

# Chiral Gross-Neveu at Finite Density Using Holography

by

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# Abstract

A gravitational dual description to the Chiral Gross-Neveu model with infinite number of fermion fields is examined at finite fermion density. Under a number of simplifying assumptions, it is determined qualitatively that above a critical density, chiral symmetry is restored. The QFT side is reviewed as well. Using a method that does not require a limit from finite temperature situation, nor bosonization techniques, it is shown that, on the QFT side, symmetry is restored above a critical value of the chemical potential.

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# Chapter 1

## Introduction

### 1.1 Background and Purpose

The usefulness of the Chiral Gross-Neveu model derives from multiple areas. One such area is condensed matter, where it is used to model quasi-one-dimensional systems. For a review, see Section 11 of [12].

It has also found a home in theoretical high energy physics. The model has a continuous chiral symmetry and is asymptotically free – two characteristics it shares with QCD. As a result, it provides a useful toy model to examine symmetry breaking and phase diagram structure. Determining the exact structure of the phase of diagram of QCD has proven to be a difficult task. At zero baryon chemical potential (i.e.  $\mu_B = 0$ ), lattice techniques have been used to gain insight [10]. When  $\mu_B \neq 0$ , lattice techniques are far more limited and a strategy that can be used is to apply holographic techniques [11]. Although there is no rigorous gravity dual to QCD, the method has been used to study strongly coupled theories to help gain an understanding of the phase diagrams of theories with similar characteristics.

The work of E. Antonyan, J. A. Harvey and D. Kutasov in [1] is one such example. They examined a particular brane configuration in Type IIA String Theory that at weak coupling and low energy reduced to a QFT that closely resembles the Chiral Gross-Neveu model with infinite number of fermion fields. At strong coupling, a gravity dual to the QFT arose. It was shown that, both at strong and weak coupling, spontaneous chiral symmetry breaking occurred. They also went on to examine the gravity dual at finite temperature and found that there existed a critical temperature,  $T_c$ , such that when  $T > T_c$ , chiral symmetry was restored.

The main purpose of this work is to analyze the same brane configuration as the one found in [1], but at zero temperature and finite fermion density. Although examined under a plethora of assumptions and simplifications, qualitative results are determined suggesting that in the gravity dual, symmetry is restored above a critical density.

Before delving into brane setup and the resulting gravity dual, the Chiral Gross-Neveu model with infinite number of fermion fields is discussed.

The topic of spontaneous chiral symmetry breaking is reviewed both in zero and finite chemical potential cases. The derivation of the effective potential (which is used to determine if chiral symmetry has been spontaneously broken) in the finite chemical potential case is determined using a method uncommon in the literature. This method is analytic and does not use a limit from the case of  $T \neq 0$ , nor bosonization techniques.

## 1.2 Outline

In Section 2, the Chiral Gross-Neveu model, with  $N$  fermion fields and  $N \rightarrow \infty$ , is examined using QFT techniques. First, it is shown that when the number of fermion fields is taken to infinity, a bosonic field may be introduced in such a way that the resulting theory can be treated classically. Doing so simplifies the computation of the effective potential. The effective potential is then computed and is used to show that the chiral symmetry is spontaneously broken. The properties of dimensional transmutation and asymptotic freedom are discussed.

Next, the same Chiral Gross-Neveu model is examined with a non-zero chemical potential,  $\mu$ . The effective potential is computed once again and it is determined that there exists a critical value of the chemical potential,  $\mu_c$ , such that if  $\mu > \mu_c$ , chiral symmetry is restored.

Section 3 reviews the work of E. Antonyan et al. in [1]. In the framework of type IIA string theory, a brane set up consisting of stacks of  $N_c$  D4-Branes,  $N_f$  D6-Branes and  $N_f$   $\overline{D6}$ -Branes is introduced. This brane setup is used for the remainder of the report. It is shown that at low energies and weak coupling (the precise meaning of “weak coupling” to be discussed later), the system can be studied as a non-local Chiral Gross-Neveu model, a QFT that at large distances approaches the Chiral Gross-Neveu model (if  $N_f = 1$ ). Results regarding spontaneous symmetry breaking of the non-local model in the vacuum are discussed.

Section 4 continues with the same brane set up but analyzed at strong coupling. The QFT methods are no longer applicable in this case. In this case, a gravity dual is appropriate; the D4-Branes are replaced by an appropriate geometry of space-time, and the D6 and  $\overline{D6}$ -Branes are probe branes in the geometry. This is the method found in [1] but is generalized in this report to include a non-zero charge density,  $D$ , due to strings stretched between the D4 and D6-Branes. It is determined that there exists a critical value of  $D_c$  such that if  $D < D_c$ , chiral symmetry is spontaneously broken and if  $D > D_c$ , chiral symmetry is restored.



## 1.2. Outline

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In section 5, the assumptions of section 4 are discussed. Interesting computations that are still left to be done and ways that results in this report can be improved are discussed as well.

## Chapter 2

# The Chiral Gross-Neveu Model

### 2.1 Definition and $N \rightarrow \infty$

The Chiral Gross-Neveu model is a field theory of  $N$  Dirac fermions in 1+1 dimensions [6]. The Lagrangian density of the model is:

$$\mathcal{L} = \bar{\Psi}_i(i\gamma^\mu\partial_\mu)\Psi_i + \frac{1}{2}g^2 [(\bar{\Psi}_i\Psi_i)^2 - (\bar{\Psi}_i\gamma^5\Psi_i)^2] \quad (2.1.1)$$

where  $i \in \{1, \dots, N\}$ ,  $\gamma^5 = \gamma^0\gamma^1$ , and  $\gamma^\mu$  are the standard gamma matrices that obey the relation  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbf{1}_{2 \times 2}$ .  $\Psi$  has 2 components, each of which is a Weyl fermion; i.e.  $\Psi = \begin{bmatrix} q_L \\ q_R \end{bmatrix}$ . The Lagrangian has a continuous  $U(1)$  symmetry:  $\Psi_i \rightarrow e^{i\theta\gamma^5}\Psi_i$ .

During manipulation of equations or computations, the following precise values of the gamma matrices are used:

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad (2.1.2)$$

Spontaneous symmetry breaking in the limit of  $N \rightarrow \infty$  and with a non-zero chemical potential are examined in this chapter. As preliminaries, effects of taking  $N$  to infinity and spontaneous symmetry breaking in the case of no chemical potential are discussed first.

To begin the analysis, it is convenient to introduce a complex field,  $\Phi$ , by considering the following Lagrangian:

$$\begin{aligned} \mathcal{L}_\Phi = & \bar{\Psi}_i(i\gamma^\mu\partial_\mu)\Psi_i + \frac{1}{2}g^2 [(\bar{\Psi}_i\Psi_i)^2 - (\bar{\Psi}_i\gamma^5\Psi_i)^2] \\ & - \left( \frac{1}{\sqrt{2}g}\Phi - \sqrt{2}gq_{Li}^\dagger q_{Ri} \right) \left( \frac{1}{\sqrt{2}g}\Phi^\dagger - \sqrt{2}gq_{Rj}^\dagger q_{Lj} \right) \end{aligned} \quad (2.1.3)$$

Note that the first line of (2.1.3) is identical to the Lagrangian density of the Chiral Gross-Neveu model. The second line appears as a Gaussian function of  $\Phi$  in the path integral formulation of QFT, and as a result the  $\Phi$  field can

## 2.2. Spontaneous Symmetry Breaking In Zero Chemical Potential

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be integrated out to recover (2.1.1) (up to an additive constant). Hence it can be concluded that (2.1.3) and (2.1.1) describe the same theory.  $\mathcal{L}_\Phi$  is used for the remainder of the analysis. Expanding, simplifying and applying some algebraic manipulation leads us to the following form of the Lagrangian:

$$\mathcal{L}_\Phi = \bar{\Psi}_i (i\gamma^\mu \partial_\mu + \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix}) \Psi_i - \frac{1}{2g^2} \Phi \Phi^\dagger \quad (2.1.4)$$

To retain the symmetry, the  $\Phi$  field must be transformed as well. The Lagrangian is invariant under the transformation:

$$\Psi_i \rightarrow e^{i\theta\gamma^5} \Psi_i \quad (2.1.5)$$

$$\Phi \rightarrow e^{-2i\theta} \Phi \quad (2.1.6)$$

This implies that if the vacuum expectation value of  $\Phi$  is non-zero, symmetry has been spontaneously broken.

Next, the fermion field is integrated out which leads to the following effective action:

$$S_{\Phi eff} = N \left[ -\frac{1}{2\lambda_{gn}} \int d^2x \Phi \Phi^\dagger - iTr \left[ \ln \left( i\gamma^\mu \partial_\mu + \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix} \right) \right] \right] \quad (2.1.7)$$

where  $\lambda_{gn} = g^2 N$  is defined as the Gross-Neveu coupling.  $\lambda_{gn}$  is a bare quantity and hence will require renormalization. As  $N \rightarrow \infty$ ,  $\lambda_{gn}$  is kept constant.

Recalling the path integral formulation of QFT, and retaining  $\hbar$ :

$$Z = \int \mathcal{D}\Phi \mathcal{D}\Phi^\dagger \exp \left[ i \frac{S(\Phi, \Phi^\dagger)}{\hbar} \right]$$

it can be seen that the number of fermion fields,  $N$ , plays the role of  $1/\hbar$ . Hence, taking  $N \rightarrow \infty$  has the same effect as  $\hbar \rightarrow 0$ , which is the classical limit. This simplifies the analysis in that the vacuum expectation value of  $\Phi$  can be determined by finding the classical value of  $\Phi$  that minimizes the potential derived from (2.1.7).

## 2.2 Spontaneous Symmetry Breaking In Zero Chemical Potential

By symmetry, it is expected that the vacuum state is invariant under translation. As a result,  $\Phi$  is a constant with respect to space and time in (2.1.7).

