Inradius Bounds for Stable, Minimal Surfaces in 3-Manifolds with Positive Scalar Curvature

by

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Abstract

Concrete topological properties of a manifold can be found by examining its geometry. Theorem 17 of this thesis, due to Myers [Mye41], is one such example of this; it gives an upper bound on the length of any minimizing geodesic in a manifold N in terms of a lower positive bound on the Ricci curvature of N, and concludes that N is compact. Our main result, Theorem 40, is of the same flavour as this, but we are instead concerned with stable, minimal surfaces in manifolds of positive scalar curvature. This result is a version of Proposition 1 in the paper of Schoen and Yau [SY83], written in the context of Riemannian geometry. It states: a stable, minimal 2-submanifold of a 3-manifold whose scalar curvature is bounded below by $\kappa > 0$ has a inradius bound of $\sqrt{\frac{8}{3}}\frac{\pi}{\sqrt{\kappa}}$, and in particular is compact.

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Chapter 1

Introduction

A major area of study in Riemannian geometry is understanding the relationship between curvature and topology of Riemannian manifolds. An important and widely used tool in this area is the minimal submanifold technique. The application of this technique is the main theme of this thesis. The main theorem in this thesis, Theorem 40 is as follows:

Theorem 40. *Let M be a stable, minimal, orientable, complete 2-dimensional submanifold of a 3-dimensional Riemannian manifold (N, g). Suppose that there is a globally defined unit length normal vector e_3 on M. Suppose that on M we have $R_N \geq \kappa > 0$.*

1. *If M has no boundary then:*
$$\operatorname{diam}(M) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

2. *If M has non-empty boundary then:*
$$d_M(p, \partial M) := \inf\{d_M(p, q) \in \mathbb{R} : q \in \partial M\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$$
for all $p \in M$.

This theorem is a version of Proposition 1 in Schoen and Yau [SY83] but stated in the context of Riemannian geometry. In this context, Theorem 40 can be seen as an example of a theorem within the framework of the minimal submanifold technique.

Chapter 2 gives an expose of this technique, starting with a discussion of the typical conditions we look for in the submanifold M. In this technique we are concerned with submanifolds M that are minimal and stable. Roughly speaking, these are submanifolds which locally minimize volume inside the ambient space N. That is, if M is deformed slightly inside N it will not decrease in volume. To give a precise definition of this we need two formulae, known as the first and second variations of volume, derived in Appendix A.

Section 2.1 introduces the first and second variations of the submanifold M and gives precise definitions of minimality and stability.

As discussed in Section 2.1 the minimal submanifold technique consists of two main types of theorems. The first assumes something about the topology of a manifold and tries to prove the existence of a stable, minimal submanifold inside the ambient space. The second assumes some curvature conditions and the existence of a particular stable, minimal submanifold and examines the second variation of the submanifold. The technique combines the two theorems together to relate the curvature of the manifold to its topology. Theorem 40 can be seen to be a theorem of the second type. Theorem 40 has two cases; the submanifold M has no boundary and the submanifold M has non-empty boundary $\partial M \neq \emptyset$. We divide our expose into these two cases; Section 2.2 deals with the case where M has non-empty boundary $\partial M \neq \emptyset$, and Section 2.3 deals with the case where M has no boundary.

Chapter 3 gives a proof of our main theorem, Theorem 40. Section 3.1 gives a brief discussion of the conditions of the theorem and Section 3.2 gives proofs of two inradius bounds, including our main theorem, Theorem 40. The proof uses the second variation formula. Associated to this formula is a Jacobi operator $\mathcal{L}$ acting on functions on M. The condition of stability implies the second variation is non-negative which in turn implies $\mathcal{L}$ has non-negative eigenvalue. Letting f be an eigenfunction for $\mathcal{L}$, the idea is then to consider a warped product manifold with warping function f and using further arguments conclude the inradius bounds in Theorem 40. The warped product manifold is a generalization of the usual product manifold and arises naturally in the study of general relativity. Appendix B gives a definition for this warped space and some curvature calculations for warped product manifolds.

The concluding chapter, Chapter 4, gives a discussion of Theorem 40 in a wider context. It begins with a discussion of how it relates to the result of Schoen and Yau [SY83], and rephrases it in their language - in terms of a bound on the “2-dimensional diameter” or “fill radius” of the ambient space. We then give a discussion of possible extensions of this result to the case where the ambient manifold is of higher dimension. This includes two conjectures under stronger curvature assumptions. The chapter concludes with recent work that shows how these conjectures, if they are true, would give strong conclusions about the topology of the manifold.

Chapter 2

The Minimal Submanifold Technique

2.1 Introduction

The main theme in this thesis is the minimal submanifold technique. In this chapter we give an exposition of some successful applications of this technique.

We begin with the basic definitions used in the technique; specifically, the definition of a minimal submanifold, what it means for a minimal submanifold to be stable, and the definition of the index - which gives a measure of the stability of a submanifold. A submanifold which is minimal and stable is a submanifold which, in some sense, locally minimizes volume inside the ambient space.

Let (N, g) be a Riemannian manifold and M a submanifold of N. To obtain a concrete definition for minimality and stability, first consider a deformation or variation of M inside N indexed by a variable $t \in (-\epsilon, \epsilon)$, which leaves M unchanged when $t = 0$. We may write this as a map $f : (-\epsilon, \epsilon) \times M \to N$, with $f(0, M) = f_0(M) = M$. Then we can consider $t \to \mathrm{vol}(f_t(M))$ as a function from $\mathbb{R}$ to $\mathbb{R}$. If $f_0(M) = M$ locally minimizes volume we know from single variable calculus that $\frac{d}{dt} \mathrm{vol}(f_t(M))|_{t=0} = 0$ and $\frac{d^2}{dt^2} \mathrm{vol}(f_t(M))|_{t=0} \geq 0$. We shall derive formulae for these expressions soon but first we need to define something called the variation vector field which measures the direction in which M changes under a variation f.

Given a variation f we define a variation vector field $V := df(\frac{\partial}{\partial t})|_{t=0}$. This is a vector field on M that describes the direction $f_t(M)$ varies inside N at each point of $f_0(M) = M$ as t increases. Since we are interested in the derivative of $t \to \mathrm{vol}(f_t(M))$ we only need to look at what $f_t(M)$ looks like locally around $t = 0$ and thus it will suffice to look only at the variation vector field V, rather than the entire function f. Variation vector fields are sections of the bundle $TN \to M$. Equivalently every section of the bundle $TN \to M$ defines a variation vector field. We use the notation $\Gamma(M, TN)$

for these sections and the notation $\Gamma_c(M, TN)$ for $V \in \Gamma(M, TN)$ with compact support and we may give these spaces a vector space structure in the obvious way. Variation vector fields encode all the information we need to know about the variation so these spaces are important in deciding whether a submanifold M is locally volume minimizing.

A variation vector field V can be split into a normal $V^N \in \Gamma(M, TM^N)$ and a tangential component $V^T \in \Gamma(M, TM)$ to M, as $V = V^N + V^T$. A variation vector field V tells us the direction that M varies inside N with respect to a variation f, so if we look only at the tangential component V^T we are not gaining any information. A variation with a tangential variation vector field amounts to a re-parameterization of M and $f_t(M)$ is unchanged as t varies. Thus is makes sense to restrict ourselves only to normal variations.

Given a normal variation vector field $V \in \Gamma(M, TN)$, we now derive forumulae for $\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0}$ and $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0}$, known as the first and second variation respectively. Here M may have possible boundary ∂M, and if so we require that the boundary is fixed in the variation. This is equivalent to saying $V = 0$ on ∂M. We state both formulae here, the proofs can be found in Appendix A. The second variation formula depends on the vector field V, but also involves the curvature of the ambient manifold N. This is where the power of the minimal submanifold technique lies; if we can guarantee the existence of a stable, minimal submanifold M of N then we may be able to gain information about the curvature of N by looking at the second variation of M.

Given a submanifold M of N and a variation $f : (-\epsilon, \epsilon) \times M \to N$ of M in N with normal variation vector field V we now state the formula for the first variation $\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0}$. We assume M is compact and orientable with possible boundary ∂M.

Proposition 1. *The first variation of a compact, orientable submanifold M (with possible boundary ∂M), of a Riemannian manifold (N, g) with normal variation vector field V (with $V = 0$ on ∂M), is given by:*

$$\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0} = -\int_M \langle V, H_M\rangle dv$$

where H_M is the mean curvature of M in N and dv is the Riemannian volume form associated with the pullback metric $f_0^* g$.

Proof. This is Proposition 54 of Appendix A. □

Remark 2. We need M to be compact and orientable in order to define the integral in Proposition 1. However, as we shall see, we can generalize the condition $\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0} = 0$ to non-compact, non-orientable submanifolds.

From examining the integral in Proposition 1 we can see that the condition that $\frac{d}{dt}\operatorname{vol}(f(M))|_{t=0} = 0$ for any normal variation vector field V is equivalent to saying that H_M is zero. This motivates the next definition.

Definition 3. A submanifold M inside N is *minimal* if $H_M = 0$ on M.

Remark 4. H_M is defined for non-compact and non-orientable manifolds so this is a generalization of the condition that the first variation is zero.

Remark 5. We call 1-dimensional minimal submanifolds geodesics. The geodesic equation for such a submanifold γ be recovered from the condition of minimality via a computation of the mean curvature; $0 = H_\gamma = \nabla_{\dot\gamma}\dot\gamma$.

Given a submanifold M and a variation $f : (-\epsilon, \epsilon) \times M \to N$ with normal variation vector field V we derive an expression for the second variation $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0}$. We assume M is compact and orientable with possible boundary ∂M.

Proposition 6. *Assuming M is minimal, i.e. $H_M = 0$, the second variation of a compact, orientable submanifold M, (with possible boundary ∂M), in N with normal variation vector field V (with $V = 0$ on ∂M), is given by:*

$$\begin{aligned}\frac{d^2}{dt^2}\operatorname{vol}\left(f_t\left(M\right)\right)|_{t=0} &= \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - 2\|\Pi^V\|^2\right) dv \\ &= \int_M \left(\|DV\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - \|\Pi^V\|^2\right) dv\end{aligned}$$

where D is the normal connection on $TN|_M \to M$, i.e. $D_X Y = \left(\nabla_X Y\right)^N$.

Proof. This is Proposition 57 of Appendix A. □

There is a curvature term in Proposition 6. This allows us to make topological statements about the ambient manifold N under particular curvature assumptions, usually some form of *positive* curvature assumption.

If M is a local minimum for the volume we must have the inequality $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} \geq 0$. We assumed that M was compact and orientable so that the integral in Proposition 6 would make sense. However if V were compactly supported it would still be well-defined. This motivates the following definition.

Definition 7 (Stability)**.** An orientable submanifold M inside N is *stable* if it is minimal and the integral in Proposition 6 is non-negative for all compactly supported normal variation vector fields $V \in \Gamma_c(M, TN)$. A non-orientable submanifold is stable if its orientable double cover is stable.

Remark 8. The above definition extends the condition $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} \geq 0$ to non-compact and non-orientable manifolds.

Definition 9. We define an index form I on compactly supported $V \in \Gamma_c(M, TN)$

$$\begin{aligned} I(V,V) &= \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - 2\|\Pi^V\|^2 \right) dv \\ &= \int_M \left(\|DV\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - \|\Pi^V\|^2 \right) dv. \end{aligned}$$

Then our condition that $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} \geq 0$ is equivalent to saying the quadratic form I is positive semi-definite on $\Gamma_c(M, TN)$.

We may use this definition of the index form I to define a number associated to the submanifold M which measures its stability. Known as the index this is defined as follows:

Definition 10. Given a minimal submanifold M of N, its *index* is defined as the maximal dimension of a subspace of $\Gamma_c(M, TN)$ on which I is negative definite.

Remark 11. Roughly speaking, the index measures the number of linearly independent directions in the space of sections for which the second variation is negative. The lower the index, the closer M is to being locally volume minimizing. We say that, the lower the index, the more stable M is, i.e, the less likely it is to increase in volume when taking a variation.

We may rewrite Definition 7 in terms of the index in Definition 10.

Definition 12 (Stability 2)**.** A minimal, orientable submanifold M in N is stable if its index is 0. Once again, a non-orientable submanifold is stable if its orientable double cover is stable.

The minimal submanifold technique consists of two main types of theorems. The first, of independent interest, assumes some topological condition on a manifold, and tries to prove the existence of a minimal submanifold with some stability, inside the ambient manifold. The second assumes we have a particular minimal submanifold and examines the second variation formula on the submanifold. For a stable, minimal submanifold the second variation must be non-negative and we can use this to gain some information about the curvature of the manifold. The idea is then to put the two theorems together.

The rest of this chapter is devoted to examples of successful applications of the minimal submanifold technique. When using this technique there are two main cases that arise: the case where the submanifold has boundary and the case where it has no boundary. We deal with each of these separately.

2.2 The Submanifold has Non-empty Boundary

The simplest example of this technique is Bonnet's theorem. Here we are concerned with a stable, minimal 1-dimensional submanifold (stable geodesic) of a complete, connected Riemannian manifold N. We immediately have a theorem of the first type in this case:

Proposition 13. *Suppose N is a complete, connected Riemannian manifold. If $p, q \in N$ then there exists a length minimizing geodesic inside N whose boundary is $\{p, q\}$.*

Remark 14. This is usually stated in the following way; suppose N is a complete, connected Riemannian manifold, then for any $p, q \in N$ there is a length minimizing geodesic connecting p and q.

