

On global properties of solutions of some nonlinear Schrödinger-type equations

by

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Abstract

The Schrödinger equation, an equation central to quantum mechanics, is a dispersive equation which means, very roughly speaking, that its solutions have a wave-like nature, and spread out over time. In this thesis, we will consider global behaviour of solutions of two nonlinear variations of the Schrödinger equation.

In particular, we consider the nonlinear magnetic Schrödinger equation for $u : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$,

$$iu_t = (i\nabla + A)^2 u + Vu + g(u), \quad u(x, 0) = u_0(x),$$

where $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the magnetic potential, $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the electric potential, and $g = \pm|u|^2 u$ is the nonlinear term. We show that under suitable assumptions on the electric and magnetic potentials, if the initial data is small enough in H^1 , then the solution of the above equation decomposes uniquely into a standing wave part, which converges as $t \rightarrow \infty$, and a dispersive part, which scatters.

We also consider the Schrödinger map equation

$$\vec{u}_t = \vec{u} \times \Delta \vec{u}$$

for $\vec{u} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{S}^2$. We obtain a global well-posedness result for this equation with radially symmetric initial data without any size restriction on the initial data. Our technique involves translating the Schrödinger map equation into a cubic, non-local Schrödinger equation via the generalized Hasimoto transform. There, we also show global well-posedness for the non-local Schrödinger equation with radially-symmetric initial data in the critical space $L^2(\mathbb{R}^2)$, using the framework of Kenig-Merle and Killip-Tao-Visan.

Preface

A version of Chapter 2 has been published. Eva Koo, Asymptotic stability of small solitary waves for nonlinear Schrödinger equations with electromagnetic potential in \mathbb{R}^3 , J. Differential Equations 250 (April 2011), no. 8, 3473-3503. I wrote the entire manuscript.

A version of Chapter 3 has been submitted for publication. This is joint-work with my supervisor Stephen Gustafson. Theorem 4 (in particular, the ruling out of possible blow up solutions) is a result of close collaboration with my supervisor. Many of the details were worked out jointly. I was responsible for adapting proofs from previous work (for example, [48] Killip-Tao-Visan 09) to our situation, and I wrote the first draft of the manuscript.

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Dedication

To all my teachers

Chapter 1

Introduction

The dynamics in many physical settings – for example, those involving the propagation of light or sound, or the evolution of quantum systems – are well-described by dispersive partial differential equations. Because of their importance, and the subtle properties of their solutions, the mathematical study of dispersive equations has attracted a lot of attention. In Section 1.1, as an introduction to the subject, we will give a relatively mild introduction to dispersive equations and dispersive effects. We will also introduce a few examples of dispersive equations that will be of interest to us. In Section 1.2, we will introduce and discuss some well-studied questions in the field of dispersive equations. In Section 1.3, we will give an overview of the rest of the thesis.

1.1 Dispersive effects and dispersive equations

1.1.1 A gentle introduction to dispersive effects

To illustrate the dispersive property, we will consider three well-studied partial differential equations of which only one is truly dispersive. In each of the three equations, let $u = u(x, t)$ be a function of space variable $x \in \mathbb{R}^n$ and time variable $t \in [0, \infty)$. We use u_t to denote the partial time derivative of u (i.e. $u_t = \frac{\partial u}{\partial t}$) and Δu to denote the Laplacian of u which is given by

$$\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}. \quad (1.1)$$

These equations are the **heat equation**

$$u_t = \Delta u, \quad (1.2)$$

the **wave equation**

$$u_{tt} = \Delta u \quad (1.3)$$

and the **Schrödinger equation**

$$u_t = i\Delta u \quad (1.4)$$

where in each case above, we will consider the function

$$u(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{C} \quad (1.5)$$

with initial data

$$u(x, 0) = u_0(x) \text{ and for the wave equation } u_t(x, 0) = v(x) \text{ as well.} \quad (1.6)$$

To keep this discussion concrete, we will limit the space dimension to $n = 1$. In this case, the heat equation models the temperature of a thin rod of infinite length, the wave equation models the height of a wave on an infinitely long string and the Schrödinger equation models a free quantum particle in one dimensional space. To gain some insight on the behaviour of solutions of such equations, we will apply the Fourier transform in the variable x to each equation. Let

$$\widehat{u}(\xi, t) := \frac{1}{2\pi} \int_{\mathbb{R}} u(x, t) e^{-ix\xi} dx \quad (1.7)$$

denote the spatial-domain Fourier transform of $u(x, t)$. Using the property

$$\widehat{u_{xx}}(\xi, t) = -\xi^2 \widehat{u}(\xi, t), \quad (1.8)$$

the equations now read

heat equation:

$$\widehat{u}_t(\xi, t) = -\xi^2 \widehat{u}(\xi, t) \quad (1.9)$$

wave equation:

$$\widehat{u}_{tt}(\xi, t) = -\xi^2 \widehat{u}(\xi, t) \quad (1.10)$$

Schrödinger equation:

$$\widehat{u}_t(\xi, t) = -i\xi^2 \widehat{u}(\xi, t). \quad (1.11)$$

Each of the equations above is an ordinary differential equation in t , and after solving them, we get

heat equation:

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0)e^{-\xi^2 t} \quad (1.12)$$

wave equation:

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0) \cos(\xi t) + \frac{\widehat{u}_t(\xi, 0)}{\xi} \sin(\xi t) \quad (1.13)$$

Schrödinger equation:

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0)e^{-i\xi^2 t}. \quad (1.14)$$

We can already read off some differences between the behaviours of the three equations from the above. For example, for the heat equation, the magnitude of each Fourier mode $|\widehat{u}(\xi, t)|$ is exponentially decreasing, while for the Schrödinger equation, the magnitude of each Fourier mode $|\widehat{u}(\xi, t)|$ is fixed in size. By Parseval's theorem, which says that

$$\|u\|_{L_x^2} := \left(\int_{\mathbb{R}} |u(x, t)|^2 dx \right)^{\frac{1}{2}} = \sqrt{2\pi} \|\widehat{u}\|_{L_\xi^2}, \quad (1.15)$$

we see that for the heat equation, the L_x^2 -norm of u is decaying in time while for the Schrödinger equation, the L_x^2 -norm of u stays constant. In fact, the heat equation is an example of a **dissipative** equation, which means that as

time evolves, the solution dies down (dissipates) to 0. On the other hand, we see that the Schrödinger equation does not dissipate. By Parseval's theorem, the L^2 -norm of the solution remains constant.

To differentiate the behaviour of the wave equation and the Schrödinger equation we can apply the inverse Fourier transform and recover $u(x, t)$ from $\widehat{u}(\xi, t)$ by

$$u(x, t) = \int_{\mathbb{R}} \widehat{u}(\xi, t) e^{ix\xi} d\xi. \quad (1.16)$$

For the sake of discussion, suppose the initial data of the wave equation are chosen so that $\widehat{u}(\xi, 0) = \delta_{\xi_0}(\xi)$ and $\widehat{u}_t(\xi, 0) = 0$ for some fixed frequency $\xi_0 \in \mathbb{R}$. Here, $\delta_{\xi_0}(\xi) = \delta(\xi - \xi_0)$ is the delta function. Then \widehat{u} is given by

$$\widehat{u}(\xi, t) = \frac{1}{2} \left(\delta_{\xi_0}(\xi) e^{i\xi_0 t} + \delta_{\xi_0}(\xi) e^{-i\xi_0 t} \right). \quad (1.17)$$

Hence, the solution u of the wave equation is given by

$$u(x, t) = \int_{\mathbb{R}} \widehat{u}(\xi, t) e^{ix\xi} d\xi = \frac{1}{2} \left(e^{i\xi_0(x+t)} + e^{i\xi_0(x-t)} \right). \quad (1.18)$$

This is a sum of two plane waves, one moving to the left at speed 1 and one moving to the right at speed 1. In fact, for general initial data $u(x, 0) = u_0(x)$ and $u_t(x, 0) = v_0(x)$, the solution of the wave equation is given by the d'Alembert formula

$$u(x, t) = \frac{1}{2} (u_0(x+t) + u_0(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy. \quad (1.19)$$

In the case where $v_0 \equiv 0$, the solution of the wave equation is

$$u(x, t) = \frac{1}{2} (u_0(x+t) + u_0(x-t)). \quad (1.20)$$

In particular, we see that as time evolves, the initial shape of the wave divides into two equal parts, where one part moves to the right and the other moves to the left at an uniform speed. This is the wave behaviour in which the original waveform move outwards at a uniform speed.

We will see next that the behaviour of the Schrödinger equation is dif-

1.1. Dispersive effects and dispersive equations

ferent from that of the heat equation and the wave equation. Now, for the Schrödinger equation, we have the solution

$$u(x, t) = \int_{\mathbb{R}} \widehat{u}(\xi, 0) e^{i(x-\xi t)\xi} d\xi \quad (1.21)$$

where

$$\widehat{u}(\xi, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x, 0) e^{-ix\xi} dx. \quad (1.22)$$

Let us again consider a special initial condition $u(x, 0)$ made up of only one Fourier mode. Such a u_0 is a plane wave which, of course, is not in the space $L^2(\mathbb{R}^2)$, the space of square-integrable functions on the real line. One typically requires that initial data u_0 be in L^2 . Nevertheless, we will consider such a u_0 for the sake of discussion. In other words, suppose

$$\widehat{u}(\xi, 0) = \delta_{\xi_0}(\xi) \text{ for some fixed } \xi_0 \in \mathbb{R}. \quad (1.23)$$

In this case, the solution of the Schrödinger equation will read

$$u(x, t) = e^{i(x-\xi_0 t)\xi_0}. \quad (1.24)$$

This is the equation of a plane wave moving at speed ξ_0 . Under the evolution of the Schrödinger equation, an initial data u_0 which consists of a single Fourier mode ξ_0 remains a function with only that Fourier mode and which moves in the physical space at a speed of ξ_0 .

Now, instead of an initial condition $u(x, 0) = u_0(x)$ consisting of a single Fourier mode, if our initial condition consists of a continuum of Fourier modes, then the part of the solution having Fourier mode ξ will move at a speed of ξ . As a result, components of the solution with higher Fourier modes will move faster than those with smaller Fourier modes. As a result, over time, the solution will spread out, and the amplitude of the solution will decay over time. However, as we showed before, the L^2 -norm remains unchanged. This spreading out of the solutions is what we call the **dispersive effect**.

There are equations other than the Schrödinger equation that are dis-

1.1. Dispersive effects and dispersive equations

persive in nature. Other dispersive equations include the **Klein-Gordon equation**

$$u_{tt} - \Delta u + m^2 u = 0, \quad (1.25)$$

and the **Airy equation** (which is the linear part of the well studied KdV equation)

$$u_t + u_{xxx} = 0. \quad (1.26)$$

Recall that we showed that for the one dimensional Schrödinger equation, the Fourier transform of the solution u is given by

$$\widehat{u}(\xi, t) = e^{-ih(\xi)t} \widehat{u}_0(\xi) \text{ where } h(\xi) = \xi^2. \quad (1.27)$$

Here, $h(\xi)$ is called the **dispersion relation**. As before, if we apply the inverse Fourier transform of $\widehat{u}(\xi, t) = e^{-ih(\xi)t} \widehat{u}_0(\xi)$, we find that the component of the solution with the Fourier mode ξ travels at a speed equal to $\frac{h(\xi)}{\xi}$. In fact, associated with the dispersion relation are two different velocities, the **phase velocity**

$$v_p = \frac{h}{\xi} \quad (1.28)$$

and the **group velocity**

$$v_g = \frac{dh}{d\xi}. \quad (1.29)$$

Here, if one imagines the solution is modulated by an overall wave envelope (wave packet) and the solution oscillates within the wave packet, then the group velocity is the speed of the wave packet while the phase velocity is the speed of the oscillations within the wave packet (see [93]). For the Schrödinger equation,

$$v_p = \xi \text{ and } v_g = 2\xi \quad (1.30)$$

and we see that the group velocity is faster than the phase velocity.

Just like for the Schrödinger equation, we can get dispersion relations for other dispersive equations. For the Airy equation, we have

$$\widehat{u}_t = i\xi^3 \widehat{u} \quad (1.31)$$

which gives

$$\widehat{u}(\xi, t) = e^{i\xi^3 t} \widehat{u}(\xi, 0), \quad (1.32)$$

so for the Airy equation, the dispersion relation is $h(\xi) = \xi^3$. The Klein-Gordon equation is second order in time. To get the dispersion relation, we look for a function $h(\xi)$ so that

$$u = e^{i(\xi x - h(\xi)t)} \quad (1.33)$$

satisfies the Klein-Gordon equation. Notice that this way of computing $h(\xi)$ is consistent with the ways in which $h(\xi)$ is computed above. Substituting (1.33) into the Klein-Gordon equation, we get that

$$-h^2(\xi) + \xi^2 + m^2 = 0 \quad (1.34)$$

which gives

$$h(\xi) = \pm \sqrt{m^2 + \xi^2}. \quad (1.35)$$

1.1.2 Variations of Schrödinger equations

In the previous subsection, we introduced the dispersive effect. There, we worked mainly with the free Schrödinger equation which is a standard model of linear dispersive equations. In this subsection, we will introduce certain variations of the free Schrödinger equation and discuss how each variation changes the behaviour of the equation.

Schrödinger equation with potential

In the previous subsection, we considered what is called the free Schrödinger equation

$$iu_t = -\Delta u \quad (1.36)$$

which models the evolution of a quantum particle in the absence of any external fields. In this subsection, we will consider the Schrödinger equation

with a potential

$$iu_t = -\Delta u + Vu \quad (1.37)$$

which models the evolution of a quantum particle under the influence of a potential $V(x)$. Here, we will assume that the potential V only varies in space but not in time.

Equation (1.37) is central to modern physics. For example, if we take V to be the Coulomb potential $V = -\frac{1}{|x|}$, then for $n = 3$, equation (1.37) models an electron moving around the nucleus. Applications of equation (1.37) range from predictions of chemical properties in chemical compounds and computations and explanations of physical properties in solid state physics (see introductory textbooks such as [36] and [1]).

To gain a glimpse of the effect of the potential on the solution, let us consider an extreme case of our potential which, because of the simplicity of the solutions, is commonly found in introductory textbooks on quantum mechanics. Consider the one-dimensional Schrödinger equation

$$iu_t = -\Delta u + V(x)u \quad (1.38)$$

where the potential is given by

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ \infty & \text{if } x \leq 0 \text{ or } x \geq 1 \end{cases}. \quad (1.39)$$

This is commonly known as the one-dimensional infinite square well. The general solution to this equation is given by

$$u(x, t) = \begin{cases} \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) e^{-i\lambda_n^2 t} & \text{where } \lambda_n = n\pi \\ 0 & \text{if } x \notin (0, 1) \end{cases}. \quad (1.40)$$

If we look at (1.40), we see that unlike solutions to the free Schrödinger equation, the solution to (1.38) does not spread out in space over time as the potential V is trapping the solutions within the region $0 < x < 1$

(i.e. infinite square well). On one hand, the free part of equation (1.38) ($u_t = i\Delta u$) tries to spread the solution out but on the other hand, the potential V confines the solution to the infinite square well. As a result of the confining part of the equation, equation (1.38) admits some stationary solutions of the form $\sin(\lambda_n x)e^{i\lambda_n^2 t}$ for $n = 1, 2, 3, \dots$. These solutions are known as standing wave solutions since other than the complex phase $e^{i\lambda_n^2 t}$, they are stationary in time. These standing waves are examples of bound states – states which remain spatially localized for all time.

The infinite square well is not the only potential that gives rise to bound states. Other less confining potentials give rise to bound states as well. For example, when $V(x) = x^2$, equation (1.38) is called the harmonic oscillator, another standard example in introductory textbooks of quantum mechanics. Computations to derive the solutions are more complicated than the infinite square well, but nevertheless, solutions of the harmonic oscillator is given by

$$u(x, t) = \sum_{n=0}^{\infty} a_n e^{-i2(n+\frac{1}{2})t} H_n(x) e^{-\frac{x^2}{2}}. \quad (1.41)$$

Here, H_n is the Hermite polynomial, a degree n polynomial whose the explicit form can be worked out for each n . In the above, each summand $e^{-i2(n+\frac{1}{2})t} H_n(x) e^{-\frac{x^2}{2}}$ is a bound state of the harmonic oscillator with energy level $E_n = 2(n + \frac{1}{2})$. See [36].

So far, we have only been considering confining potentials. Let us instead consider yet another standard example in introductory texts in quantum mechanics, which instead of a confining potential, is an attractive potential

$$V(x) = \begin{cases} -V_0 & \text{for } -1 < x < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases} \quad (1.42)$$

for some $V_0 > 0$. (As an aside, the case where $V_0 < 0$ is called a repulsive potential.) This is known as the one-dimensional finite square well. In this case, equation (1.37) admits solutions of the form

$$u(x, t) = e^{-iEt} \phi(x) \quad (1.43)$$

for both $E < 0$ and $E > 0$. When $E < 0$, solutions of the form (1.43) only exist for a finite number of $E = E_1, E_2, \dots, E_n$, where n depends on V_0 . The larger V_0 is, the larger n is. Solutions with $E < 0$ are the bound states and are highly localized in space. On the other hand, solutions with $E > 0$ are called scattering states and are not localized in space (in fact, not in L^2). Again, see [36].

For more general potential V , it turns out that if $V \rightarrow \infty$ as $|x| \rightarrow \infty$, then (1.37) only admits bound state. This is due to the potential stopping the solutions from escaping. On the other hand, if $V \rightarrow 0$ as $|x| \rightarrow \infty$, then it is possible for dispersive solutions and bound states to co-exist. See the textbooks such as [42], for example.

Nonlinear Schrödinger equations

Next, instead of adding a potential V to the free Schrödinger equation

$$iu_t = -\Delta u, \tag{1.44}$$

we will add a nonlinear term. Nonlinear versions of the Schrödinger equation arise in many applications, including optics, magnetism, and Bose-Einstein condensation (see [78] or [2]). As a standard example, we add in the nonlinearity $\lambda|u|^{p-1}u$ to get the nonlinear Schrödinger equation

$$iu_t = -\Delta u + \lambda|u|^{p-1}u. \tag{1.45}$$

Our next task is to see the effect of the nonlinear term on the equation. Without the nonlinear term, the different Fourier modes of a solution of the Schrödinger equation act independently. As the Schrödinger equation is dispersive, as time evolves, the higher Fourier modes travel faster than those with a lower Fourier mode, so the solution spreads out in space over time. With the nonlinear term, the different Fourier modes no longer evolve independently and the interaction of different Fourier modes gives rise to more complex behaviour.

By rescaling the function u , it is only necessary to consider $\lambda = \pm 1$.

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It turns out the behaviour of (1.45) is different according to the sign of λ . When $\lambda < 0$, the nonlinear term $\lambda|u|^2u$ acts like an attractive potential when u is large and it has little effect on the equation when u is small. Because the nonlinear term acts like a attractive potential, on one hand, the linear part of (1.45) $iu_t - \Delta u$ tries to disperse the solution but on the other hand, the nonlinear part $|u|^2u$ attempts to confine or even concentrate the solution. As the solution evolves in time, there is a competition between the dispersive effect and the concentration effect. When a delicate balance is achieved between the two opposite effects, equation (1.45) gives rise to a special form of solution which neither concentrates nor disperses. These special solutions are like the analogue of the standing wave of the finite square well we saw in the last section. These special solutions of (1.45) are called **solitons** or **solitary waves**.

On the other hand, when $\lambda > 0$, the nonlinear term behaves more like a repulsive potential. As a result, solutions in this case are always dispersive. Because of the difference of the behaviour between $\lambda > 0$ and $\lambda < 0$, the case where $\lambda < 0$ is called the **focusing case** while the case where $\lambda > 0$ is called the **defocusing case**.

Other than the above heuristic regarding attractive and repulsive potentials, one can also get a glimpse of the difference in behaviour of (1.45) in the signs of λ from a conserved quantity of the equation (1.45) called the energy, which is defined by

$$E(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{p+1} |u|^{p+1} \right) dx. \quad (1.46)$$

The term $\frac{1}{2} |\nabla u|^2$ in the expression for E comes from the dispersive term $-\Delta u$ while the term $\frac{\lambda}{p+1} |u|^{p+1}$ comes from the nonlinear term $|u|^{p-1}u$. From this, we see that when $\lambda < 0$, the nonlinear term $|u|^{p-1}u$ is working in the opposite direction as the dispersive term Δu , while when $\lambda > 0$, the nonlinear term works in the same direction as the dispersive term.

Schrödinger map

So far, we have considered two different variations of the free Schrödinger equation

$$u_t = i\Delta u. \quad (1.47)$$

In particular, we discussed the behaviour of the solutions with the addition of a potential V or nonlinear term $|u|^{p-1}u$. In Chapter 2, we will consider the Schrödinger equation with the addition of both the potential term and the nonlinear term. In all of these variations, the solution u is a map from $\mathbb{R}^n \times \mathbb{R}$ (n spatial dimensions and one time dimension) to \mathbb{C} . We would like to consider a variation of the Schrödinger equation where the target space \mathbb{C} is replaced by a manifold M . For our discussion, we will only consider the case where the target manifold M is the 2-sphere \mathbb{S}^2 .

In this case, the equation analogous to (1.47) is the Schrödinger map equation

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad (1.48)$$

where $\vec{u} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{S}^2$. Here, we treat the 2-sphere as a sphere embedded in \mathbb{R}^3 , i.e.

$$\mathbb{S}^2 = \{\vec{u} \in \mathbb{R}^3 : |\vec{u}| = 1\} \subset \mathbb{R}^3. \quad (1.49)$$

Hence, we view \vec{u} as

$$\vec{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \quad (1.50)$$

where

$$u_1^2 + u_2^2 + u_3^2 = 1. \quad (1.51)$$

Now for each $\vec{u} \in \mathbb{S}^2$, we define the operator $J^{\vec{u}}$ by

$$J^{\vec{u}}\vec{\xi} = \vec{u} \times \vec{\xi}. \quad (1.52)$$

If we let

$$T_{\vec{u}}\mathbb{S}^2 = \{\vec{\xi} \in \mathbb{R}^3 | \vec{u} \cdot \vec{\xi} = 0\} \quad (1.53)$$

to be the tangent plane at $\vec{u} \in \mathbb{S}^2$, we can view $J^{\vec{u}}$ as an operator that rotates vectors on the tangent plane $T_{\vec{u}}\mathbb{S}^2$ by 90° , just as multiplication by i rotates a vector in the complex plane by 90° . With this notation, the Schrödinger map equation can be written as

$$\vec{u}_t = J^{\vec{u}}\Delta\vec{u}. \quad (1.54)$$

If we compare equation (1.47) with equation (1.54), we see that they look very similar. This is why we treat equation (1.48) as an analogue of (1.47). Despite the similarity in look, there is a major difference between equation (1.47) and (1.54): equation (1.47) is a linear while equation (1.54) is nonlinear. Due to the self-interaction of a solution caused by the nonlinearity, behaviours of solutions of equation of (1.54) are more complex and are not fully understood.

The Schrödinger map equation arises from a model for ferromagnetism introduced by Landau and Lifshitz in 1935 ([54]) and can also be viewed as the continuous version of the Heisenberg model of a ferromagnet ([79]). In such a model, \vec{u} represents either (classical) the spin of an atom or the magnetization of a magnetic material. In this model, the equation for \vec{u} takes a more general form

$$\frac{\partial \vec{u}}{\partial t} = \alpha \vec{u} \times [\Delta \vec{u} + \vec{a}(\vec{u})] + \beta \vec{u} \times (\vec{u} \times [\Delta \vec{u} + \vec{a}(\vec{u})]) \quad (1.55)$$

for some parameters α and β . Here $\alpha \vec{u} \times [\Delta \vec{u} + \vec{a}(\vec{u})]$ is the precession term describing the revolution of the magnetization vector about an effective magnetic field $\Delta \vec{u} + \vec{a}(\vec{u})$, $\beta \vec{u} \times (\vec{u} \times [\Delta \vec{u} + \vec{a}(\vec{u})])$ is a dissipation term, and $\vec{a} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a vector field representing an anisotropy in the magnet. When dissipation is absent (i.e. when $\beta = 0$) and in the isotropic case ($\vec{a} = 0$), we are left with

$$\frac{\partial \vec{u}}{\partial t} = \alpha \vec{u} \times \Delta \vec{u} \quad (1.56)$$

1.2. Background

$u(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{C} \text{ or } \mathbb{R}$	$\vec{u}(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$
$u_t = i\Delta u$ (Schrödinger equation)	$\vec{u}_t = J^{\vec{u}} \Delta \vec{u}$ (Schrödinger map)
$\Delta u = 0$ (Laplace equation)	$\Delta \vec{u} + \nabla \vec{u} ^2 \vec{u} = 0$ (harmonic map)
$u_t = \Delta u$ (heat equation)	$\vec{u}_t = \Delta \vec{u} + \nabla \vec{u} ^2 \vec{u}$ (harmonic map heat flow)
$u_{tt} = \Delta u$ (wave equation)	$\vec{u}_{tt} = \Delta \vec{u} + (\nabla \vec{u} ^2 - \vec{u}_t ^2) \vec{u}$ (wave map)

Table 1.1: Classical pde's and their geometric counterparts.

which is essentially the Schrödinger map equation.

Other than the physical applications, there are mathematical reasons for studying Schrödinger maps. In fact, one can view the Schrödinger maps as a natural extension of the linear Schrödinger equation $u_t = i\Delta u$ with the target space \mathbb{C} replaced by a Kähler manifold. There are other geometric map equations that can be thought of as geometric analogues of some classical PDEs. In Table 1.1, we list the classical PDEs in the left column and their geometric counterparts on the right. These geometric map equations have been the subject of intense mathematical study (to name a few, [77], [15] for the harmonic map heat flow, [80], [52], [67] for the wave map and [16], [4] for the Schrödinger map).

We should note that in the Table 1.1, the classical PDEs are all linear but the geometric PDEs are nonlinear due to the non-trivial geometry of the target space.

1.2 Background

In this section, we will introduce several common topics in the study of nonlinear dispersive equations. They are

- local well-posedness;
- global well-posedness;

- special solutions such as solitary waves;
- stability of special solutions;
- scattering.

In the subsections, we will define the topics in this list and discuss some known results. More in-depth expositions on many of these topics can be found in books on dispersive equations such as [75], [78], [8], [26], [82] and [2].

Here, we will focus mainly on nonlinear Schrödinger equations with a power nonlinearity

$$iu_t = -\Delta u + \lambda|u|^{p-1}u \quad (1.57)$$

with initial condition

$$u(x, 0) = u_0(x) \in X \quad (1.58)$$

for $X = L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$ (to be defined in the upcoming subsection).

1.2.1 Notation

Before we begin, we should define some notation that will be useful for the later sections of this chapter. For quantities A and B , we use the notation

$$A \lesssim B \quad (1.59)$$

to mean

$$A \leq CB \quad (1.60)$$

for some constant $C > 0$. Similarly, we use the notation

$$A \lesssim_p B \quad (1.61)$$

to highlight that the constant C may depend on the parameter p .

Next, we will define various function spaces that will be important for later sections. For each $1 \leq p \leq \infty$, let $L^p(\mathbb{R}^n)$ denote the space of Lebesgue

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measurable functions f in \mathbb{R}^n up to a.e. equivalence such that the L^p norm defined by

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f|^p dx \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty \quad (1.62)$$

and

$$\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname{ess\,sup}_{\mathbb{R}^n} |f| \quad (1.63)$$

is finite.

Let $C^\infty(\mathbb{R}^n)$ be the space of infinitely differentiable functions in \mathbb{R}^n . Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i \geq 0$ for $i = 1, 2, \dots, n$. Here, α is called a multi-index. We define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. We use the notation D^α to denote the differential operator

$$D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}. \quad (1.64)$$

For each $1 \leq p < \infty$ and positive integer m , we will define the Sobolev space $W^{m,p}(\mathbb{R}^n)$ to be the closure of

$$\{u \in C^\infty(\mathbb{R}^n) \mid D^\alpha u \in L^p, \forall |\alpha| \leq m\} \quad (1.65)$$

under the norm

$$\|u\|_{m,p} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\mathbb{R}^n)}. \quad (1.66)$$

The case $p = 2$ is special in that $W^{m,2}$ is a Hilbert space (a complete normed linear space with an inner product). We will write $W^{m,2}$ as H^m .

We have so far defined H^m for positive integer m ; one can also define the space H^s for $s \in \mathbb{R}$. To do so, let the Schwartz space \mathcal{S} be defined by

$$\mathcal{S} = \{u \in C^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta u| < \infty \text{ for all multi-indices } \alpha \text{ and } \beta\}. \quad (1.67)$$

Here, for $x \in \mathbb{R}^n$ and multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, x^α denotes $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. The space $H^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ is defined to be the closure of the Schwartz

space under the norm

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^n} |(1 + |\xi|^s) \widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \quad (1.68)$$

where \widehat{u} denotes the Fourier transform of u .

Next, we will give some motivation for why these spaces are central to the study of partial differential equations. For simplicity, consider the free Schrödinger equation

$$\begin{cases} iu_t = -\Delta u \\ u(x, 0) = u_0(x) \end{cases} \quad (1.69)$$

Here, $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$ and $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$. One can think of the free Schrödinger equation as a mapping which takes an initial condition $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$ and maps it into the function $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$. However, there is a different way to think of the free Schrödinger equation. If instead, one treats u_0 as an object in some functional space X such as L^2 or H^1 and for each t , one treats $u(\cdot, t)$ as an object in some functional space Y , then the free Schrödinger equation can be thought of as a mapping from X to $t \mapsto Y$. In other words, given an initial condition $u_0 \in X$, the free Schrödinger equation returns an element in Y for each t as long as the solution is defined. Viewed this way, partial differential equations are infinite dimensional analogues of ordinary differential equations. For example, an ordinary differential equation such as

$$\begin{cases} \frac{d\vec{u}}{dt} = M\vec{u} \\ \vec{u}(0) = u_0 \in \mathbb{R}^n \end{cases} \quad (1.70)$$

where M is an $n \times n$ matrix, can be thought of as a mapping from the vector space \mathbb{R}^n to $t \mapsto \mathbb{R}^n$. The main difference between ordinary differential equations and partial differential equations is that \mathbb{R}^n is a finite dimensional vector space but functional spaces such as L^2 and H^1 are infinite dimensional. Because of this difference, behaviours of partial differential equations are more complex and less well understood than ordinary differential equa-

tions.

Let X and Y be some functional spaces. If the free Schrödinger equation

$$\begin{cases} iu_t = -\Delta u \\ u(x, 0) = u_0(x) \in X \end{cases} \quad (1.71)$$

admits an unique solution $u(x, t) \in Y$, we will define its solution operator $e^{i\Delta t}$ as the mapping from X to Y such that

$$e^{i\Delta t}u_0(x) = u(x, t). \quad (1.72)$$

1.2.2 Local well-posedness

Given an equation modelling some dynamical physical process, a beginning step to ensuring the equation is a good model is to ensure that given an initial condition u_0 , there is a solution, which exists at least up to some time $T > 0$, to the equation. One further wishes that such a solution be unique and that it depends continuously on the initial data. Roughly speaking, this is the idea of local well-posedness.

