SIMULTANEOUS MEASUREMENT OF FULL-FIELD VIBRATION MODES USING ELECTRONIC SPECKLE PATTERN INTERFEROMETRY (ESPI)

by

PETER GEORGAS

B.Sc.E., Queen’s University, 2010

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

(Mechanical Engineering)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2012

© Peter Georgas, 2012
Abstract

The natural frequencies and vibration mode shapes of flat plates are measured using ESPI, even when multiple modes are simultaneously present. The method involves measuring the surface shape of a vibrating plate at high frame rate using a Michelson interferometer and high-speed camera. The vibration is either excited by white (random) noise or by impact. Fourier analysis of the acquired data gives the natural frequencies and associated mode shapes. The analytical procedure used has the advantage that it simultaneously identifies all vibration modes with frequencies up to half the sampling frequency. In comparison, the ESPI Time-Averaged method and the traditional Chladni method both require that the plate be sinusoidally excited at each natural frequency to allow separate measurements of the associated mode shapes. Example measurements are presented to illustrate the use and capabilities of the proposed plate natural frequency and mode shape measurement method.
Preface

Parts of chapter 3 have been published. P. Georgas and G. Schajer, "Modulo-2\pi Phase Determination From Individual ESPI Images," *Optics and Lasers in Engineering*, no. Special Issue: Reliability and Variability, 2012. Specifically, Figures 20, 21, 22, 23, 24 have been reprinted with the permission of Elsevier. Dr. Gary Schajer and I co-authored this paper, describing these algorithms I had been developing.

Some similar results from chapter 4 were presented at the 2012 SEM XII International Congress & Exposition on Experimental and Applied Mechanics. P. Georgas and G. Schajer, "Simultaneous Measurement of Plate Natural Frequencies and Vibration Mode Shapes Using ESPI," in *Society for Experimental Mechanics*, Costa Mesa, 2012. Results were obtained from an experimental apparatus which I constructed. The results are attributed to the development of algorithms I had been developing.
# Table of Contents

Abstract .......................................................................................................................... ii  
Preface............................................................................................................................ iii  
Table of Contents .......................................................................................................... iv  
List of Tables .................................................................................................................. vi  
List of Figures ................................................................................................................. vii  
Acknowledgements....................................................................................................... ix  
Chapter 1 – Introduction ............................................................................................... 1  
  1.1 Introduction to Modal Analysis ............................................................................... 1  
  1.1.1 Motivation ........................................................................................................... 2  
  1.2 Existing Methods .................................................................................................... 4  
  1.2.1 Chladni Method ................................................................................................ 4  
  1.2.2 Instrumented Hammer (IH) ............................................................................. 7  
  1.2.3 Laser Doppler Vibrometry (LDV) .................................................................. 9  
  1.2.4 Time-Averaged Interferometry (TAI) .............................................................. 10  
  1.3 Research Objectives and Proposed Method ........................................................ 12  
  1.4 Research Outline ................................................................................................... 14  
Chapter 2 – Background Theory ................................................................................ 16  
  2.1 Modal Analysis ..................................................................................................... 16  
  2.2 Introduction to Laser Interferometry ..................................................................... 20  
  2.2.1 Typical Apparatus ............................................................................................ 23  
  2.3 Electronic Speckle Pattern Interferometry (ESPI) .............................................. 25  
  2.4 Phase Unwrapping ............................................................................................... 28  
  2.5 Sampling Procedure .............................................................................................. 30  
  2.6 Summary ................................................................................................................. 31  
Chapter 3 – Phase Determination Algorithms ........................................................... 32  
  3.1 Introduction to Phase Determination Algorithms ............................................ 32  
  3.2 Classical Method ................................................................................................... 33  
  3.2.1 Limitations to the Classical 4-Step Method ..................................................... 37
3.3 Other Approaches ................................................................................................. 38
3.4 Single Fringe Image Algorithm Class .................................................................... 39
  3.4.1 Four Reference Image Method (4-RIM) ......................................................... 41
  3.4.2 Two Reference Image Method (2-RIM) ......................................................... 45
  3.4.3 Reduced Two Reference Image Method (R2-RIM) .......................................... 48
  3.4.5 Other Considerations ...................................................................................... 48
3.5 Interpolated Reference Images ............................................................................... 51
3.6 Summary ............................................................................................................... 53

Chapter 4 – Experimental Validation ........................................................................... 55
4.1 Experiments Overview .......................................................................................... 55
4.2 Experiment I: Algorithm Performance ................................................................. 56
  4.2.1 Comparison of Algorithm Variations ............................................................. 57
  4.2.2 Fringe Pattern Quality and Computation Time ............................................. 59
  4.2.3 Error Estimates ............................................................................................. 64
4.3 Experiment II: Modal Analysis .............................................................................. 66
  4.3.1 Modal Analysis Apparatus ............................................................................ 67
  4.3.2 Mode Shape Recovery ................................................................................... 69
  4.3.2 Comparison with Other Methods ................................................................ 75
4.4 Discussion .............................................................................................................. 76
  4.4.1 Image Quality Discussion ............................................................................ 79

Chapter 5 – Conclusion ............................................................................................... 82
5.1 Research Summary and Contributions .................................................................. 82
5.2 Remaining Challenges and Limitations ............................................................... 83
5.3 Recommendations for Future Work ..................................................................... 85
  5.3.1 Improved Interpolated Reference Images ....................................................... 85
  5.3.2 Damping Ratio ............................................................................................. 85

References .................................................................................................................... 87
Appendix – Uncertainty Derivation ............................................................................. 90
List of Tables

Table 1: A summary of predominant attributes of existing and proposed methods ....... 12
Table 2: Experimental test parameters................................................................. 70
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Image of a typical Chladni experiment ................................................................. 5</td>
</tr>
<tr>
<td>2</td>
<td>Mode shape sketches of a square plate as recorded by Chladni ................................. 6</td>
</tr>
<tr>
<td>3</td>
<td>Mode shape construction using Fourier analysis ....................................................... 8</td>
</tr>
<tr>
<td>4</td>
<td>A LDV schematic diagram demonstrating the operation principle ................................ 9</td>
</tr>
<tr>
<td>5-1</td>
<td>A comparison of mode shapes calculated using TAI ..................................................... 11</td>
</tr>
<tr>
<td>6 a,b</td>
<td>Simulated data and its Fourier transform of a point .................................................. 18</td>
</tr>
<tr>
<td>7</td>
<td>(1,1) mode of a square plate ....................................................................................... 20</td>
</tr>
<tr>
<td>8</td>
<td>Interference behaviour of two waves with respect to phase difference ......................... 21</td>
</tr>
<tr>
<td>9</td>
<td>Typical Michelson Interferometer .............................................................................. 24</td>
</tr>
<tr>
<td>10 a,b</td>
<td>Michelson interferometer set up ESPI measurements .................................................. 26</td>
</tr>
<tr>
<td>11</td>
<td>Formation of speckle (Steinchen &amp; Yang [10]) .............................................................. 27</td>
</tr>
<tr>
<td>12</td>
<td>Typical ESPI image with a magnified section ............................................................... 28</td>
</tr>
<tr>
<td>13</td>
<td>Simulation of the unwrapping process ......................................................................... 29</td>
</tr>
<tr>
<td>14</td>
<td>Effect of phase shift on intensity (An [25]) ................................................................. 35</td>
</tr>
<tr>
<td>15</td>
<td>Phase stepping applied to static and dynamic deformations ......................................... 38</td>
</tr>
<tr>
<td>16</td>
<td>Behaviour of neighbouring pixels .............................................................................. 40</td>
</tr>
<tr>
<td>17</td>
<td>Comparison of fringe image to reference images ......................................................... 42</td>
</tr>
<tr>
<td>18</td>
<td>Neighbouring 5x5 pixel clusters for calculation of r-values ......................................... 50</td>
</tr>
<tr>
<td>19 a-d</td>
<td>Effect of different phase evaluation algorithms ............................................................ 57</td>
</tr>
<tr>
<td>20 a-d</td>
<td>Effect of box size on Fringe Pattern Quality .............................................................. 60</td>
</tr>
<tr>
<td>21</td>
<td>Effect of box size on Fringe Pattern Quality .............................................................. 60</td>
</tr>
<tr>
<td>22 a</td>
<td>The Phase-Average Intensity and b) Modulation ............................................................. 61</td>
</tr>
<tr>
<td>23</td>
<td>Effect of phase averaged intensity subtraction ............................................................. 62</td>
</tr>
<tr>
<td>24</td>
<td>Effect of modulation weighting ................................................................................... 63</td>
</tr>
</tbody>
</table>
Figure 25: Error Estimates of Approximations.......................................................... 65
Figure 26: Conventional and modified Michelson interferometer ............................... 67
Figure 27: Example Fringe Image from R2-RIM Box size 17.................................... 70
Figure 28: Average Fourier Spectrum of Selected Points ........................................... 71
Figure 29 a-g): Recovered mode shapes using explicit reference images .................... 72
Figure 30: Fringe pattern created from explicit and interpolated reference images ...... 73
Figure 31: Fourier Spectrum from explicit and interpolated reference images .......... 74
Figure 32 a-l): Mode shapes from explicit and interpolated reference images.......... 74
Figure 33 a-g): Mode shapes recovered using TAI .................................................... 75
Figure 34: Chladni mode shape at 301 Hz using pepper .......................................... 76
Figure 35: Modal shapes extracted using 2x2 binning and frame rate of 350 fps........ 79
Figure 36: Single Mode Excitation trials.................................................................. 80
Acknowledgements

I would like to thank my supervisor, Dr. Schajer, for offering guidance and support throughout my degree. I would also like to offer my gratitude to the UBC Mechanical Engineering faculty, staff, and fellow students, who have made my stay here a truly enjoyable experience.

I would like to acknowledge NSERC, ICICS, and American Stress Technologies for helping fund my research.

Finally, I would like to thank my family for supporting me throughout my stay in Vancouver. Never far in thought, their ongoing belief in me allowed me to take chances and gain new experiences.
Chapter 1 – Introduction

1.1 Introduction to Modal Analysis

The vibration characteristics of any machine or component thereof are widely studied in the world of engineering and have a profound impact on design. An entire discipline of engineering is dedicated to understanding such vibrations for diagnostic purposes alone [1]. Early signs of fatigue and failure that are discovered through vibration measurements play a large role in preventative maintenance, meaning larger and more expensive repairs occur less frequently.

