

**The critical points of lattice trees and lattice animals in high
dimensions**

by

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Abstract

We study lattice trees and lattice animals in high dimensions. Lattice trees and animals are interesting combinatorial objects used to model branched polymers in polymer science. They are also of interest in combinatorics and in the study of critical phenomena in statistical physics.

The nearest-neighbor and spread-out models on the d dimensional integer lattice \mathbb{Z}^d , have edge set consisting of pairs $\{x, y\}$ with $\|x - y\|_1 = 1$ and $0 < \|x - y\|_\infty \leq L$ with $L \geq 1$ fixed, respectively. On either graph, a *lattice animal* is a finite connected subgraph, and a *lattice tree* is an animal without cycles. Let t_n and a_n be the number of lattice trees and animals with n bonds that contain the origin, respectively. Standard subadditivity arguments provide the existence of the *growth constants* $\tau = \lim_{n \rightarrow \infty} t_n^{1/n}$ and $\alpha = \lim_{n \rightarrow \infty} a_n^{1/n}$. We are interested in the *critical points* of these models, which are the reciprocals of the corresponding growth constants.

We rigorously calculate the first three terms of a $1/d$ -expansion for the critical points of nearest-neighbor lattice trees and animals. The proof follows a recursive argument similar to the one used in [20] and [24], to obtain analogous results for the critical points of self-avoiding walks and percolation. To provide the leading terms in the expansions, we use a mean-field model, related to the Galton-Watson branching process with critical Poisson offspring distribution, and results obtained with the lace expansion. The leading terms are also calculated in the spread-out model. Then we develop expansions for the nearest-neighbor generating functions and, together with the lace expansion, obtain the first and second correction terms.

Our result gives a rigorous proof for previous work on the subject [11], [21, 36]. Given the algorithmic nature of the proof, it can be extended, with sufficient labor, to compute higher degree terms. It may provide the starting point for proving the existence of an asymptotic expansion with rational coefficients, for the critical point of nearest-neighbor lattice trees.

Preface

The work presented in this thesis is the result of my time as a doctoral student under the supervision of my advisor Professor Gordon Slade. The research problems were originally suggested by Professor Slade. We jointly worked on the publication:

- [1] Y. Mejía-Miranda and G. Slade. The growth constants of lattice trees and lattice animals in high dimensions. *Electronic Communications in Probability*, 16:129–136, 2011.

The publication is the basis for Chapter 2. The content of Section 3.2.1 and Section 3.2.2 is based on [39] and is not original work. The research presented in Section 3.1, Chapter 4 and Chapter 5, was conducted by myself and supervised by Professor Slade.

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Lastly, I would like to thank the support of CONACYT (Consejo Nacional de Ciencia y Tecnología).

Chapter 1

Introduction

This chapter gives an introduction to the terminology, context and structure of this thesis. In Section 1.1 and Section 1.2, we present some of the motivation for the present work and define the models of study. In Section 1.3 and Section 1.4, we discuss some of the known results for these models in low and high dimensions, respectively. We state our main result in Section 1.5. Finally, in Section 1.6, we explain the strategy that we will follow for the proof of our main result and, at the same time, outline the structure of this thesis.

1.1 Motivation

Lattice animals are finite connected subgraphs of the d -dimensional integer lattice \mathbb{Z}^d . Lattice trees are lattice animals that do not contain cycles. They are very interesting combinatorial objects that are used to model branched polymers in chemistry and physics [27]. They are also of interest in combinatorics, graph theory and in statistical physics for the study of critical phenomena, as we shall see shortly in Section 1.2.

Lattice trees and lattice animals are natural models for branched polymers. A *linear polymer* is a large molecule that consists of atoms or groups of atoms (called monomers) joined in a sequence with chemical bonds, see Figure 1.1a. A *branched polymer* is a linear polymer with side-chains (set of linear polymers) attached to the monomers. A common way to model a polymer is using a graph in which the vertices represent the monomers and the edges the chemical bonds. The chemical bonds connect the monomers at determined angles. This behavior is modeled by embedding the graph in a lattice, e.g., \mathbb{Z}^d . In this context, lattice trees model branched polymers that do not have cycles, see Figure 1.1b; and lattice animals model branched polymers that may have cycles, see Figure 1.1c.

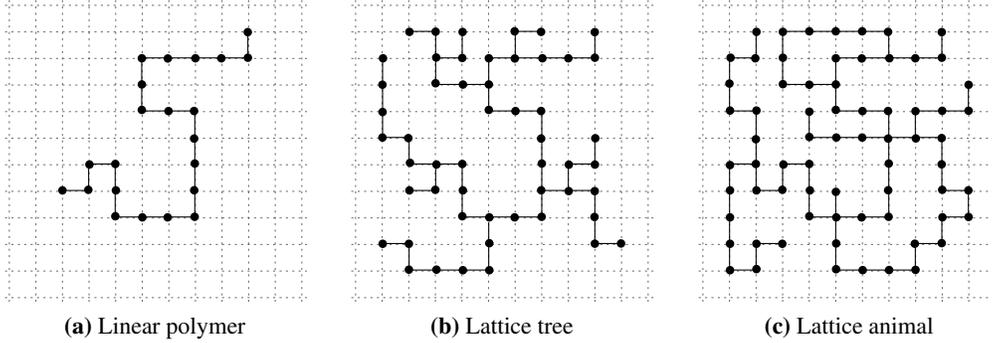


Figure 1.1: Linear and branched polymers embedded in \mathbb{Z}^2 .

1.2 The model

Our work takes place in the d -dimensional integer lattice \mathbb{Z}^d , which are the points in \mathbb{R}^d with integer coordinates. We write (x_1, x_2, \dots, x_d) for the coordinates of a vertex (point) $x \in \mathbb{Z}^d$, and the unordered pair $\{x, y\}$ for the edge (bond) joining the vertices x and y . The two graphs in which we will define the models of study have vertex set given by \mathbb{Z}^d . These are:

1. The *nearest-neighbor* graph has edge set consisting of pairs $\{x, y\}$ with

$$\|x - y\|_1 := \sum_i^d |x_i - y_i| = 1.$$

2. The *spread-out* graph has edge set consisting of pairs $\{x, y\}$ with

$$0 < \|x - y\|_\infty := \max_{1 \leq i \leq d} |x_i - y_i| \leq L, \text{ with } L \text{ fixed.}$$

These two graphs have degree $2d$ and $(2L + 1)^d - 1$, respectively. Often we discuss both graphs simultaneously, and use K to denote the degree. Also we will write $\lim_{K \rightarrow \infty}$ to simultaneously denote the limit as the dimension $d \rightarrow \infty$ for the nearest-neighbor case, and the limit as $L \rightarrow \infty$ for the spread out-case.

For $n \geq 1$, a *walk* ω of length n , or n -step walk, is a sequence $(\omega(0), \omega(1), \dots, \omega(n))$ of vertices in \mathbb{Z}^d such that $\{\omega(i), \omega(i+1)\}$ is an edge in the corresponding (nearest-neighbor or spread-out) graph for all $i = 0, \dots, n-1$. We denote the length of ω by $|\omega|$. Let $\mathcal{W}_n(x, y)$ be the set of n -step walks with $\omega(0) = x$ and $\omega(n) = y$, and let $\mathcal{W}(x, y) = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n(x, y)$ denote the set of all walks that start at x and end at y . A *cycle* of length n , or n -step cycle, is an n -step walk with $\omega(0) = \omega(n)$ and $\omega(i) \neq \omega(j)$ for all $0 < i < j < n$.

The models that we study are (bond) lattice animals and lattice trees on the nearest-neighbor and spread-out graphs. On either graph, *lattice animals* are finite connected subgraphs and *lattice*

trees are the very interesting subset of lattice animals without cycles. Figure 1.1b and Figure 1.1c show a lattice tree and a lattice animal in \mathbb{Z}^2 , respectively. We denote a lattice tree by T and a lattice animal by A . We write C to refer simultaneously to a tree and an animal, and call C a *cluster*. The number of bonds contained in the cluster C , is denoted by $|C|$. If the vertex x is contained in the cluster C , we denote by $C \ni x$.

Often in our discussion, we will settle *asymptotic* results, therefore it is convenient to explain our notation. Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we write

- $f \sim g$ to mean $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$.
- $f = o(g)$ to mean $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$.
- $f = O(g)$ to mean that there is a constant $M > 0$ and a number N such that for all $n \geq N$, we have $|f(n)| \leq M|g(n)|$.

We denote by t_n and a_n the number of lattice trees and lattice animals with n bonds that contain the origin of \mathbb{Z}^d , respectively. For instance, in the nearest-neighbor model, $t_0 = a_0 = 1$ (the single animal that contains the origin and has no edges), $t_1 = a_1 = 2d$, $t_2 = a_2 = 2d(2d - 1) + \binom{2d}{2}, \dots$. Let $t_n^{(0)}$ and $a_n^{(0)}$ denote the number of lattice trees and lattice animals with n bonds, modulo translation, respectively. Then

$$t_n = (n + 1)t_n^{(0)} \quad \text{and} \quad a_n^{(0)} \leq a_n \leq (n + 1)a_n^{(0)}, \quad (1.1)$$

where the equality on the left is due to the fact that trees with n bonds have $n + 1$ vertices (so $n + 1$ possibilities for the origin); and the expression on the right is due to the fact that animals with n bonds have at most $n + 1$ vertices (so at most $n + 1$ options for the origin).

An elementary combinatorial question is to know the values of t_n and a_n for all n , but even in two dimensions the answer is not known for large values of n . However, standard subadditivity arguments (see [32] for trees and [31] for animals), provide the existence of the *growth constants* $\tau > 0$ and $\alpha > 0$, defined by

$$\tau = \lim_{n \rightarrow \infty} \left(t_n^{(0)} \right)^{1/n} \quad \text{and} \quad \alpha = \lim_{n \rightarrow \infty} \left(a_n^{(0)} \right)^{1/n}. \quad (1.2)$$

The growth constants depend on the dimension d , and for the spread-out model, also on L . Then (1.1) and (1.2) imply that

$$\lim_{n \rightarrow \infty} t_n^{1/n} = \tau \quad \text{and} \quad \lim_{n \rightarrow \infty} a_n^{1/n} = \alpha.$$

The precise asymptotic behavior of t_n and a_n as $n \rightarrow \infty$ is believed to be given by

$$t_n \sim C_t \tau^n n^{1-\theta_t} \quad \text{and} \quad a_n \sim C_a \alpha^n n^{1-\theta_a},$$

where the constants C_t , C_a , τ and α should depend upon the lattice, but the *critical exponents* θ_t and θ_a depend only on the dimension. This lack of dependence on the details of the model is called *universality*, and models with the same critical exponents are said to be in the same *universality class*. It is also believed that lattice trees and lattice animals belong to the same universality class, so that in particular, $\theta_t = \theta_a = \theta$ in every dimension. It has been proved that $\theta_t = \frac{5}{2}$ for lattice trees in high dimensions [19]. The bounds

$$c_1 n^{-c_2 \log n} \tau^n \leq t_n^{(0)} \leq c_3 n^{-(d-1)/d} \tau^n,$$

were proved in [26] and [33] respectively. This implies

$$c_1 n^{-c_2 \log n} \tau^n \leq t_n \leq c_3 n^{1/d} \tau^n.$$

In studying t_n and a_n , it is convenient to consider their generating functions, which we call *one-point functions*. These are given by

$$g^{(t)}(z) = \sum_{n=0}^{\infty} t_n z^n \quad \text{and} \quad g^{(a)}(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (1.3)$$

or equivalently,

$$g^{(t)}(z) = \sum_{T \ni 0} z^{|T|} \quad \text{and} \quad g^{(a)}(z) = \sum_{A \ni 0} z^{|A|}, \quad (1.4)$$

where the sums are over the lattice trees and the lattice animals that contain the origin, respectively.

The *two-point functions* are defined by

$$G_z^{(t)}(x, y) = \sum_{T \ni x, y} z^{|T|} \quad \text{and} \quad G_z^{(a)}(x, y) = \sum_{A \ni x, y} z^{|A|}, \quad (1.5)$$

where the sums are over the lattice trees and the lattice animals that contain the vertices x and y , respectively. We denote $G_z^{(t)}(0, x)$ and $G_z^{(a)}(0, x)$ by $G_z^{(t)}(x)$ and $G_z^{(a)}(x)$, respectively. Due to the symmetries of the lattice \mathbb{Z}^d , the two-point functions in (1.5) can be rewritten as $G_z^{(t)}(y-x)$ and $G_z^{(a)}(y-x)$, respectively.

The *susceptibility functions* are defined as

$$\chi^{(t)}(z) = \sum_{x \in \mathbb{Z}^d} G_z^{(t)}(x) \quad \text{and} \quad \chi^{(a)}(z) = \sum_{x \in \mathbb{Z}^d} G_z^{(a)}(x). \quad (1.6)$$

The *critical points* are the radii of convergence of the susceptibility functions. They are given in terms of the growth constants by

$$z_c^{(t)} = \tau^{-1} \quad \text{and} \quad z_c^{(a)} = \alpha^{-1}.$$

The reader may have noticed that the super indices (t) and (a) in a quantity, mean that it corresponds to lattice trees and animals, respectively. However, when we omit them, the term refers to both models. For instance, z_c denotes $z_c^{(t)}$ and $z_c^{(a)}$, and $G_{z_c}(x)$ denotes $G_{z_c^{(t)}}^{(t)}$ and $G_{z_c^{(a)}}^{(a)}$.

It is predicted that the two-point function changes from having an exponential decay in $|x|$ for $z < z_c$, to a power law decay

$$G_{z_c}(x) \sim \text{const} \frac{1}{|x|^{d-2+\eta}}, \quad \text{as } |x| \rightarrow \infty.$$

where η is a universal critical exponent. Using the lace expansion, it was proved in [16] that the two-point function at the critical point is asymptotic to $\text{const}|x|^{2-d}$ as $|x| \rightarrow \infty$, for sufficiently large dimension d . This change in the properties of the model at $z = z_c$ is characteristic of a phase transition.

The critical points for lattice trees and lattice animals are the focus of our study. In the following sections of this chapter we give a brief account of several known facts for these quantities.

1.3 Growth constants in low dimensions

In low dimensions the values of the growth constants have been approximated through exact enumeration of trees and animals; however, this task is non-trivial. In [28],

$$\tau = 5.1439 \pm 0.0025 \tag{1.7}$$

was numerically estimated for the nearest-neighbor graph in \mathbb{Z}^2 , using Montecarlo methods. In [12],

$$\tau = 10.53 \pm 0.07 \tag{1.8}$$

was numerically estimated for the nearest-neighbor graph in \mathbb{Z}^3 .

In [9], the following numerical estimates are reported,

$$\log \alpha = 1.651 \pm 0.002 \implies 5.2018 \leq \alpha \leq 5.2226, \tag{1.9}$$

for the nearest-neighbor graph in \mathbb{Z}^2 , and

$$\log \alpha = 2.364 \pm 0.005 \implies 10.5804 \leq \alpha \leq 10.6867, \tag{1.10}$$

for the nearest-neighbor graph in \mathbb{Z}^3 .

On the other hand, for (bond) lattice trees and animals in the spread out model, Penrose [38]

showed that, in all dimensions $d \geq 1$,

$$\frac{K^K}{(K-1)^{K-1}} - O(K^{5/7} \log K) \leq \tau \leq \alpha \leq \frac{K^K}{(K-1)^{K-1}} \quad (1.11)$$

Notice that both the right- and left-hand sides of (1.11) are $\sim Ke$ as $K \rightarrow \infty$.

1.4 Growth constants in high dimensions

In the high dimensional case exact results for the growth constants are unavailable. Instead, expansions in the inverse dimension or $1/d$ -expansions, have been extensively studied in the physics literature. They provide approximate results in situations where more precise information is unavailable. An inverse dimension, or $1/d$ -expansion is an asymptotic expansion of some function $f(d)$ in powers of $1/d$, where d is the dimension. It is generally assumed that in such expansions the remainder is of the order of the first omitted term, but this is rarely proved due to the limitations of the employed techniques.

These type of expansions have been obtained for the closely related models of self-avoiding walks (SAW) and percolation. These models are of great relevance in the fields of combinatorics, probability theory, statistical physics and polymer chemistry. See [34], for an extensive study of self-avoiding walks, and [15] for an introduction to percolation.

Self-avoiding walks model linear polymers, see Figure 1.1a. Mathematically, a self-avoiding walk is a walk $\omega = (\omega(0), \omega(1), \dots, \omega(|\omega|))$ in \mathbb{Z}^d that satisfies $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let c_n denote the number of n -step self-avoiding walks starting at the origin. It is well known that the connective constant $\mu := \lim_{n \rightarrow \infty} c_n^{1/n}$ exists [34]. For nearest-neighbor self-avoiding walks, it was proved in [20] that the connective constant has an asymptotic expansion

$$\mu \sim \sum_{i=-1}^{\infty} \frac{m_i}{(2d)^i} \quad (\text{as } d \rightarrow \infty),$$

with $m_i \in \mathbb{Z}$ for all i . The first thirteen coefficients in this expansion are now known [7], and it appears likely that the series $\sum_i m_i x^i$ has radius of convergence equal to zero. It may however be Borel summable, and a partial result in this direction is given in [14].

The percolation model can be thought intuitively as a model for a liquid passing through a porous material, like water going through the soil in a flowerpot. Since the liquid penetrates the medium depending on the porosity, one of the basic questions that we can ask is whether it percolates. In our example, this could be whether the water ever reaches the root of the plant we are watering. To model this situation, we think of the material (soil) as given by the nearest-neighbor model or spread out model in \mathbb{Z}^d . To each bond $\{x, y\}$ we associate an independent Bernoulli random variable $n_{\{x, y\}}$, which takes the value of 1 with probability p and the value of 0 with probability $1 - p$, where p is a parameter in $[0, 1]$ that corresponds to the porosity in our example. We say that

the edge $\{x, y\}$ is *open*, if the random variable $n_{\{x, y\}} = 1$, and *closed*, otherwise. We think of an open edge as being open to the transit of the liquid.

A configuration is a realization of the random variables for all edges. We denote by \mathbb{P}_p the joint probability distribution of the bond variables. Given a configuration and two vertices x and y , we say that x and y are *connected*, denoted $x \leftrightarrow y$, if there is a path joining x and y consisting of open bonds, or if $x = y$. The open cluster $C(x)$ is the random set of vertices connected to x , we denote its cardinality by $|C(x)|$.

The principal object of study is the *percolation probability* $\theta(p)$ given by

$$\theta(p) = \mathbb{P}_p(|C(0)| = \infty).$$

The *critical probability* is defined as

$$p_c = p_c(d) := \sup\{p : \theta(p) = 0\}.$$

The critical probability p_c for nearest-neighbor bond percolation has been a much studied object in literature. It has been proved by several authors that the leading order term, as $d \rightarrow \infty$, of $p_c(d)$ is $(2d)^{-1}$. However, Bollobás and Kesten [30], Hara and Slade [17], Kohayakawa [4], Gordon [13] and Alon, Benjamini and Stacey [2] derived different bounds for the error term. Currently the best rigorous result is

$$p_c(d) = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{7}{2(2d)^3} + O((2d)^{-4}),$$

which was established in [20, 24] using the lace expansion. Moreover, this same technique was used in [23] to prove the existence of an asymptotic expansion for p_c of the form

$$p_c(d) \sim \sum_{i=1}^{\infty} \frac{q_i}{(2d)^i},$$

where the coefficients q_i are rational numbers. It is believed that this expansion has radius of convergence equal to zero, but there is no proof of this.

Some related results for spread-out models of percolation and self-avoiding walks are obtained in [22, 37, 38]. In particular, using the lace expansion, it is shown in [22] that when $d > 4$ for self-avoiding walks and $d > 6$ for percolation, and L is sufficiently large, the critical points are given by

$$1 + \frac{c}{L^d} + O\left(\frac{1}{L^{d+1}}\right),$$

where c is a model-dependent constant.

For lattice trees and lattice animals, it is shown in [18] that the one-point functions at criticality $g^{(t)}(z_c^{(t)})$ and $g^{(a)}(z_c^{(a)})$ are finite (in fact, at most 4) in sufficiently high dimensions for the nearest-

neighbor model, and for the spread-out model in dimensions $d > 8$ if L is large enough, and that in these two limits, the critical points $z_c^{(t)}$ and $z_c^{(a)}$ obey the equations

$$\lim_{K \rightarrow \infty} K z_c^{(t)} g^{(t)}(z_c^{(t)}) = \lim_{K \rightarrow \infty} K z_c^{(a)} g^{(a)}(z_c^{(a)}) = 1. \quad (1.12)$$

This is discussed for the nearest neighbor model in [16] (see, in particular, [16, (1.31)]). A more detailed discussion is found in [39].

The behavior of the growth constants τ and α in the nearest-neighbor model, as $d \rightarrow \infty$, has been extensively studied in the physics literature. For τ , the expansion

$$\tau = \sigma e \exp\left(-\frac{1}{2} \frac{1}{\sigma} - \frac{8}{3} \frac{1}{\sigma^2} - \frac{85}{12} \frac{1}{\sigma^3} - \frac{931}{20} \frac{1}{\sigma^4} - \frac{2777}{10} \frac{1}{\sigma^5} + \dots\right) \quad \text{where } \sigma = 2d - 1 \quad (1.13)$$

is reported in [11], but without a rigorous estimate for the error term. This raises the question of whether there exists an asymptotic expansion for τ of the form

$$e \sum_{i=-1}^{\infty} \frac{r_i}{(2d)^i},$$

with $r_i \in \mathbb{Q}$. For α , the series

$$\alpha = \sigma e \exp\left(-\frac{1}{2} \frac{1}{\sigma} - \left(\frac{8}{3} - \frac{1}{2e}\right) \frac{1}{\sigma^2} - \left(\frac{85}{12} - \frac{1}{4e}\right) \frac{1}{\sigma^3} - \left(\frac{931}{20} - \frac{139}{48e} - \frac{1}{8e^2}\right) \frac{1}{\sigma^4} - \left(\frac{2777}{10} + \frac{177}{32e} - \frac{29}{12e^2}\right) \frac{1}{\sigma^5} + \dots\right) \quad (1.14)$$

was derived in [21, 36], again without a rigorous error estimate; here the role of the transcendental number e is more delicate.

Expansions (1.13) and (1.14) yield the estimates for τ and α in Table 1.1. Comparing these results to (1.7)–(1.10), we notice that the more terms we consider in the expansion does not necessarily give a better estimate. However, we must recall that these expansions are for dimension d high.

	d	1 term	2 terms	3 terms	4 terms	5 terms
τ	2	6.9029	5.1328	3.9484	2.2225	0.7088
	3	12.298	11.0538	10.4448	9.6951	8.8708
α	2	6.9029	5.2388	4.0436	2.3067	0.7305
	3	12.298	11.1354	10.5297	9.7909	8.9535

Table 1.1: Estimates for τ and α for dimensions $d = 2$ and 3 , taking into account 1, 2, 3, 4 and 5 terms in expansions (1.13) and (1.14)

On the other hand, expansions (1.13) and (1.14) would imply that the critical points satisfy

$$z_c^{(t)} = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24}}{(2d)^3} + \frac{\frac{309}{16}}{(2d)^4} + \frac{\frac{619103}{5760}}{(2d)^5} + \frac{\frac{543967}{768}}{(2d)^6} \right] + \dots \quad (1.15)$$

$$z_c^{(a)} = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24} - \frac{1}{2}e^{-1}}{(2d)^3} + \frac{\frac{309}{16} - 2e^{-1}}{(2d)^4} + \frac{\frac{619103}{5760} - \frac{113}{12}e^{-1}}{(2d)^5} + \frac{\frac{543967}{768} - \frac{395}{12}e^{-1} - \frac{55}{24}e^{-2}}{(2d)^6} \right] + \dots \quad (1.16)$$

In [35] we provide a rigorous confirmation of the leading terms in (1.15)–(1.16). This is the content of Theorem 1.1 below, whose proof will be presented in Chapter 2.

Theorem 1.1. *For the nearest neighbor model as $d \rightarrow \infty$, and for the spread-out model in dimensions $d > 8$ as $L \rightarrow \infty$,*

$$z_c^{(t)} \sim \frac{1}{Ke} \quad \text{and} \quad z_c^{(a)} \sim \frac{1}{Ke}, \quad (1.17)$$

and, in these same limits,

$$\lim_{K \rightarrow \infty} g^{(t)}(z_c^{(t)}) = \lim_{K \rightarrow \infty} g^{(a)}(z_c^{(a)}) = e. \quad (1.18)$$

The main step in the proof of Theorem 1.1 is to establish a connection between the critical points and the one-point functions of lattice trees and animals, to the analogous quantities of a mean-field model. This mean-field model (see [5, 39]), is related to the Galton–Watson branching process with critical Poisson offspring distribution. The advantage of using it is that the values of the critical point and the one-point function at criticality are known. In other developments, the relation with the mean-field model is reflected by the super-Brownian scaling limits proved for lattice trees in high dimensions [8, 25]. The proof of Theorem 1.1 uses a lace expansion result by Hara and Slade [16, 18]. However, the lace expansion technique is not explicitly used. No bound on the rate of convergence is obtained for either (1.17) or (1.18).

To our knowledge, Theorem 1.1 is new for the nearest-neighbor model. Although for the spread-out model, Penrose’s result given in (1.11) is stronger, since it gives the asymptotic result (1.17) without the restriction $d > 8$ and including error estimates. In fact, if we combine (1.11) and (1.12), we get (1.18) for the spread-out model in dimensions $d > 8$.

A similar result to Theorem 1.1, was proved for the related models of nearest-neighbor (site) trees and (site) animals in [1] and [3], respectively. A nearest-neighbor *site animal* is a connected section graph of the lattice, i.e., for each pair of vertices v_1 and v_2 in a site animal, which differ by one in exactly one coordinate, the edge $e = (v_1, v_2)$ must be in the site animal. In [1] and [3], using very different methods than ours, it is proved that the corresponding growth constants are $2de - o(d)$, as $d \rightarrow \infty$.

1.5 Our main result

The main result of the thesis is the rigorous calculation of the first three terms in a $1/d$ -expansion for the critical points of nearest-neighbor lattice trees and animals. This is specifically stated in the following theorem.

Theorem 1.2. *In the nearest neighbor model, as $d \rightarrow \infty$,*

$$\begin{aligned} z_c^{(t)} &= e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24}}{(2d)^3} \right] + o((2d)^{-3}) \\ z_c^{(a)} &= e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24} - \frac{1}{2}e^{-1}}{(2d)^3} \right] + o((2d)^{-3}). \end{aligned} \tag{1.19}$$

Remark 1.3. *The coefficients in the expansions for $z_c^{(t)}$ and $z_c^{(a)}$ given in (1.19) agree with the ones predicted in (1.15) and (1.16). Moreover, in our proof, the appearance of $-\frac{1}{2}e^{-1}$ in the third term of $z_c^{(a)}$ is due to the animals that have a 4-step cycle containing the origin. This of course cannot occur in the lattice tree case.*

The proof is based on a recursive argument similar to the one used in [20] and [24], where $1/d$ -expansions for the critical points of self-avoiding walks and percolation were obtained, respectively. The procedure uses the first n terms to compute the $(n+1)^{th}$ term in the $1/d$ -expansion. The starting point is Theorem 1.1, which provides the leading term in the expansion of z_c . The following two steps in the reasoning yield the second and third terms in the expansion. They extensively use the *lace expansion* and the results obtained with this method in [16].

The lace expansion for lattice trees and animals will be discussed in Chapter 3. This technique gives, among other things, implicit formulas for the critical points of several models like self-avoiding walks, percolation, and of course lattice trees and animals. It is the core of the argument for obtaining the $1/d$ -expansions for the critical points of self-avoiding walks and percolation that we mentioned before. However, unlike these two models, the formula for z_c includes the corresponding critical one-point function $g(z_c)$. This fact increments significantly the level of difficulty of our problem. To tackle this challenge we developed a new approach that consists in

1. Finding an expansion for the one-point function $g(z_c)$, such that the resulting expression resembles the one given by the lace expansion for the two-point function. This allows us to use analogous methods to this technique to bound error terms.
2. Combining the formula for z_c and the expansion for $g(z_c)$ in a convenient way.

In the following section we describe the strategy that will lead us to the proof of Theorem 1.2.

1.6 Overview of the proof and the thesis

In this section we explain the strategy that we will follow to prove Theorem 1.2, and at the same time describe the organization of the thesis. We commence by defining the quantities needed.

The *Fourier transform* of an absolutely summable function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x}, \quad (1.20)$$

where $k \in [-\pi, \pi]^d$ and $k \cdot x = \sum_{j=1}^d k_j x_j$. The *inverse Fourier transform* is given by

$$f(x) = \int_{[-\pi, \pi]^d} \hat{f}(k) e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d}. \quad (1.21)$$

Once equipped with these definitions, we can see that the susceptibility function defined in (1.6), is also equivalent to $\chi(z) = \hat{G}_z(0)$.

The lace expansion is one of the main ingredients in the proof of Theorem 1.2. As we will discuss in Chapter 3, this technique involves a sequence of inclusion-exclusion steps and a process of resummation that results in an expansion for the two-point function, in terms of the one-point function and the Fourier transform of a function $\hat{\Pi}_z(0)$ (see equation (3.25) for trees and equation (3.41) for animals). The Fourier transform of this expansion evaluated at 0 is

$$\chi(z) = \hat{G}_z(0) = \frac{g(z) + \hat{\Pi}_z(0)}{1 - 2dz [g(z) + \hat{\Pi}_z(0)]}. \quad (1.22)$$

Using that the critical point z_c is the value of z such that the denominator in (1.22) vanishes [16, (1.30)], i.e.,

$$1 - 2dz_c (g(z_c) + \hat{\Pi}_{z_c}(0)) = 0, \quad (1.23)$$

we get the following implicit formula for the critical point z_c

$$z_c = \frac{1}{2d} \frac{1}{g(z_c) + \hat{\Pi}_{z_c}(0)}, \quad (1.24)$$

which will be one of the key ingredients in the proof of Theorem 1.2.

The demonstration of Theorem 1.2 follows a recursive argument in which the calculation of the terms in the expansion for z_c is intertwined with the computation of the terms in the expansions for the quantities $\hat{\Pi}_{z_c}(0)$ and $g(z_c)$, given in Theorem 1.4 and Theorem 1.5 below. The route that will lead us to the proof of Theorem 1.2 is as follows.

In Chapter 2, we will prove Theorem 1.1, which is the starting point of the recursive argument. Theorem 1.1 establishes the leading term for the one-point function $g(z_c)$ and the critical point z_c .

For the nearest-neighbor model, these are given by

$$g(z) = e + o(1) \quad \text{and} \quad z_c = \frac{1}{2de} + o\left(\frac{1}{2d}\right), \quad \text{as } d \rightarrow \infty.$$

It is worth noticing that Theorem 1.1 also gives analogous results for the spread-out model. However, in the remaining chapters we restrict our work to the nearest-neighbor model. In the proof of Theorem 1.1 we use the following estimates

$$\hat{\Pi}_{z_c}(0) = o(1) \quad \text{and} \quad \lim_{d \rightarrow \infty} 2dz_c g(z_c) = 1,$$

obtained in [16, (1.23)] and [16, (1.30)], respectively, using the lace expansion.

In Chapter 3, we will develop a new expansion for the one-point function that will allow us to study it at criticality, and present some fundamental facts for the lace expansion technique for lattice trees and lattice animals. In particular, we will see that the lace expansion technique gives an expansion for the two-point function whose Fourier transform, evaluated at 0, is given in (1.22).

In Chapter 4, we will compute the first term in the expansion for $\hat{\Pi}_{z_c}(0)$ given below in Theorem 1.4. Then we will calculate the second term in the expansion of the one-point function $g(z_c)$ given below in Theorem 1.5. Finally, using formula (1.24), we will get the first order correction of z_c , which is the second term in the expansion for z_c in Theorem 1.2.

In Chapter 5, all the previous information will allow us to calculate the second term in the following expansion of $\hat{\Pi}_{z_c}(0)$.

Theorem 1.4. *In the nearest neighbor model, as $d \rightarrow \infty$,*

$$\begin{aligned} \hat{\Pi}_{z_c^{(t)}}^{(t)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\ \hat{\Pi}_{z_c^{(a)}}^{(a)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e - \frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \end{aligned} \tag{1.25}$$

Then we will be able to compute the third term in the following expansion for the one-point function $g(z_c)$.

Theorem 1.5. *In the nearest neighbor model, as $d \rightarrow \infty$,*

$$\begin{aligned} g^{(t)}(z_c^{(t)}) &= e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \\ g^{(a)}(z_c^{(a)}) &= e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e - 1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \end{aligned} \tag{1.26}$$

Substituting these expansions for $\hat{\Pi}_{z_c}(0)$ and $g(z_c)$ in the formula for z_c given in (1.24), yields the second order correction of the critical point as follows.

Proof of Theorem 1.2. By (1.24), (1.25) and (1.26), for the critical point of nearest-neighbor lattice trees we have

$$\begin{aligned} z_c^{(t)} &= \frac{1}{2d \left[g^{(t)}(z_c^{(t)}) + \hat{\Pi}_{z_c^{(t)}}^{(t)}(0) \right]} = \frac{1}{2de \left[1 - \frac{3}{2d} - \frac{61}{24} \frac{1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \right]} \\ &= e^{-1} \left[\frac{1}{2d} + \frac{3}{(2d)^2} + \frac{115}{(2d)^3} \right] + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

And for the critical point of nearest-neighbor lattice animals,

$$\begin{aligned} z_c^{(a)} &= \frac{1}{2d \left[g^{(a)}(z_c^{(a)}) + \hat{\Pi}_{z_c^{(a)}}^{(a)}(0) \right]} = \frac{1}{2de \left[1 - \frac{3}{2d} - \frac{61}{24} \frac{1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \right]} \\ &= e^{-1} \left[\frac{1}{2d} + \frac{3}{(2d)^2} + \frac{115}{(2d)^3} - \frac{1}{2e} \right] + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

□

Therefore, it only remains to show Theorem 1.1, Theorem 1.4 and Theorem 1.5. This is the main content of the following chapters, which we summarize in Figure 1.2.

