Escape of Mass on Hilbert Modular Varieties

by

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Abstract

Let F be a number field, $G = \mathrm{PGL}(2, F_\infty)$, and K be a maximal compact subgroup of G. We eliminate the possibility of escape of mass for measures associated to Hecke-Maaß cusp forms on Hilbert modular varieties, and more generally on congruence locally symmetric spaces covered by G/K, hence enabling its application to the non-compact case of the Arithmetic Quantum Unique Ergodicity Conjecture. This thesis generalizes work by Soundararajan in 2010 eliminating escape of mass for congruence surfaces, including the classical modular surface $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2$, and follows his approach closely.

First, we define M, a congruence locally symmetric space covered by G/K, and articulate the details of its structure. Then we define Hecke-Maass cusp forms and provide their Whittaker expansion along with identities regarding the Whittaker coefficients. Utilizing these identities, we introduce mock $\mathcal{P}$-Hecke multiplicative functions and bound a key related growth measure following Soundararajan's paper. Finally, amassing our results, we eliminate the possibility of escape of mass for Hecke-Maaß cusp forms on M.

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To my parents

Introduction

The methodologies and techniques employed in this thesis are primarily number theoretic in nature, but the fundamental phenomenon in question, namely "escape of mass", is a commonplace concept in real analysis. Let M be a topological measure space, and $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of measures on M. There exists many notions of convergence of measures, but in this thesis, we are concerned with a specific type, namely *weak-$*$ convergence.* We say the sequence μ_j *converges to a measure μ in the weak-$*$ topology* if for every compact set $A \subseteq M$, we have

$$\int_A d\mu_j \to \int_A d\mu, \qquad \text{as } j \to \infty.$$

More simply, we say μ_j *weak-$*$ converges to μ*, or write $\mu_j \overset{\text{wk-}*}{\longrightarrow} \mu$.

If one assumes all μ_j are probability measures, i.e. $\int_M d\mu_j = 1$ and $\mu_j \overset{\text{wk-}*}{\longrightarrow} \mu$, then it is a natural question to ask:

> *Is μ still a probability measure? In other words, is $\int_M d\mu = 1$?*

In these general terms, the question is a simple exercise in real analysis. If M is compact, then it follows immediately from the definition of weak-$*$ convergence that μ is indeed a probability measure. However, if M is not compact, then the answer is: not necessarily.

Since $\{\mu_j\}_{j=1}^{\infty}$ are probability measures, it follows that $\int_M d\mu \le 1$, but it is certainly possible that this inequality is strict, or in other words, there may be *escape of mass.* A simple example demonstrating this phenomenon would be taking $M := \mathbb{R}$ and $d\mu_j := \mathbf{1}_{[j,j+1]}(x)dx$ for $j \ge 1$ where dx is the Lebesgue measure. With this choice, one can easily see that μ_j weak-$*$ converges to the zero measure, and so have "lost mass". Thus, in general, the set of probability measures is not closed in the weak-$*$ topology.

Our intention is to study this phenomenon known as "escape of mass" in a setting with number theoretic origins and implications, where further restrictions are placed on the measures μ_j and manifold M. Beginning with the classical case, let Γ be a congruence group (e.g.

SL(2, $\mathbb{Z}$)) and $\mathbb{H}^2$ denote the upper half plane of complex numbers endowed with the hyperbolic metric. The negatively curved manifold $M = \Gamma\backslash\mathbb{H}^2$ is called a *congruence surface*. It is well known that M is non-compact and so, in general, escape of mass can occur. However, we restrict our attention to probability measures of number-theoretic importance, namely those associated to Hecke-Maaß cusp forms on M (see Chapter 3 for details).

A Hecke-Maaß cusp form is a function $\phi \in L^2(\Gamma\backslash\mathbb{H}^2)$, which is a simultaneous eigenfunction of the hyperbolic Laplacian Δ, and every Hecke operator $T(n), n \in \mathbb{Z}^+$. Further, to ϕ, we can associate the probability measure

$$d\mu_\phi := \frac{|\phi(z)|^2 d\mathrm{vol}(z)}{||\phi||_{L^2}^2}$$

where $d\mathrm{vol}(z)$ is the hyperbolic area measure on $\mathbb{H}^2$. Then one may ask if escape of mass can occur for such probability measures, which is addressed in the following theorem:

Theorem (Soundararajan, 2010 [Sou10])**.** *Let $\Gamma \leq \mathrm{SL}(2,\mathbb{Z})$ be a congruence group. Suppose $\{\phi_j\}_{j=1}^\infty$ is a sequence of Hecke-Maaß cusp forms on $\Gamma\backslash\mathbb{H}^2$, and $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$ for some measure μ. Then μ is a probability measure. In other words, no escape of mass occurs.*

Utilizing the multiplicative properties of the Hecke eigenvalues of ϕ inherited from the Hecke operators $\{T(n) : n \in \mathbb{Z}^+\}$, Soundararajan was able to eliminate the possibility of escape of mass by applying deceptively simple yet ingenious elementary number theory. This key result and its methods are the basis of this thesis work and the related results.

Now, the primary application of Soundararajan's result is completing the proof of Arithmetic Quantum Unique Ergodicity (AQUE) and in fact, the problem of AQUE was the motivation to prove such a result. To describe AQUE, let us begin with the initial conjecture of Quantum Unique Ergodicity (QUE), which has been widely studied since it was first stated by Rudnick and Sarnak in 1994.

Conjecture (Quantum Unique Ergodicity, [RS94])**.** *Let M be a compact negatively curved manifold M with Laplacian Δ and volume measure* vol*. Suppose $\{\phi_j\}_{j=1}^\infty \subseteq L^2(M)$ is a sequence of eigenfunctions of Δ with eigenvalues $\lambda_j \to \infty$ and with associated probability measures*

$$d\mu_{\phi_j} = \frac{|\phi_j(z)|^2 d\mathrm{vol}(z)}{||\phi_j||_{L^2}^2}.$$

If $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$, then $\mu =$ vol.

As stated in these general terms, QUE remains elusive without further restrictions on M, and seemingly has little to do with number theory. Instead, we are concerned with the same conjecture, but further assume M is a congruence surface, giving rise to the simpler conjecture with connections to number theory:

Conjecture (Arithmetic Quantum Unique Ergodicity, [RS94])**.** *Let M be a congruence hyperbolic surface with Laplacian Δ and volume measure* vol*. Suppose $\{\phi_j\}_{j=1}^{\infty} \subseteq L^2(M)$ is a sequence of eigenfunctions of Δ with associated probability measures μ_{ϕ_j}. If $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$, then $\mu = \mathrm{vol}$.*

Not long after this conjecture, the combined works of Luo and Sarnak [LS95], and Jakobson [Jak94] onfirmed AQUE for the continuous spectrum of Δ. However, AQUE for the discrete spectrum remained unknown until in 2006, Lindenstrauss [Lin06] proved AQUE for Hecke-Maaß cusp forms ϕ except for the possibility of escape of mass.

Theorem (Lindenstrauss, 2006 [Lin06])**.** *Let M be a congruence surface. Suppose $\{\phi_j\}_{j=1}^{\infty} \subseteq L^2(M)$ is a sequence of Hecke-Maaß cusp forms with Laplace eigenvalues $\lambda_j \to \infty$. If $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$, then $\mu = c \cdot \mathrm{vol}$ for some $c \in [0, 1]$.*

If M is a compact surface, one can conclude $c = 1$ by the definition of weak-$*$ convergence, thus completing the proof of AQUE for compact congruence hyperbolic surfaces M. However, in the non-compact case, one must necessarily guarantee μ is a probability measure, i.e. eliminate escape of mass, to conclude $c = 1$ and hence complete the proof of AQUE. Several years later, the aforementioned theorem of Soundarajan proves exactly this fact yielding

Corollary (Lindenstrauss [Lin06], Soundararajan [Sou10])**.** *AQUE holds for Hecke-Maaß cusp forms on congruence surfaces.*

For a more detailed overview of the QUE conjecture, one should see an article by Sarnak [Sar11], and for further details on AQUE, there are notes by Einsiedler and Ward [EW10] or an article by Marklof [Mar06].

Now, this thesis is concerned with generalizations of Soundararajan's theorem to higher dimensional analogues of congruence hyperbolic surfaces $M = \Gamma\backslash\mathbb{H}^2$. A natural and well known analogue is a Hilbert modular variety. For a totally real number field F of degree n, let $\mathcal{O}$ denote the ring of integers. Then the group $\mathrm{SL}(2, \mathcal{O})$ acts discretely on the n-fold product of upper half planes, $(\mathbb{H}^2)^n$ via the n embeddings of F (see Chapter 2). The finite volume manifold $M = \mathrm{SL}(2, \mathcal{O})\backslash(\mathbb{H}^2)^n$ is known as the *Hilbert modular variety of F*. In the case $F = \mathbb{Q}$, the field of rational numbers, this is simply the modular surface. As in Chapter 3, Hecke-Maaß cusp forms can be defined more generally on Hilbert modular varieties, and so one can conjecture AQUE holds for Hecke-Maaß cusp forms on Hilbert modular varieties.

In fact, one can allow F to be an arbitrary number field with r_1 real embeddings and r_2 complex embeddings so $n = r_1 + 2r_2$. Replacing $(\mathbb{H}^2)^n$ by $\mathcal{X} := (\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}$ where $\mathbb{H}^3$ is hyperbolic 3-space, we may again define a finite volume manifold $\Gamma_F \backslash \mathcal{X}$ where $\Gamma_F = \mathrm{PGL}(2, \mathcal{O})$. Even more generally, one can replace Γ_F by congruence subgroups Γ of $\mathrm{PGL}(2, \mathcal{O})$ and study the manifolds $M = \Gamma \backslash \mathcal{X}$, which we call *congruence locally symmetric spaces.* Again, Hecke-Maaß cusp forms can be defined in this case, so we arrive at the following generalized conjecture.

Conjecture 1. *AQUE holds for Hecke-Maaß cusp forms on congruence locally symmetric spaces, and hence on Hilbert modular varieties as well.*

In principle, as remarked by Sarnak [Sar11], a theorem of the form of Lindenstrauss' [Lin06] can be established for this conjecture by following methods of [Lin06], [EKL06] and [BL03]. Therefore, in order to positively answer the above conjecture, one must eliminate escape of mass on congruence locally symmetric spaces as Soundararajan did for congruence surfaces. This elimination of escape of mass for congruence locally symmetric spaces is the central result of this thesis.

Theorem. *Let M be a congruence locally symmetric space, and let $\{\phi_j\}_{j=1}^{\infty} \subseteq L^2(M)$ be Hecke-Maaß cusp forms of M with associated probability measures μ_{ϕ_j}. If $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$ for some measure μ, then μ is a probability measure.*

In other words, there is no escape of mass.

In proving the main theorem above, we will very closely follow the approach of Soundararajan [Sou10] in the classical case except for one main distinction. For a Hecke-Maaß cusp form ϕ on the modular surface $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2$, the Whittaker coefficients $\{c_\sigma(\phi; n) : n \in \mathbb{Z}\}$ at a cusp σ of ϕ can be identified with its Hecke eigenvalues $\lambda_\phi(n)$; namely, for all $n \in \mathbb{Z}$,

$$c_\sigma(\phi; n) = C_\phi \cdot \lambda_\phi(n)$$

where $C_\phi \in \mathbb{C}$ is some constant. As a result, $c_\sigma(\phi; n)$ inherits the multiplicative properties of $\lambda_\phi(n)$, on which Soundararajan's result critically relies.

Now, in our case, for a Hecke-Maaß cusp form ϕ on a congruence locally symmetric space $\Gamma \backslash \mathcal{X}$ of level $\mathfrak{N}$, this identification is no longer true. Instead, the Whittaker coefficients $\{c_\sigma(\phi; \alpha) : \alpha \in (\mathfrak{D}\mathfrak{N})^{-1}\}$ at a cusp σ possess a less restrictive relation with the Hecke eigenvalues $\{\lambda_\phi(\mathfrak{n}) : \mathfrak{n} \subseteq \mathcal{O} \text{ ideal }\}$. Namely, suppose an integral ideal $\mathfrak{n}$ is composed of unramified principal prime ideals not dividing the level of Γ, so $\mathfrak{n} = (\eta)$ is itself principal. If $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$ is a unit modulo $\mathfrak{n}$, then

$$c_\sigma(\phi; \alpha\eta) = c_\sigma(\phi; \alpha) \cdot \lambda_\phi(\mathfrak{n}).$$

This thesis introduces objects related to this weaker identity, called *mock $\mathcal{P}$-Hecke multiplicative functions*, and adapts Soundararajan's argument to this scenario, producing the key technical result analogous to Theorem 3 of Soundararajan [Sou10]:

Theorem. *Let $\mathcal{P}$ be a set of unramified prime ideals of a number field F not dividing the integral ideal $\mathfrak{N}$, and f be a mock $\mathcal{P}$-Hecke multiplicative function of level $\mathfrak{N}$. If $\mathcal{P}$ has positive natural density, then*

$$\sum_{\mathbb{N}\mathfrak{n} \leq y/Y} |f(\mathfrak{n})|^2 \ll_{\mathcal{P}} \left(\frac{1 + \log Y}{\sqrt{Y}} \right) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.$$

for $1 \leq Y \leq y$.

On a separate note, while Soundararajan's proof is written for $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}^2$, it is apparent that the argument can be easily adjusted to any congruence surface $\Gamma \backslash \mathbb{H}^2$. The proof we provide shall explicitly take into account the level of Γ and hence any congruence locally symmetric space $\Gamma \backslash \mathcal{X}$.

Before embarking on the proof eliminating escape of mass, we summarize the contents of this thesis. First, in Chapter 1, we review necessary material about hyperbolic 2- and 3-space. Second, in Chapter 2, we precisely define a congruence locally symmetric space M, and describe its structure as a finite number of cusps and a compact centre. In Chapter 3, we define Hecke-Maass cusp forms on M, provide their Whittaker expansion, and show the relations between coefficients and Hecke eigenvalues. Then in Chapter 4, we establish the key technical result on mock $\mathcal{P}$-Hecke multiplicative functions. Finally, Chapter 5 is the culmination of the proof eliminating escape of mass.

Chapter 1

Hyperbolic 2- and 3-space

This section reviews necessary and well-known material regarding the geometry of hyperbolic 2- and 3-space and related group actions. The information is drawn from [Iwa02] for hyperbolic 2-space, and from [EGM98] for hyperbolic 3-space. General theory for hyperbolic n-space can be found in [BH99], [Rat94], [BP92] and [CFKP97].

1.1 Hyperbolic 2-space and $\mathrm{PGL}(2, \mathbb{R})$

To model hyperbolic 2-space, i.e. the maximally symmetric, simply connected, 2-dimensional Riemannian manifold with constant sectional curvature -1, we shall use the upper half space model, namely

$$\mathbb{H}^2 := \{z = (x, y) : x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$$

with the metric

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

We may regard $\mathbb{H}^2$ as a subset of the complex numbers $\mathbb{C}$ by identifying $(x, y) \mapsto x + iy$, i.e. $i = (0, 1)$. We also define the map

$$\begin{aligned} \mathrm{Im} : \ \mathbb{H}^2 &\to \mathbb{R}_{>0} \\ (x, y) &\mapsto y. \end{aligned}$$

The associated volume measure is given by

$$d\mathrm{vol}(z) = \frac{dA(x)dy}{y^2} \quad \text{where } dA(x) = dx, \tag{1.1}$$

and the associated Laplace-Beltrami operator is given by

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big). \tag{1.2}$$

Closely related to $\mathbb{H}^2$ is the group

$$\mathrm{PGL}(2,\mathbb{R}) := \mathrm{GL}(2,\mathbb{R})/\{\lambda I : \lambda \in \mathbb{R}^*\}$$

which acts transitively on the upper half plane $\mathbb{H}^2 \subseteq \mathbb{C}$ via fractional linear transformations:

$$g \cdot z := \begin{cases} \dfrac{az+b}{cz+d} & \det g > 0 \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} & \det g < 0 \end{cases}, \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{R}) \text{ and } z \in \mathbb{H}^2,$$

Note that this action is independent of the choice of representative for g, since scalar matrices act trivially. Since the $\mathrm{PGL}(2,\mathbb{R})$-action on $\mathbb{H}^2$ is transitive, we have that the orbit of $i = (0,1) \in \mathbb{H}^2$ is

$$\mathrm{PGL}(2,\mathbb{R}) \cdot (0,1) = \mathbb{H}^2.$$

By direct calculations, one can verify that the stabilizer of $(0,1) \in \mathbb{H}^2$ is

$$\mathrm{PO}(2,\mathbb{R}) := \{A \in \mathrm{GL}(2,\mathbb{R}) : A^T A = I\}/\{\pm I\},$$

so we may identify the quotient group $\mathrm{PGL}(2,\mathbb{R})/\mathrm{PO}(2,\mathbb{R})$ with $\mathbb{H}^2$ via the map $g\mathrm{PO}(2,\mathbb{R}) \mapsto g \cdot (0,1)$. Equipping the quotient group with the quotient topology, this identification is in fact a diffeomorphism. We summarize our conclusions in the following proposition.

Proposition 1.1. *The space* $\mathbb{H}^2$*, equipped with the metric* ds*, is diffeomorphic to* $\mathrm{PGL}(2,\mathbb{R})/\mathrm{PO}(2,\mathbb{R})$*, equipped with the natural quotient topology. Further, the diffeomorphism may be chosen so that the* $\mathrm{PGL}(2,\mathbb{R})$ *action on* $\mathbb{H}^2$ *transfers to an action by left multiplication on* $\mathrm{PGL}(2,\mathbb{R})/\mathrm{PO}(2,\mathbb{R})$.

Now, consider the space of functions $C^\infty(\mathbb{H}^2)$, on which the Laplace-Beltrami operator Δ acts naturally. The group $\mathrm{PGL}(2,\mathbb{R})$ also acts on naturally on this space: for $g \in \mathrm{PGL}(2,\mathbb{R})$ and $\phi(z) \in C^\infty(\mathbb{H}^2)$, we define

$$g \cdot \phi(z) := \phi(g \cdot z).$$

Either by direct computation or realizing Δ as an element of the center of the universal enveloping algebra of the Lie algebra associated to $\mathrm{PGL}(2,\mathbb{R})$, one may verify that the action of

Δ commutes with the action of $\mathrm{PGL}(2,\mathbb{R})$ on $C^\infty(\mathbb{H}^2)$.

Proposition 1.2. *The Laplace-Beltrami operator Δ commutes with the action of* $\mathrm{PGL}(2,\mathbb{R})$ *on $C^\infty(\mathbb{H}^2)$, which is defined by*

$$g \cdot \phi(z) = \phi(g \cdot z)$$

for $g \in PGL(2,\mathbb{R})$ and $\phi(z) \in C^\infty(\mathbb{H}^2)$.

As a final note, the $\mathrm{PGL}(2,\mathbb{R})$-action on $\mathbb{H}^2$ preserves the volume measure $d\mathrm{vol}(z)$, which one can show through direct computation.

Proposition 1.3. *The volume measure* $d\mathrm{vol}(z)$ *of* $\mathbb{H}^2$ *is invariant under the action of* $\mathrm{PGL}(2,\mathbb{R})$*, i.e.* $d\mathrm{vol}(g \cdot z) = d\mathrm{vol}(z)$ *for all* $g \in \mathrm{PGL}(2,\mathbb{R})$.

1.2 Hyperbolic 3-space and $\mathrm{PGL}(2,\mathbb{C})$

To model hyperbolic 3-space, i.e. the maximally symmetric, simply connected, 3-dimensional Riemannian manifold with constant sectional curvature -1, we shall utilize the set

$$\mathbb{H}^3 := \{z = (x, y) : x \in \mathbb{C}, y \in \mathbb{R}_{>0}\}$$

with the metric

$$ds^2 = y^{-2}(dx_1^2 + dx_2^2 + dy^2)$$

where $x = x_1 + ix_2$. We may regard $\mathbb{H}^3$ as a subset of the Hamilton's quaternions $\mathcal{H}$ by identifying $(x, y) \mapsto x + jy$, i.e. $j = (0, 1)$. We also define the map

$$\begin{aligned} \mathrm{Im} :\ \mathbb{H}^3 &\to \mathbb{R}_{>0} \\ (x, y) &\mapsto y. \end{aligned}$$

The associated volume measure is given by

$$d\mathrm{vol}(z) = \frac{dA(x)dy}{y^3} \quad \text{where } dA(x) = dx_1 dx_2, \tag{1.3}$$

and the associated Laplace-Beltrami operator is given by

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2}\Big) - y\frac{\partial}{\partial y}. \tag{1.4}$$

Closely related to $\mathbb{H}^3$ is the group

$$\mathrm{PGL}(2, \mathbb{C}) := \mathrm{GL}(2, \mathbb{C})/\{\lambda I : \lambda \in \mathbb{C}^\times\}$$

which acts on $\mathbb{H}^3 \subseteq \mathcal{H}$ via fractional linear transformations:

$$g \cdot z := (az + b)(cz + d)^{-1}, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C}) \text{ and } z \in \mathbb{H}^3,$$

where one considers the point at ∞ in a natural limiting sense. One can verify (or force by definition) that scalar matrices act trivially, and so the action is independent of the choice of representative of g.

Since the $\mathrm{PGL}(2, \mathbb{C})$-action on $\mathbb{H}^3$ is transitive, we have that the orbit of $(0, 1) \in \mathbb{H}^3$ is

$$\mathrm{PGL}(2, \mathbb{C}) \cdot (0, 1) = \mathbb{H}^3.$$

By direct calculations, one can verify that the stabilizer of $(0, 1) \in \mathbb{H}^3$ is

$$\mathrm{PU}(2, \mathbb{C}) := \{A \in \mathrm{GL}(2, \mathbb{C}) : A^H A = I\}/\{\lambda I : \lambda \in \mathbb{C}^\times, |\lambda| = 1\},$$

so we may identify the quotient group $\mathrm{PGL}(2, \mathbb{C})/\mathrm{PU}(2, \mathbb{R})$ with $\mathbb{H}^3$ via the map $g\mathrm{PU}(2, \mathbb{C}) \mapsto g \cdot (0, 1)$. Equipping the quotient group with the quotient topology, this identification is in fact a diffeomorphism. We summarize our conclusions in the following proposition.

Proposition 1.4. *The space* $\mathbb{H}^3$*, equipped with the metric* ds*, is diffeomorphic to* $\mathrm{PGL}(2, \mathbb{C})/\mathrm{PU}(2, \mathbb{C})$*, equipped with the natural quotient topology. Further, the diffeomorphism may be chosen so that the* $\mathrm{PGL}(2, \mathbb{C})$ *action on* $\mathbb{H}^3$ *transfers to an action by left multiplication on* $\mathrm{PGL}(2, \mathbb{C})/\mathrm{PU}(2, \mathbb{C})$.

Now, consider the space of functions $C^\infty(\mathbb{H}^3)$, on which the Laplace-Beltrami operator Δ acts naturally. The group $\mathrm{PGL}(2, \mathbb{C})$ also acts on naturally on this space: for $g \in \mathrm{PGL}(2, \mathbb{C})$ and $\phi(z) \in C^\infty(\mathbb{H}^3)$, we define

$$g \cdot \phi(z) := \phi(g \cdot z).$$

Either by direct computation or realizing Δ as an element of the center of the universal enveloping algebra of the Lie algebra associated to $\mathrm{PGL}(2, \mathbb{C})$, one may deduce that the action of Δ commutes with the action of $\mathrm{PGL}(2, \mathbb{C})$ on $C^\infty(\mathbb{H}^3)$.

Proposition 1.5. *The Laplace-Beltrami operator* Δ *commutes with the action of* $\mathrm{PGL}(2, \mathbb{C})$

on $C^\infty(\mathbb{H}^3)$*, which is defined by*

$$g \cdot \phi(z) = \phi(g \cdot z)$$

for $g \in PGL(2, \mathbb{C})$ *and* $\phi(z) \in C^\infty(\mathbb{H}^3)$*.*

As a final note, the $\mathrm{PGL}(2, \mathbb{C})$-action on $\mathbb{H}^3$ preserves the volume measure $d\mathrm{vol}(z)$, which one can show through direct computation or general theory.

Proposition 1.6. *The volume measure* $d\mathrm{vol}(z)$ *of* $\mathbb{H}^3$ *is invariant under the action of* $\mathrm{PGL}(2, \mathbb{C})$*, i.e.* $d\mathrm{vol}(g \cdot z) = d\mathrm{vol}(z)$ *for all* $g \in \mathrm{PGL}(2, \mathbb{R})$*.*

Chapter 2

Congruence Locally Symmetric Spaces

This chapter is dedicated to generalizing our understanding of a fundamental domain for the classical modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$ (see Figure 2.1) to a congruence locally symmetric space $M = \Gamma\backslash\mathcal{X}$.

We describe in detail a congruence locally symmetric space $M = \Gamma\backslash\mathcal{X}$ utilizing the models of hyperbolic 2- and 3-space, and fixing notation for various parametrizations of M. Much of the discussion will be focused on the cusps of M and the precise structure of the cusp stabilizer fundamental domain. Finally, we provide a complete description of a fundamental domain for M as a disjoint union consisting of a compact centre, and a finite number of cusps.

The material on congruence locally symmetric spaces is derived from Chapter 1 of [Gee88]; other sources include [Hir73] and [Fre90]. The classical case $F = \mathbb{Q}$ is discussed in [Miy89], [Iwa02] and [IK04]. Any necessary algebraic number theory can be found in [Neu99], [Lan94], [Mar77], or [ME05].

