THE EXACT TAIL ASYMPTOTICS BEHAVIOUR OF THE JOINT STATIONARY DISTRIBUTIONS OF THE GENERALIZED JOIN THE SHORTEST QUEUEING MODEL

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Abstract

Parallel queueing networks have advantage over single server queueing networks, because when some servers simultaneously serve the customers in the line, the efficiency increases. Therefore, in the real world parallel queueing servers such as computer networks and multiple parallel processors, have become common. Since then many scientists have been studying the analysis of parallel queueing networks to give the exact practical models for the real world queueing problems.

One of the topics in parallel queueing networks is the two-dimensional random walk, which recently have been studied by many scientists. The formulation for a random walk model in the first quadrant has been already studied by Fayolle, Malyshev and Iasnogorodski [19]. In this thesis I extend the formulation of a general random walk model to the half plane, including the first and fourth quadrants, and by using kernel method and Tauberian-like Theorem I investigate the exact tail asymptotic behaviour of the joint stationary distribution of the generating functions.

In addition, I apply the results of the formulation of a general random walk model in the half plane to the Generalized-JSQ model, which is a queueing system with two parallel servers that have three streams of arrivals, two of which are dedicated to each servers, and the third one joins the shorter queue. Suppose that arrivals are independent Poisson processes, and service times have identical exponential distributions. Although this queueing model has been already studied by Zhao and Grassmann [75], and M. Miyazawa, [56], in this thesis I will use a different method named kernel method to investigate the exact tail asymptotic behaviour of the generat-

ing functions. The kernel method is simpler and faster than other methods, since in this method we are not dealing with the explicit expressions in terms of generating functions, but we only discuss the dominant singularity and its location.

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I offer my best and special thanks to my parents, best friends of my life, who have supported me morally and financially throughout my years of education. Lastly, I thank my wonderful wife, who is the most valuable asset of my life.

Dedication

I lovingly dedicate this thesis to my father, who was the first teacher of my life, to my mother, whose love and kindness never ends, and also to my wonderful wife, whose love and encouragements are always with me.

Chapter 1

Introduction

1.1 Motivation and Contribution

Previously, the formulation for a random walk model in the first quadrant, $(x \geq 0, y \geq 0)$, was studied by Fayolle, Malyshev and Iasnogorodski [19]. In chapter 3 of this thesis I will extend the formulation of a general random walk model to the half plane, including the first quadrant $(x \geq 0, y \geq 0)$ and the fourth quadrant $(x \geq 0, y \leq 0)$, and by using the kernel method and Tauberian-like Theorem I investigate the exact tail asymptotic behaviour of the joint stationary distribution of the generating functions, which can be used as a reference for the one who is interested in analysing any random walk model in the half plane.

In addition, in chapter 4, I will apply the result of the formulation of a general random walk model in the half plane to a real queueing model, the Generalized-JSQ, which is a queueing system with two parallel servers that have three streams of arrivals, two of which are dedicated to each servers, and the third one joins the shorter queue upon arrival. We assume that arrivals are independent Poisson processes, and service times have identical exponential distributions. Although this queueing model already has been studied by Zhao and Grassmann [75], and M. Miyazawa, [56], in this thesis I will use a different method named the kernel method, which is a simpler and faster method for investigating the exact tail asymptotic behaviour, due to the fact that in the kernel method we are not dealing with the explicit expressions in terms of generating functions, and we only discuss the dominant singularity and its location compared with other singularities to figure out that how far we may expect to extend the radius of convergence in generating functions.

1.2 Literature Review

Parallel queueing networks have advantages over single server queueing networks, because when some servers simultaneously serve the customers in the line, the efficiency increases. Therefore, using parallel queueing servers such as computer networks and multiple parallel processors has become common. As a consequence many scientists and engineers have been studying parallel queueing networks in order to give the exact practical models for these queueing problems. We will see some of the works done in this area in the following paragraph.

