Curve-counting Invariants and Crepant Resolutions of Calabi-Yau Threefolds

by

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Abstract

The Donaldson-Thomas (DT) theory of a Calabi-Yau threefold X gives rise to subtle deformation invariants. They are considered to be the mathematical counterparts of BPS state counts in topological string theory compactified on X. Principles of physics (see [46], [50]) indicate that the string theory of a singular Calabi-Yau threefold and that of its crepant resolution ought to be equivalent, so one might expect that the DT theory of a singular Calabi-Yau threefold ought to be equivalent to that of its crepant resolution. There is some difficulty in defining DT when X is singular, but the authors of [14] have (in some generality) defined DT theory in the case where X is the coarse moduli space of an orbifold $\mathcal{X}$. The crepant resolution conjecture of [14] gives a formula determining the DT invariants of the orbifold in terms of the DT invariants of the crepant resolution. In this dissertation, we begin a program to prove the crepant resolution conjecture using Hall algebra techniques inspired by those of Bridgeland [12].

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Finally, I would like to thank my parents, whose continued support has made all the difference.

Dedicated to my friends.

Chapter 1

Introduction

Donaldson-Thomas (DT) theory of a Calabi-Yau threefold X gives rise to
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[14] gives a formula determining the DT invariants of the orbifold in terms
of the DT invariants of the crepant resolution. In this dissertation, we begin
a program to prove the crepant resolution conjecture using Hall algebra
techniques inspired by those of Bridgeland [12].

Donaldson-Thomas theory

Let Y be a smooth projective Calabi-Yau threefold. Let $K(Y)$ be the nu-
merical K-theory of Y, i.e. the quotient of the K-group of $\mathrm{coh}(Y)$ by the
kernel of the Chern character map to cohomology. The Hilbert scheme of
Y, $\mathrm{Hilb}^{\alpha}(Y)$, parametrizes quotients $\mathcal{O}_Y \to \mathcal{O}_Z$, such that the class of $\mathcal{O}_Z$
in $K(Y)$ is α. The group $K(Y)$ is filtered by the dimension of the support:

$$F_0 K(Y) \subset F_1 K(Y) \subset F_2 K(Y) \subset F_3 K(Y) = K(Y).$$

In this dissertation, we will focus on curves, i.e., $\alpha \in F_1 K(Y)$, with $\mathrm{ch}(\alpha) = (0, 0, \beta, n)$, where $\beta \in H^4(Y, \mathbb{Z})$ is a curve class, and $n \in H^6(Y, \mathbb{Z}) \cong \mathbb{Z}$ is the holomorphic Euler characteristic. In [43], an obstruction theory for this moduli space is constructed, which produces (by [5]) a virtual fundamental cycle. Donaldson-Thomas invariants are defined by integrating over the zero-dimensional virtual fundamental class:

$$DT^\alpha(Y) = \int_{[\mathrm{Hilb}^\alpha(Y)]^{vir}} 1.$$

Since the obstruction theory is symmetric, we may also express the invariants as the Euler characteristic of $\mathrm{Hilb}^\alpha(Y)$ weighted by Behrend's microlocal function [3]:

$$DT^\alpha(Y) = \sum_{n \in \mathbb{Z}} n\chi(\nu^{-1}(n)),$$

where $\nu : \mathrm{Hilb}^\alpha(Y) \to \mathbb{Z}$ is Behrend's function.

Following [35], we assemble the invariants into a partition function

$$\mathrm{DT}(Y) = \sum_{\alpha \in F_1 K(Y)} \mathrm{DT}^\alpha(Y) q^\alpha.$$

Remark 1.1. In [29], Donaldson-Thomas invariants are greatly generalized, from the case of structure sheaves of curves to that of arbitrary sheaves. The price of admission to this generality is the formidable machinery of Joyce [24, 25, 26, 27, 28]. An even more ambitious program of generalization is being lead by Kontsevich and Soibelman [31].

The Donaldson-Thomas crepant resolution conjecture

We follow [14] in our treatment of the crepant resolution conjecture.

An *orbifold CY3* is defined to be a smooth, quasi-projective, Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$ of dimension three having generically trivial sta-

bilizers and trivial canonical bundle,

$$K_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}.$$

The definition implies that the local model for $\mathcal{X}$ at a point p is $[\mathbb{C}^3/G_p]$ where $G_p \subset SL(3,\mathbb{C})$ is the (finite) group of automorphisms of p. The orbifold CY3s that appear in this dissertation will all be projective and satisfy the *hard Lefschetz condition* ([16, definition 1.1]), which in this case is equivalent ([15, lemma 24]) to the condition that all G_p are finite subgroups of $SO(3) \subset SU(3)$ or $SU(2) \subset SU(3)$.

Let X denote the coarse space of $\mathcal{X}$. A *crepant resolution* of X is a resolution of singularities $\pi : Y \to X$ such that $\pi^* K_X \cong K_Y$. Lemma 1 and proposition 1 of [48] prove that

$$R^\bullet \pi_* \mathcal{O}_Y = \mathcal{O}_X. \tag{1.2}$$

The results of [13] and [18] prove that one distinguished crepant resolution of X is

$$Y = \mathrm{Hilb}^{[\mathcal{O}_p]}(\mathcal{X}), \tag{1.3}$$

the Hilbert scheme parametrizing substacks in the class $[\mathcal{O}_p] \in F_0 K(\mathcal{X})$. The hard Lefschetz condition implies that the resolution is *semi-small* (i.e., that the fibres of π are zero- or one-dimensional), and that the singular locus of X is one-dimensional; see [8, 15]. Furthermore, [13] and [18] prove that there is a Fourier-Mukai isomorphism

$$\Psi : D^b(Y) \to D^b(\mathcal{X})$$

defined by

$$E \mapsto Rq_* p^* E$$

where

$$p : Z \to Y, \ q : Z \to \mathcal{X}$$

are the projections from the universal substack $Z \subset \mathcal{X} \times Y$ onto each factor.

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This isomorphism descends to an isomorphism of K-theory also denoted Ψ : $K(Y) \to K(\mathcal{X})$. It does not respect the filtration by dimension. However, the hard Lefschetz condition implies that the image of $F_0 K(\mathcal{X})$ is contained in $F_1 K(Y)$, under the inverse Φ of Ψ. We call the image $F_{exc} K(Y)$; its elements can be represented by formal differences of sheaves supported on the exceptional fibres of $\pi : Y \to X$. We define the multi-regular part of K-theory, $F_{mr}(\mathcal{X})$, to be the preimage of $F_1 K(Y)$ under Ψ. Its elements can be represented by formal differences of sheaves supported in dimension one where at the generic point of each curve in the support, the associated representation of the stabilizer group of that point is a multiple of the regular representation. The following filtrations are respected by Ψ:

$$F_{exc} K(Y) \subset F_1 K(Y) \subset K(Y)$$

$$F_0 K(\mathcal{X}) \subset F_{mR} K(\mathcal{X}) \subset K(\mathcal{X}).$$

Define the *exceptional* DT generating series of Y, the multi-regular generating series, and degree zero generating series of $\mathcal{X}$ to be:

$$\mathrm{DT}_{exc}(Y) = \sum_{\alpha \in F_{exc} K(Y)} DT^\alpha(Y) q^\alpha,$$

$$\mathrm{DT}_{mr}(\mathcal{X}) = \sum_{\alpha \in F_{mr} K(\mathcal{X})} DT^\alpha(\mathcal{X}) q^\alpha$$

$$\mathrm{DT}_0(\mathcal{X}) = \sum_{\alpha \in F_0 K(\mathcal{X})} DT^\alpha(\mathcal{X}) q^\alpha$$

We state the crepant resolution conjecture of [14, conjecture 1]:

Conjecture 1.4. Let $\mathcal{X}$ be an orbifold CY3 satisfying the hard Lefschetz condition. Let Y be the Calabi-Yau resolution of $\mathcal{X}$ given by equation 1.3. Then using Ψ to identify the variables, we have an equality

$$\frac{\mathrm{DT}_{mr}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})} = \frac{\mathrm{DT}(Y)}{\mathrm{DT}_{exc}(Y)}.$$

This dissertation makes progress towards proving this conjecture.

π-stable pairs

Objects of the Hilbert scheme may be viewed as two-term complexes,

$$\mathcal{O}_Y \xrightarrow{\gamma} G,$$

where the cokernel of γ must be zero, and where G may be any sheaf admitting such a map γ. The new invariants introduced in this dissertation, π-stable pairs, are a modification of this idea. They have been constructed with a view towards proving the crepant resolution conjecture, and as such, they depend on a crepant resolution $Y \xrightarrow{\pi} X$ as described in the previous section. The objects of our moduli space allow more variation in our cokernels, but less in the sheaf G. In particular, a two-term complex

$$\mathcal{O}_Y \xrightarrow{\gamma} G$$

is a π-*stable pair* (c.f. definition 2.7) if:

1. $R^\bullet \pi_* \operatorname{coker}(\gamma)$ is a zero-dimensional sheaf on X, and

2. G admits only the zero map from any sheaf P with the property above, namely that $R^\bullet \pi_* P$ is a zero-dimensional sheaf.

Remark 1.5. These pairs were inspired by, and are a generalization of, the stable pairs of Pandharipande and Thomas [39]. In fact, when $X = Y$ and π is the identity map, the above definition reduces to their definition of stable pairs.

Below, we prove that there is a finite-type algebraic space, $\pi\text{-Hilb}^\alpha$ parametrizing these objects with $[G] = \alpha \in K(Y)$. We may then define invariants

$$\pi\text{-PT}^\alpha(Y) = \sum_{n \in \mathbb{Z}} n\chi(\nu^{-1}(n)),$$

where $\nu : \pi\text{-Hilb}^\alpha \to \mathbb{Z}$ is Behrend's microlocal function. Note that if $\pi : Y \to X$ is the identity, then $\pi\text{-PT}^\alpha(Y) = \text{PT}^\alpha(Y)$, the usual Pandharipande-Thomas invariants of Y. As with Donaldson-Thomas theory, we collect the

invariants into a generating series,

$$\pi\text{-PT}(Y) = \sum_{\alpha \in F_1 K(Y)} \pi\text{-PT}^{\alpha}(Y) q^{\alpha}.$$

Main result

The following theorem rests the work of Bridgeland [12] and Joyce–Song [29], and we therefore require our Calabi-Yau threefold Y to satisfy

$$H^1(Y, \mathcal{O}_Y) = 0.$$

Theorem 1.6. Let X be a projective Calabi-Yau threefold that is the coarse space of an CY3 orbifold $\mathcal{X}$ that satisfies the hard Lefschetz condition. Let $\pi : Y \to X$ be the resolution given by equation 1.3. Then the generating series for the π-stable pair invariants and the DT invariants are related by the equation

$$\pi\text{-PT}(Y) = \frac{\text{DT}(Y)}{\text{DT}_{exc}(Y)}.$$

The aim of this dissertation is to prove this theorem. We summarize the chapters below. In chapter 2, we describe a torsion pair $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ that is crucial to our definition of π-stable pairs. We explain the similarities between π-stable pairs and PT stable pairs and objects of the Hilbert scheme. The chapter ends by establishing results about the moduli space of π-stable pairs.

In chapter 3, we recall the concept of a stability condition in the sense of Joyce. We then define the stability condition that we will use through out. The rest of the chapter is dedicated to proving that we may apply Joyce's powerful machinery.

In chapter 4, we introduce the Harder-Narasimhan filtration for our stability condition, which will be our main tool to prove the relationship between the stability condition and the torsion pair from chapter 2.

In chapter 5, we introduce the motivic Hall algebra.

In chapter 6, we introduce the infinite-type Hall algebra as a purely pedagogical tool. It helps us to give the essence of the idea of many re-

sults, without having to concern ourselves with convergence issues, which are handled in the next chapter.

In chapter 7, we introduce the Laurent Hall algebra, address the convergence issues alluded to in the previous chapter, and prove theorem 1.6.

Remark 1.7. To prove the crepant resolution conjecture, we need to prove that $\pi\text{-PT}(Y) = \text{PT}(\mathcal{X})$ and then use ([2]) Bayer's proof of the PT/DT correspondence on $\mathcal{X}$,

$$\text{PT}(\mathcal{X}) = \frac{\text{DT}_{mR}(\mathcal{X})}{\text{DT}_0(\mathcal{X})}.$$

The hope is that the Fourier-Mukai isomorphism Ψ takes π-stable pairs (as an object in $D^b(Y)$) to a PT pair on $\mathcal{X}$.

In his recent article [17], John Calabrese proves a relationship between the DT invariants of a Calabi-Yau threefold and its flop. This problem is similar in many respects to the crepant resolution conjecture studied in this thesis, and Calabrese uses many similar techniques. He constructs a torsion pair and new counting invariants which he relates to invariants on the flop via equations in the Hall algebra and the integration map. While this is very similar to our approach in outline, the actual torsion pair and counting invariants that Calabrese considers (even when adapted to the orbifold setting) are quite different from ours. It would be very interesting to find the precise relationship between the two approaches.

Chapter 2

π-stable pairs

In this section, we define π-stable pairs, and prove some basic results.

Categorical constructions

Let $\mathcal{A}$ be an abelian category. Here we recall the notion of torsion pairs.

Definition 2.1. Let $(\mathcal{P}, \mathcal{Q})$ be a pair of full subcategories of $\mathcal{A}$. We say $(\mathcal{P}, \mathcal{Q})$ is a *torsion pair* if the following conditions hold.

- $\mathrm{Hom}(T, F) = 0$ for any $T \in \mathcal{P}$ and $F \in \mathcal{Q}$.

- Any object $E \in \mathcal{A}$ fits into a unique exact sequence,

$$0 \to T \to E \to F \to 0, \tag{2.2}$$

  with $T \in \mathcal{P}$ and $F \in \mathcal{Q}$.

We borrow the following lemma from Toda [45].

Lemma 2.3. Suppose that $\mathcal{A}$ is a noetherian abelian category.

(i) Let $\mathcal{P} \subset \mathcal{A}$ be a full subcategory which is closed under extensions and quotients in $\mathcal{A}$. Then for $\mathcal{Q} = \{E \in \mathcal{A} : \mathrm{Hom}(\mathcal{P}, E) = 0\}$, the pair $(\mathcal{P}, \mathcal{Q})$ is a torsion pair on $\mathcal{A}$.

(ii) Let $\mathcal{Q} \subset \mathcal{A}$ be a full subcategory which is closed under extensions and subobjects in $\mathcal{A}$. Then for $\mathcal{P} = \{E \in \mathcal{A} : \mathrm{Hom}(E, \mathcal{Q}) = 0\}$, the pair $(\mathcal{P}, \mathcal{Q})$ is a torsion pair on $\mathcal{A}$.

Proof: We only show (i), as the proof of (ii) is similar. Take $E \in \mathcal{A}$ with $E \notin \mathbb{Q}$. Then there is $T \in \mathcal{P}$ and a non-zero morphism $T \to E$. Since $\mathcal{P}$ is

closed under quotients, we may assume that $T \to E$ is a monomorphism in $\mathcal{A}$. Take an exact sequence in $\mathcal{A}$,

$$0 \to T \to E \to F \to 0. \qquad (2.4)$$

By the noetherian property of $\mathcal{A}$ and the assumption that $\mathcal{P}$ is closed under extensions, we may assume that there is no $T \subsetneq T' \subset E$ with $T' \in \mathcal{T}$. Then we have $F \in \mathcal{Q}$ and (2.4) gives the desired sequence. $\blacksquare$

Example 2.5. Let

$$\mathcal{P} = \{0\text{-dimensional sheaves on } Y\},$$

and let

$$\mathcal{Q} = \{E \in \mathrm{coh}(Y) : \mathrm{Hom}(P, E) = 0 \text{ for all } P \in \mathcal{P}\}.$$

Lemma 2.3 easily proves that the pair $(\mathcal{P}, \mathcal{Q})$ is a torsion pair.