2.1 Symmetric Space of $G = \mathrm{PGL}(2, F_\infty)$

For the remainder of this thesis, fix a number field F of degree n (i.e. $[F : \mathbb{Q}] = n$) with r_1 real embeddings and $2r_2$ complex embeddings, so $n = r_1 + 2r_2$. For each place v of F, denote F_v to be the completion of F with respect to v. Define the groups

$$G_v := \mathrm{PGL}(2, F_v) := \mathrm{GL}(2, F_v)\Big/\left\{\eta I : \eta \in F_v^\times\right\}$$

We are interested in the $m := r_1 + r_2$ archimedean places v of F, denoted $v \mid \infty$. For $v \mid \infty$,

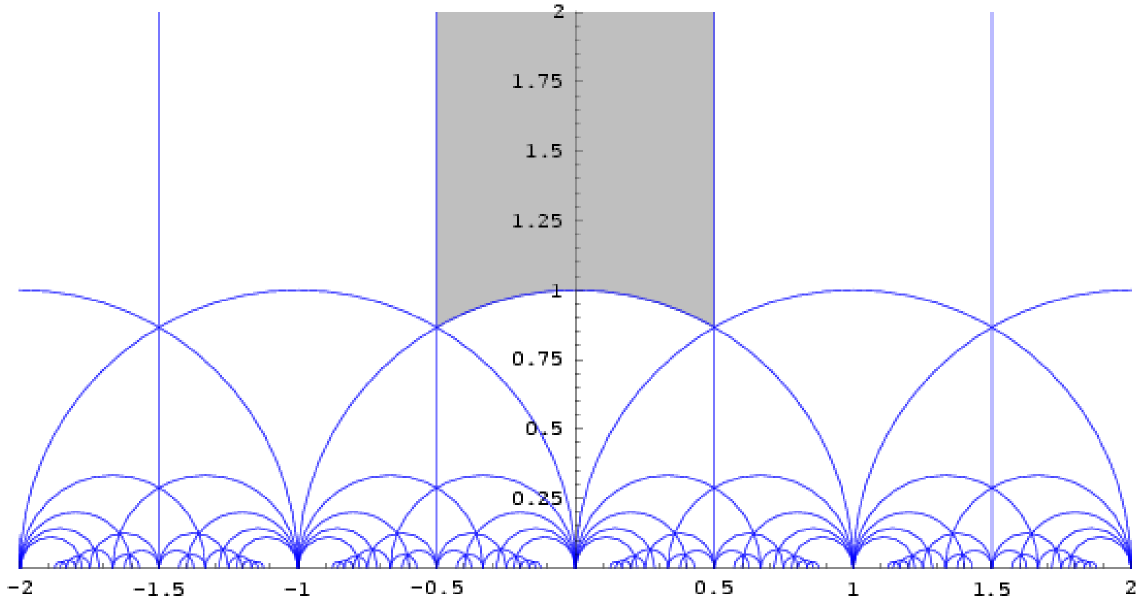

Figure 2.1: A fundamental domain for $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$

Source: © Fropuff, 2004, by permission. Retrieved 25 April 2012 from *Wikipedia*.
http://en.wikipedia.org/wiki/File:ModularGroup-FundamentalDomain-01.png

we choose an explicit maximal compact subgroup of G_v, namely

$$K_v := \begin{cases} \mathrm{PO}(2,\mathbb{R}) & v \text{ real} \\ \mathrm{PU}(2,\mathbb{C}) & v \text{ complex} \end{cases},$$

where

$$\begin{aligned} \mathrm{PO}(2,\mathbb{R}) &= \mathrm{O}(2,\mathbb{R})/Z(\mathrm{O}(2,\mathbb{R})) \\ &= \left\{A \in GL(2,\mathbb{R}) : A^T A = I\right\}/\{\pm I\}, \\ \mathrm{PU}(2,\mathbb{C}) &= \mathrm{U}(2,\mathbb{C})/Z(\mathrm{U}(2,\mathbb{C})) \\ &= \{A \in \mathrm{GL}(2,\mathbb{C}) : A^* A = I\}\,/\left\{\eta I : \eta \in \mathbb{C}^\times, |\eta| = 1\right\}. \end{aligned}$$

Define

$$G := \mathrm{PGL}(2, F_\infty) = \prod_{v|\infty} G_v, \qquad K := \prod_{v|\infty} K_v,$$

where $F_\infty = \prod_{v|\infty} F_v$. This yields the following diffeomorphism:

$$G/K = \prod_{v|\infty} G_v/K_v \cong \prod_{j=1}^{r_1} \mathbb{H}^2 \times \prod_{j=1}^{r_2} \mathbb{H}^3$$

such that through the above map, the action of G on G/K by left multiplication transfers to a component-wise action of G on $(\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}$ as described in Chapter 1. The space

$$\mathcal{X} = \mathcal{X}_F := (\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}$$

will be referred to as the *symmetric space of* G. We collect the facts and notation from Chapter 1 to provide a sufficient and consistent description of $\mathcal{X}$. Elements of $\mathcal{X}$ will be written in the following coordinates:

$$\mathbf{z} = (z_v)_{v|\infty} \in \mathcal{X} \quad \text{where } z_v = (x_v, y_v) \in \begin{cases} \mathbb{H}^2 & v \text{ real} \\ \mathbb{H}^3 & v \text{ complex} \end{cases}.$$

Often, we will decompose these coordinates so that

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \qquad \text{where } \mathbf{x} = (x_v)_{v|\infty} \in F_\infty, \text{ and } \mathbf{y} = (y_v)_{v|\infty} \in \mathbb{R}^m_{>0}.$$

With this parametrization, we define the map

$$\begin{aligned} \mathrm{Im}: \ \mathcal{X} &\to \mathbb{R}^m_{>0} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \mathbf{y}. \end{aligned}$$

so $\mathrm{Im}(\mathbf{z}) = (\mathrm{Im}_v(z_v))_{v|\infty}$. The volume measure associated to $\mathcal{X}$ is simply the product measure $d\mathrm{vol}(\mathbf{z}) = \wedge_{v|\infty} d\mathrm{vol}_v(z_v)$, or in $(\mathbf{x}, \mathbf{y})$-coordinates is given by

$$\begin{aligned} d\mathrm{vol}(\mathbf{z}) = dV_1(\mathbf{x})\, dV_2(\mathbf{y}) \text{ where } \quad dV_1(\mathbf{x}) &:= \bigwedge_{v|\infty} dA_v(x_v) \quad \text{and} \\ dV_2(\mathbf{y}) &:= \bigwedge_{v|\infty} \frac{1}{\mathbb{N}_v y_v} \cdot \frac{dy_v}{y_v}. \end{aligned} \tag{2.1}$$

Note $\mathbb{N}_v$ extends the local norm $\mathbb{N}_{\mathbb{R}}^{F_v}$ of a place $v \mid \infty$ to $\mathbb{H}^2$ for v real and to $\mathbb{H}^3$ for v complex by

$$\mathbb{N}_v z_v = \begin{cases} z_v & v \text{ real} \\ z_v \overline{z_v} & v \text{ complex} \end{cases}.$$

We shall also extend the global norm $\mathbb{N} = \mathbb{N}_{\mathbb{Q}}^{F}$ to $\mathbf{z} \in \mathcal{X}$ via the usual product formula:

$$\mathbb{N}\mathbf{z} = \prod_{v|\infty} \mathbb{N}_v z_v.$$

As a simple remark, for $r \in \mathbb{R}_{>0}$, the substitution $\mathbf{y} \mapsto r \cdot \mathbf{y} := (ry_v)_{v|\infty}$ yields

$$dV_2(r \cdot \mathbf{y}) = \bigwedge_{v|\infty} \frac{1}{\mathbb{N}_v(ry_v)} \frac{dy_v}{y_v} = \Big(\prod_{v|\infty} \frac{1}{r^{[F_v:\mathbb{R}]}} \Big) dV_2(\mathbf{y}) = r^{-n} dV_2(\mathbf{y}). \tag{2.2}$$

Finally, Δ_v shall denote the Laplacian associated to the z_v-coordinate of $\mathbf{z} \in \mathcal{X}$.

2.2 Congruence Subgroups and their Cusps

Consider the group

$$\mathrm{PGL}(2, F) := \mathrm{GL}(2, F) / \left\{ \eta I : \eta \in F^{\times} \right\}$$

which can be naturally embedded

$$\mathrm{PGL}(2, F) \hookrightarrow \prod_{v|\infty} \mathrm{PGL}(2, F_v) = G \tag{2.3}$$

via the usual embeddings $F \hookrightarrow \prod_{v|\infty} F_v$. Through this embedding, $\mathrm{PGL}(2, F)$ acts on G/K by left multiplication, and hence acts on $\mathcal{X}$. We aim to understand the action of a class of discrete subgroups of $\mathrm{PGL}(2, F)$ acting on $\mathcal{X}$. Let $\mathcal{O}$ denote the ring of integers of F, and define the distinguished group

$$\Gamma_F := \mathrm{PGL}(2, \mathcal{O}) = \left\{ A \in M_{2\times 2}(\mathcal{O}) : \det A \in \mathcal{O}^{\times} \right\} / \left\{ \eta I : \eta \in \mathcal{O}^{\times} \right\}.$$

Definition 2.1. Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}$. Define a subgroup Γ of $\Gamma_F = \mathrm{PGL}(2, \mathcal{O})$ to be a *congruence subgroup of level* $\mathfrak{N}$ if Γ contains

$$\Gamma(\mathfrak{N}) := \{ A \in \mathrm{GL}(2, \mathcal{O}) : A \equiv I \pmod{\mathfrak{N}} \} / \left\{ \eta I : \eta \in (1 + \mathfrak{N}) \cap \mathcal{O}^{\times} \right\}.$$

the *principal congruence subgroup of level* $\mathfrak{N}$. Note $\Gamma(\mathfrak{N})$ is the kernel of the reduction map

$$\mathrm{PGL}(2, \mathcal{O}) \to \mathrm{PGL}(2, \mathcal{O}/\mathfrak{N}).$$

If the level of Γ is unspecified, we simply call Γ a *congruence subgroup*.

Remark. The *Hilbert modular group*

$$\mathrm{PSL}(2,\mathcal{O}) = \{A \in M_{2\times 2}(\mathcal{O}) : \det A = 1\} \Big/ \{\pm I\}\,.$$

is a congruence subgroup of level $\mathcal{O}$.

Proposition 2.2. *Every congruence subgroup Γ of level $\mathfrak{N}$ is finite index in $\Gamma_F = \mathrm{PGL}(2,\mathcal{O})$, and discrete in $G = \prod_{v|\infty} \mathrm{PGL}(2, F_v)$.*

Proof. The finite index property follows immediately from the reduction map, since $[\Gamma_F : \Gamma(\mathfrak{N})] = \#\mathrm{PGL}(2, \mathcal{O}/\mathfrak{N}) < \infty$, so any intermediate subgroup is also finite index.

For the discreteness, it suffices to prove the result for $\Gamma = \Gamma_F$ by the finite index property. This fact is immediate as

$$\Gamma_F = \mathrm{PGL}(2,\mathcal{O}) \hookrightarrow \prod_{v|\infty} \mathrm{PGL}(2, F_v) = G$$

via $F \hookrightarrow \prod_{v|\infty} F_v$, and noting $\mathcal{O}$ embeds discretely under this map. $\square$

Suppose $\Gamma \leq \mathrm{PGL}(2,\mathcal{O})$ is a congruence subgroup of level $\mathfrak{N}$, so from (2.3), Γ possesses an action on $\mathcal{X}$. We aim to understand the structure of $M = \Gamma\backslash\mathcal{X}$, which is equivalent to the double coset space $\Gamma\backslash G/K$. We call M a *congruence locally symmetric space* (covered by G/K). To begin, we introduce the notion of a cusp.

Recall the group $\mathrm{PGL}(2, F)$ possesses a natural left action on projective linear space $\mathbb{P}^1(F)$, and thus so does a congruence subgroup Γ. We remark that, via the n embeddings of F, this action is compatible with the action on $\mathcal{X}$ but we will not require this fact.

Definition 2.3. A *cusp* σ of congruence subgroup $\Gamma \leq \mathrm{PGL}(2,\mathcal{O})$ is a Γ-orbit in $\mathbb{P}^1(F)$. One often identifies a cusp with a representative of its orbit.

For $\sigma = (\alpha : \beta) \in \mathbb{P}^1(F)$, we shall always assume both α and β are integral. Note that this choice $\sigma = (\alpha : \beta)$ is unique up to multiplication in $\mathcal{O}$, i.e. $\sigma = (\mu\alpha : \mu\beta)$ for $\mu \in \mathcal{O}$. Also, the point $(1 : 0) \in \mathbb{P}^1(F)$ is the *point at infinity* and is denoted ∞.

Proposition 2.4. *There exists an bijective correspondence between the set of cusps of $\Gamma_F = PGL(2,\mathcal{O})$ and the ideal class group $\mathscr{C}\ell(F)$ of F, namely*

$$\begin{aligned} \Gamma_F \cdot \mathbb{P}^1(F) &\to \mathscr{C}\ell(F) \\ \Gamma_F(\alpha : \beta) &\mapsto (\alpha, \beta). \end{aligned}$$

In particular, the number of cusps of Γ_F equals h_F, the class number of F.

Proof. [Gee88, p. 6] Suppose $\sigma, \tau \in \mathbb{P}^1(F)$ are contained in the same $PGL(2, \mathcal{O})$ orbit. Then there exists $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathcal{O})$ such that $\gamma \cdot \sigma = \tau$. Writing $\sigma = (\alpha : \beta)$ with $\alpha, \beta \in \mathcal{O}$, we may assume that $\beta \neq 0$. Then we see that

$$\begin{aligned}
\tau = \gamma \cdot \sigma &= \gamma \cdot (\tfrac{\alpha}{\beta} : 1) \\
&= \left(\frac{a\frac{\alpha}{\beta} + b}{c\frac{\alpha}{\beta} + d} : 1 \right) \\
&= \left(\frac{a\alpha + b\beta}{c\alpha + d\beta} : 1 \right) \\
&= (a\alpha + b\beta : c\alpha + d\beta)\,,
\end{aligned}$$

so the ideal class associated to τ is $(a\alpha + b\beta, c\alpha + d\beta)$. Since $\gamma \in PGL(2, \mathcal{O})$, this ideal equals (α, β). Thus, the map is well-defined.

Since $\mathcal{O}$ is a Dedekind domain, every ideal of $\mathcal{O}$ is generated by at most 2 elements, implying the map is surjective. It remains to prove the map is injective. Suppose $\sigma = (\alpha : \beta)$ and $\tau = (\delta : \rho)$ possess the same associated ideal class, where $\alpha, \beta, \delta, \rho \in \mathcal{O}$. Then by the definition of equivalence in the ideal class group, we may multiply by a suitable element of F and assume $(\alpha, \beta) = (\delta, \rho) = \mathfrak{a}$. Observe

$$1 \in \mathcal{O} = \mathfrak{a}\mathfrak{a}^{-1} = \alpha\mathfrak{a}^{-1} + \beta\mathfrak{a}^{-1}$$

so there exists $\alpha^*, \beta^* \in \mathfrak{a}^{-1}$ such that $\alpha\beta^* - \alpha^*\beta = 1$. Similarly, choose $\delta^*, \rho^* \in \mathfrak{a}^{-1}$ such that $\delta\rho^* - \delta^*\rho = 1$. In other words, the matrices

$$g_\sigma := \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}, \qquad g_\tau := \begin{pmatrix} \delta & \delta^* \\ \rho & \rho^* \end{pmatrix}$$

have determinant 1, and transform the cusp $\infty = (1 : 0)$ to σ and τ respectively. Notice $g_\sigma g_\tau^{-1} \in PGL(2, \mathcal{O}) = \Gamma_F$ and from our previous observations, $g_\sigma g_\tau^{-1}$ sends τ to σ. Hence, σ and τ belong to the same Γ_F-orbit, as desired. $\square$

In the above proof, the integral ideal $\mathfrak{a}$ is dependent on the choice of α and β for which $\sigma = (\alpha : \beta) \in \mathbb{P}^1(F)$. Henceforth, to remove this dependence, for $\sigma \in \mathbb{P}^1(F)$, always choose $\alpha, \beta \in \mathcal{O}$ such that $\sigma = (\alpha : \beta)$ and $|\mathbb{N}\mathfrak{a}|$ is minimum where $\mathfrak{a} = (\alpha, \beta)$. It follows that if $\epsilon \in \mathcal{O}$ divides both α and β then $\epsilon \in \mathcal{O}^\times$, so the pair α and β are now determined up to multiplication by a unit $\epsilon \in \mathcal{O}^\times$.

Consequently, the integral ideal $\mathfrak{a}$ of minimum norm is uniquely defined for each $\sigma \in \mathbb{P}^1(F)$, and shall be referred to as the *ideal associated to* $\sigma \in \mathbb{P}^1(F)$. From Proposition 2.4, we see that the same ideal $\mathfrak{a}$ is associated to any element of the orbit $\Gamma_F \cdot \sigma$.

As a separate remark, notice that even if we choose $\alpha, \beta \in \mathcal{O}$ such that $\mathfrak{a} = (\alpha, \beta)$ is the ideal associated to $\sigma \in \mathbb{P}^1(F)$, the precise definition of g_σ still depends on the choice of α and β. Nonetheless, for the remainder of this paper, we will retain the definition of g_σ using α and β generating the ideal associated to $\sigma \in \mathbb{P}^1(F)$. The dependence of g_σ on the exact choice of α and β will not be relevant as the choice only depends on F.

Corollary 2.5. *Every finite index subgroup Γ of Γ_F, and in particular every congruence subgroup, has finitely many cusps.*

Proof. Every cusp of Γ_F decomposes into at most $[\Gamma_F : \Gamma]$ cusps of Γ, and since Γ_F has finitely many cusps, so must Γ. From Proposition 2.2, the result therefore applies to congruence subgroups. $\square$

For the remainder of this thesis, we shall *fix the level to be the integral ideal* $\mathfrak{N}$.

2.3 Cusp Stabilizer

For $\sigma = (\alpha : \beta) \in \mathbb{P}^1(F)$ and a congruence subgroup Γ, define $\Gamma^{(\sigma)} := g_\sigma^{-1} \Gamma g_\sigma$, which is a subgroup of $\mathrm{PGL}(2, F)$. Under this conjugation, the cusp σ of $\Gamma\backslash\mathcal{X}$ becomes the cusp ∞ of $\Gamma^{(\sigma)}\backslash\mathcal{X}$ since $g_\sigma(\infty) = \sigma$. By direct verification, we see that the $\Gamma^{(\sigma)}$-stabilizer of $\infty = (1:0)$ is given by the upper triangular elements of $\Gamma^{(\sigma)}$. To obtain more detailed information, we shall explicitly describe the subgroup of upper triangular elements of $\Gamma(\mathfrak{N})^{(\sigma)}$ since it is finite index in $\Gamma^{(\sigma)}$.

Recall that $\det g_\sigma = 1$ and g_σ is of the form

$$\begin{pmatrix} \mathfrak{a} & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathfrak{a}^{-1} \end{pmatrix}$$

where $\mathfrak{a} = (\alpha, \beta)$ for which $\sigma = (\alpha : \beta)$ with specified $\alpha, \beta \in \mathcal{O}$. By direct computations, one can verify that

$$\Gamma(\mathfrak{N})^{(\sigma)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in 1 + \mathfrak{N},\ b \in \mathfrak{N}\mathfrak{a}^{-2}, c \in \mathfrak{N}\mathfrak{a}^2, ad - bc \in \mathcal{O}^\times \right\} \Big/ \{\eta I : \eta \in \mathcal{O}^\times\}$$

and so

$$\mathrm{stab}(\infty; \Gamma(\mathfrak{N})^{(\sigma)}) = \left\{ \begin{pmatrix} \epsilon & \theta \\ 0 & 1 \end{pmatrix} : \epsilon \in (1 + \mathfrak{N}) \cap \mathcal{O}^\times, \theta \in \mathfrak{N}\mathfrak{a}^{-2} \right\} \Big/ \{\eta I : \eta \in \mathcal{O}^\times\}. \tag{2.4}$$

We remark that now both of these definitions are *independent* of our choice of α and β for which $\sigma = (\alpha : \beta)$, since the ideal $\mathfrak{a}$ associated to σ is also independent of these choices.

For the sake of simplicity, we would like to restrict our attention to a finite index subgroup of $\mathrm{stab}(\infty; \Gamma(\mathfrak{N})^{(\sigma)})$ by restricting the additive subgroup over which b ranges.

Proposition 2.6. *Let $\sigma = (\alpha : \beta) \in \mathbb{P}^1(F)$ and Γ be a level $\mathfrak{N}$ congruence subgroup. Define*

$$\Gamma^{(\sigma)} := g_\sigma^{-1} \Gamma g_\sigma$$

where g_σ is as in Proposition 2.4. Then the $\Gamma^{(\sigma)}$-stabilizer of $\infty \in \mathbb{P}^1(F)$ contains the finite-index subgroup, depending only on $\mathfrak{N}$,

$$\Lambda = \Lambda_{\mathfrak{N}} := \left\{ \begin{pmatrix} \epsilon & \theta \\ 0 & 1 \end{pmatrix} : \epsilon \in V, \theta \in \mathfrak{N} \right\} \Big/ \{\eta I : \eta \in \mathcal{O}^\times\}.$$

where $V = V_{\mathfrak{N}} = (1 + \mathfrak{N}) \cap \mathcal{O}^\times$ is a finite index subgroup of $\mathcal{O}^\times$. Equivalently, the Γ-stabilizer of $\sigma \in \mathbb{P}^1(F)$ contains the subgoup $g_\sigma \Lambda g_\sigma^{-1}$.

Proof. From (2.4) and the preceding discussion, it is follows that Λ is a finite-index subgroup of $\mathrm{stab}(\infty; \Gamma^{(\sigma)})$ since the additive group $\mathfrak{N}$ is finite index in $\mathfrak{N}\mathfrak{a}^{-2}$ with index equal to $\mathbb{N}\mathfrak{a}^2$. To see that V has finite index in $\mathcal{O}^\times$, it suffices to note that V is the pre-image of $\{1\}$ in the reduction map

$$\mathcal{O}^\times \to (\mathcal{O}/\mathfrak{N})^\times$$

and $\{1\}$ is obviously finite index in the finite group $(\mathcal{O}/\mathfrak{N})^\times$. $\square$

With this finite index subgroup $g_\sigma \Lambda g_\sigma^{-1}$ of the cusp stabilizer $\mathrm{stab}(\sigma; \Gamma)$, we may find a domain which projects onto the fundamental domain of $\mathrm{stab}(\sigma; \Gamma) \backslash \mathcal{X}$, and thus obtain an understanding of its structure and parametrization.

Theorem 2.7. *Define $\Lambda = \Lambda_{\mathfrak{N}} \subseteq \mathrm{PGL}(2, \mathcal{O})$ as in Proposition 2.6. Then there exists pre-compact domains $\mathcal{U} = \mathcal{U}_{\mathfrak{N}} \subseteq F_\infty$ and $\mathcal{V} = \mathcal{V}_{\mathfrak{N}} \subseteq \widehat{\mathcal{Y}}$ such that the set*

$$\mathcal{D} = \mathcal{D}_{\mathfrak{N}} := \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X} : \mathbf{x} \in \mathcal{U}, \widehat{\mathbf{y}} \in \mathcal{V}, \mathbb{N}\mathbf{y} \in (0, \infty)\}$$

is a fundamental domain for $\Lambda \backslash \mathcal{X}$, where

$$\widehat{\mathbf{y}} := \Big(\frac{y_v}{(\mathbb{N}\mathbf{y})^{1/n}} \Big)_{v|\infty}, \quad \mathbb{N}\mathbf{y} := \prod_{v|\infty} \mathbb{N}_v y_v, \quad \widehat{\mathcal{Y}} := \{\mathbf{y} \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y} = 1\}.$$

Proof. [Gee88, p. 9–11] Without loss, every element $\gamma \in \Lambda_{\mathfrak{N}}$ may be written as

$$\gamma = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$$

with $\epsilon \in V_{\mathfrak{N}}, \theta \in \mathfrak{N}$ and $\zeta \in V_{\mathfrak{N}} \cap \mathcal{O}_1^\times$, where $\mathcal{O}_1^\times$ are the roots of unity of $\mathcal{O}^\times$. The matrix involving ζ is redundant, but helpful for our purposes.

We shall construct the desired fundamental domain by considering the individual action of each element. First, recall that we write $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathcal{X}$ where $\mathbf{x} \in F_\infty$ and $\mathbf{y} \in \mathbb{R}^m_{>0}$. Notice that $\mathbb{N}\widehat{\mathbf{y}} = 1$ for any $(\mathbf{x}, \mathbf{y}) \in X$.

We begin by anlayzing the action of the diagonal element for $\epsilon \in V_{\mathfrak{N}}$:

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \cdot (\mathbf{x}, \mathbf{y}) = (\epsilon \mathbf{x}, |\epsilon| \mathbf{y}).$$

where $|\epsilon| = (|\epsilon_v|)_{v|\infty}$. Note that the coordinate $|\epsilon|\mathbf{y} = (|\epsilon_v| y_v)_v$ is simply component-wise multiplication, and so if ϵ is a root of unity, $|\epsilon|\mathbf{y} = \mathbf{y}$. We wish to articulate the $V_{\mathfrak{N}}$-action on the $\mathbf{y}$-component. Consider the surjective map

$$\begin{aligned} \widehat{\mathrm{Im}} : \quad \mathcal{X} &\to \widehat{\mathcal{Y}} = \left\{ \mathbf{y} \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y} = 1 \right\} \\ (\mathbf{x}, \mathbf{y}) &\mapsto \widehat{\mathbf{y}} = \left(\frac{y_v}{(\mathbb{N}\mathbf{y})^{1/n}} \right)_{v|\infty}. \end{aligned}$$

Observe every point $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}$ is uniquely defined by the triple $(\mathbf{x}, \widehat{\mathbf{y}}, \mathbb{N}\mathbf{y})$. Since $|\mathbb{N}\epsilon| = 1$ for $\epsilon \in \mathcal{O}^\times$, the group $\mathcal{O}^\times$ acts multiplicatively on $\widehat{\mathcal{Y}}$ via

$$\epsilon \mapsto |\epsilon| = (|\epsilon_v|)_v.$$

Since the kernel of this map are the roots of unity, denoted $\mathcal{O}_1^\times$, this action may be viewed as a faithful action of $\mathcal{O}^\times / \mathcal{O}_1^\times$. Moreover, one should note that $|\mathbb{N}\epsilon| = 1$ for any $\epsilon \in \mathcal{O}^\times$. The multiplicative action of $\mathcal{O}^\times$ on $\widehat{\mathcal{Y}}$ transfers to an additive action via the bijective logarithm map

$$\begin{aligned} \log : \widehat{\mathcal{Y}} &\to \log \widehat{\mathcal{Y}} = \left\{ (a_v)_{v|\infty} \in \mathbb{R}^m : \sum_{v|\infty} \mathbf{Tr}_v a_v = 0 \right\} \\ \mathbf{y} &\mapsto \log \mathbf{y} := (\log y_v)_{v|\infty} \end{aligned}$$

since $\log(|\epsilon|\mathbf{y}) = (\log|\epsilon_v| + \log y_v)_{v|\infty}$. By Dirichlet's Unit theorem, $\mathcal{O}^\times$, and hence the finite index subgroup $V = V_{\mathfrak{N}}$, is a lattice in $\log \widehat{\mathcal{Y}}$, so we may choose a *pre-compact* fundamental domain $\log \mathcal{V}$ for the additive action $V \backslash \log \widehat{\mathcal{Y}}$. The exponential map $\exp : \log \widehat{\mathcal{Y}} \to \widehat{\mathcal{Y}}$ is a topological isomorphism, under which $\log \mathcal{V}$ becomes a *pre-compact* fundamental domain $\mathcal{V} = \mathcal{V}_{\mathfrak{N}}$ for the multiplicative action of $V \backslash \widehat{\mathcal{Y}}$. Thus, for any $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}$ we may choose $\epsilon \in V$ such that $|\epsilon|\widehat{\mathbf{y}} \in \overline{\mathcal{V}}$.

Now, we consider the action of the unipotent element: for $\theta \in \mathfrak{N}$,

$$\begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \cdot (\epsilon \mathbf{x}, |\epsilon| \mathbf{y}) = (\epsilon \mathbf{x} + \theta, |\epsilon| \mathbf{y}).$$

Since $\mathfrak{N}$ is a lattice in F_∞, a fundamental domain for the additive action $F_\infty / \mathfrak{N}$ is pre-compact. If we also include the multiplicative action of the group $\mathcal{O}_1^\times \cap V$, then we have

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \cdot (\epsilon \mathbf{x} + \theta, |\epsilon| \mathbf{y}) = (\zeta \epsilon \mathbf{x} + \zeta \theta, |\epsilon| \mathbf{y}).$$

since $|\zeta| = (|\zeta|_v)_v = (1)_v$. We see that the element only acts in the first coordinate on F_∞. Since $\mathcal{O}_1^\times \cap V$ is a finite group and a fundamental domain for $F_\infty / \mathfrak{N}$ is pre-compact , it follows that a fundamental domain $\mathcal{U} = \mathcal{U}_{\mathfrak{N}}$ for $(\mathcal{O}_1^\times \cap V) \backslash F_\infty / \mathfrak{N}$ is *pre-compact*. Thus, for $\epsilon \mathbf{x} \in F_\infty$, we may choose $\zeta \in \mathcal{O}_1^\times \cap V$ and $\theta \in \mathfrak{N}$ so that $\zeta \epsilon \mathbf{x} + \zeta \theta \in \overline{\mathcal{U}_{\mathfrak{N}}}$. To conclude, we may choose $\gamma \in \Lambda$ such that $\gamma \cdot \mathbf{z} \in \overline{\mathcal{D}}$.

