Non-Singlet Sectors of the c=1 Matrix Model

with Connections to Two Dimensional String Theory

by

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B.Sc., McGill University, 2010

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

October 2012

Abstract

The goal of this thesis is to study non-singlet sectors of the $c = 1$ matrix model in order to determine their connection with string theory in two space-time dimensions. It is well understood that the singlet sector of this matrix model is related to the closed string sector of the string theory [1, 2]. The adjoint sector has been connected to long strings stretching from an FZZT brane at infinity [3]. The goal is to find a sector which includes a large number of these long strings which condense to form an FZZT brane in line with a proposal due to Gaiotto [4]. The sectors corresponding to the $\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}$, $N^{\wedge K} \otimes \bar{N}^{\wedge K}$ and $\mathrm{Sym}^K N \otimes \bar{N}^{\wedge K}$ representations of the $U(N)$ symmetry of the matrix model are studied. Partition functions of the matrix model with a harmonic oscillator potential as well as bases for the Hilbert spaces are found for all these sectors. The first two representations are studied using a collective field approach with the inverted oscillator potential applicable to the $c = 1$ model and are found to contain large numbers of long strings which form a free bosonic gas in the large N limit and so do not condense. On the other hand, in the third sector the degrees of freedom are fermionic and a shift in the ground state energy is found in the harmonic oscillator potential, which points to the existence of a Fermi sea and the condensation of the long strings. A collective field that can be used to study this sector is proposed and the technical difficulties presented by the study of this sector are discussed, but its study is left for future work.

Preface

Chapter 2 describes the work of others needed for the rest of this thesis. In particular, section 2.5.1 is based on unpublished work done my supervisor, Assistant Professor Joanna Karczmarek, before my involvement in this work.

Chapters 3 and 4 are based on the research done during my Master's degree under the supervision of and in collaboration with Assistant Professor Joanna Karczmarek and with the support of funding from the Natural Sciences and Engineering Research Council of Canada and the Fonds de recherche du Québec – Nature et technologies.

Table of Contents

List of Figures

Chapter 1

Introduction

This thesis is a study of certain non-singlet sectors of the $c = 1$ matrix model. This study is motivated by the desire to better understand non-perturbative aspects of two dimensional string theory with the hope that this understanding can lead to the generalisation of these aspects to more realistic higher dimensional string theories. In particular, the object of our study will be FZZT branes, objects extended in the spatial dimension of the two dimensional string theory. These objects are interesting in their own right as non-perturbative objects, but they are also important in attempts to understand black holes in this string theory [3].

The idea underlying this thesis is to exploit the connections between the $c = 1$ matrix model and two dimensional string theory to study the FZZT brane through non-singlet sectors of the matrix model. This quantum mechanical model of an $N \times N$ Hermitian matrix has a $U(N)$ symmetry which allows us to write its Hilbert space as the direct sum of superselection sectors. The states of each of these sectors can be organised into subspaces isomorphic to copies of a particular representation of $U(N)$ allowing a representation to be associated to each of these sectors.

The sector corresponding to the singlet representation of $U(N)$ has been shown to describe the closed string tachyon in the string theory [1, 2]. The only closed string degree of freedom in this two dimensional string theory with a linear dilaton background is the tachyon mode of the strings, which in this case is somewhat confusingly named as this mode is massless. In this linear dilaton background, the string coupling is not spatially homogeneous. For large negative x, the strings are weakly coupled. For large positive x, there is a region of strong coupling where perturbative string theory fails. However, a tachyon condensate forms which cuts off the propagation of the perturbative tachyon mode before this strong coupling region. The primary observables for this theory are the S-matrix elements for strings entering from the weakly coupled region and reflecting back out to the weak coupling region. Figure 1.1 shows a sketch of this scattering process.

The degrees of freedom of the singlet sector of the matrix model are the eigenvalues of the matrix which behave as fermions in an inverted harmonic

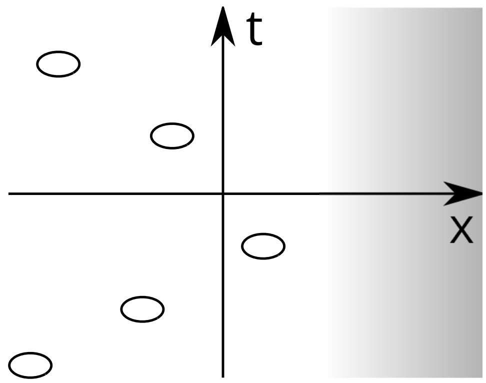

Figure 1.1: Closed string scattering, with the strong coupling region depicted at large x.

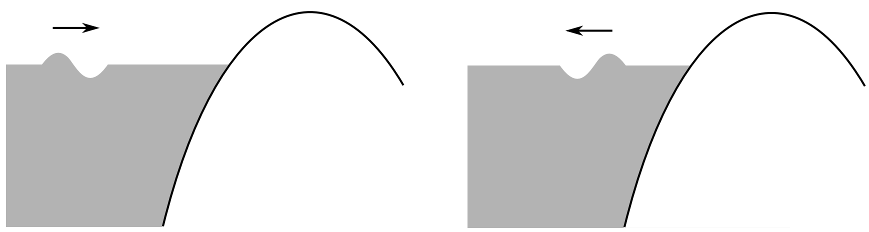

Figure 1.2: The fluctuations of the eigenvalue Fermi sea scattering off of the inverted oscillator potential. This is analogous to the closed string scattering shown in figure 1.1.

oscillator potential. These form a Fermi sea whose bosonised perturbations form a scalar field whose quanta correspond to the closed string tachyons. Figure 1.2 shows these perturbations of the Fermi sea scattering off the inverted harmonic oscillator potential analogously to the scattering process in the string theory depicted in figure 1.1. Some details of this correspondence which will be needed for the rest of this thesis will be reviewed in section 2.3.

The sector corresponding to the adjoint representation of $U(N)$ has been shown to describe long folded open strings whose endpoints are fixed to an FZZT brane pushed out to infinity in the weak coupling region [3]. The FZZT brane acts as a regulator and has no excitations except for the long strings whose energy is infinite in the limit in which the FZZT brane has been pushed to infinitely large negative x. The scattering process where an open string with a large energy stretches from the FZZT brane and then returns

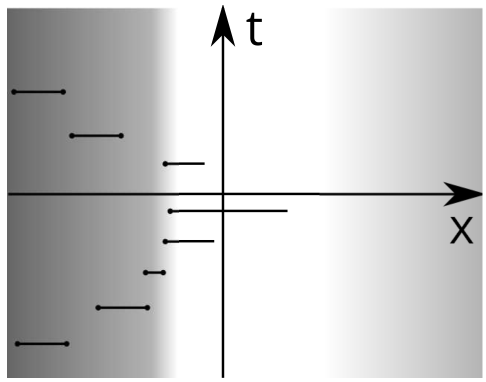

Figure 1.3: A high energy open string whose endpoints lie on an FZZT brane and stretches past the end of the brane. In the adjoint sector the FZZT brane ends at large negative x and would not be visible in this figure. Again, the strong coupling region is depicted at large positive x.

is shown in figure 1.3. The dynamics of adjoint sector of the matrix model can be reduced to a model containing only the eigenvalues of the matrix. While in the singlet sector the wavefunction is completelly symmetric in the eigenvalues, in the adjoint sector one of the eigenvalues is 'special' or 'labelled' and the wavefunction is completely symmetric in the remaining eigenvalues. This labelled eigenvalue interacts with the other eigenvalues in a way that leads to the same scattering phase as the phase for scattering of the long string described above. This correspondence is reviewed in section 2.4.

Knowing that open string degrees of freedom can be used to describe the dynamics of branes, these long strings can be studied to gain insight into the FZZT brane. Gaiotto has conjectured that it is possible to condense a large number of these long strings bringing the FZZT brane into view [4]. Move precisely, he argued that by studying a sector where there is a large number of these long strings and where the wave functions are antisymmetric under their exchange, a new Fermi sea could be found. The chemical potential of this Fermi sea could then be related to how far the brane extends into the weak coupling region. A low energy excitation above the Fermi sea would correspond to the open string tachyon degrees of freedom in the brane and a long string would correspond to a large energy excitation well above the Fermi sea. As with the closed string sector, the perturbations of this Fermi sea would correspond to bosonic degrees of freedom that could be matched to those of the open string tachyon.

To realize this idea in the matrix model, a sector in which statistics of the eigenvalues is such that a new, second Fermi sea forms must be found. Gaiotto's original proposal was to look at the sector corresponding to the $N^{\wedge K} \otimes \bar{N}^{\wedge K}$ representation of $U(N)$, where N is the fundamental representation and $V^{\wedge K}$ is the K^{th} antisymmetric tensor power of the vector space V. These will be referred to as the antisymmetric sectors. This work will also study the sectors corresponding to the $\text{Sym}^K N \otimes \text{Sym}^K \bar{N}$ representations where $\text{Sym}^K V$ is the K^{th} symmetric tensor power of the vector space V and which will be referred to as the symmetric sectors. The sectors corresponding to the $\text{Sym}^K N \otimes \bar{N}^{\wedge K}$ and $N^{\wedge K} \otimes \text{Sym}^K \bar{N}$ representations and which will be referred to as the mixed sectors will be studied as well.[1] Our computations will show that these sectors do include K strings each. However, in the symmetric and antisymmetric sectors these strings are bosonic and do not form a Fermi sea. For technical reasons described in this thesis, the mixed sectors resisted a full analysis. However, strong evidence of a Fermi sea was found in section 4.2.4, where the non-trivial ground state of the mixed sector in the harmonic oscillator potential is described. It is found to have an energy of $\frac{1}{2}K(K-1)$ above the singlet sector ground state, in agreement with the result from calculating the partition function at large N for these sectors in section 2.5.1.

In chapter 2, the background material that will be required for the rest of this thesis is reviewed. In chapter 3, the sectors of interest are studied by an approach where they are treated as particles with labels. In some cases these labels can be thought of a analogous to spin, but the analogy is not applicable in all the sectors studied. In chapter 4, a gauged matrix-vector model which reproduces the original matrix model but which makes studying the sectors of interest easier is introduced. A direct study of this model's Hilbert space leads to its partition function in the regular harmonic oscillator potential. This study of the Hilbert space allows the grounds states of these sectors to be determined. The ground state energy of the mixed sector is shifted with respect to that of the singlet sector, indicating the presence of a Fermi sea. However, the ground states of the symmetric and antisymmetric sectors are the same as that of the singlet sector, making the appearance of a Fermi sea unlikely. A collective field theory approach is used to study the symmetric and antisymmetric sectors. A study of this collective field with the harmonic oscillator potential reproduces the partition function results

[1] Note that none of these representations are irreducible. This means that the sectors associated with them could be further subdivided into non-interacting superselection sectors. This will be further explored in section 2.5.

while the inverted harmonic oscillator potential corresponding to the original c=1 matrix model leads to the identification of long strings. Consistent with the partition function computation, in both these sectors the long strings are not found to form a Fermi sea, instead they are found to be non-interacting in the large N limit. This collective field does not permit the study of the mixed sectors which are left for future work.

Chapter 2

Background

This section will expand upon the background information needed for the rest of this thesis. First the $c = 1$ matrix model as well as the sectors of interest will be defined. Then two dimensional string theory along with the linear dilaton background will be briefly reviewed. Next, the connections between this string theory and the matrix model will be explained. Following this, some results for the simplest non-singlet sector of the matrix model as well as its interpretation in the string theory will be reviewed. Finally, the non-singlet sectors of interest for the FZZT brane will be introduced and unpublished work calculating partition functions of these sectors providing motivation for this thesis will be presented. In particular, the partition functions of the non-singlet sectors of interest in the harmonic oscillator potential will be computed at large N and evidence of a shift in the ground state energy for the mixed sector will be found in equation (2.75).

2.1 $c = 1$ matrix model

The $c = 1$ matrix model is the theory consisting of an $N \times N$ Hermitian matrix with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\left[\dot{M}^2\right] - V(M), \tag{2.1}$$

where the potential is the trace of a polynomial of M.

There are two cases of interest: $V = \frac{1}{2}\mathrm{Tr}\, M^2$, the harmonic oscillator potential which is easier to study and $V = -\frac{1}{2}\mathrm{Tr}\, M^2$, the inverted harmonic oscillator potential which is the $c = 1$ model proper and which corresponds to two dimensional string theory.

In either case, this matrix model can be rewritten as a direct product of N^2 copies of the single particle theory with the potential $V(x)$ by considering the components of the matrix as independent degrees of freedom. For the harmonic oscillator potential,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\mathrm{Tr}\left[\dot{M}^2\right] - \frac{1}{2}\mathrm{Tr}\left[M^2\right] \\ &= \frac{1}{2}\sum_{i,j=1}^{N}\left[|\dot{M}_{ij}|^2 - |M_{ij}|^2\right],\end{aligned} \tag{2.2}$$

which is the Lagrangian for N real oscillators along the diagonal of the matrix and $\frac{1}{2}N(N-1)$ complex oscillators from the off-diagonal components which can be reorganised into a total of N^2 real oscillators.

This theory has a global $U(N)$ symmetry: $M \to UMU^\dagger$, with $U \in U(N)$ such that $U^\dagger = U^{-1}$.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\mathrm{Tr}\left[U\dot{M}U^\dagger U\dot{M}U^\dagger\right] - \frac{1}{2}\mathrm{Tr}\left[UMU^\dagger UMU^\dagger\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\dot{M}^2\right] - \frac{1}{2}\mathrm{Tr}\left[M^2\right].\end{aligned} \tag{2.3}$$

This implies the existence of a conserved charge and that the Hilbert space can be written as a direct sum of distinct superselection sectors carrying different values of this charge which do not interact with each other. The states in these sectors can be organised into copies of a particular representation of the $U(N)$ symmetry which allows us to associate a representation of the $U(N)$ group to each sector. These sectors can be studied by restricting the wavefunction to states transforming in the appropriate representation. Note that this symmetry mixes the N^2 degrees of freedom corresponding to the components of the matrix, so that all the sectors can include combinations of all the matrix elements.

The simplest sector is the singlet sector, where the wavefunction is invariant under the $U(N)$ symmetry. In this sector, all the matrix indices that appear in the wavefunction must be contracted. Such a wavefunction can be written as products of traces of functions of the matrix[2]. For example, in the harmonic oscillator case, from looking at the matrix model as N^2 oscillators it is clear that the ground state is the product of each of the single particle grounds states. This is proportional to

$$\prod_{i,j=1}^{N}\exp\left[-\frac{1}{2}\left|M_{ij}\right|^2\right] = \exp\left[-\frac{1}{2}\sum_{i,j=1}^{N}M_{ij}M_{ji}\right] = \exp\left[-\frac{1}{2}\mathrm{Tr}\left(M^2\right)\right]. \tag{2.4}$$

[2]Functions of the matrix are the functions $f : U(N) \to U(N)$ defined by inserting a matrix into the powers series of an analytic function $f : \mathbb{R} \to \mathbb{R}$ whose Taylor series has an infinite radius of convergence.

Thus the ground state of the matrix model with the harmonic oscillator potential is in the singlet sector.

2.2 Two dimensional string theory

The perturbation theory of two dimensional closed bosonic string theory is well studied. This section is based on a review paper written by Polchinski [1] whose notation will be used. The key fact is that a linear dilaton background allows the construction of a consistent bosonic string theory in less than the usual 26 dimensions. This means constructing a bosonic string theory in the background

$$G_{\mu\nu} = \eta_{\mu\nu} \qquad B_{\mu\nu} = 0 \qquad \Phi = \frac{1}{2}QX^1, \tag{2.5}$$

with $Q^2 = \frac{26-D}{3}$. The central charge of the matter part of the system is $c = D + 3Q^2 = 26$ as required to cancel the ghost contribution. For two dimensions, $D = 2$ and $Q^2 = 8$.

The string coupling is

$$g_s \sim e^{\Phi} = e^{\frac{1}{2}QX^1}, \tag{2.6}$$

which is position dependent. For $X^1 \to -\infty$ the coupling is small and perturbation theory can be applied to asymptotic states from the free theory. However, for $X^1 \to \infty$ perturbation theory breaks down.

In two dimensions, there is no oscillators for the vibrations of the strings remaining after gauge fixing. The only degree of freedom is the center of mass of the string, usually called the tachyon mode. The weights of the relevant vertex operators, studied in [1], give a mass shell condition for the tachyon of $k^2 = 0$, so that the "tachyon" is actually massless. A tachyon background forms, which cuts off propagation of this tachyon mode at large X^1, before the strong coupling region is reached. In summary, the degrees of freedoms of this theory are those of a spinless boson, with interactions that are weak for $X^1 \to -\infty$ and which cannot propagate to large positive X^1.

2.3 Connection between two dimensional string theory and the singlet sector of this matrix model

It has been established, for example in [1, 2], that this two dimensional theory of closed strings is equivalent to a scaling limit at large N of the singlet sector of this matrix model with the inverted harmonic oscillator potential. Proving this equivalence is beyond the scope of this thesis, but the scalar field in the matrix model whose excitations correspond to the closed string tachyon will be identified using a collective field approach.

The idea, used in for example in [5], is to introduce a collective field, $\hat{\rho}(k) = \mathrm{Tr}\left[e^{ikM}\right]$ and its Fourier transform back into position space $\rho(x) = \int \frac{\mathrm{d}k}{2\pi} e^{-ikx} \mathrm{Tr}\left[e^{ikM}\right]$. Then any trace of a function of a matrix M can be rewritten

$$\mathrm{Tr}\left[f(M)\right] = \int \mathrm{d}x \rho(x) f(x). \tag{2.7}$$

This leads us to write $\rho(x) = \mathrm{Tr}[\delta(x-M)]$, where the δ function for matrices is defined to reproduce the above result.

Then in the singlet sector, the wavefunction is a function of only this collective field. The next step is to change variables in the Hamiltonian. The potential is straightforward,

$$\pm\frac{1}{2}\mathrm{Tr}\left[M^2\right] = \pm\frac{1}{2}\int \mathrm{d}x \rho(x) x^2. \tag{2.8}$$

Using the chain rule, the kinetic term is,

$$\begin{aligned}
&\sum_{i,j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} \\
&= \int \mathrm{d}x \sum_{i,j} \frac{\partial^2 \rho(x)}{\partial M_{ij} \partial M_{ji}} \frac{\partial}{\partial \rho(x)} + \int \mathrm{d}x \mathrm{d}y \sum_{i,j} \frac{\partial \rho(x)}{\partial M_{ij}} \frac{\partial \rho(y)}{\partial M_{ji}} \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} \\
&= \int \mathrm{d}x \omega(x) \frac{\partial}{\partial \rho(x)} + \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)},
\end{aligned} \tag{2.9}$$

where $\omega(x)$ and $\Omega(x,y)$ are defined to encapsulate the terms that arise when preforming this change of variables. Calculating these terms will then be a simple exercise in taking partial derivatives.