Let $\mathcal{C} = \mathrm{coh}_{\leq 1}(Y)$ denote the full subcategory of coherent sheaves on Y whose support is of dimension no more than one. We make the following definitions:

$$\mathcal{P}_\pi = \{P \in \mathcal{C} \mid R^\bullet \pi_* P \text{ is a zero-dimensional sheaf on } X\},$$

and

$$\mathcal{Q}_\pi = \{F \in \mathcal{C} \mid \text{ for all } P \in \mathcal{P}_\pi, \mathrm{Hom}(P, F) = 0\} = \mathcal{P}_\pi^\perp.$$

Lemma 2.6. The pair $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair in $\mathcal{C}$.

Proof: By lemma 2.3, it suffices to prove that $\mathcal{P}_\pi$ is closed under extensions and quotients.

Let $P', P'' \in \mathcal{P}_\pi$, and consider the short exact sequence

$$0 \to P' \to P \to P'' \to 0.$$

We are to show that such a P must live in $\mathcal{P}_\pi$. Consider now the long exact sequence,

$$0 \to \pi_* P' \to \pi_* \mathcal{P}_\pi \to \pi_* P'' \to R^1 \pi_* P' \to R^1 \pi_* \mathcal{P}_\pi \to R^1 \pi_* P'' \to 0.$$

Since $P', P'' \in \mathcal{P}_\pi$, we know that $R^1 \pi_* P' = 0$ and $R^1 \pi_* P'' = 0$, so $R^1 \pi_* P = 0$. We also know that $\pi_* P'$ and $\pi_* P''$ are zero-dimensional sheaves, and so it is clear then that $\pi_* P$ must be so as well. This proves that $\mathcal{P}_\pi$ is closed under extensions.

Let $P \in \mathcal{P}_\pi$, and consider the quotient $P \to B \to 0$. Denote the kernel of this map by K. As before, we get a long exact sequence,

$$0 \to \pi_* K \to \pi_* P \to \pi_* B \to R^1 \pi_* K \to R^1 \pi_* P \to R^1 \pi_* B \to 0.$$

Since $P \in \mathcal{P}_\pi$, $R^1 \pi_* P = 0$, and so $R^1 \pi_* B = 0$. It remains to show that $\pi_* B$ is zero-dimensional. We know that $\pi_* K$ is zero dimensional, since it is a subsheaf of (the zero-dimensional sheaf) $\pi_* P$. The support of $R^1 \pi_* K$ is contained in the singular locus. Suppose $\dim \operatorname{supp}(R^1 \pi_* K) = 1$. Then, K must have been supported in dimension two, however this contradicts the fact that $K \in \operatorname{coh}_{\leq 1}(Y)$.

Hence $R^1 \pi_* K$ is zero dimensional. Now, $\pi_* B$ is the extension of zero-dimensional sheaves, so it too is zero-dimensional. This completes the proof that $B \in \mathcal{P}_\pi$, and that $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair.

∎

Definition 2.7. A map $\gamma : \mathcal{O}_Y \to G$ is a *π-stable pair* if $G \in \mathcal{Q}_\pi$ and $\operatorname{coker}(\gamma) \in \mathcal{P}_\pi$.

Remark 2.8. Our notion of π-stable pair is a generalization of the stable pairs of Pandharipande and Thomas [39]. In the trivial case when $X = Y$ and $\pi = $ the identity, we have that $(\mathcal{P}_\pi, \mathcal{Q}_\pi) = (\mathcal{P}, \mathcal{Q})$ of example 2.5, and the π-stable pairs are exactly PT stable pairs.

Definition 2.9. Two π-stable pairs $\gamma_1 : \mathcal{O}_Y \to G_1$ and $\gamma_2 : \mathcal{O}_Y \to G_2$ are *isomorphic* if there exists a isomorphism of sheaves $\theta : G_1 \to G_2$ making the

following diagram commute:

$$\mathcal{O}_Y \xrightarrow{\gamma_1} G_1$$

with γ_2 going to G_2 and $\theta: G_1 \to G_2$.

Remark 2.10. In [12], the tilt of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{P}, \mathcal{Q})$ of example 2.5 is denoted $\mathcal{A}^\#$, and lemma 2.3 of [12] proves that $\mathcal{A}^\#$-epimorphisms of the form $\mathcal{O}_Y \to F$ are precisely stable pairs. The abelian category generated by $\mathcal{O}_Y$ and $\mathcal{C}$ has a tilt whose epimorphisms of the form $\mathcal{O}_Y \to G$ are precisely π-stable pairs. This is analogous to the tilt used in [12]. However, since it is not strictly necessary for any of our arguments, we will not present a proof here.

We associate to every π-stable pair $\gamma : \mathcal{O}_Y \to G$ a short exact sequence

$$0 \to \mathcal{O}_C \to G \to P \to 0,$$

where $P = \operatorname{coker}(\gamma) \in \mathcal{P}_\pi$ and $\mathcal{O}_C = \mathcal{O}_Y / \ker(\gamma)$.

Proposition 2.11. Let G be a non-zero sheaf. If $\mathcal{O}_Y \to G$ is a π-stable pair, then G is not supported exclusively on exceptional curves, and $R^1\pi_*G = 0$.

Proof: Let

$$0 \to \mathcal{O}_C \to G \to P \to 0$$

be the associated short exact sequence. We will first show that $R^1\pi_*\mathcal{O}_C = 0$. Consider the short exact sequence

$$0 \to I_C \to \mathcal{O}_Y \to \mathcal{O}_C \to 0.$$

Pushing forward yields

$$0 \to \pi_*I_C \to \pi_*\mathcal{O}_Y \to \pi_*\mathcal{O}_C \to R^1\pi_*I_C \to R^1\pi_*\mathcal{O}_Y \to R^1\pi_*\mathcal{O}_C \to 0$$

which is exact since the dimension of the fibres of $Y \to X$ is at most one. Thus the vanishing of $R^1\pi_*\mathcal{O}_Y$ by equation 1.2 implies that of $R^1\pi_*\mathcal{O}_C$.

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Now consider the following long exact sequence.

$$0 \to \pi_* \mathcal{O}_C \to \pi_* G \to \pi_* P \to R^1 \pi_* \mathcal{O}_C \to R^1 \pi_* G \to R^1 \pi_* P \to 0.$$

From above, we know $R^1 \pi_* \mathcal{O}_C = 0$. As well, $P \in \mathcal{P}_\pi$ implies $R^1 \pi_* P = 0$. Thus $R^1 \pi_* G = 0$.

Now, if C consists of only exceptional curves, then $\pi_* \mathcal{O}_C$ is zero-dimensional. This implies that $\pi_* G$ is the extension of zero-dimensional sheaves, and therefore zero-dimensional. This means that $G \in \mathcal{P}_\pi$. By definition of π-stable pair, $G \in \mathcal{Q}_\pi$. By definition of $\mathcal{Q}_\pi$ the only map from an object of $\mathcal{P}_\pi$ to an object of $\mathcal{Q}_\pi$ is the zero map, hence the identity of G is the zero map, and G is the zero object. ∎

Let us introduce some terminology and results taken from [12] (modified for our purposes, since we are only interested in sheaves supported in dimension no more than one). Let $\mathcal{M}$ denote the stack of objects of $\mathrm{coh}_{\leq 1}(Y)$. It is an algebraic stack, locally of finite type over $\mathbb{C}$. Let $\mathcal{M}(\mathcal{O})$ denote the stack of framed sheaves, that is, the stack whose objects over a scheme S are pairs (E, γ) where E is a S-flat coherent sheaf on $S \times Y$, of relative dimension no more than one, together with a map $\mathcal{O}_{S \times Y} \xrightarrow{\gamma} E$. Given a morphism of schemes $f : T \to S$, and an object (F, δ) over T, a morphism in $\mathcal{M}(\mathcal{O})$ lying over f is an isomorphism

$$\theta : f^*(E) \to F$$

such that the following diagram commutes

$$
\begin{array}{ccc}
f^*(\mathcal{O}_{S \times Y}) & \xrightarrow{f^*(\gamma)} & f^*(E) \\
\downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \theta} \\
\mathcal{O}_{T \times Y} & \xrightarrow{\quad \delta \quad} & F.
\end{array}
\qquad (2.12)
$$

The symbol "can" denotes the canonical isomorphism of pullbacks.

There is a natural map

$$\mathcal{M}(\mathcal{O}) \xrightarrow{q} \mathcal{M} \qquad\qquad (2.13)$$

sending a sheaf with a section to the underlying sheaf.

The following lemmas are 2.4 and 2.5 of [12].

Lemma 2.14. The stack $\mathcal{M}(\mathcal{O})$ is algebraic and the morphism q is representable and of finite type.

Lemma 2.15. There is a stratification of $\mathcal{M}$ by locally-closed substacks

$$\mathcal{M}_r \subset \mathcal{M}$$

such that objects F of $\mathcal{M}_r(\mathbb{C})$ are coherent sheaves satisfying

$$\dim_{\mathbb{C}} H^0(Y, F) = r.$$

Furthermore, the pullback of the morphism q to $\mathcal{M}_r$ is a Zariski fibration with fibre $\mathbb{C}^r$.

Both of these are proven in [12].

Lemma 2.16. The moduli space of π-stable pairs with a fixed Hilbert polynomial is an algebraic space of finite type.

Proof: Let $\pi\text{-Hilb}^{(\beta,n)}$ denote the stack of π-stable pairs on Y whose sheaf G has Chern character $(0, 0, \beta, n)$. Notice that

$$\pi\text{-Hilb}^{(\beta,n)} \subset \mathcal{M}(\mathcal{O}).$$

Lemma 2.14 states that $\mathcal{M}(\mathcal{O})$ is an algebraic stack.

Using the forth-coming description of $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ in terms of stability, and using theorem 4.8 of [26], we know that $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ are algebraic substacks of $\mathcal{M}$. We use this fact to proven that π-Hilb is an algebraic substack of $\mathcal{M}$.

Consider the following cartesian diagram,

$$\begin{array}{ccc} \mathcal{Q}_\pi(\mathcal{O}) & \longrightarrow & \mathcal{M}(\mathcal{O}) \\ \downarrow & & \downarrow q \\ \mathcal{Q}_\pi & \longrightarrow & \mathcal{M}. \end{array}$$

Since $\mathcal{Q}_\pi$ and $\mathcal{M}(\mathcal{O})$ are algebraic stacks, their fibre product, $\mathcal{Q}_\pi(\mathcal{O})$ is an algebraic stack, too. There is a map of stacks of stack $c : \mathcal{Q}_\pi(\mathcal{O}) \to \mathcal{M}$ which takes $\gamma : \mathcal{O}_Y \to G$ to the cokernel $\mathrm{coker}(\gamma)$. Since π-Hilb fits into the following cartesian diagram,

$$\begin{array}{ccc} \pi\text{-Hilb} & \longrightarrow & \mathcal{P}_\pi \\ \downarrow & & \downarrow \\ \mathcal{Q}_\pi(\mathcal{O}) & \xrightarrow{c} & \mathcal{M}, \end{array}$$

we conclude that it too is an algebraic stack.

Now we prove π-Hilb$^{(\beta,n)}$ is of finite type. In lemma 3.16, we prove that the sheaves of Chern character $(0, 0, \beta, n)$ underlying π-stable pairs form a bounded family. In particular, this means that there exists a coherent sheaf E such that every sheaf underlying a π-stable pair G of Chern character $(0, 0, \beta, n)$ is a quotient of E. Now if we choose a polarization $\mathcal{O}(1)$ of Y, we have a finite type scheme $\mathrm{Quot}(E, P)$ where P is the Hilbert polynomial $P(t) = \beta \cdot \mathcal{O}(1)t + n$. There is a morphism $\mathrm{Quot}(E, P) \to \mathcal{M}$ that takes a quotient $E \to G$ and sends it to G. This map is of finite type. Consider the diagram:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{f} & \mathrm{Quot}(E, P) \\ f' \downarrow & & \downarrow q' \\ \mathcal{M}(\mathcal{O}) & \xrightarrow{q} & \mathcal{M} \end{array}$$

where $\mathcal{T}$ is the pullback of q' by q. The image of the map q' contains all sheaves underlying π-stable pairs with Hilbert polynomial P. This means that the image of $\mathcal{T}$ in $\mathcal{M}(\mathcal{O})$ contains all 2-term complexes $[\mathcal{O}_Y \to G]$ where G is a sheaf underlying a π-stable pair with Hilbert polynomial P.

In particular, the image of $\mathcal{T}$ contains all π-stable pairs. Now since q is of finite type, so must f be of finite type. The scheme $\mathrm{Quot}(E, P)$ is of finite type over $\mathbb{C}$, so then we may conclude that $\mathcal{T}$ is finite type over $\mathbb{C}$. Thus the image of $\mathcal{T}$ under f' is of finite type. The stack of all π-stable pairs is in this image, hence of finite type.

It remains to prove that this stack is in fact an algebraic space. It suffices to prove that all closed points of the stack $\pi\text{-Hilb}^{(\beta,n)}$ have trivial automorphism.

Let $\mathcal{O}_Y \to G$ be a π-stable pair. We will show that it has only the trivial automorphism. Consider the associated short exact sequence,

$$0 \to \mathcal{O}_C \to G \to P \to 0.$$

An automorphism of this π-stable pair leads to a diagram of the form,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C & \overset{\gamma}{\longrightarrow} & G & \overset{h}{\longrightarrow} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \bar{g}} & & \\
0 & \longrightarrow & \mathcal{O}_C & \overset{\gamma}{\longrightarrow} & G & \overset{h}{\longrightarrow} & P & \longrightarrow & 0.
\end{array}
$$

We will show that g is the identity map. Consider the following diagram, obtained by subtracting the identity from the diagram above,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C & \overset{\gamma}{\longrightarrow} & G & \overset{h}{\longrightarrow} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{zero}} & & \downarrow{\scriptstyle g-\mathrm{id}} & & \downarrow{\scriptstyle \overline{g-\mathrm{id}}} & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & G & \overset{h}{\longrightarrow} & P & \longrightarrow & 0.
\end{array}
\qquad (2.17)
$$

Since the left-most vertical arrow is zero, a diagram chase proves that the dotted morphism exists and commutes with the diagram. However, $P \in \mathcal{P}_\pi$ and $G \in \mathcal{Q}_\pi$, so this map must be zero. As a consequence, $\overline{g - \mathrm{id}} = 0$. Homological algebra then implies that $g - \mathrm{id}$ must be the zero map, forcing $g = \mathrm{id}$.

This proves that closed points of the stack of π-stable pairs have no automorphisms, and that π-stable pairs form an algebraic space. $\blacksquare$

The Behrend function identity

We state and prove a variation of [12, theorem 3.1] of Bridgeland.

Lemma 2.18. Let $\gamma : \mathcal{O}_Y \to G$ be a π-stable pair. Then there is an equality of Behrend's microlocal functions

$$\nu_{\mathcal{M}(\mathcal{O})}(\gamma) = (-1)^{\chi(G)}\nu_{\mathcal{M}}(G).$$

Proof: The case when G is a stable pair is taken care of by theorem 3.1 of [12]. Thus, we may assume that the cokernel P of $\gamma : \mathcal{O}_Y \to G$ has one-dimensional support.

Let $\mathcal{O}_C \subset G$ be the image of γ. It is the structure sheaf of a subscheme $C \subset Y$ of dimension 1. There is a line bundle L on Y such that

$$H^i(Y, G \otimes L) = 0 \qquad (2.19)$$

for all $i > 0$, and there is a divisor $H \in |L|$ such that H meets C at finitely many points, none of which are in the support of $\mathrm{coker}(\gamma)$. This claim is verified in lemma 2.22. From here, the proof is identical to Bridgeland's, but we include a portion of it to illustrate his ideas.