Remark 15. This is a standard result in Riemannian geometry, and can normally be found as a corollary to the Hopf-Rinow theorem.

This gives us a theorem of the first type. Our goal now is to get a theorem of the second type. We compute the second variation in the case that $M = \gamma$ is a unit speed geodesic $\gamma : [0, l] \to N$, and V is a normal variation of γ which is zero at the endpoints, $\gamma(0)$ and $\gamma(l)$. From Proposition 6 we have:

$$\frac{d^2}{dt^2}\operatorname{vol}\left(f_t\left(M\right)\right)|_{t=0} = \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - 2\|\Pi^V\|^2 \right) dv$$
$$= \int_0^l \left(\langle \nabla_{\dot\gamma} V, \nabla_{\dot\gamma} V\rangle - R_N(\dot\gamma, V, V\dot\gamma) - 2\langle V, \nabla_{\dot\gamma}\dot\gamma\rangle\right) ds$$
$$= \int_0^l \left(\left(\frac{\partial}{\partial s}\langle V, \nabla_{\dot\gamma} V\rangle - \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} V, V\rangle \right) - R_N(V, \dot\gamma, \dot\gamma, V) - 2\langle V, H_\gamma\rangle \right) ds$$
$$= [\langle V, \nabla_{\dot\gamma} V\rangle]_0^l - \int_0^l \left(\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} V, V\rangle - R_N(V, \dot\gamma, \dot\gamma, V)\right) ds$$
$$= -\int_0^l \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma} V + R(V, \dot\gamma)\dot\gamma, V\rangle ds \tag{2.1}$$

where the last line follows since V is zero at 0 and l.

We note that the expression in 2.1 contains a sectional curvature term so we see that the sectional curvature of N gives us information about the stability of geodesics in N.

We may combine the existence result of Proposition 13 with an examination of the second variation of a geodesic γ under some positive curvature conditions on the ambient manifold N. This is the following theorem:

Theorem 16 ([Bon55]). *Let N be a complete, connected Riemannian manifold all of whose sectional curvatures are bounded below by $\kappa > 0$. Then:*

1. *If $\gamma : [0, l] \to N$ is a stable geodesic in N we have:*

$$\operatorname{length}(\gamma) \leq \frac{\pi}{\sqrt{\kappa}}$$

2. *In particular for all $p, q \in M$ we have:*

$$d(p, q) \leq \frac{\pi}{\sqrt{\kappa}}$$

Moreover, M is compact with finite fundamental group.

Proof. Let $\gamma : [0, l] \to N$ be a stable unit speed geodesic of length l. We show $l \leq \frac{\pi}{\sqrt{\kappa}}$. Given $e \in T_pN$ with $\|e\| = 1$, $\langle e, \dot\gamma\rangle = 0$, we may obtain a parallel normal vector field e along γ by parallel transport. Let V=φe, where $\varphi : [0, l] \to \mathbb{R}$ is a smooth cut-off function with $\varphi(0) = \varphi(l) = 0$. Since

γ is length minimizing it has non-negative second variation. Therefore by 2.1

$$-\int_0^l \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + R\left(V,\dot\gamma\right)\dot\gamma, V\rangle ds \geq 0.$$

Now $\nabla_{\dot\gamma}V = \nabla_{\dot\gamma}(\varphi e) = \varphi' e + \varphi\nabla_{\dot\gamma}e = \varphi' e$ since e is parallel along γ. Repeating this argument yields $\nabla_{\dot\gamma}\nabla_{\dot\gamma}V = \varphi'' e$, and the inequality becomes:

$$-\int_0^l \langle \varphi'' e + R\left(\varphi e,\dot\gamma\right)\dot\gamma, \varphi e\rangle ds = -\int_0^l \left(\varphi'' + R\left(e,\dot\gamma,\dot\gamma,e\right)\varphi\right)\varphi ds \geq 0. \quad (2.2)$$

Now if we assume that all the sectional curvatures of M are bounded below by $\kappa > 0$, then $R\left(e,\dot\gamma,\dot\gamma,e\right) \geq \kappa$ and we have:

$$-\int_0^l \left(\varphi'' + \kappa\varphi\right)\varphi ds \geq 0.$$

Substituting $\varphi(s) = \sin\left(\frac{\pi s}{l}\right)$ gives:

$$\int_0^l \left(\frac{\pi^2}{l^2} - \kappa\right)\sin^2\left(\frac{\pi s}{l}\right) ds \geq 0..$$

Therefore $\frac{\pi^2}{l^2} - \kappa \geq 0$ and rearranging this gives $l \leq \frac{\pi}{\sqrt{\kappa}}$.

The second part of this statement follows easily from the first. Take $p, q \in N$ then by Proposition 13 there exists a length minimizing geodesic connecting p and q. Then from the above argument we have $d(p,q) = \text{length}(\gamma) \leq \frac{\pi}{\sqrt{\kappa}}$.

From this bound it follows that for $p \in N$ the map $\exp_p : B(0, \frac{\pi}{\sqrt{\kappa}}) \subseteq T_pN \to N$ is surjective. So N is equal to the image of a compact set under a continuous map and therefore it is compact.

We now show that N has a finite fundamental group. Let $\pi : \tilde N \to N$ denote the universal covering space of N with the pullback metric $\tilde g := \pi^* g$. Since N is complete and π is a local isometry one can show that $\tilde N$ is also complete and the sectional curvatures $\tilde K$ of $\tilde N$ have the same bound, $\tilde K \geq \kappa > 0$. Thus $\tilde N$ is also compact by the above argument. By the theory of covering spaces there is a one-to-one correspondence between $\pi_1(N)$ and $\pi^{-1}(p)$ for any point $p \in N$. If $\pi_1(N)$ were infinite then it would correspond to an infinite discrete set in $\tilde N$ given by $\pi^{-1}(p)$, contradicting the compactness of $\tilde N$. Thus $\pi_1(N)$ is finite. $\square$

In fact we can make slightly weaker assumptions about the curvature, as shown by Myers [Mye41].

Theorem 17 ([Mye41]). *Let N be a complete, connected Riemannian n-manifold whose Ricci tensor satisfied the following inequality for all $V \in TN$;*

$$\mathrm{Ric}\,(V,V) \geq (n-1)\,\kappa\|V\|^2$$

for some constant $\kappa > 0$. Then if $\gamma : [0,l] \to N$ is a stable geodesic in N we have:

$$\mathrm{length}(\gamma) \leq \frac{\pi}{\sqrt{\kappa}}.$$

Proof. As before suppose $\gamma : [0,l] :\to N$ is a stable unit speed geodesic of length l. Let $\{e_i\}_{i=1}^n$ be a parallel orthonormal frame along γ such that $e_n = \dot{\gamma}$, and for each $i = 1,\dots,n-1$ let $V_i = \varphi e_i$ where $\varphi = 0$ at 0 and l. The same computation as 2.2 in the proof of Bonnet's theorem (16) gives, for each $i = 1,\dots,n-1$, the following inequality:

$$-\int_0^l \left(\varphi'' + R\left(e_i,\dot{\gamma},\dot{\gamma},e_i\right)\varphi\right)\varphi ds \geq 0. \tag{2.3}$$

Summing up 2.3 gives:

$$-\int_0^l \left((n-1)\,\varphi'' + \mathrm{Ric}\left(\dot{\gamma},\dot{\gamma}\right)\varphi\right)\varphi ds. \tag{2.4}$$

Now $\sum_{i=1}^{n-1} R\left(e_i,\dot{\gamma},\dot{\gamma},e_i\right) = \mathrm{Ric}\left(\dot{\gamma},\dot{\gamma}\right) \leq (n-1)\,\kappa$. Thus replacing this term in 2.4 yields:

$$-\int_0^l \left((n-1)\,\varphi'' + (n-1)\,\kappa\varphi\right)\varphi ds \geq 0.$$

Again substituting $\varphi(s) = \sin\left(\frac{\pi s}{l}\right)$ gives:

$$\int_0^l \left((n-1)\,\frac{\pi^2}{l^2} - (n-1)\,\kappa\right)\sin^2\left(\frac{\pi s}{l}\right) ds \geq 0$$

and following a similar argument to Bonnet's theorem (16) yields $l \leq \frac{\pi}{\sqrt{\kappa}}$. □

Proposition 13 deals with 1-dimensional stable minimal submanifolds with boundary. The natural next step is to see if we can ask an analogous question in 2 dimensions. Our first formulation of this question is: given a simple closed curve Γ in $\mathbb{R}^n$, can we find a 2-dimensional submanifold (surface) with fixed boundary Γ which minimizes the area among all such submanifolds?

This is historically known as Plateau's problem. It is named after Belgian physicist Joseph Plateau (1801 - 1883) who experimented with soap films to give a physical and intuitive understanding of the problem. Plateau's experiments involved taking a wire, which represented a simple closed curve, and dipping it into soapy water. He would obtain a soapy film with the wire as its boundary. The surface obtained is a minimal surface and thus it can be seen as a solution to Plateau's problem.

In 1930 Radó [Rad30] and Douglas [Dou30] independently arrived at a mathematical solution of Plateau's problem. The precise formulation given by Douglas is as follows:

Theorem 18 ([Dou30])**.** *Given an arbitrary Jordan curve Γ in $\mathbb{R}^n$ there exists a minimal surface whose boundary is Γ.*

Remark 19. In the case where there exists a surface with finite area and boundary Γ then the surface guaranteed by Theorem 18 is a surface of minimum area. However there exists a Jordan curve Γ in $\mathbb{R}^2$ such that no surface with boundary Γ has finite area, see [Dou30], Part 27, page 320 for an example. Thus we may think of the condition that the surface be minimal as a generalization of idea that the surface minimizes area.

Remark 20. Morrey [Mor48] gives a generalization of Theorem 18 to the case when the ambient manifold is a Riemannian manifold.

Plateau's problem in full generality is concerned with the question of existence of a minimal k-dimensional Riemannian submanifold M with prescribed boundary inside an n-dimensional Riemannian manifold N. The main result in this formulation can be found in Morrey [Mor65] and builds on the argument used by Reifenberg [Rei60] in the case $N = \mathbb{R}^n$.

2.3 The Submanifold Has No Boundary

We now give an expose of some successful applications of the minimal submanifold technique in the case where the minimal submanifold M has no boundary. The first step is an existence theory for stable minimal submanifolds with no boundary. In general we will make an assumption about the topology of the ambient manifold, which will guarantee the existence of a minimal submanifold with some stability. A simple example of this is showing the existence of a 1-dimensional, stable, minimal submanifold homeomorphic to S^1 inside the ambient manifold N under the assumption $\pi_1(N) \neq 0$. First we need a definition.

Definition 21. Given a manifold N and $p \in N$ the injectivity radius $i(N, p)$ of N at p is defined as:

$$i(N,p) := \sup\{r > 0 : \exp_p \text{ is a diffeomorphism on } B(0,r) \subseteq T_pN\}.$$

And the injectivity radius of N is defined as:

$$i(N) := \inf\{i(N,p) : p \in N\}.$$

Remark 22. If $d(p, q) < i(N)$ then there exists a unique length minimizing geodesic joining p and q.

Then we have the following lemma:

Lemma 23. *Let N be a complete, compact Riemannian manifold. Then $i(N) > 0$ and if $\gamma_0, \gamma_1 : S^1 \to N$ are such that for all $s \in S^1$ we have*

$$d(\gamma_0(s), \gamma_1(s)) < i(M)$$

then γ_0 and γ_1 are homotopic.

Proof. We omit the proof. □

Now we have the following existence theorem:

Proposition 24. *Let N be a compact Riemmanian manifold with $\pi_1(N) \neq 0$. Then there exists a 1-dimensional, stable, minimal submanifold γ that is homeomorphic to S^1, i.e a stable, closed geodesic.*

Proof. We shall prove the following: a non-trivial homotopy class of curves in $\pi_1(N)$ has a shortest curve and this curve is stable and minimal.