However, this change of variables also introduces a Jacobian into the integration measure when calculating inner products. Instead of dealing with a changed inner product, this Jacobian can be absorbed into the definitions of our wavefunctions. In doing so, all of our operators transform as $\mathcal{O} \to (\mathcal{J}[\rho])^{\frac{1}{2}}\, \mathcal{O}\, (\mathcal{J}[\rho])^{-\frac{1}{2}}$, since

$$
\begin{aligned}
&\langle \Phi | \mathcal{O} | \Psi \rangle \\
&\quad = \int [\mathrm{d}M]\, \Phi^*(M) \mathcal{O} \Psi(M) \\
&\quad = \int [\mathrm{d}\rho(x)]\, \mathcal{J}[\rho] \Phi^*[\rho] \mathcal{O} \Psi[\rho] \\
&\quad = \int [\mathrm{d}\rho(x)]\, \Phi^*[\rho]\, (\mathcal{J}[\rho])^{\frac{1}{2}}\, (\mathcal{J}[\rho])^{\frac{1}{2}}\, \mathcal{O}\, (\mathcal{J}[\rho])^{-\frac{1}{2}}\, (\mathcal{J}[\rho])^{\frac{1}{2}}\, \Psi[\rho] \\
&\quad = \int [\mathrm{d}\rho(x)]\, \widetilde{\Phi}^*[\rho] \widetilde{\mathcal{O}} \widetilde{\Psi}[\rho].
\end{aligned} \tag{2.10}
$$

Applied to $\frac{\partial}{\partial \rho(x)}$ this results in a shift

$$
\begin{aligned}
\frac{\partial}{\partial \rho(x)} &\to (\mathcal{J}[\rho])^{\frac{1}{2}}\, \frac{\partial}{\partial \rho(x)}\, (\mathcal{J}[\rho])^{-\frac{1}{2}} \\
&= \frac{\partial}{\partial \rho(x)} - \frac{1}{2} (\mathcal{J}[\rho])^{-1} \frac{\partial \mathcal{J}[\rho]}{\partial \rho(x)} \\
&= \frac{\partial}{\partial \rho(x)} - \frac{1}{2} J(x).
\end{aligned} \tag{2.11}
$$

Applying this shift to the $\sum_{i,j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}}$ operator from equation (2.9) gives

$$
\sum_{i,j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} = \int \mathrm{d}x \omega(x) \frac{\partial}{\partial \rho(x)} + \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} \tag{2.12}
$$

$$
\begin{aligned}
\to &\int \mathrm{d}x \omega(x) \frac{\partial}{\partial \rho(x)} - \frac{1}{2} \int \mathrm{d}x \omega(x) J(x) \\
&+ \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} + \frac{1}{4} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) J(x) J(y) \\
&- \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \left[J(x) \frac{\partial}{\partial \rho(y)} + \frac{\partial}{\partial \rho(x)} J(y) \right].
\end{aligned} \tag{2.13}
$$

This transformation ensures that the inner product is the standard one and so the adjoint of this operator is

$$\left[\sum_{i,j}\frac{\partial}{\partial M_{ij}}\;\frac{\partial}{\partial M_{ji}}\right]^{\dagger} = -\int \mathrm{d}x\frac{\partial}{\partial\rho(x)}\omega(x) - \frac{1}{2}\int \mathrm{d}x\omega(x)J(x) \tag{2.14}$$

$$+\int \mathrm{d}x\mathrm{d}y\frac{\partial}{\partial\rho(x)}\frac{\partial}{\partial\rho(y)}\Omega(x,y) + \frac{1}{4}\int \mathrm{d}x\mathrm{d}y\Omega(x,y)J(x)J(y)$$

$$+\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[J(y)\frac{\partial}{\partial\rho(x)} + \frac{\partial}{\partial\rho(y)}J(x)\right]\Omega(x,y)$$

$$= -\int \mathrm{d}x\omega(x)\frac{\partial}{\partial\rho(x)} - \frac{1}{2}\int \mathrm{d}x\omega(x)J(x) \tag{2.15}$$

$$+\int \mathrm{d}x\mathrm{d}y\Omega(x,y)\frac{\partial}{\partial\rho(x)}\frac{\partial}{\partial\rho(y)} + \frac{1}{4}\int \mathrm{d}x\mathrm{d}y\Omega(x,y)J(x)J(y)$$

$$+\int \mathrm{d}x\mathrm{d}y\Omega(x,y)J(y)\frac{\partial}{\partial\rho(x)}$$

$$-\int \mathrm{d}x\left(\frac{\partial}{\partial\rho(x)}\omega(x)\right) + \int \mathrm{d}x\mathrm{d}y\left(\frac{\partial}{\partial\rho(x)}\frac{\partial}{\partial\rho(y)}\Omega(x,y)\right)$$

$$+\int \mathrm{d}x\mathrm{d}y\left(\frac{\partial}{\partial\rho(x)}\Omega(x,y)\right)\frac{\partial}{\partial\rho(y)}$$

$$+\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left(\left[J(y)\frac{\partial}{\partial\rho(x)} + \frac{\partial}{\partial\rho(y)}J(x)\right]\Omega(x,y)\right)$$

$$+\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\Omega(x,y)\left(\frac{\partial}{\partial\rho(y)}J(x)\right).$$

Since $\sum_{i,j}\frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial M_{ji}}$ is Hermitian, equations (2.13) and (2.15) require that

$$-\omega(x) + \int \mathrm{d}y\Omega(x,y)J(y) = 0. \tag{2.16}$$

This condition is sufficient to fix J. Substituting this condition back into equation (2.13),

$$\sum_{i,j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} = \int \mathrm{d}x \omega(x) \frac{\partial}{\partial \rho(x)} - \frac{1}{2} \int \mathrm{d}x \omega(x) J(x) \tag{2.17}$$

$$+ \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} + \frac{1}{4} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) J(x) J(y)$$

$$- \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \left[J(x) \frac{\partial}{\partial \rho(y)} + \frac{\partial}{\partial \rho(x)} J(y) \right]$$

$$= \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial}{\partial \rho(x)} \frac{\partial}{\partial \rho(y)} - \frac{1}{4} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) J(x) J(y)$$

$$- \frac{1}{2} \int \mathrm{d}x \mathrm{d}y \Omega(x,y) \frac{\partial J(y)}{\partial \rho(x)}. \tag{2.18}$$

The $\frac{\partial J(y)}{\partial \rho(x)}$ term is infinite and corresponds to an operator ordering ambiguity. It will have no effect and will be dropped in what follows.

2.3.1 Finding ω and Ω

The Ω and ω involved in this change of variables are found by taking derivatives of $\rho(x) = \int \frac{\mathrm{d}k}{2\pi} e^{-ikx} \mathrm{Tr}\left[e^{ikM}\right]$ by M. These can be evaluated using the identity

$$\frac{\partial}{\partial M_{ba}} (e^{ikM})_{cd} = (ik) \int_0^1 \mathrm{d}\alpha (e^{ik\alpha M})_{cb} (e^{ik(1-\alpha)M})_{ad}. \tag{2.19}$$

We obtain

$$\omega(x) = \frac{\partial^2 \rho(x)}{\partial M_{ij} \partial M_{ji}} \tag{2.20}$$

$$= \frac{\partial^2 \int \frac{\mathrm{d}k}{2\pi} e^{-ikx} \mathrm{Tr}\left[e^{ikM}\right]}{\partial M_{ij} \partial M_{ji}}$$

$$= 2\partial_x \int \mathrm{d}y \frac{\rho(x)\rho(y)}{y-x} \tag{2.21}$$

and

$$\Omega(x,y) = \frac{\partial \rho(x)}{\partial M_{ij}} \frac{\partial \rho(y)}{\partial M_{ji}} \tag{2.22}$$

$$= \partial_x \partial_y (\delta(x-y)\rho(x)). \tag{2.23}$$

The consistency relation from equation (2.16) is sufficient to find J:

$$\begin{aligned}
-\omega(x) + \int \mathrm{d}y \Omega(x,y) J(y) =& 0 \\
2\partial_x \int \mathrm{d}y \frac{\rho(x)\rho(y)}{y-x} =& \int \mathrm{d}y \partial_x \partial_y (\delta(x-y)\rho(x)) J(y) \\
2\partial_x \left[\rho(x) \int \mathrm{d}y \frac{\rho(y)}{y-x}\right] =& -\partial_x \left[\rho(x)\partial_x J(x)\right],
\end{aligned} \tag{2.24}$$

which is satisfied by

$$\partial_x J(x) = 2 \int \mathrm{d}y \frac{\rho(y)}{x-y}. \tag{2.25}$$

Then equation (2.18) gives

$$\begin{aligned}
&-\frac{1}{2}\sum_{i,j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} \\
&= -\frac{1}{2}\int \mathrm{d}x\mathrm{d}y \Omega(x,y) \frac{\partial}{\partial\rho(x)}\frac{\partial}{\partial\rho(y)} + \frac{1}{8}\int \mathrm{d}x\mathrm{d}y \Omega(x,y) J(x) J(y) \\
&= -\frac{1}{2}\int \mathrm{d}x \rho(x) \partial_x \frac{\partial}{\partial\rho(x)} \partial_x \frac{\partial}{\partial\rho(x)} + \frac{1}{8}\int \mathrm{d}x \rho(x) \partial_x J(x) \partial_x J(x) \\
&= -\frac{1}{2}\int \mathrm{d}x \rho(x) \partial_x \frac{\partial}{\partial\rho(x)} \partial_x \frac{\partial}{\partial\rho(x)} + \frac{1}{2}\int \mathrm{d}x \rho(x) \left(\int \mathrm{d}y \frac{\rho(y)}{x-y}\right)^2.
\end{aligned} \tag{2.26}$$

This can be simplified by using an identity from [6], which can be straightforwardly be derived in Fourier space:

$$\int \mathrm{d}x \rho(x) \left(\int \mathrm{d}y \frac{\rho(y)}{x-y}\right)^2 = \frac{\pi^2}{3}\int \mathrm{d}x \left(\rho(x)\right)^3. \tag{2.27}$$

Further, $\mathrm{Tr}\, 1 = N$, where N is the size of the matrix in our matrix model. This must be imposed as a constraint in the collective field theory, where it translates to $\int \mathrm{d}x \rho(x) = N$. This can be achieved by adding a Lagrange multiplier term $\mu\left[N - \int \mathrm{d}\rho(x)\right]$ to the Hamiltonian, then fixing μ appropriately to enforce the constraint.

Finally, an operator formalism with $\frac{\partial}{\partial\rho(x)} \to i\Pi(x)$ where $\Pi(x)$ is the canonical momentum related to $\rho(x)$ can be used.

Putting all this together gives the Hamiltonian in the collective field formalism:

$$\begin{aligned}\mathcal{H} =& \frac{1}{2}\int \mathrm{d}x\rho(x)\partial_x\Pi(x)\partial_x\Pi(x) + \frac{\pi^2}{6}\int \mathrm{d}x\,(\rho(x))^3 \\ &+ \int \mathrm{d}x\rho(x)\,(V(x)-\mu) + \mu N. \end{aligned} \tag{2.28}$$

2.3.2 Harmonic oscillator potential

In the harmonic oscillator case, where $V(x) = \frac{1}{2}\mathrm{Tr}\, M^2 = \frac{1}{2}\int \mathrm{d}\rho(x)x^2$, the Hamiltonian is

$$\begin{aligned}\mathcal{H} =& -\frac{1}{2}\int \mathrm{d}x\rho(x)\partial_x\frac{\partial}{\partial\rho(x)}\partial_x\frac{\partial}{\partial\rho(x)} + \frac{\pi^2}{6}\int \mathrm{d}x\,(\rho(x))^3 \\ &+ \int \mathrm{d}x\rho(x)\left(\frac{1}{2}x^2-\mu\right) + \mu N. \end{aligned} \tag{2.29}$$

The equivalence with string theory requires scaling the fields and coordinates with N in the large N limit. To make this explicit, the fields and coordinates must be rescaled as

$$x \to \sqrt{N}x, \tag{2.30}$$
$$\rho(x) \to \sqrt{N}\rho(x), \tag{2.31}$$
$$\Pi(x) \to \frac{1}{N}\Pi(x), \tag{2.32}$$
$$\mu \to N\mu. \tag{2.33}$$

Note that this ensures that $\int \mathrm{d}x\rho(x) \to N\left(\int \mathrm{d}x\rho(x)\right)$ so that in this rescaled system $\int \mathrm{d}x\rho(x) = 1$. $\Pi(x)$ must be rescaled to preserve the canonical commutation relations and μ must be rescaled to preserve the constraint.

After this rescaling,

$$\begin{aligned}\mathcal{H} =& -\frac{1}{2N^2}\int \mathrm{d}x\rho(x)\partial_x\frac{\partial}{\partial\rho(x)}\partial_x\frac{\partial}{\partial\rho(x)} \\ &+ N^2\left[\frac{\pi^2}{6}\int \mathrm{d}x\,(\rho(x))^3 + \int \mathrm{d}x\rho(x)\left(\frac{1}{2}x^2-\mu\right) + \mu\right]. \end{aligned} \tag{2.34}$$

Now, as all the N dependence has been made explicit, the large N limit can be taken. In the Hamiltonian at the highest order, $O(N^2)$, there is no kinetic term so the classical stationary solution to this problem must be found. This means finding $\rho_0(x)$ such that the energy is minimized,

$$0 = \left.\frac{\partial \mathcal{H}}{\partial \rho(x)}\right|_{\rho=\rho_0} \tag{2.35}$$

$$= \frac{N^2}{2}\int \mathrm{d}x \left[(\pi\rho_0(x))^2 + x^2 - 2\mu\right] \tag{2.36}$$

$$\implies \rho_0(x) = \frac{1}{\pi}\sqrt{2\mu - x^2}, \qquad x \in [-\sqrt{2\mu}, \sqrt{2\mu}]. \tag{2.37}$$

Then μ is found by insisting that

$$1 = \int \mathrm{d}\rho_0(x) \tag{2.38}$$

$$1 = \frac{1}{\pi}\int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \mathrm{d}x\sqrt{2\mu - x^2}$$

$$1 = \frac{2\mu}{\pi}\int_0^{\pi} \mathrm{d}\theta \sin^2\theta, \qquad x = \sqrt{2\mu}\cos\theta$$

$$1 = \frac{2\mu}{\pi}\frac{\pi}{2}$$

$$\mu = 1. \tag{2.39}$$

The next step is to use perturbation theory around this classical solution. The condition that $\int \mathrm{d}\rho(x) = 1$ is preserved by perturbing the classical solution by a total derivative:

$$\rho(x) = \rho_0(x) + \frac{1}{\sqrt{\pi}N}\partial_x\eta(x). \tag{2.40}$$

The momentum conjugate to $\eta(x)$ is

$$P_\eta(x) = \frac{1}{\sqrt{\pi}N}\partial_x\Pi(x). \tag{2.41}$$

Putting this together

$$\mathcal{H} = N^2\frac{1}{3\pi^2}\int \mathrm{d}x\left(2\mu - x^2\right) \tag{2.42}$$
$$+ \frac{1}{2}\int \mathrm{d}x\pi\rho_0(x)\left[(\partial_x\eta(x))^2 + (P_\eta(x))^2\right] + O(N^{-1}).$$

The first term is a shift in the energy. The second term is equivalent to the standard free scalar field after a reparametrisation of x. The higher order in N^{-1} terms include the self interactions of this scalar field.

2.3.3 Inverted harmonic oscillator potential

Building off of the study of the regular harmonic oscillator case, the inverted oscillator potential which corresponds to the string theory can be analysed. All that is required is to change the sign of the potential. The sign of μ is changed as well for convenience. Doing the same rescaling of the fields

$$\begin{aligned}\mathcal{H} =& -\frac{1}{2N^2}\int \mathrm{d}x\rho(x)\partial_x\frac{\partial}{\partial\rho(x)}\partial_x\frac{\partial}{\partial\rho(x)} \\ &+ N^2\left[\frac{\pi^2}{6}\int \mathrm{d}x\,(\rho(x))^3 + \int \mathrm{d}x\rho(x)\left(\mu - \frac{1}{2}x^2\right) - \mu\right].\end{aligned} \tag{2.43}$$

The classical solution in the large N limit is

$$0 = \left.\frac{\partial\mathcal{H}}{\partial\rho(x)}\right|_{\rho=\rho_0} \tag{2.44}$$

$$=\frac{N^2}{2}\int \mathrm{d}x\left[(\pi\rho_0(x))^2 + 2\mu - x^2\right] \tag{2.45}$$

$$\implies \rho_0(x) = \frac{1}{\pi}\sqrt{x^2-2\mu}, \qquad x\in[-\infty,-\sqrt{2\mu}]\cup[\sqrt{2\mu},\infty]. \tag{2.46}$$

Perturbing the solution in the same way,

$$\begin{aligned}\mathcal{H} =& N^2\frac{1}{3\pi^2}\int \mathrm{d}x\left(x^2-2\mu\right) \\ &+\frac{1}{2}\int \mathrm{d}x\pi\rho_0(x)\left[(\partial_x\eta(x))^2 + (P_\eta(x))^2\right] + O(N^{-1}).\end{aligned} \tag{2.47}$$

This is the scalar field corresponding to the closed string theory in two dimensions. Again, proving this correspondence by looking at higher order terms to find the self interactions and comparing scattering amplitudes is beyond the scope of this introduction. However the rest of this thesis will use a similar collective field method to identify the new degrees of freedom in the non-singlet sectors of the matrix model.