There is a short exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{s} L \to \mathcal{O}_H(H) \to 0$$

where s is the section of L corresponding to the divisor H. Tensoring it with G, and using the above assumptions yields a diagram of sheaves

$$(2.20)$$

$$
\begin{array}{ccccccccc}
& & \mathcal{O}_Y & & & & & & \\
& & \downarrow{\scriptstyle\gamma} & \searrow{\scriptstyle\delta} & & & & & \\
0 & \longrightarrow & G & \xrightarrow{\ \alpha\ } & F & \xrightarrow{\ \beta\ } & K & \longrightarrow & 0
\end{array}
$$

where $F = G \otimes L$. The support of the sheaf K is zero-dimensional, and

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disjoint from the support of $\mathrm{coker}(\gamma)$. In particular,

$$\mathrm{Hom}_Y(K, F) = 0. \tag{2.21}$$

Consider two points of the stack $\mathcal{M}(\mathcal{O})$ corresponding to the maps

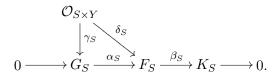$$\gamma : \mathcal{O}_Y \to G \text{ and } \delta : \mathcal{O}_Y \to F.$$

The statement of lemma 2.18 holds for the map δ because lemma 2.15 together with equation 2.19 implies that

$$q : \mathcal{M}(\mathcal{O}) \to \mathcal{M}$$

is smooth of relative dimension $\chi(F) = H^0(Y, F)$ over an open neighbourhood of the point $F \in \mathcal{M}(\mathbb{C})$. On the other hand, tensoring sheaves with L defines an automorphism of $\mathcal{M}$, so the microlocal function of $\mathcal{M}$ at the points corresponding to G and F are equal. To prove the lemma, it suffices to show that

$$(-1)^{\chi(G)} \cdot \nu_{\mathcal{M}(\mathcal{O})}(\gamma) = (-1)^{\chi(F)} \cdot \nu_{\mathcal{M}(\mathcal{O})}(\delta).$$

Consider the stack W whose S-valued points are diagrams of S-flat sheaves on $S \times Y$ of the form

$$
\begin{array}{ccccccccc}
& & \mathcal{O}_{S \times Y} & & & & & & \\
& {\scriptstyle \gamma_S}\downarrow & & \searrow{\scriptstyle \delta_S} & & & & & \\
0 \longrightarrow & G_S & \xrightarrow{\ \alpha_S\ } & F_S & \xrightarrow{\ \beta_S\ } & K_S & \longrightarrow & 0.
\end{array}
$$

There are two morphisms

$$p : W \to \mathcal{M}(\mathcal{O}), \qquad q : W \to \mathcal{M}(\mathcal{O}),$$

taking such a diagram to the maps γ_S and δ_S respectively. By passing to an open substack of W, we may assume that equation 2.21 holds for all $\mathbb{C}$-valued points of W. It follows that p and q induce injective maps on

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stabilizer groups of $\mathbb{C}$-valued points, and hence are representable.

Recall that Behrend's microlocal function satisfies the property that when $f : T \to S$ is a smooth morphism of relative dimension d, there is an identity [3, proposition 1.5]

$$\nu_T = (-1)^d f^*(\nu_S).$$

Using this identity, it will be enough to show that at the point $w \in W(\mathbb{C})$ corresponding to the diagram 2.20, the morphisms p and q are smooth of relative dimension $\chi(K)$ and 0, respectively. For the proof of these facts, see the [12, pages 11–13]. ■

Lemma 2.22. Given a π-stable pair $\gamma : \mathcal{O}_Y \to G$, we may choose a very ample divisor H on X such that its pull-back is equal to its proper transform (we denote both by $\widetilde{H}$), it satisfies $\operatorname{supp}(\operatorname{coker}\gamma) \cap \widetilde{H} = \emptyset$, $\widetilde{H} \cap \operatorname{supp} G$ is 0-dimensional, and

$$H^1(Y, G(\widetilde{H})) = 0.$$

Proof: First we collect a little notation. Let E be the exceptional locus of Y, let E' be the image of E in X. Define a subset Z to be

$$Z = \{p \in X : G|_{\pi^{-1}(p)} \text{ is one dimensional}\} \subset E'.$$

Notice that Z is a finite collection of points.

Since $\operatorname{coker}\gamma \in \mathcal{P}_\pi$, we have that $\pi(\operatorname{supp}(\operatorname{coker}\gamma))$ is zero dimensional. Moreover, $\pi(\operatorname{supp} G)$ is one dimensional. Thus we may choose an ample divisor H on X so that

$$H \cap \pi(\operatorname{supp}(\operatorname{coker}\gamma)) = \emptyset$$

and $H \cap \pi(\operatorname{supp} G)$ is zero dimensional and does not contain any of the points in Z. It follows that $\widetilde{H} \cap \operatorname{supp}(\operatorname{coker}\gamma)$ is empty, and $\widetilde{H} \cap \operatorname{supp}(G)$ is zero dimensional. Moreover, by Serre vanishing, we may assume that H is

sufficiently ample on X so that

$$H^1(X, (\pi_* G)(H)) = 0. \tag{2.23}$$

We now show that $H^1(Y, G(\widetilde{H})) = 0$. By proposition 2.11, we know that $R^1\pi_* G = 0$, since $\mathcal{O}_Y \to G$ is a π-stable pair. The sequence

$$0 \to G \to G(\widetilde{H}) \to G(\widetilde{H})|_{\widetilde{H}} \to 0$$

gives

$$\ldots \to R^1\pi_* G \to R^1\pi_* G(\widetilde{H}) \to R^1\pi_* G(\widetilde{H})|_{\widetilde{H}} \to 0.$$

However, we know $R^1\pi_* G = 0$ and $G(\widetilde{H})|_{\widetilde{H}}$ is supported on points so $R^1\pi_* G(\widetilde{H})|_{\widetilde{H}} = 0$, so $R^1\pi_* G(\widetilde{H}) = 0$. Now by the Leray spectral sequence,

$$
\begin{aligned}
H^1(Y, G(\widetilde{H})) &= H^1(X, \pi_*(G(\widetilde{H})) \\
&= H^1(X, \pi_*(G \otimes \pi^* \mathcal{O}_X(H))) \\
&= H^1(X, (\pi_*(G))(H)) \\
&= 0,
\end{aligned}
$$

where the last equality comes from equation 2.23. ∎

Chapter 3

Stability conditions

In this section, we define a stability condition on $\mathcal{C} = \mathrm{coh}_{\leq 1} Y$. We follow Joyce's treatment of stability conditions as found in section 4 of [26], though not in as great generality.

Let $N_1(Y)$ denote the abelian group of cycles of dimension one modulo numerical equivalence. We begin by quoting lemmas 2.1 and 2.2 of [12].

Lemma 3.1. An element $\beta \in N_1(Y)$ has only finitely many decompositions of the form $\beta = \beta_1 + \beta_2$ with β_i effective.

Lemma 3.2. The Chern character map induces an isomorphism

$$ch = (\mathrm{ch}_2, \mathrm{ch}_3) : F_1 K(Y) \to N_1(Y) \oplus \mathbb{Z}.$$

Define

$$\Delta = \{[E] \in F_1 K(Y) : E \in \mathcal{C}\}$$

to be the *positive* or *effective cone* of $F_1 K(Y)$.

Definition 3.3. A *stability condition* on $\mathcal{C}$ is a triple $(T, \tau, \leq)$ where $(T, \leq)$ is a set T with a total ordering $\leq$, and τ is a map $\Delta \xrightarrow{\tau} T$ from the effective cone to T, satisfying the following condition: whenever $\alpha + \beta = \gamma$ in Δ, then

$$\tau(\alpha) < \tau(\gamma) < \tau(\beta),$$

or

$$\tau(\beta) < \tau(\gamma) < \tau(\alpha),$$

or

$$\tau(\alpha) = \tau(\gamma) = \tau(\beta).$$

A triple $(T, \tau, \leq)$ is called a *weak stability condition* if it satisfies the weaker condition that whenever $\alpha + \beta = \gamma$ in Δ, $\tau(\alpha) \leq \tau(\gamma) \leq \tau(\beta)$ or $\tau(\beta) \leq \tau(\gamma) \leq \tau(\alpha)$.

Definition 3.4. A non-zero sheaf G is

1. τ-*semistable* if for all $S \subset G$, such that $S \ncong 0$, we have that $\tau(S) \leq \tau(G/S)$;

2. τ-*stable* if for all $S \subset G$, such that $S \ncong 0$, we have that $\tau(S) < \tau(G/S)$;

3. τ-unstable if it is not τ-semistable.

Lemma 3.5. Let F and G be τ-semistable sheaves, and let $F \xrightarrow{f} G$ be a map of sheaves. Then either $\tau(F) \leq \tau(G)$ or $f = 0$.

Proof: Consider the inclusion map $\iota : \mathrm{im}(f) \to G$. Since G is semistable, we know either $\mathrm{im}(f) = 0$ or $\tau(\mathrm{im}(f)) \leq \tau(G)$. Consider now the corestriction of f, $\mathrm{cor}(f) : F \to \mathrm{im}(f)$. Since F is semistable, we know that either $\mathrm{im}(f) = 0$ or $\tau(F) \leq \tau(\mathrm{im}(f))$. This implies either $\mathrm{im}(f) = 0$ or $\tau(F) \leq \tau(G)$. ∎

Now we define a stability condition on $\mathcal{C}$.

Definition 3.6. Let E be the exceptional divisor of π. Choose an ample divisor H on X, let $\widetilde{H}$ denote the total transform of H in Y. Let L be a very ample line bundle on Y.

Given a sheaf G in $\mathcal{C}$, define the π-*slope* of G to be

$$\mu_\pi(G) = \Big(\frac{\chi(G)}{\beta \cdot \widetilde{H}}, \frac{\chi(G)}{\beta \cdot L} \Big) \in (-\infty, +\infty] \times (-\infty, +\infty],$$

where $(-\infty, +\infty] \times (-\infty, +\infty]$ is ordered lexicographically, and $\beta = \beta_G$ the homology class associated to the support of G.

Theorem 3.7. The map

$$\mu_\pi : \Delta \to (-\infty, +\infty] \times (-\infty, +\infty]$$

defines a weak permissible stability condition.

Unpacking the definitions 4.1 and 4.7 of [26], we see that a weak permissible stability condition is one that satisfies three conditions:

1. (weak seesaw property) for any short exact sequence

$$0 \to A \to G \to B \to 0,$$

either $\mu_\pi(A) \leq \mu_\pi(G) \leq \mu_\pi(B)$ or $\mu_\pi(A) \geq \mu_\pi(G) \geq \mu_\pi(B)$,

2. μ_π is dominated by a permissible stability condition, and

3. the family of all μ_π-stable sheaves of a fixed Chern class is bounded.

We will prove theorem 3.7 by proving the above three properties in lemma 3.8, corollary 3.10, and lemma 3.16.

Lemma 3.8. The function μ_π satisfies the seesaw property.

Proof: Let $0 \to A \to G \to B \to 0$ be a short exact sequence of sheaves, and suppose $\mu_\pi(A) \leq \mu_\pi(G)$. Less concisely, we are supposing

$$\left(\frac{\chi(A)}{\beta_A \cdot \widetilde{H}}, \frac{\chi(A)}{\beta_A \cdot L} \right) \leq \left(\frac{\chi(G)}{\beta_G \cdot \widetilde{H}}, \frac{\chi(G)}{\beta_G \cdot L} \right),$$

from which we are to deduce that $\mu_\pi(G) \leq \mu_\pi(B)$. Before we start a case-by-case analysis, notice that $\chi(G) = \chi(A) + \chi(B)$ and $\beta_G = \beta_A + \beta_B$.

case 1: $\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} < \frac{\chi(G)}{\beta_G \cdot \widetilde{H}}$ and no denominator is zero.

Then this follows from the observation

$$\frac{a}{b} < \frac{a+c}{b+d} \Rightarrow \frac{a+c}{b+d} < \frac{c}{d},$$

provided $b, d > 0$. In particular, we assume

$$\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} \leq \frac{\chi(G)}{\beta_G \cdot \widetilde{H}}.$$

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Rewriting the second term yields

$$\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} \leq \frac{\chi(A) + \chi(B)}{(\beta_A + \beta_B) \cdot \widetilde{H}}.$$

The observation above then proves that

$$\frac{\chi(A) + \chi(B)}{(\beta_A + \beta_B) \cdot \widetilde{H}} \leq \frac{\chi(B)}{\beta_B \cdot \widetilde{H}},$$

as desired.

case 2: $\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} = \frac{\chi(G)}{\beta_G \cdot \widetilde{H}}$, $\frac{\chi(A)}{\beta_A \cdot L} \leq \frac{\chi(G)}{\beta_G \cdot L}$ and no denominator is zero.

We are given that $\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} = \frac{\chi(G)}{\beta_G \cdot \widetilde{H}}$, so

$$\chi(A)(\beta_G \cdot \widetilde{H}) = (\beta_A \cdot \widetilde{H})\chi(G).$$

Writing everything in terms of A and B,

$$\chi(A)((\beta_A + \beta_B) \cdot \widetilde{H}) = \beta_A \cdot \widetilde{H}(\chi(A) + \chi(B)),$$

which implies

$$\chi(A)(\beta_B \cdot \widetilde{H}) = \beta_A \cdot \widetilde{H}\chi_B.$$

Since we assume that all denominators are non-zero, we have

$$\frac{\chi(A)}{\beta_A \cdot \widetilde{H}} = \frac{\chi(B)}{\beta_B \cdot \widetilde{H}}.$$

So we must show that

$$\frac{\chi(G)}{\beta_G \cdot L} < \frac{\chi(B)}{\beta_B \cdot L}.$$

This follows from the same observation made in case 1.

case 3: $\beta_A \cdot \widetilde{H} = 0$. Then $+\infty = \frac{\chi(A)}{\beta_A \cdot \widetilde{H}} \leq \frac{\chi(A)+\chi(B)}{(\beta_A+\beta_B)\cdot\widetilde{H}}$. This implies that $\frac{\chi(A)+\chi(B)}{(\beta_A+\beta_B)\cdot\widetilde{H}} = +\infty$, so $(\beta_A + \beta_B) \cdot \widetilde{H} = 0$, and hence $\beta_B \cdot \widetilde{H} = 0$. This

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reduces us to Gieseker stability on Y, which we know satisfies the seesaw property.

case 4: $\beta_G \cdot \widetilde{H} = 0$. We know $\beta \cdot \pi^* H = \pi_*(\beta \cdot \pi^* H) = \pi_*(\beta) \cdot H \geq 0$, hence $\beta \cdot \widetilde{H} \geq 0$ for any effective curve class β, so we must have $\beta_A \cdot \widetilde{H} = 0$ and $\beta_B \cdot \widetilde{H} = 0$. This lands us back in the case of Gieseker stability on Y, and lemma 3.8 is proven.

∎

Let us now recall from Joyce the following facts and definitions. A stability condition τ on $\mathcal{C}$ is *permissible* if:

- $\mathcal{C}$ is τ-artinian, i.e. there exists no infinite descending chain $\cdots A_2 \subset A_1 \subset A$ in $\mathcal{C}$ such that $A_i \neq A_{i+1}$, and $\tau(A_{i+1}) \geq \tau(A_i/A_{i+1})$ for all i; and

- the substack of τ-semistable objects of a fixed Chern character of the stack parametrizing objects of $\mathcal{C}$ is a constructible substack of $\mathcal{M}$.

Thankfully, the second condition amounts to showing that the family of μ_π-semistable sheaves of a fixed Chern character is bounded. See the proof of theorem 4.30 from [26] for details.

Joyce proves that these properties are inherited by *domination*. Recall that a stability condition $\tilde{\tau}$ is said to dominate τ if for any A, B in $\mathcal{C}$ with $\tau(A) \leq \tau(B)$ then $\tilde{\tau}(A) \leq \tilde{\tau}(B)$.