Finally, we prove that distinct points $\mathbf{z}, \mathbf{z}' \in \mathcal{D}$ are not Λ-equivalent. Suppose there exists $\gamma \in \Lambda$ such that $\gamma \cdot \mathbf{z} = \mathbf{z}'$. Using the decomposition of γ as before, this implies

$$(\zeta \epsilon \mathbf{x} + \zeta \theta, |\epsilon| \mathbf{y}) = (\mathbf{x}', \mathbf{y}')$$

for some $\epsilon \in V, \theta \in \mathfrak{N}$ and $\zeta \in \mathcal{O}_1^\times \cap V$. By comparing coordinates, we see that $|\epsilon| \mathbf{y} = \mathbf{y}'$, which implies $|\epsilon| \widehat{\mathbf{y}} = \widehat{\mathbf{y}}'$. However, because $\widehat{\mathbf{y}}, \widehat{\mathbf{y}}' \in \mathcal{V}$, it must be that $|\epsilon_v| = 1$ for all $v \mid \infty$, or in other words, $\epsilon \in \mathcal{O}_1^\times \cap V_{\mathfrak{N}}$ is a root of unity. Considering the first coordinate, we see that

$$\mathbf{x}' = \zeta \epsilon \mathbf{x} + \zeta \theta \in (\mathcal{O}_1^\times \cap V_{\mathfrak{N}}) \cdot \mathbf{x} \cdot \mathfrak{N}$$

since $\zeta, \epsilon \in \mathcal{O}_1^\times \cap V$ and $\theta \in \mathfrak{N}$. As $\mathbf{x}, \mathbf{x}' \in \mathcal{U}$, it must be that $\theta = 0$ and $\zeta \epsilon = 1$. In other words, γ is the identity motion, as desired. $\square$

Remark. Since Λ is a finite-index subgroup of $\operatorname{stab}(\infty; \Gamma^{(\sigma)})$, the fundamental domain $\mathcal{D} = \mathcal{D}_{\mathfrak{N}}$ for $\Lambda \backslash X$ is composed of finitely many copies of the fundamental domain for $\operatorname{stab}(\infty; \Gamma^{(\sigma)}) \backslash X$. The above theorem then suggests that the notion of depth into the cusp ∞ is measured by the single parameter $\mathbb{N}\mathbf{y}$, i.e. a large value of $\mathbb{N}\mathbf{y}$ implies $\mathbf{y}$ is “close to infinity”. More crudely, there is only “one way to infinity”.

Corollary 2.8. *Keep notation as in Theorem 2.7. Then via the map*

$$\Lambda\backslash\mathcal{X} \twoheadrightarrow \mathrm{stab}(\infty;\Gamma^{(\sigma)})\backslash\mathcal{X},$$

the domain $\mathcal{D}$ projects onto a fundamental domain for $\mathrm{stab}(\infty;\Gamma^{(\sigma)})\backslash\mathcal{X}$*. Equivalently, via the map*

$$g_\sigma\Lambda g_\sigma^{-1}\backslash\mathcal{X} \twoheadrightarrow \mathrm{stab}(\sigma;\Gamma)\backslash\mathcal{X},$$

the domain $g_\sigma\mathcal{D}$ projects onto a fundamental domain for $\mathrm{stab}(\sigma;\Gamma)\backslash\mathcal{X}$*.*

Proof. This follows immediately from Proposition 2.6 and Theorem 2.7 since Λ is a finite index subgroup of $\mathrm{stab}(\infty;\Gamma^{(\sigma)})$. □

2.4 Distance to Cusps

As in the remark following Theorem 2.7, each cusp $\sigma \in \mathbb{P}^1(F)$ of Γ possesses a one-dimensional parameter which measures the "closeness" of a point $\mathbf{z} \in \mathcal{X}$. Motivated by and using the notation of Theorem 2.7, we define

$$\mathcal{D}(y) = \mathcal{D}_{\mathfrak{N}}(y) := \{(\mathbf{x},\mathbf{y}) \in \mathcal{D} : \mathbb{N}\mathbf{y} \in (y,\infty)\},$$

for $y \geq 0$. We note some simple properties:

- $\mathcal{D}(0) = \mathcal{D}$.
- $\mathcal{D}(y) \supseteq \mathcal{D}(y')$ for $0 < y \leq y'$.
- $\mathcal{D}(y) \to \varnothing$ as $y \to \infty$.
- $\mathcal{D}(y)$ is a fundamental domain for the action of Λ on

$$\mathcal{X}(y) := \{(\mathbf{x},\mathbf{y}) \in \mathcal{X} : \mathbb{N}\mathbf{y} \in (y,\infty)\}.$$

From Corollary 2.8, elements $\mathbf{z} \in g_\sigma\mathcal{D}(y)$ are "close" to $\sigma \in \mathbb{P}^1(F)$ when y is large. Equivalently, we have $g_\sigma^{-1}\mathbf{z} \in \mathcal{D}(y)$, which implies

$$y < \mathbb{N}(\mathrm{Im}(g_\sigma^{-1}\mathbf{z})) = \frac{\mathbb{N}\mathbf{y}}{|\mathbb{N}(\beta\mathbf{z}+\alpha)|^2}.$$

where $\sigma = (\alpha : \beta)$ and $\mathfrak{a} = (\alpha,\beta)$ is the ideal associated to σ. We see that the expression on the righthand side measures the closeness of a point $\mathbf{z}$ to $\sigma \in \mathbb{P}^1(F)$, so we define

$$\rho(\sigma; \mathbf{z}) := \frac{\mathbb{N}\mathbf{y}}{|\mathbb{N}(-\beta\mathbf{z} + \alpha)|^2}$$

for the same choice of α and β. As a special example, we have $\rho(\infty; \mathbf{z}) = \mathbb{N}\mathbf{y}$ since $\mathfrak{a} = \mathcal{O}$ and $(\alpha : \beta) = (1 : 0)$.

Observe that, since α and β generate the ideal associated to σ, the expression $\rho(\sigma; \mathbf{z})$ does *not* depend on the choice of α, β, which are determined up to multiplication by an integral unit. Because suppose we write $\sigma = (\mu\alpha : \mu\beta)$ for some $\mu \in \mathcal{O}^\times$, then

$$\frac{\mathbb{N}\mathbf{y}}{|\mathbb{N}(-\mu\beta\mathbf{z} + \mu\alpha)|^2} = \frac{\mathbb{N}\mathbf{y}}{|(\mathbb{N}\mu)\mathbb{N}(-\beta\mathbf{z} + \alpha)|^2} = \frac{\mathbb{N}\mathbf{y}}{|\mathbb{N}(-\beta\mathbf{z} + \alpha)|^2}$$

as $|\mathbb{N}\mu| = 1$. Thus, $\rho(\sigma; \mathbf{z})$ is well-defined for $\mathbf{z} \in \mathcal{X}$ and $\sigma \in \mathbb{P}^1(F)$.

Notice, by definition of $\rho(\sigma; \mathbf{z})$, it follows that

$$g_\sigma\mathcal{D}(y) = \{\mathbf{z} \in g_\sigma\mathcal{D} : \rho(\sigma; \mathbf{z}) > y\} \tag{2.5}$$

for each $\sigma \in \mathbb{P}^1(F)$. The purpose of this section is to prove several useful lemmas regarding $\rho(\sigma; \mathbf{z})$. First, we prove a simple invariance property.

Lemma 2.9. *For $\sigma \in \mathbb{P}^1(F)$ and $\mathbf{z} \in \mathcal{X}$, we have*

$$\rho(\gamma \cdot \sigma; \gamma \cdot \mathbf{z}) = \rho(\sigma; \mathbf{z})$$

for all $\gamma \in \Gamma_F = \mathrm{PGL}(2, \mathcal{O})$.

Proof. [Gee88, p. 7] Write $\sigma = (\alpha : \beta)$ with $\mathfrak{a} = (\alpha, \beta)$ and write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a, b, c, d \in \mathcal{O}$. By direct computations, we see that

$$\mathrm{Im}(\gamma \cdot \mathbf{z}) = \frac{\mathbf{y}}{|c\mathbf{z} + d|^2}, \qquad \gamma \cdot \mathbf{z} = \frac{a\mathbf{z} + b}{c\mathbf{z} + d}$$

from which it follows that

$$\begin{aligned}
-(c\alpha + d\beta)(\gamma \cdot \mathbf{z}) + a\alpha + b\beta &= -(c\alpha + d\beta)\frac{a\mathbf{z}+b}{c\mathbf{z}+d} + a\alpha + b\beta \\
&= \frac{-(c\alpha + d\beta)(a\mathbf{z}+b) + (a\alpha + b\beta)(c\mathbf{z}+d)}{c\mathbf{z}+d} \\
&= \frac{-bc\alpha - ad\beta\mathbf{z} + ad\alpha + bc\beta\mathbf{z}}{c\mathbf{z}+d} \\
&= \frac{(-\beta\mathbf{z}+\alpha)\det\gamma}{c\mathbf{z}+d}
\end{aligned}$$

Combining these calculations, we obtain the desired result for $\gamma \in \Gamma_F$. □

Next, we show that every point $\mathbf{z} \in \mathcal{X}$ must be *uniformly* close to *at least* one $\sigma \in \mathbb{P}^1(F)$.

Lemma 2.10. *There exists a positive constant $y_0 = y_0(F)$ such that, for every $\mathbf{z} \in \mathcal{X}$, there exists some $\sigma \in \mathbb{P}^1(F)$ with $\rho(\sigma; \mathbf{z}) > y_0$. Specifically, one may take $y_0 = (2^n D_F)^{-1}$ where D_F is the absolute discriminant of F.*

Proof. [Gee88, p. 8] We claim it suffices to choose $y_0 > 0$, depending only of F, such that for every $\mathbf{z} \in X$, there exists a solution $\alpha, \beta \in \mathcal{O}$ to the inequality

$$\frac{|\mathbb{N}(-\beta\mathbf{z}+\alpha)|^2}{\mathbb{N}\mathbf{y}} < \frac{1}{y_0}. \tag{2.6}$$

If this is the case, then we show $\sigma = (\alpha : \beta) \in \mathbb{P}^1(F)$ is the desired element. Note that the LHS of (2.6) is not necessarily equal to $\rho(\sigma; \mathbf{z})$ since α and β may not generate the ideal associated to σ. However, suppose $\sigma = (\alpha' : \beta')$ and $\mathfrak{a} = (\alpha', \beta')$ is the ideal associated to σ, then $\alpha = \alpha'\mu$ and $\beta = \beta'\mu$ for some $\mu \in \mathcal{O}$, and so

$$y_0 < \frac{\mathbb{N}\mathbf{y}}{|\mathbb{N}(-\beta\mathbf{z}+\alpha)|^2} = \frac{\mathbb{N}\mathbf{y}}{|(\mathbb{N}\mu)\mathbb{N}(-\beta'\mathbf{z}+\alpha')|^2} \leq \rho(\sigma; \mathbf{z})$$

because $|\mathbb{N}\mu| \geq 1$. This proves the claim, and so it suffices to prove a solution $\alpha, \beta \in \mathcal{O}$ exists to (2.6) for some $y_0 > 0$ depending only on F.

Before proving the claim, recall for each place $v \mid \infty$, there is a non-canonically associated embedding $\tau_v : F \hookrightarrow F_v$ giving rise to the place v. Further, the collection of $\{\tau_v\} \cup \{\overline{\tau}_v\}_{v|\infty}$ gives the entire list of n embeddings, $Hom(F, \mathbb{C})$. With this indexing, we define the injective map

$$\begin{aligned}
\mathcal{X} &\to \mathbb{C}^n \\
\mathbf{z} = (z_v)_{v|\infty} &\mapsto (z_\tau)_{\tau \in Hom(F,\mathbb{C})}
\end{aligned}$$

where

$$z_\tau = \begin{cases} z_v & \tau = \tau_v \text{ for some } v \\ \overline{z}_v & \tau = \overline{\tau}_v \text{ for some } v \end{cases}.$$

Using this notation, we see that for $\mathbf{z} \in \mathcal{X}$,

$$\mathbb{N}\mathbf{z} = \prod_{v|\infty} \mathbb{N}_v z_v = \prod_\tau z_\tau$$

since at complex v, $\mathbb{N}_v z_v = z_v \overline{z}_v$ and at real v, $\mathbb{N}_v z_v = z_v$.

Now, to prove the claim, we bound the LHS of (2.6) by considering the norm expressed as a product at each *embedding* $\tau : F \hookrightarrow \mathbb{C}$, as per the above. By the Triangle Inequality,

$$|-\beta_\tau z_\tau + \alpha_\tau|\, y_\tau^{-1/2} \leq |-\beta_\tau x_\tau + \alpha_\tau|\, y_\tau^{-1/2} + |\beta_\tau|\, y_\tau^{1/2}.$$

Thus, if we can choose $c_\tau, d_\tau \in \mathbb{R}_{>0}$, depending only on F, for each embedding τ such that the set of $2n$ equalities

$$|\beta_\tau x_\tau + \alpha_\tau|\, y_\tau^{-1/2} \leq c_\tau, \quad |\beta_\tau|\, y_\tau^{1/2} \leq d_\tau, \qquad \tau \in Hom(F, \mathbb{C}) \tag{2.7}$$

possesses a solution $\alpha, \beta \in \mathcal{O}$, then by applying the previous inequality, we see that

$$\frac{|\mathbb{N}(-\beta\mathbf{z} + \alpha)|^2}{\mathbb{N}\mathbf{y}} = \prod_\tau |-\beta_\tau z_\tau + \alpha_\tau|^2\, y_\tau^{-1} \leq \prod_{v|\infty} (c_\tau + d_\tau)^2$$

so we may take $y_0 = \prod_\tau (c_\tau + d_\tau)^{-2}$.

To achieve a solution to (2.7), let $\{\omega^{(k)}\}_{k=1}^n$ be an integral basis for $\mathcal{O}$ in F, and write

$$\alpha = \sum_{k=1}^n a^{(k)} \omega^{(k)}, \quad \beta = \sum_{k=1}^n b^{(k)} \omega^{(k)}$$

where $a^{(k)}, b^{(k)} \in \mathbb{Z}$ for $k = 1, \ldots, n$ are free variables. Substituting these sums in (2.7), we obtain $2n$ linear inequalities

$$\begin{aligned} &\Big| \sum_{k=1}^n a^{(k)} \cdot (y_\tau^{-1/2} \omega_\tau^{(k)}) + \sum_{k=1}^n b^{(k)} \cdot (-x_\tau y_\tau^{-1/2} \omega_\tau^{(k)}) \Big| \leq c_\tau, \quad \tau \in Hom(F, \mathbb{C}) \\ &\Big| \sum_{k=1}^n a^{(k)} \cdot 0 + \sum_{k=1}^n b^{(k)} \cdot (y_\tau^{1/2} \omega_\tau^{(k)}) \Big| \leq d_\tau, \quad \tau \in Hom(F, \mathbb{C}) \end{aligned}$$

with $2n$ variables, $\{a^{(k)}, b^{(k)} : k = 1, \ldots, n\}$. Defining the $n \times n$ matrices

$$A_{11} = (y_\tau^{-1/2} \omega_\tau^{(k)})_{\tau,k}, \quad A_{12} = (-x_\tau y_\tau^{-1/2} \omega_\tau^{(k)})_{\tau,k}, \quad A_{22} = (y_\tau^{1/2} \omega_\tau^{(k)})_{\tau,k},$$

and $n \times 1$ vectors $\mathbf{a} = (a^{(k)})_k$ and $\mathbf{b} = (b^{(k)})_k$, we see that the linear system is of the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}.$$

From Minkowksi's Theorem on linear forms [Neu99, p. 27–28], a solution exists with $a^{(k)}, b^{(k)} \in \mathbb{Z}$ to the above system provided

$$\prod_\tau c_\tau d_\tau \geq \det \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

We can easily see that the RHS

$$= \det(A_{11} A_{22}) = \det((\omega_\tau^{(k)})_{k,\tau})^2 = D_F$$

where D_F is the discriminant of F, so we may take $c_\tau = d_\tau = D_F^{1/2n}$ implying $r_0 = (2^n D_F)^{-1}$. □

Finally, we prove that a point $\mathbf{z} \in \mathcal{X}$ can be very close, in a *uniform* sense, to *at most* one $\sigma \in \mathbb{P}^1(F)$.

Lemma 2.11. *There exists sufficiently large $Y_0 = Y_0(F) > 0$ such that if, for $\mathbf{z} \in \mathcal{X}$, we have $\rho(\sigma; \mathbf{z}) > Y_0$ and $\rho(\tau; \mathbf{z}) > Y_0$ for $\sigma, \tau \in \mathbb{P}^1(F)$, then $\sigma = \tau$.*

Proof. [Gee88, p. 6] In a remark following Proposition 2.4, we noted that the ideal $\mathfrak{a}$ associated to a point $\sigma \in \mathbb{P}^1(F)$ is actually the same for any element in the orbit $\Gamma_F \cdot \sigma$. Since, by Proposition 2.4, there are only finitely many Γ_F-orbits of $\mathbb{P}^1(F)$, it follows that we may bound the norm of $\mathfrak{a}$ by some constant $C > 0$ depending only on F.

Suppose $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ and write $\sigma = (\alpha : \beta)$ where $\alpha, \beta \in \mathcal{O}$ generate the ideal associated to σ, and similarly write and $\tau = (\mu : \nu)$ where $\mu, \nu \in \mathcal{O}$ generate the ideal associated to τ.

Now, recall that by Dirichlet's theorem, the embedding

$$\begin{aligned} \mathcal{O}^\times &\to \mathbb{R}^m_{>0} \\ \epsilon &\mapsto (|\epsilon_v|)_v \end{aligned}$$

makes the subgroup $(\mathcal{O}^\times)^2$ a multiplicative lattice in the hyperplane $\widehat{\mathcal{Y}} = \{\mathbf{y}' \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y}' = 1\}$.

This implies for any $\mathbf{y}' \in \mathbb{R}^m_{>0}$, that we may find a unit $\epsilon \in \mathcal{O}^\times$ such that $\widehat{|\epsilon^2|\mathbf{y}'}$ is contained in some fundamental domain for $(\mathcal{O}^\times)^2\backslash\widehat{\mathcal{Y}}$. Since this domain is precompact in $\widehat{\mathcal{Y}}$, it is bounded away from zero and infinity. This implies that each coordinate in $\widehat{|\epsilon^2|\mathbf{y}'}$ is bounded by some constant $c = c(F) > 0$. To summarize, for each $\mathbf{y}' \in \mathbb{R}^m_{>0}$, there exists some $\epsilon \in \mathcal{O}^\times$ such that

$$|\epsilon_v^2 y_v'| \leq c \cdot (\mathbb{N}\mathbf{y}')^{1/n}, \qquad v \mid \infty$$

where $c > 0$ is some constant depending only on F.

Thus, utilizing the above observation for

$$\mathbf{y}' = \mathrm{Im}(g_\sigma^{-1}\mathbf{z})^{-1} = \Big(\frac{|\beta_v z_v + \alpha_v|^2}{y_v}\Big)_{v|\infty}$$

we have, after multiplying $\mathbf{y}'$ by an appropriate unit $\epsilon \in \mathcal{O}^\times$, for $v \mid \infty$

$$\begin{aligned}\left|\epsilon_v^2 y_v'\right| = \left|y_v^{-1}(-\epsilon_v\beta_v z_v + \epsilon_v\alpha_v)^2\right| &\leq c \cdot (\mathbb{N}\mathbf{y}')^{1/n} \\ &= c \cdot \rho(\sigma;\mathbf{z})^{-1/n} \\ &< c \cdot Y_0^{-1/n}\end{aligned}$$

since $\rho(\sigma;\mathbf{z}) = (\mathbb{N}\mathbf{y}')^{-1}$ by definition and $\rho(\sigma;\mathbf{z}) > Y_0$ by assumption. We may replace α and β by $\epsilon\alpha$ and $\epsilon\beta$ since $\rho(\sigma;\mathbf{z})$ is independent of this choice. Then the above inequality yields two inequalities:

$$\begin{aligned}|\beta_v x_v + \alpha_v|\, y_v^{-1/2} &\leq c^{1/2} \cdot Y_0^{-1/n} \qquad v \mid \infty \\ |\beta_v|\, y_v^{1/2} &\leq c^{1/2} \cdot Y_0^{-1/n} \qquad v \mid \infty\end{aligned} \tag{2.8}$$

by writing $z_v = (x_v, y_v)$. Similarly, we may obtain inequalities for μ and ν.

$$\begin{aligned}|\nu_v x_v + \mu_v|\, y_v^{-1/2} &\leq c^{1/2} \cdot Y_0^{-1/n} \qquad v \mid \infty \\ |\nu_v|\, y_v^{1/2} &\leq c^{1/2} \cdot Y_0^{-1/n} \qquad v \mid \infty\end{aligned} \tag{2.9}$$

On the other hand,

$$\alpha_v\nu_v - \beta_v\mu_v = (-\beta_v x_v + \alpha_v)y_v^{-1/2} \cdot \nu_v y_v^{1/2} - (-\nu_v x_v + \mu_v)y_v^{-1/2} \cdot \beta_v y_v^{1/2}$$

from which it follows by the Triangle Inequality, (2.8), and (2.9) that

$$|\mathbb{N}(\alpha\nu - \beta\mu)| < c^n Y_0^{-1}.$$

Thus, for $Y_0 > c^n$, we see that the algebraic integer $\alpha\nu - \beta\mu$ has norm whose absolute value is less than 1, and hence $\alpha\nu - \beta\mu = 0$ implying $\sigma = \tau$. Since $c > 0$ depends only on F, this completes the proof. $\square$

2.5 Fundamental Domain of $\Gamma\backslash\mathcal{X}$

With a relatively complete understanding of the structure of $\mathrm{stab}(\sigma;\Gamma)\backslash\mathcal{X}$ and the distance of points in $\mathcal{X}$ to cusps of Γ, we may now sufficiently describe $\Gamma\backslash\mathcal{X}$ for our purposes.

Proposition 2.12. *Let Γ be a congruence subgroup of level $\mathfrak{N}$, and let $\Omega \subseteq \mathbb{P}^1(F)$ be a set of inequivalent representatives of $\Gamma\backslash\mathbb{P}^1(F)$. Then the image of the set*

$$\mathcal{G}(y) := \bigcup_{\sigma\in\Omega} g_\sigma \mathcal{D}(y) \subseteq \mathcal{X}$$

under the map $\mathcal{X} \to \Gamma\backslash\mathcal{X}$,

*(a) for $0 < y \leq y_0$, **surjects** onto $\Gamma\backslash\mathcal{X}$, where y_0 is as in Lemma 2.10.*

*(b) for $y > Y_0$, **injects** into $\Gamma\backslash\mathcal{X}$, where Y_0 is as in Lemma 2.11, and further the union over $\sigma \in \Omega$ is disjoint.*

Proof. [Gee88, p. 9–11]

(a) It suffices to show that for $\mathbf{z} \in \mathcal{X}$, there exists $\gamma \in \Gamma$ and $\sigma \in \Omega$ such that

$$\gamma \cdot \mathbf{z} \in g_\sigma \mathcal{D}(y_0).$$

From Lemma 2.10, we have that there exists $\sigma' \in \mathbb{P}^1(F)$ such that $\rho(\sigma'; \mathbf{z}) > y_0$. Applying an appropriate element $\gamma' \in \Gamma \subseteq \Gamma_F$, by Lemma 2.9, we have $\rho(\sigma; \gamma' \cdot \mathbf{z}) > y_0$ for $\sigma = \gamma' \cdot \sigma' \in \Omega$. Then, from the proof Theorem 2.7, we may choose an appropriate element $\gamma'' \in \Lambda$ such that

$$(g_\sigma \gamma'' g_\sigma^{-1} \gamma') \cdot \mathbf{z} \in g_\sigma \mathcal{D}(y_0).$$

Since $g_\sigma \Lambda g_\sigma^{-1} \subseteq \Gamma(\mathfrak{N}) \subseteq \Gamma$, we may take $\gamma := g_\sigma \gamma'' g_\sigma^{-1} \gamma' \in \Gamma$ to obtain the desired result.

(b) From (2.5), we see that $\mathbf{z} \in g_\sigma \mathcal{D}(y)$ implies $\rho(\sigma; \mathbf{z}) > y$. Then by Lemma 2.11, the union over $\sigma \in \Omega$ is necessarily disjoint. To see that the set injects, suppose $\mathbf{z}, \gamma \cdot \mathbf{z} \in \mathcal{G}(y)$ for some $\gamma \in \Gamma$. Thus, again by (2.5), we have

$$\rho(\sigma; \mathbf{z}) > y \quad \text{and} \quad \rho(\tau; \gamma \cdot \mathbf{z}) > y$$

for some $\sigma, \tau \in \Omega$. Using Lemma 2.9, we see that $\rho(\gamma^{-1} \cdot \tau; \mathbf{z}) > y \geq Y_0$. Applying Lemma 2.11, we conclude $\gamma^{-1} \cdot \tau = \sigma$, and so $\sigma = \tau$ as Ω is a set of inequivalent representatives of $\Gamma \backslash \mathbb{P}^1(F)$. $\square$

The decomposition described in Theorem 2.13 below is best accompanied with the visual aid in Figure 2.2.

Theorem 2.13. *Let Γ be a congruence subgroup of level $\mathfrak{N}$, and let $\Omega \subseteq \mathbb{P}^1(F)$ be a set of inequivalent representatives of $\Gamma \backslash \mathbb{P}^1(F)$. For y sufficiently large, depending only on F, there exists a set $\mathcal{S}(y) \subseteq X$ such that*

(i) The set

$$\mathcal{S}(y) \sqcup \bigsqcup_{\sigma \in \Omega} g_\sigma \mathcal{D}(y) \subseteq \mathcal{X}$$

surjects onto a fundamental domain $\mathcal{F}$ for $\Gamma \backslash \mathcal{X}$.

(ii) The set

$$\bigsqcup_{\sigma \in \Omega} g_\sigma \mathcal{D}(y) \subseteq \mathcal{X}$$

injects into a fundamental domain $\mathcal{F}$ for $\Gamma \backslash \mathcal{X}$.

(iii) $\mathcal{S}(y)$ is compact

(iv) $\mathcal{S}(y) \supseteq \mathcal{S}(y')$ for $y \geq y' \geq y_0$.

(v) As $y \to \infty$, the image of $\mathcal{S}(y)$, under the map $\mathcal{X} \to \Gamma \backslash \mathcal{X}$, approaches $\Gamma \backslash \mathcal{X}$.

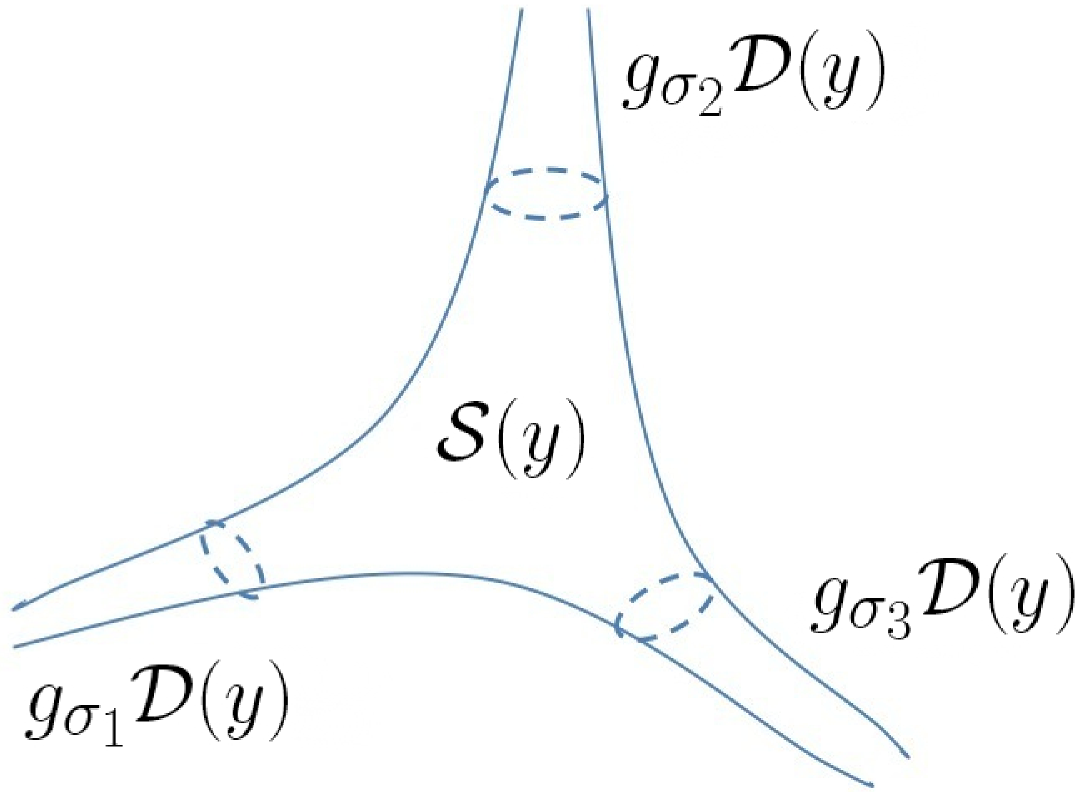

Figure 2.2: Depiction of $\Gamma\backslash\mathcal{X}$ with 3 cusps

Proof. [Gee88, p. 9–11]

For y_0 as in Lemma 2.10, define

$$\mathcal{S}(y) := \overline{\left(\mathcal{G}(y_0) \setminus \mathcal{G}(y)\right)} = \overline{\bigcup_{\sigma\in\Omega} g_\sigma(\mathcal{D}(y_0) \setminus \mathcal{D}(y))}.$$

We claim this choice of $\mathcal{S}(y)$ has the desired properties.