Let $C := [\sigma]$ be a non-trivial free homotopy class of closed curves in M. We will first show that there is a shortest curve in this class. Choose $L > 0$ large, so that there is at least one curve in C of length $\leq L$. We restrict to all such curves. Define l as

$$l := \inf_{\sigma \in C} \text{length}(\sigma) > 0.$$

Then we may take a sequence of curves $\gamma_n : S^1 = [0, 2\pi) \to M$ in C, with length(γ_n) a decreasing sequence, length$(\gamma_n) \leq L$ for all n, and length$(\gamma_n) \to l$. Now M is compact so we have that $i(M) = r > 0$ for some r. Since length$(\gamma_n) \leq L$, for each curve γ_n there exists $0 = s_{0,n} < s_{1,n} < \ldots s_{m-1,n} < s_{m,n} = 2\pi$ such that:

$$\text{length}(\gamma_n|_{[s_{j-1,n}, s_{j,n}]}) \leq \frac{r}{2}.$$

Note that m here is independent of n and in fact is any integer $m \geq \lceil \frac{2L}{r} \rceil$. Then for each j, $d(\gamma_n(s_{j-1,n}), \gamma_n(s_{j,n})) \leq l(\gamma_n|_{[s_{j-1,n},s_{j,n}]}) \leq \frac{r}{2}$ and we replace the segment of γ_n joining $\gamma_n(s_{j-1,n})$ and $\gamma_n(s_{j,n})$ with the unique length minimizing geodesic between these two points guaranteed by Remark 22. From Lemma 23 we know each of these segments are homotopic and thus the curves as whole are homotopic. We call this new curve $\bar{\gamma}_n$ then we have a sequence $\{\bar{\gamma}_n\} \subset C$ with length$(\bar{\gamma}_n) \leq$ length(γ_n) and length$(\bar{\gamma}_n) \to l$. Define $p_{j,n} := \gamma_n(s_{j,n})$, then since M is compact we may, passing to a subsequence if necessary, assume that the points $p_{0,n}, \ldots, p_{m,n}$ converge to points $p_0, \ldots p_m$. Since $d(p_{j-1,n}, p_{j,n}) \leq \frac{r}{2}$ for all n we know that $d(p_{j-1}, p_j) \leq \frac{r}{2} < r$ and we may join each p_{j-1}, p_j with the unique geodesic of shortest length between them to construct a new curve γ. We claim that $\gamma \in C$ and length$(\gamma) = l$. By Lemma 23 γ is homotopic to $\bar{\gamma}_n$ for some large n and thus $\gamma \in C$. We show length$(\gamma) = l$. The unique geodesic connecting $p_{j-1,n}$ and $p_{j,n}$ is contained in $\bar{\gamma}_n$ and since $p_{j-1,n}, p_{j,n}$ converge to p_{j-1}, p_j we know that the length of the segment $\bar{\gamma}_n|_{[s_{j-1,n},s_{j,n}]}$ converges to the distance between p_{j-1} and p_j. Since γ contains the unique geodesic realizing the distance between these two points we may sum up the lengths of each segment to obtain:

$$\text{length}(\gamma) = \sum_{j=1}^{m} \text{length}(\gamma|_{[p_{j-1},p_j]}) = \sum_{j=1}^{m} \lim_{n\to\infty} \text{length}(\bar{\gamma}_n|_{[p_{j-1,n},p_{j,n}]})$$

$$= \lim_{n\to\infty} \sum_{j=1}^{m} \text{length}(\bar{\gamma}_n|_{[p_{j-1,n},p_{j,n}]}) = \lim_{n\to\infty} \text{length}(\bar{\gamma}_n) = l.$$

Thus γ is our desired curve. We have shown there exists $\gamma \in C$ with the shortest length amongst all curves in C. So if we were to deform γ via a homotopy we cannot obtain a shorter curve. Therefore γ is locally length minimizing and thus it is minimal and stable. □

Thus we have a theorem of the first type. Now we look for a theorem of the second type. The idea is to construct a vector field on a closed geodesic, if one exists, that will shorten it. We assume our manifold is even dimensional and orientable with positive sectional curvatures.

Proposition 25 ([Syn36])**.** *Let N^n be a even dimensional, orientable manifold with positive sectional curvatures $K > 0$. Then any closed geodesic γ is unstable, that is, can be shortened by a variation.*

Proof. Let $\gamma : [0, l] \to N$ be a closed, unit speed geodesic in N of length l, and let $p_0 \in \gamma$. Consider the parallel transport P_l around γ

$$P_l : T_{\gamma(0)=p_0}N \to T_{\gamma(l)=p_0}.$$

We show there exists a non-zero vector $V \in T_{p_0}N$, orthogonal to $\dot{\gamma}$, that is fixed by P_l, i.e. $P_l V = V$. Since γ is a geodesic, P_l leaves $\dot{\gamma} \in T_{p_0}N$ unchanged and thus leaves $\dot{\gamma}^{\perp}$ invariant. We restrict P_l to this space and consider $P_l : \dot{\gamma}^{\perp} \to \dot{\gamma}^{\perp}$. Parallel transport preserves the inner product so P_l is an orthogonal linear transformation and $\det P_l = \pm 1$. Furthermore since N is orientable $\det P_l = 1$. Since $\dim \dot{\gamma}^{\perp} = n - 1$ is odd it follows that P_l has 1 as an eigenvalue and the corresponding eigenvector $V \in \dot{\gamma}^{\perp}$ is fixed by P_l.

We may assume without loss of generality that $\|V\| = 1$. By parallel transporting V around γ we obtain a well defined normal vector field $V(p)$ on γ and we may define a variation $f_s(p) := \exp_p(sV(p))$ of γ. The second variation for geodesics, as calculated in Section 2.2 is:

$$\frac{d^2}{dt^2} \operatorname{vol}(f_t(\gamma))|_{t=0} = \int_0^l \left(\|\nabla_{\dot{\gamma}} V\|^2 - R_N(V, \dot{\gamma}, \dot{\gamma}, V)\right) ds. \tag{2.5}$$

The vector field $V(p)$ is parallel along γ so $\|\nabla_{\dot{\gamma}} V\|^2 = 0$. Then 2.5 becomes:

$$\frac{d^2}{dt^2} \operatorname{vol}(f_t(\gamma))|_{t=0} = -\int_0^l R_N(V, \dot{\gamma}, \dot{\gamma}, V)\, ds.$$

V and $\dot{\gamma}$ are orthonormal and therefore $R_N(V, \dot{\gamma}, \dot{\gamma}, V)$ is equal to the sectional curvature of the plane spanned by these vectors. Since all sectional curvatures $K > 0$ are assumed to be positive we see the second variation is strictly negative. Hence γ is unstable. $\square$

Putting Proposition 24 and Proposition 25 together we obtain the following theorem, due to Synge [Syn36].

Theorem 26 ([Syn36])**.** *If N is even dimensional, compact and orientable with positive sectional curvature then it is simply connected.*

Proof. Suppose N has non-trivial fundamental group. Then since N is compact, Proposition 24 guarantees the existence of a stable, closed geodesic. But since N is even dimensional and orientable, by Proposition 25 this geodesic must be unstable, a contradiction. Therefore N is simply connected. $\square$

Micallef and Moore [MM88] give a result that can be seen as a 2-dimensional version of Synge's theorem. As with the 1-dimensional case, the first step is an existence theorem. In non-trivial work of Sacks and Uhlenbeck [SU81] they developed an analogous general existence theory for branched minimal surfaces; namely, assuming something about the topology of N they prove existence of a non-constant minimal branched two-sphere. Their result is as follows:

Theorem 27 ([SU81]). *Let N be a compact Riemannian manifold of dimension ≥ 3 such that the universal covering space of N is not contractible. Then there exists a non-constant, branched, minimal two-sphere in N.*

The condition that the universal cover of N is not contractible is equivalent to the condition that $\pi_k(N) \neq 0$ for some $k \geq 2$. Micallef and Moore [MM88] extended the result of Sacks and Uhlenbeck [SU81] as follows:

Theorem 28 ([MM88]). *If N is a compact Riemannian manifold such that $\pi_k(N) \neq 0$, where $k \geq 2$, then there exists a non-constant minimal branched two-sphere in N of index $\leq k-2$.*

Remark 29. The index is introduced in Definition 10 and gives a measure of the stability of the submanifold.

Notice that Proposition 24 states that if a manifold N has $\pi_1(N) \neq 0$ then N has a minimal, stable 1-dimensional submanifold γ that is homeomorphic to S^1. Theorem 28 can be seen as a 2-dimensional version of this. The topological condition $\pi_1(N) \neq 0$ is replaced with the condition that N is compact and $\pi_k(N) \neq 0$ for some $k \geq 2$, and the existence of a stable, minimal 1-sphere is replaced by the existence of a non-constant, branched, minimal two-sphere of index $\leq k-2$.

The second part of Synge's theorem (26), Proposition 25, states that if N is an even dimensional, orientable manifold with positive sectional curvatures $K > 0$ then a minimal 1-dimensional submanifold homeomorphic to S^1 is unstable. Again, Micallef and Moore [MM88] have a theorem analogous to this. The assumption of positive sectional curvature is changed to an assumption of positive *isotropic curvature*, the condition that N is even dimensional and orientable is changed to the condition that N has dimension $n \geq 4$, and the result that a minimal 1-dimensional manifold homeomorphic to S^1 is unstable is changed to the result that a branched minimal immersion $f : S^2 \to M$ has index at least $\frac{n}{2} - \frac{3}{2}$. The result as stated by Micallef and Moore [MM88] is as follows:

Theorem 30 ([MM88]). *Let N be an n-dimensional Riemannian manifold with positive isotropic curvature. Then any branched minimal immersion $f : S^2 \to$ has index at least $\frac{n}{2} - \frac{3}{2}$.*

Micallef and Moore [MM88] then put these two theorems together to obtain a proof of their main theorem, and we sketch a proof of it here.

Theorem 31 ([MM88]). *Let N be a compact, simply connected n-dimensional Riemannian manifold which has positive isotropic curvature, where $n \geq 4$. Then N is homeomorphic to a sphere.*

Sketch of Proof. Let k be the smallest such integer such that $\pi_k(N) \neq 0$. Since N is assumed to be simply connected, $k \geq 2$. By Theorem 28 there exists a non-constant minimal two-sphere in N whose index is $m \leq k-2$. On the other hand it follows from Theorem 30 that $m \geq \frac{n}{2} - \frac{3}{2}$ and hence $k > \frac{n}{2}$. This is a purely topological condition and Micallef and Moore [MM88] exploit this to conclude that N must be homeomorphic to a sphere. They use a number of deep theorems in topology, and we give a rough sketch of their argument here:

Since $k > \frac{n}{2}$ one may use the Hurewicz isomorphism theorem and Poincaré duality to show that we must have $k = n$, i.e. the lowest non-vanishing homotopy group of N is $\pi_n(N)$. Then one can use Whitehead's theorem to conclude that N must be a homotopy sphere, followed by the generalized Poincaré conjecture when $n \geq 4$ to conclude that N is homeomorphic to a sphere. □

The results of Sack and Uhlenbeck [SU81] and Micallef and Moore [MM88], Theorem 27 and Theorem 28 respectively, show the existence of a non-constant, minimal branched sphere in the ambient space N, after making some assumptions about the topology of N.

Schoen and Yau [SY79] and Sacks and Uhlenbeck [SU82] proved an existence theory for surfaces of an arbitrary genus $g > 0$ rather than spheres. The result as formulated by Schoen and Yau [SY79] is as follows:

Theorem 32 ([SY79]). *Suppose N is a compact Riemannian manifold, M_g is a Riemannian surface of genus $g > 0$ and $f : M_g \to N$ is a continuous map such that the induced map on the fundamental group given by $f_\sharp : \pi_1(M_g) \to \pi_1(N)$ is injective. Then there is a branched minimal immersion $h : M_g \to N$ so that $h_\sharp = f_\sharp$ on $\pi_1(M_g)$ and the induced area of h is the least among all maps with the same action on $\pi_1(N)$. If N is 3-dimensional, it follows that h is an immersion. If $\pi_2(N) = 0$, then h can be deformed from f continuously.*

In the case where the ambient manifold N is 3-dimensional we are guaranteed that the resulting map has no branch points, that is, the image of

h is an immersed submanifold. Thus we have a very strong theorem of the first type.

Next, we examine the second variation and ask if there are theorems of the second type. In the same paper as the existence results, Schoen and Yau [SY79] give such a theorem; it examines the second variation formula and concludes that if the ambient manifold N has positive scalar curvature then any stable, minimal immersed 2-dimensional submanifold has genus 0. In the wording of Schoen and Yau [SY79]:

Theorem 33 ([SY79])**.** *Let N be a compact, oriented, 3-dimensional manifold with positive scalar curvature. Then N has no compact, immersed, stable, minimal 2-dimensional submanifolds of positive genus.*

Proof. This is just a restatement of Corollary 41. □

Thus if we consider a 3-dimensional manifold N we have a theorem of each type: a theorem of the first type that guarantees the existence of a stable minimal 2-submanifold under certain conditions, and a theorem of the second type that makes the assumption that N has positive scalar curvature and shows that only genus 0 stable minimal 2-submanifolds can occur. The idea then is to make some assumptions about the topology of our manifold N, use these to guarantee the existence of a stable, minimal 2-submanifold of a non-zero genus, then use Theorem 33 to conclude that N cannot have positive scalar curvature. The precise result, due to Schoen and Yau [SY79], is as follows:

Theorem 34 ([SY79])**.** *Let N^3 be compact and oriented. Suppose either of the following two conditions hold for N:*

1 $\pi_1(N)$ contains a finitely generated non-cyclic abelian subgroup

2 $\pi_1(N)$ contains a subgroup isomorphic to the fundamental group of a surfaces of genus greater than zero.

Then N admits no metric of positive curvature. In fact, any metric having non-negative scalar curvature is flat.

Remark 35. The first condition is actually a special case of the second. If $\pi_1(N)$ contains a finitely generated non-cyclic abelian group then it has an abelian subgroup of rank 2. Then one can show there is a map $\phi : T^2 \to N$, where T^2 is the usual 2-torus, such that $\phi_\sharp : \pi_1(T^2) \to \pi_1(N)$ maps onto this subgroup injectively.

Chapter 3

Inradius Bounds for Minimal Submanifolds

3.1 Introduction

In this chapter we examine the minimal submanifold technique in the case of a stable, minimal surface M inside an ambient 3-dimensional manifold N, whose scalar curvature R_N is bounded below by a constant $\kappa > 0$.

