Modular Symbols, Eisenstein Series, and Congruences

by

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Abstract

Let E and f be an Eisenstein series and a cusp form, respectively, of the same weight $k \geq 2$ and of the same level N, both eigenfunctions of the Hecke operators, and both normalized so that $a_1 = 1$. The main result we seek is that when E and f are congruent mod a prime $\mathfrak{p}$ (which may be a prime ideal lying over a rational prime $p > 2$), the algebraic parts of the special values $L(E, \chi, j)$ and $L(f, \chi, j)$ satisfy congruences mod the same prime. More explicitly, the congruence result states that, under certain conditions,

$$\frac{\tau(\bar{\chi})L(f, \chi, j)}{(2\pi i)^{j-1}\Omega_f^{\mathrm{sgn}(E)}} \equiv \frac{\tau(\bar{\chi})L(E, \chi, j)}{(2\pi i)^j \Omega_E} \pmod{\mathfrak{p}}$$

where the sign of E is ± 1 depending on E, and $\Omega_f^{\mathrm{sgn}(E)}$ is the corresponding canonical period for f. Also, χ is a primitive Dirichlet character of conductor m, $\tau(\bar{\chi})$ is a Gauss sum, and j is an integer with $0 < j < k$ such that $(-1)^{j-1} \cdot \chi(-1) = \mathrm{sgn}(E)$. Finally, Ω_E is a $\mathfrak{p}$-adic unit which is independent of χ and j. This is a generalization of earlier results of Stevens and Vatsal for weight $k = 2$.

In this paper we construct the modular symbol attached to an Eisenstein series, and compute the special values. We give numerical examples of the congruence theorem stated above, and we sketch the proof of the congruence theorem.

Preface

The proofs of the final two theorems that appear in Chapter 6 were supplied by V. Vatsal.

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Chapter 1

Introduction

The idea that congruences between modular forms should carry over to congruences in the special values of their L-functions has been studied for decades. We seek to highlight one of the results in that direction and show how it generalizes from weight 2 to higher weight.

The method of finding special values of L-functions of cusp forms by integrating a differential form on an appropriate modular curve was known at least since the 1970s. In 1972, Manin (in [7]) proved several theorems that gave an efficient way to compute special values for weight 2 cusp forms, using what he called "distinguished classes". Today these are called *Manin symbols*. The Manin symbols have an analogue for higher weight, which was developed by Shokurov ([16]) and Merel ([9], and see also [17]). These ideas and their relevance to L-functions are discussed below.

In 1977, Mazur (in [8]) used Manin's idea, as well as a new idea that he called "modular symbols", to prove congruence formulae for weight 2 cusp forms of specific level. Then in 1982, Stevens (in [18]) proved that, given a weight 2 cusp form, a weight 2 Eisenstein series of the same level congruent to f mod some prime p, a primitive Dirichlet character χ, and certain specific conditions, the algebraic parts of $L(f, \chi, 1)$ and $L(E, \chi, 1)$ are congruent mod p. This theorem was refined by Vatsal in [21].

The paper [21] uses a method of establishing the congruences which we also use here. First, we construct a modular symbol associated to each modular form f and E. Then we show that the L-values mod p are determined by the modular symbols mod p. Finally we show that a congruence mod p between the modular forms implies the same congruence between the modular symbols, and then deduce that the L-values must also be congruent.

The construction of a modular symbol associated to a cusp form f is well-known, and after dividing by a canonical period, it takes values in a module $L_{k-2}(\mathbb{K})$ (we will make this explicit below). We can use this construction to show congruence theorems for cusp forms of higher weight (see [21]), but the case of congruences between higher-weight cusp forms and higher-weight Eisenstein series was left open. The main reason for this is that there is no good way to construct a Γ-invariant $L_{k-2}(\mathbb{C})$-valued modular symbol with the same Hecke eigenvalues as E. Instead of this, we define a modular symbol M_E mod p, and this modular symbol will be Γ-invariant when E is congruent to a cusp form mod p. The modular symbol we define is built on the maps used in [20], Example 6.4(a). We will present the necessary construction result below.

To show the final congruence, we use the fact that the two modular symbols attached to E and f have Hecke eigenvalues the same as those of E and f. If they are assumed congruent mod p, then we use a multiplicity one argument to show that the modular symbols are the same up to scaling. (We also need to show that the modular symbol M_E is not identically zero. This result is presented below.)

The organization of this paper is as follows. The material in Chapters 2 and 3 are mostly a review of results already in the literature. In Chapter 2 we define some functions related to special values of L-functions attached to modular forms, and we prove some basic properties of those functions. In Chapter 3, we define the modular symbols attached to cusp forms and show the connection to the special values of character twists of L-functions. In Chapter 4 we define the modular symbol attached to an Eisenstein series, and we prove an integrality result in more generality than just the weight 2 case. In Chapter 5 we calculate the special values of this modular symbol, and relate them to the character twists of the corresponding L-functions. In Chapter 6, we sketch the proof of the congruence theorem for the special values of character twists of a cusp form and a congruent Eisenstein series. Finally, Chapter 7 explains how to compute the "algebraic parts" of special values of L-functions attached to cusp forms; we conclude by showing some computed examples of both.

Chapter 2

Functions Connected to Special Values

Before we can state the main results later on, we need to define some functions connected to special values of L-functions and prove some results about them. The results in this section hold for general modular forms; throughout this section let f be a modular form of weight $k \geq 2$ and any level.

Let A be a ring, and let $L_n(A)$ (for a nonnegative integer n) be the symmetric polynomial algebra over A of degree n. (Thus the elements of $L_n(A)$ are homogeneous polynomials of degree n with coefficients in A.) Throughout what follows, we will always take A to be a subring of $\mathbb{C}$.

$L_n(\mathbb{C})$, for any nonnegative integer n, admits a left action of $\mathrm{GL}_2^+(\mathbb{Q})$: if $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, and $P(X,Y) \in L_n(\mathbb{C})$, then

$$\alpha | P(X,Y) = \det(\alpha)^{-n} \cdot P(aX + cY, bX + dY)$$

We will make frequent use of this action below.

Throughout what follows we will always put $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Also, let

$$D_f(s) := \int_0^\infty \tilde{f}(z) y^{s-1} dz$$

where the tilde means that we subtract $a_0(f)$, and y is the imaginary part of z. This integral will converge whenever we take s with real part big enough (it depends on the weight). The main point is:

Proposition 2.1. *In the region of convergence of the integral,*

$$D_f(s) = i \cdot \Gamma(s) \cdot (2\pi)^{-s} \cdot L(f,s)$$

This identity links the above integral to the L-function of f. For the proof, see [13], p. I-5. We can also prove:

Proposition 2.2. *Within the region of convergence of the integral, we have the formula*

$$D_f(s) = \int_i^\infty \tilde{f}(z) y^{s-1} dz + i^k \int_i^\infty (\widetilde{f|\sigma})(z) y^{k-1-s} dz$$
$$-i \left(\frac{a_0(f)}{s} + i^k \frac{a_0(\widetilde{f|\sigma})}{k-s} \right)$$

Furthermore, this formula defines a meromorphic continuation of $D_f(s)$ to the entire complex plane, with functional equation

$$D_f(s) = i^k D_{f|\sigma}(k-s).$$

Proof. Break the integral into the sum

$$D_f(s) = \int_0^i \tilde{f}(z) y^{s-1} dz + \int_i^\infty \tilde{f}(z) y^{s-1} dz.$$

The second integral converges for all s; we only need to deal with the first. First we have

$$\int_0^i \tilde{f}(z) y^{s-1} dz = \int_0^i f(z) y^{s-1} dz - i \cdot \tfrac{a_0(f)}{s}$$

Now to treat this integral, we transform z to σz. The result is

$$-\int_i^\infty f(\sigma z) \mathrm{Im}(\sigma z)^{s-1} d(\sigma z)$$
$$= -\int_i^\infty z^{k-2} (f|\sigma)(z) y^{1-s} dz$$

Now since $z = iy$ along this path, this equals

$$i^k \int_i^\infty (f|\sigma)(z) y^{k-1-s} dz$$
$$= i^k \int_i^\infty (\widetilde{f|\sigma})(z) y^{k-1-s} dz - i \cdot i^k \frac{a_0(\widetilde{f|\sigma})}{k-s}$$

The last step holds for large enough s. (The extra i comes from using $dz = idy$.) Putting all four pieces together proves the formula. Also, the remaining integral in the last step above, as well the other integral from the first step and both of the other terms, are defined for all s, proving the analytic continuation. From here the functional equation is checked simply by using the formula on both sides of the claimed identity. $\square$

With this in mind, let us now define a new function that takes values in $L_n(\mathbb{C})$:

$$F_f(s) := \int_0^\infty \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz$$

This integral will clearly converge for large enough values of s.

Proposition 2.3. *Within the region of convergence of the integral, we have the formula*

$$F_f(s) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot D_f(s+j) \cdot X^j Y^{k-2-j}$$

Furthermore, this defines a meromorphic continuation of $F_f(s)$ to the entire complex plane.

Proof. As in the proof of Proposition 2.2, we begin by splitting the integral to obtain

$$F_f(s) = \int_0^i \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz + \int_i^\infty \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz$$

The second integral converges for all s, so we can expand the polynomial and split the integral, and we obtain

$$\int_i^\infty \tilde{f}(z)(zX+Y)^{k-2}y^{s-1}dz = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot \int_i^\infty \tilde{f}(z)y^{s+j-1}dz \tag{2.1}$$

Now we need to treat the first integral. Similarly to before, the first integral becomes

$$\int_0^i f(z)(zX+Y)^{k-2}y^{s-1}dz - \int_0^i a_0(f)(zX+Y)^{k-2}y^{s-1}dz$$

The second of these is a polynomial integral, so we can evaluate it and obtain

$$-a_0(f) \cdot \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot i \cdot \frac{1}{s+j} \tag{2.2}$$

For the remaining integral, we use the transformation $z \mapsto \sigma z$, and the integral becomes

$$-\int_i^\infty \tfrac{1}{z^2} f(\sigma z)(\tfrac{-1}{z}X+Y)^{k-2}y^{1-s}dz$$

$$= -\int_i^\infty (f|\sigma)(z)(zY - X)^{k-2} y^{1-s} dz$$

(All the powers of z cancel.) Now again we split this integral into one involving $a_0(f|\sigma)$ and one involving the difference. The former, for large enough s, gives

$$-i^k a_0(f|\sigma) \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot i \cdot \frac{1}{k-j-s} \tag{2.3}$$

This follows from using the Binomial Theorem and keeping track of the various powers of i that show up, some of which come from the minus sign in the polynomial (and one of which comes from $dz = idy$).

The latter integral is

$$i^2 \int_i^\infty (\tilde{f|}\sigma)(z)(zY - X)^{k-2} y^{1-s} dz$$

This integral converges for all s, so we can expand the polynomial and split the integral to obtain

$$i^k \sum_{j=0}^{k-2} i^j \cdot \binom{k-2}{j} \cdot X^j Y^{k-2-j} \cdot \int_i^\infty (\tilde{f|}\sigma)(z) y^{k-1-s-j} dz \tag{2.4}$$

Now we put these four pieces together to obtain the following formula, which *a priori* will be valid only where the integral for $F_f(s)$ converges:

$$F_f(s) = \sum_{j=0}^{k-2} \cdot \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j}$$

$$\cdot \left[\int_i^\infty \tilde{f}(z) y^{s+j-1} dz + i^k \int_i^\infty (\tilde{f|}\sigma)(z) y^{k-1-s-j} dz - i \left(\tfrac{a_0(f)}{s+j} + i^k \tfrac{a_0(f|\sigma)}{k-j-s} \right) \right]$$

Now we use Proposition 2.2 to arrive at the desired result. It immediately shows the formula, and the analytic continuation follows from the fact that the right-hand side of the formula already has an analytic continuation. □

Now we can state the connection to the L-function of f:

Corollary 2.4. *We have*

$$F_f(s) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^{j+1} \cdot \Gamma(s+j) \cdot \tfrac{1}{(2\pi)^{s+j}} \cdot L(f, s+j) \cdot X^j Y^{k-2-j}$$

and this holds for all s.

Proof. Simply combine Proposition 2.3 and Proposition 2.1. □

The corollary shows that

$$F_f(1) = \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^{j+1} \cdot (j!) \cdot \tfrac{1}{(2\pi)^{j+1}} \cdot L(f, j+1) \cdot X^j Y^{k-2-j}$$

In other words, $F_f(1)$ is a polynomial whose coefficients encode all the special values of the L-function of f at the so-called *critical integers*, namely those strictly between 0 and k.

Now we return to Proposition 2.3. To do the proof, we expanded all the polynomials and treated their individual terms separately. However, there is another formula as well. We can prove it using the same logic as above; it is just a matter of grouping the terms differently. We state it here as a separate result.

Proposition 2.5. *We have the formula*

$$F_f(s) = \int_i^\infty \tilde{f}(z)(zX+Y)^{k-2} y^{s-1} dz - \int_i^\infty (\widetilde{f|\sigma})(z)[\sigma|(zX+Y)^{k-2}] y^{1-s} dz$$

$$+ i\left[a_0(f|\sigma) \cdot \sigma| \left(\sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot \tfrac{1}{2-s+j}\right) - a_0(f) \cdot \sum_{j=0}^{k-2} \binom{k-2}{j} \cdot i^j \cdot X^j Y^{k-2-j} \cdot \tfrac{1}{s+j}\right]$$

From here we can prove another corollary:

Corollary 2.6. *For an arbitrary base point z_0 in the upper half-plane,*

$$F_f(1) = \int_{z_0}^\infty \tilde{f}(z)(zX+Y)^{k-2} dz - \int_{z_0}^\infty (\widetilde{f|\sigma})(z)[\sigma|(zX+Y)^{k-2}] dz$$

$$-a_0(f) \cdot \int_0^{z_0} (zX+Y)^{k-2} dz + a_0(f|\sigma) \cdot \int_0^{z_0} \sigma|(zX+Y)^{k-2} dz$$

$$-\int_{z_0}^{\sigma z_0} f(z)(zX+Y)^{k-2} dz$$

Proof. If we consider the expression on the right as a function of z_0, we see that it is constant (by which we mean that it is a polynomial in X and Y whose coefficients are constants); this follows from the fact that the derivative is 0 (which is an elementary computation using the Fundamental Theorem of Calculus and the definition of how σ acts on modular forms). But by Proposition 2.5 (with $s = 1$), it is true for $z_0 = i$. Thus it is true for all z_0. □

We now prove two more lemmas which will be used in later sections.