2.4 The adjoint sector and long strings

The adjoint sector of the matrix model will be found to correspond to the presence of a single long string in the string theory, but first some general properties of the wavefunction will be needed to study these non-singlet sectors. Following [6], consider a basis of wavefunctions in a particular representation, $\Psi_\alpha(M)$, Greek indices being $U(N)$ representation indices. This means that

$$\Psi_\alpha(UMU^\dagger) = U_{\alpha\beta}\Psi_\beta(M). \tag{2.48}$$

Now consider the action of a diagonal $U(N)$ transformation D acting at a point where $M = \Lambda$ is diagonal:

$$\Psi_\alpha(\Lambda) = \Psi_\alpha(D\Lambda D^\dagger) = D_{\alpha\beta}\Psi_\beta(\Lambda), \tag{2.49}$$

which implies that $\Psi_\alpha(\Lambda)$ must be a zero weight vector of this representation. If v_α^k is a basis for the K dimensional zero weight subspace of a particular representation,

$$\Psi_\alpha(\Lambda) = \sum_{k=1}^{K} f_k(\Lambda) v_\alpha^k \tag{2.50}$$

$$\implies \Psi_\alpha(M = \Omega\Lambda\Omega^\dagger) = \sum_{k=1}^{K} f_k(\Lambda)\Omega_{\alpha\beta} v_\beta^k, \tag{2.51}$$

where the fact that any Hermitian matrix can be diagonalised is used.

Thus understanding the zero weight vectors of a representation is a necessary first step. In particular only representations with zero weight vectors need to be considered.

A traceless matrix C in the adjoint representation transforms as $U_{\alpha\beta}C_\beta = UCU^\dagger$ under the action of $U \in U(N)$. This means that for a zero weight vector C,

$$C_\alpha = D_{\alpha\beta}C_\beta \quad \implies \quad C = DCD^\dagger \tag{2.52}$$

$$C_{ij} = D_{ii}D_{jj}^* C_{ij}. \tag{2.53}$$

Since D is an arbitrary unitary diagonal matrix so that $D_{ii}^* = D_{ii}^{-1}$, this requires that C be diagonal. The condition that $\operatorname{Tr} C = 0$ means that this is an $N-1$ dimensional space of zero weight vectors.

The matrix M can be diagonalised as in equation (2.51) which showed that in the singlet sector the degrees of freedom are those of the N eigenvalues, so that $M = \Omega\Lambda\Omega^\dagger$. However, this diagonalisation does not completely fix Ω; there is a freedom to permute the eigenvalues making them indistinguishable. However, this permutation also changes the zero weight vector C. The basis of zero weight vectors can be chosen to be $v^(k)_{ij} = \delta_{ik}\delta jk - \delta iN\delta jN$. Then a transposition of the l^{th} and m^{th} eigenvalues also transposes the l^{th} and m^{th} zero weight basis vectors. Thus this new degree of freedom can be thought of as a label which distinguishes an eigenvalue.[3] See [6] for the details.

2.4.1 Collective fields in the adjoint

An alternate approach is available by using the fact that in this sector, the wavefunction must transform in the adjoint representation of $U(N)$. It must be of the form

$$\Psi(M) = \mathrm{Tr}(Af(M))\psi(\rho), \tag{2.54}$$

where A is an arbitrary matrix in the adjoint representation, not necessarily a zero weight vector.

Multipling such a wavefunction by a factor that is invariant under the action of $U(N)$ and so is in the singlet gives another wavefunction in the adjoint. Thus any wavefunction in the Hilbert space of the adjoint sector can be factored into a representative part that transforms in the adjoint times a part that transforms in the singlet, as in equation (2.54). The singlet part, as seen in section 2.3, can be written as a functional of the collective field ρ. Then the adjoint factor can be written as

$$\mathrm{Tr}(Af(M)) = \int \frac{\mathrm{d}k\mathrm{d}x}{2\pi}\mathrm{Tr}(Ae^{ikM})e^{-ikx}f(x) \tag{2.55}$$

$$= \int \mathrm{d}x\phi(x)f(x), \tag{2.56}$$

where $\phi(x) = \mathrm{Tr}(A\delta(x-M)) = \int \frac{\mathrm{d}k}{2\pi}\mathrm{Tr}(Ae^{ikM})e^{-ikx}$.

The wavefunction can be expressed as a function of two collective fields, ρ and ϕ, where the wavefunction is linear in ϕ:

$$\Psi(M) = \int \mathrm{d}x\phi(x)\Phi(\rho, x). \tag{2.57}$$

Acting with our Hamiltonian on a wavefunction of this form, its action in terms of these collective fields can be determined. The chain rule gives terms where both the derivatives of M act on the ρ variable reproducing the same terms seen for the singlet in section 2.3, as well as new terms coming from derivatives acting on ϕ.

[3] Which eigenvalue is distinguished depends of our choice of diagonalisation, the important point is that there is two sets of eigenvalues, the labelled and the unlabelled ones, and the permutations left after diagonalising do not mix them. In the adjoint, there is 1 labelled eigenvalue and $N-1$ unlabelled eigenvalues.

The same notation as for the singlet sector will be used, but with two collective fields to be summed over. $\omega(x;\rho)$ is unchanged and a new $\omega(x;\phi)$ is added. Note that in this case, $\Omega(x,y;\phi,\phi) = \frac{\partial\phi(x)}{\partial M_{ab}}\frac{\partial\phi(y)}{\partial M_{ba}}$ is not needed as the wavefunction is linear in ϕ so that $\frac{\partial}{\partial\phi(x)}\frac{\partial}{\partial\phi(y)}\Psi(\rho,\phi) = 0$.

All that is left is to calculate the relevant derivatives. This has been done before, in particular in [6] and the results will be quoted here without derivation. The Jacobian for the ϕ field will not yet be needed, since the derivatives $\frac{\partial}{\partial\phi(x)}$ will be evaluated to find the action of the Hamiltonian on $\Phi(x,\rho)$. This can be thought of as the wavefunction for the density of eigenvalues and the one impurity that is labelled by our zero weight vector.

The new terms are

$$\begin{aligned}
\omega(x;\phi) &= \frac{\partial^2\phi}{\partial M_{ij}\partial M_{ji}} \\
&= 2\partial_x \int \mathrm{d}y \frac{\phi(x)\rho(y)}{y-x} + 2\int \mathrm{d}y \frac{\rho(x)\phi(y)-\phi(x)\rho(y)}{(x-y)^2} \qquad (2.58)\\
\Omega(x,y;\rho,\phi) &= \frac{\partial\rho(x)}{\partial M_{ij}}\frac{\phi(y)}{\partial M_{ji}} = \partial_x\partial_y(\delta(x-y)\phi(x)), \qquad (2.59)
\end{aligned}$$

which when added to those from the singlet sector described in equation (2.28) give

$$\begin{aligned}
\mathcal{H}_{adj} =& -\frac{1}{2}\int \mathrm{d}x \left[\omega(x;\phi) - \int \mathrm{d}y\Omega(y,x;\rho,\phi)\left(\frac{\partial}{\partial\rho(y)} - \frac{1}{2}J(y;\rho)\right)\right]\frac{\partial}{\partial\phi(x)}, \\
=& -\int \mathrm{d}x\mathrm{d}y \left[\partial_x\left(\frac{\phi(x)\rho(y)}{y-x}\right) + \frac{\rho(x)\phi(y)-\phi(x)\rho(y)}{(x-y)^2}\right]\frac{\partial}{\partial\phi(x)} \\
& - \int \mathrm{d}x\mathrm{d}y\partial_x\partial_y(\delta(x-y)\phi(x))\left(\frac{\partial}{\partial\rho(y)} - \int \mathrm{d}z\frac{\rho(z)}{x-z}\right)\frac{\partial}{\partial\phi(x)}, \\
=& -\int \mathrm{d}x\mathrm{d}y \left[\partial_x\left(\frac{\phi(x)\rho(y)}{y-x}\right) + \frac{\rho(x)\phi(y)-\phi(x)\rho(y)}{(x-y)^2}\right]\frac{\partial}{\partial\phi(x)} \\
& - \int \mathrm{d}x\phi(x)\left(\partial_x\frac{\partial}{\partial\rho(x)} - \int \mathrm{d}y\frac{\rho(y)}{x-y}\right)\partial_x\frac{\partial}{\partial\phi(x)}, \\
=& \int \mathrm{d}x\mathrm{d}y\frac{\phi(x)\rho(y)-\rho(x)\phi(y)}{(x-y)^2}\frac{\partial}{\partial\phi(x)} - \int \mathrm{d}x\phi(x)\partial_x\frac{\partial}{\partial\rho(x)}\partial_x\frac{\partial}{\partial\phi(x)}.
\end{aligned} \qquad (2.60)$$

Acting with this on the wavefunction

$$\mathcal{H}_{adj}\int \mathrm{d}x\phi(x)\Phi(x,\rho) \tag{2.61}$$

$$=\int \mathrm{d}x\phi(x)\left[\int \mathrm{d}y\frac{\rho(y)}{(x-y)^2}\left[\Phi(x,\rho)-\Phi(y,\rho)\right]+\partial_x\Pi(x)(-i\partial_x)\Phi(x,\rho)\right]. \tag{2.62}$$

Extracting explicitly the scaling in the large N limit, according to equation (2.33), it becomes apparent that $\int \mathrm{d}y\frac{\rho(y)}{(x-y)^2}\left[\Phi(x,\rho)-\Phi(y,\rho)\right]$ scales as N^0 and $\partial_x\Pi(x)(-i\partial_x)\Phi(x,\rho)$ scales as N^{-2} and so can be dropped. Then to lowest order ρ can be replaced with ρ_0. Going straight to the inverted harmonic oscillator case and striping off the factor of $\int \mathrm{d}x\phi(x)$,

$$H_{adj}\Phi(x,\rho)=\int \mathrm{d}y\frac{\rho(y)}{(x-y)^2}\left[\Phi(x,\rho)-\Phi(y,\rho)\right]. \tag{2.63}$$

This is the same operator as found in equation (3.4) of [3] for the long string Hamiltonian. Using the same approach, $\tilde{\Phi}(x)=\rho_0(x)\Phi(x,\rho)$ and

$$H_{adj}\tilde{\Phi}(x)=-\int \mathrm{d}y\frac{\rho_0(x)\tilde{\Phi}(y)}{(x-y)^2}+\tilde{\Phi}(x)\left(\int \mathrm{d}y\frac{\rho(y)}{(x-y)^2}\right). \tag{2.64}$$

The time of flight variable τ is defined such that $\frac{\mathrm{d}x}{\mathrm{d}\tau}=\pi\rho_0$. In this case, $\rho_0=\frac{1}{\pi}\sqrt{x^2-2\mu}$, so $x=\sqrt{2\mu}\cosh\tau$, which gives

$$H_{adj}\tilde{\Phi}(\tau)=-\frac{1}{\pi}\int \mathrm{d}\tau'\frac{\tilde{\Phi}(\tau')}{4\sinh^2\left(\frac{\tau-\tau'}{2}\right)}+\tilde{\Phi}(\tau)\left(\int \mathrm{d}y\frac{\rho(y)}{(x-y)^2}\right). \tag{2.65}$$

The second term diverges and needs to be regulated. Using that

$$\int_{\sqrt{2\mu}}^{\infty}\mathrm{d}y\frac{\rho_0(y)}{(x-y)^2}=\frac{1}{\pi}\int_0^{\infty}\mathrm{d}\tau'\frac{\sinh^2\tau'}{(\cosh\tau-\cosh\tau')^2}, \tag{2.66}$$

the integral can be cut off in τ space at τ_c so that

$$\begin{aligned}
\int_{\sqrt{2\mu}}^{\infty} \mathrm{d}y \frac{\rho_0(y)}{(x-y)^2} &= \frac{1}{\pi}\int_0^{\tau_c} \mathrm{d}\tau' \frac{\sinh^2\tau'}{(\cosh\tau - \cosh\tau')^2} \\
&= \frac{1}{\pi\sinh\tau}\int_{-\tau_c}^{\tau_c} \mathrm{d}\tau' \frac{\sinh\tau'}{4\sinh^2(\frac{\tau-\tau'}{2})} \\
&= \frac{1}{\pi\sinh\tau}\int_{-\tau_c-\tau}^{\tau_c-\tau} \mathrm{d}\tau' \frac{\cosh\tau\sinh\tau' + \sinh\tau\cosh\tau'}{4\sinh^2(\frac{\tau'}{2})} \\
&= \frac{1}{\pi\tanh\tau}\int_{\frac{-\tau_c-\tau}{2}}^{\frac{\tau_c-\tau}{2}} \mathrm{d}\tau' \coth(\tau') + \frac{1}{\pi}\int_{\frac{-\tau_c-\tau}{2}}^{\frac{\tau_c-\tau}{2}} \mathrm{d}\tau' \left[1+\coth^2(\tau')\right] \\
&= \frac{1}{\pi\tanh\tau}\ln\left|\frac{\sinh(\frac{\tau_c-\tau}{2})}{\sinh(\frac{-\tau_c-\tau}{2})}\right| + \frac{1}{\pi}\left[\tau_c - 1\right] \\
&\approx \frac{1}{\pi\tanh\tau}\ln\left|\frac{\exp(\frac{\tau_c-\tau}{2})}{\exp(\frac{\tau_c+\tau}{2})}\right| + \frac{1}{\pi}\left[\tau_c - 1\right] \\
&= \frac{1}{\pi}\left[\frac{\tau}{\tanh\tau} + \tau_c - 1\right] .
\end{aligned} \tag{2.67}$$

This term is a constant shift of the energy along with a potential proportional to $\frac{\tau}{\tanh\tau}$. For large τ this is $|\tau|$ which can be interpreted as the tension of a long string.

It is shown in equation (3.9) of [3], by a Fourier transform, that the first term of (2.65) is the kinetic term required, along with the tension term found in equation (2.67), to interpret this Hamiltonian for the impurity as that of the tip of a long string stretching in from infinity. This is the long open string with its endpoints fixed at infinity on an FZZT brane discussed previously [3].

2.5 Non-singlet sectors and the FZZT brane

Gaiotto proposed condensing large numbers of these long string excitations of FZZT branes at infinity to bring the brane in to finite distance in [4]. This condensation should occur in a sector of the matrix model which contains many of these long strings in an antisymmetric representation so that a Fermi surface arises together with a tunable parameter, like μ in the singlet case, that controls the level of this surface.

There are representations which can be constructed in a similar manner to the adjoint by taking tensor products of the fundamental and anti-fundamental representations that are natural candidates. Recall that the

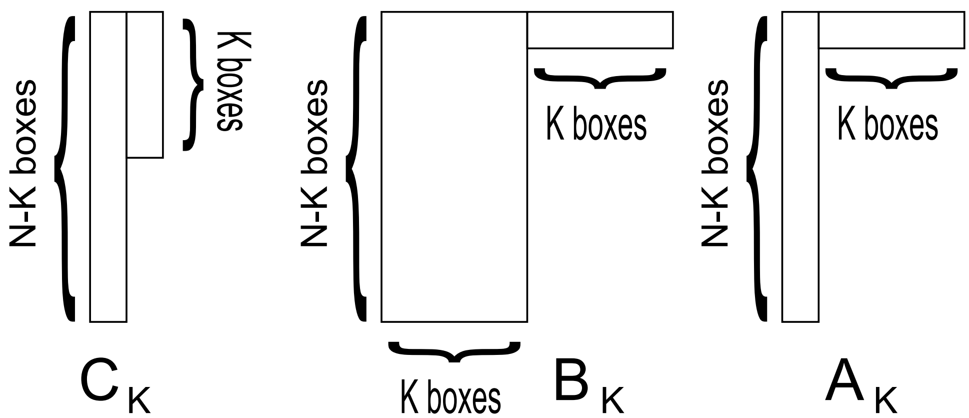

Figure 2.1: The shape of the Young tableaux of the three families of irreducible representations discussed in the text.

adjoint can be constructed by taking a tensor product of the fundamental and antifundamental representations and removing the singlet. That is that $N \otimes \bar{N} = Singlet \oplus Adjoint$. Three families of larger representations can be considered, the symmetric representations $\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}$, the antisymmetric representations $N^{\wedge K} \otimes \bar{N}^{\wedge K}$ and the mixed representations $\mathrm{Sym}^K N \otimes \bar{N}^{\wedge K}$ or equivalently $N^{\wedge K} \otimes \mathrm{Sym}^K \bar{N}$. Note that none of these are irreducible and can be expressed in terms of irreducible representations A_k, B_k and C_k, which are defined in terms of their Young tableaux in figure 2.1.

Using the Littlewood-Richardson rule

$$\begin{aligned} N^{\wedge K} \otimes \bar{N}^{\wedge K} &= \bigoplus_{p=0}^{\min(K,N-K)} C_p \\ \mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N} &= \bigoplus_{p=0}^{K} B_p \\ \mathrm{Sym}^K N \otimes \bar{N}^{\wedge K} &= A_K \oplus A_{K-1}. \end{aligned} \tag{2.68}$$

2.5.1 Partition functions

As a first step for understanding these new sectors, the work of Boulatov and Kazakov [7] will be used to calculate the partition function of the matrix model restricted to various sectors with the right side up harmonic oscillator. This is not the potential needed to reproduce the string theory, but will motivate our choice of representations due to a connection with the inverted oscillator that can be made through an analytic continuation [7].

Start with the fact from [7] that

$$Z_{sing}(q) = q^{\frac{N^2}{2}} \prod_{p=1}^{N} \frac{1}{1-q^p}, \tag{2.69}$$

where $Z_{sing}(q)$ is the partition function restricted to the singlet sector and $q = e^{-\beta}$. Note that the Hamiltonian, the temperature and all other quantities in this thesis are given in units where the frequency that was omitted from our original harmonic oscillator potential, ω, as well as c, $\hbar$ and k_B are all 1. Other systems of units could be recovered by inserting these factors as required.

The work of Boulatov and Kazakov [7] also gives a generating function for the partition function $Z_R(q)$ at finite N restricted to the sectors associated with the representations of interest:

$$\sum_{k=0}^{\infty} Z_{B_k}(q)x^k = \left(\prod_{p=1}^{N-1} \frac{1}{1-xq^p}\right) Z_{sing}(q) \tag{2.70}$$

$$\sum_{k=0}^{\infty} Z_{A_k}(q)x^k = \left(\prod_{p=1}^{N-1} (1+xq^p)\right) Z_{sing}(q). \tag{2.71}$$

There is no known simple formula for the C_k representations at finite N, but they can be studied in the large N limit.