We note that condition that the scalar curvature is non-negative also appears naturally in the theory of general relativity. In general relativity we typically deal with a 4-dimensional manifold S with a Lorentzian metric g and an initial data set (N, g, Π). The initial data set (N, g, Π) consists of; a 3-dimensional "spacelike" submanifold N, the metric g, and the second fundamental form Π of N inside S. Since N is a hypersurface of S we can think of Π being a real valued $(0, 2)$ tensor, rather then a vector valued $(0, 2)$ tensor, and we may define the following quantities:

$$\mu := \frac{1}{2}\left(R_N - \|\Pi\|^2 + \operatorname{tr}(\Pi)^2\right) \qquad J := \operatorname{div}(\Pi - \operatorname{tr}(\Pi)g).$$

Then we typically impose the dominant energy condition; $\mu \geq \|J\|$. This condition arises naturally in general relativity. Letting $\{e_i\}_{i=1}^4$ be a local orthonormal frame with e_4 normal to N we may rewrite $\operatorname{tr}(\Pi)$ as:

$$\operatorname{tr}(\Pi) = \sum_{i=1}^{3}\langle \Pi(e_i, e_i), e_4\rangle = \sum_{i=1}^{3}\langle (\nabla_{e_i} e_i)^N, e_4\rangle = \langle H_N, e_4\rangle.$$

Then if N is minimal inside S we have $\operatorname{tr}(\Pi) = \langle H_N, e_4\rangle = 0$ and the dominant energy condition becomes:

$$\frac{1}{2}\left(R_N - \|\Pi\|^2\right) \geq \|J\|$$

which implies:

$$R_N \geq 2\|J\| + \|\Pi\|^2 \geq 0.$$

Thus we have the condition $R_N \geq 0$.

The paper of Schoen and Yau [SY83] proves the existence of black holes when enough matter is condensed in a small region. Proposition 1 of their paper shows that if the curvature of a manifold is positive in some region then that region is small in sense of a “2-dimensional diameter” or “fill radius”. This chapter provides an exposition of Proposition 1 in the language of Riemannian geometry. Specifically; it gives an inradius bound for a stable, minimal surface M in a 3-dimensional manifold N in terms of a positive lower bound on the scalar curvature R_N of the ambient space. In the context of the result of Schoen and Yau [SY83], the ambient manifold N is the “spacelike” submanifold of the spacetime (S, g) and we concerned with stable, minimal 2-dimensional submanifolds M of N.

3.2 Inradius Bounds

Recall the theorem of Bonnet-Myers, Theorem 17, gives an upper bound on the length of a minimizing geodesic in an n-dimensional manifold in terms of a positive lower bound on the Ricci curvature of the manifold. Our main theorem, Theorem 40, can be seen as a 2-dimensional version of this. We give an upper bound on the inradius of a stable, minimal, 2-dimensional submanifold (surface) in a ambient manifold in terms of a positive lower bound on the scalar curvature of the ambient manifold. However, we must assume that N is 3-dimensional. We refer the reader to Chapter 4 where we discuss possible extensions of this result to higher dimensions and the implications these would have in understanding manifolds of positive curvature.

We give two inradius bounds, the first one is as follows:

Theorem 36. *Let M be a stable, minimal, orientable, complete 2-dimensional submanifold of a 3-dimensional Riemannian manifold (N, g). Suppose that there is a globally defined unit length normal vector e_3 on M. Suppose that on M we have $R_N \geq \kappa > 0$.*

1. *If M has no boundary then:*

$$\operatorname{diam}(M) \leq \frac{2\pi}{\sqrt{\kappa}}.$$

2. *If M has non-empty boundary then:*

$$d_M(p, \partial M) := \inf\{d_M(p,q) \in \mathbb{R} : q \in \partial M\} \leq \frac{2\pi}{\sqrt{\kappa}}$$

for all $p \in M$.

The proof uses the second variation formula, given in Proposition 6. This states that for normal variations V we have:

$$\frac{d^2}{dt^2}\operatorname{vol}\left(f_t\left(M\right)\right)|_{t=0} = \int_M \left(\|DV\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - \|\Pi^V\|^2\right) dv.$$

The first thing we notice about this equation is the curvature term $\sum_{i=1} R_N(e_i, V, V, e_i)$. When the submanifold is of one less dimension than the ambient space this is simply $\mathrm{Ric}_N(V, V)$. The first step is to rewrite $\mathrm{Ric}_N(V, V)$ in terms of the scalar curvatures of the manifolds, the mean curvature H_M, and the second fundamental form Π of M inside N.

We derive such an expression now. We need only that M is of one less dimension than N.

Proposition 37. *For an n-dimensional submanifold M in an $(n+1)$-dimensional manifold N we have for a normal vector $V = \varphi e_{n+1}$, where e_{n+1} is unit normal vector on M, the following equality:*

$$\mathrm{Ric}_N\left(V, V\right) = \frac{1}{2}R_N\varphi^2 - \frac{1}{2}R_M\varphi^2 + \frac{1}{2}\|H_M\|\varphi^2 - \frac{1}{2}\|\Pi^V\|^2.$$

Proof. We check the equality at an arbitrary $p \in N$. Let $\{e_i\}_{i=1}^{n+1}$ be a local orthonormal frame around p with e_{n+1} normal to M. Then calculating $\varphi^2 R_N$ around p we have:

$$\begin{aligned}
\varphi^2 R_N &= \varphi^2 \sum_{i,j=1}^{n+1} R_N\left(e_i, e_j, e_j, e_i\right) \\
&= \varphi^2 \sum_{i=1}^{n+1} R_N\left(e_i, e_{n+1}, e_{n+1}, e_i\right) + \varphi^2 \sum_{i=1, j\neq n+1}^{n+1} R_N\left(e_i, e_j, e_j, e_i\right) \\
&= \mathrm{Ric}_N\left(V, V\right) + \varphi^2 \sum_{i,j\neq n+1}^{n+1} R_N\left(e_i, e_j, e_j, e_i\right) + \varphi^2 \sum_{j\neq n+1}^{n+1} R_N\left(e_{n+1}, e_j, e_j, e_{n+1}\right) \\
&= 2\,\mathrm{Ric}_N\left(V, V\right) + \varphi^2 \sum_{i,j=1}^{n} R_N\left(e_i, e_j, e_j, e_i\right).
\end{aligned}$$

Expanding the second term via the Gauss equation we obtain:

$$\varphi^2 \sum_{i,j=1}^{n} \left(R_M\left(e_i, e_j, e_j, e_i\right) - \langle \Pi\left(e_i, e_i\right), \Pi\left(e_j, e_j\right)\rangle + \langle \Pi\left(e_i, e_j\right), \Pi\left(e_i, e_j\right)\rangle\right).$$

The first term of this equation is $\varphi^2 R_M$, the second is $-\varphi^2\|H_M\|^2$. Writing $\Pi(e_i, e_j)$ as $\Pi(e_i, e_j) = \langle \nabla_{e_i} e_j, e_{n+1}\rangle e_{n+1}$ the third term becomes $\|\Pi^V\|^2$. Therefore we have:

$$\varphi^2 R_N = 2\,\mathrm{Ric}_N(V,V) + \varphi^2 R_M - \varphi^2\|H_M\| + \|\Pi^V\|^2.$$

Rearranging yields the result. $\square$

Remark 38. In particular $\mathrm{Ric}_N(V,V) = \frac{1}{2}R_N\varphi^2 - \frac{1}{2}R_M\varphi^2 - \frac{1}{2}\|\Pi^V\|^2$ if M is minimal in N.

We now give a proof of our first diameter bound, Theorem 36. Recall Theorem 36 states that for a stable, minimal, 2-dimensional manifold M inside a 3-dimensional manifold N with $R_N \geq \kappa > 0$ we have the following inradius bounds:

1 If M has no boundary then:

$$\mathrm{diam}(M) \leq \frac{2\pi}{\sqrt{\kappa}}.$$

2 If M has non-empty boundary then:

$$d_M(p, \partial M) := \inf\{d_M(p,q) \in \mathbb{R} : q \in \partial M\} \leq \frac{2\pi}{\sqrt{\kappa}}$$

for all $p \in M$.

Proof of Theorem 36. First assume M has no boundary. Take a normal variation $V := \varphi e_3$ where φ is compactly supported on M and e_3 is a globally defined unit normal vector on M. Since M is stable and minimal from Definition 7 and Proposition 6 we have the following inequality:

$$\int_{\mathrm{supp}\,V}\left(\|DV\|^2 - \sum_{i=1} R_N(e_i, V, V, e_i) - \|\Pi^V\|^2\right) dv \geq 0.$$

Since M is of one less dimension than N we have:

$$\sum_{i=1} R_N(e_i, V, V, e_i) = \mathrm{Ric}_N(V,V) = \frac{1}{2}R_N\varphi^2 - \frac{1}{2}R_M\varphi^2 - \frac{1}{2}\|\Pi^V\|^2 \qquad (3.1)$$

where the last equality follows from Proposition 37 and the minimality of M. Let $\{e_i\}_{i=1}^3$ be a local orthonormal frame that includes our globally defined e_3. We evaluate $\|DV\|^2$.

$$\begin{aligned}\|DV\|^2 &= \sum_{i=1} \langle (\nabla_{e_i} V)^N, (\nabla_{e_i} V)^N \rangle = \sum_{i=1} \langle \nabla_{e_i} V, e_3 \rangle^2 \\ &= \sum_{i=1} \langle e_i(\varphi) e_3 + \varphi \nabla_{e_i} e_3, e_3 \rangle^2 = \sum_{i=1} e_i(\varphi)^2 .\end{aligned}$$

where the last equality follows since $\langle \nabla_{e_i} e_3, e_3 \rangle = \frac{1}{2} e_i \langle e_3, e_3 \rangle = \frac{1}{2} e_i(1) = 0$. A quick check shows that $\sum_{i=1} e_i(\varphi)^2 = \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi \rangle$. Then since $\varphi = 0$ on $\partial \operatorname{supp}\varphi$, Green's identities give:

$$\int_{\operatorname{supp}\varphi} \|DV\|^2 = \int_{\operatorname{supp}\varphi} \langle \operatorname{grad}\varphi, \operatorname{grad}\varphi \rangle = -\int_{\operatorname{supp}\varphi} (\Delta_M \varphi)\varphi. \tag{3.2}$$

Then substituting the expressions in 3.1 and 3.2 into the formula for the second variation we have:

$$\int_{\operatorname{supp}\varphi} \left(\left(-\Delta_M \varphi - \frac{1}{2} R_N \varphi + \frac{1}{2} R_M \varphi \right) \varphi - \frac{1}{2} \|\Pi^V\|^2 \right) dv \geq 0$$

which implies:

$$\int_{\operatorname{supp}\varphi} \left(-\Delta_M \varphi - \frac{1}{2} R_N \varphi + \frac{1}{2} R_M \varphi \right) \varphi \, dv \geq 0. \tag{3.3}$$

Therefore the operator $\mathcal{L}$ defined by:

$$\mathcal{L}\varphi := -\Delta_M \varphi - \frac{1}{2} R_N \varphi + \frac{1}{2} R_M \varphi \tag{3.4}$$

has non-negative first eigenvalue on the space of φ with compact support.

Now take $p, q \in M$ with $d_M(p,q) = l > 0$ and define $B := B_M(p,l) = \{x \in M : d_M(p,x) < l\}$, $\bar{B} := \{x \in M : d_M(p,x) \leq l\}$ and $\partial B := \{x \in M : d_M(p,x) = l\}$.

$\mathcal{L}$ is a self-adjoint operator and results in PDE theory (see [GT83], Theorem 8.38, page 214) guarantee the existence of an eigenfunction f for such an operator, with $f > 0$ on B and $f = 0$ on ∂B. Let $\lambda \geq 0$ be the corresponding eigenvalue for f, then multiplying the equation $\mathcal{L}f = \lambda f$ by $2f^{-1}$ and rearranging gives:

$$R_M - 2f^{-1}(\Delta_M f) = R_N + 2\lambda \geq \kappa > 0.$$

Now consider $B \times S^1$ with the warped product metric $\tilde{g} = g + f^2 d\theta^2$. Appendix B gives a discussion of warped product metrics and derives a useful computation of the scalar curvature of a warped space that we will use now. This is Proposition 66 and we use it to compute $R_{B\times S^1}$ as:

$$R_{B\times S^1} = R_B - 2f^{-1}\left(\Delta_B f\right) = R_M - 2f^{-1}\left(\Delta_M f\right) = R_N + 2\lambda \geq \kappa$$

where the second equality follows since B is just a subset of M. So $B \times S^1$ obeys the same curvature condition.

For a curve γ from p to the boundary ∂B we define the quantity $L_f(\gamma) := 2\pi \int_\gamma f ds$. Then let ds be the volume form of γ and $d\tilde{s}$ be the volume form of the warped space $\gamma \times S^1$. Expanding $d\tilde{s}$ in local coordinates we have:

$$d\tilde{s} = \sqrt{\det \tilde{g}} ds d\theta = f ds d\theta$$

and if h is constant over S^1 we have:

$$\int_{\gamma\times S^1} h d\tilde{s} = \int_{\gamma\times S^1} h f ds d\theta = \int_\gamma h f ds \int_{S^1} d\theta = 2\pi \int_\gamma h f ds. \tag{3.5}$$

In particular if $h = 1$ on M we have:

$$\mathrm{vol}(\gamma \times S^1) = \int_{\gamma\times S^1} d\tilde{s} = 2\pi \int_\gamma f ds = L_f(\gamma). \tag{3.6}$$

Since $\bar{B}$ is compact there exists a curve γ minimizing $L_f(\gamma)$ among all such curves. Without loss of generality we may assume there is no segment of γ inside ∂B and γ ends at some $y \in \partial B$. We suppose γ has unit speed parametrization as $\gamma : [0, L] \to \bar{B}$ with $\gamma(0) = p$ and $\gamma(L) = y \in \partial B$. Note that length$(\gamma) = L \geq l$.