## 2.2. Spontaneous Symmetry Breaking In Zero Chemical Potential

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First, the trace in (2.1.7) needs to be calculated. This can be accomplished by moving from position-space representation to momentum-space and noting that  $i\partial_\mu$  becomes  $p_\mu$ , where  $p$  is the momentum coordinate. So the computation proceeds as:

$$\begin{aligned} & Tr \left[ \ln \left( i\gamma^\mu \partial_\mu + \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix} \right) \right] \\ &= VT \int \frac{d^2 p}{(2\pi)^2} tr \left[ \ln \left( \gamma^\mu p_\mu + \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix} \right) \right] \end{aligned} \quad (2.2.1)$$

where  $V$  and  $T$  are infinite constants associated with space and time. The lowercase  $tr$  differentiates from  $Tr$  in that it only takes the trace of a matrix rather than an operator. To compute both the trace and natural logarithm, the eigenvalues of the matrix inside the  $\ln$  function need to be determined. Doing so leads to the following form of the integral:

$$VT \int \frac{d^2 p}{(2\pi)^2} \ln \left( \Phi\Phi^\dagger - p^2 \right) \quad (2.2.2)$$

The above integral diverges and therefore needs to be regulated. Using dimensional regularization, the integral, rather than being computed in 2 dimensions, is computed in  $d$  dimensions. The result is known and can be found in graduate level QFT textbooks such as Peskin & Schroeder. Hence we have:

$$\begin{aligned} & VT \int \frac{d^d p}{(2\pi)^d} \ln \left( \Phi\Phi^\dagger - p^2 \right) \\ &= -i(VT)\Gamma \left( \frac{-d}{2} \right) \left( \frac{\Phi\Phi^\dagger}{4\pi} \right)^{d/2} \end{aligned}$$

where  $\Gamma$  is the well known gamma function. Next, using that  $\Gamma(z+1) = z\Gamma(z)$ , defining  $\epsilon = 2 - d$ , and using that  $\Gamma(\epsilon/2) \approx \frac{2}{\epsilon} - \gamma$  for small values of  $\epsilon$  (where  $\gamma$  is the Euler–Mascheroni constant), the above can be written as:

$$= iVT \left( 1 + \frac{\epsilon}{2} \right) \left( \frac{2}{\epsilon} - \gamma \right) \left( \frac{\Phi\Phi^\dagger}{4\pi} \right) \left( 1 - \frac{\epsilon}{2} \ln \left( \frac{\Phi\Phi^\dagger}{4\pi} \right) \right)$$

which is substituted back into (2.1.7). The effective potential, leaving out terms that will go away as  $\epsilon$  approaches zero, is:

$$\begin{aligned} V_{eff} &= -\frac{S_{\Phi eff}}{VTN} \\ &= \frac{\Phi\Phi^\dagger}{2\lambda_{gn}} + \frac{\Phi\Phi^\dagger}{4\pi} \left( \ln \left( \frac{\Phi\Phi^\dagger}{4\pi} \right) - 1 - \frac{2}{\epsilon} + \gamma \right) \end{aligned} \quad (2.2.3)$$

### 2.3. Interesting Phenomena

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Renormalizing using modified minimal subtraction, the renormalized Gross-Neveu coupling,  $\lambda_{gn}$ , is defined by the relationship:

$$\frac{1}{\tilde{\lambda}_{gn}} - \frac{1}{\lambda_{gn}} = \frac{1}{2} \left( \ln \left( \frac{M^2}{4\pi} \right) - \frac{2}{\epsilon} + \gamma \right) \quad (2.2.4)$$

where  $M$  is a renormalization scale. The effective potential then takes the form:

$$V_{eff} = \frac{\Phi\Phi^\dagger}{2\tilde{\lambda}_{gn}} + \frac{\Phi\Phi^\dagger}{4\pi} \left( \ln \left( \frac{\Phi\Phi^\dagger}{M^2} \right) - 1 \right) \quad (2.2.5)$$

The effective potential, as expected, is invariant under the  $U(1)$  symmetry. It has two stationary points:  $\Phi = 0$  and  $\Phi\Phi^\dagger = M^2 e^{-2\pi/\tilde{\lambda}_{gn}}$ . The solution with smaller energy is  $\Phi\Phi^\dagger = M^2 e^{-2\pi/\tilde{\lambda}_{gn}}$ , which implies that  $\Phi \neq 0$ . So, we can conclude that symmetry is spontaneously broken.

## 2.3 Interesting Phenomena

The result above is interesting given that the Chiral Gross-Neveu model, as described by (2.1.1), does not possess any dimensionful constants. The action of the chiral Gross-Neveu model is invariant under rescaling and this would lead to the conclusion that, classically, all physical results should be independent of rescaling. Nonetheless, we find that the theory possesses a dimensionful parameter – the expectation value of  $\Phi\Phi^\dagger$  in the vacuum. This is a quantum phenomenon called “dimensional transmutation”. A better description, which provides some insight into what has taken place, is that the conformal symmetry of the model was anomalously broken.

Another important characteristic of the Chiral G-N model that can be readily seen is that it is asymptotically free – i.e. the coupling of the theory,  $\tilde{\lambda}_{gn}$ , approaches zero as the energy scale,  $M$ , approaches infinity. To see that this is so, note that the expectation value of  $\Phi\Phi^\dagger$  in the vacuum is a physical quantity, and cannot depend on  $M$ , a quantity that was introduced into the computation by hand for convenience. As a result, it must be the case that  $\Phi_c\Phi_c^\dagger = M^2 e^{-2\pi/\tilde{\lambda}_{gn}}$  is a constant with respect to  $M$ . This gives the relation between  $\tilde{\lambda}_{gn}$  and  $M$ . Taking  $M \rightarrow \infty$ , keeping  $\Phi_c\Phi_c^\dagger$  constant requires that  $\tilde{\lambda}_{gn} \rightarrow 0$ .

Some readers may be familiar with Coleman’s proof that shows spontaneous breaking of continuous symmetries in two dimensions cannot occur[7]. The motivation for the proof involves IR divergences that occur in two dimensions. Regarding the model examined in this report, these IR divergences

are avoided because  $N$  was taken to infinity and, as shown, the problem can then be treated classically. If one instead tackles the problem at large (but still finite)  $N$ , one finds as expected, symmetry is not broken [9, 10], but all results depend on  $N$  in such a way that when  $N \rightarrow \infty$ , the chiral symmetry is spontaneously broken.

## 2.4 Effective Potential in the Case of Non-Zero Chemical Potential

In the case of zero temperature and non-zero chemical potential,  $\mu \neq 0$ , the analysis can be done by subtracting the chemical potential multiplied by the number operator from the Hamiltonian and continuing as in the previous section. Note that  $\mu > 0$  is assumed. In the case of fermions, this substitution takes the form  $\mathcal{H} \rightarrow \mathcal{H} - \mu \Psi_i^\dagger \Psi_i$ , or equivalently replacing the Lagrangian:  $\mathcal{L} \rightarrow \mathcal{L} + \mu \Psi_i^\dagger \Psi_i$ . The effective potential may seem to be as easy to compute as before, but this is not quite true. Extra care must be taken as will be seen in a moment.

Noting that  $(\gamma^0)^2 = \mathbf{1}_{2 \times 2}$ , the above substitution of the Lagrangian is equivalent to making the substitution  $i\partial_0 \rightarrow i\partial_0 + \mu$  in (2.1.1) and (2.1.4). Using the same procedure as before, the integrand of (2.2.1) needs to be changed by shifting the 0th-component of the 4-moment by the constant  $\mu$  (i.e.  $p_0 \rightarrow p_0 + \mu$ ). But continuing with the evaluation of the trace, an integral over all momenta must be done, which means that shifting the momentum by a constant in the integrand has no effect. This implies that the effective potential remains identical independent of the choice of  $\mu$ . This is a surprising result, and in fact, untrue. What has gone wrong?

A common technique taught in graduate level QFT books is to multiply the Hamiltonian by  $(1 - i\epsilon)$ , where  $0 < \epsilon \ll 1$ . At the end of computations,  $\epsilon$  is taken to zero in a limit. The purpose is to exclude states that are not the vacuum. Although this step was ignored in the case of zero chemical potential, it was left out because in that particular case it would not affect the results. This time, the replacement  $\mathcal{H} \rightarrow \mathcal{H}(1 - i\epsilon)$  is made. The proper Lagrangian density that accounts for this change takes the form:

$$\begin{aligned} \mathcal{L}_\Phi = & \bar{\Psi}_i \gamma^0 \partial_0 \Psi_i \\ & + (1 - i\epsilon) \left( \bar{\Psi}_i i\gamma^1 \partial_1 \Psi_i + \bar{\Psi}_i \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix} \Psi_i - \frac{1}{2g^2} \Phi \Phi^\dagger + \mu \bar{\Psi}_i \gamma^0 \Psi_i \right) \end{aligned} \quad (2.4.1)$$

Now the computation proceeds by integrating out the fermion fields. The

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effective action, analogous to (2.1.7), is:

$$S_{\text{eff}} = N \left[ -\frac{(1-i\epsilon)}{2\lambda_{gn}} \int d^2x \Phi \Phi^\dagger - i\Theta \right] \quad (2.4.2)$$

where  $\Theta$  is a trace that can be expressed as:

$$\Theta = \text{Tr} \left[ \ln \left( (i\gamma^0 + (1-i\epsilon)\mu)\partial_0 + (1-i\epsilon) \left( i\gamma^1\partial_1 + \begin{bmatrix} -i\Phi & 0 \\ 0 & i\Phi^\dagger \end{bmatrix} \right) \right) \right] \quad (2.4.3)$$