Lemma 2.7. *Put* $\alpha := \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ *with rational entries and positive determinant. Then*

$$a_0(f|\alpha) = \frac{a^{k-1}}{d} \cdot a_0(f)$$

Proof. By definition, $(f|\alpha)(z) = (ad)^{k-1} \cdot d^{-k} \cdot f(\alpha z) = \frac{a^{k-1}}{d} \cdot f(\alpha z)$, so $a_0(f|\alpha) = \frac{a^{k-1}}{d} \cdot a_0(f)$. □

Lemma 2.8. *Put* $\alpha := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\tau := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$*, all with rational entries and positive determinant. Then*

$$F_{f|\alpha\tau}(1) = \tau^{-1}|F_{f|\alpha}(1)$$

Proof. By definition, for large enough s, we have

$$F_{f|\alpha\tau}(s) = \int_0^\infty (f|\tilde{\alpha\tau})(z)(zX+Y)^{k-2}y^{s-1}dz$$

Now use the substitution $u = \tau z = tz$, so that $z = u/t$, $dz = du/t$, the limits of integration are unchanged, and $y = \mathrm{Im}(u)/t$. We also note that

$$\begin{aligned}(f|\tilde{\alpha\tau})(z) &= (f|\alpha\tau)(z) - a_0(f|\alpha\tau) \\ &= t^{k-1}(f|\alpha)(u) - t^{k-1}a_0(f|\alpha) \\ &= t^{k-1}(f|\tilde{\alpha})(u)\end{aligned}$$

where in the second line we used Lemma 2.7 for the second term. This means that after some manipulation of terms, we see that the integral is equal to

$$\begin{aligned} t^{1-s}\int_0^\infty (f|\tilde{\alpha})(u) \cdot (\tfrac{1}{t})^{-k+2}(\tfrac{u}{t}X+Y)^{k-2}y^{s-1}du \\ = t^{1-s} \cdot \tau^{-1}|F_{f|\alpha}(s)\end{aligned}$$

where we used the definition of τ^{-1} acting on a polynomial, and the definition of $F_f(s)$.

Now the identity $F_{f|\alpha\tau}(s) = t^{1-s}\tau^{-1}|F_{f|\alpha}(s)$ has been proved for s in some right half-plane; therefore it must remain true in the analytic continuation. Now plug in $s = 1$ and we have the desired result. □

Finally, we prove one last lemma:

Lemma 2.9. *Let f be a modular form of weight $k \geq 2$, and let χ be a primitive Dirichlet character of conductor m. Then*

$$\tau(\bar{\chi})D_{f\otimes\chi}(s) = m^{1-s}\sum_{a=0}^{m-1}\bar{\chi}(a)D_{f|\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}}(s)$$

Proof. We start from the identity

$$\tau(\bar{\chi})(f\otimes\chi)(z) = \sum_{a \bmod m}\bar{\chi}(a)f(z+\tfrac{a}{m})$$

where $\tau(\bar{\chi})$ is the standard Gauss sum. To prove this, we expand the right-hand side using the Fourier expansion of f and switch the order of summation; what comes out is exactly the left-hand side.

From the above identity and the definition of $D_f(s)$, it follows that

$$\tau(\bar{\chi})D_{f\otimes\chi}(s) = \sum_{a=0}^{m-1}\bar{\chi}(a)D_{f|\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}}(s)$$

Now to complete the proof we only need to show that

$$D_{f|\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}}(s) = m^{s-1}D_{f|\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}}(s)$$

But by definition,

$$\begin{aligned} f|\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}(z) &= f|\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & m\end{pmatrix}(z)\\ &= m^{k-1}\cdot m^{-k}\cdot f|\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}(z/m)\\ &= m^{-1}f|\begin{pmatrix}1 & a/m\\ 0 & 1\end{pmatrix}(z/m)\end{aligned}$$

So again using the definitions (inside the region of convergence of the integral),

$$D_{f|\begin{pmatrix}1 & a\\ 0 & m\end{pmatrix}}(s)$$

$$= m^{-1} \int_0^{i\infty} \tilde{f}| \begin{pmatrix} 1 & a/m \\ 0 & 1 \end{pmatrix} (z/m) y^{s-1} dz$$

Now keeping in mind that $y = \mathrm{Im}(z)$ and making the substitution $u = z/m$ gives exactly the claim above. Outside the region of convergence of the integral, the same formula must hold due to the uniqueness of the analytic continuation. This finishes the proof. $\square$

Chapter 3

Modular Symbols and Cusp Forms

In this section, we let f be a normalized (meaning $a_1 = 1$) cuspidal eigenform of weight $k \geq 2$ and level Γ for some congruence subgroup Γ. Our goal in this section is to define a modular symbol M_f attached to f and show a link between M_f and the algebraic parts of special values $L(f, \chi, j)$ for a primitive character χ and an integer j with $1 \leq j \leq k-1$. (The meaning of "algebraic part" will be explained below.) The discussion in this section will closely follow that of [4], Section 4.

We first state the definition of a modular symbol, along with two other definitions that we will use below:

Definition 3.1. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of degree zero divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})]$-module (with the matrices acting on the left). We refer to a map as an A-valued *modular symbol* over a congruence subgroup Γ if the map is a Γ-homomorphism from degree zero divisors to elements of A.

Definition 3.2. Let f be as above. Then the *standard weight k modular symbol* M_f is the $L_{k-2}(\mathbb{C})$-valued modular symbol defined as follows: on divisors $\{b\} - \{a\}$ (with $a, b \in \mathbb{P}^1(\mathbb{Q})$),

$$M_f(\{b\} - \{a\}) := 2\pi i \int_a^b f(z)(zX + Y)^{k-2} dz$$

Define M_f on all other degree-zero divisors by linearity.

Definition 3.3. Let χ be a primitive Dirichlet character of conductor m. Let Φ be any A-valued modular symbol. The operator R_χ, called the *twist operator*, is defined as follows: for any degree-zero divisor D,

$$(\Phi | R_\chi)(D) := \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}^{-1} |\Phi(\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} D)$$

Remark 3.4. This operator also appears in Section 4 of [4], but our definition is slightly different. We use $\bar{\chi}$ in the definition instead of χ; also, in that paper, the authors define the twist operator using a right action of matrices. Our definition has changed to account for the fact that our matrix action is a left action. Thus we change from a right action of $\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}$ to a left action of the inverse.

Now we can state a result concerning the special values of the L-function of f:

Theorem 3.5. *Let f be as above, and let χ be a primitive Dirichlet character of conductor m. Then*

$$(M_f|R_\chi)(\{\infty\}-\{0\}) = 2\pi i\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{f\otimes\chi}(1+j)\cdot X^jY^{k-2-j}$$

Proof. By definition,

$$(M_f|R_\chi)(\{\infty\}-\{0\}) = \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix}|M_f(\{\infty\}-\{\tfrac{a}{m}\})$$

$$= 2\pi i\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/m & 0 \\ 0 & 1/m \end{pmatrix}|\int_{a/m}^{\infty} f(z)(zX+Y)^{k-2}dz$$

$$= 2\pi i\sum_{a=0}^{m-1}\bar{\chi}(a)\int_{a/m}^{\infty} f(z)(zmX-aX+Y)^{k-2}dz$$

Now we use the substitution $z \mapsto \frac{z+a}{m}$ and then the definition of matrices acting on modular forms, and the above sum is equal to

$$2\pi i\sum_{a=0}^{m-1}\bar{\chi}(a)\int_{0}^{\infty} \tfrac{1}{m}f(\tfrac{z+a}{m})(zX+Y)^{k-2}dz$$

$$= 2\pi i\sum_{a=0}^{m-1}\bar{\chi}(a)\int_{0}^{\infty} (f|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix})(z)(zX+Y)^{k-2}dz$$

By the definitions in the previous section, along with Proposition 2.3, this is equal to

$$2\pi i\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot\sum_{a}\bar{\chi}(a)D_{f|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1+j)\cdot X^jY^{k-2-j}$$

Applying Lemma 2.9 with $s = j+1$, we obtain

$$2\pi i\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{f\otimes\chi}(1+j)X^jY^{k-2-j}$$

which completes the proof. □

Corollary 3.6. *With f and χ as in the above theorem,*

$$(M_f|R_\chi)(\{\infty\}-\{0\}) = \sum_{j=0}^{k-2}(-1)^{j+1}\binom{k-2}{j}\cdot j!\cdot m^j\cdot \tfrac{\tau(\bar{\chi})L(f,\chi,1+j)}{(2\pi i)^j}\cdot X^jY^{k-2-j}$$

Proof. Combine the theorem with Proposition 2.1. □

The above corollary gives a connection between M_f and $L(f,\chi,j)$; but so far we do not have any assurances of any algebraicity properties of either. To show how we get algebraic numbers from the modular symbol M_f, we define an involution on modular symbols induced by the action of the matrix $\iota = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, which sends

$$M_f(\{b\}-\{a\}) \mapsto \iota|M_f(\{-b\}-\{-a\})$$

(the left action of ι on polynomials simply sends $Y \mapsto -Y$). Now choose a "sign" $\pm$—meaning the $+1$ or -1 eigenspace of this involution—and project the modular symbol M_f to one of these eigenspaces. We obtain a new modular symbol which we will denote $M_f^\pm$ (for one choice of sign). It is a theorem of Shimura (proved in [5], or also see [4], [8], or [21]) that there exist transcendental numbers $\Omega_f^\pm$, called *canonical periods*, such that the modular symbols $\frac{1}{\Omega_f^\pm}M_f^\pm$ both give values in $L_{k-2}(K)$, where K is the algebraic field extension generated over $\mathbb{Q}$ by the Hecke eigenvalues of f. Shimura's theorem even tells us, for a primitive character χ and a critical integer j, which sign to choose so that the number

$$\frac{\tau(\bar{\chi})L(f,\chi,j)}{(2\pi i)^{j-1}\Omega_f^\pm}$$

is algebraic. (The choice of sign is $(-1)^{j-1}\mathrm{sgn}(\chi)$.) For that choice of sign, the above expression is called the *algebraic part* of $L(f,\chi,j)$. (However, the canonical periods, and therefore the algebraic parts, are only determined up to scaling by a unit in K.)

Chapter 4

Modular Symbols and Eisenstein Series

4.1 A Basis of Eisenstein Series

We begin with a definition of our basic Eisenstein series ϕ_{k,x_1,x_2}. Following [20], Section 6, pick a positive integer $k > 2$ (unlike in that paper, here we do not assume k is even) and let $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$, and define

$$G_{k,x_1,x_2}(z) := \frac{(k-1)!}{(2\pi i)^k} \sum_{\substack{(a_1,a_2)\in\mathbb{Q}-(0,0)\\ (a_1,a_2)\equiv(x_1,x_2) \pmod{\mathbb{Z}}}} (a_1 z + a_2)^{-k}$$

This series converges absolutely and defines a holomorphic Eisenstein series of weight k. Define ϕ_{k,x_1,x_2} to be the Fourier transform of this series, in the sense of [20], Definition 3.6 (and the beginning of Section 4). More explicitly, ϕ_{k,x_1,x_2} is defined as follows. Let N be the least common denominator of x_1 and x_2 and consider the map

$$\psi_{x_1,x_2} : (\tfrac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \to \mathbb{C}^\times$$

defined by

$$\psi_{x_1,x_2}(\tfrac{a_1}{N}, \tfrac{a_2}{N}) = e^{2\pi i(a_2 x_1 - a_1 x_2)}$$

Then

$$\phi_{k,x_1,x_2}(z) = \sum_{(a_1,a_2)\in(\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2} \psi_{x_1,x_2}(a_1,a_2) G_{k,a_1,a_2}(z)$$

Our first goal is to study the special values of the L-functions attached to these Eisenstein series. For that we need to review the basic definitions of Bernoulli numbers and Bernoulli polynomials, since they will be used below. We can define the Bernoulli numbers B_n inductively by setting $B_0 = 1$, and for $n \geq 2$,

$$\sum_{j=1}^{n} \binom{n}{j} B_{n-j} = 0$$

We can then define the Bernoulli polynomials $\tilde{B}_n(x)$ by

$$\tilde{B}_n(x) = \sum_{j+0}^{n} \binom{n}{j} B_{n-j} x^j$$

(For more details including equivalent definitions of these two terms, see Appendix B of [11].) If $\lfloor \cdot \rfloor$ is the greatest integer function on real numbers, then define the the periodified Bernoulli polynomial

$$B_n(x) = \tilde{B}_n(x - \lfloor x \rfloor)$$

Now we can state our first result:

Proposition 4.1. *ϕ_{k,x_1,x_2} has Fourier expansion*

$$\phi_{k,x_1,x_2}(z) = \frac{B_k(x_1)}{k} - J(k, x_1, x_2; z) - (-1)^k J(k, -x_1, -x_2; z)$$

where

$$J(k, a, b; z) := \sum_{\substack{\kappa \equiv a \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} e^{2\pi i z m \kappa} e^{2\pi i m b}$$

For the proof, see [15].

Define, for $x \in \mathbb{Q}/\mathbb{Z}$,

$$Z(s, x) := \sum_{n=1}^{\infty} e^{2\pi i n x} n^{-s}$$

and

$$\zeta(s, x) := \sum_{\substack{m \equiv x \pmod 1 \\ m \in \mathbb{Q}^+}} m^{-s}$$

(Clearly these functions are well-defined for $x \in \mathbb{Q}/\mathbb{Z}$.) Now we have:

Proposition 4.2. *$L(\phi_{k,x_1,x_2}, s) = -\zeta(1-(k-s), x_1)Z(s, x_2) - (-1)^k \zeta(1-(k-s), -x_1)Z(s, -x_2)$.*

Proof. Consider $J(k, a, b; z)$, defined above. If we put $q = e^{2\pi i z}$, then

$$J(k, a, b; z) = \sum_{\substack{\kappa \equiv a \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} q^{m\kappa} e^{2\pi i m b}$$

So its L-function is

$$\sum_{\substack{\kappa\equiv a \pmod 1\\ \kappa\in\mathbb{Q}^+}} \kappa^{k-1}\cdot\sum_{m=1}^{\infty}(m\kappa)^{-s}e^{2\pi imb}$$

$$= \sum_{\substack{\kappa\equiv a \pmod 1\\ \kappa\in\mathbb{Q}^+}} \kappa^{k-1-s}\cdot\sum_{m=1}^{\infty}m^{-s}e^{2\pi imb}$$

$$= \zeta(1-(k-s),a)Z(s,b)$$

Now the result follows from the above proposition. □

We will need the following three properties of the two functions defined above:

Proposition 4.3. *For any positive integer n and any x as above (and B_n as above),*

$$\zeta(1-n,x) = -\frac{B_n(x)}{n}$$

This is a well-known property of the Hurwitz zeta function. See, for example, [11], p. 341.

