Localization in Weak Bialgebras
and Hopf Envelopes

by

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A thesis submitted in partial fulfilment of the requirements for
the degree of

Doctor of Philosophy

in

The Faculty of Graduate Studies
(Mathematics)

The University of British Columbia
(Vancouver)

April 2013
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Abstract

This dissertation primarily investigates the following questions. Given a weak bialgebra $H$, is it possible to invert some elements and still have a weak bialgebra structure? In such a case, are there requirements on $H$ or on the set of elements we want to invert?

We answer these questions by establishing sufficient conditions for the localization to exist. For instance, the monoid $G$ of all group-like elements in a weak bialgebra $H$ such that $G$ is almost central, i.e. $ag = gJ_g(a)$, and such that $J_g(G) \subset G$, two conditions we introduce in this thesis, forms a suitable set to be localized.

We give a constructive proof of the existence of the localization and detail its weak bialgebra structure. We also prove that it satisfies a universal property. We show these results without any requirement of centrality nor regularity for the elements we invert.

We use this work to construct interesting examples of bialgebras and weak bialgebras. We show for instance that $GL_q(2)$ has a universal property. We also build the localization of the weak bialgebra associated with a finite directed graph.

The examples we exhibit moreover show how our construction can be used in full generality and that our assumptions such as the non-regularity or non-centrality of the elements we invert are not superfluous.
We furthermore present and reformulate Manin’s Hopf envelope. This leads us to define the notion of weak Hopf envelope of a weak bialgebra, and then discuss its relationship with the localization of a weak bialgebra at the monoid of all group-like elements.

Finally, in the first part of this thesis we set up an inventory of technical results with detailed proofs for weak bialgebras and coquasi-triangular weak bialgebras. Some of these results are not published in any article and are fully presented here as a basis for future reference.
Preface

Part of the material presented in Sections 2.2 and 2.3 was used to produce:


The identification of the research program leading to the material of Sections 2.2 and 2.3 was made by Hendryk Pfeiffer and I conducted the research activities. Both authors contributed to the preparation of the manuscript.
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Acknowledgements

There are many people to whom I want to express my gratitude for their support and help over the course of these past years. I thank my advisor Patrick Brosnan for giving me the opportunity to undertake this research and for his assistance throughout my degree.
I also want to thank Jim Carrell for his help as a committee member and his assistance with administrative matters. Lastly I want to express my sincere gratitude to Hendryk Pfeiffer for introducing me to weak bialgebras, for his constant support and guidance, and for the numerous interesting discussions we have had.

I am moreover grateful to my family and friends for their help, and to my wife who has always been there to cheer me up in bad times and make good times better.

Finally, I thank the many people I have forgotten to mention but who will undoubtedly recognize themselves here.


Introduction

Weak bialgebras and weak Hopf algebras were introduced by Böhm, Nill and Szlachanyi in [BNS99] as a generalization of the well-known notions of bialgebras and Hopf algebras. Since then much research has been conducted on this topic. In particular, it has been shown that every fusion category (that is, a $k$-linear autonomous finitely semi-simple monoidal category with finite-dimensional Hom-spaces that satisfy a specific condition) is equivalent to a category of modules over a weak Hopf algebra ([Hay, Thm. 4.1], [Ost03, Thm. 4] and [ENO05, Cor. 2.22]).

Having been developed fairly recently, many questions about weak bialgebras and weak Hopf algebras remain open. In this dissertation, we are interested in the following problem. Given a weak bialgebra, how can we invert, or localize, group-like elements? In particular, what conditions should the weak bialgebra and group-like elements satisfy in order for the localization to make sense and have a weak bialgebra structure? And when the localization exists, what are its properties?

In order to answer these questions, we use, as often with weak bialgebras, methods originating in the theory of bialgebras. We give a constructive proof of the existence of the localization, which enables us to study its properties and compute concrete examples.

We also investigate Manin’s Hopf envelope. After presenting Manin’s
original result, we reformulate it using categorical language, which enables us to apply this construction to any bialgebra. We then define the notion of weak Hopf envelope and show that in specific cases it coincides with the localization relative to the monoid of all group-like elements, thus providing an alternative description of the localization.

This thesis is organized as follows. In Chapter 1 we start by introducing weak bialgebras and weak Hopf algebras and prove some of their basic properties. We then establish an inventory of technical results that are needed in subsequent chapters. We next briefly introduce string diagrams.

In Chapter 2 we first look at the localization of coquasi-triangular bialgebras and examine some examples. We then establish the localization of a weak bialgebra relative to a suitable subset of group-like elements.

Chapter 3 is devoted to the construction of the Hopf envelope. First, we present Manin’s original result. We then reformulate it for any bialgebra using categorical language. Finally, we define the notion of weak Hopf envelope and study its relationship with the localization.
Chapter 1

Fundamental Notions

In this chapter we introduce the main concepts that will be used throughout this thesis. We also present some basic examples and, in Section 1.3, give some technical results that we shall need in the following chapters. Finally, we present string diagrams and explain how they are used for computations.

1.1 Coquasi-Triangular Weak Hopf Algebras

Definition 1.1.1. A weak bialgebra \((H, \mu, \eta, \Delta, \varepsilon)\) over a field \(k\) is a vector space \(H\) such that

1. \((H, \mu, \eta)\) forms an associative algebra with multiplication \(\mu : H \otimes H \to H\) and unit \(\eta : k \to H\),

2. \((H, \Delta, \varepsilon)\) forms a coassociative coalgebra with comultiplication \(\Delta : H \to H \otimes H\) and counit \(\varepsilon : H \to k\),

3. the following compatibility conditions hold:

   - Multiplicativity of the Comultiplication:

     \[
     \Delta \circ \mu = (\mu \otimes \mu) \circ (\text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H) \circ (\Delta \otimes \Delta),
     \]  

     (1.1)
• Weak Multiplicativity of the Counit:
\[
\varepsilon \circ \mu \circ (\mu \otimes \text{id}_H) = (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta \otimes \text{id}_H)
\]
\[
= (\varepsilon \otimes \varepsilon) \circ (\mu \otimes \mu) \circ (\text{id}_H \otimes \Delta^{\text{op}} \otimes \text{id}_H),
\]
(1.2)

• Weak Comultiplicativity of the Unit:
\[
(\Delta \otimes \text{id}_H) \circ \Delta \circ \eta = (\text{id}_H \otimes \mu \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta)
\]
\[
= (\text{id}_H \otimes \mu^{\text{op}} \otimes \text{id}_H) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta),
\]
(1.3)

where \(\sigma_{V,W} : V \otimes W \to W \otimes V : v \otimes w \mapsto w \otimes v\) flips the two tensor factors. Moreover \(\mu^{\text{op}} = \mu \circ \sigma_{H,H}\) is the opposite multiplication and \(\Delta^{\text{op}} = \sigma_{H,H} \circ \Delta\) is the opposite comultiplication. We also implicitly use Mac Lane’s coherence theorem for the monoidal category [Mac71, Chap. VII], identifying \((U \otimes V) \otimes W \cong U \otimes (V \otimes W)\) as well as \(V \otimes k \cong V \cong k \otimes V\).

A homomorphism of weak bialgebras \(\varphi : H \to H'\) is a homomorphism of both unital algebra and counital coalgebra.

**Definition 1.1.2.** Let \(H\) be a weak bialgebra. The linear map

\[
\varepsilon_s = (\text{id}_H \otimes \varepsilon) \circ (\text{id}_H \otimes \mu) \circ (\sigma_{H,H} \otimes \text{id}_H) \circ (\text{id}_H \otimes \Delta) \circ (\text{id}_H \otimes \eta)
\]
(1.4)

is called the source counital map whereas

\[
\varepsilon_t := (\varepsilon \otimes \text{id}_H) \circ (\mu \otimes \text{id}_H) \circ (\text{id}_H \otimes \sigma_{H,H}) \circ (\Delta \otimes \text{id}_H) \circ (\eta \otimes \text{id}_H)
\]
(1.5)

is called the target counital map.

Their images \(H_s := \varepsilon_s(H)\) and \(H_t := \varepsilon_t(H)\) form algebras and are respectively called the source base algebra and target base algebra.
Remark 1.1.3. The name weak bialgebra is fairly self explanatory. In particular, we see that it is the compatibility between the algebra and coalgebra structures that is weakened. In contrast to a bialgebra, the multiplicativity of the counit
\[ \varepsilon \circ \mu = \varepsilon \otimes \varepsilon \]
and the comultiplicativity of the unit
\[ \Delta \circ \eta = \eta \otimes \eta \]
do not hold in general anymore and are replaced by (1.2) and (1.3) respectively. Moreover, the condition \( \varepsilon \circ \eta = 1_k \) is absent.

A weak bialgebra is a bialgebra if and only if \( \varepsilon_s = \eta \circ \varepsilon \), and if and only if \( \varepsilon_t = \eta \circ \varepsilon \).

Remark 1.1.4. Note that if \( H \) is a finite-dimensional weak bialgebra then so is \( H^* \). We say that the definition is “self-dual”.

Notation 1.1.5. In what follows we use Sweedler’s notation for the coproduct. Namely, let \( H \) be a weak bialgebra and \( a \in H \), we then write
\[ \Delta(a) = a' \otimes a'' \]
as an abbreviation of
\[ \Delta(a) = \sum_i a_{i1} \otimes a_{i2} \]
with some \( a_{i1}, a_{i2} \in H \). In other words, with Sweedler’s notation the summation symbol and indices are implied but not explicitly written.

Definition 1.1.6. Let \( (H, \mu, \eta, \Delta, \varepsilon) \) be a weak bialgebra over a field \( k \). It is called coquasi-triangular if there exists a linear form \( r : H \otimes H \rightarrow k \) called the universal \( r \)-form that satisfies the following conditions:
i) For all \(a, b \in H\)

\[
r(a \otimes b) = \varepsilon(a'b')r(a'' \otimes b'') = r(a' \otimes b')\varepsilon(b''a'').
\] (1.6)

ii) The form \(r\) has a weak convolution inverse, i.e. there exists \(r^{-1} : H \otimes H \to k\) such that

\[
\begin{align*}
    r(a' \otimes b') r^{-1}(a'' \otimes b'') &= \varepsilon(ab), \\
    r^{-1}(a' \otimes b') r(a'' \otimes b'') &= \varepsilon(ba),
\end{align*}
\] (1.7) (1.8)

iii) For all \(a, b, c \in H\), we have

\[
\begin{align*}
r(a' \otimes b')b''a'' &= a'b'r(a'' \otimes b''), \\
r(ab \otimes c) &= r(b \otimes c')r(a \otimes c''), \\
r(a \otimes bc) &= r(a' \otimes b)r(a'' \otimes c).
\end{align*}
\] (1.9) (1.10) (1.11)

Note that condition (1.9) implies that the commutativity inside \(H\) is “controlled” by the \(r\)-form, this is why one often says that \(H\) is almost commutative.

A homomorphism of coquasi-triangular weak bialgebras \(\varphi : (H, r) \to (H', r')\) is a homomorphism of weak bialgebra satisfying \(r' \circ (\varphi \otimes \varphi) = r\).

**Remark 1.1.7.** A coquasi-triangular weak bialgebra that is a bialgebra is also coquasi-triangular as a bialgebra. In this case, one can simply omit (1.6) since it is automatically satisfied in a bialgebra. Moreover

\[
r(a' \otimes b') r^{-1}(a'' \otimes b'') = \varepsilon(a)\varepsilon(b) = r^{-1}(a' \otimes b') r(a'' \otimes b'')
\] (1.12)

since in a bialgebra \(\varepsilon(ab) = \varepsilon(a)\varepsilon(b) = \varepsilon(b)\varepsilon(a) = \varepsilon(ba)\).

**Remark 1.1.8** (Dual of [NTV03, Prop. 5.2]). The category of finite-dimensional
comodules over a weak bialgebra is monoidal. If we then consider the category of finite-dimensional comodules over a coquasi-triangular weak bialgebra, it is moreover braided.

**Lemma 1.1.9.** Let \((H, r)\) be a coquasi-triangular weak bialgebra, then the coopposite weak bialgebra \((H^{\text{op,cop}}, r^{-1})\) is coquasi-triangular as well.

**Remark 1.1.10.** If we refer to \((1.10)\) in the following, this indicates either the direct use of this equality for \((H, r)\) or the use of the corresponding equality \(r^{-1}(ab \otimes c) = r^{-1}(b \otimes c')r^{-1}(a \otimes c')\) for \((H^{\text{op,cop}}, r^{-1})\). The context will every time make clear in which situation we are.

**Definition 1.1.11.** A weak Hopf algebra \((H, \mu, \eta, \Delta, \varepsilon, S)\) is a weak bialgebra \((H, \mu, \eta, \Delta, \varepsilon)\) with a linear map \(S : H \to H\), called the antipode, that satisfies :

\[
\mu \circ (S \otimes \text{id}_H) \circ \Delta = \varepsilon, \tag{1.13}
\]

\[
\mu \circ (\text{id}_H \otimes S) \circ \Delta = \varepsilon, \tag{1.14}
\]

\[
S = \mu \circ (\mu \otimes \text{id}_H \circ (S \otimes \text{id}_H) \circ (\Delta \otimes \text{id}_H) \circ \Delta. \tag{1.15}
\]

A homomorphism of weak Hopf algebras \(\varphi : H \to H'\) is a homomorphism of weak bialgebras.

**Remark 1.1.12.**

i) In this case, the Hopf algebra axioms

\[
\mu \circ (S \otimes \text{id}_H) \circ \Delta = \eta \circ \varepsilon \quad \text{and} \quad \mu \circ (\text{id}_H \otimes S) \circ \Delta = \eta \circ \varepsilon
\]

are weakened to \((1.13)\) and \((1.14)\) respectively whereas \((1.15)\) is new.

ii) The axioms of the previous definition imply that the antipode is both an algebra- and coalgebra anti-homomorphism. In other words, for a weak Hopf algebra \(H\) and \(a, b \in H\), we have \(S(ab) = S(b)S(a)\) and \(S(a') \otimes S(a'') = S(a)' \otimes S(a)'\).
iii) Note that any homomorphism of weak bialgebras \( \varphi \) between two weak Hopf algebras \((H, S)\) and \((H', S')\) automatically satisfies \( S' \circ \varphi = \varphi \circ S \).

This is proved in the next lemma.

**Lemma 1.1.13.** Let \( \varphi : H \rightarrow H' \) be a homomorphism of weak bialgebras and let \( H, H' \) be weak Hopf algebras. Then \( S' \circ \varphi = \varphi \circ S \).

**Proof.** Let \( a \in H \). Using the weak Hopf algebra axioms, we find

\[
S'(\varphi(a)) = S'(\varphi(a'))\varphi(a)''S'(\varphi(a)''')
\]

\[
= \varepsilon'_s(\varphi(a'))S'(\varphi(a)''')
\]

\[
= \varepsilon'_s(\varphi(a'))S'(\varphi(a''))
\]

\[
= \varphi(\varepsilon_s(a'))S'(\varphi(a''))
\]

\[
= \varphi(S(a')a'')S'(\varphi(a''))
\]

\[
= \varphi(S(a'))\varphi(a'')S'(\varphi(a''))
\]

\[
= \varphi(S(a'))\varphi(a'')'\varphi(a'')''
\]

\[
= \varphi(S(a'))\varepsilon'_t(\varphi(a''))
\]

\[
= \varphi(S(a'))\varphi(\varepsilon_t(a''))
\]

\[
= \varphi(S(a'))a''S(a'')
\]

\[
= \varphi(S(a)),
\]

where \( \ast \) uses that \( \varphi(1) = 1' \) and \( \varepsilon'(\varphi(a)) = \varepsilon(a) \) and thus \( \varphi(\varepsilon_s(a)) = \varepsilon'_s(a) \) and \( \varphi(\varepsilon_t(a)) = \varepsilon'_t(a) \). \( \square \)

**Example 1.1.14.** Let \( k \) be a field, \( G = (G_0, G_1) \) a groupoid with finite set of objects \( G_0 \) and \( f, f' \in G_1 \). Consider the free vector space \( k[G_1] \) on the set of morphisms \( G_1 \). One can build a weak Hopf algebra structure on \( k[G_1] \) that is called the *groupoid algebra* associated with \( G \) and then denoted \( k[G] \).
Its structure is given as follows:

\[ \mu(f \otimes f') = \begin{cases} 
  f \circ f' & \text{if target}(f') = \text{source}(f) \\
  0 & \text{otherwise}
\end{cases}, \]

\[ \eta(1) = \sum_{x \in G_0} \text{id}_x, \]

\[ \Delta(f) = f \otimes f, \]

\[ \varepsilon(f) = 1 \quad \forall f \in G_1, \]

\[ S(f) = f^{-1}. \]

Note that, due to its construction, a groupoid algebra is always cocommutative. Moreover, this is an example of the weak Hopf algebra that is not a Hopf algebra.

**Notation 1.1.15.** From now on we shall abbreviate weak bialgebra by WBA and weak Hopf algebra by WHA. Moreover, coquasi-triangular will be written “CQT”; thus a coquasi-triangular weak bialgebra will be called CQT WBA.

We now introduce a concept that will play an important role in the rest of this thesis.

**Definition 1.1.16.** Let \( H \) be a WBA. An element \( g \in H \) is called right group-like if

\[ \Delta(g) = g1' \otimes g1'' \quad \text{and} \quad \varepsilon_s(g) = 1, \quad (1.16) \]

it is called left group-like if

\[ \Delta(g) = 1'g \otimes 1''g \quad \text{and} \quad \varepsilon_l(g) = 1. \quad (1.17) \]

An element \( g \in H \) is called group-like if it is both right and left group-like. We denote the set of group-like elements of \( H \) by \( G(H) \).

**Notation 1.1.17.** In what follows we sometimes have two or more units showing up in our computations. In order to differentiate them and keep
track of which one is which one, we use subscripts. Hence we have, for example, \( \varepsilon_s(a) = 1'\varepsilon(a1'') \) and then \( 1\varepsilon_s(a) = 1_11'_2\varepsilon(a1''). \)

**Lemma 1.1.18.** The set of group-like elements \( G(H) \) of a WBA \( H \) is a monoid.

**Proof.**

i) \( 1 \in H \) is group-like.

We have \( 1_1 \cdot 1'_2 \cdot 1''_3 = 1' \otimes 1'' = \Delta(1) \) and similarly \( 1'_1 \cdot 1_2 \otimes 1''_3 \cdot 1 = \Delta(1) \). Furthermore, \( \varepsilon_s(1) = 1'\varepsilon(1_2 \cdot 1''_3) = 1 \) and \( \varepsilon_t(1) = \varepsilon((1'_1 \cdot 1_2)1'') = 1 \).

ii) If \( g, h \in G(H) \) then \( gh \in G(H) \).

We have

\[
\Delta(gh) = (gh)' \otimes (gh)'' = g'h'' \otimes g''h'' = (g1'_1)(1'_2h) \otimes (g1''_1)(1''_2h) \\
= g(1'_11'2h) \otimes g(1''_11''_2h) = g(1'h) \otimes g(1''h) = g(h1') \otimes g(h1'') \\
= (gh)1' \otimes (gh)1'',
\]

by definition of the comultiplication; multiplicativity of the comultiplication; definition of group-like; associativity; associativity and unit axiom; definition of group-like; associativity.

Similarly, \( \Delta(gh) = 1'(gh) \otimes 1''(gh) \). We furthermore have

\[
\varepsilon_s(gh) \overset{1234}{=} \varepsilon_s(\varepsilon_s(g)h) = \varepsilon(1 \cdot h) = \varepsilon(h) = 1,
\]

where we have used that \( g \) is group-like. In a similar way, \( \varepsilon_t(gh) = 1 \).

\( \square \)

**Lemma 1.1.19.** Let \( H \) be a WHA. Then every group-like element is invertible with \( g^{-1} = S(g) \) and \( G(H) \) forms a group.

**Proof.** Let \( g \in H \) be group-like. Then

\[
S(g)g = S(g)1g = S(g)\varepsilon_s(1)g = S(g)S(1')1''g \\
\overset{**}{=} S(1'g)1''g = S(g')g'' = \varepsilon_s(g) = 1,
\]

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where $*$ uses that $S$ is an algebra anti-homomorphism. One similarly finds $gS(g) = 1$ and hence $g^{-1} = S(g)$.

Let us now look at the group structure of $G(H)$. From the previous lemma we know that $1$ is group-like and that if $g$ and $h$ are in $G(H)$ then so is $gh$. It remains to prove that for $g$ group-like, so is $g^{-1}$. We have

$$
\Delta(g^{-1}) = (g^{-1})' \otimes (g^{-1})'' = g^{-1}g(g^{-1})' \otimes g^{-1}g(g^{-1)''}
= g^{-1}g(1g^{-1})' \otimes g^{-1}g(1g^{-1})'' = g^{-1}g1'(g^{-1})' \otimes g^{-1}g1''(g^{-1)''}
= g^{-1}(g1')(g^{-1})' \otimes g^{-1}(g1'')(g^{-1})'' = g^{-1}g'(g^{-1})' \otimes g^{-1}g''(g^{-1})''
= g^{-1}(gg^{-1})' \otimes g^{-1}(gg^{-1})''
= g^{-1}1' \otimes g^{-1}1'',
$$

and similarly $\Delta(g^{-1}) = 1'g^{-1} \otimes 1''g^{-1}$. Finally, we have

$$
\varepsilon_s(g^{-1}) = \varepsilon_s(1g^{-1}) = \varepsilon_s(\varepsilon_s(g)g^{-1}) = \varepsilon_s(gg^{-1}) = \varepsilon_s(1) = 1,
$$

and similarly $\varepsilon_t(g^{-1}) = 1$, hence $g^{-1}$ is group-like.

$\square$

## 1.2 Examples

In this section we introduce some examples of Hopf algebras and WHAs that will be useful later in this dissertation.

### 1.2.1 $M_q(2)$, $GL_q(2)$ and $SL_q(2)$

We present here the construction of $M_q(2)$, $GL_q(2)$ and $SL_q(2)$. The reader interested in having more details is referred to [Kas95] Chap. IV.3-6 & VIII.7.
Definition 1.2.1. Let $k$ be a field and let $q \in k$ be a non-zero element such that $q^2 \neq -1$. Let $M_q(2)$ be the quotient of the free algebra $k\{t_{11}, t_{12}, t_{21}, t_{22}\}$ by the two-sided ideal $J_q$ generated by

\begin{align*}
t_{12}t_{11} &= qt_{11}t_{12}, & t_{22}t_{12} &= qt_{12}t_{22}, \\
t_{21}t_{11} &= qt_{11}t_{21}, & t_{21}t_{22} &= qt_{21}t_{22}, \quad (1.18) \\
t_{12}t_{21} &= t_{21}t_{12}, & t_{11}t_{22} - t_{22}t_{11} &= (q^{-1} - q)t_{12}t_{21}.
\end{align*}

Remark 1.2.2. When $q = 1$ the commutative algebra we get is called $M(2)$. This algebra has a natural grading given by the degree of the polynomials and since the ideal $J_q$ is generated by quadratic elements, it induces a grading on $M_q(2)$ for which the generators $t_{11}, t_{12}, t_{21}, t_{22}$ are of degree one.

We shall now see that $M_q(2)$ has a CQT bialgebra structure.

Proposition 1.2.3 ([Kas93] Chap. IV.5 & VIII.7). The algebra $M_q(2)$ has a coquasi-triangular structure with the comultiplication and counit given by

$$
\Delta(t_{ij}) = \sum_{m=1}^{2} t_{im} \otimes t_{mj} \quad \text{and} \quad \varepsilon(t_{ij}) = \delta_{ij}.
$$

This type of coalgebra structure is often called matrix coalgebra because it is dual to a matrix algebra. Moreover, this coalgebra structure can be rewritten in the following compact form, where each entry of the matrix gives an equation:

$$
\Delta \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \otimes \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} \otimes t_{11} + t_{12} \otimes t_{21} & t_{11} \otimes t_{12} + t_{12} \otimes t_{22} \\ t_{21} \otimes t_{11} + t_{22} \otimes t_{21} & t_{21} \otimes t_{12} + t_{22} \otimes t_{22} \end{pmatrix}
$$

\[12\]
and

$$\varepsilon \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

The $r$-form $r : M_q(2) \otimes M_q(2) \to k$ is given on the generators of $M_q(2) \otimes M_q(2)$ by

$$r \begin{pmatrix} t_{11} \otimes t_{11} & t_{11} \otimes t_{12} & t_{12} \otimes t_{11} & t_{12} \otimes t_{12} \\ t_{11} \otimes t_{12} & t_{21} \otimes t_{12} & t_{22} \otimes t_{11} & t_{12} \otimes t_{22} \\ t_{21} \otimes t_{11} & t_{11} \otimes t_{22} & t_{12} \otimes t_{21} & t_{22} \otimes t_{12} \\ t_{21} \otimes t_{21} & t_{21} \otimes t_{22} & t_{22} \otimes t_{21} & t_{22} \otimes t_{22} \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$ 

Using (1.10) and (1.11) one checks that the $r$-form is 0 on the ideal defined by relations (1.18). We can therefore use equations (1.10) and (1.11) to extend the $r$-form the all $M_q(2)$.

We now introduce a special element in $M_q(2)$.

**Definition 1.2.4.** The element $\det_q = t_{11}t_{22} - q^{-1}t_{12}t_{21} = t_{22}t_{11} - qt_{12}t_{21}$ is called the *quantum determinant*.

**Remark 1.2.5.** Note that the quantum determinant is both group-like and central (one verifies this by direct computation).

**Remark 1.2.6.** We know that in a Hopf algebra every group-like element is invertible. As we shall see in Section 2.2.2 adding the inverse of $\det_q$ to $M_q(2)$ is enough, in this specific case, for the existence of the antipode.

**Definition 1.2.7.** We define

$$GL_q(2) = M_q(2)[t]/(\det_q \cdot t - 1)$$

and

$$SL_q(2) = M_q(2)/(\det_q - 1) = GL_q(2)/(t - 1).$$
Proposition 1.2.8 ([Kas95, Thm. IV.6.1]). Both $GL_q(2)$ and $SL_q(2)$ are Hopf algebras with the bialgebra structure inherited from $M_q(2)$ and the antipode given by
\[ S \left( \begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right) = \left( \begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right)^{-1} = \det_q^{-1} \left( \begin{array}{cc} t_{22} & -q t_{12} \\ -q^{-1} t_{21} & t_{11} \end{array} \right). \]

1.2.2 Groupoid Algebra of the Action Groupoid Associated with $S_3$

In this section we construct the WHA structure on the groupoid algebra of the groupoid generated by the action of the symmetric group on a set of three elements and the related structure on its dual.

Let us start by presenting how one can construct the groupoid algebra of the groupoid associated with a group action.

Proposition 1.2.9. Let $\tilde{G}$ be a group acting on a set $S$. One defines a groupoid $G = (G_0, G_1)$ in the following way. The set of objects $G_0$ is simply the set $S$. For two elements $s, s' \in S$, there is a morphism from $s$ to $s'$ if there is $g \in \tilde{G}$ such that $g \cdot s = s'$.

Proof. To prove that $G$ is actually a groupoid, we first have to check that every object has an identity morphism. This follows form the fact that $e \cdot s = s$ where $e$ is the unit element of $\tilde{G}$. Then the associativity in the composition of the morphisms follows directly from the fact that for a group action $g' \cdot (g \cdot s) = (g'g) \cdot s$. Finally, to have a groupoid, we still need every morphism to be invertible. Let $f_g : s \to s'$ be the morphism associated with the action $g \cdot s = s'$. Then $f_{g^{-1}} : s' \to s$ is the inverse morphism; indeed, $f_g \circ f_{g^{-1}} = \text{id}_{s'}$ and $f_{g^{-1}} \circ f_g = \text{id}_s$ since $g^{-1} \cdot (g \cdot s) = s$ and $g \cdot (g^{-1} \cdot s') = s'$. \qed

Definition 1.2.10. The groupoid constructed in the previous proposition is called the action groupoid associated with the action of $\tilde{G}$ on $S$. 

