm{sf}})n^{\frac{1}{4}} \underbrace{c_3^s[\lambda_3](n)c_5^s[\lambda_5](n)c_2'[\alpha_2'](n) \prod_{p\neq 2,3,5} c_p^0[\lambda_p](n)}_{C_2(n)}.$$

We carried out these computations and obtained the first few local factors C_1 and C_2 in Table 4.2 below.

Table 4.2: The Local Factors C_1 and C_2

n	$C_1(n)$	$C_2(n)$	n	$C_1(n)$	$C_2(n)$
2	2	2	13	0	0
3	$\frac{1+i\sqrt{7}}{2}$	$\frac{1-i\sqrt{7}}{2}$	14	0	0
4	0	0	15	$\frac{-3+i\sqrt{7}}{2\sqrt{2}}$	$\frac{-3-i\sqrt{7}}{2\sqrt{2}}$
5	$\sqrt{2}$	$\sqrt{2}$	16	0	0
6	0	0	17	2	2
7	0	0	18	$-\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$
8	$\frac{-1+i\sqrt{7}}{\sqrt{2}}$	$\frac{-1-i\sqrt{7}}{\sqrt{2}}$	19	0	0
9	0	0	20	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
10	0	0	21	0	0
11	0	0	22	0	0
12	$-\sqrt{2}$	$-\sqrt{2}$	23	$\frac{-3+i\sqrt{7}}{\sqrt{2}}$	$\frac{-3-i\sqrt{7}}{\sqrt{2}}$

Therefore,

$$\begin{aligned} f(\mathbf{c_1}, A_F) = {} & A_F(2)2^{\frac{1}{4}}(2)q^2 + A_F(3)3^{\frac{1}{4}}\left(\frac{1+i\sqrt{7}}{2}\right)q^3 + A_F(5)5^{\frac{1}{4}}(\sqrt{2})q^5 \\ & - A_F(2)2^{\frac{1}{4}}(1-i\sqrt{7})q^8 - 2A_F(3)3^{\frac{1}{4}}q^{12} - A_F(15)15^{\frac{1}{4}}\left(\frac{3-i\sqrt{7}}{2\sqrt{2}}\right)q^{15} \\ & + 2A_F(17)17^{\frac{1}{4}}q^{17} - 2A_F(2)2^{\frac{1}{4}}q^{18} - A_F(5)5^{\frac{1}{4}}\left(\frac{1-i\sqrt{7}}{\sqrt{2}}\right)q^{20} \\ & - A_F(23)23^{\frac{1}{4}}\left(\frac{3-i\sqrt{7}}{\sqrt{2}}\right)q^{23} + O(q^{24}), \end{aligned}$$

and

$$
\begin{aligned}
f(\mathbf{c_2}, A_F) = {} & A_F(2)2^{\frac{1}{4}}(2)q^2 + A_F(3)3^{\frac{1}{4}}\left(\frac{1-i\sqrt{7}}{2}\right)q^3 + A_F(5)5^{\frac{1}{4}}(\sqrt{2})q^5 \\
& - A_F(2)2^{\frac{1}{4}}(1+i\sqrt{7})q^8 - 2A_F(3)3^{\frac{1}{4}}q^{12} - A_F(15)15^{\frac{1}{4}}\left(\frac{3+i\sqrt{7}}{2\sqrt{2}}\right)q^{15} \\
& + 2A_F(17)17^{\frac{1}{4}}q^{17} - 2A_F(2)2^{\frac{1}{4}}q^{18} - A_F(5)5^{\frac{1}{4}}\left(\frac{1+i\sqrt{7}}{\sqrt{2}}\right)q^{20} \\
& - A_F(23)23^{\frac{1}{4}}\left(\frac{3+i\sqrt{7}}{\sqrt{2}}\right)q^{23} + O(q^{24}).
\end{aligned}
$$

Using Theorem 4.4.5 to calculate the global factors $A_F(t)$ gives the values shown in Table 4.3 below.