In the early 1960s and 1970s, the diffusion approximation method for queueing systems has been discussed by Cox and Miller [13], and later on by Gaver [30]. In 1990-1991 Adan et al. [3] introduced the compensation method which was useful for some specific two-dimensional queueing models. In 1987 and 1990 Blanc [7], and in 1988 Hooghiemstra, Keane, and Van de Ree [35] proposed a new technique which was based on the power-series expansion of the state probabilities. In 1995 B. Blaszczyszyn, A. Frey, and V. Schmidt [5] applied the factorial moment expansion in order to get the approximate formula for stationary characteristics of the multi-server queue with Markov-modulated arrival process and FIFO (first in first served) discipline. In 2005 the large deviation method was applied to modified Jackson Network by Foley and David R. McDonald [15], which is a good tool for analysing the multi-server queueing models. Later in 2008 the large deviation method was used on infinite dimensional stochastic dynamical systems by Amarjit Budhiraja, Paul Dupuis and Vasileios Maroulas [6]. In the first years of 1970s, V. A. Malyshev [60] [61] [62], and also in

1977 L. Flatto and H. P. McKean proposed the generating function method in order to get the asymptotic behaviour of the stationary probabilities of a random walk in the quarter plane containing the points with integer coordinates, $\mathbb{Z}_+^2 = \{(i,j):\ i,j = 0,1,2,...\}$. In 1994 J. W. Cohen [11] used the generating function method in two-dimensional random walk. In 1999 G. Fayolle, R. Iasnogorodski, and V. Malyshev [19] published a book called "Random Walks in the Quarter-plane", in which they have applied many different mathematical tools in order to get the explicit expressions for generating functions of the two-dimensional random walk in the quarter-plane. Halfin [34], Mitzenmacher [63], and Winston [73] have been gaining valuable results for the parallel queues. Although over the past years many researchers have published papers in the area of parallel queues, still JSQ model has lots of problems that need to be worked on. Some work has been done on the stability problems of the JSQ model including Foley and McDonald [16], Foss and Chernova [27] [28], Kurkova [44], Sharifnia [66], Suhov and Vvedenskaya [67], Tandra, Hemachandra and Manjunath [69], Vvedenskaya and Suhov [71], Vvedenskaya, Dobrushin and Karpelevich [72].

1.3 Organization

In chapter 2 of this thesis, I will give some basic definitions and concepts relating to queueing systems; In addition, I will go through the two-dimensional random walks, generating functions, stability conditions for a random walk model, and the analysis of kernel in a fundamental form. Moreover, some real world queueing examples are given for each area. In chapter 3, I will get the formula for analysis of the exact tail asymptotic behaviour of the stationary probabilities in a general two-dimensional random walk model in the first and fourth quadrants using the kernel method. In chapter 4, I use the results of chapter 3 to apply the kernel method to the Generalized-JSQ (Join the Shortest Queue) model in order to obtain the exact tail asymptotic behaviour of the Generalized-JSQ's joint stationary probability distributions, and finally in the last chapter, I will summarize my main findings and achievements and give some suggestions in the area of random walk to the motivated readers for further studies in this area.

Chapter 2

Background

2.1 Queueing Theory

In this section, I will review some definitions and basic concepts from Queueing Theory. A queueing system includes customers arriving at random times to a system and depart after receiving service. Queueing systems are classified according to :
(1) the input process, the probability distribution of arrival of customers in time;
(2) the service time distribution, the probability distribution of the time to serve the customers;
(3) the queueing discipline, the order in which customers are served;
(4) the number of servers.
A very basic queueing formula is $L = \lambda W$, where
$L =$ the average number of customers in the system,
$\lambda =$ the arrival rate of customers to the system,
$W =$ the average waiting time for each customer in the system.

2.1.1 Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is called a counting process when $N(t)$ represents the total number of events that have occurred by time t. $N(t)$ must satisfy:

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer valued.

(iii) if $s < t$, then $N(s) \leq N(t)$ (monotonicity).

(iv) For $s < t$, $N(t) - N(s)$ represents the number of events that have occurred in the interval (s, t).

2.1.2 Poisson Process

One of the simplest counting processes is the Poisson process.

Definition 2.1. *The counting process $\{N(t), t \geq 0\}$ is called a Poisson process having rate $\lambda, \lambda > 0$, if*
(i) $N(0) = 0$.
(ii) *The process has independent increments.*
(iii) *The number of events in any interval t is Poisson distributed with mean λt, for all $s, t \geq 0$.*
$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}, n = 0, 1, ...$

Definition 2.2. *A counting process is called a process with stationary increments if the distribution of the number of events occurring in any time interval only depends on the length of the interval. That is, the process has stationary increments if the number of events in the time interval $(t_1 + a, t_2 + a)$ has the same distribution as the number of events in the time interval (t_1, t_2).*