For our discussion, consider the nonlinear Schrödinger equation (NLS)

$$iu_t = -\Delta u + \lambda|u|^{p-1}u \quad (1.73)$$

with initial condition

$$u(x, 0) = u_0(x) \in X \quad \forall x \in \mathbb{R}^n \quad (1.74)$$

for some functional space X such as H^1 or L^2 . Local well-posedness concerns the following questions:

Existence: Does there exist a time $T > 0$ where there is a solution u of (1.73) defined in $C((-T, T), X)$ (i.e. a continuous function of time into the space X)?

Uniqueness: Is the solution unique?

Continuous dependence on data: Does the solution u depend continuously on the initial data?

If the answer to the above question is affirmative, one can further ask whether there is a **blow-up alternative**. What this means is that, suppose the solution u to (1.73) exists up to a maximal time $T_{\max} < \infty$; one wants to know whether

$$\lim_{t \rightarrow T_{\max}^-} \|u\|_X = \infty. \quad (1.75)$$

In other words, if a solution fails to exist at a certain point in time, one wants to know if that is due to the X -norm of the solution blowing up.

Existence theory of (1.73) for $X = H^1$ is obtained in a series of papers by authors such as [3], [72], [43] and [45]. Their results give

Theorem 1. (*local well-posedness of NLS in H^1*)

Let p_{\max} be defined by

$$p_{\max} = \begin{cases} \infty & \text{if } n = 1, 2 \\ \frac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}. \quad (1.76)$$

Suppose $1 < p < p_{\max}$, and suppose $u_0 \in H^1(\mathbb{R}^n)$. Then there exists a $T > 0$ and a unique solution u to (1.73) which depends continuously on u_0 in H^1 such that $u \in C((-T, T), H^1(\mathbb{R}^n))$. Furthermore, the following quantities

$$\textbf{mass: } M(u) := \|u\|_{L^2}^2 \quad (1.77)$$

$$\textbf{energy: } \mathcal{E}(u) := \int \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{p+1} |u|^{p+1} \right) dx \quad (1.78)$$

are conserved.

It should be noted that for functions u in $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, Δu may not be defined. When we say u is a solution of (1.73) in $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, we actually mean u satisfies the integral form (1.83) of (1.73). In other

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words, our sense of solution is weaker than the classical sense in which u is required to satisfy (1.73).

Existence theory of (1.73) for $X = L^2$ is obtained in a series of papers by authors such as [72] and [88]. Their results give

Theorem 2. (*local well-posedness of NLS in L^2*)

Let p_c be defined by

$$p_c = 1 + \frac{4}{n}. \quad (1.79)$$

Suppose $1 \leq p < p_c$, and suppose $u_0 \in L^2(\mathbb{R}^n)$. Then there exists a $T > 0$ and a unique solution u to (1.73) such that $u \in C((-T, T), L^2(\mathbb{R}^n))$. Furthermore, the mass

$$M(u) := \|u\|_{L^2} \quad (1.80)$$

is a conserved quantity.

In the former theorem, p_{\max} is the p in (1.73) such that the homogeneous H^1 -norm (\dot{H}^1) remains unchanged under the solution-invariant scaling $u(x, t) \rightarrow \mu^{\frac{2}{p-1}} u(\mu x, \mu^2 t)$ for $\mu > 0$. In other words, if u is a solution of (1.73) with $p = p_{\max}$ and $v(x, t) = \mu^{\frac{2}{p-1}} u(\mu x, \mu^2 t)$, then v is also a solution of (1.73) and

$$\|v(\cdot, t)\|_{\dot{H}^1(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\nabla v(x, t)|^2 dx \right)^{\frac{1}{2}} = \|u(\cdot, \mu^2 t)\|_{\dot{H}^1(\mathbb{R}^n)}. \quad (1.81)$$

Similarly, p_c is the p in (1.73) such that the L^2 -norm remains unchanged under the solution invariant scaling $u(x, t) \rightarrow \mu^{\frac{2}{p-1}} u(\mu x, \mu^2 t)$. In other words, let u be a solution of (1.73) with $p = p_c$ and let $v(x, t) = \mu^{\frac{2}{p-1}} u(\mu x, \mu^2 t)$, then v is also a solution of (1.73) and

$$\|v(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|u(\cdot, \mu^2 t)\|_{L^2(\mathbb{R}^n)}. \quad (1.82)$$

The case where $p < p_{\max}$ is called **H¹-subcritical** and the case where $p < p_c$ is called **L²-subcritical**.

The case where $p = p_{\max}$ is called **H¹-critical** while the case where $p = p_c$ is called **L²-critical**. The proof of existence in critical cases is more

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difficult because at the critical power ($p = p_{\max}$ or p_c) the norm (H^1 or L^2) cannot be used to control the nonlinear term $|u|^{p-1}u$. Existence theorems for (1.73) in the H^1 -critical case and L^2 -critical case were obtained by [14].

Local existence for the H^1 -subcritical case with H^1 initial data (Theorem 1) and the L^2 -subcritical case with L^2 initial data (Theorem 2) are proved using a contraction mapping argument. The idea is to reformulate the partial differential equation (1.73) into the integral form

$$u(x, t) = e^{it\Delta}u_0(x) - i \int_0^t e^{i\Delta(t-\tau)}|u(x, \tau)|^{p-1}u(x, \tau)d\tau. \quad (1.83)$$

Working with this reformation of the original problem, various dispersive estimates such as the decay estimate and Strichartz estimates are employed to complete the contraction mapping argument. Here, the decay estimate says that

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})}\|u_0\|_{L^{q'}(\mathbb{R}^n)} \quad (1.84)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $2 \leq q \leq \infty$. The decay estimate is a special property of dispersive equations. For example, if we take $q = \infty$ and $q' = 1$, the decay estimate tells us that

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n}{2}}\|u_0\|_{L^1(\mathbb{R}^n)}. \quad (1.85)$$

In the case where $\|u_0\|_{L^1(\mathbb{R}^n)}$ is finite, such an inequality gives us the rate at which the height of the function $e^{it\Delta}u_0$ is decreasing. Since the L^2 -norm of $e^{it\Delta}u_0$ is fixed, the decrease in the L^∞ -norm of $e^{it\Delta}u_0$ is due to u dispersing.

The decay estimate alone is not enough to prove many results. The reason is that the L^q -norm and the $L^{q'}$ -norm are not equivalent unless $q = q' = 2$. With the decay estimate alone, unless $p = p' = 2$, to control the L^p -norm of $e^{it\Delta}u$, we need control of the $L^{p'}$ -norm of u for which we have no estimate.

As the decay estimates alone are not enough to prove well posedness results, we need other estimates, called Strichartz estimates. Strichartz estimates are mixed space-time estimates, meaning that they control certain

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space-time integrals of the solution. Let

$$\|\cdot\|_{L_t^q L_x^r} \text{ denote } \|\|\cdot\|_{L_x^r(\mathbb{R}^n)}\|_{L_t^q(\mathbb{R})}. \quad (1.86)$$

We also define the pair (q, r) to be *admissible* if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \text{ for } q, r \in [2, \infty] \text{ and } (q, r, n) \neq (2, \infty, 2). \quad (1.87)$$

The Strichartz estimates say that for admissible (q, r) and (\tilde{q}, \tilde{r}) ,

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_x^2} \quad (1.88)$$

and

$$\left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \quad (1.89)$$

where \tilde{q}' and \tilde{r}' satisfy

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1 \text{ and } \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1. \quad (1.90)$$

Equation (1.88) is the homogeneous Strichartz estimate allowing us to handle the $e^{i\Delta t}u_0(x)$ term in (1.83) while Equation (1.89) is the inhomogeneous Strichartz estimate allowing us to handle the $\int_0^t e^{i\Delta(t-\tau)}|u(x, \tau)|^{p-1}u(x, \tau)d\tau$ term in (1.83). Such estimates originate from [76]. The inhomogeneous Strichartz estimates are developed by [94] and [13]. The end point case $((q, r) = (2, \frac{2n}{n-2})$ for $n \geq 3$) is given by [46]. With the decay estimate and Strichartz estimates, local well-posedness is shown using a contraction mapping argument.

It should be mentioned that the proofs for Theorem 1 and 2 hold for more general nonlinear terms than $|u|^{p-1}u$. However, for our discussion, we will only consider pure power nonlinearities $|u|^{p-1}u$.

Following the proofs of the local existence for the H^1 -subcritical case with initial data in H^1 , the duration of existence T depends on the dimension n , the power of nonlinearity p and the norm of the initial data $\|u_0\|_{H^1}$.

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Similarly, for the L^2 -subcritical case with initial data in L^2 , the length of time of existence T depends on the dimension n , the power of nonlinearity p and the norm of the initial data $\|u_0\|_{L^2}$. In fact, in both cases, for fixed n and p , T is mainly determined by a quantity like

$$C\|u_0\|_X^{-m} \tag{1.91}$$

for some constants C and $m > 0$ depending on n and p for $X = H^1$ or L^2 . There is local well-posedness also for the critical cases, $p = p_{\max}$ or $p = p_c$, but then the time of existence T depends on not just the norm of the initial data but also on the structure of the initial data.

1.2.3 Global well-posedness

Given an equation modelling some physical process, suppose the equation is locally well-posed, then we know given an initial condition u_0 , there exists a unique solution that depends continuously on the initial data. Local well-posedness guarantees that such a solution will exist up to time T , after which it may fail to exist. If a solution fails to exist after time T , this indicates that the equation is only a good model of the physical process up to time T . As a result, after local well-posedness of an equation is obtained, a natural question to ask is whether such a solution exists indefinitely or if it instead blows up in finite time. This is the question of global well-posedness.

In the last subsection, we saw that for the equation

$$\begin{cases} iu_t + \Delta u = \pm |u|^{p-1}u \\ u(x, 0) = u_0(x) \in X \text{ for } X = H^1(\mathbb{R}^n) \text{ or } L^2(\mathbb{R}^n) \end{cases}, \tag{1.92}$$

the local existence theorem says that for suitable ranges of p , there exists a time $T > 0$ such that there exists a solution $u(x, t)$ of (1.92) where $u(x, t) \in C((-T, T), X)$.

In this subsection, we will consider the question of whether $T = \infty$, in which case, we say that the equation is globally well-posed. In this subsection, we will restrict our discussion to the well-studied and physically

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$n = 1$	L^2 -subcritical	H^1 -subcritical
$n = 2$	L^2 -critical	H^1 -subcritical
$n = 3$	L^2 -supercritical	H^1 -subcritical

Table 1.2: L^2 - and H^1 - criticality of cubic NLS for $n = 1, 2$ and 3 .

important cubic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda |u|^2 u \\ u(x, 0) = u_0(x) \in X \text{ for } X = H^1(\mathbb{R}^n) \text{ or } L^2(\mathbb{R}^n) \end{cases} \quad (1.93)$$

for $\lambda = \pm 1$ and for space dimension $n = 1, 2$ and 3 . Table 1.2 summarizes the L^2 - and H^1 - criticality for $n = 1, 2$ and 3 .

Recall from local well-posedness, for the subcritical case, the length of existence T is a function of only $\|u_0\|_X$. Thus, if one can obtain an a priori bound on $\|u(t)\|_X$, one automatically gets global well-posedness. Indeed, suppose we know that $\|u\|_X < M$, then local existence guarantees a solution up to time $t = T(\|u_0\|_X) \equiv t_1$. We can then take the solution at $t = t_1$ as initial data and the local existence will guarantee a solution up to time $t = T(\|u(\cdot, t_1)\|_X) \equiv t_2$. This process can be iterated and the total length of existence will be given by $\sum_{j=1}^{\infty} t_j$. Now, since $\|u\|_X < M$, for each j , $t_j \geq T(M) > 0$. Hence, $\sum_{j=1}^{\infty} t_j = \infty$.

Since the L^2 norm is a conserved quantity of (1.92), we automatically get L^2 global well-posedness for $n = 1$. Similarly, as the energy

$$E(u) = \int \left(\frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} |u|^4 \right) dx \quad (1.94)$$

is a conserved quantity, in the case $\lambda = 1$ (the defocusing case), this gives

$$\int \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \right) dx = E(u_0). \quad (1.95)$$

The above gives

$$\int \frac{1}{2} |\nabla u|^2 dx < E(u_0) \quad (1.96)$$

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and thereby together with L^2 -conservation yields an upper bound on $\|u\|_{H^1}$. Therefore, for the defocusing case $\lambda = 1$, (1.93) is globally well-posed in H^1 for $n = 1, 2$ and 3 .

The situation is different for the focusing case $\lambda = -1$. In this case, we have

$$\int \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{4} |u|^4 \right) dx = E(u_0). \quad (1.97)$$

Here, we see that the nonlinear term $\frac{-1}{p+1} |u|^{p+1}$ is now acting against the dispersive term $\frac{1}{2} |\nabla u|^2$ in the expression for energy. In this case, a way to show global well-posedness is to try to use the dispersive term $\frac{1}{2} |\nabla u|^2$ to control the nonlinear term $\frac{1}{4} |u|^4$. In fact, the Gagliardo-Nirenberg inequality (see, for example, [26]) gives that for $p \leq p_{\max}$,

$$\int |u|^{p+1} dx \lesssim \left(\int |\nabla u|^2 dx \right)^\alpha \left(\int |u|^2 dx \right)^\beta \quad (1.98)$$

for $\alpha = \frac{n(p-1)}{4}$ and $\beta = \frac{2+n+(2-n)p}{4}$. As we have chosen $p = 3$, we get that for $n \leq 4$,

$$\int |u|^4 dx \lesssim \left(\int |\nabla u|^2 dx \right)^{\frac{n}{2}} \left(\int |u|^2 dx \right)^{2-\frac{n}{2}}. \quad (1.99)$$

For $n = 1$, energy conservation and the above give us

$$\begin{aligned} \int \frac{1}{2} |\nabla u|^2 dx &= E(u_0) + \frac{1}{4} \int |u|^4 dx \\ &\lesssim E(u_0) + \left(\int |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int |u|^2 dx \right)^{\frac{3}{2}}. \end{aligned}$$

Using Young's inequality in the form

$$ab \leq \frac{\epsilon^2}{2} a^2 + \frac{1}{2\epsilon^2} b^2, \quad (1.100)$$

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for sufficiently small ϵ , we get that

$$\int \frac{1}{2} |\nabla u|^2 dx \leq E(u_0) + \frac{1}{4} \left(\int |\nabla u|^2 dx \right) + C \left(\int |u|^2 dx \right)^3$$

for some constant C . and this shows

$$\int |\nabla u|^2 dx \lesssim E(u_0) + \left(\int |u|^2 dx \right)^3. \quad (1.101)$$

Since the L^2 -norm of u is a conserved quantity, this shows that $\|u\|_{H^1}$ is bounded uniformly in time, so we have global well-posedness for $n = 1$.

On the other hand, unlike $n = 1$, for the focusing case in dimensions $n = 2$ or $n = 3$, there exist solutions that blow up in finite time. [35] showed if the initial data $u_0 \in H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; |x|^2 dx)$ and $E(u_0) < 0$, then the solution u blows up in finite time. Here, $L^2(\mathbb{R}^n; |x|^2 dx)$ denotes the space of functions $v : \mathbb{R}^n \rightarrow \mathbb{C}$ with

$$\|vx\|_{L^2(\mathbb{R}^n)} < \infty. \quad (1.102)$$

The idea is to consider the quantity

$$I(t) = \int_{\mathbb{R}^n} |x|^2 |u(x, t)|^2 dx. \quad (1.103)$$

Here, if we think of $|u|^2$ as a probability density, then I is essentially the second moment. If u is a solution of (1.92), simple calculations show that

$$I'' = 16E + \lambda \frac{4n}{p+1} (p - p_c) \int |u|^{p+1} dx. \quad (1.104)$$

For $n = 2$, $n = 3$, $p = 3 \geq p_c$. Hence, in the focusing case $\lambda = -1$, the term $\lambda \frac{4n}{p+1} (p - p_c) \int |u|^{p+1} dx$ is non-positive. Hence,

$$I'' \leq 16E. \quad (1.105)$$

Since energy is a conserved quantity, the above says that I has a constant negative concavity meaning that it will go below 0 in finite time. However,

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the quantity I is a positive quantity, so the solution u must have failed to exist before this time. Ozawa-Tsutsumi ([59], [58]) later removed the assumption that the initial condition be in $L^2(\mathbb{R}^n; |x|^2 dx)$ and showed if $u_0 \in H^1$ is radially symmetric and $E(u_0) < 0$, then the solution u blows up in finite time.

For $n \leq 3$, (1.93) is H^1 -subcritical. However, (1.93) is H^1 -critical for $n = 4$ and H^1 -supercritical for $n \geq 5$. As blow-up is possible for the focusing case, we will only consider the defocusing case for which global well-posedness is at least possible. In H^1 -critical and supercritical cases, the earlier argument does not apply to show global well-posedness as the time of existence T depends not only on the $\|u_0\|_{H^1}$ but also on the structure of u_0 . Proving global well-posedness even for the defocusing case for H^1 data is more difficult in this situation. In fact, not much is known about global well-posedness about the H^1 -supercritical case. On the other hand, the critical case is much better understood due to a number of recent breakthroughs. Consider the H^1 -critical nonlinear Schrödinger equation

$$\begin{cases} iu_t = -\Delta u + |u|^{p_{\max}-1}u \\ u(x, 0) = u_0(x). \end{cases} \quad (1.106)$$

As before, p_{\max} is the H^1 -critical exponent. For example, when $n = 3$, $p_{\max} = 5$. For $n = 3$ and 4, Bourgain ([7] and [8]) showed for radial initial data $u_0 \in H^1(\mathbb{R}^3)$, solutions to (1.106) are global. [37] (a new proof for $n = 3$) and [81] (for $n \geq 5$) extended Bourgain's result to other dimensions. [22] removed the radial assumption for $n = 3$. [64] and [89] extended the result of [22] to $n = 4$ and $n \geq 5$ respectively. Equation (1.106) is defocusing. Global well-posedness and blow-up for the H^1 -critical focusing equation with radial initial data

$$iu_t = -\Delta u - |u|^{p_{\max}-1}u \quad (1.107)$$

has been worked out by [47].

For $n = 1$, (1.93) is L^2 -subcritical. On the other hand, for $n = 2$ and $n = 3$, (1.93) is L^2 -critical and L^2 -supercritical respectively. It is very

difficult, if not impossible, to show global well-posedness in L^2 for these cases. The above method will not work because the time of existence T depends not only on the size of the L^2 norm of the initial data u_0 but also on the structure of u_0 . There have been a lot of results in this direction showing global well-posedness in the defocusing case with initial data in H^s for $0 \leq s < 1$. The current state of research is to get s as low as possible. For $n = 2$, some recent results in this direction are [32] (for $s > \frac{1}{2}$), [19] (for $s > \frac{2}{5}$), [23] (for $s > \frac{1}{3}$) and finally [28] (for $s = 0$). For $n = 3$, some recent results in this direction are [20] (for $s > \frac{5}{6}$) and [21] (for $s > \frac{4}{5}$). Note that $s = 0$ corresponds to L^2 . For $n = 3$, (1.93) is L^2 -supercritical and the equation is locally ill-posed in L^2 (see [18]). On the other hand, globally well-posed in L^2 for the L^2 -critical nonlinear Schrödinger equation in $n = 3$

$$iu_t = \Delta u - |u|^{\frac{4}{3}}u \quad (1.108)$$

has been shown by [27].

1.2.4 Solitary waves and their stability

Consider the equation

$$iu_t = -\Delta u - |u|^{p-1}u. \quad (1.109)$$

This is the focusing nonlinear Schrödinger equation. As mentioned earlier, equation (1.109) admits very special solutions called solitary waves due to the competition between the dispersive effect and the concentration effect from the nonlinear term. Here, we start by looking for special solutions taking the form

$$u(x, t) = e^{it}\phi(x). \quad (1.110)$$

If we substitute the above into (1.109), we find that ϕ must satisfy

$$\phi = \Delta\phi + |\phi|^{p-1}\phi. \quad (1.111)$$

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When spatial dimension $n = 1$, the above equation is an ordinary differential equation and it turns out that in this case, it can be solved exactly and the solution is given by

$$\phi = \left(\frac{p+1}{2} \right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{p-1}{2} x \right). \quad (1.112)$$

If we apply the solution preserving scaling $u(x, t) \mapsto \lambda^{\frac{2}{p+1}} u(\lambda x, \lambda^2 t)$, we find that if we let

$$\phi_w = \omega^{\frac{1}{p+1}} \left(\frac{p+1}{2} \right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{\sqrt{\omega}(p-1)}{2} x \right), \quad (1.113)$$

then

$$u(x, t) = e^{i\omega t} \phi_w(x) \quad (1.114)$$

is a solution to (1.109).

For higher dimensions, (1.111) cannot be solved explicitly. For $n \geq 3$, [76] and [6] showed for $p < p_{\max}$, there exists at least a positive, spherically symmetric **ground state** solution of (1.111) as well as infinitely many **excited state** (sign-changing) solutions. Here, ground state solution refers to a positive solution. Existence for ground states for $n = 2$ was later obtained by [5]. [53] showed uniqueness of the ground state. [44] showed for $n \geq 2$ and $p < p_{\max}$, for each non-negative integer m , there exists a radial solution with m zeros.

Because solitary waves arise from a delicate balance between two opposing effects, intuitively, one may believe the solitary waves are unstable under perturbation. However, in many cases, the solitary waves are found to be remarkably stable (see [83] for a discussion of the stability of solitary waves). The issues regarding stability of solutions of equations modelling physical process are important. This is because unstable solutions are difficult to observe experimentally since any small perturbation in the system will destroy them.

In fact, the discovery of solitary waves is linked to their stability. These waves were first observed by John Scott Russell in 1834 as he was traveling

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along a canal where a pulse of water wave caught his attention. Russell followed the pulse of wave for miles on horseback and was surprised by the fact that the wave did not change its shape. J. Boussinesq modelled the wave by what is now known as the Boussinesq equation. Later D.J. Korteweg and G. de Vries modelled the wave by what is now called the KdV equation (Korteweg-de Vries equation)

$$u_t + u_{xxx} + (u^2)_x = 0 \quad (1.115)$$

which is a standard example of a nonlinear dispersive equation having solitary wave solutions. A more detailed account of the history can be found, for example, in the introductory chapter of [2].

We are going to discuss two different notions of stability. They are orbital stability and asymptotic stability. Very generally speaking, we say that a solution u is orbitally stable, if given an initial condition v_0 close to $u(x, 0)$, the solution v with the initial condition v_0 remains close to u . Here, the notion of what it means to be “close” remains to be defined up to the symmetries of the equation (i.e. v remains close to the symmetry orbit of u). On the other hand, we say that a solution u is asymptotically stable if given an initial condition v_0 close to $u(x, 0)$, the solution v with the initial condition v_0 approaches the symmetry orbit of u as time $t \rightarrow \infty$.

To properly define these two notions of stability, we need to define a notion of closeness. In order to do so, we need to first understand some invariances of (1.109). The following transformations leave (1.109) invariant (in other words, if u is a solution of (1.109), u remains a solution after the transformation):

$$\textbf{translation: } u(x, t) \mapsto u(x - x_0, t - t_0) \quad (1.116)$$

$$\textbf{rescaling: } u(x, t) \mapsto \mu^{\frac{2}{p-1}} u(\mu x, \mu^2 t) \quad (1.117)$$

$$\textbf{phase shift: } u(x, t) \mapsto e^{i\theta} u(x, t) \quad (1.118)$$

$$\textbf{Galilean boost: } u(x, t) \mapsto e^{i(v \cdot x - |v|^2 t)} u(x - 2vt, t) \quad (1.119)$$

With these in mind, we will attempt to define orbital stability. First, we start with a generic notion of orbital stability which turns out to be not suitable for our purposes and we will discuss the reason. Let X and Y be Banach spaces. Our first attempt is to define orbital stability by the following:

Definition 1. (the first attempt)

We say that a solution u of (1.109) is **orbitally stable** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if a solution v of (1.109) satisfies $\|v(\cdot, 0) - u(\cdot, 0)\|_X < \delta$, then $\|v(\cdot, t) - u(\cdot, t)\|_Y < \epsilon$ for all $t \geq 0$.

However, the above is not a good definition for our problem, because under this definition, no soliton will be stable under our usual choice of Banach spaces X and Y such as L^2 or H^1 . The reason is that if $u(x, t) = e^{i\omega t} Q(x)$ is a solitary wave solution of (1.109), then by the Galilean invariance,

$$v(x, t) = e^{i[(w - |v|^2)t + v \cdot x]} Q(x - 2vt) \quad (1.120)$$

is also a solution. Furthermore, $v(x, 0) = e^{iv \cdot x} Q(x)$, so

$$v(x, 0) - u(x, 0) = (e^{iv \cdot x} - 1)Q(x) \quad (1.121)$$

and for $|v|$ small, we have

$$\|v(x, 0) - u(x, 0)\|_X \ll 1 \quad (1.122)$$

for $X = L^2$ or H^1 . However,

$$\|v(\cdot, t) - u(\cdot, t)\|_Y = \|e^{i[(w - |v|^2)t + v \cdot x]} Q(x - 2vt) - e^{i\omega t} Q(x)\|_Y \quad (1.123)$$

may not be small once t becomes large enough no matter how small $|v|$ is. Because of this, we need to define orbital stability to incorporate the possible invariant transformations of the equation. As a result, we will define orbital

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stability as the following:

Definition 2. (the correct version)

We say that a solution u of (1.109) is **orbitally stable** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if a solution v of (1.109) satisfies $\|v(\cdot, 0) - u(\cdot, 0)\|_X < \delta$, then

$$\inf_{\theta, x_0} \|v(x, t) - e^{i\theta} u(x - x_0, t)\|_Y < \epsilon \quad (1.124)$$

for all $t \geq 0$.

Orbital stability results in H^1 for the ground states are obtained by [12], [91], [92], and [66]. [39] formulated an abstract framework to show stability of solitary waves. These results show that the ground state is stable if $1 < p < p_c$ and unstable if $p_c \leq p < p_{\max}$.

Next, we will introduce the concept of asymptotic stability. Consider first the linear equation

$$iu_t = -\Delta u + Vu. \quad (1.125)$$

As explained before, under suitable assumptions on the potential V , such an equation will admit bound state solutions. For our purposes, we will assume the potential is such that (1.125) admits a single bound state solution ϕ_0 with eigenvalue $-\omega_0$. In other words, ϕ_0 is a solution to the differential equation

$$\Delta \phi_0 - V \phi_0 = \omega_0 \phi_0 \quad (1.126)$$

and

$$u = e^{it\omega_0} \phi_0(x) \quad (1.127)$$

is a solution of (1.125). Next, suppose we add a cubic nonlinear term $\lambda|u|^2 u$ to (1.125) to get

$$iu_t = -\Delta u + Vu + \lambda|u|^2 u \quad (1.128)$$

where $\lambda = \pm 1$ and $V : \mathbb{R}^n \rightarrow \mathbb{R}$. It turns out that under suitable assumptions on V , such as sufficient decay as $|x| \rightarrow \infty$, equation (1.128) admits solutions of the form

$$u(x, t) = e^{iEt} Q(x). \quad (1.129)$$

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We call these solutions nonlinear bound states. Here, Q and E solve the equation

$$\Delta Q - VQ - \lambda|Q|^2Q = EQ. \quad (1.130)$$

In fact, there exists a one (complex) parameter family of these nonlinear bound states $Q[z]$. If we let z be the complex parameter, then E and Q has the form

$$E[z] = \omega_0 + o(z) \text{ and } Q[z] = z\phi_0 + q(z) \quad (1.131)$$

for sufficiently small z . Notice as z is small parameter, the size of these nonlinear bound state solutions is small.

If the initial data $u_0 = Q[z_1]$ for some sufficiently small z_1 so $Q[z_1]$ is a nonlinear bound state, then the corresponding solution u of (1.128) will be given exactly by

$$u(x, t) = e^{iE[z_1]t}Q[z_1](x). \quad (1.132)$$

Now, suppose the initial condition is given by

$$u_0 = Q[z_1] + \text{a small error}. \quad (1.133)$$

As the initial data is not exactly a bound state, the time evolution will not be given exactly by (1.132). We would like to find out how such a solution will evolve: whether the solution will disperse to zero or whether the solution will land back onto some other bound state. Asymptotic stability concerns the following question: suppose we start with an initial data $u(x, 0) = u_0(x)$ that is “close” to $Q[z_1](x)$, will the solution $u(x, t)$ of (1.128) approach a nearby nonlinear bound state $e^{iE[z_2]t}Q[z_2](x)$ as $t \rightarrow \infty$. More precisely, we say that the solutions $e^{iE[z_1]t}Q[z_1](x)$ are asymptotically stable if whenever

$$\|u_0 - Q[z_1](x)\|_X \quad (1.134)$$

is sufficiently small for some functional space X , then the solution u of

(1.128) will be given in the form

$$u(x, t) = Q[z(t)](x) + \eta(x, t) \quad (1.135)$$

where $z(t)$ approaches some limit z_+ close to z_1 as $t \rightarrow \infty$, and η disperses to 0.

The asymptotic stability of small nonlinear bound state solutions is studied by authors such as [68], [69], [61], [40], [50], [57], [50] and [49]. The case where $\Delta + V$ has two instead of one eigenvalues has also been studied by [87], [86] and [33]. Asymptotic stability of large solitary waves is also studied by [9], [10], [25], [26] and [60].

1.2.5 Scattering

To illustrate the idea of scattering, consider the linear Schrödinger equation with potential

$$iu_t = -\Delta u + Vu \quad (1.136)$$

where the potential V is nonzero near the origin but is diminishing in size away from the origin ($\lim_{|x| \rightarrow \infty} V(x) = 0$). For example, one can take V to be the Coulomb potential $V(x) = -\frac{1}{|x|}$. Now, imagine a quantum particle moving in a straight line towards the origin from very far away, passing by the region close to the origin, and moving far away again. When the particle is far away from the origin, it does not feel much of the presence of the potential, so such a quantum particle evolves like one driven by the free Schrödinger equation $iu_t = -\Delta u$. As the particle gets close to the origin, it interacts with the potential, and the particle does not act like a free particle. Again, once the particle leaves the region near the origin and gets sufficiently far, it acts like a free particle again. This is called scattering by a potential.

The topic we will discuss in this section is the analogue of the above but for the nonlinear Schrödinger equation

$$\begin{cases} iu_t = -\Delta u + \lambda|u|^{p-1}u \\ u(x, 0) = u_0(x) \end{cases} \quad (1.137)$$

Very roughly, we say that a solution u of (1.137) scatters if u behaves like a solution of the linear equation $iu_t = -\Delta u$ far back in time ($t \rightarrow -\infty$) and far in the future ($t \rightarrow \infty$). In other words, scattering happens when the nonlinear term $|u|^2 u$ “turns off” asymptotically. More rigorously, we define scattering as follows:

Definition 3. (Scattering)

Let X be $H^1(\mathbb{R}^n)$. Let u be a solution of (1.137). We say that u scatters if there exists $u_{\pm} \in X$ such that

$$\|u(t) - e^{i\Delta t} u_{\pm}\|_X \rightarrow 0 \text{ as } t \rightarrow \pm\infty. \quad (1.138)$$

Notice that when u scatters, the asymptotic linear states u_{\pm} are unique.

Next, suppose for each $u_{\pm} \in X$, there exists a unique initial data u_0 such that the solution u scatters to the states u_{\pm} , then we can define the **wave operators** W_{\pm} to be the maps from u_{\pm} to u_0 . On the other hand, suppose for each $u_0 \in X$, the solution u with $u(x, 0) = u_0$ scatters, then we say (1.137) is **asymptotically complete**. In other words, (1.137) is asymptotic complete when every initial data in X gives rise to scattering solution.