Using the relations from equation (2.68)

$$\begin{aligned}\lim_{K\to\infty} Z_{\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}} &= \sum_{k=0}^{\infty} Z_{B_k} \\ &= \left(\prod_{p=1}^{N-1} \frac{1}{1-q^p}\right) Z_{sing}(q),\end{aligned} \tag{2.72}$$

which is the partition function of $N-1$ particles in a harmonic oscillator together with the N particles in a harmonic oscillator from in the singlet.

In the same manner, the generating function for the symmetric representations is

$$\sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes \mathrm{Sym}^k \bar{N}}(q) x^k = \frac{1}{1-x}\left(\prod_{p=1}^{N-1} \frac{1}{1-xq^p}\right) Z_{sing}(q)$$
$$= \left(\prod_{p=0}^{N-1} \frac{1}{1-xq^p}\right) Z_{sing}(q). \tag{2.73}$$

Formally setting x=1 gives the partition function for the combination of all of the symmetric sectors at once. This will have to diverge since it includes the sum of an infinite number of copies of each of the sectors corresponding to the irreducible representations B_k. Equation (2.73) shows that this divergence is encapsulated in the to the presence of a zero mode at $p = 0$ in the product.

Now by relating the mixed representations to the partition functions for the A_k in equation (2.71) using the relation in equation (2.68)

$$\sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes \bar{N}^{\wedge k}}(q) x^k = (1+x)\left(\prod_{p=1}^{N-1} (1+xq^p)\right) Z_{sing}(q)$$
$$= \left(\prod_{p=0}^{N-1} (1+xq^p)\right) Z_{sing}(q). \tag{2.74}$$

At large N this can be expressed as

$$Z_{\mathrm{Sym}^K N \otimes \bar{N}^{\wedge K}}(q) = q^{\frac{1}{2}K(K-1)}\left(\prod_{p=1}^{K} \frac{1}{1-q^p}\right) Z_{sing}(q). \tag{2.75}$$

The prefactor indicates a shift in the ground state energy of this sector of $\frac{1}{2}K(K-1)$. This point towards the existence of a non-trivial ground state and a Fermi sea.

Partition functions at large N

One can show that for large N the partition function of the symmetric sectors is the same as that for antisymmetric sectors. That is that

$$\sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes \mathrm{Sym}^k \bar{N}}(q) x^k = \sum_{k=0}^{\infty} Z_{N^{\wedge k} \otimes \bar{N}^{\wedge k}}(q) x^k = \left(\prod_{p=0}^{\infty} \frac{1}{1-xq^p}\right) Z_{sing}(q), \tag{2.76}$$

which means that for large N,

$$Z_{\mathrm{Sym}^K N\otimes \mathrm{Sym}^K \bar{N}}(q) = Z_{N^{\wedge K}\otimes \bar{N}^{\wedge K}}(q) = \left(\prod_{p=1}^{K}\frac{1}{1-q^p}\right) Z_{sing}(q). \tag{2.77}$$

The rest of this section will be dedicated to calculating these partition functions directly at large N in order to demonstrate this relationship.

Boulatov and Kazakov derived that [7]

$$Z_R(q) = \int_0^{2\pi}\prod_{k=1}^{N}\frac{\mathrm{d}\theta_k}{2\pi}\left|\Delta(e^{i\theta})\right|^2\prod_{k,m=1}^{N}\frac{1}{1-qe^{i(\theta_k-\theta_m)}}\chi_R\left(\mathrm{diag}(e^{i\theta_k})\right), \tag{2.78}$$

as the formula for calculating the partition function restricted to a particular sector, where $q = e^{-\beta}$, $\Delta(e^{i\theta}) = \prod_{k<m}[e^{i\theta_k} - e^{i\theta_m}]$ and χ_R is the character of the representation associated with the sector.

This can be expressed as

$$Z_R(q) = C\int_0^{2\pi}\prod_{k=1}^{N}\frac{\mathrm{d}\theta_k}{2\pi}e^{-S(\theta)}, \tag{2.79}$$

where $S = \frac{1}{2}\sum_{k,m}V(\theta_k-\theta_m) + \ln\chi_R(\mathrm{diag}(e^{i\theta_k}))$. The two body potential is

$$\begin{aligned}
V(\theta_k-\theta_m) &= \begin{cases} -\ln\left|\frac{e^{i\theta_k}-e^{i\theta_m}}{1-qe^{i(\theta_k-\theta_m)}}\right|^2 & \text{for } k\neq m\\ 2\ln(1-q) & \text{for k=m}\end{cases} && (2.80)\\
&= -\ln\frac{\sin^2(\frac{\theta_k-\theta_m}{2})+\epsilon^2}{q\sin^2(\frac{\theta_k-\theta_m}{2})+(\frac{1-q}{2})^2} && (2.81)\\
&= \ln\left(\frac{(1-q)^2}{4\sin^2(\frac{\theta_k-\theta_m}{2})+4\epsilon^2}+q\right), && (2.82)
\end{aligned}$$

where ϵ is a regulator that will be taken to zero which removes the need for the special $k = m$ case. Thus the partition function can be interpreted as that of N particles living on a circle with an action including a two body potential.

Starting with the singlet sector, $\chi_{sing} = 1$, which leaves only the repulsive two body potential. In the large N limit the system can be described by a particle density $\rho(\theta) = N^{-1}\sum_k \delta(\theta_k - \theta)$, so that

$$S = \frac{1}{2}\sum_{k,m} V(\theta_k - \theta_m) = N^2 \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 V(\theta_1 - \theta_2)\rho(\theta_1)\rho(\theta_2). \tag{2.83}$$

Then, by symmetry, the lowest energy configuration must have constant density. The partition function can be evaluated by considering variations of this density,

$$\rho(\theta) = \frac{1}{2\pi} + \sum_{n\neq 0} c_n e^{in\theta}, \qquad\qquad c_{-n} = c_n^*. \tag{2.84}$$

Substituting this into the action gives

$$N^{-2}S = S_0 + \frac{1}{2}\sum_{n,m} c_n c_m \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 V(\theta_1 - \theta_2) e^{in\theta_1 + im\theta_2} \tag{2.85}$$

$$= S_0 + 2\pi \sum_{n>0} |c_n|^2 \int_0^{2\pi} \mathrm{d}u V(u)\cos(nu), \tag{2.86}$$

where the terms linear in c_n vanishes as they must.

We obtain the partition function,

$$Z_{sing}(q) = C\int \mathrm{d}^2 c_n e^{-S(c_n)}. \tag{2.87}$$

Evaluating the Gaussian integrals gives

$$Z_{sing}(q) = C\prod_{n>0} \frac{1}{2N^2 A_n}, \tag{2.88}$$

where

$$A_n = \int_o^{2\pi} \mathrm{d}u V(u)\cos(nu) \tag{2.89}$$

$$= \int_o^{2\pi} \mathrm{d}u \ln\left(\frac{(1-q)^2}{4\sin^2(\frac{\theta_k - \theta_m}{2}) + 4\epsilon^2} + q\right)\cos(nu) \tag{2.90}$$

$$= \frac{2\pi}{n}(1 - q^n), \tag{2.91}$$

so that

$$Z_{sing}(q) = C\prod_{n>0} \frac{n}{4\pi N^2(1-q^n)}. \tag{2.92}$$

Setting the q independent normalisation by requiring that $Z(q=0)=1$ gives

$$Z_{sing}(q)=\prod_{n>0}\frac{1}{1-q^n}. \tag{2.93}$$

To calculate the partition function for the symmetric and antisymmetric sectors, the relevant characters will be required. They can be obtained from the standard generating functions

$$\sum_{k=0}^{\infty}x^k\chi_{N^k}(U)=e^{\mathrm{Tr}\ln(1+xU)} \tag{2.94}$$

$$\sum_{k=0}^{\infty}x^k\chi_{\mathrm{Sym}^kN}(U)=e^{\mathrm{Tr}\ln(1+xU)}, \tag{2.95}$$

so that

$$\sum_{k,l=0}^{N}t^{k+l}e^{i(k-l)\omega}\chi_{N^k\otimes\bar{N}^l}=e^{\mathrm{Tr}\ln(1+te^{i\omega}U)}e^{\mathrm{Tr}\ln(1+te^{-i\omega}xU^\dagger)} \tag{2.96}$$

$$\sum_{k,l=0}^{N}t^{k+l}e^{i(k-l)\omega}\chi_{\mathrm{Sym}^kN\otimes\mathrm{Sym}^l\bar{N}}=e^{-\mathrm{Tr}\ln(1-te^{i\omega}U)}e^{-\mathrm{Tr}\ln(1-te^{-i\omega}xU^\dagger)}. \tag{2.97}$$

The $k=l$ case can be recovered by looking at the ω independent part of this.

Thus

$$\begin{aligned}\sum_{k,l=0}^{N}t^{k+l}e^{i(k-l)\omega}Z_{N^k\otimes\bar{N}^l}=&\int\prod_{k=1}^{N}\frac{\mathrm{d}\theta_k}{2\pi}e^{-\frac{1}{2}\sum_{k,m}V(\theta_k-\theta_l)}\\&\times e^{\sum_{j=1}^{N}\ln(1+te^{i(\omega+\theta_j)})}e^{\sum_{j=1}^{N}\ln(1+te^{-i(\omega+\theta_j)})}\end{aligned} \tag{2.98}$$

$$\begin{aligned}\sum_{k,l=0}^{N}t^{k+l}e^{i(k-l)\omega}Z_{\mathrm{Sym}^kN\otimes\mathrm{Sym}^l\bar{N}}=&\int\prod_{k=1}^{N}\frac{\mathrm{d}\theta_k}{2\pi}e^{-\frac{1}{2}\sum_{k,m}V(\theta_k-\theta_l)}\\&\times e^{-\sum_{j=1}^{N}\ln(1-te^{i(\omega+\theta_j)})}e^{-\sum_{j=1}^{N}\ln(1-te^{-i(\omega+\theta_j)})}.\end{aligned} \tag{2.99}$$

A shift of the θ_k removes the ω, so that these expressions are independent of ω.

Now, going to the large N limit and using the particle density formulation,

$$\pm \sum_j \ln(1 \pm te^{i\theta_j}) = \pm N \int d\theta \rho(\theta) \ln(1 \pm te^{i\theta}) \tag{2.100}$$

$$= \pm N \sum_{n \neq 0} c_n \int d\theta e^{in\theta} \ln(1 + te^{i\theta}) = N \sum_{n<0} c_n \left(\mp \frac{2\pi}{|n|} \right) (\mp t)^{|n|}$$

$$\pm \sum_j \ln(1 \pm te^{-i\theta_j}) = N \sum_{n>0} c_n \left(\mp \frac{2\pi}{|n|} \right) (\mp t)^{|n|}, \tag{2.101}$$

which means that

$$\sum_{k=0}^{\infty} Z_{N^k \otimes \bar{N}^k} t^{2k} = \tag{2.102}$$

$$C \int d^2 c_n e^{-N^2 S_0 - 2\pi N^2 \sum_{n>0} |c_n|^2 A_n - 4\pi \sum_{n>0} \frac{(-t)^n}{n} \Re(c_n)}$$

$$\sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes \mathrm{Sym}^k \bar{N}} t^{2k} = \tag{2.103}$$

$$C \int d^2 c_n e^{-N^2 S_0 - 2\pi N^2 \sum_{n>0} |c_n|^2 A_n + 4\pi \sum_{n>0} \frac{(t)^n}{n} \Re(c_n)}.$$

These are the same Gaussian integrals found in the singlet sector except for a linear term. Thus the result is

$$\sum_{k=0}^{\infty} Z_{N^k \otimes \bar{N}^k} t^{2k} = \sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes \mathrm{Sym}^k \bar{N}} t^{2k} \tag{2.104}$$

$$= Z_{sing}(q) e^{\sum_{n>0} \frac{t^{2n}}{n(1-q^n)}} = Z_{sing}(q) \prod_{k=0}^{\infty} \frac{1}{(1 - t^2 q^k)}, \tag{2.105}$$

which was the relationship to be demonstrated.

Chapter 3

The Matrix Model in Non-Singlet Sectors

This chapter will detail one method for studying the non-singlet sectors of interest, by comparing them to spin systems. Although, the Hamiltonian could not be written in terms of a standard spin-spin interaction, the Hilbert space can be understood in terms of spins and similarities to certain Calogero models are found. The first step to understanding the Hilbert space of these non-singlet sectors is to characterise the zero weight vectors of their related representations. This is presented in section 3.1. The Hilbert space of these sectors can then be understood in terms of a system of N particles with spins as explored in section 3.2. The action of Hamiltonian on this spin system is found in section 3.3 for each of the sectors of interest. The derived Hamiltonian does not correspond to that the standard spin-spin interaction Hamiltonian that might be expected for this system and could not be written in a simple way using spin picture operators. Finally, in section 3.4, some links to Calogero models are mentioned.

3.1 Zero weight vectors

The three representations of $U(N)$ that will be studied are the symmetric ones $\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}$, the antisymmetric ones $N^{\wedge K} \otimes \bar{N}^{\wedge K}$ and the mixed ones $\mathrm{Sym}^K N \otimes \bar{N}^{\wedge K}$ (which are equivalent to $N^{\wedge K} \otimes \mathrm{Sym}^K \bar{N}$).

Following [6], as noted in section 2.4, the wavefunction can be expressed as

$$\Psi_\alpha = \sum_{k=1}^{\tilde{K}} f_{(k)}(\lambda_i)\Omega_{\alpha\beta} v_\beta^{(k)}, \tag{3.1}$$

where $v^{(k)}$ are a basis of zero weight vectors of the representation, $\tilde{K}$ is the size of this basis of zero weight vectors, greek indices are representation indices and the matrix has been decomposed as $M = \Omega\Lambda\Omega^\dagger$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Thus, the first step is to find the zero weight vectors of the representations of interest.

The zero weight vectors satisfy

$$D_{\alpha\beta} v_\beta = v_\alpha,$$

where D is a diagonal matrix in $U(N)$. Then if

$$\{|i\rangle\}_{i=1\ldots N}$$

is a basis for the fundamental representation,

$$\{|i_1\rangle|i_2\rangle\ldots|i_K\rangle\langle j_1|\langle j_2|\ldots\langle j_K|\}_{i_n,j_n=1\ldots N}$$

is a basis for $V^{\otimes K}\otimes\bar{V}^{\otimes K}$. Then,

$$\{|k_1\rangle|k_2\rangle\ldots|k_K\rangle\langle k_1|\langle k_2|\ldots\langle k_K|\}_{k_n=1\ldots N} \tag{3.2}$$

are linearly independent zero weight vectors, since for $D=\operatorname{diag}(d_1,\ldots,d_N)$,

$$\begin{aligned} D_{\alpha\beta}v_\beta &\to D|k_1\rangle D|k_2\rangle\ldots D|k_K\rangle\langle k_1|D^\dagger\langle k_2|D^\dagger\ldots\langle k_K|D^\dagger \\ &= \left(|d_{k_1}|^2|d_{k_2}|^2\ldots|d_{k_K}|^2\right)\left(|k_1\rangle|k_2\rangle\ldots|k_K\rangle\langle k_1|\langle k_2|\ldots\langle k_K|\right), \end{aligned}$$

and for $D\in U(N)$, $|d_i|^2=1$. These can then be projected to the relevant subspaces to generate the zero weight vectors for the relevant representations.

This means that given a basis for V, the zero weight vectors for $V^{\otimes K}\otimes\bar{V}^{\otimes K}$ are in one-to-one correspondence with ordered lists of K elements of $\mathbb{Z}_N$. Restricting to the symmetric representation the zero weight vectors are in one-to-one correspondence with unordered lists and restricting to the antisymmetric and mixed cases, due to the antisymmetry, gives unordered lists with no repeated entries. Define $\mathcal{L}_\mathcal{S}$ as the set of the unordered lists and $\mathcal{L}_\mathcal{A}$ as the set of unordered lists with no repeated entries.

Thus,

$$\Psi_{i_1\ldots i_K,j_1\ldots j_K} = \sum_{l\in\mathcal{L}_\mathcal{S},\mathcal{L}_\mathcal{A}} f_{(l)}(\lambda_i)\,\mathbb{P}\left(\prod_{n=1}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right), \tag{3.3}$$

where $\mathbb{P}$ symmetrises or antisymmetrises over the i_n, j_n as required to be in the right representation. The $\mathbb{P}$ will be dropped from now on, but the symmetrisation is understood.

3.2 Particles with spin

Unordered list can be specified by listing each entry or by how many times a each entry comes up. So a list can be presented as a list of elements $(l_1, \ldots, l_K)$ or as a list of multiplicities for possible element in the list $(m_1, \ldots, m_N)$, such that $\sum_{i=1}^{N} m_i = K$.[4] For the mixed and antisymmetric representations there are no repeated elements, so $m_i = 0, 1$. Now notice that the $s = 1/2$ reprenstation of $SU(2)$ is two dimensional, so these unordered lists with no repeated entries can be thought of as labelling the two possible "spin" states for each of the eigenvalues. However, this is not geometric spin, it a new internal degree of freedom. Now $SL(2) \sim SU(1,1)$, is not compact so it has no finite dimensional unitary representations [8]. The equivalent $s = 1/2$ representation is infite dimensional, which corresponds to the fact that for the symmetric representation, the m_i can be any non-negative integer allowing us to make a similar identification.

This leads to an interpretation of our Hilbert space as that of N particles with positions in 1 spatial dimension and which have an internal degree of freedom analogous to spin. In particular the symmetric representations have an $SL(2)$ internal space and the antisymmetric and mixed representations have an $SU(2)$ internal space. The states are restricted to have total "spin in the z-direction" equal to K. Note that even though these m_i can be associated with the spin in the z-direction of an $s = 1/2$ representation of $SU(2)$, the values of these m_i will still be labelled by $0, 1$ for now.

3.2.1 Permutation symmetries of these particles

When a matrix is diagonalised the order of the eigenvalues in not specified and so the wavefunction must be symmetric in these eigenvalues to reflect this indistinguishability. More specifically, from a permutation $\sigma \in S_N$ a permutation matrix $P = \delta_{i\sigma(i)}$ in $U(N)$ can be constructed. Then,

$$\begin{aligned}
\Psi(\Omega P \Lambda P^\dagger \Omega^\dagger)_\alpha &= \sum_k f_k(P\Lambda P^\dagger)\Omega_{\alpha\beta} v^k_\beta = \sum_k f_k(\lambda_{\sigma(i)})\Omega_{\alpha\beta} v^k_\beta && (3.4)\\
&= \sum_k f_k(\lambda_i)(\Omega P)_{\alpha\beta} v^k_\beta = \sum_k f_k(\lambda_i)\Omega_{\alpha\beta}(P v^k)_\beta \\
&= \sum_{k,l} P^{kl} f_k(\lambda_i)\Omega_{\alpha\beta} v^l_\beta. && (3.5)
\end{aligned}$$

[4] In what follows when an l is used to mean the former and an m to mean the latter way of presenting a list.