2.1.3 Markov Chain

In this section we consider a discrete stochastic process $\{X_n, n = 0, 1, 2, ...\}$ that takes on a countable number of possible values in the state space A. If $X_n = i$, then the process is in state i at time n. The notation p_{ij} means the probability of moving from state i to j. If we have the following condition,

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0\} = P\{X_{n+1} = j|\ X_n = i\} = p_{i,j}, \quad (2.1)$$

for all states $i_0, i_1, ..., i_{n-1}, i, j$ and all $n \geq 0$, then stochastic process is called discrete-time Markov chain. Equation (2.1) indicates that the next state in Markov chain only depends on its current state, and does not depend on any previous states.

A discrete-time homogeneous Markov chain is characterized by the stochastic matrix

$$P = ||p_{i,j}||, i, j \in A,$$

such that

$$p_{i,j} \geq 0, \; \sum_j p_{i,j} = 1, \; \forall i \in A.$$

The matrix elements of P^n are denoted by $p_{i,j}^{(n)}$.

Definition 2.3. *A Markov chain is called irreducible if, for every i, j, there exists m, depending on (i, j) such that*

$$p_{i,j}^{(m)} \neq 0.$$

A Markov chain is called aperiodic if, for some $i, j \in A$, the set $\{n : p_{i,j}^{(n)} \neq 0\}$ has greatest common divisor equal to 1.

Definition 2.4. *An irreducible aperiodic Markov chain is called ergodic if, and only if, the equation*

$$\pi P = \pi,$$

where π is the vector $\pi = \{\pi_\alpha, \alpha \in A\}$, has a unique l_1-solution up to a multiplicative factor, which can be chosen

$$\sum_\alpha \pi_\alpha = 1, \; \pi_\alpha > 0.$$

The π_α's are called stationary probabilities.

A continuous-time Markov process is the continuous-time version of a Markov chain. Hence, it is a stochastic process $\{X(t) : t \geq 0\}$, which satisfies the Markovian property or memoryless property, and can only take on values from a state space set A. For $t_m > t_n > 0$, the Markov property says that the conditional probability of an event at time t_m, given the probabilities of that event for all times up to and including time t_n, is only depending on time t_n. The continuous-time Markov chain has many application in queueing systems.

2.1.4 Champan-Kolmogorov Equations

We have already defined the one-step transition probabilities, $p_{i,j}$, now we want to define the n-step transition probabilities denoted by $p_{i,j}^{(n)}$, which indicates the probability that a process in state i will be in state j after n transitions. That is,

$$p_{i,j}^{(n)} = p\{X_{n+m} = j | X_m = i\}, \; n \geq 0, \; i, j \geq 0. \tag{2.2}$$

Obviously $p_{i,j}^{(1)} = p_{i,j}$. The Chapman-Kolmogorov equations provide a method to compute the n-step transition probabilities. These equations are,

$$p_{i,j}^{(n+m)} = \sum_{k=0}^{\infty} p_{i,k}^{(n)} p_{k,j}^{(m)} \tag{2.3}$$

for all $n, m \geq 0$, and all i, j. More information about Chapman-Kolmogorov equation can be found in [65].

2.1.5 Queueing Models

In this section we provide basic information on some well-known queueing models.

1. $M/M/s$ model: The arrivals form a Poisson process, and the service times are exponentially distributed. In this model, there are s servers. The queueing parameters, W, the mean waiting time, and L, the mean queue length, are as follows,

$$W = \frac{(\sum_{j=0}^{n-1} \frac{1}{j!}(\frac{\lambda}{\mu})^j + \frac{(\frac{\lambda}{\mu}^n)}{s!(\frac{1-\lambda}{s\mu})})^{-1}(\frac{\lambda}{s\mu})\lambda^s}{\lambda\ s!\ \mu^s(1-\frac{\lambda}{\mu})^2} + \frac{1}{\mu}, \tag{2.4}$$

$$L = \frac{(\sum_{j=0}^{n-1} \frac{1}{j!}(\frac{\lambda}{\mu})^j + \frac{(\frac{\lambda}{\mu}^n)}{s!(\frac{1-\lambda}{s\mu})})^{-1}(\frac{\lambda}{s\mu})\lambda^s}{s!\ \mu^s(1-\frac{\lambda}{\mu})^2} + \frac{\lambda}{\mu}. \tag{2.5}$$

2. $M/G/1$: Like in the previous model, arrivals follow a Poisson process with rate λ, while service time has an arbitrary distribution, $G(y) = Pr\{Y_k \leq y\}$ with finite mean service time $\nu = E[Y_k]$. Also, the service rate is $\mu = \frac{1}{\nu}$. In this model there is one server.