(i) The surjectivity follows by noting

$$\mathcal{S}(y) \cup \bigcup_{\sigma\in\Omega} g_\sigma\mathcal{D}(y) \supseteq \bigcup_{\sigma\in\Omega} g_\sigma\mathcal{D}(y_0)$$

and applying part (a) of Proposition 2.12. The disjoint union follows by definition of $\mathcal{S}(y)$ and part (b) of Proposition 2.12.

(ii) This is a restatement of Proposition 2.12(b).

(iii) Observe

$$\overline{\mathcal{D}(y_0) \setminus \mathcal{D}(y)} = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{X} : \mathbf{x} \in \overline{\mathcal{U}}, \widehat{\mathbf{y}} \in \overline{\mathcal{V}}, \mathbb{N}\mathbf{y} \in [y_0, y]\}$$

where $\mathcal{U} \subseteq F_\infty$ and $\mathcal{V} \subseteq \widehat{\mathcal{Y}}$ are precompact sets defined in Proposition 2.6. The above subset of $\mathcal{X}$ is then topologically of the form $\overline{\mathcal{U}} \times \overline{\mathcal{V}} \times [y_0, y]$ and is hence compact; therefore, $\mathcal{S}(y)$, the closure of a finite union of these subsets, is itself compact.

(iv) This is immediate from the definition of $\mathcal{S}(y)$ and noting $\mathcal{D}(y) \subseteq \mathcal{D}(y')$ for $y \geq y'$.

(v) The limit of $\mathcal{S}(y)$ as $y \to \infty$ is well-defined by (iv). The desired result follows from (i), and noting $\mathcal{D}(y) \to \varnothing$ as $y \to \infty$. $\square$

Chapter 3

Hecke-Maaß Cusp Forms

In this chapter, we will define a Hecke-Maaß cusp form in several stages, and ultimately provide its Whittaker expansion with proof. We will conclude by demonstrating crucial relations satisfied by the Whittaker coefficients which form the basis of the following chapter. Sources for this discussion include [Fre90] and [Gee88], and in terms of the adèles, [Hid90]. The case $F = \mathbb{Q}$, which is fairly similar, can be found in [Iwa02], [IK04], [Miy89], or [DS05].

3.1 Maaß Forms and their Fourier Expansion

Let $\Gamma \leq \mathrm{PGL}(2, O)$ be a congruence subgroup of level $\mathfrak{N}$.

Definition 3.1 (Automorphic form)**.** A function $\phi : \mathcal{X} \to \mathbb{C}$ is an *automorphic form (with respect to* Γ*)* provided

(i) $\phi(\gamma \cdot \mathbf{z}) = \phi(\mathbf{z})$ for all $\gamma \in \Gamma$

(ii) ϕ is an eigenfunction of the Laplacian Δ_v for every $v \mid \infty$, i.e.

$$\Delta_v \phi = \lambda_v \phi$$

where $\lambda_v = s_v(1 - s_v) \in \mathbb{C}$. Denote $\boldsymbol{\lambda} = (\lambda_v)_v$ and$\mathbf{s} = (s_v)_v$.

Our interest lies in automorphic forms which satisfy certain growth conditions, namely belong to the L^2-space of $\Gamma \backslash \mathcal{X}$. Recall that, for $\mathbb{C}$-valued functions ϕ and ψ on $\Gamma \backslash \mathcal{X}$, the L^2-inner product on $\Gamma \backslash \mathcal{X}$ is given by

$$\langle \phi, \psi \rangle_{L^2} = \int_{\mathcal{F}} \phi(\mathbf{z}) \overline{\psi(\mathbf{z})} d\mathbf{z}$$

where $\mathcal{F} \subseteq X$ is a fundamental domain for $\Gamma\backslash\mathcal{X}$. We denote the L^2-norm of ϕ by $||\phi||_{L^2(\Gamma\backslash\mathcal{X})}$, or more simply as $||\phi||_{L^2}$ when Γ is understood.

Definition 3.2 (Maaß form)**.** A function $\phi : \mathcal{X} \to \mathbb{C}$ is a *Maaß form (with respect to* Γ*)* provided ϕ is an automorphic form, and also $\phi \in L^2(\Gamma\backslash\mathcal{X})$.

Remark. By Elliptic Regularity of Δ_v, it follows that $\phi \in C^\infty(\Gamma\backslash\mathcal{X})$. See [Eva10, §6.3] for details.

Henceforth, let $\phi : \mathcal{X} \to \mathbb{C}$ be a Maaß form of Γ. Now, let σ be a cusp of Γ, and define

$$\phi_\sigma(\mathbf{z}) := \phi(g_\sigma \cdot \mathbf{z})$$

where $g_\sigma \in G$ is as defined in Proposition 2.4, satisfying $g_\sigma(\infty) = \sigma$. We claim ϕ_σ is now a Maaß form of $\Gamma^{(\sigma)} = g_\sigma \Gamma g_\sigma^{-1}$. This follows from the fact that Δ_v commutes with the G_v-action on ϕ and also since $d\text{vol}(\mathbf{z})$ is invariant under the G-action. Roughly speaking, studying ϕ at the cusp σ amounts to studying ϕ_σ at the cusp ∞. Recall from Proposition 2.6 that

$$\Lambda := \left\{ \begin{pmatrix} \epsilon & \theta \\ 0 & 1 \end{pmatrix} : \epsilon \in V, \theta \in \mathfrak{N} \right\} \Big/ \{\eta I : \eta \in V\}.$$

is contained in the $\Gamma^{(\sigma)}$-stabilizer of ∞, or more generally, $\Lambda \leq \Gamma^{(\sigma)}$. In particular, we have that ϕ_σ is invariant by Λ, which by setting $\epsilon = 1 \in V$, implies

$$\phi_\sigma(\mathbf{x} + \theta, \mathbf{y}) = \phi_\sigma(\mathbf{x}, \mathbf{y}), \qquad \text{for } \theta \in \mathfrak{N}.$$

where we have written $\phi_\sigma(\mathbf{z}) = \phi_\sigma(\mathbf{x}, \mathbf{y})$ for $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. Under this additive action, $\mathfrak{N}$ is a lattice in $F_\infty = \prod_{v|\infty} F_v$, the $\mathbf{x}$-coordinate. Define the standard trace form bilinear pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : F_\infty \times F_\infty &\to \mathbb{C} \\ \langle \mathbf{w}, \mathbf{x} \rangle &\mapsto \mathbf{Tr}(\mathbf{w}\mathbf{x}) := \sum_{v|\infty} \mathbf{Tr}_v(w_v x_v) \end{aligned}$$

where $\mathbf{Tr}$ extends the global trace $\mathbf{Tr}_{\mathbb{Q}}^F$ to F_∞ by the above formula, and $\mathbf{Tr}_v$ extends the local trace $\mathbf{Tr}_{\mathbb{R}}^{F_v}$ at $v \mid \infty$ by

$$\mathbf{Tr}_v x_v = \begin{cases} x_v & v \text{ real} \\ x_v + \overline{x_v} & v \text{ complex} \end{cases}$$

This pairing yields the dual lattice $\mathfrak{N}^\vee = (\mathfrak{D}\mathfrak{N})^{-1}$ where $\mathfrak{D}$ is the absolute different ideal of F

[Lan94, §3.1]. Hence, the *Fourier series expansion of ϕ at the cusp σ* is given by

$$\phi_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}} \widehat{\phi_\sigma}(\mathbf{y}; \alpha) e(\langle \mathbf{x}, \alpha \rangle) \tag{3.1}$$

where $e(x) = \exp(2\pi i x)$ and

$$\widehat{\phi_\sigma}(\mathbf{y}; \alpha) := \int_{\mathcal{U}} \phi_\sigma(\mathbf{x}, \mathbf{y}) e(-\langle \mathbf{x}, \alpha \rangle) dV_1(\mathbf{x}),$$

with $\mathcal{U}$ being a fundamental domain for $F_\infty/\mathfrak{N}$. Note $\widehat{\phi_\sigma}(\mathbf{y}; \alpha)$ is the *α-Fourier coefficient of ϕ at cusp σ*.

Since the Maaß form $\phi_\sigma(\mathbf{z})$ is an eigenfunction for Δ_v for each $v \mid \infty$, it follows by Elliptic Regularity that $\phi_\sigma(\mathbf{z}) \in C^\infty(\mathcal{X})$, i.e. it possesses infinitely many derivatives in the coordinates x_v and y_v for all $v \mid \infty$. From general Fourier analysis, it follows that the Fourier series (3.1) converges absolutely and uniformly on compacta. We are concerned with Maaß cusp forms, whose definition depends upon this Fourier expansion.

3.2 Maaß Cusp Forms and their Whittaker Expansion

In this section, we aim to elaborate further on the Fourier expansion at the cusp σ of a Maaß form ϕ. Material on partial differential equations will be skipped but can be found in [Eva10].

Definition 3.3 (Maaß cusp form)**.** A function $\phi : \mathcal{X} \to \mathbb{C}$ is a *Maaß cusp form (with respect to Γ)* if ϕ is a Maaß form of Γ and additionally, $\widehat{\phi_\sigma}(\mathbf{y}; 0) \equiv 0$ for every cusp σ of Γ.

Henceforth, we shall assume ϕ is a Maaß cusp form of Γ. From this definition and (3.1), it follows that the Fourier expansion at a cusp σ of Γ is

$$\phi_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}} \widehat{\phi_\sigma}(\mathbf{y}; \alpha) e(\langle \mathbf{x}, \alpha \rangle). \tag{3.2}$$

Since $\phi_\sigma(\mathbf{z})$ is smooth and an eigenfunction for Δ_v, whose derivatives involve y_v, we are able to yield more information about the precise form of the Fourier coefficients $\widehat{\phi_\sigma}(\mathbf{y}; \alpha)$. For $v \mid \infty$, applying Δ_v to both sides of (3.2), we find

$$\lambda_v \phi_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}} \Delta_v(\widehat{\phi_\sigma}(\mathbf{y}; \alpha) e(\langle \mathbf{x}, \alpha \rangle)).$$

Note we may commute the derivative with the infinite sum due to the uniform convergence of the Fourier series. Now, comparing Fourier coefficients of both sides, and using the form of the Laplace operator Δ_v given in (1.2) and (1.4), one obtains a partial differential equation for each $v \mid \infty$. Therefore, the coefficient $\widehat{\phi_\sigma}(\mathbf{y};\alpha)$ with $\alpha \neq 0$ satisfies the following separable linear system of partial differential equations:

$$\begin{cases} y_v^2\Big(\dfrac{\partial^2\psi}{\partial y_v^2}(\mathbf{y}) - 4\pi^2\alpha_v^2\psi(\mathbf{y})\Big) = \lambda_v\psi(\mathbf{y}), & v \text{ real} \\ y_v^2\Big(\dfrac{\partial^2\psi}{\partial y_v^2}(\mathbf{y}) - 16\pi^2|\alpha_v|^2\psi(\mathbf{y})\Big) - y_v\dfrac{\partial\psi}{\partial y_v}(\mathbf{y}) = \lambda_v\psi(\mathbf{y}), & v \text{ complex} \end{cases} \tag{3.3}$$

where $\psi : \mathbb{R}^m_{>0} \to \mathbb{C}$.

Remark. If ϕ were more generally a Maaß form, the zeroth Fourier coefficient $\phi_\sigma(\mathbf{y};0)$ satisfies the same PDE but $\alpha_v = 0$, which results in a very different solution. In this sense, the analysis could continue along the same path, but is unnecessary for our purposes.

We present the solution to the system in (3.3) with the following lemma and proposition.

Lemma 3.4. *Let $a_v \in \mathbb{R}_{>0}$ and $\lambda_v = s_v(1-s_v) \in \mathbb{C}$. Suppose $\psi_v : y_v \to \mathbb{C}$ satisfies*

$$\begin{cases} y_v^2\Big(\psi_v''(y_v) - a_v^2\psi_v(y_v)\Big) = \lambda_v\psi_v(y_v), & v \textit{ real} \\ y_v^2\Big(\psi_v''(y_v) - a_v^2\psi_v(y_v)\Big) - y_v\psi_v'(y_v) = \lambda_v\psi_v(y_v), & v \textit{ complex} \end{cases}$$

then for some constants $c_1, c_2 \in \mathbb{C}$, depending on ψ_v, we have

$$\psi_v(y_v) = c_1\sqrt{\mathbb{N}_v(a_vy_v)}K_{1/2-s_v}(a_vy_v) + c_2\sqrt{\mathbb{N}_v(a_vy_v)}I_{1/2-s_v}(a_vy_v)$$

where $K_\nu(z)$ and $I_\nu(z)$ are the K_ν and I_ν Bessel functions.

Proof. See [Zwi98, §45] or verify using computer algebra package Maple. Information about the Bessel functions can be found in [Iwa02, Appendix B] or [Bow10]. □

For non-zero $\alpha = (\alpha_v) \in F_\infty$, set

$$a_v = 2\pi\mathbf{Tr}_v(|\alpha_v|) = \begin{cases} 2\pi|\alpha_v| & v \text{ real} \\ 4\pi|\alpha_v| & v \text{ complex} \end{cases}.$$

Then, with this choice, the above lemma corresponds to the system described in (3.3). Motivated by this choice, we shall define

$$B_v^-(s_v;y_v) := \sqrt{\mathbb{N}_vy_v}\cdot K_{1/2-s_v}(2\pi\mathbf{Tr}_v(y_v))$$
$$B_v^+(s_v;y_v) := \sqrt{\mathbb{N}_vy_v}\cdot I_{1/2-s_v}(2\pi\mathbf{Tr}_v(y_v))$$

so then by Lemma 3.4, the set of functions

$$\{B_v^+(s_v; |\alpha_v|y_v), B_v^-(s_v; |\alpha_v|y_v)\}$$

forms a basis of solutions for the corresponding ordinary differential equation from the system in (3.3).

Proposition 3.5. *Let* $\alpha = (\alpha_v)_v \in F_\infty, \boldsymbol{\lambda} = (\lambda_v)_v \in \mathbb{C}^m$. *Write* $\lambda_v = s_v(1 - s_v)$ *and set* $\mathbf{s} = (s_v)_{v|\infty}$. *For the sequence of signs* $\boldsymbol{\delta} = (\delta_v)_{v|\infty} \in \{\pm\}^m$, *define*

$$B^{\boldsymbol{\delta}}(\mathbf{s}; \mathbf{y}) := \prod_{v|\infty} B_v^{\delta_v}(s_v; y_v)$$

where $\mathbf{y} \in \mathbb{R}^m_{>0}$. *Then the set of* 2^m *functions*

$$\left\{B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha|\mathbf{y}) : \boldsymbol{\delta} \in \{\pm\}^m\right\}$$

forms a basis of smooth solutions over $\mathbb{C}$ *for the system of partial differential equations given in* (3.3).

Proof. This is a direct consequence of Lemma 3.4 and the fact that the system (3.3) may be solved by separation of variables. See [Zwi98] or [Eva10] for details on this method. □

With Proposition 3.5, we already have a strong understanding of the coefficient $\widehat{\phi_\sigma}(\mathbf{y}; \alpha)$ as it satisfies (3.3); in particular, $\widehat{\phi_\sigma}(\mathbf{y}; \alpha)$ may be written as a linear combination of $B^\delta(\mathbf{s}; |\alpha|\mathbf{y})$ for $\delta \in \{\pm\}^m$. This fact combined with invariance properties of ϕ_σ provides a relation between different Fourier coefficients, which is articulated in the lemma below.

Lemma 3.6. *Let* ϕ *be a Maaß cusp form of* Γ, *and* σ *be a cusp. By Proposition 3.5, we may write for* $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}$,

$$\widehat{\phi_\sigma}(\mathbf{y}; \alpha) = \sum_{\boldsymbol{\delta} \in \{\pm\}^m} c_{\boldsymbol{\delta}}(\alpha) B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha|\mathbf{y})$$

where $c_{\boldsymbol{\delta}}(\alpha) \in \mathbb{C}$ *also depends on* σ. *Then defining* $V \subseteq \mathcal{O}^\times$ *as in Proposition 2.6, we have*

$$c_{\boldsymbol{\delta}}(\alpha) = c_{\boldsymbol{\delta}}(\alpha\epsilon)$$

for all $\epsilon \in V$ *and* $\boldsymbol{\delta} \in \{\pm\}^m$.

Proof. Since

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda \subseteq \Gamma^{(\sigma)}$$

for $\epsilon \in V$, we have $\phi_\sigma(\epsilon \cdot \mathbf{z}) = \phi_\sigma(\mathbf{z})$. Applying the Fourier expansion (3.1) with the form above for $\widehat{\phi_\sigma}(\mathbf{y};\alpha)$, and sending $\alpha \mapsto \alpha\epsilon^{-1}$, we see for all $\mathbf{z} \in \mathcal{X}$,

$$\begin{aligned}\phi(\epsilon \cdot \mathbf{z}) &= \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}\setminus\{0\}} \sum_{\boldsymbol{\delta} \in \{\pm\}^m} c_{\boldsymbol{\delta}}(\alpha) B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha\epsilon|\mathbf{y}) \cdot e(\langle \epsilon\mathbf{x}, \alpha\rangle) \\ &= \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}\setminus\{0\}} \sum_{\boldsymbol{\delta} \in \{\pm\}^m} c_{\boldsymbol{\delta}}(\alpha\epsilon^{-1}) B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha|\mathbf{y}) \cdot e(\langle \mathbf{x}, \alpha\rangle)\end{aligned}$$

by noting $\langle \epsilon\mathbf{x}, \alpha\rangle = \langle \mathbf{x}, \epsilon\alpha\rangle$. Comparing the RHS with the same expansion for $\phi(\mathbf{z})$, and noting the linear independence of $B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha|\mathbf{y})$ as a function of $\mathbf{y}$ from Proposition 3.5, we conclude that $c_{\boldsymbol{\delta}}(\alpha) = c_{\boldsymbol{\delta}}(\alpha\epsilon^{-1})$ as required. □

Further, since $\phi_\sigma \in L^2(\Gamma^{(\sigma)}\backslash X)$, we obtain the simple lemma below bounding the mass of a Fourier coefficient deep in a cusp.

Lemma 3.7. *Let ϕ be a Maaß cusp form of Γ and σ be a cusp. Then for y sufficiently large, depending only on F,*

$$\int_{\mathcal{T}(y)} |\widehat{\phi_\sigma}(\mathbf{y};\alpha)|^2 dV_2(\mathbf{y}) \leq ||\phi_\sigma||_{L^2(\Gamma^{(\sigma)}\backslash\mathcal{X})} < \infty$$

where $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}, \sigma$ is a cusp of Γ, and using notation from Theorem 2.7,

$$\mathcal{T}(y) = \mathcal{T}_{\mathfrak{N}}(y) := \{\mathbf{y} \in \mathbb{R}^m_{>0} : \widehat{\mathbf{y}} \in \mathcal{V}, \mathbb{N}\mathbf{y} \in (y,\infty)\}.$$

Proof. By Parseval's Formula, we have

$$\int_{\mathcal{U}} |\phi_\sigma(\mathbf{x},\mathbf{y})|^2 dV_1(\mathbf{x}) = \sum_{\beta \in (\mathfrak{D}\mathfrak{N})^{-1}} |\widehat{\phi_\sigma}(\mathbf{y};\beta)|^2 \geq |\widehat{\phi_\sigma}(\mathbf{y};\alpha)|^2.$$

for $\mathbf{y} \in \mathbb{R}^m_{>0}$. For y sufficiently large, $\mathcal{D}(y)$ injects into a fundamental domain for $\Gamma^{(\sigma)}\backslash\mathcal{X}$ by Theorem 2.13. Since $\phi_\sigma \in L^2(\Gamma^{(\sigma)}\backslash\mathcal{X})$, it follows that we may integrate both sides of the above inequality over the remainder of $\mathcal{D}(y)$ given explicitly in Theorem 2.7, namely over the set $\mathcal{T}(y) = \{\mathbf{y} \in \mathbb{R}^m_{>0} : \widehat{\mathbf{y}} \in \mathcal{V}, \mathbb{N}\mathbf{y} \in (r,\infty)\}$. Thus, we obtain the simple bound

$$\int_{\mathcal{T}(y)} |\widehat{\phi_\sigma}(\mathbf{y};\alpha)|^2 dV_2(\mathbf{y}) \leq \int_{\mathcal{D}(y)} |\phi_\sigma(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}) \leq ||\phi_\sigma||_{L^2(\Gamma^{(\sigma)}\backslash\mathcal{X})} < \infty.$$

for y sufficiently large, depending only on F. $\square$

Utilizing these lemmas and the asymptotics of $B_v^{\pm}$, we can completely characterize the form of $\widehat{\phi_\sigma}(\mathbf{y};\alpha)$ for $\alpha \neq 0$. The asymptotics of $B_v^{\pm}$ are known to be

$$B_v^{\pm}(s_v; y_v) \asymp_{s_v} \exp(\pm 2\pi \mathbf{Tr}_v(y_v)) \tag{3.4}$$

as $y_v \to \infty$; see [Iwa02, Appendix B].

Theorem 3.8. *Let ϕ be a Maaß cusp form of Γ with Laplace eigenvalues $\boldsymbol{\lambda} = (\lambda_v)_{v|\infty}$. Writing $\lambda_v = s_v(1-s_v)$, we have for $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}$ and cusp σ of Γ, that*

$$\widehat{\phi_\sigma}(\mathbf{y};\alpha) = c \prod_{v|\infty} B_v^{-}(s_v; |\alpha_v| y_v)$$

for some constant $c \in \mathbb{C}$ depending on ϕ, σ and α.

Proof. As in Lemma 3.6, we know $\widehat{\phi_\sigma}(\mathbf{y};\alpha)$ is of the form:

$$\widehat{\phi_\sigma}(\mathbf{y};\alpha) = \sum_{\boldsymbol{\delta} \in \{\pm\}^m} c_{\boldsymbol{\delta}} B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha| \mathbf{y})$$

where $c_{\boldsymbol{\delta}} = c_{\boldsymbol{\delta}}(\alpha) \in \mathbb{C}$. Our goal is to show that $c_{\boldsymbol{\delta}}$ is non-zero only when $\boldsymbol{\delta} = (-,-,\ldots,-)$. By Lemma 3.6, we have

$$\widehat{\phi_\sigma}(\mathbf{y};\alpha\epsilon) = \sum_{\boldsymbol{\delta} \in \{\pm\}^m} c_{\boldsymbol{\delta}} B^{\boldsymbol{\delta}}(\mathbf{s}; |\alpha\epsilon| \mathbf{y}), \qquad \text{for } \epsilon \in V.$$

Our goal will be achieved by choosing $\epsilon \in V$ appropriately and analyzing the asymptotics of $\widehat{\phi_\sigma}(\mathbf{y};\alpha\epsilon)$ in $\mathcal{T}(y)$ for $y \geq 1$ sufficiently large. Defining

$$\boldsymbol{\kappa}^+ := \{v \mid \infty : \kappa_v = (+)\} \qquad \boldsymbol{\kappa}^- := \{v \mid \infty : \kappa_v = (-)\},$$

for $\boldsymbol{\kappa} \in \{\pm\}^m$, we choose $\boldsymbol{\kappa}$ such that $\#\boldsymbol{\kappa}^+$ is maximum and $c_{\boldsymbol{\kappa}} \neq 0$. Evidently, the choice may not be unique. Suppose, for a contradiction, that $\boldsymbol{\kappa}^+ \neq \varnothing$.

Now, since the set $\overline{\mathcal{V}} \subseteq \mathbb{R}^m_{>0}$ is compact by Proposition 2.6, the constants

$$a := \min\{y_v : \mathbf{y} \in \overline{\mathcal{V}}, v \mid \infty\}, \quad b := \max\{y_v : \mathbf{y} \in \overline{\mathcal{V}}, v \mid \infty\},$$

exist and are positive. Then for $\mathbf{y} \in \mathcal{T}(y)$, we have $\widehat{\mathbf{y}} \in \mathcal{V}$ and so

$$a(\mathbb{N}\mathbf{y})^{1/n} \leq y_v \leq b(\mathbb{N}\mathbf{y})^{1/n}. \tag{3.5}$$

In particular, this shows that as $\mathbb{N}\mathbf{y} \to \infty$, every coordinate of $\mathbf{y}$ goes to infinity. Our aim is to choose $\epsilon \in V$ appropriately with respect to $\boldsymbol{\kappa}$.

If $\boldsymbol{\kappa} = (+,+,\dots,+)$, then choose $\epsilon := 1 \in V$.