Proposition 4.4. *With n and x as above, unless $x\in\mathbb{Z}$ and $n=1$,*

$$\zeta(1-n,-x) = (-1)^n\zeta(1-n,x)$$

This follows from a well-known property of the Bernoulli polynomials. See, for example, [11], equations B.10 and B.13.

Proposition 4.5. *With n and x as above, unless $x\in\mathbb{Z}$ and $n=1$,*

$$Z(n,x)+(-1)^nZ(n,-x) = -i^n\cdot(2\pi)^n\cdot\Gamma(n)^{-1}\cdot\frac{B_n(x)}{n}$$

This follows from the definition of $Z(n,x)$ and from the Fourier expansions of the periodified Bernoulli polynomials (which can be found in [14], p. 16).

Now we will use these facts to prove a result about the polynomial $F_E(1)$ when E is of the form ϕ_{k,x_1,x_2}.

Proposition 4.6. *Let E be equal to ϕ_{k,x_1,x_2} for some integer $k > 2$ and some $x_1, x_2 \in (\mathbb{Q}/\mathbb{Z})^2$. Then for any integer j with $0 \leq j \leq k-2$, the coefficient of the X^jY^{k-2-j} term in $F_E(1)$ is*

$$\binom{k-2}{j}(-1)^j \frac{B_{k-j-1}(x_1)}{k-j-1} \cdot \frac{B_{j+1}(x_2)}{j+1}$$

except in the following cases. When $x_1 = 0$, and k is even, the coefficient of X^{k-2} will be

$$i^{k+1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}}$$

and when $x_1 = 0$ and k is odd, the coefficient of X^{k-2} is

$$i^{k+2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n x_2)}{n^{k-1}}.$$

When $x_2 = 0$ and k is even, the coefficient of Y^{k-2} will be

$$i^{k+3}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_1))}{n^{k-1}}$$

and when $x_2 = 0$ and k is odd, the coefficient of Y^{k-2} will be

$$i^{k}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(-x_1))}{n^{k-1}}$$

Remark 4.7. We have excluded the case $k = 2$ from the result above, but it is treated in [18], Section 2.5. The non-exceptional cases yield the same formula as the above proposition when $k = 2$, but the exceptional cases are different (and when $k = 2$ we exclude the case $x_1 = x_2 = 0$ entirely).

Proof. First we will deal with the exceptional cases. The first is when k is even, $j = k-2$ and $x_1 = 0$. In this case, we will have

$$L(E, k-1) = -\tfrac{1}{2}(Z(k-1, x_2) + Z(k-1, -x_2)) = -\sum_{n=1}^{\infty} \cos(2\pi n x_2) n^{1-k}$$

and the claim follows. (We are computing the coefficient using Corollary 2.4.) If k is odd instead of even, the second exceptional case is proved using a similar calculation.

The last two cases are when $j = 0$ and $x_2 = 0$. Here we can simply use the functional equation at the end of Proposition 2.2 and then this reduces to the same computations as in the first case.

It only remains to show the general case. We are looking to compute $L(E, j+1)$ for $0 \leq j \leq k-2$, and all three of the above identities apply. Starting from Proposition 4.2, we begin by applying Proposition 4.4 to the Hurwitz zeta functions. Then we take out a factor of $-\zeta(1-(k-j-1), x_1)$, which is equal to $\frac{B_{k-j-1}(x_1)}{k-j-1}$ by Proposition 4.3. Finally we apply Proposition 4.5 to the sum or difference of $Z(j+1, x_2)$ and $Z(j+1, -x_2)$ terms. When we combine the results with the formula in Corollary 2.4, the factors of 2π and the gamma factors cancel; collecting all the powers of i, we obtain exactly the desired result. $\square$

An important fact that is immediately implied by the result above is the following:

Corollary 4.8. *Let E be of the form ϕ_{k,x_1,x_2} as above. Then the real part of $F_E(1)$ is rational.*

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with all integer entries and positive determinant; then we can use the distribution law given in ([20], equation 3.9) to conclude

$$\phi_{k,x_1,x_2}|\alpha^{-1} = \det(\alpha)^{-k+2} \sum_{\substack{y=(y_1,y_2)\in(\mathbb{Q}/\mathbb{Z})^2 \\ y\alpha\equiv x \pmod{\mathbb{Z}}}} \phi_{k,y_1,y_2} \tag{4.1}$$

If α has determinant 1, this specializes to

$$\phi_{k,x_1,x_2}|\alpha = \phi_{k,ax_1+cx_2,bx_1+dx_2}$$

Remark 4.9. Any element of $\mathrm{GL}_2^+(\mathbb{Q})$ can be written as a scalar matrix times the inverse of a matrix with integral entries, so this also shows how to evaluate $\phi_{k,x_1,x_2}|\alpha$ for any matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. As an example, we will compute, for a general ϕ_{k,x_1,x_2}, the action of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$. To begin, we write the matrix as

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

By the definition of matrices acting on modular forms, the action of the scalar matrix is simply to multiply by d^{k-2}. Now we can use 4.1 directly on the second matrix:

$$\phi_{k,x_1,x_2}|\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} = d^{-k+2}\sum_{\nu=0}^{d-1}\phi_{k,\frac{x_1+\nu}{d},x_2}$$

Since we also have a scalar multiple of d^{k-2} from the action of the scalar matrix, the final result is

$$\phi_{k,x_1,x_2}|\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \sum_{\nu=0}^{d-1}\phi_{k,\frac{x_1+\nu}{d},x_2}$$

This example will be used later.

Before we continue, we introduce one last definition. Given any number field K, we let $\mathcal{E}_k(K)$ be the K-span of the Eisenstein series ϕ_{k,x_1,x_2} for all $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$.

4.2 The Map S_E

4.2.1 Definition and Basic Properties

Given a field K and an Eisenstein series $E \in \mathcal{E}_k(K)$, we would like to define a map which takes as input an element of $GL_2^+(\mathbb{Q})$ and outputs an element of $L_{k-2}(K)$. To that end we define the map S_E, which does not quite give a K-rational polynomial in all cases, but after proving some basic properties of S_E we will be able to define a new map which does give K-rational polynomials.

Corollary 2.6, proved above, leads us to the following definition:

Definition 4.10. Define $S_E : \mathrm{GL}_2^+(\mathbb{Q}) \to L_n(\mathbb{C})$ by

$$S_E(\alpha) := \int_{z_0}^{\alpha z_0} E(z)(zX+Y)^{k-2}dz$$

$$+a_0(E)\cdot\int_0^{z_0}(zX+Y)^{k-2}dz - a_0(E|\alpha)\cdot\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz$$

$$-\int_{z_0}^{\infty}\tilde{E}(z)(zX+Y)^{k-2}dz + \int_{z_0}^{\infty}(\widetilde{E|\alpha})(z)[\alpha|(zX+Y)^{k-2}]dz$$

Notice that this is well-defined because, as with the proof of the corollary, the derivative with respect to z_0 is 0, so this definition does not depend on the choice of z_0. Notice also (directly from the definition) that we can write

E as a linear combination of Eisenstein series of the form ϕ_{k,x_1,x_2} and the map S_E will respect the linearity.

It is an immediate consequence of Corollary 2.6 that

$$S_E(\sigma) = -F_E(1) \tag{4.2}$$

We now prove some basic properties of S_E. Both of the next two results were proved for the case $k = 2$ in Proposition 2.3.3 of [18].

Proposition 4.11. *S_E satisfies the relation*

$$S_E(\alpha\beta) = S_E(\alpha) + \alpha|S_{E|\alpha}(\beta)$$

Proof. If we consider the last four terms in the definition of S_E, it is a simple calculation to show that grouped together without the first term, they satisfy the relation. (We need to use the fact that the action of α on the polynomials inside the integrals commutes with integration, which we know since the integrals are absolutely convergent.) But the first term also satisfies this relation; to see this it suffices to show the identity

$$\int_{z_0}^{\beta z_0} (E|\alpha)(z)[\alpha|(zX+Y)^{k-2}]dz = \int_{\alpha z_0}^{\alpha\beta z_0} E(z)(zX+Y)^{k-2}dz$$

This is a straightforward calculation using the substitution $u = \alpha z$ on the left-hand integral (along with the definition of $\alpha^{-1}|(uX+Y)^{k-2}$). $\square$

Theorem 4.12. *Put $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (with all rational entries and positive determinant) and $M_\alpha = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}$. Then S_E satisfies the following formula: if $c = 0$, then*

$$S_E(\alpha) = a_0(E)\int_0^{b/d}(tX+Y)^{k-2}dt$$

If $c > 0$, then

$$S_E(\alpha) = a_0(E)\int_0^{a/c}(tX+Y)^{k-2}dt + a_0(E|\alpha)\int_{-d/c}^{0}\alpha|(tX+Y)^{k-2}dt - M_\alpha|F_{E|M_\alpha}(1)$$

Proof. We begin with the case $c = 0$. Starting from the definition of $S_E(\alpha)$, we split the first integral and obtain

$$S_E(\alpha) = \int_{z_0}^{\alpha z_0}\tilde{E}(z)(zX+Y)^{k-2}dz + a_0(E)\int_{z_0}^{\alpha z_0}(zX+Y)^{k-2}dz$$

$$+a_0(E)\cdot\int_0^{z_0}(zX+Y)^{k-2}dz - a_0(E|\alpha)\cdot\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz$$
$$-\int_{z_0}^{\infty}\tilde{E}(z)(zX+Y)^{k-2}dz + \int_{z_0}^{\infty}(\tilde{E|\alpha})(z)[\alpha|(zX+Y)^{k-2}]dz$$

Now since this does not depend on z_0, as explained above, we let $z_0\to i\infty$. Since $c=0$, this means $\alpha z_0\to i\infty$ as well. So all the integrals that converge in this case—namely, the first one and the last two—will vanish, and we only need to treat the other three. The goal is to show that they combine to give a polynomial not dependent on z_0, and that this polynomial is the one given above.

We can combine the first two integrals to conclude that the expression we need to find the limit of is

$$a_0(E)\int_0^{\alpha z_0}(zX+Y)^{k-2}dz - a_0(E|\alpha)\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz$$

To prove the formula in this case, it suffices to show that

$$-a_0(E|\alpha)\int_0^{z_0}\alpha|(zX+Y)^{k-2}dz = a_0(E)\int_{\alpha z_0}^{b/d}(uX+Y)^{k-2}du$$

for then we could combine the two integrals and obtain the desired result immediately. But this is simple to show: firstly, we use Lemma 2.7 to replace $a_0(E|\alpha)$ with $\frac{a^{k-1}}{d}a_0(E)$. Then we use the definition of $\alpha|(zX+Y)^{k-2}$ (along with the fact that the determinant of α is ad) along with the substitution $u=\alpha z=\frac{az+b}{d}$. From there an elementary calculation shows the desired result.

We now turn to the case $c>0$, having already proved the $c=0$ case (which we will use below). Here we make use of Proposition 4.11 along with the identity

$$\alpha = \begin{pmatrix}1/c & 0\\ 0 & 1/c\end{pmatrix}\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c & d\\ 0 & 1\end{pmatrix}$$

where $\delta = ad-bc$ is just the determinant of α.

Notice that S_E evaluated on the diagonal matrix is 0, and the action of it on the polynomials is simply to multiply everything by the scalar c^{k-2}. However, by definition of the action of a matrix on a modular form, we see that

$$E|\begin{pmatrix}1/c & 0\\ 0 & 1/c\end{pmatrix}(z) = c^{-k+2}E(z)$$

so the scalar multiples cancel in the final result.

The next step is to evaluate $S_E \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}$. This is simply an application of the $c = 0$ case above; we obtain

$$a_0(E) \int_0^{a/c} (tX + Y)^{k-2} dt$$

Next, consider the final term, which (after some matrix multiplication) is equal to

$$\alpha \cdot \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} |S_{E|\alpha \cdot \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix}} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$$

Before we consider the matrix actions at all, we use the $c = 0$ case to evaluate S_E. We conclude that this term is equal to

$$\alpha \cdot \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} |a_0 \left(E|\alpha \cdot \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} \right) \int_0^d (tX + Y)^{k-2} dt$$

If we pull out the constants and use Lemma 2.7, we get

$$\tfrac{1}{c} \cdot a_0(E|\alpha) \cdot \alpha \cdot \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} | \int_0^d (tX + Y)^{k-2} dt$$

Now consider the action only of the right-hand matrix. Using the definition of this action and the substitution $u = \frac{t-d}{c}$, an elementary calculation confirms that the above expression is equal to

$$a_0(E|\alpha) \int_{-d/c}^0 \alpha|(uX + Y)^{k-2} du$$

Now we consider the middle term, which Proposition 4.11 tells us is

$$\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} |S_{E|\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma)$$

Now we apply (4.2), and this is equal to

$$-\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} |F_{E|\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(1)$$

From here, consider that

$$\begin{pmatrix}\delta & a\\ 0 & c\end{pmatrix} = \begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & 1\end{pmatrix}$$

This means we can apply Lemma 2.8 and the expression becomes

$$-\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}|F_{E|\begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}}(1)$$

Putting all the terms together, this gives the above formula and completes the proof. $\square$

Remark 4.13. It would appear at first glance that we have not covered the case $c < 0$. However, it is clear that the matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ evaluates to 0 under S_E. Also, when we take into account the action on polynomials and the action on modular forms, the combination of the two actions will always be trivial whether the weight is odd or even. So to evaluate S_E in the $c < 0$ case we simply change the signs of all the entries in the matrix and use the $c > 0$ case:

$$S_E(\begin{pmatrix}a & b\\ c & d\end{pmatrix}) = S_E(\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix}) = a_0(E)\int_0^{a/c}(tX+Y)^{k-2}dt$$

$$+a_0(E|\begin{pmatrix}-a & -b\\ -c & -d\end{pmatrix})\int_{-d/c}^{0}\alpha|(tX+Y)^{k-2}dt - \begin{pmatrix}1 & -a\\ 0 & -c\end{pmatrix}|F_{E|\begin{pmatrix}1 & -a\\ 0 & -c\end{pmatrix}}(1)$$

4.2.2 The Involution ι

Let $\iota = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, a matrix of determinant -1. We define the action of ι on an Eisenstein series as follows: first define

$$\phi_{k,x_1,x_2}|\iota = (-1)^k\phi_{k,x_1,-x_2}$$

and then extend by linearity to general Eisenstein series.

It is elementary to check that for any numbers a and c, the following holds:

$$\iota\begin{pmatrix}1 & -a\\ 0 & c\end{pmatrix} = \begin{pmatrix}1 & a\\ 0 & c\end{pmatrix}\iota \tag{4.3}$$

We will make frequent use of this fact below.