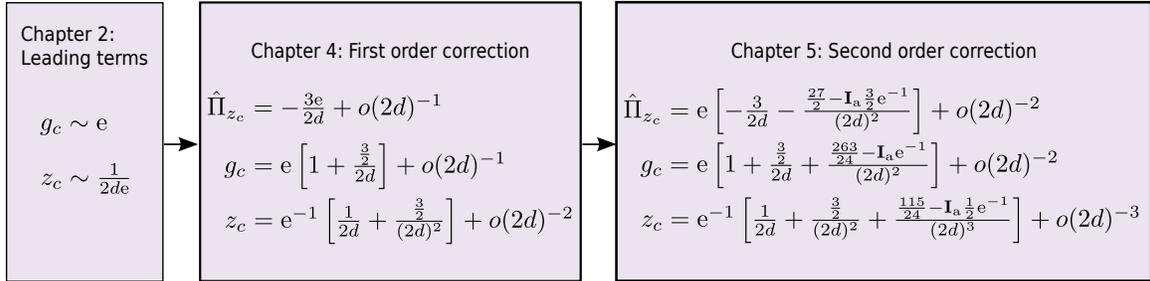


Figure 1.2: Flow of the proof of Theorem 1.2. The indicator function \mathbf{I}_a takes the value of 1 for the case of lattice animals and 0 for the case of lattice trees.

Chapter 2

Proof of leading behavior

In this chapter we present the proof of Theorem 1.1, which we published in [35]. This theorem gives the leading order behavior of z_c and $g(z_c)$ for both lattice trees and lattice animals in the nearest-neighbor model and the spread out model. In Section 2.1 we assume Lemma 2.1 and Proposition 2.2 below, and prove Theorem 1.1. In Section 2.2 we prove these two key results.

2.1 Main steps in the proof

We define

$$z_0 := \frac{1}{Ke}, \quad (2.1)$$

where K , as stated before, represents the degree of both the nearest-neighbor and spread-out graphs on \mathbb{Z}^d .

The proof of Theorem 1.1 uses the following two ingredients:

Lemma 2.1. *In all dimensions $d \geq 1$, and for the nearest-neighbor or spread-out models,*

$$z_c^{(t)} \geq z_c^{(a)} \geq z_0 = \frac{1}{Ke}.$$

Proposition 2.2. *For the nearest-neighbor model, or for the spread-out model in dimensions $d \geq 1$,*

$$\lim_{K \rightarrow \infty} g^{(t)}(z_0) = e,$$

where the notation $\lim_{K \rightarrow \infty}$ means the limit as the dimension $d \rightarrow \infty$ for the nearest-neighbor case, and the limit as $L \rightarrow \infty$ for the spread out-case.

Since, by definition, a lattice tree is a lattice animal, we have $\tau \leq \alpha$, which implies that the critical points obey

$$z_c^{(t)} \geq z_c^{(a)}.$$

In fact, the strict inequality $\tau < \alpha$ is known [10]. Therefore, the main content of Lemma 2.1 is that

$$z_c^{(a)} \geq z_0, \quad \text{or, equivalently, that} \quad \alpha \leq Ke.$$

This fact is presumably well-known, though we did not find an explicit proof in the literature. Klarner [31] proves that for 2-dimensional nearest-neighbor site animals the growth constant is at most $27/4 = 3^3/2^2$, and Penrose [38] states that this can be generalized to the upper bound $\alpha \leq K^K/(K-1)^{K-1} \sim Ke$ for bond animals on an arbitrary regular graph. We will provide an elementary proof that $\alpha \leq Ke$ in Section 2.2, both to be self-contained and because elements of the proof are also useful elsewhere in our approach. At this stage we assume Lemma 2.1 and Proposition 2.2, whose proof we deferred to Section 2.2, and show Theorem 1.1.

Proof of Theorem 1.1. We will prove that, under the hypotheses of Theorem 1.1,

$$\lim_{K \rightarrow \infty} g^{(t)}(z_c^{(t)}) = e. \quad (2.2)$$

It then follows from (1.12) that

$$z_c^{(t)} \sim z_0.$$

Lemma 2.1 then implies that

$$z_c^{(a)} \sim z_0,$$

and finally (1.12) implies that

$$\lim_{K \rightarrow \infty} g^{(a)}(z_c^{(a)}) = e,$$

which are the desired results. Thus Theorem 1.1 will follow, once we prove (2.2).

Proposition 2.2 gives that $1 = \lim_{K \rightarrow \infty} e^{-1} g^{(t)}(z_0)$ which together with (1.12) imply,

$$\lim_{K \rightarrow \infty} \left(K z_c^{(t)} g^{(t)}(z_c^{(t)}) - e^{-1} g^{(t)}(z_0) \right) = 0.$$

By adding and subtracting $\lim_{K \rightarrow \infty} K z_0 g^{(t)}(z_c^{(t)}) = \lim_{K \rightarrow \infty} e^{-1} g^{(t)}(z_c^{(t)})$, this expression can be rewritten as

$$\lim_{K \rightarrow \infty} \left[K \left(z_c^{(t)} - z_0 \right) g^{(t)}(z_c^{(t)}) + e^{-1} \left(g^{(t)}(z_c^{(t)}) - g^{(t)}(z_0) \right) \right] = 0.$$

By Lemma 2.1 and the monotonicity of $g^{(t)}$ in z , both terms in the limit are non-negative. Therefore, the two of them vanish in the limit, and in particular,

$$\lim_{K \rightarrow \infty} \left(g^{(t)}(z_c^{(t)}) - g^{(t)}(z_0) \right) = 0.$$

With Proposition 2.2, this gives (2.2) and completes the proof. \square

It remains to prove Lemma 2.1 and Proposition 2.2, and this is what we do in the next section.

2.2 The proof completed

We will make use of the following mean-field model (see [5, 39]), which is related to the Galton-Watson branching process with critical Poisson offspring distribution.

Let \mathcal{T}_n denote the set of n -edge rooted plane trees [40], and let $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$. Given $\tilde{T} \in \mathcal{T}$, we consider mappings $\varphi : \tilde{T} \rightarrow \mathbb{Z}^d$ with the property that φ maps the root to the origin, and maps each other vertex of \tilde{T} to a neighbor of its parent (nearest-neighbor or spread-out, depending on the setting); the set of such mappings is denoted $\Phi(\tilde{T})$. There is no self-avoidance constraint. By definition, for $\tilde{T} \in \mathcal{T}_n$, the cardinality of $\Phi(\tilde{T})$ is K^n . We may interpret the image of \tilde{T} under φ as a multigraph (a graph whose edges may have the same end vertices) without loops, and we refer to the pair (\tilde{T}, φ) as a *mean-field configuration*. The set of all mean-field configurations (\tilde{T}, φ) with $\tilde{T} \in \mathcal{T}_n$ is denoted \mathcal{M}_n .

Let ξ_i denote the number of children for vertex i ; this is the degree of the root when i is the root of \tilde{T} and otherwise it is the degree of i minus 1. For $n \geq 0$, let

$$f_n = \sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} = K^n \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!}, \quad (2.3)$$

where the second equality follows from the fact that $\Phi(\tilde{T})$ has K^n elements. Let $|\tilde{T}|$ denote the number of edges in \tilde{T} . Then the generating function for $\{f_n\}$ is

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{\tilde{T} \in \mathcal{T}} (Kz)^{|\tilde{T}|+1} \prod_{i \in \tilde{T}} \frac{1}{e^{\xi_i!}}. \quad (2.4)$$

In this expression notice that the factor $e^{-|\tilde{T}|-1}$ appears in the last product since it is over the vertices in \tilde{T} , which are one more than the number of edges. Moreover, since $\mathbb{P}(\tilde{T}) = \prod_{i \in \tilde{T}} (e^{\xi_i!})^{-1}$ is the probability that \tilde{T} arises as the family tree of a Galton-Watson branching process with critical Poisson offspring distribution, it follows from (2.4) that

$$\sum_{n=0}^{\infty} f_n z^n = e \sum_{\tilde{T} \in \mathcal{T}} \mathbb{P}(\tilde{T}) = e. \quad (2.5)$$

The relation with the critical Poisson branching process can easily be exploited further (see, e.g., [5, Theorem 2.1]) to obtain

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} (Kz)^n.$$

The series on the right-hand side converges if and only if $|Kez| \leq 1$, by Stirling's formula, and hence

$$\lim_{n \rightarrow \infty} f_n^{1/n} = Ke = \frac{1}{z_0}. \quad (2.6)$$

Let \mathcal{L}_n denote the set of n -bond lattice trees containing the origin; its cardinality is t_n . We will use the fact, proved in [5, (5.5)], that for every $L \in \mathcal{L}_n$,

$$\sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n: \varphi(\tilde{T})=L} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} = 1. \quad (2.7)$$

The proof of (2.7) in [5] is given for the nearest-neighbor model, but it applies without change also to the spread-out model. By summing (2.7) over $L \in \mathcal{L}_n$, we obtain

$$t_n \leq f_n, \quad (2.8)$$

and hence $\tau \leq \lim_{n \rightarrow \infty} f_n^{1/n} = Ke$. This gives the inequality $z_c^{(t)} \geq z_0$, which is weaker than the inequality $z_c^{(a)} \geq z_0$ that we seek in Lemma 2.1.

Proof of Lemma 2.1. Recall that the inequality $z_c^{(t)} \geq z_c^{(a)}$ follows from $t_n \leq a_n$, and the equality $z_0 = (Ke)^{-1}$ holds by definition, so it suffices to prove that $z_c^{(a)} \geq z_0$. By (2.6), for this it suffices to prove that

$$a_n \leq f_n. \quad (2.9)$$

To prove this, we adapt the proof of (2.7) from [5].

The first step involves a unique determination of a tree structure within a lattice animal. For this, we order all bonds in the infinite lattice lexicographically. Also, we regard a bond as an arc joining the vertices of its endpoints, and we order the two halves of this arc as minimal and maximal. These orderings are fixed once and for all. Given a lattice animal A , suppose that it contains c cycles. Choose the minimal bond whose removal would break a cycle, and remove its minimal half from the animal. Repeat this until no cycles remain. The result is a kind of lattice tree, which we will call the *cut-tree* A^* , in which c leaves are endpoints of half edges. See Figure 2.1. Recall that \mathcal{A}_n denotes the set of n -bond lattice animals that contain the origin. Let \mathcal{A}_n^* denote the set of n -bond cut-trees that can be produced from a lattice animal in \mathcal{A}_n by this procedure. By construction, lattice animals and cut-trees are in one-to-one correspondence, so \mathcal{A}_n^* has cardinality a_n .

We may regard the edges of $\tilde{T} \in \mathcal{T}$ as directed away from the root, and we write a directed edge as (i, i') . Given $A^* \in \mathcal{A}_n^*$ and $(\tilde{T}, \varphi) \in \mathcal{M}_n$, we say that $\varphi(\tilde{T}) = A^*$ if (i) each bond in A^* is the image of a unique edge in \tilde{T} under φ , and if, in addition, (ii) if (b^+, b^-) is a directed bond in A^* from which the half-bond containing b^- is removed in A^* , and if the edge of \tilde{T} that is mapped by φ to (b^+, b^-) is (i, i') , then i' is a leaf of \tilde{T} . Roughly speaking, the condition $\varphi(\tilde{T}) = A^*$ means that the mapping

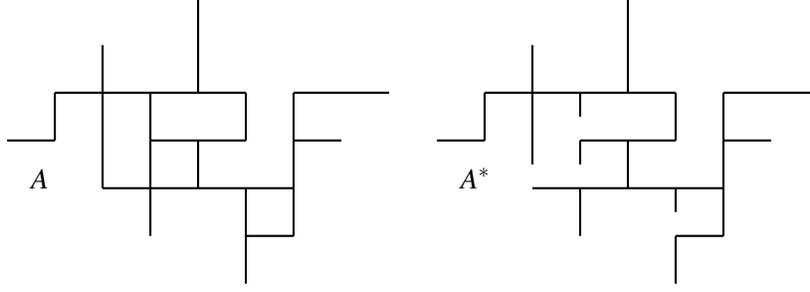


Figure 2.1: A lattice animal A and its associated cut-tree A^* .

φ “folds” \tilde{T} over A^* in such a way that the tree structure of \tilde{T} is preserved in A^* . We claim that for every $A^* \in \mathcal{A}_n^*$,

$$\sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n: \varphi(\tilde{T})=A^*} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} = 1. \quad (2.10)$$

This implies that

$$a_n = \sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n: \varphi(\tilde{T}) \in \mathcal{A}_n^*} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} \leq \sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} = f_n,$$

which is the required inequality (2.9). Thus it suffices to prove (2.10).

To prove (2.10), we adapt the proof of (2.7) from [5], as follows. Let b_0 be the degree of 0 in A^* , and given a nonzero vertex $x \in A^*$, let b_x be the degree of x in A^* minus 1 (the forward degree of x). Then the set $\{b_x : x \in A^*\}$ (with repetitions) must be equal to the set $\{\xi_i : i \in \tilde{T}\}$ (with repetitions) for any \tilde{T} such that $\varphi(\tilde{T}) = A^*$. Defining $v(A^*)$ to be the cardinality of $\{(\tilde{T}, \varphi) : \varphi(\tilde{T}) = A^*\}$, (2.10) is therefore equivalent to

$$v(A^*) = \prod_{x \in A^*} b_x!. \quad (2.11)$$

We prove (2.11) by induction on the number N of generations of A^* , i.e., the number of bonds or half-bonds in the longest self-avoiding path in A^* starting from the origin. The identity (2.11) clearly holds if $N = 0$. Our induction hypothesis is that (2.11) holds if there are $N - 1$ or fewer generations. Suppose A^* has N generations, and let $A_1^*, \dots, A_{b_0}^*$ denote the cut-trees resulting by deleting from A^* the origin and all bonds and half-bonds incident on the origin. We regard each A_a^* as rooted at the non-zero vertex in the corresponding deleted bond. It suffices to show that $v(A^*) = b_0! \prod_{a=1}^{b_0} v(A_a^*)$, since each A_a^* has fewer than N generations.

To prove this, we note that each pair (\tilde{T}, φ) with $\varphi(\tilde{T}) = A^*$ induces a set of (\tilde{T}_a, φ_a) such that $\varphi_a(\tilde{T}_a) = A_a^*$. This correspondence is $b_0!$ to 1, since (\tilde{T}, φ) is determined by the set of (\tilde{T}_a, φ_a) , up to permutation of the branches of \tilde{T} at its root. This proves $v(A^*) = b_0! \prod_{a=1}^{b_0} v(A_a^*)$, and completes the proof of the lemma. \square

Lemma 2.3. *For the nearest-neighbor or spread-out models (the latter in all dimensions $d \geq 1$), for each fixed $n \geq 0$,*

$$\lim_{K \rightarrow \infty} \frac{t_n}{K^n} = \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!}.$$

Proof. By (2.7),

$$t_n = \sum_{(\tilde{T}, \varphi) \in \mathcal{M}_n: \varphi(\tilde{T}) \in \mathcal{L}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} = \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} \sum_{\varphi \in \Phi(\tilde{T}): \varphi(\tilde{T}) \in \mathcal{L}_n} 1. \quad (2.12)$$

Given $\tilde{T} \in \mathcal{T}_n$, the cardinality of $\Phi(\tilde{T})$ is K^n , so there are at most K^n nonzero terms in the above sum over φ . On the other hand, there are at least $K(K-1) \cdots (K-n+1)$ nonzero terms. To see this, consider the mapping φ of \tilde{T} to proceed in a connected fashion to map the edges of \tilde{T} one by one to bonds in \mathbb{Z}^d , starting from the root. The first edge of \tilde{T} can be mapped to any one of K possible bonds. The second edge of \tilde{T} includes one of the vertices of the first edge, and to avoid the image of the other vertex of the first edge, it can be mapped to any one of $K-1$ possible edges. In this way, as φ proceeds from the root to map vertices of \tilde{T} into \mathbb{Z}^d , the restriction that the image contain $n+1$ distinct vertices allows K choices for the first bond, $K-1$ choices for the second bond, at least $K-2$ for the third, at least $K-3$ for the fourth, and so on. This implies that

$$K(K-1) \cdots (K-n+1) \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} \leq t_n \leq K^n \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!},$$

and the desired conclusion follows. \square

Proof of Proposition 2.2. By (2.8), (2.3) and (2.1),

$$t_n z_0^n \leq f_n z_0^n = e^{-n} \sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!},$$

which is independent of K . Also, by (2.5), $\sum_{n=0}^{\infty} f_n z_0^n = e$. Hence, by Lemma 2.3 and the dominated convergence theorem, we have

$$\lim_{K \rightarrow \infty} g^{(t)}(z_0) = \sum_{n=0}^{\infty} \lim_{K \rightarrow \infty} t_n z_0^n = \sum_{n=0}^{\infty} \left(\sum_{\tilde{T} \in \mathcal{T}_n} \prod_{i \in \tilde{T}} \frac{1}{\xi_i!} \right) e^{-n} = \sum_{n=0}^{\infty} f_n z_0^n = e.$$

\square

Chapter 3

Expansion methods

In this chapter we present the expansions for the quantities needed in the proof of Theorem 1.2. Since this theorem exclusively involves nearest-neighbor lattice trees and animals, the results obtained in this chapter are only valid for the nearest-neighbor model. However, with minor modifications, the results on the lace expansion can be extended to the spread-out model, see [39].

In Section 3.1, we develop a new expansion for the one-point function $g(z)$. This expansion is one of the key ingredients that will allow us to study the behavior of the one-point function, at criticality, in subsequent chapters.

In Section 3.2, we present the lace expansion technique for lattice trees and lattice animals. We follow the ideas developed by Hara and Slade in [16] and [39], but do not give a detailed explanation of the derivation. We will discuss the main quantities that contribute to this technique, and omit a few of the technicalities which can be consulted in [39].

3.1 Expansion for the one-point functions

In this section, we develop a new expansion for the one-point functions of lattice trees and lattice animals, simultaneously. The expansions will be obtained by a systematic use of inclusion-exclusion.

In the case of $g^{(a)}(z)$, it is convenient to separate the sum over the lattice animals depending on whether the origin is contained in a cycle or not, which we denote by $0 \in cycle$ and $0 \notin cycle$, respectively. Then we get

$$g^{(a)}(z) = \sum_{A \ni 0} z^{|A|} = \sum_{\substack{A \ni 0 \\ 0 \notin cycle}} z^{|A|} + \sum_{\substack{A \ni 0 \\ 0 \in cycle}} z^{|A|}. \quad (3.1)$$

The one-point function for trees and the sum over animals in which the origin does not belong to a cycle in (3.1), have a similar behavior. To see this, suppose that a tree T (respectively, an animal A) contains the origin and m of its $2d$ nearest neighbors. Then $T(A)$ consists of the m edges incident to the origin and m non-intersecting trees (animals) emanating from these edges. The non-intersection

restriction guarantees that, indeed, the origin is not part of a cycle for the case of lattice animals. This observation allows us to rewrite the one-point function, for both lattice trees and lattice animals (1.4), as

$$g(z) = \sum_{C \ni 0} z^{|C|} = \sum_{\substack{C \ni 0 \\ 0 \notin \text{cycle}}} z^{|C|} + \mathbf{I}_a \sum_{\substack{C \ni 0 \\ 0 \in \text{cycle}}} z^{|C|}, \quad (3.2)$$

where the clusters C are lattice trees or lattice animals depending on which model we consider, and the indicator function \mathbf{I}_a takes the value 1 for the case of lattice animals, and the value 0 for the case of lattice trees.

We will use inclusion-exclusion to expand the first term on the right-hand side of (3.2), but will not expand the second one. Let $C_0 = \{0\}$ be the cluster that only contains the origin. We count a factor of z for every one of the m edges incident to the origin, attach a cluster C_i to each one of these edges, count a factor of z for every edge in these clusters, and discard the contributions in which the attached clusters intersect C_0 or each other. See Figure 3.1 for a depiction of such a decomposition of a lattice tree and a lattice animal containing the origin.

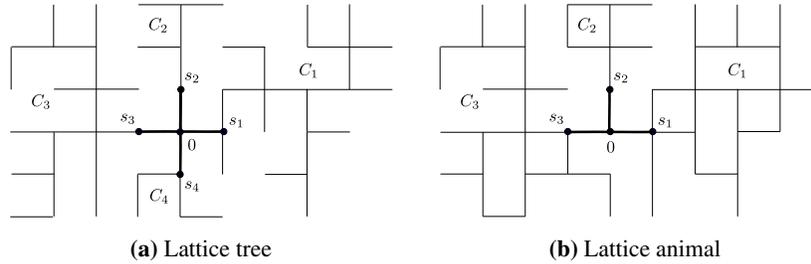


Figure 3.1: Decomposition into the edges connecting 0 to its nearest neighbors s_i and the non-intersecting clusters C_i attached to them.

This procedure yields that

$$\sum_{\substack{C \ni 0 \\ 0 \notin \text{cycle}}} z^{|C|} = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j}), \quad (3.3)$$

where $\mathcal{E} = \{e_1, e_2, \dots, e_{2d}\}$ is the set of the $2d$ nearest-neighbors of the origin ordered such that $e_i = (0, \dots, 1, 0, \dots, 0)$, with 1 located at the i -th coordinate for $1 \leq i \leq d$, and $e_i = -e_{i-d}$ for $d+1 \leq i \leq 2d$; and for $0 \leq i < j \leq m$ and a set $\vec{C} = \{C_0, \dots, C_m\}$ of $m+1$ clusters, the function $\mathcal{V}_{i,j} = \mathcal{V}_{i,j}(\vec{C})$ is given by

$$\mathcal{V}_{i,j}(\vec{C}) = \begin{cases} -1 & \text{if clusters } C_i \text{ and } C_j \text{ share a common vertex} \\ 0 & \text{if clusters } C_i \text{ and } C_j \text{ share no common vertex.} \end{cases}$$

Remark 3.1. The sum in (3.3) is a finite sum since the terms vanish for $m > 2d$. The function $\mathcal{V}_{0,j}$ ensures that the cluster C_j does not contain the origin, while $\mathcal{V}_{i,j}$, that the clusters C_i and C_j do not intersect each other, in particular, excluding the possibility that $s_i = s_j$.

We will expand the product $\prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j})$ in (3.3), and group the summands in a special way. We want to identify the factors \mathcal{V}_{ij} that have label $i = 0$, since, as we will see, the cluster $C_0 = 0$ plays an important role. We start by noticing that

Claim 3.2. For a positive integer n ,

$$\prod_{i=1}^n (1 + x_i) = 1 + \sum_{i=1}^n x_i \prod_{j=i+1}^n (1 + x_j), \quad (3.4)$$

with the convention that the empty product in the second term on the right-hand side is equal to 1.

Proof of Claim 3.2. We follow an inductive argument. For $n = 1$ the formula clearly holds. Assuming it is true for n , the case for $n + 1$ is given by

$$\begin{aligned} \prod_{i=1}^{n+1} (1 + x_i) &= (1 + x_1) \prod_{i=2}^{n+1} (1 + x_i) = \prod_{i=2}^{n+1} (1 + x_i) + x_1 \prod_{i=2}^{n+1} (1 + x_i) \\ &= 1 + \sum_{i=2}^{n+1} x_i \prod_{j=i+1}^{n+1} (1 + x_j) + x_1 \prod_{i=2}^{n+1} (1 + x_i) = 1 + \sum_{i=1}^{n+1} x_i \prod_{j=i+1}^{n+1} (1 + x_j). \end{aligned}$$

□

We can iterate the result in (3.4) by applying it to the product that appears on its right-hand side and obtain

$$\begin{aligned} \prod_{i=1}^n (1 + x_i) &= 1 + \sum_{i=1}^n x_i \prod_{j=i+1}^n (1 + x_j) = 1 + \sum_{i=1}^n x_i \left(1 + \sum_{j=i+1}^n x_j \prod_{k=j+1}^n (1 + x_k) \right) \\ &= 1 + \sum_{i=1}^n x_i + \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j \prod_{k=j+1}^n (1 + x_k). \end{aligned} \quad (3.5)$$

Using this expression we rewrite the product $\prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j})$ as

$$\begin{aligned} \prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j}) &= 1 + \sum_{0 \leq i < j \leq m} \mathcal{V}_{i,j} + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \\ &= 1 + \sum_{j=1}^m \mathcal{V}_{0,j} + \sum_{1 \leq i < j \leq m} \mathcal{V}_{i,j} + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}), \end{aligned} \quad (3.6)$$

where the set

$$A_{ij} = A_{ij}(m) = \{(i, l) : j < l \leq m\} \cup \{(k, l) : i < k < l \leq m\}. \quad (3.7)$$

In the second equality of (3.6) we have only separated the terms in the first sum that correspond to $i = 0$ and $i > 0$. Substituting (3.6) into (3.3) yields

$$\begin{aligned} \sum_{\substack{C \ni 0 \\ 0 \notin \text{cycle}}} z^{|C|} &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \\ &\times \left(1 + \sum_{j=1}^m \mathcal{V}_{0,j} + \sum_{1 \leq i < j \leq m} \mathcal{V}_{i,j} + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \right) \quad (3.8) \\ &= A(z) + B(z) + C(z) + E(z), \end{aligned}$$

where the four terms in the last line arise from distributing the product among the four terms within parentheses in the second line (in that order).

Therefore, substituting (3.8) into (3.2) yields

$$g(z) = A(z) + B(z) + C(z) + E(z) + \mathbf{I}_a \sum_{\substack{C \ni 0 \\ 0 \in \text{cycle}}} z^{|C|}, \quad (3.9)$$

which is the expansion for $g(z)$ that will allow us to study it at criticality in the following chapters.

The term $A(z)$ can be immediately computed. Indeed, by definition, and (1.4),

$$A(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] = \sum_{m=0}^{\infty} \frac{z^m}{m!} (2dg(z))^m = e^{2dzg(z)}. \quad (3.10)$$

The term $B(z)$ can be simplified by exchanging the order of the sums over the nearest-neighbors s_1, \dots, s_m and the label j , as follows

$$\begin{aligned} B(z) &= \sum_{m=1}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \sum_{j=1}^m \mathcal{V}_{0,j}, \\ &= - \sum_{m=1}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-1} \sum_{j=1}^m \sum_{s_j \in \mathcal{E}} \sum_{C_j \ni s_j, 0} z^{|C_j|} \\ &= - \sum_{m=1}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-1} m 2dG_z(0, e_1) \\ &= -2dze^{2dzg(z)} G_z(e_1) = -2dzA(z)G_z(e_1). \end{aligned} \quad (3.11)$$

The term $C(z)$, given by

$$C(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \sum_{1 \leq i < j \leq m} \mathcal{V}_{i,j}, \quad (3.12)$$

will be expanded in the proof of the next step of the recursive argument, in Chapter 4, to extract its

contribution to $g(z)$.

The term $E(z)$, given by

$$E(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{C}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \left(\sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \right), \quad (3.13)$$

will correspond to an error term in the proof of the next step of the recursive argument in Chapter 4.

3.2 The lace expansion

The lace expansion was developed by Brydges and Spencer in [6] to tackle problems related to self-avoiding walks in 5 or more dimensions. Originally, it involved an expansion and resummation procedure, but it has been expanded to other models like percolation, lattice trees and lattice animals, by using an inclusion-exclusion relation [39].

This technique will produce an expansion for the two-point function $G_z(x)$ in terms of the functions $g(z)$ and $\Pi_z(x)$ which, as we anticipated in (1.24), implicitly defines the critical point z_c . Moreover, through the derivation of such expansion the quantity $\Pi_z(0, x)$ will be clearly defined.

3.2.1 The lace expansion for lattice trees

We commence by defining the quantities needed in our discussion of the technique. Recall that the two-point function for lattice trees $G_z^{(t)}(x) = G_z^{(t)}(0, x)$ is given in (1.5) by

$$G_z^{(t)}(x) = \sum_{T \ni 0, x} z^{|T|}, \quad (3.14)$$

where the sum is over the lattice trees that contain both vertices 0 and x . A tree T with this characteristic consists of a path connecting 0 to x and mutually non-intersecting subtrees emanating from every vertex on this path. Due to the topology of T this path is unique, we refer to it as the *backbone*, and to the subtrees emanating from the backbone as the *ribs*. See Figure 3.2 for a depiction of such a decomposition of a tree T that contains 0 and x .

This observation allows us to rewrite $G_z^{(t)}(x)$ in terms of walks ω from 0 to x , and non-intersecting ribs emanating from the vertices of ω . Recall that $\mathcal{W}(0, x)$ is set of walks ω from 0 to x . Given $\omega \in \mathcal{W}(0, x)$, let $\vec{R} = \{R_0, \dots, R_{|\omega|}\}$ be the set of $|\omega| + 1$ mutually non-intersecting trees (ribs) R_j attached to the vertices of ω .

To guarantee that the ribs $\vec{R} = \{R_a, \dots, R_b\}$ are mutually non-intersecting, we consider

$$K[a, b] := \prod_{a \leq i < j \leq b} \left(1 + \mathcal{U}_{ij}(\vec{R}) \right), \quad (3.15)$$

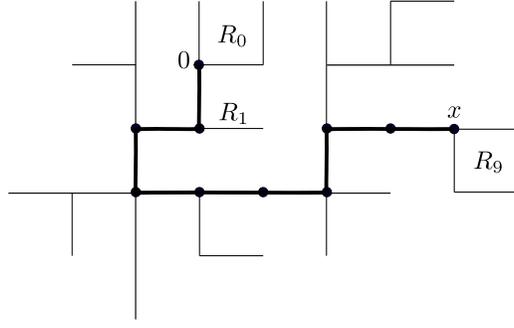


Figure 3.2: Decomposition of a lattice tree T into the backbone from 0 to x (bold), and the ribs emanating from it $\vec{R} = \{R_0, \dots, R_9\}$.

where the function $\mathcal{U}_{ij}(\vec{R})$ is defined by

$$\mathcal{U}_{ij}(\vec{R}) = \begin{cases} -1 & \text{if ribs } R_i \text{ and } R_j \text{ share a common vertex} \\ 0 & \text{if ribs } R_i \text{ and } R_j \text{ share no common vertex.} \end{cases} \quad (3.16)$$

The product (3.15) is equal to 1 when all the ribs satisfy the desired property and it vanishes otherwise.

Then the two-point function (3.14) can be written

$$G_z^{(t)}(x) = \sum_{\omega \in \mathcal{W}(0,x)} z^{|\omega|} \sum_{R_0 \ni 0} \dots \sum_{R_{|\omega|} \ni x} z^{|R_0| + \dots + |R_{|\omega|}|} K[0, |\omega|],$$

where the sum over ω is over all simple random walks from 0 to x , the sums over R_i are over trees that contain the vertex $\omega(i)$, and $\vec{R} = \{R_0, \dots, R_{|\omega|}\}$ is the set of ribs emanating from the walk ω . Note that $K[0, |\omega|]$ guarantees that only the self-avoiding walks (paths) ω contribute to the sum. In this expression we will abbreviate

$$\sum_{R_0 \ni 0} \dots \sum_{R_{|\omega|} \ni x} z^{|R_0| + \dots + |R_{|\omega|}|} = \prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|},$$

where the equal sign means equality when both sides are applied to a function of the ribs. Therefore, the two-point function can be rewritten as

$$G_z^{(t)}(x) = \sum_{\omega \in \mathcal{W}(0,x)} z^{|\omega|} \left(\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right) K[0, |\omega|]. \quad (3.17)$$

In order to rewrite the product (3.15), we define

Definition 3.3. Let $[a, b]$ be an interval of non-negative integers a, b .

1. An edge is a pair $\{i, j\}$ ($i < j$) of elements of $[a, b]$, which we will also denote by ij .
2. A graph Γ is a set of edges. The set of all graphs on $[a, b]$ is denoted by $\mathcal{B}[a, b]$.
3. A connected graph Γ on $[a, b]$, is a graph in which both a and b are endpoints of edges in Γ , and if in addition for each $c \in (a, b)$ there are $i, j \in [a, b]$ such that $i < c < j$ with either $\{i, j\} \in \Gamma$, or $\{i, c\} \in \Gamma$ and $\{c, j\} \in \Gamma$. Equivalently, Γ is connected if, as intervals, $\cup_{ij \in \Gamma} [i, j] = [a, b]$. The set of all connected graphs on $[a, b]$ is denoted by $\mathcal{G}[a, b]$.

Remark 3.4. The definition of connectivity does not correspond to the usual notion of path-connectivity in graph theory. The graphic representation of the graphs we are considering is depicted in Figure 3.3.

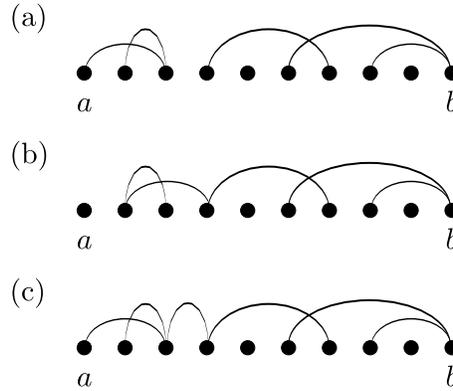


Figure 3.3: Graphs in which the edge ij is represented by an arc joining i and j . Graphs (a) and (b) are not connected, the former has an edge containing a while the latter does not. Graph (c) is connected.

We set $K[a, a] = 1$, then for $a < b$ the expression (3.15) is equivalent to

$$K[a, b] = \sum_{\Gamma \in \mathcal{B}[a, b]} \prod_{ij \in \Gamma} \mathcal{U}_{ij}, \quad (3.18)$$

where we have omitted the dependence on \vec{R} . For the case of connected graphs we consider an analogous quantity to $K[a, b]$. We set $J[a, a] = 1$, and for $a < b$ we define

$$J[a, b] = \sum_{\Gamma \in \mathcal{G}[a, b]} \prod_{ij \in \Gamma} \mathcal{U}_{ij}, \quad (3.19)$$

where the sum is restricted to the connected graphs.