When studying vibrations, two parameters are commonly of interest: resonant frequencies and their respective mode shapes [2]. The first refers to the rate of oscillation and the latter to the particular shape the deformation takes. To complicate matters, an object will often be vibrating in a superposition of modes. It is important to note, however, that these parameters are not random but instead depend on the physical characteristics of the component and its boundary conditions. This is why the first string on a guitar continues to sound like an E (330 Hz) after successive plucks. The frequency content can be altered by changing the material (physical properties) of the string or changing how the string is coupled to the guitar body (boundary conditions). In actuality, this guitar string is vibrating in a superposition of modes with E being the mode of lowest
frequency and greatest amplitude. To illustrate this principle, the string can be made to sound like a high E (660 Hz) by pressing lightly on a fret half way down the neck, damping out the lowest mode (i.e. E) and allowing the vibration of higher modes. This is an example of changing the boundary conditions. While a guitar string is a rudimentary one-dimension example, these behaviours also exist analogously in both the second and third dimensions. The work presented here focuses on plate vibrations – a two-dimensional case.

In a procedure similar to the way a musician changes the physical properties of a string to tune a guitar (applying more/less tension), an engineer can use the vibration characteristics to improve upon a component design by changing its physical properties in an effort to either promote or suppress certain modes. Modal analysis is the tool that makes these improvements possible. Objects that are smaller than can be seen with the naked eye to skyscrapers can all be subjected to modal analyses in an effort to make these structures as safe, comfortable, robust, etc. as possible.

1.1.1 Motivation

In order to make any design recommendations, it is first necessary to have a reliable measurement procedure. This way a certain component behaviour or new design can be fairly evaluated against a standard behaviour or existing design. After examination
of existing modal analysis methods, five identifiable desirable features tend to emerge which contribute to reliability and confidence in measurement. These are:

- Full-field image of the mode shape
- Quantitative data of the mode shape amplitudes
- Sign retention of the mode shape amplitudes
- Simultaneous calculation of mode shapes
- Non-contact observation

*Full-field*-ness refers to the mode shape not consisting of discretely sampled points. If the mode shape is thought of a jigsaw puzzle, the general trends of the mode shape can be gleaned from a subset of the pieces in place, but the fine detail comes when the puzzle is completely assembled. Discretely sampling points can be time consuming and to compensate, often relatively few points end up being sampled. If points can be sampled simultaneously in the full-field, with a CCD camera for example, then the spatial resolution is only limited by the camera resolution. *Quantitative*-ness refers to the ability to quantify the amplitudes in the mode shape. *Sign retention* indicates if the amplitudes are positive or negative, revealing which parts of the plate are in or out of phase with others. *Simultaneous calculation* is the ability to compute existing mode shapes individually from data consisting of some superposition of modes. Finally, *non-contact* observation indicates that the sensor need not be physically attached to the object being measured. This is important when doing so is difficult or impossible in practice (i.e.
object too hot, too small, etc.). These features, when taken as a whole, would provide the most comprehensive and useful set of data to date. An examination of some existing methods is presented in the following sections. Only the continuous scanning laser Doppler Vibrometry method approaches fulfilling all these features.

1.2 Existing Methods

Over the course of the last 250 years, a few solutions have been developed to measure mode shapes. Reflecting the available technology of the time, these solutions were the result of brilliant insight and ingenuity. Four of the more commonly encountered methods are discussed here to provide reference.

1.2.1 Chladni Method

This method is the oldest known and perhaps most illustrative of resonant vibration. Ernst Chladni, an 18th century German physicist, realized that he could excite individual frequency modes by drawing a violin bow across the edge of a clamped plate. Figure 1 shows a typical setup for a Chladni experiment.
By mounting the plate horizontally and sprinkling a fine powder across its surface, the vibrations cause the powder to migrate away from regions undergoing large accelerating to small ones. The powder eventually comes to rest at areas on the plate where the acceleration is so small it can no longer cause any migration. These regions are the nodal lines corresponding to that vibrational mode shape. A set of Chladni’s original sketches of square plate mode shapes is shown in Figure 2 [3].
While this method is indeed very elegant, it has some limitations which make it impractical for modern use. Although the patterns are full-field, the magnitudes as well of the signs of the amplitudes are unquantifiable. It is also impossible measure a particular mode when the plate is vibrating with multiple modes, and so modes must be excited one

Figure 2: Mode shape sketches of a square plate as recorded by Chladni (Chladni [3])
at a time. In addition, the plate must lie horizontal and the use of a powder makes this method relatively messy. In spite of these limitations, a better solution would not be developed for nearly two centuries when precision electronics made possible a greater variety of measurements with both more accuracy and speed.

### 1.2.2 Instrumented Hammer (IH)

The instrumented hammer method measures the output from a fleet of accelerometers in response to a hammer impulse [4]. In control theory, this configuration is called “single input multiple output” (SIMO). Alternatively a single accelerometer and a sequence of hammer impulses at known positions can be used. This is the” multiple input single output” (MISO) configuration and is mathematically equivalent to SIMO. In either case, the input impulse signal is required in the calculation and is accomplished by using a hammer outfitted with a force transducer on its tip, i.e. an instrumented hammer.

A Fourier analysis is then used to extract the frequency content for each of the test points. Mode shapes are constructed by selecting a frequency in the available spectrum, and mapping the amplitudes of that frequency for each point. Due to the fact that the Fourier analyses provide information about all the frequencies in range, the mode shapes are said to be calculated simultaneously. In contrast, the Chladni method requires
resonant excitation of single modes. Figure 3 is an illustration of how mode shapes might be constructed from a modal analysis conducted on a beam [5].

![Figure 3: Mode shape construction using Fourier analysis [5]](image)

This is a trusted method that scales up with size very effectively, often being used on large buildings. Importantly, the measured mode shapes are quantitative and the sign information is retained. One disadvantage, however, is that many points need to be individually sampled to acquire good spatial resolution and thus this is not a full-field method. Additionally, the hammer and accelerometers physically interact with the system.
1.2.3 Laser Doppler Vibrometry (LDV)

Laser Doppler Vibrometry has many features in common with the IH method. In its simplest form, a set of points on the test surface is typically measured one at a time in order to acquire frequency content. From the frequency content, modes shapes are constructed as in the IH method. The principal difference is that where the IH method uses accelerometers to measure vibrations, LDV measures the Doppler frequency shift of a reflected laser beam incident on a point [6]. This point is illustrated in Figure 4. The velocity can then be determined directly from that frequency shift.

*Figure 4: A LDV schematic diagram demonstrating the operation principle*
The non-contact nature of LDV makes it particularly attractive in situations where applying accelerometers to the test object is impractical. By the same token, LDV shares similar disadvantages associated with the IH method. This is namely a reduced spatial resolution.

Continuous scanning laser Doppler vibrometers show increased functionality by measuring points quasi-simultaneously. A mirror is actuated which directs the beam to sweep across an area. Each sweep constitutes a single “simultaneous” measurement of the field. Current resolution is typically on the order of hundreds of points [7]. Limitations here are speckle noise and the rate at which the mirrors can be actuated to redirect the beam.

1.2.4 Time-Averaged Interferometry (TAI)

The final method discussed here is Time-Averaged Interferometry. Electronic Speckle Pattern Interferometry (ESPI) is used to take full-field (~ 0.3 Megapixels) time-averaged exposures over many periods of the vibrating test object [8]. Similar holographic methods also exist [9]. Two exposures are taken, the second of which has the inclusion of a π phase shift in the reference beam. Computations on the difference in these exposures produce fringe patterns which resemble typical ESPI fringe patterns. Unlike typical ESPI fringe patterns, however, the fringes are not uniformly spaced. Here,
the displacements correspond to the roots of the zeroth order Bessel function [10]. Due to the unequal spacing of the zeroes of the Bessel function and the lack of sign information, it is necessary to keep track of the fringe order relative to the zero order fringe. The zeroth order fringe is the brightest fringe in the pattern and is usually readily identifiable. The very grainy texture of TAI images also makes it difficult to extract results at each pixel. Figure 5 a-f) presents example TAI images in which the lowest order fringes are indicated by white arrows [3], [11].

Figure 5 a-l): A comparison of mode shapes calculated using the time-averaged method (left) and Chladni method (right). White arrows identify the lowest order fringe in the Chladni diagrams (adapted from Steiznig & Schajer [11] and Chladni [3]).

TAI is attractive because it produces full-field images and is non-contact. This has the potential to provide the most complete picture of the vibration. The biggest detraction
is that while the vibration amplitudes are technically quantitative even with the loss of sign, it is a difficult calculation to automate and doing so by hand is impractical when considering data sets of any significant size.

1.3 Research Objectives and Proposed Method

Table 1 is a summary of the attributes exhibited by existing methods. It is the aim of the research presented in this thesis to develop a new method of modal analysis which encapsulates the set of desirable attributes and perhaps most importantly can be applied in situ.

Table 1: A summary of predominant attributes of existing and proposed methods

<table>
<thead>
<tr>
<th></th>
<th>Full-field</th>
<th>Quantitative</th>
<th>Sign retention</th>
<th>Simultaneous calculation</th>
<th>Non-contact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chladni</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>Instrumented Hammer</td>
<td>✗</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>Laser Doppler Vibrometry</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Time-Averaged Interferometry</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Proposed Method</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

The proposed method will try to take advantage of the useful pieces of technology and mitigate any limitations. This is clearer in some cases than others. For example, IH and non-continuous scanning LDV are not full-field because the sampling of discrete points is necessarily done one at a time. Accordingly, a logical design choice is to use ESPI with a CCD camera, allowing the sampling of a complete surface simultaneously.
At the same time, the method by which LDV and IH compute mode shapes is very appealing.

These insights as well as some others pointed to a solution that uses an ESPI apparatus to sample points, and then uses a calculation method similar to LDV and IH. Unlike the time-averaged ESPI method, it is the intention here to make use of its more traditional application: measuring static surface displacements. The use of ESPI would fulfill the following requirements: Full-field-ness, Quantitative-ness, Sign Retention, and the Non-contact observation. The idea of sampling displacements over a period of time as in LDV or IH fulfills the simultaneous calculation requirement. The samples themselves are of the form of 512 ESPI images.