Now for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, define the map

$$S_E^\iota(\alpha) := (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota)$$

We will need the following lemma:

Lemma 4.14. *S_E^ι satisfies the relation*

$$S_E^\iota(\alpha\beta) = S_E^\iota(\alpha) + \alpha|S_{E|\alpha}^\iota(\beta)$$

Proof. We compute directly from the definition and Proposition 4.11:

$$\begin{aligned} S_E^\iota(\alpha\beta) = (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\beta\iota) &= (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota\iota^{-1}\beta\iota) \\ &= (-1)^{k-1}\iota|[S_{E|\iota}(\iota^{-1}\alpha\iota) + \iota^{-1}\alpha\iota|S_{E|\alpha\iota}(\iota^{-1}\beta\iota)] \\ &= (-1)^{k-1}\iota|S_{E|\iota}(\iota^{-1}\alpha\iota) + (-1)^{k-1}\alpha|\iota|S_{(E|\alpha)|\iota}(\iota^{-1}\beta\iota) \\ &= S_E^\iota(\alpha) + \alpha|S_{E|\alpha}^\iota(\beta) \end{aligned}$$

which is the desired result. □

4.2.3 The Map ξ_E and Rationality

Define the map

$$\xi_E := \tfrac{1}{2}(S_E + S_E^\iota)$$

The main result about ξ_E is the following:

Proposition 4.15. *For any number field K, any E in $\mathcal{E}_k(K)$ and any $\alpha \in GL_2^+(\mathbb{Q})$, $\xi_E(\alpha) \in L_{k-2}(K)$.*

Proof. We will show this for an arbitrary Eisenstein series of the form ϕ_{k,x_1,x_2} and then the result will follow by linearity. Put $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Without loss of generality, we can assume the matrix has integer entries and $c \geq 0$ (because it is equal to such a matrix times a scalar matrix). First notice that

$$\iota^{-1}\alpha\iota = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

Now we will use Theorem 4.12 (and the subsequent remark).

First we see that if $c = 0$, the result is clear (since both summands in the definition of ξ_E are clearly K-rational in this case). So we may assume $c > 0$. Now it suffices to show that the expression

$$\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} |F_{E|\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}}(1) - \iota \begin{pmatrix} 1 & -a \\ 0 & c \end{pmatrix} |F_{E|\iota|\begin{pmatrix} 1 & -a \\ 0 & c \end{pmatrix}}$$

gives a K-rational polynomial. By Equation (4.3) it suffices to show that

$$F_{E|\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}}(1) - \iota|F_{E|\iota|\begin{pmatrix} 1 & -a \\ 0 & c \end{pmatrix}}$$

gives a K-rational polynomial.

For now, assume k is even. We proceed by using the distribution law (4.1) along with our assumption that $E = \phi_{k,x_1,x_2}$. For the first term, we have

$$E| \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} = \sum_{\nu=0}^{c-1} \phi_{k, \frac{x_1+\nu}{c}, x_2+a\frac{x_1+\nu}{c}}$$

(To see this more clearly, separate the matrix into the product $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$; for the action of the former, refer to Remark 4.9.) By Proposition 4.6, the only time the corresponding polynomial yields a non-rational term will be if $\frac{x_1+\nu}{c} = 0$ or if $x_2 + a\frac{x_1+\nu}{c} = 0$. In the former case, we obtain a term of the form

$$X^{k-2} \cdot i^{k+1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}}$$

In the latter case, we obtain a term of the form

$$Y^{k-2} \cdot i^{k+3}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_2/a))}{n^{k-1}}$$

(unless $a = 0$, in which case we obtain $c - 1$ different terms similar to the above but with $-x_2/a$ replaced by $(x_1+j)/c$ for each j between 0 and $c-1$).

Now we treat the second term. We first use the definition of $E|\iota$ and then the distribution law to obtain

$$E|\iota| \begin{pmatrix} 1 & -a \\ 0 & c \end{pmatrix} = \sum_{\nu=0}^{c-1} \phi_{k, \frac{x_1+\nu}{c}, -x_2-a\frac{x_1+\nu}{c}}$$

The first subscript here is the same as the first one above, and the second subscript here is the opposite of the second one above, so they will be zero for the same values of x_1 and x_2 as above. So using the same logic as above, if the first subscript is 0, we get a term of the form

$$X^{k-2} \cdot i^{k+1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_2))}{n^{k-1}}$$

which, because the cosine function is even and because the action of ι is trivial on this monomial, will cancel out the corresponding term from the first polynomial. If the second subscript is 0, we get a term of the form

$$Y^{k-2} \cdot i^{k+3}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(-x_2/a))}{n^{k-1}}$$

(again, unless $a = 0$, in which case we get a sum of terms exactly equal to the sum in the first term), which will cancel out the corresponding term from the first polynomial, by the same reasoning as above. Since we have now shown that all non-rational terms in the formula for ξ_E cancel, we have shown the desired result for even k. For odd k, the calculation is exactly the same, except that the action of ι on polynomials includes a factor of -1, but the sine function (instead of the cosine used above) is odd. So the conclusion is the same. $\square$

4.2.4 The Map ξ'_E

Our definition and the subsequent computation with ξ_E leads us to consider another map

$$\xi'_E := \tfrac{1}{2i}(S_E - S_E^{\iota})$$

Under this definition, we have

$$S_E = \xi_E + i\xi'_E$$

We wish to do a similar computation as in the previous section, using the explicit formula to compute $\xi'_E(\alpha)$ for a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and positive determinant (as before, we can extend to rational entries by using multiplication by a scalar matrix). As in the above computation, we start by letting $E = \phi_{k,x_1,x_2}$ for some $x_1, x_2 \in \mathbb{Q}/\mathbb{Z}$ and then we can extend by linearity.

By definition, and by the remark following Theorem 4.12,

$$S_E(\alpha) - S_E^{\iota}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = S_E(\alpha) - (-1)^{k-1}\iota|S_{E|\iota}(\begin{pmatrix} -a & b \\ c & -d \end{pmatrix})$$

If $c = 0$, the two terms cancel. This is because the terms $a_0(E)$ are the same in both (the action of ι does not change it in the second term when k is even, and multiplies it by -1 when k is odd) which makes the sum equal to

$$a_0(E) \cdot \left[\int_0^{b/d} (tX+Y)^{k-2} dt + (-1)^k \int_0^{-b/d} (tX-Y)^{k-2} dt\right]$$

Now it is clear, using the transformation $t \mapsto -t$ in the second integral, that the two terms must cancel.

Now we suppose $c > 0$ (as before, we can reduce the $c < 0$ case to this case). The explicit formula has three terms. Breaking up the computation term-by-term, the first term will be

$$a_0(E) \cdot \left[\int_0^{a/c} (tX+Y)^{k-2} dt + (-1)^k \int_0^{-a/c} (tX-Y)^{k-2} dt\right]$$

which, similarly to the above, is 0.

The corresponding calculation for the second term will also be zero; here we need to know that $a_0(E|\alpha) = (-1)^k a_0(E|\alpha\iota)$, which is clear when E is of the form ϕ_{k,x_1,x_2}, since the constant term only depends on x_1, which is unchanged by the ι-action.

That leaves the third term; so (using Equation (4.3)) it remains to compute

$$F_{E|\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}}(1) - (-1)^{k-1}\iota|F_{E|\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}\iota}(1)$$

and then apply the action of $\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}$ to get the final result.

As in the computation for ξ_E, the distribution law tells us that

$$E|\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} = \sum_{\nu=0}^{c-1} \phi_{k,\frac{x_1+\nu}{c},x_2+a\frac{x_1+\nu}{c}}$$

Now we will carry out the rest of the computation using Proposition 4.6. The proposition tells us that for each summand above, the X^jY^{k-2-j}-term is

$$\binom{k-2}{j}(-1)^j \frac{B_{k-j-1}(\frac{x_1+\nu}{c})}{k-j-1} \cdot \frac{B_{j+1}(x_2 + a\frac{x_1+\nu}{c})}{j+1} X^jY^{k-2-j}$$

except for the exceptional cases which we will deal with below. For now let us treat the non-exceptional cases. When we act by ι on the above Eisenstein series and the polynomial part before using Proposition 4.6, we end up with terms corresponding to the above, of the form

$$\binom{k-2}{j}(-1)^j \frac{B_{k-j-1}(\frac{x_1+\nu}{c})}{k-j-1} \cdot \frac{B_{j+1}(-(x_2+a\frac{x_1+\nu}{c}))}{j+1} X^j(-Y)^{k-2-j}$$

In general, we have $B_n(-x) = (-1)^n B_n(x)$, and if $k-2-j$ is even, then $j+1$ is odd, and vice versa (since k is even). This means that in all cases, the two corresponding terms will be equal and will cancel when we subtract them.

So it remains to compute the two exceptional cases: the X^{k-2} term when the first subscript is zero, and the Y^{k-2} term when the second subscript is zero. We show the result for the even weight case; the odd weight case is exactly the same. Looking at the terms from the distribution law above, and again using Proposition 4.6, the Y^{k-2} term will be

$$\sum_{\nu=0}^{c-1} Y^{k-2} \cdot \delta_{x_2+a\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

(where here δ means 1 if the subscript is an integer and 0 otherwise). By similar logic, the X^{k-2} term is

$$-\sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(x_2))}{n^{k-1}}$$

This means we have shown the following:

Proposition 4.16. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ of even weight, and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ as above, $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is*

$$(aX+cY)^{k-2} \cdot \sum_{\nu=0}^{c-1} \delta_{x_2+a\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

$$-\sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-2}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\cos(2\pi n(x_2))}{n^{k-1}}$$

If instead E has odd weight, then $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is

$$(aX + cY)^{k-2} \cdot \sum_{\nu=0}^{c-1} \delta_{x_2 + a\frac{x_1+\nu}{c}} \cdot i^{k-1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(\frac{x_1+\nu}{c}))}{n^{k-1}}$$

$$- \sum_{\nu=0}^{c-1} X^{k-2} \cdot \delta_{\frac{x_1+\nu}{c}} \cdot i^{k-1}(k-2)! \cdot (2\pi)^{-(k-1)} \cdot \sum_{n=1}^{\infty} \frac{\sin(2\pi n(x_2))}{n^{k-1}}$$

There is a more succinct way to phrase the above formula. If we put, for $E = \phi_{k,x_1,x_2}$ and k even,

$$\mathcal{C}(E) := i^{k-2} \cdot X^{k-2} \cdot \delta_{x_1} \cdot (k-2)! \cdot (2\pi)^{-(k-1)} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x_2)}{n^{k-1}}$$

and for k odd,

$$\mathcal{C}(E) := i^{k-1} \cdot X^{k-2} \cdot \delta_{x_1} \cdot (k-2)! \cdot (2\pi)^{-(k-1)} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x_2)}{n^{k-1}}$$

and extend the definition by linearity to other Eisenstein series, we can state:

Corollary 4.17. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ as above, $\xi'_E(\alpha)$ is 0 when $c = 0$, and when $c \neq 0$ it is*

$$\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} | \left(\sigma | \mathcal{C}(E| \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \sigma) - \mathcal{C}(E| \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}) \right)$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This result has another corollary which will be useful below:

Corollary 4.18. *For an Eisenstein series $E = \phi_{k,x_1,x_2}$ and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$\xi'_E(\alpha) = \alpha | \mathcal{C}(E|\alpha) - \mathcal{C}(E)$$

Proof. We first check this property on generators of $\mathrm{SL}_2(\mathbb{Z})$. For $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the result follows directly from the formula in the above corollary (with $a = 0$ and $c = 1$). For $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we wish to show that ξ'_E evaluates to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} |\mathcal{C}(\phi_{k,x_1,x_1+x_2}) - \mathcal{C}(\phi_{k,x_1,x_2})$$

Now if $x_1 \neq 0$, both terms above vanish by the definition of $\mathcal{C}(E)$. But if $x_1 = 0$, then the infinite sum in both terms is identical, so since the polynomial term on the left is also unchanged, the two terms will cancel. Since the above corollary states that $\xi'_E(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ is in fact 0, this shows the desired result for the two generators of $\mathrm{SL}_2(\mathbb{Z})$.

To show the result in general, we use the fact that

$$\xi'_E(\alpha\beta) = \xi'_E(\alpha) + \alpha|\xi'_{E|\alpha}(\beta)$$

for any $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$. (This is a consequence of the definition, Proposition 4.11, and Lemma 4.14.) If the desired property is satisfied for the two matrices α and β, then

$$\begin{aligned} \xi'_E(\alpha\beta) &= \alpha|\mathcal{C}(E|\alpha) - \mathcal{C}(E) + \alpha|[\beta|\mathcal{C}(E|\alpha\beta) - \mathcal{C}(E|\alpha) \\ &= \alpha\beta|\mathcal{C}(E|\alpha\beta) - \mathcal{C}(E) \end{aligned}$$

This shows the desired result for all of $\mathrm{SL}_2(\mathbb{Z})$. □

4.3 Primes Dividing the Denominators of Values of ξ_E

In this section we wish to prove the following:

Lemma 4.19. *Suppose E is an Eisenstein series of the form $E = \phi_{k,x/N,y/N}$. Then for any $\alpha \in SL_2(\mathbb{Z})$, $\xi_E(\alpha)$ is a polynomial whose coefficients' denominators are divisible only by primes dividing N and primes less than or equal to $k+1$.*

Proof. We begin by showing that this is true when α is one of the two generators of $\mathrm{SL}_2(\mathbb{Z})$. We begin by using Theorem 4.12 and Proposition 4.1 to compute

$$\xi_E(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \frac{B_k(x/N)}{k} \int_0^1 (tX+Y)^{k-2} dt$$

Since the Bernoulli polynomial's coefficients only have denominators divisible by primes at most $k+1$ (a fact that follows from, for example, [11], equations B.5 and B.7), it is clear that our claim holds for this generator.

The other generator is $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. To evaluate ξ_E on this generator we use the equation $S_E = \xi_E + i\xi_E'$, the equation $S_E(\sigma) = -F_E(1)$, Proposition 4.6, and the computation in the proof of Proposition 4.16. The result is that $\xi_E(\sigma)$ is a rational polynomial whose coefficients are 0 or are given by the values of Bernoulli polynomials evaluated on x/N and y/N divided by positive integers less than k. So the denominators of these coefficients must be divisible only by primes dividing N and primes less than k. This proves the claim for both generators.

Now to complete the proof, we use the fact that for any $\alpha, \beta \in \mathrm{SL}_2(\mathbb{Z})$,

$$\xi_E(\alpha\beta) = \xi_E(\alpha) + \alpha|\xi_{E|\alpha}(\beta) \tag{4.4}$$

(which follows from Proposition 4.11, Lemma 4.14, and the definitions). We need to know that $E|\alpha$ satisfies the same hypotheses as E, which is clearly true; and we need to know that the matrix action of α does not introduce any new denominators, which is clearly true since α has integer entries and determinant 1. This proves the desired result. $\square$

Remark 4.20. Going carefully through the steps of the proof, we see that if $k+1$ is prime, and relatively prime to N, then the only place this appears as a factor of any of the denominators is from the constant term of the Bernoulli polynomial B_k. We will use this fact below.