The trace, as before, is evaluated by moving to momentum-space and changing the trace to an integral. The value of the trace then becomes:

$$\Theta = VT \int \frac{d^2p}{(2\pi)^2} \ln \left( (1-i\epsilon)^2 \Phi \Phi^\dagger - (p_0 + (1-i\epsilon)\mu)^2 + (1-i\epsilon)^2 p_1^2 \right) \quad (2.4.4)$$

which is equal to:

$$\begin{aligned} \Theta &= VT \int \frac{d^2p}{(2\pi)^2} \ln \left( \Phi \Phi^\dagger - \left( \frac{p_0}{(1-i\epsilon)} + \mu \right)^2 + p_1^2 \right) \\ &\quad + VT \int \frac{d^2p}{(2\pi)^2} \ln ((1-i\epsilon)^2) \end{aligned}$$

which can be further simplified by noting that for small  $\epsilon$ ,  $1/(1-i\epsilon) \sim 1+i\epsilon$ . Also, the second term is independent of any parameters of the problem and also approaches 0 as  $\epsilon \rightarrow 0$ . So finally we have following nice form of our trace:

$$\Theta = VT \int \frac{d^2p}{(2\pi)^2} \ln \left( \Phi \Phi^\dagger - ((1+i\epsilon)p_0 + \mu)^2 + p_1^2 \right) \quad (2.4.5)$$

Doing the integral over  $p_0$  first, we replace the variable of  $p_0$  with a new variable  $z$ , defined as:  $z(p_0) = \Phi \Phi^\dagger - ((1+i\epsilon)p_0 + \mu)^2 + p_1^2$ . The new form of the integral is now:

$$\Theta = \frac{VT}{-2(1+i\epsilon)} \int \frac{dp_1}{2\pi} \int \frac{dz}{2\pi} \frac{\ln(z)}{\sqrt{-z + \Phi \Phi^\dagger + p_1^2}} \quad (2.4.6)$$

Let us define  $\Theta_0$  as the value of  $\Theta$  in the case of  $\mu = 0$ . Since the logarithmic function has a branch cut, the precise values of  $z$  as  $p_0$  takes on the values from  $-\infty$  to  $\infty$  must be examined.

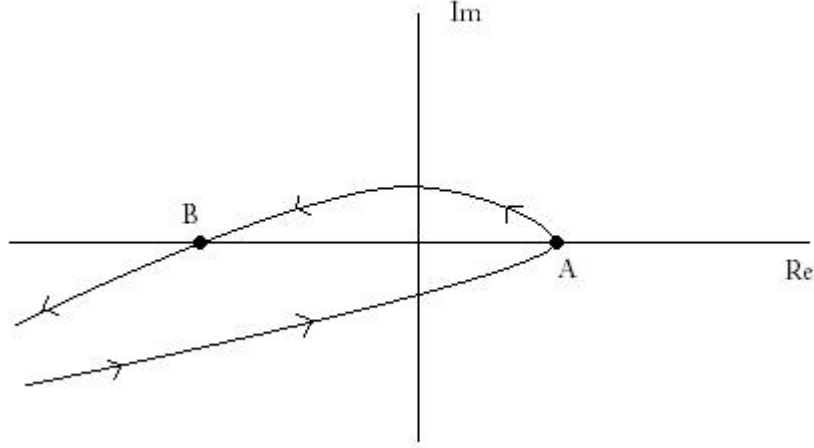


Figure 2.1: The contour  $z(p_0)$  plotted on the complex plane

Figure 2.1 is an example of the contour  $z(p_0)$  plotted on the complex plane. The contour intersects the real axis at the two points marked 'A' and 'B'. At point A,  $p_0 = -\mu$  and  $z = \Phi\Phi^\dagger + p_1^2 + O(\epsilon^2)$ ; at point B,  $p_0 = 0$  and  $z = \Phi\Phi^\dagger + p_1^2 - \mu^2$ . If the point B lies to the right of the imaginary axis, then the contour can be deformed to be identical to the case of  $\mu = 0$  without changing the value of the integral. On the other hand, if point B lies to the left of the imaginary axis, such a deformation of the contour can no longer be done due to the branch cut of the logarithmic function. It is this second case that requires further computation, so we take  $\Phi\Phi^\dagger + p_1^2 - \mu^2 < 0$  for any particular value  $p_1$ . If  $\mu^2 < \Phi\Phi^\dagger$ , then no values of  $p_1$  exist to satisfy the condition, and therefore  $\Theta$  is unchanged from the  $\mu = 0$  case. In the case of  $\mu = 0$ , points A and B are identical and the contour does not go above the real axis of the complex plane.

To do the integral, we will only determine the difference from the  $\mu = 0$  case and then add it to our previous results. The contour is deformed in a way that does not change the final result, but can be easily compared to the zero chemical potential case.

Figure 2.2 shows the new contour on the complex plane. It has been split into three pieces, labeled '(a)', '(b)' and '(c)', so that each part can be examined separately. Special care needs to be taken when computing the integral over this contour. Note that  $\ln(re^{i\theta}) = \ln(r) + i\theta$ .

When the integral of (2.4.6) is done over the contours (a) and (c) and the results added together, the contour is identical to the zero chemical potential



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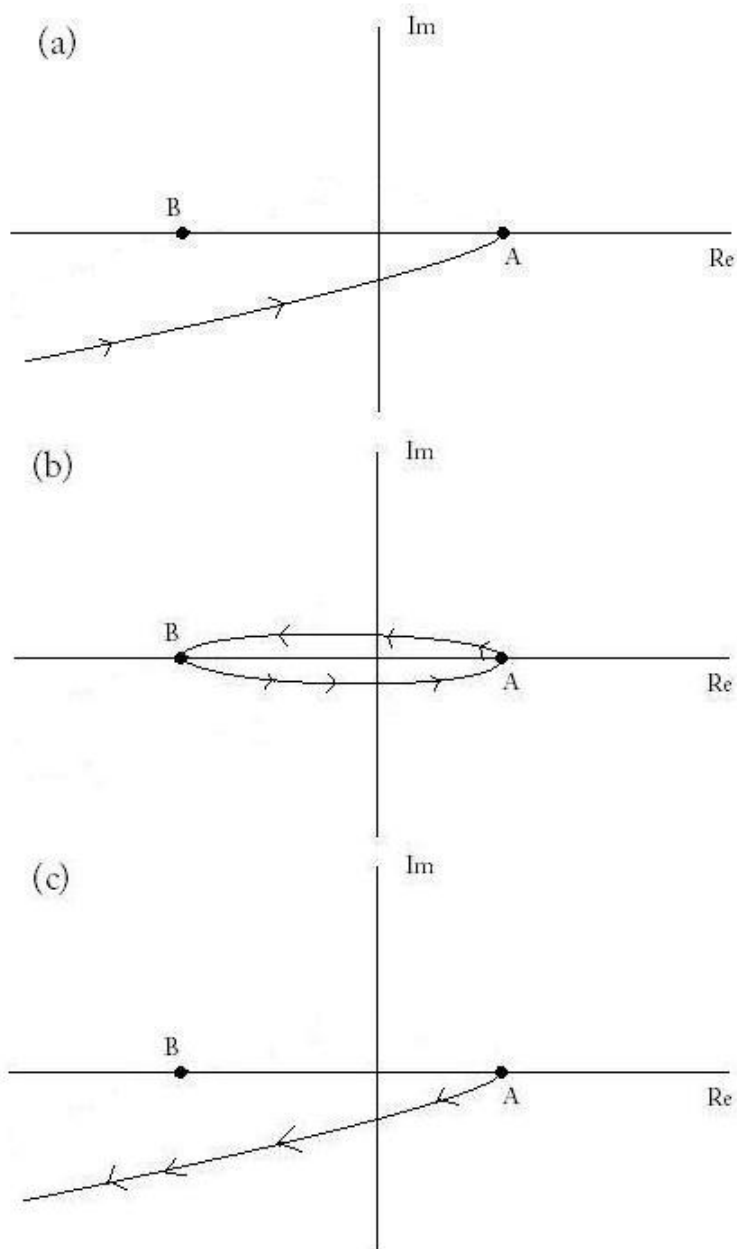


Figure 2.2: The new contour  $z(p_0)$

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case plus an extra term from the integral over (c) due to the contour having rotated a full rotation around the origin. Hence:

$$\begin{aligned}\Delta\Theta &= \Theta - \Theta_0 \\ &= \frac{VT}{-2(1+i\epsilon)} \int \frac{dp_1}{2\pi} \int_{-\infty}^{\Phi\Phi^\dagger+p_1^2} dz \frac{-2\pi i}{2\pi \sqrt{-z + \Phi\Phi^\dagger + p_1^2}} \\ &\quad + \text{Integral over contour (b)}\end{aligned}$$

We have taken  $\epsilon$  to zero in the interval of the integral. Making a change of variables to  $k = -z + \Phi\Phi^\dagger + p_1^2$ , we get the following form of  $\Delta\Theta$ :

$$\begin{aligned}\Delta\Theta &= \frac{VT}{-2(1+i\epsilon)} \int \frac{dp_1}{2\pi} \int_0^\infty \frac{dk}{2\pi} \frac{-2\pi i}{\sqrt{k}} \\ &\quad + \text{Integral over contour (b)}\end{aligned}$$

The first term in the above expression does not depend on  $\Phi\Phi^\dagger$  and therefore would only provide an additive constant to the effective potential. Its value has no effect on determining if symmetry is broken and hence can be dropped from the computation. This leaves only the integral over contour (b).