There are challenges, however, in doing this. ESPI commonly uses a set of four phase-stepped images for each state of deformation in order to determine the relative deformation between the states [12]. Phase-stepping can be accomplished through a variety of means, but is often done using a piezoelectric actuator (PZT) out of practicality [13]. When trying to measure a moving surface, however, there is not enough time to actuate the PZT and take multiple images. By the time actuation occurs, the surface has deformed into a completely new state, and the data are compromised. This is to make no mention of the timing difficulty associated with stepping a PZT at the same time as running a high speed camera.
A majority of the novel work presented here focuses on solving this problem. One approach to resolving this issue is eliminating the need for phase stepping altogether. As a result of exploring this avenue, a class of algorithms has been developed which enables the calculation of displacement from a single ESPI fringe image with a set of phase-stepped reference images [14]. The reference images may be acquired before vibration is induced, but due to the nature of this experiment, these reference images may be interpolated from the set of the ESPI images constituting the samples. The need for this arises in cases where the static reference images cannot be taken because the object or machine being measured cannot be turned off.

If these algorithms prove successful, the ability to sample displacements instantaneously will be realized. Mode shapes can be reconstructed to a high degree of accuracy and up to frequencies of near half the frame rate of the camera as described by the Nyquist-Shannon sampling theorem [15]. The camera used for experiments in this paper operates in the order of hundreds of Hz, but cameras that operate on the order of hundreds of kHz are commercially available [16].

1.4 Research Outline

Background theory describing ESPI and Modal Analysis in more detail is presented in Chapter 2. A reader with a strong understanding of both subjects may wish
to proceed directly to Chapter 3, where theory regarding the algorithms developed specifically for this research can be found. In the subsequent chapters, experiments are described detailing results obtained from a working apparatus of the proposed method. These results will then be evaluated against results obtained from some of the alternative methods described in Chapter 1.
Chapter 2 – Background Theory

The various methods of performing modal analysis share some common elements, but none are precisely alike. In this chapter and the next, the theory explaining the proposed vibration mode measurement method will be presented. The theory behind previous methods will not be discussed in further detail, but a great deal of the content that lies ahead in the following two chapters provides technical insight to those methods. The current chapter is designed to inform the reader of established theory on interferometry. Chapter 3 – Phase Determination Algorithms is where the contributions of the author are described in detail.

2.1 Modal Analysis

Before any discussion of interferometry is presented, a broader look at modal analysis should be considered in order to provide context to the rest of the theory. First, it is important to get an appreciation for the intimate relationship between the time domain and frequency domain. The time domain is generally how we might perceive the universe – an object moving from point-A to point-B in certain amount of time. On the other hand, the frequency domain is far less tangible. A sinusoid in the time domain is represented concisely as a single peak in the frequency domain, marked at the frequency, amplitude, and phase of the sinusoid. A more powerful realization is that any signal can be
constructed as the infinite sum of sine waves with different frequencies. Fortunately, Fourier developed a method for breaking down a time domain signal and representing it in the frequency domain – the **Fourier Transform**.

This is intriguing in the context of modal analysis, because data from an object sinusoidally vibrating could be represented similarly by a single peak in the frequency domain. Furthermore, if \( n \) vibrations at different frequencies were present, then \( n \) peaks occur in the frequency spectrum. The accelerometers from the IH method output *acceleration versus time* data and the photo-detectors from the LDV method output *velocity versus time* data. In the time domain, these signals are messy, but in the frequency domain, the signals are far more accessible. In the same vein, the proposed method will measure *displacement versus time* data, and Fourier analysis will once again be a helpful tool for extracting the desired information.

Figure 6 shows a set of highly idealized simulated data of some arbitrary surface point of a vibrating object and a Fourier transform of those data. It should be evident from the figure that the data in the frequency domain are more concise and useful for further calculation, even though they contain the same information as presented above in the time domain. The interpretation of this data from the Fourier transform is that the object is vibrating in a superposition of at least three modes, one at each 5 Hz, 13 Hz, and 19 Hz.
By computing the full Fourier transform for a sample of points, the frequencies can be identified quickly. With the frequencies identified, the discrete Fourier transform for individual frequencies can be implemented on a pixel by pixel basis:

$$A(x, y, f) = \sum_{j}^{n} d[j] e^{-i2\pi f}$$  \hspace{1cm} (1)
where $A(x,y,f)$ is the amplitude for pixel $(x,y)$ for frequency $f$, $d[j]$ is a displacement datum at frame $j$, and $F$ is the sampling frequency.

Mode shapes are simply a two dimensional map of the peak amplitudes for a given frequency. Regions of the object vibrating with a high displacement (also velocity and acceleration) produce appropriately large peaks in the Fourier transform. The same argument applies with regions of low vibrations. Taking this idea to its logical conclusion, regions where there is locally no displacement are called *nodes* and will not present themselves as peaks in the frequency domain. The absence of a peak at a given frequency, however, does not imply the absence of peaks at that frequency elsewhere on the plate. If peaks are found elsewhere, we may like to think of a peak with amplitude *zero* as actually existing for the sake of completeness.

When this amplitude map is plotted in three dimensions, the result resembles Figure 7 - the $(1,1)$ mode of a square plate extracted from simulated data. Although not shown in Figure 7, the Fourier transform actually returns a pair of complex numbers for each peak and a sign (i.e. positive or negative) is attributed to each peak. It is then possible to identify from the complex angle, the regions of the object are vibrating in phase and out of phase with respect to each other. In Figure 7, red and blue regions have opposite phase.
2.2 Introduction to Laser Interferometry

An interferometer is a device that takes advantage of how light waves mutually interact in order to make practical measurements [12]. When set up properly, dimensional measurements accurate to fractions of a wavelength (nm scale) are achievable. It is easiest to understand the phenomenon by considering the wave-like nature of light.
When two monochromatic light waves interact, they mutually reinforce each other if they are in phase and cancel out if they are out of phase [17]. Figure 8 demonstrates this principle by considering the effect of varying the phase difference of two interfering sinusoidal waves.

![Figure 8: Effect of relative phase on the light intensity of the interference pattern produced by the interference of two coherent waves.](image)

The interaction is modeled mathematically by adding the complex amplitudes at any point. First, each wave is modeled by a sinusoid in complex notation by:

\[
A_1 = |A_1|e^{i\phi_1} \quad A_2 = |A_2|e^{i\phi_2}
\]

(2)
where $A$ is the amplitude, $|A|$ is the magnitude of the amplitude, and $\phi$ is the phase corresponding to the distance from the source to the recording medium. When the two waves interact, their complex amplitudes add together. The result is:

$$A_{1+2} = A_1 + A_2$$

$$A_{1+2} = |A_1| e^{i\phi_1} + |A_2| e^{i\phi_2}$$ (3)

The intensity of the wave is the square of the magnitude:

$$I = |A|^2 = (A)(A^*)$$ (4)

where $A^*$ is the complex conjugate of $A$. Applying equation (4) to equation (3), the intensity recorded by a camera is as follows:

$$I_{1+2} = |A_{1+2}|^2$$

$$I_{1+2} = (A_1 + A_2)(A_1^* + A_2^*)$$

$$I_{1+2} = A_1A_1^* + A_2A_2^* + A_1A_2^* + A_1^*A_2$$

$$I_{1+2} = I_1 + I_2 + |A_1||A_2| e^{i\phi_1} e^{-i\phi_2} + |A_1||A_2| e^{-i\phi_1} e^{i\phi_2}$$

$$I_{1+2} = I_1 + I_2 + 2\sqrt{I_1I_2} \cos(\phi_1 - \phi_2)$$ (5)

This result reveals some interesting properties of wave interference. The most useful one is that the intensity as read by a camera is constant for a fixed phase
difference, as might have been predicted in Figure 8. If a single light source is used, then the phase difference corresponds to the difference in distances that each path of light takes to the point of measurement. Since the intensity is periodic with every additional displacement of one wavelength $\lambda$, the optical path difference is found by:

$$d = \frac{\lambda}{2\pi} (\phi_1 - \phi_2)$$

(6)

One complication that arises is that the cosine function has repeated values. That is, for any given recorded intensity there are two solutions on modulo-$2\pi$. Without additional information, it is impossible to determine which solution is the correct one. Exploring this topic has constituted a majority of the novel work presented in this thesis as well as much research from others in the field. Accordingly, a chapter has been dedicated to this topic alone: Chapter 3 – Phase Determination Algorithms. With this anticipation in mind, this chapter will proceed under the premise that one can indeed determine which of the two solutions is correct and therefore accurately calculate phase difference.

2.2.1 Typical Apparatus

While many configurations exist, one of the oldest and simplest is the Michelson interferometer - shown in Figure 9 [12]. The Michelson configuration is ideal for measuring out of plane displacements. Coherent light emanating from the Laser Source is
divided by a 50% beam splitter, reflecting 50% of the light towards Mirror 1 and transmitting 50% through to Mirror 2. It is customary to call the beam incident on the test surface/mirror the object beam and the beam incident on the stationary surface/mirror the reference beam (i.e. Mirror 2 and Mirror 1 respectively). The reflected light from each mirror is recombined at the beam splitter and then continues to the recording medium, in the case the CCD Camera.

Figure 9: Typical Michelson Interferometer

The intensity read by the camera is constant as long as the geometry stays fixed. When Mirror 2, however, is subjected to translations in the direction shown by the red arrow in Figure 9, the intensity varies sinusoidally as described by equations (5). To
obtain the physical distance corresponding to the translations of Mirror 2, the optical path difference is calculated using equation (6) once before translation and once after. The difference in these values is the physical displacement of Mirror 2. This is the quantity that is sought after in the proposed method.

As mentioned in section 2.1, the phase for any single path from the source to the camera depends primarily on the distance from the source to the camera. The other factor is the index of refraction of the medium through which the light travels. As long as the index is constant throughout the apparatus, the effect cancels out. In the presence of air currents (e.g. from ventilation systems) or local temperature fluctuations the index is not constant and this can be a significant source of noise. Another source of noise is vibration transferred from the ground to the table where the interferometer is mounted.

It is commonplace to mount the apparatus on a floating optical table to combat unwanted perturbations and enclose the system with a physical barrier to reduce the effect of air currents and temperature fluctuations.

### 2.3 Electronic Speckle Pattern Interferometry (ESPI)

The interferometer in Figure 9 is well suited to measure displacements for the single point of laser incidence. In order to perform full-field measurements, however, the laser must be expanded through a divergent lens as shown Figure 10.a,b) [18]. A CCD
camera, which can have millions of light intensity sensors, is used to image the surface. In a practical sense, it is akin to having a unique laser and sensor for every point on the plate (up to the resolution of the camera).