3. $M/G/\infty$: It is like the $M/G/1$ model except that in this model there is infinite number of servers instead of one server.

2.2 Transition Probabilities

In this section we introduce the transition probabilities for the first and fourth quadrants of a two-dimensional random walk plane.
A two-dimensional transition probability is the probability of moving from state, (m, n), to another state, $(m + i, n + j)$, which is independent of m and n. The maximum step in any dimension is ± 1. Transition probabilities depend only on step sizes, except for

boundaries. Therefore, we define $p_{i,j}$ as follows,

$$\bar{P}_{(m,n),(m+i,n+j)} = \begin{cases} p_{i,j} & \text{if } m \geq 1 \text{ and } n \geq 1,\ -1 \leq i,j \leq 1, \\ p^{(0)}_{i,j} & \text{if } (m,n) = (0,0),\ -1 \leq i,j \leq 1, \\ p^{(1)}_{i,j} & \text{if } m \geq 1 \text{ and } n = 0,\ -1 \leq i,j \leq 1, \\ p^{(2)}_{i,j} & \text{if } m = 0 \text{ and } n \geq 1,\ -1 \leq i,j \leq 1, \\ p^{(-)}_{i,j} & \text{if } m \geq 1 \text{ and } n \leq -1,\ -1 \leq i,j \leq 1, \\ p^{(-2)}_{i,j} & \text{if } m = 0 \text{ and } n \leq -1,\ -1 \leq i,j \leq 1, \end{cases}$$

where $p_{i,j},\ p^{(0)}_{i,j},\ p^{(1)}_{i,j},\ p^{(2)}_{i,j},\ p^{(-2)}_{i,j}$, and $p^{(-)}_{i,j}$ are non negative real numbers in $[0,1]$. Since in this thesis we are dealing with discrete-time Markov chains which are homogeneous random walks, we have

$$\sum_{i,j=0,1} p^{(0)}_{i,j} = 1, \quad \sum_{i=0,\pm1,j=0,1} p^{(1)}_{i,j} = 1, \quad \sum_{i,j=0,1} p^{(-)}_{i,j} = 1,$$
$$\sum_{i=0,1,j=0,\pm1} p^{(2)}_{i,j} = 1, \quad \sum_{i=0,\pm1,j=0,\pm1} p_{i,j} = 1. \quad \sum_{i,j=0,1} p^{(-2)}_{i,j} = 1.$$

2.3 Random Walk in the Quarter Plane

Discrete-time Markov chains which are homogeneous two-dimensional random walks, have three main properties as follows,

1. The state space is two-dimensional, and consists of points with non-negative integer coordinates. So the state space is $A = \mathbb{Z}^2_+ = \{(i,j) : i,j \geq 0 \text{ are integers}\}$

2. Because of the boundaries, we partition $\mathbb{Z}^2_+$ as follows

$$\mathbb{Z}^2_+ = S \cup S^{(1)} \cup S^{(2)} \cup \{(0,0)\},$$

where

$S = \{(i,j) : i,j > 0\}$, $\quad S^{(1)} = \{(i,0) : i > 0\}$, $\quad S^{(2)} = \{(0,j) : j > 0\}$.

3. We assume that jumps are bounded. Therefore,

$$p_{(i,j)(i+\alpha,j+\beta)} = 0, \quad \text{unless } -1 \leq \alpha, \beta \leq 1,$$

Next we introduce an important theorem which indicates the stability conditions for a random walk model in first quadrant ($x \geq 0, y \geq 0$).