Table 4.3: The Global Factors $A_F(t)$

t	$A_F(t)$
2	$-r8^{-\frac{1}{4}}$
3	$r2^{-\frac{1}{2}}3^{-\frac{1}{4}}$
5	$r5^{-\frac{1}{4}}$
15	$-r15^{-\frac{1}{4}}$
17	0
23	$r23^{-\frac{1}{4}}$

Notice that the global factor $A_F(17)$ is zero because the coefficient c_{68} of g is zero. We apply the following linear combinations:

$$
\begin{aligned}
f &= \frac{\sqrt{2}}{i\sqrt{7}}(f(\mathbf{c_2}, A_F) - f(\mathbf{c_1}, A_F)) \\
&= 2^{\frac{1}{2}}A_F(3)3^{\frac{1}{4}}q^3 + 2A_F(2)8^{\frac{1}{4}}q^8 + A_F(15)15^{\frac{1}{4}}q^{15} + 2A_F(5)5^{\frac{1}{4}}q^{20} \\
&\quad + 2A_F(23)23^{\frac{1}{4}}q^{23} + O(q^{24})
\end{aligned}
$$

and

$$
\begin{aligned}
h &= \sqrt{2}(f(\mathbf{c_1}, A_F) + f(\mathbf{c_2}, A_F)) \\
&= 4A_F(2)2^{\frac{3}{4}}q^2 + A_F(3)2^{\frac{1}{2}}3^{\frac{1}{4}}q^3 + 4A_F(5)5^{\frac{1}{4}}q^5 - 2A_F(2)2^{\frac{3}{4}}q^8 \\
&\quad - 4A_F(3)2^{\frac{1}{2}}3^{\frac{1}{4}}q^{12} - 3A_F(15)15^{\frac{1}{4}}q^{15} + 4A_F(17)2^{\frac{1}{2}}17^{\frac{1}{4}}q^{17} - 4A_F(2)2^{\frac{3}{4}}q^{18} \\
&\quad - 2A_F(5)5^{\frac{1}{4}}q^{20} - 6A_F(23)23^{\frac{1}{4}}q^{23} + O(q^{24}).
\end{aligned}
$$

After substituting the values of $A_F(t)$ obtained above, we get

$$f = g = q^3 - 2q^8 - q^{15} + 2q^{20} + 2q^{23} + O(q^{24})$$

and

$$h = -4q^2 + q^3 + 4q^5 + 2q^8 - 4q^{12} + 3q^{15} + 4q^{18} - 2q^{20} - 6q^{23} + O(q^{24}).$$

Therefore, the modular forms g and h form a basis for the space $S_{\frac{3}{2}}(\Gamma_0(60), F)$.

In what follows, we generate more examples of the modular forms g and h which form a basis for the space $S_{\frac{3}{2}}(\Gamma_0(4N), F)$, using the functions we created in Sage to implement calculations of h (as described following Theorem 3.7) and g (as described in Section 1). These functions are available from the author upon request. The open source mathematical software package may be found at **http** : //**www.sagemath.org**/.

Example 1 We compute a basis $\{g1, h1\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(44), F1)$ with $F1$ being the newform in $S_2^{\text{new}}(\Gamma_0(11))$ corresponding to the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$.

```
sage: F1=Newforms(11)[0]

sage: g1=shimura_lift_in_kohnen_subspace(F1,11,40)

sage: g1
```

$$-q^3 + q^4 + q^{11} + q^{12} - q^{15} - 2q^{16} - q^{20} + q^{23} + q^{27} + q^{31} + O(q^{40})$$

```
sage: h1=shimura_lift_not_in_kohnen_subspace(F1,11,40)

sage: h1
```

$$2q-2q^4-2q^5+2q^{12}-4q^{14}+4q^{15}+2q^{20}+4q^{22}-4q^{23}+4q^{26}-4q^{31}-2q^{33}-4q^{34}-2q^{37}+4q^{38}+O(q^{40})$$

Example 2 We compute a basis $\{g2, h2\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(84), F2)$ with $F2$ being the newform in $S_2^{\text{new}}(\Gamma_0(21))$ corresponding to the elliptic curve $y^2 + xy = x^3 - 4x - 1$.

```
sage: F2=Newforms(21)[0]

sage: g2=shimura_lift_in_kohnen_subspace(F2,21,40)
```

```
sage: g2
```

$$q^3 - q^7 - 2q^{19} - 2q^{24} + q^{27} + 2q^{28} + 2q^{31} + O(q^{40})$$

```
sage: h2=shimura_lift_not_in_kohnen_subspace(F2,21,40)

sage: h2
```

$$q^3-4q^6+3q^7+8q^{10}-4q^{12}-8q^{13}-2q^{19}+4q^{21}+2q^{24}+q^{27}+2q^{28}-6q^{31}+8q^{33}+O(q^{40})$$