In other words, a permutation of the eigenvalues is equivalent to a permutation of the zero weight vectors. However, in the representations of study, such a permutation of the zero weight vectors is trivial due to the symmetries of the representation. In our case the action of the permutation is

$$\begin{aligned}\sum_{l\in\mathcal{L}} f_{(l)}(\lambda_{\sigma(i)}) \prod_{n=1}^{K} \Omega_{i_n l_n}\Omega^{\dagger}_{l_n j_n} &= \sum_{l\in\mathcal{L}} f_{(l)}(\lambda_i) \prod_{n=1}^{K} \Omega_{i_n \sigma(l_n)}\Omega^{\dagger}_{\sigma(l_n) j_n} \\ &= \sum_{l\in\mathcal{L}} f_{\sigma^{-1}[(l)]}(\lambda_i) \prod_{n=1}^{K} \Omega_{i_n l_n}\Omega^{\dagger}_{l_n j_n}, \end{aligned} \tag{3.6}$$

which means that the labels are indistinguishable and that the only observables are the number of labels and how those labels are distributed amongst the indistinguishable eigenvalues. Also, swapping two eigenvalues that have the same number of labels, only changes the wavefunction by a sign since the symmetrisation over the representation indices (the i_n and j_n) can be used to return the list to its original order. This causes no change for the symmetric representations, a factor of $[\text{sign}(\sigma)]^2 = 1$ for the antisymmetric representations and a factor of $\text{sign}(\sigma)$ for the mixed representations.

In addition, this change of variables to the eigenvalues requires a Jacobian which is antisymmetric in the eigenvalues and so picks up a sign under the swap of any two eigenvalues [1, 2, 6]. This means that for all cases the unlabelled eigenvalues are fermionic while the labelled eigenvalues are fermionic for the symmetric and antisymmetric representations and bosonic for the mixed representations. However, it should be noted that in two dimensions particle statistics are not as in higher dimensions. These exchange symmetries can be used to order the particles by increasing position and then to distinguish them by this ordering so they are not fundamentally undistinguishable. The Hamiltonian must be studied to find the spectrum and whether particles appear bosonic or fermionic cannot be relied on to constrain the spectrum.

3.3 The Hamiltonian

The next step is to find the action of the Hamiltonian on this Hilbert space in the spin picture starting with the Hamiltonian from equation (17) of [6]

$$H = \sum_{i=1}^{N} \left(\frac{1}{2} \left(P_i^{\lambda} \right)^2 + V(\lambda_i) \right) + \frac{1}{2} \sum_{i\neq j=1}^{N} \frac{\Pi_{ij}\Pi_{ji}}{(\lambda_i - \lambda_j)^2} = H_\lambda + H_\Pi. \tag{3.7}$$

H_λ describes the dynamics of N particles in a potential while H_Π describes interactions involving spin flips.

$\Pi_{ij}\Omega_{kl} = \delta_{il}\Omega_{kj}$ and $\Pi_{ij}\Omega^\dagger_{kl} = -\delta_{kj}\Omega^\dagger_{il}$ [6] can be used to compute the action of H_Π on the wavefunction from equation 3.3:

$$\begin{aligned}
&\Pi_{ij}\Pi_{ji}\left(\prod_{n=1}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right) &(3.8)\\
&= \Pi_{ij}\sum_{r=1}^{K}\left(\prod_{\substack{n=1\\n\neq r}}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right)\left(\delta_{jl_r}\Omega_{i_r i}\Omega^\dagger_{l_r j_r} - \delta_{l_r i}\Omega_{i_r l_r}\Omega^\dagger_{j j_r}\right)\\
&= \mathbb{A} + \mathbb{B},\\
\mathbb{A} &= \sum_{r=1}^{K}\left(\prod_{\substack{n=1\\n\neq r}}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right)\\
&\times\left(\delta_{jl_r}\delta_{ii}\Omega_{i_r j}\Omega^\dagger_{l_r j_r} - \delta_{jl_r}\delta_{l_r j}\Omega_{i_r i}\Omega^\dagger_{i j_r} - \delta_{l_r i}\delta_{il_r}\Omega_{i_r j}\Omega^\dagger_{j j_r} + \delta_{l_r i}\delta_{jj}\Omega_{i_r l_r}\Omega^\dagger_{i j_r}\right)\\
&= 2\sum_{r=1}^{K}\left(\prod_{\substack{n=1\\n\neq r}}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right)\left(\delta_{jl_r}\Omega_{i_r l_r}\Omega^\dagger_{l_r j_r} - \delta_{jl_r}\Omega_{i_r i}\Omega^\dagger_{i j_r}\right), &(3.9)
\end{aligned}$$

$$\begin{aligned}
\mathbb{B} &= \sum_{\substack{r,1=1\\r\neq s}}^{K}\left(\prod_{\substack{n=1\\s\neq n\neq r}}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right)\left(\delta_{jl_r}\Omega_{i_r i}\Omega^\dagger_{l_r j_r} - \delta_{l_r i}\Omega_{i_r l_r}\Omega^\dagger_{j j_r}\right)\\
&\qquad\times\left(\delta_{il_s}\Omega_{i_s j}\Omega^\dagger_{l_s j_s} - \delta_{l_s j}\Omega_{i_s l_s}\Omega^\dagger_{i j_s}\right)\\
&= \sum_{\substack{r,1=1\\r\neq s}}^{K}\left(\prod_{\substack{n=1\\s\neq n\neq r}}^{K}\Omega_{i_n l_n}\Omega^\dagger_{l_n j_n}\right) &(3.10)\\
&\times\Big(\delta_{il_r}\delta_{jl_s}(\Omega_{i_s l_r}\Omega^\dagger_{l_r j_r}\Omega_{i_r l_s}\Omega^\dagger_{l_s j_s} + \Omega_{i_r l_r}\Omega^\dagger_{l_r j_s}\Omega_{i_s l_s}\Omega^\dagger_{l_s j_r})\\
&\quad -\delta_{jl_r}\delta_{l_r l_s}(\Omega_{i_r i}\Omega^\dagger_{l_r j_r}\Omega_{i_s l_s}\Omega^\dagger_{i j_s} + \Omega_{i_r l_r}\Omega^\dagger_{i j_r}\Omega_{i_s i}\Omega^\dagger_{l_s j_s})\Big),
\end{aligned}$$

where the fact that i and j will be summed over has been used to rename $i \leftrightarrow j$. $\mathbb{B}$ can be further simplified by using the implicit symmetrisation over the indices i_n and j_n encapsulated in the omitted operator $\mathbb{P}$:

$$\mathbb{B} = 2\alpha \sum_{\substack{r,1=1 \\ r\neq s}}^{K} \left(\prod_{\substack{n=1 \\ n\neq r}}^{K} \Omega_{i_n l_n} \Omega^\dagger_{l_n j_n} \right) \left(\delta_{il_r}\delta_{jl_s}\Omega_{i_r l_r}\Omega^\dagger_{l_r j_r} - \delta_{jl_r}\delta_{l_r l_s}\Omega_{i_r i}\Omega^\dagger_{ij_r} \right), \tag{3.11}$$

where α is $+1$, -1 or 0 for the symmetric, antisymmetric and mixed representations respectively.

Each $\Omega_{i_n l_n}\Omega_{l_n j_n}$ is a label telling us which are the eigenvalues selected in the unordered list parametrising our zero weight vectors. As discussed in section 3.2, this list can be expressed as an internal degree of freedom similar to spin for each eigenvalue. So a state $|m_1 \dots m_N\rangle$ encodes the information from the Ω part of the wavefunction in these representations. Then the relationship translating between these two ways of understanding the matrix model,

$$\prod_{n=1}^{K} \Omega_{i_n l_n}\Omega^\dagger_{l_n j_n} = \langle \Omega | m_1 \dots m_N \rangle, \tag{3.12}$$

can be used to translate the action of the Hamiltonian on the wavefunction written explicitly in terms of Ω and the eigenvalues into an action of the Hamiltonian on eigenvalues with an internal degree of freedom.

3.3.1 Symmetric representations

For the symmetric representations, the m_i can be any non-negative integer, subject to the constraint that their sum has to be K.

The Hamiltonian is

$$\begin{aligned} &H_\Pi \Psi_{i_1 \dots i_K, j_1 \dots j_K} \qquad\qquad (3.13) \\ &= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{(\lambda_i - \lambda_j)^2} \left[\sum_{r=1}^{K} \left(\prod_{\substack{n=1 \\ n\neq r}}^{K} \Omega_{i_n l_n}\Omega^\dagger_{l_n j_n} \right) \left(\delta_{jl_r}\Omega_{i_r l_r}\Omega^\dagger_{l_r j_r} - \delta_{jl_r}\Omega_{i_r i}\Omega^\dagger_{ij_r} \right) \right. \\ &\qquad \left. + \sum_{\substack{r,1=1 \\ r\neq s}}^{K} \left(\prod_{\substack{n=1 \\ n\neq r}}^{K} \Omega_{i_n l_n}\Omega^\dagger_{l_n j_n} \right) \left(\delta_{il_r}\delta_{jl_s}\Omega_{i_r l_r}\Omega^\dagger_{l_r j_r} - \delta_{jl_r}\delta_{l_r l_s}\Omega_{i_r i}\Omega^\dagger_{ij_r} \right) \right], \end{aligned}$$

which becomes

$$
\begin{aligned}
H_\Pi = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(\lambda_i-\lambda_j)^2} \Bigg[\sum_{r=1}^{K} \left(\delta_{jl_r}\left(1-J_i^+J_j^-\right)\right) \\
+ \sum_{\substack{r,s=1\\ r\neq s}}^{K} \left(\delta_{il_r}\delta_{jl_s} - \delta_{jl_r}\delta_{jl_s}J_i^+J_j^-\right)\Bigg] \\
= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(\lambda_i-\lambda_j)^2} \left[\left(m_j\left(1-J_i^+J_j^-\right) + m_i m_j - m_j(m_j-1)J_i^+J_j^-\right)\right] \\
= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{m_j(1+m_i) - m_j^2(J_i^+J_j^-)}{(\lambda_i-\lambda_j)^2},
\end{aligned}
\tag{3.14}
$$

where that $\sum_{r=1}^{K}\delta_{jl_r} = m_j$ was used. Note that $J^\pm$ are normalised so that $J^\pm|m_i\rangle = |m_i \pm 1\rangle$ and $J^-|0\rangle = 0$. This means that they are not normalised the same way as the usual spin ladder operators.

This looks a little confusing, but it is an interaction between two particles that mixes spins states and where a unit of spin can move from one particle to another. It also preserves the sum of the m_i thus keeping the state in the same representation as required by the $U(N)$ symmetry.

3.3.2 Antisymmetric representations

For the antisymmetric representations $\delta_{l_rl_s} = 0$ for $r \neq s$, since we have no repeated elements in the list.

The Hamiltonian is,

$$
\begin{aligned}
&H_\Pi \Psi_{i_1\dots i_K, j_1\dots j_K} \\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(\lambda_i-\lambda_j)^2} \Bigg[\sum_{r=1}^{K} \left(\prod_{\substack{n=1\\ n\neq r}}^{K} \Omega_{i_nl_n}\Omega^\dagger_{l_nj_n}\right) \left(\delta_{jl_r}\Omega_{i_rl_r}\Omega^\dagger_{l_rj_r} - \delta_{jl_r}\Omega_{i_ri}\Omega^\dagger_{ij_r}\right) \\
&\quad - \sum_{\substack{r,1=1\\ r\neq s}}^{K} \left(\prod_{\substack{n=1\\ n\neq r}}^{K} \Omega_{i_nl_n}\Omega^\dagger_{l_nj_n}\right) \left(\delta_{il_r}\delta_{jl_s}\Omega_{i_rl_r}\Omega^\dagger_{l_rj_r}\right)\Bigg],
\end{aligned}
\tag{3.15}
$$

which can be reinterpreted as

$$
\begin{aligned}
H_\Pi &= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(\lambda_i-\lambda_j)^2}\left[\sum_{r=1}^{K}\delta_{jl_r}\left(1-J_i^+J_{l_r}^-\right)-\sum_{\substack{r,1=1\\ r\neq s}}^{K}\left(\delta_{il_r}\delta_{jl_s}\right)\right]\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{m_j\left(1-J_i^+J_j^-\right)-m_im_j}{(\lambda_i-\lambda_j)^2}\\
&= \sum_{\substack{i,j=1\\ m_i=0\\ m_j=1}}^{N} \frac{\left(1-J_i^+J_j^-\right)}{(\lambda_i-\lambda_j)^2}.
\end{aligned}
\tag{3.16}
$$

Again the $J^{\pm}$ here are normalised so that $J^{\pm}|m_i\rangle = |m_i \pm 1\rangle$ as well as $J^-|0\rangle = 0$ and $J^+|1\rangle = 0$. In this case this is actually the usual normalisation since for the spin-$\frac{1}{2}$ ladder operators $J_i^+|m_i\rangle = \sqrt{(1-m_i)(1+m_i)}|m_i+1\rangle$ and $J_i^-|m_i\rangle = \sqrt{m_i(2-m_i)}|m_i-1\rangle$, which is the same thing.

Again, this interaction mixes the two spin states and can move a unit of spin between particles, but it only allows interactions between particles in with different spins. Again, in this interaction the sum of the m_i is preserved as the $U(N)$ symmetry ensures that the Hamiltonian commutes with this observable.

3.3.3 Mixed representations

Finally the mixed representations are similar to the antisymmetric representations and again $\delta_{l_rl_s} = 0$ for $r \neq s$, since we can have no repeated elements in the list.

The Hamiltonian is

$$
\begin{aligned}
&H_\Pi\Psi_{i_1\ldots i_K,j_1\ldots j_K}\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{1}{(\lambda_i-\lambda_j)^2}\left[\sum_{r=1}^{K}\left(\prod_{\substack{n=1\\ n\neq r}}^{K}\Omega_{i_nl_n}\Omega^\dagger_{l_nj_n}\right)\left(\delta_{jl_r}\Omega_{i_rl_r}\Omega^\dagger_{l_rj_r}-\delta_{jl_r}\Omega_{i_ri}\Omega^\dagger_{ij_r}\right)\right].
\end{aligned}
\tag{3.17}
$$

This can reinterpreted as

$$H_\Pi = \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{(\lambda_i - \lambda_j)^2} \left[\sum_{r=1}^{K} \delta_{jl_r} \left(1 - J_i^+ J_{l_r}^-\right) \right]$$
$$= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{m_j \left(1 - J_i^+ J_j^-\right)}{(\lambda_i - \lambda_j)^2}, \tag{3.18}$$

where that the $J^\pm$ are normalised the same as for the antisymmetric case.

Again, this interaction mixes the two spin states and can move a unit of spin between particles. Again in this interaction the sum of the m_i is preserved, so the wavefunction stays in the same sector as expected.

3.4 A comment on Calogero models

The inverse square potential in one spatial dimension is quite well studied and is know as a Calogero model amongst other names [9]. It turns out that this potential leads to essentially free particles, but with generalised statistics. In fact there is even a version of this model that includes spins and looks like it should be relevant to our interests since it can be related to a matrix model. However, the interaction term found here does not exactly match the Calogero case and so no direct comparisons can be made. Further study of the links between these models would be interesting, but will be left for future work.

Chapter 4

Gauged Matrix-Vector Model

Gaiotto proposed introducing vector degrees of freedom to make restricting our Hilbert space to these non-singlet sectors easier [4]. In his proposal, the extra degrees of freedom introduced by doing so are be eliminated by elevating the $U(N)$ symmetry to a gauge symmetry. These new vectors introduce new states invariant under the action of the $U(N)$ symmetry, which can be described by new collective fields. In section 4.2, the Hilbert spaces of the non-singlet sectors of interest are studied by listing the new invariant states involving these vectors. This leads to a new approach for calculating the partition functions of these non-singlet sectors, extending the results from section 2.5.1. In particular, the ground state of each of the sectors of interest is found. For the symmetric and antisymmetric sectors, it is found to be the singlet sector ground state. However, for the mixed sectors, in section 4.2.4, a non-trivial ground state is found with an energy above that of the singlet sector ground state, providing a strong indication that these sectors could contain the FZZT brane. In section 4.3, a collective field that allows the study of the symmetric and antisymmetric sectors is introduced. The action of the Hamiltonian in terms of this collective field is found which leads to the identification of a free field in the harmonic oscillator potential and of long strings in the inverted oscillator potential. However, these long strings are not found to form a Fermi sea, in agreement with our expectations from the partition functions. In section 4.4, other possible collective fields are proposed. However, due to technical challenges arising from the non-trivial nature of the mixed sector ground state, expressing the Hamiltonian in terms of these collective fields for the mixed sector is left for future work.

4.1 Introducing the matrix-vector model

Introducing vector degrees of freedom and gauging the model to remove the new degrees of freedom simplifies the study of the non-singlet sectors of interest. Adding two new commuting vector degrees of freedom allows the symmetric representations to be reproduced. The antisymmetric representations will require two anticommuting vectors and the mixed representations need one of each.