Lemma 3.5. For any $a < b$

$$K[a, b] = J[a, b] + K[a+1, b] + \sum_{c=a+1}^{b-1} J[a, c]K[c+1, b]. \quad (3.20)$$

Proof of Lemma 3.5. The sum on the right hand side of (3.18) can be over the connected graphs on $[a, b]$, whose contribution is $J[a, b]$, and the disconnected graphs on $[a, b]$. In the latter case, the contribution of the graphs for which a is not in an edge is $K[a + 1, b]$. For the disconnected graphs Γ that have an edge containing a , let $c(\Gamma)$ be the largest value of $c \in (a, b)$ such that the set of edges in Γ with both ends in the interval $[a, c]$ forms a connected graph on $[a, c]$. Then the sum over these graphs Γ factorizes into sums over connected graphs on $[a, c]$, which yield the factor $J[a, c]$, and arbitrary graphs on $[c + 1, b]$, which yield the factor $K[c + 1, b]$. \square

Substituting the expression (3.20), with $a = 0$ and $b = |\omega|$, into (3.17) yields

$$\begin{aligned}
G_z^{(t)}(x) &= \sum_{R_0 \ni 0} z^{|R_0|} \delta_{0,x} + \sum_{\substack{\omega \in \mathcal{W}(0,x) \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega|] \\
&+ \sum_{\substack{\omega \in \mathcal{W}(0,x) \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] K[1, |\omega|] \\
&+ \sum_{\substack{\omega \in \mathcal{W}(0,x) \\ |\omega| \geq 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{c=1}^{|\omega|-1} J[0, c] K[c + 1, |\omega|].
\end{aligned} \tag{3.21}$$

The first term on the right hand side results from the walks with length 0 and, by (1.4), is equal to $g^{(t)}(z) \delta_{0,x}$. The other summands are due to the walks whose length is at least 1. We denote the second term by

$$\Pi_z^{(t)}(x) := \sum_{\substack{\omega \in \mathcal{W}(0,x) \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J[0, |\omega|]. \tag{3.22}$$

Let $D : \mathbb{Z}^d \rightarrow \mathbb{R}$ be the one-step transition probability function for a simple random walk in the nearest-neighbor model of \mathbb{Z}^d , i.e.,

$$D(x) = \begin{cases} (2d)^{-1} & \text{if } \|x\| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.23}$$

The *convolution* of the absolutely summable functions $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ and $g : \mathbb{Z}^d \rightarrow \mathbb{R}$ is given by

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x - y). \tag{3.24}$$

Therefore, the third term on the right hand side of (3.21) can be written as

$$\sum_{\substack{\omega \in \mathcal{W}(0,x) \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] K[1, |\omega|] = g^{(t)}(z) \left(2dzD * G_z^{(t)} \right)(x),$$

and the fourth term can be expressed as

$$\sum_{\substack{\omega \in \mathcal{W}'(0,x) \\ |\omega| \geq 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \sum_{c=1}^{|\omega|-1} J[0,c] K[c+1, |\omega|] = \left(\Pi_z^{(t)} * 2dzD * G_z^{(t)} \right) (x).$$

Hence, the expression in (3.21) is equal to

$$G_z^{(t)}(x) = g^{(t)}(z) \delta_{0,x} + \Pi_z^{(t)}(x) + g^{(t)}(z) \left(2dzD * G_z^{(t)} \right) (x) + \left(\Pi_z^{(t)} * 2dzD * G_z^{(t)} \right) (x), \quad (3.25)$$

which is the lace expansion for lattice trees. Taking the Fourier transform of the right-hand side of this equation, and using that the Fourier transform of a convolution of functions is given by the product of their Fourier transforms, i.e.,

$$\widehat{f * g} = \hat{f} \hat{g}, \quad (3.26)$$

yields

$$\hat{G}_z^{(t)}(k) = g^{(t)}(z) + \hat{\Pi}_z^{(t)}(k) + g^{(t)}(z) 2dz\hat{D}(k) \hat{G}_z^{(t)}(k) + \hat{\Pi}_z^{(t)}(k) 2dz\hat{D}(k) \hat{G}_z^{(t)}(k),$$

where we have used that $\hat{\delta} \equiv 1$. Solving for $\hat{G}_z^{(t)}(k)$ we get

$$\hat{G}_z^{(t)}(k) = \frac{g^{(t)}(z) + \hat{\Pi}_z^{(t)}(k)}{1 - 2dzg^{(t)}(z)\hat{D}(k) - 2dz\hat{D}(k)\hat{\Pi}_z^{(t)}(k)}. \quad (3.27)$$

The Fourier transform of the transition probability function $D(x)$ (3.23), is given by

$$\hat{D}(k) = \frac{1}{d} \sum_{j=1}^d \cos(k_j), \quad \text{which implies that } \hat{D}(0) = 1. \quad (3.28)$$

Therefore, evaluating (3.27) at $k = 0$, yields

$$\hat{G}_z^{(t)}(0) = \frac{g^{(t)}(z) + \hat{\Pi}_z^{(t)}(0)}{1 - 2dzg^{(t)}(z) - 2dz\hat{\Pi}_z^{(t)}(0)},$$

which agrees with equation (1.22).

In order to fully exploit expression (3.27), we will rewrite the function $\Pi_z^{(t)}(x)$ in a more manageable way. The quantity $\Pi_z^{(t)}(x)$ was defined in (3.22) in terms of $J[0, |\omega|]$, which involves a sum over connected graphs. Therefore, we focus on connected graphs. Figure 3.4 depicts two graphs of this kind, we observe that in the graph Γ , depicted in Figure 3.4 (a), the removal of the edges $\{1, 3\}$, $\{3, 4\}$ or $\{6, 9\}$ does not disconnect it, but if any other edge is suppressed it becomes disconnected. On the other hand, in the graph Γ' , depicted in (b), the absence of any of its edges disconnects it. This idea of minimal connectedness is captured below in the notion of a *lace*.

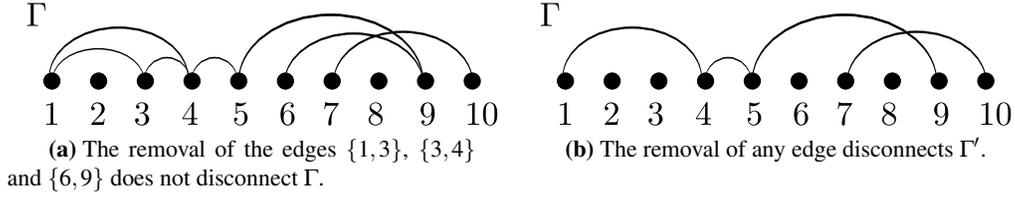


Figure 3.4: A connected graph and a minimal connected graph

Definition 3.6. A lace L is a minimal connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted by $\mathcal{L}[a, b]$, and the set of laces on $[a, b]$ which consist of exactly N edges is denoted by $\mathcal{L}^{(N)}[a, b]$.

For example, the graph depicted in Figure 3.4 (a), is not a lace but the one in (b) is. The set $\mathcal{L}^{(1)}[a, b]$ consists of the lace $L = \{ab\}$, the one whose only edge is ab . The set $\mathcal{L}^{(2)}[a, b]$ consists of laces of the form $L = \{aj_1, j_1b\}$ with $a < j_1 < b$, and $L = \{aj_1, i_2b\}$ with $a < i_2 < j_1 < b$. See Figure 3.5 for a graphical representation of these laces.

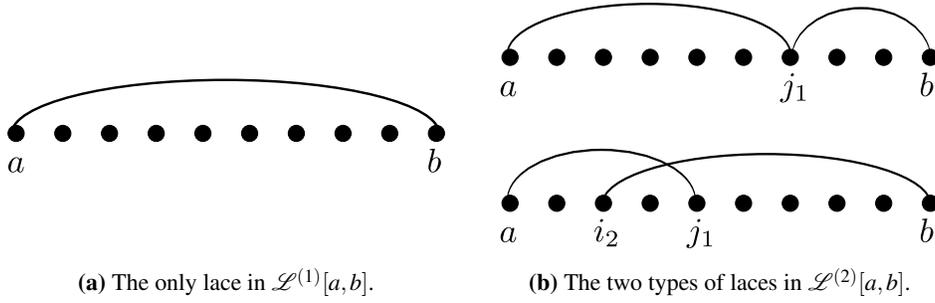


Figure 3.5: Laces in $\mathcal{L}^{(1)}[a, b]$ and $\mathcal{L}^{(2)}[a, b]$.

For $N \geq 3$, the laces $L \in \mathcal{L}^{(N)}[a, b]$ are of the form $L = \{i_1j_1, \dots, i_Nj_N\}$, with

$$a = i_1 < i_2, \quad i_{k+1} \leq j_k < i_{k+2} \quad \text{for } k = 1 \dots N-2, \quad i_N \leq j_{N-1} < j_N = b.$$

There are 2^{N-1} different topologies for the laces in $\mathcal{L}^{(N)}[a, b]$, depending on whether $i_{k+1} = j_k$ or $i_{k+1} < j_k$, like in the case of Figure 3.5 (b).

Given a connected graph $\Gamma \in \mathcal{G}[a, b]$ we can associate a *unique* lace L_Γ to it. This lace L_Γ consists of the edges i_1j_1, i_2j_2, \dots , whose endpoints are determined inductively as follows:

$$j_1 = \max \{j : aj \in \Gamma\}, \quad i_1 = a,$$

$$j_{k+1} = \max \{j : \exists i \leq j_k \text{ such that } ij \in \Gamma\}, \quad i_{k+1} = \min \{i : i \leq j_{k+1} \in \Gamma\}.$$

The induction finishes when $j_{k+1} = b$. Given a lace L , the *compatible* edges $\mathcal{C}(L)$ to L , is the set of

all edges $ij \notin L$ such that $L \cup \{ij\} = L$. See Figure 3.6 for an example of a connected graph Γ and its extracted lace L_Γ .

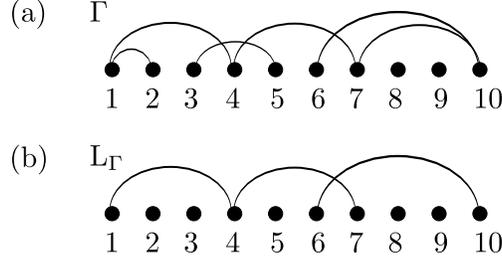


Figure 3.6: (a) A connected graph Γ . (b) The associated lace $L = L_\Gamma$. Edges $\{1, 2\}$, $\{3, 5\}$ and $\{7, 10\}$ are compatible with L .

Now we are ready to rewrite $\Pi_z^{(t)}(x)$ in a more convenient way. The core of the lace expansion is that the sum over connected graphs in the term $J[a, b]$, which was defined in (3.19), can be done by first summing over laces and then, given a lace, summing over all connected graphs whose associated lace (obtained by the above procedure) coincides with the given lace, see [39] for more details on this important result. This yields,

$$J[a, b] = \sum_{L \in \mathcal{L}[a, b]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}). \quad (3.29)$$

For $N \geq 1$, let

$$J^{(N)}[a, b] := \sum_{L \in \mathcal{L}^{(N)}[a, b]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}) \quad (3.30)$$

and

$$\Pi_z^{(t, N)}(x) := (-1)^N \sum_{\substack{\omega \in \mathcal{W}^{(0, x)} \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J^{(N)}[0, |\omega|]. \quad (3.31)$$

The fact that the set $\mathcal{L}[a, b] = \bigsqcup_{N \geq 1} \mathcal{L}^{(N)}[a, b]$ (disjoint union) and (3.29), imply

$$J[a, b] = \sum_{N \geq 1} J^{(N)}[a, b], \quad (3.32)$$

which together with (3.22) yield that the function $\Pi_z^{(t)}(x)$ is given by the alternating series

$$\Pi_z^{(t)}(x) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(t, N)}(x). \quad (3.33)$$

3.2.2 The lace expansion for lattice animals

In this section we present the lace expansion for lattice animals which, as we will see, is an extension of the lace expansion for lattice trees that we studied in Section 3.2.1. We start by giving the definitions of the quantities involved in our discussion.

The two-point function for lattice animals $G_z^{(a)}(x) = G_z^{(a)}(0, x)$ was given in (1.5) by

$$G_z^{(a)}(x) = \sum_{A \ni 0, x} z^{|A|}, \quad (3.34)$$

where the sum is over the lattice animals that contain both vertices 0 and x . An animal A with this characteristic contains a path connecting 0 to x ; however, unlike the lattice tree case, this path is not necessarily unique. To deal with this difficulty we need the following definitions.

Definition 3.7. *Let A be an animal containing the vertices x and y .*

1. *The animal A has a double connection from x to y , if there are two bond-disjoint self avoiding walks in A between x and y (the walks may share a common vertex, but not a common bond), or if $x = y$. The set of all animals having a double connection between x and y is denoted by $\mathcal{D}_{x,y}$.*
2. *A bond $\{x, y\}$ in A is pivotal for the connection from x to y , if its removal would disconnect the animal into two connected components with x contained in one of them and y in the other.*

An animal $A \ni x, y$ that is not an element of $\mathcal{D}_{x,y}$, has at least one pivotal bond for the connection from x to y . To establish an order among these edges, we define the *first* pivotal bond to be the first edge for which there is a double connection between x and one of the endpoints of this bond. This endpoint is the *first* endpoint of the first pivotal bond. To determine the *second* pivotal bond, the role of x is played by the second endpoint of the first pivotal bond, and so on.

For a lattice animal A that contains x and y , the *backbone* is the set of pivotal bonds for the connection from x to y . Notice that the backbone is not necessarily connected. The *ribs* are the connected components that remain after the backbone is removed from A . By definition, the ribs do not have any pivotal bond for the connection from x to y , so they have a double connection between the end-points of the corresponding pivotal bonds. See Figure 3.7 for an example of an animal decomposed into its backbone and ribs.

Let B be an arbitrary finite ordered set of directed bonds

$$B = ((u_1, v_1), \dots, (u_{|B|}, v_{|B|})),$$

and let $v_0 = 0$ and $u_{|B|+1} = x$. Then we can write the two-point function (3.34) in terms of the

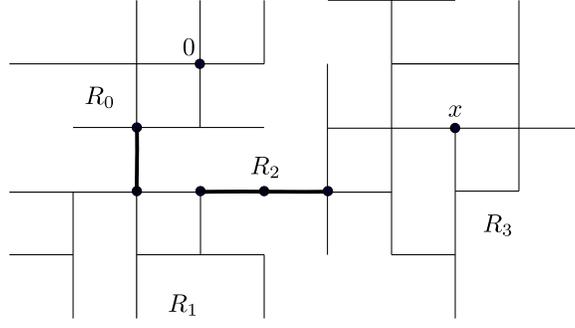


Figure 3.7: Decomposition of a lattice animal A into the backbone from 0 to x (bold), and the ribs emanating from it $\vec{R} = \{R_0, R_1, R_2, R_3\}$. Note that R_2 only contains the vertex in the backbone.

backbone B and the mutually nonintersecting ribs $\vec{R} = \{R_0, \dots, R_{|B|}\}$, as

$$G_z^{(a)}(x) = \sum_{B:|B|\geq 0} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] K[0, |B|], \quad (3.35)$$

where $K[0, |B|]$ is defined as in (3.15), with $\mathcal{U}_{ij}(\vec{R})$ as in (3.16) for the new notion of ribs \vec{R} .

We recall the definition of $J[0, |B|]$ in (3.19), and define

$$\tilde{\Pi}_z^{(a)}(x) := \sum_{B:|B|\geq 1} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] J[0, |B|], \quad (3.36)$$

and

$$h_z^{(a)}(y-x) := \sum_{R \in \mathcal{D}_{x,y}} z^{|R|}. \quad (3.37)$$

Notice that $\mathcal{D}_{x,x}$ is the set of animals that contain x and that $h_z^{(a)}(0) = g^{(a)}(z)$.

An argument similar to the one used to derive (3.25), using (3.20), yields the lace expansion for lattice animals

$$G_z^{(a)}(x) = h_z^{(a)}(x) + \tilde{\Pi}_z^{(a)}(x) + \left(h_z^{(a)} * 2dzD * G_z^{(a)} \right)(x) + \left(\tilde{\Pi}_z^{(a)} * 2dzD * G_z^{(a)} \right)(x). \quad (3.38)$$

To make this expression look like (3.25), we define

$$\Pi_z^{(a,0)}(x) := (1 - \delta_{0,x}) \sum_{R \in \mathcal{D}_{0,x}} z^{|R|}, \quad (3.39)$$

and

$$\Pi_z^{(a)}(x) := \Pi_z^{(a,0)}(x) + \tilde{\Pi}_z^{(a)}(x). \quad (3.40)$$

Then

$$h_z^{(a)}(x) = g^{(a)}(z)\delta_{0,x} + \Pi_z^{(a,0)}(x),$$

which implies that (3.38) can be written as

$$G_z^{(a)}(x) = g^{(a)}(z)\delta_{0,x} + \Pi_z^{(a)}(x) + g^{(a)}(z) \left(2dzD * G_z^{(a)} \right) (x) + \left(\Pi_z^{(a)} * 2dzD * G_z^{(a)} \right) (x). \quad (3.41)$$

We use relation (3.26) to compute the Fourier transform of this expression, and get

$$\hat{G}_z^{(a)}(k) = g^{(a)}(z) + \hat{\Pi}_z^{(a)}(k) + g^{(a)}(z)2dz\hat{D}(k)\hat{G}_z^{(a)}(k) + \hat{\Pi}_z^{(a)}(k)2dz\hat{D}(k)\hat{G}_z^{(a)}(k).$$

Isolating $\hat{G}_z^{(a)}(k)$ in this expression yields

$$\hat{G}_z^{(a)}(k) = \frac{g^{(a)}(z) + \hat{\Pi}_z^{(a)}(k)}{1 - 2dzg^{(a)}(z)\hat{D}(k) - 2dz\hat{D}(k)\hat{\Pi}_z^{(a)}(k)}. \quad (3.42)$$

This result and the fact that $\hat{D}(0) = 1$, as established in (3.28), implies that

$$\hat{G}_z^{(a)}(0) = \frac{g^{(a)}(z) + \hat{\Pi}_z^{(a)}(0)}{1 - 2dzg^{(a)}(z) - 2dz\hat{\Pi}_z^{(a)}(0)},$$

which, like in the lattice tree case, agrees with equation (1.22).

Our aim is to rewrite $\Pi_z^{(a)}(x)$ in a more manageable way. We proceed similarly to what we did for the corresponding quantity for lattice trees. For $N \geq 1$, we define $\Pi_z^{(a,N)}$ by substituting $J[0, |B|]$ by $J^{(N)}[0, |B|]$ in (3.36),

$$\Pi_z^{(a,N)}(x) := (-1)^N \sum_{B:|B|\geq 1} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] J^{(N)}[0, |B|], \quad (3.43)$$

where $J^{(N)}[0, |B|]$ is defined as in (3.30). This is analogous to the lattice tree quantity $\Pi_z^{(t,N)}$ given in (3.31). The fact that $J[0, |B|] = \sum_{N \geq 1} J^{(N)}[0, |B|]$, as established in (3.32), together with equations (3.36) and (3.43) yields

$$\tilde{\Pi}_z^{(a)}(x) = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(a,N)}(x).$$

Finally, by (3.40), we get

$$\Pi_z^{(a)}(x) = \sum_{N=0}^{\infty} (-1)^N \Pi_z^{(a,N)}(x). \quad (3.44)$$

Chapter 4

Proof of first order correction

In this chapter we compute the first order correction for the critical points of lattice trees and animals in the nearest-neighbor case. This result corresponds to the second term in the expansion of z_c in Theorem 1.2, which we specifically state in Lemma 4.1 below.

In Section 4.1, we give the proof of Lemma 4.1 after assuming the expansions for $\hat{\Pi}_{z_c}(0)$ and $g(z_c)$ given below in Lemma 4.2 and Lemma 4.3, respectively.

In Section 4.2, we introduce the quantities involved in the demonstrations of Lemma 4.2 and Lemma 4.3, and give some estimates on them.

In Section 4.3, we show the expansion for $\hat{\Pi}_{z_c}(0)$ in Lemma 4.2. We do it for lattice trees in Section 4.3.1, and for lattice animals in Section 4.3.2.

Finally, in Section 4.4 we show the expansion for $g(z_c)$ in Lemma 4.3, simultaneously, for lattice trees and animals.

We remind the reader that we will only work in the nearest-neighbor model. So all the results in this chapter and the coming one, only apply to this case.

4.1 Main ingredients

Our goal is to show that

Lemma 4.1. *For nearest-neighbor lattice trees and animals, the critical point obeys*

$$z_c = \frac{e^{-1}}{2d} + \frac{\frac{3}{2}e^{-1}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (4.1)$$

In the proof of Lemma 4.1 there are two main ingredients. The first one is the following lemma, which we prove in Section 4.3.

Lemma 4.2. *For nearest-neighbor lattice trees and animals, as $d \rightarrow \infty$,*

$$\hat{\Pi}_{z_c}(0) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.2)$$

This result will allow us to show, in Section 4.4, the second ingredient for the proof of Lemma 4.1. This states that the one-point function at criticality obeys

Lemma 4.3. *For nearest-neighbor lattice trees and animals, as $d \rightarrow \infty$,*

$$g(z_c) = e + \frac{\frac{3}{2}e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.3)$$

Assuming Lemma 4.2 and Lemma 4.3, the demonstration of Lemma 4.1 is as follows.

Proof of Lemma 4.1. Substituting the expansions given in (4.27) and (4.53), into the formula that the lace expansion produces for the critical point in (1.24), yields that

$$z_c = \frac{1}{2d} \frac{1}{g(z_c) + \hat{\Pi}_{z_c}(0)} = \frac{1}{2de} \frac{1}{1 - \frac{\frac{3}{2}e}{2d} + o\left(\frac{1}{2d}\right)} = \frac{1}{2de} \left[1 + \frac{\frac{3}{2}e}{2d} + o\left(\frac{1}{2d}\right) \right], \text{ as } d \rightarrow \infty,$$

which is the desired expression. \square

Therefore, it only remains to prove Lemma 4.2 and Lemma 4.3. Before accomplishing this task we need to define and bound the quantities involved in the demonstrations. This is the purpose of the following section.

4.2 Useful estimates

In this section, we formulate and estimate the essential quantities used in the proofs of Lemma 4.2 and Lemma 4.3. These are given in Proposition 4.5 and Claim 4.6 below.

Recall the convolution of the functions f and g , given in (3.24). We will denote by f^{*l} the convolution of l factors of f , i.e.,

$$f^{*l}(x) := \underbrace{(f * f * \cdots * f)}_l(x).$$

Armed with this notation and the definition of the transition probability $D(x)$ in (3.23), we can see that the quantity $D^{*2m}(0)$ is the probability that a random walk that starts at the origin returns to it after $2m$ steps. Bound [24, (3.12)] establishes that for positive integers m there is a positive constant

c_m , independent of the dimension d , such that the inequality below holds

$$D^{*2m}(0) = \int_{[-\pi, \pi]^d} \hat{D}(k)^{2m} \frac{d^d k}{(2\pi)^d} \leq \frac{c_m}{(2d)^m}, \quad (4.4)$$

where the equality is given by the inverse Fourier transform and relation (3.26).

The *infrared bound* for nearest-neighbor lattice trees and lattice animals, given in [16, (1.25)], establishes that for dimensions $d \geq d_0$, where d_0 is sufficiently large, there is a positive constant c independent of z and d , such that for $0 \leq z \leq z_c$,

$$0 \leq \hat{G}_z(k) \leq \frac{cd}{\|k\|_2^2}, \quad (4.5)$$

where $\|k\|_2 := (k_1^2 + \dots + k_d^2)^{1/2}$ is the Euclidean norm.

Proposition 4.4. *In the case of nearest-neighbor lattice trees and lattice animals, if m and n are non-negative integers and the dimension $d > \max\{d_0, 4n\}$, then there is a constant $K_{m,n}$, whose value only depends on m and n , such that for all $0 \leq z \leq z_c$,*

$$\sup_x D^{*m} * G_z^{*n}(x) \leq \frac{K_{m,n}}{(2d)^{m/2}}. \quad (4.6)$$

Proof of Proposition 4.4. Using the inverse Fourier transform (1.21) and relation (3.26), we get

$$D^{*m} * G_z^{*n}(x) = \int_{[-\pi, \pi]^d} \hat{D}(k)^m \hat{G}_z(k)^n e^{-ik \cdot x} \frac{d^d k}{(2\pi)^d}.$$

The Cauchy-Schwarz inequality implies

$$D^{*m} * G_z^{*n}(x) \leq \left(\int_{[-\pi, \pi]^d} |\hat{D}(k)|^{2m} \frac{d^d k}{(2\pi)^d} \right)^{1/2} \left(\int_{[-\pi, \pi]^d} \hat{G}_z(k)^{2n} \frac{d^d k}{(2\pi)^d} \right)^{1/2}, \quad (4.7)$$

where we have used that the left-hand side of this expression is nonnegative, and that so is $\hat{G}_z(k)$ for $0 \leq z \leq z_c^{(t)}$, according to (4.5). By (3.28), we also know that the Fourier transform $\hat{D}(k)$ is real, which implies that $|\hat{D}(k)|^{2m} = \hat{D}(k)^{2m}$. Then estimate (4.4) gives the desired result, as long as we show that the second factor on the right-hand side of (4.7) is finite independently of d . This is what we do next.

The infrared bound (4.5) implies

$$\int_{[-\pi, \pi]^d} \hat{G}_z(k)^{2n} \frac{d^d k}{(2\pi)^d} \leq c^{2n} \int_{[-\pi, \pi]^d} \left(\frac{d}{\|k\|_2^2} \right)^{2n} \frac{d^d k}{(2\pi)^d} \quad (4.8)$$

To deal with the right-hand side of this expression, we use that for $A > 0$ and $j > 0$,

$$\frac{1}{A^j} = \frac{1}{\Gamma(j)} \int_0^\infty u^{j-1} e^{-uA} du. \quad (4.9)$$

Thus taking $A = \frac{\|k\|_2^2}{d}$ in (4.9) and using Fubini theorem, yield

$$\begin{aligned} c^{2n} \int_{[-\pi, \pi]^d} \left(\frac{d}{\|k\|_2^2} \right)^{2n} \frac{d^d k}{(2\pi)^d} &= \frac{c^{2n}}{\Gamma(2n)} \int_{[-\pi, \pi]^d} \int_0^\infty u^{2n-1} e^{-u \frac{\|k\|_2^2}{d}} du \frac{d^d k}{(2\pi)^d} \\ &= \frac{c^{2n}}{\Gamma(2n)} \int_0^\infty u^{2n-1} \left(\int_{-\pi}^\pi e^{-u \frac{k^2}{d}} \frac{dk}{2\pi} \right)^d du \\ &\leq \frac{c^{2n}}{\Gamma(2n)} \int_0^\infty u^{2n-1} \left(\int_{-\pi}^\pi e^{-u \frac{k^2}{d} p} \frac{dk}{2\pi} \right)^{\frac{d}{p}} du, \end{aligned}$$

where we have used convexity in the last inequality with $p \geq 1$. Let $d_1 = \max\{d_0, 4n\} + 1$, clearly $d_1 \leq d$. If we choose p such that $\frac{d}{p} = d_1$, then the expression in the last line becomes

$$\begin{aligned} \frac{c^{2n}}{\Gamma(2n)} \int_0^\infty u^{2n-1} \left(\int_{-\pi}^\pi e^{-u \frac{k^2}{d_1}} \frac{dk}{2\pi} \right)^{d_1} du &= \frac{c^{2n}}{\Gamma(2n)} \int_{[-\pi, \pi]^{d_1}} \int_0^\infty u^{2n-1} e^{-u \frac{\|k\|_2^2}{d_1}} du \frac{d^{d_1} k}{(2\pi)^{d_1}} \\ &= c^{2n} \int_{[-\pi, \pi]^{d_1}} \left(\frac{d_1}{\|k\|_2^2} \right)^{2n} \frac{d^{d_1} k}{(2\pi)^{d_1}}, \end{aligned}$$

where in the last equality we have used (4.9) in a reversed way. This proves that the right-hand side of (4.8) is non-increasing in the dimension d . Now, we show that it is finite for the dimension $d = d_1$. A change to hyper spherical coordinates gives

$$c^{2n} \int_{[-\pi, \pi]^{d_1}} \left(\frac{d_1}{\|k\|_2^2} \right)^{2n} \frac{d^{d_1} k}{(2\pi)^{d_1}} \leq (d_1 c)^{2n} N_\Theta^{d_1} \int_0^R r^{d_1-1-4n} dr = (d_1 c)^{2n} N_\Theta^{d_1} \frac{R^{d_1-4n}}{d_1-4n} < \infty,$$

where $R := \sqrt{d_1} \pi$ is big enough to contain the hypercube $[-\pi, \pi]^{d_1}$, and $N_\Theta^{d_1}$ denotes the value of the integrals over the angular coordinates. Notice that the integral on the radial variable r is finite since $d_1 > 4n$. Hence, (4.6) holds with

$$K_{m,n} = c_m^{1/2} (d_1 c)^{2n} N_\Theta^{d_1} \frac{R^{d_1-4n}}{d_1-4n}.$$

□

Let k be a non-negative integer and let C be a cluster (a tree or an animal) containing the vertices x and y . We denote by

$$\{x \overset{k}{\leftrightarrow} y\} \quad (4.10)$$

point functions. See Figure 4.1a for a depiction of $S_z^{(m,n)}(y-x)$.

Proposition 4.5 below is the estimate on $S_z^{(m,n)}(y-x)$ used to bound error terms. Essentially, it states that the quantity $S_z^{(m,n)}(x)$ is of order at most $O((2d)^{m/2})$, independently of n and x . See Figure 4.1 for a depiction on the use of $S_z^{(m,n)}$ and Proposition 4.5 for bounding different diagrams.

Proposition 4.5. *Let m and n be positive integers, if the dimension $d > \max\{d_0, 4m\}$, then there is a constant $C_{m,n}$, whose value only depends on m and n , such that for all $z \leq z_c$,*

$$S_z^{(m,n)} := \sup_x S_z^{(m,n)}(x) \leq \frac{C_{m,n}}{(2d)^{m/2}}. \quad (4.13)$$

Proof of Proposition 4.5. We commence by analyzing $G_z^{(k)}(x,y)$ for $k \geq 1$, which is the particular case $S_z^{(k,1)}(y-x)$. If in (4.11) we neglect the self avoidance restriction among the k steps of the path connecting x and y , and treat the ribs emanating from them as independent, we get

$$G_z^{(k)}(x,y) \leq (2dzg(z)D)^{*k} * G_z(x-y). \quad (4.14)$$

This bound and the definition of $S_z^{(m,n)}(x)$ in (4.12) imply

$$\begin{aligned} S_z^{(m,n)}(x) &= \sum_{k_1+\dots+k_n=m} G_z^{(k_1)} * \dots * G_z^{(k_n)}(x) \\ &\leq \sum_{k_1+\dots+k_n=m} \left((2dzg(z)D)^{*k_1} * G_z \right) * \dots * \left((2dzg(z)D)^{*k_n} * G_z \right) (x) \\ &= \sum_{k_1+\dots+k_n=m} (2dzg(z)D)^{*m} * G_z^{*n}(x) = \tilde{C}_{m,n} (2dzg(z)D)^{*m} * G_z^{*n}(x), \end{aligned}$$

where $\tilde{C}_{m,n}$ is the corresponding combinatorial factor, its exact value is not important only the fact that it depends on m and n . By (1.12), since d is large enough, we know that for $z \leq z_c$,

$$2dzg(z) \leq 2dzg(z_c) \leq 2, \quad (4.15)$$

which together with Proposition 4.4 imply

$$S_z^{(m,n)}(x) \leq \tilde{C}_{m,n} \frac{2^m K_{m,n}}{(2d)^{m/2}}.$$

□

The following quantity is another essential ingredient in coming demonstrations. For vertices x and y , we define the negative quantity

$$W_z(x,y) := \sum_{C_0 \ni x} \sum_{C_1 \ni y} z^{|C_0|+|C_1|} \mathcal{W}_{0,1}, \quad (4.16)$$

where the sums are over the clusters C_0 and C_1 (both trees or animals) that contain the vertices x and y , respectively; and the factor $\mathcal{U}_{0,1} = \mathcal{U}_{0,1}(C_0, C_1) := -\mathbf{I}_{\{C_0 \cap C_1 \neq \emptyset\}}$, is minus the indicator function that the clusters C_0 and C_1 have a vertex in common. We will denote $W_z(0, x)$ by $W_z(x)$. Notice that due to the symmetries of the lattice, $W_z(x, y) = W_z(y - x)$. Claim 4.6 below, gives an estimate on $W_z(x, y)$ which is sufficient for the proofs in this chapter; however, it will be extended in the following chapter.

Claim 4.6. *For nearest-neighbor lattice trees and animals, let x be a nearest neighbor of the origin, then as $d \rightarrow \infty$,*

$$W_{z_c}(x) = -\frac{2e^2}{2d} + o\left(\frac{1}{2d}\right). \quad (4.17)$$

Proof of Claim 4.6. For the case $W_{z_c}(x) = W_{z_c}^{(a)}(x)$, i.e., the clusters C_0 and C_1 involved in the sums of (4.16) are animals, we distinguish two cases:

- (a) The origin and x belong to a cycle that is contained in either animal C_0 or animal C_1 .
- (b) Both animals C_0 and C_1 , do not have any cycle that contains the origin and x .

For part (a), suppose that C_0 is the cluster that contains such a cycle. The smallest cycles that contain 0 and x have length 4. The other cycles containing both vertices have length 6 or more and, by Proposition 4.5, their contribution to $W_{z_c}^{(a)}(x)$ is of order $O((2d)^3)$ and can be neglected. The number of 4-step cycles that contain 0 and x is $(2d - 2)$. The contribution to $W_{z_c}^{(a)}(x)$ of clusters with such cycles is bounded by

$$2(2d - 2)(z_c^{(a)})^4 g^{(a)}(z_c^{(a)})^5, \quad (4.18)$$

where the factor 2 is obtained when inverting the roles of C_0 and C_1 ; one of the factors $g^{(a)}(z)$ corresponds to treating cluster C_1 as independent of C_2 ; and the remaining factors are obtained when treating the ribs emanating from the vertices in the 4-step cycles as independent of each other.