So γ minimizes L_f amongst all curves from p to y and therefore from the computation in 3.6 we see that vol $\left(\gamma \times S^1\right)$ under the warped metric is also minimized amongst all such curves. Then if we take a normal variation vector field V along γ and extend it to a field along $\gamma\times S^1$ such that $V(s,\theta) = V(s)$, the second variation of $\gamma\times S^1$ will be non-negative. We consider vector fields V of the form $V = \varphi e$ where e is a unit length, parallel, normal vector field on γ and φ is a smooth cut-off function on γ with $\operatorname{supp}\varphi \subseteq [0, L-\epsilon]$ for $0 < \epsilon < L$.

From the second variation formula for $\gamma\times S^1$ inside $B\times S^1$ and a similar argument to that used in 3.1 and 3.2 we have following inequality:

$$\int_{\gamma([0,L-\epsilon])\times S^1} \left(\langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle - \frac{1}{2} R_{B\times S^1}\varphi^2 + \frac{1}{2} R_{\gamma\times S^1}\varphi^2\right) d\tilde{s} \geq 0.$$

Remark 39. This is the place we are using the fact that M is 2-dimensional. γ is 1-dimensional and in order to get the above inequality we need $\gamma \times S^1$ to be a submanifold of $M \times S^1$ of one less dimension.

Since $R_{B\times S^1} \geq \kappa$ on B we can replace the $R_{B\times S^1}$ term to give:

$$\int_{\gamma([0,L-\epsilon])\times S^1} \left(\langle \operatorname{grad} \varphi, \operatorname{grad} \varphi \rangle - \frac{1}{2}\kappa\varphi^2 + \frac{1}{2}R_{\gamma\times S^1}\varphi^2 \right) d\tilde{s} \geq 0.$$

By Proposition 66 we have $R_{\gamma\times S^1} = R_\gamma - 2f^{-1}\left(\Delta_\gamma f\right) = -2f^{-1}\left(\Delta_\gamma f\right)$ where the last equality follows as $R_\gamma = 0$, since γ is 1-dimensional. This yields:

$$\int_{\gamma([0,L-\epsilon])\times S^1} \left(\langle \operatorname{grad} \varphi, \operatorname{grad} \varphi \rangle - \frac{1}{2}\kappa\varphi^2 - f^{-1}\left(\Delta_\gamma f\right)\varphi^2 \right) d\tilde{s} \geq 0. \tag{3.7}$$

Since all of these functions are constant over S^1 we may use the equality in 3.5 to write this as an integral over $\gamma\left([0, L-\epsilon]\right)$. Parameterizing the resulting integral by s and dividing through by 2π we can pull this back to an integral on $[0, L-\epsilon]$, this gives:

$$\int_0^{L-\epsilon} \left(\left(\varphi'\right)^2 f - \frac{1}{2}\kappa\varphi^2 f - f''\varphi^2 \right) ds \geq 0.$$

or

$$\int_0^{L-\epsilon} \left(-\left(\varphi'\right)^2 f + \frac{1}{2}\kappa\varphi^2 f + f''\varphi^2 \right) ds \leq 0.$$

Since $\left(\varphi'\right)^2 f > 0$ we may change the coefficient of this term to 2 giving:

$$\int_0^{L-\epsilon} \left(-2\left(\varphi'\right)^2 f + \frac{1}{2}\kappa\varphi^2 f + f''\varphi^2 \right) ds \leq 0.$$

Making the substitution $\varphi = f^{-\frac{1}{2}}\phi$ where ϕ is another function with $\phi = 0$ at 0 and $L - \epsilon$, we have:

$$\int_0^{L-\epsilon} \left(-2\left(\phi'\right)^2 + \frac{1}{2}\kappa\phi^2 - \frac{1}{2}\phi^2\left(f'\right)^2 f^{-2} + 2\phi\phi' f^{-1} f' + \phi^2 f'' f^{-1} \right) ds \leq 0. \tag{3.8}$$

Expanding out $\frac{d}{ds}\left(f^{-1}f'\phi^2\right)$ gives:

$$\frac{d}{ds}\left(f^{-1}f'\phi^2\right) = -\phi^2\left(f'\right)^2 f^{-2} + 2\phi\phi' f^{-1} f' + \phi^2 f'' f^{-1}$$

and substituting this into the inequality in 3.8 yields:

$$\int_0^{L-\epsilon} \left(-2\left(\phi'\right)^2 + \frac{1}{2}\kappa\phi^2 + \frac{1}{2}\phi^2\left(f'\right)^2 f^{-2} + \frac{d}{ds}\left(f^{-1}f'\phi^2\right) \right) ds \leq 0.$$

The fundamental theorem of calculus gives:

$$\int_0^{L-\epsilon} \frac{d}{ds}\left(f^{-1}f'\phi^2\right) = \left[f^{-1}f'\phi^2\right]_0^{L-\epsilon} = 0.$$

Since $\phi^2\left(f'\right)^2 f^{-2} \geq 0$ we may get rid of this term, which yields:

$$\int_0^{L-\epsilon} \left(-2\left(\phi'\right)^2 + \frac{1}{2}\kappa\phi^2 \right) ds \leq 0.$$

Integration by parts gives $\int_0^{L-\epsilon} \phi''\phi ds = -\int_0^{L-\epsilon}\left(\phi'\right)^2 ds$, therefore we have:

$$\int_0^{L-\epsilon} \left(2\phi''\phi + \frac{1}{2}\kappa\phi^2 \right) ds \leq 0.$$

This implies the operator $\mathcal{L}_0\phi = 2\phi'' + \frac{1}{2}\kappa\phi$ has non-positive eigenvalues. Choosing the eigenfunction $\phi(s) = \sin\left(\frac{\pi s}{L-\epsilon}\right)$ and computing its eigenvalue gives:

$$\left(-\frac{2\pi^2}{\left(L-\epsilon\right)^2} + \frac{1}{2}\kappa \right) \leq 0$$

which implies:

$$L - \epsilon \leq \frac{2\pi}{\sqrt{\kappa}}.$$

Since $0 < \epsilon < L$ was arbitrary we have $L \leq \frac{2\pi}{\sqrt{\kappa}}$. Then since $l \leq L$ we have $d_M\left(p,q\right) = l \leq \frac{2\pi}{\sqrt{\kappa}}$. Finally since $p, q \in M$ were arbitrary it follows that $\operatorname{diam}\left(M\right) \leq \frac{2\pi}{\sqrt{\kappa}}$.

Now assume M has non-empty boundary $\partial M \neq \emptyset$. We use the notation $\bar{M} = M \cup \partial M$ where $M \cap \partial M = \emptyset$. We assume $\bar{M}$ is topologically complete, that is; all closed and bounded sets are compact. Take $p \in M$, then we show:

$$d_M\left(p, \partial M\right) := \inf\{d_M\left(p,q\right) \in \mathbb{R} : q \in \partial M\} \leq \frac{2\pi}{\sqrt{\kappa}}.$$

Suppose $l := d_M\left(p, \partial M\right) > 0$. Then $B := B_M\left(p, l\right) = \{x \in \bar{M} : d_M\left(p,x\right) < l\}$ is contained entirely within M and since $\bar{B} := \{x \in \bar{M} : d_M\left(p,x\right) \leq l\}$ is compact we may use the same argument as Theorem 36, starting from 3.4, and replacing B appropriately. Thus we have $l \leq \frac{2\pi}{\sqrt{\kappa}}$ and the bound is proven. $\square$

In fact the bound in Theorem 36 can be improved. This is our main theorem, Theorem 40.

Theorem 40. *Let M be a stable, minimal, orientable, complete 2-dimensional submanifold of a 3-dimensional Riemannian manifold (N, g). Suppose that there is a globally defined unit length normal vector e_3 on M. Suppose that on M we have $R_N \geq \kappa > 0$.*

1. *If M has no boundary then:*

$$\operatorname{diam}(M) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

2. *If M has non-empty boundary then:*

$$d_M(p, \partial M) := \inf\{d_M(p,q) \in \mathbb{R} : q \in \partial M\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$$

   for all $p \in M$.

Proof. We first assume that M has no boundary, $\partial M = \emptyset$. We begin from 3.7 in Theorem 36, with the added condition $0 < 3\epsilon < L$. Recall 3.7 states that for φ with $\operatorname{supp} \varphi \subseteq [0, L-\epsilon]$, we have the following stability inequality:

$$\int_{\gamma([0,L-\epsilon])\times S^1} \left(\langle \operatorname{grad} \varphi, \operatorname{grad} \varphi \rangle - \frac{1}{2}\kappa\varphi^2 - f^{-1}(\Delta_\gamma f)\varphi^2 \right) d\tilde{s} \geq 0.$$

Since $\varphi = 0$ on the boundary of $\gamma([0, L-\epsilon])$ we may use Green's identity to give:

$$\int_{\gamma\times S^1} \left(-(\Delta_{\gamma\times S^1}\varphi) - \frac{1}{2}\kappa\varphi - f^{-1}(\Delta_\gamma f)\varphi \right) \varphi d\tilde{s} \geq 0. \tag{3.9}$$

Now we compute $\Delta_{\gamma\times S^1}\varphi$. We let $\{E_1 = \frac{\partial}{\partial s}, E_2 = f^{-1}\frac{\partial}{\partial \theta}\}$ be an orthonormal frame on $\gamma([0, L-\epsilon]) \times S^1$. From the connection identities in Proposition 64 we have:

$$\begin{aligned}\Delta_{\gamma\times S^1}\varphi &= \operatorname{div}_{\gamma\times S^1} \operatorname{grad}\varphi = \langle \nabla_{E_1} \operatorname{grad}\varphi, E_1\rangle + \langle \nabla_{E_2} \operatorname{grad}\varphi, E_2\rangle \\ &= \operatorname{div}_\gamma \operatorname{grad}\varphi + \langle (\operatorname{grad}\varphi)(f) f^{-1} E_2, E_2\rangle = \Delta_\gamma\varphi + \langle \operatorname{grad}\varphi, \operatorname{grad} f\rangle f^{-1}.\end{aligned}$$

Therefore 3.9 becomes:

$$-\int_{\gamma\times S^1} \left((\Delta_\gamma\varphi) + \langle \operatorname{grad}\varphi, \operatorname{grad} f\rangle f^{-1} + \frac{1}{2}\kappa\varphi + f^{-1}(\Delta_\gamma f)\varphi \right) \varphi d\tilde{s} \geq 0.$$

From the above inequality we see the operator $\mathcal{L}_0$

$$\mathcal{L}_0\varphi := (\Delta_\gamma\varphi) + \langle \operatorname{grad}\varphi, \operatorname{grad} f\rangle f^{-1} + \frac{1}{2}\kappa\varphi + f^{-1}\left(\Delta_\gamma f\right)\varphi$$

has non-positive eigenvalue on the space of functions with $\operatorname{supp}\varphi \subseteq [0, L-\epsilon]$. Let g be an eigenfunction of $\mathcal{L}_0$ with $g > 0$ on $(0, L-\epsilon)$, $g(0) = g(L-\epsilon) = 0$ and eigenvalue $\lambda_0 \leq 0$. Then parameterizing γ by s and using primes to denote $\frac{d}{ds}$ we expand the inequality $\mathcal{L}_0\left(g\right)g^{-1} = \lambda_0 \leq 0$, to obtain:

$$g^{-1}g'' + f^{-1}f'' + f^{-1}g^{-1}g'f' + \frac{1}{2}\kappa \leq 0. \tag{3.10}$$

Now let ϕ be a function with $\operatorname{supp}\phi \subseteq [\epsilon, L-2\epsilon]$, multiply both sides of 3.10 by ϕ^2 and integrate over $[\epsilon, L-2\epsilon]$. Integrating the first term by parts gives:

$$\begin{aligned}\int_\epsilon^{L-2\epsilon} g^{-1}g''\phi^2 ds &= [g^{-1}g\phi^2]_\epsilon^{L-2\epsilon} - \int_\epsilon^{L-2\epsilon} g'\left(-g^{-2}g'\phi^2 + 2\phi\phi' g^{-1}\right) ds \\ &= \int_\epsilon^{L-2\epsilon} g^{-2}\left(g'\right)^2\phi^2 ds - \int_\epsilon^{L-2\epsilon} 2\phi\phi' g^{-1}g' ds\end{aligned}$$

similarly,

$$\int_\epsilon^{L-2\epsilon} f^{-1}f''\phi^2 ds = \int_\epsilon^{L-2\epsilon} f^{-2}\left(f'\right)^2\phi^2 ds - \int_\epsilon^{L-2\epsilon} 2\phi\phi' f^{-1}f' ds.$$

Thus after rearranging, we have:

$$\begin{aligned}\int_\epsilon^{L-2\epsilon} g^{-2}\left(g'\right)^2\phi^2 + f^{-2}\left(f'\right)^2\phi^2 + f^{-1}g^{-1}f'g'\phi^2 + \frac{1}{2}\kappa\phi^2 ds& \\ \leq \int_\epsilon^{L-2\epsilon} 2\phi\phi'\left(g^{-1}g' + f^{-1}f'\right) ds.&\end{aligned} \tag{3.11}$$