4.4 The Eisenstein Series Associated to a Pair of Dirichlet Characters

Definition 4.21. Let ε_1 and ε_2 be two Dirichlet characters mod N_1 and N_2 respectively; we do not assume they are primitive, but we assume that N_1 and N_2 are not both 1 and that the product of the two characters is odd when k is odd and even when k is even. Then for any integer $k \geq 2$, we define the Eisenstein series

$$E(k, \varepsilon_1, \varepsilon_2; z) := \sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x)\bar{\varepsilon_1}(y)\phi_{(k, \frac{x}{N_2}, \frac{y}{N_1})}(N_2 z)$$

Let K be the field generated over $\mathbb{Q}$ by the values of the two characters. Then $E(k, \varepsilon_1, \varepsilon_2) \in \mathcal{E}_k(K)$. In this section we compute the Fourier expansion and the L-function of $E(k, \varepsilon_1, \varepsilon_2)$.

Define, for any Dirichlet character ψ mod m,

$$\hat{\psi}(n) := \sum_{a=0}^{m-1} \psi(a) e^{2\pi i a n/m}$$

Recall the definition

$$J(k, a, b; z) := \sum_{\substack{\kappa \equiv a \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \cdot \sum_{m=1}^{\infty} e^{2\pi i z m \kappa} e^{2\pi i m b}$$

We used this definition earlier (in Proposition 4.1) to state the Fourier expansion of ϕ_{k,x_1,x_2}. Now, to help compute the Fourier expansion of $E(k, \varepsilon_1, \varepsilon_2)$, we compute

$$\sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x) \bar{\varepsilon_1}(y) J(k, x/N_2, y/N_1; N_2 z)$$

$$= \sum_{m=1}^{\infty} \left(\sum_{y=0}^{N_1-1} \bar{\varepsilon_1}(y) e^{2\pi i z m \kappa} e^{2\pi i m (y/N_1)} \right) \left(\sum_{x=0}^{N_2-1} \sum_{\substack{\kappa \equiv x/N_2 \pmod 1 \\ \kappa \in \mathbb{Q}^+}} \kappa^{k-1} \varepsilon_2(x) e^{2\pi i m N_2 z \kappa} \right)$$

$$= \sum_{m=1}^{\infty} \hat{\bar{\varepsilon_1}}(m) \cdot \sum_{x=0}^{N_2-1} \sum_{\substack{\kappa \equiv x \pmod{N_2} \\ \kappa \in \mathbb{Z}^+}} \kappa^{k-1} \cdot \tfrac{1}{N_2^{k-1}} \cdot \varepsilon_2(x) e^{2\pi i m z \kappa}$$

$$= N_2^{1-k} \sum_{m=1}^{\infty} \hat{\bar{\varepsilon_1}}(m) \sum_{\kappa=1}^{\infty} \kappa^{k-1} \varepsilon_2(\kappa) e^{2\pi i z m \kappa}$$

$$= N_2^{1-k} \sum_{n=1}^{\infty} \left(\sum_{\substack{m\kappa = n \\ m, \kappa \in \mathbb{Z}^+}} \hat{\bar{\varepsilon_1}}(m) \kappa^{k-1} \varepsilon_2(\kappa) \right) q^n$$

where in the last line, $q = e^{2\pi i z}$.

Since $\bar{\varepsilon_1}\varepsilon_2$ has the same sign as $(-1)^k$, it follows that the corresponding sum for $(-1)^k J(k, -x/N_2, -y/N_1)$ will be the same as for $J(k, x/N_2, y/N_1)$. Using Proposition 4.1, this proves

Proposition 4.22. *Let $E = E(\varepsilon_1, \varepsilon_2)$ as above. Then*

$$L(E, s) = -2N_2^{1-k} L(\hat{\hat{\varepsilon}}_1, s) L(\varepsilon_2, s - k + 1)$$

where the L-functions on the right are Dirichlet L-functions.

4.5 Modular Symbols Attached to Eisenstein Series

From now on we will add the following hypotheses on our Eisenstein series E. We assume that it is of the form $-\frac{N_2^{k-1}}{2} E(k, \varepsilon_1, \varepsilon_2)$ (in light of the above computation for the Fourier expansion, the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We recall the definition of a modular symbol. Let $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ be the group of degree zero divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})]$-module. We refer to a map as an A-valued *modular symbol* over a congruence subgroup Γ if the map is a Γ-homomorphism from degree zero divisors to elements of A.

Suppose that r is the greatest integer such that $\mathfrak{p}^r$ divides all the constant terms at the cusps of E. By assumption, r is positive. Now we define a map

$$M_E : \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to L_{k-2}(K^+/\mathfrak{p}^r \mathcal{O}_{K,\mathfrak{p}})$$

where K^+ means we are thinking of K as an additive group only, and $\mathcal{O}_{K,\mathfrak{p}}$ is the localization of $\mathcal{O}_K$ at the prime ideal $\mathfrak{p}$. The map is defined as follows:

$$M_E(\{b\} - \{a\}) = \xi_E(\gamma_b) - \xi_E(\gamma_a)$$

where γ_b and γ_a are elements of $\mathrm{SL}_2(\mathbb{Z})$ that map the cusp at infinity to the cusps b and a, respectively.

Our goal is to show:

Theorem 4.23. *Let E, K, $\mathcal{O}_K$, $\mathfrak{p}$, $\mathcal{O}_{K,\mathfrak{p}}$, p, and r be as above. Let N_1 and N_2 be the moduli of ε_1 and ε_2, respectively, and suppose that p is relatively prime to N, the least common multiple of N_1 and N_2. Then M_E is a modular symbol, over the same congruence subgroup Γ for which E is modular, taking values in $L_{k-2}(\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}^r \mathcal{O}_{K,\mathfrak{p}})$.*

Proof. To begin with, we must show that the map is well-defined. In other words, the matrices γ_b and γ_a are defined only up to multiplication by the stabilizer of ∞ on the right, and the stabilizer of the specific cusp on the left. We will show that for any cusp, choosing a different γ does not change the value of $\xi_E(\gamma) \bmod \mathfrak{p}^r$.

Let γ_a be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ that sends ∞ to a cusp a. Let α be another such matrix that stabilizes a, so that $\alpha\gamma_a$ also sends ∞ to a. But $\alpha\gamma_a = \gamma_a(\gamma_a^{-1}\alpha\gamma_a)$, and $\gamma_a^{-1}\alpha\gamma_a$ stabilizes ∞. So now it suffices to show that for any choice of integer n, $\gamma_a \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ gives the same values mod $\mathfrak{p}^r$ as γ_a does when evaluating ξ_E on them. Now by Equation (4.4) (and the definition of ξ_E),

$$\begin{aligned} \xi_E(\gamma_a \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}) - \xi_E(\gamma_a) &= \gamma_a | \xi_{E|\gamma_a}(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}) \\ &= a_0(E|\gamma_a) \int_0^n \gamma_a | (tX+Y)^{k-2} dt \end{aligned}$$

where for the last line we have used Theorem 4.12 and the definition of ξ_E. By choice of r, it is now clear (since $p > k$ and the action of γ_a introduces no denominators) that the coefficients of the last expression are $\mathfrak{p}$-adic integers divisible by $\mathfrak{p}^r$, which suffices to show that M_E is well-defined.

The next step is to show that this is a Γ-homomorphism. This is a simple computation using Equation (4.4): if a and b are any cusps, and $\gamma \in \Gamma$,

$$\begin{aligned} M_E(\gamma(\{b\} - \{a\})) &= \xi_E(\gamma\gamma_b) - \xi_E(\gamma\gamma_a) \\ &= \gamma|\xi_E(\gamma_b) - \gamma|\xi_E(\gamma_a) \end{aligned}$$

which is the desired result. (We have used the fact that $E|\gamma = E$.)

The last step is to show that M_E takes values with coefficients not just in K, but with denominators not divisible by $\mathfrak{p}$. This is a consequence of the fact that E is a half-integer multiple of an algebraic integer linear combination of Eisenstein series satisfying Lemma 4.19, and the fact that $p > k$ and p and N are relatively prime. This shows the theorem when $p \neq k+1$.

To finish the proof, we must show this still holds in the case $p = k+1$, when p appears exactly once in the denominator of each of the constant coefficients of $B_k(x/N_2)$ (for $0 \le x \le N_2 - 1$). But now we use the definition of $E(k, \varepsilon_1, \varepsilon_2)$ to see that

$$a_0(E) = \sum_{x=0}^{N_2-1} \sum_{y=0}^{N_1-1} \varepsilon_2(x)\bar{\varepsilon_1}(y) B_k(x/N_2)/k$$

and so the constant coefficients will cancel when we take the sum of character values. A similar calculation shows the same result for $a_0(E|\alpha)$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. This shows that p does not appear in the denominators of $\xi_E(\alpha)$ even when $p = k+1$, completing the proof. $\square$

4.6 Hecke Operators and Modular Symbols

In this section E will be an Eisenstein series satisfying the same assumptions as in the above theorem. So far we have not discussed the action of Hecke operators on modular symbols. In this section, now that we have defined the modular symbol M_E, we prove a result concerning the action of the Hecke operators on it.

First we need the general definition of the Hecke operators. Following [4], we define them using double coset operators. Let g be a matrix with positive determinant and integer entries, and let $\tilde{\Gamma}$ be any congruence subgroup. The double coset $\tilde{\Gamma} g \tilde{\Gamma}$ can be written as a finite disjoint union of right cosets of the form $\tilde{\Gamma} g_j$. We now write, for any modular symbol Φ,

$$\Phi | T(g) = \sum_j \Phi | g_j$$

where for a degree-zero divisor D, $(\Phi|g_j)(D) = g_j^{-1}|\Phi(gD)$ (as we did before, we change the definition in [4] to account for the fact that our matrix action on polynomials is a left action). For any prime ℓ, the Hecke operator T_ℓ arises from the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix}$. We also have, for any positive integer d, the diamond operator $\langle d \rangle$ which arises from any element of $\Gamma_0(N)$ whose lower right entry is congruent to d mod N. (Here N is the level of the modular form we are acting on.)

This shows that the Hecke operators act on modular symbols in a similar way to modular forms: as a sum of actions by matrices. For a cusp form f and the corresponding modular symbol M_f, it is a fact stated in [4] (and easily proved from the definitions) that for a matrix $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$M_f | \alpha = M_{f|\alpha}$$

We will now prove the corresponding result for an Eisenstein series E:

Lemma 4.24. *Let E be an Eisenstein series as in the above theorem, let M_E be the associated modular symbol, and let $\alpha \in GL_2^+(\mathbb{Q})$. Then for any degree-zero divisor of the form $\{b\} - \{a\}$,*

$$(M_E|\alpha)(\{b\} - \{a\}) = \xi_{E|\alpha}(\gamma_b) - \xi_{E|\alpha}(\gamma_a)$$

where γ_b and γ_a are elements of $SL_2(\mathbb{Z})$ that map the cusp at infinity to the cusps b and a, respectively.

Proof. This is a straightforward computation from the definitions (and also Proposition 4.11 and Lemma 4.14):

$$(M_E|\alpha)(\{b\} - \{a\}) = \alpha^{-1}|M_E(\{\alpha b\} - \{\alpha a\})$$
$$= \alpha^{-1}|[\xi_E(\alpha\gamma_b) - \xi_E(\alpha\gamma_a)]$$
$$= \alpha^{-1}|[\xi_E(\alpha) + \alpha|\xi_{E|\alpha}(\gamma_b) - \xi_E(\alpha) - \alpha|\xi_{E|\alpha}(\gamma_a)$$
$$= \xi_{E|\alpha}(\gamma_b) - \xi_{E|\alpha}(\gamma_a)$$

□

Combining the above lemma with the fact that the Hecke operators can be expressed as the sum of right actions of matrices, we arrive at the following:

Corollary 4.25. *Let E be an Eisenstein series as above, and suppose that E is a simultaneous eigenfunction for the Hecke operators T_ℓ (ℓ prime) and $\langle d\rangle$. Then M_E is also a simultaneous eigenfunction for the Hecke operators with the same eigenvalues as E.*

Proof. As implied above, this follows from the lemma and the definition of the Hecke operators. We also use the fact that the ξ_E map respects summing different Eisenstein series (in the sense that for two Eisenstein series $E_1, E_2 \in \mathcal{E}_k(K)$, $\xi_{E_1} + \xi_{E_2} = \xi_{E_1+E_2}$) and also scalar multiplication. These facts show that for any Hecke operator T with eigenvalue a_T,

$$a_T\xi_E = \xi_{E|T} = \sum_j \xi_{E|g_j}$$

and then the definitions and the lemma show that for any cusps a and b,

$$(M_E|T)(\{b\} - \{a\}) = \sum_j (\xi_{E|g_j}(\gamma_b) - \xi_{E|g_j}(\gamma_a)) = a_T M_E(\{b\} - \{a\})$$

□

Chapter 5

Twisted Special Values

In this section we keep the same assumptions on E as we had at the end of the previous section, which we restate here. We assume that it is of the form $-\frac{N_2^{k-1}}{2}E(k,\varepsilon_1,\varepsilon_2)$ (the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We know from the previous section that we can associate to E a modular symbol M_E. Let χ be a primitive Dirichlet character of conductor m. We recall the definition of the *twist operator* R_χ on modular symbols. If Φ is any modular symbol, then for any degree-zero divisor D,

$$(\Phi|R_\chi)(D) := \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}^{-1} |\Phi(\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} D)$$

As in [4], we refer to the "special values" attached to a modular symbol as the evaluation of that modular symbol on the divisor $\{\infty\} - \{0\}$.

5.1 Twisted Special Values on Boundary Symbols

Definition 5.1. Let $\mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ be the group of divisors on the rational cusps of the upper half-plane. Let A be a $\mathbb{Q}[M_2(\mathbb{Z}) \cap \mathrm{GL}_2^+(\mathbb{Q})]$-module. We refer to a map as an A-valued *boundary symbol* over a congruence subgroup Γ if the map is a Γ-homomorphism from divisors to elements of A.

Comparing this definition with that of a modular symbol, it is clear that all boundary symbols are modular symbols. Therefore we can apply the twist operator to a boundary symbol when we restrict the boundary symbol to degree-zero divisors.