Theorem 1.1 yields that this expression is of order

$$2(2d - 2)(z_c^{(a)})^4 g^{(a)}(z_c^{(a)})^5 = O((2d)^{-3}). \quad (4.19)$$

Therefore, the contribution to $W_{z_c}^{(a)}(x)$ from the animals of type (a) is negligible. We only need to show that the contribution to $W_{z_c}^{(a)}(x)$ from the animals of type (b), agrees with (4.17).

The clusters C_0 and C_1 , both trees or type (b) animals, only contribute to $W_z(x)$ if they have a vertex in common, say y . Therefore, there is a path ω connecting 0 to x , passing through y , that consists in a path in C_0 connection 0 to x , and a path in C_1 connecting y to x . This path ω can be chosen uniquely. The simplest path ω consists of the bond $\{0, x\}$, which implies that $y = x$ or $y = 0$ depending on whether $\{0, x\}$ is contained in either C_0 or C_1 , respectively. Then this cluster

(that is, C_0 , respectively, C_1) consists of the edge $\{0, x\}$ and two non intersecting subclusters, one emanating from 0 and the other one emanating from x , which we denote by C_0^* and C_1^* , respectively. Discounting the contribution due to the event in which both clusters C_0 and C_1 contain $\{0, x\}$ yields the upper bound

$$W_z(x) \leq -2zg(z) \sum_{\substack{C_0^* \ni 0 \\ C_1^* \ni x}} z^{|C_0^*|+|C_1^*|} (1 + \mathcal{W}_{0,1}^*) + z^2 \left[\sum_{\substack{C_0^* \ni 0 \\ C_1^* \ni x}} z^{|C_0^*|+|C_1^*|} (1 + \mathcal{W}_{0,1}^*) \right]^2,$$

where $\mathcal{W}_{0,1}^* = -1$ if the subclusters C_0^* and C_1^* have a common vertex, and 0 otherwise. See Figure 4.2 for a depiction of this bound.

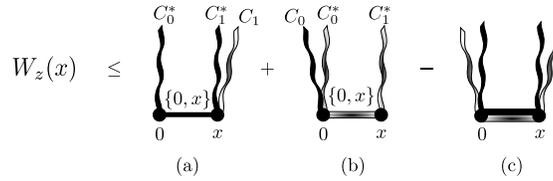


Figure 4.2: Curved lines represent (sub)clusters, the ones with the same shading are non intersecting. The straight line corresponds to the edge $\{0, x\}$. (a) $C_0 = C_0^* \cup \{0, x\} \cup C_1^*$ is independent of C_1 . (b) C_0 is independent of $C_1 = C_0^* \cup \{0, x\} \cup C_1^*$. (c) Intersection of both situations (a) and (b).

Distributing the product in the first term of the previous expression, together with the definitions of $W_z(x)$ and $g(z)$, and the fact that the factor $1 + \mathcal{W}_{0,1}^*$ in the second term is at most 1, give the upper bound

$$W_z(x) \leq -2zg(z)^3 - 2zg(z)W_z(x) + z^2g(z)^4.$$

Thus

$$W_z(x) \leq \frac{-2zg(z)^3 + z^2g(z)^4}{1 + 2zg(z)}.$$

Theorem 1.1 yields

$$W_z(x) \leq -\frac{2e^2}{2d} + o\left(\frac{1}{2d}\right), \text{ as } d \rightarrow \infty.$$

For a lower bound, we neglect the non-intersection of the subclusters C_0^* and C_1^* , this yields the first term in the inequality below. The second term, is due to the fact that the remaining paths ω , connecting 0 and x , have length at least 3. See Figure 4.3 for a depiction of this situation.

$$W_z(x) \geq -2zg(z)^3 - S_z^{(3,2)}(x).$$

where the last equality is due to Theorem 1.1 and Claim 4.6. This completes the proof of (4.20) for the case of lattice trees.

Lattice animals. Now we analyze the two-point function for lattice animals, which was defined in (3.35), in terms of the lace expansion, as

$$G_z^{(a)}(x) = \sum_{B:|B|\geq 0} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] K[0, |B|].$$

We separate the sum over backbones B in the cases $|B| = 0$, $|B| = 1$, $|B| = 2$ and $|B| \geq 3$. We commence with the case $|B| = 1$ which, as we will see, is the only one that contributes to $G_z^{(a)}(x)$ up to order $(2d)^{-1}$.

For the case $|B| = 1$, the backbone is given for some directed edge (u, v) . Then the contribution to $G_z^{(a)}(x)$ is

$$\sum_{(u,v)} z \sum_{\substack{R_0 \in \mathcal{D}_{0,u} \\ R_1 \in \mathcal{D}_{v,x}}} z^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}),$$

which, distinguishing whether the backbone (u, v) is given by the edge $(0, x)$, becomes

$$z \sum_{\substack{R_0 \ni 0 \\ R_1 \ni x}} z^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}) + z \sum_{\substack{u,v \\ (u,v) \neq (0,x)}} 2dD(v-u) \sum_{\substack{R_0 \in \mathcal{D}_{0,u} \\ R_1 \in \mathcal{D}_{v,x}}} z^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}). \quad (4.22)$$

The first term agrees with the corresponding expression for lattice trees when the backbone $|\omega|$ has length 1, then a similar argument to (4.21) together with Theorem 1.1 and Claim 4.6, yield that it is $e(2d)^{-1} + o((2d)^{-1})$, which gives the desired result (4.20). To show that the second term's contribution is negligible, notice that if the backbone is not $(0, x)$, then the double connections between 0 and u , and between v and x , require at least eight edges. Therefore, this term is bounded by

$$\begin{aligned} & z \sum_{\substack{u,v \\ (u,v) \neq (0,x)}} 2dD(v-u) \sum_{\substack{R_0 \in \mathcal{D}_{0,u} \\ R_1 \in \mathcal{D}_{v,x}}} z^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}) \\ & \leq 2dz \sum_{i+j+k+l=8} \sum_{\substack{u,v \\ (u,v) \neq (0,x)}} D(v-u) G_z^{(a,i)}(u) G_z^{(a,j)}(u) G_z^{(a,k)}(x-v) G_z^{(a,l)}(x-v) \\ & \leq 2dz \sum_{i+j+k+l=8} \sup_u G_z^{(a,i)}(u) \sup_v G_z^{(a,k)}(v) D * G_z^{(a,j)} * G_z^{(a,l)}(x) \\ & \leq \frac{2dz}{(2d)^{9/2}} \sum_{i+j+k+l=8} \left(2dzg^{(a)}(z) \right)^{j+l} C_{i,1} C_{k,1} K_{1+j+l,2}. \end{aligned}$$

The last inequality is due to Proposition 4.4 and Proposition 4.5. Theorem 1.1 yields that the con-

tribution of this term at criticality is of order $o((2d)^{-4})$, as $d \rightarrow \infty$, and can be neglected. It only remains to prove that the cases $|B| = 0$, $|B| = 2$ and $|B| \geq 3$ produce error terms.

When $|B| = 0$, the contribution to $G_z^{(a)}(x)$ is

$$\sum_{R \in \mathcal{D}_{0,x}} z^{|R|},$$

where the sum is over the animals R containing 0 and x that have a double connection from 0 to x , i.e., there are two self-avoiding walks in R between 0 and x that share no common bond. These two walks form a cycle containing 0 and x . This situation is very similar to part (a) in the proof of Claim 4.6, the difference is that now there is only one animal containing both vertices, which implies that we need to omit the factors $2g^{(a)}(z_c^{(a)})$ from (4.18). Then Theorem 1.1 and Proposition 4.5 yield

$$\sum_{R \in \mathcal{D}_{0,x}} (z_c^{(a)})^{|R|} = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (4.23)$$

When $|B| = 2$, the contribution to $G_z^{(a)}(x)$ is

$$\sum_{B:|B|=2} z^2 \left[\prod_{i=0}^2 \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] K[0, 2]. \quad (4.24)$$

Due to the topology of \mathbb{Z}^d , the fact that x is at distance 1 from the origin, and that the backbone has length 3, at least one of the three double connections must have a cycle (of length 4 or more) containing the corresponding pivotal end points. The contribution of the double connections with such cycles will be bounded by the term $3 \sum_{j+k=2} G_z^{(a,2+j)}(u_1) G_z^{(a,k)}(u_1)$ below, the factor 3 takes into account the three possibilities for the double connection.

$$\begin{aligned} & \sum_{B:|B|=2} z^2 \left[\prod_{i=0}^2 \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] K[0, 2] \\ &= \sum_{\substack{u_1, v_1 \\ u_2, v_2}} (2dz)^2 D(v_1 - u_1) D(v_2 - u_2) \left[\prod_{i=0}^2 \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] K[0, 2] \\ &\leq 3 \sum_{\substack{u_1, v_1 \\ u_2, v_2}} (2dz)^2 D(v_1 - u_1) D(v_2 - u_2) \left(\sum_{j+k=2} G_z^{(a,2+j)}(u_1) G_z^{(a,k)}(u_1) \right) G_z^{(a)}(v_1 - u_2) G_z^{(a)}(x - v_2) \\ &\leq 3 (2dz)^2 \sum_{j+k=2} \sup_u G_z^{(a,2+j)}(u) D^{*2} * G_z^{(a,k)} * \left(G_z^{(a)} \right)^{*2}(x), \end{aligned} \quad (4.25)$$

where the last line was obtained by exchanging the sums over the labels j and k and the vertices u_1 ,

v_1 , u_2 and v_2 ; the convolution is obtained after summing over u_1 , v_1 , u_2 and v_2 , in that order. Then inequality (4.14) gives the bound

$$3(2dz)^2 \sum_{j+k=2} S_z^{(2+j,1)} (2dzg(z))^k D^{*(2+k)} * (G_z)^{*3}(x).$$

Theorem 1.1, Proposition 4.4 and Proposition 4.5 yield that this expression, at criticality, is

$$3 \left(2dz_c^{(a)}\right)^2 \frac{1}{(2d)^3} \sum_{j+k=2} \left(2dz_c^{(a)} g^{(a)}(z_c^{(a)})\right)^k C_{2+j,1} K_{2+k,3} = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (4.26)$$

Finally, when $|B| \geq 3$, the contribution to $G_z^{(a)}(x)$ is bounded by $G_{z_c^{(a)}}^{(a,3)} \leq S_{z_c^{(a)}}^{(3,1)}$, which is of order $O((2d)^{-2/3})$, by Proposition 4.5, and can be neglected. \square

4.3 Proof of Lemma 4.2

The goal of this section is to show Lemma 4.2, which states that for nearest-neighbor lattice trees and lattice animals,

$$\hat{\Pi}_{z_c}(0) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right), \text{ as } d \rightarrow \infty. \quad (4.27)$$

In the proof we use extensively Proposition 4.5 and Claim 4.6. For convenience we have separated the proof in two parts, the one for lattice trees in Section 4.3.1, and the one for lattice animals in Section 4.3.2.

4.3.1 Lattice trees

Proof of Lemma 4.2 for lattice trees. Recall that the Fourier transform $\hat{\Pi}_{z_c^{(t)}}^{(t)}(0)$ is given in (3.33) by

$$\hat{\Pi}_{z_c^{(t)}}^{(t)}(0) = -\hat{\Pi}_{z_c^{(t)}}^{(t,1)}(0) + \sum_{N \geq 2} (-1)^N \hat{\Pi}_{z_c^{(t)}}^{(t,N)}(0). \quad (4.28)$$

It is known (see [16, (1.23)]) that there is a $c > 0$ such that for all $N \geq 1$ and all $z \in [0, z_c^{(t)}]$,

$$0 \leq \hat{\Pi}_{z_c^{(a)}}^{(a,N)}(0) \leq c^N d^{-N}, \quad (4.29)$$

and this implies that the second summand in (4.28) is of order $O((2d)^{-2})$. Therefore, it is enough to prove

$$-\hat{\Pi}_{z_c^{(t)}}^{(t,1)}(0) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.30)$$

The quantity $\hat{\Pi}_{z_c}^{(t,1)}(0)$ is the Fourier transform of the function $\Pi_z^{(t,1)}(0,x)$, at criticality, and is given by

$$\hat{\Pi}_{z_c}^{(t,1)}(0) = \sum_x \Pi_{z_c}^{(t,1)}(x). \quad (4.31)$$

The function $\Pi_z^{(t,1)}(x)$ corresponds to the term in the lace expansion whose lace has only one edge. It was defined in (3.31) and Figure 3.5 (a) depicts a graphical representation of it. Below it is given in more detail

$$\begin{aligned} \Pi_z^{(t,1)}(x) &= - \sum_{\substack{\omega \in \mathcal{W}^{(0,x)} \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] J^{(1)}[0, |\omega|] \\ &= - \sum_{\substack{\omega \in \mathcal{W}^{(0,x)} \\ |\omega| \geq 1}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \mathcal{U}_{0,|\omega|} \prod_{\substack{0 \leq s < t \leq |\omega| \\ (s,t) \neq (0,|\omega|)}} (1 + \mathcal{U}_{st}), \end{aligned} \quad (4.32)$$

the factor $\mathcal{U}_{0,|\omega|}$ forces the first and last ribs to have a common vertex, say y , and the final product avoids any other intersection among the ribs.

Notice that the sum in (4.31) is over all the vertices, in particular, when $x = 0$, the walk ω in (4.32) forms a cycle that contains the origin, and automatically there is an intersection between the first and last ribs, which implies that the factor $\mathcal{U}_{0,|\omega|} = -1$. Similarly, when the vertex $x \neq 0$, the paths connecting 0 and x , 0 and y , and x and y , form a cycle. See Figure 4.4 for a depiction of these cycles.

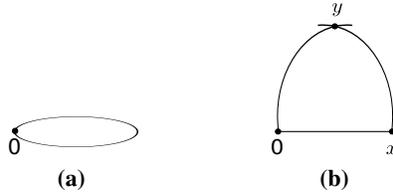


Figure 4.4: (a) Cycle formed in $\Pi_z^{(t,1)}(0)$. (b) Cycle formed in $\Pi_z^{(t,1)}(x)$, for $x \neq 0$.

We will denote by $\hat{\Pi}_z^{(t,1,n)}(x)$ the contribution to $\hat{\Pi}_z^{(t,1)}(x)$ by the cases in which the cycle formed in (4.32) has length n . Proposition 4.5 imply

$$\begin{aligned} \left| \hat{\Pi}_z^{(t,1,\geq n)}(x) \right| &\leq \sum_{j+k+l=n} \sum_x \sum_y G_z^{(t,j)}(y) G_z^{(t,k)}(y-x) G_z^{(t,l)}(x) \\ &= \sum_{j+k+l=n} G_z^{(t,j)} * G_z^{(t,k)} * G_z^{(t,l)}(0) \\ &= S_z^{(n,3)}(0) \leq O\left((2d)^{-n/2}\right). \end{aligned} \quad (4.33)$$

Therefore, when computing $\hat{\Pi}_{z^c}^{(t,1)}(0)$ up to order $(2d)^{-1}$, we only need to study the cases in which the cycles formed in (4.32) have length 2 since, due to the topology of the lattice \mathbb{Z}^d , all other cases form cycles that have length at least 4, and (4.33) implies that their contribution is of order $O((2d)^{-2})$ and, therefore, negligible.

The following are the only two scenarios in which the cycles in (4.32) have length 2.

- (a) The vertex $x = 0$. In this case the walk ω has length 2, it connects 0 to one of its $2d$ nearest neighbors and returns to the origin.
- (b) The vertex x is one of the $2d$ nearest neighbors of the origin. In this case the walk ω is of length 1, it connects 0 to x and the ribs on both end-points, R_0 and R_1 , intersect. Since the cycle has length 2, the presence of the edge $\{0, x\}$ in either R_0 or R_1 is necessary, and in either case the cycle is given by $(0, x, 0)$.

Let $\Pi_z^{(t,1,n)}(x)$ denote the contribution to $\Pi_z^{(t,1)}(x)$ when the cycles formed in (4.32) have length n . We are interested in $\Pi_z^{(t,1,2)}(0)$ and $\Pi_z^{(t,1,2)}(x)$ due to cases (a) and (b), respectively.

The contribution to $\hat{\Pi}_z^{(t,1)}(0)$ due to (a). Using that $\mathcal{U}_{0,2} = -1$ in (4.32), we have

$$\Pi_z^{(t,1,2)}(0) = 2dz^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}). \quad (4.34)$$

Since the factors $(1 + \mathcal{U}_{0,1})$ and $(1 + \mathcal{U}_{1,2})$ are at most 1, we get the upper bound

$$\Pi_z^{(t,1,2)}(0) \leq 2dz^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} = 2dz^2 g^{(t)}(z)^3. \quad (4.35)$$

Theorem 1.1 imply

$$\Pi_{z^c}^{(t,1,2)}(0) \leq \frac{e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.36)$$

On the other hand, since the terms $\mathcal{U}_{i,j}$ are 0 or -1, we have

$$\prod_{0 \leq i < j \leq n} (1 + \mathcal{U}_{i,j}) \geq 1 + \sum_{0 \leq i < j \leq n} \mathcal{U}_{i,j}. \quad (4.37)$$

Hence, the product

$$(1 + \mathcal{U}_{0,1})(1 + \mathcal{U}_{1,2}) \geq 1 + \mathcal{U}_{0,1} + \mathcal{U}_{1,2},$$

which yields the lower bound

$$\begin{aligned}\Pi_z^{(t,1,2)}(0) &\geq 2dz^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} (1 + \mathcal{U}_{0,1} + \mathcal{U}_{1,2}) \\ &= 2dz^2 \left[g^{(t)}(z)^3 + 2g^{(t)}(z)W_z^{(t)}(e_1) \right].\end{aligned}\tag{4.38}$$

Hence, Theorem 1.1 and Claim 4.6 imply

$$\Pi_{z_c^{(t)}}^{(t,1,2)}(0) \geq \frac{e}{2d} + o\left(\frac{1}{2d}\right),\tag{4.39}$$

which together with (4.36) give

$$\Pi_{z_c^{(t)}}^{(t,1,2)}(0) = \frac{e}{2d} + o\left(\frac{1}{2d}\right).\tag{4.40}$$

The contribution to $\hat{\Pi}_z^{(t,1)}(0)$ due to (b). This is

$$\sum_{x: \|x\|_1=1} \Pi_z^{(t,1,2)}(x) = 2d\Pi_z^{(t,1,2)}(0, e_1) = -2dz \sum_{\substack{R_0 \ni 0 \\ R_1 \ni e_1}} z^{|R_0|+|R_1|} \mathcal{U}_{0,1} = -2dz W_z^{(t)}(e_1).\tag{4.41}$$

Then Theorem 1.1 and Claim 4.6 yield

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,2)}(x) = \frac{2e}{2d} + o\left(\frac{1}{2d}\right).\tag{4.42}$$

Finally, adding (4.40) and (4.42) gives the desired result (4.27) for lattice trees, namely,

$$\hat{\Pi}_{z_c^{(t)}}^{(t)}(0) = -\hat{\Pi}_{z_c^{(t)}}^{(t,1)}(0) + o\left(\frac{1}{2d}\right) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right).$$

□

4.3.2 Lattice animals

In this section we complete the proof of Lemma 4.2 by addressing the lattice animal case. We will see that the analysis, up to some details, is very similar to the one for lattice trees that we exposed in Section 4.3.1.

Proof of Lemma 4.2 for lattice animals. Recall that the Fourier transform $\hat{\Pi}_{z_c^{(a)}}^{(a)}(0)$ is given in (3.44) by

$$\hat{\Pi}_{z_c^{(a)}}^{(a)}(0) = \hat{\Pi}_{z_c^{(a)}}^{(a,0)}(0) - \hat{\Pi}_{z_c^{(a)}}^{(a,1)}(0) + \sum_{N \geq 2} (-1)^N \hat{\Pi}_{z_c^{(a)}}^{(a,N)}(0).\tag{4.43}$$

It is known (see [16, (1.23)]) that there is a $c > 0$ such that for all $N \geq 1$ and all $z \in [0, z_c^{(a)}]$,

$$0 \leq \hat{\Pi}_{z_c^{(a)}}^{(a,N)}(0) \leq c^N d^{-(N \vee 1)}, \quad (4.44)$$

and this implies that the third summand in (4.43) is of order $O((2d)^{-2})$ and, therefore, can be neglected. We will prove that the first two terms on the right-hand side of (4.43), satisfy

$$\hat{\Pi}_{z_c^{(a)}}^{(a,0)}(0) = O((2d)^{-2}) \quad \text{and} \quad -\hat{\Pi}_{z_c^{(a)}}^{(a,1)}(0) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right), \quad (4.45)$$

which give the desired result (4.27) for lattice animals.

The term $\hat{\Pi}_{z_c^{(a)}}^{(a,0)}(0)$ is the Fourier transform of the function $\Pi_z^{(a,0)}(x)$, at criticality, which was defined in (3.39). It is given by

$$\hat{\Pi}_z^{(a,0)}(0) = \sum_x \Pi_z^{(a,0)}(x) = \sum_x (1 - \delta_{0,x}) \sum_{R \in \mathcal{D}_{0,x}} z^{|R|}.$$

Notice that, by definition, an animal $R \in \mathcal{D}_{0,x}$ contains two self-avoiding walks between 0 and x that do not have common bonds. These walks form a cycle, of length 4 or more, containing 0 and x . Therefore Proposition 4.5 implies that, for $z \leq z_c^{(a)}$,

$$\hat{\Pi}_z^{(a,0)}(0) \leq \sum_x \sum_{i+j=4} G_z^{(a,i)}(0,x) G_z^{(a,j)}(0,x) = \sum_{i+j=4} G_z^{(a,i)} * G_z^{(a,j)}(0) = S_z^{(4,2)}(0) \leq O((2d)^{-2}).$$

On the other hand, the analysis of the term $\hat{\Pi}_z^{(a,1)}(0)$ is analogous to the corresponding one for lattice trees. It is given by

$$\hat{\Pi}_z^{(a,1)}(0) = \sum_x \Pi_z^{(a,1)}(x) = \Pi_z^{(a,1)}(0) + \sum_{x: \|x\|_1=1} \Pi_z^{(a,1)}(x) + \sum_{x: \|x\|_1 \geq 2} \Pi_z^{(a,1)}(x), \quad (4.46)$$

where the function

$$\Pi_z^{(a,1)}(x) = - \sum_{B: |B| \geq 1} z^{|B|} \left[\prod_{i=0}^{|B|} \sum_{R_i \in \mathcal{D}_{v_i, u_{i+1}}} z^{|R_i|} \right] \mathcal{U}_{0,|B|} \prod_{\substack{0 \leq s < t \leq |B| \\ (s,t) \neq (0,|B|)}} (1 + \mathcal{U}_{st}), \quad (4.47)$$

with $v_0 = 0$ and $u_{|B|+1} = x$. The factor $\mathcal{U}_{0,|B|}$ in the previous expression forces a common vertex, say y , between the first and last ribs, and the last product forbids any other intersection. This implies that the paths connecting 0 to y , 0 to x and x to y form a cycle containing 0 and x . We denote by $\hat{\Pi}_z^{(a,1,n)}(0)$, the contribution to $\hat{\Pi}_z^{(a,1)}(0)$ by the cases in which the cycle formed in (4.47) has length n . An argument similar to the one used to obtain (4.33), gives that when these cycles have length n

or more, the contribution to $\hat{\Pi}_z^{(a,1)}(0)$, for $z \leq z_c^{(a)}$, is at most of order

$$\left| \hat{\Pi}_z^{(a,1,\geq n)}(0) \right| \leq O\left((2d)^{-n/2}\right). \quad (4.48)$$

Therefore, when computing the Fourier transform $\hat{\Pi}_{z_c^{(a)}}^{(a,1)}(0)$ up to order $(2d)^{-1}$, we only need to study the cases in which the cycles formed in (4.47) have length 2. All other cases form cycles that have length at least 4, then (4.48) implies that their contribution is of order $O((2d)^{-2})$, and can be neglected.

This observation implies that the contribution of the third term on the right-hand side of (4.46) is negligible, since any cycle containing 0 and x (which is at distance 2 or more from the origin) is of length at least 6. Hence, we will only analyze the first and second summands in (4.46), for the cases in which the cycles formed in (4.47) have length 2. Let $\Pi_z^{(a,1,n)}(x)$ denote the contribution to $\Pi_z^{(a,1)}(x)$ when the cycles formed in (4.47) have length n . We are interested in $\Pi_z^{(a,1,2)}(0)$ and $\Pi_z^{(a,1,2)}(x)$, for $\|x\|_1 = 1$.

When the vertex $x = 0$, the cycles of length 2 are formed by a path connecting the origin with one of its neighbors and ending at 0. These cycles are at the same time the backbones $B = ((u_1, v_1), (u_2, v_2))$, with $u_1 = 0$, $v_1 = u_2$ given by a nearest neighbor of the origin, and $u_2 = 0$. This implies that the sums over animals R_i with a double connection between v_i and u_{i+1} in (4.47), get simplified to sums over animals containing the vertex $v_i = u_{i+1}$. Let

Then we have

$$\Pi_z^{(a,1,2)}(0) = 2dz^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}), \quad (4.49)$$

which agrees with the analogous quantity for lattice trees given in (4.34). Since Theorem 1.1 and Claim 4.6 give equal results for lattice trees and animals, a procedure similar to the one used to study $\Pi_{z_c^{(t)}}^{(t,1,2)}(0)$, in (4.36) and (4.39), yields

$$\Pi_{z_c^{(a)}}^{(a,1,2)}(0) = \frac{e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.50)$$

For the second term on (4.46), the cycles of length 2 consist of the backbone $B = (0, x)$, and the intersection of the ribs emanating from 0 and x through one edge. All other possibilities in which the backbone consists of a bond different from $(0, x)$ or has more than one edge, produce cycles of length 4 or more, and are negligible. In the case of $B = (0, x)$, the sums over animals R_i with a double connection between 0 and x that appear in (4.47), get simplified to sums over animals that

contain the vertices 0 or x . This implies that

$$\sum_{x: \|x\|_1=1} \Pi_z^{(a,1,2)}(0,x) = 2d\Pi_z^{(a,1,2)}(0,e_1) = -2dz \sum_{\substack{R_0 \ni 0 \\ R_1 \ni x}} z^{|R_0|+|R_1|} \mathcal{W}_{0,1} = -2dz W_z^{(a)}(e_1), \quad (4.51)$$

which agrees with the corresponding quantity for lattice trees given in (4.41). Theorem 1.1 and Claim 4.6 imply

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(a)}}^{(a,1,2)}(0,x) = \frac{2e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.52)$$

Finally, adding (4.50) and (4.52), we get the desired result (4.27) for lattice animals, namely,

$$\hat{\Pi}_{z_c^{(a)}}^{(a,1)}(0) = -\hat{\Pi}_{z_c^{(a)}}^{(a,1)}(0) + o\left(\frac{1}{2d}\right) = -\frac{3e}{2d} + o\left(\frac{1}{2d}\right).$$

□

4.4 Proof of Lemma 4.3

The goal of this section is to show Lemma 4.3, which states that for nearest-neighbor lattice trees and lattice animals,

$$g(z_c) = e + \frac{3e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.53)$$

Except for minor adjustments for the case of lattice animals, the proof will be simultaneously done for both models. In our reasoning, the expansion for $\hat{\Pi}_z(0)$ obtained in Section 4.3, and estimates Proposition 4.5, Claim 4.6 and Lemma 4.7 from Section 4.2, will be fundamental.

Proof of Lemma 4.3. By (3.9), we know that

$$g(z_c) = A(z_c) + B(z_c) + C(z_c) + E(z_c) + \mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} (z_c)^{|A|}.$$

We will show that for nearest-neighbor lattice trees and lattice animals, as $d \rightarrow \infty$,

$$\begin{aligned}
A(z_c) &= e + \frac{3e}{2d} + o\left(\frac{1}{2d}\right) \\
B(z_c) &= -\frac{e}{2d} + o\left(\frac{1}{2d}\right) \\
C(z_c) &= -\frac{\frac{e}{2}}{2d} + o\left(\frac{1}{2d}\right) \\
E(z_c) &= o\left(\frac{1}{2d}\right) \\
\mathbf{I}_a \sum_{\substack{A \geq 0 \\ 0 \in \text{cycle}}} (z_c)^{|A|} &= o\left(\frac{1}{2d}\right).
\end{aligned} \tag{4.54}$$

Adding these quantities gives the desired result (4.53); hence, it only remains to show (4.54).

A key ingredient for the proof, is to relate $\hat{\Pi}_{z_c}(0)$ to the one-point function $g(z_c)$. To tackle this task, we first define the quantity

$$M(z) := 2dz\hat{\Pi}_z(0), \tag{4.55}$$

which will play an important role in the expansion of $g(z)$. Theorem 1.1 and Lemma 4.2 yield

$$M(z_c) = -\frac{3}{2d} + o\left(\frac{1}{2d}\right) \quad \text{as } d \rightarrow \infty. \tag{4.56}$$

The term $A(z_c)$. Recall that $A(z)$ is given, in (3.10), by

$$A(z) = e^{2dzg(z)}.$$

Then (1.23) and (4.55), yield that the exponent

$$2dz_c g(z_c) = 1 - 2dz_c \hat{\Pi}_{z_c}(0) = 1 - M(z_c),$$

which implies that

$$A(z_c) = e^{1-M(z_c)}. \tag{4.57}$$

Therefore,

$$\lim_{d \rightarrow \infty} 2d(A(z_c) - e) = \lim_{d \rightarrow \infty} 2de \left(e^{-M(z_c)} - 1 \right) = \lim_{d \rightarrow \infty} -2deM(z_c) \lim_{d \rightarrow \infty} \frac{e^{-M(z_c)} - 1}{-M(z_c)} = 3e,$$

where in the last equality we used (4.56) and the fact that the limit of the ratio is 1, since it is the

derivative of the function e^x evaluated at 0. Then we have the desired result (4.54) for $A(z_c)$, i.e.,

$$A(z_c) = e + \frac{3e}{2d} + o\left(\frac{1}{2d}\right). \quad (4.58)$$

The term $B(z_c)$. Recall that $B(z)$ was defined in (3.11) as $B(z) = -2dzA(z)G_z(e_1)$. Using (4.58) together with Theorem 1.1 and Lemma 4.7, yield the desired result (4.54) for $B(z_c)$, i.e.,

$$B(z_c) = -2dz_cA(z_c)G_{z_c}(e_1) = -\frac{e}{2d} + o\left(\frac{1}{2d}\right).$$

The term $C(z_c)$. Recall that $C(z)$ is given in (3.12) by

$$C(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \sum_{1 \leq i < j \leq m} \mathcal{V}_{i,j}.$$

We first exchange the sums over $s_1, \dots, s_m \in \mathcal{E}$ and $1 \leq i < j \leq m$ and get

$$C(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} \left[\sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} z^{|C_j|} \right] \mathcal{V}_{i,j}.$$

We distinguish whether the vertices s_i and s_j are equal, denoted by $\mathbf{I}_{s_i=s_j}$, or not, denoted by $\mathbf{I}_{s_i \neq s_j}$,

$$C(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} (\mathbf{I}_{s_i=s_j} + \mathbf{I}_{s_i \neq s_j}) \left[\sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} (z_c)^{|C_j|} \right] \mathcal{V}_{i,j}.$$

In the case that $s_i = s_j$, automatically the factor $\mathcal{V}_{i,j} = -1$, which implies that the sums over C_i and C_j are independent and can be reduced to one-point functions. This observation yields

$$\begin{aligned} C(z) &= - \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \frac{(m-1)m}{2} 2dg(z)^2 \\ &\quad + \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} \mathbf{I}_{s_i \neq s_j} \sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} z^{|C_j|} \mathcal{V}_{i,j} \\ &= -\frac{1}{2} 2d(zg(z))^2 e^{2dzg(z)} + C_{error}(z), \end{aligned} \quad (4.59)$$

where

$$C_{error}(z) := \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} \mathbf{I}_{s_i \neq s_j} \sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} z^{|C_j|} \mathcal{V}_{i,j}. \quad (4.60)$$

Notice that the first term in the last line of (4.59) is

$$-\frac{1}{2} 2d(zg(z))^2 e^{2dzg(z)} = -\frac{1}{2} 2d(zg(z))^2 A(z).$$

Theorem 1.1 and (4.58) imply that, at criticality, this expression is

$$-\frac{1}{2}2d(z_c g(z_c))^2 A(z_c) = -\frac{1}{2}e + o\left(\frac{1}{2d}\right),$$

which is the desired result (4.54) for the term $C(z_c)$. Thus it only remains to prove that $C_{error}(z_c)$ is an error term (hence its name).

The factor $\mathcal{V}_{i,j}$ in (4.60) forces an intersection between the clusters C_i and C_j , say y . The simplest ways in which an intersection happens is either C_i or C_j contain the couple of edges $\{0, s_i\}$ and $\{0, s_j\}$, or C_i contains the bond $\{0, s_i\}$ and C_j contains $\{0, s_j\}$. In case that s_i and s_j are perpendicular, we also need to consider the possibility that either C_i or C_j contain the couple of edges $\{s_i, s_i + s_j\}$ and $\{s_j + s_i, s_i\}$, or C_i contains the bond $\{s_i, s_i + s_j\}$ and C_j contains $\{s_i + s_j, s_j\}$. All other intersections between C_i and C_j require at least 4 edges and, hence, their contribution is bounded by $S_z^{(4,2)}(s_j - s_i) \leq O((2d)^{-2})$, by Proposition 4.5. Therefore, ignoring the non-intersection restriction of the three ribs emanating from the endpoints of these 2 edges and the overlapping of these events, yields

$$\mathbf{I}_{s_i \neq s_j} \sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} z^{|C_j|} |\mathcal{V}_{i,j}| \leq 6z^2 g(z)^4 + O((2d)^{-2}).$$

Substituting this result into (4.60), gives

$$\begin{aligned} |C_{error}(z)| &\leq \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{\substack{s_i, s_j \in \mathcal{E} \\ s_i \neq s_j}} [6z^2 g(z)^3 + O((2d)^{-2})] \\ &= \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \frac{(m-1)m}{2} 2d(2d-1) [6z^2 g(z)^3 + O((2d)^{-2})] \\ &\leq \frac{1}{2} (2dz)^2 A(z) [6z^2 g(z)^3 + O((2d)^{-2})]. \end{aligned}$$

Theorem 1.1 and (4.58) imply that, at criticality, this expression is of order $o((2d)^{-1})$, as $d \rightarrow \infty$.