In actuality, the physics is more complex. Speckle is the product of diffuse surface reflections. This is in contrast to specular reflection (as from a mirror) where a ray of light reflects from the surface at an angle equal to the angle of incidence. If the incident wavelength of light is approximately the same size as the surface irregularities, diffuse reflection occurs and light scatters from each illuminated point in all directions [10]. Then, for an arbitrary point in space, the intensity is the complex sum of the contribution from all illuminated points. Figure 11 shows how speckle is formed from surface

Figure 10 a,b): Conventional Michelson interferometer set up ESPI measurements. a) incident light paths to the two mirrors, b) reflected light paths from the two mirrors.
reflections [10]. The pattern in free space is called *objective* since it is independent of the observer. When an imaging system is used, however, the recorded image depends on the optical equipment (specifically camera aperture) and is so called *subjective* [19]. The aperture, in addition to controlling the amount of light entering the camera, acts to restrict admission of spatial frequencies. Accordingly, a small aperture produces large speckles and vice versa.

*Figure 11: Formation of speckle (Steinchen & Yang [10])*

Due to the randomness of the surface irregularities, the reconstructed phase of any point in space is also random. This is evident by examining an ESPI image. Figure 12 shows a typical ESPI image taken by a CCD camera as well as a magnified section exhibiting characteristic randomness.
2.4 Phase Unwrapping

Examining equation (5) and (6) reveals a subtle problem. Consider that if the one arm of the interferometer translates by exactly $\lambda/2$ (causing a path difference change of $\lambda$), then the camera records the same intensity and the apparent translation is naught. Based on one pixel alone, it is only possible to calculate the modulo-$2\pi$ phase difference. For many purposes, the modulo-$2\pi$ information is sufficient since an experienced user can read what is called the fringe pattern – a cosine map of the modulo-$2\pi$ difference - and extract the important information. The problem, however, is that it fails to represent the true displacements. For modal analysis, a true displacement map is required and the large quantity of ESPI images demands that an automated procedure solve this dilemma.
The process of converting wrapped data to unwrapped data is called *phase unwrapping* [10], [20].

More specifically, phase unwrapping is the stitching together of the $2\pi$ discontinuities in the modulo-$2\pi$ phase map. Figure 13 shows simulated phase maps of the same data in both wrapped and unwrapped format. Cross sections at the red line are shown below each map. An important feature to note is the magnitude of the scales between the wrapped and unwrapped data.

*Figure 13: Simulation of the unwrapping process for the (1,1) mode shape of a square plate simply supported at its edges.*

Unwrapping is relatively easy to do by hand, even in the presence of considerable noise. Establishing a robust computer algorithm, on the other, can be a complicated
matter. Although an important part of the modal analysis process, it is not the aim of the research done here to improve upon unwrapping algorithms. For this reason, the experimental validation uses an established algorithm [21].

When multiple ESPI images are taken in succession and displacement maps are calculated from the phase unwrapping procedure, a final step is needed in order to establish a consistent frame of reference throughout all the images. Depending on what the displacement is at the location where the unwrapping begins, the zero-displacement point can vary somewhat. In the experiments described in this thesis, a rectangular plate is clamped at its centre. Knowing that the clamping site is the true zero-displacement point, the whole displacement map needs to be shifted by a constant value (unique to that image only) to achieve this agreement. The constant shift that is applied must agree with the boundary conditions and will therefore change from case to case.

2.5 Sampling Procedure

While ESPI is the tool which creates the effect which makes displacement measurements possible, it is the digital camera which quantifies and records the effect. In the experiments presented in this paper, vibrations on the order of 10s to 100s of Hz are being measured and as per the Nyquist-Shannon sampling theorem, the plate
displacements need to be measured at frequencies no less than twice those being measured [15].

A high speed CCD camera is used for this purpose. Using such a camera, a known quantity of images can be acquired at a fixed rate. This is critical since it means that the exact time that each image is acquired is known, which ultimately allows the Fourier analyses to be performed.

2.6 Summary

The use of a high speed camera makes possible the acquisition of hundreds (or even thousands) of phase maps in mere seconds and importantly with the known times of when the images were acquired. The phase unwrapping process, when applied to each phase map, produces a set of displacement maps at these known times. Recalling from section 2.1 that the goal is displacement versus time data, it would appear that this has been achieved. Also recall, however, that it was assumed that we could calculate phase displacements on modulo-2π from the raw intensity data. The resolution of this matter is presented in chapter 3.
Chapter 3 – Phase Determination Algorithms

The phase determination algorithms presented here have been designed with a very specific objective in mind – to measure surface displacements on a dynamic surface. Procedures for measuring the displacement between a set of phase stepped images and a single fringe image taken a later time will be introduced. Reference images may be acquired using the conventional method using phase stepping when the object is static. In order to avoid the requirement that the object be static at any point during the test, however, a second, interpolation method is proposed which establishes the quadrature image through statistical calculations. This process is discussed in Section 3.5 – Interpolated Quadrature Images

3.1 Introduction to Phase Determination Algorithms

ESPI hinges on the ability to calculate phase difference between interfering beams of light indirectly, using the intensity of light recorded by some sensor. Because the intensity varies sinusoidally with phase, a single intensity measurement produces multiple possible phase difference, two of which are unique solutions on modulo-2π. Through the process of phase unwrapping, the modulo-2π solutions form a two dimensional map which can then be synthesized to form a true displacement map.
A great deal of research has been carried out in an effort to develop algorithms that make experimentation more convenient and provides data that describes the system with ever improving accuracy [13], [22]. The features and limitations of these algorithms in turn specify the scope of their application. As of yet, a modal analysis procedure featuring the five key attributes outlined in the introduction has not been demonstrated. In this chapter, the classical phase determination algorithm will be discussed followed by the presentation of a novel class of algorithms. The intention of the new work is to overcome the previously mentioned shortcomings, with the ultimate goal of use in a practical in situ modal analysis experiment.

3.2 Classical Method

Given that the information is incomplete with merely a single ESPI image, the classical method requires additional data to resolve the phase difference discrepancy. Simplifying equation (4):

\[ A \equiv I_1 + I_2 \quad B \equiv 2\sqrt{I_1I_2} \quad \Delta \varphi \equiv \varphi_1 - \varphi_2 \]

\[ I = A + B \cos(\Delta \phi) \]  

(7)
where \( I \) is the intensity read by the camera, \( A \) is the sum of the intensities of the two beams, \( B \) is the amplitude modulation, and \( \Delta \phi \) is the phase difference between the two paths.

From equation (7), there are three unknowns \( \{ A, B, \Delta \phi \} \) and so three equations are required for a solution. To construct the three equations, a common tactic is introducing a known variable to equation (7) which can be manipulated by the experimenter. Any time the variable is adjusted, a new equation is produced. It follows that two such adjustments along with the original equation completes the necessary set of three. Introducing the new variable by changing the intensities of either beam is impractical, but a piezo-electric actuator (PZT) can introduce small and controllable phase shifts to one arm of the beam. The modification to equation (7) as a result of the inclusion of the phase shift is given by:

\[
I = A + B \cos(\Delta \phi + \psi)
\]

(8)

where \( \psi \) is the known shift. For ease of calculations, \( \psi \) is often a multiple of \( \pi/2 \), but it is not strictly necessary [23], [24]. The set of equations is commonly referred to as phase-stepped:

\[
I_1 = A + B \cos(\Delta \phi)
\]

(9.a)

\[
I_2 = A + B \cos(\Delta \phi + \pi/2)
\]

(9.b)
\[ I_3 = A + B \cos(\Delta \phi + \pi) \]  
\[ I_4 = A + B \cos(\Delta \phi + 3\pi/2) \]

This same idea is shown graphically in Figure 14. Note how what is recorded is the intensity and it is the sequence in which the intensity values are recorded that allows the phase to be determined.

Performing the algebra, and solving for \( \Delta \phi \):

\[
\frac{I_4 - I_2}{I_1 - I_3} = \frac{A + B \sin(\Delta \phi) - A + B \sin(\Delta \phi)}{A + B \cos(\Delta \phi) - A + B \cos(\Delta \phi)}
\]

\[
= \frac{\sin(\Delta \phi)}{\cos(\Delta \phi)}
\]
\[ \Delta \phi = \arctan \left( \frac{I_4 - I_2}{I_1 - I_3} \right) \]  \hspace{1cm} (10)

When \( \Delta \phi \) is found both before and after some surface deformation of an object, the difference is related to the actual displacement by:

\[ d = \frac{\lambda}{2\pi} (\Delta \phi_1 - \Delta \phi_2) \]  \hspace{1cm} (11)

where \( d \) is the displacement of the plate, and \( \lambda \) is the wavelength of light. Equation (11) is identical to equation (5) in form. In equation (5), phase represents the path length of each beam. In equation (11), \( \Delta \phi \) is the difference in path lengths for each state (i.e. before/after deformation). Because the length of the reference beam is unchanged, the phase value corresponding to the reference beam gets implicitly canceled in the calculations. Therefore, the displacement of the object is given by the change in phase of the object beam alone, and the reference beam provides a standard throughout any state of deformation against which the object beam phase is compared interferometrically.
3.2.1 Limitations to the Classical 4-Step Method

When the surface is static in each state being measured, no serious problem exists with the classical method and it still provides effective and reliable measurements. The greatest challenge is working with the PZT. Its inherent non-linearity, susceptibility to mechanical perturbations, and difficulty in calibration lead to additional sources of error [1].

The problems with the PZT can be overcome, but the pertinent problem with the classical method arises when object surface is moving, as is the case in modal analysis. Most cameras cannot record images fast enough to take four images within a time period small enough that the measured surface does not move significantly between images. Any surface motion during the measurement period distorts the desired 90 degree phase steps between images and correspondingly distorts the resulting phase evaluation. This property is called here phase step latency. Figure 15 demonstrates the difference between applying the phase stepping algorithm to a static state and to a dynamic state.
3.3 Other Approaches

There have been many attempts to improve upon the classical method, generally with specific applications in mind [13], [22]. Among the group, a common goal is reducing the number of images required in either state of deformation. Considering the example in Figure 15, if fewer images are used, then the accumulated error due to phase step latency can be significantly smaller.