To do the integral over contour (b), we set the branch cut to be the positive real axis. The contour begins close to point A above the branch cut and finishes close to point A below the branch cut. Noticing that the integral simplifies considerably since only the  $i\theta$  term of the logarithm remains non-zero, the  $\Delta\Theta$  has the form:

$$\begin{aligned}\Delta\Theta &= \frac{VT}{-2(1+i\epsilon)} \int \frac{dp_1}{2\pi} \int_0^{\Phi\Phi^\dagger+p_1^2} dz \frac{2\pi i}{2\pi \sqrt{-z + \Phi\Phi^\dagger + p_1^2}} \\ &= \frac{iVT}{-(1+i\epsilon)} \int \frac{dp_1}{2\pi} \sqrt{\Phi\Phi^\dagger + p_1^2}\end{aligned}$$

The next step is to integrate over the variable  $p_1$ . Any differences from the  $\mu = 0$  case were determined to occur only if  $\Phi\Phi^\dagger + p_1^2 - \mu^2 < 0$ . This means that the integral over  $p_1$  must be done over the interval  $|p_1| < \sqrt{\mu^2 - \Phi\Phi^\dagger}$ . The result is:

$$\Delta\Theta = \frac{iVT}{-4\pi} \left( 2\mu\sqrt{\mu^2 - \Phi\Phi^\dagger} + \Phi\Phi^\dagger \ln \left( \frac{\mu + \sqrt{\mu^2 - \Phi\Phi^\dagger}}{\mu - \sqrt{\mu^2 - \Phi\Phi^\dagger}} \right) \right) \quad (2.4.7)$$

Note that the limit  $\epsilon \rightarrow 0$  has been taken. The results can be substituted back into (2.4.2) and the effective potential derived from there. Writing the effective potential in terms of  $\Theta_0$  and  $\Delta\Theta$ , the previous result of (2.2.5) can

## 2.5. Symmetry Breaking in the case of $\mu \neq 0$

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be used to determine the new effective potential. Given that  $\mu^2 > \Phi\Phi^\dagger$ , we have:

$$\begin{aligned}
 V_\mu &= -\frac{S_{\Phi eff}}{VTN} \\
 &= -\frac{1}{VT} \left[ -\frac{1}{2\lambda_{gn}} \int d^2x \Phi\Phi^\dagger - i(\Theta_0 + \Delta\Theta) \right] \\
 &= \frac{1}{VT} \left[ \frac{1}{2\lambda_{gn}} \int d^2x \Phi\Phi^\dagger + i\Theta_0 \right] + \frac{i\Delta\Theta}{VT} \\
 &= V_{\mu=0} + \frac{i\Delta\Theta}{VT}
 \end{aligned}$$

where  $V_{\mu=0}$  is the effective potential of the  $\mu = 0$  case, (2.2.5). We stick to the same renormalization scheme as used in the previous section, (2.2.4). So, finally, we can express our effective potential as:

$$V_\mu = \begin{cases} V_{\mu=0} + \frac{i\Delta\Theta}{VT} & \text{if } \Phi\Phi^\dagger < \mu^2 \\ V_{\mu=0} & \text{if } \Phi\Phi^\dagger > \mu^2 \end{cases} \quad (2.4.8)$$

## 2.5 Symmetry Breaking in the case of $\mu \neq 0$

Now that the effective potential has been determined explicitly as a function of  $\mu$  and  $\Phi\Phi^\dagger$ , its critical points and the values of  $V_\mu$  at those critical points must be determined. If there exists a value of  $\mu$  for which the global minimum of  $V_\mu$  occurs at  $\Phi = 0$ , then symmetry has been restored at that particular chemical potential. Note that the field  $\Phi$  only appears in  $V_\mu$  as  $\Phi\Phi^\dagger$  and so we treat  $V_\mu$  as a function of  $\Phi\Phi^\dagger$ . The piece-wise function (2.4.8) is continuous at the point of  $\Phi\Phi^\dagger = \mu^2$ , but its rate of change with respect to  $\Phi\Phi^\dagger$  is not. Hence,  $\Phi\Phi^\dagger = \mu^2$  is a critical point. Also,  $\Phi\Phi^\dagger = 0$  is a critical point because it is an endpoint of the interval upon which  $V_\mu$  is defined. On the interval  $\Phi\Phi^\dagger \in (0, \mu^2)$ , the second derivative of  $V_\mu$  with respect to  $\Phi\Phi^\dagger$  can be computed directly, and it is:

$$\begin{aligned}
 \frac{\partial^2 V_\mu}{\partial(\Phi\Phi^\dagger)^2} &= \frac{-\mu}{4\pi(\mu^2 - \Phi\Phi^\dagger)^{3/2}} \\
 &\quad - \frac{1}{4\pi \left( \mu + \sqrt{\mu^2 - \Phi\Phi^\dagger} \right) \sqrt{\mu^2 - \Phi\Phi^\dagger}} \quad (2.5.1)
 \end{aligned}$$

Noting that  $\mu > 0$  and  $\Phi\Phi^\dagger < \mu^2$ , we can conclude that the second derivative of the effective potential is negative. This implies that for  $\Phi\Phi^\dagger \in (0, \mu^2)$ ,

## 2.5. Symmetry Breaking in the case of $\mu \neq 0$

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any critical points will be local maximums and therefore do not need to be considered as potential candidates for global minimum.

From the  $\mu = 0$  case, we know that  $V_{\mu=0}$  has a global minimum at the critical value  $\Phi_c \Phi_c^\dagger = M^2 e^{-2\pi/\tilde{\lambda}_{gn}}$ . Hence, it is convenient to consider two cases separately:  $\mu^2 < \Phi_c \Phi_c^\dagger$  and  $\mu^2 > \Phi_c \Phi_c^\dagger$ .

In the case of  $\mu^2 < \Phi_c \Phi_c^\dagger$ , there are 3 critical points:  $\Phi \Phi^\dagger = 0$ ,  $\Phi \Phi^\dagger = \mu^2$  and  $\Phi \Phi^2 = \Phi_c \Phi_c^2$ . We know that  $V_{eff}(\Phi_c \Phi_c^2) < V_{eff}(\mu^2)$  because  $\Phi \Phi^2 = \Phi_c \Phi_c^2$  is a global minimum on the interval  $\Phi \Phi^\dagger > \mu^2$  and  $V_\mu$  is continuous. Furthermore,  $V_{eff}(\Phi_c \Phi_c^2) < V_{eff}(0)$  is known to be true by explicit computation using (2.4.8). Hence, it can be concluded that in this case chiral symmetry remains broken.

Moving on to the case of  $\mu^2 > \Phi_c \Phi_c^\dagger$ , there are only two critical points:  $\Phi \Phi^\dagger = 0$  and  $\Phi \Phi^\dagger = \mu^2$ . Symmetry is restored whenever  $V_\mu(0) < V_\mu(\mu^2)$ . This inequality is readily solved using the form of  $V_\mu$  that has been computed. The solution is  $\mu^2 > M^2 e^{(3-2\pi/\tilde{\lambda}_{gn})}$ .

Concluding, we have found that there exists a critical value of the chemical potential,  $\mu_c^2 = M^2 e^{(3-2\pi/\tilde{\lambda}_{gn})}$ , such that for  $\mu > \mu_c$ , symmetry is restored.

## Chapter 3

# The Brane Setup and Previous Results

In this section, parts of [1] that are relevant to the discussion here will be reviewed. In particular, the brane configuration in [1] is identical to the configuration used in this report. The conclusions regarding chiral symmetry breaking will be reviewed here as well.

The analysis takes place in the framework of type IIA string theory. A stack of  $N_c$  colour D4-Branes intersect  $N_f$  stacked flavour D6-Branes and  $N_f$  stacked flavour  $\overline{\text{D6}}$ -Branes. The directions that these branes span are indicated in Table 3.1 below:

	0	1	2	3	4	5	6	7	8	9
D4:	x	x	x	x	x					
D6, $\overline{\text{D6}}$ :	x	x				x	x	x	x	x

Table 3.1: Directions spanned by each brane type

In the above table, an 'x' indicates that the brane spans that particular direction. The stacks of D6 and  $\overline{\text{D6}}$ -Branes are parallel but separated in the  $x^4$  direction by a distance  $L$ . The values of the  $x^2$  and  $x^3$  coordinates of the D6 and  $\overline{\text{D6}}$ -Branes are equivalent.

The limit in which the configuration was studied is  $N_c \rightarrow \infty$  and  $g_s \rightarrow 0$ , with  $N_c g_s$  held fixed. Also,  $L \gg l_s$ . This has the effect that the coupling of  $6-6$ ,  $\overline{6}-\overline{6}$  and  $6-\overline{6}$  strings goes to zero and therefore become non-dynamical.

The effective action of the low-energy limit can be written as:

$$\begin{aligned}
 S = & \int d^5x \left[ -\frac{1}{4g_5^2} F_{MN}^2 \right] \\
 & + \int d^5x \left[ \delta^3(\vec{x} - \vec{x}_1) q_L^\dagger \bar{\sigma}^\mu (i\partial_\mu + A_\mu) q_L + \delta^3(\vec{x} - \vec{x}_2) q_R^\dagger \sigma^\mu (i\partial_\mu + A_\mu) q_R \right]
 \end{aligned}
 \tag{3.0.1}$$

where  $(x^0, x^1, \vec{x})$  are the coordinates that parametrize the worldvolume of the colour D4-Branes.  $x^0$  and  $x^1$  parametrize the directions parallel to the worldvolume of the D6-Branes and  $\vec{x} = (x^2, x^3, x^4)$  parametrize the remaining spatial directions.  $\vec{x}_1$  is the coordinate at which the flavour branes are placed and  $\vec{x}_2$  is the coordinate at which the flavour anti-branes are placed.  $F_{MN}$  is the electromagnetic tensor associated with the gauge field due to low energy 4 – 4 strings.  $q_L$  and  $q_R$  are left-moving and right-moving weyl fermions that come from the low energy 4 – 6 and 4 –  $\bar{6}$  strings.  $\sigma$  and  $\bar{\sigma}$  are defined such that terms in the free Dirac Lagrangian have correct coefficients. Their values can be determined from the equality  $q_L^\dagger \bar{\sigma}^\mu \partial_\mu q_L + q_R^\dagger \sigma^\mu \partial_\mu q_R = \bar{\Psi} \gamma^\mu \partial_\mu \Psi$ .