The goal of this section is to show that for any $L_{k-2}(\mathbb{C})$-valued boundary symbol B, and any primitive character χ of conductor m, we have $(B|R_\chi)(\{\infty\} - \{0\}) = 0$. For that result, we will need the following:

Lemma 5.2. *Let P be a homogeneous polynomial in X and Y of degree $k-2$. Suppose P is fixed under the $SL_2(\mathbb{Z})$-action of a nontrivial subgroup of the stabilizer (in $SL_2(\mathbb{Z})$) of the cusp at infinity. Then P is of the form CX^{k-2} for some constant C.*

Proof. In general, P is of the form $CX^{k-2} + a_{k-3}X^{k-3}Y + \cdots + a_1XY^{k-3} + a_0Y^{k-2}$. Meanwhile, an element of the group that we are assuming fixes P is of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some n (and since the group is assumed nontrivial there is at least one such n not equal to 0). The action of a matrix of this form fixes X and sends $Y \mapsto nX + Y$. So the action on P gives a polynomial of the form

$$CX^{k-2} + a_{k-3}X^{k-3}(nX+Y) + \cdots + a_1X(nX+Y)^{k-3} + a_0(nX+Y)^{k-2}$$

Collecting all the X^{k-2} terms together and using our hypothesis that P is fixed under this action, we obtain the equation

$$a_{k-3}n + a_{k-4}n^2 + \cdots + a_1n^{k-3} + a_0n^{k-2} = 0$$

However, if this equation is true for one nonzero n, then by hypothesis it is also true for all multiples of n. Choosing enough multiples to obtain $k-2$ equations, we get a system of equations in $a_{k-3}, a_{k-2}, \ldots, a_1, a_0$ whose coefficient matrix is invertible (since its determinant will be a constant times a Van der monde determinant). This is enough to show that $a_{k-3}, a_{k-2}, \ldots, a_1, a_0$ are all zero, which completes the proof. □

Lemma 5.3. *Let P be a homogeneous polynomial in X and Y of degree $k-2$. Suppose P is fixed under the $SL_2(\mathbb{Z})$-action of a nontrivial subgroup of the stabilizer (in $SL_2(\mathbb{Z})$) of the cusp at a rational number a/c. Then P is of the form $C(aX+cY)^{k-2}$ for some constant C.*

Proof. A matrix in $\mathrm{SL}_2(\mathbb{Z})$ that stabilizes the cusp at a/c must be of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for some $b, d \in \mathbb{Z}$ such that $ad - bc = 1$, and some $n \in \mathbb{Z}$. Now for a matrix of this form to fix P, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}|P = P$$

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} |P = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} |P$$

This means that $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} |P$ is a polynomial satisfying the hypotheses of Lemma 5.2 and therefore is of the form CX^{k-2} for a constant C. Therefore,

$$\begin{aligned} P &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} |CX^{k-2} \\ &= C(aX + cY)^{k-2} \end{aligned}$$

which is the desired result. □

Theorem 5.4. *Let B be an $L_{k-2}(\mathbb{C})$-valued boundary symbol for a congruence subgroup Γ, and let χ be a primitive Dirichlet character of conductor m. Then $(B|R_\chi)(\{\infty\} - \{0\}) = 0$.*

Proof. From the definitions, we have

$$(B|R_\chi)(\{\infty\} - \{0\}) = \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} (\{\infty\} - \{0\}))$$

$$\sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\{\infty\}) - \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix} |B(\{\tfrac{a}{m}\})$$

where we may split the sum because B is a boundary symbol.

The key observation now is that if γ is a matrix in Γ that stabilizes a cusp α, then it must also fix the polynomial $B(\alpha)$ under the matrix action. (This follows directly from the Γ-homomorphism property.) So we may treat each of the sums above using Lemmas 5.2 and 5.3.

For the first sum, we use Lemma 5.2 to conclude that $B(\{\infty\})$ is of the form C_1X^{k-2}. That polynomial is fixed under the action of $\begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix}$, so the first sum is a constant times X^{k-2} times a sum of character values, and thus is zero.

For the second sum, we use Lemma 5.3 to conclude that $B(\{\frac{a}{m}\})$ is of the form $C_2(aX + mY)^{k-2}$. When we apply the action of $\begin{pmatrix} 1 & -a/m \\ 0 & 1/m \end{pmatrix}$, the resulting polynomial is C_2Y^{k-2}. So the second sum is a constant times Y^{k-2} times a sum of character values, and thus is also zero. This shows the desired result. □

5.2 Twisted Special Values Associated to E

Now we are able to connect the modular symbol M_E with the special values of $L(E,\chi,j)$, for a primitive Dirichlet character χ, at the critical integers.

Theorem 5.5. *Let E, k, ε_1, ε_2, N_1, N_2, $\mathfrak{p}$, p, and r be as above, and let χ be a primitive Dirichlet character of conductor m, with m relatively prime to both p and N, the least common multiple of N_1 and N_2. Then*

$$(M_E|R_\chi)(\{\infty\}-\{0\}) \equiv \tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{E\otimes\chi}(1+j)\cdot X^jY^{k-2-j}$$

where the equivalence is mod $\mathfrak{p}^r$ where $\mathfrak{p}$ is understood to be an ideal of the ring of integers of $K[\chi]$ localized at a prime above $\mathfrak{p}$.

Proof. We begin by computing from the definitions:

$$(M_E|R_\chi)(\{\infty\}-\{0\}) = \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix}1 & -a/m\\ 0 & 1/m\end{pmatrix}|M_E(\{\infty\}-\{\tfrac{a}{m}\})$$

$$= \sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix}m & -a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/m & 0\\ 0 & 1/m\end{pmatrix}|[\xi_E(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\xi_E(\gamma_{a/m})]$$

where $\gamma_{a/m}$ is a matrix of the form $\begin{pmatrix}a & b_a\\ m & d_a\end{pmatrix}$ of determinant 1, i.e. it is an element of $\mathrm{SL}_2(\mathbb{Z})$ that carries ∞ to $\frac{a}{m}$. Since $\xi_E(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}) = 0$, the sum is equal to

$$-m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix}m & -a\\ 0 & 1\end{pmatrix}|\xi_E(\gamma_{a/m})$$

Now we claim that

$$-\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix}m & -a\\ 0 & 1\end{pmatrix}|\xi'_E(\gamma_{a/m}) = 0$$

To prove the claim, we define a map B_E on divisors on cusps: for a cusp α,

$$B_E(\alpha) := \gamma_\alpha|\mathcal{C}(E|\gamma_\alpha)$$

where $\mathcal{C}(E)$ is as defined in the previous section, and we extend linearly to other divisors. It is elementary to check that this map is a Γ-homomorphism, where Γ is the congruence subgroup that stabilizes E; so now we can use Theorem 5.4 to conclude that

$$(B_E|R_\chi)(\{\infty\} - \{0\}) = 0$$

But computing from the definitions and Corollary 4.18, we see that

$$0 = (B_E|R_\chi)(\{\infty\} - \{0\}) = \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |[B_E(\{\infty\}) - B_E(\{\tfrac{a}{m}\})]$$

$$= \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |[B_E(\{\infty\}) - B_E(\{\tfrac{a}{m}\})]$$

$$= \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |[\mathcal{C}(E) - \gamma_{a/m}|\mathcal{C}(E|\gamma_{a/m})]$$

$$= - \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |\xi'_E(\gamma_{a/m})$$

which proves the claim.

The claim shows that

$$-m^{k-2} \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |\xi_E(\gamma_{a/m}) = -m^{k-2} \sum_{a=0}^{m-1} \bar{\chi}(a) \begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} |S_E(\gamma_{a/m})$$

From here we carry out the computation using Theorem 4.12, which says that

$$S_E(\gamma_{a/m}) = a_0(E) \int_0^{a/m} (tX+Y)^{k-2} dt + a_0(E|\gamma_{a/m}) \int_{-d_{a/m}}^{0} \gamma_{a/m}|(tX+Y)^{k-2} dt$$

$$- \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} |F_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}^{(1)}$$

The first two terms of the sum are $\mathfrak{p}$-integral and divisible by $\mathfrak{p}^r$, by our assumption on E. (In order to know this, we need to know that the integrals contain no terms with negative p-adic valuation. That is implied by the hypotheses that p and m are relatively prime and also that $p > k - 1$.) The matrix action in the definition of the twist operator will not change these facts since it introduces no denominators.

We now wish to treat the sum

$$m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}|F_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1)$$

$$= m^{k-2}\sum_{a=0}^{m-1}\bar{\chi}(a)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}|\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot D_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1+j)\cdot X^jY^{k-2-j}$$

by Proposition 2.3. Then we apply the definition of matrices acting on polynomials and switch the order of summation to obtain

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot\sum_{a}\bar{\chi}(a)D_{E|\begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix}}(1+j)\cdot X^jY^{k-2-j}$$

Now we use Lemma 2.9 with $s=j+1$; the resulting sum is

$$\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot i^j\cdot m^j\cdot D_{E\otimes\chi}(1+j)X^jY^{k-2-j}$$

Since the first two terms reduce to 0 mod $\mathfrak{p}^r$, this is exactly the desired result. $\square$

Corollary 5.6. *With all notation the same as in the above theorem,*

$$(M_E|R_\chi)(\{\infty\}-\{0\})\equiv\sum_{j=0}^{k-2}(-1)^{j+1}\binom{k-2}{j}\cdot j!\cdot m^j\cdot\tfrac{\tau(\bar{\chi})L(E,\chi,1+j)}{(2\pi i)^{j+1}}\cdot X^jY^{k-2-j}$$

where the equivalence is mod $\mathfrak{p}^r$ where $\mathfrak{p}$ is understood to be an ideal of the ring of integers of $K[\chi]$ localized at a prime above $\mathfrak{p}$.

Proof. Combine the above theorem with Proposition 2.1. $\square$

5.3 An Explicit Formula

In this section we will give an explicit formula to compute the twisted special values that appeared in the above section; it will be used in the next two chapters. Recall that for any E, we can find two Dirichlet characters ε_1 and ε_2 (not necessarily nontrivial or primitive) such that

$$L(E,s)=L(\varepsilon_1,s)L(\varepsilon_2,s-k+1)$$

(For full details, see [10], chapter 4, in particular section 4.7.)

Now let χ be a nontrivial primitive Dirichlet character. Since χ is totally multiplicative, we have

$$L(E,\chi,s) = L(\chi\varepsilon_1, s)L(\chi\varepsilon_2, s-k+1)$$

We can evaluate $L(E,\chi,s)$ at the critical integers simply by evaluating the Dirichlet L-functions on the right-hand side of the above equation. To evaluate $L(\chi\varepsilon_2, s-k+1)$, we can use the following standard formula (see, for example, [18], p. 91): if ψ is a Dirichlet character mod m, then

$$L(\psi, 1-n) = \tfrac{-1}{n} \cdot m^{n-1} \cdot \sum_{a=1}^{m} \psi(a) B_n(\tfrac{a}{m}).$$

This formula can also be rephrased as $\frac{-1}{n} \cdot B_n(\psi)$, where

$$B_n(\psi) = m^{n-1} \cdot \sum_{a=1}^{m} \psi(a) B_n(\tfrac{a}{m})$$

To evaluate $L(\chi\varepsilon_1, s)$ we must combine the above formula with the standard functional equation for Dirichlet L-functions: if we set $\kappa(\psi)$ to be 0 for odd characters and 1 for even characters, then for primitive ψ,

$$L(\psi, s)\Gamma(\tfrac{s+\kappa}{2})(\tfrac{m}{\pi})^{(s+\kappa)/2} = \frac{\tau(\psi)}{i^\kappa\sqrt{q}} \cdot L(\overline{\psi}, 1-s)\Gamma(\tfrac{1-s+\kappa}{2})(\tfrac{m}{\pi})^{(1-s+\kappa)/2}$$

Unraveling these two formulas, as well as using other standard formulas for Dirichlet L-functions (all of which can be found in [11], chapter 10), produces the following:

Proposition 5.7. *Let E, ε_1, and ε_2 be as above, and let N_1 and N_2 be the conductors of ε_1 and ε_2 (though the characters need not be primitive). Let χ be a primitive Dirichlet character of conductor m. Let $\tilde{\varepsilon_1}$ and $\tilde{\varepsilon_2}$ be the primitive characters that induce ε_1 and ε_2 respectively, with conductors $\tilde{N}_1$ and $\tilde{N}_2$ respectively. Let j be an integer strictly between 0 and the weight of E such that $\varepsilon_1\chi$ has the same sign as $(-1)^j$. Then*

$$\tau(\bar{\chi})L(E,\chi,j)/(2\pi i)^j$$

$$= (-i)^j\tau(\bar{\chi})\tau(\tilde{\varepsilon_1}\chi) \cdot C_j \cdot B_j(\overline{\tilde{\varepsilon_1}\chi}) \cdot B_{k-j}(\tilde{\varepsilon_2}\chi) \cdot \prod_{q|mN_1} (1 - \tfrac{\tilde{\varepsilon_1}(q)\chi(q)}{q^j}) \cdot \prod_{q|mN_2} (1 - \tilde{\varepsilon_2}(q)\chi(q)q^{k-j-1})$$

where if j is odd,

$$C_j = (-i) \cdot 2^{-j} \cdot (m\tilde{N}_1)^{-j} \cdot (\tfrac{j-1}{2})!^{-1} \cdot [(\tfrac{-1}{2})(\tfrac{-3}{2}) \cdots (\tfrac{2-j}{2})]^{-1} \cdot \tfrac{1}{j(k-j)}$$

and if j is even,

$$C_j = 2^{-j} \cdot (m\tilde{N}_1)^{-j} \cdot (\tfrac{j}{2}-1)!^{-1} \cdot [(\tfrac{-1}{2})(\tfrac{-3}{2}) \cdots (\tfrac{1-j}{2})]^{-1} \cdot \tfrac{1}{j(k-j)}$$

Remark 5.8. For critical integers not meeting the condition that $\varepsilon_1\chi$ has the same sign as $(-1)^j$, $L(E,\chi,j)$ is zero due to a trivial zero arising from the Dirichlet L-functions.

Corollary 5.9. *With all notation as in the above proposition, including the definition of C_j, we have $D_{E\otimes\chi}(j)$ equal to*

$$i \cdot j! \cdot \tau(\tilde{\varepsilon_1}\chi) \cdot C_j \cdot B_j(\overline{\tilde{\varepsilon_1}\chi}) \cdot B_{k-j}(\tilde{\varepsilon_2}\chi) \cdot \prod_{q|mN_1} (1 - \tfrac{\tilde{\varepsilon_1}(q)\chi(q)}{q^j}) \cdot \prod_{q|mN_2} (1 - \tilde{\varepsilon_2}(q)\chi(q)q^{k-j-1})$$

Proof. Use the above proposition and Proposition 2.1. □

5.4 The Sign of E and the Action of ι

We keep the same assumptions on E as earlier in this section. Recall the matrix $\iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and consider the degree-zero divisor

$$\Lambda_E(\chi) := \sum_{a=0}^{m-1} \bar{\chi}(a)(\{\infty\} - \{\tfrac{a}{m}\})$$

For any $a, b \in \mathbb{P}^1(\mathbb{Q})$, the action of ι on a degree-zero divisor $\{b\} - \{a\}$ is

$$(\{b\} - \{a\})^\iota = \{-b\} - \{-a\}$$

(see, for example, [1] or [20]). We are going to obtain identities involving the twisted special values computed above by considering the polynomial $M_E(\Lambda_E(\chi)^\iota)$ for an arbitrary primitive character χ.