Otherwise, choose $\epsilon \in V$ such that

$$|\epsilon_v| > 1, v \in \boldsymbol{\kappa}^+ \qquad \text{and} \qquad |\epsilon_v| < 1, v \in \boldsymbol{\kappa}^-,$$

which is possible because by Dirichlet's Unit Theorem, V is an $(m-1)$-dimensional lattice in $\{\mathbf{y} \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y} = 1\}$. See [Lan94, p. 104–108] or [Neu99, §1.7] for details on this choice. Observe for $\boldsymbol{\delta} \in \{\pm\}^m$ from (3.4), we have

$$B^{\boldsymbol{\delta}}(\mathbf{s}; |\epsilon\alpha|\mathbf{y}) \asymp_{\mathbf{s},\alpha} \exp\Big(2\pi\Big(\sum_{v\in\boldsymbol{\delta}^+} \mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v) - \sum_{v\in\boldsymbol{\delta}^-} \mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v)\Big)\Big) \tag{3.6}$$

as $\mathbb{N}\mathbf{y} \to \infty$ for $\mathbf{y} \in \mathcal{T}(y)$. From (3.5), it follows that

$$\begin{aligned}
\frac{B^{\delta}(\mathbf{s}; |\epsilon\alpha|\mathbf{y})}{B^{\kappa}(\mathbf{s}; |\epsilon\alpha|\mathbf{y})} &\ll_{\mathbf{s},\alpha} \exp\Big(4\pi\Big(\sum_{v\in\boldsymbol{\delta}^+\setminus\boldsymbol{\kappa}^+} \mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v) - \sum_{v\in\boldsymbol{\delta}^-\setminus\boldsymbol{\kappa}^-} \mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v)\Big)\Big) \\
&\ll_{\mathbf{s},\alpha} \exp\Big(4\pi(\mathbb{N}\mathbf{y})^{1/n}\underbrace{\Big(\sum_{v\in\boldsymbol{\delta}^+\setminus\boldsymbol{\kappa}^+} \mathbf{Tr}_v(|\epsilon_v\alpha_v|b) - \sum_{v\in\boldsymbol{\delta}^-\setminus\boldsymbol{\kappa}^-} \mathbf{Tr}_v(|\epsilon_v\alpha_v|a)\Big)}_{(*)}\Big)
\end{aligned}$$

If $\boldsymbol{\delta} \neq \boldsymbol{\kappa}$ and $c_{\boldsymbol{\delta}} \neq 0$, then $\boldsymbol{\delta}^\pm \setminus \boldsymbol{\kappa}^\pm \subseteq \boldsymbol{\kappa}^\mp$, and, by maximality of $\boldsymbol{\kappa}$, the set $\boldsymbol{\delta}^- \setminus \boldsymbol{\kappa}^- \neq \varnothing$. Then by choice of ϵ, if we replace ϵ by a sufficiently large power of itself (independent of $\mathbf{y}$), we may assume the quantity in $(*)$ is negative for all $\boldsymbol{\delta} \neq \boldsymbol{\kappa}$. In other words, for $\boldsymbol{\delta} \neq \boldsymbol{\kappa}$ such that $c_{\boldsymbol{\delta}} \neq 0$, we have shown

$$B^{\boldsymbol{\delta}}(\mathbf{s}; |\epsilon\alpha|\mathbf{y}) = B^{\boldsymbol{\kappa}}(\mathbf{s}; |\epsilon\alpha|\mathbf{y}) \cdot o(1)$$

as $\mathbb{N}\mathbf{y} \to \infty$. Thus, we may write

$$\widehat{\phi_\sigma}(\mathbf{y}; \epsilon\alpha) = B^{\kappa}(\mathbf{s}; |\epsilon\alpha|\mathbf{y})\Big(c_{\boldsymbol{\kappa}} + \sum_{\delta\neq\kappa} c_{\boldsymbol{\delta}} \cdot o(1)\Big)$$

as $\mathbb{N}\mathbf{y} \to \infty$ for $\mathbf{y} \in \mathcal{T}(y)$. From this equation, it follows by (3.6) and (3.5) that for $\mathbb{N}\mathbf{y}$

sufficiently large,

$$
\begin{aligned}
|\widehat{\phi_\sigma}(\mathbf{y};\epsilon\alpha)| &\geq \frac{|c_{\boldsymbol{\kappa}}|}{2}|B^{\kappa}(\mathbf{s};|\epsilon\alpha|\mathbf{y})| \\
&\gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp\Big(2\pi\sum_{v\in\boldsymbol{\kappa}^+}\mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v) - 2\pi\sum_{v\in\boldsymbol{\kappa}^-}\mathbf{Tr}_v(|\epsilon_v\alpha_v|y_v)\Big) \\
&\gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp\Big(2\pi(\mathbb{N}\mathbf{y})^{1/n}\underbrace{\Big(\sum_{v\in\boldsymbol{\kappa}^+}\mathbf{Tr}_v(|\epsilon_v\alpha_v|a) - \sum_{v\in\boldsymbol{\kappa}^-}\mathbf{Tr}_v(|\epsilon_v\alpha_v|b)\Big)}_{(**)}\Big)
\end{aligned}
$$

Again, by choice of ϵ, if we replace ϵ by a sufficiently large power of itself (independent of $\mathbf{y}$), we may assume the quantity in $(**)$ is positive. More simply, for some constant $\eta > 0$, we have

$$
|\widehat{\phi_\sigma}(\mathbf{y};\epsilon\alpha)| \gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp(\eta(\mathbb{N}\mathbf{y})^{1/n}).
$$

for $\mathbf{y} \in \mathcal{T}(y)$ and $\mathbb{N}\mathbf{y}$ sufficiently large. Using this bound, we see that for y sufficiently large,

$$
\begin{aligned}
\int_{\mathcal{T}(y)} |\widehat{\phi_\sigma}(\mathbf{y};\alpha)|^2 dV_2(\mathbf{y}) &\gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}| \int_{\mathcal{T}(y)} \exp(2\eta(\mathbb{N}\mathbf{y})^{1/n}) dV_2(\mathbf{y}) \\
&\gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp(2\eta\cdot y^{1/n})\int_{\mathcal{T}(y)} dV_2(\mathbf{y}).
\end{aligned} \tag{3.7}
$$

Finally, sending $\mathbf{y} = (y_v)_v \mapsto y^{1/n}\cdot\mathbf{y} = (y^{1/n}\cdot y_v)_v$, the set $\mathcal{T}(y)$ is mapped bijectively to $\mathcal{T}(1)$ and from (2.2), we have $dV_2(\mathbf{y}) \mapsto y^{-1}\cdot dV_2(\mathbf{y})$. Thus, the RHS of (3.7) is

$$
\gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp(2\eta\cdot y^{1/n})y^{-1}\int_{\mathcal{T}(1)} dV_2(\mathbf{y}) \gg_{\mathbf{s},\alpha} |c_{\boldsymbol{\kappa}}|\exp(2\eta\cdot y^{1/n})y^{-1}.
$$

Taking $y \to \infty$, we deduce from (3.7) that

$$
\int_{\mathcal{T}(y)} |\widehat{\phi_\sigma}(\mathbf{y};\alpha)|^2 dV_2(\mathbf{y}) \to \infty
$$

contradicting Lemma 3.7. This completes the proof. □

In light of Theorem 3.8 and (3.1), we make the following definitions.

Definition 3.9 (Local Whittaker Function)**.** For $v \mid \infty$, define the *local Whittaker function at* $v \mid \infty$ to be

$$
W_v(s_v;z_v) := \sqrt{y_v}K_{1/2-s_v}(2\pi\mathbf{Tr}_v(y_v))\cdot e(\mathbf{Tr}_v(x_v))
$$

where $s_v \in \mathbb{C}$ and $z_v = (x_v, y_v) \in F_v \times \mathbb{R}_{>0}$. Observe that

$$W_v(s_v; z_v) = B_v^-(s_v; y_v)e(\mathbf{Tr}_v(x_v)).$$

Definition 3.10 (Whittaker Function)**.** Define the *(unramified) Whittaker function of* G to be

$$W(\mathbf{s}; \mathbf{z}) := \prod_{v|\infty} W_v(s_v; z_v)$$

where $\mathbf{s} = (s_v)_{v|\infty} \in \mathbb{C}^m, \mathbf{z} = (z_v)_{v|\infty} \in \mathcal{X}$. From the previous definition and Theorem 3.8, we may define $c_\sigma(\phi; \alpha) \in \mathbb{C}$ to be such that

$$\widehat{\phi_\sigma}(\mathbf{y}; \alpha) = c_\sigma(\phi; \alpha)|\mathbb{N}\alpha|^{-1/2}W(\mathbf{s}; \alpha \cdot \mathbf{z}), \quad \text{for } \mathbf{z} \in \mathcal{X}$$

where $\alpha \cdot \mathbf{z} := (\alpha_v x_v, |\alpha_v| y_v)_{v|\infty} \in F_\infty \times \mathbb{R}^m_{>0}$. The constant $c_\sigma(\phi; \alpha)$ shall be called the *α-Whittaker coefficient of ϕ at the cusp σ*. Note the normalization factor $|\mathbb{N}\alpha|^{-1/2}$ is not necessary, but chosen to better suit later multiplicative relations.

Corollary 3.11 (Whittaker Expansion of Maaß cusp forms)**.** *Let ϕ be a Maaß cusp form of Γ with Laplace eigenvalues $\boldsymbol{\lambda} = (\lambda_v)_{v|\infty}$, and let σ be a cusp of Γ. Writing $\lambda_v = s_v(1 - s_v)$, we have that for $\mathbf{z} \in \mathcal{X}$,*

$$\phi_\sigma(\mathbf{z}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}} c_\sigma(\phi; \alpha)|\mathbb{N}\alpha|^{-1/2}W(\mathbf{s}; \alpha \cdot \mathbf{z})$$

where $c_\sigma(\phi; \alpha) \in \mathbb{C}$ and $\alpha \cdot \mathbf{z} := (\alpha_v x_v, |\alpha_v| y_v)_{v|\infty} \in \mathcal{X}$ for $\alpha \in F$. The infinte sum is absolutely and uniformly convergent.

Proof. This is an immediate consequence of Theorem 3.8 and (3.1); the convergence properties come from the fact that this sum is the Fourier expansion of a smooth function. $\square$

Remark. The expansion of $\phi_\sigma(\mathbf{z})$ given in Corollary 3.11 is known as the *Whittaker expansion of ϕ at the cusp σ.*

3.3 Hecke Operators

We wish to define Hecke operators acting on a Hecke-Maaß cusp form ϕ, so by Atkin-Lehner theory, we may assume that ϕ is a Hecke-Maaß cusp form with respect to

$$\Gamma_0(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}) : c \equiv 0 \pmod{\mathfrak{N}} \right\} \Big/ \{\eta I : \eta \in \mathcal{O}^\times\}.$$

for some ideal $\mathfrak{N}$. Relevant discussion can be found in [Iwa02, §8.5] and [Hid90]. As a result, we will *assume without loss that* $\Gamma = \Gamma_0(\mathfrak{N})$ *is our fixed congruence subgroup of level* $\mathfrak{N}$ *for the rest of this chapter.*

Defining Hecke operators can take several approaches with some more natural than others; for example, Hida [Hid90] takes an adèlic perspective. In our case, we shall only need Hecke operators defined a specific subset of ideals, namely

$$\mathcal{N}_{\mathfrak{N}} := \{\mathfrak{n} \subseteq \mathcal{O} : \mathfrak{p} \mid \mathfrak{n} \implies \mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}\}$$

where

$$\mathcal{P}_{\mathfrak{N}} := \{\mathfrak{p} \subseteq \mathcal{O} : \mathfrak{p} \text{ unramified principal prime ideal and } \mathfrak{p} \nmid \mathfrak{N}\}.$$

To begin, we shall define Hecke operators for powers of prime ideals $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$. Since the primes are principal, we may take an explicit approach for defining Hecke operators, which has little difference from the usual Hecke operators defined over $F = \mathbb{Q}$. As a result, we will quote material from sources discussing classical Hecke operators over $\mathbb{Q}$, such as [IK04] and [Iwa02]. More thorough and general proofs over $\mathbb{Q}$ can be found in [Miy89].

Definition 3.12 ($\mathfrak{p}^k$-Hecke operator)**.** Let $k \geq 0$ and $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$ have uniformizer ϖ, i.e. $\mathfrak{p} = (\varpi)$. Suppose $\phi : \Gamma_0(\mathfrak{N})\backslash\mathcal{X} \to \mathbb{C}$. Then we define $T(\mathfrak{p}^k)\phi : \mathcal{X} \to \mathbb{C}$ by

$$(T(\mathfrak{p}^k)\phi)(\mathbf{z}) := \frac{1}{\sqrt{\mathbb{N}\mathfrak{p}^k}} \sum_{j=0}^{k} \sum_{\rho \in \mathcal{O}/\mathfrak{p}^j} \phi\Big(\begin{pmatrix} \varpi^{k-j} & \rho \\ 0 & \varpi^j \end{pmatrix} \cdot \mathbf{z} \Big)$$

where the inner sum over ρ is over a set of inequivalent representatives of $\mathcal{O}/\mathfrak{p}^j$. The operator $T(\mathfrak{p}^k)$ is called the $\mathfrak{p}^k$-*Hecke operator.*

Remark. This definition is independent of the choice of representatives for $\mathcal{O}/\mathfrak{p}^j$ and uniformizer ϖ as $\Gamma = \Gamma_0(\mathfrak{N})$ contains all elements of the form

$$\begin{pmatrix} \epsilon & \theta \\ 0 & 1 \end{pmatrix}, \text{ where } \epsilon \in \mathcal{O}^\times, \theta \in \mathcal{O}.$$

First, we note that the Hecke operator is in fact an action on $\Gamma_0(\mathfrak{N})$-invariant functions, and further on the space of Maaß cusp forms.

Proposition 3.13 (§14.6 of [IK04])**.** *The Hecke operator $T(\mathfrak{p})$ for $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$ acts on the space of $\Gamma_0(\mathfrak{N})$-invariant functions, i.e. $\{\phi : \Gamma_0(\mathfrak{N})\backslash\mathcal{X} \to \mathbb{C}\}$.*

Corollary 3.14 (§8.5 of [Iwa02])**.** *The Hecke operator $T(\mathfrak{n})$ for $n \in \mathcal{N}_{\mathfrak{N}}$ is an L^2-bounded linear operator acting on the space of Maaß cusp forms of $\Gamma_0(\mathfrak{N})$.*

Now, we describe the multiplicative relations of Hecke operators on the space of Maaß cusp forms of $\Gamma_0(\mathfrak{N})$. We shall first discuss the collection of $\{T(\mathfrak{p}^k)\}_{k=0}^{\infty}$ for a fixed prime ideal $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$. Note that the normalization factor $(\mathbb{N}\mathfrak{p}^k)^{-1/2}$ affects these relations, and in other sources, is often replaced by 1 or $(\mathbb{N}\mathfrak{p}^k)^{-1}$. Our choice is primarily to retain consistentcy with [Sou10]. It is a simple elementary number theory exercise to adjust the relations according to these factors.

Proposition 3.15 (Proposition 14.9 of [IK04])**.** *Let $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$ be given. Then for $k \geq 2$,*

$$T(\mathfrak{p}^k) = T(\mathfrak{p}^{k-1})T(\mathfrak{p}) - T(\mathfrak{p}^{k-2})$$

on the space of Maaß cusp forms of $\Gamma_0(\mathfrak{N})$.

The second relation demonstrates that Hecke operators for distinct prime ideals $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}_{\mathfrak{N}}$ commute.

Proposition 3.16 (Proposition 14.9 of [IK04])**.** *Let $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}_{\mathfrak{N}}$ be distinct prime ideals. Then*

$$T(\mathfrak{p})T(\mathfrak{q}) = T(\mathfrak{q})T(\mathfrak{p})$$

on the space of Maaß cusp forms of $\Gamma_0(\mathfrak{N})$.

Thus, we may extend our definition of Hecke operators to all ideals of $\mathcal{N}_{\mathfrak{N}}$ multiplicatively.

Definition 3.17 ($\mathfrak{n}$-Hecke operators)**.** For $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$ are relatively prime, i.e. $(\mathfrak{m}, \mathfrak{n}) = (1) = \mathcal{O}$, define

$$T(\mathfrak{m}\mathfrak{n}) := T(\mathfrak{m})T(\mathfrak{n}).$$

With this definition, we may collect our results, and summarize them in the following theorem.

Theorem 3.18 (Proposition 14.9 of [IK04], or §8.5 of [Iwa02])**.** *On the space of Maaß cusp forms of $\Gamma_0(\mathfrak{N})$, the Hecke operators $\{T(\mathfrak{n}) : \mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}\}$ satisfy the following multiplicative relations:*

(i) $T(\mathcal{O}) = id$.

(ii) $T(\mathfrak{m})T(\mathfrak{n}) = \sum_{\mathfrak{d}|(\mathfrak{m},\mathfrak{n})} T\left(\frac{\mathfrak{m}\mathfrak{n}}{\mathfrak{d}^2}\right)$ *for* $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$.

3.4 Hecke-Maaß Cusp Forms and their Whittaker Coefficients

We are now in a position to define the key object of interest: Hecke-Maaß cusp forms. In order to possess well-defined Hecke operators, recall that we have assumed $\Gamma = \Gamma_0(\mathfrak{N})$ for some ideal $\mathfrak{N}$.

Definition 3.19 (Hecke-Maaß cusp form)**.** A function $\phi : \mathcal{X} \to \mathbb{C}$ is a *Hecke-Maaß cusp form (with respect to* $\Gamma_0(\mathfrak{N})$*)* provided

- ϕ is a Maaß cusp form
- ϕ is an eigenfunction of every Hecke operator $T(\mathfrak{n})$ for all ideals $\mathfrak{n}$.

More explicitly, ϕ satisfies all of the following:

(i) ϕ is $\Gamma_0(\mathfrak{N})$-invariant, i.e. $\phi(\gamma \cdot \mathbf{z}) = \phi(\mathbf{z})$ for all $\gamma \in \Gamma_0(\mathfrak{N})$ and $\mathbf{z} \in \mathcal{X}$.

(ii) ϕ is an eigenfunction of Δ_v for every $v \mid \infty$, i.e.

$$\Delta_v \phi(\mathbf{z}) = \lambda_v \phi(\mathbf{z})$$

where $\lambda_v = s_v(1 - s_v) \in \mathbb{C}$. Denote $\boldsymbol{\lambda} = (\lambda_v)_v$ and $\mathbf{s} = (s_v)_v$.

(iii) $\phi \in L^2(\Gamma_0(\mathfrak{N})\backslash X)$, i.e.

$$||\phi||_{L^2}^2 = \int_{\mathcal{F}} |\phi(\mathbf{z})|^2 d\text{vol}(\mathbf{z}) < \infty$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(\mathfrak{N})\backslash\mathcal{X}$.

(iv) The zeroth Fourier coefficient of ϕ at every cusp σ of $\Gamma_0(\mathfrak{N})$ vanishes, i.e. $\widehat{\phi_\sigma}(\mathbf{y}; 0) = 0$ for all cusps σ of $\Gamma_0(\mathfrak{N})$.

(v) ϕ is an eigenfunction of every Hecke operator $T(\mathfrak{n})$ for all ideals $\mathfrak{n}$, i.e.

$$T(\mathfrak{n})\phi(\mathbf{z}) = \lambda_\phi(\mathfrak{n})\phi(\mathbf{z})$$

where $\lambda_\phi(\mathfrak{n}) \in \mathbb{C}$ is the $\mathfrak{n}$-*Hecke eigenvalue of* ϕ.

Remark. In the previous section, Hecke operators $T(\mathfrak{n})$ were defined for certain ideals $\mathfrak{n}$, namely $\mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$, albeit one can define them for all ideals $\mathfrak{n}$. In the literature, Hecke-Maaß cusp forms are simultaneous eigenforms of all Hecke operators, but as already noted, we shall only require $T(\mathfrak{n})$ for $\mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$.

The Hecke eigenvalues $\lambda_\phi(\mathfrak{n})$ for $\mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$ naturally inherit the multiplicative properties of their Hecke operators as described in the previous section.

Proposition 3.20. *Let ϕ be a Hecke-Maaß cusp form with respect to $\Gamma_0(\mathfrak{N})$. Then*

(i) $\lambda_\phi(\mathcal{O}) = 1$.

(ii) $\lambda_\phi(\mathfrak{p}^k) = \lambda_\phi(\mathfrak{p}^{k-1})\lambda_\phi(\mathfrak{p}) - \lambda_\phi(\mathfrak{p}^{k-2})$ *for* $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$ *and* $k \geq 2$.

(iii) $\lambda_\phi(\mathfrak{m})\lambda_\phi(\mathfrak{n}) = \sum_{\mathfrak{d}|(\mathfrak{m},\mathfrak{n})} \lambda_\phi\Big(\frac{\mathfrak{m}\mathfrak{n}}{\mathfrak{d}^2}\Big)$ *for* $\mathfrak{m}, \mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$.

Proof. This is immediate from Proposition 3.15 and Theorem 3.18. Note (ii) is a specific case of (iii). □

For a Maaß cusp form ϕ, recall that we have the following Whittaker expansion from Corollary 3.11.

$$\phi_\sigma(\mathbf{z}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}\setminus\{0\}} c_\sigma(\phi;\alpha)|\mathbb{N}\alpha|^{-1/2} W(\mathbf{s};\alpha\cdot\mathbf{z})$$

where σ is a cusp of $\Gamma_0(\mathfrak{N})$. If we additionally assume that ϕ is a Hecke-Maaß cusp form, then the Whittaker coefficients $c_\sigma(\phi;\alpha)$ are closely related to the Hecke eigenvalues of ϕ.

Proposition 3.21. *Let ϕ be a Hecke-Mass cusp form of $\Gamma_0(\mathfrak{N})$ and σ be a cusp. For $\mathfrak{p} \in \mathcal{P}_{\mathfrak{N}}$, let ϖ be a uniformizer for $\mathfrak{p}$. If $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$ is a unit modulo $\mathfrak{p}$, then*

$$c_\sigma(\phi;\alpha\varpi^k) = c_\sigma(\phi;\alpha)\cdot\lambda_\phi(\mathfrak{p}^k) \quad \textit{for } k \geq 1.$$

Proof. We will follow the argument structure of [IK04, §14.6]. Applying $T(\mathfrak{p}^k)$ and then g_σ to $\phi(z)$, we utilize the Whittaker expansion from Corollary 3.11 to deduce

$$\begin{aligned}
&(N\mathfrak{p})^{k/2}(g_\sigma\cdot T(\mathfrak{p}^k))\phi(\mathbf{z}) \\
&= \sum_{j=0}^{k}\sum_{\rho\in\mathcal{O}/\mathfrak{p}^j} \phi_\sigma\Big(\frac{\varpi^{k-j}\mathbf{z}+\rho}{\varpi^j}\Big) \\
&= \sum_{\beta\in(\mathfrak{D}\mathfrak{N})^{-1}\setminus\{0\}} \frac{c_\sigma(\phi;\beta)}{|\mathbb{N}\beta|^{1/2}} \sum_{j=0}^{k}\sum_{\rho\in\mathcal{O}/\mathfrak{p}^j} W\Big(\mathbf{s};\beta\cdot\frac{\varpi^{k-j}\mathbf{z}+\rho}{\varpi^j}\Big)
\end{aligned} \tag{3.8}$$

We prove that the RHS of (3.8) is actually the Whittaker expansion of $T(\mathfrak{p}^k)\phi_\sigma(\mathbf{z})$. To do this, we determine the non-zero terms of (3.8) by considering the inner sum over ρ for a fixed non-zero $\beta \in (\mathfrak{D}\mathfrak{N})^{-1}$ and $0 \leq j \leq k$. We remark that, in the following arguments, the re-arranging of terms is valid, as the Whittaker expansion is absolutely convergent due to the infinite differentiability of ϕ_σ. Notice by Definition 3.9 and Definition 3.10, we have

$$\begin{aligned}
&\sum_{\rho \in \mathcal{O}/\mathfrak{p}^j} W\Big(\mathbf{s}; \beta \cdot \frac{\varpi^{k-j}\mathbf{z} + \rho}{\varpi^j}\Big) \\
&= \sum_{\rho \in \mathcal{O}/\mathfrak{p}^j} W\Big(\mathbf{s}; |\beta\varpi^{k-2j}| \cdot \mathbf{y}\Big) e\Big(\mathbf{Tr}\Big(\beta \cdot \frac{\varpi^{k-j}\mathbf{x} + \rho}{\varpi^j}\Big)\Big) \\
&= W\Big(\mathbf{s}; |\beta\varpi^{k-2j}|\mathbf{y}\Big) e\Big(\mathbf{Tr}\Big(\beta\varpi^{k-2j}\mathbf{x}\Big)\Big) \sum_{\rho \in \mathcal{O}/\mathfrak{p}^j} e\Big(\mathbf{Tr}\Big(\frac{\beta\rho}{\varpi^j}\Big)\Big)
\end{aligned} \tag{3.9}$$

For this last sum, we claim

$$\sum_{\rho \in \mathcal{O}/\mathfrak{p}^j} e\Big(\mathbf{Tr}\Big(\frac{\beta\rho}{\varpi^j}\Big)\Big) = \begin{cases} (\mathbb{N}\mathfrak{p})^j & \nu_\mathfrak{p}(\beta) \geq j \\ 0 & \text{else} \end{cases} \tag{3.10}$$

where $\nu_\mathfrak{p} : F \to \mathbb{Z}$ is the $\mathfrak{p}$-adic valuation. Again, we note that $\mathcal{O}/\mathfrak{p}^j$ is a set of inequivalent representatives rather than a set of cosets. Since $\mathfrak{p} \nmid \mathfrak{N}$ we may assume, without loss, that the representatives ρ belong to $\mathfrak{N}$. Observe that the map

$$\begin{aligned}
\mathcal{O} &\to \mathbb{C}_1^\times \\
\rho &\mapsto e(\mathbf{Tr}(\tfrac{\beta\rho}{\varpi^j}))
\end{aligned}$$

is a character of $\mathcal{O}/\mathfrak{p}^j$. If $\nu_\mathfrak{p}(\beta) \geq j$, then $\beta\varpi^{-j} \in (\mathfrak{D}\mathfrak{N})^{-1}$ implying $\mathbf{Tr}(\beta\varpi^{-j}\rho) \in \mathbb{Z}$ for all $\rho \in \mathfrak{N}$ by definition of the absolute different. Thus, the character is trivial for $\rho \in \mathfrak{N}$ and hence for all the representatives ρ in the sum. Further noting $\#\mathcal{O}/\mathfrak{p}^j = (\mathbb{N}\mathfrak{p})^j$ gives the result in this case. If $\nu_\mathfrak{p}(\beta) < j$, then since $\mathfrak{p} \in \mathcal{P}_\mathfrak{N}$ is unramified and $\mathfrak{p} \nmid \mathfrak{N}$, the character is non-trivial (evaluate at any $\rho \notin \mathfrak{p}$), so by orthgonality of characters, the sum is zero.

Thus, from (3.10), the terms involving $e(\mathbf{Tr}(\beta\varpi^{k-2j}\mathbf{x}))$ in (3.9) are non-zero only if $\beta\varpi^{k-2j} \in (\mathfrak{D}\mathfrak{N})^{-1}$. As a result, the RHS of (3.8) is indeed the Whittaker expansion of $T(\mathfrak{p}^k)\phi_\sigma(\mathbf{z})$.

Now, our goal is to collect terms in (3.9) and determine the coefficient of $e(\mathbf{Tr}(\alpha\mathbf{x}))$ for a given $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$ which is a unit modulo $\mathfrak{p}$. For $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$, define

$$A_\alpha = \{(\beta, j) : \alpha = \beta\varpi^{k-2j}, \beta \in (\mathfrak{D}\mathfrak{N})^{-1}\mathfrak{p}^j\}.$$

Then by (3.8), (3.9) and orthogonality of Fourier coefficients, we see that

$$(\mathbb{N}\mathfrak{p})^{k/2}(g_\sigma\widehat{T(\mathfrak{p}^k)}\phi)(\mathbf{y};\alpha) = W\Big(\mathbf{s};|\alpha|\mathbf{y}\Big)\sum_{(\beta,j)\in A_\alpha}\frac{c_\sigma(\phi;\beta)}{|\mathbb{N}\beta|^{1/2}}\cdot\mathbb{N}\mathfrak{p}^j, \tag{3.11}$$

so it remains to determine the pairs $(\beta,j)\in A_\alpha$. Evidently, for a given $\alpha\in(\mathfrak{D}\mathfrak{N})^{-1}$, the value of j determines $\beta\in(\mathfrak{D}\mathfrak{N})^{-1}$ by the formula $\beta:=\alpha\varpi^{2j-k}$. Since α is a $\mathfrak{p}$-adic unit by assumption, we see that

$$\nu_\mathfrak{p}(\beta)=2j-k.$$

On the other hand, from our previous observations, $\nu_\mathfrak{p}(\beta)\geq j$, and so $2j-k\geq j$ implying $j=k$. Thus, A_α consists of exact one element $(\alpha\varpi^k,k)$. Substituting this result into (3.11), we see that

$$(g_\sigma\widehat{T(\mathfrak{p}^k)}\phi)(\mathbf{y};\alpha)=c_\sigma(\phi;\alpha\varpi^k)|\mathbb{N}\alpha|^{-1/2}W(\mathbf{s};|\alpha|\mathbf{y}).$$

On the other hand, $g_\sigma T(\mathfrak{p}^k)\phi=\lambda_\phi(\mathfrak{p}^k)\phi_\sigma$, and so we also have

$$(g_\sigma\widehat{T(\mathfrak{p}^k)}\phi)(\mathbf{y};\alpha)=\lambda_\phi(\mathfrak{p}^k)c_\sigma(\phi;\alpha)|\mathbb{N}\alpha|^{-1/2}W(\mathbf{s};|\alpha|\mathbf{y}).$$

Comparing the previous two equations, we have the desired result. □

Corollary 3.22. *Let ϕ be a Hecke-Mass cusp form of $\Gamma_0(\mathfrak{N})$ and σ be a cusp. For $\mathfrak{n}\in\mathcal{N}_\mathfrak{N}$, choose $\eta\in\mathfrak{n}$ such that $\mathfrak{n}=(\eta)$. If $\alpha\in(\mathfrak{D}\mathfrak{N})^{-1}$ is a unit modulo $\mathfrak{n}$, then*

$$c_\sigma(\phi;\alpha\eta)=c_\sigma(\phi;\alpha)\cdot\lambda_\phi(\mathfrak{n}).$$

Proof. This follows immediately from Proposition 3.21 and Proposition 3.20. □

Corollary 3.22 provides significant information about the Whittaker coefficients. In the classical case of $F=\mathbb{Q}$ for [Sou10], it is the statement of Weak Multiplicity One; in other words, all Whittaker coefficients correspond directly to Hecke eigenvalues, up to a constant depending only on ϕ. However, for a general number field F, the Whittaker coefficients $c_\sigma(\phi;\alpha)$ behave differently for Hecke operators of ramified ideals or ideals dividing $\mathfrak{N}$, so the coefficients cannot be exactly identified with Hecke eigenvalues. Nonetheless, the relation in Corollary 3.22 between Whittaker coefficients and Hecke eigenvalues will fundamentally drive the material in the following chapter and the proof of the main result.