Define the following stability condition:

$$\delta(G) = -\dim \operatorname{supp} G \in \mathbb{Z}.$$

In [26], it is proven that the abelian category $\operatorname{coh}(Y)$ is δ-artinian. To prove that $\mathcal{C} = \operatorname{coh}_{\leq 1}(Y)$ is μ_π-artinian, we will show that μ_π is dominated by δ.

Lemma 3.9. The stability condition μ_π is dominated by the stability condition δ.

Proof: Let $\mu_\pi(A) \leq \mu_\pi(B)$. We need to show that this implies that $\delta(A) \leq \delta(B)$. Expanding, we have that

$$\left(\frac{\chi(A)}{\beta_A \cdot \widetilde{H}}, \frac{\chi(A)}{\beta_A \cdot L}\right) \leq \left(\frac{\chi(B)}{\beta_B \cdot \widetilde{H}}, \frac{\chi(B)}{\beta_B \cdot L}\right).$$

We proceed with a case-by-case analysis.

case 1: the denominators are non-zero. Since we have restricted our attention to sheaves supported in dimension ≤ 1, it follows that neither A nor B is 0-dimensional. Thus, they are both one dimensional, and $\delta(A) = \delta(B)$. In particular, $\delta(A) \leq \delta(B)$.

case 2: $\beta_B \cdot \widetilde{H} = 0$ and $\beta_A \cdot \widetilde{H} \neq 0$. Then $\dim \operatorname{supp} A \geq 1 \geq \dim \operatorname{supp} B$. So $\delta(A) \leq \delta(B)$.

case 3: Both $\beta_B \cdot \widetilde{H} = 0$ and $\beta_A \cdot \widetilde{H} = 0$. Then $\mu_\pi(A) \leq \mu_\pi(B)$ amounts to regular Geiseker stability, which is dominated by δ as demonstrated by Joyce [26, §4.4].

■

Corollary 3.10. The category $\mathcal{C}$ is μ_π-artinian.

Proof: As mentioned above, this follows from the previous lemma since it is inherited by domination. ■

To finish the proof that μ_π is a permissible weak stability condition, it remains only to prove the family of all μ_π-semistable sheaves of a fixed Chern class is bounded. This is proven in the next section.

Boundedness

Let us recall the basic results concerning boundedness (cf. [23]).

Definition 3.11. A subcategory $\mathcal{U}$ of $\operatorname{coh}(Y)$ is bounded if there exists a scheme S of finite type and a sheaf U on $X \times S$ such that for every object U_i of $\mathcal{U}$, there exists a closed point $s_i \in S$ such that $U_i \cong U|_{X \times \{s_i\}}$.

Notice that this definition still makes sense if we have a set of isomorphism classes of sheaves instead of a category.

Definition 3.12. Let Y be a scheme, let $\mathcal{O}(1)$ be an ample line bundle, and let m be an integer. A sheaf F on Y is m-regular if, for all $i > 0$,

$$H^i(Y, F(m - i)) = 0.$$

A proof for the following may be found in [30], as well as in [37].

Lemma 3.13. If F is m-regular, then the following statements are true:

1. F is m'-regular for all $m' \geq m$.

2. $F(m)$ is globally generated.

3. For all $n \geq 0$, the natural map $H^0(Y, F(m)) \otimes H^0(Y, \mathcal{O}(n)) \to H^0(Y, F(n+m))$ is surjective.

Definition 3.14. The Mumford-Castelnuovo regularity of a sheaf F is the number $\text{reg}(F) = \inf\{m \in \mathbb{Z} : F \text{ is } m\text{-regular }\}$.

Lemma 3.15. Let $\mathcal{U}$ be a category of sheaves on Y. The following statements are equivalent.

1. $\mathcal{U}$ is bounded.

2. The set of Hilbert polynomials of objects U_i of $\mathcal{U}$ is finite, and there is an integer N such that for all objects U_i of $\mathcal{U}$, $\text{reg}\, U_i < N$.

3. The set of Hilbert polynomials of objects U_i of $\mathcal{U}$ is finite, and there exists a sheaf F such that each object of $\mathcal{U}$ is isomorphic to a quotient of F.

Proof of this lemma may be found in [20].

Now we prove the boundedness result we require.

Lemma 3.16. The family of π-stable pairs $\mathcal{O}_Y \xrightarrow{\gamma} G$ with a fixed Chern class is bounded.

Proof: To each such π-stable pair there is an associated a short exact sequence,

$$0 \to \mathcal{O}_C \to G \to P \to 0,$$

where $\mathcal{O}_C$ is the image of the map γ, and P is the cokernel. We will show that the family of possibilities for $\mathcal{O}_C$ and the family of possible Ps are both bounded families. Once this is established, it is clear that the family of sheaves underlying a π-stable pair is a bounded family of sheaves.

First we will consider the family of possibilities for $\mathcal{O}_C$. To show that this family is bounded, we will show:

1. the Hilbert polynomials of this family take on only a finite number of values; and

2. there exists a single sheaf that surjects onto each member of this family.

The second requirement is trivially satisfied, since each member of this family is the structure sheaf of a subscheme of Y, and hence, admits a surjective maps from $\mathcal{O}_Y$. It remains to find upper and lower bounds for the coefficients of the Hilbert polynomial of a general element from this family.

In contrast to PT-theory, the support of $\mathcal{O}_C$ is not equal to the support of G, since P is not necessarily zero-dimensional. However, we still have $\beta_G = \beta_C + \beta_P$, where all β are effective. We know that there are only finitely many decompositions of β_G into the sum of two effective curve classes. This forces an upper and lower bound on the linear coefficient of the Hilbert polynomial of $\mathcal{O}_C$. It remains to find upper and lower bounds for the Euler characteristic of $\mathcal{O}_C$ (the constant coefficient of the Hilbert polynomial).

The Leray spectral sequence proves that $\chi(P) = \chi(R^\bullet \pi_* P) \geq 0$, the inequality following from the fact that $R^\bullet \pi_* P$ is zero dimensional. Now, $\chi(G) = \chi(P) + \chi(\mathcal{O}_C)$, and $\chi(P) \geq 0$ implies $\chi(\mathcal{O}_C) \leq \chi(G)$. This gives us an upper bound, since the Chern character, and hence, the Euler characteristic, of G is fixed. For the lower bound, let $\alpha_G = (\beta_G, n_G)$ be the Chern character of G, and let $\alpha_C = (\beta_C, n_C)$ be the Chern character of C.

In general, if $\mathrm{Hilb}^{(\beta,n)}$ is non-empty (say one of its points represents a curve J), then $\dim \mathrm{Hilb}^{(\beta,n+k)} \geq 3k$ since we get a $3k$-dimensional space

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of curves coming from the curve J with k "wandering points." This line of reasoning tells us that

$$\dim \text{Hilb}^{(\beta_C, n_G)} \geq 3(n_G - n_C).$$

Rearranging this yields

$$n_C \geq n_G - \frac{1}{3} \dim \text{Hilb}(\beta_C, n_G).$$

This gives us a lower bound for n_C, which completes the proof that the corresponding family is bounded.

Now to show that the family of cokernels is bounded, we will show

1. there is a common upper-bound to the index of regularity; and

2. the Hilbert polynomials of this family take on only a finite number of values

Using the Leray spectral sequence again, we note that for all $P \in \mathcal{P}_\pi$,

$$H^1(Y, P) = H^1(X, \pi_* P) = 0,$$

thus, all $P \in \mathcal{P}_\pi$ are 1-regular. To show this family is bounded, it remains to find upper and lower bounds for the coefficients of the Hilbert polynomial of a general object.

As above, there are only a finite number of options for the support curve of P. This yields upper and lower bounds on the linear coefficient of the Hilbert polynomial.

We know that $\chi(G) = \chi(P) + \chi(\mathcal{O}_C)$. Since $\chi(\mathcal{O}_C)$ is bounded, and $\chi(G)$ is fixed, so too must $\chi(P)$ be bounded.

This completes the proof that the sheaves underlying a π-stable pair of fixed K-class forms a bounded family of sheaves. ∎

This also completes the proof that μ_π is a permissible weak stability condition.

Chapter 4

The torsion pair and the stability condition

In this section, we show that $\mathcal{P}_\pi$ may be conveniently expressed in terms of the stability condition, and similarly for $\mathcal{Q}_\pi$. First we give a rapid introduction to the modern Harder-Narasimhan property, a generalization of the Harder-Narasimhan filtration of [22].

Definition 4.1. A weak stability condition $(T, \tau, \leq)$ on $\mathcal{C}$ is said to have the *Harder-Narasimhan property* if for every sheaf G, there exists a unique filtration of G

$$0 = HN_\tau(G)_0 \subset HN_\tau(G)_1 \subset \cdots \subset HN_\tau(G)_{N-1} \subset HN_\tau(G)_N = G$$

(where the inclusions are strict) such that the quotients

$$Q_i = HN_\tau(G)_i / HN_\tau(G)_{i-1}$$

are τ-semistable and

$$\tau(Q_i) > \tau(Q_{i+1})$$

for all $i > 0$. When it is clear from context, most of the notation will be suppressed, and we will denote the Harder-Narasimhan filtration of G with respect to τ by $0 \subset G_1 \subset G_2 \subset \cdots \subset G_{N-1} \subset G_N = G$. The G_i are called the *filtered objects* of the Harder-Narasimhan filtration, and the Q_i are called the *quotient objects*.

We borrow the following definition and theorem from Joyce [26]. In [24, §9], Joyce proves that the category of coherent sheaves satisfies assumptions

3.7 of [26]. This is enough for us to conclude that the assumptions are also true of $\mathcal{C}$ the category of coherent sheaves supported in dimension one or less.

Theorem 4.2 ([26], Theorem 4.4)**.** Let $(T, \tau, \leq)$ be a weak stability condition on an abelian category $\mathcal{A}$. If $\mathcal{A}$ is Noetherian and τ-artinian, then $(T, \tau, \leq)$ has the Harder-Narasimhan property.

Corollary 4.3. The weak stability condition μ_π on $\mathcal{C}$ has the Harder-Narasimhan property.

Proof: The category $\mathcal{C}$ is Noetherian because it is a subcategory of the category of coherent sheaves, which is Noetherian. Corollary 3.10 proves that $\mathcal{C}$ is μ_π-artinian. ∎

When we refer to the Harder-Narasimhan filtration in what follows, we will always be referring to the filtration with respect to the stability condition μ_π.

We present some notation before we state and prove the main result of this section. Recall that our slope function μ_π takes values in the lexico-graphically ordered set $(-\infty, +\infty] \times (-\infty, +\infty]$. To avoid awkwardly writing the ordered pairs $(+\infty, +\infty)$ and $(+\infty, 0)$ through-out, let us denote

$$\infty := (+\infty, +\infty),$$

and

$$\frac{\infty}{2} := (+\infty, 0).$$

Given an interval $I \subset (-\infty, +\infty] \times (-\infty, +\infty]$, we define $\mathrm{SS}(I) \subset \mathcal{C}$ to be the full subcategory of zero objects together with those one-dimensional sheaves whose Harder-Narasimhan quotients have μ_π-value in the interval I. If $a, b \in (-\infty, +\infty] \times (-\infty, +\infty]$ such that $a < b$, then we denote the closed interval between a and b by $a \leq \square \leq b$, and similarly for open, half-open, etc. intervals.

Lemma 4.4.

$$\mathcal{P}_\pi = \mathrm{SS}(\square \geq \frac{\infty}{2}),$$

and

$$\mathcal{Q}_\pi = \mathrm{SS}(\square < \frac{\infty}{2}).$$

Proof: First we will show that $\mathcal{P}_\pi \subset \mathrm{SS}(\square \geq \frac{\infty}{2})$.

Case 1: let $P \in \mathcal{P}_\pi$ be semi-stable. We will show that $P \in \mathrm{SS}(\square \geq \frac{\infty}{2})$. Since P is semi-stable, it suffices to show that $\mu_\pi(P) \geq \frac{\infty}{2}$. Now $P \in \mathcal{P}_\pi$ implies that $\chi(P) \geq 0$. By ampleness of L, we know $\beta_P \cdot L \geq 0$. Hence $\mu_\pi(P) \geq (+\infty, 0)$.

Case 2: Let $P \in \mathcal{P}_\pi$ be general, let the following be its Harder-Narasimhan (HN) filtration,

$$0 = P_0 \subset P_1 \subset \cdots \subset P_{N-1} \subset P_N = P,$$

and let $Q_i = \frac{P_i}{P_{i-1}}$ be the ith quotient; we must show that $\mu_\pi(Q_i) \geq \frac{\infty}{2}$. Since $P_N = P \in \mathcal{P}_\pi$ and $\mathcal{P}_\pi$ is closed under quotients, it follows that $Q_N \in \mathcal{P}_\pi$. By definition of the HN filtration, Q_N is semi-stable, hence $\mu_\pi(Q_N) \geq \frac{\infty}{2}$ and $Q_N \in \mathrm{SS}(\square \geq \frac{\infty}{2})$ by the previous case. Another defining property is that $\mu_\pi(Q_1) > \mu_\pi(Q_2) > \cdots > \mu_\pi(Q_N)$. Hence, $\mu_\pi(Q_i) \geq \frac{\infty}{2}$ for all i, in other words, $P \in \mathrm{SS}(\square \geq \frac{\infty}{2})$.

Now we will show that $\mathrm{SS}(\square \geq \frac{\infty}{2}) \subset \mathcal{P}_\pi$.

Case 1: Let $G \in \mathrm{SS}(\square \geq \frac{\infty}{2})$ be semi-stable. In this case, $G \in \mathrm{SS}(\square \geq \frac{\infty}{2})$ implies that $\mu_\pi(G) \geq \frac{\infty}{2}$. Since $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair, for every sheaf there exists a uniquely associated short exact sequence,

$$0 \to A \to G \to B \to 0$$

where $A \in \mathcal{P}_\pi$ and $B \in \mathcal{Q}_\pi$. By the above, we know that $A \in \mathrm{SS}(\square \geq \frac{\infty}{2})$. If A itself was semistable, then we could conclude immediately that $\mu_\pi(A) \geq \frac{\infty}{2}$. Using this argument, and inducting on the length of the HN filtration of A, we may prove that $\mu_\pi(Q_i) \geq \mu_\pi(A_i) \geq \mu_\pi Q_{i+1}$, where Q_i denotes the ith quotient object of the HN filtration of A, and A_i denotes the ith filtered object. This implies that $\mu_\pi(A) \geq \frac{\infty}{2}$ in general.

We know that $\mu_\pi(G) \geq \frac{\infty}{2} = (+\infty, 0)$ so $\chi(G) \geq 0$ by non-negativity of denominators in μ_π. Similarly, $\chi(A) \geq 0$. By the see-saw property and the semi-stability of G, we know that $\mu_\pi(A) \leq \mu_\pi(G) \leq \mu_\pi(B)$, hence $\mu_\pi(B) \geq \frac{\infty}{2}$ and as before we have $\chi(B) \geq 0$.

Notice that G must be supported on a fibre of π because $\mu_\pi(G) \in \{+\infty\} \times (0, +\infty]$. Hence B is also supported on a fibre. We claim that this forces $H^0(B) = 0$. For suppose there was a non-zero map $\mathcal{O}_Y \to B$. This would yield non-trivial $0 \to \mathcal{O}_C \to B$ where C is the support of the map $\mathcal{O}_Y \to B$. From the proof of proposition 2.11, we know that $R^1\pi_*\mathcal{O}_C = 0$, which implies that $\mathcal{O}_C \in \mathcal{P}_\pi$ which contradicts the definition of $\mathcal{Q}_\pi$. Hence $H^0(B) = 0$.

However, $\chi(B) \geq 0$ so $\dim H^1(B) \leq 0$. This implies that $H^1(B) = 0$, which implies that $R^1\pi_*B = 0$ (by the theorem of cohomology and base-change) and hence $B \in \mathcal{P}_\pi$. Since $B \in \mathcal{Q}_\pi$ we conclude that $B = 0$ and $G = A \in \mathcal{P}_\pi$.