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Figure 1.1: Action groupoid $G_{S_3}$

We now construct the groupoid algebra of the action groupoid associated with the action by permutation of the symmetric group $S_3$ on the set $S = \{1, 2, 3\}$.

Example 1.2.11. Consider the action of $S_3$ on $S = \{1, 2, 3\}$ and the action groupoid $G_{S_3}$ associated with it. Between any two objects $i, j$ there are two morphisms “in each direction” and two morphisms between $i$ and itself. This is summarized in Figure 1.1. For $i, j, k \in S$, we denote the morphism from $j$ to $i$ associated to the transposition $(ij)$ by $e_{ij} : j \to i$ and the one associated to the 3-cycle $(123)$ or $(132)$ by $\overline{e}_{ij} : j \to i$. Similarly, the morphism $e_{ii} : i \to i$ is associated with $(jk)$ whereas $\overline{e}_{ii} : i \to i$ is associated with the identity morphism.

Let $k$ be a field; we know from Example 1.1.14 that the groupoid algebra $k[G_{S_3}]$ of the action groupoid $G_{S_3}$ has a WHA structure. In our case, this
structure is given by:

\[
\begin{align*}
\mu(e_{ij} \otimes e_{kl}) &= \delta_{jk} e_{il}, & \mu(\bar{e}_{ij} \otimes e_{kl}) &= \delta_{jk} \bar{e}_{il}, \\
\mu(e_{ij} \otimes \bar{e}_{kl}) &= \delta_{jk} \bar{e}_{il}, & \mu(\bar{e}_{ij} \otimes \bar{e}_{kl}) &= \delta_{jk} e_{il}, \\
\eta(1) &= \bar{e}_{11} + \bar{e}_{22} + \bar{e}_{33}, & \\
\Delta(e_{ij}) &= e_{ij} \otimes e_{ij}, & \Delta(\bar{e}_{ij}) &= \bar{e}_{ij} \otimes \bar{e}_{ij}, \\
\varepsilon(e_{ij}) &= 1, & \varepsilon(\bar{e}_{ij}) &= 1 \forall i,j \in S, \\
S(e_{ij}) &= e_{ji}, & S(\bar{e}_{ij}) &= \bar{e}_{ji}.
\end{align*}
\]

As mentioned in Example 1.1.14, this groupoid algebra is cocommutative.

Later in this thesis, we shall look at coquasi-triangular weak bialgebras and weak Hopf algebras. Starting with a finite-dimensional cocommutative WHA, one can take the dual to obtain a commutative, and thus trivially coquasi-triangular, weak Hopf algebra.

In the present case, since \(\{e_{11}, \bar{e}_{11}, \ldots, \bar{e}_{33}\}\) forms a basis of \(G_{S_3}\), its dual \(\{e_{11}^*, \bar{e}_{11}^*, \ldots, \bar{e}_{33}^*\}\) forms a basis of \(k[G_{S_3}]^*\). Then the WHA structure on \(k[G_{S_3}]^*\) is given by:

- **Multiplication**: \(m(f \otimes g)(a) = \Delta^*(f \otimes g)(a) = f(a')g(a'')\),
- **Unit**: \(u(1) = \varepsilon^*(1) = \varepsilon\),
- **Comultiplication**: \(cm(f)(a \otimes b) = \mu^*(f)(a \otimes b) = f(ab)\),
- **Counit**: \(cu(f) = \eta^*(f) = f(1)\),
- **Antipode**: \(S^*(f)(a) = f(S(a))\),

with \(f, g \in G_{S_3}^*, \ a, b \in G_{S_3}\). Note that \(k[G_{S_3}]^*\) is commutative since by definition of \(\Delta\) we have that

\[
m(f \otimes g)(a) = (f \otimes g)\Delta(a) = (f \otimes g)(a \otimes a) = f(a)g(a)
\]
and the field \( k \) is of course commutative. Hence \( k[G_S]^\ast \) is coquasi-triangular with the trivial \( r \)-form.

## 1.3 Technical Results about WBAs

In this section we present some technical results we shall use later in this thesis. Many of the results presented here are scattered around the literature while others are used but not published in any paper. Hence, out of completeness, and as a basis for future reference, we prove here the lemmata and propositions we shall need in the next chapters.

**Lemma 1.3.1.** Let \( H \) be a WBA, \( a,b \in H \). We have

\[
\varepsilon_s(1') \otimes 1'' = 1' \otimes 1'' \quad \text{and} \quad 1' \otimes \varepsilon_s(1'') = 1' \otimes 1'', \tag{1.19}
\]
\[
\varepsilon(a) = \varepsilon(ab) \quad \text{and} \quad \varepsilon(a) = \varepsilon(ab), \tag{1.20}
\]
\[
\varepsilon_s(a) \varepsilon(b) = \varepsilon(ab) \quad \text{and} \quad \varepsilon_s(a) \varepsilon(b) = \varepsilon(ab). \tag{1.21}
\]

**Proof.**

i) We have \( \varepsilon_s(1') \otimes 1'' = 1' \varepsilon((1')_2 \otimes 1'') = 1' \otimes 1'' \), and similarly \( 1' \otimes \varepsilon_s(1'') = 1' \otimes 1'' \).

ii) We have \( \varepsilon(\varepsilon(a)) = \varepsilon((1')_2 \otimes 1'') = 1' \varepsilon((1'')_2 \otimes 1'') = 1' \otimes 1'' \). We similarly prove that \( \varepsilon(a \varepsilon(b)) = \varepsilon(ab) \).

iii) Equalities \( 1.21 \) are direct consequences of \( 1.19 \) and \( 1.2 \).

**Lemma 1.3.2.** Let \( H \) be a WBA and \( a \in H \). Then

\[
\varepsilon_s(a) = 1' \varepsilon_s(1'') \tag{1.22},
\]
\[
\Delta \circ \varepsilon_s(a) = 1' \otimes \varepsilon_s(a)1'' \tag{1.23},
\]
\[
1' \otimes 1'' \varepsilon_s(a) = \varepsilon_s(a)' \otimes \varepsilon_s(a)'' \tag{1.24},
\]
\[
a' \otimes \varepsilon_s(a'') = a1' \otimes \varepsilon_s(1'') \tag{1.25}.
\]
\[ \varepsilon_s(a') \otimes \varepsilon_s(a'') = \varepsilon_s(\varepsilon_s(a')) \otimes \varepsilon_s(a''), \]  
(1.26) 

\[ \varepsilon_t(a') \otimes \varepsilon_t(a'') = \varepsilon_t(\varepsilon_t(a')) \otimes \varepsilon_t(\varepsilon_t(a'')). \]  
(1.27) 

Proof. i) Using (1.2), we have

\[
1' \varepsilon_s(a) 1'' = 1_2' \varepsilon_s(1_1' \varepsilon_s(a) 1_2'') \\
= 1_2' \varepsilon_s(a) \varepsilon_s(1_1' 1_2'') \\
= \varepsilon_s(a) \varepsilon_s(1_1' 1_2'') \\
= 1' \varepsilon_s(a) \\
= \varepsilon_s(a).
\]

ii) Using (1.3) we find

\[
\Delta \circ \varepsilon_s(a) = \Delta(1' \varepsilon_s(a) 1'') \\
= 1' \otimes 1'' \varepsilon_s(a) \\
= 1_1' \varepsilon_s(a) 1_2'' \\
= 1' \otimes \varepsilon_s(a) 1''.
\]

iii) Using (1.3) we find

\[
1' \otimes 1'' \varepsilon_s(a) = 1_1' \otimes 1_2'' \varepsilon_s(a) \\
= 1' \otimes 1'' \varepsilon_s(a) \\
= (1')' \otimes (1')'' \varepsilon_s(a) \\
= \varepsilon_s(a)' \otimes \varepsilon_s(a)''.
\]
iv) We have

\[
a' \otimes \varepsilon_s(a'') = a' \otimes 1' \varepsilon_s(a''1''')
\]

\[
= (a_{11}')\varepsilon_s((a_{11})''1''') \otimes 1'_2
\]

\[
= a'1'_1 \varepsilon_s(a''1''') \otimes 1'_2
\]

\[
\leq a'1'_1 \varepsilon_s(a''1''') \varepsilon_s(1''1''') \otimes 1'_2
\]

\[
= a'(1'_1)\varepsilon_s((a''(1'_1)''') \varepsilon_s(1''1''') \otimes 1'_2
\]

\[
= (a_{11}')\varepsilon_s((a_{11})'' \varepsilon_s(1''1''') \otimes 1'_2
\]

\[
= a1'_1 \otimes 1'_2 \varepsilon_s(1''1''')
\]

\[
= a1'_1 \otimes \varepsilon_s(1''').
\]

v) Using (1.3) again, we find

\[
\varepsilon_s(a') \otimes \varepsilon_s(a'') = 1' \otimes 1'' \varepsilon_s(a''1''') = 1'_1 \otimes 1''1''_2 \varepsilon_s(a''1''')
\]

\[
= 1'_1 \varepsilon_s(1''1''_2) \varepsilon_s(a''1''') = 1'_1 \varepsilon_s(1''1''_2) \otimes 1''1''_3 \varepsilon_s(a1''3)
\]

\[
= \varepsilon_s(1'_1) \otimes 1''1''_2 \varepsilon_s(a1''3) = \varepsilon_s(1') \otimes 1'' \varepsilon_s(a1''3)
\]

\[
= \varepsilon_s(\varepsilon_s(a')) \otimes \varepsilon_s(a'').
\]

We prove in a similar way that \(\varepsilon_t(a') \otimes \varepsilon_t(a'') = \varepsilon_t(a') \otimes \varepsilon_t(\varepsilon_t(a''))\).

**Lemma 1.3.3.** Let \((H, r)\) be a CQT WBA. Then

\[
r(a \otimes 1) = r(1 \otimes a) = \varepsilon(a). \quad (1.28)
\]

**Proof.** We have

\[
\varepsilon(a) = \varepsilon(1 \cdot a) \leq r(1 \otimes a')r^{-1}(1'' \otimes a'') = r(1'_1 \cdot 1'' \otimes a'')r^{-1}(1'' \otimes a'')
\]

\[
\leq r(1'_2 \otimes a'')r(1'_1 \otimes a'')r^{-1}(1'' \otimes a'') \leq r(1'_2 \otimes a') \varepsilon(1' \cdot a''
\]

\[
= r(1 \otimes a).
\]
Similarly, $\varepsilon(a) = r(a \otimes 1)$.

The next lemma will give us the tools required to prove equality (1.42).

**Lemma 1.3.4.** Let $H$ be a WBA and $a, b \in H$. Then

\begin{align*}
\varepsilon_t(a') \otimes \varepsilon_s(a'') &= \varepsilon_t(a'') \otimes \varepsilon_s(a'), \quad (1.29) \\
\varepsilon_s(a') \otimes \varepsilon_t(a'') &= \varepsilon_s(a'') \otimes \varepsilon_t(a'), \quad (1.30) \\
\varepsilon_t(a') \varepsilon(a''b) &= \varepsilon_t(ab), \quad (1.31) \\
\varepsilon(ab') \varepsilon_s(b'') &= \varepsilon_s(ab), \quad (1.32) \\
\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') &= \varepsilon_t(a') \varepsilon(a''b') \otimes \varepsilon_s(b''), \quad (1.33) \\
\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') &= \varepsilon_t(a') \otimes \varepsilon_s(b') \varepsilon(a''b''), \quad (1.34) \\
\varepsilon_t((ab)') \otimes \varepsilon_s((ab)'') &= \varepsilon(ab') \varepsilon_t(a'') \otimes \varepsilon_s(b''). \quad (1.35)
\end{align*}

**Proof.**  

i) We have

\[
\varepsilon_t(a') \otimes \varepsilon_s(a'') = \varepsilon(1' a') 1''_1 \otimes 1''_2 \varepsilon(a'' 1''_2) = \varepsilon(1' a'' 1''_2) = \varepsilon_t(a'') \otimes \varepsilon_s(a').
\]

Using this, we straightforwardly get that $\varepsilon_s(a') \otimes \varepsilon_t(a'') = \varepsilon_s(a'') \otimes \varepsilon_t(a')$.

ii) Here we have

\[
\varepsilon_t(a') \varepsilon(a''b) = \varepsilon(1' a') 1'' \varepsilon(a'') = \varepsilon(1' a'') 1'' = \varepsilon_t(ab).
\]

We similarly prove that $\varepsilon(ab') \varepsilon_s(b'') = \varepsilon_s(ab)$.
iii) We have

\[ \varepsilon_t((ab)') \otimes \varepsilon_s((ab)''') = \varepsilon_t(a'b') \otimes \varepsilon_s(a''b'') \]
\[ \overset{\diamond}{=} \varepsilon_t(a')\varepsilon(a''b') \otimes \varepsilon_s(a''b''') \]
\[ = \varepsilon_t(a')\varepsilon(a''b') \otimes \varepsilon_s(b'''), \]

where \( \diamond \) follows from \( \text{(1.31)} \) and \( \text{(1.32)} \).

iv) We have

\[ \varepsilon_t((ab)') \otimes \varepsilon_s((ab)''') = \varepsilon_t(a'b') \otimes \varepsilon_s(a''b'') \]
\[ = \varepsilon_t(a')\varepsilon_t(b') \otimes \varepsilon_s(a''b') \]
\[ = \varepsilon_t(a')\varepsilon_t(b') \otimes \varepsilon_s(a''b') \]
\[ = \varepsilon_t(a') \otimes \varepsilon_s(b') \varepsilon_t(b'''). \]

We similarly prove that \( \varepsilon_t((ab)') \otimes \varepsilon_s((ab)''') = \varepsilon(a'b')\varepsilon_t(a''b'') \otimes \varepsilon_s(b''). \)

\[ \square \]

The next lemma will help us prove \( \text{(1.45)} \).

**Lemma 1.3.5.** Let \( H \) be a WBA and \( a, b \in H \). Then

\[ a\varepsilon_t(b) = \varepsilon(a'b)a'', \quad \text{(1.36)} \]
\[ \varepsilon_s(ab) = b'\varepsilon_t(ab''), \quad \text{(1.37)} \]
\[ \varepsilon(ac') \varepsilon(bc'') = \varepsilon(a\varepsilon_t(c)') \varepsilon(b\varepsilon_t(c)''), \quad \text{(1.38)} \]
\[ \varepsilon_t(a') \otimes \varepsilon_t(a'') = \varepsilon_t(\varepsilon_t(a')) \otimes \varepsilon_t(\varepsilon_t(a'')), \quad \text{(1.39)} \]
\[ \varepsilon(ab') \varepsilon_s(\varepsilon_t(a'')) = \varepsilon(a'b)\varepsilon_s(a''). \quad \text{(1.40)} \]

If \( H \) is moreover CQT with \( r \)-form \( r \), then

\[ r(a' \otimes b)\varepsilon_s(\varepsilon_t(a'')) = r(a \otimes b')\varepsilon_s(b''). \quad \text{(1.41)} \]
Proof. \hspace{1em} i) We have
\[
\begin{align*}
\varepsilon_t(a) &= \varepsilon((a \varepsilon_t(b))')(a \varepsilon_t(b))'' \\
&= \varepsilon((a \varepsilon_t(1')b')')(a \varepsilon_t(1'')b'')'' \\
&= \varepsilon((a1')')(a(1'b))'' \\
&= \varepsilon(a''')b'' \\
&= \varepsilon((a1)'b')(a1'')'' \\
&= \varepsilon(a'b)'''.
\end{align*}
\]
We similarly prove that \(\varepsilon_s(a)b = b'\varepsilon(ab'')\).

ii) We have
\[
\begin{align*}
\varepsilon(ac')\varepsilon(bc'') &= \varepsilon(a \varepsilon_t(b)c) = \varepsilon(a \varepsilon_s(b)\varepsilon_s(c)) = \varepsilon(a \varepsilon_t(c)')\varepsilon(b \varepsilon_t(c)'').
\end{align*}
\]

iii) We have
\[
\begin{align*}
\varepsilon_t(a') \otimes \varepsilon_t(a'') &= \varepsilon(1' a') 1''_1 \otimes \varepsilon(1' a'') 1''_2 \\
&= \varepsilon(1' \varepsilon_t(a))' 1''_1 \otimes \varepsilon(1' \varepsilon_t(a))'' 1''_2 \\
&= \varepsilon_t(\varepsilon_t(a')') \otimes \varepsilon_t(\varepsilon_t(a'')').
\end{align*}
\]

iv) We have
\[
\begin{align*}
\varepsilon(ab') \varepsilon_s(\varepsilon_t(b'')) &= \varepsilon(a \varepsilon_t(b')) \varepsilon_s(\varepsilon_t(b'')) \\
&= \varepsilon(a \varepsilon_t(\varepsilon_t(b'))) \varepsilon_s(\varepsilon_t(\varepsilon_t(b''))) \\
&= \varepsilon(a \varepsilon_t(b')) \varepsilon_s(\varepsilon_t(\varepsilon_t(b''))) \\
&= \varepsilon(a \varepsilon_t(b')) \varepsilon_s(\varepsilon_t(b'')) \\
&= \varepsilon_s(a \varepsilon_t(b))
\end{align*}
\]
v) We have

\[ r(a' \otimes b) \varepsilon_s(\varepsilon_t(a'')) = r(a' \otimes b') \varepsilon_s(\varepsilon_t(a'')) \]

\[ = r(a' \otimes b') \varepsilon_s(\varepsilon_t(a''')) \]

\[ = r(a \otimes b') \varepsilon_s(b''). \]

Lemma 1.3.6. Let \((H, r)\) be a CQT WBA and \(a, b \in H\). Then

\[ \varepsilon_t(a') \otimes \varepsilon_s(b') r(a'' \otimes b'') = r(a'' \otimes b'') \varepsilon_t(a'') \varepsilon_s(a'') \]  

(1.42)

\[ r(a \otimes \varepsilon_t(b)) = \varepsilon(ab) \]  

(1.43)

\[ r(a \otimes \varepsilon_s(b)) = \varepsilon(ba) \]  

(1.44)

\[ r(\varepsilon_s(a) \otimes b) = \varepsilon(b \varepsilon_s(a)) \]  

(1.45)

\[ r(\varepsilon_t(a) \otimes b) = \varepsilon(\varepsilon_t(a)b) \]  

(1.46)

\[ r^{-1}(a \otimes \varepsilon_s(b)) = \varepsilon(a \varepsilon_s(b)) \]  

(1.47)

\[ r^{-1}(\varepsilon_t(a) \otimes b) = \varepsilon(ba). \]  

(1.48)

Proof. i) We have

\[ \varepsilon_t(a') \otimes \varepsilon_s(b') r(a'' \otimes b'') \]

\[ = \varepsilon_t(a') \otimes \varepsilon_s(b') \varepsilon(\varepsilon_s(b'')) r(a''' \otimes b''') \]

(1.46)

\[ = \varepsilon_t((a'b')') \otimes \varepsilon_s((a'b'')') r(a'' \otimes b'') \]

(1.46)

\[ = r(a' \otimes b') \varepsilon_t((b''a''') \otimes \varepsilon_s((b''a''))) \]

(1.35)

\[ = r(a' \otimes b') \varepsilon_t((b''a'') \otimes \varepsilon_s(a''')) \]

(1.46)

\[ = r(a' \otimes b') \varepsilon_t(b'') \otimes \varepsilon_s(a''). \]
ii) We have

\[
\begin{align*}
r(a \otimes \varepsilon_t(b)) &= r(a \otimes \varepsilon(1'b)1'') = \varepsilon(1'b)r(a \otimes 1'') \\
&= \varepsilon(\varepsilon_t(a'))\varepsilon(\varepsilon_t(1'b)r(a'' \otimes 1'')) \\
&= \varepsilon(\varepsilon_t(a'))\varepsilon(\varepsilon_t(1'b))r(a'' \otimes 1'') \\
&= \varepsilon(\varepsilon_t(1'b))r(a'' \otimes 1'') \\
&= \varepsilon(ab).
\end{align*}
\]

Similarly, one has that \(r(a \otimes \varepsilon_s(b)) = \varepsilon(ba)\).

iii) We have

\[
\begin{align*}
r(\varepsilon_s(a) \otimes b) &= r(1'\varepsilon(a1'') \otimes b) \\
&\cong \varepsilon(\varepsilon_s(a)\varepsilon_t(1''))r(1' \otimes b) \\
&= \varepsilon(1'\varepsilon(a1'')\varepsilon(1'1'')1'1'')r(1' \otimes b) \\
&= \varepsilon(1'1'')\varepsilon(1'1'')\varepsilon(a1'')r(1' \otimes b) \\
&= \varepsilon(\varepsilon_t(1'')\varepsilon_s(a))r(1' \otimes b) \\
&= \varepsilon(\varepsilon_t(1'')\varepsilon_s(a))r(1' \otimes b) \\
&= \varepsilon(\varepsilon_t(1'')\varepsilon_s(a))r(1' \otimes b) \\
&= \varepsilon(\varepsilon_s(1'')\varepsilon_s(a))r(1' \otimes b) \\
&= \varepsilon(\varepsilon_s(1'')\varepsilon_s(a))r(1' \otimes b) \\
&= \varepsilon(b'\varepsilon_s(a))\varepsilon(b') \\
&= \varepsilon(b'\varepsilon_s(a))\varepsilon(b'),
\end{align*}
\]

where \(\cong\) follows from (1.19) and (1.20) and \(\ast\) from (1.20) and (1.28). We similarly prove that \(r(\varepsilon_t(a) \otimes b) = \varepsilon(\varepsilon_t(a)b)\).
iv) The equality $r^{-1}(a \otimes \varepsilon_s(b)) = \varepsilon(a \varepsilon_s(b))$ follows from Lemma \ref{lem:1.1.9} and \ref{eq:1.46}. On the other hand, $r^{-1}(\varepsilon_t(a) \otimes b) = \varepsilon(ba)$ follows from Lemma \ref{lem:1.1.9} and \ref{eq:1.44}.

\begin{proof}
In this proof, we indicate by $\ast$ the equalities where we use that $g$ is 25
\end{proof}

**Notation 1.3.7.** In what follows we often take many times the comultiplication of an element. In order to make it easy to read we slightly modify Sweedler’s notation (cf. Notation \ref{not:1.1.5}); namely, we use roman numbers for “orders” higher than three, thus $a^{m'''}$ becomes $a^{IV}$ whereas $a^{m''''}$ is written $a^{V}$. Using this convention we have, for example,

$$
(\Delta \otimes \Delta)(\Delta \otimes \Delta)(\Delta \otimes \Delta) \cdot (\Delta \otimes \Delta \otimes \Delta)(a' \otimes a'' \otimes a''') = a' \otimes a'' \otimes a''' \otimes a^{IV} \otimes a^{V} \otimes a^{VI}.
$$

We now observe some facts about group-like elements in WBAs that will be useful later in this thesis.

**Proposition 1.3.8.** Let $(H, r)$ be a CQT WBA, $a, b \in H$ and $g \in G(H)$ a group-like element. Then

\begin{enumerate}
  \item $\varepsilon(a g) = \varepsilon(a), \quad (1.49)$
  \item $\Delta(g a) = a'g \otimes a''g, \quad (1.50)$
  \item $r^{-1}(a' \otimes g)r(a'' \otimes g) = \varepsilon(a), \quad (1.51)$
  \item $r(a' \otimes g)r^{-1}(a'' \otimes g) = \varepsilon(a), \quad (1.52)$
  \item $r(a \otimes g')r(b \otimes g'') = r(a' \otimes g)r(b' \otimes g)\varepsilon(b''a'''), \quad (1.53)$
  \item $r^{-1}(a \otimes g')r^{-1}(b \otimes g'') = \varepsilon(a'b')r^{-1}(a'' \otimes g)r^{-1}(b'' \otimes g). \quad (1.54)$
\end{enumerate}

If $(H, r)$ is a CQT bialgebra, then we moreover have

$$
r(a \otimes g^{-1}) = r^{-1}(a \otimes g) \quad \text{and} \quad r(g^{-1} \otimes a) = r^{-1}(g \otimes a). \quad (1.55)
$$

\begin{proof}
In this proof, we indicate by $\ast$ the equalities where we use that $g$ is

\end{proof}
group-like.

i) We have

\[ \varepsilon(ag) \varepsilon(a \varepsilon_{t}(g)) = \varepsilon(a \cdot 1) = \varepsilon(a). \]

ii) We find

\[ \Delta(ag) = a'g' \otimes a''g'' = a'1'g \otimes a''1''g \]
\[ = (a1)'g \otimes (a1)''g = (a' \otimes a'')(g \otimes g) \]
\[ = a'g \otimes a''g. \]

iii) We have

\[ r^{-1}(a' \otimes g)r(a'' \otimes g) = r^{-1}(a' \otimes g)\varepsilon((a''1')\varepsilon((a''1'')r(aIV \otimes g) \]
\[ = r^{-1}(a' \otimes g)\varepsilon(a''1')\varepsilon(a''1'')r(aIV \otimes g) \]
\[ = r^{-1}(a' \otimes g)\varepsilon(a''1')r(a'' \otimes 1'')r(aIV \otimes g) \]
\[ = r^{-1}(a' \otimes g)\varepsilon(a''1')r(a'' \otimes 1'') \]
\[ = r^{-1}(a' \otimes g)\varepsilon(a's(1'))r(a'' \otimes 1'') \]
\[ = r^{-1}(a' \otimes g)r^{-1}(a'' \otimes \varepsilon_{s}(1'))r(a'' \otimes 1'') \]
\[ = r^{-1}(a' \otimes g)r^{-1}(a'' \otimes 1')r(a'' \otimes 1'') \]
\[ = r^{-1}(a' \otimes 1')r(a'' \otimes 1'') \]
\[ = r^{-1}(a' \otimes g')r(a'' \otimes g'') \]
\[ \varepsilon(ga) \varepsilon(\varepsilon_{s}(g)a) \varepsilon(1a) \]
\[ = \varepsilon(a). \]

iv) Similar to iii).
v) Here we have

\[ r(a \otimes g')r(b \otimes g'') = r(a \otimes g1')r(b \otimes g1'') \]

\[ \overset{11}{=} r(a' \otimes g)r(a'' \otimes 1')r(b' \otimes g)r(b'' \otimes 1'') \]

\[ \overset{11}{=} r(a' \otimes g)r(b' \otimes g)r(b''a'' \otimes 1) \]

\[ \overset{12}{=} r(a' \otimes g)r(b' \otimes g)\varepsilon(b''a''). \]

vi) Similar to v).

vii) First, note that

\[ r(a' \otimes g)r(a'' \otimes g^{-1}) \overset{1.11}{=} r(a \otimes gg^{-1}) = r(a \otimes 1) = \varepsilon(1 \cdot a) = \varepsilon(a), \]

and that using [1.11] we similarly find that \( r(a' \otimes g^{-1})r(a'' \otimes g) = \varepsilon(a) \).