Since scattering concerns asymptotic behaviour of solutions, when a solution u scatters, it has to be a global solution. When $p > p_{\max}$, solutions may not even be locally well-posed. As a result, scattering cannot occur for all initial data when p is too large. On the other hand, scattering requires the nonlinearity to “turn off” as $t \rightarrow \pm\infty$. When p is too small, the nonlinear term may not decay fast enough for its effect to “disappear”. In other words, scattering occurs when p is of intermediate size. It turns out that for the defocusing case ($\lambda > 0$) in H^1 , scattering occurs when $p_c < p \leq p_{\max}$ ([73], [74], [34], [22], [64] and [89]).

1.3 Main results of the thesis

In this chapter, we have touched upon some basic properties of dispersive equations. We started from the free Schrödinger equation as a standard

model of a dispersive equation. We then introduced various variations of the free Schrödinger equation, and discussed how each variation changes the behaviour of solutions. We have also introduced and discussed some of the main mathematical questions, such as the existence, stability and asymptotic behaviours of the solutions.

In Chapter 2, we study the asymptotic stability of bound states of the nonlinear Schrödinger equation with a magnetic potential in \mathbb{R}^3 . Here, we will consider the equation

$$i\psi_t = H\psi + \lambda|\psi|^2\psi \quad (1.139)$$

where the operator

$$H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V \quad (1.140)$$

and $\lambda = \pm 1$. Here, $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a vector (magnetic) potential modelling the magnetic field and $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a scalar (electric) potential modelling an electric field. Without the nonlinear term, the equation

$$i\psi_t = H\psi \quad (1.141)$$

models the time evolution of a quantum particle (such as an electron) in the presence of a magnetic and electric field. The addition of the nonlinear term allows different linear modes of the solution interact with each other and the time evolution of the equation is more complex.

We will consider the case where H admits a single eigenvalue e_0 with the eigenfunction ϕ_0 . In other words,

$$H\phi_0 = e_0\phi_0 \quad (1.142)$$

and hence, $e^{-ie_0t}\phi_0(x)$ is a solution of the linear equation (1.141). We will show in Chapter 2 that under suitable assumptions on A and V , such as sufficient decay as $|x| \rightarrow \infty$, equation (1.139) admits nonlinear bound states

solutions of the form

$$u(x, t) = e^{-iEt}Q(x). \quad (1.143)$$

In fact, we will show that, just like the case where $A = 0$, there exists a one (complex) parameter family of these nonlinear bound states and if we let z be the complex parameter, then E and Q has the form

$$E[z] = e_0 + o(z) \quad \text{and} \quad Q[z] = z\phi_0 + q(z) \quad (1.144)$$

for sufficiently small z .

Lemma 1.3.1. *(Existence and decay of nonlinear bound states) For each sufficiently small $z \in \mathbb{C}$, there is a corresponding eigenfunction $Q[z] \in H^2$ solving the nonlinear eigenvalue problem*

$$HQ + g(Q) = EQ \quad (1.145)$$

with the corresponding eigenvalue $E[z] = e_0 + o(z)$ and $Q[z] = z\phi_0 + q(z)$ with

$$q(z) = o(z^2), \quad DQ[z] = (1, i)\phi_0 + o(z) \quad \text{and} \quad D^2Q[z] = o(1) \quad \text{in } H^2 \quad (1.146)$$

where we denote

$$DQ[z] = (D_1Q[z], D_2Q[z]) = \left(\frac{\partial}{\partial z_1}Q[z], \frac{\partial}{\partial z_2}Q[z] \right), \quad \text{and} \quad z = z_1 + iz_2. \quad (1.147)$$

Furthermore, Q has exponential decay in the sense that

$$e^{\beta|x|}Q \in H^1 \cap L^\infty \quad (1.148)$$

for some $\beta > 0$ (independent of z).

Then we will describe a result regarding asymptotic stability of these small nonlinear bound states.

Theorem 3. *(Asymptotic stability of nonlinear bound states) Under various assumptions to be described in chapter 2. For $0 \leq t < \infty$, every solution*

1.3. Main results of the thesis

ψ of equation (1.139) with initial data ψ_0 sufficiently small in H^1 can be uniquely decomposed as

$$\psi(t) = Q[z(t)] + \eta(t), \quad (1.149)$$

with differentiable $z(t) \in \mathbb{C}$ and $\eta(t) \in H^1$ satisfying $\langle i\eta, D_1 Q[z] \rangle = 0$, $\langle i\eta, D_2 Q[z] \rangle = 0$ and

$$\|\eta\|_X \lesssim \|\psi_0\|_{H^1}, \quad \|\dot{z} + iE[z]z\|_{L_t^1} \lesssim \|\psi_0\|_{H^1}^2. \quad (1.150)$$

Furthermore, as $t \rightarrow \infty$,

$$z(t) \exp\left(i \int_0^t E[z(s)] ds\right) \rightarrow z_+, \quad E[z(t)] \rightarrow E(z_+) \quad (1.151)$$

for some $z_+ \in \mathbb{C}$ and

$$\|\eta(t) - e^{-itH} \eta_+\|_{H_x^1} \rightarrow 0 \quad (1.152)$$

for some $\eta_+ \in H_x^1 \cap \text{Range}(P_c)$.

My work builds on the work of [68], [61] and [40]. The works [68] and [61] consider nonlinear bound states that are small in both the H^1 -norm and a weighted L^2 -norm, while the work [40] considers nonlinear bound states that are small in only the H^1 -norm. As a result, my result is closer to that of [40]. However, all of the previous results consider only the scalar potential case ($A \equiv 0$) while I consider both the scalar and vector potential. More discussion will be given in Chapter 2.

In Chapter 3, we will consider the Schrödinger map equation discussed in Section 1.1.2 of this chapter. There, we will consider the equation

$$\begin{cases} \vec{u}_t = \vec{u} \times \Delta \vec{u} \\ \vec{u}(x, 0) = \vec{u}_0(x) \end{cases} \quad (1.153)$$

where $\vec{u} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{S}^2$. We consider the question of global well-posedness for radial solutions for $n = 2$. In particular, we will prove the following

theorem in Chapter 3.

Theorem 4. (*Global well-posedness of 2D radial Schrödinger map into \mathbb{S}^2*)
Suppose $\vec{u}(x, 0) = \vec{u}_0(x)$ is radial and $\vec{u}_0 - \hat{k} \in H^2(\mathbb{R}^2)$. Then

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad \text{with} \quad \vec{u}(r, 0) = \vec{u}_0(r), \quad r = |x| \quad (1.154)$$

has a unique global solution $\vec{u} \in L^\infty([0, \infty); H^2(\mathbb{R}^2))$.

Results similar to Theorem 4 under the extra assumption that the energy,

$$\mathcal{E}(\vec{u}) := \frac{1}{2} \|\nabla \vec{u}(t)\|_{L^2(\mathbb{R}^2)}^2, \quad (1.155)$$

is small has been obtained over the decade or so. [16] showed that for $n = 2$ if $\mathcal{E}(\vec{u}_0)$ sufficiently small, radial solutions to (1.153) are global. A very recent result by [4] showed that for $n = 2$, suppose the initial data u_0 satisfies $\vec{u}_0 - Q \in H^s$ for all $s > 0$ for some $Q \in \mathbb{S}^2$ and $\mathcal{E}(\vec{u}_0)$ is small, then the solution \vec{u} to (3.1) is global and $\vec{u} - Q \in H^s$ for all $s > 0$. Our result is the first that shows global well-posedness in $n = 2$ without the assumption of solutions having small energy. More discussions will be given in Chapter 3.

Chapter 2

Asymptotic stability of small solitary waves for nonlinear Schrödinger equations with electromagnetic potential in \mathbb{R}^3

2.1 An overview

The goal of this chapter is to prove Theorem 3 stated in Section 1.3. We will start with an overview that will put our result in perspective with the known results. For this, let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that $-\Delta + V$ has an eigenvalue e_0 with the corresponding eigenfunction ϕ_0 . Now, consider the nonlinear Schrödinger equation

$$i\psi_t = -\Delta\psi + V\psi + |\psi|^2\psi. \quad (2.1)$$

Such nonlinear Schrödinger equations find numerous physical applications, for example, in Bose-Einstein condensates and nonlinear optics.

As mentioned in Chapter 1, under suitable assumptions on V , equation (2.1) admits a one-(complex)-parameter family of nonlinear bound states solutions $Q[z]$ and the corresponding eigenvalue $E[z]$ for sufficiently small z . Further analysis on the structure of Q and E reveals that the first order

2.2. Our result

dependence of Q and E on z is given by

$$Q[z] = z\phi_0 + O(z^3) \quad \text{and} \quad E[z] = e_0 + O(z). \quad (2.2)$$

As z is a small parameter, we say that $Q[z]$ emerges (bifurcates) from the zero solution along the eigenfunction ϕ_0 of the linear operator $-\Delta\psi + V\psi$ under the perturbation of the nonlinear term $|\psi|^2\psi$.

Asymptotic stability of these nonlinear bound states has been studied by various authors. As described in Chapter 1, for the case where $-\Delta + V$ has exactly one eigenvalue, asymptotic stability has been proved by authors such as [68], [61], [40], [50] and [57]. In the more complicated case where $-\Delta + V$ has more than one eigenvalue, the nonlinear bound states with lowest eigenvalue (ground states) may still be asymptotically stable. This situation has been studied by authors such as [87], [86], [70] and [33].

2.2 Our result

The previous results on asymptotic stability of bound states of equation (2.1) are for scalar potentials $V : \mathbb{R}^n \rightarrow \mathbb{R}$. The goal here is to extend these results with the addition of a vector potential. In particular, we consider the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t\psi = (-\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V)\psi + g(\psi) \\ \psi(x, 0) = \psi_0(x) \in H^1(\mathbb{R}^3) \end{cases} \quad (2.3)$$

for $\psi(x, t) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$, where

$$g(\psi) = \pm|\psi|^2\psi. \quad (2.4)$$

Here, $A(x) = (A_1(x), A_2(x), A_3(x)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the magnetic potential (also known as the vector potential) and $V(x) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the electric potential (also known as the scalar potential). Equation (2.3) can be equiv-

2.2. Our result

alently written as

$$i\partial_t\psi = (i\nabla + A)^2\psi + V\psi + g(\psi) \quad (2.5)$$

by replacing V with $V - |A|^2$. We will only consider potentials $A(x)$ and $V(x)$ which decay to 0 as $|x| \rightarrow \infty$.

Equation (2.3) describes a charged quantum particle subject to external electric and magnetic fields, and a self-interaction (nonlinearity).

Just as equation (2.1), under certain assumptions on V and A , equation (2.3) admits standing wave solutions (or nonlinear bound states) of the form

$$\psi(x, t) = e^{-iEt}Q(x). \quad (2.6)$$

The existence of standing waves to equation (2.3) for certain electric and magnetic potentials was first proved in [31].

Here we consider small solutions of the form (2.6) which bifurcate from zero along an eigenvalue of the linear Hamiltonian operator

$$H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V. \quad (2.7)$$

Physical intuition suggests that the ground-state standing wave (the one corresponding to the lowest eigenvalue E) should remain stable when the self-interaction (nonlinearity) is turned on, and should become *asymptotically stable* (that is, nearby solutions should relax to the ground state by radiating excess energy to infinity – see below for a more precise statement). The main goal of this chapter is to prove asymptotic stability of the ground state, in the energy space (H^1), and in the presence of both the electric and magnetic field.

Remark 1. Our argument should also go through for nonlinearities $g(\psi) = \pm|\psi|^{p-1}\psi$ for $\frac{7}{3} \leq p < 5$, or combinations of these. For concreteness, we will work with $g(\psi) = \pm|\psi|^2\psi$.

In order to study equation (2.3), we need the operator H to be self-adjoint. To ensure this, we make the following assumption,

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Assumption 1. (Self-adjointness assumption) We assume that each component of A is a real-valued function in $L^q + L^\infty$ for some $q > 3$, that $\nabla \cdot A \in L^2 + L^\infty$, and that V is a real-valued function in $L^2 + L^\infty$.

Then by Theorem X.22 of [62], the operator H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.

We will only consider the case where H has only one eigenvalue. More precisely, we make the following assumption.

Assumption 2. (Spectral assumption) We assume that H supports only one eigenvalue $e_0 < 0$, which is nondegenerate. We also assume 0 is not a resonance of H (see e.g. [30] for the definition of resonance).

By the above assumption, H supports only one eigenvalue $e_0 < 0$. Let $\phi_0 > 0$ be the positive, L^2 -normalized eigenfunction corresponding to the eigenvalue e_0 of H .

We need the following assumption to show the existence and exponential decay of the nonlinear bound states.

Assumption 3. (Assumptions for existence and exponential decay of nonlinear bound states) We assume

$$\|A\|_{L^q + L^\infty(|x| > R)} + \|V_-\|_{L^2 + L^\infty(|x| > R)} \rightarrow 0 \text{ as } R \rightarrow \infty \quad (2.8)$$

for some $q > 3$.

Under the above assumptions, standing waves Q for E near e_0 bifurcate from the zero solution along ϕ_0 , we have the following lemma on the existence and decay of nonlinear bound states.

Lemma 2.2.1. (*Existence and decay of nonlinear bound states*) For each sufficiently small $z \in \mathbb{C}$, there is a corresponding eigenfunction $Q[z] \in H^2$ solving the nonlinear eigenvalue problem

$$HQ + g(Q) = EQ \quad (2.9)$$

with the corresponding eigenvalue $E[z] = e_0 + o(z)$, and $Q[z](x) = z\phi_0 +$

2.2. Our result

$q[z](x)$ with

$$q(z) = o(z^2), \quad DQ[z] = (1, i)\phi_0 + o(z) \quad \text{and} \quad D^2Q[z] = o(1) \quad \text{in } H^2 \quad (2.10)$$

where we denote

$$DQ[z] = (D_1Q[z], D_2Q[z]) = \left(\frac{\partial}{\partial z_1}Q[z], \frac{\partial}{\partial z_2}Q[z] \right), \quad \text{and } z = z_1 + iz_2. \quad (2.11)$$

Furthermore, Q has exponential decay in the sense that

$$e^{\beta|x|}Q \in H^1 \cap L^\infty \quad (2.12)$$

for some $\beta > 0$ (independent of z).

Next, we need assumptions on A and V which ensure our linear Schrödinger evolution obeys some dispersive estimates. For $f, g \in L^2(\mathbb{R}^3, \mathbb{C})$, define the real inner product $\langle f, g \rangle$ by

$$\langle f, g \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^3} \bar{f} g dx \right). \quad (2.13)$$

Denote $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and fix $\sigma > 4$. Let P_c be the projection onto the continuous spectral subspace of H . Following [30], we have:

Assumption 4. (Strichartz estimates assumption) We assume that for all $x, \xi \in \mathbb{R}^3$,

$$|A(x)| + \langle x \rangle |V(x)| \lesssim \langle x \rangle^{-1-\varepsilon}, \quad (2.14)$$

$$\langle x \rangle^{1+\varepsilon'} A(x) \in \dot{W}^{\frac{1}{2}, 6}(\mathbb{R}^3), \quad (2.15)$$

and

$$A \in C^0(\mathbb{R}^3) \quad (2.16)$$

for some $\varepsilon > 0$ and some $\varepsilon' \in (0, \varepsilon)$.

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Define the space-time norm

$$\|\psi\|_X = \|\langle x \rangle^{-\sigma} \psi\|_{L_t^2 H_x^1} + \|\psi\|_{L_t^3 W_x^{1, \frac{18}{5}}} + \|\psi\|_{L_t^\infty H_x^1}.$$

We can now state the main result, which says that all H^1 -small solutions converge to a solitary wave (nonlinear bound state) as $t \rightarrow \infty$:

Theorem 5. (*Asymptotic stability of small solitary waves*) *Let assumptions 1, 2, 3 and 4 hold. For $0 \leq t < \infty$, every solution ψ of equation (2.3) with initial data ψ_0 sufficiently small in H^1 can be uniquely decomposed as*

$$\psi(t) = Q[z(t)] + \eta(t), \quad (2.17)$$

with differentiable $z(t) \in \mathbb{C}$ and $\eta(t) \in H^1$ satisfying $\langle i\eta, D_1 Q[z] \rangle = 0$, $\langle i\eta, D_2 Q[z] \rangle = 0$ and

$$\|\eta\|_X \lesssim \|\psi_0\|_{H^1}, \quad \|\dot{z} + iE[z]z\|_{L_t^1} \lesssim \|\psi_0\|_{H^1}^2. \quad (2.18)$$

Furthermore, as $t \rightarrow \infty$,

$$z(t) \exp\left(i \int_0^t E[z(s)] ds\right) \rightarrow z_+, \quad E(z(t)) \rightarrow E(z_+) \quad (2.19)$$

for some $z_+ \in \mathbb{C}$ and

$$\|\eta(t) - e^{-itH} \eta_+\|_{H_x^1} \rightarrow 0 \quad (2.20)$$

for some $\eta_+ \in H_x^1 \cap \text{Range}(P_c)$.

For comparison, consider the nonlinear Schrödinger equation with just a scalar potential V ,

$$i\partial_t \psi = (-\Delta + V)\psi + g(\psi) \quad (2.21)$$

for the same nonlinearity g as above, which is a special case of equation (2.3) with $A = 0$. The corresponding asymptotic stability result for (2.21) was obtained in dimension three in [40], in dimension one in [57] and in

dimension two in [50, 56].

2.3 Discussion and outline of the proof

In this section, we will give an outline of the proof and discuss difficulties encountered when proving the results. The detailed proof will be given in the next section.

There are three main parts to the proof. The first part is to show the existence of nonlinear bound state solutions of the form (2.6) of equation (2.3). Substituting (2.6) into (2.3), we see that Q satisfies

$$(i\nabla + A)^2 Q + VQ = EQ - g(Q) \text{ where } g(Q) = \pm|Q|^2 Q. \quad (2.22)$$

For our stability argument, it is essential to have sufficient decay and regularity for the standing wave Q . We will show that under Assumption 3, standing waves Q with E near e_0 bifurcate from the zero solution along ϕ_0 , and such standing waves decay exponentially at ∞ . This result is stated in Lemma 2.2.1 and is proven by an contracting mapping argument. The exponential decay as $|x| \rightarrow \infty$ of Q is shown by showing that Q is uniformly bounded in H^1 with a local exponential weight. The detailed proof of Lemma 2.2.1 will be given in Section 2.4.1.

The second part of the proof is to establish various estimates used to show the main result. Our approach for showing Theorem 5 will be similar to that used by [40] for showing corresponding results for $A = 0$. There, [40] uses the Strichartz estimates

$$\|e^{it(\Delta-V)} P_c \phi\|_{\tilde{X}} \lesssim \|\phi\|_{H^1} \quad (2.23)$$

and

$$\left\| \int_{-\infty}^t e^{i(t-s)(\Delta-V)} P_c F(s) ds \right\|_{\tilde{X}} \lesssim \|F\|_{L_t^2 W^{1, \frac{6}{5}}} \quad (2.24)$$

where $\tilde{X} = L_t^\infty H^1 \cap L_t^2 W^{1,6} \cap L_t^2 L^{6,2}$, which are known to hold for a class of scalar potentials V . Our approach will use the Strichartz estimates for H from [30]. However, the proof of [30] of the inhomogeneous Strichartz

2.3. Discussion and outline of the proof

estimates

$$\left\| \int_{-\infty}^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}} \quad (2.25)$$

for $H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V$ uses a lemma from [17] which does not hold for the endpoint case $(q, p) = (2, 6)$ or $(\tilde{q}, \tilde{p}) = (2, 6)$. To overcome the lack of endpoint Strichartz estimates, we will use estimates in weighted spaces, as in [57] and [56]. The extension of these weighted-space estimates in the presence of a vector potential turns out to be somewhat involved, and is the most difficult and novel part of the work. We will establish these estimates in Subsection 2.4.2.

The last part is the actual proof of our main result which can be found in Subsection 2.4.3. The strategy is as follows: if we substitute

$$\psi = Q[z(t)](x) + \eta(x, t) \quad (2.26)$$

into (2.3), after some manipulations of terms, we get that η satisfies the equation

$$i\partial_t \eta = H\eta + F \quad \text{where} \quad F = \pm|Q+\eta|^2(Q+\eta) \mp |Q|^2 Q - iDQ(\dot{z} + iEz). \quad (2.27)$$

Here, for $z = z_1 + iz_2$ and for $w \in \mathbb{C}$, denote

$$DQ[z]w = \frac{\partial}{\partial z_1} Q[z] \operatorname{Re} w + \frac{\partial}{\partial z_2} Q[z] \operatorname{Im} w. \quad (2.28)$$

A key idea, as in [40], is to choose $z(t)$, at each time t , so that the orthogonality conditions

$$\langle iD_1 Q, \eta \rangle = \langle iD_2 Q, \eta \rangle = 0 \quad (2.29)$$

hold. Further manipulations then show that the quantity

$$|\dot{z} + iEz| \lesssim |\langle 2Q|\eta|^2 + \overline{Q}\eta^2 + |\eta|^2 \eta, DQ \rangle| (1 + \|\eta\|_{L^2}) \quad (2.30)$$

is quadratic in η (terms linear in η have cancelled out).

To prove the main theorem, we are faced with two tasks:

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1. First, we would like to show that if $\psi(0)$ is small in H^1 , then η remains small for all future time, and indeed scatters.
2. Second, we would like to show $|\dot{z} + iEz|$ stays small in L_t^1 -norm which implies convergence of $z(t) \exp(i \int_0^t E[z(s)] ds)$ as $t \rightarrow \infty$.

Roughly speaking, our strategy is to find some space-time norm X such that the following holds

$$\|\eta\|_X \lesssim \|\eta(0)\|_{H^1} + \|\eta\|_X^a \text{ for some } a > 1. \quad (2.31)$$

Once (2.31) has been shown, smallness of η for all time follows by a continuity argument. The idea is that since η starts out small, the constraint (2.31) posts a limit on how large η can get and hence, η has to stay small for all future times. Once the smallness of η has been shown, we can use (2.30) to complete the second task. More precisely, we need X to provide time-decay of η at the level of L_t^2 , so that (2.30) will control $\|\dot{z} + iEz\|_{L_t^1}$.

The main tools used to complete the tasks are Strichartz-type estimates which we briefly explained before. If we write (2.27) in the integral form, it becomes

$$\eta = e^{-itH} \left(\eta(0) - i \int_0^t e^{isH} F(s) ds \right). \quad (2.32)$$

One may attempt to use Strichartz-type estimates on (2.32) to obtain (2.31). However, such an attempt will fail as η may contain a component of the “discrete spectrum” (i.e. eigenfunction ϕ_0) of H which does not have decay properties needed for Strichartz-type estimates to hold. Instead, let P_c denote the projection onto the continuous spectral subspace of H and let $\eta_c = P_c \eta$, then η_c satisfies

$$\eta_c = e^{-itH} P_c \eta(0) - i \int_0^t e^{-i(t-s)H} P_c F(s) ds. \quad (2.33)$$

Using Lemma 2.2 of [40], it can be shown that $\|\eta\|_Y \lesssim \|\eta_c\|_Y$ for any reasonable space Y since η satisfies the orthogonality condition (2.29). We then

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have

$$\|\eta\|_X \lesssim \|\eta_c\|_X \leq \|e^{-itH} P_c \eta(0)\|_X + \left\| \int_0^t e^{-i(t-s)H} P_c F(s) ds \right\|_X. \quad (2.34)$$

Our strategy will be to use Strichartz estimates on (2.34) to obtain (2.31). For Strichartz estimates, we mean space-time bounds on the evolution operator e^{-itH} such as

$$\|e^{-itH} P_c f\|_{L_t^q L_x^p} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (2.35)$$

and

$$\left\| \int_0^t e^{-i(t-s)H} P_c F(s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}. \quad (2.36)$$

Under sufficient decay and regularity of A and V , [30] showed the above estimates hold for (p, q) and (\tilde{p}, \tilde{q}) satisfying

$$\frac{2}{q} + \frac{3}{p} = \frac{3}{2} \quad \text{with } 2 \leq p < 6. \quad (2.37)$$

Here, (2.35) is useful for bounding terms like $\|e^{-itH} P_c \eta(0)\|_X$ in (2.34) and (2.36) is useful for bounding terms like $\left\| \int_0^t e^{-i(t-s)H} P_c F(s) ds \right\|_X$.

However, (2.35) and (2.36) are not enough to finish the proof. There are two major obstacles:

1. First, since Q has no decay in time, Q cannot be in L_t^q for $q < \infty$. To control the right hand side of (2.30) in L_t^1 , we need to control terms of the form $\|\eta\|_{L_t^2 L_x^p}$ for some p , but this is not covered by (2.35) or (2.36).
2. Second, to control terms like $\|\eta^3\|_{L_t^q L_x^p}$ on the right hand side of (2.34), we will use Gagliardo-Nirenberg inequality

$$\|u\|_{L_x^p} \lesssim \|\nabla u\|_{L_x^2}^\theta \|u\|_{L_x^q}^{1-\theta} \quad \text{for appropriate values of } p, q \text{ and } \theta. \quad (2.38)$$

What this means is that we need to control $\nabla \eta$ in some norm as well.

The first obstacle can be overcome as follows. While Strichartz estimates

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(2.35) and (2.36) for $L_t^2 L_x^6$ is not available, a weighted version of (2.35) for $L_t^2 L_x^2$ is available from [30] and a similar weighted version of (2.36) can be derived from results in [30].

The second obstacle is more difficult to overcome. Here, our approach is to derive versions of (2.35) and (2.36) for $L_t^q W_x^{1,p}$ as well as similar estimates for the weighted of (2.35) and (2.36). This requires some work. Our strategy is to show

$$\|u\|_{W^{1,p}} \sim \|H_1^{\frac{1}{2}} u\|_{L^p} \quad \text{and} \quad \|u\|_{\langle x \rangle^s H^1} \sim \|H_1^{\frac{1}{2}} u\|_{\langle x \rangle^s L^2}. \quad (2.39)$$

for some operator H_1 which commutes with H . Once (2.39) is achieved, the estimates we want can be obtained by commuting H_1 through both sides of the expressions (2.35) and (2.36). Of all the estimates, the estimate

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 H_x^1} \lesssim \|\langle x \rangle^\sigma F\|_{L_t^2 H_x^1} \quad (2.40)$$

is the most difficult to achieve. The reason is the operator H_1 does not commute with the factor $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. As a result, (2.40) does not trivially follow from (2.39).

2.4 Detailed proof

2.4.1 Existence and decay of standing waves

The following is the proof for Lemma 2.2.1, the existence and exponential decay of nonlinear bound states.

Proof of existence of nonlinear bound states:

For each small $z \in \mathbb{C}$, we look for a solution

$$Q = z\phi_0 + q \quad \text{and} \quad E = e_0 + e' \quad (2.41)$$

of

$$(-\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V)Q + g(Q) = EQ \quad (2.42)$$

with $(\phi_0, q) = 0$ and $e' \in \mathbb{R}$ small.

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Let $H_0 = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V - e_0$. If we substitute $Q = z\phi_0 + q$ and $E = e_0 + e'$ into equation (2.42), we get

$$H_0 q + g(z\phi_0 + q) = e'(z\phi_0) + e'q. \quad (2.43)$$

Projecting equation (2.43) on the ϕ_0 and ϕ_0^\perp directions, we get

$$e'z = (\phi_0, g(z\phi_0 + q)) \quad (2.44)$$

and

$$H_0 q = -P_c g(z\phi_0 + q) + e'q. \quad (2.45)$$

Now, let

$$K = \{(q, e') \in H_\perp^2 \times \mathbb{R} \mid \|q\|_{H^2} \leq |z|^2, |e'| \leq |z|\} \quad (2.46)$$

for sufficiently small $z \in \mathbb{C}$ where $H_\perp^2 = \{q \in H^2 \mid (q, \phi_0) = 0\}$. Also, define the map $M : (q_0, e'_0) \mapsto (q_1, e'_1)$ by

$$g_0 := g(z\phi_0 + q_0), \quad (2.47)$$

$$ze'_1 := (\phi_0, g_0) \quad (2.48)$$

and

$$q_1 := H_0^{-1}(-P_c g_0 + e'_0 q_0). \quad (2.49)$$

Now if $(q_0, e'_0) \in K$, we have

$$|ze'_1| = |(\phi_0, g_0)| = |(\phi_0, g(z\phi_0 + q_0))| = |(\phi_0, |z\phi_0 + q_0|^2(z\phi_0 + q_0))| \lesssim O(z^3) \quad (2.50)$$

and

$$\|q_1\|_{H^2} \lesssim \|-P_c g_0 + e'_0 q_0\|_{L^2} \leq \|g_0\|_{L^2} + |e'_0| \|q_0\|_{H^2} \lesssim O(z^3). \quad (2.51)$$

Therefore, $|e'_1| \lesssim O(z^2)$ and $\|q_1\|_{H^2} \lesssim O(z^3)$. This shows that M maps K

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into K for sufficiently small z .

Next, we would like to show that M is a contraction mapping. Let $(a_1, b_1) := M(q_0, e'_0)$ and $(a_2, b_2) := M(q_1, e'_1)$ with $g_j = g(z\phi_0 + q_j)$ for $j = 0, 1$. Then

$$\begin{aligned}
 |z(b_2 - b_1)| &= |(\phi_0, g_0 - g_1)| \\
 &= |(\phi_0, g(z\phi_0 + q_0) - g(z\phi_0 + q_1))| \\
 &= |(\phi_0, |z\phi_0 + q_0|^2(z\phi_0 + q_0) - |z\phi_0 + q_1|^2(z\phi_0 + q_1))| \\
 &\lesssim \int \phi_0(|z|^2\phi_0^2 + |q_0|^2 + |q_1|^2)|q_0 - q_1| \lesssim |z|^2\|q_0 - q_1\|_{L^2}.
 \end{aligned}$$

As $a_i = H_0^{-1}(-P_c g_{i-1} + e'_{i-1} q_{i-1})$ for $i = 1, 2$ and $\|H_0^{-1}\|_{L^2 \rightarrow H^2} \leq \infty$, we have

$$\begin{aligned}
 \|a_1 - a_2\|_{H^2} &\lesssim \|P_c(g_1 - g_0) + e'_0 q_0 - e'_1 q_1\|_{L^2} \\
 &\lesssim \|g_1 - g_0\|_{L^2} + |e'_0 - e'_1| \|q_0\|_{L^2} + |e'_1| \|q_0 - q_1\|_{L^2}.
 \end{aligned}$$

Since

$$\begin{aligned}
 \|g_1 - g_0\|_{L^2} &= \|g(z\phi_0 + q_1) - g(z\phi_0 + q_0)\|_{L^2} \\
 &\lesssim |z|^2 \|\phi_0^2(q_1 - q_2)\|_{L^2} + |z| \|\phi_0(q_1^2 - q_2^2)\|_{L^2} + \|q_1^3 - q_2^3\|_{L^2} \\
 &\lesssim |z|^2 \|\phi_0^2\|_{L^3} \|q_1 - q_2\|_{L^6} + |z| \|\phi_0\|_{L^6} \|q_1 + q_2\|_{L^6} \|q_1 - q_2\|_{L^6} \\
 &\quad + \|(|q_1|^2 + |q_1 q_2| + |q_2|^2)\|_{L^4} \|q_1 - q_2\|_{L^4},
 \end{aligned}$$

together, we have

$$\|a_1 - a_2\|_{H^2} \lesssim |z| \|q_1 - q_2\|_{H^2} + |z|^2 |e'_0 - e'_1|. \quad (2.52)$$

Hence, M is a contraction mapping for z sufficiently small. Now by the contraction mapping theorem, there exists a unique fixed point (q, e') satisfying $\|q\|_{H^2} = O(z^3)$ and $|e'| = O(z^2)$ as $z \rightarrow 0$. The statements about derivatives of Q and E with respect to z follow by differentiating (2.43) with respect to z and applying the contraction mapping principle again.