The term $E(z_c)$. Recall that $E(z)$ was defined in (3.13) as

$$E(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \left(\sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \right).$$

The fact that the factors $\mathcal{V}_{p,q}$ are equal to 0 or to -1, implies that the product $\prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \leq 1$. This observation yields the bound

$$E(z) \leq \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \left(\sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \right). \quad (4.61)$$

We decompose the last factor in this expression as follows

$$\sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} = \sum_{1 \leq j < l \leq m} \mathcal{V}_{0,j} \mathcal{V}_{0,l} + \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \mathcal{V}_{0,j} \mathcal{V}_{k,l} + \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l}, \quad (4.62)$$

where the first summand corresponds to $i = k = 0$, the second one to $i = 0$ and $k > i$, and the third one to $i \geq 1$. We substitute this expression back into (4.61) and distribute the product among the summands, to get

$$E(z) \leq E^1(z) + E^2(z) + E^3(z), \quad (4.63)$$

where

$$E^1(z) := \sum_{m \geq 2} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \sum_{1 \leq j < l \leq m} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \right], \quad (4.64)$$

$$E^2(z) := \sum_{m \geq 2} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \right], \quad (4.65)$$

$$E^3(z) := \sum_{m \geq 3} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \right]. \quad (4.66)$$

We will analyze the terms $E^1(z)$, E^2 and $E^3(z)$, separately; and show that, at criticality, they are of order $o((2d)^{-1})$, as $d \rightarrow \infty$. In each case we start by exchanging the sums over the labels i , j , k and l , and the vertices s_2, \dots, s_m .

The term $E^1(z_c)$. By definition, $E^1(z)$ is equivalent to

$$\begin{aligned} E^1(z) &= \sum_{m \geq 2} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_j, s_l \in \mathcal{E}} G_z(s_j) G_z(s_l) \\ &= \sum_{m \geq 2} \frac{z^m}{m!} (2dg(z))^{m-2} \frac{m(m-1)}{2} (2dG_z(e_1))^2 \\ &= \frac{1}{2} (2dzG_z(e_1))^2 e^{2dzg(z)} = \frac{1}{2} (2dzG_z(e_1))^2 A(z). \end{aligned}$$

By Theorem 1.1, Lemma 4.7 and (5.50), we have

$$E^1(z_c) = \frac{\frac{1}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \quad (4.67)$$

and can be neglected at this stage.

The term $E^2(z_c)$. By definition, $E^2(z)$ is equivalent to

$$E^2(z) = \sum_{m \geq 2} \frac{z^m}{m!} (2dg(z))^{m-3} \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \left[\sum_{s_j, s_k, s_l \in \mathcal{E}} \sum_{\substack{C_j \ni s_j, C_k \ni s_k \\ C_l \ni s_l}} z^{|C_j|+|C_k|+|C_l|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \right]. \quad (4.68)$$

The cardinality of the set of labels j, k and l satisfy $r := |\{j, k, l\}| = 2$ or 3 . In both cases, the number of possibilities for the labels is $\tilde{C}_r \binom{m}{r}$, where the coefficient \tilde{C}_r is a combinatorial factor, independent of d , that can be determined for $r = 2, 3$. Substituting $\tilde{C}_r \binom{m}{r}$ in (4.68) produces

$$\begin{aligned} & \sum_{m \geq r} \frac{z^m}{m!} (2dg(z))^{m-r} \tilde{C}_r \binom{m}{r} \mathbf{I}_{|\{j,k,l\}|=r} \left[\sum_{s_j, s_k, s_l \in \mathcal{E}} \sum_{\substack{C_j \ni s_j, C_k \ni s_k \\ C_l \ni s_l}} z^{|C_j|+|C_k|+|C_l|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \right] \\ &= \frac{\tilde{C}_r}{r!} A(z) z^r \mathbf{I}_{|\{j,k,l\}|=r} \left[\sum_{s_j, s_k, s_l \in \mathcal{E}} \sum_{\substack{C_j \ni s_j, C_k \ni s_k \\ C_l \ni s_l}} z^{|C_j|+|C_k|+|C_l|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \right], \end{aligned} \quad (4.69)$$

with $r = 2, 3$. We will show that the contribution in both cases $r = 2, 3$, is of order $o((2d)^{-1})$, as $d \rightarrow \infty$. In our analysis we estimate the following quantity for $r = 2, 3$,

$$\mathbf{I}_{|\{j,k,l\}|=r} \sum_{s_j, s_k, s_l \in \mathcal{E}} \sum_{\substack{C_j \ni s_j, C_k \ni s_k \\ C_l \ni s_l}} z^{|C_j|+|C_k|+|C_l|} \mathcal{V}_{0,j} \mathcal{V}_{k,l}. \quad (4.70)$$

If the labels obey $r = |\{j, k, l\}| = 2$, then either $j = k \neq l$ or $j = l \neq k$. Then the number of options for the labels is $2 \binom{m}{2}$. We only analyze the case $j = k \neq l$ since the other one is analogous. For convenience we suppose that $j = k = 1$ and $l = 2$. The vertices s_1 and s_2 are either the same or different nearest-neighbors of the origin, i.e., the cardinality of the set $\{s_1, s_2\}$ is $|\{s_1, s_2\}| = 1$ or $|\{s_1, s_2\}| = 2$.

If $|\{s_1, s_2\}| = 1$, automatically the factor $\mathcal{V}_{1,2} = -1$. Then (4.70) becomes

$$\sum_{s_1, s_2 \in \mathcal{E}} \mathbf{I}_{s_1=s_2} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1|+|C_2|} \mathcal{V}_{0,1} \mathcal{V}_{1,2} = -2dg(z) \sum_{C_1 \ni e_1} z^{|C_1|} \mathcal{V}_{0,1} = 2dg(z) G_z(e_1). \quad (4.71)$$

Substituting this result into (4.69) and using Theorem 1.1, Lemma 4.7 and (5.50), we get that the contribution to $E^2(z_c)$ is

$$2dA(z_c) z_c^2 g(z_c) G_{z_c}(e_1) = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \quad (4.72)$$

and can be neglected at this stage.

If $|\{s_1, s_2\}| = 2$, the simplest way in which the term $\mathcal{V}_{0,1} \mathcal{V}_{1,2}$ is non zero is when the clus-

ter C_1 contains the edges $\{0, s_1\}$ and $\{0, s_2\}$, or the cluster C_1 contains $\{0, s_1\}$ and C_2 contains $\{0, s_2\}$, all other intersections require at least four edges and their contribution is bounded by $S_z^{(4,2)} \leq O((2d)^{-2})$. Therefore, (4.70) is bounded by

$$\sum_{s_1, s_2 \in \mathcal{E}} \mathbf{1}_{s_1 \neq s_2} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1|+|C_2|} \mathcal{V}_{0,1} \mathcal{V}_{1,2} \leq 2(2d)^2 (2z^2 g(z)^4 + O((2d)^{-2})).$$

Replacing this expression into (4.69) and using Theorem 1.1, Lemma 4.7 and (5.50), we get that the contribution to $E^2(z_c)$ is of order

$$A(z_c) z_c^2 (2d)^2 (2z_c^2 g(z_c)^4 + O((2d)^{-2})) = o\left(\frac{1}{2d}\right), \text{ as } d \rightarrow \infty. \quad (4.73)$$

When the labels satisfy $r = |\{j, k, l\}| = 3$, to determine the number of options for j, k and l with $1 \leq j \leq m, 1 \leq k < l \leq m$ and $j \neq k \neq l$, we choose three labels among m , and notice that j can be any of these and once j is determined, so are k and l . Therefore, the desired quantity is $3 \binom{m}{3}$. For convenience, we suppose that $j = 1, k = 2$ and $l = 3$. Hence, (4.70) becomes

$$\begin{aligned} & \left[\sum_{s_1 \in \mathcal{E}} \sum_{C_1 \ni s_1} z^{|C_1|} \mathcal{V}_{0,1} \right] \left[\sum_{s_2, s_3 \in \mathcal{E}} \sum_{C_2 \ni s_2, C_3 \ni s_3} z^{|C_2|+|C_3|} \mathcal{V}_{2,3} \right] \\ &= -2d G_z(e_1) \left[\sum_{s_2, s_3 \in \mathcal{E}} \sum_{C_2 \ni s_2, C_3 \ni s_3} z^{|C_2|+|C_3|} \mathcal{V}_{2,3} \right]. \end{aligned} \quad (4.74)$$

In the last factor, we distinguish whether the vertices s_2 and s_3 are equal. When $s_2 = s_3$, the factor $\mathcal{V}_{2,3}$ is automatically -1 and does not affect the sums, which become a product of two one-point functions. When $s_2 \neq s_3$ the intersection between C_2 and C_3 requires at least two edges. These observations imply that (4.74) is bounded by

$$2d G_z(e_1) \left[2dg(z_c)^2 + 2d(2d-1) S_z^{(2,2)} \right].$$

Therefore, replacing this expression into (4.69) and using Theorem 1.1, Proposition 4.5, Lemma 4.7 and (4.58), yield that the contribution to $E^2(z_c)$, when $|\{j, k, l\}| = 3$, is of order $o((2d)^{-1})$, as $d \rightarrow \infty$.

The term $E^3(z)$. We exchange the order of the sums over the labels i, j, k and l , and the vertices s_2, \dots, s_m in (4.66), to get the equivalent expression

$$E^3(z) = \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-4} \sum_{\substack{1 \leq i < j \leq m \\ (k,l) \in A_{i,j}}} \left[\sum_{s_i, \dots, s_l \in \mathcal{E}} \sum_{C_i \ni s_i \dots C_l \ni s_l} z^{|C_i|+\dots+|C_l|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \right]. \quad (4.75)$$

The set of labels satisfies $r := |\{i, j, k, l\}| = 3$ or 4. In both cases, the number of possibilities for

the labels is $\tilde{C}_r \binom{m}{r}$, where the coefficient \tilde{C}_r is a combinatorial factor, independent of d , that can be determined for $r = 3, 4$. Substituting $\tilde{C}_r \binom{m}{r}$ in (4.76) produces

$$\begin{aligned} & \sum_{m \geq r} \frac{z^m}{m!} (2dg(z))^{m-r} \tilde{C}_r \binom{m}{r} \mathbf{I}_{|\{i,j,k,l\}|=r} \left[\sum_{s_i, \dots, s_l \in \mathcal{C}} \sum_{C_i \ni s_i \dots C_l \ni s_l} z^{|C_i| + \dots + |C_l|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \right] \\ &= \frac{\tilde{C}_r}{r!} A(z) z^r \mathbf{I}_{|\{i,j,k,l\}|=r} \left[\sum_{s_i, \dots, s_l \in \mathcal{C}} \sum_{C_i \ni s_i \dots C_l \ni s_l} z^{|C_i| + \dots + |C_l|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \right], \end{aligned} \quad (4.76)$$

for $r = 3, 4$. We will show that the contribution in both cases is of order $o((2d)^{-1})$, as $d \rightarrow \infty$. In our analysis we estimate the following quantity for $r = 3, 4$,

$$\mathbf{I}_{|\{i,j,k,l\}|=r} \sum_{\substack{C_i \ni s_i, C_j \ni s_j \\ C_k \ni s_k, C_l \ni s_l}} z^{|C_i| + |C_j| + |C_k| + |C_l|} \mathcal{V}_{i,j} \mathcal{V}_{k,l}. \quad (4.77)$$

The condition $r = |\{i, j, k, l\}| = 3$ is satisfied when $k = i$, $k = j$ or $l = j$. In all cases, we choose three labels from a set of m and order them, this order automatically determines which one corresponds to i , j , k and l . Hence, the number of options for the labels is $3 \binom{m}{3}$. We will study the case $k = i$, since the other two are analogous. For convenience we assume that $i = k = 1$, $j = 2$ and $l = 3$. We distinguish two possibilities for the vertices s_1, s_2 and s_3 , $|\{s_1, s_2, s_3\}| = 1$ and $|\{s_1, s_2, s_3\}| = 2, 3$.

If $|\{s_1, s_2, s_3\}| = 1$, then automatically the factor $\mathcal{V}_{1,2} \mathcal{V}_{1,3} = 1$ because the three clusters C_1, C_2 and C_3 contain the same vertex s_1 . Therefore, (4.77) becomes

$$\sum_{s_1, s_2, s_3 \in \mathcal{C}} \mathbf{I}_{|\{s_1, s_2, s_3\}|=1} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1| + |C_2| + |C_3|} \mathcal{V}_{1,2} \mathcal{V}_{1,3} = 2dg(z)^3. \quad (4.78)$$

Substituting this expression into (4.76) and using Theorem 1.1 and (4.58), yield that the contribution to $E^3(z_c)$ is

$$\frac{3}{3!} 2dA(z_c) (z_c g(z_c))^3 = \frac{1}{2} e + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \quad (4.79)$$

and can be neglected at this stage.

If $|\{s_1, s_2, s_3\}| = 2$ or 3 , then at least two of the vertices are different, say $s_1 \neq s_2$. Using that $|\mathcal{V}_{1,3}| \leq 1$, we get the following bound for (4.77),

$$\begin{aligned} & \sum_{s_1, s_2, s_3 \in \mathcal{C}} \mathbf{I}_{|\{s_1, s_2, s_3\}| \geq 2} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1| + |C_2| + |C_3|} \mathcal{V}_{1,2} \mathcal{V}_{1,3} \leq 3(2d)^3 g(z) \sum_{C_1 \ni e_1, C_2 \ni e_2} z^{|C_1| + |C_2|} |\mathcal{V}_{1,2}| \\ & \leq 3(2d)^3 g(z) \left(6z^2 g(z)^2 + S_z^{(4,2)} \right), \end{aligned} \quad (4.80)$$

where in the last inequality the terms in braces correspond to the intersection between C_1 and C_2 requiring 2, and 4 or more edges, respectively. Substituting this result into (4.76) yields that this term is bounded by

$$\frac{3}{2}z^3 A(z_c)(2d)^3 g(z) \left(6z^2 g(z)^2 + S_z^{(4,2)}\right) = o((2d)^{-1}), \text{ as } d \rightarrow \infty,$$

where in the equality we have used Theorem 1.1, Proposition 4.5 and (4.58).

Now we study the case when i, j, k and l are all different. To determine the number of possibilities for the labels we chose four labels from a set of m and order them. Then i is the smallest one by definition, j has the remaining 3 options, and once j is determined, so are k and l . Hence, the desired quantity is $3\binom{m}{4}$. For convenience we assume that $i = 1, k = 2, j = 3$ and $l = 4$. Then in this case, (4.77) is given by

$$\begin{aligned} \sum_{s_1, \dots, s_4 \in \mathcal{E}} \left[\sum_{\substack{C_1 \ni s_1 \\ C_2 \ni s_2}} z^{|C_1|+|C_2|} \mathcal{V}_{1,2} \right] \left[\sum_{\substack{C_3 \ni s_3 \\ C_4 \ni s_4}} z^{|C_3|+|C_4|} \mathcal{V}_{3,4} \right] &= \left[\sum_{s_1, s_2 \in \mathcal{E}} \sum_{\substack{C_1 \ni s_1 \\ C_2 \ni s_2}} z^{|C_1|+|C_2|} \mathcal{V}_{1,2} \right]^2 \\ &\leq \left[2dg(z_c)^2 + 2d(2d-1)S_z^{(2,2)} \right]^2, \end{aligned} \quad (4.81)$$

where in the inequality we have applied the same reasoning that we employed when analyzing the last factor in (4.74). Replacing this expression into (4.76) yields the bound,

$$\frac{3}{4!}z^4 A(z) \left[2dg(z_c)^2 + 2d(2d-1)S_z^{(2,2)} \right]^2.$$

By Theorem 1.1, Proposition 4.5 and (4.58), this expression, at criticality, is of order $o((2d)^{-1})$, as $d \rightarrow \infty$. This completes the proof that $E(z)$ is an error term.

The term $\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} z^{|A|}$. Due to the topology of \mathbb{Z}^d , a cycle that contains the origin has length 4 or more, this implies

$$\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} z^{|A|} \leq \sum_x \sum_{i+j=4} G_z^{(a,i)}(x) G_z^{(a,j)}(x) = \sum_{i+j=4} G_z^{(a,i)} * G_z^{(a,j)}(0) = S_z^{(4,2)}(0),$$

and Proposition 4.5 gives

$$\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} (z_c)^{|A|} = o\left(\frac{1}{2d}\right) \text{ as } d \rightarrow \infty.$$

This completes the proof of Lemma 4.3. □

Chapter 5

Proof of second order correction

The goal of this chapter is to prove Theorem 1.4, which states that

$$\begin{aligned}\hat{\Pi}_{z_c^{(t)}}^{(t)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\ \hat{\Pi}_{z_c^{(a)}}^{(a)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e - \frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right),\end{aligned}$$

and Theorem 1.5, which states that

$$\begin{aligned}g^{(t)}(z_c^{(t)}) &= e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \\ g^{(a)}(z_c^{(a)}) &= e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e - 1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).\end{aligned}$$

These demonstrations complete the proof of Theorem 1.2, which is the main goal of this thesis.

In Section 5.1, we calculate the next term in the expansions for $W_{z_c}(x)$ and $G_{z_c}(x)$. These extended expansions will allow us to prove Theorem 1.4, in Section 5.2, and Theorem 1.5, in Section 5.3.

5.1 Update of useful estimates

In this section, we calculate the next term in the expansions for $W_{z_c}(x)$ and $G_{z_c}(x)$ in Claim 5.1 and Lemma 5.2, respectively, which extends the results in Claim 4.6 and Lemma 4.7. The proof uses the new expansions for z_c and $g(z_c)$ that we computed in Lemma 4.1 and Lemma 4.3, respectively. On this occasion, we also estimate the quantity $W_{z_c}(x)$, when x is at distance 2 from the origin; however, in this case, the results for z_c and $g(z_c)$ in Theorem 1.1 are sufficient.

Recall that $W_z(x)$ was defined in (4.16) as

$$W_z(x) = \sum_{C_0 \ni 0} \sum_{C_1 \ni x} z^{|C_0|+|C_1|} \mathcal{U}_{0,1},$$

where $\mathcal{U}_{0,1} = -1$ if the clusters C_0 and C_1 share a common vertex, and 0 otherwise.

Claim 5.1. *For nearest-neighbor lattice trees and animals, let e_i and e_j be two nearest neighbors of the origin such that they are perpendicular. Then as $d \rightarrow \infty$,*

$$W_{z_c}(e_i) = -\frac{2e^2}{2d} - \frac{11e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \quad (5.1)$$

$$W_{z_c}(2e_i) = -\frac{3e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \quad (5.2)$$

$$W_{z_c}(e_i + e_j) = -\frac{6e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.3)$$

Proof of Claim 5.1. For the lattice animal case, we distinguish whether 0 and x belong to a cycle contained in one of the animals C_0 or C_1 . A similar argument to the one preceding (4.19), yields that the contribution to $W_{z_c}(x)$ when this condition is satisfied is at most of order $o((2d)^{-2})$. Therefore, for the rest of the proof we assume that both animals, C_0 and C_1 , do not have any cycle containing the 0 and $x = e_i, 2e_i$ or $e_i + e_j$.

Clusters C_0 and C_1 contribute to $W_z(x)$ if and only if they have a vertex in common, say y . Then there is a path connecting 0 and y contained in C_0 , and a path connecting y and x contained in C_1 , which we denote ω^0 and ω^1 , respectively. The paths ω^0 and ω^1 can be uniquely determined. The union of ω^0 and ω^1 form a path connecting 0 to x (passing through y), which we call ω . We write $W_z^n(x)$ to refer to the contribution to $W_z(x)$ by all the configurations whose path ω has length n .

Proof of Equation 5.1. If the path ω has length 1, then $\omega = (0, e_i)$. This means that the edge $\{0, e_i\}$ is contained in either C_0 or C_1 , say in C_0 . In this case, C_0 consists of the edge $\{0, e_i\}$ and two non intersecting subclusters, C_0^* and C_1^* , the first one emanating from 0 and the second one emanating from e_i . Exchanging the roles of C_0 and C_1 , and subtracting the contribution due to the event in which both clusters C_0 and C_1 contain the bond $\{0, e_i\}$, yield

$$W_z^1(e_i) = -2zg(z) \sum_{\substack{C_0^* \ni 0 \\ C_1^* \ni x}} z^{|C_0^*|+|C_1^*|} (1 + \mathcal{U}_{0,1}^*) + z^2 \left[\sum_{\substack{C_0^* \ni 0 \\ C_1^* \ni x}} z^{|C_0^*|+|C_1^*|} (1 + \mathcal{U}_{0,1}^*) \right]^2,$$

where $\mathcal{U}_{0,1}^* = -1$ if the subclusters C_0^* and C_1^* have a common vertex, and 0 otherwise. Distributing the products in this expression we get

$$W_z^1(e_i) = -2zg(z)^3 - 2zg(z)W_z(x) + z^2 [g(z)^2 + W_z(x)]^2.$$

By Theorem 1.1, Lemma 4.1 and Claim 4.6 we have

$$W_{z_c}^1(e_i) = -\frac{2e^2}{2d} - \frac{7e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.4)$$

If the path ω has length 3, then it is of the form $\omega = (0, w, w + e_i, e_i)$, for some vertex w perpendicular to e_i . There are $2d - 2$ of such paths and each of them has four possibilities for y . If we treat the five ribs emanating from the vertices in ω^0 and ω^1 as independent, we have the lower bound

$$W_{z_c}^3(e_i) \geq -4(2d - 2)z_c^3 g(z_c)^5 = -\frac{4e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty,$$

where the last equality is due to Theorem 1.1. For an upper bound, we use inclusion-exclusion and subtract from the lower bound the contribution when there are pairwise intersections among the ribs that belong to the same path, either ω^0 or ω^1 . This gives

$$W_{z_c}^3(e_i) \leq -4(2d - 2)z_c^3 g(z_c)^3 [g(z_c)^2 + 4W_{z_c}(e_i) + 2W_{z_c}(e_i + w)].$$

On the right-hand side of this expression appears $W_z(e_i + w)$, which is precisely one of the quantities we want to compute. At this point we only determine that it is of order $O((2d)^{-1})$. Since $|e_i + w| = 2$, Proposition 4.5 implies that

$$|W_{z_c}(e_i + w)| \leq \sum_{j+k=2} G_{z_c}^{(i)} * G_{z_c}^{(j)}(e_i + w) = S_{z_c}^{(2,2)}(e_i + w) \leq \frac{C_{2,2}}{2d}. \quad (5.5)$$

This result, Theorem 1.1 and Claim 4.6, yield

$$W_{z_c}^3(e_i) \leq -\frac{4e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

Therefore, both bounds yield

$$W_{z_c}^3(e_i) = -\frac{4e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

The contribution to $W_z(e_i)$ when the path ω has length 5 or more, is bounded by

$$|W_{z_c}^{\geq 5}(e_i)| \leq \sum_y \sum_{i+j=5} G_{z_c}^{(i)}(y) G_{z_c}^{(j)}(e_i - y) = S_{z_c}^{(5,2)}(e_i) = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty,$$

where the last equality is due to Proposition 4.5.

Finally, we have that, as $d \rightarrow \infty$,

$$W_{z_c}(e_i) = W_{z_c}^1(e_i) + W_{z_c}^3(e_i) + W_{z_c}^{\geq 5}(e_i) = -\frac{2e^2}{2d} - \frac{11e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

This completes the proof of equation (5.1).

Proof of Equation 5.2. If the path ω has length 2, then $\omega = (0, e_i, 2e_i)$. There are three possibilities for y . Treating the four ribs emanating from the vertices in ω^0 and ω^1 as independent, yields the lower bound

$$W_{z_c}^2(2e_i) \geq -3z_c^2 g(z_c)^4 = -\frac{3e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty,$$

where the equality is due to Theorem 1.1. For an upper bound, we use inclusion-exclusion and subtract from the lower bound the contribution by the pairwise intersections among the ribs that belong to the same path, either ω^0 or ω^1 . This gives

$$W_{z_c}^2(2e_i) \leq -3z_c^2 g(z_c)^2 [g(z_c)^2 + 2W_{z_c}(e_i) + W_{z_c}(2e_i)]. \quad (5.6)$$

On the right-hand side of this expression appears $W_{z_c}(2e_i)$, which is precisely the quantity we want to compute. At this point we only determine that is of order $O((2d)^{-1})$. Since $|2e_i| = 2$, Proposition 4.5 implies that

$$|W_{z_c}(2e_i)| \leq \sum_{j+k=2} G_{z_c}^{(i)} * G_{z_c}^{(j)}(2e_i) = S_{z_c}^{(2,2)}(2e_i) \leq \frac{C_{2,2}}{2d}. \quad (5.7)$$

This result, Theorem 1.1 and Claim 4.6, yield

$$W_{z_c}^2(2e_i) \leq -\frac{3e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

If the path ω has length 4, then it is of the form $\omega = (0, w, w + e_i, w + 2e_i, 2e_i)$, for some vertex w perpendicular to e_i . There are $2d - 2$ of such paths and each of them has five possibilities for y . If we treat the six ribs emanating from the vertices in ω^0 and ω^1 as independent, we get the bound

$$|W_{z_c}^4(2e_i)| \leq 5(2d - 2)z_c^4 g(z_c)^6 = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty$$

where the equality is due to Theorem 1.1.

The contribution to $W_z(2e_i)$ when the path ω has length 6 or more, is bounded by

$$|W_{z_c}^{\geq 6}(2e_i)| \leq \sum_y \sum_{i+j=6} G_{z_c}^{(i)}(y) G_{z_c}^{(j)}(2e_i - y) = S_{z_c}^{(6,2)}(2e_i) \leq O((2d)^{-2}),$$

by Proposition 4.5.

Finally, we have that, as $d \rightarrow \infty$,

$$W_{z_c}(2e_i) = W_{z_c}^2(2e_i) + W_{z_c}^4(2e_i) + W_{z_c}^{\geq 6}(2e_i) = -\frac{3e^2}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right),$$

which completes the proof of equation (5.2).

Proof of Equation 5.3. The argument is similar to the one we use in the proof of Equation 5.2. The main difference is that when the path ω has length 2, there are two possibilities for it, $\omega = (0, e_i, e_i + e_j)$ and $\omega = (0, e_j, e_i + e_j)$. For each path ω there are three options for y . This explains the appearance of 6 instead of 3 for the coefficient of $(2d)^{-2}$. The other thing that must be modified, is that the upper bound for $W_{z_c}^2(e_i + e_j)$, analogous to (5.6), will contain the quantity $W_{z_c}(e_i + e_j)$, instead of $W_{z_c}(2e_i)$. Using that $W_{z_c}(2e_i)$ is of order $O((2d)^{-1})$, as estimated in (5.5), gives the result. \square

Claim 5.1 will be fundamental in the proofs of Theorem 1.4 and Theorem 1.5. For showing the latter, we first update the expansion of the two-point function $G_{z_c}(x)$, with $\|x\|_1 = 1$, in Lemma 5.2 below.

Lemma 5.2. *For nearest-neighbor lattice trees and animals, let x be a nearest neighbor of the origin, then as $d \rightarrow \infty$,*

$$G_{z_c}(x) = \frac{e}{2d} + \frac{\frac{7}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.8)$$

Proof of Lemma 5.2. Lattice trees. In the definition of $G_z^{(t)}(x)$ given in (3.17), let $x = e_1 = (1, 0, \dots, 0)$. We separate the sum over walks in the cases $|\omega| = 1$, $|\omega| = 3$ and $|\omega| \geq 5$, which gives the first, second and third terms in the expression below, respectively,

$$\begin{aligned} G_z^{(t)}(0, x) &= z \sum_{\substack{R_0 \ni 0 \\ R_1 \ni x}} z^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}) \\ &\quad + (2d-2)z^3 \sum_{\substack{R_0 \ni 0, R_1 \ni e_2 \\ R_2 \ni y, R_3 \ni x}} z^{|R_0|+|R_1|+|R_2|+|R_3|} \prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{i,j}) \\ &\quad + G_z^{(t,5)}(0, x), \end{aligned} \quad (5.9)$$

where in the second summand $e_2 = (0, 1, 0, \dots, 0)$, and $y = e_1 + e_2$.

The first summand on the right-hand side of (5.9), evaluated at criticality, is given by

$$\begin{aligned} z_c^{(t)} \sum_{\substack{R_0 \ni 0 \\ R_1 \ni x}} \left(z_c^{(t)}\right)^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}) &= z_c^{(t)} g^{(t)}(z_c^{(t)})^2 + z_c^{(t)} W_{z_c^{(t)}}^{(t)}(x) \\ &= \frac{e}{2d} + \frac{\frac{5}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \text{ as } d \rightarrow \infty, \end{aligned} \quad (5.10)$$

where the second equality is due to Lemma 4.1, Lemma 4.3 and Claim 5.1.

To analyze the second summand on the right-hand side of (5.9), we use that the product

$$\prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{i,j}) \leq 1,$$

which together with Theorem 1.1 yield the upper bound

$$(2d-2) \left(z_c^{(t)} \right)^3 g^{(t)}(z_c^{(t)})^4 = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \text{ as } d \rightarrow \infty.$$

For a lower bound, we use inequality (4.37) and get

$$\begin{aligned} & (2d-2) z^3 \sum_{\substack{R_0 \ni 0, R_1 \ni e_2 \\ R_2 \ni y, R_3 \ni x}} z^{|R_0|+|R_1|+|R_2|+|R_3|} \prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{i,j}) \\ & \geq (2d-2) z^3 \sum_{\substack{R_0 \ni 0, R_1 \ni e_2 \\ R_2 \ni y, R_3 \ni x}} z^{|R_0|+|R_1|+|R_2|+|R_3|} \left(1 + \sum_{0 \leq i < j \leq 3} \mathcal{U}_{i,j} \right) \\ & = (2d-2) z^3 \left[g^{(t)}(z)^4 + 4g^{(t)}(z)^2 W_z^{(t)}(x) + 2g^{(t)}(z)^2 W_z^{(t)}(x+y) \right]. \end{aligned}$$

By Theorem 1.1 and Claim 5.1, the last expression, at criticality, is

$$\frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

Hence, both bounds yield that, as $d \rightarrow \infty$,

$$(2d-2) \left(z_c^{(t)} \right)^3 \sum_{\substack{R_0 \ni 0, R_1 \ni e_2 \\ R_2 \ni y, R_3 \ni x}} \left(z_c^{(t)} \right)^{|R_0|+|R_1|+|R_2|+|R_3|} \prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{i,j}) = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.11)$$

The third summand on the right-hand side of (5.9), satisfies

$$G_{z_c^{(t)}}^{(t,5)}(x) = S_{z_c^{(t)}}^{(5,1)}(x) = o\left(\frac{1}{(2d)^2}\right), \quad (5.12)$$

by Proposition 4.5.

Finally, adding the contributions of (5.10), (5.11) and (5.12) gives the desired result (5.8) for lattice trees, namely,

$$G_{z_c^{(t)}}^{(t)}(x) = \frac{e}{2d} + \frac{7}{2} \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

Lattice animals. Now we analyze the two-point function for lattice animals $G_z^{(a)}$, which was defined

in (3.35) in terms of the lace expansion. Recall that in the proof of Lemma 4.7 we analyze it according to the length of the backbone, $|B| = 0, 1, 2$ and ≥ 3 . By (4.23) and (4.26), we know that the contribution to $G_{z_c}^{(a)}(x)$ when $|B| = 0$ or $|B| = 2$, is of order $o((2d)^{-2})$. On this occasion we will study the cases $|B| = 1, 3, 4$ and $|B| \geq 5$, and see that only the cases $|B| = 1$ and $|B| = 3$ contribute to $G_{z_c}^{(a)}(x)$ up to order $(2d)^{-2}$.

When $|B| = 1$, (4.22) gives the two terms that contribute to $G_{z_c}^{(a)}(x)$ in this case. By the discussion below (4.22), we know that one of them is of order $o((2d)^{-4})$. The other term is

$$\begin{aligned} z_c^{(a)} \sum_{\substack{R_0 \ni 0 \\ R_1 \ni x}} \left(z_c^{(a)} \right)^{|R_0|+|R_1|} (1 + \mathcal{U}_{0,1}) &= z_c^{(a)} g^{(a)}(z_c^{(a)})^2 + z_c^{(a)} W_{z_c}^{(a)}(x) \\ &= \frac{e}{2d} + \frac{\frac{5}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \end{aligned} \tag{5.13}$$

where the second equality is due to Lemma 4.1, Lemma 4.3 and Claim 5.1.

For $|B| = 3$, if the backbone is one of the $2d - 2$ walks of the form $\omega = (0, y, y + x, x)$, for some nearest-neighbor y of the origin perpendicular to x , then the double connections between the end-points of these vertices, are simplified to sums over non-intersecting animals containing them. The contribution in this case agrees with the analogous quantity for lattice trees given in the second summand of (5.9). The only difference is that now, the sums are over animals instead of trees. Since the results from Theorem 1.1 and Claim 5.1 are the same for lattice trees and animals, the analysis applied to obtain (5.11), can also be used in this situation to get that, as $d \rightarrow \infty$,

$$(2d - 2) \left(z_c^{(a)} \right)^3 \sum_{\substack{R_0 \ni 0, R_1 \ni e_2 \\ R_2 \ni y, R_3 \ni x}} \left(z_c^{(a)} \right)^{|R_0|+|R_1|+|R_2|+|R_3|} \prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{i,j}) = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

This expression together with (5.13) give the desired result (5.8) for lattice animals, namely,

$$G_{z_c}^{(a)}(x) = \frac{e}{2d} + \frac{\frac{7}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

It only remains to prove that the contribution when backbone is not of form $\omega = (0, y, y + x, x)$, is negligible.

If the backbone $|B| = 3$ but is not given by any of the 3-step walks connecting 0 to x , or $|B| = 4$, then necessarily one of the double connections between the endpoints of the vertices conforming the backbone has a cycle, of length four or more, containing the corresponding backbone vertices. These two situations are very similar to the one analyzed in the proof of Claim 4.6, for the lattice animal case when $|B| = 2$. In fact, following the same argument, gives bounds for both cases analogous to the one given in (4.25). The main difference when $|B| = 3$ and $|B| = 4$, is that the factor $(2dz)^2$ gets replaced by $(2dz)^3$ and $(2dz)^4$, there are three and four functions D instead of

two, and three and four two-point functions $G_z^{(a)}$ instead of two, respectively. In both cases, we obtain that the contribution is of order $o((2d)^{-2})$, which is analogous to (4.26).