In order to provide natural definitions for the following chapter, we rearrange the Whittaker expansion in a more practical form via the following lemma.

Lemma 3.23. *Let ϕ be a Hecke-Mass form of $\Gamma_0(\mathfrak{N})$ and σ be a cusp. Then*

$$c_\sigma(\phi;\alpha) = c_\sigma(\phi;\epsilon\alpha)$$

for $\epsilon \in V$ and $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$, where $V \subseteq \mathcal{O}^\times$ is as in Proposition 2.6.

Proof. Since

$$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda \subseteq \Gamma_0(\mathfrak{N})^{(\sigma)}$$

for $\epsilon \in V$, we have $\phi_\sigma(\epsilon \cdot \mathbf{z}) = \phi_\sigma(\mathbf{z})$. Applying the Whittaker expansion, and sending $\alpha \mapsto \alpha\epsilon^{-1}$, we see

$$\phi(\epsilon \cdot \mathbf{z}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}} c_\sigma(\phi;\alpha) W(\mathbf{s};\alpha\epsilon \cdot \mathbf{z}) = \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}} c_\sigma(\phi;\alpha\epsilon^{-1}) W(\mathbf{s};\alpha \cdot \mathbf{z})$$

Comparing with $\phi(\mathbf{z})$, we deduce $c_\sigma(\phi;\alpha) = c_\sigma(\phi;\alpha\epsilon^{-1})$. □

Fix a finite set of representatives ξ for $\mathcal{O}^\times/V$ and a single generator $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}$ for every principal fractional ideal $\mathfrak{n}$ of $(\mathfrak{D}\mathfrak{N})^{-1}$. Then, by Lemma 3.23 we may define the $\mathfrak{n}$-*Whittaker ideal coefficient (of ϕ at cusp σ for ξ)*

$$c_\sigma^{(\xi)}(\phi;\mathfrak{n}) := c_\sigma(\phi;\xi\alpha)$$

yielding the following corollary.

Corollary 3.24. *Let ϕ be a Hecke-Mass form of $\Gamma_0(\mathfrak{N})$ and σ be a cusp. Then*

$$\phi_\sigma(\mathbf{z}) = \sum_{\xi \in \mathcal{O}^\times/V} \sum_{\substack{\mathfrak{n} \subseteq (\mathfrak{D}\mathfrak{N})^{-1} \\ \mathfrak{n} = (\alpha) \neq (0)}} c_\sigma^{(\xi)}(\phi;\mathfrak{n}) \Big(\sum_{\epsilon \in V} W(\mathbf{s};\epsilon\xi\alpha \cdot \mathbf{z}) \Big)$$

where the infinite sums are absolutely and uniformly convergent. Further, if $\mathfrak{n} \in \mathcal{N}_\mathfrak{N}$ and $\mathfrak{m}$ is a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$ such that $\mathfrak{m}$ and $\mathfrak{n}$ are coprime, then

$$c_\sigma^{(\xi)}(\phi;\mathfrak{m}\mathfrak{n}) = c_\sigma^{(\xi)}(\phi;\mathfrak{m})\lambda_\phi(\mathfrak{n}) \quad \text{for } \xi \in \mathcal{O}^\times/V.$$

Proof. The Whittaker ideal expansion is immediate from Lemma 3.23. The absolute and uniform convergence is inherited from Corollary 3.11. The last property is a restatement of Corollary 3.22. □

Chapter 4

Mock $\mathcal{P}$-Hecke Multiplicative Functions

The purpose of this chapter is to abstract functions satisfying the properties of Whittaker coefficients and Hecke eigenvalues derived in Proposition 3.20 and Corollary 3.24, and then analyze the relevant growth measures of these functions which shall arise in the proof eliminating escape of mass. The structure and approach will closely follow Soundararajan [Sou10] with which we will provide direct comparisons to our results.

4.1 Statement of Main Theorem

Throughout this chapter, the level $\mathfrak{N}$ is an integral ideal, and $\mathcal{P}$ shall denote a fixed subset of *unramified* prime ideals of F *not dividing* $\mathfrak{N}$. Denote the set of *$\mathcal{P}$-friable ideals* by

$$\mathcal{N} = \mathcal{N}(\mathcal{P}) := \{\mathfrak{n} \subseteq \mathcal{O} : \mathfrak{p} \mid \mathfrak{n} \implies \mathfrak{p} \in \mathcal{P}\}.$$

Note the ideal $(1) = \mathcal{O} \in \mathcal{N}$ vacuously and $(0) \notin \mathcal{N}$. We begin with a definitions motivated by the properties from Proposition 3.20.

Definition 4.1 ($\mathcal{P}$-Hecke Multiplicative)**.** For a subset $\mathcal{P}$ of unramified prime ideals of F, a function $f_{\mathcal{P}} : \mathcal{N} \to \mathbb{C}$ is *$\mathcal{P}$-Hecke multiplicative (of level $\mathfrak{N}$)* if

(i) $f_{\mathcal{P}}(\mathcal{O}) = 1$

(ii) $f_{\mathcal{P}}(\mathfrak{m}) f_{\mathcal{P}}(\mathfrak{n}) = \sum_{\mathfrak{d} \mid (\mathfrak{m},\mathfrak{n})} f_{\mathcal{P}}\left(\frac{\mathfrak{m}\mathfrak{n}}{\mathfrak{d}^2}\right)$

Remark. If $\mathcal{P} = \mathcal{P}_{\mathfrak{N}} = \{\mathfrak{p} \subseteq \mathcal{O} : \mathfrak{p} \text{ unramified principal prime ideals}, \mathfrak{p} \nmid \mathfrak{N}\}$, then for a Hecke-Maaß cusp form of $\Gamma_0(\mathfrak{N})$, the Hecke eigenvalues $\lambda_\phi : \mathcal{N} \to \mathbb{C}$ are $\mathcal{P}$-Hecke multiplicative by Proposition 3.20.

Remark. In the proof of the main result, we shall only require two special cases of property (ii): namely, when $\mathfrak{m} = \mathfrak{p}^k$ and $\mathfrak{n} = \mathfrak{p} \in \mathcal{P}$,

$$f_{\mathcal{P}}(\mathfrak{p}^{k+1}) = f_{\mathcal{P}}(\mathfrak{p}^k) f_{\mathcal{P}}(\mathfrak{p}) - f_{\mathcal{P}}(\mathfrak{p}^{k-1}). \tag{4.1}$$

and when $(\mathfrak{m}, \mathfrak{n}) = \mathcal{O}$, we have $f_{\mathcal{P}}(\mathfrak{m}) f_{\mathcal{P}}(\mathfrak{n}) = f_{\mathcal{P}}(\mathfrak{m}\mathfrak{n})$.

The next definition is motivied by the property of Whittaker coefficients in Corollary 3.24, and the fact that Hecke eigenvalues of a Maaß form are $\mathcal{P}$-Hecke multiplicative for an appropriate choice of $\mathcal{P}$.

Definition 4.2 (Mock $\mathcal{P}$-Hecke Multiplicative)**.** Let $\mathcal{P}$ be a set of unramified prime ideals not dividing $\mathfrak{N}$. Suppose f is a $\mathbb{C}$-valued function on the fractional ideals of $(\mathfrak{D}\mathfrak{N})^{-1}$. Then we say f is *mock $\mathcal{P}$-Hecke Multiplicative (of level $\mathfrak{N}$)* if there exists a $\mathcal{P}$-Hecke multiplicative function such that

$$f(\mathfrak{m}\mathfrak{n}) = f(\mathfrak{m}) f_{\mathcal{P}}(\mathfrak{n})$$

for $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m}$ a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$ with $\mathfrak{m}$ and $\mathfrak{n}$ coprime.

Remark. Let ϕ be a Hecke-Maaß cusp form of $\Gamma_0(\mathfrak{N})$, σ be a cusp, and $\xi \in \mathcal{O}^\times / V$. If we define $\mathcal{P} = \mathcal{P}_{\mathfrak{N}} = \{\mathfrak{p} \subseteq \mathcal{O} : \mathfrak{p} \text{ unramified principal prime ideal}, \mathfrak{p} \nmid \mathfrak{N}\}$ as before, and set

$$f(\mathfrak{n}) := \begin{cases} c_\sigma^{(\xi)}(\phi; \mathfrak{n}) & \mathfrak{n} \text{ principal fractional ideal of } (\mathfrak{D}\mathfrak{N})^{-1} \\ 0 & \text{else} \end{cases}$$

then, by Corollary 3.24, f is mock $\mathcal{P}$-Hecke mutliplicative where $f_{\mathcal{P}}(\mathfrak{n}) = \lambda_\phi(\mathfrak{n})$ for $\mathfrak{n} \in \mathcal{N}_{\mathfrak{N}}$.

For a fixed mock $\mathcal{P}$-Hecke multiplicative function f of level $\mathfrak{N}$, our aim is to understand the decay of

$$\sum_{\mathbb{N}\mathfrak{n} \leq y/Y} |f(\mathfrak{n})|^2$$

where $1 \leq Y \leq y$. We shall consider Y to vary, and y to be fixed. We present the main technical theorem.

Theorem 4.3. *Let $\mathcal{P}$ be a set of unramified prime ideals of F not dividing $\mathfrak{N}$, and f be a mock $\mathcal{P}$-Hecke multiplicative function of level $\mathfrak{N}$. If $\mathcal{P}$ has positive natural density, then*

$$\sum_{\mathbb{N}\mathfrak{n}\leq y/Y} |f(\mathfrak{n})|^2 \ll_{\mathcal{P}} \left(\frac{1+\log Y}{\sqrt{Y}}\right) \sum_{\mathbb{N}\mathfrak{n}\leq y} |f(\mathfrak{n})|^2.$$

for $1 \leq Y \leq y$.

This chapter is dedicated to the proof of this result, which is the analogue to Theorem 3 of [Sou10]. Let $y > 1$ be a fixed value henceforth. For $Y \geq 1$, define

$$\mathcal{F}(Y) = \mathcal{F}(Y; y) := \frac{\sum_{\mathbb{N}\mathfrak{n}\leq y/Y} |f(\mathfrak{n})|^2}{\sum_{\mathbb{N}\mathfrak{n}\leq y} |f(\mathfrak{n})|^2}$$

Theorem 4.3 then equivalently asserts that

$$\mathcal{F}(Y) \ll_{\mathcal{P}} \frac{1+\log Y}{\sqrt{Y}} \tag{4.2}$$

provided $\mathcal{P}$ has positive natural density. This chapter's goal is to establish this fact. Observe

- $\mathcal{F}(Y) \leq 1$ for all $Y \geq 1$.
- $\mathcal{F}(Y) = 0$ for $Y > y$.
- $\mathcal{F}$ is a decreasing function of Y.

For convenience, we shall extend the domain of f to all fractional ideals of F by defining $f(\mathfrak{t}) := 0$ if $\mathfrak{t}$ is not a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$.

4.2 Preliminary Lemmas

We begin with a simple lemma, utilized frequently by Soundararajan throughout [Sou10]; this is essentially an immediate consequence of the Hecke relations of $f_{\mathcal{P}}$.

Lemma 4.4. *Let f be mock $\mathcal{P}$-Hecke multiplicative of level $\mathfrak{N}$. Suppose $\mathfrak{p} \in \mathcal{P}$ and $\mathfrak{n}$ is a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$. Then*

$$f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}) = \begin{cases} f(\mathfrak{n}\mathfrak{p}) & \mathfrak{p} \nmid \mathfrak{n} \\ f(\mathfrak{n}\mathfrak{p}) + f(\mathfrak{n}\mathfrak{p}^{-1}) & \mathfrak{p} \mid \mathfrak{n} \end{cases} \tag{4.3}$$

so in particular

$$|f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p})| \leq |f(\mathfrak{n}\mathfrak{p})| + |f(\mathfrak{n}\mathfrak{p}^{-1})|, \tag{4.4}$$

and

$$|f(\mathfrak{n}\mathfrak{p})| \leq |f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p})| + |f(\mathfrak{n}\mathfrak{p}^{-1})|. \tag{4.5}$$

Similarly,

$$f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}^2) = \begin{cases} f(\mathfrak{n}\mathfrak{p}^2) & \mathfrak{p} \nmid \mathfrak{n} \\ f(\mathfrak{n}\mathfrak{p}^2) + f(\mathfrak{n}) & \mathfrak{p} \parallel \mathfrak{n} \\ f(\mathfrak{n}\mathfrak{p}^2) + f(\mathfrak{n}) + f(\mathfrak{n}\mathfrak{p}^{-2}) & \mathfrak{p}^2 \mid \mathfrak{n} \end{cases} \tag{4.6}$$

so in particular

$$|f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}^2)| \leq |f(\mathfrak{n}\mathfrak{p}^2)| + |f(\mathfrak{n})| + |f(\mathfrak{n}\mathfrak{p}^{-2})|. \tag{4.7}$$

and

$$|f(\mathfrak{n}\mathfrak{p}^2)| \leq |f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}^2)| + |f(\mathfrak{n})| + |f(\mathfrak{n}\mathfrak{p}^{-2})|. \tag{4.8}$$

Proof. Note that since $\mathfrak{p} \nmid \mathfrak{D}\mathfrak{N}$, the ideal $\mathfrak{n}$ is integral in $\mathcal{O}_{\mathfrak{p}}$, so it is valid to only consider cases where $\mathfrak{p}$ does or does not divide $\mathfrak{n}$. Write $\mathfrak{n} = \mathfrak{p}^k\mathfrak{m}$ where $k \geq 0$ and $\mathfrak{m}$ is a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$ prime to $\mathfrak{p}$. Then the mock $\mathcal{P}$-Hecke multiplicativity of f, and Hecke multiplicativity of $f_{\mathcal{P}}$ from (4.1) gives

$$\begin{aligned} f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}) &= f(\mathfrak{m})f_{\mathcal{P}}(\mathfrak{p}^k)f_{\mathcal{P}}(\mathfrak{p}) \\ &= f(\mathfrak{m}) \cdot \begin{cases} f_{\mathcal{P}}(\mathfrak{p}) & k = 0 \\ f_{\mathcal{P}}(\mathfrak{p}^{k+1}) + f_{\mathcal{P}}(\mathfrak{p}^{k-1}) & k \geq 1 \end{cases} \\ &= \begin{cases} f(\mathfrak{n}\mathfrak{p}) & k = 0 \\ f(\mathfrak{n}\mathfrak{p}) + f(\mathfrak{n}\mathfrak{p}^{-1}) & k \geq 1 \end{cases} \end{aligned}$$

as required. Similarly, we have

$$\begin{aligned} f(\mathfrak{n})f_{\mathcal{P}}(\mathfrak{p}^2) &= f(\mathfrak{m})f_{\mathcal{P}}(\mathfrak{p}^k)f_{\mathcal{P}}(\mathfrak{p}^2) \\ &= f(\mathfrak{m}) \cdot \begin{cases} f_{\mathcal{P}}(\mathfrak{p}^2) & k = 0 \\ f_{\mathcal{P}}(\mathfrak{p}^3) + f_{\mathcal{P}}(\mathfrak{p}) & k = 1 \\ f_{\mathcal{P}}(\mathfrak{p}^{k+1}) + f_{\mathcal{P}}(\mathfrak{p}^{k-1}) & k \geq 2 \end{cases} \\ &= \begin{cases} f(\mathfrak{n}\mathfrak{p}^2) & k = 0 \\ f(\mathfrak{n}\mathfrak{p}^2) + f(\mathfrak{n}) & k = 1 \\ f(\mathfrak{n}\mathfrak{p}) + f(\mathfrak{n}\mathfrak{p}^{-1}) & k \geq 2 \end{cases} \end{aligned}$$

as required. $\square$

The next lemma gives an easy bound of the magnitude of $f_{\mathcal{P}}$ in relation to $\mathcal{F}$ at prime ideals $\mathfrak{p} \in \mathcal{P}$. To be brief, the bound is derived from a combination of Lemma 4.4 and an application of Cauchy-Schwarz.

Lemma 4.5. *Suppose* $\mathfrak{p} \in \mathcal{P}$. *If* $\mathbb{N}\mathfrak{p} \leq y$, *then*

$$|f_{\mathcal{P}}(\mathfrak{p})| \leq \frac{2}{\mathcal{F}(\mathbb{N}\mathfrak{p})^{1/2}}$$

and, if $\mathbb{N}\mathfrak{p} \leq \sqrt{y}$, *then*

$$|f_{\mathcal{P}}(\mathfrak{p})| \leq \frac{2}{\mathcal{F}(\mathbb{N}\mathfrak{p}^2)^{1/4}}.$$

Proof. Compare with Lemma 3.1 of [Sou10].

Let $q := \mathbb{N}\mathfrak{p}$ and $\mathfrak{n}$ be any fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$. For the first bound, observe that (4.4) of Lemma 4.4 implies $|f_{\mathcal{P}}(\mathfrak{p})f(\mathfrak{n})|^2 \leq 2(|f(\mathfrak{n}\mathfrak{p})|^2 + |f(\mathfrak{n}\mathfrak{p}^{-1})|^2)$ by Cauchy-Schwarz. Applying this inequality, we find

$$|f_{\mathcal{P}}(\mathfrak{p})|^2 \sum_{\mathbb{N}\mathfrak{n} \leq y/q} |f(\mathfrak{n})|^2 \leq 2 \sum_{\mathbb{N}\mathfrak{n} \leq y/q} (|f(\mathfrak{n}\mathfrak{p})|^2 + |f(\mathfrak{n}\mathfrak{p}^{-1})|^2).$$

Since $\mathbb{N}\mathfrak{p} = q$, notice $\mathbb{N}\mathfrak{n}\mathfrak{p} \leq y$ and $\mathbb{N}\mathfrak{n}\mathfrak{p}^{-1} \leq y$ provided $\mathbb{N}\mathfrak{n} \leq y/q$, so the sum on the RHS is

$$\leq 4 \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2$$

as required.

Similarly, for the second bound, equation (4.4) of Lemma 4.4 implies $|f_{\mathcal{P}}(\mathfrak{p})f(\mathfrak{n})|^2 \leq 3(|f(\mathfrak{n}\mathfrak{p}^2)^2 + |f(\mathfrak{n})|^2 + |f(\mathfrak{n}\mathfrak{p}^{-2})|^2)$ by Cauchy-Schwarz. Applying this inequality, we find

$$|f_{\mathcal{P}}(\mathfrak{p}^2)|^2 \sum_{\mathbb{N}\mathfrak{n} \leq y/q^2} |f(\mathfrak{n})|^2 \leq 3 \sum_{\mathbb{N}\mathfrak{n} \leq y/q^2} (|f(\mathfrak{n}\mathfrak{p}^2)|^2 + |f(\mathfrak{n})|^2 + |f(\mathfrak{n}\mathfrak{p}^{-2})|^2).$$

Since $\mathbb{N}\mathfrak{p} = q$, notice the norms of $\mathfrak{n}\mathfrak{p}^2, \mathfrak{n}$ and $\mathfrak{n}\mathfrak{p}^{-2}$ are $\leq y$ provided $\mathbb{N}\mathfrak{n} \leq y/q^2$, so the sum on the RHS is

$$\leq 9 \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.$$

Combining the two above inequalities, we conclude

$$|f_{\mathcal{P}}(\mathfrak{p}^2)|^2 \leq \frac{9}{\mathcal{F}(\mathbb{N}\mathfrak{p}^2)}$$

Then by the Hecke relations given in (4.1),

$$|f_{\mathcal{P}}(\mathfrak{p})|^2 \leq |f_{\mathcal{P}}(\mathfrak{p}^2)| + 1 \leq \frac{3}{\mathcal{F}(\mathbb{N}\mathfrak{p}^2)^{1/2}} + 1$$

Since $\mathcal{F}(\mathbb{N}\mathfrak{p}^2) \leq 1$, we have $1 \leq \mathcal{F}(\mathbb{N}\mathfrak{p}^2)^{-1/2}$, yielding the result. □

Proposition 4.6. *Suppose $\mathfrak{a} \in \mathcal{N}$ is a square-free integral ideal. Then*

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{a}|\mathfrak{n}}} |f(\mathfrak{n})|^2 \leq \tau(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} (1 + |f_{\mathcal{P}}(\mathfrak{p})|^2) \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2$$

where $\tau(\mathfrak{a})$ denotes the number of integral ideals dividing $\mathfrak{a}$. Furthermore,

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{a}^2|\mathfrak{n}}} |f(\mathfrak{n})|^2 \leq \tau_3(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} (2 + |f_{\mathcal{P}}(\mathfrak{p})|^2) \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}^2) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2$$

where $\tau_3(\mathfrak{a})$ denotes the number of ways of writing $\mathfrak{a}$ as a product of three integral ideals.

Proof. Compare with Proposition 3.2 of [Sou10].

Let $\mathfrak{m}$ be a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$. Using (4.5) from Lemma 4.4, we may find by induction that

$$|f(\mathfrak{m}\mathfrak{a})| \leq \sum_{\mathfrak{s}\mathfrak{t}=\mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{s})||f(\mathfrak{m}\mathfrak{t}^{-1})|$$

where the sum runs over integral ideals $\mathfrak{s}$ and $\mathfrak{t}$. Note that in the above inductive argument, we utilize that $\mathfrak{a}$ is square-free to simplify a product of distinct prime divisors of $\mathfrak{a}$ into $f_{\mathcal{P}}(\mathfrak{s})$. Now, from the above inequality, it follows by Cauchy-Schwarz that

$$|f(\mathfrak{m}\mathfrak{a})|^2 \leq \tau(\mathfrak{a}) \sum_{\mathfrak{s}\mathfrak{t}=\mathfrak{a}} |f(\mathfrak{s})|^2 |f(\mathfrak{m}\mathfrak{t}^{-1})|^2$$

where $\tau(\mathfrak{a})$ denotes the number of integral ideal divisors of $\mathfrak{a}$. Summing this inequality over all $\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})$ and commuting the sums, we have

$$\begin{aligned}
\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{a}|\mathfrak{n}}} |f(\mathfrak{n})|^2 &= \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})} |f(\mathfrak{m}\mathfrak{a})|^2 \\
&\leq \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})} \tau(\mathfrak{a}) \sum_{\mathfrak{s}\mathfrak{t}=\mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{s})|^2 |f(\mathfrak{m}\mathfrak{t}^{-1})|^2 \\
&= \tau(\mathfrak{a}) \sum_{\mathfrak{s}\mathfrak{t}=\mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{s})|^2 \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})} |f(\mathfrak{m}\mathfrak{t}^{-1})|^2.
\end{aligned}$$

Recall that $f(\mathfrak{m}\mathfrak{t}^{-1}) = 0$ for $\mathfrak{t} \nmid \mathfrak{m}$ by convention. Thus, for $\mathfrak{t} \mid \mathfrak{a}$ we have $\mathbb{N}\mathfrak{t} \geq 1$ so we may bound the inner sum as follows:

$$\begin{aligned}
\sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})} |f(\mathfrak{m}\mathfrak{t}^{-1})|^2 &= \sum_{\substack{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}) \\ \mathfrak{t} \mid \mathfrak{m}}} |f(\mathfrak{m}\mathfrak{t}^{-1})|^2 \\
&= \sum_{\mathbb{N}\mathfrak{m}' \leq y/(Y \cdot \mathbb{N}\mathfrak{a}\mathfrak{t})} |f(\mathfrak{m}')|^2 \\
&\leq \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a})} |f(\mathfrak{m})|^2 \\
&= \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.
\end{aligned}$$

which is independent of $\mathfrak{t}$. Using this bound in the previous inequality, we find

$$\sum_{\substack{\mathfrak{n} \leq y/Y \\ \mathfrak{a} \mid \mathfrak{n}}} |f(\mathfrak{n})|^2 \leq \tau(\mathfrak{a}) \left(\sum_{\mathfrak{s} \mid \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{s})|^2 \right) \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.$$

For the sum over $\mathfrak{s} \mid \mathfrak{a}$, we use the Hecke multiplicativity of $f_{\mathcal{P}}$ to deduce

$$\sum_{\mathfrak{s} \mid \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{s})|^2 = \prod_{\mathfrak{p} \mid \mathfrak{a}} (1 + f_{\mathcal{P}}(\mathfrak{p}))^2$$

because $\mathfrak{a}$ is square-free. Substituting this into the former inequality, we have the desired first bound.