For notation's sake we make the following definitions:

$$X := g^{-2}(g')^2 + f^{-2}(f')^2 \qquad\qquad Y := f^{-1}g^{-1}f'g'.$$

Then $(X+Y)\phi^2$ gives the first three terms in the integrand of 3.11. We claim:

$$2\phi\phi'\left(g^{-1}g' + f^{-1}f'\right) \leq \frac{4}{3}\left(\phi'\right)^2 + (X+Y)\phi^2. \tag{3.12}$$

We show this now. After expanding and rearranging the following inequality:

$$0 \leq \left(\sqrt{\frac{4}{3}} \phi' - \frac{1}{\sqrt{\frac{4}{3}}} \phi \left(g^{-1} g' + f^{-1} f' \right) \right)^2$$

we obtain:

$$2\phi\phi' \left(g^{-1} g' + f^{-1} f' \right) \leq \frac{4}{3} (\phi')^2 + \frac{3}{4} (X + 2Y) \phi^2.$$

Thus it suffices to show:

$$\frac{3}{4}(X + 2Y) \leq (X + Y) \Leftrightarrow 2Y \leq X.$$

This follows immediately from expanding and rearranging the following inequality:

$$(g^{-1} g' - f^{-1} f')^2 \geq 0.$$

Thus we have proven 3.12. Substituting this into 3.11 and cancelling we obtain:

$$\int_{\epsilon}^{L-2\epsilon} \frac{1}{2} \kappa \phi^2 ds \leq \int_{\epsilon}^{L-2\epsilon} \frac{4}{3} \left(\phi' \right)^2 ds.$$

Integration by parts gives $\int_{\epsilon}^{L-2\epsilon} (\phi')^2 \, ds = -\int_{\epsilon}^{L-2\epsilon} \phi'' \phi ds$, making this substitution and rearranging gives:

$$\int_{\epsilon}^{L-2\epsilon} \left(\frac{1}{2} \kappa \phi + \frac{4}{3} \phi'' \right) \phi ds \leq 0.$$

Choosing $\phi(s) = \sin\left(\frac{\pi(s-\epsilon)}{L-3\epsilon}\right)$ and following the same argument as Theorem 36 gives:

$$\frac{1}{2}\kappa - \frac{4}{3} \frac{\pi^2}{(L - 3\epsilon)^2} \leq 0$$

which implies:

$$L - 3\epsilon \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

Since $0 < 3\epsilon < L$ was arbitrary we have $L \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$. Then since $l \leq L$ we have $d_M(p, q) = l \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$. Finally since $p, q \in M$ were arbitrary it follows that $\operatorname{diam}(M) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$.

Now assume that M has non-empty boundary $\partial M \neq \emptyset$. Using a similar argument to Theorem 36 we may conclude that for any point $p \in M$ we have:

$$d_M(p, \partial M) := \inf\{d_M(p,q) \in \mathbb{R} : q \in \partial M\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

□

Corollary 41. *A stable, minimal surface M without boundary in a 3-manifold N whose scalar curvature R_N is bounded below by a positive constant $\kappa > 0$ is homeomorphic to either S^2 or $\mathbb{R}P^2$*

Proof. First we assume M is orientable. Since we have a bound on the diameter it follows that M is compact. Thus we may take a variation $V = \varphi e_3$ whose support is all of M. From 3.3 we have

$$\int_M \left(-\Delta_M \varphi - \frac{1}{2} R_N \varphi + \frac{1}{2} R_M \varphi\right) \varphi \, dv \geq 0.$$

Choosing $\varphi \equiv 1$ this becomes:

$$\int_M \left(-\frac{1}{2} R_N + \frac{1}{2} R_M\right) dv \geq 0.$$

Therefore, after rearranging, using the lower bound on the scalar curvature of N, and the Gauss-Bonnet theorem, we have:

$$0 < \frac{1}{2} \int_M \kappa \, dv < \frac{1}{2} \int_M R_N \, dv \leq \frac{1}{2} \int_M R_M \, dv = \int_M K_M \, dv = 2\pi\chi(M).$$

Now $\chi(M) = 2 - 2g > 0$ and thus we must have $g = 0$. Therefore M is homeomorphic to S^2. Now suppose M is not orientable. Let $\pi : \bar{M} \to M$ be the orientable double cover of M with pullback metric $\bar{g} := \pi^* g$. Then from the definition of minimal and stable for non-orientable submanifolds we have that the inequality 3.3 applies to $\bar{M}$ with scalar curvature $\bar{R}_M = R_M \circ \pi$ and $\bar{R}_N := R_N \circ \pi$. Following the same argument as above we have $\bar{M} \cong S^2$ and thus $M \cong \mathbb{R}P^2$. □

Theorem 36 uses a second variation argument to draw geometric conclusions about stable, minimal surfaces in 3-manifolds that have a lower positive bound on the scalar curvature. Viewing Theorem 36 (2) as a theorem of the second type in the context of the minimal submanifold technique when the submanifold has boundary, we may put it together with the theorem of the first type - the existence theorem for Plateau's problem (Theorem 18) - to draw topological conclusions about 3-manifolds with positive scalar curvature:

Theorem 42 (Schoen and Yau)**.** *A compact,* 3*-dimensional, manifold* N *whose universal cover is contractible cannot have* $R_N > 0$*.*

This is an unpublished result of Schoen and Yau. Note the result in Theorem 42 is for a 3-dimensional manifold. The following extension is still an open question in Riemannian geometry: can an n-dimensional, $n \geq 4$, manifold N whose universal cover is contractible have $R_N > 0$?

Chapter 4

Conclusion

The main theorem in this thesis, Theorem 40, states that if M is a stable, minimal, orientable, surface in a 3-dimensional Riemannian manifold N with scalar curvature R_N bounded below by a positive constant $\kappa > 0$, we have the following inradius bounds:

1. If M has no boundary then:

$$\operatorname{diam}(M) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

2. If M has non-empty boundary then:

$$d_M(p, \partial M) := \inf\{d_M(p,q) \in \mathbb{R} : q \in \partial M\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$$

for all $p \in M$.

Theorem 40 is a version of Proposition 1 in Schoen and Yau [SY83] stated in the language of Riemannian geometry. While Theorem 40 gives an inradius bound for stable minimal surfaces in N, the result of Schoen and Yau [SY83] gives a bound on the "2-dimensional diameter" or "fill radius" of N. We define this now:

Definition 43. Let γ be a smooth, simple closed curve in N which bounds a disk in N. Set $N_r(\gamma)$ to be:

$$N_r(\gamma) := \{x \in N : d_N(x, \gamma) \leq r\}.$$

The fill radius (fillrad(γ)) of γ is defined as:

$$\operatorname{fillrad}(\gamma) := \begin{cases} \sup\{r : d_N(\gamma, \partial N) > r \text{ and } \gamma \text{ doesn't bound a disk in } N_r(\gamma)\} & \partial N \neq \emptyset \\ \sup\{r : \gamma \text{ doesn't bound a disk in } N_r(\gamma)\} & \partial N = \emptyset. \end{cases}$$

The fill radius of N is defined as:

$$\text{fillrad}(N) := \sup\{\text{fillrad}(\gamma) : \gamma \text{ as above }\}.$$

Then we may rewrite Theorem 40 in terms of the fill radius:

Theorem 44. *Let N be a complete Riemannian 3-dimensional manifold with scalar curvature R_N bounded below by $R_N \geq \kappa > 0$. Then if γ is a smooth, simple closed curve in N which bounds a disk in N we have:*

$$\text{fillrad}(\gamma) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

And in particular:

$$\text{fillrad}(N) := \sup\{\text{fillrad}(\gamma) : \gamma \textit{ as above}\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

Proof. Let γ be a smooth, simple closed curve in N that bounds a disk in N and let $r := \text{fillrad}(\gamma)$. Since we know γ bounds a disk in N we may use the generalization of Plateau's problem, Theorem 18, given by Morrey [Mor48], to conclude that there exists stable minimal disk M with boundary γ. From Theorem 40 we have the following bound for every point $p \in M$:

$$d_M(p, \partial M) := \inf\{d_M(p, q) \in \mathbb{R} : q \in \partial M\} \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}. \tag{4.1}$$

Now γ does not bound a disk in $N_r(\gamma)$ so therefore there is a point $p \in M \cap (N \backslash N_r(\gamma))$. Then from 4.1 we have $d_M(p, \partial M) = d_M(p, \gamma) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$. On the other hand $p \in N \backslash N_r(M)$ and therefore $d_N(p, \gamma) > r$. Then we have:

$$r < d_N(p, \gamma) \leq d_M(p, \gamma) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}.$$

So $\text{fillrad}(\gamma) < \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$, and therefore, taking the supremum, we may conclude $\text{fillrad}(N) \leq \sqrt{\frac{8}{3}} \frac{\pi}{\sqrt{\kappa}}$. □

The result as stated in Proposition 1 of Schoen and Yau [SY83] makes slightly weaker assumptions on the curvature. Their result is as follows:

Theorem 45 ([SY83]). *Let N be a complete 3-dimensional Riemannian manifold such that the operator $-\Delta_N + \frac{1}{2}R_N$ has its first Dirichlet eigenvalue bounded below by $\Lambda > 0$. Then if γ is a smooth simple closed curve in N which bounds a disk in N we have:*

$$\operatorname{fillrad}(\gamma) \leq \frac{2}{\sqrt{3}}\frac{\pi}{\sqrt{\Lambda}}.$$

And in particular:

$$\operatorname{fillrad}(N) \leq \frac{2}{\sqrt{3}}\frac{\pi}{\sqrt{\Lambda}}.$$

Suppose we have that $R_N \geq \kappa$. Then let λ be the first Dirichlet eigenvalue of the operator $-\Delta_N + \frac{1}{2}R_N$, with eigenfunction f. We have $\lambda f = -\Delta_N f + \frac{1}{2}R_N f$. Multiplying by f and integrating yields:

$$\begin{aligned}
\lambda \int_N f^2 dv &= \int_N (-\Delta_N f) f dv + \frac{1}{2}\int_N R_N f^2 dv \\
&= \int_N \langle \operatorname{grad} f, \operatorname{grad} f\rangle dv + \frac{1}{2}\int_N R_N f^2 dv \\
&\geq \int_N \langle \operatorname{grad} f, \operatorname{grad} f\rangle dv + \frac{1}{2}\kappa \int_N f^2 dv \\
&\geq \frac{1}{2}\kappa \int_N f^2 dv
\end{aligned}$$

where the second equality follows from Green's identity. This implies that $\lambda \geq \frac{1}{2}\kappa$. Then taking $\Lambda = \frac{1}{2}\kappa$ in Theorem 45 gives Theorem 44.

Theorem 44 is interesting in its own right but is also important in a wider context. As we have seen in Theorem 42, it can be applied to give topological information about manifolds with positive scalar curvature. Note the result in Theorem 42 is for a 3-dimensional manifold. This leads naturally to the question: can an n-dimensional, $n \geq 4$, manifold N whose universal cover is contractible have $R_N > 0$? This is still an open question in Riemannain geometry, but would presumably follow if an extension of Theorem 44 to the case where N is n-dimensional could be proven.

If it is true, proving a result like Theorem 44 in the general case of an n-dimensional ambient manifold N will require new techniques, since the proof in 3-dimensions makes essential use of the fact that M is of codimension 1 and the dimension of N is three. Perhaps a more approachable problem is to prove such a fill radius bound under a stronger curvature assumption. To this effect two conjectures have been made. The first conjecture involves an assumption involving two-positive Ricci curvature. A manifold N is said to have two-positive Ricci curvature greater or equal to $\kappa > 0$ if the sum of the two smallest eigenvalues of the Ricci curvature is greater or equal to $\kappa > 0$. The first conjecture is as follows:

Conjecture 46. *Let (N, g) be a complete Riemannian n-manifold with two-positive Ricci curvature bounded below by κ, for a constant $\kappa > 0$. If γ is a smooth, simple closed curve in N which bounds a disk in N then:*

$$\text{fillrad}(\gamma) \leq C(\kappa).$$

The second conjecture involves positive isotropic curvature. It is as follows:

Conjecture 47. *Let (N, g) be a complete Riemmanian n-manifold with positive isotropic curvature bounded below by κ, for a constant $\kappa > 0$. If γ is a smooth, simple closed curve in N which bounds a disk in N then:*

$$\text{fillrad}(\gamma) \leq C(\kappa).$$

Remark 48. If a manifold N has either two-positive Ricci curvature bounded below by $\kappa > 0$ or positive isotropic curvature bounded below by $\kappa > 0$, then it immediately has scalar curvature R_N bounded below by $\kappa > 0$.

If these conjectures are true then topological information can be obtained about the manifold N. A recent result of Ramachandran and Wolfson [RW10] shows that if the universal cover $\tilde{N}$ has bounded fill radius then the fundamental group of N is "virtually free". Virtually free is a purely group theoretic notion, defined as follows:

Definition 49. A group G is said to be virtually free if it possesses a finite index subgroup that is a free group.

The result as stated by Ramachandran and Wolfson [RW10] is as follows:

Theorem 50 ([RW10])**.** *Let N be a closed Riemannian n-manifold. Suppose that the universal cover $\pi : \tilde{N} \to N$ is given the pullback metric $\tilde{g} := \pi^* g$. If $(\tilde{N}, \tilde{g})$ has bounded fill radius then the fundamental group of N is virtually free.*

Thus if Conjecture 46 or Conjecture 47 is true we may conclude that a manifold with two-postive Ricci curvature bounded below by $\kappa > 0$ or with positive isotropic curvature bounded below by $\kappa > 0$, has a virtually free fundamental group.