Proof of exponential decay:

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Lemma 2.4.1. For $\varepsilon > 0$, define the exponential weight function χ_R by

$$\chi_{R,\varepsilon} = \begin{cases} e^{\varepsilon(|x|-R)} - 1 & \text{if } R < |x| \leq 2R, \\ e^{\varepsilon(3R-|x|)} - 1 & \text{if } 2R < |x| < 3R, \\ 0 & \text{else} \end{cases} \quad (2.53)$$

Suppose for $\varepsilon > 0$ small enough, $f \in H^1$ satisfies

$$\|\chi_{R,\varepsilon} f\|_{H^1} \leq C \quad (2.54)$$

for some constant C independent of R , then

$$e^{\varepsilon'|x|} f \in H^1 \quad (2.55)$$

for some $\varepsilon' > 0$.

Proof. For $R > 0$, $\|\chi_{R,\varepsilon} f\|_{H^1} \leq C$ implies that

$$\|(e^{\varepsilon(|x|-R)} - 1)f\|_{H^1[\frac{3}{2}R, 2R]} \leq C. \quad (2.56)$$

Since $f \in H^1$,

$$\|e^{\varepsilon(|x|-R)} f\|_{H^1[\frac{3}{2}R, 2R]} \leq C + \|f\|_{H^1} \leq C'. \quad (2.57)$$

$e^{\frac{1}{2}\varepsilon R} \leq e^{\varepsilon(|x|-R)}$ for $|x| \in [\frac{3}{2}R, 2R]$, so

$$\|e^{\frac{1}{2}\varepsilon R} f\|_{H^1[\frac{3}{2}R, 2R]} \leq C'. \quad (2.58)$$

So

$$\|e^{(\frac{1}{2}(\frac{1}{2}\varepsilon))(2R)} f\|_{H^1[\frac{3}{2}R, 2R]} \leq C'. \quad (2.59)$$

Let $\varepsilon' = (\frac{1}{2}(\frac{1}{2}\varepsilon))$. Using $e^{\varepsilon' 2R} \geq e^{\varepsilon'|x|}$ for $|x| \in [\frac{3}{2}R, 2R]$, we get that

$$\|e^{\varepsilon'|x|} f\|_{H^1[\frac{3}{2}R, 2R]} \leq C' \quad (2.60)$$

for some constant C' independent of R .

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Let $\varepsilon'' = \frac{1}{2}\varepsilon'$. Then

$$\|e^{\varepsilon''|x|}f\|_{H^1(|x|>1)}^2 = \sum_{k=0}^{\infty} \|e^{\varepsilon''|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]}^2. \quad (2.61)$$

Now, for each k , since $e^{\varepsilon'} = e^{\varepsilon''} e^{\varepsilon''}$, taking $R = \frac{2^{2k+1}}{3^{k+1}}$ in (2.60), we have

$$\begin{aligned} C' &\geq \|e^{\varepsilon'|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]} \\ &= \|e^{\varepsilon''|x|}e^{\varepsilon''|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]} \\ &\geq e^{(\varepsilon''\frac{2^{2k}}{3^k})} \|e^{\varepsilon''|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]}. \end{aligned}$$

This means that,

$$\|e^{\varepsilon''|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]} \leq C' e^{-(\varepsilon''\frac{2^{2k}}{3^k})} \quad (2.62)$$

Therefore,

$$\begin{aligned} \|e^{\varepsilon''|x|}f\|_{H^1(|x|>1)}^2 &= \sum_{k=0}^{\infty} \|e^{\varepsilon''|x|}f\|_{H^1[\frac{2^{2k}}{3^k}, \frac{2^{2(k+1)}}{3^{k+1}}]}^2 \\ &\leq C'^2 \sum_{k=0}^{\infty} e^{-\varepsilon''\frac{2^{2k+1}}{3^k}} \\ &< \infty \end{aligned}$$

□

By Lemma 2.4.1, to show that $\|e^{\alpha|x|}Q\|_{H^1} < \infty$ for some $\alpha > 0$, it suffices to show that $\|\chi_{R,\varepsilon}Q\|_{H^1} \leq C$ for some constant C independent of R . Here, $\chi_{R,\varepsilon}$ is the exponential weight function as in Lemma 2.4.1.

Consider the bilinear form

$$\mathcal{E}(\psi, \phi) = (\nabla\psi, \nabla\phi) + i \int (2\bar{\psi}A \cdot \nabla\phi + \bar{\psi}(\nabla \cdot A)\phi) dx + \int V\bar{\psi}\phi dx \quad \text{for } \psi, \phi \in H^1 \quad (2.63)$$

associated to the magnetic Schrödinger operator $-\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V$.

Then

$$\begin{aligned}\mathcal{E}(\psi, \psi) &= (\nabla\psi, \nabla\psi) + i \int (2\bar{\psi}A \cdot \nabla\psi + \bar{\psi}(\nabla \cdot A)\psi)dx + \int V\bar{\psi}\psi dx \\ &= (\nabla\psi, \nabla\psi) + 2\operatorname{Im}(\int \bar{\psi}A \cdot \nabla\psi dx) + \int V\bar{\psi}\psi dx\end{aligned}$$

Set

$$b := \lim_{R \rightarrow \infty} \inf \{ \mathcal{E}(\phi, \phi) \mid \phi \in H^1, \|\phi\|_2 = 1, \phi(x) = 0 \text{ for } |x| < R \}. \quad (2.64)$$

We will show that $b \geq 0$ by contradiction. Suppose $b < 0$. Then there exists a sequence $\phi_{R_j} \in H^1$ with $R_j \rightarrow \infty$, satisfying $\|\phi_{R_j}\|_2 = 1$, $\phi_{R_j}(x) = 0$ for $|x| < R_j$, and $\mathcal{E}(\phi_{R_j}, \phi_{R_j}) < \delta$ for some fixed $\delta < 0$.

Suppose $V \in L^\infty$, then

$$\int V \overline{\phi_{R_j}} \phi_{R_j} dx \leq \|V\|_\infty \|\phi_{R_j}\|_2^2 = \|V\|_\infty. \quad (2.65)$$

Suppose $V \in L^2$, then

$$\begin{aligned}\int V \overline{\phi_{R_j}} \phi_{R_j} dx &\leq \|V\|_2 \|\phi_{R_j}\|_4^2 \\ &\lesssim \|V\|_2 \|\phi_{R_j}\|_2^{\frac{1}{2}} \|\nabla \phi_{R_j}\|_2^{\frac{3}{2}} \\ &\lesssim \tilde{\delta} (\|\nabla \phi_{R_j}\|_2^{\frac{3}{2}})^{\frac{4}{3}} + \frac{1}{\tilde{\delta}} (\|V\|_2 \|\phi_{R_j}\|_2^{\frac{1}{2}})^4 \\ &= \tilde{\delta} \|\nabla \phi_{R_j}\|_2^2 + \frac{1}{\tilde{\delta}} \|V\|_2^4.\end{aligned}$$

Hence,

$$\int V \overline{\phi_{R_j}} \phi_{R_j} dx \lesssim \tilde{\delta} \|\nabla \phi_{R_j}\|_2^2 + \frac{1}{\tilde{\delta}} \|V\|_{L^\infty + L^2}^4 \quad \text{where } \tilde{\delta} \text{ is sufficiently small.} \quad (2.66)$$

Similarly, suppose $A \in L^\infty$, then

$$|(\phi_{R_j}, A \cdot \nabla \phi_{R_j})| \leq \|A\|_\infty \|\phi_{R_j}\|_2 \|\nabla \phi_{R_j}\|_2 = \|A\|_\infty \|\nabla \phi_{R_j}\|_2. \quad (2.67)$$

2.4. Detailed proof

On the other hand, suppose $A \in L^{(3+\tilde{\varepsilon})}$, then

$$\begin{aligned}
|(\phi_{R_j}, A \cdot \nabla \phi_{R_j})| &\leq \|A\|_{(3+\tilde{\varepsilon})} \|\phi_{R_j}\|_{\frac{2(3+\tilde{\varepsilon})}{1+\tilde{\varepsilon}}} \|\nabla \phi_{R_j}\|_2 \\
&\lesssim \|A\|_{(3+\tilde{\varepsilon})} \|\phi_{R_j}\|_2^{\frac{5}{2} + \frac{3(1+\tilde{\varepsilon})}{2(3+\tilde{\varepsilon})}} \|\nabla \phi_{R_j}\|_2^{\frac{5}{2} - \frac{3(1+\tilde{\varepsilon})}{2(3+\tilde{\varepsilon})}} \\
&= \|A\|_{(3+\tilde{\varepsilon})} \|\nabla \phi_{R_j}\|_2^{\frac{5}{2} - \frac{3(1+\tilde{\varepsilon})}{2(3+\tilde{\varepsilon})}}.
\end{aligned}$$

Hence,

$$|(\phi_{R_j}, A \cdot \nabla \phi_{R_j})| \lesssim \|A\|_{L^{(3+\tilde{\varepsilon})} + L^\infty} (\|\nabla \phi_{R_j}\|_2 + \|\nabla \phi_{R_j}\|_2^{\frac{5}{2} - \frac{3(1+\tilde{\varepsilon})}{2(3+\tilde{\varepsilon})}}), \quad (2.68)$$

in which $\frac{5}{2} - \frac{3(1+\tilde{\varepsilon})}{2(3+\tilde{\varepsilon})}$ is strictly less than 2 for $\tilde{\varepsilon} > 0$.

Since $\text{supp}(\phi_{R_j}) \subset \{|x| \geq R_j\}$, by the assumption $\|V_-\|_{(L^2+L^\infty)(|x|>R_j)} \rightarrow 0$ and $\|A\|_{(L^{3++}+L^\infty)(|x|>R_j)} \rightarrow 0$, $\int V_- |\phi_{R_j}|^2 dx$ and the negative part of $\text{Im} \int \overline{\phi_{R_j}} A \cdot \nabla \phi_R$ converge to 0. Hence, the negative part of the energy converges to 0, a contradiction. Thus $b \geq 0$. So there exists $\delta(R)$ with $\delta(R) \rightarrow b \geq 0$ as $R \rightarrow \infty$, such that for any $\phi \in H^1$ satisfying $\phi(x) = 0$ for $|x| < R$, we have

$$\mathcal{E}(\phi, \phi) \geq \delta(R) \|\phi\|_2^2. \quad (2.69)$$

For $\phi \in H^1$, we have

$$\begin{aligned}
\delta(R) \|\chi_R \phi\|_2^2 &\leq \mathcal{E}(\chi_R \phi, \chi_R \phi) \\
&= (\nabla \chi_R \phi, \nabla \chi_R \phi) - 2 \text{Im} \left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx \right) + \int V \overline{\chi_R \phi} \chi_R \phi dx.
\end{aligned}$$

If we expand the factor $\nabla \chi_R \phi$, we get that

$$(\nabla \chi_R \phi, \nabla \chi_R \phi) = (\phi \nabla \chi_R, \phi \nabla \chi_R) + 2(\phi \nabla \chi_R, \chi_R \nabla \phi) + (\chi_R \nabla \phi, \chi_R \nabla \phi)$$

and since $\text{Im}(\int |\phi|^2 A \cdot \chi_R^2 \nabla \chi_R) = 0$

$$\begin{aligned}
-2 \text{Im} \left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx \right) &= -2 \text{Im} \left(\int \chi_R^2 \overline{\phi} A \cdot \nabla \phi \right) - 2 \text{Im} \left(\int |\phi|^2 A \cdot \chi_R^2 \nabla \chi_R \right) \\
&= -2 \text{Im} \left(\int \chi_R^2 \overline{\phi} A \cdot \nabla \phi \right).
\end{aligned}$$

Since

$$2(\phi \nabla \chi_R, \chi_R \nabla \phi) + (\chi_R \nabla \phi, \chi_R \nabla \phi) - 2 \operatorname{Im} \left(\int \chi_R^2 \bar{\phi} A \cdot \nabla \phi \right) + \int V \overline{\chi_R \phi} \chi_R \phi dx$$

is nothing but $\mathcal{E}(\chi_R^2 \phi, \phi)$, we have

$$\begin{aligned} \delta(R) \|\chi_R \phi\|_2^2 &\leq \mathcal{E}(\chi_R^2 \phi, \phi) + \|\phi \nabla \chi_R\|_2^2 \\ &= (\chi_R^2 \phi, H_0 \phi) + e_0 \|\chi_R \phi\|_2^2 + \|\phi \nabla \chi_R\|_2^2 \end{aligned}$$

where $H_0 = -\Delta + i(A \cdot \nabla + \nabla \cdot A) + V - e_0$.

From direct calculation, we see that for $R > 0$,

$$|\nabla \chi_R| \lesssim \varepsilon(\chi_R + 1), \quad (2.70)$$

so

$$\|\phi \nabla \chi_R\|_2^2 \lesssim \varepsilon^2 \|\phi(\chi_R + 1)\|_2^2. \quad (2.71)$$

Putting everything together, we have

$$\delta(R) \|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + (e_0 + \varepsilon^2) \|\chi_R \phi\|_2^2 + \varepsilon^2 \|\phi\|_2^2.$$

Since $e_0 < 0$ and $\lim_{R \rightarrow \infty} \delta(R) \geq 0$, for ε small enough and R sufficiently large, $\delta(R) - e_0 - \varepsilon^2$ is positive and bounded away from zero. Therefore, we have

$$\|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \varepsilon^2 \|\phi\|_2^2. \quad (2.72)$$

Next,

$$\begin{aligned} \|\chi_R \nabla \phi\|_2^2 &\leq \|\nabla(\chi_R \phi)\|_2^2 + \|\phi \nabla \chi_R\|_2^2 \\ &\lesssim \mathcal{E}(\chi_R \phi, \chi_R \phi) + 2 \operatorname{Im} \left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx \right) - \int V \overline{\chi_R \phi} \chi_R \phi dx \\ &\quad + \varepsilon^2 \|\phi\|_2^2 \end{aligned}$$

Since

$$\operatorname{Im}\left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx\right) \leq \|A\|_{L^\infty(|x| \geq R)} \|\chi_R \phi\|_{L^2} \|\nabla(\chi_R \phi)\|_{L^2} \quad (2.73)$$

and

$$\begin{aligned} \operatorname{Im}\left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx\right) &\leq \|A\|_{L^3(|x| \geq R)} \|\chi_R \phi\|_{L^6} \|\nabla(\chi_R \phi)\|_{L^2} \\ &\leq \|A\|_{L^3(|x| \geq R)} \|\chi_R \phi\|_{H^1} \|\nabla(\chi_R \phi)\|_{L^2}, \end{aligned}$$

we have that

$$\begin{aligned} \operatorname{Im}\left(\int \overline{\chi_R \phi} A \cdot \nabla \chi_R \phi dx\right) &\leq \|A\|_{(L^\infty + L^3)(|x| \geq R)} \|\chi_R \phi\|_{H^1} \|\nabla(\chi_R \phi)\|_{L^2} \\ &\leq \|A\|_{(L^\infty + L^3)(|x| \geq R)} \|\chi_R \phi\|_{H^1}^2. \end{aligned}$$

Therefore,

$$\|\chi_R \nabla \phi\|_2^2 \lesssim \mathcal{E}(\chi_R \phi, \chi_R \phi) + \|A\|_{(L^\infty + L^3)(|x| \geq R)} \|\chi_R \phi\|_{H^1}^2 + \|\chi_R \phi\|_2^2 + \varepsilon^2 \|\phi\|_2^2. \quad (2.74)$$

Now using $\mathcal{E}(\chi_R \phi, \chi_R \phi) = (\chi_R^2 \phi, H_0 \phi) + e_0 \|\chi_R \phi\|_2^2$ and $\|\chi_R \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \varepsilon^2 \|\phi\|_2^2$, we have that

$$\|\chi_R \nabla \phi\|_2^2 \lesssim (\chi_R^2 \phi, H_0 \phi) + \|A\|_{(L^\infty + L^3)(|x| \geq R)} \|\chi_R \phi\|_{H^1}^2 + \varepsilon^2 \|\phi\|_2^2. \quad (2.75)$$

Since

$$\|\nabla(\chi \phi)\|_{L^2} = \|\phi \nabla \chi_R\|_{L^2} + \|\chi_R \nabla \phi\|_{L^2} \lesssim \varepsilon \|\phi(\chi_R + 1)\|_{L^2} + \|\chi_R \nabla \phi\|_{L^2}, \quad (2.76)$$

putting everything together, we have that

$$\|\chi_R \phi\|_{H^1}^2 \lesssim (\chi_R \phi, \chi_R H_0 \phi) + \varepsilon^2 \|\phi\|_2^2 + \|A\|_{(L^\infty + L^3)(|x| \geq R)} \|\chi_R \phi\|_{H^1}^2, \quad (2.77)$$

so for R sufficiently large,

$$\|\chi_R \phi\|_{H^1}^2 \lesssim (\chi_R \phi, \chi_R H_0 \phi) + \varepsilon^2 \|\phi\|_2^2. \quad (2.78)$$

2.4. Detailed proof

If we let $\phi = \phi_0$ and use that $H_0\phi_0 = 0$, we have

$$\|\chi_R\phi_0\|_{H^1}^2 \lesssim \|\phi_0\|_2^2 = 1. \quad (2.79)$$

Next, let $\phi = q$. Using that $H_0q = -P_c g(z\phi_0 + q) + e'q$, we get

$$\begin{aligned} \|\chi_Rq\|_{H^1}^2 &\lesssim (\chi_Rq, \chi_RH_0q) + \varepsilon^2\|q\|_2^2 \\ &\lesssim (\chi_Rq, \chi_R(-P_cg(z\phi_0 + q) + e'q)) + \varepsilon^2\|q\|_2^2 \\ &\lesssim \|\chi_R^2 q g(z\phi_0 + q)\|_1 + e'\|\chi_Rq\|_2^2 + \varepsilon^2\|q\|_2^2. \end{aligned}$$

As $g(z) = |z|^2z$, we have

$$\begin{aligned} &\|\chi_R^2 q g(z\phi_0 + q)\|_1 \\ &\lesssim |z|^3\|\chi_R^2 q \phi_0^3\|_1 + |z|^2\|\chi_R^2 q^2 \phi_0^2\|_1 + |z|\|\chi_R^2 q^3 \phi_0\|_1 + \|\chi_R^2 q^4\|_1 \\ &\lesssim |z|^3\|\chi_R^2 \phi_0^3\|_2\|q\|_2 + |z|^2\|\chi_R^2 \phi_0^2\|_2\|q^2\|_2 + |z|\|\chi_R^2 \phi_0\|_2\|\chi_Rq^3\|_2 \\ &\quad + \|\chi_R^2 q^2\|_2\|q^2\|_2 \\ &\leq o(z^2). \end{aligned}$$

Hence,

$$\|\chi_Rq\|_{H^1}^2 \leq o(z^2) \quad (2.80)$$

by (2.79) and $\|q\|_{H^2} = o(z^2)$.

Next if we substitute $\phi = Dq$, and use that

$$H_0Dq = -P_cDg(z\phi_0 + q) + qDe' + e'Dq, \quad (2.81)$$

we get

$$\begin{aligned} &\|\chi_RDq\|_{H^1}^2 \\ &\lesssim (\chi_RDq, \chi_RH_0Dq) + \varepsilon^2\|Dq\|_2^2 \\ &\lesssim (\chi_RDq, \chi_R(-P_cDg(z\phi_0 + q) + qDe' + e'Dq)) + \varepsilon^2\|Dq\|_2^2 \\ &\lesssim \|\chi_R^2 Dq Dg(z\phi_0 + q)\|_1 + \|\chi_R^2 Dq q De'\|_1 + e'\|\chi_RDq\|_2^2 + \varepsilon^2\|q\|_2^2. \end{aligned}$$

2.4. Detailed proof

Here, the first term $\|\chi_R^2 Dq Dg(z\phi_0 + q)\|_1$ is bounded by

$$\begin{aligned}
& \|\chi_R^2 Dq Dg(z\phi_0 + q)\|_1 \\
& \lesssim \|\chi_R^2 Dq \phi_0 |z\phi_0 + q|^2\|_1 \\
& \lesssim z^2 \|\chi_R^2 Dq \phi_0^3\|_1 + z \|\chi_R^2 Dq \phi_0^2 q\|_1 + \|\chi_R^2 \phi_0 q^2\|_1 \\
& \lesssim z^2 \|\chi_R Dq\|_{H^1} \|\chi_R \phi_0\|_{H^1} \|\phi_0\|_{H^1}^2 + z \|Dq\|_{H^1} \|q\|_{H^1} \|\chi_R \phi_0\|_{H^1}^2 \\
& \leq o(z^2),
\end{aligned}$$

and the second term $\|\chi_R^2 Dq q De'\|_1$ is bounded by

$$\begin{aligned}
\|\chi_R^2 Dq q De'\|_1 & \leq \|\chi_R Dq\|_3 \|\chi_R q\|_3 \|De'\|_3 \\
& \lesssim \|\chi_R Dq\|_{H^1} \|\chi_R q\|_{H^1} \|De'\|_{H^1} \\
& \leq o(z^2).
\end{aligned}$$

Therefore,

$$\|\chi_R Dq\|_{H^1}^2 \leq o(z^2). \quad (2.82)$$

Hence, by Lemma 2.4.1 and $Q = z\phi_0 + q$, we have $\|e^{\beta|x|}Q\|_{H^1} \leq \infty$ and $\|e^{\beta|x|}DQ\|_{H^1} \leq \infty$ for some $\beta > 0$.

Next, we would like to show $\|e^{\beta|x|}Q\|_{L^\infty} \leq \infty$ by bounding $\|\Delta(e^{\beta|x|}Q)\|_{L^{\frac{3}{2}+}}$. Since $\|\Delta(e^{\beta|x|}Q)\|_{L^\infty(|x|\leq 1)} < \infty$ already holds, it remains to show $\|\Delta(e^{\beta|x|}Q)\|_{L^{\frac{3}{2}+}(|x|>1)} < \infty$. Let $\gamma = \frac{\beta}{3}$. Using the equation for Q , we get

$$\begin{aligned}
& \|\Delta(e^{\gamma|x|}Q)\|_{L^{\frac{3}{2}+}(|x|>1)} \\
& \lesssim \|(\Delta e^{\gamma|x|})Q\|_{L^{\frac{3}{2}+}(|x|>1)} + \|(\nabla e^{\gamma|x|}) \cdot (\nabla Q)\|_{L^{\frac{3}{2}+}(|x|>1)} \\
& \quad + \|e^{\gamma|x|}A \cdot \nabla Q\|_{L^{\frac{3}{2}+}(|x|>1)} + \|e^{\gamma|x|}[(\nabla \cdot A) + V]Q\|_{L^{\frac{3}{2}+}(|x|>1)} \\
& \quad + \|e^{\gamma|x|}g(Q)\|_{L^{\frac{3}{2}+}(|x|>1)} + \|e^{\gamma|x|}EQ\|_{L^{\frac{3}{2}+}(|x|>1)}.
\end{aligned}$$

Let f and g be such that $\Delta e^{\gamma|x|} = f(x)e^{\gamma|x|}$ and $\nabla e^{\gamma|x|} = g(x)e^{\gamma|x|}$. We can bound the first two terms loosely by

$$\|(\Delta e^{\gamma|x|})Q\|_{L^{\frac{3}{2}+}(|x|>1)} \lesssim \|e^{-\frac{2}{3}\gamma|x|}f(x)\|_{L^{6+}(|x|>1)} \|e^{\beta|x|}Q\|_{L^2} \quad (2.83)$$

2.4. Detailed proof

and

$$\begin{aligned}
& \|(\nabla e^{\gamma|x|}) \cdot (\nabla Q)\|_{L^{\frac{3}{2}+}(|x|>1)} \\
& \lesssim \|e^{-\frac{1}{3}\beta|x|}g(x)\|_{L^{6+}(|x|>1)} \|e^{\frac{2}{3}\beta|x|}(\nabla Q)\|_{L^2} \\
& \lesssim \|e^{\frac{2}{3}\beta|x|}Q\|_{H^1} + \|e^{-\frac{1}{3}\beta|x|}g(x)\|_{L^\infty(|x|>1)} \|e^{\beta|x|}Q\|_{L^2}.
\end{aligned}$$

In a similar way, we can also bound $\|e^{\gamma|x|}g(Q)\|_{L^{\frac{3}{2}+}(|x|>1)}$ and $\|e^{\gamma|x|}EQ\|_{L^{\frac{3}{2}+}(|x|>1)}$.

Next, for $\|e^{\gamma|x|}A \cdot \nabla Q\|_{L^{\frac{3}{2}+}(|x|>1)}$, we have

$$\begin{aligned}
& \|e^{\gamma|x|}A \cdot \nabla Q\|_{L^{\frac{3}{2}+}(|x|>1)} \\
& \leq \|A\|_{L^{3+}+L^\infty} (\|e^{\frac{1}{3}\beta|x|}\nabla Q\|_{L^3(|x|>1)} + \|e^{\frac{1}{3}\beta|x|}\nabla Q\|_{L^{\frac{3}{2}}(|x|>1)}) \\
& \lesssim \|e^{\frac{-2}{3}\beta|x|}e^{\beta|x|}(\nabla Q)\|_{L^3} + \|e^{\beta|x|}\nabla Q\|_{L^2}.
\end{aligned}$$

We already shown above that $\|e^{\beta|x|}\nabla Q\|_{L^2} < \infty$. To bound $\|e^{\frac{-2}{3}\beta|x|}e^{\beta|x|}(\nabla Q)\|_{L^3}$, let $h = e^{\beta|x|}(\nabla Q)$ and from above, we know that $h \in L^2$. Now, consider the set

$$M = \{x | (e^{-\frac{2}{3}\beta|x|}|h|)^3 > |h|^2\} = \{x | |h| > e^{2\beta|x|}\}. \quad (2.84)$$

Clearly,

$$\begin{aligned}
\|e^{\frac{-2}{3}\beta|x|}e^{\beta|x|}(\nabla Q)\|_{L^3(M^c)} &= \|e^{\frac{-2}{3}\beta|x|}h\|_{L^3(M^c)} \leq \| |h|^{\frac{2}{3}} \|_{L^3} = \|h\|_{L^2}^{\frac{2}{3}} < \infty.
\end{aligned} \quad (2.85)$$

On the other hand, inside M , $|e^{\beta|x|}(\nabla Q)| > e^{2\beta|x|}$ and hence, $|\nabla Q| > e^{\beta|x|}$. Then

$$\begin{aligned}
\|e^{\frac{-2}{3}\beta|x|}e^{\beta|x|}(\nabla Q)\|_{L^3(M)} &\leq \|e^{\frac{-2}{3}\beta|x|}|\nabla Q|^2\|_{L^3} \\
&\leq \| |\nabla Q|^2 \|_{L^3} \\
&= \|\nabla Q\|_{L^6}^3 \\
&\lesssim \|\nabla Q\|_{H^1}^3.
\end{aligned}$$

Hence, we have

$$\|\Delta(e^{\gamma|x|}Q)\|_{L^{\frac{3}{2}+}} < \infty. \quad (2.86)$$

2.4. Detailed proof

By Sobolev embedding, we have

$$\|e^{\gamma|x|}Q\|_{L^\infty} < \infty. \quad (2.87)$$

2.4.2 Linear estimates

In this section, we will prove the following theorem.

Theorem 6. *We say that (p, q) is Strichartz admissible if*

$$\frac{2}{q} + \frac{3}{p} = \frac{3}{2} \quad \text{with } 2 \leq p < 6. \quad (2.88)$$

If (q, p) and (\tilde{p}, \tilde{q}) are Strichartz admissible, then

$$\begin{aligned} & \left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q W_x^{1,p}} + \|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 H_x^1} \\ & \lesssim \min(\|\langle x \rangle^\sigma F\|_{L_t^2 H_x^1}, \|F\|_{L_t^{\tilde{q}'} W_x^{1,\tilde{p}'}}). \end{aligned}$$

To prove theorem 6, we need a few preparatory lemmas. The following lemmas 2.4.2 and 2.4.3 are from [30]:

Lemma 2.4.2. *(Non-endpoint Strichartz estimates) Under assumptions 4 and 2, if (p, q) and (\tilde{p}, \tilde{q}) are Strichartz admissible, we have*

$$\|e^{itH} P_c f\|_{L_t^q L_x^p} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \quad (2.89)$$

and

$$\left\| \int_{-\infty}^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}. \quad (2.90)$$

Notice that the above does not include the L_t^2 -norm. Fix $\sigma > 4$.