Then

\[ r^{-1}(a \otimes g) \overset{*}{=} r^{-1}(a' \otimes g)\varepsilon(a''g) \]

\[ = r^{-1}(a' \otimes g)\varepsilon(a'')\varepsilon(g) \]

\[ \overset{*}{=} r^{-1}(a' \otimes g)\varepsilon(a'') \]

\[ = r^{-1}(a' \otimes g)r(a'' \otimes g)r(a''' \otimes g^{-1}) \]

\[ \overset{1.8}{=} \varepsilon(ga')r(a'' \otimes g^{-1}) \]

\[ = \varepsilon(g)\varepsilon(a')r(a'' \otimes g^{-1}) \]

\[ \overset{*}{=} r(a \otimes g^{-1}), \]

where \(*\) uses that \( g \) is group-like. The other equality is proved in a similar fashion.

\[ \square \]

27
In this section we briefly introduce a very convenient tool for depicting morphisms in any monoidal category called “string diagrams”. Here we want to apply this tool to the structure maps $\mu, \eta, \Delta, \varepsilon$, and $S$ of WBAs and WHAs. We therefore work in the monoidal category $\text{Vect}_k$ where $k$ is a field.

One of the great advantages of this method is that it fully uses the extension for symmetric monoidal categories of Mac Lane’s coherence theorem [Mac71, Chap. VII]. In other words, many computations are made easier because this notation implicitly uses the fact that we can identify $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ as well as $k \otimes V \cong V \cong V \otimes k$. As we shall see later in this section, using string diagrams makes it particularly easy to see when two elements are equal and which properties or equalities can be used.

String diagrams, even though widely used, do not often appear in published articles. One “historical” reason is that it used to be technically difficult for publishers to smoothly insert them in written texts. More importantly these diagrams take lots of space and thus considerably expand the length of papers; this is the main reason why we mostly use equations and not diagrams in this thesis.

Due to their widespread use and importance, we nonetheless introduce them in an informal manner, focusing on how they are used, and show a few computations. The reader interested in a formal and systematic introduction to string diagrams is referred to [JS91] and [JS].

**Remark 1.4.1.** When working in a monoidal category $\mathcal{C}$ there are two main ways of producing new morphisms from old ones; by composing them or by taking their tensor product. Suppose that in $\mathcal{C}$ we have the following objects and morphisms

$$a: A \to B \otimes B, \quad b: B \to C \otimes D, \quad c: B \otimes C \to C, \quad d: D \otimes C \to D.$$
It is sometimes unclear whether two expressions such as

\[ f_1 = (\text{id}_B \otimes c \otimes d) \circ (\text{id}_B \otimes \text{id}_B \otimes b \otimes \text{id}_C) \circ (a \otimes \text{id}_B \otimes \text{id}_C)(A \otimes B \otimes C) \]

and

\[ f_2 = (\text{id}_B \otimes \text{id}_C \otimes d) \circ (\text{id}_B \otimes c \otimes \text{id}_D \otimes \text{id}_C) \circ (a \otimes b \otimes \text{id}_C)(A \otimes B \otimes C) \]

are equal or not. When drawing their diagrams we obtain

![Diagram for \( f_1 \)]

![Diagram for \( f_2 \)]

for \( f_1 \) and \( f_2 \) respectively. The convention is to use lines for objects and circles with labels for morphisms (the lines are only labeled if there is a possible ambiguity). We moreover read them from top to bottom and a blank space between two objects or morphisms represent a tensor product. We see that these two diagrams are a “deformation” of each other and thus we intuitively want them to represent the same element.

Now that we have had a first glance at string diagrams and understand the basic idea behind them, let us see more precisely how we construct them.

**Construction 1.4.2.** We are going to define string diagrams as classes of graphs satisfying specific properties.

Let \( a < b \) be real numbers. A **plane graph (between levels \( a \) and \( b \))** is a topological graph \( \Gamma \) embedded in \( \mathbb{R} \times [a, b] \) such that every elements \( x \in \Gamma \cap \{a, b\} \) are nodes and that they belong to the closure of exactly one edge. The
nodes on level $a$ are called the \textit{domain} whereas the nodes on level $b$ are the \textit{codomain}. By convention these nodes (called \textit{outer nodes}) are not represented on diagrams.

Since we want to read these diagrams from top to bottom, we consider \textit{oriented} plane graphs. We moreover don’t want turn backs and thus require the graphs to be \textit{progressive}, i.e. the projection on the vertical axis $p_2 : \mathbb{R} \times [a, b] \to [a, b]$ is injective on each edge.

As mentioned above we want to be able to deform the diagrams without changing the elements they represent. A \textit{deformation} of progressive plane graph is a continuous function $h : \Gamma \times [0, 1] \to \mathbb{R} \times [a, b]$ such that for each $t \in [0, 1]$ the function $h(-, t) : \Gamma \to \mathbb{R} \times [a, b]$ is a homeomorphism. We then say that two progressive plane graphs are equivalent if there exists a deformation between them. We thus consider classes of progressive plane graphs under this equivalence relation.

If the domain of a diagram $\Gamma$ equals the codomain of $\Omega$ we can compose them and the diagram of $\Gamma \circ \Omega$ is simply the juxtaposition of the two diagrams with $\Omega$ on top, appropriately connected to $\Gamma$ at the bottom.

The tensor product $\Gamma \otimes \Omega$ of two diagrams $\Gamma$ and $\Omega$ is formed by putting them side by side (on the same level).

Many conventions are employed when using string diagrams. Here we summarize the most important ones when working with a weak bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ over a field $k$.

\textbf{Convention 1.4.3.}  
\begin{itemize}
  \item \textbf{Domain and Codomain.}\n  
As already indicated the nodes of the domain and codomain (which are simply the identity morphism on one element) are not indicated.

  \item \textbf{The Field $k$.}\n  
The field is not indicated in the diagrams, in other words we implicitly identify $k \otimes H \cong H \cong H \otimes k$.\n\end{itemize}
• Unit and counit.
  The unit is depicted by
  \[
  \bullet
  \]
  whereas the counit is depicted by
  \[
  \circ.
  \]

• Multiplication and Comultiplication.
  In order to make string diagrams more readable, the multiplication is
  pictured by
  \[
  \begin{array}{cc}
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \end{array}
  \]
  instead of
  \[
  \mu
  \]
  and the comultiplication by
  \[
  \begin{array}{cc}
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \quad & \quad \\
  \end{array}
  \]
  instead of
  \[
  \Delta
  \]

As an example, we are going to prove equation (1.25) using string diagrams. Before that, let us see how some of the basic WBA axioms of Definition 1.1.1 are expressed in term of diagrams.

**Remark 1.4.4.** Let \((H, \mu, \eta, \Delta, \varepsilon)\) be a WBA over a field \(k\) and let \(a, b, c \in H\). Let us see how we depict some axioms using string diagrams (these will be used in the following proof):
SD1 The unit axiom $1a = a = a1$ is represented by

\[
\begin{array}{c}
\bullet \\
|
\end{array}
\quad = 
\quad |
\quad = 
\begin{array}{c}
\bullet \\
|
\end{array}.
\]

SD2 The counit axiom $\varepsilon(a') \otimes a'' = a = a' \otimes \varepsilon(a'')$ is represented by

\[
\begin{array}{c}
|
\end{array}
\quad = 
\quad |
\quad = 
\begin{array}{c}
|
\end{array}.
\]

SD3 The coassociativity axiom $(a')' \otimes (a'')' \otimes a'' = a' \otimes (a'')' \otimes (a'')''$ is represented by

\[
\begin{array}{c}
|
\end{array}
\quad = 
\quad |
\quad = 
\begin{array}{c}
|
\end{array}.
\]

SD4 The multiplicativity of the comultiplication $(ab)' \otimes (ab)'' = a'b' \otimes a''b''$ is represented by

\[
\begin{array}{c}
|
\end{array}
\quad = 
\quad |
\quad = 
\begin{array}{c}
|
\end{array}.
\]

Note that there is no vertex in the middle of the right-hand side picture.

SD5 The weak multiplicativity of the counit $\varepsilon(abc) = \varepsilon(ab')\varepsilon(b'c) = \varepsilon(ab'')\varepsilon(b'c)$ is depicted by
We are now ready to prove equation (1.25).

**Lemma 1.4.5.** Let $H$ be a WBA and $x \in H$. Then $x' \otimes \varepsilon_s(x'') = x'1' \otimes \varepsilon_s(1'')$.

**Proof.** Indeed, we have

\[ \begin{array}{c}
\varepsilon_s \\
\star \\
SD_1 \\
SD_4 \\
SD_5 \\
SD_3 \\
\varepsilon_s \\
\end{array} = \begin{array}{c}
\varepsilon_s \\
\star \\
SD_1 \\
SD_4 \\
SD_5 \\
SD_3 \\
\varepsilon_s \\
\end{array} = \begin{array}{c}
\varepsilon_s \\
\star \\
\varepsilon_s \\
\end{array} ,
\]

where $\star$ follows from the definition of $\varepsilon_s$. \qed
Chapter 2

Localization in a Coquasi-Triangular WBA

In this chapter we shall introduce the localization of a WBA relative to a monoid of group-like elements. More precisely, starting with a CQT WBA \((H,r)\) and \(G\) a sub monoid of the set of group-like elements \(G(H)\), we want to construct the WBA \(H[G^{-1}]\) and WBA homomorphism \(\varphi : H \to H[G^{-1}]\) having the following universal property. For any WBA homomorphism \(\psi : H \to \bar{H}\) such that \(\psi(g)\) is invertible for any \(g \in G\), there exists a unique WBA homomorphism \(\bar{\psi} : H[G^{-1}] \to \bar{H}\) such that the following diagram commutes

\[
\begin{array}{ccc}
H & \xrightarrow{\varphi} & H[G^{-1}] \\
\downarrow{\psi} & & \downarrow{\bar{\psi}} \\
\bar{H} & \end{array}
\]

This generalizes the well-known notion of localization of a non-commutative ring \(R\) relative to a multiplicatively closed subset \(S\).
2.1 Ring of Fractions

The main sources we use for this section are [Ste75, Chap. II.1] and [Row88, Chap. 3.1]. Many other sources exist but the advantage of these ones is to prove the results in the most general case; in particular the set $S$ can contain non-regular elements.

**Definition 2.1.1.** Let $R$ be a ring and $S \subset R$ a subset. We call $S$ multiplicatively closed if

\[ s, t \in S \text{ implies } st \in S \quad \text{and} \quad 1 \in S. \]

**Definition 2.1.2.** Let $R$ be a ring and $S$ be a multiplicatively closed set. We define the right ring of fractions of $R$ with respect to $S$ as a ring $R[S^{-1}]$ together with a ring homomorphism $\varphi : R \to R[S^{-1}]$ satisfying:

- $\varphi(s)$ is invertible for any $s \in S$,
- every element in $R[S^{-1}]$ is of the form $\varphi(a)\varphi(s)^{-1}$ with $a \in R, s \in S$,
- $a \in \ker\varphi$ if and only if $as = 0$ for some $s \in S$.

Left rings of fractions are defined in a similar way.

Rings of fractions have the following universal property.

**Proposition 2.1.3** ([Ste75, Prop. II.1.1.1]). Suppose $R[S^{-1}]$ exists, then for every ring homomorphism $\psi : R \to \bar{R}$ such that $\psi(s)$ is invertible for every $s \in S$, there exists a unique homomorphism $\tilde{\psi} : R[S^{-1}] \to \bar{R}$ such that the following diagram commutes

\[
\begin{array}{ccc}
R & \overset{\varphi}{\longrightarrow} & R[S^{-1}] \\
\downarrow{\psi} & & \downarrow{\tilde{\psi}} \\
\bar{R} & & \bar{R}
\end{array}
\]
Corollary 2.1.4. When it exists, a ring of fractions \( R[S^{-1}] \) is unique up to a unique isomorphism.

As one would expect, the ring of fractions does not always exist. We shall now see a condition characterizing its existence.

Proposition 2.1.5 ([Ste75 Prop. II.1.1.4] or [Row88 Thm. 3.1.4]). Let \( R \) be a ring and \( S \) multiplicatively closed set of \( R \).

i) The ring of fractions \( R[S^{-1}] \) exists if and only if \( S \) satisfies:

(S1) if \( s_1 \in S \) and \( a_1 \in R \), there exist \( s_2 \in S \) and \( a_2 \in R \) such that
\[
 s_1a_2 = a_1s_2,
\]
(S2) if \( s_1a = 0 \) for \( s_1 \in S \), \( a \in R \), then there exists \( s_2 \in S \) such that
\[
 as_2 = 0.
\]

ii) When \( R[S^{-1}] \) exists, it is of the form

\[
 R[S^{-1}] = R \times S / \sim
\]

where \( \sim \) is the equivalence relation for which \( (a_1, s_1) \sim (a_2, s_2) \) if there exist \( u_1, u_2 \in R \) such that
\[
 a_1u_1 = a_2u_2 \in R \quad \text{and} \quad (2.1)
\]

\[
 s_1u_1 = s_2u_2 \in S. \quad (2.2)
\]

Then, writing \( \frac{a}{s} \) for the equivalence class containing \((a, s)\), the addition is given by

\[
 \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1v_1 + a_2v_2}{s_1v_1} \quad (2.3)
\]

where condition S1 ensures the existence of \( v_1 \in S \) and \( v_2 \in R \) such that \( s_1v_1 = s_2v_2 \in S \). Furthermore, the multiplication is given by

\[
 \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1v_1}{s_2v_2} \quad (2.4)
\]
where $S_1$ is used again to find $v_1 \in R$ and $v_2 \in S$ such that $s_1v_1 = a_2v_2$.

**Remark 2.1.6.**  

i) From the previous proposition we see that condition $S_1$ ensures that two fractions can always be put on the same denominator. They then represent the same class if one can find a common denominator for which the numerators are equal.

ii) The condition $S_2$ will be nearly exclusively used in the following way. 
Let $s_1 \in S$ and $a_1, a_2 \in R$ be such that $s_1a_1 = s_1a_2$. Then we have $0 = s_1a_1 - s_1a_2 = s_1(a_1 - a_2)$ and thus by condition $S_2$ there exists $s_2 \in S$ such that $(a_1 - a_2)s_2 = 0$. Hence we have an element $s_2 \in S$, such that $a_1s_2 = a_2s_2$. We can therefore reformulate condition $S_2$ as follows:

$(S_2')$ if $s_1a_1 = s_1a_2$ with $s_1 \in S$ and $a_1, a_2 \in R$, there exists $s_2 \in S$ such that $a_1s_2 = a_2s_2$.

Indeed, suppose that $S_2'$ holds and that $s_1a = 0$ with $s_1 \in S$ and $a \in R$. Then, we have in particular that $s_1a = 0 = s_10$ and thus by $S_2'$ there exists $s_2 \in S$ such that $a_1s_2 = 0s_2 = 0$. Thus we have just proved that conditions $S_2$ and $S_2'$ are equivalent.

iii) Note that the definitions of the addition (2.3) and multiplication (2.4) are independent of the choice of $v_1$ and $v_2$.

**Definition 2.1.7.** A multiplicatively closed set $S$ satisfying the conditions $S_1$ and $S_2$ of the preceding Proposition is called a *right denominator set* or *right Ore set* in reference to [Ore31].

**Example 2.1.8** (Commutative Case). Let $R$ be a commutative ring and $S$ a multiplicatively closed set. In this case both conditions $S_1$ and $S_2$ are trivially satisfied by taking $s_2 = s_1$ and $a_2 = a_1$. 

An important special case is when one takes $S$ to be the set of all regular elements (i.e. non-zero-divisors) in $R$; then $R[S^{-1}]$ is called the *classical (right) ring of fractions* or *total (right) ring of fractions.*
2.2 Localization in Coquasi-Triangular Bialgebras

In the first part of this section we follow \cite{Hay92} and use the theory presented above to construct the localization of a coquasi-triangular bialgebra at a submonoid of group-like elements. In order to make the generalization of this construction as easy as possible, we reorganize the order and complete many proofs of the above reference.

2.2.1 Construction of the Localization

Let us introduce a definition that will play a central role in our construction.

Definition 2.2.1. Let \((B, r)\) be a coquasi-triangular bialgebra and let \(g\) be a group-like element of \(B\). We define the linear map \(I_g : B \to B\) by

\[
I_g(a) = r(a' \otimes g) a'' r^{-1}(a''' \otimes g)
\]

for any \(a \in B\).

Lemma 2.2.2. The linear map \(I_g\) defined in the previous definition is a CQT bialgebra homomorphism.

Proof. First, using relation \((1.10)\), the fact that \(g\) is a group-like and Lemma \((1.1.9)\), we have

\[
\begin{align*}
I_g(ab) &= r((ab)' \otimes g)(ab)' r^{-1}((ab)'' \otimes g) \\
&= r(a' \otimes g) r(b' \otimes g) a'' b'' r^{-1}(a''' \otimes g) r^{-1}(b''' \otimes g) \\
&= I_g(a) I_g(b).
\end{align*}
\]
Moreover, the unit \(1\) being group-like we get

\[
\mathcal{I}_g(1) = r(1 \otimes g)1r^{-1}(1 \otimes g) = 1\varepsilon(1 \cdot g) = 1,
\]

where we have used that \(g\) is group-like. Let us now check that \(\mathcal{I}_g\) is a coalgebra homomorphism. We have

\[
(\mathcal{I}_g \otimes \mathcal{I}_g)(\Delta(a)) = r(a' \otimes g)a''r^{-1}(a''' \otimes g)
\]

\[
\otimes r(a^{IV} \otimes g)a^{V}r^{-1}(a^{VI} \otimes g)
\]

\[
= r(a' \otimes g)a''\varepsilon(ga''') \otimes a^{IV}r^{-1}(a^{V} \otimes g)
\]

\[
= r(a' \otimes g)a'' \otimes a'''r^{-1}(a^{IV} \otimes g)
\]

\[
= \Delta(\mathcal{I}_g(a)),
\]

and

\[
\varepsilon(\mathcal{I}_g(a)) = r(a' \otimes g)\varepsilon(a'')r^{-1}(a''' \otimes g) = r(a' \otimes g)r^{-1}(a'' \otimes g)
\]

\[
= \varepsilon(ag) = \varepsilon(a)\varepsilon(g)
\]

\[
= \varepsilon(a).
\]

Thus \(\mathcal{I}_g\) is a bialgebra homomorphism. Using properties of the \(r\)-form we find

\[
r(\mathcal{I}_g(a) \otimes \mathcal{I}_g(b)) = r(a' \otimes b')r(a'' \otimes b'')r^{-1}(b''' \otimes a''')
\]

\[
= r(a' \otimes b')(a'' \otimes b'')r^{-1}(b''' \otimes a''')
\]

\[
= r(a' \otimes b')r(b''a'' \otimes g)r^{-1}(b'''a''' \otimes g)
\]

\[
= r(a' \otimes b')\varepsilon(b''a'')
\]

\[
= r(a' \otimes b')\varepsilon(b'')\varepsilon(a'' \otimes g)
\]

\[
= r(a \otimes b),
\]

and hence \(\mathcal{I}_g\) is a CQT bialgebra homomorphism.
Proposition 2.2.3.  

i) For each $g \in G(B)$, the endomorphism $I_g$ is a CQT bialgebra automorphism; its inverse being given by

\[ I_g^{-1}(a) = r^{-1}(a' \otimes g) a'' r(a''' \otimes g). \]

ii) Let $G \subset G(B)$ be a monoid of group-like elements. The morphism

\[ J : G \to \text{Aut}(B) : g \mapsto I_g \]

is a monoid anti-homomorphism.

Proof.  

i) It is straightforward to verify that $I_g^{-1}$ is the inverse of $I_g$; it then follows directly that $I_g^{-1}$ is also a CQT bialgebra homomorphism and thus $I_g$ is an automorphism.

ii) This follows from (1.11) and Lemma 1.3.3.

We now give some useful properties of $I_g$.

Proposition 2.2.4. Let $(B, r)$ be a CQT bialgebra, $a, b \in B$, $g, h \in G(B)$. Then

i) 

\[ g I_g(a) = ag, \quad (2.5) \]

\[ J_g^{-1}(a) g = ga. \quad (2.6) \]

ii) 

\[ J_g(h) = h, \quad (2.7) \]

iii) $ga = 0$ if and only if $ag = 0$,

iv) the monoid $G(B)$ is commutative.
Proof. i) We have

\[ gI_g(a) = r(a' \otimes g)ga''r^{-1}(a''' \otimes g) \]
\[ = a'g r(a'' \otimes g)r^{-1}(a''' \otimes g) \]
\[ = a'g \varepsilon(a''g) \]
\[ = a'g \varepsilon(a'') \varepsilon(g) \]
\[ = ag. \]

Using the definition of \( I_g^{-1} \) and \ref{19} again, one similarly finds that \( I_g^{-1}(a)g = ga \).

ii) Using that \( h \) is group-like, we get

\[ I_g(h) = r(h \otimes g)hr^{-1}(h \otimes g) = h \varepsilon(hg) = h. \]

iii) Suppose \( ga = 0 \) then

\[ 0 = I_g(ga) = I_g(g)I_g(a) \overset{\ref{27}}{=} gI_g(a) \overset{\ref{25}}{=} ag. \]

Similarly \( ag = 0 \) implies \( ga = 0 \).

iv) This follows directly from i) and ii).

\[ \square \]

This Proposition enables us to prove the next

**Lemma 2.2.5.** Let \((B,r)\) be a CQT bialgebra and \(G\) be a sub monoid of \(G(B)\). Then seeing \(B\) as a ring and \(G\) as a multiplicatively closed set, they satisfy conditions S1 and S2 of Prop. \ref{2.1.5}

Proof. Using the notations of Prop. \ref{2.1.5} let \( s_1 = g \in G \) and \( a_1 = a \in B \). By \ref{2.5} we have \( gI_g(a) = ag \), i.e. \( a_2 = I_g(a) \) and \( s_2 = g \). Concerning condition S2, suppose \( ga = 0 \). Then \( ag = 0 \) by Prop. \ref{2.2.4} iii), i.e. we can simply take

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Remark 2.2.6. This result is important since it will enable us to define an equivalence relation ∼ on $B \times G$ as well as a ring structure on $(B \times G)/\sim$ in analogy to Prop. 2.1.5. We shall, out of completeness, give all the details of the proof that $(B, G)/\sim$ forms an algebra.

We now have all the tools required to prove the following

**Theorem 2.2.7.** Let $(B, r)$ be a CQT bialgebra and $G$ a sub monoid of $G(B)$. Define the equivalence relation ∼ on $B \times G$ as follows: $(a_1, g_1) \sim (a_2, g_2)$ if there exist $u_1, u_2 \in B$ such that

\begin{align*}
a_1 u_1 &= a_2 u_2 \in B \quad \text{and} \\
g_1 u_1 &= g_2 u_2 \in G.
\end{align*}

(2.8)

(2.9)

Let us denote by $\frac{a}{g}$ the equivalence class containing $(a, g)$. Then $B[G^{-1}] := (B \times G)/\sim$ has a unital algebra structure given by

\begin{align*}
\frac{a}{g} + \frac{b}{h} &= \frac{ah + bg}{gh}, \\
\frac{a}{g} \cdot \frac{b}{h} &= \frac{a\mathfrak{g}(b)}{gh},
\end{align*}

(2.10)

(2.11)

where $\frac{1}{1}$ is the unit and scalar multiplication is given by $\alpha \cdot \frac{a}{g} = \frac{\alpha a}{g}$ for $\alpha \in k$.

In order to make the proof easy to read, we are going to break it down into small parts.

**Lemma 2.2.8.** Let $(B, r)$, $G$ and ∼ be as in Thm. 2.2.7. Then the relation ∼ is indeed an equivalence relation.

**Proof.** For the reflexivity just take $u_1 = u_2 = 1$ and for the symmetry simply exchange $u_1$ and $u_2$. 

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Let us now prove the transitivity of $\sim$. Suppose $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$ and $\frac{a_2}{g_2} \sim \frac{a_3}{g_3}$, i.e. there exist $u_1, u_2, v_1, v_2$ such that

\[
\begin{align*}
  a_1 u_1 &= a_2 u_2 \\
  g_1 u_1 &= g_2 u_2
\end{align*}
\]

and

\[
\begin{align*}
  a_2 v_2 &= a_3 v_3 \\
  g_2 v_2 &= g_3 v_3
\end{align*}
\]

Consider $g_2 u_2 \in G$ and $g_2 v_2 \in G$; by S1 (which is satisfied by Lem. [2.2.5]) there exist $w_1, w_2$ such that $g_2 u_2 w_1 \in G$. Then by S2' there exists $h \in G$ such that

\[
  u_2 w_1 h = v_2 w_2 h. \tag{2.12}
\]

Hence

\[
\begin{align*}
  a_1 u_1 w_1 h &= a_2 u_2 w_1 h \\&= a_2 v_2 w_2 h = a_3 v_3 w_2 h \\
  g_1 u_1 h &= g_2 u_2 w_1 h = g_2 v_2 h = g_3 v_3 h,
\end{align*}
\]

with $g_1 u_1 w_1 h$ belonging to $G$ since both $g_2 u_2 w_1$ and $h$ do. 

We now prove a technical result that will enable us to substantially simplify subsequent proofs.

**Lemma 2.2.9.** Let $(B, r)$ and $G$ be as in Thm. [2.2.7]. Then $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$ if and only if there exist $v_1, v_2 \in G$ (and not simply in $B$) such that $a_1 v_1 = a_2 v_2$ and $g_1 v_1 = g_2 v_2 \in G$.

**Proof.** If we have $v_1$ and $v_2$ in $G$ such that $a_1 v_1 = a_2 v_2$ and $g_1 v_1 = g_2 v_2 \in G$ then clearly $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$.

Now suppose that $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$; we have to show that we can find $v_1, v_2$ in $G$ such that

\[
  a_1 v_1 = a_2 v_2 \quad \text{and} \quad g_1 v_1 = g_2 v_2 \in G.
\]

By definition there exist $u_1, u_2 \in B$ such that

\[
\begin{align*}
  a_1 u_1 &= a_2 u_2 \\
  g_1 u_1 &= g_2 u_2
\end{align*}
\]

and

\[
\begin{align*}
  a_2 v_2 &= a_3 v_3 \\
  g_2 v_2 &= g_3 v_3
\end{align*}
\]
By equations (2.5) and (2.7) we get \( u_1 g_1 = g_1 \mathcal{I}_{g_1}(u_1) = \mathcal{I}_{g_1}(g_1 u_1) = g_1 u_1 \); similarly \( g_2 u_2 = u_2 g_2 \). Hence \( v_1 := u_1 g_1 g_2 \) and \( v_2 := u_2 g_1 g_2 \) belong to \( G \). Then \( a_1 v_1 = a_1 u_1 g_1 g_2 = a_2 u_2 g_1 g_2 = a_2 v_2 \) and \( g_1 v_1 = g_1 u_1 g_1 g_2 = g_2 u_2 g_1 g_2 = g_2 v_2 \in G \).

**Remark 2.2.10.** Note that the definition of the equivalence relation only requires \( v_1, v_2 \) to be in \( B \); the previous lemma tells us that these elements can actually be taken in \( G \).

This property is known to hold if the elements we invert are central or are all regular [Bea99, Lem. A.5.3]. Thus the previous lemma shows that it is enough for these elements to be “almost central”, i.e. to commute up to a homomorphism, for this property to hold.