By the above, we have

$$\Lambda_E(\chi)^\iota = \sum_a \bar{\chi}(a)(\{\infty\} - \{-\tfrac{a}{m}\})$$

and so

$$\Lambda_E(\chi)^\iota = \chi(-1) \sum_a \bar{\chi}(-a)(\{\infty\} - \{-\tfrac{a}{m}\})$$

This means

$$M_E(\Lambda_E(\chi)^\iota) = \mathrm{sgn}(\chi) M_E(\Lambda_E(\chi))$$

Now recall that E is of the form $E(k, \varepsilon_1, \varepsilon_2)$. We define the *sign* of E to be $(-1)^{k-1}\varepsilon_1(-1)$. (In weight 2 this is in accordance with the definition in [18].)

Lemma 5.10. *For any E as above and any degree-zero divisor d,*

$$M_E(d^\iota) = sgn(E) \cdot \iota | M_E(d)$$

Proof. If $\{\alpha\}$ is a cusp, ι sends it to $\{-\alpha\}$. If we let $\gamma_\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ mapping $\{\infty\}$ to $\{\alpha\}$, then an element mapping $\{\infty\}$ to $\{-\alpha\}$ is $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \iota^{-1}\gamma_\alpha\iota$. We will show that

$$\xi_E(\iota^{-1}\gamma_\alpha\iota) = \mathrm{sgn}(E) \cdot \iota | \xi_E(\gamma_\alpha)$$

and then the definition of M_E will imply the lemma. To show the identity claimed above, we simply compute directly:

$$\begin{aligned} \xi_E(\iota^{-1}\gamma_\alpha\iota) &= \tfrac{1}{2}(S_E(\iota^{-1}\gamma_\alpha\iota) + S_E^\iota(\iota^{-1}\gamma_\alpha\iota)) \\ &= \tfrac{1}{2}(S_E(\iota^{-1}\gamma_\alpha\iota) - \iota | S_{E|\iota}(\gamma_\alpha)) \end{aligned}$$

Because the action of ι on Eisenstein series and on polynomials is an involution, this is equal to

$$\begin{aligned} &\tfrac{1}{2}\iota|(\iota|S_{(E|\iota)|\iota}(\iota^{-1}\gamma_\alpha\iota) - S_{E|\iota}(\gamma_\alpha)) \\ &\qquad = -\iota|\xi_{E|\iota}(\gamma_\alpha) \end{aligned}$$

Now using the fact that $E = E(k, \varepsilon_1, \varepsilon_2)$ and the definition of the ι-action on E, we see that $E|\iota = (-1)^k \varepsilon_1(-1)E$, so this shows the claim and hence the lemma. $\square$

If we combine the lemma with the equation immediately before it, we see that we now have two different ways of computing $M_E(\Lambda_E(\chi)^\iota)$, so the results are equal:

$$\mathrm{sgn}(\chi) M_E(\Lambda_E(\chi)) = \mathrm{sgn}(E) \cdot \iota | M_E(\Lambda_E(\chi))$$

On the right-hand side, $(-1)^k\iota$ acts as the involution on polynomials $Y \mapsto -Y$. So any term with Y raised to an even power will be fixed by the involution, and any term with Y raised to an odd power will be negated by it. This shows the following:

Proposition 5.11. *Let E and χ be as above, and consider the polynomial $M_E(\Lambda_E(\chi))$. If sgn(E) = sgn(χ), then the coefficients of the terms X^jY^{k-2-j} with $0 \leq j \leq k-2$ with j odd are all zero. If sgn$(E) \neq$ sgn(χ), then the coefficients of the terms X^jY^{k-2-j} with $0 \leq j \leq k-2$ with j even are all zero.*

Remark 5.12. In weight 2, where there is only one term, a constant times X^0Y^0, this proposition implies that $M_E(\Lambda_E(\chi))$ can be nonzero only if $\mathrm{sgn}(E) = \mathrm{sgn}(\chi)$, and is always zero when the signs do not match. This was already known in weight 2—see, for example, [18].

Chapter 6

The Congruence Theorem

The above results were obtained in order to show congruence results concerning the special values of the L-functions of a cusp form and a congruent Eisenstein series. (We will explain below what it means for two modular forms to be congruent mod a prime.) The final results we seek are listed below for the sake of completeness. Some key points of the proofs are merely sketched and are also given for even weight only. Full details, including generalization to the odd weight case, will be published separately in joint work with V. Vatsal.

In this section we keep the same assumptions on E as we had at the end of the previous section, which we restate here. We assume that it is of the form $-\frac{N_2^{k-1}}{2}E(k,\varepsilon_1,\varepsilon_2)$ (the coefficient is a normalizing factor so that $a_1 = 1$). We also make two more assumptions on E. To state them, let K be the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of E, with ring of integers $\mathcal{O}_K$. We suppose that there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ such that at any cusp, the constant term of the Fourier expansion has positive $\mathfrak{p}$-adic valuation. Finally, if we let p be the unique rational prime lying under $\mathfrak{p}$, we suppose $p > k$.

We now begin with the following:

Theorem 6.1. *Let E be as above. Then the modular symbol M_E is not identically zero mod $\mathfrak{p}$.*

Proof. We will show that there exists a character χ such that the χ-twisted special value of M_E is indivisible by $\mathfrak{p}$. To do this, we shall use the calculation of special values above, together with a result of Friedman and Washington on indivisibility of twisted Bernoulli numbers. Let ℓ denote an odd prime which we will specify later, and let χ denote a primitive Dirichlet character of ℓ-power conductor ℓ^n with n large. We assume that χ has order ℓ^{n-1}. Thus χ is a character of the cyclotomic $\mathbb{Z}_\ell$-extension of $\mathbb{Q}$. Then we may apply Proposition 5.7 and Corollary 5.6 to determine the $\mathfrak{p}$-adic divisibility of the coefficients of the χ-twisted special value of M_E. Combining these results shows that the coefficient of X^jY^{k-j-2} is the product of three types of term: elementary explicit constants, Euler products over primes $q|N_1N_2$, and twisted Bernoulli numbers $B_j(\overline{\tilde{\varepsilon_1}\chi})$ and $B_{k-j}(\tilde{\varepsilon_2}\chi)$. We want to show that for suitable ℓ and large n, that all these quantities are $\mathfrak{p}$-adic units. This is obvious in the case of the elementary constants for all $\ell \neq p$ and any n, since $p > k$. In the case of the Euler factors, it is evident that, since the ℓ-power roots of unity are distinct modulo $\mathfrak{p}$, there are only finitely many primes ℓ and integers n such that terms of the form $1 - \frac{\tilde{\varepsilon_1}(q)\chi(q)}{q^j}$ or $1 - \tilde{\varepsilon_2}(q)\chi(q)q^{k-j-1}$ are congruent to zero modulo $\mathfrak{p}$. It remains to deal with the Bernoulli numbers, and this we do by appealing to a theorem of Friedman and Washington. First consider the case when the characters ϵ_1 and ϵ_2 are both odd. In this case the result is most easily phrased for the coefficient of Y^{k-2}, so that $j = 0$. Then we are dealing with $L(E, \chi, 1)$, and the Bernoulli numbers are $B_1(\overline{\tilde{\varepsilon_1}\chi}) \cdot B_{k-1}(\tilde{\varepsilon_2}\chi)$. We want to know that these are $\mathfrak{p}$-adic units for suitable ℓ and n and χ. But this follows directly from [3], especially the remark at the end of the proof of Lemma 3 on page 432, and a brief translation of the notation from section 1 of that paper. (One could also deduce this result from the properties of p-adic L-functions, which reduce the case of B_{k-j} to B_1, and then apply the well-known results of Ferrero and Washington.) Thus there exists χ such that the χ-twisted special value of M_E is indivisible by $\mathfrak{p}$, and M_E is nonzero modulo $\mathfrak{p}$. If on the other hand ϵ_1 and ϵ_2 are even, one has to look at $L(E, \chi, j)$ for even j. The argument is similar and we omit it. $\square$

Now let f be a normalized (meaning $a_1 = 1$) cuspidal eigenform of the same weight and level as E. We will also assume that f is congruent to E mod $\mathfrak{p}^r$ in the following sense. If $f = \sum a_n q^n$ and $E = \sum b_n q^n$ are given by the standard Fourier expansions in terms of $q = e^{2\pi i z}$, we say that E and f are congruent modulo $\mathfrak{p}^r$ if $a_n \equiv b_n \pmod{\mathfrak{p}^r}$ for all $n \geq 1$ and $\mathfrak{p}^r$ divides the constant terms of the Fourier expansions of E at all cusps. Here we understand that $\mathfrak{p}$ is a prime of residue characteristic p in the ring of integers of a number field K containing the Fourier coefficients of E and f.

With this definition, we can finally state our main result:

Theorem 6.2. *Let f and E be a cusp form and an Eisenstein series respectively, of the same even weight $k \geq 2$ and level N, with $E \equiv f \pmod{\mathfrak{p}}$, and $(N, p) = 1$, and $p > k$. Fix canonical periods $\Omega_f^{\pm}$ for f. Then there exists a $\mathfrak{p}$-adic unit Ω_E such that the following holds:*

Let χ be a primitive Dirichlet character of conductor m, with m prime to both N and p. Then for all positive integers $j < k$ with $(-1)^{j-1} \cdot \chi(-1) =$

$sgn(E)$, we will have

$$\frac{\tau(\bar{\chi})L(f,\chi,j)}{(2\pi i)^{j-1}\Omega_f^{sgn(E)}} \equiv \frac{\tau(\bar{\chi})L(E,\chi,j)}{(2\pi i)^j\Omega_E} \pmod{\mathfrak{p}^r}$$

Proof. Let M_f denote the modular symbol associated to f in Chapter 3. Let $M_f = M_f^+ + M_f^-$ denote the decomposition of M_f into eigenspaces for the involution ι, and write $M_f^\pm = N_f\Omega_f^\pm$, where $\Omega_f^\pm$ are periods of f selected so that the modular symbols $N_f^\pm$ are K-rational. We may normalize $\Omega_f^\pm$ so that the modular symbols $N_f^\pm$ actually take values in $\mathcal{O}_{K,\mathfrak{p}}$, where $\mathcal{O}$ denotes the ring of integers of K, and such that $N_f^\pm$ are nonzero modulo $\mathfrak{p}$. Thus the periods $\Omega_f^\pm$ are determined up to some $\mathfrak{p}$-adic unit.

Now let M_E denote the modular symbol on $\Gamma_1(N)$ with values in $L_n(\mathcal{O}/\mathfrak{p}^r)$ associated to E that was constructed in Chapter 4 above. According to the theorem above, M_E is nonzero. Furthermore, we have proven in Chapter 5 that M_E is an eigenvector for ι with eigenvalue given by the sign of E. Now let $\overline{N}_f^{\mathrm{sgn}(E)}$ denote the reduction of $N_f^{\mathrm{sgn}(E)}$ modulo $\mathfrak{p}^r$. Then we would like to compare the modular symbols $\overline{N}_f^{\mathrm{sgn}(E)}$ and M_E. We cannot do this directly, and indeed, in general one expects that these modular symbols will be different. However, we can salvage the situation as follows. If Γ is any congruence subgroup and A is a Γ-module, then there is a map which we denote $\delta = \delta_A$ from the space of A-valued modular symbols for $\Gamma_1(N)$ to the cohomology group $H^1(\Gamma_1(N), A)$, as explained in [4], and that the kernel of δ is the group of A-valued boundary symbols. Applying the map δ to the modular symbols considered above, we get cohomology classes $\delta(\overline{N}_f^{\mathrm{sgn}(E)})$ and $\delta(M_E)$ with values in $L_n(\mathcal{O}/\mathfrak{p}^r)$. Then we claim that

$$c \cdot \delta(\overline{N}_f^{\mathrm{sgn}(E)}) = \delta(M_E)$$

where c is a unit in $\mathcal{O}/\mathfrak{p}^r$.

Let us admit this claim for the moment and see how to complete the proof. The claim implies that $\delta(c\overline{N}_f^{\mathrm{sgn}(E)} - M_E) = 0$, hence $c\overline{N}_f^{\mathrm{sgn}(E)} - M_E$ is a boundary symbol. But we have computed the χ-twisted special values of boundary symbols in Section 6.1, and all these special values vanish. It follows that the χ-twisted special values of M_E and $c\overline{N}_f^{\mathrm{sgn}(E)}$ are equal, which evidently implies the theorem, if we take Ω_E to be a fixed lift of the unit c to $\mathcal{O}$.

It remains therefore to prove the claim. In the case of weight two, it turns out that the statement is equivalent to a multiplicity one statement for the étale part of a certain group scheme occurring as a subgroup of $J_1(N,p)[p]$, which is proved in [22], Theorem 2.12 (and see also [21], Theorem 2.7). The case of weight k may be reduced to that of weight 2 and $J_1(Np)$ by using Hida theory as developed in [6], Section 4. We omit the details. It is relevant however to point out that the computed value of the sign $\mathrm{sgn}(E)$ is crucial to distinguish the étale and multiplicative parts of the subgroup schemes in question. We note also that Hida assumes $p \geq 5$; this assumption holds in our case since $p > k \geq 4$.

$\square$

Chapter 7

Computing Special Values

Before proceeding, we pause for a quick word on terminology. The term "modular symbol" has two different definitions in the literature. One definition, the one used above, is the one used in [4], [8], [18], and [21], among others. The other definition, which is not equivalent to the first one, is used in such places as [1], [9], [17], and [20]. It is these latter sources that give us the computational method discussed in this chapter. However, in this paper we do not wish to use the same terminology to refer to two different objects. So the objects called "modular symbols", in the sense of Cremona, Merel, and Stein, will be referred to below as *modular paths*.

7.1 Special Values Related to Cusp Forms

From Chapter 3, we know that the special values of $L(f, \chi, j)$ at the critical integers $1, 2, \ldots, k-1$ and a primitive character χ can all be thought of as the product of a transcendental "canonical period" and an algebraic number. But we have yet to provide a method that allows us to explicitly compute the algebraic parts of these special values. We will do this below, using two key ideas: the theory of modular paths, and the existence of an integration pairing between these modular paths and modular forms. We will briefly describe these two ideas here and explain how they come together to give a way to compute algebraic parts of special values of L-functions. For the sake of simplicity, for this section only we let $\Gamma = \Gamma_0(N)$.