When $|B| \geq 5$, then necessarily there is a path of length at least five connecting 0 and x . Then the contribution in this case is bounded by

$$G_{z_c}^{(a,5)}(x) = S_{z_c}^{(5,1)}(x) \leq O\left((2d)^{-5/2}\right),$$

where the inequality is due to Proposition 4.5. □

5.2 Proof of Theorem 1.4

The goal of this section is to show Theorem 1.4, which states that for nearest-neighbor lattice trees and lattice animals,

$$\begin{aligned} \hat{\Pi}_{z_c}^{(t)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\ \hat{\Pi}_{z_c}^{(a)}(0) &= -\frac{3e}{2d} - \frac{\frac{27}{2}e - \frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \end{aligned} \tag{5.14}$$

In the proof, we use extensively Proposition 4.5 and Claim 5.1, together with the new results for z_c and $g(z_c)$ in Lemma 4.1 and Lemma 4.3, respectively. For convenience we have separated the proof in two parts, the one for lattice trees in Section 5.2.1, and the one for lattice animals in Section 5.2.2.

5.2.1 Lattice trees

Proof of Theorem 1.4 for lattice trees. Recall that the Fourier transform $\hat{\Pi}_{z_c}^{(t)}(0)$ is given, in (4.28), by

$$\hat{\Pi}_{z_c}^{(t)}(0) = -\hat{\Pi}_{z_c}^{(t,1)}(0) + \hat{\Pi}_{z_c}^{(t,2)}(0) + \sum_{N \geq 3} (-1)^N \hat{\Pi}_{z_c}^{(t,N)}(0). \tag{5.15}$$

By (4.29), the third summand in this expression is of order $O((2d)^{-3})$ and can be neglected. We will prove that the first and second terms on the right-hand side of (5.15) satisfy

$$-\hat{\Pi}_{z_c}^{(t,1)}(0) = -\frac{3e}{2d} - \frac{\frac{49}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \tag{5.16}$$

and

$$\hat{\Pi}_{z_c}^{(t,2)}(0) = \frac{11e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \tag{5.17}$$

Substituting (5.16) and (5.17) into (5.15) gives the desired result (5.14) for the case of lattice trees. Notice that the main difference with the previous step in our analysis, is that now we also need to study the next term in the series of $\hat{\Pi}_{z_c}^{(t)}(0)$.

Proof of Equation 5.16. It is convenient to decompose the quantity $\hat{\Pi}_{z_c}^{(t,1)}(0)$ as

$$\hat{\Pi}_{z_c}^{(t,1)}(0) = \Pi_{z_c}^{(t,1)}(0) + \sum_{x:\|x\|_1=1} \Pi_{z_c}^{(t,1)}(x) + \sum_{x:\|x\|_1=2} \Pi_{z_c}^{(t,1)}(x) + \sum_{x:\|x\|_1 \geq 3} \Pi_{z_c}^{(t,1)}(x). \quad (5.18)$$

By (4.33), the contribution of the first three terms when the cycles formed in (4.32) have length 6 or more, and the contribution of the last term is bounded by

$$\left| \hat{\Pi}_{z_c}^{(t,1,\geq 6)}(0) \right| = O((2d)^{-3}). \quad (5.19)$$

Therefore, we only need to study the first three terms in (5.18), when the cycles formed in (4.32) have length 2 and 4.

First term in (5.18). For $\Pi_{z_c}^{(t,1)}(0)$, we will study $\Pi_{z_c}^{(t,1,2)}(0)$ and $\Pi_{z_c}^{(t,1,4)}(0)$.

In the case of $\Pi_{z_c}^{(t,1,2)}(0)$, if in (4.34) the product $(1 + \mathcal{U}_{0,1})(1 + \mathcal{U}_{1,2})$ is expanded, then

$$\begin{aligned} \Pi_{z_c}^{(t,1,2)}(0) &= 2dz^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} (1 + \mathcal{U}_{0,1} + \mathcal{U}_{1,2} + \mathcal{U}_{0,1}\mathcal{U}_{1,2}) \\ &= 2dz^2 \left[g^{(t)}(z)^3 + 2g^{(t)}(z)W_z^{(t)}(e_1) \right. \\ &\quad \left. + \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,1}\mathcal{U}_{1,2} \right]. \end{aligned} \quad (5.20)$$

We study in more detail the last term in this expression. The simplest way in which it contributes is when the rib R_1 contains the edge $\{0, e_1\}$, then automatically the factors $\mathcal{U}_{0,1}$ and $\mathcal{U}_{1,2}$ are -1 and do not interfere. In this case, we can think of R_1 consisting of the edge $\{0, e_1\}$ and the two non-intersecting subribs R_1^0 and R_1^1 emanating from 0 and e_1 , respectively. Then we have

$$\begin{aligned} \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,1}\mathcal{U}_{1,2} &= zg^{(t)}(z)^2 \sum_{R_1^0 \ni 0, R_1^1 \ni e_1} z^{|R_1^0|+|R_1^1|} (1 + \mathcal{U}_{0,1}^*) \\ &\quad + \sum_{\substack{R_0, R_2 \ni 0 \\ R_1 \ni e_1, R_1 \not\ni \{0, e_1\}}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,1}\mathcal{U}_{1,2}, \end{aligned} \quad (5.21)$$

where $\mathcal{U}_{0,1}^* := -1$ if the subribs R_1^0 and R_1^1 have a common vertex, and 0 otherwise. The second summand of this expression is of order $o((2d)^{-1})$ as $d \rightarrow \infty$. The reason is that if the edge $\{0, e_1\}$ is not present in R_1 , then either R_1 reaches R_0 and R_2 , which requires at least three edges and can be bounded by $g^{(t)}(z)^2 G_z^{(t,3)}(e_1)$; or R_0 and R_2 reach R_1 , which requires at least 2 edges (one from R_0 and another one from R_2), and this contribution can be bounded by $z^2 g^{(t)}(z)^5$. These bounds evaluated at $z = z_c^{(t)}$, together with Theorem 1.1 and Proposition 4.5, yield the desired order $o((2d)^{-1})$.

We distribute the product by $(1 + \mathcal{U}_{0,1}^*)$ in the first term of (5.21) and get

$$\begin{aligned} \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni 0}} \binom{z_c^{(t)}}{z_c}^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,1} \mathcal{U}_{1,2} &= z_c^{(t)} g^{(t)}(z_c^{(t)})^4 + z_c^{(t)} g^{(t)}(z_c^{(t)})^2 W_{z_c^{(t)}}^{(t)}(e_1) + o\left(\frac{1}{2d}\right) \\ &= z_c^{(t)} g^{(t)}(z_c^{(t)})^4 + o\left(\frac{1}{2d}\right) \text{ as } d \rightarrow \infty, \end{aligned}$$

where the second equality is due to Theorem 1.1 and Claim 4.6. Substituting this result into (5.20), and using Claim 4.6, Lemma 4.3 and Lemma 4.1, yield

$$\begin{aligned} \Pi_{z_c^{(t)}}^{(t,1,2)}(0) &= 2d \binom{z_c^{(t)}}{z_c}^2 \left[g^{(t)}(z_c^{(t)})^3 + 2g^{(t)}(z_c^{(t)}) W_{z_c^{(t)}}^{(t)}(e_1) + z_c^{(t)} g^{(t)}(z_c^{(t)})^4 + o\left(\frac{1}{2d}\right) \right] \\ &= \frac{e}{2d} + \frac{\frac{9}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned} \quad (5.22)$$

The contribution to $\Pi_z^{(t,1)}(0)$ when the cycles formed in (4.32) have length 4, is given by

$$\Pi_z^{(t,1,4)}(0) = \sum_{\substack{\omega \in \mathcal{W}^{(0,0)}, \\ |\omega|=4}} z^4 \left[\prod_{i=0}^4 \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \prod_{\substack{0 \leq k < l \leq 4 \\ (k,l) \neq (0,4)}} (1 + \mathcal{U}_{k,l}). \quad (5.23)$$

The walks in this sum have the form $\omega = (0, e_i, e_i + e_j, e_j, 0)$, where e_i and e_j are perpendicular nearest-neighbors of the origin, denoted by $e_i \perp e_j$. There are $2d(2d-2)$ of these walks. Hence, for the walk $\omega = (0, e_1, e_1 + e_2, e_2, 0)$ we have

$$\Pi_z^{(t,1,4)}(0) = 2d(2d-2)z^4 \left[\prod_{i=0}^4 \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \prod_{\substack{0 \leq k < l \leq 4 \\ (k,l) \neq (0,4)}} (1 + \mathcal{U}_{k,l}). \quad (5.24)$$

To determine an upper bound for this quantity we use that the product $\prod (1 + \mathcal{U}_{k,l}) \leq 1$, and Theorem 1.1, to get

$$\Pi_{z_c^{(t)}}^{(t,1,4)}(0) \leq 2d(2d-2) \binom{z_c^{(t)}}{z_c}^4 g^{(t)}(z_c^{(t)})^5 = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.25)$$

To obtain a lower bound for (5.24), we apply inequality (4.37) to the last product in it and get

$$\begin{aligned} \Pi_z^{(t,1,4)}(0) &\geq 2d(2d-2)z^4 \left[\prod_{i=0}^4 \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \left(1 + \sum_{\substack{0 \leq k < l \leq 4 \\ (k,l) \neq (0,4)}} \mathcal{U}_{k,l} \right) \\ &= 2d(2d-2)z^4 \left[g^{(t)}(z)^5 + 6g^{(t)}(z)^3 W_z^{(t)}(e_1) + 3g^{(t)}(z)^3 W_z^{(t)}(e_1 + e_2) \right]. \end{aligned}$$

Estimate (5.5) together with Theorem 1.1 and Claim 4.6, yield

$$\Pi_{z_c^{(t)}}^{(t,1,4)}(0) \geq \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.26)$$

Therefore, bounds (5.25) and (5.26) imply

$$\Pi_{z_c^{(t)}}^{(t,1,4)}(0) = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.27)$$

Adding (5.22) and (5.27) yields

$$\Pi_{z_c^{(t)}}^{(t,1)}(0) = \frac{e}{2d} + \frac{11e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.28)$$

Second summand in (5.18). In this case the vertex x is a nearest neighbor of the origin. By (4.41) and Claim 5.1, the contribution by the cycles with length 2 is given by

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,2)}(x) = -2dz_c^{(t)} W_{z_c^{(t)}}^{(t)}(e_1) = \frac{2e}{2d} + \frac{14e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.29)$$

The contribution by the cycles with length 4 is given by

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,4)}(x) = 2d\Pi_z^{(t,1,4)}(0, e_1) = - \sum_{\substack{\omega \in \mathcal{W}(0, e_1) \\ |\omega|=3}} z^3 \left[\prod_{i=0}^3 \sum_{R_i \ni \omega_i} z^{|R_i|} \right] \mathcal{U}_{0,3} \prod_{\substack{0 \leq k < l \leq 3 \\ (k,l) \neq (0,3)}} (1 + \mathcal{U}_{k,l}),$$

where the sum is over all 3-step walks connecting 0 to e_1 of the form $\omega = (0, y, y + e_1, e_1)$, with y a nearest-neighbor of the origin perpendicular to e_1 . There are $2d - 2$ of these walks. Therefore, the fact that the final product $\prod (1 + \mathcal{U}_{k,l}) \leq 1$, Theorem 1.1 and Claim 4.6, yield the upper bound

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,4)}(x) \leq -2d(2d-2) \left(z_c^{(t)}\right)^3 g^{(t)}(z_c^{(t)})^2 W_{z_c^{(t)}}^{(t)}(e_1) = \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

A lower bound can be achieved by using inequality (4.37),

$$\begin{aligned} \sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,4)}(x) &\geq -2d(2d-2)z^3 \left[\prod_{i=0}^3 \sum_{R_i \ni \omega_i} z^{|R_i|} \right] \mathcal{U}_{0,3} \left(1 + \sum_{\substack{0 \leq k < l \leq 3 \\ (k,l) \neq (0,3)}} \mathcal{U}_{k,l} \right) \\ &= -2d(2d-2)z^3 g^{(t)}(z)^2 W_z^{(t)}(e_1) \\ &\quad - 2d(2d-2)z^3 \left[\prod_{i=0}^3 \sum_{T_i \ni \omega_i} z^{|R_i|} \right] \mathcal{U}_{0,3} (\mathcal{U}_{0,1} + \mathcal{U}_{0,2} + \mathcal{U}_{1,2} + \mathcal{U}_{1,3} + \mathcal{U}_{2,3}). \end{aligned} \quad (5.30)$$

Applying Lemma 4.3, Lemma 4.1 and Claim 4.6 to the first term, and using that the second one evaluated at criticality is of order $o((2d)^{-2})$, which we will prove below, we get

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,4)}(x) \geq \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

Therefore, these upper and lower bounds imply

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1,4)}(x) = \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.31)$$

Adding (5.29) and (5.31), gives

$$\sum_{x: \|x\|_1=1} \Pi_{z_c^{(t)}}^{(t,1)}(x) = \frac{2e}{2d} + \frac{16e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.32)$$

To see that the second term in (5.30) is of order $o((2d)^{-2})$, we distribute the product of $\mathcal{U}_{0,3}$ among the factors $\mathcal{U}_{0,1}, \mathcal{U}_{0,2}, \dots, \mathcal{U}_{2,3}$, and study the summand containing $\mathcal{U}_{0,3}\mathcal{U}_{0,1}$, which is analogous to the others. This term is given by

$$-2d(2d-2)z^3 g^{(t)}(z) \sum_{\substack{R_0 \ni 0, R_1 \ni y \\ R_3 \ni e_1}} z^{|R_0|+|R_1|+|R_3|} \mathcal{U}_{0,3} \mathcal{U}_{0,1}.$$

Intersections between the trees R_0 and R_3 , and R_0 and R_1 , require at least one edge each, so if we treat the ribs emanating from these bonds as independent of each other, we get the bound

$$\sum_{\substack{R_0 \ni 0, R_1 \ni y \\ R_3 \ni e_1}} z^{|R_0|+|R_1|+|R_3|} \mathcal{U}_{0,3} \mathcal{U}_{0,1} \leq 4z^2 g^{(t)}(z)^5, \quad (5.33)$$

which together with Theorem 1.1 imply the desired result, namely,

$$\begin{aligned} & \left| -2d(2d-2) \left(z_c^{(t)}\right)^3 \left[\prod_{i=0}^3 \sum_{R_i \ni \omega_i} \left(z_c^{(t)}\right)^{|R_i|} \right] \mathcal{U}_{0,3} (\mathcal{U}_{0,1} + \mathcal{U}_{0,2} + \mathcal{U}_{1,2} + \mathcal{U}_{1,3} + \mathcal{U}_{2,3}) \right| \\ & \leq 5 \cdot 4 \cdot 2d(2d-2) \left(z_c^{(t)}\right)^5 g^{(t)}(z_c^{(t)})^6 = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

Third summand in (5.18). In this case the vertices x are at distance 2 from the origin, they are of the form $x = 2e_i$ or $x = e_i + e_j$, with e_i and e_j perpendicular nearest-neighbors of the origin.

When $x = 2e_i$, the shortest walk that connects it to the origin is given by $\omega = (0, e_i, 2e_i)$. If there is an intersection between the ribs containing 0 and x , without intersecting the rib emanating from e_i , a cycle of length at least 6 must exist. Moreover, all other walks connecting 0 and x in which the first and last rib intersect, form a cycle of length greater than 6. Hence, the contribution of this case

is included in the error term (5.19) and is of order $O((2d)^{-3})$.

Therefore, we only need to analyze the cases in which $x = e_i + e_j$. The cycles have length 4, they consist of a 2-step walk connecting x to the origin, and two more edges belonging to either the first or last rib. There are $2d(2d-2)/2$ of the points x and for each one of them, there are two 2-step walks connecting them to the origin, namely, $\omega = (0, e_i, e_i + e_j)$ and $\omega = (0, e_j, e_i + e_j)$. This implies

$$\sum_{x=e_i+e_j} \Pi_z^{(t,1,4)}(x) = -2d(2d-2)z^2 \sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni e_1+e_2}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,2} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}). \quad (5.34)$$

The analysis of the sum is very similar to one we do for the quantity $W_z^{(t)}(e_1 + e_2)$ in the proof of Claim 5.1, if we treat the tree R_1 independently. The only difference is that, since R_0 and R_2 must avoid intersecting R_1 , there is only one walk along which these two trees intersect instead of two, namely, $(0, e_2, e_1 + e_2)$. Therefore, we have the lower bound

$$\sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni e_1+e_2}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,2} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) \geq \frac{1}{2} W_z^{(t)}(e_1 + e_2) g^{(t)}(z).$$

To get an upper bound for this sum, we subtract the cases in which R_1 intersects one of the trees R_0 or R_2 , this event requires of at least an edge. This gives

$$\sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni e_1+e_2}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,2} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) \leq \frac{1}{2} W_z^{(t)}(e_1 + e_2) g^{(t)}(z) - \frac{1}{2} W_z^{(t)}(e_1 + e_2) z g^{(t)}(z)^2.$$

These two bounds together with Theorem 1.1 and Claim 5.1 imply

$$\sum_{\substack{R_0 \ni 0, R_1 \ni e_1 \\ R_2 \ni e_1+e_2}} \left(z_c^{(t)} \right)^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,2} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) = -\frac{3e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

Substituting this result into (5.34) and Theorem 1.1, yield

$$\sum_{x=e_i+e_j} \Pi_{z_c}^{(t,1,4)}(x) = \frac{3e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.35)$$

Finally, replacing (5.28), (5.32) and (5.35) into (5.18) gives the desired result (5.16), namely,

$$\hat{\Pi}_{z_c}^{(t,1)}(0) = \frac{3e}{2d} + \frac{49e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

Proof of Equation 5.17. The quantity $\hat{\Pi}_{z_c}^{(t,2)}(0)$ is the Fourier transform of the function $\Pi_z^{(t,2)}(x)$, at

criticality, it is given by

$$\hat{\Pi}_z^{(t,2)}(0) = \sum_x \Pi_z^{(t,2)}(x). \quad (5.36)$$

The quantity $\Pi_z^{(t,2)}(x)$ is defined, in (3.31), by

$$\Pi_z^{(t,2)}(x) = \sum_{\substack{\omega \in \mathcal{W}^{(0,x)} \\ |\omega| \geq 2}} z^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z^{|R_i|} \right] \left[\sum_{L \in \mathcal{L}^{(2)}[0,|\omega|]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}) \right], \quad (5.37)$$

where the set $\mathcal{L}^{(2)}[0,|\omega|]$ consists of the following two types of laces:

(a) $L = \{0j, j|\omega|\}$, with $0 < j < |\omega|$.

(b) $L = \{0j, i|\omega|\}$, with $0 < i < j < |\omega|$.

Recall that in Figure 3.5 (b) there is a graphical representation of these laces. We will analyze $\Pi_z^{(t,2)}(x)$ according to the length of the walk ω . We will denote by $\Pi_z^{(t,2,n)}(x)$, the contribution to $\Pi_z^{(t,2)}(x)$ when $|\omega| = n$ on the right-hand side of (5.37).

When the walk ω has length 2, the only lace in $\mathcal{L}^{(2)}[0,2]$ is of type (a). It is given by $L = \{01, 12\}$ and has no compatible edges. This implies that

$$\sum_x \Pi_z^{(t,2,2)}(x) = \sum_x \sum_{\substack{\omega \in \mathcal{W}^{(0,x)} \\ |\omega|=2}} z^2 \sum_{\substack{R_0 \ni \omega(0), R_1 \ni \omega(1) \\ R_2 \ni \omega(2)}} z^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{0,1} \mathcal{U}_{1,2}. \quad (5.38)$$

This quantity is non-zero when the vertex $x = 0$ or x is at distance two from the origin.

When $x = 0$, the walk is of the form $\omega = (0, e_i, 0)$, where e_i is one of the $2d$ nearest-neighbor vertices of the origin. The simplest way in which there are intersections among the ribs R_0 and R_1 , and R_1 and R_2 , is when R_1 consists of the edge $\{0, e_i\}$ and two non-intersecting subribs R_0^* and R_1^* , emanating from 0 and e_i respectively. In this situation the product $\mathcal{U}_{0,1} \mathcal{U}_{1,2} = 1$ and does not affect R_0 and R_1 , which implies that the sums over these trees become one-point functions. Then we have the lower bound

$$\begin{aligned} \Pi_{z_c}^{(t,2,2)}(0) &\geq 2d(z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \sum_{R_0^* \ni 0, R_1^* \ni e_i} (z_c^{(t)})^{|R_0^*|+|R_1^*|} (1 + \mathcal{U}_{0,1}^*) \\ &= 2d(z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \left[g^{(t)}(z_c^{(t)})^2 + W_{z_c^{(t)}}^{(t)}(e_i) \right] = \frac{\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \end{aligned}$$

where $\mathcal{U}_{0,1}^*$ is minus the indicator that subribs R_0^* and R_1^* have a common vertex, and the last equality is given by Theorem 1.1 and Claim 4.6.

For an upper bound, in the previous expression we use that $1 + \mathcal{U}_{0,1}^* \leq 1$, to get that the contribution when R_1 contains the edge $\{0, e_i\}$ is bounded above by $2d(z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^4$. When $\{0, e_i\} \notin R_1$, either both ribs R_0 and R_2 contain the edge $\{0, e_i\}$ or one of them does not, say R_0 . In the first case,

we treat the subribs emanating from $\{0, e_i\}$ as independent to get one-point functions. In the second case, the intersection among R_0 and R_1 requires at least three bonds. These observations imply the upper bound

$$\begin{aligned} \Pi_{z_c^{(t)}}^{(t,2,2)}(0) &\leq 2d(z_c^{(t)})^2 \left[z_c^{(t)} g^{(t)}(z_c^{(t)})^4 + (z_c^{(t)})^2 g^{(t)}(z_c^{(t)})^5 + g^{(t)}(z_c^{(t)}) \sum_{j+k=3} G_{z_c^{(t)}}^{(t,j)} * G_{z_c^{(t)}}^{(t,k)}(e_i) \right] \\ &= \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \end{aligned}$$

where the last equality is given by Theorem 1.1 and Proposition 4.5.

When x is one of the $2d(2d-1)$ vertices at distance two from the origin, i.e., $x = e_i + e_j$ with e_i and e_j nearest-neighbors of the origin with $e_j \neq -e_i$. If $e_i \neq e_j$, the two walks connecting 0 to x are $\omega = (0, e_i, x)$ and $\omega = (0, e_j, x)$. In this case the simplest intersection among the ribs R_0 and R_1 , and R_1 and R_2 , requires that either R_0 or R_1 have the bond $\{0, e_i\}$, and that either R_1 or R_2 have the bond $\{e_i, x\}$. To get a lower bound, we treat the subribs emanating from these bonds as independent and use inclusion-exclusion to subtract the possible intersections among them, this yields

$$\begin{aligned} \sum_{x: \|x\|_1=2} \Pi_{z_c^{(t)}}^{(t,2,2)}(x) &\geq 4(2d)(2d-2)(z_c^{(t)})^4 g^{(t)}(z_c^{(t)}) \left[g^{(t)}(z_c^{(t)})^4 - 2W_{z_c^{(t)}}^{(t)}(e_i) z_c^{(t)} g^{(t)}(z_c^{(t)})^2 \right] \\ &= \frac{4e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \end{aligned}$$

where the last equality is given by Theorem 1.1 and Claim 4.6.

For an upper bound, if $\{0, e_i\}$ is not present in R_0 and R_1 , or $\{e_i, x\}$ is not present in R_1 and R_2 , then an intersection among the corresponding ribs requires at least three edges. This implies

$$\begin{aligned} \sum_{x: \|x\|_1=2} \Pi_{z_c^{(t)}}^{(t,2,2)}(x) &\leq 4(2d)(2d-1)(z_c^{(t)})^4 g^{(t)}(z_c^{(t)})^4 \\ &\quad + 2(2d)(2d-1)(z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \sum_{j+k=3} G_{z_c^{(t)}}^{(t,j)} * G_{z_c^{(t)}}^{(t,k)}(e_i) \\ &\quad + 2d(2d-1)(z_c^{(t)})^2 \left[\sum_{j+k=3} G_{z_c^{(t)}}^{(t,j)} * G_{z_c^{(t)}}^{(t,k)}(e_i) \right]^2 \\ &= \frac{4e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \end{aligned}$$

where the last equality is given by Theorem 1.1 and Proposition 4.5.

Therefore, when $|\omega| = 2$ the contribution to (5.36) is

$$\sum_x \Pi_{z_c^{(t)}}^{(t,2,2)}(x) = \Pi_{z_c^{(t)}}^{(t,2,2)}(0) + \sum_{x: \|x\|_1=2} \Pi_{z_c^{(t)}}^{(t,2,2)}(x) = \frac{5e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.39)$$

When the walk ω has length 3, the laces of type (a) in $\mathcal{L}^{(2)}[0, 2]$ are $L = \{01, 13\}$ and $L = \{02, 23\}$. There is only one lace of type (b), namely $L = \{02, 13\}$. Due to the symmetries of \mathbb{Z}^d , the two type (a) laces give the same contribution, so we only study the contribution to $\Pi_z^{(t, 2, 3)}(x)$ in the case of $L = \{01, 13\}$, which is

$$\sum_x \Pi_z^{(t, 2, 3)}(x) = \sum_x \sum_{\substack{\omega \in \mathcal{W}(0, x) \\ |\omega|=3}} z^3 \sum_{\substack{R_0 \ni \omega(0), R_1 \ni \omega(1) \\ R_2 \ni \omega(2), R_3 \ni \omega(3)}} z^{|R_0|+|R_1|+|R_2|+|R_3|} \mathcal{U}_{0,1} \mathcal{U}_{1,3} (1 + \mathcal{U}_{1,2}) (1 + \mathcal{U}_{2,3}). \quad (5.40)$$

This quantity is non-zero when x is at distance 1 and 3 from the origin.

When $\|x\|_1 = 1$, the walks ω are of the form $\omega = (0, x, y, x)$, for y a nearest-neighbor of x (including the origin); and $\omega = (0, e_i, e_i + x, x)$, for a nearest-neighbor of the origin e_i different from x . In the first case, when $\omega = (0, x, y, x)$, there are $2d$ of these walks for every vertex x and, since $\omega(1) = x = \omega(3)$, the factor $\mathcal{U}_{1,3} = -1$ and does not affect (5.40). The fact that $(1 + \mathcal{U}_{1,2})(1 + \mathcal{U}_{2,3}) \leq 1$, Theorem 1.1 and Claim 5.1, yield that the contribution to (5.40), at criticality, is bounded above by

$$\begin{aligned} -(2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \sum_{R_0 \ni 0, R_1 \ni x} (z_c^{(t)})^{|R_0|+|R_1|} \mathcal{U}_{0,1} &= -(2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 W_{z_c^{(t)}}^{(t)}(x) \\ &= \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

For a lower bound, we use inequality (4.37). Then Theorem 1.1 and Claim 5.1 yield that the contribution to (5.40), at criticality, is bounded below by

$$\begin{aligned} -(2d)^2 (z_c^{(t)})^3 \sum_{\substack{R_0 \ni \omega(0), R_1 \ni \omega(1) \\ R_2 \ni \omega(2), R_3 \ni \omega(3)}} z^{|R_0|+|R_1|+|R_2|+|R_3|} \mathcal{U}_{0,1} (1 + \mathcal{U}_{1,2} + \mathcal{U}_{2,3}) \\ \geq -(2d)^2 (z_c^{(t)})^3 \left[g^{(t)}(z_c^{(t)})^2 W_{z_c^{(t)}}^{(t)}(x) + 2W_{z_c^{(t)}}^{(t)}(x)^2 \right] &= \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

Recall to count this result twice to take into account the contribution of the other lace, $L = \{02, 23\}$.

In the second case, when $\omega = (0, e_i, e_i + x, x)$, using the fact that $|\mathcal{U}_{0,1}(1 + \mathcal{U}_{1,2})(1 + \mathcal{U}_{2,3})| \leq 1$, Theorem 1.1 and Claim 5.1, yield that the contribution to (5.40), at criticality, is bounded by

$$\begin{aligned} (2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \sum_{R_1 \ni e_i, R_3 \ni x} z^{|R_1|+|R_3|} |\mathcal{U}_{1,3}| &= (2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \left| W_{z_c^{(t)}}^{(t)}(x - e_i) \right| \\ &= o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \end{aligned}$$

If $\|x\|_1 = 3$, an intersection between the ribs R_1 and R_3 , without intersecting R_2 , requires at least four edges. This observation together with Theorem 1.1 and Proposition 4.5, yield that the

contribution to (5.40), at criticality, is bounded by

$$\begin{aligned}
& (2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)}) \sum_{\substack{R_1 \ni \omega(1), R_2 \ni \omega(2) \\ R_3 \ni \omega(3)}} (z_c^{(t)})^{|R_1|+|R_2|+|R_3|} |\mathcal{U}_{1,3} (1 + \mathcal{U}_{1,2}) (1 + \mathcal{U}_{2,3})| \\
& \leq (2d)^2 (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^2 \sum_{j+k=4} G_{z_c^{(t)}}^{(t,j)} * G_{z_c^{(t)}}^{(t,k)}(x - e_i) = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.
\end{aligned} \tag{5.41}$$

Now we analyze the type (b) lace $L = \{02, 13\}$. The contribution to $\Pi_z^{(t,2,3)}(x)$ is

$$\sum_x \Pi_z^{(t,2,3)}(x) = \sum_x \sum_{\substack{\omega \in \mathcal{W}^{(0,x)} \\ |\omega|=3}} z^3 \sum_{\substack{R_0 \ni \omega(0), \dots, \\ R_3 \ni \omega(3)}} z^{|R_0|+\dots+|R_3|} \mathcal{U}_{0,2} \mathcal{U}_{1,3} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) (1 + \mathcal{U}_{2,3}). \tag{5.42}$$

This expression is non-zero when x is at distance 1 and 3 from the origin.

When $\|x\|_1 = 1$, the walks ω are of the form $\omega = (0, x, 0, x)$, and $\omega = (0, e_i, e_i + y, x)$, for a nearest-neighbor of the origin e_i different from x , and $y = -e_i$ or $y = x$. In the first case, since $\omega(0) = 0 = \omega(2)$ and $\omega(1) = x = \omega(3)$, the factor $\mathcal{U}_{0,2} \mathcal{U}_{1,3} = 1$ and does not affect (5.42). The fact that $(1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) (1 + \mathcal{U}_{2,3}) \leq 1$ and Theorem 1.1, yield that the contribution to (5.40), at criticality, is bounded above by

$$2d (z_c^{(t)})^3 g^{(t)}(z_c^{(t)})^4 = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

For a lower bound, we use inequality (4.37). Then Theorem 1.1 and Claim 5.1 yield that the contribution to (5.42), at criticality, is bounded below by

$$\begin{aligned}
2dz^3 \sum_{\substack{R_0 \ni 0, \dots, \\ R_3 \ni x}} z^{|R_0|+\dots+|R_3|} (1 + \mathcal{U}_{0,1} + \mathcal{U}_{1,2} + \mathcal{U}_{2,3}) & \geq 2dz^3 \left[g^{(t)}(z_c^{(t)})^4 + 3W_{z_c^{(t)}}^{(t)}(x) \right] \\
& = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.
\end{aligned}$$

In the second case, when $\omega = (0, e_i, e_i + y, x)$, the fact that $|\mathcal{U}_{0,2} (1 + \mathcal{U}_{0,1}) (1 + \mathcal{U}_{1,2}) (1 + \mathcal{U}_{2,3})| \leq 1$, Theorem 1.1 and Claim 5.1, yield that the contribution to (5.40), at criticality, is bounded by

$$\begin{aligned}
2d(2d-1)z^3 g^{(t)}(z_c^{(t)})^2 \sum_{R_1 \ni e_i, R_3 \ni x} (z_c^{(t)})^{|R_1|+|R_3|} |\mathcal{U}_{1,3}| & \leq 2d(2d-1)z^3 g^{(t)}(z_c^{(t)})^2 \left| W_{z_c^{(t)}}^{(t)}(x - e_i) \right| \\
& = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.
\end{aligned}$$

If $\|x\|_1 = 3$, the lace $L = \{02, 13\}$ forces an intersection between the ribs R_1 and R_3 , without intersecting R_2 . An argument similar to (5.41), yields that the contribution in this case is of order $o((2d)^{-2})$.

Therefore, when $|\omega| = 3$ the contribution to (5.36) is

$$\sum_x \Pi_{z_c}^{(t,2,3)}(x) = \sum_{x:\|x\|_1=1} \Pi_{z_c}^{(t,2,3)}(x) + \sum_{x:\|x\|_1=3} \Pi_{z_c}^{(t,2,3)}(x) = \frac{5e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.43)$$

When the walk ω has length 4, the laces of type (a) in $\mathcal{L}^{(2)}[0, 2]$ are $L = \{01, 14\}$, $L = \{02, 24\}$ and $L = \{03, 34\}$. The laces of type (b) are $L = \{02, 14\}$, $L = \{03, 14\}$ and $L = \{03, 24\}$. First, we will analyze the lace $L = \{02, 24\}$, and then show that the remaining laces' contributions are of order $((2d)^{-2})$ as $d \rightarrow \infty$.