For convenience, define the 't Hooft coupling by:

$$\lambda = g_s N_c = \frac{g_5^2 N_c}{(2\pi)^2}$$

The theory was then analyzed in two separate regimes: i)  $\lambda \ll L$ , which can be studied as a Quantum Field Theory; and ii)  $\lambda \gg L$ , which was studied in the gravity dual.

In the case of  $\lambda \ll L$ , i.e. weak coupling, the gauge fields in (3.0.1) are integrated out and only the leading terms in  $\lambda/L$  are kept. This results in the following (Wick rotated) effective action:

$$\begin{aligned} S_{eff} = & i \int d^2x \left( q_L^\dagger \bar{\sigma}^\mu \partial_\mu q_L + q_R^\dagger \sigma^\mu \partial_\mu q_R \right) \\ & + \frac{g_5^2}{4\pi^2} \int d^2x d^2y G(x-y, L) \left( q_L^\dagger(x) \cdot q_R(y) \right) \left( q_R^\dagger(y) \cdot q_L(x) \right) \end{aligned} \quad (3.0.2)$$

The function  $G(x, L)$  has the form:

$$G(x, L) = \frac{1}{(x^2 + L^2)^{\frac{3}{2}}}$$

At this point it is instructive to show how this effective action is related to the Chiral Gross-Neveu model.  $G(x, L)$  is a positive function that has a maximum at  $x = (0, 0)$  and decreases to zero as  $x^2$  approaches infinity. This is a similar structure to the  $\delta$ -function. At distances large compared to  $L$ ,  $G(x, L)$  can be replaced with the appropriate proportionality constant multiplied by the  $\delta$ -function. Given that:

$$\int d^2x G(x, L) = \frac{2\pi}{L}$$

the effective action, at length scales large compared to  $L$ , takes the form:

$$S_{eff} = \int d^2x \left[ q_L^\dagger i\bar{\sigma}^\mu \partial_\mu q_L + q_R^\dagger i\sigma^\mu \partial_\mu q_R \right. \\ \left. + \frac{2\pi\lambda}{N_c L} \left( q_L^\dagger(x) \cdot q_R(x) \right) \left( q_R^\dagger(x) \cdot q_L(x) \right) \right] \quad (3.0.3)$$

Setting  $N_f = 1$ , the above action describes the chiral Gross-Neveu model with a UV cutoff of  $\Lambda \sim \frac{1}{L}$ . The coupling in the above action is related to the coupling in (2.1.1) by:

$$g^2 = \frac{\pi\lambda}{N_c L}$$

E. Antonyan, et al. go on to show that the vacuum expectation of  $q_L^\dagger(x) \cdot q_R(y)$ , a quantity that is not invariant under the chiral symmetry, is non-zero. As a result, they conclude that the chiral symmetry is spontaneously broken. Furthermore, a fermion mass scale,  $m_f$ , is dynamically generated:

$$m_f \simeq \Lambda e^{-L/\lambda}$$

In the case of strong coupling,  $\lambda \gg L$ , spontaneously broken symmetry was also determined to occur. The reasoning that led to this conclusion is repeated in the next section, but generalized to include a non-zero charge density of fermions.

# Chapter 4

## The Gravity Dual

### 4.1 Overview of Approach

The QFT approximation does not hold if either the D6-Branes are moved closer together or  $g_s N_c$  is increased. In this regime, where  $\lambda/L \gg 1$ , the situation can be analyzed in a gravity dual. The D4-Branes are replaced by a particular geometry of the space-time and the D6-Branes are analyzed by inspecting their DBI action in the geometry. Only one pair of D6 and  $\overline{D6}$ -Branes are considered – i.e.  $N_f = 1$ .

In this analysis,  $U(1)$  gauge fields living on the D6 and  $\overline{D6}$ -Branes are inspected as well. Strings stretched between the D4-Branes and the D6-Branes are used to source a charge for the gauge fields.

The analysis begins by using the geometry of the space, the dilaton and the DBI action to determine the action of the D6-Branes. The action of the stretched strings is then determined using the Nambu-Goto action. Adding the two pieces together results in the total action of the system. Further, this action can be used to determine the equations of motion of both the embedding of the D6-Branes and the gauge fields living on them.

It will be found that there are two classes of solutions: 1) where the D6 and  $\overline{D6}$ -Branes are not connected, which is associated with the symmetric solution; and 2) D6 and  $\overline{D6}$ -Branes are connected by a wormhole-like structure, breaking the chiral symmetry. The Helmholtz Free Energy will be derived for both classes of solutions and compared at various values of the charge density (to simplify the computation, the parameter that will be increased is the density rather than the chemical potential). This will inform us if the symmetry is restored at high density.

### 4.2 The Total Action

The near-horizon metric and dilaton of the stack of D4-branes are:



## 4.2. The Total Action

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$$ds^2 = \left(\frac{U}{R}\right)^{\frac{3}{2}} [\eta_{\mu\nu} dx^\mu dx^\nu - (dx^4)^2] - \left(\frac{U}{R}\right)^{-\frac{3}{2}} (dU^2 + U^2 d\Omega_4^2) \quad (4.2.1)$$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{\frac{3}{4}}$$

where  $\alpha' = 1$  and  $R$  is defined as:

$$R^3 = \pi g_s N_c = \pi \lambda \quad (4.2.2)$$

The action of the system will have two parts: one will be the DBI action associated with the D6-Branes and the other will be strings stretched between the the D6-Branes and  $U = 0$ .

We first consider the metric of the D6-brane. It is stretched in the directions (01) and wraps the spherical coordinates  $\Omega_4$ . Furthermore, it forms a curve in the  $(U, x^4)$  plane, which we parametrize by  $\sigma$ . The curve is described by the two functions  $U(\sigma)$  and  $x^4(\sigma)$ . The metric of the D6-Brane takes the following form:

$$\begin{aligned} d\tilde{s}^2 = & \left(\frac{U}{R}\right)^{\frac{3}{2}} [(dx^0)^2 - (dx^1)^2] \\ & - \left[ \left(\frac{U}{R}\right)^{\frac{3}{2}} \left(\frac{\partial x^4}{\partial \sigma}\right)^2 - \left(\frac{U}{R}\right)^{-\frac{3}{2}} \left(\frac{\partial U}{\partial \sigma}\right)^2 \right] d\sigma^2 - \left(\frac{U}{R}\right)^{-\frac{3}{2}} U^2 d\Omega_4^2 \end{aligned} \quad (4.2.3)$$

To determine the DBI action, we need to compute  $\sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab})}$ , where  $G_{ab}$  is associated with  $d\tilde{s}^2$ , and  $F_{ab} = \partial_a A_b - \partial_b A_a$ . We assume that only terms that are concerned with electrostatics are non-zero. Furthermore, by symmetry in directions other than  $\sigma$ , we find that  $F_{0\sigma}$  is the only non-zero part of the electromagnetic tensor. Since we are concerned with static solutions,  $F_{0\sigma} = -\partial_\sigma A_0$ .

We define  $\tilde{A}_a = 2\pi\alpha' A_a$ , and:

$$(G_{ab} + 2\pi\alpha' F_{ab}) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} G_{00} & 0 & -\partial_\sigma \tilde{A}_0 \\ 0 & G_{11} & 0 \\ \partial_\sigma \tilde{A}_0 & 0 & G_{\sigma\sigma} \end{bmatrix} \text{ and } B \text{ is the metric associated with the}$$

$\Omega_4$  coordinates. We have that:

## 4.2. The Total Action

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$$\begin{aligned} \det(A) &= -G_{11} \left( G_{00} G_{\sigma\sigma} - (\partial_\sigma \tilde{A}_0)^2 \right) \\ &= \left( \frac{U}{R} \right)^{\frac{3}{2}} \left( \left( \frac{U}{R} \right)^3 \left[ \left( \frac{\partial x^4}{\partial \sigma} \right)^2 - \left( \frac{U}{R} \right)^{-3} \left( \frac{\partial U}{\partial \sigma} \right)^2 \right] - (\partial_\sigma \tilde{A}_0)^2 \right) \end{aligned}$$

Integrating over the Quantum Field directions and over the  $\Omega_4$  coordinates, the DBI action is found to be:

$$\begin{aligned} S_{D6} &= (-\tau_6 V_{1+1} V_4) \tag{4.2.4} \\ &\times \int d\sigma U \left( \frac{U}{R} \right)^{-3} \left\{ \left( \frac{U}{R} \right)^3 \left[ \left( \frac{\partial x^4}{\partial \sigma} \right)^2 + \left( \frac{U}{R} \right)^{-3} \left( \frac{\partial U}{\partial \sigma} \right)^2 \right] - (\partial_\sigma \tilde{A}_0)^2 \right\}^{\frac{1}{2}} \end{aligned}$$

where  $\tau_6 = T_6/g_s$ ,  $V_4$  is the volume of the unit 4-sphere and  $V_{1+1}$  is the volume of 1+1 dimensional Minkowski space.

The electromagnetic field is sourced by strings that stretch between the D6-Brane and  $U = 0$ . We assume that all the strings are stretched between the point of the D6-Brane that is closest to  $U = 0$ , which we refer to as  $U = U_0$ . We begin by considering the action of one such string and then generalizing to a distribution of strings. The string worldsheet stretches in  $x^0$  and the  $U$  directions. The metric on its worldsheet is:

$$ds^2 = \left( \frac{U}{R} \right)^{\frac{3}{2}} (dx^0)^2 - \left( \frac{U}{R} \right)^{-\frac{3}{2}} dU^2 \tag{4.2.5}$$

Assuming that the string stretches between  $U = 0$  and ends at  $U = U_0$ , it will have the shortest length if it stretches only in the  $U$ -direction. We find that the Nambu-Goto action evaluates to:

$$S_1 = -TU_o \int dx^0,$$

where  $T = 1/2\pi\alpha'$  is the string tension.