Similarly, for the second bound, let $\mathfrak{m}$ be a fractional ideal of $(\mathfrak{D}\mathfrak{N})^{-1}$. Using (4.8) from Lemma 4.4, we may find by induction that

$$|f(\mathfrak{m}\mathfrak{a}^2)| \leq \sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)| |f(\mathfrak{m}\mathfrak{t}^{-2})|$$

where the sum runs over integral ideals $\mathfrak{r}.\mathfrak{s}$ and $\mathfrak{t}$. Again, in the above inductive argument, we utilize that $\mathfrak{a}$ is square-free to simplify a product of distinct prime divisors of $\mathfrak{a}$ into $f_{\mathcal{P}}(\mathfrak{r}^2)$. Then from the above inequality, it follows by Cauchy-Schwarz that

$$|f(\mathfrak{m}\mathfrak{a}^2)|^2 \leq \tau_3(\mathfrak{a}) \sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 |f(\mathfrak{m}\mathfrak{t}^{-2})|^2$$

where $\tau_3(\mathfrak{a})$ denotes the number of integral ideal triples $(\mathfrak{r}, \mathfrak{s}, \mathfrak{t})$ such that $\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}$. Summing this inequality over all $\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)$ and commuting the sums, we have

$$\begin{aligned}\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{a}^2 | \mathfrak{n}}} |f(\mathfrak{n})|^2 &= \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)} |f(\mathfrak{m}\mathfrak{a}^2)|^2 \\ &\leq \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)} \tau_3(\mathfrak{a}) \sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 |f(\mathfrak{m}\mathfrak{t}^{-2})|^2 \\ &= \tau_3(\mathfrak{a}) \sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)} |f(\mathfrak{m}\mathfrak{t}^{-2})|^2.\end{aligned}$$

Recall that $f(\mathfrak{m}\mathfrak{t}^{-2}) = 0$ for $\mathfrak{t}^2 \nmid \mathfrak{m}$ by convention. Thus, for $\mathfrak{t} \mid \mathfrak{a}$ we have $\mathbb{N}\mathfrak{t} \geq 1$ so we may bound the inner sum as follows:

$$\begin{aligned}\sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)} |f(\mathfrak{m}\mathfrak{t}^{-2})|^2 &= \sum_{\substack{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2) \\ \mathfrak{t}^2 | \mathfrak{m}}} |f(\mathfrak{m}\mathfrak{t}^{-2})|^2 \\ &= \sum_{\mathbb{N}\mathfrak{m}' \leq y/(Y \cdot \mathbb{N}(\mathfrak{a}\mathfrak{t})^2)} |f(\mathfrak{m}')|^2 \\ &\leq \sum_{\mathbb{N}\mathfrak{m} \leq y/(Y \cdot \mathbb{N}\mathfrak{a}^2)} |f(\mathfrak{m})|^2 \\ &= \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}^2) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.\end{aligned}$$

which is independent of $\mathfrak{t}$. Using this bound in the previous inequality, we find

$$\sum_{\substack{\mathfrak{n} \leq y/Y \\ \mathfrak{a}^2 | \mathfrak{n}}} |f(\mathfrak{n})|^2 \leq \tau_3(\mathfrak{a}) \left(\sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 \right) \mathcal{F}(Y \cdot \mathbb{N}\mathfrak{a}^2) \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2. \tag{4.9}$$

For the sum over triples $\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}$, we see that

$$\sum_{\mathfrak{r}\mathfrak{s}\mathfrak{t} = \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 = \sum_{\mathfrak{r} | \mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 \tau(\mathfrak{a}\mathfrak{r}^{-1})$$

Since $f_{\mathcal{P}}$ is Hecke-multiplicative and $\mathfrak{a}$ is square-free, we may write each term in the above sum as

$$|f_{\mathcal{P}}(\mathfrak{r}^2)|^2 \tau(\mathfrak{a}\mathfrak{r}^{-1}) = \prod_{\mathfrak{p} | \mathfrak{r}} |f_{\mathcal{P}}(\mathfrak{p}^2)|^2 \prod_{\mathfrak{p} | \mathfrak{a}\mathfrak{r}^{-1}} 2.$$

Using this identity and the former equation. we deduce

$$\sum_{\mathfrak{rst}=\mathfrak{a}} |f_{\mathcal{P}}(\mathfrak{r}^2)|^2 = \prod_{\mathfrak{p}|\mathfrak{a}} (2 + |f_{\mathcal{P}}(\mathfrak{p}^2)|^2).$$

Substituting this into the RHS of (4.9), we have the desired result. □

4.3 A Large Set of Prime Ideals

Thus far, we have not made any assumption about the size of $\mathcal{P}$, our set of prime ideals. In this section, we shall assume $\mathcal{P}$ has positive natural density in the set of prime ideals of F. Specifically, we assume that the following limit

$$\delta = \delta(\mathcal{P}) := \lim_{t\to\infty} \frac{\#\{\mathfrak{p} \in \mathcal{P} : \mathbb{N}\mathfrak{p} \leq t\}}{\#\{\mathfrak{p} \text{ prime ideal} : \mathbb{N}\mathfrak{p} \leq t\}}$$

exists and is positive, i.e. $\delta > 0$. It is well known [Lan94, p. 315] that this is equivalent to

$$\#\{\mathfrak{p} \in \mathcal{P} : \mathbb{N}\mathfrak{p} \leq t\} \sim \frac{\delta t}{\log t}.$$

In Chapter 5, we will require the natural density of $\mathcal{P}_{\mathfrak{N}}$.

Proposition 4.7. *The set of ideals*

$$\mathcal{P}_{\mathfrak{N}} = \{\mathfrak{p} \subseteq \mathcal{O} : \mathfrak{p} \text{ unramified principal prime ideal with } \mathfrak{p} \nmid \mathfrak{N}\}$$

has natural density equal to $1/h_F$ *where* h_F *is the class number of* F.

Proof. Without loss, we may show the set of principal prime ideals of F has natural density $1/h_F$ as this differs from $\mathcal{P}_{\mathfrak{N}}$ by only finitely many prime ideals. Let L be the Hilbert class field of F. Recall a prime ideal $\mathfrak{p}$ is principal in F if and only if it splits completely in L [Neu99, p. 409], which occurs if and only if its associated Frobenius element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(L/F)$ is trivial. Thus, by Chebatorev's density theorem [Hei67], the set of principal prime ideals of F has natural density equal to $\frac{\#\{1\}}{\#\mathrm{Gal}(L/F)} = \frac{1}{h_F}$. □

Now, we return to our generic set of prime ideals $\mathcal{P}$ with positive natural density. For $Y \geq 1$, define

$$\mathcal{P}(Y) := \{\mathfrak{p} \in \mathcal{P} : \mathbb{N}\mathfrak{p} \in [\sqrt{Y}/2, \sqrt{Y}]\}.$$

Since $\mathcal{P}$ has positive density δ, we have

$$\#\mathcal{P}(Y) \geq \frac{\delta\sqrt{Y}}{2\log Y}, \quad \text{for } Y \geq c_{\mathcal{P}} \tag{4.10}$$

where $c_{\mathcal{P}}$ is some positive constant. In other words, we have a large set of primes, $\mathcal{P}(Y)$.

We wish to appropriately divide this large set $\mathcal{P}(Y)$ according to values related to our mock $\mathcal{P}$-Hecke multiplicative function f. The following construction mimics the discussion in [Sou10] preceding Proposition 3.3. We recall from Lemma 4.5 that for $\mathbb{N}\mathfrak{p} \leq \sqrt{Y}$,

$$|f_{\mathcal{P}}(\mathfrak{p})| \leq \frac{2}{\mathcal{F}(Y)^{1/4}}.$$

Therefore, setting

$$J := \left[\frac{1}{4\log 2}\log(1/\mathcal{F}(Y))\right] + 3 \tag{4.11}$$

we see that $0 \leq |f_{\mathcal{P}}(\mathfrak{p})| \leq 2^{J-1}$ for all $\mathfrak{p} \in \mathcal{P}(Y)$. We may thus partition $\mathcal{P}(Y)$ into sets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_J$ such that

$$\mathcal{P}_0 = \mathcal{P}_0(Y) = \{\mathfrak{p} \in \mathcal{P}(Y) : |f_{\mathcal{P}}(\mathfrak{p})| \leq 2^{-1}\}$$

and for $1 \leq j \leq J$,

$$\mathcal{P}_j = \mathcal{P}_j(Y) = \{\mathfrak{p} \in \mathcal{P}(Y) : 2^{j-2} < |f_{\mathcal{P}}(\mathfrak{p})| \leq 2^{j-1}\}.$$

For $k \geq 1$, define $\mathcal{N}_0(k)$ to be the set of *fractional* ideals of $(\mathfrak{D}\mathfrak{N})^{-1}$ divisible by at most k distinct *squares of prime ideals in* $\mathcal{P}_0$. For $1 \leq j \leq J$, define $\mathcal{N}_j(k)$ to be the set of *fractional* ideals of $(\mathfrak{D}\mathfrak{N})^{-1}$ divisible by at most k distinct prime ideals in $\mathcal{P}_j$. The notion of divisibilty is well-defined as $\mathcal{P}$ consists of *unramified* prime ideals.

Proposition 4.8. *Keep the notations above. For $2 \leq k \leq |\mathcal{P}_0|/4$, we have*

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} |f(\mathfrak{n})|^2 \leq \frac{4k}{|\mathcal{P}_0|} \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.$$

Further, if $1 \leq j \leq J$, and $1 \leq k \leq |P_j|/4 - 1$, we have

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \leq \frac{2^{12}k^2}{2^{4j}|\mathcal{P}_j|^2} \sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2.$$

Proof. Compare with Proposition 3.3 of [Sou10].

Since $|f_{\mathcal{P}}(\mathfrak{p})| \le 1/2$ for $\mathfrak{p} \in \mathcal{P}_0$, we have by the Hecke relations $|f_{\mathcal{P}}(\mathfrak{p}^2)| = |f_{\mathcal{P}}(\mathfrak{p}^2) - 1| \ge 3/4$. Thus,

$$\begin{aligned}
\sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} |f(\mathfrak{n})|^2 \Big(\sum_{\substack{\mathfrak{p} \in \mathcal{P}_0 \\ \mathfrak{p}^2 \nmid \mathfrak{n}}} |f_{\mathcal{P}}(\mathfrak{p}^2)|^2 \Big) &\ge \sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} |f(\mathfrak{n})|^2 \Big(\sum_{\substack{\mathfrak{p} \in \mathcal{P}_0 \\ \mathfrak{p}^2 \nmid \mathfrak{n}}} \frac{9}{16} \Big) \\
&\ge \frac{9}{16} (|\mathcal{P}_0| - k) \sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} |f(\mathfrak{n})|^2 \qquad (4.12) \\
&\ge \frac{27}{64} |\mathcal{P}_0| \sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} |f(\mathfrak{n})|^2.
\end{aligned}$$

In the last step, we have noted $k \le |\mathcal{P}_0|/4$. To achieve an upper bound on the LHS of (4.12), we first claim that for $\mathfrak{p} \in \mathcal{P}_0(\mathfrak{n})$ and $\mathfrak{p}^2 \nmid \mathfrak{n}$, we have $|f(\mathfrak{n}) f_{\mathcal{P}}(\mathfrak{p}^2)| \le |f(\mathfrak{n}\mathfrak{p}^2)|$. For $\mathfrak{p} \nmid \mathfrak{n}$, we have equality by (4.6) of Lemma 4.4. For $\mathfrak{p} \parallel \mathfrak{n}$, again by (4.6), we see $f(\mathfrak{n})(f_{\mathcal{P}}(\mathfrak{p}^2) - 1) = f(\mathfrak{n}\mathfrak{p}^2)$. Since $|f_{\mathcal{P}}(\mathfrak{p}^2)| \le 1/2$, it follows that $|f(\mathfrak{n}) f_{\mathcal{P}}(\mathfrak{p}^2)| \le |f(\mathfrak{n})(f_{\mathcal{P}}(\mathfrak{p}^2) - 1)| = |f(\mathfrak{n}\mathfrak{p}^2)|$. This proves the claim. Thus, the LHS of (4.12) is

$$\le \sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_0(k)}} \sum_{\substack{\mathfrak{p} \in \mathcal{P}_0 \\ \mathfrak{p}^2 \nmid \mathfrak{n}}} |f(\mathfrak{n}\mathfrak{p}^2)|^2.$$

In the above sum, $\mathfrak{p} \in \mathcal{P}_0$, we have $\mathbb{N}\mathfrak{p}^2 \le Y$, and so $\mathbb{N}\mathfrak{n}\mathfrak{p}^2 \le y$. Also, $\mathfrak{n} \in \mathcal{N}_0(k)$ implies that the product $\mathfrak{n}\mathfrak{p}^2$ is divisible by at most $k+1$ distinct squares of prime ideals of $\mathcal{P}_0$. Thus, in the sum, the terms are of the form $|f(\mathfrak{m})|^2$ where $\mathbb{N}\mathfrak{m} \le y$ and $\mathfrak{m}$ is divisible by at most $k+1$ distinct squares of prime ideals in $\mathcal{P}_0$; furthermore, each such term appears at most $k+1$ times. Therefore, the above is

$$= \sum_{\mathbb{N}\mathfrak{m} \le y} |f(\mathfrak{m})|^2 \Big(\sum_{\substack{\mathfrak{m} = \mathfrak{n}\mathfrak{p}^2 \\ \mathbb{N}\mathfrak{n} \le y/Y, \mathfrak{n} \in \mathcal{N}_0(k) \\ \mathfrak{p} \in \mathcal{P}_0, \mathfrak{p}^2 \nmid \mathfrak{n}}} 1 \Big) \le (k+1) \sum_{\mathbb{N}\mathfrak{m} \le y} |f(\mathfrak{m})|^2. \qquad (4.13)$$

For the second assertion, we argue similarly. Since $|f_{\mathcal{P}}(\mathfrak{p})| \ge 2^{j-2}$ for $\mathfrak{p} \in \mathcal{P}_j$, by Hecke multiplictativity, we have $|f_{\mathcal{P}}(\mathfrak{p}_1\mathfrak{p}_2)| = |f_{\mathcal{P}}(\mathfrak{p}_1) f_{\mathcal{P}}(\mathfrak{p}_2)| \ge 2^{2j-4}$ for distinct $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}_j$. Therefore,

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \Big(\frac{1}{2} \sum_{\substack{\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}_j \\ \mathfrak{p}_1 \ne \mathfrak{p}_2 \\ \mathfrak{p}_1 \nmid \mathfrak{n},\ \mathfrak{p}_2 \nmid \mathfrak{n}}} |f_{\mathcal{P}}(\mathfrak{p}_1\mathfrak{p}_2)|^2 \Big) \ge \sum_{\substack{\mathbb{N}\mathfrak{n} \le y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \Big(\sum_{\substack{\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}_j \\ \mathfrak{p}_1 \ne \mathfrak{p}_2 \\ \mathfrak{p}_1 \nmid \mathfrak{n},\ \mathfrak{p}_2 \nmid \mathfrak{n}}} 2^{4j-9} \Big) \qquad (4.14)$$

The inner sum on the RHS counts the number of ordered pairs of distinct prime ideals in $\mathcal{P}_j$ not dividing $\mathfrak{n}$. By definition, $\mathfrak{n} \in \mathcal{N}_j(k)$ is divisible by at most k distinct prime ideals of $\mathcal{P}_j$, and so there are at least $|\mathcal{P}_j| - k$ prime ideals of $\mathcal{P}_j$ not dividing $\mathfrak{n}$. Thus, there are $2\binom{|\mathcal{P}_j|-k}{2}$ terms in the inner sum, so the RHS

$$\begin{aligned}
&\geq 2^{2j-9} \cdot 2\binom{|\mathcal{P}_j| - k}{2} \sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \\
&= 2^{2j-9}(|\mathcal{P}_j| - k)(|\mathcal{P}_j| - k - 1) \sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \\
&\geq 2^{2j-9} \cdot \frac{9}{16}|\mathcal{P}_j|^2 \sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2
\end{aligned} \tag{4.15}$$

since $2 \leq k \leq |\mathcal{P}_j|/4 - 1$.

As for an upper bound, notice that since f is mock $\mathcal{P}$-Hecke multiplicative, the LHS of (4.14)

$$= \frac{1}{2} \sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} \sum_{\substack{\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}_j \\ \mathfrak{p}_1 \neq \mathfrak{p}_2 \\ \mathfrak{p}_1 \nmid \mathfrak{n},\ \mathfrak{p}_2 \nmid \mathfrak{n}}} |f(\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2)|^2.$$

In the above sum, since $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}_j$, we have $\mathbb{N}\mathfrak{p}_\ell \leq \sqrt{Y}$ for $\ell = 1, 2$, and so $\mathbb{N}\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2 \leq y$. Also, $\mathfrak{n} \in \mathcal{N}_j(k)$ implies that the product $\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2$ is divisible by at most $k+2$ distinct prime ideals of $\mathcal{P}_j$. Thus, in the sum, the terms are of the form $|f(\mathfrak{m})|^2$ where $\mathbb{N}\mathfrak{m} \leq y$ and $\mathfrak{m}$ is divisible by at most $k+2$ distinct prime ideals in $\mathcal{P}_j$; furthermore, each such term appears at most $2\binom{k+2}{2}$ times. Therefore, the above expression is

$$\leq \binom{k+2}{2} \sum_{\mathbb{N}\mathfrak{m} \leq y} |f(\mathfrak{m})|^2 \leq 3k^2 \sum_{\mathbb{N}\mathfrak{m} \leq y} |f(\mathfrak{m})|^2.$$

Combining this result with (4.15), we have

$$\sum_{\substack{\mathbb{N}\mathfrak{n} \leq y/Y \\ \mathfrak{n} \in \mathcal{N}_j(k)}} |f(\mathfrak{n})|^2 \leq \frac{2^{13}k^2}{3 \cdot 2^{2j}|\mathcal{P}_j|^2} \sum_{\mathbb{N}\mathfrak{m} \leq y} |f(\mathfrak{m})|^2 \leq \frac{2^{12}k^2}{2^{2j}|\mathcal{P}_j|^2} \sum_{\mathbb{N}\mathfrak{m} \leq y} |f(\mathfrak{m})|^2$$

as desired. □

We have thus established the necessary facts to prove Theorem 4.3, the key technical result.

4.4 Proof of Theorem 4.3

Proof of Theorem 4.3. Compare with section 4 of [Sou10].

We shall utilize the notation of this chapter, and in particular, for a fixed value $y \geq 1$, recall that we defined

$$\mathcal{F}(Y) = \mathcal{F}(Y; y) := \frac{\sum_{\mathbb{N}\mathfrak{n} \leq y/Y} |f(\mathfrak{n})|^2}{\sum_{\mathbb{N}\mathfrak{n} \leq y} |f(\mathfrak{n})|^2}$$

for $Y \geq 1$. We again remark that $\mathcal{F}(Y)$ is a decreasing function of Y, $\mathcal{F}(Y) \leq 1$, and $\mathcal{F}(Y) = 0$ for $Y \geq y$. We aim to show (4.2) holds. We claim it suffices to show

$$\mathcal{F}(Y) \leq C \left(\frac{1 + \log Y}{\sqrt{Y}} \right)$$

for $Y \geq y_0$, where C and y_0 are positive constants depending only on $\mathcal{P}$. Replacing C by $\max\{C, \sqrt{y_0}\}$, the above inequality holds for $Y \geq 1$ since $\mathcal{F}(Y) \leq 1$ for $Y \geq 1$, thus proving the claim. To show the desired inequality, we shall take

$$y_0 := c_{\mathcal{P}} + 2 \quad \text{ and } C := \frac{2^{24}}{\delta^2} \tag{4.16}$$

where $\delta = \delta_{\mathcal{P}} \in (0, 1]$ is the natural density of $\mathcal{P}$, and $c_{\mathcal{P}}$ is as chosen in (4.10).

Suppose, for a contradiction, that there exists $Y \geq y_0$ such that

$$\mathcal{F}(Y) > C \left(\frac{1 + \log Y}{\sqrt{Y}} \right). \tag{4.17}$$

Since $F(Y) = 0$ for $Y > y$, we may choose $Y \in [1, y]$ maximal with respect to the property that Y satisfies the above inequality and no value larger than $Y + 1$ does. Since $Y \geq y_0 \geq c_{\mathcal{P}}$, we have $\#\mathcal{P}(Y) \geq \delta\sqrt{Y}/(2\log Y)$ by (4.10). As in the previous section, we divide $\mathcal{P}(Y)$ into the sets $\mathcal{P}_j$ for $0 \leq j \leq J$ where J is defined in (4.11), so $\mathcal{P}(Y) = \sqcup_{j=0}^{J} \mathcal{P}_j$. It follows by the Pigeonhole Principle, either $\#\mathcal{P}_0 \geq \delta\sqrt{Y}/(4\log Y)$ or $\#\mathcal{P}_j \geq \delta\sqrt{Y}/(4J\log Y)$ for some $1 \leq j \leq J$. The arguments below are divided by these two distinguished cases.

Case 1: $\#\mathcal{P}_0 \geq \delta\sqrt{Y}/(4\log Y)$.

Set $K := [(\#\mathcal{P}_0)\mathcal{F}(Y)/8]$. Then since Y satisfies (4.17), we have

$$K \geq \frac{\delta\sqrt{Y}}{4\log Y} \cdot C \left(\frac{1 + \log Y}{\sqrt{Y}} \right) \cdot \frac{1}{8} - 1 \geq \frac{\delta C}{32} - 1 \geq 2^{18}$$

since $C \geq 2^{24}$. On the other hand, we can easily see that $K \leq (\#\mathcal{P}_0)/4$ since $\mathcal{F}(Y) \leq 1$. Thus, we may apply Proposition 4.8 using K to conclude

$$\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y\\ \mathfrak{n}\in\mathcal{N}_0(K)}} |f(\mathfrak{n})|^2 \leq \frac{1}{2}\mathcal{F}(Y)\sum_{\mathbb{N}\mathfrak{n}\leq y}|f(\mathfrak{n})|^2$$

by noting $4K/(\#\mathcal{P}_0) \leq \frac{1}{2}\mathcal{F}(Y)$ from the definition of K. From this inequality, it follows that

$$\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y\\ \mathfrak{n}\notin\mathcal{N}_0(K)}} |f(\mathfrak{n})|^2 \geq \frac{1}{2}\mathcal{F}(Y)\sum_{\mathbb{N}\mathfrak{n}\leq y}|f(\mathfrak{n})|^2. \tag{4.18}$$

If $\mathfrak{n}\notin\mathcal{N}_0(K)$, then $\mathfrak{n}$ must be divisible by at least $K+1$ squares of prime ideals in $\mathcal{P}_0$. There are $\binom{\#\mathcal{P}_0}{K+1}$ integral ideals that are products of exactly $K+1$ prime ideals from $\mathcal{P}_0$, and we denote this set of ideals as $\mathcal{P}_0(K+1)$. Each of these ideals has norm exceeding $(\sqrt{Y}/2)^{K+1}$ since $\mathcal{P}_0\subseteq\mathcal{P}(Y)$. A fractional ideal $\mathfrak{n}\notin\mathcal{N}_0(K)$ must be divisible by the square of one of these ideals, say $\mathfrak{a}\in\mathcal{P}_0(K+1)$. To summarize, $\mathfrak{a}^2\mid\mathfrak{n}$ and $\mathfrak{a}$ is a square-free integral ideal composed entirely of prime ideals in $\mathcal{P}_0\subseteq\mathcal{P}$. Thus, by Proposition 4.6, we have

$$\begin{aligned}\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y\\ \mathfrak{a}^2|\mathfrak{n}}} |f(\mathfrak{n})|^2 &\leq \tau_3(\mathfrak{a})\prod_{\mathfrak{p}|\mathfrak{a}}(2+|f_{\mathcal{P}}(\mathfrak{p})|^2)\mathcal{F}(Y\cdot\mathbb{N}\mathfrak{a}^2)\sum_{\mathbb{N}\mathfrak{n}\leq y}|f(\mathfrak{n})|^2\\ &\leq 3^{K+1}\cdot 3^{K+1}\cdot\mathcal{F}(Y\cdot(Y/4)^{K+1})\sum_{\mathbb{N}\mathfrak{n}\leq y}|f(\mathfrak{n})|^2.\end{aligned}$$

In the above, we noted (i) $\tau_3(\mathfrak{a}) = 3^{K+1}$ as $\mathfrak{a}$ is a product of exactly $K+1$ distinct prime ideals, (ii) $|f_{\mathcal{P}}(\mathfrak{p})|^2\leq 1/2\leq 1$ by definition of $\mathfrak{p}\in\mathcal{P}_0$, and (iii) $\mathcal{F}$ is a decreasing function and $\mathbb{N}\mathfrak{a}^2\geq(Y/4)^{K+1}$. Summing this inequality over all $\mathfrak{a}\in\mathcal{P}_0(K+1)$, by our previous observations, we deduce

$$\begin{aligned}\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y\\ \mathfrak{n}\notin\mathcal{N}_0(K)}} |f(\mathfrak{n})|^2 &\leq \sum_{\mathfrak{a}\in\mathcal{P}_0(K+1)}\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y\\ \mathfrak{a}^2|\mathfrak{n}}}|f(\mathfrak{n})|^2\\ &\leq\binom{\#\mathcal{P}_0}{K+1}3^{K+1}\cdot 3^{K+1}\mathcal{F}(Y\cdot(Y/4)^{K+1})\sum_{\mathbb{N}\mathfrak{n}\leq y}|f(\mathfrak{n})|^2.\end{aligned}\tag{4.19}$$

By some simple combinatorial bounds, Stirling's formula, and our choice of K, we have

$$\binom{\#\mathcal{P}_0}{K+1}\leq\frac{(\#\mathcal{P}_0)^{K+1}}{(K+1)!}<\left(\frac{e(\#\mathcal{P}_0)}{K+1}\right)^{K+1}<\left(\frac{24}{\mathcal{F}(Y)}\right)^{K+1}.$$

Further, by the maximality of Y and (4.17), we know

$$\begin{aligned}\mathcal{F}(Y \cdot (Y/4)^{K+1}) &\le C\left(\frac{1+\log(Y^{K+2}\cdot 4^{-(K+1)})}{Y^{(K+2)/2}\cdot 4^{-(K+1)/2}}\right)\\ &\le 2^{K+1}C\left(\frac{1+(K+2)\log Y}{Y^{(K+2)/2}}\right)\\ &\le 2^{K+1}C\left(\frac{1+\log Y}{Y^{1/2}}\right)^{K+2}\\ &\le 2^{K+1}C\left(\frac{\mathcal{F}(Y)}{C}\right)^{K+2}.\end{aligned}$$

Combining the last two inequalities into (4.19), we deduce that

$$\begin{aligned}\sum_{\substack{\mathbb{N}\mathfrak{n}\le y/Y\\ \mathfrak{n}\notin\mathcal{N}_0(K)}} |f(\mathfrak{n})|^2 &\le \left(\frac{24}{\mathcal{F}(Y)}\right)^{K+1} 3^{K+1}\cdot 3^{K+1}\cdot 2^{K+1}C\left(\frac{\mathcal{F}(Y)}{C}\right)^{K+2}\sum_{\mathbb{N}\mathfrak{n}\le y}|f(\mathfrak{n})|^2\\ &\le \left(\frac{432}{C}\right)^{K+1}\mathcal{F}(Y)\sum_{\mathbb{N}\mathfrak{n}\le y}|f(\mathfrak{n})|^2\\ &< \frac{1}{2}\mathcal{F}(Y)\sum_{\mathbb{N}\mathfrak{n}\le y}|f(\mathfrak{n})|^2\end{aligned}$$

since $C \ge 2^{24}$ and $K \ge 2^{18}$, hence contradicting (4.18).

Case 2: $\#\mathcal{P}_j \ge \delta\sqrt{Y}/(4J\log Y)$ for some $1 \le j \le J$.

Set $K := [2^{2j-9}(\#\mathcal{P}_j)\mathcal{F}(Y)^{1/2}]$. By the contradiction assumption (4.17) and noting $C \ge 2^{24}$, we see that

$$J \le 3 + \frac{\log\mathcal{F}(Y)^{-1}}{4\log 2} \le 3 + \frac{\log(Y^{1/2}/C)}{4\log 2} \le \frac{\log Y}{4}.$$

With this bound on J and (4.17), we have

$$K \ge \frac{2^{2j-9}\delta\sqrt{Y}}{4J\log Y}\cdot C^{1/2}\left(\frac{1+\log Y}{\sqrt{Y}}\right)^{1/2} - 1 \ge \frac{2^{-7}\delta C^{1/2}Y^{1/4}}{(\log Y)^{3/2}} - 1.$$

by also noting $j \ge 1$. If we additionally observe that $C = 2^{24}\cdot\delta^{-2}$ and $Y \ge 2$, we see that the RHS is

$$\ge \frac{2^{-7}\cdot 2^{12}\cdot 1}{1} - 1 \ge 2^4,$$

so $K \ge 2^4$.