One effort which achieved moderate success was an algorithm described Kao et al. called the DC-(5,1) [26]. This algorithm uses a single fringe image, but is tailored specifically for use with five phase-stepped reference images. The novel algorithm class presented here extends these ideas further, enabling reliable modulo $2\pi$ evaluations both

Figure 15: Phase stepping applied to static and dynamic deformations
more compactly and efficiently. These improvements ultimately make the new algorithms suitable for modal analysis applications as intended.

### 3.4 Single Fringe Image Algorithm Class

In light of phase step latency, it would appear that PZT phase stepping is unsuited for the purposes of this research. On the other hand, without the use of phase stepping, a new source of additional information is required to resolve the modulo-$2\pi$ two solution discrepancies. The first key insight here is to expand the region of interest from a single pixel to a local cluster of pixels. In a small enough cluster, the local plate deformation will impart nearly identical phase changes. While each individual pixel has two unique solutions, one and only one of those solutions is common to all in the cluster. Figure 16 shows how two simulated neighbouring pixels might behave.
An important feature is the acquisition of phase-stepped reference images. In one variant of the proposed procedure, reference images are acquired before vibrations are induced and the surface is static. As shown in Figure 15, phase stepping is not problematic for a static surface and thus does not invalidate the implications of this idea. In the second procedure variant, additional work is performed to extract reference images from the set of 512 fringe images. No static state is required because reference images are not explicitly acquired, but rather interpolated from within the data set.
The following class of algorithms developed throughout this section will use a group of four phase stepped reference images (either explicit or interpolated) and a single ESPI fringe image captured from a dynamic surface. In actuality, as few as two reference images could be used, but the ease at which phase stepped images can be acquired while the surface is static means that acquiring four is neither problematic nor time consuming, and the data prove to be more robust as a result without dramatically increasing computation time.

3.4.1 Four Reference Image Method (4-RIM)

While determining both solutions and sifting through the data works in principle, it is not entirely quantitative and very time consuming. This method is improved upon by using a statistical approach. The fringe image is compared to each reference image to determine how much it is like to each reference image. The reference images provide a comprehensive description of how any change of the relative path differences in the interferometer affects the fringe patterns at the local scope. Therefore, the set of comparisons indicate what surface displacement must have occurred in order to produce the resultant fringe image. For example, if the fringe image has a strong resemblance to the $\pi/2$ phase stepped reference image, it can be inferred with a high degree of confidence that the displacement of the plate must have undergone a modulo-$2\pi$ phase displacement of $\pi/2$. 

41
Figure 17 shows this idea using simulated data of nine pixels forming a cluster. Notice how if only the top left pixel of the fringe image is considered, the $\pi/2$ and $3\pi/2$ solutions are indistinguishable. As soon as the pixel to its right is taken into consideration, the $\pi/2$ solution is in agreement, whereas the $3\pi/2$ is not. Using a larger set of pixels in the comparison provides robustness to the algorithm. The apparatus is imperfect and some pixels are expected to be faulty. As the number of pixels considered increases, it becomes increasingly unlikely that enough pixels give erroneous data such that the aggregate result is corrupted.

![Figure 17: Comparison of fringe image to reference images](image-url)

The example presented is trivial, and fails to explain what happens if the fringe image is at a $\pi/4$ displacement for example or any other displacement in the $[0,2\pi]$ continuum. Using the Pearson product-moment correlation coefficient (often called the \textit{r-value} or
regression coefficient), the data from the fringe image can be quantifiably compared to each reference image and thereby eliminating any guesswork. The r-value calculation is defined as:

\[ r \equiv \frac{\text{cov}(I,J)}{\sigma_I \sigma_J} \]  (12)

where \( \text{cov}(I,J) \) is the covariance between the fringe image data \( J \) and the reference image \( I \), and \( \sigma \) is the standard deviation of a data set [27].

Schmitt and Hunt described that the r-value varied sinusoidally with respect to the phase difference between two ESPI images [28]. When an r-value was calculated, the arccosine function was employed to retrieve the phase difference, but unique solutions were limited to \([-\pi/2, \pi/2]\], and no solution to this matter was provided.

The proposed algorithms aims to extend the range of solutions to the full modulo-2\( \pi \) by considering the set of r-values obtained from the fringe image with each reference image. To show this, consider that the r-value between the fringe image and first reference image varies as:

\[ r_1 = \cos(\varphi) \]  (13)

where \( \varphi \) is the phase between the two images. Calculating the r-value between the fringe image and second reference image which has the additional \( \pi/2 \) phase step gives:
\[ r_2 = \cos(\varphi + \frac{\pi}{2}) \]
\[ r_2 = -\sin(\varphi) \]  
(14)

Combing equations (13) and (14) to solve for \( \varphi \):

\[ \varphi = \text{atan2}(\neg r_2, r_1) \]  
(15)

where \text{atan2} is the two argument arctan function, which returns angles on the complete \([-\pi, \pi]\), instead of \([-\pi/2, \pi/2]\) as with the more familiar one argument arctan function. Equation (15) describes the solution that requires the minimum of data: two reference images and one fringe image. In many applications, the phase map produced by this procedure can be satisfactory. The ease at which reference images can be acquired, however, coupled with the benefits of data redundancy and noise averaging which improve phase evaluation accuracy suggest that additional reference images are worth obtaining. Because the fourth and second reference images are \(\pi\) phase stepped, \(r_4\) is the negative of \(r_2\). The same applies for \(r_1\) and \(r_3\). The solution using four reference images with \(\pi/2\) phase steps is therefore:

\[ \varphi = \text{atan2}(r_4 - r_2, r_1 - r_3) \]  
(16)

where the four \(r\)-values are computed similarly as in equations (13) and (14). Equation (16) is called the \textit{Four Reference Image Method (4-RIM)} since it requires correlating the fringe image to each of the four reference images.
3.4.2 Two Reference Image Method (2-RIM)

A disadvantage with the 4-RIM method is the relatively lengthy computation time associated with doing four such correlations rather than two as in equation (15). Noticing that two $r$-values are subtracted from the others after correlation, reversing the sequence of operations (i.e. subtraction of images followed by correlation) offers potential advantages. First, it would maintain the data redundancy of the 4-RIM method, but it should also take only marginally longer than correlating the fringe image to two untreated reference images. The r-values of the fringe image and subtracted image pairs are defined as:

\[
r_{42} \equiv \frac{cov(I_4 - I_2, J)}{(\sigma_{I_4-I_2})\sigma_J} \quad r_{13} \equiv \frac{cov(I_1 - I_3, J)}{(\sigma_{I_1-I_3})\sigma_J}
\]  \hspace{1cm} (17)

In order to validate this variation is called the Two Reference Image Method (2-RIM), it needs to be shown that the ratio of $r$-values between the fringe image and the subtracted image pairs (i.e. $r_{42}$ and $r_{13}$) is the ratio as the ratio that would be used in the 4-RIM procedure. This ensures that the two argument arctan function returns the same answer. Mathematically, the following needs to be shown:

\[
\frac{r_{42}}{r_{13}} = \frac{r_4 - r_2}{r_1 - r_3}
\]  \hspace{1cm} (18)
In addition, the sign of $r_{42}$ needs to be same sign as $r_4 - r_2$ in order to ensure that the solution is in the correct arctan quadrant. This also applies to $r_{13}$.

First, equation (12) can be expanded:

$$r_4 - r_2 = \frac{\text{cov}(I_4, J)}{\sigma_4 \sigma_J} - \frac{\text{cov}(I_2, J)}{\sigma_2 \sigma_J}$$  \hspace{1cm} (19)

Next, it is assumed that the standard deviation of the intensities of the pixels in the cluster is constant through reference images. This follows because of two reasons: 1. The number of pixels being considered is statistically significant; 2. The cluster occupies the same spot in each image, and so the intensity/modulation of the incident light will be consistent in that region. This assumption is explored mathematically in the section 3.4.5 Other Considerations. This approximation is given by:

$$\sigma_l \equiv \sigma_{I_1} \approx \sigma_{I_2} \approx \sigma_{I_3} \approx \sigma_{I_4}$$  \hspace{1cm} (20)

By a similar reasoning, the following property which will become useful later should also hold:

$$\sigma_{\Delta I} \equiv \sigma_{I_4 - I_2} \approx \sigma_{I_1 - I_3}$$  \hspace{1cm} (21)

Using the approximation in equation (20), the standard deviations can be collected from equation (19):
\[ r_4 - r_2 = \frac{1}{\sigma_i \sigma_f} (\text{cov}(I_4, I) - \text{cov}(I_2, I)) \]  
(22)

Applying an established mathematical property of covariance:

\[ r_4 - r_2 = \frac{\text{cov}(I_4 - I_2, I)}{\sigma_i \sigma_f} \]  
(23)

Similarly, \( r_1 - r_3 \) can be shown as:

\[ r_1 - r_3 = \frac{\text{cov}(I_1 - I_3, I)}{\sigma_i \sigma_f} \]  
(24)

Dividing equation (23) by (24):

\[ \frac{r_4 - r_2}{r_1 - r_3} = \frac{\text{cov}(I_4 - I_2, I)}{\text{cov}(I_1 - I_3, I)} \]  
(25)

Then, making use of the assumption in equation (21):

\[ \frac{r_4 - r_2}{r_1 - r_3} = \frac{\text{cov}(I_4 - I_2, I)}{\sigma_f \sigma_{\Delta I}} \frac{\sigma_f \sigma_{\Delta I}}{\text{cov}(I_1 - I_3, I)} = \frac{r_{42}}{r_{13}} \]  
(26)

as required. Now considering equations (23) and (24) and noting that standard deviation is always positive, it can be seen that the sign of the covariance must reflect the sign of the subtracted \( r \)-values. This sign agreement ensures that when the two argument arctan function is used, the answer is returned in the correct quadrant.
3.4.3 Reduced Two Reference Image Method (R2-RIM)

In the derivation of the 2-RIM, an interesting result pops up. From equations (25) and (26), the standard deviations are introduced in order to clearly make the connection to the r-values \( r_{42} \) and \( r_{13} \). From the assumption from equation (21), the standard deviation terms cancel out, suggesting that that only the covariance needs to be calculated in order to make equally accurate calculations. This variation is called the Reduced Two Reference Image Method (R2-RIM). This algorithm does not actually rely on the second assumption and in addition to certainly being faster, it should also be more accurate.