The endpoint of the string which is on the D6-Brane acts as a charge for the gauge field on it. Defining the value of this charge as  $q$ , the full string action for one string is:

$$S = \int dx^0 q A_0 - TU_o \int dx^0 \tag{4.2.6}$$

Due to symmetric concerns, the distribution of strings must be constant in the directions of the brane that is not parametrized by  $\sigma$ . A parameter,

## 4.2. The Total Action

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$\rho$ , is introduced to describe the density of strings. The total action of the system, including both DBI action of D6-Branes and the string action, is:

$$\begin{aligned}
 S &= -(\tau_6 V_{1+1} V_4) & (4.2.7) \\
 &\times \left[ \int d\sigma U^4 \left(\frac{U}{R}\right)^{-3} \left\{ \left(\frac{U}{R}\right)^3 \left(\frac{\partial x^4}{\partial \sigma}\right)^2 + \left(\frac{\partial U}{\partial \sigma}\right)^2 - \left(\partial_\sigma \tilde{A}_0\right)^2 \right\}^{\frac{1}{2}} \right. \\
 &\quad \left. - q\rho \tilde{A}_0(U_0) + \rho U_o \right]
 \end{aligned}$$

Note that various constants were absorbed into the definition of  $\rho$  to achieve this form of the total action. Also, the action is invariant under reparametrization of  $\sigma$ .

From the action we can determine the equation of motion associated with  $x^4$ . For simplification, we ignore the Nambu-Goto action term,  $\rho U_o$ , when determining the embedding of the D6-Branes. The effect is that although the mass of the strings will be included in the Free Energy, the back-reaction on the D6-Branes – being pulled down at  $U = U_0$  – is ignored. The equation of motion is found to be:

$$0 = \frac{\partial}{\partial \sigma} \left[ \frac{U^4 \frac{\partial x^4}{\partial \sigma}}{\left\{ \left(\frac{U}{R}\right)^3 \left(\frac{\partial x^4}{\partial \sigma}\right)^2 + \left(\frac{\partial U}{\partial \sigma}\right)^2 - \left(\partial_\sigma \tilde{A}_0\right)^2 \right\}^{\frac{1}{2}}} \right]$$

which can further be integrated to get,

$$B = \frac{U^4 \frac{\partial x^4}{\partial \sigma}}{\left\{ \left(\frac{U}{R}\right)^3 \left(\frac{\partial x^4}{\partial \sigma}\right)^2 + \left(\frac{\partial U}{\partial \sigma}\right)^2 - \left(\partial_\sigma \tilde{A}_0\right)^2 \right\}^{\frac{1}{2}}} \quad (4.2.8)$$

where  $B$  is a constant of integration.

There are two classes of solutions that will be discussed separately. The first is  $B \neq 0$ , which describes the chiral symmetry-breaking solution. It will be shown that in this case, the two D6-Branes are connected by a wormhole like structure. The second case is that of  $B = 0$  – the symmetric solution. In this case the D6-Branes have a constant value of  $x^4$ , and therefore are two flat branes. Both solutions are generalizations of the ones found by Antonyan et. al.

### 4.3 Broken Symmetry Solution, or $B \neq 0$

Varying the total action by adding a small piece to the gauge field on the D6-Brane, i.e.  $A_0 \rightarrow A_0 + \delta A_0$ , results in the change of total action:

$$\delta S = \int d\sigma \left( \frac{\partial \mathcal{L}}{\partial \tilde{A}_0} - \frac{\partial}{\partial \sigma} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \right) \right) \delta \tilde{A}_0 + \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \delta \tilde{A}_0 \Bigg|_{\sigma=-\infty}^{\sigma=\infty} \quad (4.3.1)$$

where  $\pm\infty$  is used to represent the maximum and minimum values of  $\sigma$  in the case that  $\sigma$  is not defined over all  $\mathfrak{R}$ . Remembering that we wish to examine the Free Energy as a function of the charge, the last term in the above expression is problematic since there is no guarantee that  $\delta A(\sigma = \pm\infty)$  will be equal to zero as only the charge is taken to be constant. This will be dealt with at a later point when the Free Energy is derived. For the moment, it is assumed that the equations of motion derived from the first term of the variation of the action are applicable.

The equation of motion for the gauge field is:

$$\frac{\partial}{\partial \sigma} \left( \frac{UR^3 \partial_\sigma \tilde{A}_0}{\left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial \sigma} \right)^2 + \left( \frac{\partial U}{\partial \sigma} \right)^2 - \left( \partial_\sigma \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}} \right) = \delta(\sigma) \rho q$$

where, without loss of generality, we have made the assumption that  $U(\sigma = 0) = U_0$ . Integrating the equation of motion gives us:

$$\frac{UR^3 \partial_\sigma \tilde{A}_0}{\left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial \sigma} \right)^2 + \left( \frac{\partial U}{\partial \sigma} \right)^2 - \left( \partial_\sigma \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}} = \Theta(\sigma) \rho q + c_1 \quad (4.3.2)$$

where  $\Theta$  is the Heaviside step function and  $c_1$  is a constant of integration. Equation (4.3.2) is analogous to Gauss's Law. The exact value of  $c_1$  will be determined later. Lets also define  $D = \rho q + c_1$ .

As mentioned previously,  $\sigma$  can be redefined by reparameterization. The majority of the analysis will be simplified if we set  $\sigma = U$ . This choice of  $\sigma$  does not necessarily cover the entire D6-Brane. The equations that govern the gauge field and the embedding of the D6-Brane now become:

$$B = \frac{U^4 \frac{\partial x^4}{\partial U}}{\left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial U} \right)^2 + 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}}$$

and

$$D = \frac{UR^3 \partial_U \tilde{A}_0}{\left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial U} \right)^2 + 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}}$$

from which it can be determined that:

$$\frac{\partial x^4}{\partial U} = \frac{BR^3}{\sqrt{-B^2 R^3 U^3 + D^2 U^6 + R^6 U^8}} \quad (4.3.3)$$

and

$$\frac{\partial \tilde{A}_0}{\partial U} = \frac{DU^{\frac{3}{2}}}{\sqrt{-B^2 R^3 + D^2 U^3 + R^6 U^5}} \quad (4.3.4)$$

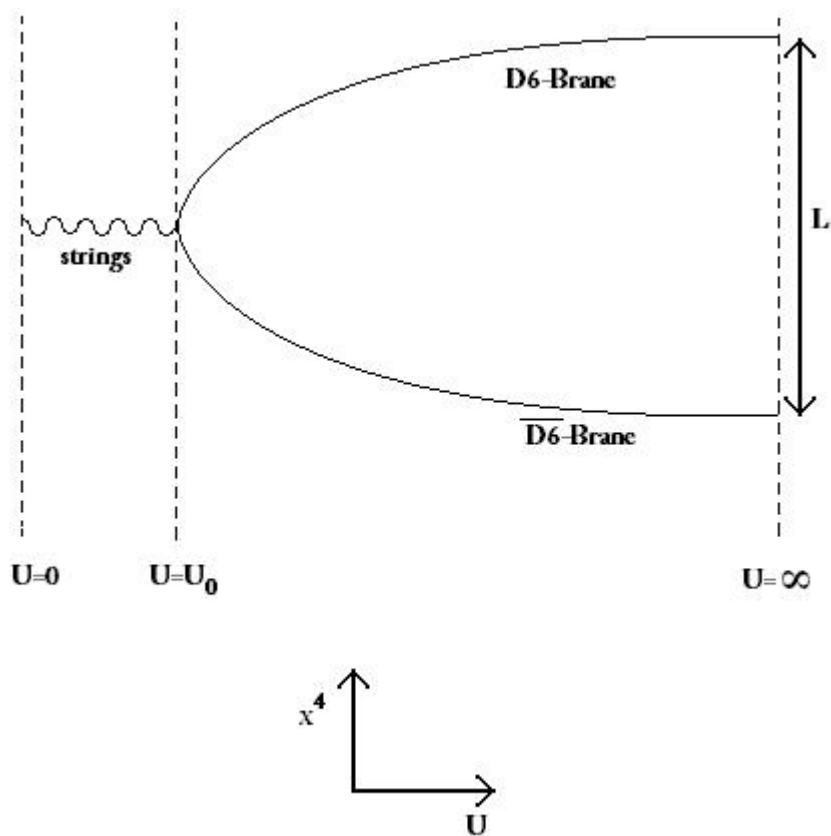
The derivative of  $x^4$  with respect to  $U$  gives us the embedding of the D6-Brane. As  $U \rightarrow \infty$ ,  $\partial_U x^4$  goes to 0, and therefore the embedding approaches that of the flat embedding,  $B = 0$ . The smallest value of  $U$  for which the embedding is defined,  $U_0$ , is given by the solution to the equation  $0 = -B^2 R^3 + D^2 U_0^3 + R^6 U_0^5$ . As  $U \rightarrow U_0$ ,  $\partial_{x^4} U \rightarrow 0$ . Assuming the same solutions for the  $\overline{\text{D6}}$ -Brane as for the D6-Brane, they can be connected by a wormhole-like structure at  $U_0$  as mentioned before. To make the connection between the branes smooth, and also to treat both the left and right-handed fermions symmetrically, we should choose  $c_1$  such that (4.3.2) is symmetric across  $\sigma = 0$ . The appropriate choice is  $c_1 = -\frac{1}{2}\rho q$ . This assures that the D6 and  $\overline{\text{D6}}$ -branes are mirror copies of one another.