Lemma 2.4.3. *(Weighted homogeneous L_t^2 estimates) Under assumptions 4 and 2, we have*

$$\|\langle x \rangle^{-\sigma} e^{-itH} f\|_{L_t^2 L_x^2} \lesssim \|f\|_{L_x^2}, \quad (2.91)$$

and

$$\sup_{\lambda \geq 0} \langle \lambda \rangle \|\langle x \rangle^{-\sigma} (H - (\lambda^2 + i0))^{-1} \langle x \rangle^{-\sigma}\|_{L^2 \rightarrow L^2} \lesssim 1. \quad (2.92)$$

2.4. Detailed proof

The weighted resolvent estimate of lemma 2.4.3 implies weighted inhomogeneous estimates for the linear evolution:

Lemma 2.4.4. (*Weighted L_t^2 inhomogeneous estimates*) Under the assumptions of lemma 2.4.3,

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c \langle x \rangle^{-\sigma} F(s) ds\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^2 L_x^2}. \quad (2.93)$$

Proof. For simplicity we may restrict to times $t \geq 0$. By Plancherel, we have

$$\begin{aligned} & \|\chi_{\{t \geq 0\}} \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) ds\|_{L_t^2} \\ &= \left\| \int_0^\infty e^{it\tau} \langle x \rangle^{-\sigma} \left(\int_0^t e^{i(t-s)(H+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) ds \right) dt \right\|_{L_\tau^2} \end{aligned}$$

Next, change the order of the ds and dt integral and use that

$$\begin{aligned} & \int_s^\infty dt e^{it(H-\tau+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) \\ &= \frac{1}{i} (H - \tau + i\varepsilon)^{-1} e^{it(H-\tau+i\varepsilon)} \Big|_{t=s}^{t=\infty} P_c \langle x \rangle^{-\sigma} F(s) \\ &= \frac{-1}{i} (H - \tau + i\varepsilon)^{-1} e^{is(H-\tau+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s), \end{aligned}$$

we get

$$\begin{aligned} & \|\chi_{\{t \geq 0\}} \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) ds\|_{L_t^2} \\ &= \left\| \langle x \rangle^{-\sigma} \int_0^\infty ds e^{-is(H+i\varepsilon)} \frac{-1}{i} (H - \tau + i\varepsilon)^{-1} e^{is(H-\tau+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) \right\|_{L_\tau^2} \\ &= \left\| \langle x \rangle^{-\sigma} (H - \tau + i\varepsilon)^{-1} P_c \langle x \rangle^{-\sigma} \int_0^\infty ds e^{-is\tau} F(s) \right\|_{L_\tau^2}. \end{aligned}$$

2.4. Detailed proof

If we take the L_x^2 -norm of both sides, we get

$$\begin{aligned}
& \|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)(H+i\varepsilon)} P_c \langle x \rangle^{-\sigma} F(s) ds\|_{L_t^2 L_x^2} \\
& \lesssim \|\langle x \rangle^{-\sigma} (H - \tau + i\varepsilon)^{-1} P_c \langle x \rangle^{-\sigma} \int_0^\infty ds e^{-is\tau} F(s)\|_{L_\tau^2 L_x^2} \\
& \lesssim \sup_\tau \|\langle x \rangle^{-\sigma} (H - \tau + i\varepsilon)^{-1} P_c \langle x \rangle^{-\sigma}\|_{L^2 \rightarrow L^2} \|\int_0^\infty ds e^{-is\tau} F(s)\|_{L_\tau^2 L_x^2} \\
& \lesssim \|F\|_{L_t^2 L_x^2} \text{ by Plancherel and Lemma 2.4.3.}
\end{aligned}$$

Now sending ε to 0, we have

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c \langle x \rangle^{-\sigma} F(s) ds\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^2 L_x^2} \quad (2.94)$$

as needed. \square

Lemma 2.4.5. (*Mixed Strichartz weighted estimates*) *Let (q, p) and (\tilde{p}, \tilde{q}) be Strichartz admissible. Then*

$$\begin{aligned}
& \left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q L_x^p} + \|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 L_x^2} \\
& \lesssim \min(\|\langle x \rangle^\sigma F\|_{L_t^2 L_x^2}, \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{p}'}}).
\end{aligned}$$

Proof. First,

$$\left\| \int_0^\infty e^{-isH} P_c F(s) ds \right\|_{L_x^2}^2 = \left(\int_0^\infty e^{-isH} P_c F(s) ds, \int_0^\infty e^{-itH} P_c F(s) dt \right).$$

Moving the integrals through the inner product and rearranging the terms,

we get

$$\begin{aligned}
& \left\| \int_0^\infty e^{-isH} P_c F(s) ds \right\|_{L_x^2}^2 \\
&= \int_0^\infty ds (P_c F(s), \int_0^\infty e^{-i(t-s)H} P_c F(s) dt) \\
&= \int_0^\infty ds (\langle x \rangle^\sigma P_c F(s), \langle x \rangle^{-\sigma} \int_0^\infty e^{-i(t-s)H} P_c F(s) dt) \\
&\quad \text{by Hölder inequality} \\
&\leq \| \langle x \rangle^\sigma P_c F(s) \|_{L_s^2 L_x^2} \| \langle x \rangle^{-\sigma} \int_0^\infty e^{-i(t-s)H} P_c F(s) dt \|_{L_s^2 L_x^2} \\
&\quad \text{and by lemma 2.4.4} \\
&\lesssim \| \langle x \rangle^\sigma P_c F(s) \|_{L_s^2 L_x^2}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\| \int_0^\infty e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^p L_x^q} &= \| e^{itH} \int_0^\infty e^{-isH} P_c F(s) ds \|_{L_t^p L_x^q} \\
&\lesssim \left\| \int_0^\infty e^{-isH} P_c F(s) ds \right\|_{L_x^2} \quad \text{by lemma 2.4.2} \\
&\lesssim \| \langle x \rangle^\sigma F(s) \|_{L_s^2 L_x^2}.
\end{aligned}$$

Now, by a lemma of Christ-Kiselev (see [17]), we have

$$\left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^p L_x^q} \lesssim \| \langle x \rangle^\sigma F(s) \|_{L_s^2 L_x^2}. \quad (2.95)$$

Next, let $\langle x \rangle^\sigma g(x, t) \in L_t^2 L_x^2$. Then

$$\begin{aligned}
& \int_0^\infty (\langle x \rangle^\sigma g(x, t), \langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s) ds) dt \\
&= \int_0^\infty (g(x, t), \int_0^\infty e^{i(t-s)H} P_c F(s) ds) dt
\end{aligned}$$

Moving the integrals through the inner product and rearranging the terms,

we get

$$\begin{aligned}
& \int_0^\infty (\langle x \rangle^\sigma g(x, t), \langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s) ds) dt \\
&= \int_0^\infty ds \left(\int_0^\infty e^{i(s-t)H} P_c g(x, t) dt, F(s) \right) \\
&\quad \text{by Hölder inequality} \\
&\leq \left\| \int_0^\infty e^{i(s-t)H} P_c g(x, t) dt \right\|_{L_t^q L_x^p} \|F(s)\|_{L_t^{q'} L_x^{p'}} \\
&\lesssim \|\langle x \rangle^\sigma g\|_{L_x^2 L_t^2} \|F(s)\|_{L_t^{q'} L_x^{p'}}
\end{aligned}$$

Hence,

$$\|\langle x \rangle^{-\sigma} \int_0^\infty e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 L_x^2} \lesssim \|F(s)\|_{L_t^{q'} L_x^{p'}}. \quad (2.96)$$

Again, by the lemma of Christ-Kiselev, we have

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 L_x^2} \lesssim \|F(s)\|_{L_t^{q'} L_x^{p'}}. \quad (2.97)$$

Now by lemma 2.4.2 and lemma 2.4.4, we have shown lemma 2.4.5. \square

Lemma 2.4.6. (*Derivative Strichartz estimates*) Let $p \geq 2$ and let

$$H_1 = H + K = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V + K \quad (2.98)$$

for a sufficiently large number K . Then H_1 is a positive operator on L^p , and

$$\|\phi\|_{W^{1,p}} \sim \|H_1^{\frac{1}{2}} \phi\|_{L^p}. \quad (2.99)$$

From this, it follows that

$$\|e^{-itH} f\|_{L_t^q W_x^{1,p}} \lesssim \|f\|_{H_x^1} \quad (2.100)$$

and

$$\left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q W_x^{1,p}} \lesssim \|F\|_{L_t^{\tilde{q}'} W_x^{1,\tilde{p}'},} \quad (2.101)$$

for Strichartz admissible (q, p) and (\tilde{p}, \tilde{q}) .

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Proof. We would like to first show

$$\|\phi\|_{W^{1,p}} \sim \|H_1^{\frac{1}{2}}\phi\|_{L^p} \text{ for } \phi \in W^{1,p}. \quad (2.102)$$

Clearly $\|\phi\|_{W^{0,p}} = \|\phi\|_{L^p} = \|H_1^0\phi\|_{L^p}$. As shown in the appendix of [51], if K is large enough, H_1 is a positive operator on L^p , and

$$\|\phi\|_{W^{2,p}} \sim \|H_1\phi\|_{L^p}. \quad (2.103)$$

By Theorem 1 of [24], there exist positive numbers ε and C , such that H_1^{it} is a bounded operator on L^p for $-\varepsilon \leq t \leq \varepsilon$ and $\|H_1^{it}\| \leq C$. Therefore the hypothesis of Section 1.15.3 of [85] holds and we have that

$$[D(H_1), D(H_1^0)]_{\frac{1}{2}} = D(H_1^{\frac{1}{2}}). \quad (2.104)$$

Using that $D(H_1) = W^{2,p}$, $D(H_1^0) = L^p$ and $[W^{2,p}, L^p]_{\frac{1}{2}} = W^{1,p}$, we find that

$$D(H_1^{\frac{1}{2}}) = W^{1,p}. \quad (2.105)$$

Now by Section 1.15.2 of [85], $H_1^{\frac{1}{2}}$ is an isomorphic mapping from $D(H_1^{\frac{1}{2}}) = W^{1,p}$ onto L^p . Therefore, we have

$$\|\phi\|_{W^{1,p}} \sim \|H_1^{\frac{1}{2}}\phi\|_{L^p}. \quad (2.106)$$

Finally,

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_t^q W_x^{1,p}} &= \left\| \left\| \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{W_x^{1,p}} \right\|_{L_t^q} \\ &\sim \left\| \left\| H_1^{\frac{1}{2}} \int_0^t e^{i(t-s)H} P_c F(s) ds \right\|_{L_x^p} \right\|_{L_t^q} \\ &= \left\| \left\| \int_0^t e^{i(t-s)H} P_c H_1^{\frac{1}{2}} F(s) ds \right\|_{L_x^p} \right\|_{L_t^q} \\ &\lesssim \|H_1^{\frac{1}{2}} F\|_{L_t^{\tilde{q}'} L_x^{\tilde{q}'}} \\ &\sim \|F\|_{L_t^{\tilde{q}'} W_x^{1,\tilde{p}'}} \end{aligned}$$

□

For $s \in \mathbb{R}$, denote the norm $\|\phi\|_{\langle x \rangle^s L^2}$ by

$$\|\phi\|_{\langle x \rangle^s L^2} = \|\langle x \rangle^{-s} \phi\|_{L^2} \quad (2.107)$$

and the norm $\|\phi\|_{\langle x \rangle^s H^1}$ by

$$\|\phi\|_{\langle x \rangle^s H^1} = \|\phi\|_{\langle x \rangle^s L^2} + \|\nabla \phi\|_{\langle x \rangle^s L^2}. \quad (2.108)$$

Next we need derivative version of the weighted estimates of Lemma 2.4.4 - this is given in Lemma (2.4.9) below. First, we need two preparatory lemmas.

Lemma 2.4.7. *For $t > 0$, let $A_t(x) = \frac{1}{\sqrt{t}} A(\frac{x}{\sqrt{t}})$ and $V_t(x) = \frac{1}{t} V(\frac{x}{\sqrt{t}})$. Let*

$$\tilde{H} = -\Delta + 2iA_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t}K + 1. \quad (2.109)$$

Then there exists $T > 0$ such that $\sup_{t>T} \|\tilde{H}^{-1}\|_{L^2 \rightarrow H^2} < \infty$.

Proof. Take $t \geq 1$. For $\phi \in L^2$, let $h = \tilde{H}^{-1}\phi$. Then

$$\begin{aligned} \|\phi\|_2^2 &= \left((-\Delta + 2iA_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t}K + 1)h, \right. \\ &\quad \left. (-\Delta + 2iA_t \cdot \nabla + i(\nabla \cdot A_t) + V_t + \frac{1}{t}K + 1)h \right) \\ &= \|\Delta h\|_2^2 + \|h\|_2^2 + \|A_t \cdot \nabla h\|_2^2 + 2\|\nabla h\|_2^2 + F \\ &\gtrsim \|\Delta h\|_2^2 + \|h\|_2^2 + F \end{aligned}$$

where F denotes the rest of the terms, and recall that $q > 3$. We would like

2.4. Detailed proof

to show that every term in F is bounded by $\|h\|_{H^2}^2$. Here,

$$\begin{aligned}
 |F| &\leq 2\|(\Delta h)(A_t \cdot \nabla h)\|_1 + 2\|(\Delta h)(\nabla \cdot A_t + V_t + \frac{1}{t}K)h\|_1 \\
 &\quad + 2\|[A_t(\nabla \cdot A_t + V_t + \frac{1}{t}K)] \cdot (\nabla h)h\|_1 \\
 &\quad + 2\|A_t \cdot (\nabla h)h\|_1 + 2\|(A_t + V_t + \frac{1}{t}K)^2 h^2\|_1
 \end{aligned}$$

Here,

$$\begin{aligned}
 \|(\Delta h)(A_t \cdot \nabla h)\|_1 &\lesssim \frac{1}{\sqrt{t}} \|\Delta h\|_2 \|A(\frac{\cdot}{\sqrt{t}})\|_{L^\infty+L^q} (\|\nabla h\|_2 + \|\nabla h\|_2^{\frac{2q}{q-2}}) \\
 &\quad \text{where } \frac{2q}{q-2} < 6 \\
 &\lesssim \frac{1}{\sqrt{t}} \|\Delta h\|_2 \|A(\frac{\cdot}{\sqrt{t}})\|_{L^\infty+L^q} (\|\nabla h\|_2 + \|\Delta h\|_2^{\frac{3}{q}} \|\nabla h\|_2^{\frac{q-3}{q}}) \\
 &\lesssim t^{-\frac{(q-3)}{2q}} \|\Delta h\|_2 \|A\|_{L^\infty+L^q} (\|\nabla h\|_2 + \|\Delta h\|_2^{\frac{3}{q}} \|\nabla h\|_2^{\frac{q-3}{q}}),
 \end{aligned}$$

$$\begin{aligned}
 &\|(\Delta h)((\nabla \cdot A_t) + V_t + \frac{1}{t}K)h\|_1 \\
 &\lesssim \frac{1}{t} \|\Delta h\|_2 (\|(\nabla \cdot A)(\frac{\cdot}{\sqrt{t}})\|_{L^\infty+L^2} + \|V(\frac{\cdot}{\sqrt{t}})\|_{L^\infty+L^2} + K) (\|h\|_2 + \|h\|_\infty) \\
 &\lesssim t^{\frac{-1}{4}} \|\Delta h\|_2 (\|\nabla \cdot A\|_{L^\infty+L^2} + \|V\|_{L^\infty+L^2} + K) (\|h\|_2 + \|h\|_2^{\frac{1}{4}} \|\Delta h\|_2^{\frac{3}{4}}).
 \end{aligned}$$

Similar bounds hold for the other terms of F . We conclude that

$$\|\phi\|_2^2 \geq (1 + o(1)) \|h\|_{H^2}^2 \text{ as } t \rightarrow \infty. \quad (2.110)$$

Hence, for all t large enough, we have

$$\|h\|_{H^2}^2 \lesssim \|\phi\|_2^2. \quad (2.111)$$

□

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Lemma 2.4.8. *Let H_1 be as in lemma 2.4.6. For $\phi \in L^2$ and $t > 0$, we have*

$$\|\nabla(H_1 + t)^{-1}\phi\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2}}\|\phi\|_{L^2}. \quad (2.112)$$

Proof. For $\phi \in L^2$, let $\psi = (H_1 + t)^{-1}\phi$. For t bounded away from zero, define $\widehat{\psi}$ by $\psi(x) = \frac{1}{t}\widehat{\psi}(\sqrt{t}x)$. Then $\Delta\psi(x) = \Delta\widehat{\psi}(\sqrt{t}x)$, $\nabla\psi(x) = \frac{1}{\sqrt{t}}\nabla\widehat{\psi}(\sqrt{t}x)$ and $V(x)\psi(x) = \frac{1}{t}V(x)\widehat{\psi}(\sqrt{t}x)$ and

$$(\tilde{H}\widehat{\psi})(\sqrt{t}x) = \phi(x). \quad (2.113)$$

Replacing x by $\frac{x}{\sqrt{t}}$ and inverting \tilde{H} , we get

$$\widehat{\psi}(x) = \tilde{H}^{-1}\phi\left(\frac{x}{\sqrt{t}}\right). \quad (2.114)$$

Hence,

$$\psi(x) = \frac{1}{t}[\tilde{H}^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x) \quad (2.115)$$

and

$$\nabla\psi(x) = \frac{1}{\sqrt{t}}[\nabla(\tilde{H})^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x). \quad (2.116)$$

By Lemma 2.4.7, $\|\tilde{H}^{-1}\|_{L^2 \rightarrow L^2}$ is uniformly bounded for $t \geq T$. Therefore,

$$\begin{aligned} \|\nabla\psi(x)\|_2 &= \left\| \frac{1}{\sqrt{t}}[\nabla\tilde{H}^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)](\sqrt{t}x) \right\|_2 \\ &= t^{-\frac{3}{4}-\frac{1}{2}}\|\nabla\tilde{H}^{-1}\phi\left(\frac{\cdot}{\sqrt{t}}\right)\|_2 \\ &\lesssim t^{-\frac{3}{4}-\frac{1}{2}}\|\nabla\tilde{H}^{-1}\|_{L^2 \rightarrow L^2}\|\phi\left(\frac{\cdot}{\sqrt{t}}\right)\|_2 \\ &= t^{-\frac{1}{2}}\|\phi\|_2 \end{aligned}$$

Therefore, for $t \geq T$,

$$\|\nabla(H_1 + t)^{-1}\phi\|_2 \lesssim t^{-\frac{1}{2}}\|\phi\|_2 \quad (2.117)$$

and the lemma follows. \square

Lemma 2.4.9. *(Derivative weighted estimates) Let H_1 be as in lemma*

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2.4.6. We have

$$\|\phi\|_{\langle x \rangle^s H^1} \sim \|H_1^{\frac{1}{2}} \phi\|_{\langle x \rangle^s L^2} \text{ for } s \in \mathbb{R}. \quad (2.118)$$

From this, it follows that

$$\|\langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H} P_c F(s) ds\|_{L_t^2 H_x^1} \lesssim \|\langle x \rangle^\sigma F\|_{L_t^2 H_x^1}. \quad (2.119)$$

Proof. Since $\|f\|_{\langle x \rangle^s H^1} = \|\langle x \rangle^{-s} f\|_{L^2} + \|\nabla \langle x \rangle^{-s} f\|_{L^2}$, to show the lemma, it suffices to show

$$\|\langle x \rangle^{-s} H_1^{-\frac{1}{2}} \langle x \rangle^s\|_{L^2 \rightarrow L^2} < \infty \quad (2.120)$$

and

$$\|\nabla \langle x \rangle^{-s} H_1^{-\frac{1}{2}} \langle x \rangle^s\|_{L^2 \rightarrow L^2} < \infty. \quad (2.121)$$

The second bound above is the harder of the two. We will show the second bound and the first one follows by a similar argument. First,

$$\nabla \langle x \rangle^{-s} H_1^{-\frac{1}{2}} \langle x \rangle^s \phi = \nabla H_1^{-\frac{1}{2}} \phi + \nabla \langle x \rangle^{-s} [H_1^{-\frac{1}{2}}, \langle x \rangle^s] \phi \quad (2.122)$$

Now $\nabla H_1^{-\frac{1}{2}}$ is bounded from L^2 to L^2 since $H_1^{-\frac{1}{2}}$ maps from L^2 to H^1 while ∇ maps from H^1 to L^2 .

For the second term, we use $H_1^{-\frac{1}{2}} = \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1}$ and $[(H_1 + t)^{-1}, \langle x \rangle^s] = (H_1 + t)^{-1} [H_1 + t, \langle x \rangle^s] (H_1 + t)^{-1}$ to get

$$\nabla \langle x \rangle^{-s} [H_1^{-\frac{1}{2}}, \langle x \rangle^s] = \nabla \langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} [H_1 + t, \langle x \rangle^s] (H_1 + t)^{-1} \quad (2.123)$$

Recall that

$$H_1 = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V + K, \quad (2.124)$$

so

$$[H_1 + t, \langle x \rangle^s] = (-\Delta \langle x \rangle^s) - 2(\nabla \langle x \rangle^s) \cdot \nabla + 2iA \cdot (\nabla \langle x \rangle^s).$$

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Let $g(x) = (-\Delta\langle x \rangle^s) + 2iA \cdot (\nabla\langle x \rangle^s)$ and $h(x) = -2(\nabla\langle x \rangle^s)$. Then

$$\nabla\langle x \rangle^{-s}[H_1^{-\frac{1}{2}}, \langle x \rangle^s] = \nabla\langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} (g(x) + h(x) \cdot \nabla) (H_1 + t)^{-1}. \quad (2.125)$$

Since $g(x) \lesssim \langle x \rangle^{s-1}$, we rewrite the $g(x)$ -part of the above as

$$\begin{aligned} & \nabla\langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} g(x) (H_1 + t)^{-1} \\ = & \nabla \int_0^\infty \frac{dt}{\sqrt{t}} \langle x \rangle^{-s} g(x) (H_1 + t)^{-1} (H_1 + t)^{-1} \\ & + \nabla\langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} [H_1 + t, g(x)] (H_1 + t)^{-1} (H_1 + t)^{-1} \end{aligned}$$

The first part of the above sum is bounded. For the second part, writing $[H_1 + t, g(x)] = \tilde{g}(x) + \tilde{h}(x) \cdot \nabla$ as before, we can iterate the above process until $\tilde{g}(x) \lesssim 1$. Since $h(x) \lesssim \langle x \rangle^{s-1}$, so by the similar argument, we have

$$\begin{aligned} & \nabla\langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} h(x) \cdot \nabla (H_1 + t)^{-1} \\ = & \nabla \int_0^\infty \frac{dt}{\sqrt{t}} \langle x \rangle^{-s} h(x) (H_1 + t)^{-1} \nabla (H_1 + t)^{-1} \\ & + \nabla\langle x \rangle^{-s} \int_0^\infty \frac{dt}{\sqrt{t}} (H_1 + t)^{-1} [H_1 + t, h(x)] (H_1 + t)^{-1} \nabla (H_1 + t)^{-1} \end{aligned}$$

As before, the first part of the above sum is bounded. For the second part, $[H_1 + t, g(x)] = \tilde{g}(x) + \tilde{h}(x) \cdot \nabla$ as before, we can iterate the above process until $\tilde{h}(x) \lesssim 1$. As a result, it suffices to consider

$$\int_0^\infty \frac{dt}{\sqrt{t}} ((H_1 + t)^{-1})^m \quad (2.126)$$

and

$$\int_0^\infty \frac{dt}{\sqrt{t}} ((H_1 + t)^{-1} \nabla (H_1 + t)^{-1})^m \quad (2.127)$$

for $m \geq 1$. Now by lemma 2.4.8, both of the expressions above are bounded in L^2 . \square

Now, to prove theorem 6, apply lemma 2.4.6 and 2.4.9 to lemma 2.4.5,

we get the result.

Finally, we need a lemma from [40] for the projection operator P_c onto the continuous spectral subspace.

Lemma 2.4.10. (*Continuous spectral subspace comparison*) *Let the continuous spectral subspace $\mathcal{H}_c[z]$ be defined as*

$$\mathcal{H}_c[z] = \{\eta \in L^2 \mid \langle i\eta, D_1 Q[z] \rangle = \langle i\eta, D_2 Q[z] \rangle = 0\}. \quad (2.128)$$

Then there exists $\delta > 0$ such that for each $z \in \mathbb{C}$ with $|z| \leq \delta$, there is a bijective operator $R[z] : \text{Ran } P_c \rightarrow \mathcal{H}_c[z]$ satisfying

$$P_c|_{\mathcal{H}_c[z]} = (R[z])^{-1}. \quad (2.129)$$

Moreover, $R[z] - I$ is compact and continuous in z in the operator norm on any space Y satisfying $H^2 \cap W^{1,1} \subset Y \subset H^{-2} + L^\infty$.

The proof of lemma 2.4.10 is given in lemma 2.2 of [40]. We will use lemma 2.4.10 with $Y = L^2$.

2.4.3 Proof of the main theorem

Lemma 2.2.1 gives the following corollary which will form part of the main theorem.

Lemma 2.4.11. (*Best decomposition*) *There exists $\delta > 0$ such that any $\psi \in H^1$ satisfying $\|\psi\|_{H^1} \leq \delta$ can be uniquely decomposed as*

$$\psi = Q[z] + \eta \quad (2.130)$$

where $z \in \mathbb{C}$, $\eta \in H^1$, $\langle i\eta, D_1 Q[z] \rangle = \langle i\eta, D_2 Q[z] \rangle = 0$, and $|z| + \|\eta\|_{H^1} \lesssim \|\psi\|_{H^1}$.

The proof of lemma 2.4.11 is essentially an application of the implicit function theorem on the equation $B(z) = 0$ with

$$B(z) = (B_1(z), B_2(z)), \quad B_j = \langle i(\psi - Q[z]), D_j Q[z] \rangle \quad \text{for } j = 1, 2. \quad (2.131)$$

2.4. Detailed proof

Details can be found in lemma 2.3 of [40].

Now, we prove theorem 5.

Proof. Substitute

$$\psi(t) = Q[z(t)] + \eta(t) \quad (2.132)$$

into equation (2.3) to get

$$i(DQ\dot{z} + \partial_t\eta) = HQ + H\eta + g(Q + \eta)$$

where for $w \in \mathbb{C}$, we denote $DQ[z]w = D_1Q[z] \operatorname{Re} w + D_2Q[z] \operatorname{Im} w$. Since $HQ + g(Q) = EQ$ and $DQ[z]iz = iQ[z]$ (since $Q[e^{i\alpha}z] = e^{i\alpha}Q[z]$ for $\alpha \in \mathbb{R}$), we have

$$\begin{aligned} i\partial_t\eta &= H\eta - iDQ\dot{z} + EQ - g(Q) + g(Q + \eta) \\ &= H\eta - iDQ(\dot{z} + iEz) - g(Q) + g(Q + \eta). \end{aligned}$$

We can write this as

$$i\partial_t\eta = H\eta + F \quad (2.133)$$

where

$$F = g(Q + \eta) - g(Q) - iDQ(\dot{z} + iEz). \quad (2.134)$$

In integral form,

$$\eta(t) = e^{-itH}(\eta(0) - i \int_0^t e^{isH} F(s) ds). \quad (2.135)$$

Let $\eta_c = P_c\eta$. Then

$$\eta_c = e^{-itH} P_c\eta(0) - i \int_0^t e^{i(t-s)H} P_c F(s) ds. \quad (2.136)$$

2.4. Detailed proof

Then for fixed $\sigma > 4$, since $\eta = \text{Re}[z]\eta_c$, we have

$$\begin{aligned}
\|\eta\|_X &\lesssim \|\eta_c\|_X \\
&\lesssim \|\eta(0)\|_{H_x^1} + \left\| \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \overline{Q}\eta^2 - |\eta|^2\eta) ds \right\|_X \\
&\quad + \left\| \int_0^t e^{-i(s-t)H} P_c(2Q|\eta|^2 + \overline{Q}\eta^2 + |\eta|^2\eta) ds \right\|_X \\
&\lesssim \|\eta(0)\|_{H_x^1} + \left\| \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \overline{Q}\eta^2 - |\eta|^2\eta) ds \right\|_X \\
&\quad + \|Q\eta^2\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}} + \|\eta^3\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}}.
\end{aligned}$$

For $\|Q\eta^2\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}}$, we have

$$\begin{aligned}
\|Q\eta^2\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}} &= \|Q\eta^2\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} + \|\nabla(Q\eta^2)\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} \\
&\lesssim \|(|Q| + |\nabla Q|)\eta^2\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} + \|Q\eta\nabla\eta\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} \\
&\lesssim \|Q\|_{L_t^\infty W_x^{1,6}} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}}^2 + \|Q\|_{L_t^\infty L_x^6} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}} \|\nabla\eta\|_{L_t^3 L_x^{\frac{18}{5}}} \\
&\lesssim \|Q\|_{W_x^{1,6}} \|\eta\|_X^2.
\end{aligned}$$

For $\|\eta^3\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}}$, we have

$$\begin{aligned}
\|\eta^3\|_{L_t^{\frac{3}{2}} W_x^{1, \frac{18}{13}}} &= \|\eta^3\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} + \|\nabla\eta^3\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} \\
&\lesssim \|\eta^3\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} + \|\eta^2\nabla\eta\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}} \\
&\leq \|\eta^2\|_{L_t^3 L_x^{\frac{9}{4}}} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}} + \|\eta^2\|_{L_t^3 L_x^{\frac{9}{4}}} \|\nabla\eta\|_{L_t^3 L_x^{\frac{18}{5}}} \\
&\leq \|\eta\|_{L_t^6 L_x^{\frac{9}{2}}}^2 \|\eta\|_{L_t^3 W_x^{1, \frac{18}{5}}}.
\end{aligned}$$

Now, using $\|\eta\|_{L_x^{\frac{9}{2}}} \lesssim \|\nabla\eta\|_{L_x^2}^{\frac{1}{2}} \|\eta\|_{L_x^{\frac{18}{5}}}^{\frac{1}{2}}$, we get

$$\|\eta\|_{L_t^6 L_x^{\frac{9}{2}}} \lesssim \|\nabla\eta\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}}^{\frac{1}{2}}. \quad (2.137)$$

2.4. Detailed proof

So

$$\|\eta^3\|_{L_t^{\frac{3}{2}}W_x^{1,\frac{18}{13}}} \lesssim \|\nabla\eta\|_{L_t^\infty L_x^2} \|\eta\|_{L_t^3W_x^{1,\frac{18}{5}}}^2 \lesssim \|\eta\|_X^3. \quad (2.138)$$

Together we have

$$\begin{aligned} \|\eta\|_X &\lesssim \|\eta(0)\|_{H_x^1} + \left\| \int_0^t e^{-i(s-t)H} P_c(F(s) - 2Q|\eta|^2 - \overline{Q}\eta^2 - |\eta|^2\eta) ds \right\|_X \\ &\quad + \|Q\|_{W_x^{1,6}} \|\eta\|_X^2 + \|\eta\|_X^3 \\ &\lesssim \|\eta(0)\|_{H_x^1} + \|(F(s) - 2Q|\eta|^2 - \overline{Q}\eta^2 - |\eta|^2\eta)\|_{L_t^2\langle x \rangle^{-\sigma}H_x^1} \\ &\quad + \|\eta\|_X^2 + \|\eta\|_X^3 \end{aligned}$$

Next, for $g(\psi) = |\psi|^2\psi$,

$$\begin{aligned} &\|(F - 2Q|\eta|^2 - \overline{Q}\eta^2 - |\eta|^2\eta)\|_{L_t^2\langle x \rangle^{-\sigma}H_x^1} \\ &= \|Q^2\overline{\eta} + 2|Q|^2\eta - iDQ(\dot{z} + iEz)\|_{L_t^2\langle x \rangle^{-\sigma}H_x^1} \\ &\lesssim \|\langle x \rangle^{2\sigma}Q^2\|_{W_x^{1,\infty}} \|\eta\|_{L_t^2\langle x \rangle^\sigma H_x^1} + \|DQ\|_{\langle x \rangle^{-\sigma}H_x^1} \|\dot{z} + iEz\|_{L_t^2}. \end{aligned}$$

Next, we would like to bound $(\dot{z} + iEz)$. Recall that we imposed

$$\langle i\eta, \frac{\partial}{\partial z_1}Q[z] \rangle = 0 \quad \text{and} \quad \langle i\eta, \frac{\partial}{\partial z_2}Q[z] \rangle = 0 \quad (2.139)$$

through Lemma 2.4.11. By Gauge covariance of Q , we have

$$Q[e^{i\alpha}z] = e^{i\alpha}Q[z]. \quad (2.140)$$

So for $z = z_1 + iz_2$,

$$Q[z] = e^{i\alpha}\tilde{Q}[|z|^2] \quad \text{where} \quad \alpha = \tan^{-1}\left(\frac{z_2}{z_1}\right). \quad (2.141)$$

Here $\tilde{Q} : \mathbb{R}^+ \rightarrow \mathbb{R}$. So

$$\partial_{z_1}Q = \partial_{z_1}(e^{i\alpha})\tilde{Q} + 2z_1e^{i\alpha}\tilde{Q}' = e^{i\alpha}i(\partial_{z_1}\alpha)\tilde{Q} + 2z_1e^{i\alpha}\tilde{Q}' = i(\partial_{z_1}\alpha)Q + 2z_1e^{i\alpha}\tilde{Q}' \quad (2.142)$$

and

$$\partial_{z_2} Q = \partial_{z_2} (e^{i\alpha}) \tilde{Q} + 2z_2 e^{i\alpha} \tilde{Q}' = e^{i\alpha} i(\partial_{z_2} \alpha) \tilde{Q} + 2z_2 e^{i\alpha} \tilde{Q}' = i(\partial_{z_2} \alpha) Q + 2z_2 e^{i\alpha} \tilde{Q}'. \quad (2.143)$$