The lace $L = \{02, 24\}$ contributes only for $x = 0$. In this case the significant walks are of the form $\omega = (0, e_i, 0, e_j, 0)$ with e_i and e_j nearest neighbors of the origin. There are $(2d)^2$ of such walks and due to their topology the factors $\mathcal{U}_{0,2}$ and $\mathcal{U}_{2,4}$ are automatically -1. Hence, treating the ribs emanating from the walk as independent and applying Theorem 1.1, gives the upper bound

$$(2d)^2 (z_c^{(t)})^4 g^{(t)}(z_c^{(t)})^5 = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

For a lower bound we subtract from this term the possible contributions from the intersections of the ribs with the term $W_{z_c}^{(t)}(e_1)$, this together with Theorem 1.1 and Claim 5.1 yield

$$(2d)^2 (z_c^{(t)})^4 \left(g^{(t)}(z_c^{(t)})^5 + 4g^{(t)}(z_c^{(t)})^3 W_{z_c}^{(t)}(e_1) \right) = \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

Adding this contribution to (5.39) and (5.43) we get the desired result for $\hat{\Pi}_{z_c}^{(t,2)}(0)$ in (5.17), namely.

$$\hat{\Pi}_{z_c}^{(t,2)}(0) = \frac{11e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

Next we prove that the remaining terms' contributions is of order $o((2d)^{-2})$, as $d \rightarrow \infty$.

The other walks connecting 0 to 0 of length four, are 4-step cycles containing the origin. If we substitute one of the factors $\mathcal{U}_{0,2}$ or $\mathcal{U}_{2,4}$ by 1, say $\mathcal{U}_{0,2}$, and use that there are $2d(2d-2)/2$ of such cycles, we get that the contribution to $\Pi_{z_c}^{(t,2,4)}(0)$ is bounded by

$$2 \frac{2d(2d-2)}{2} (z_c^{(t)})^4 g^{(t)}(z_c^{(t)})^3 \left| W_{z_c}^{(t)}(e_1 + e_2) \right| = o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty,$$

where the first factor 2 is given by exchanging the role of $\mathcal{U}_{0,2}$ and $\mathcal{U}_{2,4}$, $W_{z_c}^{(t)}(e_1 + e_2)$ takes into account the intersection of two of the ribs R_2 and R_4 that $\mathcal{U}_{2,4}$ forces, and the equality is given by Theorem 1.1 and Claim 5.1.

For the same lace $L = \{02, 24\}$, when $x \neq 0$. We will analyze just one case to illustrate how

these terms can be bounded. If x is at distance two from the origin, then the walk may look like in Figure 5.1. Notice that the factor $\mathcal{U}_{2,4}$ forces an intersection among the corresponding ribs R_2 and R_4 , then we can find paths of lengths i and j that connect 0 to y , and y to x , respectively, with $i + j \geq 2$. This contribution can be bounded by

$$\begin{aligned} \sum_{i+j=2} \sum_x \sum_y G_z^{(t,4)}(x) G_z^{(t,i)}(y) G_z^{(t,j)}(x-y) &= \sum_{i+j=2} G_z^{(t,4)} * G_z^{(t,i)} * G_z^{(t,j)}(0) \\ &\leq 3 \left(2dzg^{(t)}(z)\right)^6 D^{*6} * G_{z_c}^{(t,*i)}(0) \quad (5.44) \\ &\leq 3 \left(2dzg^{(t)}(z)\right)^6 \frac{C_{6,3}}{(2d)^3}, \end{aligned}$$

where we used (4.14) for the first inequality, and Proposition 4.5 for the second one. Then Theorem 1.1 gives that this expression at criticality is of order $o((2d)^{-2})$, as $d \rightarrow \infty$.

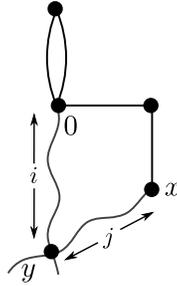


Figure 5.1: $L = \{02, 24\}$ and $\|x\|_1 = 2$.

The other laces for the 4-walk ω , have an edge between points at distance three, for example, in $L = \{01, 14\}$ the edge 14 is present. Moreover, for these laces, there are compatible edges of length two, in our example these are 13 and 24. This implies that the part of the walk that lies between the points 1 and 4 is self-avoiding. Since the edge of 14 forces an intersection between the corresponding ribs, a cycle of length four or more is created. These observations will allow us to conclude that all laces, except $L = \{02, 24\}$, give a contribution of order $O((2d)^{-3})$, which we can neglect. We will only explain $L = \{01, 14\}$ and a few other cases depicted in Figure 5.2.

The following is a standard step in the lace expansion technique. We only outline the main ideas behind it, so we encourage the curious reader to consult [39] for more details. We construct a walk ω , with the minimum amount of edges to satisfy the intersections among its emanating ribs, according to the lace on the left in Figure 5.2. We add bonds (dash line in picture), if necessary, to create such intersections. As we have noticed, the parts of the path that lie between the end points of the edges in the lace are self-avoiding, therefore, some cycles are created in order to satisfy the intersections required by the lace. The contribution by these walks is bounded by a convolution of

n two-point functions taking at least m steps (m and n depend on the lace), like we did in (5.44). We separate (factor out) the cycles and bound them by $S_z^{(p,q)}$, then Proposition 4.5 yields the contribution on the right hand side.

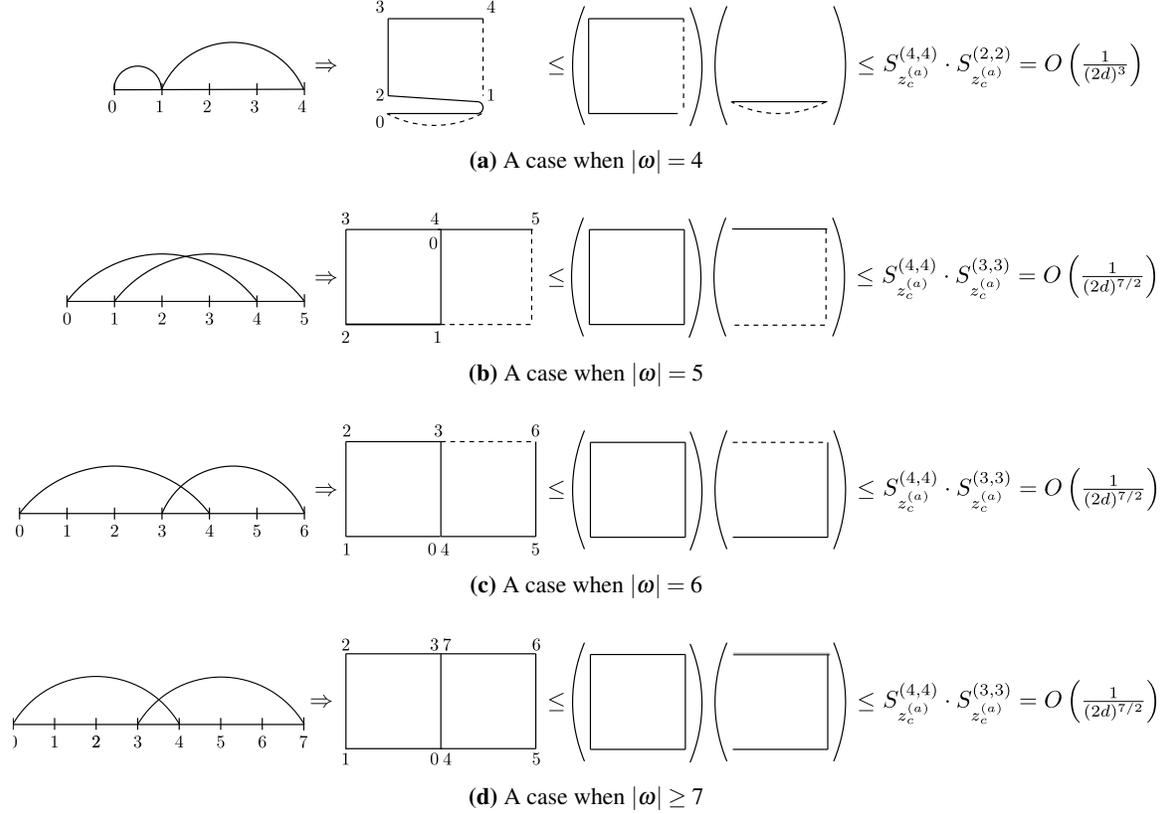


Figure 5.2: Bounds for different laces and different sizes of $|\omega|$. Dashed lines represent edges that must be added in order to create the intersections required by the lace on the left.

□

5.2.2 Lattice animals

In this section we prove Theorem 1.4 for the case of lattice animals. Moreover, the discrepancy between the expansions in Theorem 1.2 is explained.

Proof of Theorem 1.4 for lattice animals. Recall that the Fourier transform $\hat{\Pi}_{z_c}^{(a)}(0)$ is given in (3.44) by

$$\hat{\Pi}_{z_c}^{(a)}(0) = \hat{\Pi}_{z_c}^{(a,0)}(0) - \hat{\Pi}_{z_c}^{(a,1)}(0) + \hat{\Pi}_{z_c}^{(a,2)}(0) + \sum_{N \geq 3} (-1)^N \hat{\Pi}_{z_c}^{(a,N)}(0). \quad (5.45)$$

By (4.44), the fourth term in this expression is of order $O((2d)^{-3})$, and can be neglected. We will

prove that the first three terms on the right-hand side of (5.45) satisfy, as $d \rightarrow \infty$,

$$\hat{\Pi}_{z_c}^{(a,0)}(0) = \frac{\frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \quad (5.46)$$

$$-\hat{\Pi}_{z_c}^{(a,1)}(0) = -\frac{3e}{2d} - \frac{\frac{49}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \quad (5.47)$$

$$\hat{\Pi}_{z_c}^{(a,2)}(0) = \frac{11e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.48)$$

Substituting (5.46), (5.47) and (5.48) into (5.45), gives the desired result (5.14).

Remark 5.3. *It is the term $\hat{\Pi}_{z_c}^{(a,0)}(0)$ in (5.46), which is absent for lattice trees, that gives the new term in (1.19) for lattice animals. Results in (5.47) and (5.48) coincide with the corresponding ones for lattice trees in (5.16) and (5.17).*

The term $\hat{\Pi}_{z_c}^{(a,0)}(0)$. It is defined in (3.39). It is bounded above by summing z^4 over $x \neq 0$ and the cycles of length four, that contain the vertices 0 and x , plus the contribution of the animals with cycles of length six and more that contain 0 and x , which is bounded by $S_z^{(6,2)}(0)$. There are $2d(2d-2)/2$ of cycles with length four, and each such cycle has three possibilities for x . If we treat the four ribs emanating from the vertices in the four-step cycle as independent, then, by Theorem 1.1 and Proposition 4.5, we get

$$\hat{\Pi}_{z_c}^{(a,0)}(0) \leq \frac{3}{2}2d(2d-2)(z_c^{(a)}g^{(a)}(z_c^{(a)}))^4 + S_{z_c}^{(6,2)}(0) = \frac{\frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

From the term corresponding to the four-step cycles we subtract the contribution of the possible intersections among the ribs, to get the lower bound

$$\begin{aligned} \hat{\Pi}_{z_c}^{(a,0)}(0) &\geq \frac{3}{2}2d(2d-2)(z_c^{(a)})^4 \left[g^{(a)}(z_c^{(a)})^4 + 4g^{(a)}(z_c^{(a)})^2 W_{z_c}^{(a)}(e_1) + 2g^{(a)}(z_c^{(a)})^2 W_{z_c}^{(a)}(e_1 + e_2) \right] \\ &= \frac{\frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \end{aligned}$$

where the equality is due to Theorem 1.1 and Claim 5.1. This completes the proof of equation (5.46).

Remark 5.4. *The sum over cycles is the reason why the factor e is not present in the expression for $\hat{\Pi}_{z_c}^{(a,0)}(0)$. Cycles, unlike walks, have the same amount of edges and vertices. Hence, there are the same amount of ribs emanating from its vertices as there are edges. Each one of these ribs, when treated as independent, contributes with a factor of $g(z_c) \sim e$ which gets canceled by the factor e^{-1} in $z_c \sim (2de)^{-1}$ associated to each one of the edges in the cycle.*

Terms $-\hat{\Pi}_{z_c}^{(a,1)}(0)$ and $\hat{\Pi}_{z_c}^{(a,2)}(0)$. First we notice that when the backbone is given by walks, then the

double connections between the endpoints of the edges in the backbone become sums over animals that contain such vertices. Thus the reasoning that we did for lattice trees in the previous section can be applied to this situation and yield the results in (5.47) and (5.48).

On the other hand, if the backbone is not given by a walk, then some of the double connections between the endpoints of the vertices in the backbone contain cycles of length four or more. Similar ideas used in our discussion for $G_{z_c}(x)$ and $\hat{\Pi}_{z_c}^{(a)}(0)$ in Section 4.2 and Section 5.2.2, respectively, yield that the contribution in this case is of order $(2d)^{-2}$, as $d \rightarrow \infty$.

□

5.3 Proof of Theorem 1.5

The goal of this section is to show Theorem 1.5, which states that, as $d \rightarrow \infty$,

$$\begin{aligned} g^{(t)}(z_c^{(t)}) &= e + \frac{3e}{2d} + \frac{263e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \\ g^{(a)}(z_c^{(a)}) &= e + \frac{3e}{2d} + \frac{263e - 1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \end{aligned} \tag{5.49}$$

This result completes the demonstration of Theorem 1.2.

The proof will be simultaneously done for both lattice trees and lattice animals, except for minor adjustments for the latter case. In our reasoning, the expansion for $\hat{\Pi}_z(0)$ obtained in Section 5.2, Claim 5.1 and Lemma 5.2, will be fundamental.

Proof of Theorem 1.5. We will show that for nearest-neighbor lattice trees, as $d \rightarrow \infty$,

$$A(z_c^{(t)}) = e + \frac{3e}{2d} + \frac{45e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \tag{5.50}$$

while for nearest-neighbor lattice animals,

$$A(z_c^{(a)}) = e + \frac{3e}{2d} + \frac{45e - 3}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \tag{5.51}$$

and for both lattices trees and animals,

$$\begin{aligned}
B(z_c) &= -\frac{e}{2d} - \frac{8e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\
C(z_c) &= -\frac{\frac{1}{2}e}{2d} - \frac{\frac{15}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\
E(z_c) &= \frac{\frac{95}{24}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\
\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} z_c^{|A|} &= \frac{\frac{1}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).
\end{aligned} \tag{5.52}$$

By (3.9), this gives the desired result

$$g^{(t)}(z_c^{(t)}) = A(z_c^{(t)}) + B(z_c^{(t)}) + C(z_c^{(t)}) + E(z_c^{(t)}) = e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right),$$

and

$$\begin{aligned}
g^{(a)}(z_c^{(a)}) &= A(z_c^{(a)}) + B(z_c^{(a)}) + C(z_c^{(a)}) + E(z_c^{(a)}) + \mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} (z_c^{(a)})^{|A|} \\
&= e + \frac{\frac{3}{2}e}{2d} + \frac{\frac{263}{24}e - 1}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).
\end{aligned}$$

It remains to show (5.50), (5.51) and (5.52).

The term $A(z_c)$. We commence studying $A(z)$, which is given in (4.57) by

$$A(z_c) = e^{1-M(z_c)} = e \left(1 - M(z_c) + \frac{M(z_c)^2}{2!} + O(M(z_c)^3) \right). \tag{5.53}$$

We update the expansion for $M(z_c)$ by substituting the new results for $\hat{\Pi}_{z_c}^{(t)}(0)$ and $\hat{\Pi}_{z_c}^{(a)}(0)$ given in Theorem 1.4, into (4.55). In the case of lattice trees, we get

$$M(z_c^{(t)}) = -\frac{3}{2d} - \frac{18}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty,$$

and for lattice animals,

$$M(z_c^{(a)}) = -\frac{3}{2d} - \frac{18 - \frac{3}{2}e^{-1}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

Substituting these expansions in (5.53) yields the desired results (5.50) and (5.51). For the remainder of the proof, only the first two terms of the expansions in (5.50) and (5.51) will be used. Since these

terms agree, we will only refer to (5.50) for both lattice trees and lattice animals.

The term $B(z_c)$. Recall that $B(z_c)$ is given in (3.11) by

$$B(z_c) = -2dz_c A(z_c) G_{z_c}(e_1).$$

The desired result (5.52) for $B(z_c)$ is obtained when substituting (5.50), and the new expansions for z_c and $G_{z_c}(e_1)$ given in Lemma 4.1 and Lemma 5.2, respectively, into this expression.

The term $C(z_c)$. Recall that $C(z)$ is given in (4.59) by

$$C(z) = -\frac{1}{2}2d(zg(z))^2 A(z) + C_{error}(z), \quad (5.54)$$

where

$$C_{error}(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} (2dg(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} \mathbf{I}_{s_i \neq s_j} \sum_{C_i \ni s_i} z^{|C_i|} \sum_{C_j \ni s_j} z^{|C_j|} \gamma_{i,j}.$$

In the proof of Lemma 4.3 in Section 4.4, $C_{error}(z_c)$ was indeed an error term. However, we will see that on this occasion, its contribution cannot be neglected.

For the first term on the right-hand side of (5.54), Theorem 1.1, (5.50) and Lemma 4.1, yield

$$-\frac{1}{2}2d(z_c g(z_c))^2 A(z_c) = -\frac{\frac{1}{2}\mathbf{e}}{2d} - \frac{\frac{9}{2}\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.55)$$

For $C_{error}(z)$, we consider separately when the vertices s_i and s_j are parallel and perpendicular, we denoted the latter situation by $s_1 \perp s_2$. Hence,

$$\begin{aligned} C_{error}(z) &= z^2 \sum_{m=2}^{\infty} \frac{1}{m!} (2dzg(z))^{m-2} \frac{m(m-1)}{2} \sum_{s_1, s_2 \in \mathcal{E}} [\mathbf{I}_{s_2 = -s_1} + \mathbf{I}_{s_1 \perp s_2}] \left[\sum_{\substack{C_1 \ni s_1 \\ C_2 \ni s_2}} z^{|C_1|+|C_2|} \gamma_{1,2} \right] \\ &= \frac{1}{2} z^2 e^{2dzg(z)} \left[2d \sum_{\substack{C_1 \ni e_1 \\ C_2 \ni -e_1}} z^{|C_1|+|C_2|} \gamma_{1,2} + 2d(2d-2) \sum_{\substack{C_1 \ni e_1 \\ C_2 \ni e_2}} z^{|C_1|+|C_2|} \gamma_{1,2} \right] \\ &= \frac{1}{2} z^2 A(z) [2dW_z(2e_1) + 2d(2d-2)W_z(e_1 + e_2)]. \end{aligned}$$

By Theorem 1.1, (5.50) and Claim 5.1, we get that, as $d \rightarrow \infty$,

$$C_{error}(z_c) = -\frac{3\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.56)$$

Finally, replacing (5.55) and (5.56) into (5.54), yields the desired result

$$C(z_c) = -\frac{1}{2d}e - \frac{15}{2} \frac{e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty.$$

The term $E(z_c)$. Recall that $E(z)$ is given in (3.13) by

$$E(z) = \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{C}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \left(\sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \prod_{(p,q) \in A_{k,l}} (1 + \mathcal{V}_{p,q}) \right),$$

We will rewrite it in a more convenient expression for its study.

We iterate the result in (3.5) by applying it to the product that appears on its right-hand side, to obtain

$$\begin{aligned} \prod_{i=1}^n (1 + x_i) &= 1 + \sum_{i=1}^n x_i + \sum_{1 \leq i < j \leq n} x_i x_j \prod_{j < k \leq n} (1 + x_k) \\ &= 1 + \sum_{i=1}^n x_i + \sum_{1 \leq i < j \leq n} x_i x_j \left(1 + \sum_{j < k \leq n} x_k + \sum_{j < k < l \leq n} x_k x_l \prod_{l < p \leq n} (1 + x_p) \right) \\ &= 1 + \sum_{i=1}^n x_i + \sum_{1 \leq i < j \leq n} x_i x_j + \sum_{1 \leq i < j < k \leq n} x_i x_j x_k \\ &\quad + \sum_{1 \leq i < j < k < l \leq n} x_i x_j x_k x_l \prod_{l < p \leq n} (1 + x_p). \end{aligned} \tag{5.57}$$

Using this result, we can decompose the product $\prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j})$ as

$$\begin{aligned} \prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{i,j}) &= 1 + \sum_{1 \leq j \leq m} \mathcal{V}_{0,j} + \sum_{1 \leq i < j \leq m} \mathcal{V}_{i,j} + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \\ &\quad + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \\ &\quad + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \sum_{(s,t) \in A_{p,q}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t} \prod_{(u,v) \in A_{s,t}} (1 + \mathcal{V}_{u,v}), \end{aligned} \tag{5.58}$$

where the sets $A_{i,j}$ are as defined in (3.7). If we compare this expression to the one given in (3.6), we see that the last three terms in (5.58) correspond to the one used for defining $E(z)$ in (3.13). Moreover, the fourth summand on the right-hand side of (5.58) was decomposed in (4.62). Substituting (4.62) and the last two terms on the right-hand side of (5.58) into the definition of $E(z)$ in (3.13),

and distributing the product among the summands, yield

$$\begin{aligned}
E(z) &= \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right] \left(\sum_{1 \leq j < l \leq m} \mathcal{V}_{0,j} \mathcal{V}_{0,l} + \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \right. \\
&\quad + \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \\
&\quad \left. + \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \sum_{(s,t) \in A_{p,q}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t} \prod_{(u,v) \in A_{s,t}} (1 + \mathcal{V}_{u,v}) \right) \\
&= E^1(z) + E^2(z) + E^3(z) + E^4(z) + E^5(z),
\end{aligned} \tag{5.59}$$

where $E^1(z)$, $E^2(z)$, and $E^3(z)$ are as in (4.64), (4.65) and (4.66), respectively, and

$$E^4(z) := \sum_{m=3}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right], \tag{5.60}$$

$$\begin{aligned}
E^5(z) &:= \sum_{m=4}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \left[\sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \right. \\
&\quad \left. \times \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \sum_{(s,t) \in A_{p,q}} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t} \prod_{(u,v) \in A_{s,t}} (1 + \mathcal{V}_{u,v}) \right].
\end{aligned} \tag{5.61}$$

To analyze $E^1(z)$, $E^2(z)$ and $E^3(z)$, we will recall several results for these terms that were obtained in the proof of Lemma 4.3 in Section 4.4.

The term $E^1(z_c)$. By (4.67), we know that

$$E^1(z_c) = \frac{\frac{1}{2}\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \tag{5.62}$$

The term $E^2(z_c)$. Recall equation (4.69) in the proof of Lemma 4.3. For this equation we analyzed the cases when $r = |\{j, k, l\}| = 2$ and $r = |\{j, k, l\}| = 3$; counted that the number of possibilities for the labels j, k and l that obey these conditions are $2\binom{m}{2}$ and $3\binom{m}{3}$, respectively; estimated (4.70) and substituted this result into (4.69) to obtain the contribution in each case. Here, we will follow this same strategy using the updated values for the quantities involved.

Assume that the labels obey $|\{j, k, l\}| = 2$, then the vertices s_1 and s_2 satisfy $|\{s_1, s_2\}| = 1$ or $|\{s_1, s_2\}| = 2$. In the first case, by (4.72), we know that the contribution is

$$\frac{\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \tag{5.63}$$

When $|\{s_1, s_2\}| = 2$, the simplest way in which the term $\mathcal{V}_{0,1}\mathcal{V}_{1,2}$ is positive is when the cluster C_1 contains the edges $\{0, s_1\}$ and $\{0, s_2\}$, or the cluster C_1 contains $\{0, s_1\}$ and C_2 contains $\{0, s_2\}$. The analysis when $s_1 \perp s_2$ (perpendicular) and $s_1 = 2s_2$ is very similar to the one we did in Claim 5.1 for studying the quantities $W_z(e_i + e_j)$ and $W_z(2e_i)$, respectively. The difference is that the path connecting s_1 and s_2 does not belong to the cluster C_2 ; and in the case of $s_1 \perp s_2$, since the path must contain the origin, there is only one two-step walk connecting the vertices s_1 and s_2 instead of two. Hence, the contribution is captured in the positive quantity $-W_z(e_1 + e_2)/3$. This implies that (4.70) is given by

$$\sum_{s_1, s_2 \in \mathcal{E}} \mathbf{1}_{s_1 \neq s_2} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1|+|C_2|} \mathcal{V}_{0,1} \mathcal{V}_{1,2} = -\frac{1}{3} 2d(2d-1) W_z(e_1 + e_2). \quad (5.64)$$

Substituting this result into (4.69), and applying Theorem 1.1, (5.50) and Claim 5.1, give that the contribution to $E^2(z_c)$ is

$$-\frac{1}{3} 2d(2d-1) A(z_c) z_c^2 W_z(e_i + e_j) = \frac{2e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.65)$$

For the case $r = |\{j, k, l\}| = 3$, by (4.74) and (4.69), we know that the contribution to $E^2(z)$ is

$$-\frac{1}{2} 2dz^3 A(z_c) G_z(e_1) \left[\sum_{s_2, s_3 \in \mathcal{E}} \sum_{C_2 \ni s_2, C_3 \ni s_3} z^{|C_2|+|C_3|} \mathcal{V}_{2,3} \right]. \quad (5.66)$$

There are three possibilities for vertices s_2 and s_3 in this expression, namely, $s_2 = s_3$, $s_2 = -s_3$ and $s_2 \perp s_3$. Notice that in the first case, the factor $\mathcal{V}_{2,3}$ is automatically -1 and does not affect the sums. These three cases correspond to the three terms in the expression below,

$$\sum_{s_2, s_3 \in \mathcal{E}} \sum_{C_2 \ni s_2, C_3 \ni s_3} z^{|C_2|+|C_3|} \mathcal{V}_{2,3} = -2dg(z)^2 + 2dW_z(2e_1) + 2d(2d-2)W_z(e_1 + e_2). \quad (5.67)$$

Replacing this expression in (5.66) and applying Theorem 1.1, (5.50), Claim 5.1 and Lemma 5.2, yield that the contribution to $E^2(z_c)$ is

$$\frac{\frac{1}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.68)$$

Finally, adding (5.63), (5.65) and (5.68) yields that, as $d \rightarrow \infty$,

$$E^2(z_c) = \frac{\frac{7}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.69)$$

The term $E^3(z)$. We will follow a similar strategy to the one used previously for $E^2(z)$. Recall

equation (4.76) in the proof of Lemma 4.3. For this equation we analyzed the cases when $r = |\{i, j, k, l\}| = 3$ and 4; counted that the number of possibilities for the labels i, j, k and l that obey these conditions are $3\binom{m}{3}$ and $3\binom{m}{4}$, respectively; estimated (4.77) and substituted this result into (4.76) to obtain the contribution in each case.

The case $r = |\{i, j, k, l\}| = 3$. We assume that $i = k = 1, j = 2$ and $l = 3$, and study the situations $|\{s_1, s_2, s_3\}| = 1, 2, 3$.

If $|\{s_1, s_2, s_3\}| = 1$, by (4.79) we know that the contribution to $E^3(z_c)$ is

$$\frac{\frac{1}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \quad (5.70)$$

If $|\{s_1, s_2, s_3\}| = 2$, then two of the vertices are different and two are equal, say $s_1 \neq s_2 = s_3$. Using that $|\mathcal{V}_{2,3}| \leq 1$, yields the following bound for (4.77)

$$\sum_{s_1, s_2, s_3 \in \mathcal{E}} \mathbf{I}_{|\{s_1, s_2, s_3\}|=2} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1|+|C_2|+|C_3|} |\mathcal{V}_{1,2} \mathcal{V}_{1,3}| = 2d(2d-1)[-3g(z)W_z(e_1 + e_2)],$$

where the minus sign in the braces guarantees that $W_z(e_1 + e_2)$ is a positive quantity. Theorem 1.1 and Claim 5.1, imply that this expression at criticality is of order $O(1)$. Therefore, if we substitute it in (4.76), the presence of the factor z^3 together with Theorem 1.1 and (5.50) gives a contribution of order $O((2d)^{-3})$, and thus negligible.

If $|\{s_1, s_2, s_3\}| = 3$, then the simplest way in which the factor $\mathcal{V}_{1,2} \mathcal{V}_{1,3}$ is non zero is when the clusters C_1, C_2 and C_3 also contain the origin. Other intersections require at least four edges, if we treat the ribs emanating from these bonds, then the sum (4.77) is bounded by

$$\sum_{s_1, s_2, s_3 \in \mathcal{E}} \mathbf{I}_{|\{s_1, s_2, s_3\}|=3} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1|+|C_2|+|C_3|} \mathcal{V}_{1,2} \mathcal{V}_{1,3} \leq (2d)^3 \left[G_z(e_1)^3 + 2S_z^{(2,2)} z^2 g(z) \right]. \quad (5.71)$$

The right-hand side of this expression at criticality is of order $O(1)$, by Lemma 5.2 and Proposition 4.5. Hence, when substituted into (4.76), the factor z^3 and Theorem 1.1 together with (5.50), yield that the contribution is of order $O((2d)^{-3})$ and thus negligible.

The case $r = |\{i, j, k, l\}| = 4$. We assume that $i = 1, k = 2, j = 3$ and $l = 4$. By (4.81) and (5.67), (4.77) is given by

$$\left[\sum_{s_1, s_2 \in \mathcal{E}} \sum_{\substack{C_1 \ni s_1 \\ C_2 \ni s_2}} z^{|C_1|+|C_2|} \mathcal{V}_{1,2} \right]^2 = [-2dg(z)^2 + 2dW_z(2e_1) + 2d(2d-2)W_z(e_1 + e_2)]^2.$$

In this expression we are only interested in the terms of order $O((2d)^2)$, which is $(2d)^2 g(z)^4$, by Theorem 1.1 and Claim 5.1. Replacing this expression into (4.76), and applying equation (5.50),

yield that the contribution to $E^3(z_c)$, as $d \rightarrow \infty$, is

$$\frac{3}{4!} z_c^4 A(z_c) (2d)^2 g(z_c)^4 = \frac{\frac{1}{8}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right). \quad (5.72)$$

Therefore, this result and (5.70) yield

$$E^3(z_c) = \frac{\frac{5}{8}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.73)$$

The term $E^4(z_c)$. In the expression (5.60) that defines this term, we distinguish whether the labels i, k , or p , in $\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q}$, are equal to 0. According to this condition we decompose the factors $\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q}$ into the following four cases:

- (1) $\mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{0,q}$ for $1 \leq j < l < q \leq m$,
- (2) $\mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q}$ for $1 \leq j < l \leq m$ and $1 \leq p < q \leq m$,
- (3) $\mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q}$ for $1 \leq j \leq m$, $1 \leq k < l \leq m$ and $(p, q) \in A_{k,l}$,
- (4) $\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q}$ for $1 \leq i < j \leq m$, $(k, l) \in A_{i,j}$ and $(p, q) \in A_{k,l}$.

We substitute this result in the definition of $E^4(z)$, given in (5.60), and exchange the order of sums over i, \dots, q and s_1, \dots, s_m , to obtain

$$E^4(z) = E^{4,1}(z) + E^{4,2}(z) + E^{4,3}(z) + E^{4,4}(z),$$

where

$$\begin{aligned} E^{4,1}(z) &:= \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-3} \sum_{1 \leq j < l < q \leq m} \sum_{s_j, s_l, s_q \in \mathcal{E}} \left[\sum_{\substack{C_j \ni s_j, C_l \ni s_l \\ C_q \ni s_q}} z^{|C_j|+|C_l|+|C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{0,q} \right] \\ E^{4,2}(z) &:= \sum_{m \geq 2} \frac{z^m}{m!} (2dg(z))^{m-4} \sum_{\substack{1 \leq j < l \leq m \\ 1 \leq p < q \leq m}} \sum_{s_j, s_l, s_p, s_q \in \mathcal{E}} \left[\sum_{\substack{C_j \ni s_j, C_l \ni s_l \\ C_p \ni s_p, C_q \ni s_q}} z^{|C_j|+|C_l|+|C_p|+|C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q} \right] \\ E^{4,3}(z) &:= \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-5} \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \sum_{(p,q) \in A_{k,l}} \sum_{s_j, \dots, s_q \in \mathcal{E}} \left[\sum_{C_j \ni s_j, \dots, C_q \ni s_q} z^{|C_j|+\dots+|C_q|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right] \\ E^{4,4}(z) &:= \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-6} \sum_{1 \leq i < j \leq m} \sum_{\substack{(k,l) \in A_{i,j} \\ (p,q) \in A_{k,l}}} \sum_{s_i, \dots, s_q \in \mathcal{E}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i|+\dots+|C_q|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right]. \end{aligned}$$

We will study each of these terms separately and show that, as $d \rightarrow \infty$,

$$\begin{aligned} E^{4,1}(z) &= o\left(\frac{1}{(2d)^2}\right), & E^{4,2}(z) &= -\frac{\frac{1}{2}\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\ E^{4,3}(z) &= o\left(\frac{1}{(2d)^2}\right), & E^{4,4}(z) &= -\frac{\frac{1}{6}\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) \\ E^5(z_c) &= o\left(\frac{1}{(2d)^2}\right). \end{aligned} \tag{5.74}$$

Notice that if we assume (5.74), and add them to the contributions of $E^1(z_c)$, $E^2(z_c)$ and $E^3(z_c)$ given in (5.62), (5.69) and (5.73), respectively, we get that as, $d \rightarrow \infty$,

$$\begin{aligned} E(z_c) &= E^1(z_c) + E^2(z_c) + E^3(z_c) + E^{4,1}(z_c) + E^{4,2}(z_c) + E^{4,3}(z_c) + E^{4,4}(z_c) \\ &= \frac{\left(\frac{1}{2} + \frac{7}{2} + \frac{5}{8} - \frac{1}{2} - \frac{1}{6}\right)\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right) = \frac{\frac{95}{24}\mathbf{e}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \end{aligned}$$

which is the desired result for $E(z_c)$ in (5.52). Therefore, it only remains to prove (5.74).

The term $E^{4,1}(z_c)$. In this case, the number of labels j, l and q that satisfy $1 \leq j < l < q \leq m$ is $\binom{m}{3}$. For convenience we assume $j = 1, l = 2$ and $q = 3$, then

$$\sum_{s_1, s_2, s_3 \in \mathcal{L}} \left[\sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1|+|C_2|+|C_3|} \mathcal{V}_{0,1} \mathcal{V}_{0,2} \mathcal{V}_{0,3} \right] = [-2dG_z(e_1)]^3.$$

Substituting this expression in the definition of $E^{4,1}(z_c)$ we get

$$E^{4,1}(z_c) = \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z_c))^{m-3} \binom{m}{3} [-2dG_{z_c}(e_1)]^3 = -\frac{1}{3!} A(z_c) [2dz_c G_{z_c}(e_1)]^3.$$

Theorem 1.1, (5.50) and Lemma 5.2, give that this term is of order $O((2d)^{-3})$ and, hence, can be neglected.