Consider u_i, $\bar{u}_i$, $u_i^\dagger$, $\bar{u}_i^\dagger$, v_i, $\bar{v}_i$, $v_i^\dagger$ and $\bar{v}_i^\dagger$ with

$$\begin{gathered}
[u_i^\dagger, u_j] = \delta_{ij} \quad [u_i, u_j] = 0 \quad [\bar{u}_i^\dagger, \bar{u}_j] = \delta_{ij} \quad [\bar{u}_i, \bar{u}_j] = 0 \quad [u_i, \bar{u}_j] = 0 \\
\{v_i^\dagger, v_j\} = \delta_{ij} \quad \{v_i, v_j\} = 0 \quad \{\bar{v}_i^\dagger, \bar{v}_j\} = \delta_{ij} \quad \{\bar{v}_i, \bar{v}_j\} = 0 \quad \{v_i, \bar{v}_j\} = 0 \\
[u_i, v_j] = 0 \quad \text{etc.}
\end{gathered} \tag{4.1}$$

transforming as

$$\begin{aligned}
u_i &\to U_{ij} u_j \qquad & \bar{u}_i &\to (U^\dagger)_{ij} \bar{u}_j \\
v_i &\to U_{ij} v_j \qquad & \bar{v}_i &\to (U^\dagger)_{ij} \bar{v}_j.
\end{aligned} \tag{4.2}$$

A wavefunction in one of the symmetric sectors of this matrix-vector model will be a function of $u^\dagger$, $\bar{u}^\dagger$ and M. Then, restricting to the $\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}$ representation is equivalent to restricting to the eigenspace of the number operators $u^\dagger u$ and $\bar{u}^\dagger \bar{u}$ with eigenvalue K for each [4]. That is that the number of u's and $\bar{u}$'s determine the representation.

The antisymmetric representations can be obtained in a similar way using $v^\dagger$ and $\bar{v}^\dagger$ while for the mixed representations $u^\dagger$ and $\bar{v}^\dagger$ (or equivalently $v^\dagger$ and $\bar{u}^\dagger$) must be used.

That this gauged model reproduces the correct Hilbert space can be seen by looking at the form of the wavefunctions in a particular gauge. This will be shown for the symmetric representations as the others are similar. The wavefunction must look like a sum of terms of the form

$$u^\dagger f_1(M) \bar{u}^\dagger \dots u^\dagger f_K(M) \bar{u}^\dagger g(M) |0\rangle, \tag{4.3}$$

where $g(M)$ is invariant under the action of $U(N)$ (that is that it is in the singlet sector) and $|0\rangle$ is the ground state for the u and $\bar{u}$ part of the Hilbert space. If a particular gauge is fixed, acting on this with $\langle 0| (\prod_{n=1}^K u_{i_n} \bar{u}_{j_n})$ strips the vector degrees of freedom and leaves a wavefunction that depends only on the matrix M. Due to the commutation relations of the vectors, this wavefunction will be in the appropriate sector. The particular wavefunction obtained will change under gauge transformations as the dual space state $\langle 0| (\prod_{n=1}^K u_{i_n} \bar{u}_{j_n})$ is not gauge invariant, but it will always be in the correct sector.

Alternatively, a gauge where the matrix is diagonal can be used so that $u^\dagger f(M)\bar{u}^\dagger = \sum_{i=1}^{N} f(\lambda_i) u_i^\dagger \bar{u}_i^\dagger$. The wavefunction in equation (4.3) looks like

$$\sum_{i_1 \ldots i_K=1}^{N} f_1(\lambda_{i_1}) \ldots f_K(\lambda_{i_K}) g(\lambda_1 \ldots \lambda_N) \left(\prod_{k=1}^{K} u_{i_k}^\dagger \bar{u}_{i_k}^\dagger \right). \tag{4.4}$$

This is a sum of terms like those in equation (3.3), except that the $\Omega_{i_n l_n} \Omega^\dagger_{l_n j_n}$ are replaced with $u_{l_k}^\dagger \bar{u}_{l_k}^\dagger$. Again, the vectors keep track of the representation indices, which in this case means that they act as the labels that specify the distinguished eigenvalues.

These arguments for the equivalence of these Hilbert spaces are adapted from [4]. See that work for further discussion.

4.2 States and partition functions in the matrix-vector model

4.2.1 Singlet

The matrix-vector model will be used to study the Hilbert space of the sectors of interest and obtain partition functions that can be compared to those obtained in section 2.5.1, starting with the singlet. As in section 2.5.1, the matrix model with the harmonic oscillator potential will be studied as its spectrum is easier to understand. The connection to the two dimensional string theory is through the similarities this potential has with the inverted harmonic oscillator. This was connection was evident in the singlet sector in section 2.3, where a collective field found in the harmonic oscillator potential, with a non-trivial classical solution and perturbations equivalent to a free field at large N, lead directly to a collective field in the inverted oscillator potential with similar properties. A more formal argument using an analytical continuation is made in [7].

First, note that the ground state of the matrix system is

$$e^{-\frac{1}{2}\sum_{i,j=1}^{N} |M_{ij}|^2} = e^{-\frac{1}{2}\mathrm{Tr}(M^2)}, \tag{4.5}$$

since the components of the matrix model each correspond to decoupled harmonic oscillators. This state is in the singlet sector so it must be the ground state of the matrix model restricted to the singlet sector.

States of the form

$$(\mathrm{Tr} M^{a_1}) \ldots (\mathrm{Tr} M^{a_n}) e^{-\frac{1}{2}\mathrm{Tr} M^2} \tag{4.6}$$

form a complete set of states for the singlet sector. However, this set is not linearly independent, since a matrix satisfies its own characteristic polynomial and as this polynomial involves only traces of the matrix with powers of the matrix up to M^N, $\mathrm{Tr}(M^a)$ for $a > N$ can be related to traces of lower powers of M by taking the trace of the characteristic polynomial evaluated at M multiplied by M^{a-N}. This is called the trace relation.

Taking the trace relation into account, the states

$$(\mathrm{Tr}M)^{p_1}(\mathrm{Tr}M^2)^{p_2}\dots(\mathrm{Tr}M^N)^{p_N}e^{-\frac{1}{2}\mathrm{Tr}M^2}, \quad \text{for } p_i \geq 0, \tag{4.7}$$

form a basis for the singlet sector.

For a standard single particle harmonic oscillator system, the energy eigenfunctions are

$$\left(x^n + O(x^{n-1})\right)e^{-\frac{1}{2}x^2}. \tag{4.8}$$

The energy of the eigenstate is determined by the highest power of x and the terms with lower powers of x can be determined by requiring orthogonality with the lower energy eigenstates.

Similarly, the basis states in equation (4.7) can be associated with energy eigenstates by a Gram-Schmidt procedure ensuring their orthogonality, since the components of the matrix M correspond to decoupled harmonic oscillators. The energy of this associated energy eigenstate is determined by counting the powers of M in the basis state, $E = \sum_{i=1}^{N} i \cdot p_i + \frac{1}{2}N^2$. This means that counting the basis states from equation (4.7) is sufficient to compute the partition function:

$$Z_{sing}(q) = q^{\frac{1}{2}N^2}\prod_{p=1}^{N}\frac{1}{1-q^p}, \tag{4.9}$$

in agreement with the result from equation (2.69).

4.2.2 Symmetric representations

Constructing gauge invariant wavefunctions from $u^\dagger$, $\bar{u}^\dagger$ and M gives states of the form

$$(u^\dagger M^{a_1}\bar{u}^\dagger)\dots(u^\dagger M^{a_n}\bar{u}^\dagger)\times(\text{Singlet}), \tag{4.10}$$

where (Singlet) represents a wavefunction from the singlet sector as studied in the previous subsection. Thus the Hilbert space of the symmetric sector corresponding to the $\mathrm{Sym}^K N \otimes \mathrm{Sym}^K \bar{N}$ representation can be written as the tensor product of the singlet Hilbert space and the symmetric sector Hilbert space modulo the singlet sector Hilbert space, which will be referred to as $\mathcal{S}_K$. Then the partition functions of the matrix-vector model restricted to these sectors are a product of the singlet partition function and a contribution found by looking at how the states in $\mathcal{S}_K$ effect the energy. $\mathcal{S}_K$ contains the eigenstates of $u^\dagger u$ with eigenvalue K, that is the states with K u's. Looking at all states of the form of equation (4.10) leads to $\mathcal{S} = \sum_{K=0}^{\infty} \mathcal{S}_K$. When studying these $\mathcal{S}_K$ spaces, factors in the singlet can be omitted by choosing an appropriate representative.

Again, the characteristic polynomial gives us trace relations between these states. $u^\dagger M^a \bar{u}^\dagger$ for $a \geq N$ can be related to such factors with lower powers of M times traces of M which are in the singlet.

Thus,

$$(u^\dagger \bar{u}^\dagger)^{p_0} (u^\dagger M \bar{u}^\dagger)^{p_1} (u^\dagger M^2 \bar{u}^\dagger)^{p_2} \dots (u^\dagger M^N \bar{u}^\dagger)^{p_N}, \quad \text{for } p_i \geq 0, \tag{4.11}$$

forms a basis for $\mathcal{S}$.

For the same reasons as in the singlet case, these states can be associated with energy eigenstates by adding appropriate terms of lower order in M. This means that this basis times the basis for the singlet sector in equation (4.7) is in one-to-one correspondence with a basis of energy eigenstates for the direct sum of the symmetric sectors and counting these states along with their powers of M is sufficient to calculate the factor contributed to the partition functions by these sectors.

The basis in equation (4.11) makes looking at fixed K difficult, but is useful for calculating partition functions at large K for comparison to section 2.5.1. A basis for $\mathcal{S}_K$, more appropriate for fixed K, will be presented shortly.

For large K, but finite N, arbitrary states can be constructed by specifying p_i for $i > 0$ then setting p_0 to the large value required to fix K. Thus

$$Z_{K\to\infty}(q) = \left(\prod_{p=1}^{N-1} \frac{1}{1-q^p} \right) \times Z_{sing}(q), \tag{4.12}$$

which agrees with the result in equation (2.72).

The partition function restricted to the sector corresponding to the direct sum of all the symmetric representations can be obtained by looking at all the states in equation (4.11).

$$\sum_{k=0}^{\infty} Z_k(q) = \left(\prod_{p=0}^{N-1} \frac{1}{1-q^p} \right) \times Z_{sing}(q), \tag{4.13}$$

which matches equation (2.73), including the divergence from the $p = 0$ mode coming from the fact that each sector corresponding to an irreducible representation is included an infinite number of times in $\mathcal{S}$.

The states

$$(u^\dagger M^{a_1} \bar{u}^\dagger)(u^\dagger M^{a_2} \bar{u}^\dagger) \dots (u^\dagger M^{a_K} \bar{u}^\dagger), \quad \text{for } 0 \le a_i \le N, \tag{4.14}$$

form a basis for $\mathcal{S}_K$, as long as $a_i \le a_j$ for $i < j$ to avoid over counting. Counting powers of M gives the energy these states add to the omitted singlet factors. Thus, the spectrum of a symmetric sector with fixed K modulo the singlet sector is equivalent that of K bosons in a harmonic oscillator potential where where no boson is allowed to have more than N units of energy. For finite N this doesn't lead to a nice formula for the partition function. For large N, using the standard way of rewriting the non-decreasing list $(a_1 \dots a_K)$ by instead counting the spacing between each quanta commonly used when considering K bosons in a harmonic oscillator potential,

$$Z_{N\to\infty}(q) = \left(\prod_{p=1}^{K} \frac{1}{1-q^p} \right) \times Z_{sing}(q), \tag{4.15}$$

which matches equation (2.77).

Note that while a simple formula for the partition function at finite N and K was not found, an explicit basis for the states in the symmetric sectors can be constructed by taking a cartesean product of the states in equations (4.15) and (4.7). These basis states can be matched to energy eigenstates for the harmonic oscillator potential effectively giving a recipe for diagonalising the Hamiltonian restricted to this sector. In particular note that the ground state for this sector is

$$(u^\dagger \bar{u}^\dagger)^K e^{-\frac{1}{2}\mathrm{Tr}M^2}, \tag{4.16}$$

which is degenerate with the ground state for the singlet. Stripping the vector degrees of freedom to return to the original matrix model using $\langle 0 | (\prod_{n=1}^{K} u_{i_n} \bar{u}_{j_n})$, as discussed in section 4.1, shows that the ground states of the symmetric sectors are the singlet ground state.

4.2.3 Antisymmetric representations

For the antisymmetric sectors, wavefunctions are constructed using $v^\dagger$, $\bar{v}^\dagger$ and M leading to representative states of $\mathcal{A} = \sum_{K=0}^{\infty} \mathcal{A}_K$, where $\mathcal{A}_K$ is the Hilbert space of the antisymmetric sector $N^{\wedge K} \otimes \bar{N}^{\wedge K}$ modulo the singlet sector Hilbert space, of the form

$$(v^\dagger M^{a_1} \bar{v}^\dagger) \dots (v^\dagger M^{a_K} \bar{v}^\dagger). \tag{4.17}$$

The anticommuting nature of $v^\dagger$ and $\bar{v}^\dagger$ leads to new relations between states in addition to the ones noted for the symmetric sectors. In particular, the representations with $K = k$ and $K = N - k$ are isomorphic and they are trivial for $K > N$, requiring new relations between states not present in the symmetric case.

Writing our states in a gauge where M is diagonal gives

$$\sum_{i_1 \dots i_K = 1}^{N} \left(\prod_{k=1}^{K} \lambda_{i_k}^{a_k} \right) \left(\prod_{k=1}^{K} v_{i_k}^\dagger \bar{v}_{i_k}^\dagger \right). \tag{4.18}$$

Since the $v_i^\dagger$ anticommute, $(v_i^\dagger)^2 = 0$ and the terms where $i_j = i_k$ for $j \neq k$ vanish.

All the states vanish for $K > N$, as required by the fact that the antisymmetric representations are trivial for $K > N$, since there are only N different $v_i^\dagger$.

When writing states in $\mathcal{A}$, a representative state of the form from equation (4.17) has been chosen to represent the equivalence classes. However, multiplying the states in $\mathcal{A}$ by factors in the singlet gives another state in the same equivalence class. On the other hand, using relations to be developed below, these state can be related to other states of the form of those in equation (4.17). Thus states that on the surface appear different are in fact in the same equivalence class of $\mathcal{A}$. This requires identifying these states in order to avoid over counting them.

The starting point for these relations is

$$\begin{aligned} &\left[\sum_{j_1 \neq j_2 \neq \dots j_A = 1}^{N} \left(\prod_{a=1}^{A} \lambda_{j_a}^{p_a} \right) \right] \left(\sum_{i_1 \neq i_2 \neq \dots i_K = 1}^{N} \prod_{k=1}^{K} v_{i_k}^\dagger \bar{v}_{i_k}^\dagger \right) \\ &= \sum_{B=0}^{\min(A, N-K)} \sum_{i_1 \neq \dots i_K \neq j_1 \neq \dots j_B = 1}^{N} \left(\prod_{b=1}^{B} \lambda_{j_b}^{p_b} \right) \left(\prod_{c=1}^{A-B} \lambda_{i_c}^{p_c} \right) \left(\prod_{k=1}^{K} v_{i_k}^\dagger \bar{v}_{i_k}^\dagger \right). \end{aligned} \tag{4.19}$$

Define the sets of states $\mathcal{A}_K(B,C) \subset \mathcal{A}_K$ as the states of the form

$$\sum_{i_1 \neq \ldots i_K \neq j_1 \neq \ldots j_B = 1}^{N} \left(\prod_{b=1}^{B} \lambda_{j_b}^{p_b}\right)\left(\prod_{c=1}^{C} \lambda_{i_c}^{p_c}\right)\left(\prod_{k=1}^{K} v_{i_k}^{\dagger} \bar{v}_{i_k}^{\dagger}\right), \tag{4.20}$$

for $p_i > 0$. Note that $\mathcal{A}_K(B,C)$ is defined such that it contains only the zero vector for $B > N-K$ or $C > K$. $\mathcal{A}_K = \bigoplus_{C=1}^{K} \mathcal{A}_K(0,C)$, as can be seen from the form of the states of $\mathcal{A}_K$ in equation (4.18).

Equation (4.19) relates states in $\mathcal{A}_K(0,0)$ to a sum of states in $\mathcal{A}_K(B,0)$, $\mathcal{A}_K(B-1,1), \ldots$ and $\mathcal{A}_K(0,B)$. A similar identity relating states in $\mathcal{A}_K(0,D)$ to a sum of states in $\mathcal{A}_K(B,D)$, $\mathcal{A}_K(B-1,D+1), \ldots$ and $\mathcal{A}_K(0,D+B)$ exists, as can be seen by multiplying a state in $\mathcal{A}_K(0,C)$ by a factor in the singlet. Equivalently, these can be seen as relating a state in $\mathcal{A}_K(B,D)$ to a sum of states in the sets $\mathcal{A}_K(B,C)$, with $C < D$.

Using these this relation repeatedly, with progressively smaller D, any state in $\mathcal{A}_K(0,C)$ can be related to states in $\bigoplus_{B=0}^{\min(C,N-K)} \mathcal{A}_K(B,0)$. In other words, $\mathcal{A}_K(0,C) \subset \bigoplus_{B=0}^{\min(C,N-K)} \mathcal{A}_K(B,0)$ and

$$\mathcal{A}_K = \bigoplus_{C=1}^{K} \mathcal{A}_K(0,C) \subset \bigoplus_{B=1}^{N-K} \mathcal{A}_K(B,0). \tag{4.21}$$

This also goes through inverting the first and second arguments of $\mathcal{A}_K(\cdot,\cdot)$, giving

$$\bigoplus_{B=1}^{N-K} \mathcal{A}_K(B,0) \subset \bigoplus_{C=1}^{K} \mathcal{A}_K(0,C) \implies \bigoplus_{C=1}^{K} \mathcal{A}_K(0,C) = \bigoplus_{B=1}^{N-K} \mathcal{A}_K(B,0). \tag{4.22}$$

Finally, the map

$$\sum_{\substack{i_1 \neq \ldots i_K \neq \\ j_1 \neq \ldots j_B = 1}}^{N} \left(\prod_{b=1}^{B} \lambda_{j_b}^{p_b}\right)\left(\prod_{k=1}^{K} v_{i_k}^{\dagger} \bar{v}_{i_k}^{\dagger}\right) \to \sum_{i_1 \neq \ldots \neq i_{N-K}}^{N} \left(\prod_{b=1}^{B} \lambda_{i_b}^{p_b}\right)\left(\prod_{k=1}^{N_K} v_{i_k}^{\dagger} \bar{v}_{i_k}^{\dagger}\right) \tag{4.23}$$

provides an isomorphism from $\mathcal{A}_{N-K}$ to $\bigoplus_{B=1}^{N-K} \mathcal{A}_K(B,0)$.

Putting this together, it can be seen explicitly that

$$\mathcal{A}_K \cong \mathcal{A}_{N-K}, \tag{4.24}$$

as follows from the fact that the representations associated with the antisymmetric sectors for K and $N-K$ are isomorphic.