3.4.5 Other Considerations

2-RIM and R2-RIM hinge on an assumption that above went unevaluated. To determine the effect of the approximation, the uncertainty that is introduced as a result is computed. The assumption in equation (20) can be rewritten as:

\[
\frac{\sigma_{I_i}}{\sigma_{I_1}} \approx 1 \rightarrow \frac{\sigma_{I_i}}{\sigma_{I_1}} = 1 + \delta_i
\]

where \( \delta_i \) is the error in the approximation. Note that \( \delta_1 = 0 \). It follows that the error introduced as a result of the approximation is:
\[
\frac{\delta_R}{R} = \frac{\text{cov}(I_4,I)\,\delta_4}{\sigma_4} + \frac{\text{cov}(I_2,I)\,\delta_2}{\sigma_2} + \frac{\text{cov}(I_3,I)\,\delta_3}{\sigma_3}
\]

(28)

where R is the ratio \((r_4-r_2)/(r_1-r_3)\). For a complete derivation of this formula, consult the Appendix.

In addition to the application of these algorithms, some investigation has gone into preparing the data in the ESPI images to work more effectively with the algorithms. This consisted of two procedures; 1. Subtracting off the average pixel intensities; 2. After procedure 1, weight the data favouring pixels with high modulation.

The motivation for conditioning the data as per the first procedure is the standard deviation term:

\[
\sigma = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}
\]

(29)

The standard deviation depends on the average of the intensities of pixels in a local cluster which changes as different clusters are considered. This average is not to be confused with the average intensity of an individual pixel (i.e. the A term in equation (8)) around which phase changes cause modulation of intensity. Figure 18 shows two neighbouring clusters, indicated by colours red and green. The difference between average intensities of each cluster is caused only by the pixels not in common to both groups. The deviation from these averages, however, would ideally be the same for both
calculations. Due to unevenness of the object illumination, this is not case. By subtracting off the average intensity on a pixel by pixel basis (i.e A), the collective average of the pixels is consistently near 0. More importantly, the variation from cluster to cluster is considerably lower and the intensity deviation from the average of the common pixels is more consistent from one cluster to the next. The result is an expected reduction of noise from the calculation of \( r_1 \) to \( r_2 \).

![Figure 18: Neighbouring 5x5 pixel clusters for calculation of \( r \)-values](image)

The second procedure is more intuitive. Data weighting is commonplace when some of the data within a group is considered more reliable than other data. In this case, pixels with high modulation (i.e. the B term in equation (8)) are considered the highly reliable data and provide the basis for establishing weights.
3.5 Interpolated Reference Images

In conventional ESPI, reference images are explicitly acquired which provide a standard against which displacements can be measured. To eliminate the need to explicitly acquire static reference images, some new standard would need to be established. Here, the first image measured in the 512 dynamic ESPI images will act as that standard and be used as the initial reference images, leaving the quadrature images to be extracted from the rest of the set.

In actuality, only one quadrature needs to be constructed. The third quadrature image has a phase step of \( \pi \) from the first, as does the fourth from the second. To get the third and fourth images then, the pixel data from the first and second images are simply inverted about the average intensity \( (A) \) – which can be acquired by averaging all 512 images.

The problem is therefore reduced to a matter of establishing the second quadrature image – phase stepped \( \pi/2 \) from the first – by interpolating data from the 512 dynamic images. Consider a local region on the plate, perhaps the red box in Figure 19.a). When sufficiently large vibrations occur (at least \( \pi/2 \) everywhere across the plate), then an image must exist where the local phase difference is just \( \pi/2 \). The image can be identified by correlating the data from each of the 512 image in that region to the data from the initial image. An r-value of 0 indicates a phase difference of +/- \( \pi/2 \). Plotting the r-values
might produce a result as in Figure 19.b). Interpolation of data from images around the zero crossing may be helpful.

![Diagram showing burst of images and r-values showing cross-over point](image)

*Figure 19: a) Burst of images and b) r-values showing cross-over point*

This procedure is repeated for the next box over (the green box in Figure 19.a) with the goal of building the quadrature image in a composite fashion. While it does not matter if this quadrature image is shifted $+\pi/2$ or $-\pi/2$ from the first as a whole, each piece added to the image must agree with previously established data. For the first piece of data added, the sign will arbitrarily be chosen as “positive” $\pi/2$. This may not be the case in actuality. From there, all the new data to be included will also have to agree in sign with the original choice. This is achieved by correlating overlapping data from adjacent boxes. If they do agree in sign, then the overlapping data will be nearly identical.
and a correlation coefficient of near +1 is computed. The new data are then added to the quadrature image as is. In the case they do not agree in sign, then the overlapping data must be in complete anti-phase, and a correlation coefficient is near -1 is computed. The new data are simply inverted about its pixel average intensity (A) before inclusion in the quadrature image.

In order to ensure large enough displacements at the clamping site (where no actual displacement is expected), a ramp signal is sent to the PZT during the acquisition of the sequence of images. This signal need not be calibrated particularly accurately. As a result of the ramp, however, a low frequency peak is expected in the Fourier spectra.

### 3.6 Summary

A class of algorithms which explores variations on a theme (i.e. use of correlation coefficient) to calculate phase difference between a single ESPI fringe image and a set of phase-stepped reference images – both explicitly taken and interpolated from within the set of fringe images - has been presented. The successful application of this class bridges the gap left open in Chapter 2. In Chapter 4 – Experimental Validation, experiments are described which evaluate the effectiveness of these algorithms. The uncertainty introduced as a result of the first approximation will also be evaluated to
ensure reasonable behaviour. Then the algorithms will be employed as part of a larger modal analysis experiment, in order to demonstrate how the research presented here is useful in practice and how this proposed modal analysis procedure compares against standards defined by existing methods.
Chapter 4 – Experimental Validation

4.1 Experiments Overview

Two experiments are performed to validate the theory that has been presented. The first evaluates the effectiveness of the algorithm class by using hole-drilling data from a standard ESPI apparatus. The variations of the algorithms will be compared against each other as well as the classical 4-step phase-stepping method.

The second experiment is intended to demonstrate a practical modal analysis test by building an apparatus and using the novel algorithms to process the data. Using the same apparatus, mode shapes will be acquired using the Chladni method and Time-Average Interferometry as described in Chapter 1 – Introduction to provide a suitable comparison. It is the intention here to show that this proposed procedure offers a more complete and quantitative measurement of the plate’s vibrations than existing methods and that this is made possible by the success of the developed algorithms.
4.2 Experiment I: Algorithm Performance

In order to test the algorithms, ESPI images are acquired from a Michelson interferometer, configured to measure in-plane surface displacements in a laboratory procedure for determining residual stresses by the hole-drilling method [29], [18], [30]. The method by which ESPI images were obtained and their function are not important for these purposes, but rather the acquisition of representative ESPI images was the goal.

Four ESPI reference images were acquired, phase-stepped at $\pi/2$ intervals, followed by a surface deformation and the subsequent acquisition of four ESPI fringe images, also stepped at $\pi/2$ intervals. The surface deformations are caused by the release of residual stresses in the test object by drilling a small hole. In Figure 20 as well as subsequent figures, a white circle has been superimposed over the drilling site because the local surface was removed during drilling and thus all the corresponding pixels are entirely uncorrelated [14].
4.2.1 Comparison of Algorithm Variations

Figure 20a) shows the ESPI fringe pattern evaluated using the classical phase angle procedure, using four $\pi/2$ stepped reference images and four $\pi/2$ stepped fringe images. The grainy texture of the image is typical of ESPI fringe patterns. It is caused by both the random nature of the speckle pattern, and the fact that phase is calculated on a pixel by pixel basis.
Figure 20.b) is produced using the 4-RIM variation. Here, a single fringe image (the first in sequence of four) is used a correlated using a 9x9 box size against each of the four reference images. It can be seen that the fringe pattern closely resembles the solution calculated using the classical 4-image phase-stepping method, but with a coarser grain. This occurs because individual pixels affect their neighbouring pixels in the correlation computation, whereas in the four-step method, they behave entirely independent of one another. Accordingly, there is a spatial smoothing effect occurring and erroneous pixels tend to regress to the correct solution.

Figure 20.c) shows the solution calculated the 2-RIM variation, again using a box size of 9x9 pixels as in Figure 20.b). The quality is somewhat poorer as is to be expected since the 2-RIM is achievable through an approximation that inherently introduces some error. On the other hand, this method is considerably faster than the 4-RIM variation.

Finally, Figure 20.d) shows results from the yet faster R2-RIM variation. There is no obvious reduction in quality from the 2-RIM method. The generally good quality of the (R)2-RIM variations suggests that the approximations used to develop these algorithms are acceptable.

The new algorithms are expected to take longer to compute than the 4-step method, since they involve the summation of several summed quantities (i.e. covariance, standard deviation) within a box around each pixel. By contrast, the 4-step method considers quantities on a pixel by pixel basis only. Through careful programming,
however, intermediate sums can be stored such that data from overlapping boxes in the computation need not be recalculated. Rather, only non-overlapping data are introduced. The result is a process that is nearly independent of box dimensions and competitive with the 4-step method in terms of computation time. Using a typical 2009 laptop computer, the approximate computation times for the algorithms used to generated the fringe patterns in Figure 20.c-d) are 0.4s, 0.3s, and 0.2s respectively, compared to 0.1s for the 4-step method shown in Figure 20.a).

4.2.2 Fringe Pattern Quality and Computation Time

The textures of Figure 20.b-d) are of considerably courser texture than Figure 20.a). This is again due to the effect that pixel data have on their neighbours as it gets considered in the correlation computations. The use of smaller box sizes produce increasingly fine textures, approaching a result that resembles the appearance from the 4-step method.

Figure 21 illustrates the effect of box size on image quality. The figure combines horizontal sub-sections of fringe patterns computed using a variety of box sizes, starting with 2x2 at the top and reaching 21x21 at the bottom. An increased spatial smoothing effect is noticed which is the by-product of correlation using bigger and bigger boxes. Noise from pixel to neighbouring pixel is reduced at the expense of decreased spatial
resolution. It is therefore imperative that the dimensions of the box are kept well below the width of any fringe to be resolved. This is in accordance with the assumption that all pixels within a box undergo approximately the same phase difference. Even within these guidelines, however, there is still some leeway. The fringes in Figure 21 are easily recognizable for the range of box sizes shown even when the box size is comparable to the fringe spacing.