**Proof of Thm. 2.2.7.** i) The addition is well-defined.

Let \( \frac{a_1}{g_1} \sim \frac{a_2}{g_2} \) and \( \frac{b_1}{h_1} \sim \frac{b_2}{h_2} \); by Lemma 2.2.9 there exist \( u_1, u_1, v_1, v_2 \in G \) such that

\[
\begin{align*}
    a_1 u_1 &= a_2 u_2 & \text{and} & & b_1 v_1 &= b_2 v_2 \\
    g_1 u_1 &= g_2 u_2 & & h_1 v_1 &= h_2 v_2.
\end{align*}
\]

We want to show that

\[
\frac{a_1}{g_1} + \frac{b_1}{h_1} = \frac{a_1 h_1 + b_1 g_1}{g_1 h_1} \sim \frac{a_2 h_2 + b_2 g_2}{g_2 h_2} = \frac{a_2}{g_2} + \frac{b_2}{h_2}.
\]

We have

\[
(a_1 h_1 + b_1 g_1)u_1 v_1 = a_1 h_1 u_1 v_1 + b_1 g_1 u_1 v_1 \\
= a_1 u_1 h_1 v_1 + b_1 v_1 g_1 u_1 \\
= a_2 u_2 h_2 v_2 + b_2 v_2 g_2 u_2 \\
= a_2 h_2 u_2 v_2 + b_2 g_2 u_2 v_2 \\
= (a_2 h_2 + b_2 g_2)u_2 v_2,
\]

where we have used the commutativity of \( G \) (cf. Prop. 2.2.4.iv). More-
over we have that
\[(g_1 h_1)u_1 v_1 = g_1 u_1 h_1 v_1 = g_2 u_2 h_2 v_2 = (g_2 h_2)u_2 v_2 \in G.\]

ii) One verifies in a straightforward manner that \(B[G^{-1}]\) is a vector space.

iii) The multiplication is well-defined.

As above, take \(\frac{a_1}{g_1} \sim \frac{a_2}{g_2}\) and \(\frac{b_1}{h_1} \sim \frac{b_2}{h_2}\); we want to show that
\[
\frac{a_1}{g_1} \cdot \frac{b_1}{h_1} = \frac{a_1}{g_1} \frac{b_1}{h_1} = \frac{a_2}{g_2} \cdot \frac{b_2}{h_2}.
\]

Then
\[
\begin{align*}
a_1 g_1 (b_1)u_1 v_1 & \overset{\text{Def.}}{=} a_1 u_1 g_1 (b_1)(v_1) \overset{\star}{=} a_2 u_2 g_1 u_1 (b_1) v_1 \\
& \overset{\text{Def.}}{=} a_2 u_2 g_1 u_1 (b_1) g_1 u_1 (v_1) = a_2 u_2 g_1 u_1 (b_1 v_1) \\
& = a_2 u_2 g_2 u_2 (b_2 v_2) = a_2 u_2 g_2 u_2 (b_2) v_2 \\
& = a_2 u_2 g_2 (b_2) v_2 \\
& \overset{\text{Def.}}{=} a_2 g_2 (b_2) u_2 v_2,
\end{align*}
\]

where \(*\) follows from Prop. 2.2.3.ii). Moreover, we have
\[(g_1 h_1)u_1 v_1 = g_1 u_1 h_1 v_1 = g_2 u_2 h_2 v_2 = (g_2 h_2)u_2 v_2 \in G,
\]

where we have obviously used the commutativity of the monoid \(G(B)\).

iv) One verifies in a straightforward way the compatibility of the multiplication with the scalars.

v) The multiplication is distributive.
We have
\[
\frac{a}{g} \left( \frac{b}{h} + \frac{c}{k} \right) = \frac{a \cdot bk + ch}{hk} = \frac{a \mathcal{I}_g(bk + ch)}{ghk} = \frac{a \mathcal{I}_g(bk) + a \mathcal{I}_g(ch) \mathcal{I}_g(bk + ch)}{ghk} = \frac{a \mathcal{I}_g(bk) + a \mathcal{I}_g(c)h}{ghk} = \frac{(a \mathcal{I}_g(b)k + a \mathcal{I}_g(c)gh)\mathcal{I}_g(bk + ch)}{ghgk} = \frac{a \mathcal{I}_g(b) + a \mathcal{I}_g(c)}{gh} + \frac{a \mathcal{I}_g(b)g + a \mathcal{I}_g(c)gh}{ghkg} = \frac{a \mathcal{I}_g(b)}{gh} + \frac{a \mathcal{I}_g(c)}{ghk} = \frac{a \cdot b}{gh} + \frac{a \cdot c}{ghk},
\]
Left distributivity is proved in a similar way.

vi) The multiplication is associative.
Indeed, we have
\[
\left( \frac{a}{g} \cdot \frac{b}{h} \right) \frac{c}{k} = \frac{a \mathcal{I}_g(b)}{gh} \frac{c}{k} = \frac{a \mathcal{I}_g(b) \mathcal{I}_g(c)}{ghk} \mathcal{I}_g(bk + ch) = \frac{a \mathcal{I}_g(b) \mathcal{I}_g(c)}{ghk} = \frac{a \mathcal{I}_g(b) \mathcal{I}_g(c)}{ghk} \mathcal{I}_g(bh) = \frac{a \mathcal{I}_g(b) \mathcal{I}_g(c)}{ghk} \mathcal{I}_g(bh + ck) = \frac{a \mathcal{I}_g(b) \mathcal{I}_g(c)}{ghk} \mathcal{I}_g(b \cdot c),
\]
where * uses the commutativity of $G(B)$.

vii) The element $\frac{1}{1}$ is the unit.
Indeed,
\[
\frac{a}{g} \cdot \frac{1}{1} = \frac{a \mathcal{I}_g(1)}{g \cdot 1} = \frac{a \cdot 1}{g} = \frac{a}{g},
\]
where we have used that $1$ is group-like. Similarly, $\frac{1}{1} \cdot \frac{a}{g} = \frac{a}{g}$.

\[
\n\]
Before building the CQT bialgebra structure of $B[\frac{G}{G-1}]$, let us introduce a notation.

**Notation 2.2.11.** Let $B$ be a CQT bialgebra, $G$ a monoid of group-likes and $B[\frac{G}{G-1}]$ its localization. We denote by $\rho$ the homomorphism given by

$$\rho : B \to B[\frac{G}{G-1}] : a \mapsto \frac{a}{1}.$$ 

**Theorem 2.2.12.** Let $(B, r)$ be a CQT bialgebra and $G$ a sub monoid of $G(B)$. Then $B[\frac{G}{G-1}]$ has a coquasi-triangular bialgebra structure. Its algebra structure is given by Thm. 2.2.7 whereas the comultiplication, counit and $r$-form are given by

$$\Delta \left( \frac{a}{g} \right) = \frac{a'}{g} \otimes \frac{a''}{g},$$

$$\varepsilon \left( \frac{a}{g} \right) = \varepsilon(a),$$

$$r \left( \frac{a}{g} \otimes \frac{b}{h} \right) = r(a' \otimes b' \otimes h r^{-1}(a'' \otimes h) r^{-1}(g \otimes b) r(g \otimes h).$$

**Remark 2.2.13.** Note that we could prove this theorem by using the fact that the localization has an algebra structure (Thm. 2.2.7) and then exploit its universal property to build a CQT bialgebra structure. Such a proof would use that the morphisms $\Delta$ and $\varepsilon$ are algebra homomorphism, a property that does not hold in the weak case. Such a proof would thus not generalize to the weak case and we therefore do not use this argument here.

**Proof of Thm. 2.2.12**

i) The comultiplication is well-defined.

Let $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$, i.e. there exist $u_1, u_2 \in G$ such that

$$a_1 u_1 = a_2 u_2,$$

$$g_1 u_1 = g_2 u_2.$$
Applying $\Delta$ on both sides of $a_1u_1 = a_2u_2$ we get $a'_1u_1 \otimes a''_1u_1 = a'_2u_2 \otimes a''_2u_2$. Next applying $\rho \otimes \rho$ on both sides of the latter, we find

$$\frac{a'_1u_1}{1} \otimes \frac{a''_1u_1}{1} = \frac{a'_2u_2}{1} \otimes \frac{a''_2u_2}{1}. \tag{2.16}$$

Now we have

$$\Delta \left( \frac{a_1}{g_1} \right) = \frac{a'_1}{g_1} \otimes \frac{a''_1}{g_1} = \frac{a'_1u_1}{g_1u_1} \otimes \frac{a''_1u_1}{g_1u_1} = \left( \frac{a'_1u_1}{1} \otimes \frac{a''_1u_1}{1} \right) \frac{1}{g_1u_1} \otimes \frac{1}{g_1u_1} = \left( \frac{a'_2u_2}{1} \otimes \frac{a''_2u_2}{1} \right) \frac{1}{g_2u_2} \otimes \frac{1}{g_2u_2} = \frac{a'_2}{g_2} \otimes \frac{a''_2}{g_2} = \Delta \left( \frac{a_2}{g_2} \right).$$

ii) The counit is well-defined.

Again take $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$, we have

$$\varepsilon \left( \frac{a_1}{g_1} \right) = \varepsilon(a_1) = \varepsilon(a_1) \cdot 1 = \varepsilon(a_1)\varepsilon(u_1) = \varepsilon(a_1u_1) = \varepsilon(a_2) \varepsilon(u_2) = \varepsilon(a_2) \cdot 1 = \varepsilon(a_2) = \varepsilon \left( \frac{a_2}{g_2} \right).$$

iii) The comultiplication is coassociative.

Let $a \in B$, we have $(\text{id} \otimes \Delta) \circ (\Delta(a)) = a' \otimes (a'')' \otimes (a'')'' \ast (a')' \otimes (a'')'' \otimes a'' = (\Delta \otimes \text{id}) \circ (\Delta(a))$ by coassociativity of $B$. Then by applying $\rho \otimes \rho \otimes \rho$ on both sides of $\ast$, we find

$$\frac{a'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{(a'')''}{1} = \frac{(a')'}{1} \otimes \frac{(a'')''}{1} \otimes \frac{a''}{1}. \tag{2.17}$$
Now we have

\[
(id \otimes \Delta) \circ \left( \Delta \left( \frac{a}{g} \right) \right) = \frac{a'}{g} \otimes \frac{(a'')'}{g} \otimes \frac{(a'')''}{g} = \left( \frac{a'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{(a'')''}{1} \right) \frac{1}{g} \otimes \frac{1}{g} \otimes \frac{1}{g}
\]

\[
\left( \frac{a'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{(a'')''}{1} \right) \frac{1}{g} \otimes \frac{1}{g} \otimes \frac{1}{g}
\]

\[
\left( \frac{(a')'}{g} \otimes \frac{(a'')'}{g} \otimes \frac{a''}{g} \right)
\]

\[
= (\Delta \otimes id) \circ \left( \Delta \left( \frac{a}{g} \right) \right).
\]

iv) The comultiplication is additive.

We have

\[
\Delta \left( \frac{a}{g} + \frac{b}{h} \right) = \Delta \left( \frac{ah + bg}{gh} \right)
\]

\[
= \frac{a'h}{gh} \otimes \frac{a''h}{gh} + \frac{b'g}{gh} \otimes \frac{b''}{gh}
\]

\[
= \frac{a'}{g} \otimes \frac{a''}{h} + \frac{b'}{h} \otimes \frac{b''}{h}
\]

\[
= \Delta \left( \frac{a}{g} \right) + \Delta \left( \frac{b}{h} \right),
\]

where we have used in \( \star \) that \( G \) is commutative.

v) The counit is additive.

Indeed,

\[
\varepsilon \left( \frac{a}{g} + \frac{b}{h} \right) = \varepsilon \left( \frac{ah + bg}{gh} \right) = \varepsilon (ah + bg)
\]

\[
= \varepsilon (a) \varepsilon (h) + \varepsilon (b) \varepsilon (g) = \varepsilon (a) + \varepsilon (b)
\]

\[
= \varepsilon \left( \frac{a}{g} \right) + \varepsilon \left( \frac{b}{h} \right),
\]

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where we have used that \( \varepsilon(g) = 1 = \varepsilon(h) \).

vi) One verifies using straightforward computations that the comultiplication and counit are \( k \)-linear maps.

vii) \( B[G^{-1}] \) is a counital coalgebra.

It remains to show that \( (\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta \). We have

\[
(\text{id} \otimes \varepsilon) \circ \Delta \left( \frac{a}{g} \right) = (\text{id} \otimes \varepsilon) \left( \frac{a'}{g} \otimes \frac{a''}{g} \right) = \frac{a'}{g} \varepsilon(a'') = \frac{\varepsilon(a'')a'}{g} = \frac{a}{g},
\]

where we have used the counit axiom in \( B \). The other equality is proved in a similar way.

We now have to verify the axioms regarding to the compatibility of the algebra and coalgebra structures.

viii) The comultiplication preserves the multiplication.

We have to prove that \( \Delta \circ \mu = (\mu \otimes \mu) \circ (\text{id} \otimes \sigma \otimes \text{id}) \circ (\Delta \otimes \Delta) \),

where \( \sigma : B[G^{-1}] \otimes B[G^{-1}] \to B[G^{-1}] \otimes B[G^{-1}] : \frac{a}{g} \otimes \frac{b}{h} \mapsto \frac{b}{h} \otimes \frac{a}{g} \) is the morphism flipping two adjacent tensor factors. We get

\[
\Delta \circ \mu \left( \frac{a}{g} \otimes \frac{b}{h} \right) = \Delta \left( \frac{a \mathcal{J}_g(b)}{gh} \right) = \frac{(a \mathcal{J}_g(b))'}{gh} \otimes \frac{(a \mathcal{J}_g(b))''}{gh} = \frac{a' \mathcal{J}_g(b')}{gh} \otimes \frac{b'' \mathcal{J}_g(b'')}{gh} = \frac{a' \mathcal{J}_g(b')}{gh} \cdot \frac{b'' \mathcal{J}_g(b'')}{gh} = (\mu \otimes \mu) \circ (\text{id} \otimes \sigma \otimes \text{id}) \circ (\Delta \otimes \Delta) \left( \frac{a}{g} \otimes \frac{b}{h} \right),
\]

where \( * \) holds since \( \mathcal{J}_g \) is CQT bialgebra homomorphism by Lemma 2.2.2.
ix) The counit preserves the multiplication.
Indeed,
\[
\varepsilon \left( \frac{a}{g} \cdot \frac{b}{h} \right) = \varepsilon \left( \frac{a \mathcal{J}_g(b)}{gh} \right) = \varepsilon(a \mathcal{J}_g(b))
\]
\[
= \varepsilon(a) \varepsilon(\mathcal{J}_g(b)) \star \varepsilon(a) \varepsilon(b)
\]
\[
= \varepsilon \left( \frac{a}{g} \right) \varepsilon \left( \frac{b}{h} \right),
\]
where again \(\star\) holds since \(\mathcal{J}_g\) is CQT bialgebra homomorphism.

x) The comultiplication preserves the unit.
Indeed,
\[
\Delta \left( \frac{1}{1} \right) = \frac{1}{1} \otimes \frac{1''}{1} = \frac{1}{1} \otimes \frac{1}{1}
\]
since 1 is group-like in \(B\).

xi) The counit preserves the unit.
Indeed,
\[
\varepsilon \left( \frac{1}{1} \right) = \varepsilon(1) = 1_k.
\]

Hence we have just proved that \(B[G^{-1}]\) is a bialgebra. We now want to show it has an \(r\)-form.

xii) The \(r\)-form is well-defined.
Let \(\frac{a_1}{g_1} \sim \frac{a_2}{g_2}\) and \(\frac{b_1}{h_1} \sim \frac{b_2}{h_2}\), i.e. there exist \(u_1, u_2, v_1, v_2 \in G\) such that
\[
\begin{align*}
a_1 u_1 &= a_2 u_2 \\
g_1 u_1 &= g_2 u_2 \\
b_1 v_1 &= b_2 v_2 \\
h_1 v_1 &= h_2 v_2.
\end{align*}
\]
Then using that \(u_1, u_2, v_1, v_2\) are group-likes and that the monoid of group-like elements \(G(B)\) is commutative by Proposition 2.2.4 iv), we
have

\[
\begin{align*}
& r \left( \frac{a_1 \otimes b_1}{g_1 h_1} \right) = \\
& = r(a_1' \otimes b_1') r^{-1}(a_1'' \otimes h_1) r^{-1}(g_1 \otimes b_1') r(g_1 \otimes h_1) \\
& = r(a_1' \otimes b_1') r^{-1}(a_1'' \otimes h_1) r^{-1}(g_1 \otimes b_1') r(g_1 \otimes h_1) \\
& \hspace{1em} \cdot r(a_1'' \otimes v_1) r^{-1}(a_1'' \otimes v_1) r^{-1}(g_1 \otimes v_1) r(g_1 \otimes v_1) \\
& = r(a_1' \otimes b_1') r(a_1'' \otimes v_1) r^{-1}(a_1'' \otimes h_1) r^{-1}(a_1'' \otimes v_1) \\
& \hspace{1em} \cdot r^{-1}(g_1 \otimes v_1) r^{-1}(g_1 \otimes v_1) r(g_1 \otimes h_1) \\
& = r(a_1' \otimes b_1' \otimes h_1 v_1) r^{-1}(a_1'' \otimes h_1 v_1) r^{-1}(g_1 \otimes h_1 v_1) \\
& = r(a_1' \otimes (b_1 v_1)'') r^{-1}(u_1 \otimes (b_1 v_1)''') r^{-1}(u_1 \otimes (b_1 v_1)''') r^{-1}(g_1 \otimes (b_1 v_1)') \\
& \hspace{1em} \cdot r^{-1}(a_1'' \otimes h_1 v_1) r^{-1}(u_1 \otimes h_1 v_1) r^{-1}(u_1 \otimes h_1 v_1) r(g_1 \otimes h_1 v_1) \\
& = r(a_1'' \otimes h_2 v_2) r^{-1}(u_1 \otimes h_2 v_2) r(u_1 \otimes h_2 v_2) r(u_1 \otimes h_2 v_2) r(g_2 \otimes h_2 v_2) \\
& r(a_1'' \otimes h_2 v_2) r^{-1}(g_2 \otimes h_2 v_2) r^{-1}(a_1'' \otimes h_2 v_2) r(g_2 \otimes h_2 v_2) \\
& = r(a_1' \otimes b_2' v_2) r^{-1}(g_2 \otimes b_2' v_2) r^{-1}(a_1'' \otimes h_2 v_2) r(g_2 \otimes h_2 v_2) \end{align*}
\]
\[ r \left( \frac{a_2 \otimes b_2}{g_2 \otimes h_2} \right), \]

where \( \ast \) and \( \blacklozenge \) indicate the terms that “cancel” each other.

xiii) Equation 1.11 holds for \( B[G^{-1}] \).

Indeed, we have

\[
\begin{align*}
&= r \left( \frac{a \otimes b}{g \otimes h} \cdot \frac{c}{k} \right) = r \left( \frac{a \otimes b I_h(c)}{g \otimes h k} \right) \\
&= r(a' \otimes (b I_h(c'))) r^{-1}(a'' \otimes h k) r^{-1}(g \otimes (b I_h(c'))) r(g \otimes h k) \\
&= r(a' \otimes b' I_h(c')) r^{-1}(a'' \otimes h k) r^{-1}(g \otimes b' I_h(c')) r(g \otimes h k) \\
&= r(a' \otimes b' I_h(c')) r(a'' \otimes I_h(c'')) r^{-1}(a''' \otimes h) r^{-1}(a'''' \otimes k) \\
&\quad \cdot r^{-1}(g \otimes h) r(a'' \otimes h) r(a'''' \otimes h) r(a''' \otimes h), \tag{2.18}
\end{align*}
\]

where \( \ast \) uses that \( I_h \) is a bialgebra homomorphism and that \( G(B) \) is commutative. Now considering the three terms indicated by \( \blacklozenge \), we get

\[
\begin{align*}
&= r^{-1}(g \otimes I_h(c')) r(a'' \otimes I_h(c'')) r^{-1}(a''' \otimes h) = \\
&= r^{-1}(g \otimes I_h(c')) r^{-1}(g \otimes h) r(g \otimes h) \\
&\quad \cdot r^{-1}(a'' \otimes h) r(a''' \otimes h) r(a'''' \otimes h) r(a''' \otimes h) \\
&= r^{-1}(g \otimes h I_h(c')) r(g \otimes h) r^{-1}(a'' \otimes h) \\
&\quad \cdot r(a''' \otimes h I_h(c'')) r^{-1}(a'''' \otimes h) \\
&= r^{-1}(g \otimes c' h) r(g \otimes h) r^{-1}(a'' \otimes h) \\
&\quad \cdot r(a''' \otimes c'' h) r^{-1}(a'''' \otimes h) \\
&= r^{-1}(g \otimes c') r^{-1}(g \otimes h) r(g \otimes h) r^{-1}(a'' \otimes h) \\
&\quad \cdot r(a''' \otimes c'' h) r^{-1}(a'''' \otimes h) \\
&= r^{-1}(g \otimes c') r^{-1}(g \otimes h) r(g \otimes h) r^{-1}(a'' \otimes h) \\
&\quad \cdot r(a''' \otimes c'' h) r^{-1}(a'''' \otimes h) \\
&\quad \cdot r(a''' \otimes c'' h) r(a'''' \otimes h) r^{-1}(a'''' \otimes h) \\
\end{align*}
\]

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\[ r^{-1}(g \otimes c') r^{-1}(a'' \otimes h) r(a''' \otimes c''). \]

Hence, plugging these terms back into 2.18, we get

\[
\begin{align*}
2.18 &= r(a' \otimes b'') r^{-1}(a'' \otimes h) r(a''' \otimes c'') r^{-1}(a'IV \otimes k) \\
&\quad \cdot r^{-1}(g \otimes c') r^{-1}(g \otimes b') r(g \otimes h) r(g \otimes k) \\
&= r(a' \otimes b'') r^{-1}(a'' \otimes h) r^{-1}(g \otimes b') r(g \otimes h) \\
&\quad \cdot r(a''' \otimes c'') r^{-1}(a'IV \otimes k) r^{-1}(g \otimes c') r(g \otimes k) \\
&= r \left( \frac{a'}{g} \otimes \frac{b'}{h} \right) r \left( \frac{a''}{g} \otimes \frac{c''}{k} \right).
\end{align*}
\]

xiv) We similarly have that \( r \left( \frac{a}{g} \cdot \frac{b \otimes c}{h} \right) = r \left( \frac{b}{h} \otimes \frac{c'}{k} \right) \) for \( x = g \in B \) and \( y = b \in G \), we get

\[
\begin{align*}
r^{-1}(g \otimes b') gb'' &= b' gr^{-1}(g \otimes b'').
\end{align*}
\]

Then applying \( \rho : B \rightarrow B[G^{-1}] : a \mapsto \frac{a}{1} \) on both sides of this equality, it becomes

\[
\begin{align*}
r^{-1}(g \otimes b') \frac{gb''}{1} &= \frac{b'g}{1} r^{-1}(g \otimes b'')
\end{align*}
\]

which can be rewritten

\[
\begin{align*}
r^{-1}(g \otimes b') \frac{g}{1} \cdot \frac{b''}{1} &= \frac{b'}{1} \cdot \frac{g}{1} r^{-1}(g \otimes b'').
\end{align*}
\]
Now, multiplying on both the right and the left by \( \frac{1}{g} \) and noting that
\[
 \begin{align*}
 r \left( \frac{1}{g} \otimes \frac{b'}{h} \right) & = r(1 \otimes (b'')r^{-1}(1 \otimes 1)r^{-1}(g \otimes (b')')r(g \otimes 1) \\
 & = \varepsilon((b'')r)(1 \otimes (b')')\varepsilon(g) \\
 & = r^{-1}(g \otimes b'),
\end{align*}
\]
we find
\[
 r \left( \frac{1}{g} \otimes \frac{b'}{h} \right) \frac{b''}{1} \cdot \frac{1}{g} = \frac{1}{g} \cdot \frac{b'}{1} r \left( \frac{1}{g} \otimes \frac{b''}{1} \right).
\] (2.19)

Similarly one gets for \( a \in B, h \in G \)
\[
 r \left( \frac{a''}{1} \otimes \frac{1}{h} \right) \frac{1}{h} = \frac{1}{h} \cdot \frac{a''}{1} r \left( \frac{a''}{1} \otimes \frac{1}{h} \right),
\] (2.20)

moreover using equation 1.9 one has
\[
 r \left( \frac{a'}{1} \cdot \frac{b''}{1} \right) \frac{b''}{1} \cdot \frac{a''}{1} = \frac{a'}{1} \cdot \frac{b''}{1} r \left( \frac{a''}{1} \otimes \frac{b''}{1} \right).
\] (2.21)

Then, we get
\[
 r \left( \frac{a'}{g} \otimes \frac{b'}{h} \right) \frac{b''}{1} \cdot \frac{a''}{g} =
\]
\[
 = r \left( \frac{a'}{1} \cdot \frac{1}{g} \otimes \frac{b'}{1} \cdot \frac{1}{h} \right) \frac{b''}{1} \cdot \frac{a''}{g}
\]
\[
 r \left( \frac{1}{g} \otimes \frac{b'}{1} \cdot \frac{1}{h} \right) r \left( \frac{a'}{1} \otimes \frac{b''}{1} \cdot \frac{1}{h} \right) \frac{b''}{1} \cdot \frac{a''}{g}
\]
\[
 r \left( \frac{1}{g} \otimes \frac{1}{h} \right) r \left( \frac{a'}{1} \otimes \frac{b''}{1} \right) r \left( \frac{a''}{1} \otimes \frac{1}{h} \right) \frac{b''}{1} \cdot \frac{1}{h} \cdot \frac{1}{g}
\]
\[
 r \left( \frac{1}{g} \otimes \frac{b'}{1} \right) r \left( \frac{1}{g} \otimes \frac{1}{h} \right) r \left( \frac{a'}{1} \otimes \frac{b''}{1} \right) r \left( \frac{a''}{1} \otimes \frac{1}{h} \right) \frac{b''}{1} \cdot \frac{1}{h} \cdot \frac{1}{g}
\] (2.20) (2.21)
\[
\begin{align*}
&= r \left( \frac{1}{g} \otimes \frac{b'}{1} \right) r \left( \frac{1}{g} \otimes \frac{1}{h} \right) r \left( \frac{a''}{1} \otimes \frac{b'''}{1} \right) r \left( \frac{a'''}{1} \otimes \frac{1}{h} \right) \frac{a'}{1} \cdot \frac{b''}{1} \cdot \frac{1}{g} \cdot \frac{1}{h} \\
&= r \left( \frac{1}{g} \otimes \frac{b''}{1} \right) r \left( \frac{1}{g} \otimes \frac{1}{h} \right) r \left( \frac{a''}{1} \otimes \frac{b'''}{1} \right) r \left( \frac{a'''}{1} \otimes \frac{1}{h} \right) \frac{a'}{1} \cdot \frac{1}{g} \cdot \frac{b'}{1} \cdot \frac{1}{h} \\
&= r \left( \frac{1}{g} \otimes \frac{b''}{1} \cdot \frac{1}{h} \right) r \left( \frac{a''}{1} \otimes \frac{b'''}{1} \cdot \frac{1}{h} \right) \frac{a'}{1} \cdot \frac{b'}{1} \cdot \frac{1}{g} \cdot \frac{1}{h} \\
&= r \left( \frac{a''}{1} \cdot \frac{b''}{1} \otimes \frac{1}{h} \right) \frac{a'}{1} \cdot \frac{b'}{1} \cdot \frac{1}{g} \cdot \frac{1}{h};
\end{align*}
\]

where we have used that \( \frac{1}{g} \) and \( \frac{1}{h} \) commute since \( g \) and \( h \) do and that \( \mathcal{J}_g(1) = 1 = \mathcal{J}_h(1) \).

\[ \square \]

**Proposition 2.2.14.** Let \((B, r)\) and \(G \subset G(B)\) be as above. Then \(\rho : B \to B[G^{-1}] : a \mapsto \frac{a}{1}\) is a CQT bialgebra homomorphism and \(B\) enjoys the following universal property: for any CQT bialgebra homomorphism \(\psi : B \to \bar{B}\) such that \(\psi(g)\) is invertible for any \(g \in G\), there exists a unique CQT bialgebra homomorphism \(\bar{\psi} : B[G^{-1}] \to \bar{B}\) such that

\[
\begin{array}{ccc}
B & \xrightarrow{\rho} & B[G^{-1}] \\
\downarrow{\psi} & & \downarrow{\bar{\psi}} \\
\bar{B} & & \bar{B}.
\end{array}
\]

commutes.