Thus, looking at $K \leq \frac{1}{2}N$ is sufficient. Equation (4.19) also leads to more relations between states. These come from applying it with $A > N-K$. Then there is no term from $\mathcal{A}_K(A,0)$ in the relation, as this set contains only zero. In other words, states in $\mathcal{A}_K(0,0)$ can be related to a sum of states in $\mathcal{A}_K(N-K, A-N+K)$, $\mathcal{A}_K(N-K+1, A-N+K-1)$, ..., $\mathcal{A}_K(0,A)$. Through the creative use of relations like that in equation (4.19), these can all be reduced to states in $\bigoplus_{C=1}^{K} \mathcal{A}_K(0,C)$ and this gives new non-trivial relations.[5]

However, this only removes states with more powers of M than $N-K$. If $K \leq \frac{1}{2}N$, $N-K \geq \frac{1}{2}N$, so these relations all involve some states with more powers of M than $\frac{1}{2}N$. There are a large number of these relations and we have found no way of efficiently listing them in a way that would allow us to compute the partition function in closed form.

This means that partition functions can only be determined up to terms of order $q^{\frac{1}{2}N}$, which is sufficient to study the finite energy excitations at large N. In this case, the antisymmetric sectors lead to the same results as for the symmetric sectors, that is that

$$Z(q) = \left(\prod_{p=1}^{\min(K,N-K)} \frac{1}{1-q^p} + O(q^{\frac{1}{2}N}) \right) \times Z_{sing}(q). \tag{4.25}$$

Where the large N limit matches equation (2.76) from the introduction.

The ground state in this sector is

$$(v^\dagger \bar{v}^\dagger)^K e^{-\frac{1}{2}\mathrm{Tr}M^2}, \tag{4.26}$$

which is again degenerate with the singlet sector ground state. Once more, stripping the vectors shows that the ground states of the antisymmetric sectors of the original matrix model are the singlet ground state.

4.2.4 Mixed representations

In the mixed representations, wavefunctions can be constructed using $u^\dagger$, $\bar{v}^\dagger$ and M, leading to states for $\mathcal{M} = \sum_{K=0}^{\infty} \mathcal{M}_K$, where $\mathcal{M}_K$ is the Hilbert space of the mixed sector $\mathrm{Sym}^K N \otimes \bar{N}^{\wedge K}$ modulo the singlet sector Hilbert space, of the form

$$(u^\dagger M^{a_1} \bar{v}^\dagger) \dots (u^\dagger M^{a_K} \bar{v}^\dagger). \tag{4.27}$$

[5] If this seems too abstract, this fact can easily be verified small N and K.

Since $u^\dagger$ commutes and $\bar{v}^\dagger$ anticommutes, factors of $u^\dagger M^a \bar{v}^\dagger$ anticommute. This means that the wavefunction vanishes unless $a_i \neq a_j$ for $i \neq j$.

Thus,

$$(u^\dagger \bar{v}^\dagger)^{p_0}(u^\dagger M \bar{v}^\dagger)^{p_1}(u^\dagger M^2 \bar{v}^\dagger)^{p_2} \dots (u^\dagger M^N \bar{v}^\dagger)^{p_N}, \quad \text{for } p_i = \{0,1\}, \tag{4.28}$$

forms a basis for $\mathcal{M}$. This leads us to

$$\sum_{k=0}^{\infty} Z_{\mathrm{Sym}^k N \otimes N^{\wedge K}}(q) x^k = \prod_{p=0}^{\infty}(1 + x q^p), \tag{4.29}$$

reproducing the result from equation 2.74.

The states

$$(u^\dagger M^{a_1} \bar{u}^\dagger)(u^\dagger M^{a_2} \bar{u}^\dagger) \dots (u^\dagger M^{a_K} \bar{u}^\dagger), \quad \text{for } 0 \leq a_i \leq N, \tag{4.30}$$

with the requirement that $a_i < a_j$ for $i < j$ form a basis for $\mathcal{M}_K$. Now, since these factors anticommute and so are not allowed to repeat, this is the same as K fermions in a harmonic oscillator potential where no fermion is allowed to have more than N units of energy. For large N, using the standard way of rewriting the increasing list $(a_1 \dots a_K)$ by instead counting the spacing between each quanta, that is commonly used when considering K fermions in a harmonic oscillator potential,

$$Z_{N \to \infty}(q) = q^{\frac{1}{2}K(K-1)} \left(\prod_{p=1}^{K} \frac{1}{1 - q^p} \right) \times Z_{sing}(q) \tag{4.31}$$

which matches equation (2.75) as expected.

Note that due to the anticommuting of these factors $(u^\dagger \bar{v}^\dagger)^K = 0$ for $K > 1$. This means that the ground state in this sector must be

$$u^\dagger \bar{v}^\dagger u^\dagger M \bar{v}^\dagger \dots u^\dagger M^{K-1} \bar{v}^\dagger e^{-\frac{1}{2}\mathrm{Tr} M^2}. \tag{4.32}$$

This particularly interesting because the shift of the ground state energy found in this mixed sector is consistent with the picture of a large number of long strings condensing to form a Fermi sea and this ground state explicitly describes this Fermi sea.

4.3 Collective fields in this matrix-vector model

The collective field approach introduced in section 2.3 will used to describe this matrix vector model. Two gauge invariant collective fields, $\rho(x) =$

$\mathrm{Tr}[\delta(x-M)]$ and $\phi(x)=u^\dagger\delta(x-M)\bar{u}^\dagger$ can be constructed for each of the types of sectors. Since the computation for each sector is very similar, the symmetric sector, with u and $\bar{u}$, will be presented. However, wherever the sectors differ, primarily in the signs of certain terms, will be noted. For the symmetric or antisymmetric sectors $\phi(x)$ is a commuting variable making the interpretation of the collective field straightforward. However, for the mixed representation, the collective field $\phi(x)=u^\dagger\delta(x-M)\bar{v}^\dagger$ is anticommuting and cannot be studied within the confines of the standard collective field approach.

Using the standard collective field techniques introduced in 2.3 to write the action of the Hamiltonian in terms of the collective fields, the first step is to calculate $\omega(x;\rho)$ and $\omega(x;\phi)$ as well as all the $\Omega(x,y;\cdot,\cdot)$. These are the same as for the adjoint, as in section 2.4, except for the new $\Omega(x,y;\phi,\phi)$ that can now be calculated using that $\phi(x)=\int\frac{\mathrm{d}k}{2\pi}e^{-ikx}\bar{u}e^{ikM}u$ and

$$\frac{\partial}{\partial M_{ba}}(e^{ikM})_{cd}=(ik)\int_0^1\mathrm{d}\alpha(e^{ik\alpha M})_{cb}(e^{ik(1-\alpha)M})_{ad}. \tag{4.33}$$

This new term is

$$\begin{aligned}
\Omega(x,y;\phi,\phi) &= \frac{\partial\phi(x)}{\partial M_{ab}}\frac{\partial\phi(y)}{\partial M_{ba}} \\
&= \int\frac{\mathrm{d}k\mathrm{d}q}{(2\pi)^2}\int_0^1\mathrm{d}\alpha\mathrm{d}\beta e^{-i(kx+qy)}(ik)(iq)u^\dagger_c\bar{u}^\dagger_d u^\dagger_e\bar{u}^\dagger_f \\
&\qquad (e^{ik\alpha M})_{ca}(e^{ik(1-\alpha)M})_{bd}(e^{iq\beta M})_{eb}(e^{iq(1-\beta)M})_{af} \\
&= \pm\int\frac{\mathrm{d}k\mathrm{d}q}{(2\pi)^2}\mathrm{d}w\mathrm{d}z(ik)(iq)\phi(w)\phi(z) \\
&\qquad \int_0^1\mathrm{d}\alpha\mathrm{d}\beta e^{-i(kx+qy)}e^{i(k\alpha+q(1-\beta))z}e^{i(k(1-\alpha)+q\beta)w} \\
&= \pm\int\frac{\mathrm{d}k\mathrm{d}q}{(2\pi)^2}\mathrm{d}w\mathrm{d}z\frac{\phi(w)\phi(z)}{(z-w)^2}\quad[\delta(x-z)\delta(y-z)+\delta(x-w)\delta(y-w) \\
&\qquad\qquad -\delta(x-z)\delta(y-w)-\delta(x-w)\delta(y-z)] \\
&= \pm2\left(\int\mathrm{d}z\frac{\phi(x)\phi(z)}{(x-z)^2}\delta(x-y)-\frac{\phi(x)\phi(y)}{(x-y)^2}\right),
\end{aligned} \tag{4.34}$$

where the $\pm$ is a $+$ if both vectors, $u^\dagger$ and $\bar{u}^\dagger$, are commuting, which corresponds to the symmetric sectors, whereas it is a $-$ if they are both anticommuting as for the antisymmetric sectors. The rest of the ω and Ω terms are independent of whether commuting or anticommuting vectors are used.

Now gather all the useful terms, together with their N–scaling under

$$x \to \sqrt{N}x, \tag{4.35}$$

$$\rho(x) \to \sqrt{N}\rho(x), \tag{4.36}$$

$$\phi(x) \to \sqrt{N}\phi(x). \tag{4.37}$$

These are:

$$\omega(x;\rho) = 2\partial_x \int \mathrm{d}y \frac{\rho(x)\rho(y)}{y-x} \qquad \sqrt{N} \tag{4.38}$$

$$\omega(x;\phi) = 2\partial_x \int \mathrm{d}y \frac{\phi(x)\rho(y)}{y-x} + 2\int \mathrm{d}y \frac{\rho(x)\phi(y)-\phi(x)\rho(y)}{(x-y)^2} \qquad \sqrt{N} \tag{4.39}$$

$$\Omega(x,y;\rho,\rho) = \partial_x\partial_y(\delta(x-y)\rho(x)) \qquad \frac{1}{N} \tag{4.40}$$

$$\Omega(x,y;\rho,\phi) = \partial_x\partial_y(\delta(x-y)\phi(x)) \qquad \frac{1}{N} \tag{4.41}$$

$$\Omega(x,y;\phi,\phi) = \pm\left[-2\frac{\phi(x)\phi(y)}{(x-y)^2} + 2\int \mathrm{d}z \frac{\phi(x)\phi(z)}{(z-x)^2}\delta(x-y)\right] \qquad 1. \tag{4.42}$$

Following the standard collective field approach [10], where Π is a vector of the momentum conjugate to our fields and where sums over our fields (ϕ and ρ) are implied:

$$-(\frac{\partial}{\partial M})^2 = -i\omega\Pi + \Pi\Omega\Pi. \tag{4.43}$$

The resulting Hamiltonian does not appear Hermitian, as the Jacobian of the change of variables to collective fields has changed the inner product. Following the procedure from section 2.3, $\Pi \to \Pi + \frac{1}{2}iJ$ to take into account the Jacobian and return to the standard inner product. This gives

$$-(\frac{\partial}{\partial M})^2 = -i\omega\Pi + \frac{1}{2}\omega J + \Pi\Omega\Pi + i\Pi\Omega J - \frac{1}{4}J\Omega J, \tag{4.44}$$

which is Hermitian if J is chosen such that $-\omega + \Omega J = 0$.

From this Hermiticity condition,

$$
\begin{aligned}
0 &= -\omega(x;\rho) + \int \mathrm{d}y \Omega(x,y;\rho,\rho) J(y;\rho) + \int \mathrm{d}y \Omega(x,y;\rho,\phi) J(y,\phi) \\
0 &= 2\partial_x \int \mathrm{d}y \frac{\rho(x)\rho(y)}{x-y} + \int \mathrm{d}y \partial_x \partial_y (\delta(x-y)\rho(x)) J(y;\rho) \\
&\quad + \int \mathrm{d}y \partial_x \partial_y (\delta(x-y)\phi(x)) J(y,\phi) \\
\partial_x J(x;\rho) &= 2 \int \mathrm{d}y \frac{\rho(y)}{x-y} - \frac{\phi(x)}{\rho(x)} \partial_x J(x;\phi), \qquad (4.45) \\
0 &= -\omega(x;\phi) + \int \mathrm{d}y \Omega(x,y;\rho,\phi) J(y;\rho) + \int \mathrm{d}y \Omega(x,y;\phi,\phi) J(y,\phi) \\
0 &= 2\partial_x \int \mathrm{d}y \frac{\phi(x)\rho(y)}{x-y} + 2 \int \mathrm{d}y \frac{\phi(x)\rho(y) - \rho(x)\phi(y)}{(x-y)^2} - \partial_x(\phi(x)\partial_x J(y;\rho)) \\
&\quad \pm 2 \int \mathrm{d}y \frac{\phi(x)\phi(y)}{(x-y)^2} (J(x,\phi) - J(y;\phi)) \\
0 &= \partial_x \left(\frac{[\phi(x)]^2}{\rho(x)} \partial_x J(y;\phi) \right) \pm 2 \int \mathrm{d}y \frac{\phi(x)\phi(y)}{(x-y)^2} (J(x,\phi) - J(y;\phi)) \\
&\quad + 2 \int \mathrm{d}y \frac{\phi(x)\rho(y) - \rho(x)\phi(y)}{(x-y)^2} \\
0 &= O(\frac{1}{\sqrt{N}}) J(\phi) + O(\sqrt{N}) J(\phi) + O(\sqrt{N}) \\
J(x;\phi) &= \pm \frac{\rho(x)}{\phi(x)} + O(\frac{1}{N}), \qquad (4.46) \\
\partial_x J(x;\rho) &= 2 \int \mathrm{d}y \frac{\rho(y)}{x-y} \mp \frac{\phi(x)}{\rho(x)} \partial_x \left(\frac{\rho(x)}{\phi(x)} \right) + O(N^{-\frac{3}{2}}). \qquad (4.47)
\end{aligned}
$$

Thus the Hermiticity condition leads to a simple $J(x;\phi)$ at this order and gives the same $J(x;\rho)$ as seen in the singlet sector with corrections at a lower order in N.

With this J,

$$
\begin{aligned}
-(\frac{\partial}{\partial M})^2 &= \frac{1}{4}\omega J + \Pi\Omega\Pi \\
&= \int \mathrm{d}x\rho(x)\left(\int \mathrm{d}y\frac{\rho(y)}{x-y}\right)^2 \mp \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{\phi(x)\rho(y)}{x-y}\partial_x\left(\frac{\rho(x)}{\phi(x)}\right)\right] \\
&\quad \pm \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{\phi(x)\rho(y)}{x-y}\partial_x\left(\frac{\rho(x)}{\phi(x)}\right)\right] \\
&\quad \pm \frac{1}{2}\int \frac{\mathrm{d}x\mathrm{d}y}{(x-y)^2}\left[\frac{[\rho(x)]^2\phi(y)}{\phi(x)} - \rho(x)\rho(y)\right] + O(N^0) \\
&= \frac{\pi^2}{3}\int \mathrm{d}x[\rho(x)]^3 \\
&\quad \pm \frac{1}{2}\int \frac{\mathrm{d}x\mathrm{d}y}{(x-y)^2}\left[\frac{[\rho(x)]^2\phi(y)}{\phi(x)} - \rho(x)\rho(y)\right] + O(N^0).
\end{aligned} \tag{4.48}
$$

The next step is to use the Hamiltonian to find the classical solution. Pulling out explicit N dependence gives

$$
H = -\frac{1}{2}(\frac{\partial}{\partial M})^2 - \frac{1}{2}M^2 \tag{4.49}
$$

$$
\begin{aligned}
&= \frac{\pi^2}{6}N^2\int \mathrm{d}x[\rho(x)]^3 - \frac{1}{2}N^2\int \mathrm{d}x\rho(x)x^2 + N^2\mu\left[\int \mathrm{d}x\rho(x) - 1\right] \\
&\quad \pm \frac{1}{4}N\int \frac{\mathrm{d}x\mathrm{d}y}{(x-y)^2}\left[\frac{[\rho(x)]^2\phi(y)}{\phi(x)} - \rho(x)\rho(y)\right] + O(N^0).
\end{aligned} \tag{4.50}
$$

Solving at order N^2 gives $\rho_0(x) = \frac{1}{\pi}\sqrt{x^2 - 2\mu}$. Then expanding

$$
\rho(x) = \rho_0(x) + \frac{1}{N}\partial_x\eta, \tag{4.51}
$$

leads to a solution for $\phi_0(x)$ at order N.

Substituting this expansion into the Hamiltonian,

$$
\begin{aligned}
H =& N^2\int \mathrm{d}x\left[\frac{\pi^2}{6}[\rho_0(x)]^3 - \frac{1}{2}\rho_0(x)x^2\right] \mp \frac{1}{4}N\int \frac{\mathrm{d}x\mathrm{d}y}{(x-y)^2}\rho_0(x)\rho_0(y) \\
&\pm N\int \frac{\mathrm{d}x\mathrm{d}y}{(x-y)^2}\frac{[\rho_0(x)]^2\phi(y)}{4\phi(x)} - N^2\mu + O(N^0).
\end{aligned} \tag{4.52}
$$

To find the classical solution, the configuration of minimal energy must be found:

$$\frac{\partial H}{\partial \phi(z)} = \pm N \left[- \int \frac{\mathrm{d}y}{(z-y)^2} \frac{[\rho_0(z)]^2 \phi(y)}{4[\phi(z)]^2} + \int \frac{\mathrm{d}x}{(x-z)^2} \frac{[\rho_0(x)]^2}{4\phi(x)} \right] + O(N^0), \tag{4.53}$$

which vanishes if $\phi_0(x) = \kappa\rho_0(x)$ so that

$$\rho_0(x) = \frac{1}{\pi}\sqrt{x^2 - 2\mu} \tag{4.54}$$

$$\phi_0(x) = \kappa\rho_0(x). \tag{4.55}$$

ϕ can be expanded around this classical solution as

$$\phi(x) = \kappa\rho_0(x) + \frac{\sqrt{2}\kappa}{\sqrt{N}}\varphi(x). \tag{4.56}$$

The factor of $\frac{1}{\sqrt{N}}$ is consistent with associating $\phi(x)$ with open strings. The closed string coupling constant is the square of the open string one and here $\frac{1}{N}$ takes the role of the closed string coupling. Note that in this case the canonical momentum conjugate to φ, P_φ is

$$P_\varphi = \frac{\sqrt{2}\kappa}{\sqrt{N}}\Pi_\phi(x), \tag{4.57}$$

where $\Pi_\phi(x)$ is the canonical momentum conjugate to ϕ.