Example 3 We compute a basis $\{g3, h3\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(132), F3)$ with $F3$ being the newform in $S_2^{\text{new}}(\Gamma_0(33))$ corresponding to the elliptic curve $y^2 + xy = x^3 + x^2 - 11x$.

```
sage: F3=Newforms(33)[0]

sage: g3=shimura_lift_in_kohnen_subspace(F3,33,40)

sage: g3
```

$$q^3 + q^{11} + 2q^{12} - 2q^{15} - 4q^{20} - 2q^{23} - q^{27} + O(q^{40})$$

```
sage: h3=shimura_lift_not_in_kohnen_subspace(F3,33,40)

sage: h3
```

$$3q^3-8q^5+3q^{11}-2q^{12}+8q^{14}+2q^{15}-4q^{20}+2q^{23}-8q^{26}-3q^{27}+8q^{33}+16q^{38}+O(q^{40})$$

Example 4 We compute a basis $\{g4, h4\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(140), F4)$ with $F4$ being the newform in $S_2^{\text{new}}(\Gamma_0(35))$ corresponding to the elliptic curve $y^2 + y = x^3 + x^2 + 9x + 1$.

```
sage: F4=Newforms(35,names='a')[0]

sage: g4=shimura_lift_in_kohnen_subspace(F4,35,40)

sage: g4
```

$$-2q^4 + 2q^{11} + 2q^{15} - 2q^{35} - 4q^{36} + 2q^{39} + O(q^{40})$$

```
sage: h4=shimura_lift_not_in_kohnen_subspace(F4,35,40)

sage: h4
```

$$-4q-8q^9+4q^{11}+8q^{14}-4q^{15}+8q^{16}-4q^{21}+4q^{25}+12q^{29}+8q^{30}-4q^{35}-4q^{39}+O(q^{40})$$

Example 5 We compute a basis $\{g5, h5\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(148), F5)$ with $F5$ being the newform in $S_2^{\text{new}}(\Gamma_0(37))$ corresponding to the elliptic curve $y^2 + y = x^3 + x^2 - 23x - 50$.

```
sage: F5=Newforms(37)[1]

sage: g5=shimura_lift_in_kohnen_subspace(F5,37,40)

sage: g5
```

$$q^8 - q^{15} + q^{19} - q^{20} - q^{24} - q^{35} + q^{39} + O(q^{40})$$

```
sage: h5=shimura_lift_not_in_kohnen_subspace(F5,37,40)

sage: h5
```

$$2q^2 - 2q^5 - 2q^6 - 2q^{13} + 2q^{14} + 2q^{15} + 2q^{17} + 2q^{19} - 2q^{22} - 4q^{32} - 2q^{35} + 2q^{37} - 2q^{39} + O(q^{40})$$

Example 6 We compute a basis $\{g6, h6\}$ for the space $S_{\frac{3}{2}}(\Gamma_0(924), F6)$ with $F6$ being the newform in $S_2^{\text{new}}(\Gamma_0(231))$ corresponding to the elliptic curve $y^2 + xy + y = x^3 + x^2 - 34x + 62$.

```
sage: F6=Newforms(231,names='a')[0]

sage: g6=shimura_lift_in_kohnen_subspace(F6,231,40)

sage: g6
```

$$4q^8 + 2q^{11} - 4q^{32} + 4q^{35} + O(q^{40})$$

```
sage: h6=shimura_lift_not_in_kohnen_subspace(F6,231,40)

sage: h6
```

$$8q^2 - 4q^8 + 2q^{11} - 8q^{18} - 8q^{21} + 16q^{29} - 16q^{30} - 12q^{32} + 4q^{35} + O(q^{40})$$

Bibliography

[1] M. Alsina and P. Bayer. *Quaternion Orders, Quadratic Forms, and Shimura Curves*, volume 22 of *CRM Monograph Series*. American Mathematical Society, 2004.

[2] E. Baruch and Z. Mao. Central value of automorphic L-functions. *GAFA, Geom. funct. anal.*, 17:333–384, 2007.