**Proof.** Using Lemma 1.3.3 it is straightforward to verify that \(\rho\) is a CQT bialgebra homomorphism. Then \(\bar{\psi}\) is defined by \(\bar{\psi}\left(\frac{a}{g}\right) = \psi(a)\psi(g)^{-1}\) and is obviously uniquely determined by \(\psi\). We then verify by direct computation that \(\bar{\psi}\) is bialgebra homomorphism. The fact that \(\bar{\psi}\) preserves the \(r\)-from directly follows from (1.55). \[ \square \]
Remark 2.2.15. Note that the homomorphism $\rho : B \to B[G^{-1}]$ satisfies the conditions of Definition 2.1.2. In other words, $B[G^{-1}]$ is the ring of fractions of $B$ with respect to $G$ (on which we have built a CQT bialgebra structure).

Corollary 2.2.16. When the localization exists, it is unique up to a unique isomorphism.

2.2.2 Example : $GL_q(2)$

Let us now apply this construction to the coquasi-triangular bialgebra $M_q(2)$ introduced in section 1.2.1.

Example 2.2.17. Let $G$ be the monoid generated by the group-like element $\det_q = t_{11}t_{22} - q^{-1}t_{12}t_{21} = t_{22}t_{11} - q t_{12}t_{21}$. The coquasi-triangular bialgebra $M_q(2)[G^{-1}]$ exists and its structure is given, for $a, b \in M_q(2)$ and $g, h \in G$, by

\[
\begin{align*}
\frac{a}{g} + \frac{b}{h} &= \frac{ah + bg}{gh}, & \text{the zero is : } & \frac{0}{1}, \\
\frac{a}{g} \cdot \frac{b}{h} &= \frac{ab}{gh}, & \text{the unit is : } & \frac{1}{1}, \\
\Delta \left( \frac{a}{g} \right) &= \frac{a'}{g} \otimes \frac{a''}{g}, & \varepsilon \left( \frac{a}{g} \right) &= \varepsilon(a'), \\
r \left( \frac{a}{g} \otimes \frac{b}{h} \right) &= r(a' \otimes b'')r^{-1}(a'' \otimes h)r^{-1}(g \otimes b')r(g \otimes h).
\end{align*}
\]

Remark 2.2.18. i) Note that the multiplication in $GL_q(2)$ as defined above follows from Remark 2.1.6 iii) and the fact that the monoid $G$ is in the centre of $M_q(2)$ since $\det_q$ is central. Indeed, using the centrality of $g$ we have $gb = bg$ and thus using (2.4) in Proposition 2.1.5 we find that

\[
\frac{a}{g} \cdot \frac{b}{h} = \frac{ab}{gh}.
\]
ii) Recall from Definition 1.2.7 that $GL_q(2) = M_q(2)[t]/(\det_q t - 1)$. It is this latter description that we are going to use in the proof of the next proposition.

**Proposition 2.2.19.** The coquasi-triangular bialgebras $GL_q(2)$ and $M_q(2)[G^{-1}]$ are isomorphic.

**Remark 2.2.20.** Note that one could prove that $GL_q(2)$ is isomorphic to the localization $M_q(2)[G^{-1}]$ by verifying that $GL_q(2)$ satisfies the universal property of the localization. Nevertheless, as our goal is here to illustrate the construction of the localization presented in the previous section, we are going to use explicitly construct an isomorphism between $GL_q(2)$ and $M_q(2)[G^{-1}]$.

**Proof of Prop. 2.2.19.** In order to prove this, we are going to define two homomorphisms inverse of each other. First, for $a \in M_q(2)$, let us define

$$
\varphi : GL_q(2) \longrightarrow M_q(2)[G^{-1}]
$$

$\begin{align*}
a & \mapsto \frac{a}{1} \\
t & \mapsto \frac{1}{\det_q t}.
\end{align*}
$

Then note that since the monoid $G$ is generated by $\det_q$, any $g \in G$ is of the form $\alpha \cdot \det_q^n$ with $\alpha \in k$ and $n \in \mathbb{N}_0$. We can thus define

$$
\psi : M_q(2)[G^{-1}] \longrightarrow GL_q(2)
$$

$\begin{align*}
a & \longmapsto a \\
\frac{1}{\det_q} & \longmapsto t^n.
\end{align*}
$

We now prove that $\varphi$ and $\psi$ are well-defined bialgebra homomorphisms and inverse of each other.
i) $\varphi$ is well-defined.
   Indeed,
   \[
   \varphi(\det_q t - 1) = \varphi(\det_q) \varphi(t) - \varphi(1) = \frac{\det_q}{1} \frac{1}{\det_q} - \frac{1}{1} = \frac{\det_q - 1}{\det_q - 1} = \frac{1}{1} - \frac{1}{1} = 0.
   \]

ii) $\varphi$ is linear.
   Let $\alpha, \beta \in k$ and $a, b \in M_q(2)$. Then
   \[
   \varphi(\alpha a + \beta b) = \frac{\alpha a + \beta b}{1} = \frac{\alpha a + \beta b}{1} \cdot \frac{1}{1} = \alpha a \frac{1}{1} + \beta b \frac{1}{1} = \alpha \varphi(a) + \beta \varphi(b).
   \]

iii) $\varphi$ is an algebra homomorphism.
   First, note that $\varphi(1) = \frac{1}{1}$. Moreover
   \[
   \varphi(a) \varphi(b) = \frac{a}{1} \cdot \frac{b}{1} = \frac{ab}{1} = \varphi(ab)
   \]
   and similarly $\varphi(a) \varphi(t^n) = \varphi(at^n)$.

iv) $\varphi$ is a coalgebra homomorphism.
   First, we have $\varepsilon(\varphi(a)) = \varepsilon \left(\frac{a}{1}\right) = \varepsilon(a)$. Furthermore
   \[
   \Delta(\varphi(a)) = \Delta \left(\frac{a}{1}\right) = \frac{a^\prime}{1} \otimes \frac{a^\prime}{1} = (\varphi \otimes \varphi) \circ \Delta(a).
   \]

v) $\varphi$ preserves the $r$-form.
Indeed,

\[ r(\varphi(at^m) \otimes \varphi(bt^n)) = r \left( \frac{a}{\det_q^m} \otimes \frac{b}{\det_q^n} \right) \]

\[ = r(a' \otimes b'')r^{-1}(a'' \otimes \det_q^n)r^{-1}(\det_q^m \otimes b')r(\det_q^m \otimes \det_q^n) \]

\[ = r(a' \otimes b'')r(a'' \otimes t^n)r(t^m \otimes b')r(t^m \otimes t^n) \]

\[ = r(at^m \otimes bt^n), \]

where the last equality follows from (1.10) and (1.11). 

vi) \( \psi \) is well-defined. 
Let \( \frac{a}{\det_q^m} \sim \frac{b}{\det_q^n} \), by Lemma 2.2.9 there exist \( u_1, u_2 \in G \) such that 

\[ au_1 = bu_2 \quad \text{and} \quad \det_q^m u_1 = \det_q^n u_2. \]

Since \( u_1, u_2 \in G \) we know that there exist \( r, s \in \mathbb{N}_0 \) such that \( u_1 = \det_q^r \) and \( u_2 = \det_q^s \). Using the above equalities we get 

\[ a\det_q^r = b\det_q^s \quad \text{and} \quad \det_q^m \det_q^r = \det_q^n \det_q^s. \]

Since \( \{ t_{i1}^{j1} t_{12}^{j2} t_{21}^{j3} t_{22}^{j4} \}_{i,j,k,l \geq 0} \) forms a basis of \( M_4(2) \) by [Kas95, Thm. IV.4.1], it follows that \( m + r = n + s \). We then have 

\[ \psi \left( \frac{b}{\det_q^n} \right) = bt^n = b\det_q^s t^{m+s} = a\det_q^r t^{n+s} = at^{n+s-r} = at^m = \psi \left( \frac{a}{\det_q^m} \right). \]

vii) \( \psi \) is linear. 
This immediately follows from the construction of \( \psi \). 

viii) \( \psi \) is an algebra homomorphism.
Indeed, we have
\[
\psi \left( \frac{a}{\det_q^m} \cdot \frac{b}{\det_q^n} \right) = \psi \left( \frac{ab}{\det_q^{m+n}} \right) = abt^{m+n} = at^m b t^n \\
= \psi \left( \frac{a}{\det_q^m} \right) \psi \left( \frac{b}{\det_q^n} \right).
\]

Moreover, we have \( \psi \left( \frac{1}{1} \right) = 1 \).

ix) \( \psi \) is a coalgebra homomorphism.

Using that \( t \) is group-like, we have
\[
(\psi \otimes \psi) \circ \Delta \left( \frac{a}{\det_q^m} \right) = (\psi \otimes \psi) \left( \frac{a'}{\det_q^m} \otimes \frac{a''}{\det_q^m} \right) \\
= a't^m \otimes a''t^m = \Delta(a) \Delta(t^m) = \Delta(at^m) = \Delta \circ \psi \left( \frac{a}{\det_q^m} \right).
\]

Using again that \( t \) is group-like, we find for the counit that
\[
\varepsilon \circ \psi \left( \frac{a}{\det_q^m} \right) = \varepsilon(at^m) = \varepsilon(a) \varepsilon(t^m) = \varepsilon(a) = \varepsilon \left( \frac{a}{\det_q^m} \right).
\]

x) \( \psi \) preserves the \( r \)-form.

Similar to vi).

xi) \( \psi \circ \varphi = \text{id}_{GL_q(2)} \).

Indeed, we have
\[
\psi(\varphi(a)) = \psi \left( \frac{a}{1} \right) = a \quad \text{and} \quad \psi(\varphi(t^m)) = \psi \left( \frac{1}{\det_q^m} \right) = t^m.
\]

xii) \( \varphi \circ \psi = \text{id}_{M_q(2)[G^{-1}]} \).

We have
\[
\varphi \circ \psi \left( \frac{a}{\det_q^m} \right) = \varphi(at^m) = \frac{a}{1} \frac{1}{\det_q^m} = \frac{a}{\det_q^m}.
\]
Hence, we have thus just proved that \( \varphi \) and \( \psi \) are bialgebra isomorphisms.

\( \square \)

**Remark 2.2.21.** As noted in Remark 2.2.18 i) the centrality of the monoid \( G \) implies that we don’t have to use the morphism \( \mathcal{I}_{\text{det}_q} \) when defining the multiplication in \( M_q(2)[G^{-1}] \). Note that this fact alone doesn’t mean that \( \mathcal{I}_g \) is the identity.

In the present case we do have \( \mathcal{I}_{\text{det}_q} = \text{id}_{M_q(2)} \) and this follows from both centrality and regularity of the elements of \( G \). Indeed, then for any \( g \in G \) and \( a \in M_q(2) \) we have by (2.5) that \( g\mathcal{I}_g(a) = ag \), and thus \( g\mathcal{I}_g(a) - ag = 0 \). Then, by centrality of \( g \) we get \( g(\mathcal{I}_g(a) - a) = 0 \). Since \( g \) is non-zero and not a zero-divisor (since \( M_q(2) \) itself has no zero-divisor) this forces \( \mathcal{I}_g(a) = a \).

In the next example we shall take a monoid \( G \) that is regular but not central and see that \( \mathcal{I}_g \) is in general not the identity.

### 2.2.3 Example : Sweedler’s 4-dimensional Hopf Algebra

This Hopf algebra was introduced by Sweedler in his seminal book [Swe69].

**Theorem 2.2.22.** Let \( H_4 \) be the algebra generated by two elements \( g \) and \( y \) subject to the relations

\[
g^2 = 1, \quad y^2 = 0 \quad \text{and} \quad gy = -yg.
\]

Then \( H_4 \) has a Hopf algebra structure if we define the comultiplication \( \Delta \) by

\[
\Delta(g) = g \otimes g \quad \text{and} \quad \Delta(y) = y \otimes 1 + g \otimes y,
\]

the counit \( \varepsilon \) by

\[
\varepsilon(g) = 1 \quad \text{and} \quad \varepsilon(y) = 0,
\]
and the antipode \( S \) by

\[
S(g) = g^{-1} \quad \text{and} \quad S(y) = yg.
\]

Moreover, note that \( \{1, g, y, yg\} \) is a basis of the underlying vector space.

As noted for example by Majid, it turns out that this Hopf algebra has a coquasi-triangular structure.

**Proposition 2.2.23** ([Maj95, p. 51]). Sweedler’s 4-dimensional Hopf algebra \( H_4 \) is coquasi-triangular; its \( r \)-form is given by

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & \alpha & \alpha \\
0 & 0 & -\alpha & \alpha
\end{pmatrix},
\]

where \( \alpha \in k \) is arbitrary.

**Remark 2.2.24.** The only group-likes in \( H_4 \) being 1 and \( g \), the monoid of group-like elements is \( G = \{1, g\} \). Note that since \( gy = -yg \), the monoid \( G \) is not in the centre of \( H_4 \).

The elements of \( G \) are nonetheless regular. Indeed, let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in k \) and suppose that

\[
g(\alpha_1 1 + \alpha_2 g + \alpha_3 y + \alpha_4 yg) = g(\beta_1 1 + \beta_2 g + \beta_3 y + \beta_4 yg),
\]

then

\[
\alpha_2 1 + \alpha_1 g - \alpha_4 y - \alpha_3 yg = \beta_2 1 + \beta_1 g - \beta_4 y - \beta_3 yg
\]

and thus

\[
\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2, \quad \alpha_3 = \beta_3, \quad \alpha_4 = \beta_4,
\]

since \( \{1, g, y, yg\} \) is a basis of the underlying vector space. Hence we have just proved that \( ga = gb \) implies \( a = b \) for any \( a, b \in H_4 \) and thus \( g \) is left-
regular. We similarly show that $g$ is right-regular and therefore regular.

To sum up, our monoid is in this case regular but not central.

**Remark 2.2.25.** In order to construct the localization $H_4[G^{-1}]$ we need to compute $\mathcal{I}_1$ and $\mathcal{I}_g$. The morphism $\mathcal{I}_1$ is the identity by Lemma 1.3.3. Then using equation (1.12) to compute $r^{-1}$ and the definition of $\mathcal{I}_g$ (Def. 2.2.1) we get

$\mathcal{I}_g(1) = 1, \quad \mathcal{I}_g(g) = g, \quad \mathcal{I}_g(y) = -y, \quad \mathcal{I}_g(yg) = -yg.$

We thus see that in this case $\mathcal{I}_g \neq \text{id}_{H_4}$.

Finally note that since the elements of $G$ are already invertible in $H_4$, we get that $H_4 \cong H_4[G^{-1}]$. The main purpose of this example is not so much to compute the localization $H_4[G^{-1}]$ as to show that $\mathcal{I}_g$ is not always the identity.

### 2.2.4 Example : Monoid Algebra

In this example we are going to construct the localization of the monoid algebra $k[M]$ at the monoid generated by all basis vectors. One special feature of this example is that the bialgebra $k[M]$ has non-regular group-like elements; this will be established in Remark 2.2.29.

Let us first note the following property of monoids.

**Lemma 2.2.26.** Let $(M, \cdot, 1)$ be a finite monoid. Let $g \in M$ and $\varphi : M \to M : a \mapsto ga$ the multiplication on the left by $g$. Then $g$ is invertible if and only if $\varphi$ is a bijection of sets.

**Proof.** First assume that $g$ is invertible. Then the map $\psi : M \to M : a \mapsto g^{-1}a$ is the inverse of $\varphi$ and thus $\varphi$ is bijective.

Now assume that $\varphi$ is bijective. Then $\varphi^{-1}(1)$ exists and is unique. It turns
out that this element is the inverse of \( g \). Indeed, \( g\varphi^{-1}(1) = \varphi(\varphi^{-1}(1)) = 1 \).

Moreover
\[
\varphi(\varphi^{-1}(1)g) = g\varphi^{-1}(1)g = 1g = g = \varphi(1)
\]
which implies, by injectivity of \( \varphi \), that \( \varphi^{-1}(1)g = 1 \).

\[ \square \]

**Corollary 2.2.27.** Let \( M \) be a finite monoid that is not a group. Then there exist \( g, h, k \in M \) such that \( h \neq k \) and \( gh = gk \).

**Proof.** Indeed, \( M \) being not a group, there exists a non-invertible element \( g \in M \). Then, by the preceding lemma, the left multiplication by \( g \) is not bijective. The monoid \( M \) being finite, this amounts to saying that it is not injective, i.e. there exist \( h, k \in M \) such that \( h \neq k \) and \( gh = gk \).

\[ \square \]

**Proposition 2.2.28.** Let \((M, \cdot, 1)\) be a finite monoid that is not a group, then the monoid algebra \( k[M] \) has a bialgebra structure. Denoting by \( e_1, e_a, e_b \) the basis vectors of \( k[M] \) associated with \( 1, a, b \in M \), the bialgebra structure is given by

\[
\mu(e_a \otimes e_b) = e_{ab}, \quad \Delta(e_a) = e_a \otimes e_a, \\
\eta(1) = e_1, \quad \varepsilon(e_a) = 1.
\]

**Proof.** This checked by direct computations.

\[ \square \]

**Remark 2.2.29.** Let \( M \) be a monoid that is not a group. Then, in contrast to the previous examples, the bialgebra \( k[M] \) has non-regular group-likes. Indeed, let \( g \) be a non-invertible element in \( M \). By Corollary 2.2.27 there exist \( h \) and \( k \) different from each other and such that \( gh = gk \). Therefore \( e_g(e_h - e_k) = e_{gh} - e_{gk} = 0 \) even though both \( e_g \) and \( e_h - e_k \) are non-zero. The element \( e_g \) is thus a group-like and a zero-divisor.

We now want to construct the localization of \( k[M] \). In order to do so, we need a coquasi-triangular structure.
Let us take the monoid $M$ to be commutative, then so is $k[M]$, i.e. $k[M]$ is CQT with a trivial $r$-form. Then taking $G = \{e_a | a \in M\}$ to be the monoid of group-like elements generated by the $e_a$’s, we can then construct the localization $k[M][G^{-1}]$ using Theorem 2.2.7. Its bialgebra structure is given, for $a, b, c, d \in M$, by

$$
\frac{e_a + e_c}{e_b} = \frac{e_a e_d + e_c e_b}{e_b e_d}, \quad \text{the zero is: } \frac{0}{e_1},
\frac{e_a}{e_b} \cdot \frac{e_c}{e_d} = \frac{e_a e_c}{e_b e_d}, \quad \text{the unit is: } \frac{e_1}{e_1},
\Delta \left( \frac{e_a}{e_b} \right) = \frac{e_a}{e_b} \otimes \frac{e_a}{e_b}, \quad \varepsilon \left( \frac{e_a}{e_b} \right) = \varepsilon(e_a).
$$

The natural question that arises here is whether this localization is isomorphic to another object we know. The answer to this question is in the next

**Proposition 2.2.30.** Let $M$ be as above and let $\overline{M}$ be the group defined by generators and relations as follows. The set of generators is $\{a | a \in M\} \cup \{\overline{a} | a \in M\}$ and the relations $\{a\overline{a} = 1 = \overline{a} a | a \in M\}$. Then the bialgebras $k[M] \cong k[M][G^{-1}]$ are isomorphic.

**Proof.** Let $\varphi : k[M][G^{-1}] \to k[\overline{M}]$ be defined by $\varphi \left( \frac{e_a}{e_b} \right) = e_a e_b$. We are going to show that this is an isomorphism.

i) $\varphi$ is well-defined.

Suppose $\frac{e_a}{e_b} \sim \frac{e_c}{e_d}$, i.e. there exist $e_u, e_v \in G$ such that

$$e_a e_u = e_c e_v \quad \text{and} \quad e_b e_u = e_d e_v.$$
Then
\[ \varphi \left( \frac{e_a}{e_b} \right) = e_a e_b = e_a e_u e_b = e_a e_b = e_b e_a = e_b e_d = e_c e_d = \varphi \left( \frac{e_c}{e_d} \right) \]

ii) \( \varphi \) is linear by construction.

iii) \( \varphi \) is an algebra homomorphism.

First note that \( \varphi \left( \frac{e_1}{e_1} \right) = e_1 e_1 = e_1 \). Moreover, we have
\[
\varphi \left( \frac{e_a}{e_b} \cdot \frac{e_c}{e_d} \right) = \varphi \left( \frac{e_a e_c}{e_b e_d} \right) = e_a e_c e_d e_b = e_a e_b e_c e_d = \varphi \left( \frac{e_a}{e_b} \right) \varphi \left( \frac{e_c}{e_d} \right),
\]
where * uses the commutativity of \( k[M] \).

iv) \( \varphi \) is a coalgebra homomorphism.

Indeed,
\[
\varepsilon \circ \varphi \left( \frac{e_a}{e_b} \right) = \varepsilon(e_a e_b) = \varepsilon(e_a) \varepsilon(e_b) = \varepsilon(e_a) = \varepsilon \left( \frac{e_a}{e_b} \right),
\]
where * uses that \( \varepsilon(e_b) = 1 \) (since \( \varepsilon(e_b) = 1 \)). Moreover
\[
(\varphi \otimes \varphi) \circ \Delta \left( \frac{e_a}{e_b} \right) = \varphi \left( \frac{e_a}{e_b} \right) \otimes \varphi \left( \frac{e_a}{e_b} \right) = e_a e_b \otimes e_a e_b = \Delta(e_a) \Delta(e_b) = \Delta(e_a e_b) = \Delta \left( \varphi \left( \frac{e_a}{e_b} \right) \right).
\]

v) \( \varphi \) is an isomorphism.

First note that \( \varphi \left( \frac{e_a}{e_b} \right) = e_a \) and \( \varphi \left( \frac{e_a}{e_b} \right) = e_a \) for any \( a \in M \). For the injectivity, suppose that \( \varphi \left( \frac{e_a}{e_b} \right) = \varphi \left( \frac{e_a}{e_d} \right) \), i.e.
\(e_a e_b = e_d e_c\). Then
\[e_a e_b = e_d e_c\quad\text{and}\quad e_b e_b = e_1 = e_d e_d,
\]
which is equivalent to saying that \(\frac{e_a}{e_b} \sim \frac{e_c}{e_d}\); thus \(\varphi\) is injective.

\[\square\]

In order to make this section more concrete, let us take a finite commutative monoid and compute its localization.

**Example 2.2.31.** Consider the commutative monoid \(M = (\mathbb{Z}/4\mathbb{Z}, \cdot, 1)\) obtained by taking the factor ring \((\mathbb{Z}/4\mathbb{Z}, +, \cdot)\) and “forgetting” its additive structure. In other words we have, by abuse of notation, \(\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}\). Let us denote the basis vectors of the complex monoid algebra \(\mathbb{C}[M]\) by \(\{e_0, e_1, e_2, e_3\}\) and consider the monoid of group-like elements \(G = \{e_0, e_1\}\). The monoid \(G\) contains zero-divisors since \(e_0(e_1 - e_2) = e_0 - e_0 = 0\).

Using Theorem 2.2.7 we can compute the localization \(\mathbb{C}[M][G^{-1}] = \mathbb{C}[M] \times G/\sim\). Let
\[x = \sum_{i=0}^{3} x_i e_i\quad\text{and}\quad y = \sum_{i=0}^{3} y_i e_i\]
be elements of \(\mathbb{C}[M]\) with \(x_i, y_i \in \mathbb{C}\) for all \(i\). Then \(\frac{x}{g} \sim \frac{y}{h}\) with \(g, h \in G\) if and only if there exist \(u, v \in G\) such that
\[xu = yv\quad\text{and}\quad gu = hv.
\]
Since in our case \(G\) has only two elements, we can simply test the different possibilities for \(g, h, u\) and \(v\). We then obtain that \(\frac{x}{g} \sim \frac{y}{h}\) if and only if
\[x_0 + x_1 + x_2 + x_3 = y_0 + y_1 + y_2 + y_3.
\]
Therefore, as a complex vector space, \(\mathbb{C}[M][G^{-1}] \cong \mathbb{C}\).
The next remark will give us a tool to produce new examples from known ones.

**Remark 2.2.32.**

i) Let \( g_1 \in B_1 \) and \( g_2 \in B_2 \) be group-like elements in the bialgebras \( B_1 \) and \( B_2 \), respectively. Then \( g_1 \otimes g_2 \) is a group-like element of \( H_1 \otimes H_2 \).

ii) If \((B_1, r_1), (B_2, r_2)\) are two CQT bialgebras then the bialgebra \( B_1 \otimes B_2 \) has an \( r \)-form given by

\[
r : (B_1 \otimes B_2) \otimes (B_1 \otimes B_2) \to k : (x_1 \otimes x_2) \otimes (y_1 \otimes y_2) \mapsto r_1(x_1 \otimes y_1)r_2(x_2 \otimes y_2).
\]

iii) Let \((B_1, r_1), (B_2, r_2)\) be CQT bialgebras and \( G_1 \subset B_1, G_2 \subset B_2 \) be monoids of group-like elements. Let moreover \( B = B_1 \otimes B_2 \). Then \( G = \{g_1 \otimes g_2 \mid g_1 \in G_1, g_2 \in G_2\} \subset B \) is a monoid of group-like elements. Using the universal property of the localization, we find that

\[
B[G^{-1}] \cong B_1[G_1^{-1}] \otimes B_2[G_2^{-1}]
\]

as bialgebras.

Using the preceding remark, we can now build new examples.

**Example 2.2.33.** Let \( H_4 \) be Sweedler’s Hopf algebra over \( \mathbb{C} \) (cf. Section 2.2.3) and \( G_1 = \{1, g\} \subset H_4 \) be a monoid of group-like elements. Let moreover \( \mathbb{C}[M] \) and \( G_2 = \{e_0, e_1\} \) be as in Example 2.2.31.

Then the bialgebra \( B = H_4 \otimes \mathbb{C}[M] \) is CQT and \( G = G_1 \otimes G_2 \) is monoid of group-like elements; the localization is given by \( B[G^{-1}] \cong H_4 \otimes \mathbb{C} \cong H_4 \).

Note that in this example \( G \) contains elements such as \( g \otimes e_0 \) that are not regular nor central.
2.3 Localization in a Coquasi-Triangular WBA

In this section we shall see how we can generalize to the weak case the localization in a CQT bialgebra presented in the previous section.

We want to use again the two conditions S1 and S2 of Proposition 2.1.5 which would give us the ring structure for the localization. In the bialgebra case, it is the morphism \( I_g \) introduced in Definition 2.2.1 that enabled us to use this result. Therefore, our first task here is to check whether the “essential” properties of \( I_g \) remain true in a CQT WBA; note in particular that the definition of group-like is now more general (cf. Definition 1.1.16).

2.3.1 Construction of the Localization

We are going to construct the localization in a similar fashion to Section 2.2.