The term $E^{4,2}(z_c)$. Recall that $E^{4,2}(z)$ is given by

$$E^{4,2}(z) = \sum_{m \geq 2} \frac{z^m}{m!} (2dg(z))^{m-4} \sum_{\substack{1 \leq j < l \leq m \\ 1 \leq p < q \leq m}} \sum_{s_j, s_l, s_p, s_q \in \mathcal{L}} \left[\sum_{\substack{C_j \ni s_j, C_l \ni s_l \\ C_p \ni s_p, C_q \ni s_q}} z^{|C_j|+|C_l|+|C_p|+|C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q} \right].$$

The cardinality of the set of labels obeys $r := |\{j, l, p, q\}| = 2, 3, 4$. In every case, the number of possibilities for the labels is $\tilde{C}_r \binom{m}{r}$, where the coefficient \tilde{C}_r is a combinatorial factor, independent

of d , that can be determined for $r = 2, 3, 4$. Substituting $\tilde{C}_r \binom{m}{r}$ in the definition of $E^{4,2}(z)$ produces

$$\begin{aligned} & \sum_{m \geq r} \frac{z^m}{m!} (2dg(z))^{m-r} \tilde{C}_r \binom{m}{r} \mathbf{I}_{|\{j,l,p,q\}|=r} \sum_{s_j, s_l, s_p, s_q \in \mathcal{E}} \left[\sum_{C_j \ni s_j, \dots, C_q \ni s_q} z^{|C_j| + \dots + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q} \right] \\ &= \frac{\tilde{C}_r}{r!} z^r A(z) \mathbf{I}_{|\{j,l,p,q\}|=r} \sum_{s_j, s_l, s_p, s_q \in \mathcal{E}} \left[\sum_{C_j \ni s_j, \dots, C_q \ni s_q} z^{|C_j| + \dots + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q} \right]. \end{aligned} \quad (5.75)$$

with $r = 2, 3, 4$. In our analysis we estimate the following expression for $r = 2, 3, 4$.

$$\mathbf{I}_{|\{i,j,k,l,p,q\}|=r} \sum_{s_j, s_l, s_p, s_q \in \mathcal{E}} \left[\sum_{\substack{C_j \ni s_j, C_l \ni s_l \\ C_p \ni s_p, C_q \ni s_q}} z^{|C_j| + |C_l| + |C_p| + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q} \right]. \quad (5.76)$$

When $r = |\{j, l, p, q\}| = 2$, necessarily $p = j$ and $q = l$, which implies that the number of possibilities for the labels j and l is $\binom{m}{2}$. For convenience we assume that $j = 1$ and $l = 2$. The factors $\mathcal{V}_{0,1}$ and $\mathcal{V}_{0,2}$ force an intersection of the clusters C_1 and C_2 at the origin, this implies that automatically the factor $\mathcal{V}_{1,2} = -1$. Hence, (5.76) becomes

$$\sum_{s_1, s_2 \in \mathcal{E}} \left[\sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1| + |C_2|} \mathcal{V}_{0,1} \mathcal{V}_{0,2} \mathcal{V}_{1,2} \right] = -[2dG_z(e_1)]^2.$$

By (5.50) and Lemma 5.2, the contribution to $E^{4,2}(z_c)$ is

$$-\frac{1}{2} z_c^2 e^{2dz_c g(z_c)} [2dG_{z_c}(e_1)]^2 = -\frac{\frac{1}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.77)$$

When $r = |\{j, l, p, q\}| = 3$, the number of terms $\mathcal{V}_{0,j} \mathcal{V}_{0,l} \mathcal{V}_{p,q}$ whose labels satisfy this condition is $4\binom{m}{3}$. If we choose 3 labels out of m and order them, then j gets the smallest label, l has two possible options and in each case there are two more choices. For convenience we assume that $j = 1, l = 2, p = 1$ and $q = 3$. Hence, (5.76) becomes

$$\begin{aligned} & \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1| + |C_2| + |C_3|} \mathcal{V}_{0,1} \mathcal{V}_{0,2} \mathcal{V}_{1,3} \\ &= 2dG_z(e_1) \sum_{s_1, s_3 \in \mathcal{E}} \sum_{C_1 \ni s_1, C_3 \ni s_3} z^{|C_1| + |C_3|} \mathcal{V}_{0,1} \mathcal{V}_{1,3} \\ &= 2dG_z(e_1) \left[2dg(z)G_z(e_1) - \frac{1}{3}2d(2d-1)W_z(e_1 + e_2) \right] \end{aligned}$$

where the last equality is due to (4.71) and (5.64). Therefore, substituting this expression into (5.75)

gives

$$-\frac{4}{3!}A(z)z^3(2d)^2G_z(e_1) \left[g(z)G_z(e_1) - \frac{1}{3}(2d-1)W_z(e_1+e_2) \right].$$

By Theorem 1.1, (5.50), Claim 5.1 and Lemma 5.2, the product $z_c^3(2d)^2G_{z_c}(e_1)$ is of order $O((2d)^{-2})$ and the terms in braces are of order $O((2d)^{-1})$. Therefore, their contribution to $E^{4,2}(z_c)$ can be neglected.

When $r = |\{j, l, p, q\}| = 4$ the number of terms $\mathcal{V}_{0,j}\mathcal{V}_{0,l}\mathcal{V}_{p,q}$ whose labels satisfy this condition is $3\binom{m}{4}$. If we choose 4 labels out of m and order them, then i gets the smallest label, j has three possible options and once j is determined, so are k and l . For convenience we assume $j = 1, l = 2, p = 3$ and $q = 4$. Hence, (5.76) becomes

$$\begin{aligned} & \sum_{s_1, s_2, s_3, s_4 \in \mathcal{C}} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3, C_4 \ni s_4}} z^{|C_1|+|C_2|+|C_3|+|C_4|} \mathcal{V}_{0,1} \mathcal{V}_{0,2} \mathcal{V}_{3,4} \\ &= (-2dG_z(e_1))^2 \sum_{s_3, s_4 \in \mathcal{C}} \sum_{C_3 \ni s_3, C_4 \ni s_4} z^{|C_3|+|C_4|} \mathcal{V}_{3,4} \\ &= (-2dG_z(e_1))^2 \left[-2dg(z)^2 + 2dW_z(2e_1) + 2d(2d-2)W_z(e_1+e_2) \right], \end{aligned}$$

where the last equality is due to (5.67). Notice that, at criticality, Lemma 5.2 implies that the first factor is of order $O(1)$; and Theorem 1.1 and Claim 5.1, yield that the factor within square braces is of order $O((2d)^{-1})$. Therefore, when substituting this expression in (5.75), the presence of the factor z_c^4 together with (5.50) and Theorem 1.1, yield that the contribution is of order $O((2d)^{-5})$, and thus negligible.

Therefore, (5.77) and the previous reasoning yield that

$$E^{4,2}(z_c) = -\frac{\frac{1}{2}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty, \quad (5.78)$$

which is the desired result (5.74) for $E^{4,2}(z_c)$.

The term $E^{4,3}(z_c)$. We will show that $E^{4,3}(z_c)$ is of order $o((2d)^{-2})$, as $d \rightarrow \infty$. Recall that this term is given by

$$E^{4,3}(z) = \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-5} \sum_{\substack{1 \leq j \leq m \\ 1 \leq k < l \leq m}} \sum_{(p,q) \in A_{k,l}} \sum_{s_j, \dots, s_q \in \mathcal{C}} \left[\sum_{C_j \ni s_j, \dots, C_q \ni s_q} z^{|C_j|+\dots+|C_q|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right].$$

The cardinality of the set of labels satisfies $r := |\{j, k, l, p, q\}| = 3, 4$ or 5 . In each case, the number of possibilities for the labels is $\tilde{C}_r \binom{m}{r}$, where the coefficient \tilde{C}_r is a combinatorial factor, independent of d , that can be determined for all $r = 3, 4, 5, 6$. Substituting $\tilde{C}_r \binom{m}{r}$ in the definition of $E^{4,3}(z)$

produces

$$\begin{aligned}
& \sum_{m \geq r} \frac{z^m}{m!} (2dg(z))^{m-r} \tilde{C}_r \binom{m}{r} \mathbf{I}_{|\{j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{E}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right] \\
&= \frac{\tilde{C}_r}{r!} A(z) z^r \mathbf{I}_{|\{j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{E}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right],
\end{aligned} \tag{5.79}$$

with $r = 3, 4, 5$. In our analysis we will bound the following expression

$$\mathbf{I}_{|\{i,j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{E}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{0,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right]. \tag{5.80}$$

When $r = |\{j, k, l, p, q\}| = 3$, then $k < l$, $p = k < q$ or $p = l < q$, and j has three options. Suppose that $j = k = p = 1$, $l = 2$ and $q = 3$. If we distinguish the cases when $|\{s_1, s_2, s_3\}| = 1, 2$ and 3 , then (5.80) is bounded by

$$\begin{aligned}
& 6 \sum_{s_1, s_2, s_3 \in \mathcal{E}} \left[\sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1| + |C_2| + |C_3|} |\mathcal{V}_{0,1} \mathcal{V}_{1,2} \mathcal{V}_{1,3}| \right] \\
& \leq 6 \left[2dg(z)^2 G_z(e_1) + 2(2d)^2 g(z) W_z(e_1 + e_2) + (2d)^3 \left(G_z(e_1)^3 + 2S_z^{(2,2)} z^2 g(z) \right) \right],
\end{aligned}$$

for the last summand we recall the discussion above equation (5.71). By Theorem 1.1, Claim 5.1, Lemma 4.7 and Proposition 4.5, the expression above is of order $O(1)$. Hence, if we substitute it in (5.79), the the presence of the factor z_c^3 and Theorem 1.1 and (5.50), yields a negligible contribution of order $O((2d)^{-3})$.

When $r = |\{j, k, l, p, q\}| = 4$, then either j is different from k, l, p and q , or j is equal to one of them. In the first case, the factor $\mathcal{V}_{0,j}$ in (5.80) only affects the sum over j , which becomes $G_z(e_1)$; and the other sums behave like in equation (4.77) for the case when the set of labels has cardinality three. By (5.70) and the discussion below it, we know that the contribution is of order $O((2d)^{-2})$, and when multiplied by $G_z(e_1) = O((2d)^{-1})$, we get that the contribution to $E^{4,3}(z_c)$ is of order $O((2d)^{-3})$, and thus negligible. For the second case, when j is equal to one of the other labels, (say $j = k = 1, l = 2, p = 3$ and $q = 4$), (5.80) is bounded by

$$\begin{aligned}
& 4 \left[\sum_{s_1, s_2 \in \mathcal{E}} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1| + |C_2|} \mathcal{V}_{0,1} \mathcal{V}_{1,2} \right] \left[\sum_{s_3, s_4 \in \mathcal{E}} \sum_{C_3 \ni s_3, C_4 \ni s_4} z^{|C_3| + |C_4|} \mathcal{V}_{3,4} \right] \\
& \leq 4 \left[- (2d)^2 W_z(e_1 + e_2) \right] \left[2dg(z)^2 - 2dW_z(2e_1) - (2d)^2 W_z(e_1 + e_2) \right]
\end{aligned}$$

where in the inequality we have used (5.64) and (5.67). Theorem 1.1 and Claim 5.1, yield that the first factor is of order $O(1)$ and the second one, $O(2d)$. Therefore, if we substitute this expression into (5.79), then the factor z^4 together with (5.50) and Theorem 1.1, yields a contribution of order $O((2d)^{-3})$ that we can neglect.

When $r = |\{j, k, l, p, q\}| = 5$, (5.80) is given by

$$\begin{aligned} & \sum_{s_1, \dots, s_5 \in \mathcal{L}} \left[\sum_{C_1 \ni s_1, \dots, C_5 \ni s_5} z^{|C_1| + \dots + |C_5|} \mathcal{V}_{0,1} \mathcal{V}_{2,3} \mathcal{V}_{4,5} \right] \\ &= 2dG_z(e_1) \left[\sum_{C_2 \ni s_2, C_3 \ni s_3} z^{|C_2| + |C_3|} \mathcal{V}_{2,3} \right]^2 \\ &= 2dG_z(e_1) \left[-2dg(z)^2 + 2dW_z(2e_1) + 2d(2d-2)W_z(e_1 + e_2) \right]^2, \end{aligned}$$

where the last equality is due to (5.67). Theorem 1.1, Claim 5.1 and Lemma 5.2, yield that this expression is of order $O((2d)^2)$. Then if we substitute it in (5.79), the factor z^5 together with (5.50) and Theorem 1.1, yield a contribution of order $O((2d)^{-3})$ and thus negligible.

The term $E^{4,4}(z_c)$. Recall that $E^{4,4}(z)$ is given by

$$E^{4,4}(z) = \sum_{m \geq 3} \frac{z^m}{m!} (2dg(z))^{m-6} \sum_{\substack{1 \leq i < j \leq m \\ (k,l) \in A_{i,j} \\ (p,q) \in A_{k,l}}} \sum_{s_i, \dots, s_q \in \mathcal{L}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right].$$

The cardinality of the set of labels obeys $r := |\{i, j, k, l, p, q\}| = 3, 4, 5, 6$. In every case, the number of possibilities for the labels is $\tilde{C}_r \binom{m}{r}$, where the coefficient \tilde{C}_r is a combinatorial factor, independent of d , that can be determined for all $r = 3, 4, 5, 6$. Substituting $\tilde{C}_r \binom{m}{r}$ in the definition of $E^{4,4}(z)$ produces

$$\begin{aligned} & \sum_{m \geq r} \frac{z^m}{m!} (2dg(z))^{m-r} \tilde{C}_r \binom{m}{r} \mathbf{I}_{|\{i,j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{L}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right] \\ &= \frac{\tilde{C}_r}{r!} A(z) z^r \mathbf{I}_{|\{i,j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{L}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right], \end{aligned} \quad (5.81)$$

with $r = 3, 4, 5, 6$. We will show that the case $r = 3$ contributes to $E^{4,4}(z)$, and the contributions of the other terms are of order $o((2d)^{-2})$, as $d \rightarrow \infty$. In our analysis we estimate the following sums, for $r = 3, 4, 5, 6$,

$$\mathbf{I}_{|\{i,j,k,l,p,q\}|=r} \sum_{s_i, \dots, s_q \in \mathcal{L}} \left[\sum_{C_i \ni s_i, \dots, C_q \ni s_q} z^{|C_i| + \dots + |C_q|} \mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \right]. \quad (5.82)$$

When $r = |\{i, j, k, l, p, q\}| = 3$, necessarily i is the smallest label, $i < j < l$ and $k = i$, $p = j$ and $q = l$. This implies that the number of possibilities for the labels is $\binom{m}{3}$. For convenience we assume that $i = k = 1$, $j = s = 2$ and $l = t = 3$. We distinguish three cases for the vertices s_1 , s_2 and s_3 , when $s_1 = s_2 = s_3$, when only two of them are equal, say $s_1 \neq s_2 = s_3$, and when the three of them are different. In the first case, the factor $\mathcal{V}_{1,2}\mathcal{V}_{1,3}\mathcal{V}_{2,3} = -1$ and does not interfere in the sum (5.82), which becomes

$$\sum_{s_1, s_2, s_3 \in \mathcal{E}} \mathbf{I}_{s_1=s_2=s_3} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1|+|C_2|+|C_3|} \mathcal{V}_{1,2}\mathcal{V}_{1,3}\mathcal{V}_{2,3} = -2dg(z)^3.$$

Substituting this result into (5.81) and using (5.50) and Theorem 1.1, we get that the contribution to $E^{4,4}(z_c)$ is

$$-\frac{1}{3!}A(z_c)2d(z_c g(z_c))^3 = -\frac{\frac{1}{6}e}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right), \text{ as } d \rightarrow \infty. \quad (5.83)$$

In the second case, when only two of the vertices are equal, say $s_1 \neq s_2 = s_3$, there are $2d(2d-1)$ possibilities for the vertices s_1 and s_2 , and the factor $|\mathcal{V}_{2,3}| = 1$. Then the fact that $|\mathcal{V}_{1,3}| \leq 1$ implies that (5.82) is bounded by

$$\begin{aligned} & 3(2dg(z))^3 \sum_{s_1, s_2, s_3 \in \mathcal{E}} \mathbf{I}_{s_1 \neq s_2 = s_3} \sum_{\substack{C_1 \ni s_1, C_2 \ni s_2 \\ C_3 \ni s_3}} z^{|C_1|+|C_2|+|C_3|} |\mathcal{V}_{1,2}\mathcal{V}_{1,3}\mathcal{V}_{2,3}| \\ & \leq 3(2dg(z))^3 2d(2d-1)g(z) \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1|+|C_2|} |\mathcal{V}_{1,2}| \\ & \leq 3(2d)^5 g(z)^4 [-W_z(2e_1) - W_z(e_1 + e_2)], \end{aligned}$$

the two summands in the last inequality are positive quantities that correspond to $s_1 = -s_2$ and $s_1 \perp s_2$, respectively. Substituting this result into (5.81) we get the bound

$$\frac{3}{3!}A(z)(2d)^5 g(z)^4 z^3 [-W_z(2e_1) - W_z(e_1 + e_2)].$$

By Theorem 1.1, (5.50) and Claim 5.1, we know that the product $(2d)^5 g(z_c)^4 z_c^3$ is of order $O((2d)^{-2})$ and that, so are the terms braces. This gives a contribution of order $O((2d)^{-4})$ that can be neglected.

When the vertices $s_1 \neq s_2 \neq s_3$, using that $|\mathcal{V}_{1,3}| \leq 1$, the sum (5.82) is bounded by the expression in (5.71). Then by the discussion below (5.71), we know that the contribution is negligible.

When $r = |\{i, j, k, l, p, q\}| = 4$, suppose that the labels are 1,2,3 and 4. There are 16 possible arrangements for the labels, which we show in Table 5.1. They can be reduced to the two cases:

- (i) Three labels are equal and the other three are different from the first ones and among them, e.g., $i = k = p = 1$, $j = 2$, $l = 3$ and $q = 4$. There are 4 arrangements of this type.

- (ii) Two pairs of equal labels and a pair of distinct labels, e.g., $i = k = 1$, $j = p = 2$, $l = 3$ and $q = 4$. There are 12 arrangements of this type.

i	j	k	l	p	q	i	j	k	l	p	q
1	2	1	3	1	4	1	3	1	4	2	3
1	2	1	3	2	4	1	3	1	4	2	4
1	2	1	3	3	4	1	3	2	3	2	4
1	2	1	4	2	3	1	3	2	3	3	4
1	2	1	4	3	4	1	3	2	4	3	4
1	2	2	3	2	4	1	4	2	3	2	4
1	2	2	3	3	4	1	4	2	3	3	4
1	2	2	4	3	4	1	4	2	4	3	4

Table 5.1: Possibilities for labels i, j, k, l, p, q

We explain the main ideas to bound (5.81), when the factors $|\mathcal{V}_{ij}\mathcal{V}_{k,l}\mathcal{V}_{p,q}|$ in (5.82) are given by $|\mathcal{V}_{1,2}\mathcal{V}_{1,3}\mathcal{V}_{1,4}|$ and $|\mathcal{V}_{1,2}\mathcal{V}_{1,3}\mathcal{V}_{2,4}|$. Our reasoning can be extended to the remaining cases of type (i) and (ii).

We distinguish the four possibilities $|\{s_1, s_2, s_3, s_4\}| = 1, 2, 3, 4$. Since the factors $A(z)z^4$ in (5.81) are $O((2d)^{-4})$, by Theorem 1.1 and (5.50), it is enough to prove that (5.82) is at most $O(2d)$ for each case.

If $|\{s_1, s_2, s_3, s_4\}| = 1$, the products $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$ and $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}|$ are equal to 1, and the sums in (5.82) reduce to $2dg(z_c)^4 = O(2d)$, by Theorem 1.1.

If $|\{s_1, s_2, s_3, s_4\}| = 2$, we decompose the products $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$ and $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}|$ into a factor that involves the two different vertices, and the remaining two factors. We bound these last two factors by 1, so their corresponding sums are bounded by $g(z_c)^2 = O(1)$. The remaining sums are equal to $2d|W_z(2e_1)| + 2d(2d - 2)|W_z(e_1 + e_2)|$ (see (5.67) for the case $s_1 \neq s_2$), which is of order $O(1)$ by Claim 5.1.

If $|\{s_1, s_2, s_3, s_4\}| = 3$, we decompose $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$ and $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}|$ into two factors involving the three distinct vertices, and one remaining factor. We bound the latter factor by 1, and the corresponding sum becomes $g(z_c) = O(1)$, by Theorem 1.1. The remaining sums are bounded by $E^3(z_c)$ for the case $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$, and by $E^3(z_c)$ or $(E^2(z_c))^2$ for the case $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}|$. By (5.69) and (5.73), we know that in both cases the contribution is at most $O((2d)^{-2})$.

If $|\{s_1, s_2, s_3, s_4\}| = 4$, an example of the required intersections for $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$ not to vanish is depicted in Figure 5.3a. By taking into account all possibilities for non-vanishing $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}|$ and $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}|$, we draw the crude conclusion that at least eight bonds are needed to achieve the required intersections, and this leads to an upper bound $O(2d)^{-4}$. We explain this conclusion for the case $|\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}| = 1$ depicted in Figure 5.3a. We think that each one of the four factors z , in (5.81), is located on one of the four edges $\{0, s_1\}, \dots, \{0, s_4\}$. The rib R_i emanating from vertex s_i together with $\{0, s_i\}$, are bounded by $G_z^{(1)} * G_z^{(m_i)}$, for $i = 2, 3, 4$. Similarly, rib R_1 and edge

$\{0, s_1\}$ are bounded by $G_z^{(1)} * G_z^{(m_5)} * G_z^{(m_6)} * G_z^{(m_7)}$. The steps m_2, \dots, m_7 satisfy $m_2 + \dots + m_7 \geq 4$, because the minimal case in which the ribs intersect is, when the edges $\{0, s_i\}$ are present in the corresponding rib R_i , or when all four edges $\{0, s_1\}, \dots, \{0, s_4\}$ are in R_1 , see Figure 5.3b. Therefore, the total number of steps present in all the two-point functions is at least 8. Then an application of Proposition 4.5 yields a contribution of order $O((2d)^4)$, and thus negligible.

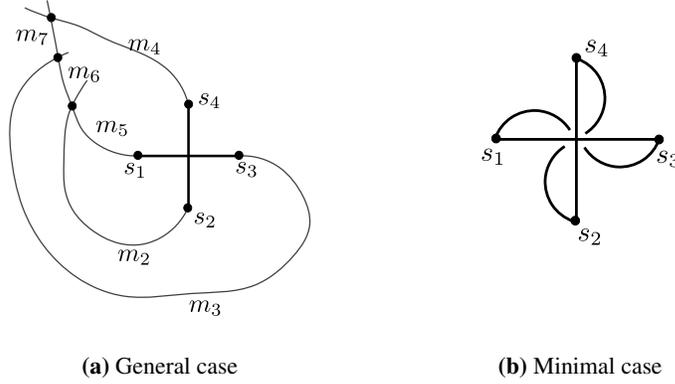


Figure 5.3: The case $s_1 \neq s_2 \neq s_3 \neq s_4$

When $r = |\{i, j, k, l, p, q\}| = 5$, one of the factors $\mathcal{V}_{i,j}$'s has labels that do not repeat, say $i = 1$ and $j = 2$, and the other two, share one of the labels. Then the sums over the labels that do not repeat, in our case $i = 1$ and $j = 2$, can be factor out from (5.82), which can be bounded by

$$\sum_{s_1, s_2 \in \mathcal{C}} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1|+|C_2|} |\mathcal{V}_{1,2}| \leq 2dg(z)^2 - 2(2d)^2 W_z(e_1 + e_2),$$

where the first term on the right-hand side corresponds to $s_1 = s_2$ and the second one bounds when to $s_1 \neq s_2$. By Theorem 1.1 and Claim 5.1, this expression is $O(2d)$.

On the other hand, the remaining sums coincide with the case of $E^3(z)$ when its number of labels is three. By (4.78) and (4.80), we know that these sums are bounded by

$$2dg(z)^3 + 3(2d)^3 g(z) \left(6z^2 g(z)^2 + S_z^{(4,2)} \right),$$

which, by Theorem 1.1 and Proposition 4.5, is $O(2d)$.

The factors $A(z)z^5$ in (5.81) are $O((2d)^{-5})$, by Theorem 1.1 and (5.50). Then our previous observations yield that the contribution of (5.81) to $E^{4,4}(z)$, for $r = 5$, is $O((2d)^{-3})$ and can be neglected.

When $r = |\{i, j, k, l, p, q\}| = 6$, the sums in (5.82) are given by

$$\begin{aligned} \sum_{s_1, \dots, s_6 \in \mathcal{L}} \sum_{C_1 \ni s_1, \dots, C_6 \ni s_6} z^{|C_1| + \dots + |C_6|} |\mathcal{V}_{1,2} \mathcal{V}_{3,4} \mathcal{V}_{5,6}| &= \left[\sum_{s_1, s_2 \in \mathcal{L}} \sum_{C_1 \ni s_1, C_2 \ni s_2} z^{|C_1| + |C_2|} |\mathcal{V}_{1,2}| \right]^3 \\ &= [2dg(z)^2 - 2dW_z(2e_1) - 2d(2d-2)W_z(e_1 + e_2)]^3, \end{aligned}$$

which is $O((2d)^3)$, by Theorem 1.1 and Claim 5.1. When substituting this result into (5.81), since $A(z)z^6$ is $O((2d)^{-6})$, by Theorem 1.1 and (5.50), we get that the contribution of (5.81) to $E^{4,4}(z)$, for $r = 6$, is $O((2d)^{-3})$ and can be neglected.

The term $E^5(z_c)$. If in the definition of $E^5(z)$ in (5.61), we use that the product $\prod_{(u,v) \in A_{s,t}} (1 + \mathcal{V}_{u,v}) \leq 1$, then we get the bound

$$|E^5(z)| \leq \sum_{m=4}^{\infty} \frac{z^m}{m!} \sum_{s_1, \dots, s_m \in \mathcal{L}} \sum_{C_1 \ni s_1} z^{|C_1|} \dots \sum_{C_m \ni s_m} z^{|C_m|} \sum_{0 \leq i < j \leq m} \sum_{(k,l) \in A_{i,j}} \sum_{(p,q) \in A_{k,l}} \sum_{(s,t) \in A_{p,q}} |\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t}|. \quad (5.84)$$

We denote the cardinality of the label set by $r = |\{i, j, k, l, p, q, s, t\}|$, so $r \in \{4, 5, 6, 7, 8\}$, and exchange the sums over vertices and labels. As in the previous discussion for $E^{4,4}(z)$, this allows us to rewrite the upper bound of (5.84) in the form

$$|E^5(z)| \leq A(z) \sum_{r=4}^8 \sum_i \alpha_{r,i} E^{(5,r,i)}(z), \quad (5.85)$$

where the sum over i is a finite sum, the $\alpha_{r,i}$ are constants whose values are immaterial, and each $E^{(5,r,i)}$ is of the form

$$E^{(5,r,i)}(z) = \sum_{s_1, \dots, s_r \in \mathcal{L}} \sum_{S_1 \ni s_1} z_c^{|S_1|} \dots \sum_{S_r \ni s_r} z_c^{|S_r|} \mathcal{V}^{(r,i)}, \quad (5.86)$$

with $\mathcal{V}^{(r,i)}$ a product of 4 factors of \mathcal{V}_{ab} having r distinct labels in all. Since $A(z_c) = O(1)$ (as observed in (5.50)), it suffices to show that each $E^{(5,r,i)}(z_c)$ is $O(2d)^{-3}$.

For $r = 4, 5$ or 6 , we substitute one of the factors in $|\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t}|$ by 1, with the restriction that the remaining three factors have at least four different labels. The sums involving the replaced factor yield 1 or $2dg(z_c)$ or $(2dg(z_c))^2$ depending on whether this factor has 0, 1 or 2 distinct labels from the remaining three factors; all three cases are $O(1)$ by Theorem 1.1. The sums involving the other three factors reduce to $|E^{4,4}(z)|$ when the set of labels has cardinality four, five or six, which by our previous discussion, are of order $o((2d)^{-2})$, as $d \rightarrow \infty$.

For $r = 7$ or 8 , we consider three factors in the product $|\mathcal{V}_{i,j} \mathcal{V}_{k,l} \mathcal{V}_{p,q} \mathcal{V}_{s,t}|$ that have six different labels and bound the fourth factor by 1. The sums involving the fourth factor yield $2dg(z_c)$ and

$(2dg(z_c))^2$ for $r = 7$ and $r = 8$, respectively. By Theorem 1.1, in both cases the contribution is $O(1)$. By our discussion on $E^{4,4}(z)$ when the set of labels has cardinality six, the sums involving the six distinct labels is $O(2d)^{-3}$. This completes the proof of (5.74).

The term $\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} z^{|A|}$. Finally, for the lattice animal case we analyze

$$\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} z^{|A|},$$

where the sum is over all animals that have a cycle containing 0. A calculation similar to the one we did to get

$$\hat{\Pi}_{z_c^{(a)}}^{(a,0)}(0) = \frac{\frac{3}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right),$$

in (5.46), yields the desired result. The only difference with $\hat{\Pi}_{z_c^{(a)}}^{(a,0)}(0)$ is the factor 3, which arose from the sum over x (the three remaining vertices in the 4-cycles containing 0). In this situation there is not such sum and therefore, that factor is 1, i.e.,

$$\mathbf{I}_a \sum_{\substack{A \ni 0 \\ 0 \in \text{cycle}}} (z_c^{(a)})^{|A|} = \frac{\frac{1}{2}}{(2d)^2} + o\left(\frac{1}{(2d)^2}\right).$$

This completes the proof of Theorem 1.5, and, therefore, the proof of Theorem 1.2. □

Chapter 6

Conclusions

The work presented in this thesis lies in the fields of combinatorics, probability theory, statistical physics and is inspired by polymer chemistry. We study (bond) lattice animals and lattice trees in the nearest-neighbor and spread-out models in \mathbb{Z}^d . Lattice animals are finite connected subgraphs of \mathbb{Z}^d , and lattice trees are animals without cycles. They are very interesting combinatorial objects that are easy to understand but in general, difficult to analyze. For instance, the basic question of computing the number of trees and animals of fixed size that contain the origin is unsolved even in two dimensions. However, extensive enumerations in low dimensions have been accomplished in the combinatorics literature [29]. In polymer chemistry, lattice trees and lattice animals are used to model branched polymers [27]. In statistical physics, lattice trees and lattice animals are examples of models that exhibit a phase transition, with a very interesting behavior at criticality [39]. They share this characteristic with the related models of self-avoiding walks and percolation.

The focus of our study are the critical points $z_c^{(t)}$ and $z_c^{(a)}$ of lattice trees and lattice animals, respectively. They are the reciprocal of the growth constants $\tau = \lim_{n \rightarrow \infty} t_n^{1/n}$ and $\alpha = \lim_{n \rightarrow \infty} a_n^{1/n}$, where t_n and a_n are the number of lattice trees and lattice animals with n bonds, that contain the origin. In this work we focus in the high dimensional case, in which exact results are not available, but instead $1/d$ -expansions are studied.

The main result of this thesis, is the rigorous calculation of the first three terms of a $1/d$ -expansion for the critical points for lattice trees and lattice animals for the nearest-neighbor model. As a side result, we have also calculated three terms of a $1/d$ -expansion for the one-point functions at criticality. The proof follows a recursive argument similar to the one used in [20] and [24], in which some of the terms of a $1/d$ -expansion were obtained for self-avoiding walk and percolation.

The main technique that we apply is the lace expansion. This method gives an implicit formula for the critical points. However, unlike the cases of self-avoiding walks and percolation, the formula given by the lace expansion includes the one-point function at criticality. We overcome this difficulty by developing an expansion for the one-point function, in a way that resembles the lace expansion

in order to exploit some of the results given by this technique.

In Chapter 2, we obtain the leading order terms for the critical points and one-point functions. The chapter is based on our publication [35]. In the proof we use results given by the lace expansion but with no further application of the technique. Instead, we use a mean-field model related to the Galton-Watson branching process with critical Poisson offspring distribution. This connection was suggested by [25].

In Chapter 3, we develop a new expansion for the one-point functions, which in the subsequent chapters becomes a key ingredient in our proof. We also derive the lace expansion for lattice trees and animals following the ideas in [39]. In particular, we explain how to obtain an implicit formula for the critical points, given in terms of the one-point function and the quantity $\hat{\Pi}_{z_c}(0)$.

Finally, in Chapter 4 and Chapter 5, we obtain the first and second order correction terms, respectively. In the proof we use the lace expansion extensively, together with the expansion for the one-point function that we develop in Chapter 3. The calculation of the terms in the expansions for the critical points are intertwined with the calculations of the terms in expansions for the one-point function and $\hat{\Pi}_{z_c}(0)$.

When computing $1/d$ -expansion it is generally assumed that in such expansions the remainder is of the order of the first omitted term, but this is rarely proved due to the limitations of the employed techniques. Our method allows us to do this in a rigorous way, confirming previous results by [11] for lattice trees, and [21, 36] for lattice animals, in which bounds on the errors are not present. Our research also shows the power of the lace expansion for handling these kinds of problems. Our method is essentially algorithmic, with sufficient labor could be extended to compute higher order terms of the $1/d$ -expansions for the critical points and the critical one-point functions. Probably, some of the calculations could be done using a computer. Moreover, it may provide the starting point for proving the existence of an asymptotic expansion, with rational coefficients, for the critical point of nearest-neighbor lattice trees in a similar fashion as in [20] and [23], for self-avoiding walks and percolation, respectively. We expect that the ideas in our proof could be applied in the analysis of the critical point of q -lattice animals. A q -lattice animal A is an animal whose boundary (edges incident to A but not in it) are weighted by the factor q . They can be thought as an interpolating lattice animals and the percolation. Specifically, we aim to achieve a lower bound for the critical point in terms of q .

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