So

$$\begin{aligned} 0 &= \langle i\eta, -z_2 \partial_{z_1} Q + z_1 \partial_{z_2} Q \rangle = \langle \eta, -z_2 (\partial_{z_1} \alpha) Q + z_1 (\partial_{z_2} \alpha) Q \rangle \\ &= (-z_2 (\partial_{z_1} \alpha) + z_1 (\partial_{z_2} \alpha)) \langle \eta, Q \rangle = \langle \eta, Q \rangle. \end{aligned}$$

Now differentiate $\langle i\eta, \frac{\partial}{\partial z_1} Q[z] \rangle = 0$ and $\langle i\eta, \frac{\partial}{\partial z_2} Q[z] \rangle = 0$ with respect to t and substitute $i\partial_t \eta = H\eta + F$, we get

$$\begin{aligned} 0 &= \langle i\partial_t \eta, \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle \\ &= \langle H\eta + F, \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle \end{aligned}$$

Recall that $F = g(Q + \eta) - g(Q) - iDQ(\dot{z} + iEz)$. Therefore, we have

$$\begin{aligned} 0 &= \langle H\eta + g(Q + \eta) - g(Q) - iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle \\ &= \langle (H\eta + \frac{\partial}{\partial \varepsilon} g(Q + \varepsilon\eta)|_{\varepsilon=0}) + (g(Q + \eta) - g(Q) - \partial_\varepsilon^0 g(Q + \varepsilon\eta)) \\ &\quad - iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q \dot{z} \rangle \end{aligned}$$

From the above, we get that

$$\begin{aligned}
& \langle (g(Q + \eta) - g(Q) - \partial_\varepsilon^0 g(Q + \varepsilon\eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \\
= & \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle \\
& + \langle (H\eta + \partial_\varepsilon^0 g(Q + \varepsilon\eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \\
& + \langle i\eta, D \frac{\partial}{\partial z_j} Q\dot{z} \rangle.
\end{aligned}$$

Let $\mathcal{H}\eta = H\eta + \partial_\varepsilon^0 g(Q + \varepsilon\eta)$. By the symmetry of \mathcal{H} and differentiating equation (2.9) by z_j , we have

$$\begin{aligned}
\langle \mathcal{H}\eta, \frac{\partial}{\partial z_j} Q \rangle = \langle \eta, \mathcal{H} \frac{\partial}{\partial z_j} Q \rangle &= \langle \eta, E \frac{\partial}{\partial z_j} Q \rangle + \langle \frac{\partial}{\partial z_j} E \rangle \langle \eta, Q \rangle \\
&= \langle \eta, E \frac{\partial}{\partial z_j} Q \rangle = \langle i\eta, iE \frac{\partial}{\partial z_j} Q \rangle \\
&= \langle i\eta, E \frac{\partial}{\partial z_j} DQiz \rangle
\end{aligned}$$

using $\langle \eta, Q \rangle = 0$ and $DQ[z]iz = iQ[z]$. So

$$\begin{aligned}
& \langle (g(Q + \eta) - g(Q) - \partial_\varepsilon^0 g(Q + \varepsilon\eta)), \frac{\partial}{\partial z_j} Q[z] \rangle \\
= & \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, E \frac{\partial}{\partial z_j} DQiz \rangle + \langle i\eta, D \frac{\partial}{\partial z_j} Q\dot{z} \rangle \\
= & \langle -iDQ(\dot{z} + iEz), \frac{\partial}{\partial z_j} Q[z] \rangle + \langle i\eta, (D \frac{\partial}{\partial z_j} Q)(\dot{z} + iEz) \rangle
\end{aligned}$$

For $g(\psi) = |\psi|^2\psi$,

$$\partial_\varepsilon^0 g(Q + \varepsilon\eta) = Q^2\bar{\eta} + 2|Q|^2\eta. \quad (2.144)$$

Therefore,

$$\begin{aligned} g(Q + \eta) - g(Q) - \partial_\varepsilon^0 g(Q + \varepsilon \eta) &= |Q + \eta|^2(Q + \eta) - |Q|^2Q - Q^2\bar{\eta} - 2|Q|^2\eta \\ &= 2Q|\eta|^2 + \bar{Q}\eta^2 + |\eta|^2\eta \end{aligned}$$

Since

$$\left\langle \frac{\partial}{\partial z_j} Q, i \frac{\partial}{\partial z_k} Q \right\rangle = j - k + o(1), \quad (2.145)$$

we have that

$$|\dot{z} + iEz| \lesssim |\langle 2Q|\eta|^2 + \bar{Q}\eta^2 + |\eta|^2\eta, DQ \rangle| (1 + \|\eta\|_{L^2}). \quad (2.146)$$

Therefore,

$$\begin{aligned} & \|\dot{z} + iEz\|_{L_t^2} \\ & \lesssim \|\langle 2Q|\eta|^2 + \bar{Q}\eta^2 + |\eta|^2\eta, DQ \rangle\|_{L_t^2} (1 + \|\eta\|_{L_t^\infty L_x^2}) \\ & \lesssim (\|QDQ|\eta|^2\|_{L_t^2 L_x^1} + \|DQ|\eta|^2\eta\|_{L_t^2 L_x^1}) (1 + \|\eta\|_{L_t^\infty L_x^2}) \\ & \leq (\|QDQ\|_{L_t^\infty L_x^2} \|\eta\|_{L_t^4 L_x^4}^2 + \|DQ\|_{L_t^\infty L_x^4} \|\eta\|_{L_t^6 L_x^4}^3) (1 + \|\eta\|_{L_t^\infty L_x^2}) \\ & \leq (\|QDQ\|_{L_t^\infty L_x^2} \|\eta\|_{L_t^\infty H^1}^{\frac{1}{2}} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}}^{\frac{3}{2}} + \|DQ\|_{L_t^\infty L_x^4} \|\eta\|_{L_t^\infty H^1}^{\frac{5}{4}} \|\eta\|_{L_t^7 L_x^{\frac{42}{13}}}^{\frac{7}{4}}) \\ & \quad (1 + \|\eta\|_{L_t^\infty L_x^2}) \\ & \lesssim \|\eta\|_X^2 + \|\eta\|_X^4 \end{aligned}$$

For $\|\eta\|_{L_t^4 L_x^4}$, we used

$$\|\eta\|_{L_x^4} \lesssim \|\nabla \eta\|_{L_x^2}^{\frac{1}{4}} \|\eta\|_{L_x^{\frac{18}{5}}}^{\frac{3}{4}}. \quad (2.147)$$

For $\|\eta\|_{L_t^6 L_x^4}$, we used

$$\|\eta\|_{L_x^4} \lesssim \|\nabla \eta\|_{L_x^2}^{\frac{5}{12}} \|\eta\|_{L_x^{\frac{42}{13}}}^{\frac{7}{12}}. \quad (2.148)$$

2.4. Detailed proof

Putting the preceding estimates together we have

$$\|\eta\|_X \lesssim \|\eta(0)\|_{H^1} + \|\langle x \rangle^{2\sigma} Q^2\|_{L_x^\infty} \|\eta\|_X + \|\eta\|_X^2 + \|\eta\|_X^4, \quad (2.149)$$

and since $\|\langle x \rangle^{2\sigma} Q^2\|_{L_x^\infty} \ll 1$,

$$\|\eta\|_X \leq C[\|\eta(0)\|_{H^1} + \|\eta\|_X^2 + \|\eta\|_X^4] \quad (2.150)$$

for some constant $C \geq 1$.

Now, let X_T be the norm defined by

$$\|\psi\|_{X_T} = \|\langle x \rangle^{-\sigma} \psi\|_{L_t^2([0,T], H_x^1)} + \|\psi\|_{L_t^3([0,T], W_x^{1, \frac{18}{5}})} + \|\psi\|_{L_t^\infty([0,T], H_x^1)}$$

Fix the initial condition $\|\psi(0)\|_X$ to be small enough so that

$$\|\eta(0)\|_{H^1} \leq \frac{1}{20C^2}. \quad (2.151)$$

Let

$$T_1 = \sup\{T > 0 : \|\eta\|_{X_T} \leq \frac{1}{10C}\} > 0. \quad (2.152)$$

Then for $0 \leq T \leq T_1$,

$$\|\eta\|_{X_T} \leq \frac{1}{20C} + \frac{1}{10^2 C} + \frac{1}{10^4 C^3} \leq \frac{1}{15C}, \quad (2.153)$$

showing that $T_1 = \infty$.

Next, we would like to bound $\|\dot{z} + iEz\|_{L_t^1}$. We have

$$\begin{aligned} & \|\dot{z} + iEz\|_{L_t^1} \\ & \lesssim \|\langle 2Q|\eta|^2 + \overline{Q}\eta^2 + |\eta|^2\eta, DQ\rangle(1 + \|\eta\|_{L_x^2})\|_{L_t^1} \\ & \lesssim (\|QDQ|\eta|^2\|_{L_t^1 L_x^1} + \|DQ|\eta|^2\eta\|_{L_t^1 L_x^1})(1 + \|\eta\|_{L_t^\infty L_x^2}) \\ & \leq (\|\langle x \rangle^{2\sigma} QDQ\|_{L_t^\infty L_x^\infty} \|\langle x \rangle^{-2\sigma} \eta^2\|_{L_t^1 L_x^1} + \|\langle x \rangle^\sigma DQ\|_{L_t^\infty L_x^\infty} \|\langle x \rangle^{-\sigma} \eta^3\|_{L_t^1 L_x^1}) \\ & \quad (1 + \|\eta\|_{L_t^\infty L_x^2}) \end{aligned}$$

2.4. Detailed proof

Here, the factor $\|\langle x \rangle^{-\sigma} \eta^3\|_{L_t^1 L_x^1}$ can be bounded by

$$\|\langle x \rangle^{-\sigma} \eta^3\|_{L_t^1 L_x^1} \leq \|\langle x \rangle^{-\sigma} \eta\|_{L_t^2 L_x^2} \|\eta\|_{L_t^4 L_x^4}^2 \lesssim \|\langle x \rangle^{-\sigma} \eta\|_{L_t^2 L_x^2} \|\eta\|_{L_t^\infty H^1}^{\frac{1}{2}} \|\eta\|_{L_t^3 L_x^{\frac{18}{5}}}^{\frac{3}{2}}.$$

Putting everything together, we have

$$\|\dot{z} + iEz\|_{L_t^1} \lesssim \|\eta\|_X^2 + \|\eta\|_X^4$$

Therefore, $|\partial_t(e^{i \int_0^t E(s) ds} z(t))| = |\dot{z} + iEz| \in L_t^1$. This means that $\lim_{t \rightarrow \infty} e^{i \int_0^t E(s) ds} z(t)$ exists. Since $|e^{i \int_0^t E(s) ds} z(t)| = |z|$, $\lim_{t \rightarrow \infty} |z(t)|$ exists. Furthermore, E is continuous and $E(z) = E(|z|)$, so $\lim_{t \rightarrow \infty} E(z(t))$ exists.

Finally, let $H = -\Delta + 2iA \cdot \nabla + i(\nabla \cdot A) + V$. So

$$\eta_c(t) = e^{itH}(\eta_c(0) - i \int_0^t e^{-isH} P_c F(s) ds). \quad (2.154)$$

By Strichartz estimates as above, we have

$$\left\| \int_S^T e^{-isH} P_c F(s) ds \right\|_{H^1} \lesssim \|F\|_X \rightarrow 0 \quad (2.155)$$

as $T > S \rightarrow \infty$. Therefore, $\int_0^\infty e^{-isH} P_c F ds$ converges in H^1 , and

$$\lim_{t \rightarrow \infty} e^{-itH} \eta_c(t) = \eta_c(0) - i \int_0^\infty e^{-isH} P_c F(s) ds =: \eta_+ \quad (2.156)$$

for some $\eta_+ \in H^1$. From this, we get that $\eta_c(t)$ converges to 0 weakly in H_1 . Now, by compactness of $R[z(t)] - I$, we have that $\eta_d(t) := \eta(t) - \eta_c(t) = (R[z(t)] - I)\eta_c(t)$ converges to 0 strongly in H^1 . Therefore

$$\|\eta(t) - e^{itH} \eta_+\|_{H^1} \rightarrow 0. \quad (2.157)$$

□

Chapter 3

Global well-posedness of two dimensional radial Schrödinger maps into the 2-sphere

In this chapter, we will discuss a result obtained jointly with Stephen Gustafson. The main result is Theorem 4 stated in Section 1.3.

3.1 Known results and our result

Consider the two dimensional Schrödinger map equation

$$\begin{cases} \vec{u}_t = \vec{u} \times \Delta \vec{u} \\ \vec{u}(x, 0) = \vec{u}_0(x), \vec{u}(0) - \widehat{k} \in H^2(\mathbb{R}^2) \end{cases} \quad (3.1)$$

where $\vec{u} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{S}^2$. Recall that we treat the \mathbb{S}^2 as a sphere embedded in \mathbb{R}^3 , i.e.

$$\mathbb{S}^2 = \{\vec{u} \in \mathbb{R}^3 : |\vec{u}| = 1\} \subset \mathbb{R}^3. \quad (3.2)$$

Hence, we view \vec{u} as

$$\vec{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \quad (3.3)$$

where

$$u_1^2 + u_2^2 + u_3^2 = 1. \quad (3.4)$$

Conserved quantities of this equation are the L^2 -mass

$$\|\vec{u}(t) - \widehat{k}\|_{L^2(\mathbb{R}^2)}^2 = \|\vec{u}_0 - \widehat{k}\|_{L^2(\mathbb{R}^2)}^2 \quad (3.5)$$

and the energy

$$\mathcal{E}(\vec{u}) = \frac{1}{2} \|\nabla \vec{u}(t)\|_{L^2(\mathbb{R}^2)}^2 = \mathcal{E}(\vec{u}_0). \quad (3.6)$$

We will consider the problem of whether (3.1) is globally wellposed. In other words, we would like to know whether the solution \vec{u} exists indefinitely or blows up in finite time. In an attempt to gain some understanding on whether blow up is possible, we will see how the energy behaves under a scaling preserving the solution. If \vec{u} is a solution of (3.1), then for $\lambda > 0$,

$$\vec{v}(x) = \vec{u}(\lambda x, \lambda^2 t) \quad (3.7)$$

is again a solution of (3.1). We would like to compare the energy \mathcal{E} of \vec{v} to that of \vec{u} . The energy scales differently according to the space dimension n and is given by

$$\mathcal{E}(\vec{v}) = \lambda^{2-n} \mathcal{E}(\vec{u}). \quad (3.8)$$

For example, for space dimensions $n = 1, 2, 3$, we have that:

$$\text{For } n = 1, \quad \mathcal{E}(\vec{v}) = \lambda \mathcal{E}(\vec{u}). \quad (3.9)$$

$$\text{For } n = 2, \quad \mathcal{E}(\vec{v}) = \mathcal{E}(\vec{u}). \quad (3.10)$$

$$\text{For } n = 3, \quad \mathcal{E}(\vec{v}) = \frac{1}{\lambda} \mathcal{E}(\vec{u}). \quad (3.11)$$

Here, as $\lambda \rightarrow \infty$, $\vec{u}(\lambda x, \lambda^2 t)$ undergoes a horizontal compression. From the energy scaling, we see that for $n = 1$, it costs a lot of energy to concentrate solutions. As energy is a conserved quantity, the heuristic is that it is hard for solutions to concentrate, so solutions are expected to be global. The case $n = 1$ is called **energy subcritical**. On the other hand, for $n = 3$, it costs very little energy to concentrate solutions, so blow up solutions are expected. The case $n = 3$ is called **energy supercritical**. For $n = 2$, the energy remains unchanged after the scaling. The case $n = 2$ is called

energy critical. In this case, the scaling argument does not give us a heuristic of whether blow up solutions are possible. It turns out that blow up can indeed occur for $n = 2$. Very recently, Merle-Raphaël-Rodnianski '11 [55] showed that within a special class of solutions to (3.1) known as **1-equivariant maps**, there are blown up solutions. More specifically, let

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e^{\alpha R} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.12)$$

For an integer m , an **m -equivariant map** $\vec{u} : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ is a map of the form

$$\vec{u}(r, \theta) = e^{m\theta R} \vec{v}(r) \quad (3.13)$$

with $\vec{v}(0) = -\hat{k}$ and $\vec{v}(\infty) = \hat{k}$.

The Schrödinger map (3.1) preserves m -equivariance and a radial solution is an example of an m -equivariant map with $m = 0$. It turns out that any m -equivariant map with the boundary conditions ($\vec{v}(0) = -\hat{k}$ and $\vec{v}(\infty) = \hat{k}$) defined above will have energy at least $4\pi|m|$. Various results concerning m -equivariant maps are known. For example, [41] showed that if $m \geq 3$ and if the energy of the initial data is slightly larger than $4\pi|m|$, then the solution is global. [55] showed if $m = 1$, then there exists a set of smooth initial data with $\mathcal{E} > 4\pi$ whose solutions blow up in finite time. The result of [55] tells us that even smooth initial data can lead to blow up solutions.

Now, let us look at some more general known results regarding global well-posedness of (3.1). For $n = 1$, [16] showed that if $\mathcal{E}(\vec{u}_0)$ is finite, then (3.1) is global. In the same paper, they also showed that for $n = 2$, if $\mathcal{E}(\vec{u}_0)$ sufficiently small, the radial solutions to (3.1) are global. [4] showed that for $n = 2$, suppose the initial data u_0 satisfies $\vec{u}_0 - Q \in H^s$ for all $s > 0$ for some $Q \in \mathbb{S}^2$ and $\mathcal{E}(\vec{u}_0)$ is small, then the solution \vec{u} to (3.1) is global and $\vec{u} - Q \in H^s$ for all $s > 0$.

Our result shows that for $n = 2$, any radial solution \vec{u} to (3.1) is global. This is the first result that showed global well-posedness in $n = 2$ without

the smallness of energy assumption. Our result is as follows:

Theorem 7. *(Global well-posedness of 2D radial Schrödinger map into \mathbb{S}^2)*
 Suppose $\vec{u}(x, 0) = \vec{u}_0(|x|)$ is radial and $\vec{u}_0 - \widehat{k} \in H^2(\mathbb{R}^2)$. Then

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad \text{with} \quad \vec{u}(r, 0) = \vec{u}_0(r) \quad (3.14)$$

has a unique global solution $\vec{u} \in L^\infty([0, \infty); H^2(\mathbb{R}^2))$.

3.2 Discussion and outline of the proof

In this section, we will give an outline to the proof of Theorem 7. Finer details of the proof will be given in the next section.

The strategy to obtain Theorem 7 is to transform (3.1) into a more familiar equation and to show global well-posedness for the transformed equation. After this, one then shows that global well-posedness of the transformed equation implies global well-posedness of (3.1). The transformation we use is the generalized Hasimoto transformation used by [16] to show global well-posedness for the Schrödinger map solutions in 1D and small solutions in 2D. It transforms our radial Schrödinger map equation into the equation

$$iq_t = -\Delta q + \frac{1}{r^2}q + \left(\int_r^\infty \frac{|q(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|q|^2 \right) q \quad (3.15)$$

for a complex-valued function $q = q(r, t)$. If we look at (3.15), we see that it is made up of the linear part

$$iq_t = -\Delta q + \frac{1}{r^2}q, \quad (3.16)$$

the local nonlinear part

$$-\frac{1}{2}|q|^2q \quad (3.17)$$

as well as the non-local nonlinear part

$$\left(\int_r^\infty \frac{|q(\rho, t)|^2}{\rho} d\rho \right) q. \quad (3.18)$$

3.2. Discussion and outline of the proof

The last term is called the non-local term because its value at $r = r_0$ depends not only on the value of q at r_0 but also on the value of q at other r 's as well. We will call (3.15) the non-local nonlinear Schrödinger equation (shortened to be NLNLS below). To show global well-posedness of Schrödinger map equation, we will show global well-posedness of the NLNLS.

Recall that the original Schrödinger map equation is

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \quad \text{with} \quad \vec{u}(r, 0) = \vec{u}_0(r) \quad (3.19)$$

where

$$\vec{u}_0 - \widehat{k} \in H^2(\mathbb{R}^2). \quad (3.20)$$

As we will see from the details of the generalized Hasimoto transformation, we have $|\vec{u}_r| = |q|$. Hence, $\nabla \vec{u}_0 \in L^2$ implies $q_0 \in L^2$. As a result, we would like to show global well-posedness for

$$iq_t = -\Delta q + \frac{1}{r^2}q + \left(\int_r^\infty \frac{|q(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|q|^2 \right) q \quad (3.21)$$

with

$$q(r, 0) = q_0 \in L^2(\mathbb{R}^2). \quad (3.22)$$

To formulate a strategy in showing global well-posedness for NLNLS, let us compare this equation with the more familiar cubic nonlinear Schrödinger equation (shortened to be NLS below)

$$iu_t = -\Delta u \pm |u|^2 u \quad (3.23)$$

with radial initial data

$$u(x, 0) = u_0(r), \quad r = |x|. \quad (3.24)$$

Recall that this equation is defocusing with the $+$ sign, and focusing with the $-$ sign. For $n = 2$, this equation is L^2 -critical and so it is highly non-trivial to show global well-posedness with L^2 initial data (and indeed it is false in the focusing case if the L^2 norm is sufficiently large).

3.2. Discussion and outline of the proof

It turns out that NLNLS and NLS share a lot of similarities. For example, both equations satisfy the mass conservation (conservation of the L^2 norm). Here, mass conservation of NLNLS arises from the conservation of the energy \mathcal{E} of \vec{u} . Since $|\vec{u}_r| = |q|$, $\|\nabla \vec{u}(t)\|_{L^2(\mathbb{R}^2)}^2 = \|\nabla \vec{u}(0)\|_{L^2(\mathbb{R}^2)}^2$ gives

$$\|q(t)\|_{L^2(\mathbb{R}^2)} = \|q(0)\|_{L^2(\mathbb{R}^2)}. \quad (3.25)$$

As mentioned above, NLS is L^2 -critical. One can also check that NLNLS is also L^2 -critical. To do so, we just observe that if q is a solution to NLNLS, then q_λ defined by

$$q_\lambda(r, t) = \lambda q(\lambda r, \lambda^2 t) \text{ for } \lambda > 0 \quad (3.26)$$

is also a solution and that

$$\|q_\lambda\|_{L^2(\mathbb{R}^2)} = \|q\|_{L^2(\mathbb{R}^2)}, \quad (3.27)$$

which says that NLNLS is L^2 -critical.

Despite the above similarities, there is a major difference between the cubic NLS and NLNLS: the quantity

$$E(u(t)) := \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla u(x, t)|^2 \pm \frac{1}{4} |u(x, t)|^4 \right), \quad (3.28)$$

known as the energy, is a conserved quantity for the NLS while NLNLS has no equivalent conserved quantity. Of course, if we only assume L^2 initial data for NLS, then the energy may not be defined for the solution. However, it turns out that energy conservation plays a big role in showing global well-posedness of NLS for even L^2 data as we will see below.

To study the global well-posedness of NLNLS, we look at results of global well-posedness for NLS. Since the two equations are similar, it may be possible to adapt a method of showing global-posedness of NLS to NLNLS. As for NLS, Killip-Tao-Vsian [48] showed global well-posedness for NLS in 2D with radial L^2 initial data in the defocusing case, and in the focusing case when the L^2 -norm is below a certain level. It turns out the method used

there can be adapted to the case of NLNLS with some significant changes, in particular due to the presence of the non-local term and the absence of energy conservation.

As an understanding of Killip-Tao-Visan's method is important for understanding the proof of our result, in the next subsection, we will outline their method.

3.2.1 Outline of Killip-Tao-Visan's proof of global well-posedness of NLS for radial L^2 initial data in 2D

The idea behind Killip-Tao-Visan's method is a proof by contradiction. Following a method developed by Kenig-Merle [47], Tao-Visan-Zhang [84] showed that if there is *any* solution which fails to scatter in the sense that

$$\int_I \int_{\mathbb{R}^2} |u(x, t)|^4 dx dt = \infty \quad (3.29)$$

where I is its time interval of existence (so this includes solutions which blow-up), then there exists such a non-scattering solution, of minimal L^2 -norm, with interval of existence I and functions

$$N : I \rightarrow \mathbb{R}^+ \text{ and } C : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad (3.30)$$

such that for each $t \in I$ and $\eta > 0$,

$$\int_{|x| \geq C(\eta)/N(t)} |u(x, t)|^2 dx \leq \eta \text{ and } \int_{|\xi| \geq C(\eta)N(t)} |\widehat{u}(\xi, t)|^2 d\xi \leq \eta. \quad (3.31)$$

Here, one can think of $\frac{1}{N(t)}$ as the spatial scale of u and $N(t)$ as the frequency scale.

Killip-Tao-Visan refined the above and showed that one can assume $N : I \rightarrow \mathbb{R}^+$ belongs to one of the following three scenarios:

Soliton-like solution:

$$I = \mathbb{R} \text{ and } N(t) = 1 \text{ for all } t \in \mathbb{R} \quad (3.32)$$

Self-similar solution:

$$I = (0, \infty) \text{ and } N(t) = t^{-\frac{1}{2}} \text{ for all } t \in I \quad (3.33)$$

Inverse cascade:

$$I = \mathbb{R}, \liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0 \text{ and } \sup_{t \in \mathbb{R}} N(t) < \infty. \quad (3.34)$$

These solutions are referred as the three enemies by Killip-Tao-Vsian.

The solutions u above have a lot of structure. For example, the soliton-like solution exists forever and is localized in space ($N(t) = 1$) while the self-similar solution concentrates and blows up in one direction in time and spreads out in the other direction of time ($I = (0, \infty)$ and $N(t) = t^{-\frac{1}{2}}$). In fact, Killip-Tao-Visan showed this structure to be incompatible with the NLS equation. They showed global well-posedness by ruling out each of the three enemies case by case.

It turns out that the method showing the existence of the three enemies when global well-posedness fails is quite general and can be applied to the NLNLS case with some modifications. However, the method used to rule out the three enemies depends on energy conservation and does not apply to the NLNLS case.

Indeed, energy conservation is a key tool in ruling out these highly structured solutions. As the initial data u_0 is only L^2 , energy is not a defined quantity. However, because the special solution u has the property (3.31), one can in fact control the H^1 -norm of u . The idea is that if one is able to improve the regularity of the solution and show that the solution is in fact in H^1 , then energy will be defined and this has implications on how such a solution should behave which are incompatible with the highly structured solutions.

We will provide a very brief outline of how solutions are ruled out. For this discussion, we consider only the defocusing NLS. This is because, as we will see, NLNLS has a defocusing character.

Ruling out the self-similar case

For the self-similar case,

$$N(t) = t^{-\frac{1}{2}} \quad \text{and} \quad I = (0, \infty). \quad (3.35)$$

Here, the frequency scale $N(t)$ is decreasing in time and it is possible to show that $u \in H^s$ for all $s > 0$. With this, the H^1 global well-posedness for defocusing NLS says that such a solution is global, but this is not compatible with the time of existence $I = (0, \infty)$.

The method above relies on the H^1 global well-posedness theory of NLS which relies heavily on the energy, so this method will not adapt well to the NLNLS case.

Ruling out the inverse cascade case

For the inverse cascade case,

$$N(t) \lesssim 1, \quad \liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0 \quad \text{and} \quad I = \mathbb{R}. \quad (3.36)$$

As the frequency scale $N(t)$ is bounded, it is possible to show that $u \in L_t^\infty H_x^s$ for all $s \geq 0$. With this, energy conservation and the Gagliardo-Nirenberg inequality show that $\|\nabla u\|_{L_x^2}$ will be bounded away from 0. However, this is not compatible with

$$\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0 \quad (3.37)$$

as it can also be shown that $\|\nabla u\|_{L_x^2} \rightarrow 0$ along any sequence of t where $N(t) \rightarrow 0$. This is a contradiction.

Just as in the previous case, the above method relies on energy conservation to show the contradiction, so this will not adapt well to the NLNLS case either.

Ruling out the soliton case

For the soliton-like case,

$$I = \mathbb{R} \quad \text{and} \quad N(t) = 1. \quad (3.38)$$

The idea is to consider the quantity

$$M(t) = 2 \operatorname{Im} \int_{\mathbb{R}^2} \bar{u} x \cdot \nabla u dx. \quad (3.39)$$

Here, the quantity $M(t)$ is formally $\frac{d}{dt} \int_{\mathbb{R}^2} x^2 |u|^2 dx$, the time derivative of the variance of $|u|^2$. However, the above quantity may not be finite for the solution u . To make the above well defined, one has to add a smooth cutoff function $\phi_R(r)$ which is zero outside a disk of some large radius $R > 0$ and consider instead

$$M_R(t) = 2 \operatorname{Im} \int_{\mathbb{R}^2} \phi_R \bar{u} x \cdot \nabla u dx. \quad (3.40)$$

Again, one can show $u \in L_t^\infty H_x^1$, so on one hand,

$$|M_R(t)| \lesssim R \|u\|_2 \|\nabla u\|_2 \lesssim R \quad (3.41)$$

but on the other hand, it can be shown that

$$\frac{d}{dt} M_R(t) = 8\mathcal{E}(u) + \text{error terms} \quad (3.42)$$

where the error terms are of size comparable to

$$\|u\|_{L^2(r>R)}^2, \quad \|\nabla u\|_{L^2(r>R)}^2 \quad \text{and} \quad \|u\|_{L^4(r>R)}^2. \quad (3.43)$$

Hence, using (3.31) with $N(t) = 1$, and by choosing R large enough, the error terms can be made to be much smaller than \mathcal{E} , so

$$\frac{d}{dt} M_R(t) \geq \mathcal{E} > 0. \quad (3.44)$$

This contradicts with $|M_R(t)| \lesssim R$.

Notice in the above, the preserved quantity, energy, is used to obtain a lower bound of the first time derivative of the quantity M_R . To adapt the above to the NLNLS case requires an alternative way to obtain such a lower bound.

3.2.2 Discussion of our proof of global well-posedness of NLNLS for radial L^2 initial data in 2D

As mentioned in the previous subsection, just like in the NLS case, it can be shown that when global well-posedness and scattering for NLNLS fails, then there exists a solution u with the structure of one of the three enemies. As in the NLS case, to show global well-posedness, we have to rule out the three enemies. The following is a brief discussion of how this is done. The complete proof will be given in the next section.

As mentioned in the previous section, the method used for the NLS case to rule out self-similar solutions and inverse cascade solutions cannot be adapted to our case due to the lack of a conserved quantity equivalent to the energy for NLNLS. However, the method for ruling out the soliton case is more general and has a chance of being adaptable to the NLNLS. To rule out the soliton and the self-similar case, we consider the quantity

$$M_R(t) = 2 \operatorname{Im} \int_{\mathbb{R}^2} \phi_R \bar{q} x \cdot \nabla q dx \quad (3.45)$$

just as in the NLS case. Recall that in the NLS case, conservation of energy is used to obtain a lower bound on the quantity $\frac{d}{dt} M_R(t)$ in attempt to reach a contradiction. Due to the lack of energy in NLNLS, we are forced to obtain such a lower bound by using more delicate estimates on various norms of the solution u . In the end, such a lower bound is obtained and soliton-like and self-similar solutions are ruled out.