Case 2: Let $G \in \mathrm{SS}(\square \geq \frac{\infty}{2})$ be general. We need to show that $G \in \mathcal{P}_\pi$. Let

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{N-1} \subset G_N = G$$

be the HN filtration of G, and let $Q_i = \frac{G_i}{G_{i-1}}$ denote the corresponding semistable quotients. By assumption, $\mu_\pi(Q_i) \geq \frac{\infty}{2}$; notice that $G_1 = Q_1$, so we have that G_1 is semistable and $\mu_\pi(G_1) \geq \frac{\infty}{2}$. The previous case then proves that for all i, $Q_i \in \mathcal{P}_\pi$, so in particular $Q_1 = G_1 \in \mathcal{P}_\pi$. Now consider the following short exact sequence

$$0 \to G_1 \to G_2 \to Q_2 \to 0$$

We know $G_1 \in \mathcal{P}_\pi$ and $Q_2 \in \mathcal{P}_\pi$ hence $G_2 \in \mathcal{P}_\pi$. Proceeding inductively, we conclude $G \in \mathcal{P}_\pi$.

This completes the proof that $\mathcal{P}_\pi = \mathrm{SS}(\square \geq \frac{\infty}{2})$.

Now we prove that $\mathcal{Q}_\pi = \mathrm{SS}(\square < \frac{\infty}{2})$. First, we will show that $\mathrm{SS}(\square < \frac{\infty}{2}) \subset \mathcal{Q}_\pi$.

Case 1: Let $G \in \mathrm{SS}(\square < \frac{\infty}{2})$ be semi-stable. There is a unique short

exact sequence associated to G

$$0 \to A \to G \to B \to 0$$

such that $A \in \mathcal{P}_\pi$ and $B \in \mathcal{Q}_\pi$. We will show that $A = 0$. Since G is semi-stable, either $A = 0$ or $\mu_\pi(A) \leq \mu_\pi(G)$. Suppose that $\mu_\pi(A) \leq \mu_\pi(G)$. Notice $\mu_\pi(G) < \frac{\infty}{2}$ so $\mu_\pi(A) < \frac{\infty}{2}$ in this case. However, the proof that $\mathcal{P}_\pi = \mathrm{SS}(\square \geq \frac{\infty}{2})$ tells us that if $A \in \mathcal{P}_\pi$ then either $A = 0$ or $\mu_\pi(A) \geq \frac{\infty}{2}$. Thus, $A = 0$.

Case 2: Let $G \in \mathrm{SS}(\square < \frac{\infty}{2})$ be general. As above, we have that each of the quotients associated to the HN filtration is an element of $\mathcal{Q}_\pi$. Since $\mathcal{Q}_\pi$ is closed under extension, we then get that each G_i is in $\mathcal{Q}_\pi$ hence $G \in \mathcal{Q}_\pi$.

Next, we will show that $\mathcal{Q}_\pi \subset \mathrm{SS}(\square < \frac{\infty}{2})$.

Case 1: Let $B \in \mathcal{Q}_\pi$ be semi-stable. Assume $B \neq 0$. We need to show that $\mu_\pi(B) < \frac{\infty}{2}$. This is equivalent to showing that either B is not supported on a fibre, or that $\chi(B) < 0$. Assume that B is supported on a fibre. As before, in this case $H^0(B) = 0$. Notice that if $H^1(B) = 0$ then $B \in \mathcal{P}_\pi$ and so $B = 0$. Since we have assumed $B \neq 0$, this forces us to assume $H^1(B) \neq 0$. Hence $\chi(B) < 0$ and $\mu_\pi(B) < \frac{\infty}{2}$.

Case 2: Let $B \in \mathcal{Q}_\pi$ be general. Let

$$0 = B_0 \subset B_1 \subset \cdots \subset B_{N-1} \subset B_N = B$$

be the HN filtration of B, and let Q_i be the associated quotients. By definition of the HN filtration, $B_1 = Q_1$ is semi-stable. Since $\mathcal{Q}_\pi$ is closed under taking sub-objects, we know that $B_1 \in \mathcal{Q}_\pi$. By the previous case, we know that $\mu_\pi(B_1) < \frac{\infty}{2}$. However, by definition of the HN filtration, we know that $\mu_\pi(Q_1) > \mu_\pi(Q_2) > \cdots > \mu_\pi(Q_N)$. Hence, $\mu_\pi(Q_i) < \frac{\infty}{2}$ for all i, i.e., $B \in \mathrm{SS}(\square < \frac{\infty}{2})$.

$\blacksquare$

Before moving on, let us collect a functorial property of the Harder-Narasimhan filtration (see [42]). It will tell us something about how it is

preserved by morphisms.

Let G be a coherent $\mathcal{O}_Y$-module, let $G_i \subset G$ denote the Harder-Narasimhan subsheaves of G. For $\alpha \in (-\infty, +\infty] \times (-\infty, +\infty]$, we define
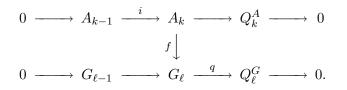
$$
G^{(\alpha)} = \begin{cases}
G_k & \text{if } \mu_\pi(Q_k^G) \geq \alpha > \mu_\pi(Q_{k-1}^G) \\
G & \text{if } \mu_\pi(Q_i^G) \geq \alpha \text{ for all } i \\
0 & \text{if } \mu_\pi(Q_i^G) < \alpha \text{ for all } i.
\end{cases}
$$

Lemma 4.5. If $f : A \to G$ is a map of sheaves on Y, then $f(A^{(\alpha)}) \subset G^{(\alpha)}$.

Proof: First, we prove by induction that if $f(A_k) \subset G_\ell$ and ℓ is minimal with this property, then $\mu_\pi(Q_\ell^G) \geq \mu_\pi(Q_k^A)$.

For $k = 0$, there is nothing to prove. For $k = 1$, suppose we have $f(A_1) \subset G_\ell$ where ℓ is minimal with respect to this property. Then we get a non-zero map $A_1 \to Q_\ell^G$. By lemma 3.5, we have that $\mu_\pi(A_1) \leq \mu_\pi(Q_\ell^G)$.

For general k, suppose we have $f(A_k) \subset G_\ell$, ℓ minimal. Consider the following short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_{k-1} & \overset{i}{\longrightarrow} & A_k & \longrightarrow & Q_k^A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & & & \\
0 & \longrightarrow & G_{\ell-1} & \longrightarrow & G_\ell & \overset{q}{\longrightarrow} & Q_\ell^G & \longrightarrow & 0.
\end{array}
$$

If the composition $q \circ f \circ i$ is zero, then we can define a non-zero map $Q_k^A \to Q_\ell^G$ which implies that $\mu_\pi(Q_k^A) \leq \mu_\pi(Q_\ell^G)$, as desired.

If the composition $q \circ f \circ i$ is non-zero, then we know that $f(A_{k-1}) \subset G_\ell$ and $f(A_{k-1}) \not\subset G_\ell$; by induction, we conclude that $\mu_\pi(Q_\ell^G) \geq \mu_\pi(Q_{k-1}^A) > \mu_\pi(Q_k^A)$, as desired.

So: $f(A_k) \subset G_\ell$, where ℓ is minimal with this property, implies that $\mu_\pi(Q_\ell^G) \geq \mu_\pi(Q_k^A)$. We will finish proving the lemma using this fact.

Fix an $\alpha \in (-\infty, +\infty] \times (-\infty, +\infty]$. By definition, $A^{(\alpha)} = A_k$ for all indices k such that $(Q_k^A) \geq \alpha > \mu_\pi(Q_{k-1}^A)$. Fix such an index k. Choose the ℓ minimal such that $f(A_k) \subset G_\ell$. Notice that we now have $\mu_\pi(Q_\ell^G) \geq \mu_\pi(Q_k^A) \geq \alpha$, and thus $G_\ell \subset G^{(\alpha)}$ by definition. ∎

The following lemma will be useful later.

Lemma 4.6. $SS(a \leq \square \leq b)$ is closed under extension.

Proof: Let A and $B \in SS(a \leq \square \leq b)$. Consider

$$0 \to A \xrightarrow{i} G \xrightarrow{j} B \to 0.$$

First we show that $G \in SS(a \leq \square)$. We know that $A = A^{(a)}$, so $i(A^{(a)}) = i(A) \subset G^{(a)}$. If we choose k minimal such that $\mu_\pi(Q_k^G) < a$, then we also know that $G^{(a)} = G_{k-1}$. Notice that the possibilities for $\frac{G}{G_{k-1}}$ are limited: either it is the zero sheaf, or it is an element of $SS(\square < a)$.

The inclusion $i(A) \subset G_{k-1}$ implies that $B = \frac{G}{i(A)}$ maps onto $\frac{G}{G_{k-1}}$; denote the quotient map by ρ. Now consider the chain of inclusions and identities

$$\rho(B^{(a)}) \subset \left(\frac{G}{G_{k-1}}\right)^{(a)} \subset \frac{G}{G_{k-1}} = \rho(B);$$

by assumption, $\rho(B^{(a)}) = \rho(B)$, so all of the inclusions are actually equalities. In particular,

$$\left(\frac{G}{G_{k-1}}\right)^{(a)} = \frac{G}{G_{k-1}},$$

which means either $\frac{G}{G_{k-1}}$ is zero, or it is an element of $SS(a \leq \square)$. By the above, we see that the only possibility is $\frac{G}{G_{k-1}} = 0$ which means that $G = G^{(a)}$, i.e. $G \in SS(a \leq \square)$.

Now we prove that $G \in SS(\square \leq b)$. Let

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{N-1} \subset G_N = G$$

denote the Harder-Narasimhan filtration of G, and let Q_i be the associated quotients. Since $G_1 = Q_1$ and $\mu_\pi(Q_1) > \mu_\pi(Q_i)$ for all $i > 1$, it suffices to prove that $\mu_\pi(Q_1) \leq b$. Consider the short-exact sequence

$$0 \to A \cap G_1 \to G_1 \to j(G_1) \to 0.$$

Since the quotients associated to the Harder-Narasimhan filtration are semi-

stable, and since $G_1 = Q_1$, we know that G_1 is semi-stable. This implies that $\mu_\pi(G_1) \le \mu_\pi(j(G_1))$, hence it suffices to prove that $\mu_\pi(j(G_1)) \le b$. Let B' be any subsheaf of B; we will show that $\mu_\pi(B') \le b$. We know that for any $c > b$,

$$B'^{(c)} \subset B^{(c)} = 0,$$

which implies that for all $c > b$, $B'^{(c)} = 0$. Hence $B' \in \text{SS}(\square \le b)$ and $\mu_\pi(B') \le b$. This allows us to conclude that $\mu_\pi(G_1) \le b$.

■

Chapter 5

The motivic Hall algebra

Here we provide a quick summary of the constructions and results of Bridgeland's papers [11], [12] (which came into existence as a gentle introduction to part of Joyce's theory of motivic Hall algebras [24], [25], [26], [27]).

Let S be a stack, locally of finite type over $\mathbb{C}$ and with affine stabilizers.

Definition 5.1. The *relative Grothendieck group $K(\mathrm{St}\,/S)$ of stacks over S* is the $\mathbb{Q}$-vector space spanned by symbols

$$[T \xrightarrow{m} S]$$

(where T is a finite-type stack and m is a morphism), subject to the following relations.

a) $[T \to S] = [U \to S] + [F \to S]$ where U is an open substack of T and F is the corresponding closed complement.

b) $[T_1 \xrightarrow{s \circ f} S] = [T_2 \xrightarrow{s \circ g} S]$, if $T_1 \xrightarrow{f} B$ and $T_2 \xrightarrow{g} B$ are Zariski fibrations[1] over B with identical fibres, and $B \xrightarrow{s} S$ is a morphism of stacks.

c) $[T \xrightarrow{a} S] = [T' \xrightarrow{b} S]$ if there exists a commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\ c\ } & T' \\
& \searrow^{a} \quad \swarrow_{b} & \\
& S &
\end{array}
$$

such that the associated map on $\mathbb{C}$-points $T(\mathbb{C}) \xrightarrow{c} T'(\mathbb{C})$ is an equivalence of categories.

---

[1] a Zariski-local product space

The vector space $K(\mathrm{St}\,/S)$ is a $K(\mathrm{St}\,/\mathrm{Spec}\,\mathbb{C})$-module, whose action we now describe. Let $\big[A \to \mathrm{Spec}\,\mathbb{C}\big] = \big[A\big] \in K(\mathrm{St}\,/\mathrm{Spec}\,\mathbb{C})$, and let $\big[T \xrightarrow{m} S\big] \in K(\mathrm{St}\,/S)$. Then we define

$$\big[A\big] \cdot \big[T \xrightarrow{m} S\big] = \big[A \times_{\mathrm{Spec}\,\mathbb{C}} T \xrightarrow{f} S\big],$$

where f is the composition of the projection $A \times_{\mathrm{Spec}\,\mathbb{C}} T \to T$ and the map $T \to S$.

We are most interested in the case where $S = \mathcal{C}$, the stack of coherent sheaves on Y supported in dimension one or less. We denote this stack by $\mathcal{M}$, and we denote $K(\mathrm{St}\,/\mathcal{M})$ by $\mathrm{H}(\mathcal{C})$. The vector space $\mathrm{H}(\mathcal{C})$ is the motivic Hall algebra; let us justify the name by endowing it with the structure of an algebra. First, we define $\mathcal{M}^{(2)}$ to be the stack of short-exact sequence of sheaves on $\mathcal{M}$. Now, given $\big[A \to \mathcal{M}\big]$ and $\big[B \to \mathcal{M}\big]$ we define the convolution product $\big[A \to \mathcal{M}\big] * \big[B \to \mathcal{M}\big]$ to be $\big[Z \to \mathcal{M}\big]$ where $Z \xrightarrow{c \circ g} \mathcal{M}$ is defined by the following Cartesian diagram:

$$
\begin{array}{ccccc}
Z & \xrightarrow{g} & \mathcal{M}^{(2)} & \xrightarrow{c} & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle (l,r)} & & \\
A \times B & \longrightarrow & \mathcal{M} \times \mathcal{M}. & &
\end{array}
$$

The morphisms l, r are the "left hand" and "right hand" morphisms, which project a short exact sequence to its left-most (resp. right-most) non-zero entry. The morphism c is the "centre" morphism. Intuitively, given families of sheaves $A \to \mathcal{M}$ and $B \to \mathcal{M}$ their product in the Hall algebra is the family $Z \to \mathcal{M}$ parametrizing extensions of objects of B by objects of A.

The motivic Hall algebra is useful tool. It holds enough information to allow us to retrieve Euler characteristics, yet is flexible enough to produce decompositions of elements in terms of extensions. We will describe an "integration" map on $\mathrm{H}(\mathcal{C})$ taking values in the ring of polynomials. Equations among elements of $\mathrm{H}(\mathcal{C})$ will be integrated to yield equations of polynomials. This entire framework will then be souped-up to incorporate Laurent series, and our theorem will be the result of applying the souped-up integration

map to equations in the souped-up Hall algebra.

In [11], Bridgeland introduces *regular elements.* Let $K(\mathrm{Var}/\mathbb{C})$ denote the *relative Grothendieck group of varieties over* $\mathbb{C}$ (cf. definition 5.1). Let $\mathbb{L}$ denote the element $[\mathbb{A}^1 \to \mathbb{C}]$, the Tate motive. Consider the maps of commutative rings,

$$K(\mathrm{Var}/\mathbb{C}) \to K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}] \to K(\mathrm{St}/\mathbb{C}),$$

and recall from [11] that $\mathrm{H}(\mathcal{C})$ is algebra over $K(\mathrm{St}/\mathbb{C})$. Define a $K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-module

$$\mathrm{H}_{\mathrm{reg}}(\mathcal{C}) \subset \mathrm{H}(\mathcal{C})$$

to be the span of classes of maps $[V \xrightarrow{f} \mathcal{M}]$ with V a variety. We call an element of $\mathrm{H}(\mathcal{C})$ *regular* if it lies in this submodule. The following result is theorem 5.1 of [11].