$x^4$  as a function of  $U$  can be determined by integrating  $\partial_U x^4$ .  $L$ , the separation of the D6 and  $\overline{\text{D6}}$ -branes at  $U \rightarrow \infty$  in the  $x^4$  direction, is then given by:

$$L(B, D) = 2 \int_{U_0}^{\infty} dU \frac{BR^3}{\sqrt{-B^2 R^3 U^3 + D^2 U^6 + R^6 U^8}} \quad (4.3.5)$$

The factor of 2 appears because the choice of  $U$  as the parameter of the embedding only covers half of the brane solution. Note that  $L$  depends on both  $B$  and  $D$ . Remembering that  $L$  is a parameter in the dual field theory, it must be kept constant in this analysis. Since we will be comparing energies of solutions at various values of the charge ( $D$  or  $\rho$ ),  $B$  must be varied at the same time. Another words, what must be determined is whether or not symmetry is restored along contours of constant  $L$ .

Returning to the earlier issue of determining the Helmholtz Free Energy from the action, we proceed by the procedure outlined in [5]. In (4.3.1), it



The D6 and  $\overline{\text{D6}}$ -Branes are connected at  $U = U_0$ . The diagram also shows the strings that are stretched from  $U = 0$  to  $U = U_0$ . Note that at  $U = \infty$ , the branes become parallel and have a difference in the  $x^4$  coordinate of  $L$ .

Figure 4.1: A graphical representation of the  $B \neq 0$  situation

was determined that the term:

$$\frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \delta \tilde{A}_0 \Bigg|_{\sigma=-\infty}^{\sigma=\infty} = (\tau_6 V_{1+1} V_4) D \left[ \delta \tilde{A}_0(\sigma = \infty) + \delta \tilde{A}_0(\sigma = -\infty) \right] \quad (4.3.6)$$

was an issue if we are to hold the charge constant. This implies that the charge projection operator is  $(-\tau_6 V_{1+1} V_4) D \left[ \tilde{A}_0(\sigma = \infty) + \tilde{A}_0(\sigma = -\infty) \right]$  and should be added to the action to compute the Helmholtz Free Energy. The effect of this is that the equations of motion used for the gauge field on the D6-Branes are correct if  $D$  or  $\rho$  is assumed to be held constant. Furthermore, the action (as well as the Hamiltonian) is not invariant under a gauge transformation; i.e. a constant added to  $\tilde{A}_0$  would change the value of the action. Upon adding the charge projection operator to the action, that is no longer the case.

To compute the Free Energy, beginning with (4.2.7) and using the identity:

$$\begin{aligned} \rho q \tilde{A}_0(\sigma = 0) &= D \left[ \tilde{A}_0(\sigma = \infty) + \tilde{A}_0(\sigma = -\infty) \right] \\ &+ D \left[ - \int_0^\infty d\sigma \partial_\sigma \tilde{A}_0 + \int_{-\infty}^0 d\sigma \partial_\sigma \tilde{A}_0 \right] \end{aligned}$$

the following form of the total action is found:

$$\begin{aligned} S &= -(\tau_6 V_{1+1} V_4) \\ &\times \left[ \int d\sigma U^4 \left( \frac{U}{R} \right)^{-3} \left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial \sigma} \right)^2 + \left( \frac{\partial U}{\partial \sigma} \right)^2 - \left( \partial_\sigma \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}} \right. \\ &- D \left[ \tilde{A}_0(\sigma = \infty) + \tilde{A}_0(\sigma = -\infty) \right] \\ &\left. - D \left[ - \int_0^\infty d\sigma \partial_\sigma \tilde{A}_0 + \int_{-\infty}^0 d\sigma \partial_\sigma \tilde{A}_0 \right] + \rho U_o \right] \end{aligned}$$

Next, the projection charge operator is added to the action and the parametrization parameter is set to  $U$ . Then, the Free Energy is determined. (Note that, as mentioned earlier, the variable  $U$  only parametrizes half of the brane solution.  $U \in (U_0, \infty)$  covers either  $\sigma \in (0, \infty)$  or  $\sigma \in (-\infty, 0)$ .) Doing so

#### 4.4. Symmetric Solution, or $B = 0$

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gives us the following form of the Free Energy:

$$F = (\tau_6 V_1 V_4) \left[ 2 \int_{U_0}^{\infty} dU (UR^3) \left\{ \left( \frac{U}{R} \right)^3 \left( \frac{\partial x^4}{\partial U} \right)^2 + 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}} + 2D \int_{U_0}^{\infty} dU \partial_U \tilde{A}_0 + \rho U_o \right] \quad (4.3.7)$$

The constant  $V_1$  appears instead of  $V_{1+1}$  because there is no integration over the time coordinate when computing the Free Energy. Finally, equations (4.3.3) and (4.3.4) can be used to substitute for derivatives of  $x^4$  and  $\tilde{A}_0$  with respect to  $U$ , to get the final form of the Helmholtz Free Energy:

$$F_{broken} = (\tau_6 V_1 V_4) \left[ \int_{U_0}^{\infty} dU \frac{2U^{7/2} R^6 + 2D^2 U^{\frac{3}{2}}}{\sqrt{-B^2 R^3 + D^2 U^3 + R^6 U^5}} + \rho U_o \right] \quad (4.3.8)$$

#### 4.4 Symmetric Solution, or $B = 0$

As mentioned, the case of  $B = 0$  is such that the brane embedding is vertical in the  $(x^4, U)$  plane – i.e.  $x^4$  is a constant. This implies that the D6 and  $\overline{\text{D6}}$ -Branes are not connected; also, if any strings act as a charge sourcing the gauge field on the branes, it must connect to the D6/ $\overline{\text{D6}}$ -Brane at  $U = 0$ . Therefore the strings that act as a charge source are of zero length and massless.

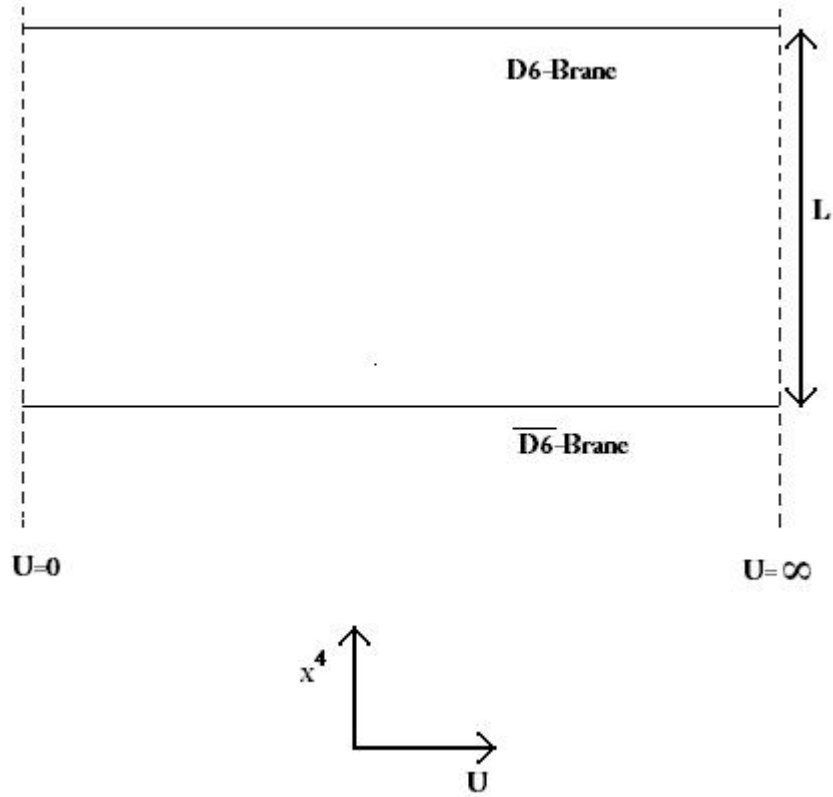
The total action described by equation (4.2.7) is also the correct action for a single D6/ $\overline{\text{D6}}$ -Brane in this case. Since there are two separate branes of equal action in the symmetric case, the action of (4.2.7) must be multiplied by two. Setting  $U_0 = 0$ ,  $\sigma = U$  (in this case, this choice of parametrization does cover an entire brane) and  $\partial_U x^4 = 0$ , the total action in the symmetric case is:

$$S = -2 (\tau_6 V_{1+1} V_4) \left[ \int dU (UR^3) \left\{ 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}} - q \rho_{sym} \tilde{A}_0(U=0) \right] \quad (4.4.1)$$

$\rho_{sym}$  is used to differentiate from the  $\rho$  used in the  $B \neq 0$  case, as the two values may not be equal.

Now we proceed to analyze the equation of motion of the gauge field living on the D6-Brane to determine how much charge is required to source





The embeddings of the branes are such that the  $x^4$  coordinate of each brane is constant and are separated by  $L$  in the  $x^4$  direction.

Figure 4.2: The D6 and  $\overline{\text{D6}}$ -Branes in the  $B = 0$  case.

#### 4.4. Symmetric Solution, or $B = 0$

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it (the case of the  $\overline{D6}$ -brane is identical). For  $U > 0$ , the equation of motion of  $\tilde{A}_0$  is:

$$\frac{\partial}{\partial U} \left( \frac{UR^3 \partial_U \tilde{A}_0}{\left\{ 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}} \right) = 0$$

Which can be integrated to get the equation:

$$\frac{UR^3 \partial_U \tilde{A}_0}{\left\{ 1 - \left( \partial_U \tilde{A}_0 \right)^2 \right\}^{\frac{1}{2}}} = c_2$$

where  $c_2$  is a constant of integration.  $c_2$  is actually the charge,  $q\rho_{sym}$ , that sits at  $U = 0$  in analogy with Gauss's Law. Both the symmetric and broken-symmetry solutions should possess the same behaviour at  $U \rightarrow \infty$ . The behaviour of  $\partial_U \tilde{A}_0$  will be identical only if  $c_2 = D$ .