Theorem 2.1. *Let M, $M^{(1)}$ and $M^{(2)}$ be as follows*

$$\begin{cases} M = (M_x, M_y) = (\sum ip_{ij}, \sum jp_{ij}) \\ M^{(1)} = (M_x^{(1)}, M_y^{(1)}) = (\sum ip_{ij}^{(1)}, \sum jp_{ij}^{(1)}) \\ M^{(2)} = (M_x^{(2)}, M_y^{(2)}) = (\sum ip_{ij}^{(2)}, \sum jp_{ij}^{(2)}) \end{cases}$$

then when $M \neq 0$, a random walk is ergodic if, and only if, one of the following conditions holds,

(i) $\begin{cases} M_x < 0,\ M_y < 0 \\ M_x M_y^{(1)} - M_y M_x^{(1)} < 0, \\ M_y M_x^{(2)} - M_x M_y^{(2)} < 0; \end{cases}$

(ii) $M_x < 0,\ M_y \geq 0,\ M_y M_x^{(2)} - M_x M_y^{(2)} < 0;$

(iii) $M_x \geq 0,\ M_y < 0,\ M_x M_y^{(1)} - M_y M_x^{(1)} < 0;$

Proof. The probabilistic proof of this theorem is given in [19]. □

2.4 Generating Functions

In this section we introduce the generating functions and the fundamental form for a random walk in the quarter plane which play an important role in the kernel analysis.

As stated in section 2.3, because of the boundaries, the state space of two-dimensional random walk can be written as the union of disjoint classes as follows,

$$\mathbb{Z}_+^2 = S \cup S^{(1)} \cup S^{(2)} \cup \{(0,0)\},$$

As discussed earlier, two states of the same class have the same transition probabilities. Let X_n denote the state of the random walk at time n. Now we define two complex variables, u_1, and u_2, one for each direction. The variable u is the vector of complex variables in $\mathbb{C}^2$ as

$$u = (u_1, u_2),\ \ |u_i| = 1,\ \ u_i \in \mathbb{C},\ \ i = 1, 2,$$

and the jump generating functions are as follows

$$\begin{aligned}
P_1(u) &= E[u^{X_{n+1}-X_n} | X_n = z \in S],\\
P_2(u) &= E[u^{X_{n+1}-X_n} | X_n = z \in S^{(1)}],\\
P_3(u) &= E[u^{X_{n+1}-X_n} | X_n = z \in S^{(2)}],\\
P_4(u) &= E[u^{X_{n+1}-X_n} | X_n = z \in \{(0,0)\}].
\end{aligned} \tag{2.6}$$

Here

$$u^z = \prod_{n=1}^{2} u_n^{z_n},$$

where u_n, z_n, $n = 1, 2$, are coordinates of the vector u and z respectively. Therefore we have

$$\begin{aligned}
E[u^{X_{n+1}}] =& E[u^{X_n} u^{X_{n+1}-X_n}]\\
=& E[u^{X_n} 1_{\{X_n \in S\}}] P_1(u) + E[u^{X_n} 1_{\{X_n \in S^{(1)}\}}] P_2(u) +\\
& E[u^{X_n} 1_{\{X_n \in S^{(2)}\}}] P_3(u) + E[u^{X_n} 1_{\{X_n \in \{(0,0)\}\}}] P_4(u).
\end{aligned} \tag{2.7}$$

Now we introduce the generating function

$$\begin{aligned}
\pi_1(u) &= \sum_{z \in S} \pi_z u^z,\\
\pi_2(u) &= \sum_{z \in S^{(1)}} \pi_z u^z,\\
\pi_3(u) &= \sum_{z \in S^{(2)}} \pi_z u^z,\\
\pi_4(u) &= \pi_{0,0},
\end{aligned} \tag{2.8}$$

where π_z indicates the stationary probability of being in state z.

Using (2.8) and taking the limit as $n \to \infty$ in (2.7), we get

$$\sum_{r=1}^{4}[1 - P_r(u)]\pi_r(u) = 0. \tag{2.9}$$

We can rewrite (2.9) as

$$-h(x,y)\pi(x,y) = h_1(x,y)\pi(x) + h_2(x,y)\tilde{\pi}(y) + \pi_{0,0}h_0(x,y), \tag{2.10}$$

which is called fundamental form, where

$$\begin{cases} \pi(x,y) = \sum\limits_{i,j=1}^{\infty} \pi_{i,j}x^{i-1}y^{j-1}, \\ \pi(x) = \sum\limits_{i\geq 1} \pi_{i,0}x^{i-1}, \\ \tilde{\pi}(y) = \sum\limits_{j\geq 1} \pi_{0,j}y^{j-1}, \end{cases} \tag{2.11}$$

and

$$\begin{aligned} h(x,y) &= xy(\sum_{i=-1}^{1}\sum_{j=-1}^{1} p_{i,j}x^i y^j - 1) \\ &= a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \end{aligned} \tag{2.12}$$