7.1.1 Modular Paths

First we describe the theory of modular paths. The theory was first developed by Manin (see [7]) and then later by Shokurov in [16] and Merel in [9] (and for a good discussion of the theory, see also [17], chapter 8).

Let $\mathbb{M}_2$ be the free abelian group generated by the set of (formal) symbols of the form $\{\alpha, \beta\}$ (order does matter, in spite of the set notation) subject to the relation

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$$

and mod any torsion. (Here $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$.)

Now define $\mathbb{M}_k$ to be the tensor product $L_{k-2}(\mathbb{Z}) \otimes \mathbb{M}_2$. We need to define an action of $\mathrm{SL}_2(\mathbb{Z})$ on this set. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \{\alpha, \beta\} = \{\tfrac{a\alpha+b}{c\alpha+d}, \tfrac{a\beta+b}{c\beta+d}\}$$

$$g(P(X,Y) \otimes \{\alpha, \beta\}) = (g|P)(X,Y) \otimes g\{\alpha, \beta\}$$

where the last line refers to the two separate actions. For a congruence subgroup Γ, the set of *weight k modular paths* for Γ will simply be the set $\mathbb{M}_k$ mod the left action of matrices in Γ.

It is not at all obvious, but it turns out to be true, that we can write down a finite $\mathbb{Z}$-basis for this space. Such a basis can be explicitly written down in terms of linear combinations of Manin symbols. Define the *Manin symbol* $[P, g]$ as the modular symbol $g(P \otimes \{0, \infty\})$. Since our space of modular paths is a quotient by the action of $\Gamma_0(N)$, we may interpret g as a representative of a (right) coset of $\Gamma_0(N)$. Note that by ([17], Proposition 3.10), we may even interpret g as an element of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ and the Manin symbol is still well-defined.

Since $\Gamma_0(N)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, we see that we can write down a finite list of Manin symbols that generate the other Manin symbols—namely, symbols of the form $[X^jY^{k-2-j}, (c:d)]$ with $j \in \mathbb{Z}$, $0 \le j \le k-2$, and $(c:d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. The number of Manin symbols we get this way is clearly $k-1$ times the order of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

Denote these "generating Manin symbols" by $x_1, x_2, \ldots, x_K$. Some of these can be written as linear combinations of other ones, as shown below.

Define a right action of $\mathrm{SL}_2(\mathbb{Z})$ on Manin symbols as follows:

$$[P, g]h = [h^{-1}P, gh]$$

Now we may state the following, originally due to Manin:

Proposition 7.1 ([17], Theorem 8.4)**.** *Let* $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. *Then for any Manin symbol x, we have*

$$x + x\sigma = 0$$

$$x + x\tau + x\tau^2 = 0$$

Furthermore, the space $\mathbb{M}_k(\Gamma_0(N))$ is the quotient of the free abelian group on $x_1, x_2, \ldots, x_K$ by these relations and by any torsion.

(The theorem is in greater generality than stated above, but here we only need it for $\Gamma_0(N)$. There is a third relation that appears when we consider other congruence subgroups.)

The proof of ([17], Theorem 8.3) assures us that any modular path can be written as a $\mathbb{Z}$-linear combination of Manin symbols. So to write down a $\mathbb{Z}$-basis for a modular path space, we need only write down a list of linear combinations of Manin symbols that are linearly independent and generate the Manin symbols by $\mathbb{Z}$-linear combinations.

The above proposition allows us to write down a relation matrix which, when put into echelon form, gives a $\mathbb{Q}$-basis for the Manin symbols. To pass from that to what we need, we can run a form of the Euclidean algorithm that will give us, in the end, a list of $\mathbb{Q}$-linear combinations of Manin symbols whose $\mathbb{Z}$-span includes all the original Manin symbols. Not only does this mean that our desired $\mathbb{Z}$-basis exists, but every step of the process just outlined can be made explicit for a given weight and level.

7.1.2 The Integration Pairing

The next step is to define the integration pairing, which makes use of the modular paths we have just defined. Before doing this, we should mention the action of the Hecke operators on modular paths.

On Manin symbols, the Hecke operators act as a sum of the right actions of a prescribed set of matrices, often called Heilbronn matrices; for full details, see ([17], Proposition 8.8 and equation (8.3.3)). These are enough to tell us how to write down the action of a Hecke operator on a Manin symbol, and because our explicit bases are all in terms of Manin symbols, this allows us to do all the computations with Hecke operators that we need. We will not go into any more detail here, but further details about the Hecke action can be found in ([17], Section 8.3).

Now we define the integration pairing:

$$\langle \cdot, \cdot \rangle : \mathbb{M}_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$$

$$\langle P(X,Y) \otimes \{\alpha, \beta\}, g \rangle = \int_{\alpha}^{\beta} P(z,1) g(z) dz$$

The integration pairing satisfies the following important properties. First of all, the Hecke operators are self-adjoint with respect to it ([17], Theorem 8.21). Second of all, we have:

Proposition 7.2. *Let j be a critical integer for $L(f,s)$. Then*

$$\begin{aligned} L(f,j) &= \frac{(-2\pi i)^{j-1}}{(j-1)!}\int_0^{i\infty} f(z)z^{j-1}dz \\ &= \frac{(-2\pi i)^{j-1}}{(j-1)!}\langle X^{j-1}Y^{k-j-1}\otimes\{0,\infty\},f\rangle \end{aligned}$$

Proof. The equality of the second and third parts is by the definition of the integration pairing. For the equality of the first of the first and second parts, we use Proposition 2.1 and the definition of $D_f(s)$. ☐

So the special values we seek can be written in terms of this pairing.

For any particular f, we can view this pairing as a map $\mathbb{M}_k(\Gamma)\to\mathbb{C}$. This map actually respects the splitting of $\mathbb{M}_k(\Gamma)$ into $+1$ and -1 eigenspaces for the star involution (whose action is defined in [17]). With notation as above, we know that f is an eigenfunction for all the Hecke operators. It turns out that we can pick two elements of $\mathbb{M}_k(\Gamma)\otimes K$ (where K is the same field defined above)—call them $v^{\pm}$—such that within each eigenspace of the star involution, $v^{\pm}$ is the only simultaneous eigenvector for the Hecke operators with the same eigenvalue as f.

From the self-adjointness of the Hecke operators, it now follows that the above map on modular paths, projected onto either eigenspace, can be evaluated by multiplying the value of the pairing $\langle v^{\pm},f\rangle$ times the projection of the desired modular path onto that eigenvector. This is a more explicit way of describing the canonical periods first described in Chapter 3. The only issue is one of scaling; so far the eigenvectors $v^{\pm}$ are only defined up to multiplication by a scalar.

We can fix this by considering the pairing of f with each element of the $\mathbb{Z}$-basis described in the previous section. For each eigenspace, we search for one element v such that the other basis elements are algebraic integer multiples of $\langle v,f\rangle$. When we find this element, the corresponding value of the pairing is our canonical period.

Now we can compute special values by taking the modular path $X^{s-1}Y^{k-s-1}\otimes\{0,\infty\}$ and projecting it onto the appropriate (properly scaled) eigenvector to obtain an algebraic multiple of the canonical period. This is what is called the *algebraic part* of the special value of the L-function.

As mentioned before, we can now also compute the algebraic parts of the special values of $L(f,\chi,s)$ for a primitive Dirichlet character χ. What makes this possible is the identity

$$(f\otimes\chi)(z)=\frac{\tau(\chi)}{m}\sum_{b \bmod m}\overline{\chi}(-b)f(z+\tfrac{b}{m})$$

where $\tau(\chi)$ is the standard Gauss sum. This identity is equivalent to the one used in proving Lemma 2.9. Now using this, we compute:

$$L(f,\chi,s) = \frac{(-2\pi i)^{s-1}}{(s-1)!} \cdot \langle X^{s-1}Y^{k-s-1} \otimes \{0,\infty\}, f \otimes \chi\rangle$$

$$= \frac{\chi(-1)(-2\pi i)^{s-1}}{\tau(\overline{\chi})(s-1)!} \int_0^{i\infty} z^{s-1} \left[\sum_{b \bmod m} \overline{\chi}(-b) f(z+\tfrac{b}{m}) \right] dz$$

$$= \frac{\chi(-1)(-2\pi i)^{s-1}}{\tau(\overline{\chi})(s-1)!} \sum_{b \bmod m} \overline{\chi}(b) \int_0^{i\infty} z^{s-1} f(z-\tfrac{b}{m}) dz$$

$$= \frac{\chi(-1)(-2\pi i)^{s-1}}{\tau(\overline{\chi})(s-1)!} \sum_{b \bmod m} \overline{\chi}(b) \int_{-b/m}^{i\infty} f(z)(z+\tfrac{b}{m})^{s-1} dz$$

$$= \frac{\chi(-1)(-2\pi i)^{s-1}}{\tau(\overline{\chi})(s-1)!} \langle \left[\sum_{b \bmod m} \overline{\chi}(b)((X+\tfrac{b}{m}Y)^{s-1}Y^{k-s-1} \otimes \{\tfrac{-b}{m},\infty\}) \right], f\rangle$$

This gives us a modular path that we can once again project to the appropriate eigenvector to obtain the algebraic part of $L(f,\chi,s)$. Notice that we need only compute the canonical periods once, and then this computation can be done for any χ.

7.2 Computations of Special Values

Below are some explicit computations of twisted special values for two specific examples of a cusp form and a congruent Eisenstein series (mod a prime p specified below). The special values associated to the cusp forms have been computed using the method above; those for the Eisenstein series were computed using Proposition 5.7.

Proposition 5.11 helps explain the way the data tables are set up. In the polynomial from Corollary 5.6, the special value $L(E,\chi,j)$ appears in the coefficient of $X^{j-1}Y^{k-j-1}$. So when the signs match, it is the *even* j where the special values are always zero, and when they do not match, it is the *odd* special values that are always zero.

Both of the Eisenstein series in these tables have weight 4 and sign -1. Therefore, when χ is odd, $L(E,\chi,2) = 0$, while $L(E,\chi,1)$ and $L(E,\chi,3)$ may be nonzero. On the other hand, when χ is even, $L(E,\chi,2)$ is the only special value which might not be zero. Comparing with the special values associated to the cusp forms, each time we have projected to the

-1 eigenspace of ι—matching the sign of E. But which values of $L(f, \chi, j)$ project to 0 and which do not depends on whether χ is odd or even (in accordance with Shimura's theorem), and the special values that do not project to 0 correspond exactly with which special values $L(E, \chi, j)$ are nonzero.

In the first 3 tables, f is the unique newform of weight 4 and level $\Gamma_0(5)$. E is a congruent Eisenstein series whose L-function is the product of Dirichlet L-functions $L(\epsilon_5, s)L(\epsilon_1, s-3)$, where ϵ_j refers to the principal character mod j. In the tables, m refers to the conductor of a primitive quadratic character χ. In the first and third tables, the character is odd; in the second table it is even. In the last column, p refers to the prime such that $E \equiv f \pmod{p}$; in this case $p = 13$ (and we exclude characters with $13|m$).

In the last 3 tables, f is the unique newform of weight 4 and level $\Gamma_0(7)$. E is the Eisenstein series whose L-function is given by $L(\epsilon_7, s)L(\epsilon_1, s-3)$. This time $p = 5$ (and we exclude characters with $5|m$). The "ratio mod p" is the ratio of the third column to the second column.

Table 7.1: Comparison of Twisted Special Values: $k=4$, $N=5$, $j=1$

m	$\frac{\tau(\overline{\chi})L(f,\chi,1)}{\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,1)}{2\pi i}$	Ratio mod p
3	100	-2/45	12
4	-100	-1/10	12
7	300	-48/35	12
8	800	-9/5	12
11	-2400	-12/5	12
15	-400	-16	12
19	-8800	-44/5	12
20	-1400	-30	12
23	5900	-432/5	12
24	-10800	-184/5	12

Table 7.2: Comparison of Twisted Special Values: $k=4$, $N=5$, $j=2$

m	$\frac{\tau(\overline{\chi})L(f,\chi,2)}{2\pi i\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,2)}{(2\pi i)^2}$	Ratio mod p
8	0	13/200	N/A
12	0	13/75	N/A
17	0	208/425	N/A
21	-300/7	64/175	12
24	-50/3	18/25	12
28	0	-208/175	N/A
29	-400/29	432/725	12
33	0	624/275	N/A
37	0	52/37	N/A

Table 7.3: Comparison of Twisted Special Values: $k=4$, $N=5$, $j=3$

m	$\frac{\tau(\overline{\chi})L(f,\chi,3)}{(2\pi i)^2\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,3)}{(2\pi i)^3}$	Ratio mod p
3	-10/9	-7/3375	12
4	-5/8	-31/8000	12
7	-30/49	-72/6125	12
8	-5/4	-189/16000	12
11	-240/121	-186/15125	12
15	-8/9	-8/225	12
19	-880/361	-682/45125	12
20	-7/4	-3/80	12
23	-590/529	-4536/66125	12
24	-15/8	-713/18000	12

Table 7.4: Comparison of Twisted Special Values: $k = 4$, $N = 7$, $j = 1$

m	$\frac{\tau(\overline{\chi})L(f,\chi,1)}{\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,1)}{2\pi i}$	Ratio mod p
3	49	-2/63	4
4	-147	-1/7	4
7	49	-8/7	4
8	-539	-12/7	4
11	-1568	-24/7	4
19	6713	-66/7	4
23	-6272	-576/7	4
24	11368	-276/7	4

Table 7.5: Comparison of Twisted Special Values: $k = 4$, $N = 7$, $j = 2$

m	$\frac{\tau(\overline{\chi})L(f,\chi,2)}{2\pi i\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,2)}{(2\pi i)^2}$	Ratio mod p
8	49/8	3/49	4
12	0	25/147	N/A
13	0	100/637	N/A
17	0	400/833	N/A
21	56/3	8/21	4
24	0	75/98	N/A
28	7/2	8/7	4
29	784/29	864/1421	4
33	0	1200/539	N/A
37	0	2400/1813	N/A

Table 7.6: Comparison of Twisted Special Values: $k = 4$, $N = 7$, $j = 3$

m	$\frac{\tau(\overline{\chi})L(f,\chi,3)}{(2\pi i)^2\Omega^-}$	$\frac{\tau(\overline{\chi})L(E,\chi,3)}{(2\pi i)^3}$	Ratio mod p
3	7/18	-19/9261	4
4	21/32	-43/10976	4
7	1/2	-4/343	4
8	77/128	-129/10976	4
11	112/121	-516/41503	4
19	959/722	-99/6517	4
23	448/529	-12384/181447	4
24	203/144	-437/10976	4

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