Theorem 5.2. The submodule of regular elements is closed under the convolution product:

$$\mathrm{H}_{\mathrm{reg}}(\mathcal{C}) * \mathrm{H}_{\mathrm{reg}}(\mathcal{C}) \subset \mathrm{H}_{\mathrm{reg}}(\mathcal{C}),$$

and is therefore a $K(\mathrm{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-algebra. Moreover the quotient

$$\mathrm{H}_{\mathrm{sc}}(\mathcal{C}) = \mathrm{H}_{\mathrm{reg}}(\mathcal{C})/(\mathbb{L} - 1)\,\mathrm{H}_{\mathrm{reg}}(\mathcal{C}),$$

is a commutative $K(\mathrm{Var}/\mathbb{C})$-algebra.

Bridgeland equips $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})$ with a Poisson bracket, defined by

$$\{f, g\} = \frac{f * g - g * f}{\mathbb{L} - 1}.$$

The integration map I is defined on $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})$. Now we work toward the polynomial ring in which it takes values.

Recall that $K(Y)$ is the numerical K-theory of Y. Recall $\Delta \subset F_1 K(Y)$ is the effective cone of $F_1 K(Y)$, that is, the collection of elements of the form $[F]$ where F is a one-dimensional sheaf. Define a ring $\mathbb{C}[\Gamma]$ to be the vector

space spanned by symbols x^α for $\alpha \in \Delta$ and defining the multiplication by

$$x^\alpha \cdot x^\beta = x^{\alpha+\beta}.$$

We equip $\mathbb{C}[\Delta]$ with the trivial Poisson bracket. We are now ready for the following theorem.

Theorem 5.3 (5.1 of [12]). There exists a Poisson algebra homomorphism

$$I : \mathrm{H}_{\mathrm{sc}}(\mathcal{C}) \to \mathbb{C}[\Delta]$$

such that

$$I\big(\big[Z \xrightarrow{f} \mathcal{M}_\alpha\big]\big) = \chi(Z, f^*(\nu))x^\alpha,$$

where $\nu : \mathcal{M} \to \mathbb{Z}$ is Behrend's microlocal function of $\mathcal{M}$, and $\mathcal{M}_\alpha$ denotes the component of $\mathcal{M}$ with fixed Chern character α.

Chapter 6

Equations in the infinite-type Hall algebra and the fake proof

For the sake of exposition only, we follow [12] and [17] by introducing an infinite-type version of the Hall algebra. This has the benefit of allowing non-finite-type stacks, but the devastating draw-back of not admitting an integration map. We use it because it will allow us to temporarily work without having to think about convergence of power series. Also, many of the arguments will be used again later. We end this chapter with a fake proof of our main result. It is our hope that this fake proof helps the reader to navigate the true one in the following chapter.

The *infinite-type Hall algebra* is defined by considering symbols as in definition 5.1, but with T assumed only to be *locally* of finite type over $\mathbb{C}$, and use relations as before, except that we do not use relation (a). (Admitting relation (a) in this case would make every infinite-type Hall algebra trivial). We denote it by $\mathrm{H}_\infty(\mathcal{C})$.

Given a substack $\mathcal{N} \subset \mathcal{M}$, we let

$$1_\mathcal{N} = [\mathcal{N} \xrightarrow{i} \mathcal{M}]$$

denote the inclusion $i : \mathcal{N} \to \mathcal{M}$. Pulling back the morphism (2.13) to $\mathcal{N} \subset \mathcal{M}$ gives a stack denoted $\mathcal{N}(\mathcal{O})$ with a morphism $\mathcal{N}(\mathcal{O}) \xrightarrow{q} \mathcal{N}$, and hence an element

$$1_\mathcal{N}^\mathcal{O} = [\mathcal{N}(\mathcal{O}) \xrightarrow{q} \mathcal{M}] \in \mathrm{H}_\infty(\mathcal{C}).$$

For example, $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ are full subcategories of $\mathcal{C}$, and define substacks of $\mathcal{M}$ (see lemma 2.16), which we abusively denote with the same letters, $\mathcal{P}_\pi, \mathcal{Q}_\pi \subset \mathcal{M}$. These substacks define elements of the infinite-type Hall algebra,

$$1_{\mathcal{P}_\pi}, 1_{\mathcal{Q}_\pi} \in \mathrm{H}_\infty(\mathcal{C}).$$

Other examples include

$$\mathcal{H} = [\mathrm{Hilb}(Y) \to \mathcal{M}],$$

$$\mathcal{H}_{exc} = [\mathrm{Hilb}_{\mathrm{exc}} \to \mathcal{M}],$$

and

$$\mathcal{H}^\pi = [\pi\text{-}\mathrm{Hilb}(Y) \to \mathcal{M}] \in \mathrm{H}_\infty(\mathcal{C}),$$

where $\mathrm{Hilb}_{\mathrm{exc}}$ denotes the Hilbert scheme of curves supported on fibres of π, and the map to $\mathcal{M}$ is given by taking $\mathcal{O}_Y \to G$ to G. Note that all Hilbert schemes are restricted to the components parametrizing sheaves G of dimension one.

Lemma 6.1.

$$1_{\mathcal{C}} = 1_{\mathcal{P}_\pi} * 1_{\mathcal{Q}_\pi}$$

This lemma reflects the fact that $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair.

Proof: Form the following Cartesian diagram:

$$
\begin{array}{ccccc}
Z & \xrightarrow{\ f\ } & \mathcal{M}^{(2)} & \xrightarrow{\ b\ } & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle (a_1, a_2)} & & \\
\mathcal{P}_\pi \times \mathcal{Q}_\pi & \xrightarrow{\ i\ } & \mathcal{M} \times \mathcal{M}. & &
\end{array}
\qquad (6.2)
$$

By lemma $A.1$ [11], the groupoid of T-valued points of Z can be described as follows. The objects are short exact sequences of T-flat sheaves on $T \times Y$ of the form

$$0 \to A \to G \to B \to 0$$

such that A and B define families of sheaves on Y lying in the subcategories $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ respectively. The morphisms are isomorphisms of short exact

sequences. The composition,

$$g = b \circ f : Z \to \mathcal{M}$$

sends a short exact sequence to the object G. Since $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ are sub-categories of $\mathcal{C}$, it follows that the composition factors through $\mathcal{C}$. This morphism induces an equivalence on $\mathbb{C}$-valued points because of the torsion pair property: every object G of $\mathcal{C}$ fits into a unique short exact sequence of the form (6.2). Thus, the identity follows from the relations in the infinite Hall algebra. ∎

We will need a framed version of the previous lemma.

Lemma 6.3.
$$1_\mathcal{C}^\mathcal{O} = 1_{\mathcal{P}_\pi}^\mathcal{O} * 1_{\mathcal{Q}_\pi}^\mathcal{O}.$$

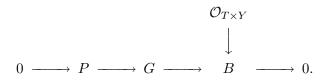**Proof:** Form the following Cartesian diagram:

$$
\begin{array}{ccccccc}
U & \xrightarrow{\ \ p\ \ } & V & \xrightarrow{\ \ j\ \ } & \mathcal{M}^{(2)} & \xrightarrow{\ \ b\ \ } & \mathcal{M} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (a_1,a_2)} & & \\
\mathcal{P}_\pi(\mathcal{O}) \times \mathcal{Q}_\pi(\mathcal{O}) & \xrightarrow{q \times \mathrm{id}} & \mathcal{P}_\pi \times \mathcal{Q}_\pi(\mathcal{O}) & \longrightarrow & \mathcal{M} \times \mathcal{M}. & &
\end{array}
$$

Then $1_{\mathcal{P}_\pi}^\mathcal{O} * 1_{\mathcal{Q}_\pi}^\mathcal{O}$ is represented by the composite map $b \circ j \circ p : U \to \mathcal{M}$. Note that, by lemma 2.5 of [12], the map $\mathcal{P}_\pi(\mathcal{O}) \to \mathcal{P}_\pi$ is a Zariski fibration with fibre $H^0(P)$ over a point $P \in \mathcal{P}_\pi$. By pullback, the same is true of the morphism p.

The groupoid of T-valued points of V can be described as follows. The objects are short exact sequences of T-flat sheaves on $T \times Y$ of the form

$$0 \to P \to G \to B \to 0$$

such that P and B define flat families of objects in $\mathcal{P}_\pi$ and $\mathcal{Q}_\pi$ respectively, together with a map $\mathcal{O}_{T \times Y} \to B$. We represent the objects of V as diagrams

of the form:

$$\mathcal{O}_{T \times Y}$$
$$\downarrow$$
$$0 \longrightarrow P \longrightarrow G \longrightarrow B \longrightarrow 0.$$

Consider the stack Z from lemma 6.1 with its map $Z \to \mathcal{M}$. Form the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\;h\;} & \mathcal{C}(\mathcal{O}) \\
\downarrow & & {\scriptstyle q}\downarrow \\
Z & \xrightarrow{\;g\;} & \mathcal{C}.
\end{array}
$$

Since g induces an equivalence on $\mathbb{C}$-valued points, so does h, so that the element $1_{\mathcal{C}}^{\mathcal{O}}$ can be represented by the map $q \circ h$.

We represent the objects of W as diagrams of the form:

$$\mathcal{O}_{T \times Y}$$
$$\downarrow{\scriptstyle \delta}$$
$$0 \longrightarrow P \longrightarrow G \xrightarrow{\;\beta\;} B \longrightarrow 0.$$

Setting $\gamma = \beta \circ \delta$ defines a map of stacks $W \to V$, which is a Zariski fibration with fibre an affine model of the vector space $H^0(P)$ over a sheaf P.

Now, since $H^1(P) = 0$, $U \to V$ is a Zariski fibration with fibre $H^0(P)$, hence they represent the same element of the Hall algebra, namely $\mathcal{Q}_\pi$. $\blacksquare$

Lemma 6.4.
$$1_{\mathcal{C}}^{\mathcal{O}} = \mathcal{H} * 1_{\mathcal{C}}$$

This is lemma 4.3 of [12]. Intuitively, this amounts to the fact that every map $\mathcal{O} \xrightarrow{\gamma} G$ factors uniquely into a surjection $\mathcal{O} \to \mathrm{im}(\gamma)$ and an inclusion $\mathrm{im}(\gamma) \to G$. The following lemma is a restriction of the previous to the substack $\mathcal{P}_\pi$.

Lemma 6.5.
$$1_{\mathcal{P}_\pi}^{\mathcal{O}} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi}$$

Proof: Form the Cartesian diagram:

$$
\begin{array}{ccccc}
Z & \longrightarrow & \mathcal{M}^{(2)} & \xrightarrow{\ b\ } & \mathcal{M} \\
\big\downarrow & & \big\downarrow{\scriptstyle (a_1,a_2)} & & \\
\mathcal{H}_{exc} \times \mathcal{P}_\pi & \xrightarrow{\ i\ } & \mathcal{M} \times \mathcal{M}. & &
\end{array}
$$

The groupoid of T-valued points of Z may be described as follows. The objects are short exact sequences of T-flat sheaves of $T \times Y$

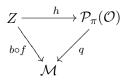$$0 \to A \to G \to B \to 0$$

such that $B \in \mathcal{P}_\pi$, and A is supported on exceptional fibres, together with an epimorphism $\mathcal{O}_Y \to A$. We can represent these objects as diagrams of the form

$$
\begin{array}{c}
\mathcal{O}_{T \times Y} \\
\big\downarrow{\scriptstyle \gamma} \\
0 \longrightarrow A \xrightarrow{\ \alpha\ } G \xrightarrow{\ \beta\ } B \longrightarrow 0.
\end{array}
$$

Since $\mathcal{O}_Y \to A$ is an epimorphism, we know that A is of the form $\mathcal{O}_C$ for some one-dimensional subscheme C of Y. By the proof of proposition 2.11, we know that $R\pi_* \mathcal{O}_C = 0$. Since $A = \mathcal{O}_C$ has exceptional support, it follows that $\pi_* A$ is a zero-dimensional sheaf, hence $A \in \mathcal{P}_\pi$. Since $B \in \mathcal{P}_\pi$ by design, and since $\mathcal{P}_\pi$ is closed under taking extensions, we conclude that in any such short exact sequence, $G \in \mathcal{P}_\pi$. There is a map $h : Z \to \mathcal{P}_\pi(\mathcal{O})$ sending the above diagram to the composite map

$$\delta = \alpha \circ \gamma : \mathcal{O}_{T \times Y} \to G.$$

This morphism h fits into a commuting diagram of stacks

$$
\begin{array}{ccc}
Z & \xrightarrow{\ h\ } & \mathcal{P}_\pi(\mathcal{O}) \\
 & {\scriptstyle b \circ f} \searrow \quad \swarrow {\scriptstyle q} & \\
 & \mathcal{M} &
\end{array}
$$

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We argue that the map h then induces an equivalence on $\mathbb{C}$-valued points. Suppose $\mathcal{O}_{T \times Y} \xrightarrow{\delta} G$ is an arbitrary map of sheaves, with G defining a family of sheaves in $\mathcal{P}_\pi$. Then we get the following diagram:

$$\mathcal{O}_{T \times Y}$$
$$\downarrow \gamma$$
$$0 \longrightarrow \mathrm{im}(\delta) \longrightarrow G \longrightarrow \mathrm{coker}(\delta) \longrightarrow 0.$$

Since $G \in \mathcal{P}_\pi$ we know that the one-dimensional component of its support is exceptional, hence $\mathrm{im}(\delta)$ is also exceptional, so that $\mathcal{O}_{T \times Y} \to \mathrm{im}(\delta)$ defines a family of objects in $\mathrm{Hilb}_{\mathrm{exc}}$. As well, we know that $\mathcal{P}_\pi$ is closed under taking quotients, so $\mathrm{coker}(\delta)$ is in $\mathcal{P}_\pi$. This completes the proof. $\blacksquare$

Morally, the next lemma is similar to lemma 6.4 since $\mathcal{H}^\pi$ may be thought of as the surjections $\mathcal{O}_Y \to G$ in a tilt of the abelian category generated by $\mathcal{O}$ and $\mathcal{C}$. We provide a direct proof since we have not constructed this tilt.

Lemma 6.6.
$$1_{\mathcal{Q}_\pi}^{\mathcal{O}} = \mathcal{H}^\pi * 1_{\mathcal{Q}_\pi}.$$

Proof: Form the following Cartesian diagram:

$$
\begin{array}{ccccc}
Z & \xrightarrow{\ f\ } & \mathcal{M}^{(2)} & \xrightarrow{\ b\ } & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle (a_1,a_2)} & & \\
\mathcal{H}^\pi \times \mathcal{Q}_\pi & \xrightarrow{\ i\ } & \mathcal{M} \times \mathcal{M} & &
\end{array}
$$

The groupoid of T-valued points of Z is described as follows. The objects are short exact sequences of T-flat sheaves on $T \times Y$

$$0 \to A \to G \to B \to 0$$

with the property that $G \in \mathcal{Q}_\pi$, together with a map $\mathcal{O}_{T \times Y} \to A$ that pulls back to a π-stable pair $\mathcal{O}_Y \to A_t$ for every $t \in T$. We can represent these objects as diagrams of the form:

$$\mathcal{O}_{T \times Y} \tag{6.7}$$

$$\downarrow \gamma$$

$$0 \longrightarrow A \overset{\alpha}{\longrightarrow} G \overset{\beta}{\longrightarrow} B \longrightarrow 0.$$

Since A and B are objects of $\mathcal{Q}_\pi$, and $\mathcal{Q}_\pi$ is closed under extensions, we conclude $G \in \mathcal{Q}_\pi$. Thus, there is a map $h : Z \to \mathcal{Q}_\pi(\mathcal{O})$ sending the above diagram to the composite map

$$\delta = \alpha \circ \gamma : \mathcal{O}_{T \times Y} \to G.$$

This map h fits into a commuting diagram of stacks

$$Z \overset{h}{\longrightarrow} \mathcal{Q}_\pi(\mathcal{O}) \tag{6.8}$$

$$b \circ f \searrow \quad \swarrow q$$

$$\mathcal{M}$$

The map h then induces an equivalence on $\mathbb{C}$-valued points because of the following argument. Let $\mathcal{O}_Y \overset{\sigma}{\to} G$ be an arbitrary map, with $G \in \mathcal{Q}_\pi$. We need to produce a diagram,

$$\mathcal{O}_Y$$

$$\downarrow \gamma$$

$$0 \longrightarrow A \overset{\alpha}{\longrightarrow} G \overset{\beta}{\longrightarrow} B \longrightarrow 0,$$

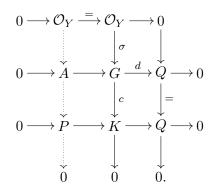with $\mathcal{O}_Y \to A$ a π-stable pair, $B \in \mathcal{Q}_\pi$, and $\alpha \circ \gamma = \sigma$.