[3] M. Bertolini and H. Darmon. A rigid analytic Gross-Zagier formula and arithmetic applications. *Ann. of Math.*, 146 (1):111–147, 1997.

[4] S. Böcherer and R. Schulze-Pillot. Vector valued theta series and Waldspurger's theorem. *Abh. Math. Sem. Univ. Hamburg*, 64:211–233, 1994.

[5] D. Bump. *Automorphic Forms and Representations*, volume 55. Cambridge Studies in Advanced Mathematics, 1996.

[6] C. Cornut and V Vatsal. CM points and quaternion algebras. *Doc. Math.*, 10:263–309 (electronic), 2005.

[7] C. Cornut and V. Vatsal. Nontriviality of Rankin-Selberg L-functions and CM points. *L-functions and Galois Represenations, Ed. Burns, Buzzard and Nekovar*, Cambridge University Press:121–186, 2007.

[8] Y. Flicker. Automorphic forms on covering groups of $GL(2)$. *Invent. Math*, 57:119–182, 1980.

[9] B. Gross. Heights and the special values of L-series. *CMS Conf. Proc.*, 7:115–187, 1987.

[10] A. Hamieh. Ternary quadratic forms and half-integral weight modular forms. *LMS Journal of Computation and Mathematics*, 15:418–435, 2012.

[11] H. Jacquet. *Automorphic Forms on $GL(2)$ (Part II)*, volume 278 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972.

[12] H. Jacquet and Langlands R. *Automorphic Forms on $GL(2)$*, volume 114 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1970.

[13] N. Koblitz. *Introduction to Elliptic Curves and Modular Forms*, volume 97 of *Graduate texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[14] W. Kohnen. Newforms of half-integral weight. *J. Reine Angew. Math.*, 333:32–72, 1982.

[15] K. Martin and D. Whitehouse. Central L-values and toric periods for GL(2). *IMRN*, 2009(1):141–191, 2008.

[16] T. Miyake. On automorphic forms on GL_2 and Hecke operators. *Ann. of Math.*, 94:174–189, 1971.

[17] A. Pacetti and G. Tornaria. Shimura correspondence for level p^2 and the central values of L-series. *J. Number Theory*, 124:396–414, 2007.

[18] A. Pizer. An algorithm for computing modular forms on $\Gamma_0(n)$. *J. Algebra*, 64:340–390, 1980.

[19] M. Ratner. Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups. *Duke Math. J.*, 77:275–382, 1995.

[20] G. Shimura. On modular forms of half-integral weight. *Annals of Math.*, 97:440–481, 1973.

[21] G. Shimura. Special values of the zeta functions associated with hilbert modular forms. *Duke Math. J.*, 45:637–679, 1978.

[22] T. Shintani. On construction of holomorphic cusp forms of half integral weight. *Nagoya Math J.*, 58:83–126, 1975.

[23] J. Sturm. Theta series of weight $\frac{3}{2}$. *J. Number Theory*, 14:353–361, 1982.

[24] V. Vatsal. Uniform distribution of Heegner points. *Invent. Math.*, 148: 1–46, 2002.

[25] V. Vatsal. Special values of anticyclotomic L-functions. *Duke Math. J.*, 116 (2):219–261, 2003.

[26] M.-F Vignéras. *Arithmétique des algèbres de quaternions*, volume 800. Springer Lecture Notes, 1980.

[27] J. Voight. The arithmetic of quaternion algebras. Online notes, 2010. URL `http://www.cems.uvm.edu/~jvoight/crmquat/book/quat-modforms-041310.pdf`.

[28] J-L. Waldspurger. Correspondance de Shimura. *J. Math. Pures et Appl.*, 59:1–132, 1980.

[29] J-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures et Appl.*, 60:375–484, 1981.

[30] J-L. Waldspurger. Correspondance de Shimura et quaternions. *Forum Math.*, 3:219–307, 1991.

[31] J.L. Waldspurger. Sur les valeurs de certaines fonctions L automorphes en leur centre de symmétrie. *Compos. Math.*, 54:174–242, 1985.

[32] S. Zhang. Gross-Zagier formula for GL(2). *Asian J. Math.*, 5(2):183–290, 2001.

[33] S. Zhang. Heights of Heegner points on Shimura curves. *Ann. of Math.*, 153:27–147, 2001.