To find the Free Energy, the same steps are repeated as the broken-symmetry case. Namely, the charge projection operator must be added to the action of the D6-Brane and the  $\overline{D6}$ -Brane. The action of the D6-Brane is half of (4.4.1) (the case of the  $\overline{D6}$ -Brane is identical). As in [5], boundary conditions must be chosen in a way that all the charges are on the brane:

$$\left. \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \right|_{U=\infty} = D, \quad \left. \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \right|_{U=0} = 0$$

The above implies that the boundary term when varying the action is:

$$\left. \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \tilde{A}_0}{\partial \sigma} \right)} \delta \tilde{A}_0 \right|_{U=0}^{U=\infty} = \tau_6 V_{1+1} V_4 D \delta \tilde{A}_0 (U = \infty)$$

which implies that the charge projection operator is  $-\tau_6 V_{1+1} V_4 D \tilde{A}_0 (U = \infty)$  for the D6-Brane. The  $\overline{D6}$ -Brane has a charge projection operator that is identical, except for the fact that it references the gauge field living on it (the two gauge fields are now independent since the branes are not connected).

Proceeding as in the  $B \neq 0$  case, the Helmholtz Free Energy is found to be:

$$F_{sym} = (\tau_6 V_1 V_4) \left[ \int_0^\infty dU \frac{2U^{7/2} R^6 + 2D^2 U^{\frac{3}{2}}}{\sqrt{D^2 U^3 + R^6 U^5}} \right] \quad (4.4.2)$$

## 4.5 Restored Symmetry

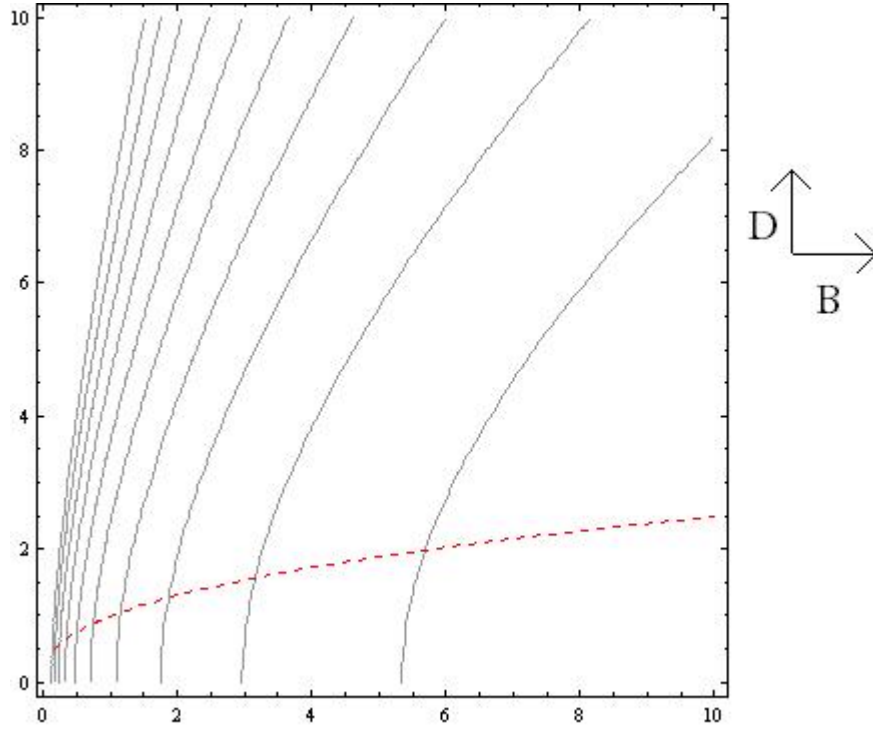
The difference of the Free Energy between the two solutions is proportional to:

$$\begin{aligned}
 \Delta F(D, B) &= F_{sym} - F_{broken} & (4.5.1) \\
 &\sim \int_0^{U_0} dU \frac{2U^{7/2}R^6 + 2D^2U^{\frac{3}{2}}}{\sqrt{D^2U^3 + R^6U^5}} \\
 &\quad + \int_{U_0}^{\infty} dU \left( 2U^{7/2}R^6 + 2D^2U^{\frac{3}{2}} \right) \\
 &\quad \times \left( \frac{1}{\sqrt{D^2U^3 + R^6U^5}} - \frac{1}{\sqrt{-B^2R^3 + D^2U^3 + R^6U^5}} \right) - 2DU_0
 \end{aligned}$$

where the charge constant,  $q$ , has been set to 1.

In Figure 4.3, various contours of  $L(B, D)$  are plotted as solid lines. The contour  $\Delta F(D, B) = 0$  is plotted as a dotted line.  $R = 2$  is used for the numerical plot.  $(B, D)$  points below the  $\Delta F(D, B) = 0$  lines are those for which  $\Delta F(D, B) < 0$ , and therefore they describe physical situations in which the broken chiral symmetry solution is preferred. Likewise, the points above the line represent physical situations in which the density is high enough to restore the chiral symmetry.

As mentioned earlier,  $L(B, D)$  must be kept constant if one wishes to only change the density in the dual field theory. Contours of  $L$  are therefore field theories where the only varying parameter is the charge density. As  $D$  increases, each contour of  $L$  crosses the  $\Delta F(D, B) = 0$  contour. This exactly implies that for any particular value of  $L$ , there exists a critical value of  $D$ , say  $D_c$ , for which if  $D > D_c$ , chiral symmetry is restored.



The dotted line is the contour of  $\Delta F = 0$ . The solid lines are contours of various constant values of  $L$ . The solid line furthest right is  $L = 1.36$ . Moving left, the value of  $L$  increases. The solid line furthest to the left is  $L = 2.89$ . The values of  $R = 2$  and  $q = 1$  were used in creating the plot.

Figure 4.3: Plot of various contours on the  $(B, D)$  plane

## Chapter 5

# Conclusion and Discussion

Using the gravity dual to the non-local chiral Gross-Neveu model, it was determined that when the system exceeds a particular fermion density, the chiral symmetry re-emerges. This was determined under assumptions that perhaps may be removed in future calculations. Some of these assumptions are discussed below.

The assumption that strings stretching from the stack of D4-Branes to the D6 and  $\overline{\text{D6}}$ -Branes all occupy the same space does not hold true upon closer inspection. The ends of the strings ending on the D6-Branes would act as point charges with respect to the gauge field living on these Branes. As a result, the strings would repel one another. For a more precise computation, the string endpoints ending on the D6 and  $\overline{\text{D6}}$ -Branes should be allowed to spread out over values of  $\sigma$  other than  $\sigma = 0$ . Although the endpoints repel one another such that there is a preference for the strings to be separated, the further a string is from  $\sigma = 0$ , the larger the distance between its endpoints and therefore the string become more massive. This is motivation to think that perhaps the assumption that the string endpoints are at  $\sigma = 0$  is an appropriate approximation.

The effect on the D6 and  $\overline{\text{D6}}$ -Branes' embeddings due to the string mass was ignored in computations. Once again, although the computations become more complex, this assumption may be relaxed for more precise results. If analyzed under the assumption that all the charges on the D6 and  $\overline{\text{D6}}$ -Branes sit at a single point, as was done in this report, one will find that in case of broken symmetry, the D6 and  $\overline{\text{D6}}$ -Branes meet at an angle and form a kink rather than the smooth solution that was determined. If the charge density that is induced due to stretched strings is allowed to spread on the D6 and  $\overline{\text{D6}}$ -Branes, the kink solution would disappear and a smooth wormhole-like solution would be found once again.

For the gravity dual part of the analysis, this report only manages to shed light on the qualitative properties of chiral symmetry breaking. This is due to the complex forms and difficulty of determining inverses of some of the equations encountered – particularly (4.3.5) and (4.5.1). A potential improvement to the results found in this report is to determine the precise

charge density at which symmetry is restored. This would allow for further comparison between the QFT and gravity sides of the problem.

# Bibliography

- [1] E. Antonyan, J. A. Harvey and D. Kutasov, “The Gross-Neveu Model from String Theory,” arXiv:hep-th/0608149v1
- [2] E. Antonyan, J.A. Harvey, S. Jensen and D. Kutasov, “NJL and QCD from string theory,” arXiv:hep-th/0604017
- [3] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, “Supergravity and The Large N Limit of Theories With Sixteen Supercharges,” arXiv:hep-th/9802042v3
- [4] M. Rozali, H. Shieh, M. Van Raamsdonk and J. Wu, “Cold Nuclear Matter In Holographic QCD,” arXiv:0708.1322v4 [hep-th]
- [5] S. Nakamura, “Comments on Chemical Potentials in AdS/CFT,” arXiv:0711.1601v3 [hep-th]
- [6] D. J. Gross and A. Neveu, “Dynamical Symmetry Breaking In Asymptotically Free Field Theories,” Phys. Rev. D 10, 3235 (1974).
- [7] S. Coleman, “There are no Goldstone Bosons in Two Dimensions,” Comm. Math. Phys. 31, 259 (1973).
- [8] E. Witten, “Chiral Symmetry, The  $1/N$  Expansion, And The SU(N) Thirring Model,” Nucl. Phys. B 145, 110 (1978).
- [9] I. Affleck, “On The Realization Of Chiral Symmetry In (1+1)-Dimensions,” Nucl. Phys. B 265, 448 (1986).
- [10] M. A. Stephanov, “QCD phase diagram: An overview,” arXiv:hep-lat/0701002v1
- [11] N. Evans, “Holographic Description of the QCD Phase Diagram And Out of Equilibrium Dynamics”, arXiv:1209.0626v1 [hep-ph]
- [12] M. Thies, “From relativistic quantum fields to condensed matter and back again: Updating the Gross-Neveu phase diagram”, arXiv:hep-th/0601049

*Bibliography*

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- [13] C. V. Johnson, 2003, “D-Branes,” University Press, Cambridge, 548 p.