Proposition 2.3.1. Let \((H, r)\) be a CQT WBA and \(g \in G(B)\) a group-like element. Then the morphism \( \mathcal{J}_g : H \to H \) introduced in Definition 2.2.1 and defined by

\[
\mathcal{J}_g(a) = r(a' \otimes g)a''r^{-1}(a''' \otimes g)
\]

is an automorphism of CQT WBA. Its inverse is given by

\[
\mathcal{J}_g^{-1}(a) = r^{-1}(a \otimes g)a''r(a''' \otimes g).
\]

Proof. In this proof we indicate by * the equalities where we use that \( g \) is group-like. First note that \( \mathcal{J}_g : H \to H \) is linear by construction. The fact that \( \mathcal{J}_g^{-1} \) is the inverse of \( \mathcal{J}_g \) as a map of sets results from (1.51), (1.52) and coassociativity. Let us now prove that \( \mathcal{J}_g \) is a CQT WBA homomorphism.

i) \( \mathcal{J}_g \) preserves the unit.

We have

\[
\mathcal{J}_g(1) = r(1' \otimes g)1''r^{-1}(1''' \otimes g)
\]

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\[
\begin{align*}
&= r(\varepsilon_s(1') \otimes g)1''r^{-1}(\varepsilon_l(1'') \otimes g) \\
&\equiv \varepsilon(g\varepsilon_s(1'))1''\varepsilon(g1''') \\
&= \varepsilon(\varepsilon_s(g)\varepsilon_s(1'))1''\varepsilon(\varepsilon_s(g)1''') \\
&\equiv \varepsilon(1\varepsilon_s(1_3'))1''\varepsilon(1_31''') \\
&= \varepsilon(\varepsilon_s(1'))1''\varepsilon(1''') \\
&\equiv 1,
\end{align*}
\]

where \( \diamond \) follows from (1.45) and (1.48).

ii) \( \mathcal{I}_g \) preserves the product.

Let \( a, b \in H \), we have

\[
\begin{align*}
\mathcal{I}_g(ab) &= r((ab) \otimes g)(ab)''r^{-1}((ab)''' \otimes g) \\
&= r(a'b' \otimes g)a''b''r^{-1}(a'''b''' \otimes g) \\
&\overset{1.10}{=} r(a'b' \otimes g)a''b''r^{-1}(a''' \otimes g')r^{-1}(b'' \otimes g'') \\
&\overset{1.10}{=} r(b' \otimes g')r(a' \otimes g')a''b''r^{-1}(a''' \otimes g')r^{-1}(b'' \otimes g'') \\
&\overset{\diamond}{=} r(b' \otimes g)r(a' \otimes g)a''b''r^{-1}(a''' \otimes g')r^{-1}(b'' \otimes g')r^{-1}(a' \otimes g)b''r^{-1}(b'' \otimes g') \\
&= \mathcal{I}_g(a)\mathcal{I}_g(b),
\end{align*}
\]

where we have used (1.53) and (1.54) for equality \( \diamond \).

iii) \( \mathcal{I}_g \) preserves the counit.

Let \( a \in H \), then

\[
\begin{align*}
\varepsilon(\mathcal{I}_g(a)) &= r(a' \otimes g)\varepsilon(a'')r^{-1}(a''' \otimes g) \\
&= r(a' \otimes g)r^{-1}(a'' \otimes g) \\
&\overset{1.32}{=} \varepsilon(a).
\end{align*}
\]
iv) \( \mathcal{J}_g \) preserves the coproduct.

For \( a \in H \) we have

\[
\Delta(\mathcal{J}_g(a)) = r(a' \otimes g) a'' \otimes a''' r^{-1}(a^{IV} \otimes g) \\
= r(a' \otimes g) \varepsilon(a'') \otimes a^{IV} r^{-1}(a^{V} \otimes g) \\
= r(a' \otimes g) a'' r^{-1}(a'' \otimes g) \otimes r(a^{IV} \otimes g) a^{V} r^{-1}(a^{VI} \otimes g) \\
= (\mathcal{J}_g \otimes \mathcal{J}_g) \Delta(a).
\]

v) \( \mathcal{J}_g \) preserves the \( r \)-form.

For \( a, b \in H \) we have

\[
r \circ (\mathcal{J}_g(a) \otimes \mathcal{J}_g(b)) = r(r(a' \otimes g) a'' r^{-1}(a'' \otimes g) \otimes r(b' \otimes g) b'' r^{-1}(b'' \otimes g)) \\
= r(a' \otimes g) r(a'' \otimes b'') r^{-1}(a'' \otimes g) r(b' \otimes g) r^{-1}(b'' \otimes g) \\
= r(a' \otimes g \mathcal{J}_g(b)) r^{-1}(a'' \otimes g) \\
= r(a' \otimes bg) r^{-1}(a'' \otimes g) \\
= r(a' \otimes b) r(a'' \otimes g) r^{-1}(a'' \otimes g) \\
= r(a' \otimes b) \varepsilon(a'') \\
= r(a \otimes b).
\]

\[\square\]

**Proposition 2.3.2.** Let \( (H, r) \) be a CQT WBA and \( G \subset G(H) \) a monoid of group-like elements. Then

\[ \mathcal{J} : G \rightarrow \text{Aut}(H) : g \mapsto \mathcal{J}_g \]

is an anti-morphism of monoids.
Proof. Let \( a \in H \). Let us first check that \( I \) preserves the unit. Indeed

\[
I_1(a) = r(a' \otimes 1)a''r^{-1}(a''' \otimes 1) \varepsilon(a') \varepsilon(a'') = \text{id}(a).
\]

Let us now verify that \( I_{gh} = I_h \circ I_g \), i.e. that \( I \) “anti-preserves” multiplication. Let \( g, h \in G(H) \), then

\[
I_h(I_g(a)) = r(a' \otimes g)r(a'' \otimes h)a'''r^{-1}(a''\otimes h)r^{-1}(a' \otimes g)
\]
\[
= r(a' \otimes gh)a''r^{-1}(a''' \otimes gh)
\]
\[
= I_{gh}(a).
\]

The next proposition will enable us to prove that conditions S1 and S2 of Prop. 2.1.5 are satisfied by \((H, r)\) and \( G \subset G(H) \) a monoid of group-like elements.

**Proposition 2.3.3.** Let \((H, r)\) be a CQT WBA, \( a \in H \) and \( g \in G(H) \) a group-like element. Then

\[ ag = gI_g(a). \] (2.26)

*Proof. We have*

\[
gI_g(a) = r(a' \otimes g)ga''r^{-1}(a''' \otimes g)
\]
\[
= r(a' \otimes g)g1''a'''r^{-1}(a'' \otimes g)
\]
\[
= r(a' \otimes g)\varepsilon(1' a'')g1''a'''r^{-1}(a'' \otimes g)
\]
\[
= r(a' \otimes g)\varepsilon(1' a'')g1''a'''r^{-1}(a'' \otimes g)
\]
\[
= r(a' \otimes g)\varepsilon(1')g1''a'''r^{-1}(a'' \otimes g)
\]
\[
= r(a' \otimes g)g1''a'''r^{-1}(a'' \otimes g)
\]
\[
= r(a' \otimes g)g''a''r^{-1}(a''' \otimes g)
\]
\[
= a'g'r(a'' \otimes g')r^{-1}(a''' \otimes g)
\]
\[
= a'g'r(a'' \otimes g'')r^{-1}(a''' \otimes g)
\]

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\[ a'1' gr(a'' \otimes 1'') r^{-1}(a''' \otimes g) \]
\[ = a'1' gr(a'' \otimes 1'') r(a''' \otimes g) r^{-1}(a''') \]
\[ = a'1' gr(a'' \otimes 1'') r(a''' \otimes g) r^{-1}(a''') \]
\[ = (a1)' g(1') r(a''' \otimes g) r^{-1}(a''') \]
\[ = 1' g(a') \]

where \(*\) holds because \(g\) is group-like.

**Lemma 2.3.4.** Let \((H, r)\) be a CQT WBA and \(G\) a monoid of group-like elements such that \(J_g(G) \subset G\) for all \(g \in G\). Then conditions S1 and S2 of Prop. 2.1.5 hold.

**Proof.** S1: for \(a \in H, g \in G\), we have \(g J_g(a) = a g\) by (2.26).

S2: let \(a \in H, g \in G\) be such that \(g a = 0\). Then

\[ 0 = J_g(g a) = J_g(J_g^{-1}(a) g) = J_g(J_g^{-1}(a) J_g(g)) = a J_g(g) \]

Thanks to these results we can now use Proposition 2.1.5 and we get the following

**Theorem 2.3.5.** Let \((H, r)\) be a CQT WBA and \(G\) be a monoid of group-like elements such that \(J_g(G) \subset G\) for all \(g \in G\). Define the equivalence relation \(\sim\) on \(H \times G\) as follows: \((a_1, g_1) \sim (a_2, g_2)\) if there exist \(u_1, u_2 \in H\) such that

\[ a_1 u_1 = a_2 u_2 \in H \quad \text{and} \quad g_1 u_1 = g_2 u_2 \in G. \quad (2.27) \]

Let us denote by \(a/g\) the equivalence class containing \((a, g)\). Then \(H[G^{-1}] := (H \times G)/\sim\) has a weak bialgebra structure. Its unital algebra structure is
given by
\[
\frac{a}{g} + \frac{b}{h} = \frac{a\mathcal{J}_g(h) + bg}{hg},
\]

\[
\frac{a}{g} \cdot \frac{b}{h} = \frac{a\mathcal{J}_g(b)}{hg},
\]

where $\frac{1}{1}$ is the unit and scalar multiplication is given by $\alpha \cdot \frac{a}{g} = \frac{\alpha a}{g}$ for $\alpha \in k$ and its counital coalgebra structure is given by

\[
\Delta \left( \frac{a}{g} \right) = \frac{a'}{g} \otimes \frac{a''}{g} \quad \text{and} \quad \varepsilon \left( \frac{a}{g} \right) = \varepsilon_H(a).
\]

Before proving this theorem let us note that Lemma 2.2.8, which proves that $\sim$ is an equivalence relation, still holds in the weak case. Concerning Lemma 2.2.9 which states that $u_1$ and $u_2$ of (2.27) can be taken in $G$, the result is still valid (as we shall see in the next lemma) even though we cannot simply use the same proof. Indeed, in the weak case we don’t know whether the morphism $\mathcal{J}_g$ is the identity on group-like elements or not.

**Lemma 2.3.6.** Let $(H,r)$ and $G$ be as in Thm. 2.3.5. Then $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$ if and only if there exist $v_1, v_2 \in G$ such that $a_1v_1 = a_2v_2$ and $g_1v_1 = g_2v_2 \in G$.

**Proof.** Obviously, if there exist $v_1, v_2 \in G$ such that $a_1v_1 = a_2v_2$ and $g_1v_1 = g_2v_2 \in G$ then $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$.

Now suppose that $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$; we have to show that we can find $v_1, v_2$ in $G$ (and not simply in $H$) such that

\[
a_1v_1 = a_2v_2 \quad \text{and} \quad g_1v_1 = g_2v_2 \in G.
\]

By definition there exist $u_1, u_2 \in H$ such that

\[
a_1u_1 = a_2u_2 \quad \text{and} \quad g_1u_1 = g_2u_2 \in G.
\]

Then, $g_1\mathcal{J}_g(g_1u_1) \sim g_1u_1g_1$ and hence by S2’ there exist $w_1 \in G$ such that
\( I g_1( g_1 u_1) w_1 = u_1 g_1 w_1 \), which belongs to \( G \) since both \( I g_1( g_1 u_1) \) and \( w_1 \) do. Similarly \( g_2 I g_2( g_2 u_2) = g_2 u_2 g_2 \) implies the existence of \( w_2 \in G \) such that \( I g_2( g_2 u_2) w_2 = u_2 g_2 w_2 \in G \). Therefore both \( v_1 := u_1 g_1 w_1 I g_1 w_1 ( g_2 w_2) \) and \( v_2 := u_2 g_2 w_2 g_1 w_1 \) are in \( G \). Then

\[
\begin{align*}
a_1 v_1 &= a_1 u_1 g_1 w_1 I g_1 w_1 ( g_2 w_2) = a_2 u_2 g_1 w_1 I g_1 w_1 ( g_2 w_2) = a_2 u_2 g_2 w_2 g_1 w_1 = a_2 v_2 \\
g_1 v_1 &= g_1 u_1 g_1 w_1 I g_1 w_1 ( g_2 w_2) = g_2 u_2 g_1 w_1 I g_1 w_1 ( g_2 w_2) = g_2 u_2 g_2 w_2 g_1 w_1 = g_2 v_2.
\end{align*}
\]

\[\square\]

**Notation 2.3.7.** In analogy to the bialgebra case, we denote by \( \rho \) the homomorphism

\[
\rho : H \to H[ G^{-1} ] : a \mapsto \frac{a}{1}.
\]

**Proof of Thm. 2.3.5.**

i) One proves in a similar way to the bialgebra case that \( H[ G^{-1} ] \) is a ring.

ii) \( H[ G^{-1} ] \) is a \( k \)-vector space.

We have, for \( \alpha, \beta \in k \),

\[
\begin{align*}
\frac{a}{g} + \frac{b}{h} &= \frac{\alpha a}{g} + \frac{\alpha b}{h} = (\alpha a) I g( h) + (\alpha b) g \\
&= \alpha \left( \frac{a I g( h)}{h g} + \frac{b g}{h g} \right) = \alpha \left( \frac{a I g( h)}{g} + \frac{b g}{h g} \right) \\
&= \alpha \left( \frac{a}{g} + \frac{b}{h} \right).
\end{align*}
\]

Moreover

\[
1 \cdot \frac{a}{g} = \frac{1}{g} \cdot a = \frac{a}{g},
\]

\[
(\alpha \beta) \frac{a}{g} = (\alpha \beta) a = \frac{\alpha (\beta a)}{g} = \frac{\beta a}{g} = \alpha \left( \frac{\beta a}{g} \right),
\]

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\[(\alpha + \beta)\frac{a}{g} = \frac{(\alpha + \beta)a}{g} = \frac{\alpha a + \beta a}{g} = \frac{\alpha a}{g} + \frac{\beta a}{g} = \frac{\alpha}{g} + \frac{\beta a}{g}.
\]

iii) \(H[G^{-1}]\) is a unital algebra.

The only verification that needs to be done is, for \(\alpha \in k\),

\[
\left(\frac{a}{g}\right) \frac{b}{h} = \frac{a}{g} \left(\frac{b}{h}\right) = \alpha \left(\frac{a \cdot b}{g \cdot h}\right).
\]

And indeed,

\[
\left(\frac{a}{g}\right) \frac{b}{h} = \frac{\alpha a \cdot b}{g \cdot h} = \frac{\alpha a \mathcal{J}_g(b)}{h g} = \frac{a \mathcal{J}_g(\alpha b)}{h g} = \frac{a \cdot \alpha b}{g} = \frac{\alpha (b)}{g h}
\]

where we have used the linearity of \(\mathcal{J}_g\). Moreover,

\[
\left(\frac{a}{g}\right) \frac{b}{h} = \frac{\alpha a \cdot b}{g \cdot h} = \frac{\alpha a \mathcal{J}_g(b)}{h g} = \alpha \left(\frac{b}{g \cdot h}\right).
\]

iv) The comultiplication is well-defined.

Let \(\frac{a_1}{g_1} \sim \frac{a_2}{g_2}\), i.e. there exist \(u_1, u_2 \in G\) such that

\[
a_1 u_1 = a_2 u_2 \quad \text{and} \quad g_1 u_1 = g_2 u_2.
\]

Then

\[
\Delta \left(\frac{a_1}{g_1}\right) = \frac{a_1'}{g_1} \otimes \frac{a_1''}{g_1} = \frac{a_1' u_1}{g_1 u_1} \otimes \frac{a_1'' u_1}{g_1 u_1}
\]

\[
= \left(\frac{a_1' u_1}{1} \otimes \frac{a_1'' u_1}{1}\right) \cdot \left(\frac{1}{g_1 u_1} \otimes \frac{1}{g_1 u_1}\right)
\]

\[
\otimes \left(\frac{a_2' u_2}{1} \otimes \frac{a_2'' u_2}{1}\right) \cdot \left(\frac{1}{g_2 u_2} \otimes \frac{1}{g_2 u_2}\right)
\]

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\[
\frac{a'_2 u_2}{g_2 u_2} \otimes \frac{a''_2 u_2}{g_2 u_2} = \frac{a'_2}{g_2} \otimes \frac{a''_2}{g_2} = \Delta \left( \frac{a_2}{g_2} \right),
\]

where equality \( \Diamond \) holds because for \( g \) group-like we have \( \Delta(a g) = a' g \otimes a'' g \) by (1.50) and thus \( a'_1 u_1 \otimes a''_1 u_1 = \Delta(a_1 u_1) = \Delta(a_2 u_2) = a'_2 u_2 \otimes a''_2 u_2 \).

We then apply \( \rho \otimes \rho \) to this last equality.

v) The comultiplication is \( k \)-linear.

For \( \alpha, \beta \in k \), we have

\[
\Delta \left( \frac{a}{g} \right) = \Delta \left( \frac{\alpha a}{g} \right) = \frac{\alpha a'}{g} \otimes \frac{a''}{g} = \alpha \frac{a'}{g} \otimes \frac{a''}{g} = \alpha \Delta \left( \frac{a}{g} \right).
\]

Moreover

\[
\Delta \left( \frac{a + b}{h} \right) = \Delta \left( \frac{a \varpi(h) + b g}{h g} \right)
\]

\[
= \frac{(a \varpi(h) + b g)'}{h g} \otimes \frac{(a \varpi(h) + b g)''}{h g} = \frac{a' \varpi(h)}{h g} \otimes \frac{a'' \varpi(h)}{h g} + \frac{(b g)'}{h g} \otimes \frac{(b g)''}{h g}
\]

\[
= \Delta \left( \frac{a}{g} \right) + \Delta \left( \frac{b}{h} \right)
\]

where \( * \) follows from the linearity of the comultiplication in \( H \) and \( \Diamond \) from the fact that for a group-like \( g \), we have \( \Delta(a g) = a' g \otimes a'' g \) by
(1.50) and $ag = g\mathcal{J}_g(a)$ by (2.26).

vi) The counit is well-defined.

Take $\frac{a_1}{g_1} \sim \frac{a_2}{g_2}$ as before. Then

$$\varepsilon \left( \frac{a_1}{g_1} \right) = \varepsilon(a_1) \stackrel{\text{[1.49]}}{=} \varepsilon(a_1 u_1) = \varepsilon(a_2 u_2) \stackrel{\text{[1.49]}}{=} \varepsilon(a_2) = \varepsilon \left( \frac{a_2}{g_2} \right).$$

vii) The counit is $k$-linear.

Indeed, we have

$$\varepsilon \left( \frac{\alpha a}{g} \right) = \varepsilon \left( \frac{\alpha a}{g} \right) = \varepsilon(\alpha a) = \alpha \varepsilon(a) = \alpha \varepsilon \left( \frac{a}{g} \right),$$

and

$$\varepsilon \left( \frac{a}{g} + \frac{b}{h} \right) = \varepsilon \left( \frac{a \mathcal{J}_g(h) + bg}{hg} \right) = \varepsilon(a \mathcal{J}_g(h) + bg)$$
$$= \varepsilon(a \mathcal{J}_g(h)) + \varepsilon(bg) = \varepsilon(a \varepsilon_t(\mathcal{J}_g(h))) + \varepsilon(b \varepsilon_t(g))$$
$$= \varepsilon(a \cdot 1) + \varepsilon(b \cdot 1) = \varepsilon(a) + \varepsilon(b)$$
$$= \varepsilon \left( \frac{a}{g} \right) + \varepsilon \left( \frac{b}{h} \right),$$

where we have used that $\mathcal{J}_g(h)$ and $g$ are group-like.

viii) $(H[G^{-1}], \Delta, \varepsilon)$ is a counital coalgebra.

We have

$$(\varepsilon \otimes \text{id}) \circ \Delta \left( \frac{a}{g} \right) = \varepsilon \left( \frac{a'}{g} \right) \frac{a''}{g} = \varepsilon(a') \frac{a''}{g} = \frac{\varepsilon(a') a''}{g} = \frac{a}{g},$$

and

$$(\text{id} \otimes \varepsilon) \circ \Delta \left( \frac{a}{g} \right) = \frac{a'}{g} \varepsilon \left( \frac{a''}{g} \right) = \varepsilon(a'') \frac{a'}{g} = \frac{\varepsilon(a'') a'}{g} = \frac{a}{g}.$$
Moreover, noting that \((a')' \otimes (a'')'' \otimes a'' = a' \otimes (a'')' \otimes (a'')''\) becomes

\[
\frac{(a')'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{a''}{1} = \frac{a'}{1} \otimes \frac{a''}{1} \otimes \frac{a'''}{1} = \frac{a'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{(a'')''}{1}
\]

once we apply \(\rho \otimes \rho \otimes \rho\) to it, we find

\[
(\Delta \otimes \text{id}) \circ \Delta \left( \frac{a}{g} \right) = \Delta \left( \frac{a'}{g} \right) \otimes \frac{a''}{g} = \frac{(a')'}{1} \otimes \frac{(a'')'}{1} \otimes \frac{a''}{1} = \frac{a'}{g} \otimes \frac{a}{g} = \Delta \left( \frac{a}{g} \right).
\]

ix) Multiplicativity of the coproduct.

We have

\[
\Delta \left( \frac{a \cdot b}{g \cdot h} \right) = \Delta \left( \frac{a \cdot \mathcal{I}_g(b)}{h \cdot g} \right) = \frac{(a \cdot \mathcal{I}_g(b))'}{h \cdot g} \otimes \frac{(a \cdot \mathcal{I}_g(b))''}{h \cdot g} = \frac{a' \cdot b'}{g} \otimes \frac{a'' \cdot b''}{h}
\]

\[
= \Delta \left( \frac{a}{g} \right) \Delta \left( \frac{b}{h} \right).
\]

x) Weak comultiplicativity of the unit.

Applying \(\rho \otimes \rho \otimes \rho\) to \(1' \otimes 1'' \otimes 1''' = 1'_1 \otimes 1''_1 1'_2 \otimes 1''_2\) we get

\[
\frac{1'}{1} \otimes \frac{1''}{1} \otimes \frac{1'''}{1} = \frac{1'_1}{1} \otimes \frac{1''_1 1'_2}{1} \otimes \frac{1''_2}{1}.
\]
Then we have
\[
\frac{1'}{1} \otimes \frac{1''}{1} \otimes \frac{1'''}{1} = \frac{1'}{1} \otimes \frac{1''}{1} \otimes \frac{1'''}{1} \tag{2.3.81}
\]
\[
= \frac{1'}{1} \otimes \frac{1''}{1} \cdot \frac{1'}{1} \otimes \frac{1''}{1}.
\]

Similarly we find that
\[
\frac{1'}{1} \otimes \frac{1''}{1} \otimes \frac{1'''}{1} = \frac{1'}{1} \otimes \frac{1''}{1} \cdot \frac{1'}{1} \otimes \frac{1''}{1}.
\]

xi) Weak multiplicativity of the unit.

We have
\[
\varepsilon\left(\frac{a \cdot b \cdot c}{g \cdot h \cdot k}\right) = \varepsilon\left(\frac{a \mathcal{J}_g(b) \cdot \frac{c}{k}}{h g}\right) = \varepsilon\left(\frac{a \mathcal{J}_g(b \mathcal{J}_h(c))}{k h g}\right)
\]
\[
= \varepsilon(a \mathcal{J}_g(b) \mathcal{J}_h(c)) = \varepsilon(a \mathcal{J}_g(b')) \varepsilon(\mathcal{J}_g(b') \mathcal{J}_h(c))
\]
\[
= \varepsilon(a \mathcal{J}_g(b')) \varepsilon(\mathcal{J}_g(b'') \mathcal{J}_h(c)) = \varepsilon(a \mathcal{J}_g(b')) \varepsilon(\mathcal{J}_g(b'' \mathcal{J}_h(c)))
\]
\[
= \varepsilon(a \mathcal{J}_g(b')) \varepsilon\left(\frac{b'' \mathcal{J}_h(c)}{h}ight) = \varepsilon\left(\frac{a \mathcal{J}_g(b')}{h}ight) \varepsilon\left(\frac{b'' \mathcal{J}_h(c)}{kh}\right)
\]
\[
= \varepsilon\left(\frac{a \cdot b'}{h} \cdot \frac{c}{k}\right) \varepsilon\left(\frac{b''}{h} \cdot \frac{c}{k}\right),
\]

where * holds since \( \mathcal{J}_g \) preserves the counit of \( H \) by Proposition 2.3.1

In a similar fashion, we find that
\[
\varepsilon\left(\frac{a \cdot b \cdot c}{g \cdot h \cdot k}\right) = \varepsilon\left(\frac{a \cdot b''}{g \cdot h}\right) \varepsilon\left(\frac{b'}{h} \cdot \frac{c}{k}\right).
\]

\[
\square
\]

**Proposition 2.3.8.** Let \((H, r)\) and \(G \subset G(H)\) be as above. Then \(\rho : H \to H[G^{-1}] : a \mapsto \frac{a}{1}\) is a WBA homomorphism and \(H\) has the following universal property: for any WBA homomorphism \(\psi : H \to \bar{H}\) such that \(\psi(g)\) is invertible for any \(g \in G\), there exists a unique WBA homomorphism
\( \tilde{\psi} : H[G^{-1}] \to \bar{H} \) such that

\[
\begin{array}{c}
H \xrightarrow{\rho} H[G^{-1}] \\
\downarrow \psi \quad \downarrow \tilde{\psi} \\
\bar{H} \\
\end{array}
\]

commutes.

The proof follows directly from the bialgebra case (cf. Proposition 2.2.14).

**Remark 2.3.9.** Note that the homomorphism \( \rho : H \to H[G^{-1}] \) of the preceding proposition satisfies the conditions of Definition 2.1.2. In other words, \( H[G^{-1}] \) is the ring of fractions of \( H \) with respect to \( G \) (on which we have built a WBA structure).

**Corollary 2.3.10.** When the localization exists, it is unique up to a unique isomorphism.

### 2.3.2 Example: Monoid Generated by a Central Group-like Element

In this section we shall see that a similar phenomenon to the one observed in Section 2.2.2 also holds in the weak case; more precisely, if \( H \) is a WBA and the monoid \( G \) is generated by a central group-like element \( g \), then the localization has the form \( H[t]/(gt - 1) \).

**Proposition 2.3.11.** Let \( H \) be a WBA and let \( g \) be a central group-like element. Assume that \( G \) is the monoid freely generated by \( g \), i.e. \( G = \{1, g, g^2, \ldots \} \). Then the localization is given by

\[
H[G^{-1}] \cong H[t]/(gt - 1).
\]

**Proof.** We know that the localization \( H[G^{-1}] \) exists and that in this case it
has a weak bialgebra structure given by, for \(a_1, a_2 \in H\) and \(g_1, g_2 \in G\),

\[
\begin{align*}
\frac{a_1}{g_1} + \frac{a_2}{g_2} &= \frac{a_1 g_2 + a_2 g_1}{g_1 g_2} & \text{since } g_1 g_2 = g_2 g_1, \\
\frac{a_1}{g_1} \cdot \frac{a_2}{g_2} &= \frac{a_1 a_2}{g_1 g_2} & \text{since } g_1 a_2 = a_2 g_1 \text{ and } g_1 g_2 = g_2 g_1, \\
\frac{1}{1} &= \text{the unit,} \\
\Delta \left( \frac{a_1}{g_1} \right) &= \frac{a_1'}{g_1} \otimes \frac{a_2'}{g_1}, & \text{and } \varepsilon \left( \frac{a_1}{g_1} \right) = \varepsilon_H(a_1).
\end{align*}
\]

First, let us show that as an algebra, the localization of \(H\) at \(G\) is given by \(H[t]/(gt - 1)\). We do this by proving that \(H[t]/(gt - 1)\) satisfies the universal property of the localization. Let \(\psi : H \to H'\) be an algebra homomorphism with \(\psi(g)\) invertible for any \(g \in G\). Let us check that there exists a unique algebra homomorphism \(\tilde{\psi} : H[t]/(gt - 1) \to H'\) such that

\[
\begin{array}{c}
H \xrightarrow{i} H[t]/(gt - 1) \\
\downarrow \psi \downarrow \tilde{\psi} \\
H' \end{array}
\]

commutes, with \(i : H \to H[t]/(gt - 1)\) given by \(i(a) = a\).