However, we are unable to obtain the more delicate estimates required to bound $\frac{d}{dt} M_R(t)$ from below for the inverse cascade case. The reason is that such delicate estimates require fine control on the structure of the solution u . In particular, we need to have very explicit knowledge of the structure

of the spatial scale $\frac{1}{N(t)}$ (and the frequency scale $N(t)$). However, for the inverse cascade case, such knowledge is not available. To get around this difficulty, instead of using M_R , we consider a different quantity

$$P(t) = \text{Im} \int_0^\infty (\bar{q}q_r)\psi(r)rdr \quad (3.46)$$

for some function ψ which tends to zero at the origin, and tends to one at infinity. It turns out that with this quantity P , we are able to construct arguments to rule out the inverse cascade case just like in the soliton and self-similar case. Details on how this is done will be given in the coming section.

3.3 Proof of our result

We will provide details of the proof of our main result in this section. There are five parts to the proof:

- Part 1:

The goal of this part is to reduce the global well-posedness problem of the Schrödinger map equation into the global well-posedness problem of the NLNLS equation by transforming the Schrödinger map equation into the NLNLS equation through the generalized Hasimoto transformation. In later parts of the proof, we will establish global well-posedness of NLNLS. In order to be able to translate the result back to the original Schrödinger map equation, we will also need to show we can translate a solution of NLNLS into a solution of the Schrödinger map equation. The result of this part is Proposition 1.

- Part 2:

The goal of this part is to develop the local well-posedness theory which part 3 relies on. The result of this part is Proposition 2.

- Part 3:

The goal of this part is to show if global well-posedness and scattering of NLNLS fails, then NLNLS admits solutions, called almost periodic

solutions, having explicit structures in terms of the spatial scale and frequency scale. It is further shown that in this case, NLNLS admits solutions of the soliton-type, inverse cascade type or self-similar type (the three enemies) as discussed in the previous section. The result of this part is Proposition 3.

- Part 4:

The goal of this part is to show the three enemies given in the previous part have more regularity than originally entitled to due to the extra structure. The result of this part is Proposition 4.

- Part 5:

The goal of this part is to rule out the possibility of the three enemies. The details of this part will be given in Subsection 3.3.5.

Once the three enemies have been ruled out, by the contrapositive of proposition 3, NLNLS must be global. Then by Proposition 1, the Schrödinger map under consideration must be global. This shows Theorem 7.

The propositions for the five parts are given below:

Proposition 1. *There is a map $\vec{u} \mapsto q = q[\vec{u}]$ from radial maps with $\vec{u}(r) - \widehat{k} \in H^2(\mathbb{R}^2)$ to complex radial functions $q(r)$ with $w(x) := e^{i\theta}q(r) \in H^1(\mathbb{R}^2)$ ((r, θ) polar coordinates on \mathbb{R}^2) such that if $\vec{u}(r, t)$ is a (radial) solution of (3.1), then $q(r, t) = q[\vec{u}]$ is a (radial) solution of (3.15). Further, the H^1 and H^2 norms of $\nabla \vec{u}$ and $w = e^{i\theta}q$ are comparable:*

$$\begin{cases} \|w(t)\|_{H^1(\mathbb{R}^2)} \lesssim \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)} + \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)}^2 \\ \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)} \lesssim \|w(t)\|_{H^1(\mathbb{R}^2)} + \|w(t)\|_{H^1(\mathbb{R}^2)}^2. \end{cases} \quad (3.47)$$

$$\begin{cases} \|w(t)\|_{H^2(\mathbb{R}^2)} \lesssim \|\nabla \vec{u}(t)\|_{H^2(\mathbb{R}^2)} + \|\nabla \vec{u}(t)\|_{H^1(\mathbb{R}^2)}^3 \\ \|\nabla \vec{u}(t)\|_{H^2} \lesssim \|w(t)\|_{H^2(\mathbb{R}^2)} + \|w(t)\|_{H^1(\mathbb{R}^2)}^3. \end{cases} \quad (3.48)$$

Moreover, the map $\vec{u} \mapsto q$ is one-to-one: given two radial maps \vec{u}^A and \vec{u}^B as above, if the corresponding associated complex functions agree, $q^A \equiv q^B$, then so do the original maps, $\vec{u}^A \equiv \vec{u}^B$.

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Here, we consider $w(x, t) = e^{i\theta} q(r, t)$ to handle the term $\frac{q}{r^2}$ in NLNLS. The corresponding equation for w is

$$iw_t = -\Delta w + \left(\int_{\rho > |x|}^{\infty} \frac{|w(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|w|^2 \right) w. \quad (3.49)$$

Proposition 1 will be proved in the Section 3.3.1.

Proposition 2. 1. For each $q_0 \in L^2$, (3.15) has a unique solution $q \in C(I; L^2) \cap L^4_{loc}(I; L^4)$ on a maximal (and non-empty) time interval $I = (T_{min}, T_{max}) \ni 0$ (possibly $T_{min} = -\infty$ and/or $T_{max} = \infty$), which conserves the L^2 norm.

2. If $T_{max} < \infty$, then $\|q\|_{L^4_t([0, T_{max}]; L^4)} = \infty$ (an analogous statement holds for T_{min}).

3. If $T_{max} = \infty$ and $\|q\|_{L^4_t([0, \infty); L^4)} < \infty$, then q scatters as $t \rightarrow +\infty$ (an analogous statement holds for $t \rightarrow -\infty$).

4. The solution at each time depends continuously on the initial data. Further, the solution has the “stability” property as in Lemma 1.5 of [48].

5. If $\|q_0\|_{L^2}$ is sufficiently small, the solution is global ($I = (-\infty, \infty)$) and $\|q\|_{L^4_t(\mathbb{R}; L^4)} < \infty$.

Proposition 2 will be proved in the Section 3.3.2.

Proposition 3. If there is any L^2 data for which global well-posedness (or merely scattering) for (3.15) with radial L^2 initial data fails, then

1. there exists a non-zero solution u of NLS with interval of existence I and functions

$$N : I \rightarrow \mathbb{R}^+ \text{ and } C : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad (3.50)$$

such that for each $t \in I$ and $\eta > 0$,

$$\int_{|x| \geq C(\eta)/N(t)} |u(x, t)|^2 dx \leq \eta \quad (3.51)$$

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and

$$\int_{|\xi| \geq C(\eta)N(t)} |\widehat{u}(\xi, t)|^2 d\xi \leq \eta. \quad (3.52)$$

2. We may assume q falls into one of the following three cases

- *soliton-type solution*: $I = \mathbb{R}$ and $N(t) \equiv 1$
- *self-similar-type solution*: $I = (0, \infty)$ and $N(t) = t^{-1/2}$
- *inverse cascade-type solution*: $I = \mathbb{R}$, $N(t) \lesssim 1$, $\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0$

Proposition 3 will be proved in the Section 3.3.3.

Proposition 4. *If a solution q of NLNLS belongs to one of the soliton-type, the self-similar-type or the inverse cascade-type, then*

$$w(x, t) := e^{i\theta} q(r, t) \in H^s(\mathbb{R}^2) \quad (3.53)$$

for every $s \geq 0$ and $t \in I$. Furthermore in the soliton and inverse cascade cases,

$$w \in L_t^\infty H^s(\mathbb{R}^2) \text{ for each } s \geq 0. \quad (3.54)$$

Proposition 4 will be proved in the Section 3.3.4.

3.3.1 Equating the Schrödinger map equation and the NLNLS equation

The goal of this section is to prove Proposition 1. The idea is to use the generalized Hasimoto transformation to translate the Schrödinger map equation into the NLNLS equation. The generalized Hasimoto transformation works as follows. First, given a solution \vec{u} of (3.1), for each fixed time t , we will build a frame $\{\widehat{e}_1(r, t), \widehat{e}_2(r, t)\}$ in the tangent space $T_{\vec{u}(r, t)}\mathbb{S}^2$. We will show how this is done below.

As \vec{u}_r and \vec{u}_t are in $T_{\vec{u}(r, t)}\mathbb{S}^2$, we can express them as

$$\vec{u}_r = q_1 \widehat{e}_1 + q_2 \widehat{e}_2 \text{ and } \vec{u}_t = p_1 \widehat{e}_1 + p_2 \widehat{e}_2 \quad (3.55)$$

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where q_1, q_2, p_1 and p_2 are real valued functions of r . Now, define

$$q = q_1 + iq_2 \text{ and } p = p_1 + ip_2. \quad (3.56)$$

Then q and p are complex valued functions of r . Here, as \vec{u} evolves over time, q does so as well. We will show here that a particular choice of the frame \hat{e}_1, \hat{e}_2 leads to the NLNLS equation.

Building the frames

Following [16], given a radial map $\vec{u}(r) \in \hat{k} + H^k$, we want to construct a unit tangent vector field, parallel transported along the curve $\vec{u}(r) \in \mathbb{S}^2$:

$$\hat{e}(r) \in T_{\vec{u}(r)}\mathbb{S}^2, \quad |\hat{e}| \equiv 1, \quad D_r \hat{e}(r) \equiv 0, \quad (3.57)$$

where here D denotes covariant differentiation of tangent vector fields: given $\vec{\xi}(s) \in T_{\vec{u}(s)}\mathbb{S}^2$,

$$D_s \vec{\xi}(s) = P_{T_{\vec{u}(s)}\mathbb{S}^2} \partial_s \vec{\xi}(s) = \partial_s \vec{\xi}(s) + (\partial_s \vec{u}(s) \cdot \vec{\xi}(s)) \vec{u}(s) \in T_{\vec{u}(s)}\mathbb{S}^2. \quad (3.58)$$

Since we have fixed the boundary condition (at infinity) $\vec{u}(r) \rightarrow \hat{k}$ as $r \rightarrow \infty$ (at least in the L^2 sense), we fix a unit vector in $T_{\hat{k}}\mathbb{S}^2$, say $\hat{i} = (1, 0, 0)$ to be the boundary condition for \hat{e} (at infinity) and write

$$\hat{e}(r) = \hat{i} + \tilde{e}(r), \quad \vec{u}(r) = \hat{k} + \tilde{u}(r) \quad (3.59)$$

so that the parallel transport equation $D_r \hat{e} \equiv 0$ becomes

$$\tilde{e}_r = -(\tilde{u}_r \cdot [\hat{i} + \tilde{e}(r)])(\hat{k} + \tilde{u}) = -(\tilde{u}_1)_r \hat{k} - (\tilde{u} \cdot \tilde{e}) \vec{u} - (\tilde{u}_r \cdot \hat{i}) \tilde{u}, \quad (3.60)$$

which we will therefore solve in from infinity as

$$\tilde{e}(r) = -\tilde{u}_1(r) \hat{k} + \int_r^\infty \left\{ (\tilde{u}(s) \cdot \tilde{e}(s)) \vec{u}(s) - (\tilde{u}_r(s) \cdot \hat{i}) \tilde{u}(s) \right\} ds =: M(\tilde{e})(r) \quad (3.61)$$

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by finding a fixed point of the map M in the space $X_R^2 := L_{rdr}^2([R, \infty); \mathbb{R}^3)$ for R large enough. To this end, we need the simple estimate

Lemma 3.3.1.

$$\left\| \int_r^\infty f(s) ds \right\|_{X_R^2} \leq \|f\|_{L_{rdr}^1([R, \infty))} =: \|f\|_{X_R^1}. \quad (3.62)$$

Proof. First by Hölder, for $r \geq R$,

$$\left| \int_r^\infty f(s) ds \right| = \left| \int_r^\infty \frac{1}{s} f(s) s ds \right| \leq \frac{1}{r} \|f\|_{X_R^1}. \quad (3.63)$$

Next, setting $F(r) := \int_r^\infty f(s) ds$ so $F' = -f$, we have $F^2(r) = 2 \int_r^\infty F(s) f(s) ds$, so changing order of integration and using (3.63),

$$\begin{aligned} \|F\|_{X_R^2}^2 &= 2 \int_R^\infty r dr \int_r^\infty F(s) f(s) ds \leq 2 \int_R^\infty |F(s)| |f(s)| ds \int_R^s r dr \\ &\leq \int_R^\infty |F(s)| s |f(s)| s ds \leq \sup_{r \geq R} (r |F(r)|) \|f\|_{X_R^2} \leq \|f\|_{X_R^1}^2 \|f\|_{X_R^2} \end{aligned} \quad (3.64)$$

and the proof is completed by taking square roots. \square

Now we may use Lemma 3.3.1 to estimate the map M :

$$\begin{aligned} \|M(\tilde{e})\|_{X_R^2} &\leq \|\tilde{u}\|_{X_R^2} + \| |\tilde{u}(s)| (|\tilde{e}(s)| + |\tilde{u}_r(s)|) \|_{X_R^1} \\ &\leq \|\tilde{u}\|_{X_R^2} + \|\tilde{u}\|_{X_R^2} \|\tilde{e}\|_{X_R^2} + \|\tilde{u}\|_{X_R^2} \|\tilde{u}_r\|_{X_R^2}. \end{aligned}$$

Since $\tilde{u} \in H^1(\mathbb{R}^2)$, there is R_0 such that for $R \geq R_0$, $\|\tilde{u}\|_{X_R^2} < 1/3$ and $\|\tilde{u}\|_{X_R^2} \|\tilde{u}_r\|_{X_R^2} < 1/3$, so

$$\|\tilde{e}\|_{X_R^2} \leq 1 \implies \|M(\tilde{e})\|_{X_R^2} \leq 1, \quad (3.65)$$

that is, M sends the unit ball in X_R^2 to itself. Also, for any $\tilde{e}^A, \tilde{e}^B \in X_R^2$,

$$\|M(\tilde{e}^A) - M(\tilde{e}^B)\|_{X_R^2} \leq \|\tilde{u}\|_{X_R^2} \|\tilde{e}^A - \tilde{e}^B\|_{X_R^2} < \frac{1}{3} \|\tilde{e}^A - \tilde{e}^B\|_{X_R^2}, \quad (3.66)$$

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so M is a contraction on the unit ball in X_R^2 , hence has a unique fixed point there.

Using $\tilde{u} \in H^2(\mathbb{R}^2)$, it follows from (3.60), that $\tilde{e}_r \in X_R^2$, $\tilde{e}/r \in X_R^2$, and, after differentiating once, $\tilde{e}_{rr} \in X_R^2$. In particular, \tilde{e} is continuously differentiable, so a genuine solution of (3.60).

Now we may simply solve the initial value problem for the linear ODE (3.60) from $r = R$ (with value $\tilde{e}(R)$) down to $r = 0$ to get \tilde{e} on $(0, \infty)$. Estimates as above imply that $\tilde{e} \in H^2(\mathbb{R}^2)$ (and in particular is continuous, and defined at $r = 0$). It is easily shown that if, in addition, $\tilde{u} \in H^3(\mathbb{R}^2)$, then $\tilde{e} \in H^3(\mathbb{R}^2)$.

So we have constructed a solution $\widehat{e}(r) = \widehat{i} + \tilde{e}(r)$ of $D_r \widehat{e} \equiv 0$. It then follows directly from this ODE that $\partial_r(\vec{u}(r) \cdot \widehat{e}(r)) \equiv 0$ and $\partial_r(\widehat{e} \cdot \widehat{e}) \equiv 0$ and hence that $\widehat{e}(r) \in T_{\vec{u}(r)}\mathbb{S}^2$ and $|\widehat{e}(r)| \equiv 1$. So we have (3.57).

The generalized Hasimoto transformation

Recall that we have

$$\vec{u}_r = q_1 \widehat{e}_1 + q_2 \widehat{e}_2 \quad \text{and} \quad \vec{u}_t = p_1 \widehat{e}_1 + p_2 \widehat{e}_2 \quad (3.67)$$

and

$$q = q_1 + iq_2 \quad \text{and} \quad p = p_1 + ip_2. \quad (3.68)$$

We would like to find an equation governing the evolution of q .

To do so, we rewrite (3.1) as

$$\vec{u}_t = -J^{\vec{u}} \left(D_r^{\vec{u}} + \frac{1}{r} \right) \vec{u}_r. \quad (3.69)$$

Expressing the above in terms of q and p , we get

$$p = -i(\partial_r + \frac{1}{r})q. \quad (3.70)$$

We would like to eliminate p from the above. To do so, we use that the fact

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that $D_r^{\vec{u}} \vec{u}_t = D_t^{\vec{u}} \vec{u}_r$ and this gives

$$\partial_r p = (\partial_t + iT)q \text{ where } D_t^{\vec{u}} \widehat{e}_1 = T \widehat{e}_2. \quad (3.71)$$

If we take partial derivative with respect to r on both sides of (3.70) and eliminate $\partial_r p$ with the above, we get that

$$q_t + iTq = -i(\Delta + \frac{1}{r^2})q. \quad (3.72)$$

Other than T which is yet to be determined, the above is an evolution equation for q . Further computation shows that T satisfies the equation

$$(T + \frac{1}{2}|q|^2)_r = -\frac{1}{r}|q|^2 \quad (3.73)$$

which we can integrate to get

$$T = -\frac{1}{2}|q|^2 + \int_r^\infty \frac{|q(\rho)|^2}{\rho} d\rho. \quad (3.74)$$

Putting everything together, we arrive at our evolution equation for q

$$iq_t = -\Delta q + \frac{1}{r^2}q + \left(\int_r^\infty \frac{|q(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|q|^2 \right) q. \quad (3.75)$$

Here, (3.75) is the result of the Schrödinger map equation after the generalized Hasimoto transformation.

Equivalence of norms

We have

$$\vec{u}_r = q_1 \widehat{e} + q_2 J \widehat{e} =: q \circ \widehat{e} \quad (3.76)$$

(the last equality just defines a convenient notation), so

$$|q| = |\vec{u}_r|, \quad (3.77)$$

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and since $D_r \widehat{e} \equiv 0$,

$$q_r \circ \widehat{e} = D_r(q \circ \widehat{e}) = D_r \vec{u}_r = \vec{u}_{rr} + |\vec{u}_r|^2 \vec{u} \quad (3.78)$$

so

$$|q_r| \leq |\vec{u}_{rr}| + |\vec{u}_r|^2, \quad |\vec{u}_{rr}| \leq |q_r| + |q|^2. \quad (3.79)$$

Setting $w(x) = e^{i\theta} q(r)$, and taking norms:

$$\|w\|_{H^1(\mathbb{R}^2)} \lesssim \|q_r\|_{L^2} + \|q/r\|_{L^2} \lesssim \|\vec{u}_{rr}\|_{L^2} + \|\vec{u}_r\|_{L^4}^2 + \|\vec{u}_r/r\|_{L^2} \lesssim \|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)} + \|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)}^2 \quad (3.80)$$

(using a Sobolev inequality at the end). And in the opposite direction,

$$\|\nabla \vec{u}\|_{H^1(\mathbb{R}^2)} \lesssim \|\vec{u}_{rr}\|_{L^2} + \|\vec{u}_r/r\|_{L^2} \lesssim \|q_r\|_{L^2} + \|q\|_{L^4}^2 + \|q/r\|_{L^2} \lesssim \|w\|_{H^1(\mathbb{R}^2)} + \|w\|_{H^1(\mathbb{R}^2)}^2. \quad (3.81)$$

These last two inequalities give (3.47). Taking another covariant derivative in r and proceeding in a similar way yields (3.48).

One to one

Suppose $\vec{u}^A(r)$ and $\vec{u}^B(r)$ are two maps in $\widehat{k} + H^2(\mathbb{R}^2)$, and let $\widehat{e}^A(r)$, $\widehat{e}^B(r)$, and $q^A(r)$, $q^B(r)$ be the corresponding unit tangent vector fields, and complex functions (respectively) constructed as above. If we also denote $\widehat{f} := J\widehat{e}$, we have the linear ODE system

$$\frac{d}{dr} \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix} = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1 & 0 & 0 \\ -q_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix} =: A(q) \begin{pmatrix} \vec{u} \\ \widehat{e} \\ \widehat{f} \end{pmatrix}. \quad (3.82)$$

Suppose now that $q^A(r) \equiv q^B(r) =: q(r)$. Then we have

$$W := \begin{pmatrix} \vec{u}^A \\ \widehat{e}^A \\ \widehat{f}^A \end{pmatrix} - \begin{pmatrix} \vec{u}^B \\ \widehat{e}^B \\ \widehat{f}^B \end{pmatrix} \in H^2(\mathbb{R}^2), \quad W_r = A(q)W. \quad (3.83)$$

Applying the estimate of Lemma 3.3.1, we find

$$\|W\|_{L^2_{rdr}[R,\infty)} \leq C\|W_r\|_{L^1_{rdr}[R,\infty)} \leq C\|q\|_{L^2_{rdr}[R,\infty)}\|W\|_{L^2_{rdr}[R,\infty)}. \quad (3.84)$$

Choosing R large enough so that $\|q\|_{L^2_{rdr}[R,\infty)} < 1/C$, we conclude $W \equiv 0$ on $[R, \infty)$. Then standard uniqueness for initial value problems for linear ODE implies $W(r) \equiv 0$ for all r . \square

Together, the above steps complete the proof of Proposition 1.

3.3.2 Local theory of NLNLS

Much of the result of Proposition 2 follows from [14], [13] (also see [26]). Following [26], part 1, 2, 3 and 5 of Proposition 2 holds for the equation

$$\begin{cases} iu_t = -\Delta u + f(u) \\ u(x, 0) = u_0(x) \in L^2(\mathbb{R}^2) \end{cases} \quad (3.85)$$

where the nonlinearity $f : L^2 \cap L^4(\mathbb{R}^2) \rightarrow L^4(\mathbb{R}^2)L^{\frac{4}{3}}(\mathbb{R}^2)$ satisfies

$$f(0) = 0 \quad (3.86)$$

and

$$\|f(u) - f(v)\|_{L^{\frac{4}{3}}(I)L^{\frac{4}{3}}(\mathbb{R}^2)} \lesssim (\|u\|_{L^4L^4} + \|v\|_{L^4L^4})\|u - v\|_{L^4L^4} \quad (3.87)$$

for any $I \in \mathbb{R}$ and $u, v \in L^4(I, L^4(\mathbb{R}^2))$. For example, the nonlinear term $f(u) = |u|^2u$ in NLS satisfies (3.86) and (3.87). Adapting this to our case, suppose q is a solution to NLNLS, we will let $w(x, t) = e^{i\theta}q(r, t)$. The equation for w is equation (3.49). We will let

$$g(w) = \left(\int_{|\rho| > |x|}^{\infty} \frac{|w(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|w|^2 \right) w. \quad (3.88)$$

3.3. Proof of our result

Then g satisfies (3.86). To check that g satisfies (3.87), we use a Hardy-type inequality for radial functions

$$\|f(r)\|_{L^p} \lesssim \|rf_r\|_{L^p}, \quad 1 \leq p < \infty \quad (3.89)$$

which gives

$$\begin{aligned} \|w \int_{|y| \geq |x|} \frac{|w(y)|^2}{|y|^2} dy\|_{L_x^{4/3}} &\lesssim \|w\|_{L_x^4} \left\| \int_{|y| \geq r} \frac{|w(y)|^2}{|y|^2} dy \right\|_{L_x^2} \\ &\lesssim \|w\|_{L_x^4} \left\| r \frac{\partial}{\partial r} \int_0^\infty \frac{dr}{r} \int_0^{2\pi} |w(r, \theta)|^2 d\theta \right\|_{L_x^2} \\ &= \|w\|_{L_x^4} \left\| \int_0^{2\pi} |w(r, \theta)|^2 d\theta \right\|_{L_x^2} \lesssim \|w\|_{L_x^4}^3. \end{aligned} \quad (3.90)$$

Using the above, we get

$$\left\| w_1 \int_{|y| \geq r} \frac{|w_1(y)|^2}{|y|^2} dy - w_2 \int_{|y| \geq r} \frac{|w_2(y)|^2}{|y|^2} dy \right\|_{L_{x,t}^{4/3}} \quad (3.91)$$

$$\lesssim \left[\|w_1\|_{L_{x,t}^4}^2 + \|w_2\|_{L_{x,t}^4}^2 \right] \|w_1 - w_2\|_{L_{x,t}^4}. \quad (3.92)$$

Since the cubic term $\frac{1}{2}|w|^2 w$ satisfies (3.87) by the Hölder inequality, we see that g satisfies (3.87).

Finally, part 4 of Proposition 2 follows from [84, Lemma 3.6] by adapting their proof with our nonlinearity g defined by (3.88).

3.3.3 Reduction to the three enemies

The goal of this section is to show Proposition 3. A version of Proposition 3 for NLS has been proven: part 1 of Proposition 3 for NLS has been proven by [84] and part 2 by [48].

Consider the equation

$$iu_t = \Delta u + f(u). \quad (3.93)$$

When $f(u) = |u|^2 u$, equation (3.93) is the defocusing NLS equation and when $f(u) = \left(\int_r^\infty \frac{|u(\rho, t)|^2}{\rho} d\rho - \frac{1}{2}|u|^2 \right) u$, equation (3.93) is the NLNLS equation.

tion with $u = e^{i\theta}q$. In both cases, the nonlinearity of (3.93) is cubic. Furthermore, in both cases, (3.93) has the same invariances such as translation, phase, Galilean transform and scaling. We should mention that for the NLS case, the u in (3.93) is radial while for the NLNLS u is not. Our strategy in proving Proposition 3 is to follow the proofs in [84] and [48] line by line and only modify lines of their proofs to account for the differences between NLNLS and NLS.

The proof of [84] on part 1 of proposition 3 for NLS depends very little on the exact structure of $f(u)$ other than that it satisfies the various invariances mentioned above. In places where estimates on f is needed, equation 3.90 can handle the task.

The proof of [48] on part 2 of Proposition 3 for NLS depends even less on the structure of $f(u)$ other than that it satisfies the various invariances mentioned above. As a result, our proof for part 2 of Proposition 3 follows line by line from that in [48].

3.3.4 Extra regularity

The goal of this subsection is to prove Proposition 4. A version of Proposition 4 has been proven by [48] for NLS. As in Section 3.3.3, our strategy is to follow their prove line by line and only modify the parts needed to account for the differences between NLS and NLNLS, mainly in places where the nonlinearity or the radial symmetry come into play.

The proof comes in two parts. The first part proves (3.53) concerning the regularity of self-similar solutions while the second part proves (3.54) concerning the regularity of the soliton and inverse-cascade solutions.

Regularity of self-similar solutions

Let us briefly outline Killip-Tao-Visan's proof of (3.53) for NLS. Readers looking for more details should read Section 5 of [48]. Here, the idea is to prove

$$\mathcal{M}(A) \lesssim_{s,u} A^{-s} \tag{3.94}$$

3.3. Proof of our result

for every $s > 0$ where

$$\mathcal{M}(A) := \sup_{T>0} \|u_{>AT^{-\frac{1}{2}}}(T)\|_{L_x^2(\mathbb{R}^2)}. \quad (3.95)$$

To achieve this, Killip-Tao-Visan considered two more quantities

$$\mathcal{S}(A) := \sup_{T>0} \|u_{>AT^{-\frac{1}{2}}}\|_{L_{t,x}^4([T,2T] \times \mathbb{R}^2)} \quad (3.96)$$

and

$$\mathcal{N}(A) := \sup_{T>0} \|P_{>AT^{-\frac{1}{2}}}(F(u))\|_{L_{t,x}^{\frac{4}{3}}([T,2T] \times \mathbb{R}^2)} \quad (3.97)$$

for $A > 0$. In the above, F is the nonlinear term, so $F(u) = |u|^2 u$ for the NLS case.

Then following mass conservation, properties of self-similar solutions and basic estimates, one gets that for all $A > 0$

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u 1 \quad (3.98)$$

and

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A). \quad (3.99)$$

To show (3.94), Killip-Tao-Visan proved the following lemma:

- (Lemma 5.3 of [48])

$$\mathcal{N}(A) \lesssim_u \mathcal{S}\left(\frac{A}{8}\right) \sqrt{A} + A^{-\frac{1}{4}} [\mathcal{M}\left(\frac{A}{8}\right) + \mathcal{N}\left(\frac{A}{8}\right)] \quad (3.100)$$

for all $A > 100$.

- (Lemma 5.4 of [48])

$$\lim_{A \rightarrow \infty} \mathcal{M}(A) = \lim_{A \rightarrow \infty} \mathcal{S}(A) = \lim_{A \rightarrow \infty} \mathcal{N}(A) = 0 \quad (3.101)$$

- (Lemma 5.5 of [48]) Let $0 < \eta < 1$. If A is sufficiently large, then

$$\mathcal{M}(A) \leq \eta \mathcal{S}\left(\frac{A}{16}\right) + A^{-\frac{1}{10}} \quad (3.102)$$

3.3. Proof of our result

item (Corollary 5.6 of [48]) For any $A > 0$,

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u A^{-\frac{1}{10}}. \quad (3.103)$$

Equation (3.94) follows by iterating (3.103) using (3.100).

We would like to adapt Killip-Tao-Visan's proof to our case for $w(x, t) = e^{i\theta}q(r)$ where q satisfies the NLNLS equation (3.75). As w is not radial and the NLNLS has a non-local term, a few places of Killip-Tao-Visan's proof has to be changed to accommodate for this. We will now highlight the changes.

First, the key in showing (3.100) is a decomposition of w into high-, medium-, and low-frequency components:

$$w = w_{>(A/8)T^{-1/2}} + w_{\sqrt{AT}^{-1/2} < \cdot \leq (A/8)T^{-1/2}} + w_{\leq \sqrt{AT}^{-1/2}}. \quad (3.104)$$

Here, since \mathcal{N} depends only the projection of the nonlinearity onto high frequencies $P_{>AT^{-\frac{1}{2}}}(F(u))$, for the cubic nonlinear $F(u) = |u|^2u$, under (3.104), any term made up of solely low frequency terms will not contribute to \mathcal{N} . The non-local nonlinearity behaves well with respect to frequency decomposition as well. Denoting

$$I(f)(r) := \int_r^\infty f(\rho) \frac{d\rho}{\rho} \quad (3.105)$$

for a radial function $f(r)$, we have $x \cdot \nabla I = rI_r = -f$, so

$$\widehat{f} = -\nabla_\xi \cdot \xi \widehat{I} = -|\xi|^{-1} \partial_{|\xi|} |\xi|^2 \widehat{I} \quad (3.106)$$

and

$$\widehat{I}(|\xi|) = \frac{1}{|\xi|^2} \int_{|\xi|}^\infty \widehat{f}(|\eta|) |\eta| d|\eta|. \quad (3.107)$$

Hence if f is frequency localized in a particular disk, so is $I(f)$. So after decomposing w as in (3.104), one can assume, exactly as in Killip-Tao-Visan, that each term of the resulting expansion of the high frequency projection of the nonlinearity, $P_{>AT^{-1/2}}(wI(|w|^2))$, must somewhere include the high frequency component $w_{>(A/8)T^{-1/2}}$. As an aside, we should also note that

3.3. Proof of our result

this decomposition preserves the form of function $w(x) = e^{i\theta}q(r)$ (each term is $e^{i\theta}$ multiplying a radial function).

The estimates in Lemma 5.3 then carry over, using (3.90) as needed, with one exception: the use of the bilinear Strichartz inequality to estimate nonlinear terms containing two low-frequency factors. The problem occurs in the non-local nonlinear term when the high-frequency factor falls outside the integral, as in

$$w_{>(A/8)T^{-1/2}} I(|w_{\leq \sqrt{A}T^{-1/2}}|^2). \quad (3.108)$$

This term does not involve a (local) product of a low-frequency and a high-frequency “approximate solution” of the Schrödinger equation, and so it is unclear how to apply the bilinear Strichartz estimate to it.