On the other hand, for $\mathfrak{p} \in \mathcal{P}_j$, we have

$$2^{2j-4} \leq |f_{\mathcal{P}}(\mathfrak{p})|^2 \leq 4/\mathcal{F}(Y)^{1/2}$$

The left inequality follows by definition of $\mathcal{P}_j$, and the right inequality follows by Lemma 4.5 and noting $\mathbb{N}\mathfrak{p}^2 \leq Y$. Rewriting this inequality, we see $\mathcal{F}(Y)^{1/2} \leq 2^{-2j+6}$, and so

$$K \leq 2^{2j-9}(\#\mathcal{P}_j)\mathcal{F}(Y)^{1/2} \leq (\#\mathcal{P}_j)/4.$$

Thus, we may apply Proposition 4.8 using K to conclude

$$\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y \\ \mathfrak{n}\in\mathcal{N}_j(K)}} |f(\mathfrak{n})|^2 \leq \frac{1}{2}\mathcal{F}(Y) \sum_{\mathbb{N}\mathfrak{n}\leq y} |f(\mathfrak{n})|^2.$$

In the above, we have noted $\frac{2^{12}K^2}{2^{4j}(\#\mathcal{P}_j)^2} \leq \frac{1}{2}\mathcal{F}(Y)$ from the definition of K. From this inequality, it follows that

$$\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y \\ \mathfrak{n}\notin\mathcal{N}_j(K)}} |f(\mathfrak{n})|^2 \geq \frac{1}{2}\mathcal{F}(Y) \sum_{\mathbb{N}\mathfrak{n}\leq y} |f(\mathfrak{n})|^2. \tag{4.20}$$

If $\mathfrak{n} \notin \mathcal{N}_j(K)$, then the fractional ideal $\mathfrak{n}$ of $(\mathfrak{D}\mathfrak{N})^{-1}$ must be divisible by at least $K+1$ distinct prime ideals in $\mathcal{P}_j$. There are $\binom{\#\mathcal{P}_j}{K+1}$ integral ideals that are products of exactly $K+1$ prime ideals from $\mathcal{P}_j$, and we denote this set of ideals by $\mathcal{P}_j(K+1)$. Each of these ideals has norm exceeding $(\sqrt{Y}/2)^{K+1}$ since $\mathcal{P}_j \subseteq \mathcal{P}(Y)$. A fractional ideal $\mathfrak{n} \notin \mathcal{N}_j(K)$ must be divisible by one of these ideals, say $\mathfrak{a}$. To summarize, $\mathfrak{a} \mid \mathfrak{n}$ and $\mathfrak{a} \in \mathcal{P}_j(K+1)$ is a square-free integral ideal composed entirely of prime ideals in $\mathcal{P}_j \subseteq \mathcal{P}$. Thus, by Proposition 4.6, we have

$$\begin{aligned}\sum_{\substack{\mathbb{N}\mathfrak{n}\leq y/Y \\ \mathfrak{a}|\mathfrak{n}}} |f(\mathfrak{n})|^2 &\leq \tau(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} (1+|f_{\mathcal{P}}(\mathfrak{p})|^2)\mathcal{F}(Y\cdot N\mathfrak{a}) \\ &\leq 2^{K+1}\cdot(2^{2j-1})^{K+1}\mathcal{F}(Y\cdot(Y/4)^{(K+1)/2}) \sum_{\mathbb{N}\mathfrak{n}\leq y} |f(\mathfrak{n})|^2.\end{aligned}$$

In the above, we noted (i) $\tau(\mathfrak{a}) = 2^{K+1}$ as $\mathfrak{a}$ is a product of exactly $K+1$ distinct prime ideals, (ii) $|f_{\mathcal{P}}(\mathfrak{p})|^2 \leq 2^{2j-2}$ by definition of $\mathfrak{p} \in \mathcal{P}_j$, and (iii) $\mathcal{F}$ is a decreasing function and $\mathbb{N}\mathfrak{a} \geq (Y/4)^2$. Summing this inequality over all $\mathfrak{a} \in \mathcal{P}_j(K+1)$, by our previous observations, we deduce

$$
\begin{aligned}
\sum_{\substack{\mathbb{N}\mathfrak{n}\le y/Y\\ \mathfrak{n}\notin\mathcal{N}_j(K)}} |f(\mathfrak{n})|^2 &\le \sum_{\mathfrak{a}\in\mathcal{P}_j(K+1)} \sum_{\substack{\mathbb{N}\mathfrak{n}\le y/Y\\ \mathfrak{a}|\mathfrak{n}}} |f(\mathfrak{n})|^2 \\
&\le \binom{\#\mathcal{P}_0}{K+1} 2^{2j(K+1)} \mathcal{F}(Y\cdot(Y/4)^{(K+1)/2}) \sum_{\mathbb{N}\mathfrak{n}\le y} |f(\mathfrak{n})|^2.
\end{aligned}
\tag{4.21}
$$

By some simple combinatorial bounds, Stirling's formula, and our choice of K, we have

$$
\binom{\#\mathcal{P}_j}{K+1} \le \frac{(\#\mathcal{P}_j)^{K+1}}{(K+1)!} < \left(\frac{e(\#\mathcal{P}_j)}{K+1}\right)^{K+1} < \left(\frac{3}{2^{2j-9}\mathcal{F}(Y)^{1/2}}\right)^{K+1}.
$$

Further, by the maximality of Y, we know

$$
\begin{aligned}
\mathcal{F}(Y\cdot(Y/4)^{(K+1)/2}) &\le C\left(\frac{1+\log(Y^{(K+3)/2}\cdot 2^{-(K+1)})}{Y^{(K+3)/4}\cdot 2^{-(K+1)/2}}\right) \\
&\le 2^{(K+1)/2} C\left(\frac{1+\frac{1}{2}(K+3)\log Y}{Y^{(K+3)/4}}\right) \\
&\le 2^{(K+1)/2} C\left(\frac{1+\log Y}{Y^{1/2}}\right)^{(K+3)/2} \\
&\le 2^{(K+1)/2} C\left(\frac{\mathcal{F}(Y)}{C}\right)^{(K+3)/2}.
\end{aligned}
$$

Combining the last two inequalities into (4.21), we deduce that

$$
\begin{aligned}
\sum_{\substack{\mathbb{N}\mathfrak{n}\le y/Y\\ \mathfrak{n}\notin\mathcal{N}_j(K)}} |f(\mathfrak{n})|^2 &\le \left(\frac{3\cdot 2^{-2j+9}}{\mathcal{F}(Y)^{1/2}}\right)^{K+1} 2^{2j(K+1)}\cdot 2^{(K+1)/2} C\left(\frac{\mathcal{F}(Y)}{C}\right)^{(K+3)/2} \sum_{\mathbb{N}\mathfrak{n}\le y} |f(\mathfrak{n})|^2 \\
&\le \left(\frac{2^{12}}{C}\right)^{K+1} \mathcal{F}(Y) \sum_{\mathbb{N}\mathfrak{n}\le y} |f(\mathfrak{n})|^2 \\
&< \frac{1}{2}\mathcal{F}(Y) \sum_{\mathbb{N}\mathfrak{n}\le y} |f(\mathfrak{n})|^2
\end{aligned}
$$

since $C \ge 2^{24}$ and $K \ge 2^4$, hence contradicting (4.20). This completes the proof. $\square$

Remark. If one wishes to determine a constant $C_0 = C_0(\mathcal{P}) > 0$ such that

$$\mathcal{F}(Y) \leq C_0 \cdot \frac{1 + \log Y}{\sqrt{Y}}$$

for all $Y \geq 1$ (instead of just for sufficiently large Y as per the above proof), then one may simply take $C_0 := \max\{\sqrt{y_0}, C\}$, where y_0 and C are as in (4.16), because $\mathcal{F}(Y) \leq 1$ for all $Y \geq 1$.

Chapter 5

Elimination of Escape of Mass

We may now return to the motivating problem: elimination of escape of mass. Let F be a number field and let $\Gamma = \Gamma_0(\mathfrak{N})$. As before, $\mathcal{X}$ is the product of hyperbolic 2- and 3-spaces associated to F, and our interest lies with the quotient space $\Gamma\backslash\mathcal{X}$.

Suppose $\{\phi_j\}_{j=1}^{\infty}$ are a sequence of Hecke-Maaß cusp forms of Γ, with associated probability measures

$$d\mu_{\phi_j} = \frac{|\phi_j(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z})}{||\phi_j||_{L^2}^2}$$

on $\Gamma\backslash\mathcal{X}$. If the probability measures μ_{ϕ_j} weak-$*$ converge to a measure μ, then:

Is μ still a probability measure? In other words, is $\mu(\Gamma\backslash\mathcal{X}) = 1$?

In this chapter, we will retain this setup and answer this question in the affirmative.

5.1 Decay High in the Cusp

From Theorem 2.13, we have gained a very clear understanding of the structure of $\Gamma\backslash\mathcal{X}$. Using a single parameter $y \geq 1$, we may divide a fundamental domain $\mathcal{F}$ of $\Gamma\backslash\mathcal{X}$ into one compact set $\mathcal{S}(y)$, and a finite collection of non-compact cusps $\{g_\sigma\mathcal{D}(y) : \sigma \in \Gamma\backslash\mathbb{P}^1(F)\}$ of the form $\{\text{compact}\} \times \mathbb{R}_{>0}$. This decomposition is best characterized by Figure 2.2.

If the measure μ supposedly lost mass, then the mass must have "escaped" into a cusp. In other words, one should expect that, for some cusp σ, the quantity $\mu_{\phi_j}(g_\sigma\mathcal{D}(y))$ does *not* go to zero as $y \to \infty$, for all sufficiently large j. Thus, in order to eliminate escape of mass, it suffices to show that, for fixed j, the measure μ_{ϕ_j} possesses some uniform decay in every cusp $g_\sigma\mathcal{D}(y)$ as $y \to \infty$. The following proposition addresses this key issue by applying Theorem 4.3 to the Fourier coefficients of a Hecke-Maaß cusp form.

Proposition 5.1. *Let F be a number field, ϕ be a Hecke-Maaß cusp form for $\Gamma = \Gamma_0(\mathfrak{N})$, and σ be a cusp of $\Gamma\backslash\mathcal{X}$. Then for $Y \geq 1$, we have*

$$\frac{1}{||\phi||_{L^2}^2} \int_{g_\sigma \mathcal{D}(Y)} |\phi(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}) \ll_{\mathfrak{N}} \frac{1+\log Y}{\sqrt{Y}}.$$

Proof. Compare with Proposition 2 of [Sou10].

Keep notation as in Chapters 2 and 3. We may assume that Y is sufficiently large, say $Y \geq Y_0 \geq 1$ where Y_0 depends only on F, so that Theorem 2.13 holds for $y \geq Y_0$. First, we apply Parseval's Formula to $\phi_\sigma(\mathbf{z}) = \phi(g_\sigma \mathbf{z})$ using Corollary 3.11, yielding

$$\begin{aligned} \int_{g_\sigma \mathcal{D}(Y)} |\phi(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}) &= \int_{\mathcal{D}(Y)} |\phi_\sigma(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}) \\ &= \int_{\mathcal{D}(Y)} \sum_{\alpha \in (\mathfrak{D}\mathfrak{N})^{-1}\backslash\{0\}} |c_\sigma(\phi;\alpha)|^2 \frac{|W(\mathbf{s};\alpha\mathbf{z})|^2}{|\mathbb{N}\alpha|} d\mathrm{vol}(\mathbf{z}). \end{aligned} \tag{5.1}$$

The integral on the LHS converges since $Y \geq Y_0$, and by Theorem 2.13, $\mathcal{D}(Y)$ injects into a fundamental domain of $\Gamma^{(\sigma)}\backslash\mathcal{X}$ on which ϕ_σ is L^2-integrable. The sum on the RHS converges absolutely by Parseval's formula, so we may rearrange terms arbitrarily. In particular, as in Corollary 3.24, we reindex the Whittaker coefficients by non-zero principal fractional ideals $\mathfrak{n} = (\alpha)$ of $(\mathfrak{D}\mathfrak{N})^{-1}$, and a finite set of representatives ξ of $\mathcal{O}^\times/V$. Thus, the RHS of (5.1)

$$= \int_{\mathcal{D}(Y)} \sum_{\xi \in \mathcal{O}^\times/V} \sum_{\substack{\mathfrak{n} \subseteq (\mathfrak{D}\mathfrak{N})^{-1} \\ \mathfrak{n}=(\alpha)\neq(0)}} |c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2 \frac{\mathcal{W}(\mathbf{s};\alpha\xi\cdot\mathbf{z})}{\mathbb{N}\mathfrak{n}} d\mathbf{z} \tag{5.2}$$

where

$$\mathcal{W}(\mathbf{s};\mathbf{z}) = \sum_{\epsilon \in V} |W(\mathbf{s};\epsilon\mathbf{z})|^2.$$

Note $|W(\mathbf{s};\mathbf{z})| = |W(\mathbf{s};\mathbf{y})|$ by Definition 3.10, so it follows that $\mathcal{W}(\mathbf{s};\mathbf{z}) = \mathcal{W}(\mathbf{s};\mathbf{y})$. Now, recall

$$\mathcal{D}(Y) = \{(\mathbf{x},\mathbf{y}) \in F_\infty \times \mathbb{R}_{>0}^m : \mathbf{x} \in \mathcal{U}, \mathbf{y} \in \mathcal{T}(Y)\}$$

where $\mathcal{T}(Y) = \{\mathbf{y} \in \mathbb{R}_{>0}^m : \widehat{\mathbf{y}} \in \mathcal{V}, \mathbb{N}\mathbf{y} \in (Y,\infty)\}$, so we may write (5.2) as an iterated integral $d\mathrm{vol}(\mathbf{z}) = dV_1(\mathbf{x})dV_2(\mathbf{y})$ as in (2.1). Using this parametrization, we find that (5.2) is

$$= \int_{\mathcal{U}} \int_{\mathcal{T}(Y)} \sum_{\xi \in \mathcal{O}^\times/V} \sum_{\substack{\mathfrak{n} \subseteq (\mathfrak{D}\mathfrak{N})^{-1} \\ \mathfrak{n}=(\alpha)\neq(0)}} |c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2 \frac{\mathcal{W}(\mathbf{s};|\alpha\xi|\mathbf{y})}{\mathbb{N}\mathfrak{n}} \, dV_2(\mathbf{y}) \, dV_1(\mathbf{x}).$$

By Tonelli's theorem, we may arbitrarily swap integrals and sums as we please. If we swap the integral $dV_2(\mathbf{y})$ and both sums, and send $\mathbf{y} = (y_v)_v \mapsto (y_v/|\alpha_v \xi_v|) = \mathbf{y}/|\alpha\xi|$, then $\mathcal{T}(Y)$ maps bijectively to $\alpha\xi \cdot \mathcal{T}(Y)$, and $dV_2(\mathbf{y}) \to \mathbb{N}\mathfrak{n} \cdot dV_2(\mathbf{y})$ similar to (2.2). Thus, the above expression

$$= \int_{\mathcal{U}} \sum_{\xi \in \mathcal{O}^\times/V} \sum_{\substack{\mathfrak{n} \subseteq (\mathfrak{D}\mathfrak{N})^{-1} \\ \mathfrak{n} = (\alpha) \neq (0)}} |c_\sigma^{(\xi)}(\phi; \mathfrak{n})|^2 \int_{\alpha\xi \cdot \mathcal{T}(Y)} \mathcal{W}(\mathbf{s}; \mathbf{y}) \, dV_2(\mathbf{y}) \, dV_1(\mathbf{x}). \tag{5.3}$$

Observe that the quantity $\alpha\xi$ is determined up to multiplication by a unit in V, but since $\mathcal{W}(\mathbf{s}; \epsilon \cdot \mathbf{y}) = \mathcal{W}(\mathbf{s}; \mathbf{y})$ for $\epsilon \in V$, this choice is irrelevant. Our general aim is swap back the same integral and sum, but the domain of integration is not immediately clear. Let us consider the inner most integral $dV_2(\mathbf{y})$ of (5.3) for a given $\alpha \in (\mathfrak{D}\mathfrak{N})^{-1} \setminus \{0\}$ and $\xi \in \mathcal{O}^\times/V$.

By definition of $\mathcal{W}$, we have

$$\int_{\alpha\xi \cdot \mathcal{T}(Y)} \mathcal{W}(\mathbf{s}; \mathbf{y}) d\mathbf{y} = \sum_{\epsilon \in V} \int_{\alpha\xi \cdot \mathcal{T}(Y)} |W(\mathbf{s}; \epsilon \mathbf{y})|^2 d\mathbf{y}. \tag{5.4}$$

Substituting $\mathbf{y} = (y_v)_v \mapsto (y_v/|\epsilon_v|)$, we have $\alpha\xi \cdot \mathcal{T}(Y)$ maps to $\alpha\xi\epsilon \cdot \mathcal{T}(Y)$ and $dV_2(\mathbf{y}) \mapsto |\mathbb{N}\epsilon| dV_2(\mathbf{y}) = dV_2(\mathbf{y})$. Thus, the RHS

$$= \sum_{\epsilon \in V} \int_{\alpha\xi\epsilon \cdot \mathcal{T}(Y)} |W(\mathbf{s}; \mathbf{y})|^2 d\mathbf{y}. \tag{5.5}$$

Now notice, for $\epsilon \in V$,

$$\alpha\xi\epsilon \cdot \mathcal{T}(Y) = \{\mathbf{y} \in \mathbb{R}^m_{>0} : \widehat{\mathbf{y}} \in \widehat{\alpha}\xi\epsilon \cdot \mathcal{V}, \mathbb{N}\mathbf{y} \in (Y \cdot \mathbb{N}\mathfrak{n}, \infty)\}$$

since $\widehat{\xi\epsilon} = \xi\epsilon$. From the proof of Theorem 2.7, the collection of sets $\{\epsilon \cdot \mathcal{V} : \epsilon \in V\}$ are disjoint, and their union is the set $\widehat{\mathcal{Y}} = \{\mathbf{y} \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y} = 1\}$. As $\widehat{\alpha}\xi$ has norm 1, and $\widehat{\mathcal{Y}}$ is a group under multiplication, it follows that the sets $\{\widehat{\alpha}\xi\epsilon \cdot \mathcal{V} : \epsilon \in V\}$ are also disjoint, and their union is $\widehat{\mathcal{Y}}$. Thus, we have shown

$$\bigsqcup_{\epsilon \in V} \alpha\xi\epsilon \cdot \mathcal{T}(Y) = \{\mathbf{y} \in \mathbb{R}^m_{>0} : \mathbb{N}\mathbf{y} \in (Y \cdot \mathbb{N}\mathfrak{n}, \infty)\} = \bigsqcup_{\epsilon \in V} \epsilon \cdot \mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n}).$$

As a result, the expression in (5.5) and hence the LHS of (5.4)

$$= \sum_{\epsilon \in V} \int_{\epsilon \cdot \mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n})} |W(\mathbf{s};\mathbf{y})|^2 dV_2(\mathbf{y}) = \sum_{\epsilon \in V} \int_{\mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n})} |W(\mathbf{s};\epsilon\mathbf{y})|^2 dV_2(\mathbf{y})$$
$$= \int_{\mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n})} \mathcal{W}(\mathbf{s};\mathbf{y}) dV_2(\mathbf{y}),$$

by doing the substitution $\mathbf{y} \mapsto \epsilon\mathbf{y}$ and swapping the sum and integral again.

With this observation, we see that (5.3)

$$= \int_{\mathcal{U}} \sum_{\xi \in \mathcal{O}^\times / V} \sum_{\substack{\mathfrak{n} \subseteq (\mathfrak{D}\mathfrak{N})^{-1} \\ \mathfrak{n} = (\alpha) \neq (0)}} |c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2 \int_{\mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n})} \mathcal{W}(\mathbf{s};\mathbf{y})\, dV_2(\mathbf{y})\, dV_1(\mathbf{x}).$$

Swapping the sum over ideals and inner integral back, notice $\mathcal{W}(\mathbf{s};\mathbf{y})$ has a contribution in the amount $|c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2$ if and only if $\mathbf{y} \in \mathcal{T}(Y \cdot \mathbb{N}\mathfrak{n})$. This occurs equivalently when $\mathbf{y} \in \mathcal{T}(Y_0)$ and $\mathbb{N}\mathbf{y} \geq Y \cdot \mathbb{N}\mathfrak{n}$. Thus, the above equation

$$= \int_{\mathcal{U}} \sum_{\xi \in \mathcal{O}^\times / V} \int_{\mathcal{T}(Y_0)} \mathcal{W}(\mathbf{s};\mathbf{y}) \sum_{\substack{\mathbb{N}\mathfrak{n} \leq \mathbb{N}\mathbf{y}/Y \\ \mathfrak{n} = (\alpha) \neq (0)}} |c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2\, dV_2(\mathbf{y})\, dV_1(\mathbf{x}). \tag{5.6}$$

For each $\xi \in \mathcal{O}^\times/V$, define f and $\mathcal{P} = \mathcal{P}_\mathfrak{N}$ as in the remark following Definition 4.2, so $\mathcal{P}$ has natural density $1/h_F$ according to Proposition 4.7. Thus, we may apply Theorem 4.3 (with $y = \mathbb{N}\mathbf{y}/Y_0$ and $Y = Y/Y_0$) to find that (5.6) is

$$\ll_\mathfrak{N} \frac{\log(eY/Y_0)}{\sqrt{Y/Y_0}} \int_{\mathcal{U}} \sum_{\xi \in \mathcal{O}^\times / V} \int_{\mathcal{T}(Y_0)} \mathcal{W}(\mathbf{s};\mathbf{y}) \sum_{\substack{\mathbb{N}\mathfrak{n} \leq \mathbb{N}\mathbf{y}/Y_0 \\ \mathfrak{n} = (\alpha) \neq (0)}} |c_\sigma^{(\xi)}(\phi;\mathfrak{n})|^2 dV_2(\mathbf{y}) dV_1(\mathbf{x}).$$

Notice the remaining double integral is the same expression as (5.6) with $Y = Y_0$, so we may unwind all of our steps and see that the above, and hence (5.1) is

$$\ll_\mathfrak{N} \frac{\log(eY)}{\sqrt{Y}} \int_{\mathcal{D}(Y_0)} |\phi_\sigma(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}).$$

Since $Y_0 \geq 1$ depends only on F, it may be absorbed into the implicit constant. Finally, by Theorem 2.13, the set $g_\sigma \mathcal{D}(Y_0)$ injects into a fundamental domain for $\Gamma_0(\mathfrak{N})\backslash X$, so the above integral is bounded by the L^2-norm of ϕ, as required. $\square$

5.2 No Escape of Mass

As per the discussion in the previous section, with a uniform decay in the cusps, we may finally eliminate the possibility of escape of mass for probability measures of Hecke-Maaß cusp forms. The culmination of this thesis work is embodied in the following theorem and its proof.

Theorem 5.2. *Let F be a number field, and $\mathfrak{N}$ be an integral ideal. Suppose $\{\phi_j\}_{j=1}^{\infty}$ are Hecke-Maaß cusp forms on $\Gamma_0(\mathfrak{N})\backslash X$ with probability measures μ_{ϕ_j}. Suppose $\mu_{\phi_j} \to \mu$ weak-$*$, that is to say,*

$$\frac{1}{||\phi_j||_{L^2}^2}\int_A |\phi_j(\mathbf{z})|^2 d\mathrm{vol}(\mathbf{z}) \to \int_A d\mu.$$

for every compact set A contained in a fundamental domain of $\Gamma_0(\mathfrak{N})\backslash X$.

Then μ is a probability measure. In other words, no escape of mass occurs.

Proof. Compare with Theorem 1 of [Sou10].

Keeping notation consistent with Theorem 2.13, we may choose Y sufficiently large (depending on F) such that

$$\mathcal{F} = \mathcal{S}(Y) \sqcup \bigsqcup_{\sigma\in\Omega} g_\sigma \mathcal{D}(Y)$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(\mathfrak{N})\backslash X$. In other words, $\mathcal{F}$ may be written as a compact centre $\mathcal{S}(Y)$ and a finite union of cusps $g_\sigma\mathcal{D}(Y)$. We shall analyze the convergence of probability measures on each of these sets.

On the compact set $\mathcal{S}(Y)$, by definition of weak-$*$ convergence, we have that

$$\int_{\mathcal{S}(Y)} d\mu_{\phi_j} = \int_{\mathcal{S}(Y)} d\mu + o_j(1)$$

as $j \to \infty$. On the other hand, from Proposition 5.1, we have

$$\int_{g_\sigma\mathcal{D}(Y)} d\mu_{\phi_j} \ll_{\mathfrak{N}} \frac{\log(eY)}{\sqrt{Y}}.$$

Notice that this bound is independent of ϕ_j. Combining these two observations and noting μ_{ϕ_j} are probability measures, we deduce

$$\begin{aligned}
1 &= \int_{\mathcal{F}} d\mu_{\phi_j} \\
&= \int_{\mathcal{S}(Y)} d\mu_{\phi_j} + \sum_{\sigma\in\Omega}\int_{g_\sigma\mathcal{D}(Y)} d\mu_{\phi_j} \\
&= \int_{\mathcal{S}(Y)} d\mu + o_j(1) + \sum_{\sigma\in\Omega} O_{\mathfrak{N}}\Big(\frac{\log(eY)}{\sqrt{Y}}\Big).
\end{aligned}$$

where $\Omega \subseteq \mathbb{P}^1(F)$ is an inequivalent set of representatives of cusps of $\Gamma_0(\mathfrak{N})$. Taking $j \to \infty$, and $Y \to \infty$, we see that

$$1 = \lim_{Y \to \infty} \int_{\mathcal{S}(Y)} d\mu = \int_{\mathcal{F}} d\mu$$

since by Theorem 2.13(v), $\mathcal{S}(Y) \to \mathcal{F}$ as $Y \to \infty$. This completes the proof. $\square$

Remark. In applying Proposition 5.1, for a fixed ϕ_j, we only require some decay in the cusp as $Y \to \infty$, as long as it is uniform with respect to ϕ_j.

Conclusion

We have eliminated the possibility of escape of mass occurring for Hecke-Maaß cusp forms on congruence locally symmetric spaces, and hence on Hilbert modular varieties. As intended, this result can be applied to become a complete proof of AQUE for congruence locally symmetric spaces with a proof of the following conjecture.

Conjecture 2. *Let M be a congruence locally symmetric space with volume measure* vol. *Suppose $\{\phi_j\}_{j=1}^{\infty} \subseteq L^2(M)$ is a sequence of Hecke-Maaß cusp forms with Laplace eigenvalues $\boldsymbol{\lambda}^{(j)} = (\lambda_v^{(j)})_{v|\infty}$ such that $\lambda_v^{(j)} \to \infty$ for some $v \mid \infty$. If $\mu_{\phi_j} \xrightarrow{wk\text{-}*} \mu$, then $\mu = c \cdot \mathrm{vol}$ for some $c \in [0,1]$.*

Corollary. *Assume Conjecture 2 holds. Then by Theorem 5.2, AQUE holds for Hecke-Maaß cusp forms on congruence locally symmetric spaces. In other words, Conjecture 1 holds.*

Conjecture 2 is the desired analogue of Lindenstrauss' result [Lin06] for congruence surfaces, and as already noted, should be able to be shown by following methods of [Lin06], [EKL06] and [BL03].

While any proof eliminating escape of mass will likely require knowledge of the structure of congruence locally symmetric spaces as in Chapter 2, the heart of our proof lies in the key technical result, Theorem 4.3, on mock $\mathcal{P}$-Hecke multiplicative functions. This mysterious argument about Whittaker coefficients and their multiplicative relations, pioneered by Soundararajan, is non-trivial but involves little more than elementary number theory techniques. It leaves an interested reader still feeling unaware of the "true" reason for no escape of mass on congruence locally symmetric spaces. An analogous proof written in terms of the adèles may be potentially more revealing and would be a worthy investigation.

Another direction of work is a generalization of AQUE to higher rank locally symmetric spaces such as $\mathrm{PGL}(n, \mathbb{R})$ for $n \geq 3$. The equidistribution result analogous to Lindenstrauss' [Lin06] has been established by Silberman and Venkatesh in [SV07] and [SV11]. However, it again remains to eliminate the possibility of escape of mass in the non-compact case. The methods employed in this thesis do not immediately extend to this scenario due to complications with the Whittaker expansion, and so further study is necessary. On the other hand, if one can produce an adèlic proof as previously mentioned, such an argument may more naturally extend to the higher rank case.

These future research objectives have relevant and meaningful impacts in the pursuit and understanding of AQUE and its implications to a diverse set of fields, such as number theory, ergodic theory, and dynamical systems. Significant and deeper investigations concerning these puzzling questions on escape of mass will certainly be required to achieve these goals.

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