Consider the cokernel K of σ. Since $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$ is a torsion pair, we know that K fits into a short exact sequence

$$0 \to P \to K \to Q \to 0$$

where $P \in \mathcal{P}_\pi$ and $Q \in \mathcal{Q}_\pi$. Let $G \overset{c}{\to} K$ be the canonical map from G to the cokernel K of σ, and let $G \overset{d}{\to} Q$ be the composition of $G \to K$ and

$K \to Q$. Define A to be the kernel of d. Consider the following diagram.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_Y & \overset{=}{\longrightarrow} & \mathcal{O}_Y & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow \\
0 & \longrightarrow & A & \longrightarrow & G & \overset{d}{\longrightarrow} & Q & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle =} \\
0 & \longrightarrow & P & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
& & 0 & & 0 & & 0.
\end{array}
$$

We know that $Q \in \mathcal{Q}_\pi$, and a diagram chase proves that the dotted vertical morphisms exist and that P is the cokernel of $\mathcal{O}_Y \to A$. The sheaf A is a subsheaf of $G \in \mathcal{Q}_\pi$, and $\mathcal{Q}_\pi$ is closed under taking subsheaves, so $A \in \mathcal{Q}_\pi$. This proves that $\mathcal{O}_Y \to A$ is a π-stable pair. This proves that the map h is surjective on $\mathbb{C}$-valued points. It remains to show that h is injective on $\mathbb{C}$-valued points.

Now let $\gamma : \mathcal{O}_Y \to G$ be an arbitrary map with $G \in \mathcal{Q}_\pi$. Since $\mathcal{Q}_\pi$ is closed under subobjects, the image of γ is an object of $\mathcal{Q}_\pi$, $\mathrm{im}(\gamma) \in \mathcal{Q}_\pi$. Now we can use the description of $\mathcal{Q}_\pi$ in terms of stability to show that $\frac{G}{\mathrm{im}(\gamma)}$ is also in $\mathcal{Q}_\pi$. This proves that the map h is injective on $\mathbb{C}$-valued points. This completes the proof.

∎

We end this section by giving a fake proof of theorem 1.6 that depends on a fake integration map. In fact, no such integration map is known to exist, but if there was one, the proof our theorem would be simpler. As it stands, we have a chapter dedicated to convergence issues to get around the fact that no such integration map exists on the infinite type Hall algebra. It is our hope that this fake proof will make the true one easier to follow.

Fake proof: By lemma 6.6, we have

$$
\mathcal{H}^\pi * 1_{\mathcal{Q}_\pi} = 1_{\mathcal{Q}_\pi}^{\mathcal{O}}. \tag{6.9}
$$

Lemma 4.3 of [12] proves

$$\mathcal{H} * 1_{\mathcal{C}} = 1_{\mathcal{C}}^{\mathcal{O}}.$$

Using lemma 6.1, we may rewrite $\mathcal{H} * 1_{\mathcal{C}}$:

$$\mathcal{H} * 1_{\mathcal{C}} = \mathcal{H} * 1_{\mathcal{P}_{\pi}} * 1_{\mathcal{Q}_{\pi}}.$$

Lemma 6.3 allows us to write

$$1_{\mathcal{C}}^{\mathcal{O}} = 1_{\mathcal{P}_{\pi}}^{\mathcal{O}} * 1_{\mathcal{Q}_{\pi}}^{\mathcal{O}}.$$

Putting these together yields

$$\mathcal{H} * 1_{\mathcal{P}_{\pi}} * 1_{\mathcal{Q}_{\pi}} = 1_{\mathcal{P}_{\pi}}^{\mathcal{O}} * 1_{\mathcal{Q}_{\pi}}^{\mathcal{O}}.$$

Applying lemma 6.5

$$\mathcal{H} * 1_{\mathcal{P}_{\pi}} * 1_{\mathcal{Q}_{\pi}} = \mathcal{H}_{exc} * 1_{\mathcal{P}_{\pi}} * 1_{\mathcal{Q}_{\pi}}^{\mathcal{O}}.$$

Using equation 6.9, we get

$$\mathcal{H} * 1_{\mathcal{P}_{\pi}} * 1_{\mathcal{Q}_{\pi}} = \mathcal{H}_{exc} * 1_{\mathcal{P}_{\pi}} * \mathcal{H}^{\pi} * 1_{\mathcal{Q}_{\pi}}.$$

Now, for reasons we will explain in the next section, $1_{\mathcal{P}_{\pi}}$ and $1_{\mathcal{Q}_{\pi}}$ are invertible in the Hall algebra. We may therefore cancel the copies of $1_{\mathcal{Q}_{\pi}}$ and isolate $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_{exc} * 1_{\mathcal{P}_{\pi}} * \mathcal{H}^{\pi} * 1_{\mathcal{P}_{\pi}}^{-1}.$$

The elements $\mathcal{H}, \mathcal{H}_{exc}, \mathcal{H}^{\pi}$ all lie in the subalgebra $\mathrm{H}_{\mathrm{reg}}(\mathcal{C})$ since they are represented by schemes. As we will see in the next section, conjugation by $1_{\mathcal{P}_{\pi}}$ induces a Poisson homomorphism of $\mathrm{H}_{\mathrm{reg}}(\mathcal{C})$ of the form: identity + terms expressed in the Poisson bracket. Since the Poisson bracket of the polynomial ring is trivial, these terms vanish when we apply the fake integration map, and we are left with

$$I(\mathcal{H}) = I(\mathcal{H}_{exc}) \cdot I(\mathcal{H}^{\pi}).$$

Up to signs arising from lemma 2.18, the "polynomials" $I(\mathcal{H})$, $I(\mathcal{H}_{exc})$, and $I(\mathcal{H}^\pi)$ are the generating series of $\mathrm{DT}(Y)$, $\mathrm{DT}_{exc}(Y)$, and π-$\mathrm{PT}(Y)$, respectively, and we see that the above equation is the formula claimed in theorem 1.6.

or

$$\frac{I(\mathcal{H})}{I(\mathcal{H}_{exc})} = I(\mathcal{H}^\pi)$$

which is exactly the statement of our theorem.   ⊔

The true proof will follow precisely these steps, fully justified, and with the appropriate convergence arguments. The next chapter describes the Laurent Hall algebra, which does have an integration map.

Chapter 7

Equations in the Laurent Hall algebra and the true proof

Laurent subsets

In this section, we will formally modify the algebra $H(\mathcal{C})$ and its integration map so that the modified integration map on the modified algebra takes valued in power series. This section is a summary of sections 5.2 and 5.3 of [12].

Definition 7.1. A subset $S \subset \Delta$ is *Laurent* if for all $\beta \in N_1(Y)$, the collection of elements of the form $(\beta, n) \in S$ is such that n is bounded below.

Let Φ denote the set of all Laurent subsets. It has the following properties:

1. if $S, T \in \Phi$ then so it $S + T = \{\alpha + \beta : \alpha \in S, \beta \in T\}$

2. if $S, T \in \Phi$, and $\alpha \in \Delta$ then there are only finitely many ways to write $\alpha = \beta + \gamma$ such that $\beta \in S$ and $\gamma \in T$.

Given a ring A graded by Δ, $A = \bigoplus_{\gamma \in \Delta} A_\gamma$, we can use the Laurent subset to define a new algebra, which we will denote A_Φ. Elements of A_Φ are of the form

$$a = \sum_{\gamma \in S} a_\gamma$$

where $S \in \Phi$, and $a_\gamma \in A_\gamma \subset A$. Given an element $a \in A_\Phi$ as above, we define $\rho_\gamma(a) = a_\gamma \in A$. (Here, our notation differs from [12], since we are using the symbol π for the map $\pi : Y \to X$.) The projection operator ρ allows us to define a product $*$ on A_Φ by:

$$\rho_\gamma(a * b) = \sum_{\gamma = \alpha + \beta} \rho_\alpha(a) * \rho_\beta(b).$$

A_Φ admits a natural topology that may be identified by declaring a sequence $(a_j)_{j \in \mathbb{N}} \subset A_\Phi$ to be convergent if for any $(\beta, n) \in \Delta$, there exists an integer K such that for all $m < n$

$$i, j > K \Rightarrow \rho_{(\beta, m)}(a_i) = \rho_{(\beta, m)}(a_j).$$

Lemma 7.2. If A is a $\mathbb{C}$-algebra and $a \in A_\Phi$ satisfies $\rho_0(a) = 0$ then any series

$$\sum_{j \geq 1} c_j a^j$$

with coefficients $c_j \in \mathbb{C}$ is convergent in the topological ring A_Φ.

See Lemma 5.3 of [12] for a proof.

Given two Δ-graded algebras A and B, and a morphism $f : A \to B$ that preserves the Δ-grading, we get an induced continuous map

$$f_\Phi : A_\Phi \to B_\Phi$$

by defining

$$\rho_\gamma(f_\Phi(a)) = f(\rho_\gamma(a)).$$

Applying this process to the map of Δ-graded algebras $I : \mathrm{H_{sc}}(\mathcal{C}) \to \mathbb{C}[\Delta]$ yields a continuous map $I_\Phi : \mathrm{H_{sc}}(\mathcal{C})_\Phi \to \mathbb{C}[\Delta]_\Phi$. We call $\mathrm{H_{sc}}(\mathcal{C})_\Phi$ the *Laurent Hall algebra*; it, too, is equipped with a Poisson bracket.

Definition 7.3. A morphism of stacks $f : W \to \mathcal{C}$ is Φ-*finite* if

a) $W_\alpha = f^{-1}(\mathcal{C}_\alpha)$ is of finite type for all $\alpha \in \Delta$, and

b) there is a Laurent subset $S \subset \Delta$ such that W_α is empty unless $\alpha \in S$.

A Φ-finite morphism of stacks $f : W \to \mathcal{C}$ defines an element of $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})_\Phi$ by the formal sum

$$\sum_{\alpha \in S} [W_\alpha \xrightarrow{f} \mathcal{C}].$$

In [12], it is shown that $[\mathrm{Hilb} \to \mathcal{M}]$ is Φ-finite, and hence defines an element in $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})_\Phi$.

Lemma 7.4. The maps

$$\mathrm{Hilb}^\pi \to \mathcal{C}, \qquad \mathrm{Hilb} \to \mathcal{C}, \qquad \mathrm{Hilb}_{exc} \to \mathcal{C},$$

are Φ-finite. The corresponding elements $\mathcal{H}^\pi, \mathcal{H}$ and $\mathcal{H}_{exc}$ of $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})_\Phi$ satisfy

$$I_\Phi(\mathcal{H}^\pi) = \sum_{(\beta,n) \in \Delta} (-1)^n \, \pi\text{-PT}(\beta, n) x^\beta q^n = \pi\text{-PT}(Y)(x, -q),$$

where we have written $x^\beta = q^{(\beta,0)}$ and $q = q^{(0,1)}$. Similarly,

$$I_\Phi(\mathcal{H}) = \sum_{(\beta,n) \in \Delta} (-1)^n \, \mathrm{DT}(\beta, n) x^\beta q^n = \mathrm{DT}(Y)(x, -q)$$

$$I_\Phi(\mathcal{H}_{exc}) = \sum_{(\beta,n) \in \Delta, \pi_* \beta = 0} (-1)^n DT(\beta, n) x^\beta q^n = \mathrm{DT}_{exc}(Y)(x, -q).$$

Proof: Lemma 2.16 proves that π-Hilb is locally of finite type, and is of finite type once the Chern character is fixed. As well, the set of elements $\alpha \in \Delta$ for which π-Hilb$^\alpha$ is non-empty is Laurent. To prove this, it suffices to show that for any curve class β, there exists an integer N such that for any $n < N$, the moduli space π-Hilb$^{(\beta,n)}$ is empty. Fix a curve class β, and consider all π-stablepairs $\mathcal{O}_Y \to G$ in that class. There is the associated short exact sequence,

$$0 \to \mathcal{O}_C \to G \to P \to 0$$

and since there are only finitely many decompositions $\beta = \beta_1 + \beta_2$ of β into a sum of effective curve classes, we lose no generality in fixing the curve class of $\mathcal{O}_C$ and P. Now the structure sheaf $\mathcal{O}_C$ lives in a Hilbert scheme, and the set of elements $(\beta_1, n_1) \in \Delta$ for which $\mathrm{Hilb}^{(\beta_1, n_1)}$ is non-empty is Laurent, so

there is a "minimal" Euler characteristic of $\mathcal{O}_C$, which we denote by N_1. As for P, the Leray spectral sequence shows that $\chi(P) \geq 0$ (see lemma 3.16). This proves that we may take $N = N_1$, and for any $n < N$, $\pi\text{-Hilb}^{(\beta,n)}$ is empty.

The formulae then follow from lemma 2.18 and Behrend's description of DT invariants as a weighted Euler characteristic.

Hilb_{exc} is a subscheme of Hilb, so the desired properties follow from [12, lemma 5.5]. ∎

Lemma 7.5. Let $I \subset (-\infty, +\infty] \times (-\infty, +\infty]$ be an interval bounded from below. Then

$$1_{SS(I)} \to \mathcal{C}$$

is Φ-finite.

Proof: Since this is holds for Gieseker stability ([23, theorem 3.3.7]), it suffices to prove that for any $b = (b_1, b_2) \in (-\infty, +\infty] \times (-\infty, +\infty]$ there exists a number M such that the family of all G with $\mu_\pi(G) \geq b$ satisfies $\mu(G) \geq M$. Here, μ stands for Gieseker slope stability, namely

$$\mu(G) = \frac{\chi(G)}{\beta \cdot L}.$$

Case 1: $\chi(G) > 0$

Here, we have

$$0 < \frac{\chi(G)}{\beta \cdot L} = \mu(G),$$

so we may take $M = 0$ in this case.

Case 2: $\chi(G) < 0$

Now, since $\beta \cdot \widetilde{H} \leq \beta \cdot L$ and $\chi(G) < 0$, we have

$$b_1 \leq \frac{\chi(G)}{\beta \cdot \widetilde{H}} \leq \frac{\chi(G)}{\beta \cdot L}.$$

In this case, we may take $M = b_1$.

Case 3: $\chi(G) = 0$

In this case, $\mu_\pi(G)$ is either 0 or $+\infty$, so we may take $M = 0$ in this case.

Case-by-case analysis reveals that we may use $M = \min\{0, b_1\}$.   ∎

Equations in the Laurent Hall algebra

In this section, following [12] we establish equations in $\mathrm{H_{sc}}(\mathcal{C})_\Phi$, and ultimately prove theorem 1.6.