We can get around this problem by replacing the use of bilinear Strichartz with an application Shao’s Strichartz estimate for radial functions [65]

$$\|P_N e^{it\Delta} f\|_{L_{x,t}^q(\mathbb{R} \times \mathbb{R}^2)} \lesssim N^{1-4/q} \|f\|_{L^2(\mathbb{R}^2)}, \quad q > 10/3 \quad (3.109)$$

plus a Bernstein estimate.

Remark 2. Note that (3.109) is for radial functions, while our functions are of the form $w(x) = e^{i\theta}q(r)$. In fact it is easily checked that Shao’s argument applies also for such functions – it is essentially a matter of replacing the Bessel function J_0 with J_1 , which has the same spatial asymptotics (and better behaviour at the origin). The same is true for the weighted Strichartz estimate [48, Lemma 2.7], which is also used in the [48] argument we are following.

Indeed, since $I(|w_{\leq M}|^2)$ is frequency-localized below M , applying Hölder, Shao, Bernstein, and Hardy, we have, for any $10/3 < q < 4$

$$\begin{aligned} \|IP_N e^{it\Delta} f\|_{L_{x,t}^{4/3}} &\lesssim \|I\|_{L_{x,t}^{\frac{4q}{3q-4}}} \|P_N e^{it\Delta} f\|_{L_{x,t}^q} \lesssim M^{\frac{4}{q}-1} \|I\|_{L_t^{\frac{4q}{3q-4}} L_x^{\frac{4q}{q+4}}} N^{1-\frac{4}{q}} \|P_N f\|_{L^2} \\ &= \left(\frac{M}{N}\right)^{\frac{4}{q}-1} \|w_{\leq M}\|_{L_t^{\frac{8q}{3q-4}} L_x^{\frac{8q}{q+4}}}^2 \|P_N f\|_{L^2}, \end{aligned} \quad (3.110)$$

and the middle factor is a Strichartz norm, so is bounded by a constant. By this argument, using also the inhomogeneous version of (3.109) (which follows in the usual way), and replacing P_N by $P_{\geq N}$ (which follows easily by summing over dyadic frequencies), we can finally arrive at the nonlinear estimate (3.100), albeit with a slower decay factor $A^{-(2/q-1/2)}$ replacing $A^{-1/4}$ (notice $0 < 2/q - 1/2 < 1/10$). This lower power does not matter, however, and the remaining estimates, (3.101), (3.102) and (3.103), carry through, establishing (3.53).

Regularity of soliton and inverse-cascade solutions

Let us briefly outline [48]'s proof of (3.54) for NLS. Readers looking for more details should read Section 6 and 7 of [48]. Here, the idea is to split the solution $u(t)$ into incoming and outgoing waves and express the solution u at time t as a sum of incoming waves integrated over the past and outgoing waves integrated over the future following the Duhamel formula.

In Section 6 of [48] defined the projection P^+ onto outgoing spherical waves to be

$$[P^+f](x) = \frac{1}{2}(2\pi)^{-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} H_0^{(1)}(|\xi||x|) J_0(|\xi||y|) f(y) d\xi dy. \quad (3.111)$$

Here, $H_0^{(1)}$ is the Hankel function of the first kind and order zero and J_0 is the Bessel function of the first kind. The projection P^- onto incoming spherical waves are defined similarly.

Our proof for (3.54) essentially follow that of [48] but we need to modify the definitions of P^+ and P^- as our function $w = e^{i\theta}q(r)$ is not radial. These projections are defined analogously for functions $w(x) = e^{i\theta}q(r)$ by simply replacing the Bessel (and Hankel) functions of order zero with those of order one: $J_0 \rightarrow J_1$, $H_0^\alpha \rightarrow H_1^\alpha$. It is easily checked that these (new) projections obey the kernel estimates listed in Proposition 6.2 of Killip-Tao-Visan, essentially because J_1 and H_1 have the same behaviour as J_0 and H_0 away from the origin [48, eqns. (77), (79)]. (At the origin, J_1 is *better* behaved, while H_1 is worse – though this plays no role in the estimates.)

Given this, the subsequent estimates of Section 7 of Killip-Tao-Visan all carry over to our case, as above using (3.90) where needed to estimate the non-local nonlinearity, to establish $w \in L_t^\infty H_x^s$ for any $s > 0$.

This shows (3.54) and completes the proof of Proposition 4.

3.3.5 Nonexistence of the three enemies

We will rule out each of the three enemies in this section. As before, we will let

$$w(x, t) = e^{i\theta} q(r, t). \quad (3.112)$$

We will use a lower bound which follows easily from the compactness:

Lemma 3.3.2.

$$\|\nabla w(\cdot, t)\|_{L^2(\mathbb{R}^2)}^2 \sim \|q_r(\cdot, t)\|_{L^2}^2 + \|q(\cdot, t)/r\|_{L^2}^2 \gtrsim N^2(t). \quad (3.113)$$

Proof. First rescale $q(r, t) = N(t)v(N(t)r, t)$, and set $\tilde{w}(x, t) = e^{i\theta}v(r, t)$, so that the estimate we seek is $\|\nabla \tilde{w}(\cdot, t)\|_{L^2} \gtrsim 1$. If this fails, then for some sequence $\{t_n\}$, $\tilde{w}_n(x) := \tilde{w}(x, t_n)$, satisfies $\|\nabla \tilde{w}_n\|_{L^2(\mathbb{R}^2)} \rightarrow 0$. Since $\|\tilde{w}_n\|_{L^2} = \text{const.}$, we can extract a subsequence (still denoted \tilde{w}_n) with $\tilde{w}_n \rightarrow 0$ weakly in H^1 , and strongly in L^2 on disks. By the compactness, on the other hand, for any $0 < \eta$, $\|\tilde{w}_n\|_{L^2(\{|x| > C(\eta)\})} < \eta$, a contradiction. \square

The soliton case

Here $I = \mathbb{R}$ and $N(t) \equiv 1$.

The main tool is a spatially localized version of the virial identity

$$\frac{d^2}{dt^2} \frac{1}{2} \int_0^\infty r^2 |q(r, t)|^2 r \, dr = \int_0^\infty \left\{ 4|q_r|^2 + 4\frac{|q|^2}{r^2} + |q|^4 \right\} r \, dr \quad (3.114)$$

. For a smooth cut-off function

$$\psi(r) \geq 0, \quad \psi \equiv 1 \text{ on } [0, 1], \quad \psi \equiv 0 \text{ on } [2, \infty), \quad (3.115)$$

3.3. Proof of our result

and a fixed radius $R > 0$, define $\phi_R(r) := \psi(r/R)$, and the quantity

$$I_R(q) := \int_0^\infty r \operatorname{Im}(\bar{q} q_r) \phi_R r \, dr, \quad (3.116)$$

a function of time. By straightforward calculation we have

Lemma 3.3.3.

$$\begin{aligned} \frac{d}{dt} I_R(q) = 2 \int_0^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \frac{1}{4} |q|^4 \right. \\ + \left(|q_r|^2 + \frac{|q|^2}{r^2} + \frac{1}{4} |q|^4 \right) (\phi_R - 1) \\ + \left(|q_r|^2 - \frac{3}{4} \frac{|q|^2}{r^2} - \frac{1}{8} |q|^4 \right) r (\phi_R)_r \\ \left. - \frac{5}{4} \frac{|q|^2}{r^2} r^2 (\phi_R)_{rr} - \frac{1}{4} \frac{|q|^2}{r^2} r^3 (\phi_R)_{rrr} \right\} r \, dr. \end{aligned} \quad (3.117)$$

From Proposition 4 we have for each $s \geq 0$, and for all t ,

$$\|w(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^2)} \leq C_s. \quad (3.118)$$

Fix $\eta > 0$, and let $R = 2C(\eta)$ so that, since $N(t) \equiv 1$,

$$\int_{|x| > R/2} |w(x, t)|^2 dx < \eta \quad (3.119)$$

for all t . Multiplying w by a cut-off function $1 - \psi(2r/R)$, and interpolating between (3.119) and (3.118) with $s = 2$ (and using a Sobolev inequality) yields

$$\int_R^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \frac{1}{4} |q|^4 \right\} r dr \sim \int_{|x| \geq R} \left\{ |\nabla w|^2 + \frac{1}{4} |w|^4 \right\} dx \lesssim \eta^{1/2}, \quad (3.120)$$

and so using $|1 - \phi_R|$, $|r(\phi_R)_r|$, $|r^2(\phi_R)_{rr}|$, $|r^3(\phi_R)_{rrr}| \lesssim 1$ in (3.117), we arrive at

$$\frac{d}{dt} I_R(q) \geq 2 \int_0^\infty \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \frac{1}{4} |q|^4 \right\} r \, dr - C\eta^{1/2}. \quad (3.121)$$

By Lemma 3.3.2 then, since $N(t) \equiv 1$, and for η chosen small enough,

$$\frac{d}{dt} I_R(q) \gtrsim 1. \quad (3.122)$$

On the other hand,

$$|I_R(q)| \lesssim R \|q\|_{L^2} \|q_r\|_{L^2} \lesssim RC_1. \quad (3.123)$$

These last two inequalities are in contradiction for sufficiently large t , and so the soliton-type blowup is ruled out.

The self-similar case

Here $I = (0, \infty)$, and $N(t) = t^{-1/2}$.

Again we use (3.117), but in this case, we need a stronger bound on the Sobolev norms – in fact, bounds which match Lemma 3.3.2. Such bounds follow from the regularity estimate of [48], in the self-similar case, as adapted to our non-local nonlinearity in Section 3.3.4:

Lemma 3.3.4. *For any $s \geq 0$,*

$$\sup_{t \in (0, \infty)} \int_{|\xi| > At^{-1/2}} |\widehat{w}(\xi, t)|^2 d\xi \leq C_s A^{-s}, \quad A > A_0(s). \quad (3.124)$$

As a consequence,

$$\|w(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim t^{-s/2} = [N(t)]^s. \quad (3.125)$$

Indeed, after re-scaling $w(x, t) = N(t) \tilde{w}(N(t)x, t)$, equation (3.124) reads

$$\int_{|\xi| > A} |\widehat{\tilde{w}}(\xi, t)|^2 d\xi \leq C_s A^{-s} \quad (3.126)$$

for all t , from which follows $\|\tilde{w}\|_{\dot{H}^s} \lesssim 1$, and thus (undoing the scaling) (3.125).

Now fix a small $\eta > 0$, and large T . A (localized) interpolation (just as in the soliton case) between (3.125) with $s = 2$ and the L^2 smallness from

compactness, gives

$$\int_{2C(\eta)/N(t)}^{\infty} \left\{ |q_r|^2 + \frac{|q|^2}{r^2} + \frac{1}{4}|q|^4 \right\} r \, dr \lesssim \eta^{1/2} \|w\|_{\dot{H}^2(\mathbb{R}^2)} \lesssim \eta^{1/2} (N(t))^2. \quad (3.127)$$

Using this, with η small enough, and Lemma 3.3.2, in (3.117), we find, for $t < T$, and $R = 2C(\eta)/N(T) > 2C(\eta)/N(t)$,

$$\frac{d}{dt} I_R(q) \gtrsim N^2(t) = \frac{1}{t}, \quad (3.128)$$

and hence for $T \gg 1$,

$$I_R(q)(T) \gtrsim I_R(q)(1) + \int_1^T \frac{dt}{t} \gtrsim \log(T). \quad (3.129)$$

On the other hand

$$|I_R(q)(T)| \lesssim R \|q(T)\|_{L^2} \|q(T)\|_{\dot{H}^1} \lesssim \frac{C(\eta)}{N(T)} N(T) = C(\eta). \quad (3.130)$$

The last two inequalities are in contradiction for T large enough, and so the self-similar-type blowup is ruled out.

The inverse-cascade case

Here $I = \mathbb{R}$, $N(t) \lesssim 1$, and $\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow \infty} N(t) = 0$.

The main tool is a variant of the Morawetz identity. Set

$$\psi(r) := \begin{cases} 4r - r^2 & 0 < r \leq 1 \\ 6 - \frac{4}{r} + \frac{1}{r^2} & 1 < r < \infty \end{cases}. \quad (3.131)$$

It is easily checked that for $r \in (0, \infty)$,

- $\psi \in C^3$
- $0 < \psi < 6$
- $\psi_r > 0$
- $\alpha(r) := \frac{1}{2}\psi_r + \frac{3}{2}\frac{\psi}{r} - r\psi_{rr} - \frac{1}{2}r^2\psi_{rrr} > 0$

- $\beta(r) := \frac{\psi}{r} - \psi_r > 0$.

Set

$$P(q) := \int_0^\infty \operatorname{Im}(\bar{q}q_r)\psi(r)r \, dr. \quad (3.132)$$

For solutions of (3.15), an elementary computation gives:

Lemma 3.3.5.

$$\frac{d}{dt}P(q) = \int_0^\infty \{ 2\psi_r|q_r|^2 + \alpha(r)\frac{|q|^2}{r^2} + \left(\frac{1}{2}\beta(r) + \frac{1}{4}\left(\frac{\psi}{r} + \psi_r\right)\right)|q|^4 \} r \, dr > 0. \quad (3.133)$$

Note that since $|\psi(r)| \lesssim 1$,

$$|P(q)| \lesssim \|q\|_{L^2}\|q_r\|_{L^2} \lesssim \|q_r\|_{L^2}. \quad (3.134)$$

Next recall that for some sequences $t_n \rightarrow -\infty$, $T_n \rightarrow +\infty$, $N(t_n) \rightarrow 0$ and $N(T_n) \rightarrow 0$. It then follows easily from the definition of $N(t)$ that $\|q_r(t_n)\|_{L^2} \rightarrow 0$ and $\|q_r(T_n)\|_{L^2} \rightarrow 0$. Hence by (3.134),

$$P(q(t_n)) \rightarrow 0, \quad P(q(T_n)) \rightarrow 0. \quad (3.135)$$

If $P(q_0) \geq 0$, then (3.133) implies $P(q(t)) > 0$ and increasing for $t > 0$, while if $P(q_0) < 0$, then (3.133) implies $P(q(t)) < 0$ and increasing for $t < 0$. In either case, (3.135) is contradicted. This rules out the inverse cascade-type blowup.

Chapter 4

Concluding chapter

In this thesis, we showed two results. First, we established asymptotic stability of small ground state solutions to the three dimensional nonlinear magnetic Schrödinger equation

$$iu_t = (i\nabla + A)^2 u + Vu + g(u), \quad u(x, 0) = u_0(x) \quad (4.1)$$

for the case where the operator $(i\nabla + A)^2 + V$ has exactly one eigenvalue. Recall that when $g \equiv 0$, equation (4.1) models a quantum particle in the presence of a electric potential V and a magnetic potential A . Here, $g(u) = |u|^2 u$ is the nonlinear term. In the absence of g , equation (4.1) is linear and the time evolution of its solution is well understood. In the presence of g , self-interactions of the solution make the behaviour of the solution more complex and much less well-understood.

When $A \equiv 0$, asymptotic stability results have been established by many authors (such as [68], [10], [87], [86], [70], [40], [50] and [57]) for cases where $-\Delta + V$ has one or more eigenvalues. Our result is the first with the presence of a magnetic potential. Stability results are important from an applications perspective. For example, if a bound solution is not stable, it would be difficult, if not impossible, to observe it experimentally or simulate it numerically. The reason is that any imprecision in the initial conditions would result in a state of the system far away from the bound state.. Furthermore, our result can be viewed as an attempt to partially understand the time evolution of solutions more generally. Viewed slightly differently, our result gives the asymptotic behaviour of a solution with initial data small in H^1 . Our result states that if the initial data is sufficiently small in H^1 , then as $t \rightarrow \infty$, the solution u will be composed of a bound state and a dispersive

part.

One natural extension to our result is to consider the more complex case where $(i\nabla + A)^2 + V$ has two or more eigenvalues. Another extension is to consider other types of nonlinearities such as a convolution type nonlinearity

$$g(u) = (F * |u|^2)u := \left(\int_{\mathbb{R}^3} F(x - y) |u(y)|^2 dy \right) u,$$

arising in a Hartree-type equation.

Second, we established global well-posedness for the Schrödinger map equation

$$\vec{u}_t = \vec{u} \times \Delta \vec{u} \tag{4.2}$$

into the 2-sphere for radially symmetric initial data in two dimensions. Equation (4.2) models the time evolution of magnetization in an isotropic magnetic material in the absence of external magnetic field and energy dissipation. When the spatial dimension is one, it is known by [16] that H^1 solutions of (4.2) are global. However, when spatial dimensions are higher than one, global behaviours of solutions to (4.2) are not well understood. For two dimensions, it has been known by [16] that solutions to (4.2) with sufficiently small energy are global. However, it is not clear what the long term behaviours of solutions with arbitrary sized energy are. Our result shows that radial solutions of arbitrary sized energy are global. Very recently, [55] showed, in two dimensions, certain solutions of (4.2) blow up in finite time. However, much work still remains to be done in this area before the complete picture of the global behaviours of solutions to (4.2) can be understood. In particular, one would like to understand the conditions on initial data that lead to global solutions as well as conditions that lead to blow up solutions. This is still an open problem.

Another extension is to consider the Schrödinger map equation in higher dimension such as $n = 3$. A crude scaling argument suggests that blow up solutions should be possible. However, so far, the construction of blow up solutions is still an open problem.

Bibliography

- [1] Ashcroft, Neil W.; Mermin, N. David; *Solid state physics*, Holt, Rinehart and Winston (1976).
- [2] Angulo Pava, Jaime; *Nonlinear dispersive equations. Existence and stability of solitary and periodic travelling wave solutions*, Mathematical Surveys and Monographs, 156. American Mathematical Society, Providence, RI, 2009.
- [3] Baillon, Jean-Bernard; Cazenave, Thierry; Figueira, Mário; *Équation de Schrödinger non linéaire*, C. R. Acad. Sci. Paris Sr. A-B 284 (1977), no. 15, A869-A872.
- [4] Bejenaru, I; Ionescu, A. D; Kenig, C. E; Tataru, D; *Global Schrödinger maps in dimensions $d \geq 2$: small data in the critical Sobolev spaces*, Ann. of Math. (2) 173 (2011), no. 3, 1443-1506.
- [5] Berestycki, Henri; Gallout, Thierry; Kavian, Otared; *Équations de champs scalaires euclidiens non linéaires dans le plan. (Nonlinear Euclidean scalar field equations in the plane)*, C. R. Acad. Sci. Paris Sr. I Math. 297 (1983), no. 5, 307-310.
- [6] Berestycki, H; Lions, P.-L; *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82 (1983), no. 4, 313-345.
- [7] Bourgain, J; *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. 12 (1999), no. 1, 145-171.
- [8] Bourgain, J; *Global solutions of nonlinear Schrödinger equations*,

- American Mathematical Society Colloquium Publications, 46. American Mathematical Society, Providence, RI, 1999.
- [9] Buslaev, V. S.; Perel'man, G. S; *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton*, (Russian) *Algebra i Analiz* 4 (1992), no. 6, 63–102; translation in *St. Petersburg Math. J.* 4 (1993), no. 6, 1111-1142.
 - [10] Buslaev, V. S.; Perel'man, G. S; *On the stability of solitary waves for nonlinear Schrödinger equations*, *Nonlinear evolution equations*, 7598, Amer. Math. Soc. Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995.
 - [11] Cazenave, Thierry; *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
 - [12] T. Cazenave, P.L. Lions; *Orbital stability of standing waves for some nonlinear Schrödinger equations*, *Comm. Math. Phys.* 85 (1982), 549-561.
 - [13] Cazenave, Thierry; Weissler, Fred B; *The Cauchy problem for the nonlinear Schrödinger equation in H^1* , *Manuscripta Math.* 61 (1988), no. 4, 477-494.
 - [14] T. Cazenave, F.B. Weissler; *Some remarks on the nonlinear Schrödinger equation in the subcritical case. New methods and results in nonlinear field equations*, (Bielefeld, 1987), 5969, *Lecture Notes in Phys.*, 347, Springer, Berlin, 1989.
 - [15] Chang, Kung-Ching; Ding, Wei Yue; Ye, Rugang; *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, *J. Differential Geom.* 36 (1992), no. 2, 507-515.
 - [16] Chang, H.-H; Shatah, J; Uhlenback, K; *Schrödinger maps*, *Comm. Pure Appl. Math.* 53 (2000), 590-602.

- [17] Christ, M; Kiselev, A; *Maximal functions associated to filtrations*, J. Fun. Anal. 179 (2001), 409-425.
- [18] Christ, Michael; Colliander, James; Tao, Terence; *Ill-posedness for nonlinear Schrödinger and wave equations*, arXiv:math/0311048.
- [19] Colliander, James; Grillakis, Manoussos; Tzirakis, Nikolaos; *Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on R^2* , Int. Math. Res. Not. IMRN 2007, no. 23.
- [20] Colliander, J; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T.; *Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation*, Math. Res. Lett. 9 (2002), no. 5-6, 659-682.
- [21] Colliander, J; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T.; *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on R^3* , Comm. Pure Appl. Math. 57 (2004), no. 8, 987-1014.
- [22] Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T.; *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in R^3* , Ann. of Math. (2) 167 (2008), no. 3, 767-865.
- [23] Colliander, James; Roy, Tristan; *Bootstrapped Morawetz estimates and resonant decomposition for low regularity global solutions of cubic NLS on R^2* , Commun. Pure Appl. Anal. 10 (2011), no. 2, 397-414.
- [24] Cowling, M. G.; *Harmonic analysis on semigroups*, Ann. of Math. (2) 117 (1983), no. 2, 267-283.
- [25] Cuccagna, Scipio; *Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. 54 (2001), no. 9, 1110-1145.
- [26] Cuccagna, Scipio; *On asymptotic stability of ground states of NLS*, Rev. Math. Phys. 15 (2003), no. 8, 877-903.
- [27] Dodson, Benjamin; *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*, arXiv:1104.1114.

- [28] Dodson, Benjamin; *Global well-posedness and scattering for the defocusing, L^2 -critical, nonlinear Schrödinger equation when $d = 2$* , arXiv:1006.1375.
- [29] Dodson, Benjamin; *Improved almost Morawetz estimates for the cubic nonlinear Schrödinger equation*, Commun. Pure Appl. Anal. 10 (2011), no. 1, 127-140.
- [30] Erdogan, M. Burak; Goldberg, Michael; Schlag, Wilhelm; *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math. 21 (2009), no. 4, 687-722.
- [31] Esteban, M; Lions, P.L.; *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, PDE and Calculus of Variations, in Honor of E. De Giorgi, Birkhuser, 1990.
- [32] Fang, Y. F; Grillakis, Manoussos G; *On the global existence of rough solutions of the cubic defocusing Schrödinger equation in R^{2+1}* , J. Hyperbolic Differ. Equ. 4 (2007), no. 2, 233-257.
- [33] Gang, Zhou; Sigal, I. M; *Asymptotic stability of nonlinear Schrödinger equations with potential*, Rev. Math. Phys. 17 (2005), no. 10, 1143-1207.
- [34] Ginibre, J.; Velo, G; *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J. Math. Pures Appl. (9) 64 (1985), no. 4, 363-401.
- [35] Glassey, R. T; *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. 18 (1977), no. 9, 1794-1797.
- [36] Griffiths, David J; *Introduction to Quantum Mechanics*, 1st edition, Prentice Hall, 1994.
- [37] Grillakis, M. G; *On nonlinear Schrödinger equations*, Comm. Partial Differential Equations 25 (2000), 1827-1844.

- [38] Grillakis, Manoussos; Shatah, Jalal; Strauss, Walter; *Stability theory of solitary waves in the presence of symmetry*, I. J. Funct. Anal. 74 (1987), no. 1, 160-197.
- [39] Grillakis, Manoussos; Shatah, Jalal; Strauss, Walter; *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. 94 (1990), no. 2, 308-348.
- [40] Gustafson, Stephen; Nakanishi, Kenji; Tsai, Tai-Peng; *Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equation with small solitary waves*, IMRN 2004, no. 66, 3559-3584.
- [41] Gustafson, Stephen; Nakanishi, Kenji; Tsai, Tai-Peng; *Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on R^2* , Comm. Math. Phys. 300 (2010), no. 1, 205-242.
- [42] Gustafson, Stephen J.; Sigal, Israel Michael; *Mathematical concepts of quantum mechanics*, Universitext. Springer-Verlag, Berlin, 2003.
- [43] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case*, J. Funct. Anal. 32 (1979), no. 1, 1-32.
- [44] Jones, C.; Küpper, T; *On the infinitely many solutions of a semilinear elliptic equation*, SIAM J. Math. Anal. 17 (1986), no. 4, 803-835.
- [45] Kato, T; *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), no. 1, 113-129.
- [46] Keel, Markus; Tao, Terence; *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955-980.
- [47] Kenig, Carlos E.; Merle, Frank; *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. 166 (2006), no. 3, 645-675.

- [48] Killip, Rowan; Tao, Terence; Visan, Monica; *The cubic nonlinear Schrödinger equation in two dimensions with radial data*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 6, 1203-1258.
- [49] Kirr, E.; Mizrak, Ö; *Asymptotic stability of ground states in 3D nonlinear Schrödinger equation including subcritical cases*, J. Funct. Anal. 257 (2009), no. 12, 3691-3747.
- [50] Kirr, E; Zarnescu, A; *On the asymptotic stability of bound states in 2D cubic Schrödinger equation*, Comm. Math. Phys. 272 (2007), no. 2, 443-468.
- [51] Koo, Eva; *Asymptotic stability of small solitary waves for nonlinear Schrödinger equations with electromagnetic potential in R^3* , J. Differential Equations 250 (2011), no. 8, 3473-3503.
- [52] Krieger, J; Schlag, W; Tataru, D; *Renormalization and blow up for charge one equivariant critical wave maps*, Invent. Math. 171 (2008), no. 3, 543-615.
- [53] Kwong, Man Kam; *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbb{R}^n* , Arch. Rational Mech. Anal. 105 (1989), no. 3, 243-266.
- [54] Landau, L.D.; Lifshitz, E.M; *Theory of the dispersion of magnetic permeability in ferromagnetic bodies*, Phys. Z. Sowietunion 8, 153, (1935).
- [55] Merle, Frank; Raphaël, Pierre; Rodnianski, Igor; *Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map*, C. R. Math. Acad. Sci. Paris 349 (2011), no. 5-6, 279-283.
- [56] Mizumachi, Tetsu; *Asymptotic stability of small solitons for 2D Nonlinear Schrödinger equations with potential*, J. Math. Kyoto Univ. 47, no. 3 (2007), 599-620.
- [57] Mizumachi, Tetsu; *Asymptotic stability of small solitons to 1D NLS with potential*, J. Math. Kyoto Univ. 48, no. 3 (2008), 471-497.

- [58] Ogawa, Takayoshi; Tsutsumi, Yoshio; *Blow-up of H^1 solution for the nonlinear Schrödinger equation*, J. Differential Equations 92 (1991), no. 2, 317-330.
- [59] Ogawa, Takayoshi; Tsutsumi, Yoshio; *Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. 111 (1991), no. 2, 487-496.
- [60] Perelman, Galina; *Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations*, Comm. Partial Differential Equations 29 (2004), no. 7-8, 1051-1095.
- [61] Pillet, Claude-Alain; Wayne, C. Eugene; *Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations*, J. Differential Equations 141 (1997), no. 2, 310-326.
- [62] Reed, Michael; Simon, Barry; *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.
- [63] Reed, Michael; Simon, Barry; *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.
- [64] Ryckman, E; Visan, M; *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in R^{1+4}* , Amer. J. Math. 129 (2007), 1-60.
- [65] Shao, Shuanglin; *Sharp linear and bilinear restriction estimates for paraboloids in the cylindrically symmetric case*, Rev. Mat. Iberoam. 25 (2009), no. 3, 1127-1168.
- [66] Shatah, Jalal; Strauss, Walter; *Instability of nonlinear bound states*, Comm. Math. Phys. 100 (1985), no. 2, 173-190.
- [67] Sterbenz, Jacob; Tataru, Daniel; *Regularity of wave-maps in dimension $2+1$* , Comm. Math. Phys. 298 (2010), no. 1, 231-264.

- [68] Soffer, A; Weinstein, M.I; *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys. 133 (1990), no. 1, 119-146.
- [69] Soffer, A.; Weinstein, M. I; *Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data*, J. Differential Equations 98 (1992), no. 2, 376-390.
- [70] Soffer, A; Weinstein, M.I; *Selection of the ground state in the nonlinear Schrödinger equation*, Rev. Math. Phys. 16, no. 8 (2004), 977-1071.
- [71] Strauss, Walter A; *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), no. 2, 149-162.
- [72] Strauss, Walter A; *Everywhere defined wave operators. Nonlinear evolution equations*, pp. 85-102, Academic Press, New York, 1978.
- [73] Strauss, Walter A; *Nonlinear scattering theory at low energy*, J. Funct. Anal. 41 (1981), no. 1, 110-133.
- [74] Strauss, Walter A; *Nonlinear scattering theory at low energy: sequel*, J. Funct. Anal. 43 (1981), no. 3, 281-293.
- [75] Strauss, Walter A; *Nonlinear wave equations*, CBMS Regional Conference Series in Mathematics, 73. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1989.
- [76] Strichartz, Robert S; *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977), no. 3, 705-714.
- [77] Struwe, Michael; *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. 60 (1985), no. 4, 558-581.
- [78] Sulem, Catherine; Sulem, Pierre-Louis; *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.

- [79] Sulem, P.-L.; Sulem, C.; Bardos, C; *On the continuous limit for a system of classical spins*, Comm. Math. Phys. 107 (1986), no. 3, 431-454.
- [80] Tao, Terence; *Global regularity of wave maps. II. Small energy in two dimensions*, Comm. Math. Phys. 224 (2001), no. 2, 443-544.
- [81] Tao, Terence; *Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data*, New York J. Math. 11 (2005), 57-80.
- [82] Tao, Terence; *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, 106, (2006).
- [83] Tao, Terence; *Why are solitons stable*, Bull. Amer. Math. Soc. (N.S.) 46 (2009), no. 1, 1-33.
- [84] Tao, Terence; Visan, Monica; Zhang, Xiaoyi; *Minimal-mass blowup solutions of the mass-critical NLS*, Forum Math. 20 (2008), no. 5, 881-919.
- [85] Triebel, H; *Interpolation Theory, Function Spaces, Differential Operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995.
- [86] Tsai, T.-P; *Asymptotic dynamics of nonlinear Schrödinger equations with many bound states*, J. Differential Equations 192 (2003), no. 1, 225-282.
- [87] Tsai, T.-P; Yau, H.-T; *Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersive-dominated solutions*, Comm. Pure Appl. Math. 55 (2002), no. 2, 153-216.
- [88] Tsutsumi, Y; *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987), no. 1, 115-125.
- [89] Visan, Monica; *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions*, Duke Math. J. 138 (2007), no. 2, 281-374.

Bibliography

- [90] Weinstein, Michael I; *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. 87 (1982/83), no. 4, 567-576.
- [91] Weinstein, Michael I; *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. 16 (1985), no. 3, 472-491.
- [92] Weinstein, Michael I; *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. 39 (1986), no. 1, 51-67.
- [93] Whitham, G. B; *Linear and Nonlinear Waves*, John Wiley & Sons (1974).
- [94] Yajima, Kenji; *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. 110 (1987), no. 3, 415-426.