Since we want \(\tilde{\psi} \circ i = \psi\), we are forced to define \(\tilde{\psi}\) by \(\tilde{\psi}(a) := \psi(a)\) for any \(a \in H\). Moreover, \(t\) being the inverse of \(g\), we have to set \(\tilde{\psi}(t) := \psi(g)^{-1}\) in order for \(\tilde{\psi}\) to be well-defined. In other words, \(\tilde{\psi}\) is completely determined by \(\psi\). Furthermore, \(\tilde{\psi}\) is indeed well-defined since

\[
\tilde{\psi}(gt - 1) = \psi(g)\tilde{\psi}(t) - \tilde{\psi}(1) = \psi(g)\psi(g)^{-1} - 1 = 0.
\]

As a second step, we use this algebra isomorphism to define a weak bialgebra structure on \(H[t]/(gt - 1)\). Using the above universal property for \(H[t]/(gt - 1)\) with \(i : H \to H[t]/(gt - 1) : a \mapsto a\) and \(H[G^{-1}]\) with \(\rho : H \to H[G^{-1}] : a \mapsto a\).
\( a \mapsto \frac{a}{1} \), we get the following commutative diagram

\[
\begin{array}{c}
H[t]/(gt - 1) \\
\downarrow \rho \\
H \\
\downarrow i \\
H[t]/(gt - 1)
\end{array}
\]

\[
\begin{array}{c}
H[G^{-1}] \\
\downarrow \bar{i} \\
H[t]/(gt - 1)
\end{array}
\]

with

\[ \bar{\rho} : H[t]/(gt - 1) \to H[G^{-1}] : at^n \mapsto \frac{a}{g^n} \]

and

\[ \bar{i} : H[G^{-1}] \to H[t]/(gt - 1) : \frac{a}{g^n} \mapsto at^n. \]

By the universal property \( \bar{i} \) and \( \bar{\rho} \) are inverse of each other and thus algebra isomorphisms.

Using that \( \bar{i} \) is moreover a coalgebra homomorphism, we then define the coalgebra structure of \( H[t]/(gt - 1) \) by

\[
\Delta(at^n) = \Delta \left( \frac{a}{g^n} \right) = (\bar{i} \otimes \bar{i}) \circ \Delta \left( \frac{a}{g^n} \right) = (\bar{i} \otimes \bar{i}) \left( \frac{a'}{g'^n} \otimes \frac{a''}{g''} \right)
\]

for the comultiplication whereas for the counit we have

\[
\varepsilon(at^n) = \varepsilon \left( \frac{a}{g^n} \right) = \varepsilon \left( \frac{a}{g^n} \right) = \varepsilon(a)
\]

and

\[
\varepsilon(t^n) = \varepsilon \left( \frac{1}{g^n} \right) = \varepsilon \left( \frac{1}{g^n} \right) = \varepsilon_H(1).
\]

\[ \Box \]

**Remark 2.3.12.** i) First, note that since \( H \) is a WBA, \( \varepsilon_H(1) \neq 1 \) in
ii) Second, this construction can be seen in the following way. Since we know that \( H[G^{-1}] \cong H[t]/(gt - 1) \) as algebras, we then define the coalgebra structure on \( H[t]/(gt - 1) \) such that the morphism \( \tilde{\rho} \) becomes a coalgebra homomorphism as well and thus \( H[G^{-1}] \) becomes isomorphic to \( H[t]/(gt - 1) \) as WBAs.

iii) Finally, note that this construction can be extended to the case where \( G \) is generated by finitely many central group-like elements.

**Remark 2.3.13.** This example illustrates that if the monoid \( G \) is in the centre of \( H \), then the localization \( H[G^{-1}] \) exists even if \( H \) is not coquasi-triangular. Indeed, the centrality of \( G \) is enough for conditions S1 and S2 of Proposition 2.1.5 to be satisfied and then for the localization to exist.

### 2.3.3 Example: Weak Bialgebra Associated with a Finite Directed Graph

In this section we use the theory developed in [Pfe11] to construct the weak bialgebra associated with a finite graph and then localize it. As we focus here on the localization, the reader interested in knowing more about this type of weak bialgebra is referred to Sections 3 and 6 of the above article.

**Construction 2.3.14.** Let \( \mathcal{G} \) be a finite directed graph whose set of vertices is denoted by \( \mathcal{G}^0 \) and the set of edges by \( \mathcal{G}^1 \subseteq \mathcal{G}^0 \times \mathcal{G}^0 \). Every edge \( p = (v_0, v_1) \in \mathcal{G}^1 \) has a source and a target vertex, denoted respectively by \( \sigma(p) = v_1 \in \mathcal{G}^0 \) and \( \tau(p) = v_0 \in \mathcal{G}^0 \). We also set \( \sigma(v) = \tau(v) = v \) for all \( v \in \mathcal{G}^0 \). The paths of length \( m \), for any \( m \in \mathbb{N} \), are denoted by

\[
\mathcal{G}^m = \left\{ (p_1, \ldots, p_m) \in (\mathcal{G}^1)^m \mid \sigma(p_j) = \tau(p_{j+1}) \text{ for all } 1 \leq j \leq m - 1 \right\}.
\]

The concatenation of two paths \( p \in \mathcal{G}^l \) and \( q \in \mathcal{G}^m \) with \( \sigma(p) = \tau(q) \) is
denoted $pq \in \mathcal{G}^{l+m}$. We write $\mathbb{C}\mathcal{G}^m$, with $m \in \mathbb{N}_0$, for the free $\mathbb{C}$-vector space on the set $\mathcal{G}^m$.

The vector space $H[\mathcal{G}] = \prod_{m \in \mathbb{N}_0} (\mathbb{C}\mathcal{G}^m)^* \otimes \mathbb{C}\mathcal{G}^m$ has then a WBA structure given by

$$
\mu([p|q]_m \otimes [r|s]_l) = \delta_{\sigma(p),\tau(r)} \delta_{\sigma(q),\tau(s)} [pr|qs]_{m+l},
\eta(1) = \sum_{j,l \in \mathcal{G}^0} [j|l]_0,
\Delta([p|q]_m) = \sum_{r \in \mathcal{G}^m} [p|r]_m \otimes [r|q]_m,
\varepsilon([p|q]_m) = \delta_{pq},
$$

for all $p, q \in \mathcal{G}^m$, $r, s \in \mathcal{G}^l, m, l \in \mathbb{N}_0$. Note that the coproduct is taken in the category $\text{Vect}_\mathbb{C}$ of $\mathbb{C}$-vector spaces and that we write $[p|q]_m$ for the homogeneous element $p \otimes q \in (\mathbb{C}\mathcal{G}^m)^* \otimes \mathbb{C}\mathcal{G}^m$. As usual, we write $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$ for all $p, q \in \mathcal{G}^m, m \in \mathbb{N}_0$.

Observe that as an algebra, $H[\mathcal{G}] \cong k(\mathcal{G} \times \mathcal{G})$ is the path algebra of the directed graph $\mathcal{G} \times \mathcal{G}$, and thus is graded with respect to path length $m \in \mathbb{N}_0$. As a coalgebra, it is a direct sum of matrix coalgebras; one for each path length.

Now that we have constructed a weak bialgebra structure on $H[\mathcal{G}]$, let us look at its localization.

**Example 2.3.15.** Let us consider the graph $\mathcal{G}$ given by

$$
\begin{array}{c}
\bullet \\
\text{0} \\
\text{1}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\bullet \\
\text{0} \\
\text{1}
\end{array}
$$

Since in this graph, the two vertices are connected by one edge “in each direction”, we can denote a path $p \in \mathcal{G}^m$ of length $m$ by a sequence of $m + 1$
vertices, namely, \( p = (i_0, \ldots, i_m) \in (G^0)^{m+1} \). The source of this path is \( \sigma(p) = i_m \) and its target \( \tau(p) = i_0 \).

Let \( H[G] \) be the WBA constructed for the graph \( G \) above. The multiplication of two elements consist simply in the concatenation of the paths in each component (and is zero if the paths can not be concatenated). The unit is \( \eta(1) = [(0)(0)]_0 + [(0)(1)]_0 + [(1)(0)]_0 + [(1)(1)]_0 \) and the counit is 1 if the two paths are equal and zero otherwise.

Consider the element of \( g \in H[G] \) given by

\[
g = [(0, 1, 0)(0, 1, 0)]_2 + [(1, 0, 1)(1, 0, 1)]_2 - [(0, 1, 0)(1, 0, 1)]_2 - [(1, 0, 1)(0, 1, 0)]_2.
\]

One checks by direct computation that \( g \) is both central and group-like in \( H[G] \).

Let \( G \) be the monoid generated by \( g \). Since \( G \) is in the centre of \( H[G] \), we know that we can localize \( H[G] \) relative to \( G \). Of course, one can use the structure of the localization given by Theorem 2.3.5 and see elements of \( H[G][G^{-1}] \) as fractions. In the present case though, it may be simpler to apply Proposition 2.3.11 and consider the localization as the polynomial algebra \( H[G][t]/(gt - 1) \).

This illustrates how weak bialgebra of fractions can sometimes “fold” into a polynomial algebra.

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Chapter 3

Hopf Envelope

In this chapter we first follow the works of Takeuchi [Tak71], Manin [Man88] and Pareigis [Par] and construct the so-called Hopf envelope of a bialgebra. We then define the more general notion of weak Hopf envelope and briefly discuss its relationship with the localization.

3.1 Manin’s Hopf Envelope

In [Man88, Chap. 7] Manin studies specific quotients of matrix bialgebras and builds a Hopf algebra $H(B)$ having the following universal property. Let $B$ be a certain quotient of a matrix-bialgebra. There exists a Hopf algebra $H(B)$ and a bialgebra homomorphism $i : B \rightarrow H(B)$ such that for any Hopf algebra $H$ and any bialgebra homomorphism $f : B \rightarrow H$ there exists a unique Hopf algebra homomorphism $\bar{f} : H(B) \rightarrow H$ satisfying $f = \bar{f} \circ i$, i.e. such that

\[
\begin{array}{ccc}
B & \xrightarrow{i} & H(B) \\
\downarrow f & & \downarrow \bar{f} \\
H & & \\
\end{array}
\]

commutes. To construct such a Hopf algebra, Manin uses a technique similar to the one developed by Takeuchi in [Tak71].
We are first going to give this result as presented in [Man88, Chap. 7]. In the next section, we shall then give an alternative formulation in a slightly more general context and exhibit some examples.

Let \( Z \) be a family of symbols \((t_{ij})_{1 \leq i,j \leq N}\). Let \( k \) be a field and \( A \) be the matrix bialgebra generated by \( Z \), i.e. the free algebra over \( Z \) with coefficients in \( k \) and comultiplication given by

\[
\Delta(t_{ij}) = \sum_{1 \leq m \leq N} t_{im} \otimes t_{mj}
\]

and counit by

\[
\varepsilon(t_{ij}) = \delta_{ij}.
\]

Let \( R = \{C_{ij}^{mn} \mid 1 \leq i,j,m,n \leq N\} \subset A \) be a set with the \( C_{ij}^{mn} \)'s defined by

\[
C_{ij}^{mn} = \sum_{1 \leq q,r \leq N} c_{ij}^{qr} t_{qm} t_{rn} - \sum_{1 \leq q,r \leq N} t_{iq} t_{jr} c_{ij}^{qr},
\]

where the \( c_{ij}^{mn} \)'s are elements of the ground field \( k \). We can then take the quotient of \( A \) by \((R)\), the two-sided ideal generated by the \( C_{ij}^{mn} \)'s. It turns out that this ideal is also a coideal and therefore the quotient is a bialgebra. Let \( B := A/(R) \).

Note that the construction of such a quotient of a matrix bialgebra is nowadays referred as the “FRT construction” because it was studied systematically by Faddeev, Reshetikhin and Takhtadjian in [RTF89].

Another good introduction on the topic is [Kas95, Chap. VIII.6].

Define a series of families of symbols by \( Z_0 = Z \), \( Z_1 = \{t_{ij}^{(1)} \mid 1 \leq i,j \leq N\} \), \( Z_2 = \{t_{ij}^{(2)} \mid 1 \leq i,j \leq N\} \), … Let \( \mathcal{H} = k\{Z_0, Z_1, \ldots\} \) be the free algebra generated by \( Z_0, Z_1, \ldots \) and let \( H(B) \) be \( \mathcal{H} \) quotiented by the ideal generated by the following relations (where all the sums are taken over \( m \)).
running from 1 to \( N \):

\[
R_n = \begin{cases} 
\text{elements of } R \text{ written for } Z_n & \text{if } n \equiv 0 \mod 2 \\
\text{elements of } R^{op,cop} \text{ written for } Z_n & \text{if } n \equiv 1 \mod 2 
\end{cases}
\]

(3.1)

\[
\sum_m t^{(n)}_{im} t^{(n+1)}_{mj} - 1, \quad \sum_m t^{(n+1)}_{im} t^{(n)}_{mj} - 1, \quad \text{for } n \equiv 0 \mod 2
\]

(3.2)

\[
\sum_m t^{(n)}_{mi} t^{(n+1)}_{jm} - 1, \quad \sum_m t^{(n+1)}_{mi} t^{(n)}_{jm} - 1, \quad \text{for } n \equiv 1 \mod 2.
\]

(3.3)

Define \( i : B \to H(B) \) as the inclusion of \( B \) in “shift” 0, i.e. \( i(B) = Z_0/(R) \).

Note that here the superscripts over the \( t \)’s indicates which family the symbol belongs to and is not a power.

The next theorem ensures that this construction makes sense and that \( H(B) \) has the desired universal property.

**Theorem 3.1.1** ([Man88 Thm. 3, Chap. 7]). a) Relations (3.1) - (3.3) generate a coideal \( \bar{R} \) with respect to the comultiplication \( \bar{\Delta} : \bar{H} \to \bar{H} \bar{\otimes} \bar{H} \) given by

\[
\bar{\Delta}(t^{(n)}_{ij}) = \begin{cases} 
\sum_m t^{(n)}_{im} \otimes t^{(n)}_{mj} & \text{for } n \equiv 0 \mod 2 \\
\sum_m t^{(n)}_{mj} \otimes t^{(n)}_{im} & \text{for } n \equiv 1 \mod 2 
\end{cases}
\]

and thus \( \bar{\Delta} \) induces a comultiplication \( \bar{\Delta} : H(B) \to H(B) \bar{\otimes} H(B) \).

Together with \( \varepsilon(t^{(n)}_{ij}) = 1 \) this makes \( H(B) \) a bialgebra and \( i : B \to H(B) \) a bialgebra homomorphism.

b) The map \( \bar{S} : \bar{H} \to \bar{H} \), defined by \( \bar{S}(t^{(n)}_{ij}) = t^{(n+1)}_{ij} \) and \( \bar{S}(ab) = \bar{S}(b)\bar{S}(a) \)
for any \( a,b \in \bar{H} \), satisfies \( \bar{S}(\bar{R}) \subset \bar{R} \). It thus induces a linear map \( S : H(B) \to H(B) \) which is the antipode of \( H(B) \).

c) For any bialgebra homomorphism \( f : B \to H \), where \( H \) is a Hopf algebra, there exists a unique Hopf algebra homomorphism \( \bar{f} : H(B) \to H(B) \).
such that
\[
\begin{array}{c}
B \\ \downarrow f \\
H
\end{array}
\quad \xrightarrow{i} \quad
\begin{array}{c}
H(B) \\
\end{array}
\quad \xrightarrow{j} \quad
\begin{array}{c}
H
\end{array}
\]

commutes.

We shall explain (and reword in a more general context) this construction in detail in Section 3.2.1. Nevertheless, let us summarise the main steps of this procedure right now.

Consider the bialgebra $B$. The first step is to form the coproduct $B := B \amalg B^{\text{op,cop}} \amalg B \amalg \ldots$ in the category of bialgebras. This corresponds to constructing $H$ and quotienting by (3.1). Second, define the "pre-antipode" $\overline{S} : \overline{B} \rightarrow \overline{B} : t_{ij} \mapsto t_{ij}^{(n+1)}$. In other words, $\overline{S}$ increases the "shift" of $t_{ij}^{(n)}$ by one. This is part b) in Theorem 3.1.1. Third, form the ideal generated by the relations
\[
\{(\mu \circ (\overline{S} \otimes \text{id}_B) \circ \Delta - \eta \circ \varepsilon)(x), \ (\mu \circ (\text{id}_B \otimes \overline{S}) \circ \Delta - \eta \circ \varepsilon)(x) \mid x \in \overline{B}\},
\]
and then quotient by this ideal. This corresponds to dividing by relations (3.2) and (3.3) in Manin’s work. Part a) of the previous theorem assures that it is a coideal as well and part c) that the newly formed $H(B)$ has the desired universal property.

### 3.2 Reformulation and Examples

Our goal is to have a description of the Hopf envelope that can be used for any bialgebra. To this end we reformulate Manin’s result using categorical language. We shall then construct examples of Hopf envelopes.
3.2.1 Categorical Reformulation of Manin’s Theorem

In order to be able to use Manin’s result in more general situations, we de-
scribe this construction in categorical language. We get the following slightly
more general

**Theorem 3.2.1.** Let $B$ be a bialgebra. Then there exists a Hopf algebra $H(B)$ and a bialgebra homomorphism $i : B \rightarrow H(B)$ satisfying the following
universal property: for any Hopf algebra $H$ and bialgebra homomorphism $f : B \rightarrow H$, there exists a unique Hopf algebra homomorphism $\overline{f} : H(B) \rightarrow H$ making the following diagram commute

\[
\begin{array}{ccc}
B & \xrightarrow{i} & H(B) \\
\downarrow f & & \downarrow \overline{f} \\
\downarrow \text{id} & & \downarrow \text{id} \\
B^{\text{op},\text{cop}} & \xrightarrow{i_{n+1}} & H.
\end{array}
\]

**Definition 3.2.2.** The Hopf algebra $H(B)$ of the theorem is called the *Hopf envelope* of $B$.

**Proof.** Define a series of bialgebras by $B_0 := B$ and $B_{n+1} := B_{n}^{\text{op},\text{cop}}$, $n \in \mathbb{N}_0$, hence when $n$ is even $B_n$ is simply $B$ and when $n$ is odd $B_n$ is the
opposite-coopposite bialgebra of $B$ (i.e. $B$ with the opposite multiplication
and opposite comultiplication).

Then, let $\overline{B} := \Pi_{n \in \mathbb{N}_0} B_n$ be the coproduct of the $B_n$’s with injections $i_n : B_n \rightarrow \overline{B}$. Using the universal property of the coproduct, we know that there exists a unique bialgebra homomorphism $\overline{S} : \overline{B} \rightarrow \overline{B}^{\text{op},\text{cop}}$ making the diagram

\[
\begin{array}{ccc}
B_n & \xrightarrow{i_n} & \overline{B} \\
\downarrow \text{id} & & \downarrow \overline{S} \\
B_{n+1}^{\text{op},\text{cop}} & \xrightarrow{i_{n+1}} & \overline{B}^{\text{op},\text{cop}}
\end{array}
\]  

(3.4)

commute for any $n \in \mathbb{N}_0$.  

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Let $I$ be the two-sided ideal in $\overline{B}$ generated by

$$\{(\mu \circ (\overline{S} \otimes \text{id}_B) \circ \Delta - \eta \circ \varepsilon)(x), \ (\mu \circ (\text{id}_B \otimes \overline{S}) \circ \Delta - \eta \circ \varepsilon)(x) \mid x \in \overline{B}\}.$$

**Remark 3.2.3.** Note that here it is not possible to use exclusively the categorical language and define the quotient by $I$ as a coequaliser in the category of bialgebras since the morphism $\mu \circ (\overline{S} \otimes \text{id}_B) \circ \Delta$ defining $I$ is not an algebra homomorphism but only a linear map.

**Notation 3.2.4.** In order to make the subsequent computations easy to read, we use the following notations. We write $\varepsilon_n := \varepsilon_{B_n}$, $\Delta_n := \Delta_{B_n}$ and in general, the subscript $n$ indicates $B_n$ and no subscript indicates $\overline{B}$ (e.g. $\Delta = \Delta_{\overline{B}}$). Moreover, the subscript $k$ indicate that we are in the field, e.g. $\mu_k$ is the multiplication in the field $k$.

Let us now return to the proof. In order to take the quotient by $I$ and get a bialgebra, we first have to show that $I$ is also a two-sided coideal, i.e. $I \subset \ker \varepsilon_{\overline{B}}$ and $\Delta_{\overline{B}}(I) \subset I \otimes \overline{B} + \overline{B} \otimes I$.

It is enough to check these conditions on generating elements of $I$, i.e. elements $y \in I$ of the form $y = i_n(x)$ for some $x \in B_n$, $n \in \mathbb{N}_0$. So, let $x$ be in some $B_n$. Then,

$$\varepsilon(\mu(\overline{S} \otimes \text{id}_B) \Delta i_n(x)) = \mu_k(\varepsilon \overline{S} \otimes \varepsilon)(i_n \otimes i_n)\Delta_n(x)$$

$$= \mu_k(\varepsilon i_{n+1} \text{id}_{B_n} \otimes \varepsilon i_n)\Delta_n(x)$$

$$= \varepsilon i_{n+1} \text{id}_{B_n}(x)$$

$$= \varepsilon_{n+1}(x)$$

$$= \varepsilon_n(x)$$

$$= \varepsilon(\eta \varepsilon i_n(x)),$$

where $\ast$ holds since $B_n = (B, \mu, \eta, \Delta, \varepsilon) = B_{n+1}^{op,cop}$ if $n$ is even and $B_n =$
\((B, \mu^{op}, \eta, \Delta^{op}, \varepsilon) = B_{n+1}^{op, cop}\) if \(n\) is odd.

Similarly we have \(\varepsilon(\mu(\text{id}_B \otimes S) \Delta i_n(x)) = \varepsilon(\eta \varepsilon i_n(x))\). Moreover, using Sweedler’s notation and \(\sigma(v \otimes w) := w \otimes v\) for the map flipping the two tensor factors, we have

\[
\Delta(\mu(S \otimes \text{id}) \Delta i_n(x)) = (\mu \otimes \mu)(\text{id} \otimes \sigma \otimes \text{id})(\Delta \otimes \Delta)(S \otimes \text{id})(i_n \otimes i_n)\Delta_n(x)
\]

\[
= (\mu \otimes \mu)(\text{id} \otimes \sigma \otimes \text{id})(\sigma(S \otimes S) \Delta \otimes \Delta)(i_n \otimes i_n)\Delta_n(x)
\]

\[
= (\mu \otimes \mu)(\text{id} \otimes \sigma \otimes \text{id})(Si_n(x'') \otimes Si_n(x') \otimes i_n(x''') \otimes i_n(x^{IV}))
\]

\[
= Si_n(x'') i_n(x''') \otimes Si_n(x') i_n(x^{IV})
\]

\[
= (S \ast \text{id})(x'') \otimes Si_n(x') i_n(x'''),
\]

where \((S \ast \text{id})\) is the convolution product of \(S\) with \(\text{id}\). Then adding the \(-\eta \varepsilon i_n(x)\) term, we get

\[
\Delta(\mu(S \otimes \text{id}) \Delta i_n(x) - \eta \varepsilon i_n(x))
\]

\[
= (S \ast \text{id})(x'') \otimes Si_n(x') i_n(x''') - \Delta \eta \varepsilon i_n(x)
\]

\[
= ((S \ast \text{id}) - \eta \varepsilon)(x'') \otimes Si_n(x') i_n(x''')
\]

\[
+ (\eta \varepsilon)(x'') \otimes Si_n(x') i_n(x''') - \Delta \eta \varepsilon i_n(x)
\]

\[
= ((S \ast \text{id}) - \eta \varepsilon)(x'') \otimes Si_n(x') i_n(x''')
\]

\[
+ \eta(1) \otimes Si_n(x') i_n(x'') - \eta(1) \otimes \eta \varepsilon i_n(x)
\]

\[
= ((S \ast \text{id}) - \eta \varepsilon)(x'') \otimes Si_n(x') i_n(x''')
\]

\[
+ \eta(1) \otimes (S \ast \text{id} - \eta \varepsilon) i_n(x)
\]

\[
\in I \otimes B + B \otimes I,
\]

hence \(I\) is a two-sided coideal. Since \(I\) is also an ideal by definition, the quotient \(H(B) = \overline{H}/I\) is a bialgebra and the canonical projection \(\pi : \overline{B} \rightarrow \overline{H}/I\) is a bialgebra homomorphism.

Next we show that \(H(B)\) is actually a Hopf algebra with antipode induced
by \( \overline{S} \). To this end we first have to verify that \( \overline{S}(I) \subset I \). On the one hand, we have

\[
\overline{S}((\overline{S} \ast \text{id})i_n(x)) = \overline{S}(\mu(\overline{S} \otimes \text{id})\Delta_n(x)) \\
= \mu \sigma(\overline{S}^2 \otimes \overline{S})(i_n \otimes i_n)\Delta_n(x) \\
= \mu \sigma(\overline{S} \otimes \text{id})(i_{n+1} \otimes i_{n+1})\Delta_n(x) \\
= \mu(\text{id} \otimes \overline{S})(i_{n+1} \otimes i_{n+1})\sigma\Delta_n(x) \\
= \mu(\text{id} \otimes \overline{S})(i_{n+1} \otimes i_{n+1})\Delta_{n+1}(x) \\
= (\text{id} \ast \overline{S})i_{n+1}(x)
\]

and on the other hand

\[
\overline{S}(\eta\varepsilon i_n(x)) = \overline{S}(\eta(1))\varepsilon_n(x) = \overline{S}(\eta(1))\varepsilon_{n+1}(x) = \overline{S}(\eta\varepsilon i_{n+1}(x)).
\]

Hence

\[
\overline{S}((\overline{S} \ast \text{id} - \eta\varepsilon)i_n(x)) = (\text{id} \ast \overline{S} - \eta\varepsilon)i_{n+1}(x) \in I.
\]

Together with the symmetric statement, this shows that \( \overline{S}(I) \subset I \). Then, since \( \ker \pi = I \), we have that \( I \subset \ker(\pi \circ \overline{S}) \). Therefore the morphism \( \pi \circ \overline{S} : \overline{B} \to H(B) \) factors through \( H(B) \) and there is a unique morphism \( S : H(B) \to H(B) \) such that \( S \circ \pi = \pi \circ \overline{S} \), or in terms of diagrams, such that

\[
\begin{array}{ccc}
\overline{B} & \xrightarrow{\pi} & H(B) \\
\overline{S} \downarrow & & \downarrow \quad S \\
\overline{B}^{\text{op}, \text{cop}} & \xrightarrow{\pi} & H(B)^{\text{op}, \text{cop}}
\end{array}
\]  

(3.5)

commutes.