Moreover recently Gadgil and Seshadri [GS09] have shown the following:

Theorem 51 ([GS09]). *Let N be a smooth, orientable, closed n-manifold such that $\pi_1(N)$ is a free group on k generators and $\pi_i(N) = 0$ for $2 \leq i \leq \frac{n}{2}$. If $n \neq 4$ or $k = 1$ then N is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$.*

If Conjecture 47 is true, Theorem 51 can be used along with Theorem 50 to say some more about the topology of manifolds with positive isotropic curvature. We give an outline of this argument now.

Suppose Conjecture 47 is true. Let N be an n-dimensional manifold with isotropic curvature bounded below by $\kappa > 0$. Then the universal cover $\tilde{N}$ of N with the pullback metric $\tilde{g} = \pi^* g$ also has isotropic curvature bounded below by $\kappa > 0$. Then if Conjecture 47 is true $\tilde{N}$ has bounded fill radius and therefore by Theorem 50 the fundamental group of N is virtually free.

If the fundamental group of N is virtually free it has a subgroup that is a free group with finite index k, and we may find a finite cover $\bar{N}$ of N with fundamental group $\pi_1(\bar{N})$ isomorphic to this free group. If $\bar{N}$ is given the pull-back metric from N then it also has isotropic curvature bounded below by $\kappa > 0$. Then from the work done by Micallef and Moore [MM88] we may use the argument given in Theorem 31 to conclude that $\pi_i(\bar{N}) = 0$ for $2 \leq i \leq \frac{n}{2}$. Then $\bar{N}$ almost satisfies the hypothesis of Theorem 51. If we have that either $n \neq 4$ or that $k = 1$, (i.e. $\pi_1(\bar{N}) = \mathbb{Z}$), then we may conclude that $\bar{N}$ is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$. Thus from the assumption of positive isotropic curvature we would have a fairly strong topological result: N has a finite cover $\bar{N}$ that is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$.

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Appendix A

The First and Second Variation Formulae

A stable, minimal submanifold M in N is a manifold which locally minimizes volume inside the ambient manifold. To obtain a concrete definition for this we consider a deformation or variation of M inside N indexed by a variable $t \in (-\epsilon, \epsilon)$, which leaves M unchanged when $t = 0$. We may write this as a map $f : (-\epsilon, \epsilon) \times M \to N$, with $f(0, M) = f_0(M) = M$. Then we can consider $t \to \mathrm{vol}(f_t(M))$ as a function from $\mathbb{R}$ to $\mathbb{R}$. If $f_0(M) = M$ locally minimizes volume we know from single variable calculus that $\frac{d}{dt}\mathrm{vol}(f_t(M))|_{t=0} = 0$ and $\frac{d^2}{dt^2}\mathrm{vol}(f_t(M))|_{t=0} \geq 0$. We shall derive formulae for these expressions soon but first we need a lemma from linear algebra.

Lemma 52. *If $A(t)$ is a smooth function of t into the space of invertible matrices and $A(t_0) = \mathrm{id}$, then*

$$\frac{d}{dt}\det A(t)\,|_{t=t_0} = \mathrm{tr}\,A'(t_0) = \sum_{i=1} A'(t_0)_{ii}$$

Proof. Let ω be the standard n-form on $\mathbb{R}^n$, then if $\{e_i\}$ is the standard basis we have:

$$\begin{aligned}
\frac{d}{dt}\det A(t)\,|_{t=t_0} &= \frac{d}{dt}\omega\left(A(t)\,e_1, \ldots, A(t)\,e_n\right)|_{t=t_0} \\
&= \sum_{i=1} \omega\left(A(t_0)\,e_1, \ldots, A'(t_0)\,e_i, \ldots, A(t_0)\,e_n\right) \\
&= \sum_{i=1} \omega\left(e_1, \ldots, A'(t_0)\,e_i, \ldots, e_n\right) \\
&= \sum_{i=1} A'(t_0)_{ii}\,\omega\left(e_1, \ldots, e_n\right) \\
&= \sum_{i=1} A'(t_0)_{ii}
\end{aligned}$$

where the third inequality follows since $A(t_0) = \mathrm{id}$. $\square$

Lemma 53. *If $B(t)$ is a smooth function of t into the space of invertible matrices then*

$$\frac{d}{dt}\det B(t) = \det B(t) \sum_{i,j=1} B^{-1}(t)_{ij} B'(t)_{ji}$$

where $B^{-1}(t) := B(t)^{-1}$.

Proof. We check this at a point $t = t_0$. Define $A(t) := B^{-1}(t_0) B(t)$, then

$$\begin{aligned}
&\frac{d}{dt}\det A(t)|_{t=t_0} = \frac{d}{dt}\det\left(B^{-1}(t_0) B(t)\right)|_{t=t_0} \\
&= \frac{d}{dt}\left(\det B^{-1}(t_0)\right)\left(\det B(t)\right)|_{t=t_0} \\
&= \det B^{-1}(t_0)\frac{d}{dt}\det B(t)|_{t=t_0} = \frac{1}{\det B(t_0)}\frac{d}{dt}\det B(t)|_{t=t_0}
\end{aligned}$$

Now $A(t)$ satisfies the conditions of Lemma 52, therefore:

$$\begin{aligned}
&\frac{1}{\det B(t_0)}\frac{d}{dt}\det B(t)|_{t=t_0} = \frac{d}{dt}\det A(t)|_{t=t_0} = \sum_{i=1} A'(t_0)_{ii} \\
&= \sum_{i=1}\left(B^{-1}(t_0) B'(t_0)\right)_{ii} = \sum_{i,j=1} B^{-1}(t_0)_{ij} B'(t_0)_{ji}
\end{aligned}$$

multiplying through by $\det B(t_0)$ we have our result. □

Given a normal variation vector field $V \in \Gamma(M, TN)$, we now derive forumulae for $\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0}$ and $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0}$, known as the first and second variation formulae. Here M may have possible boundary ∂M and if so we require that the boundary is fixed in the variation. This is equivalent to saying $V = 0$ on ∂M.

We are measuring the volume of M immersed in N, so we give it the pullback metric $f_t^* g$, thus the volume element of the metric induced on M by f_t is given by $dv_t = \sqrt{\det g(t)}dx$. The volume of $f_t(M)$ is then:

$$\operatorname{vol}(f_t(M)) = \int_M \sqrt{\det g(t)}dx$$

Then the first variation is as follows:

Proposition 54. *The first variation of a compact, orientable submanifold M (with possible boundary ∂M) of a Riemannian manifold (N, g) with normal variation vector field V (with $V = 0$ on ∂M), is given by:*

$$\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0} = -\int_M \langle V, H_M\rangle dv$$

where H_M is the mean curvature of M in N and dv is the Riemannian volume form associated with the pullback metric f_0^*g.

Proof. We wish to calculate $\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0}$, we have from Lemma 53:

$$\frac{d}{dt}\sqrt{\det g(t)} = \frac{1}{2}(\det g(t))^{-\frac{1}{2}}\frac{d}{dt}\det g(t) = \frac{1}{2}\sqrt{\det g(t)}\sum_{i,j=1} g^{-1}(t)_{ij}\, g'(t)_{ji}$$

Then letting $g = \det g(t)$ and suppressing the dependence on t we may write this in a more compact form:

$$\frac{d}{dt}\sqrt{g} = \frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\dot{g}_{ij}$$

where we have used the fact that $g^{-1}(t)$ is symmetric and renamed i, j.

Now if we denote $f_i = df\left(\frac{\partial}{\partial x^i}\right)$, then the induced metric on $f_t(M)$ is given by $g_{ij} = \langle f_i, f_j\rangle$. Then we have:

$$\dot{g}_{ij} = \frac{\partial}{\partial t}\langle f_i, f_j\rangle = \langle \nabla_t f_i, f_j\rangle + \langle f_i, \nabla_t f_j\rangle$$

So at $t = 0$ we have:

$$\frac{d}{dt}\sqrt{g}|_{t=0} = \sqrt{g}\sum_{i,j=1} g^{ij}\langle \nabla_t f_i, f_j\rangle = \sqrt{g}\sum_{i,j=1} g^{ij}\langle \nabla_{f_i} V, f_j\rangle = \sqrt{g}\operatorname{div}_M V$$

where the second equality follows since $[f_i, V] = df[\partial_i, \partial_t] = df(0) = 0$.

Then since the tangential component of V is zero we may write $V = V^N$ and we have:

$$\begin{aligned}\operatorname{div}_M V = \operatorname{div}_M V^N &= \sum_{i=1}\langle \nabla_{e_i} V^N, e_i\rangle = -\sum_{i=1}\langle V^N, \nabla_{e_i} e_i\rangle \\ &= -\sum_{i=1}\langle V, (\nabla_{e_i} e_i)^N\rangle = -\langle V, \sum_{i=1}(\nabla_{e_i} e_i)^N\rangle = -\langle V, H_M\rangle\end{aligned}$$

where H_M is the mean curvature vector of M inside N. Therefore

$$\frac{d}{dt}\operatorname{vol}(f_t(M))|_{t=0} = \int_M \operatorname{div}_M V^N\sqrt{g}dx = -\int_M \langle V, H_M\rangle dv$$

□

From examining the integral in Proposition 54 we can see that the condition that $\frac{d}{dt}\operatorname{vol}(f(M))|_{t=0} = 0$ for any normal variation vector field V is equivalent to saying that H_M is zero. This motivates the next definition.

Definition 55. A submanifold M inside N is *minimal* if $H_M = 0$.

Remark 56. H_M is defined for non-compact and non-orientable manifolds so this is a generalization of the condition that the first variation is zero.

Given a minimal submanifold M of N and a variation $f : (-\epsilon, \epsilon) \times M \to N$ with normal variation vector field V we now derive an expression for the second variation $\frac{d^2}{dt^2} \operatorname{vol}(f_t(M))|_{t=0}$. Again we assume that M is compact and orientable.

Proposition 57. *Assuming M is minimal, i.e. $H_M = 0$, the second variation of a compact, orientable submanifold M (with possible boundary ∂M) in N with normal variation vector field V (with $V = 0$ on ∂M), is given by:*

$$
\begin{aligned}
\frac{d^2}{dt^2} \operatorname{vol}\left(f_t\left(M\right)\right)|_{t=0} &= \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - 2\|\Pi^V\|^2 \right) dv \\
&= \int_M \left(\|DV\|^2 - \sum_{i=1} R_N\left(e_i, V, V, e_i\right) - \|\Pi^V\|^2 \right) dv
\end{aligned}
$$

where D is the normal connection on $TN|_M \to M$, i.e. $D_X Y = \left(\nabla_X Y\right)^N$.

Proof. We want to compute $\frac{d^2}{dt^2} \operatorname{vol}\left(f_t\left(M\right)\right)|_{t=0} = \int_M \frac{d^2}{dt^2} \sqrt{\det g(t)}|_{t=0} dx$, thus we begin by computing $\frac{d^2}{dt^2}\sqrt{g}$ where $g = \det g(t)$.

$$
\begin{aligned}
\frac{d^2}{dt^2}\sqrt{g} &= \frac{d}{dt}\left(\frac{d}{dt}\sqrt{g}\right) = \frac{d}{dt}\left(\frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\dot{g}_{ij}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\dot{g}_{ij}\right)\sum_{i,j=1} g^{ij}\dot{g}_{ij} + \frac{1}{2}\sqrt{g}\sum_{i,j=1}\dot{g}^{ij}\dot{g}_{ij} + \frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\ddot{g}_{ij} \\
&= \frac{1}{4}\sqrt{g}\left(\sum_{i,j=1} g^{ij}\dot{g}_{ij}\right)^2 + \frac{1}{2}\sqrt{g}\sum_{i,j=1}\dot{g}^{ij}\dot{g}_{ij} + \frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\ddot{g}_{ij}
\end{aligned}
$$

If G is the matrix g_{ij} then $\left(GG^{-1}\right) = \mathrm{id}$, so $\frac{d}{dt}\left(GG^{-1}\right) = 0$ therefore, by the product rule:

$$\frac{dG}{dt}G^{-1} + G\frac{dG^{-1}}{dt} = 0$$

so

$$\frac{dG^{-1}}{dt} = -G^{-1}\frac{dG}{dt}G^{-1}$$

and therefore $\dot{g}^{ij} = -\sum_{k,l=1} g^{ik}\dot{g}_{kl}g^{lj}$. Thus we have:

$$\frac{d^2}{dt^2}\sqrt{g} = \frac{1}{4}\sqrt{g}\left(\sum_{i,j=1} g^{ij}\dot{g}_{ij}\right)^2 - \frac{1}{2}\sqrt{g}\sum_{i,j,k,l=1} g^{ik}\dot{g}_{kl}g^{lj}\dot{g}_{ij} + \frac{1}{2}\sqrt{g}\sum_{i,j=1} g^{ij}\ddot{g}_{ij} \tag{A.1}$$

We evaluate A.1 at $t = 0$. Examining the first term we have:

$$\begin{aligned}\frac{1}{4}\left(\sum_{i,j=1} g^{ij}\dot{g}_{ij}\right)^2 &= \frac{1}{4}\left(2\sum_{i,j=1} g^{ij}\langle\nabla_i V, f_j\rangle\right)^2 = \frac{1}{4}\left(2\operatorname{div}_M V\right)^2 = \left(\operatorname{div}_M V\right)^2 \\ &= \left(\operatorname{div}_M V^N + \operatorname{div} V^T\right)^2 = \left(-\langle V, H_M\rangle + \operatorname{div} V^T\right)^2 = 0\end{aligned}$$

where the last equality follows since $H_M = 0$ and $V^T = 0$.