![Image Quality of R2-RIM Versus Box Size](image)

**Figure 21**: Effect of box size on Fringe Pattern Quality. Several fringe patterns computed using the R2-RIM and varying box sizes are spliced together to show the spatial smoothing dependence on box size. Reprinted with permission of Elsevier
The data in the ESPI images can be treated to reduce the effect of measurement noise through procedures outlined in Chapter 3. The first treatment is to subtract the phase-average intensity (A) across the image, plotted in Figure 22.a). The second treatment involves weighting the data using the modulation (B), plotted in Figure 3.b). The fact that neither (A) nor (B) are uniform, and in fact quite noisy provides some justification for the two treatments. It should be noted that the triangular region in the upper right corner of each field was corrupted during experimentation and should be ignored. The remainder of the image is unaffected.

![Figure 22: a) The Phase-Average Intensity and b) Modulation. Reprinted with permission of Elsevier](image)

The effect of subtracting off the average intensity (A) can be seen by comparing the treatment to an untreated fringe image. Reference images need not undergo the
subtraction explicitly in the (R)2-RIM variations, since the average intensity is removed when the image pairs are mutually subtracted. Figure 23 shows the results obtained by treating only one half of a fringe image. The left is the untreated side and is of noticeably inferior quality to the right. The improved quality of the fringe pattern having undergone (A) subtraction confirms its effectiveness. Indeed, the fringe images in Figure 20 have also been treated this way, helping to explain their good quality.

Figure 23: A fringe pattern with average intensity subtracted from the data on the right half. Noticeable image quality improvement is observed. A 9x9 box size is used. Reprinted with permission of Elsevier
As with Figure 23, the effect of B modulation weighting is presented in Figure 24. The expectations were that weighting the data to favour pixels with high modulation (i.e. well-behaved pixels) would help compensate for poorly behaved pixels. As seen in Figure 24, however, the weighting does little to enhance the image quality. The process of computing and implementing the weighting also nearly doubles the computation time. With nearly no improvement of image quality and the dramatic increase of computation time, this is not considered an effective way of reducing noise.

Figure 24: A fringe pattern with modulation weighting to the data on the right half. Little or no image quality improvement is observed. A 9x9 box size is used. Reprinted with permission of Elsevier
4.2.3 Error Estimates

To quantify the effectiveness of conditioning the images as well as verifying the soundness of the assumptions made in the algorithm derivations, the uncertainties (defined in equation (27)) are calculated the conventional way using the formula in equation (28). The uncertainty refers to the error introduced as a result of implementing the approximations to use the R2-RIM algorithm. Figure 25 is a plot of the computed errors, and three trends are visible. First, the correlation coefficient uncertainty decreases with box size. Second, subtracting off the average intensity greatly reduces the uncertainty rather significantly. Finally, there is little improvement due to the additional implementation of modulation weighting, as expected. These tendencies along with empirically gathered ESPI fringe images help justify the effectiveness of the approximations.
Keeping the uncertainty low produces quality fringe patterns, and this is very important for the processing of the fringe patterns for modal analyses. Certainly, unwrapping algorithms can be fickle, and poorly constructed fringe patterns are unlikely to be unwrapped without error. If the deformation maps are consistently damaged, the chances that clear mode shapes can be extracted quickly diminish. In addition, even with successfully unwrapped deformation maps, noisy individual pixel data will present as undesirable frequency content in the Fourier spectra. Since vibrational modes present as
frequency peaks in the spectra, excess noise can make the identification of modes less certain.

4.3 Experiment II: Modal Analysis

In this second experiment, a modal analysis test on a rectangular plate is demonstrated. The objective is to not only prove that this procedure works, but that it compares favourably to other modal analysis procedures. The results are furthermore intended to demonstrate the practicality of this method and the significance of the developed algorithms. Results from trials using both explicitly acquired reference images as well interpolated reference images are presented.

The steps, whose details are explored in depth in Chapter 2 and Chapter 3 are:

1. Acquisition of four phase stepped reference images
2. Excitation of the test plate via acoustic white noise
3. Acquisition of 512 ESPI fringe images taken at 650 frames per second
   a. Using explicitly acquire reference images
   b. Using interpolated reference images
4. Phase determination through the use of the novel algorithm class
5. Phase unwrapping to construct true displacement maps
6. Fourier analysis to determine resonant mode frequencies and shapes
In independent experiments, mode shapes are acquired using the Chladni method and Time-Averaged Interferometry to provide suitable comparison.

### 4.3.1 Modal Analysis Apparatus

A modified Michelson interferometer is illuminated using a coherent laser source (50mW, λ=532nm) as shown in Figure 26.b). A high-speed CCD camera (Allied Vision Technologies GE 680; resolution: 640x480 pixels, frame rate: 650fps with 4x1 binning) is used to image the interferograms. Operating at a frame rate of 650 fps, vibrational modes with frequencies up to 325 Hz can be measured [15]. The interference itself is created from the diffuse reflections of the test surface and piezo-actuated reference plate. Both are painted matte white to reflect the greatest amount of light. The whole apparatus is mounted on a floating optical table to provide immunity from external disturbances.

![Figure 26: a) conventional Michelson interferometer and b) modified Michelson interferometer](image)
The test plate studied in these experiments is made of brass and has dimensions of 170 x 100 x 0.92 mm. This plate has seven vibrational modes with frequencies of 325 Hz and lower. Vibrations are induced acoustically using a loudspeaker with white noise input. Other excitation methods such as piezo-electric and electromagnetic could also be suitable. Because the plate is made of brass and therefore non-magnetic, acoustic excitation is the convenient non-contacting choice. Furthermore, electromagnetic and piezo-electric excitation are limiting in that only one point is targeted. As such, the point needs to be selected to avoid the nodal lines of the mode shapes. If this condition is not met, those particular mode shapes will either be excited poorly or not at all. Acoustic excitation overcomes this problem by exciting an extended area. In this apparatus, the speaker occupies an area of about 10% of the plate, and is located in the upper left corner. Knowing that only frequencies up to 325 Hz are measurable by the interferometer, a white noise signal was produced using MATLAB software with a cut-off frequency of 500 Hz to avoid aliasing errors. The lower half of the white noise frequency band was amplified in order to balance the amplitudes of the modes. While not strictly necessary for general measurements, this proved to be helpful for these demonstrational purposes. The presence of frequencies past 500 Hz in the white noise only serves to introduce unwanted noise in the measurements.

Modifications to the conventional interferometer design were also to optimize the precision of the instrument. The modifications are shown by the differences in Figure 26.
a) and b). The conventional Michelson configuration proved to be ineffective due to the escaping of light within the system. Because of the diffuse character of the surface reflections from the plate specimen, less than 1% of the emitted laser light returns to the camera. Under these conditions, the intensity of the light from internal reflections from the beam splitter in Figure 26.a) is of the same order as the intensity of light coming back from the test surface. The reflected light seriously contaminates measurements of the desired light returning from the test plate. This artifact is overcame by the use of two beam splitters to enable outward and returning light beams to pass through separate optical surfaces and therefore not interact. A second issue when using a single beam splitter is that it does not permit balancing of the reflected light intensities from the test specimen and the reference plate. Some deviation from this ratio is permissible, but the modulation suffers as can be shown from equation (7). The separate beam splitters shown in Figure 26.b) allow the intensities of the two reflected light beams to be balanced. It was found that transmission/reflection ratios of 65/32 and 10/90 for Beam Splitters 1 and 2 respectively were appropriate.

### 4.3.2 Mode Shape Recovery

The first step to recovering the mode shapes once all the ESPI images are acquired is the calculation of phase difference for each of the 512 ESPI fringe images in the burst. The test parameters are presented below in Table 2. Figure 27 is an example
fringe image from the burst, calculated using the R2-RIM variation using the explicitly acquired reference images.

Table 2: Test parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exposure</td>
<td>0.3 ms</td>
</tr>
<tr>
<td>Intensity Gain</td>
<td>x4</td>
</tr>
<tr>
<td>Box Size</td>
<td>17 x 17</td>
</tr>
<tr>
<td>Frames</td>
<td>512</td>
</tr>
<tr>
<td>Frame Rate</td>
<td>650 fps</td>
</tr>
<tr>
<td>Binning</td>
<td>4 x 1</td>
</tr>
<tr>
<td>Algorithm</td>
<td>R2-RIM</td>
</tr>
</tbody>
</table>

Figure 27: Example Fringe Image from R2-RIM Box size 17
Once the phase maps are constructed for all the images, the unwrapping algorithm is applied and the result is displacement versus time data for each pixel in the CCD array [21]. A representative group of points across the surface (~100) is selected and the Fourier spectrum acquired and averaged to find the plate natural frequencies. This spectrum is shown in Figure 28.

![Fourier Analysis of Displacement Data](image)

*Figure 28: Average Fourier Spectrum of Selected Points*

This spectrum suggests the presence of seven vibrational modes. With the resonant frequencies now identified, equation (1) is applied to the data from each pixel in order to construct the mode shapes. These seven mode shapes are shown in Figure 29. The coloured diagrams show both the shape and sign of the vibrations for each of the mode shapes.
Figure 29 a-g): Recovered mode shapes using the proposed method with explicit reference image at frequencies of 48 Hz, 67 Hz, 82 Hz, 96 Hz, 204 Hz, 264 Hz, and 301 Hz respectively (Georgas, Schajer [31])
A similar experiment was performed, but here the mode shapes were calculated using both explicitly acquired reference images and interpolated reference images. Figure 30 shows a sample fringe patterns extracted using explicitly measured reference images versus reference images acquired through interpolation. The fringe patterns are essentially identical apart from a few local artifacts. While these artifacts can at times be tolerated, their frequent occurrence will lead to erroneous results. Further work should be done in order to increase robustness and hopefully eliminate these artifacts altogether. Corresponding Fourier spectra and recovered mode shapes are shown in Figure 31 and Figure 32 respectively.

Figure 30: Fringe pattern created using a) explicit reference images and b) interpolated reference images
**Figure 31:** Fourier Spectrum calculated from explicit reference images versus interpolated reference images

**Figure 32 a-l:** Mode shapes calculated from explicit reference images versus interpolated reference images. The images are labeled at 48 Hz, 67 Hz, 81 Hz, 96 Hz, 264 Hz, and 301 Hz respectively.
4.3.2 Comparison with Other Methods

The mode shapes recovered using the proposed method can now be compared with results obtained from other modal analysis techniques. Figure 33 shows a set of results acquired from the same test plate, but using Time-Average Interferometry. The diagrams show similar mode shapes to those in Figure 29 and Figure 32, but characteristically lack the sign information.