Since \( \overline{S} \) descends to the quotient, and that the ideal \( I \) by which we mod out is generated precisely by the relations making \( \overline{S} \) the antipode, it is straightforward that \( S : H(B) \to H(B)^{\text{op}, \text{cop}} \) is the antipode of \( H(B) \), in other words, that \( H(B) \) is a Hopf algebra.
We finally have to prove that \( H(B) \) has the universal property stated above. Let \( H \) be a Hopf algebra and \( f : B \to H \) a bialgebra homomorphism. We have to show that there exists a unique Hopf algebra homomorphism \( \overline{f} : H(B) \to H \) making the following diagram commute

\[ \begin{array}{ccc}
B & \xrightarrow{i} & H(B) \\
\downarrow{f} & & \downarrow{\overline{f}} \\
H & & H,
\end{array} \]

where \( i : H \to H(B) \) is the composite \( i = \pi \circ i_0 \).

Define the bialgebra homomorphisms \( f_n : B_n \to H \) by \( f_n = S^n_H f \) for all \( n \in \mathbb{N}_0 \); in particular, \( f_0 = f \). Then, by the universal property of the coproduct, there exists a unique bialgebra homomorphism \( f' : \overline{B} = \amalg B_n \to H \) such that \( f'i_n = f_n \) for all \( n \in \mathbb{N}_0 \).

We now show that \( I \subset \ker f' \) and therefore \( f' \) factors through the quotient \( H(B) \). Let \( x \in B_n \), then

\[
\begin{align*}
f'(\overline{S} \ast \text{id} i_n(x)) &= f' \mu(\overline{S} \otimes \text{id}) \Delta_i(x) \\
&= f' \mu(\overline{S} \otimes \text{id})(i_n \otimes i_n) \Delta_n(x) \\
&= \mu(f'\overline{S}i_n \otimes f'i_n) \Delta_n(x) \\
&= \mu(f'i_{n+1} \otimes f'i_n) \Delta_n(x) \\
&= \mu(f_{n+1} \otimes f_n) \Delta_n(x) \\
&= \mu(S_H f_n \otimes f_n) \Delta_n(x) \\
&= \mu(S \otimes \text{id}) \Delta_n f_n(x) \\
&= (S \ast \text{id}) f_n(x) \\
&= \eta \varepsilon f'i_n(x) \\
&= f' \eta \varepsilon i_n(x).
\end{align*}
\]

Together with the symmetric statement, this proves that \( I \subset \ker f' \). Thus
there exists a unique bialgebra homomorphism $\overline{f} : H(B) \to H$ such that $f' = \overline{f}\pi$. Hence we get the following commutative diagram

\[
\begin{array}{ccc}
B = B_0 \xrightarrow{i_0} \overline{B} = \amalg B \xrightarrow{\pi} H(B) = \overline{B}/I \\
\downarrow f' \quad \downarrow f \\
H
\end{array}
\]

Since every bialgebra homomorphism between Hopf algebras preserves the antipode, we know that $\overline{f}$ is indeed a Hopf algebra homomorphism.

\[\square\]

**Remark 3.2.5.** At the beginning of the proof on page 92 we took the index “$n$” to be in $\mathbb{N}_0$ and we constructed a Hopf algebra. Note that if we want to solve this universal problem in the full subcategory of Hopf algebra with bijective antipodes we then need to take $n \in \mathbb{Z}$. Indeed, since the antipode is given by $S(x^{(n)}) = x^{(n+1)}$, if we take $n \in \mathbb{Z}$ we can then construct the inverse of the antipode by defining $S^{-1}(x^{(n)}) = x^{(n-1)}$. In other words, the antipode is bijective.

Similarly, if one is interested in the full subcategory of Hopf algebras with antipode of order $2m$, then one takes $n \in \mathbb{Z}_{2m}$ since this forces the order of the antipode to be $2m$.

### 3.2.2 Example : Hopf Envelope of $M_q(2)$

Let us consider the bialgebra $M_q(2)$ introduced in Section 1.2.1; how does its Hopf envelope $H(M_q(2))$ look like? Following the procedure described in the preceding section, we have the following steps.

1. Define a series of bialgebras by $B_n = M_q(2)$ if $n$ is even and $B_n = M_q(2)^{op,cop}$ if $n$ is odd. Then form the bialgebra $\overline{B}$ as the coproduct of these bialgebras, i.e.

\[
\overline{B} = \coprod_{n \in \mathbb{N}_0} B_n.
\]
2. Using the universal property of the coproduct, we define the unique bialgebra homomorphism $S : B \to B^{\text{op},\text{cop}}$ making the diagram

\[
\begin{array}{c}
B_{n} \xrightarrow{i_{n}} B \\
\downarrow \text{id} \\
B_{n+1}^{\text{op},\text{cop}} \xrightarrow{i_{n+1}} B^{\text{op},\text{cop}}
\end{array}
\]

commute. Explicitly we have $S(t_{ij}^{(n)}) = t_{ij}^{(n+1)}$, where the superscript $(n)$ indicates the “shift” inside $B$ in which $t_{ij}$ is (and is not a power).

3. Let us now define the ideal $I$. It is generated by

\[
\{ (\mu \circ (S \otimes \text{id}_{B}) \circ \Delta - \eta \circ \varepsilon)(x), \ (\mu \circ (\text{id}_{B} \otimes S) \circ \Delta - \eta \circ \varepsilon)(x) \mid x \in B \}.
\]

On the generators of $B$ this becomes

\[
\mu \circ (S \otimes \text{id}) \circ \Delta \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} = \\
= \mu \circ (S \otimes \text{id}) \left[ \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} \otimes \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} \right] \\
= \mu \left[ \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n+1)} \otimes \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} \right] \\
= \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n+1)} \cdot \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)},
\]

and similarly

\[
\mu \circ (\text{id} \otimes S) \circ \Delta \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} = \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n)} \cdot \left( \begin{array}{cc}
t_{11} & t_{12} \\ t_{21} & t_{22}
\end{array} \right)^{(n+1)}.
\]
We moreover have
\[
\eta \circ \varepsilon \left( \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \right)^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

In other words, we see that once we quotient by \( I \), we have
\[
\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}^{(n)} - 1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}^{(n)}.
\]

4. Finally we have \( H(M_q(2)) = \mathcal{B}/I = \Pi_{n \in \mathbb{N}_0} B_n/I \). The question is of course to know whether this Hopf envelope corresponds to another bialgebra we know. The answer is given in the following remark and proposition.

**Remark 3.2.6.**

i) In what follows we are going to look at the quantum determinant in different shifts. More precisely we write \( \det_q^{(0)} = t_{11}^{(0)} t_{22}^{(0)} - q^{-1} t_{12}^{(0)} t_{21}^{(0)} \) and then
\[
\det_q^{(1)} = S(\det_q^{(0)}) = S(t_{11}^{(0)} t_{22}^{(0)} - q^{-1} t_{12}^{(0)} t_{21}^{(0)})
= t_{22}^{(1)} t_{11}^{(1)} - q^{-1} t_{21}^{(1)} t_{12}^{(1)},
\]

since \( B_1 = M_q(2)^{\text{op}, \text{cop}} \) and the antipode is an anti-homomorphism.

ii) Since \( \det_q^{(0)} \) is group-like when we apply the relations of the ideal \( \mu \circ (\mathcal{S} \otimes \text{id}_B) \circ \Delta = \eta \circ \varepsilon \) and \( \mu \circ (\text{id}_B \otimes \mathcal{S}) \circ \Delta = \eta \circ \varepsilon \) as defined above, we find that \( \det_q^{(0)} \det_q^{(1)} = 1 = \det_q^{(1)} \det_q^{(0)} \), in other words \( \det_q^{(1)} \) is the inverse of \( \det_q^{(0)} \). We similarly have that \( \det_q^{(2)} = (\det_q^{(1)})^{-1} \). Since both \( \det_q^{(0)} \) and \( \det_q^{(2)} \) are inverses of \( \det_q^{(1)} \) and inverses are unique in a monoid, we infer that \( \det_q^{(0)} = \det_q^{(2)} \). By recurrence we then get that
\[
\det_q^{(0)} = \det_q^{(2)} = \cdots = \det_q^{(2n)}
\]
and similarly
\[ \det_q^{(1)} = \det_q^{(3)} = \cdots = \det_q^{(2n+1)}. \]

iii) Using the relations defining the ideal \( I \) on the generating elements of \( M_q(2) \), we find
\[
\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}^{(n+1)} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}^{(n)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}^{(n+1)}.
\]

Using the formula for the inverse matrix and taking \( n = 0 \) in the above, we conclude that
\[
\begin{align*}
t^{(1)}_{11} &= \det_q^{(1)} t^{(0)}_{22}, & t^{(1)}_{12} &= \det_q^{(1)} (-qt^{(0)}_{12}), \\
t^{(1)}_{21} &= \det_q^{(1)} (-q^{-1}t^{(0)}_{21}), & t^{(1)}_{22} &= \det_q^{(1)} t^{(0)}_{11}.
\end{align*}
\]

By recurrence we then find that, for any \( n \in \mathbb{N}_0 \),
\[
\begin{align*}
t^{(2n)}_{11} &= t^{(0)}_{22}, & t^{(2n)}_{12} &= q^{2n} t^{(0)}_{12}, \\
t^{(2n)}_{21} &= q^{-2n} t^{(0)}_{21}, & t^{(2n)}_{22} &= t^{(0)}_{11},
\end{align*}
\]
and
\[
\begin{align*}
t^{(2n+1)}_{11} &= t^{(1)}_{11}, & t^{(2n+1)}_{12} &= q^{2n+1} t^{(1)}_{12} \\
&= \det_q^{(1)} t^{(0)}_{22}, & = -q^{2n+1} \det_q^{(1)} t^{(0)}_{12}, \\
t^{(2n+1)}_{21} &= q^{-2n} t^{(1)}_{21}, & t^{(2n+1)}_{22} &= t^{(1)}_{22} \\
&= -q^{-2n+1} \det_q^{(1)} t^{(0)}_{21}, & = \det_q^{(1)} t^{(0)}_{11}.
\end{align*}
\]

**Proposition 3.2.7.** The Hopf envelope of \( M_q(2) \) is isomorphic to \( GL_q(2) \cong M_q(2)[t]/(\det_q t - 1) \).
Remark 3.2.8. Note that one could prove that $GL_q(2)$ is isomorphic to the Hopf envelope $H(M_q(2))$ by using abstract arguments, namely by verifying that $GL_q(2)$ satisfies the proper universal property. Nevertheless, as our goal is here to illustrate the construction of the Hopf envelope, we are going to use explicitly construct an isomorphism between $GL_q(2)$ and $H(M_q(2))$.

Proof of Prop. 3.2.7. In order to show that there is an isomorphism, we define two homomorphisms and show they are inverse of each other; consider the homomorphisms defined by

$$
\varphi : GL_q(2) \longrightarrow H(M_q(2))
$$

$$
t_{ij} \mapsto t_{ij}^{(0)}
$$

$$
t \mapsto \det_q^{(1)}
$$

and, for $n \in \mathbb{N}_0$,

$$
\psi : H(M_q(2)) \longrightarrow GL_q(2)
$$

$$
t_{11}^{(2n)} \mapsto t_{11} \quad t_{11}^{(2n+1)} \mapsto t_{22} t
$$

$$
t_{12}^{(2n)} \mapsto q^{2n}t_{12} \quad t_{12}^{(2n+1)} \mapsto -q^{2n+1}t_{12} t
$$

$$
t_{21}^{(2n)} \mapsto q^{-2n}t_{21} \quad t_{21}^{(2n+1)} \mapsto -q^{-(2n+1)}t_{21} t
$$

$$
t_{22}^{(2n)} \mapsto t_{22} \quad t_{22}^{(2n+1)} \mapsto t_{11} t.
$$

We now prove that $\varphi$ and $\psi$ are well-defined bialgebra homomorphisms and inverse of each other.

i) $\varphi$ is well-defined and linear.

Indeed,

$$
\varphi(\det_q t - 1) = \varphi(\det_q)\varphi(t) - \varphi(1) = \det_q^{(0)}\det_q^{(1)} - 1 = 0.
$$

Moreover, $\varphi$ is linear by construction.
ii) \( \varphi \) is an algebra homomorphism.

This follows directly from the fact that \( \varphi \) is defined on the generators of \( GL_q(2) \).

iii) \( \varphi \) is a coalgebra homomorphism.

We have

\[
(\varphi \otimes \varphi) \circ \Delta(t_{ij}) = (\varphi \otimes \varphi)(t_{i1} \otimes t_{1j} + t_{i2} \otimes t_{2j}) \\
= t_{i1}^{(0)} \otimes t_{1j}^{(0)} + t_{i2}^{(0)} \otimes t_{2j}^{(0)} \\
= \Delta(t_{ij}^{(0)}) \\
= \Delta \circ \varphi(t_{ij}),
\]

and

\[
(\varphi \otimes \varphi) \circ \Delta(t) = (\varphi \otimes \varphi)(t \otimes t) \\
= \text{det}_q^{(1)} \otimes \text{det}_q^{(1)} \\
= \Delta(\text{det}_q^{(1)}) \\
= \Delta \circ \varphi(t),
\]

where we have used that \( \text{det}_q^{(1)} \) is group-like. Concerning the counit we have

\[
\varepsilon(\varphi(t_{ij})) = \varepsilon(t_{ij}^{(0)}) = \varepsilon(t_{ij}),
\]

by construction of the counit of the coproduct. We moreover have

\[
\varepsilon(\varphi(t)) = \varepsilon(\text{det}_q^{(1)}) = 1 = \varepsilon(t)
\]

since both \( t \) and \( \text{det}_q^{(1)} \) are group-like.

iv) \( \psi \) is linear and well-defined.

The well-definition of \( \psi \) follows directly from Remark 3.2.6 whereas its linearity is by construction.
v) \( \psi \) is a bialgebra homomorphism.

The morphism \( \psi \) is an algebra homomorphism by construction. One moreover checks by direct computation that \((\psi \otimes \psi) \circ \Delta(t^{(n)}_{ij}) = \Delta \circ (\psi(t^{(n)}_{ij}))\) and \(\varepsilon(t_{ij}) = \varepsilon(\psi(t^{(n)}_{ij}))\).

vi) \( \psi \circ \varphi = \text{id}_{GL_q(2)} \).

We have \( \psi(\varphi(t_{11})) = \psi(t^{(0)}_{11}) = t_{11} \) and similarly \( \psi \circ \varphi = \text{id} \) for \( t_{12}, t_{21}, t_{22} \). Concerning \( t \), we get

\[
\psi(\varphi(t)) = \psi(\det_q^{(1)})
\]

\[
\begin{align*}
&= \psi(t_{22}^{(1)}(1) - q^{-1}t_{21}^{(1)}(1)) \\
&= (t_{11}t)(t_{22}t) - q^{-1}(t_{21}t_1)(t_{12}t) \\
&\overset{*}= (t_{11}t_{22} - q^{-1}t_{12}t_{21})t^2 \\
&= t,
\end{align*}
\]

where \( * \) uses that \( t \) is central and that \( t_{12}t_{21} = t_{21}t_{12} \).

vii) \( \varphi \circ \psi = \text{id}_{M_q(2)[q^{-1}]} \).

Using Remark 3.2.6, we have, for any \( n \),

\[
\begin{align*}
\varphi(\psi(t^{(2n)}_{11})) &= \varphi(t_{11}) = t^{(0)}_{11} = t^{(2n)}_{11} , \\
\varphi(\psi(t^{(2n)}_{12})) &= \varphi(q^{2n}t_{12}) = q^{2n}t^{(0)}_{12} = t^{(2n)}_{12} , \\
\varphi(\psi(t^{(2n)}_{21})) &= \varphi(q^{-2n}t_{21}) = q^{-2n}t^{(0)}_{21} = t^{(2n)}_{21} , \\
\varphi(\psi(t^{(2n)}_{22})) &= \varphi(t_{22}) = t^{(0)}_{22} = t^{(2n)}_{22} , \\
\varphi(\psi(t^{(2n+1)}_{11})) &= \varphi(t_{22}t) = t^{(0)}_{22} \det_q^{(1)} = t^{(2n+1)}_{11} , \\
\varphi(\psi(t^{(2n+1)}_{12})) &= \varphi(-q^{2n+1}t_{12}t) = -q^{2n+1}t^{(0)}_{12} \det_q^{(1)} = t^{(2n+1)}_{12} , \\
\varphi(\psi(t^{(2n+1)}_{21})) &= \varphi(-q^{-(2n+1)}t_{21}t) = -q^{-(2n+1)}t^{(0)}_{21} \det_q^{(1)} = t^{(2n+1)}_{21} , \\
\varphi(\psi(t^{(2n+1)}_{22})) &= \varphi(t_{11}t) = t^{(0)}_{11} \det_q^{(1)} = t^{(2n+1)}_{22} ;
\end{align*}
\]

hence we have just proved that \( \varphi \) and \( \psi \) are isomorphisms.

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3.2.3 Example: Hopf Envelope of a Monoid Algebra

In this section we are going to look at the Hopf envelope of a monoid algebra \( k[M] \).

**Remark 3.2.9.** We know by Proposition 2.2.28 that for \((M, \cdot, 1)\) a finite monoid, its monoid algebra \( k[M] \) has a bialgebra structure. Denoting by \( e_1, e_a, e_b \) the basis vectors of \( k[M] \) associated with \( 1, a, b \in M \), this bialgebra structure is given by

\[
\mu(e_a \otimes e_b) = e_{ab}, \quad \Delta(e_a) = e_a \otimes e_a,
\]

\[
\eta(1) = e_1, \quad \varepsilon(e_a) = 1.
\]

Knowing that \( k[M] \) has a bialgebra structure, we can now take its Hopf envelope.

**Construction 3.2.10.** The Hopf envelope \( H(k[M]) \) is built as follows. Define, for any \( n \in \mathbb{N}_0 \), \( k[M]_{2n} = k[M] \) and \( k[M]_{2n+1} = k[M]^{op, cop} \). Then \( H(k[M]) = \Pi_{n \geq 0} k[M]_n/I \) where \( I \) is given on generators by

\[
I = (e^{(n+1)}_a \cdot e^{(n)}_a - e^{(n)}_1, e^{(n)}_a \cdot e^{(n+1)}_a - e^{(n)}_1) \quad (3.7)
\]

with \( a \in M \). Using these relations and the fact that the inverse is unique, we find that \( e^{(n+2)}_a = (e^{(n+1)}_a)^{-1} = ((e^{(n)}_a)^{-1})^{-1} = e^{(n)}_a \).

We similarly find that

\[
e^{(0)}_a = e^{(2)}_a = \ldots = e^{(2n)}_a \quad \text{and} \quad e^{(1)}_a = e^{(3)}_a = \ldots = e^{(2n+1)}_a.
\]

**Proposition 3.2.11.** Let \( \overline{M} \) be the group defined by generators and relations as follows. The set of generators is \( \{a \mid a \in M\} \cup \{\overline{a} \mid a \in M\} \) and the
relations \( \{a\bar{a} = 1 = \bar{a}a \mid a \in M \} \). Then the Hopf algebra \( k[\bar{M}] \) is isomorphic to \( H(k[M]) \).

**Proof.** Define \( \varphi : k[\bar{M}] \to H(k[M]) \) by \( \varphi(e_a) = e_a^{(0)} \) and \( \varphi(e_a) = e_a^{(1)} \) with \( a \in M \); we are going to show that it is an isomorphism.

i) \( \varphi \) is well-defined.
   Indeed, if \( a \in M \) turns out to be invertible with \( ab = 1 = ba \) then
   \[
   \varphi(e_b) = e_b^{(0)} = (e_a^{(0)})^{-1} = e_a^{(1)} = \varphi(e_a).
   \]

ii) \( \varphi \) is linear by construction.

iii) \( \varphi \) is an algebra homomorphism.
   We have
   \[
   \varphi(e_a e_b) = \varphi(e_{ab}) = e_{ab}^{(0)} = e_a^{(0)} e_b^{(0)} = \varphi(e_a) \varphi(e_b).
   \]
   Similarly \( \varphi(e_a e_b) = \varphi(e_a) \varphi(e_b) \) and \( \varphi(e_a e_b) = \varphi(e_a) \varphi(e_b) \). Moreover, \( \varphi(e_1) = e_1^{(0)} \).

iv) \( \varphi \) is a coalgebra homomorphism.
   We have
   \[
   (\varphi \otimes \varphi) \circ \Delta(e_a) = \varphi(e_a) \otimes \varphi(e_a) = e_a^{(0)} \otimes e_a^{(0)} = \Delta(e_a^{(0)}) = \Delta(\varphi(e_a))
   \]
   for any \( a \in M \). Moreover \( \varepsilon(\varphi(e_a)) = \varepsilon(e_a^{(0)}) = \varepsilon(e_a) \) by definition of the comultiplication of the coproduct.

   We have just proved that \( \varphi \) is a bialgebra homomorphism.

v) \( \varphi \) is bijective.
   First note that \( \varphi \) is clearly surjective. Now suppose that \( \varphi(e_a) = \varphi(e_b) \).
   Then \( e_a^{(0)} = e_b^{(0)} \) and thus \( e_a = e_b \). Similarly \( \varphi(e_a) = \varphi(e_b) \) implies \( e_a = e_b \) and \( \varphi(e_a) = \varphi(e_b) \) implies \( e_a = e_b \).
Remark 3.2.12. In the previous proposition we have proved that $k[M] \cong H(k[M])$. In other words, for a monoid $M$, its “groupification” $\overline{M}$, its monoid algebra $k[M]$ and $H(k[M])$ the Hopf envelope of $k[M]$, the following diagram commutes

$$
\begin{array}{c}
\begin{array}{ccc}
k[M] & \overset{H(-)}{\longrightarrow} & H(k[M]) \cong k[\overline{M}]\\
M & \overset{(-)}{\longrightarrow} & \overline{M}
\end{array}
\end{array}
$$

where $k[-]$ is the “monoid algebra” functor taking a monoid to its monoid algebra, $(-)$ is the “groupification” functor and $H(-)$ is the Hopf algebra functor.

Remark 3.2.13. We have seen in Section 2.2.4 that $k[M][G^{-1}] \cong k[\overline{M}]$ for $M$ commutative. Thus, in that case, we get that $k[M][G^{-1}] \cong k[\overline{M}] \cong H(k[M])$; in other words, the localization and Hopf envelope are isomorphic as Hopf algebras.

### 3.3 Weak Hopf Envelope and Relationship with the Localization

In this section we define the notion weak Hopf envelope using the universal property of the Hopf envelope. We also discuss the relationship between the weak Hopf envelope and the localization of a weak bialgebra.

**Definition 3.3.1.** Let $H$ be a WBA, then its *weak Hopf envelope* is defined as a WHA $W(H)$ together with a WBA homomorphism $i : H \rightarrow W(H)$ having the following universal property: for any WHA $H'$ and WBA homomorphism $f : H \rightarrow H'$, there exists a unique WHA homomorphism $\tilde{f}$ making the
The weak Hopf envelope, when it exists, is unique up to a unique isomorphism.

Let us now look at the relationship between the localization of a WBA $H$ relative to a monoid of group-likes $G$ and the weak Hopf envelope $W(H)$ of $H$.

**Remark 3.3.3.** We know by Lemma 1.1.19 that in a WHA all the group-likes are invertible. Therefore the WBA homomorphism $i : H \to W(H)$ factors through the localization $H[G^{-1}]$, i.e. we have the following commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{i} & W(H) \\
\downarrow{f} & & \downarrow{\overline{i}} \\
H[G^{-1}] & \xrightarrow{\overline{i}} & W(H), \\
\end{array}
\]

where $\overline{i} \left( \frac{a}{g} \right) = i(a)i(g)^{-1}$. As a consequence of the universal property of the weak Hopf envelope, we see that if the localization $H[G^{-1}]$ turns out to be a WHA, then it is isomorphic to the weak Hopf envelope $W(H)$, in other words, $H[G^{-1}] \cong W(H)$.

**Example 3.3.4.** We have seen in Sections 2.2.3 and 3.2.3 that for a commutative monoid $M$, we have an isomorphism $k[M][G^{-1}] \cong k[M] \cong H(k[M])$ between the localization and the (weak) Hopf envelope. By the previous remark, we can interpret this as follows. Since $\overline{M}$ is a group, its group algebra $k[\overline{M}]$ has a Hopf algebra structure. Therefore, the localization and the (weak) Hopf envelope are isomorphic.
Remark 3.3.3 provides a new description of the localization $H[G^{-1}]$ when it is a WHA. The question that naturally arises then is to know whether the localization of $H$ relative to the monoid of all group-like elements $G$ is always isomorphic to the weak Hopf envelope of $H$. As we shall see in the next example, the answer is in general no.

**Example 3.3.5 (Rad80, Example 2).** Consider the two-dimensional vector space $V$. We can construct on $V$ the coalgebra structure dual to $C$, viewed as a 2-dimensional algebra over $\mathbb{R}$, in the following way. Let $\{1, i\}$ be the basis of $V$. Define the comultiplication by $\Delta(1) = 1 \otimes 1 - i \otimes i$ and $\Delta(i) = 1 \otimes i + i \otimes 1$ and the counit by $\varepsilon(1) = 1$ and $\varepsilon(i) = 0$; then $V$ has a coalgebra structure. The tensor algebra $H = T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$ has then a bialgebra structure and does not have an antipode. The element $1_H$ is the only group-like and is obviously invertible.

Since in this case the monoid of all group-like elements of $H$ is just $G = \{1_H\}$, which clearly satisfies conditions S1 and S2 of Proposition 2.1.5, we can construct the localization $H[G^{-1}]$. It is immediate that $H[G^{-1}] \cong H$ and thus the localization is a (weak) bialgebra. This implies that the localization $H[G^{-1}]$ does not agree with the (weak) Hopf envelope $W(H)$. 

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Conclusion

In this section we discuss the results obtained in this dissertation as well as the new questions they open.

The central goal of this thesis was to define and construct the localization of a weak bialgebra relative to a suitable set of group-like elements as well as to study its properties. This question was addressed in Chapter 2; indeed, Theorem 2.3.5 proves the existence and gives the structure of the localization in detail. Together with Proposition 2.1.5 it provides the technical conditions on the group-like elements for the localization to exist. Through the various examples presented, we now better understand the properties of some already known bialgebras and weak bialgebras.

With this new construction, several new questions arise. On a technical level, one potential improvement of our theorem would be to “weaken”, or even completely drop, the condition \( \mathcal{I}_g(G) \subset G \) for every element \( g \) in the monoid \( G \). This could probably be achieved by studying the properties of the morphism \( \mathcal{J} \) in detail. Similarly to the bialgebra case, it would be very interesting to know when this morphism is the identity.

As we have seen in Section 3.3, the localization of a weak bialgebra relative to the monoid of all group-like elements coincides at times with the weak Hopf envelope. The natural question that comes up is to know pre-
cisely when this holds, in other words, to establish a sufficient condition on the weak bialgebra for the localization and the weak Hopf envelope to be isomorphic as weak Hopf algebras. This question is so far open, and being able to answer it would greatly improve our understanding of both the localization and the weak Hopf envelope.

Concerning the weak Hopf envelope, we have presently no construction that holds in general. It would thus be very interesting to develop such a construction in the weak case. This would moreover enable us to study its properties in a precise way. In a broader research program this could be applied to the recently developed combinatorial characterization of fusion categories of \cite{Pfe11} in the following manner. Consider a finite graph $G$ and the WBA structure on the path algebra $H[G] = k(G \times G)$ of the graph $G \times G$. Then quotient this weak bialgebra by relations of the form $RTT - TTR$ obtained by a solution $R$ to a weak generalization of the quantum Yang-Baxter equation (as in \cite{Pfe11}). Next, taking a finite-dimensional quotient of the weak Hopf envelope, we could systematically construct numerous new fusion categories. This method would moreover provide a completely new approach to the difficult question of the classification of these categories.

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