We now examine the third term in A.1, first we compute $\ddot{g}_{ij}$ at $t = 0$

$$\begin{aligned}\ddot{g}_{ij} =& \langle\nabla_t\nabla_t f_i, f_j\rangle + \langle\nabla_t f_i, \nabla_t f_j\rangle + \langle\nabla_t f_i, \nabla_t f_j\rangle + \langle f_i, \nabla_t\nabla_t f_j\rangle \\ =& \langle\nabla_t\nabla_i V, f_j\rangle + 2\langle\nabla_i V, \nabla_j V\rangle + \langle\nabla_t\nabla_j V, f_i\rangle \\ =& \langle R_N\left(V, f_i\right)V, f_j\rangle + \langle\nabla_i\nabla_t V, f_j\rangle + 2\langle\nabla_i V, \nabla_j V\rangle + \langle R_N\left(V, f_j\right)V, f_i\rangle \\ &+ \langle\nabla_j\nabla_t V, f_i\rangle\end{aligned}$$

Therefore, if $\{e_i\}$ is a local orthonormal frame in M we may write this as:

$$\begin{aligned}&\sum_{i,j=1} g^{ij}\ddot{g}_{ij} \\ &= 2\sum_{i,j=1} g^{ij}\langle\nabla_i V, \nabla_j V\rangle - 2\sum_{i,j=1} g^{ij}R_N\left(f_i, V, V, f_j\right) + 2\operatorname{div}_M\left(\nabla_t V\right) \\ &= 2\|\nabla V\|^2 - 2\sum_{i=1} R_N\left(e_i, V, V, e_i\right) + 2\operatorname{div}_M\left(\nabla_t V\right)\end{aligned}$$

where ∇ is the pullback connection on $TN|_M \to M$ and ∇V is the total covariant derivative of V on M.

Finally we compute the second term in A.1. Writing it in terms of $\{e_i\}$

$$\begin{aligned}&\sum_{i,j,k,l=1} g^{ik}\dot{g}_{kl}g^{lj}\dot{g}_{ij} \\ &= \sum_{i,j,l,k=1} g^{ik}\left(\langle\nabla_k V, f_l\rangle + \langle f_k, \nabla_l V\rangle\right)g^{lj}\left(\langle\nabla_i V, f_j\rangle + \langle f_i, \nabla_j V\rangle\right) \\ &= \sum_{i,j=1}\left(\langle\nabla_{e_i}V, e_j\rangle + \langle e_i, \nabla_{e_j}V\rangle\right)^2 = \sum_{i,j=1}\left(-\langle V, \nabla_{e_i}e_j\rangle - \langle\nabla_{e_i}e_j, V\rangle\right)^2 \\ &= 4\sum_{i,j=1}\langle V, \nabla_{e_i}e_j\rangle^2 = 4\sum_{i,j=1}\langle V, \left(\nabla_{e_i}e_j\right)^N\rangle^2 = 4\|\Pi^V\|^2\end{aligned}$$

Thus making the substitutions in A.1 at $t = 0$ we may compute $\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} = \int_M \frac{d^2}{dt^2}\sqrt{g}|_{t=0}dx$ as:

$$\begin{aligned}&\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0}\\&= \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N(e_i, V, V, e_i) + \operatorname{div}_M(\nabla_t V) - 2\|\Pi^V\|^2 \right) dv\end{aligned}$$

where Π^V is the V component of the second fundamental form Π.

Now V is zero on the boundary, thus using the divergence theorem to compute the third term of this we have:

$$\int_M \operatorname{div}_M(\nabla_t V) = \int_{\partial M} \langle \nabla_t V, v\rangle dv_{\partial M} = 0.$$

where v is the outward pointing unit normal field to the boundary. Making this substitution gives the first equation:

$$\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} = \int_M \left(\|\nabla V\|^2 - \sum_{i=1} R_N(e_i, V, V, e_i) - 2\|\Pi^V\|^2 \right) dv \tag{A.2}$$

Looking at the first term and splitting up $\nabla_{e_i}V$ as $\nabla_{e_i}V = (\nabla_{e_i}V)^N + (\nabla_{e_i}V)^T$ we have

$$\begin{aligned}\|\nabla V\|^2 &= \sum_{i=1}\langle (\nabla_{e_i}V)^N, (\nabla_{e_i}V)^N\rangle + \sum_{i=1}\langle (\nabla_{e_i}V)^T, (\nabla_{e_i}V)^T\rangle\\&= \sum_{i=1}\|D_{e_i}V\|^2 + \sum_{i=1}\sum_{j=1}\langle \nabla_{e_i}V, e_j\rangle^2 = \|DV\|^2 + \sum_{i,j=1}\langle V, \nabla_{e_i}e_j\rangle^2\\&= \|DV\|^2 + \|\Pi^V\|^2\end{aligned}$$

Therefore substituting this into the A.2 we obtain:

$$\frac{d^2}{dt^2}\operatorname{vol}(f_t(M))|_{t=0} = \int_M \left(\|DV\|^2 - \sum_{i=1} R_N(e_i, V, V, e_i) - \|\Pi^V\|^2 \right) dv$$

$\square$

This motivates the following definition:

Definition 58. An orientable submanifold M inside N is *stable* if it is minimal and the integral in Proposition 57 is non-negative for all compactly supported normal variation vector fields $V \in \Gamma_c(M, TN)$. A non-orientable submanifold is stable if its orientable double cover is stable.

Remark 59. Again, the above definition generalizes the condition $\frac{d^2}{dt^2} \operatorname{vol}(f_t(M))|_{t=0} \geq 0$ to non-compact and non-orientable manifolds.

Appendix B

Warped Product Metrics

Given two Riemannian manifolds one can construct the product manifold and endow it with a metric in a natural way. This idea is generalized with the notion of a warped product metric, where a smooth function $f > 0$ is taken on the first manifold and used to warp the overall structure. This idea is very important in general relativity. In this section we provide a quick exposition of some of the curvature calculations for a warped product.

Definition 60. Suppose (B, g_B) and (F, g_F) are Riemannian manifolds and let $f > 0$ be a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ with the metric tensor:

$$g = p_1^* g_B + (f \circ p_1)^2 \, p_2^* g_F$$

where $p_1 : B \times F \to B$, $p_2 : B \times F \to F$ are the usual projections.

Remark 61. The usual Riemannian product manifold $B \times F$ can be obtained from the above definition by taking $f = 1$ on B.

Let $M = B \times_f F$ be such a warped product, then TM splits up canonically as $TM = TB \oplus TF$. We make the identifications $TB = TB \oplus \{0\}$ and $TF = \{0\} \oplus TF$ and we say $(X, 0)$ is the lift of $X \in TB$, similarly for TF.

We now give an expose of some of the curvature computations for a warped space $B \times_f F$. We begin with calculation of the gradient of a function h that is constant in F.

Proposition 62. *If $h : B \to \mathbb{R}$ is a function on B, then the gradient* $\mathrm{grad}_{B \times_f F}$ *of the lift $h \circ p_1 : B \times F \to \mathbb{R}$ is the lift to $B \times_f F$ of the gradient* grad_B *of h.*

Proof. Take $V = (0, V) \in TB \oplus TF = TM$. Then $\langle \mathrm{grad}(h \circ p_1), V \rangle = V(h \circ p_1) = dp_1(V)h = 0$ since $dp_1(V) = 0$. Therefore $\mathrm{grad}_{B \times_f F}(h \circ p_1) \in TB \oplus \{0\} \subseteq TM$, we must now check that it is the lift of $\mathrm{grad}_B h$. This amounts to showing that for $X = (X, 0) \in TB \oplus TF = TM$ we have

$\langle dp_1 \operatorname{grad}_{B\times_f F}(h\circ p_1), dp_1 X\rangle = \langle \operatorname{grad}_B h, dp_1 X\rangle$. We show this now:

$$\begin{aligned}\langle dp_1 \operatorname{grad}_{B\times_f F}(h\circ p_1), dp_1 X\rangle &= \langle \operatorname{grad}_{B\times_f F}(h\circ p_1), X\rangle \\ &= X(h\circ p_1) = dp_1(X)h = \langle \operatorname{grad}_B h, dp_1 X\rangle\end{aligned}$$

This concludes the proof. □

Remark 63. So if our function h is constant in F there should be no confusion if we drop the subscripts and simply write $\operatorname{grad} h$ for the gradient of h.

We now give calculations for the Levi-Civita connection on $B \times_f F$, writing it in terms of the connections on B and F.

Proposition 64. *We write ∇ for the Levi-Civita connection on $M = B \times_f F$, and ∇^B, ∇^F for the Levi-Civita connections on B and F respectively. For $X, Y \in TB \oplus \{0\}$ and $V, W \in \{0\} \oplus TF$ we have:*

1. *$\nabla_X Y$ is the lift of $\nabla^B_X Y$ on B.*
2. *$\nabla_X V = \nabla_V X = X(f)f^{-1}V$*
3. *$dp_1(\nabla_V W) = -(\langle V, W\rangle f^{-1}) \operatorname{grad} f$*
4. *$dp_2(\nabla_V W) = \nabla^F_V W$*

Proof. The proof can be found in [O'N83], page 206. □

We may use Proposition 64 to compute the curvature of the warped space as follows:

Proposition 65. *Let $M = B \times_f F$ be a warped product with Levi-Civita connection ∇ and Riemannian curvature tensor R. We write R_B and R_F for the curvature tensors of B and F respectively. If $X, Y, Z \in TB \oplus \{0\}$ and $U, V, W \in \{0\} \oplus TF$ then:*

1. *$R(X, Y) Z$ is the lift of $R_B(X, Y) Z$ on B*
2. *$R(V, X) Y = -f^{-1}H^f(X, Y) V$ where H^f is the Hessian of f*
3. *$R(X, Y) V = R(V, W) X = 0$*
4. *$R(X, V) W = f^{-1}\langle V, W\rangle \nabla_X \operatorname{grad} f$*
5. *$R(V, W) U = R_F(V, W) U - f^{-2}\|\operatorname{grad} f\|^2 (\langle V, U\rangle W - \langle W, U\rangle V)$*

Proof. The proof can be found in [O'N83], page 210. □

Proposition 66. *Let M be a Riemmanian n-manifold and $f > 0$ a smooth function on M. Consider the warped product $M \times_f S^1$, where S^1 is given the usual metric. Then*

$$R_{M \times_f S^1} = R_M - 2\left(\Delta_M f\right) f^{-1}$$

Proof. Take $p \in M \times S^1$ and let $\{e_i\}_{i=1}^{n+1}$ be a local orthonormal frame for $T_pM \times S^1$ with $e_i \in TM$ for $i = 1, \ldots n$, $e_{n+1} \in TS^1$ and $\nabla_{e_i} e_j|_p = 0$ for all i, j. We use the notation R for the curvature tensor of $M \times_f S^1$ and we compute $\mathrm{Ric}_{M \times_f S^1}(e_i, e_i)$, when $i \leq n$

$$\begin{aligned}
\mathrm{Ric}_{M \times_f S^1}(e_i, e_i) &= \sum_{k=1}^{n+1} R\left(e_k, e_i, e_i, e_k\right) \\
&= \sum_{k=1}^{n} R_M\left(e_k, e_i, e_i, e_k\right) + R\left(e_{n+1}, e_i, e_i, e_{n+1}\right) \\
&= \mathrm{Ric}_M\left(e_i, e_i\right) - f^{-1} H^f\left(e_i, e_i\right)
\end{aligned}$$

and when $i = n + 1$

$$\mathrm{Ric}_{M \times_f S^1}(e_{n+1}, e_{n+1}) = \sum_{k=1}^{n} R\left(e_k, e_{n+1}, e_{n+1}, e_k\right) = -f^{-1} \sum_{k=1}^{n} \langle \nabla_{e_k} \mathrm{grad} f, e_k \rangle$$

where we have used Proposition 65. Now $\langle \nabla_{e_k} \mathrm{grad} f, e_k \rangle = e_k \langle \mathrm{grad} f, e_k \rangle - \langle \mathrm{grad} f, \nabla_{e_k} e_k \rangle = e_k e_k(f) - \nabla_{e_k} e_k(f) = H^f(e_k, e_k)$ Therefore computing $R_{M \times_f S^1}$

$$\begin{aligned}
R_{M \times_f S^1} &= \sum_{i=1}^{n+1} \mathrm{Ric}_{M \times_f S^1}(e_i, e_i) \\
&= \sum_{i=1}^{n} \mathrm{Ric}_{M \times_f S^1}(e_i, e_i) + \mathrm{Ric}_{M \times_f S^1}(e_{n+1}, e_{n+1}) \\
&= \sum_{i=1}^{n} \Big(\mathrm{Ric}_M(e_i, e_i) - f^{-1} H^f(e_i, e_i)\Big) - f^{-1} \sum_{k=1}^{n} H^f(e_k, e_k) \\
&= R_M - 2 f^{-1} \sum_{i=1}^{n} H^f(e_i, e_i)
\end{aligned}$$

Now $H^f(e_i, e_i) = e_i e_i(f) - \nabla_{e_i} e_i(f) = e_i e_i(f)$ since $\nabla_{e_i} e_i = 0$ at p. Therefore:

$$R_{M \times_f S^1} = R_M - 2f^{-1} \sum_{i=1}^{n} e_i e_i(f) = R_M - 2(\Delta_M f) f^{-1}$$

$\square$