*Figure 33 a-g): Mode shapes recovered using Time-Averaged Interferometry at frequencies of 48 Hz, 67 Hz, 82 Hz, 96 Hz, 204 Hz, 264 Hz, and 301 Hz respectively*
Mode shapes obtained using the Chladni method offer no more information than TAI, and often less, since only the nodal lines are revealed. Presented here is just the 301 Hz vibrational mode, corresponding to mode shape g) in both Figure 29 and Figure 33.

![Figure 34: Chladni mode shape at 301 Hz using pepper](image)

**4.4 Discussion**

The success of the proposed phase evaluation algorithms is evidenced both by results from example ESPI images from hole-drilling measurements and from the modal analysis test. It is clear that results can be obtained which rival the classical 4-step method in both quality and computation time, and through the use of significantly less
data. Indeed, the results from the modal analysis would not be possible without exceptional performance of the algorithms used to recover the phase difference.

Comparing the mode shapes obtained from the proposed method in Figure 29 a-g) to those from Chladni in Figure 34 and Time-Averaged Interferometry in Figure 33 a-g) reveals the effectiveness of the proposed method. All the mode shapes contain quantitative data, in contrast to the qualitative data seen in both Chladni and TAI. Significantly, no human interpretation is required to process the mode shapes in the proposed method to achieve quantitative results.

Next, examining in particular the modes in Figure 33 a,b,c,e), it can be observed that mode a) resembles mode b) and to a lesser extent mode c) resembles mode e). This false resemblance occurs because sign information is lost in TAI calculations. The same is true in the Chladni patterns. In Figure 29 a,b,c,e), however, the differences are clearly displayed. Only in these results is it apparent that the modes in a) and b) are different – in a) opposite ends of the plate are out of phase and in b) they are in phase. The same argument applies to the modes portrayed in c) and e). The retention of sign in the proposed method is one of the distinct advantages. The sign indicates which regions of the plate are vibrating in and out of phase with respect to others.

One final advantage is that all the modes in the proposed method were acquired simultaneously. Neither a-priori information nor a frequency search is necessary as is the case with Chladni and TAI, where individual modes need to be excited.
The remaining two features that are outlined in Table 1 (i.e. non-contactivity and full-fieldness) are represented in the proposed method as well as Chladni and TAI. The advantages over the IH and LDV, whose results are not presented, are also outlined in Table 1.

Modes extracted from interpolated reference images, shown Figure 32 g-1) also offer very promising results. The quality of both fringe patterns and mode shapes are comparable to the case where reference images are explicitly measured. The algorithm demonstrates reasonably good immunity to erroneous pixels. While the fringe pattern in Figure 30.b) does have a few errors in the middle of the image, they are ultimately not fatal. The statistical nature of the algorithm provides some flexibility, albeit less than when explicit reference images are taken. More importantly, the extracted mode shapes are of extremely high quality. The absence of the fifth mode in the sequence is attributable to the difficulty of balancing the amplitudes of the modes during white noise excitation. Since the vibrational amplitudes are low, modes occasionally fail to excite to a degree that is detectable.

The computation time for the interpolated reference image method is approximately twice as long as the explicit method. The development of the interpolated reference image method is relatively new, however, and has not yet been optimized for speed. While it will always take longer than the explicit method, with some work, it is expected that the interpolated method will not be prohibitively more time consuming.
4.4.1 Image Quality Discussion

Examining the mode shapes extracted using the proposed method, an unusual vertical texture is frequently observed. This is caused by binning pixels. With the camera used, the frame rate could be increased from 205fps to 650 fps by applying a vertical binning of x4. After computation of the mode shape, the map needs to be stretched to fit the actual dimensions of the test plate. To see this, another experiment was performed using 2x2 binning, and a frame rate of 350 fps. Only four modes are extractable in this case and are shown below in Figure 35. Because the binning is the same in both the horizontal and vertical directions, the mode shape is stretched equally in both directions and the vertical texture seen before disappears.

Figure 35: Modal shapes extracted using 2x2 binning and frame rate of 350 fps
The Fourier spectra and the mode shapes presented in the earlier modal analysis test exhibit considerable noise. This is in part due to the low amplitude vibrations which negatively affects the signal to noise ratio. The amplitudes are restricted to just a few wavelengths, but the underlying problem is velocity of the plate. Experiments can be performed which excite just individual modes, in which case higher amplitudes and relatively noise free data is observed.

Figure 36: Single Mode Excitation trials

In Figure 36, two trials are shown: 48 Hz mode excitation and 301 Hz mode excitation. The 48 Hz mode trial exhibits very large amplitudes and exceptionally clean data. The 96 Hz mode was harmonically excited by the 48 Hz, owing to its presence in
the spectrum. In contrast, the largest amplitudes that could be cleanly computed from the much faster 301 Hz mode are only a couple wavelengths as before. The data is still relatively clean because white noise is not being fed into the system.

This experiment goes to show the effect that increasing the signal to noise ratio will have on data quality.
Chapter 5 – Conclusion

5.1 Research Summary and Contributions

A new class of algorithms has been developed which enable the phase determination from a single ESPI image rather than the traditional use of four. While some loss of image quality occurs, as is to be expected, the reduction is not prohibitive and has proven to be effective when applied to experimental data. The reconstruction from a single fringe image makes the algorithm particularly well suited to measuring dynamic surfaces – a property for which other algorithms tend to fall short.

The successes of the new algorithms have permitted the development of a new modal analysis method which features improvements over established methods. This method combines full-field capabilities of Chladni and Time Average Interferometry while improving on these by providing quantitative measurements and avoidance of need for resonant excitation. Like both Instrumented Hammer and Laser Doppler Vibrometry, the modes are measured simultaneous. In the proposed method, however, the spatial resolution is far higher. The extracted mode shapes compare well with independent measurements taken using the Chladni method and TAI. Importantly, it has been shown that modal analysis tests can be performed \textit{in situ} without needing to turn off the test object, making the system a highly practical alternative to currently available methods.
This is also done with little or no loss of quality compared to the case where reference images are explicitly acquired. This is therefore the recommended procedure.

5.2 Remaining Challenges and Limitations

There are currently some limitations with the proposed method. One such limitation is that the displacement amplitudes which the system can resolve are relatively small (i.e. micron scale). This leaves the system susceptible to disturbances. Improving the apparatus such that higher amplitudes can be tolerated will provide some immunity from these disturbances by increasing the signal to noise ratio. The algorithms are capable of handling higher amplitudes, but rather this issue more closely depends on the availability of laser light.

A more powerful laser would have many positive implications for this method. First, the exposure time can be decreased. Long exposure times cause problems similar to phase step latency, whereby the plate undergoes such a large displacement during the exposure that the data become corrupted. A more powerful laser is also capable of illuminating a larger test object, extending the functionality of this method. The higher laser power can be a safety concern and appropriate working procedures must be used.

A second limitation which has been discussed throughout this thesis is the frame rate of the camera. The measurable modes are those with frequencies up to half the
sampling frequency (i.e. frame rate). Accordingly, as the frame rate increases, the number of measurable modes increases. A high frame rate will likely assist the interpolated reference image method even greater than the explicit reference image method. This is because the interpolated reference image method needs to actively search for its reference images from within burst fringe images. When more samples are taken per vibration period for a given mode, the interpolations required to produce the second π/2-stepped reference image become increasingly small, and accordingly their effect nears negligibility as desired.

In addition to increased frame rate, a camera with greater resolution would also improve performance. Increased resolution offers many advantages. The increased spatial resolution ensures that speckles are larger than pixels, leading to better image quality. This is often violated in the experiments presented in this thesis. Furthermore, with both increased resolution and a specialized high speed camera, binning will no longer be necessary, and the unusual vertical line textures in the mode shapes will be eliminated.
5.3 Recommendations for Future Work

5.3.1 Improved Interpolated Reference Images

There still remain some features of the interpolated reference image method which leave room for improvement. Notably, because the successful phase determination of one part of the image relies on earlier computed areas in the image, the algorithm is less robust than when reference images are explicitly taken. Additional work on the algorithms could provide some immunity from erroneous data, and this could provide considerably more robustness. This will likely be an important step when more powerful hardware is used, and amplitudes of greater magnitude are involved.

In addition, the algorithm has not yet been optimized for computation time. Because correlation computations are required across the field of the image, similar measures may be taken to the ones used in the phase determination algorithms. The storing of intermediate sums drastically increases the speed, extending the practicality of this method.

5.3.2 Damping Ratio

In addition to frequency and mode shape, another important parameter in vibration measurements is the damping ratio. This quantity describes how vibrations...
decay. This can be found directly from the Fourier spectra, but at this moment, noise makes these measurements unreliable. Further improvements in the apparatus and algorithms may make this possible.
References


Appendix – Uncertainty Derivation

First, let

\[ R \equiv \frac{r_3 - r_2}{r_1 - r_3} = \frac{\text{cov}(l_4, f)}{\sigma_{l_4} \sigma_f} - \frac{\text{cov}(l_2, f)}{\sigma_{l_2} \sigma_f} \]

Next, the assumption is applied which introduces error:

\[ R + \delta_R = \frac{\text{cov}(l_4, f) \sigma_{l_3}}{\sigma_{l_4} \sigma_f} - \frac{\text{cov}(l_2, f) \sigma_{l_3}}{\sigma_{l_2} \sigma_f} \]

The introduced term can be written as:

\[ R + \delta_R = \frac{\text{cov}(l_4, f)}{\sigma_{l_4} \sigma_f} \left(1 + \delta_4\right) - \frac{\text{cov}(l_2, f)}{\sigma_{l_2} \sigma_f} \left(1 + \delta_2\right) \]

Canceling out \( \sigma_3 \) and collecting the error terms in brackets:

\[ R + \delta_R = \frac{\text{cov}(l_4, f)}{\sigma_{l_4} \sigma_f} - \frac{\text{cov}(l_2, f)}{\sigma_{l_2} \sigma_f} + \left(\delta_4 \frac{\text{cov}(l_4, f)}{\sigma_{l_4}} + \delta_2 \frac{\text{cov}(l_2, f)}{\sigma_{l_2}}\right) \]

The uncertainty by percent is then:

\[ \delta_R = \frac{\text{cov}(l_4, f)}{\sigma_4} \delta_4 + \frac{\text{cov}(l_2, f)}{\sigma_2} \delta_2 + \frac{\text{cov}(l_3, f)}{\sigma_3} \delta_3 \]