The next step is to calculate order N^0 terms in the Hamiltonian, keeping in mind that powers of N need to be absorbed into P_φ and P_η. The partial derivatives needed for the expansion are

$$\frac{\partial^2 H}{\partial\phi(w)\partial\phi(z)}\begin{pmatrix}\rho=\rho_0\\ \phi=\kappa\rho_0\end{pmatrix} = \tag{4.58}$$

$$\mp \frac{N}{2\kappa^2}\left[\frac{1}{(z-w)^2} - \frac{1}{\rho_0(z)}\delta(z-w)\int \mathrm{d}y\frac{\rho_0(y)}{(z-y)^2}\right] + O(N^0)$$

$$\frac{\partial H}{\partial\rho(z)} = \frac{1}{2}N^2\left[\pi^2[\rho(z)]^2 - x^2 + 2\mu\right] \tag{4.59}$$

$$\pm \frac{1}{2}N\left[\int \frac{\mathrm{d}x}{(x-z)^2}\frac{\rho(z)\phi(x)}{\phi(z)} - \int \frac{\mathrm{d}x}{(x-z)^2}\rho(x)\right] + O(N^0) \tag{4.60}$$

$$\frac{\partial^2 H}{\partial\rho(w)\partial\rho(z)}\begin{pmatrix}\rho=\rho_0\\ \phi=\kappa\rho_0\end{pmatrix} = \pi^2 N^2 \rho_0(z)\delta(z-w) \pm \frac{1}{2}N\left[\int \frac{\mathrm{d}x}{(x-z)^2}\frac{\rho_0(x)}{\rho_0(z)}\delta(z-w) - \frac{1}{(z-w)^2}\right] + O(N^0) \tag{4.61}$$

$$\frac{\partial^2 H}{\partial\phi(w)\partial\rho(z)}\begin{pmatrix}\rho=\rho_0\\ \phi=\kappa\rho_0\end{pmatrix} = \pm\frac{N}{2\kappa}\left[\frac{1}{(w-z)^2} - \delta(w-z)\int\frac{\mathrm{d}x}{(x-z)^2}\frac{\rho_0(x)}{\rho_0(z)}\right] + O(N^0). \tag{4.62}$$

Thus the Hamiltonian is

$$\begin{aligned}
H =& N^2\int \mathrm{d}x\left[\frac{\pi^2}{6}[\rho_0(x)]^3 - \frac{1}{2}\rho_0(x)x^2\right] - N^2\mu \\
&+ \int \mathrm{d}z\mathrm{d}w \frac{\partial^2 H}{\partial\rho(w)\partial\rho(z)}\frac{1}{N^2}\partial_w\eta(w)\partial_z\eta(z) \\
&+ \int \mathrm{d}z\mathrm{d}w \frac{\partial^2 H}{\partial\phi(w)\partial\phi(z)}\frac{1}{N}\partial_w\varphi(w)\partial_z\varphi(z) \\
&+ \int \mathrm{d}z\mathrm{d}w\Omega(z,w;\rho,\rho)\Pi_\rho(z)\Pi_\rho(w) \\
&+ \int \mathrm{d}z\mathrm{d}w\Omega(z,w;\phi,\phi)\Pi_\phi(z)\Pi_\phi(w) + O(\frac{1}{\sqrt{N}}),
\end{aligned} \tag{4.63}$$

$$\begin{aligned}
H =& N^2\int \mathrm{d}x\left[\frac{\pi^2}{6}[\rho_0(x)]^3 - \frac{1}{2}\rho_0(x)x^2\right] - N^2\mu \\
&+ \frac{1}{2}\int \mathrm{d}x\pi^2\rho_0(x)\partial_x\eta(x)\partial_x\eta(x) + \frac{1}{2N^2}\int \mathrm{d}x\rho_0(x)\partial_x\Pi_\rho(x)\partial_x\Pi_\rho(x) \\
&\mp \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{1}{(x-y)^2} - \frac{1}{\rho_0(x)}\delta(x-y)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right]\varphi(x)\varphi(y) \\
&\mp \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{\rho_0(x)\rho_0(y)}{(x-y)^2} - \rho_0(x)\delta(x-y)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right]P_\varphi(x)P_\varphi(y),
\end{aligned} \tag{4.64}$$

$$H = H_{Class.} + H_\eta \pm H_\varphi, \tag{4.65}$$

where terms of order lower that $O(N^0)$ have been dropped. The part involving η and Π_ρ gives a standard kinetic term for the η field in time of flight coordinates. The part involving the φ field needs to be explored further.

Note that for the symmetric sectors, H_φ comes with a plus sign, but for the antisymmetric the opposite sign is found in our Hamiltonian. This seems to indicate a serious problem which may indicate that this collective field approach has problems that have not been understood. None the less, it can be shown that H_φ can be simplified by appropriate changes of variables and will be studied in the next section.

4.3.1 Fluctuations in the second collective field

Start with, from equation (4.64),

$$
\begin{aligned}
H_\varphi = & -\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{1}{(x-y)^2}-\frac{1}{\rho_0(x)}\delta(x-y)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right]\varphi(x)\varphi(y)\\
& -\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\left[\frac{\rho_0(x)\rho_0(y)}{(x-y)^2}-\rho_0(x)\delta(x-y)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right]P_\varphi(x)P_\varphi(y).
\end{aligned}
\tag{4.66}
$$

This is the Marchesini-Onofri Kernel discussed in the appenix C of [5]. It can be understood by going to time of flight coordinates, where $\frac{\mathrm{d}x}{\mathrm{d}q}=\pi\rho_0$ and $\Pi_\phi(x)=\frac{1}{\rho_0}\Pi_\phi(q)$. Using notation from [5][6] and the right-side up harmonic oscillator potential so that $\rho_0(x)=\frac{1}{\pi}\sqrt{1-x^2}$,

$$
H_\varphi = \frac{1}{2}\int \mathrm{d}q\Big[\varphi(q)\left(-1-i|\partial_q|\right)\varphi(q)+P_\varphi(q)\left(-1-i|\partial_q|\right)P_\varphi(q)\Big]. \tag{4.67}
$$

Then $\varphi(q)$ can be Fourier transformed on $[0,\pi]$ using sin's:

$$
\varphi(q) = \sum_{n=1}^{\infty}\varphi_n\sin(nq) \tag{4.68}
$$

$$
P_\varphi(q) = \sum_{n=1}^{\infty}P_n\sin(nq) \tag{4.69}
$$

$$
H_\varphi = \sum_{n=1}^{\infty}\left[(n-1)\,\varphi_n^2+(n-1)\,P_n^2\right] \tag{4.70}
$$

$$
\varphi_n \to \sqrt{n-1}\varphi_n
$$

$$
P_n \to \frac{1}{\sqrt{n-1}}P_n
$$

$$
H_\varphi = \sum_{n=1}^{\infty}\left[P_n^2+(n-1)^2\,\varphi_n^2\right]. \tag{4.71}
$$

H_φ has a spectrum of $E=n-1$ for $n=1,2,...$ This corresponds to the results found in section 4.2 for $\mathcal{S}$ and $\mathcal{A}$ associated with the symmetric and

[6] $|\partial_q|$ is defined by the Fourrier transform, so that $-i|\partial_q|e^{ikq} = |k|e^{ikq}$ and $-i|\partial_q|\sin(nq)=n\sin(nq)$.

antisymmetric sectors except for the $-$ sign appearing before H_φ in the antisymmetric case. This sign means that the spectrum for the antisymmetric sectors calculated with this collective field does not agree with the results from the direct study of the Hilbert space in section 4.2.3 or the partition function calculated in section 2.5.1. This sign is not understood and seems to indicate a problem with this collective field method in the antisymmetric sectors.

The zero energy mode is $\sin(q)$, corresponding to factors of $(u^\dagger \bar{u}^\dagger)$, since

$$\int \mathrm{d}q \sin(q)\varphi(q) = \pi \int \mathrm{d}q\rho_0(q)\varphi(q) = \pi \int \mathrm{d}x\varphi(x) \propto u^\dagger \bar{u}^\dagger, \tag{4.72}$$

which have no effect on the energy. Apart from this zero mode, this is the spectrum of a free field. Thus the perturbations of this second collective field form a new free field.

It is also worth noting that the presence of this zero mode is not special to the harmonic oscillator case. $\rho_0(y)$ is a zero eigenmode of the operator present in the general Hamiltonian,

$$\int \mathrm{d}y \left[\frac{1}{(x-y)^2} - \frac{1}{\rho_0(x)}\delta(x-y)\int \mathrm{d}z \frac{\rho_0(z)}{(x-z)^2}\right]. \tag{4.73}$$

4.3.2 Inverted harmonic oscillator

The same approach can be used to study the inverted harmonic oscillator potential as well. Start by defining τ so that $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \pi\rho_0$. In this case, $\rho_0 = \frac{1}{\pi}\sqrt{x^2 - 2\mu}$, so $x = \sqrt{2\mu}\cosh\tau$ and

$$\begin{aligned} H_\varphi = &-\frac{1}{2}\pi \int \mathrm{d}\tau \mathrm{d}y \left[\frac{\rho_0(x)}{(x-y)^2} - \delta(x-y)\int \mathrm{d}z \frac{\rho_0(z)}{(x-z)^2}\right]\varphi(\tau)\varphi(y) \\ &-\frac{1}{2}\pi \int \mathrm{d}\tau \mathrm{d}y \left[\frac{\rho_0(x)}{(x-y)^2} - \delta(x-y)\int \mathrm{d}z \frac{\rho_0(z)}{(x-z)^2}\right]P_\varphi(\tau)\rho_0(y)P_\varphi(y). \end{aligned} \tag{4.74}$$

This is the operator in equation (3.6) from [3] similar to the one in equation (2.64) from the introduction. The variables in the above equation were changed to facilitate comparison, in the end using τ everywhere will be more useful:

$$\begin{aligned}H_\varphi =&\frac{1}{2}\pi\int_0^\infty \mathrm{d}\tau\varphi(\tau)\left[-\frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\tau'\frac{\varphi(\tau')}{4\sinh^2(\frac{\tau-\tau'}{2})}+\varphi(\tau)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right]\\ &+\frac{1}{2}\pi\int_0^\infty \mathrm{d}\tau P_\varphi(\tau)\left[-\frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\tau'\frac{P_\varphi(\tau')}{4\sinh^2(\frac{\tau-\tau'}{2})}+P_\varphi(\tau)\int \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}\right].\end{aligned} \tag{4.75}$$

The second term diverges and needs to be regulated. Start with

$$\int_{\sqrt{2\mu}}^\infty \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2}=\frac{1}{\pi}\int_0^\infty \mathrm{d}\tau'\frac{\sinh^2\tau'}{(\cosh\tau-\cosh\tau')^2}, \tag{4.76}$$

then the integral can be cut-off in τ space at τ_c to get

$$\begin{aligned}\int_{\sqrt{2\mu}}^\infty \mathrm{d}z\frac{\rho_0(z)}{(x-z)^2} &= \frac{1}{\pi}\int_0^{\tau_c} \mathrm{d}\tau'\frac{\sinh^2\tau'}{(\cosh\tau-\cosh\tau')^2}\\ &=\frac{1}{\pi\sinh\tau}\int_{-\tau_c}^{\tau_c}\mathrm{d}\tau'\frac{\sinh\tau'}{4\sinh^2(\frac{\tau-\tau'}{2})}\\ &=\frac{1}{\pi\sinh\tau}\int_{-\tau_c-\tau}^{\tau_c-\tau}\mathrm{d}\tau'\frac{\cosh\tau\sinh\tau'+\sinh\tau\cosh\tau'}{4\sinh^2(\frac{\tau'}{2})}\\ &=\frac{1}{\pi\tanh\tau}\int_{\frac{-\tau_c-\tau}{2}}^{\frac{\tau_c-\tau}{2}}\mathrm{d}\tau'\coth(\tau')+\frac{1}{\pi}\int_{\frac{-\tau_c-\tau}{2}}^{\frac{\tau_c-\tau}{2}}\mathrm{d}\tau'\left[1+\coth^2(\tau')\right]\\ &=\frac{1}{\pi\tanh\tau}\ln\left|\frac{\sinh(\frac{\tau_c-\tau}{2})}{\sinh(\frac{-\tau_c-\tau}{2})}\right|+\frac{1}{\pi}\left[\tau_c-1\right]\\ &\approx\frac{1}{\pi\tanh\tau}\ln\left|\frac{\exp(\frac{\tau_c-\tau}{2})}{\exp(\frac{\tau_c+\tau}{2})}\right|+\frac{1}{\pi}\left[\tau_c-1\right]\\ &=\frac{1}{\pi}\left[\frac{\tau}{\tanh\tau}+\tau_c-1\right].\end{aligned} \tag{4.77}$$

Substituting into the Hamiltonian and using the notation from [3] for the Fourier transform of the first term,

$$\begin{aligned}H_\varphi =&\frac{1}{2}\int_0^\infty \mathrm{d}\tau\varphi(\tau)\left[\frac{-i\pi\partial_\tau}{\tanh(-i\pi\partial_\tau)}+\tau_c-1-\frac{\tau}{\tanh\tau}\right]\varphi(\tau)\\ &+\frac{1}{2}\int_0^\infty \mathrm{d}\tau P_\varphi(\tau)\left[\frac{-i\pi\partial_\tau}{\tanh(-i\pi\partial_\tau)}+\tau_c-1-\frac{\tau}{\tanh\tau}\right]P_\varphi(\tau).\end{aligned} \tag{4.78}$$

This is exactly the eigenvalue problem considered in [11], where it is found that there exists eigenfunctions

$$\pi\hat{\epsilon}h(\tau) = \left(\frac{-i\pi\partial_\tau}{\tanh(-i\pi\partial_\tau)} - \frac{\tau}{\tanh\tau}\right)h(\tau) \tag{4.79}$$

for any $\hat{\epsilon}$.

Following the approach of the previous section, $E = \tau_c - 1 + \hat{\epsilon}$. This spectrum is unbounded below, but also has a infinite constant added. Considering a renormalised energy ϵ, as in [3], replicates their conclusions. In other words, this is the long string spectrum, but the long strings do not condense. At lowest order, they create a free gas of bosonic strings.

4.3.3 Other classical solutions

The classical solution ϕ_0 found above did not give a correction to the energy, which is consistent with the direct study of the symmetric and antisymmetric sector's Hilbert spaces, where the ground states were found to be degenerate with the singlet sector's ground state. In other words, there was no expectation of a Fermi sea or a shift in the ground state energy for these sectors. Such a shift was only found for the mixed sector, which could not be studied with this collective field.

However, it is possible that there are other classical solutions of this model. To find such new configurations, new non-trivial solutions to a non-linear integral equation would need to be found, which belongs to a class of problems for which no good methodology has been developed. If they exist, there is nothing preventing these other solutions from describing configurations where the long strings condense and the FZZT brane is present.

4.4 Other collective fields

The collective fields, $\rho(x) = \mathrm{Tr}[\delta(x-M)]$ and $\phi(x) = u^\dagger\delta(x-M)\bar{u}^\dagger$, studied in section 4.3, can be used to construct all the wavefunctions in the direct sum of all of the symmetric sectors. A different collective field $\zeta(x) = u^\dagger\delta(x-M)u$ along with $\rho(x)$ could be used to study the symmetric and mixed sectors. $\zeta(x) = v^\dagger\delta(x-M)v$ could be used for the antisymmetric sectors. Since the action of these fields commute with $u^\dagger u$ and $v^\dagger v$, they would allow the study of finite fixed K sectors. In addition, unlike the field used in section 4.3, they can be used to study the mixed representations.

This type of field can be used to create any state in the Hilbert space of these fixed K sectors by acting on the ground state of the relevant sector, modulo any factors of in the singlet sector that can be accounted for with the ρ collective field. For the symmetric and antisymmetric sectors this state is $(u^\dagger \bar{u}^\dagger)^K$ and $(v^\dagger \bar{v}^\dagger)^K$ respectively. Since these have no M dependence, the change of variables to the collective field proceeds much the same, except that $\Omega(x, y; \phi, \phi)$ would have some extra terms due to the non-trivial commutation relation between u and $u^\dagger$.

For the mixed representations, $(u^\dagger \bar{v}^\dagger)^K$ vanishes due to the anticommuting nature of $u^\dagger \bar{v}^\dagger$. Instead, $(u^\dagger \bar{v}^\dagger)(u^\dagger M \bar{v}^\dagger) \ldots (u^\dagger M^{K-1} \bar{v}^\dagger)$ must be used with ζ to construct it. This state has an M dependence, so the action of $\frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}}$ on it, as well as all the relevant cross terms must be taken into account. Further analysis of the mixed representations is left for future work.

Chapter 5

Conclusions

In chapter 3, non-singlet sectors of the $c = 1$ matrix model were described by N particles with K labels to be distributed among them. These labels can be understood as an internal degree of freedom for each particle. The labels in the antisymmetric and mixed sectors correspond to an $SU(2)$ internal space in the fundamental representation analogous to spin-$\frac{1}{2}$, where there are K units of spin to be distributed to the N particles. The labels in the symmetric sector can be understood in terms of an $SL(2)$ internal space, in a representation where each particle can carry an arbitrary number of quanta of a spin analogue and where there are K quanta of this spin analogue to be distributed. However, writing the Hamiltonian in terms of these internal degrees of freedom did not lead to a standard spin-spin interaction term. The potential found resembles that of a Calogero model, which presents an interesting topic for further study.

In chapter 4, a matrix-vector model was introduced to more efficiently study the representations of interest. This lead to the direct computation of partition functions in the harmonic oscillator potential which were found to match our expectations from section 2.5.1. In particular it was found that the ground states of the symmetric and antisymmetric sectors were degenerate with that of the singlet sector, but that the mixed sector included a shift in the ground state energy indicative of a Fermi sea. In fact this ground state was explicitly constructed and found to be consistent with a the presence of a Fermi sea in the harmonic oscillator potential. A collective field was then introduced which enabled the study of the symmetric and antisymmetric sectors. This lead to a second free field in the lowest order in N perturbations of this second collective field with the harmonics oscillator potential. In the inverted oscillator potential, this collective field described an arbitrary number of the long strings which were not found to condense or form a Fermi sea. Since this collective field did not permit the study of the mixed sectors, a new collective field was proposed that would allow its study, but the development of this collective field is left to future work.

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