Lemma 7.6. Let $\mu \in (-\infty, +\infty] \times (-\infty, +\infty]$ such that $\mu < \frac{\infty}{2}$. Then the following equality holds in $\mathrm{H_{sc}}(\mathcal{C})_\Phi$:

$$1_{\mathrm{SS}(\mu \leq \square \leq \infty)} = 1_{\mathcal{P}_\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}.$$

Proof: Form the following Cartesian diagram:

$$
\begin{array}{ccccc}
Z & \xrightarrow{\quad f \quad} & \mathcal{M}^{(2)} & \xrightarrow{\quad c \quad} & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle (l,r)} & & \\
\mathcal{P}_\pi \times \mathrm{SS}(\mu \leq \square < \frac{\infty}{2}) & \longrightarrow & \mathcal{M} \times \mathcal{M} & &
\end{array}
$$

T-valued points of Z are short exact sequences $0 \to A \to G \to B \to 0$ of T-flat sheaves on $T \times Y$ such that A defines a family of objects in $\mathcal{P}_\pi$ and B a family in $\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})$. By lemma 4.4, we know $\frac{\infty}{2} \leq \mu_\pi(A) \leq \infty$. Now by [12, lemma6.2], we know that G defines a family of objects in $\mathrm{SS}(\mu \leq \square \leq \infty)$.

Now let $G \in \mathrm{SS}(\mu \leq \square \leq \infty)$. If $G \in \mathcal{P}_\pi$ or $\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})$ then we are done, since then G will be an extension where one term is zero (recall that all $SS(a < \square < b)$ include the zero objects). Otherwise, let

$$0 = G_0 \subset G_1 \subset \ldots \subset G_{N-1} \subset G_N = G$$

be the Harder-Narasimhan filtration of G, let Q_i be the associated quotients. Then there exists an index $j \in \mathbb{N}, 1 < j < N$ such that for all $i < j, \mu_\pi(Q_i) \geq \frac{\infty}{2}$ and $\mu_\pi(Q_j) < \frac{\infty}{2}$. Finally, by the uniqueness of the Harder-Narasimhan filtration, we have that $G_j \in \mathcal{P}_\pi$ and $G/G_j \in \mathrm{SS}(\mu \leq \Box < \frac{\infty}{2})$.

$\blacksquare$

Remark 7.7. The proof of this lemma is strikingly similar to the proof of Lemma 6.1. This is no coincidence. The above is actually just a minor refinement of Lemma 6.1 which says that we may cut off the tail end of $\mathcal{Q}_\pi$ and have the corresponding result still hold. As we go on, we will be less explicit about the proofs of lemmas when the argument has been already made in the infinite-type case.

Lemma 7.8. Let $\mu \in (-\infty, +\infty] \times (-\infty, +\infty]$. Then, as $\mu \to -\infty$, we have

$$\mathcal{H} * 1_{\mathrm{SS}(\mu \leq \Box \leq \infty)} - 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \Box \leq \infty)} \to 0.$$

Proof: Fix $(\beta, n) \in \Delta$, and consider (β, m) for $m < n$. There are only finitely many decompositions $(\beta, n) = (\beta_1, n_1) + (\beta_2, n_2)$ such that both $\rho_{(\beta_1, n_1)}(\mathcal{H})$ and $\rho_{(\beta_2, n_2)}(1_{\mathrm{SS}(\mu \leq \Box \leq \infty)})$ are non-zero. This follows from the fact that there are only finitely many decompositions $\beta = \beta_1 + \beta_2$ with both β_i effective. Now for each fixed β, there exist finitely many n such that both $\rho_{(\beta_1, n_1)}(\mathcal{H})$ and $\rho_{(\beta_2, n_2)}(1_{\mathrm{SS}(\mu \leq \Box \leq \infty)})$ are non-zero).

By the boundedness of the Hilbert scheme, we may assume that μ is small enough so that for any of the decompositions, $\beta = \beta_1 + \beta_2$, all points $\mathcal{O}_Y \to A$ of $\mathrm{Hilb}^{(\beta_1, n_1)}$ satisfy $A \in \mathrm{SS}(\mu \leq \Box \leq \infty)$. Consider a diagram of sheaves,

$$
\begin{array}{ccccccccc}
 & & & & \mathcal{O}_Y & & & & \quad\quad (7.9) \\
 & & & & \downarrow{\scriptstyle\gamma} & & & & \\
0 & \longrightarrow & A & \overset{\alpha}{\longrightarrow} & G & \overset{\beta}{\longrightarrow} & B & \longrightarrow & 0
\end{array}
$$

with $\mathcal{O}_Y \to A$ in $\mathrm{Hilb}^{(\beta_1, n_1)}$ and $\mathrm{ch}([G]) = (\beta, n)$.

Now $G \in \mathrm{SS}(\mu \leq \Box \leq \infty)$ if and only if $B \in \mathrm{SS}(\mu \leq \Box \leq \infty)$. Since

Bridgeland proves [12, prop 6.5]

$$\rho_{(\beta,n)}(\mathcal{H} * 1_{\mathrm{SS}(\mu \leq \square \leq \infty)}) = \rho_{(\beta,n)}(1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square \leq \infty)}),$$

the claim is proven. ∎

Lemma 7.10. Let $\mu \in (-\infty, +\infty] \times (-\infty, +\infty)$. Then, as $\mu \to -\infty$, we have

$$\mathcal{H}^{\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0.$$

Proof: Fix $(\beta, n) \in \Delta$, and consider (β, m) for $m < n$. There are only finitely many decompositions $(\beta, n) = (\beta_1, n_1) + (\beta_2, n_2)$ such that both $\rho_{(\beta_1, n_1)}(\mathcal{H}^{\pi})$ and $\rho_{(\beta_2, n_2)}(1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})})$ are non-zero.

By the boundedness of the moduli space of π-stable pairs, we may assume that μ is small enough so that for any decompositions, $\beta = \beta_1 + \beta_2$, all points $\mathcal{O}_Y \to A$ of π-Hilb$^{(\beta_1, n_1)}$ satisfy $A \in \mathrm{SS}(\mu \leq \square < \frac{\infty}{2})$. Consider a diagram of sheaves,

$$\begin{array}{ccc} & \mathcal{O}_Y & \hspace{3cm} (7.11) \\ & \downarrow{\scriptstyle \gamma} & \\ 0 \longrightarrow A \xrightarrow{\alpha} G \xrightarrow{\beta} B \longrightarrow 0 & & \end{array}$$

with $\mathcal{O}_Y \to A$ in π-Hilb(β_1, n_1) and $[G] = (\beta, n)$. Using that $\mathrm{SS}(I) * \mathrm{SS}(I) \subset \mathrm{SS}(I)$, we see that $B \in \mathrm{SS}(\mu \leq \square < \frac{\infty}{2})$ if and only if $G \in \mathrm{SS}(\mu \leq \square < \frac{\infty}{2})$. Now since $A \in \mathcal{Q}_\pi$ and $B \in \mathcal{Q}_\pi$, we have that $G \in \mathcal{Q}_\pi$. Composing the map $\mathcal{O}_Y \to A$ with the map $A \to G$, yields a map $\mathcal{O}_Y \to G$; this represents an object of $1^{\mathcal{O}}_{\mathcal{Q}_\pi}$. The proof of lemma 6.6, that

$$1^{\mathcal{O}}_{\mathcal{Q}_\pi} = \mathcal{H}^{\pi} * 1_{\mathcal{Q}_\pi},$$

can be easily adapted to now prove that

$$\rho_{(\beta,n)}(\mathcal{H}^{\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}) = \rho_{(\beta,n)}(1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}),$$

using lemma 4.6. This completes the proof that

$$\mathcal{H}^\pi * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0$$

as $\mu \to -\infty$. ∎

Proposition 7.12. We have the following equality in the Laurent Hall algebra, $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})_\Phi$:

$$\mathcal{H} * 1_{\mathcal{P}_\pi} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi.$$

Proof: Using $1_{\mathrm{SS}(\mu \leq \square \leq \infty)} = 1_{\mathcal{P}_\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}$ and $1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square \leq \infty)} = 1^{\mathcal{O}}_{\mathcal{P}^1_\pi} * 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}$, we can rewrite

$$\mathcal{H} * 1_{\mathrm{SS}(\mu \leq \square \leq \infty)} - 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square \leq \infty)} \to 0$$

as

$$\mathcal{H} * 1_{\mathcal{P}_\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - 1^{\mathcal{O}}_{\mathcal{P}_\pi} * 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0,$$

as $\mu \to -\infty$.

Multiplying $\mathcal{H}^\pi * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0$ on the left by $1^{\mathcal{O}}_{\mathcal{P}_\pi}$, and rewriting using $1^{\mathcal{O}}_{\mathcal{P}_\pi} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi}$ yields

$$\mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - 1^{\mathcal{O}}_{\mathcal{P}_\pi} * 1^{\mathcal{O}}_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0$$

as $\mu \to -\infty$. Hence

$$\mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} - \mathcal{H} * 1_{\mathcal{P}_\pi} * 1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})} \to 0$$

as $\mu \to -\infty$. Since $1_{\mathrm{SS}(\mu \leq \square < \frac{\infty}{2})}$ is invertible, we can cancel it from both sides:

$$\mathcal{H} * 1_{\mathcal{P}_\pi} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi.$$

∎

The proof of theorem 1.6

We first collect results. The next proposition is theorem 3.11 of [29], and is a very deep result whose proof depends on all the full power of the formalism of [24, 25, 26, 27, 28].

Proposition 7.13. For each slope $\mu \in ((-\infty, -\infty), (+\infty, +\infty)]$, we can write

$$1_{\mathrm{SS}(\mu)} = \exp(\epsilon_\mu) \in \mathrm{H}(\mathcal{C})_\Phi$$

with $\nu_\mu = [\mathbb{C}^*] \cdot \epsilon_\mu \in \mathrm{H}_{\mathrm{reg}}(\mathcal{C})_\Phi$ a regular element.

Proof: The proof is identical to that of [12, theorem 6.3]. Bridgeland uses Joyce's machinery, which applies in our case just as it does in his. ∎

The following corollary corresponds to Bridgeland's 6.4 [12].

Corollary 7.14. For any $\mu \in ((-\infty, -\infty), (+\infty, +\infty)]$, the element $1_{\mathrm{SS}(\mu)} \in \mathrm{H}(\mathcal{C})_\Phi$ is invertible, and the automorphism

$$\mathrm{Ad}_{1_{\mathrm{SS}(\mu)}} : \mathrm{H}(\mathcal{C})_\Phi \to \mathrm{H}(\mathcal{C})_\Phi$$

preserves the subring of regular elements. The induced Poisson automorphism of $\mathrm{H}_{\mathrm{sc}}(\mathcal{C})$ is given by

$$\mathrm{Ad}_{1_{\mathrm{SS}(\mu)}} = \exp\{\eta_\mu, -\}.$$

Proof: The proof of this is identical to that of corollary 6.4 of [12]. ∎

Now we can prove theorem 1.6. We have

$$\mathcal{H} * 1_{\mathcal{P}_\pi} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi.$$

Rearranging yields

$$\mathcal{H} = \mathcal{H}_{exc} * 1_{\mathcal{P}_\pi} * \mathcal{H}^\pi * (1_{\mathcal{P}_\pi})^{-1}.$$

By lemma 4.4, we can write $1_{\mathcal{P}_\pi} = \mathrm{SS}(\frac{\infty}{2} \leq \square \leq \infty)$, and by lemma 6.2 of

[12], we can write

$$\mathrm{SS}(\frac{\infty}{2} \leq \square \leq \infty) = \prod_{\frac{\infty}{2} \leq \mu \leq \infty} 1_{\mathrm{SS}(\mu)}.$$

In [12, lemma 6.2], it is explained that given an interval $J \subset (-\infty, +\infty] \times (-\infty, +\infty]$ that is bounded below, and an increasing sequence of finite subsets

$$V_1 \subset V_2 \subset \ldots \subset J$$

the sequence $1_{\mathrm{SS}(V_j)}$ in converges to $1_{\mathrm{SS}(J)}$, where $1_{\mathrm{SS}(V_j)}$ is defined to be

$$\prod_{v \in V_j} 1_{\mathrm{SS}(v)},$$

where the product is taken in descending order of slope. So, letting J denote the interval of slopes between $\frac{\infty}{2}$ and ∞, including $\frac{\infty}{2}$ and excluding ∞, we can write

$$\mathcal{H} =$$

$$\mathcal{H}_{exc} * \lim_{\text{finite } V \subset J} 1_{\mathrm{SS}(\mu_N)} * \ldots * 1_{\mathrm{SS}(\mu_1)} * \mathcal{H}^\pi * (1_{\mathrm{SS}(\mu_1)})^{-1} * \cdots * (1_{\mathrm{SS}(\mu_N)})^{-1},$$

where μ_i enumerate all the elements of V. Using proposition and corollary , we can rewrite

$$\mathcal{H} =$$

$$\mathcal{H}_{exc} * \lim_{\text{finite } V \subset J} \exp(\{\eta_{\mu_N}, \exp\{\eta_{\mu_{N-1}}, \ldots \exp\{\eta_{\mu_1}, -\} \ldots\}(\mathcal{H}^\pi).$$

Now hitting this equation with the integration map yields

$$I_\Phi(\mathcal{H}) =$$

$$I_\Phi(\mathcal{H}_{exc}) \cdot I_\Phi\Big(\lim_{\text{finite } V \subset J} \exp(\{\eta_{\mu_N}, \exp\{\eta_{\mu_{N-1}}, \ldots \exp\{\eta_{\mu_1}, -\} \ldots\}(\mathcal{H}^\pi)\Big).$$

The integration map commutes with limits since it is continuous, thus

$$I_\Phi(\mathcal{H}) =$$

$$I_\Phi(\mathcal{H}_{exc}) \cdot \lim_{\text{finite } V \subset J} I_\Phi\big(\exp(\{\eta_{\mu_N}, \exp(\{\eta_{\mu_{N-1}}, \dots \exp(\{\eta_{\mu_1}, -\}) \dots\})(\mathcal{H}^\pi)\big).$$

Now, the Poisson bracket is a commutator which is trivial in the ring of Laurent series, so it vanishes after applying the integration map, and we are left with

$$I_\Phi(\mathcal{H}) = I_\Phi(\mathcal{H}_{exc}) \cdot I_\Phi(\mathcal{H}^\pi).$$

Applying lemma 7.4, we get

$$\mathrm{DT}(Y)(x, -q) = \mathrm{DT}_{exc}(Y)(x, -q) \cdot \pi\text{-}\mathrm{PT}(Y)(x, -q)$$

and substituting q for $-q$ yields

$$\mathrm{DT}(Y) = \mathrm{DT}_{exc}(Y) \cdot \pi\text{-}\mathrm{PT}(Y),$$

which is what we set out to prove. ∎

Chapter 8

Conclusion

Summary of results

In this dissertation, we introduced the notion of a π-stable pair, which may be thought of as a stable pair whose cokernel is allowed to be one-dimensional provided it is contained in the exceptional divisor of $Y \xrightarrow{\pi} X$ and has vanishing higher derived push-forwards.. Following ideas from [12], we described π-stable pairs in terms of a torsion pair $(\mathcal{P}_\pi, \mathcal{Q}_\pi)$. Then, we constructed a stability condition, and established a relationship between the stability condition and the torsion pair. Finally, after reviewing the construction of the various Hall algebras, we established equations of elements in the Laurent Hall algebra which integrate to give us the desired equality of generating series.

Future research

This dissertation is progress toward proving the DT crepant resolution conjecture. The complete proof might take the following form. Let us consider the case of global quotients for simplicity, i.e., stacks of the form $\mathcal{X} = M/G$. In this case, the derived category of $\mathcal{X}$, $D^b(\mathcal{X})$, is isomorphic to the G-equivariant derived category of the master space M, $D_G^b(M)$. By the celebrated results of Bridgeland, King, and Reid [13], we know that there is an equivalence Ψ between the derived category of Y and the G-equivariant derived category of M. The next step will be to show that this equivalence (or one closely related) takes π-stable pairs on Y to stable pairs on $\mathcal{X}$.

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