$$\begin{aligned} h_1(x,y) &= x(\sum_{i=-1}^{1}\sum_{j=0}^{1} p_{i,j}^{(1)}x^i y^j - 1) \\ &= (p_{-1,0}^{(1)} + p_{0,0}^{(1)}x + p_{1,0}^{(1)}x^2) + (p_{-1,1}^{(1)} + p_{0,1}^{(1)}x + p_{1,1}^{(1)}x^2)y - x, \end{aligned} \tag{2.13}$$

$$\begin{aligned} h_2(x,y) &= y(\sum_{i=0}^{1}\sum_{j=-1}^{1} p_{i,j}^{(2)}x^i y^j - 1) \\ &= (p_{0,-1}^{(2)} + p_{0,0}^{(2)}y + p_{0,1}^{(2)}y^2) + (p_{1,-1}^{(2)} + p_{1,0}^{(2)}y + p_{1,1}^{(2)}y^2)x - y, \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
h_0(x,y) &= x(\sum_{i=0}^{1}\sum_{j=0}^{1} p^{(0)}_{i,j} x^i y^j - 1) \\
&= (p^{(0)}_{0,0} + p^{(0)}_{1,0}x + p^{(0)}_{0,1}y + p^{(0)}_{1,1}xy) - 1, \qquad (2.15)
\end{aligned}
$$

with

$$a(x) = -(p_{-1,1} + p_{0,1}x + p_{1,1}x^2), \qquad (2.16)$$
$$b(x) = -(p_{-1,0} + (p_{0,0} - 1)x + p_{1,0}x^2), \qquad (2.17)$$
$$c(x) = -(p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2), \qquad (2.18)$$
$$\tilde{a}(y) = -(p_{1,-1} + p_{1,0}y + p_{1,1}y^2), \qquad (2.19)$$
$$\tilde{b}(y) = -(p_{0,-1} + (p_{0,0} - 1)y + p_{0,1}y^2), \qquad (2.20)$$

2.4.1 Queueing Examples

Here we give some examples of random walks in queueing systems and their corresponding generating functions.

1. The Symmetric Join the Shortest Queue Model

In this queueing model, customers arrive to system with the rate of λ. Upon arrival, customers determine which server has the shorter queue, and then join the shorter one. Also service rates of customers from both servers have the same value of μ. If the length of the both server's queues are the same, an arriving customer will join either queue with the probability of 1/2. In this queueing model, the random walk probabilities are as follows,

$$
\begin{aligned}
&p_{-1,1} = \mu,\ p_{0,-1} = \mu,\ p_{1,-1} = \lambda, \\
&p^{(1)}_{-1,1} = 2\mu,\ p^{(1)}_{0,1} = \lambda, \\
&p^{(2)}_{0,-1} = \mu,\ p^{(2)}_{1,-1} = \lambda,\ p^{(2)}_{0,0} = \mu, \\
&p^{(0)}_{0,1} = \lambda,\ p^{(0)}_{0,0} = 2\mu.
\end{aligned}
$$

For this model the generating functions are

$$\begin{aligned}
h(x,y) &= xy - \mu y^2 - \mu x - \lambda x^2,\\
h_1(x,y) &= 2\mu y + \lambda xy - x,\\
h_2(x,y) &= \mu y + \mu + \lambda x - y,\\
h_0(x,y) &= 2\mu + \lambda y - 1.
\end{aligned}$$

2. Symmetric Join the Shorter Queue with Coupled processor

This model is similar to the Symmetric JSQ with the difference that whenever the length of Q_i is zero, the service rate for the other queue will be changed from μ_j to $\mu_j + \mu_i^*$. In this queueing model when x-axis represents min $\{Q_1, Q_2\}$, and y-axis represents the $|Q_2 - Q_1|$, the transition probabilities of random walk are as follows,

$$\begin{aligned}
&p_{1,-1} = \lambda,\ p_{0,-1} = \mu,\ p_{-1,1} = \mu,\\
&p^{(1)}_{0,1} = \lambda,\ p^{(1)}_{-1,1} = 2\mu,\\
&p^{(2)}_{1,-1} = \lambda,\ p^{(2)}_{0,-1} = \mu + \mu^*,\ p^{(2)}_{0,0} = 1 - (\lambda + \mu + \mu^*),\\
&p^{(0)}_{0,1} = \lambda,\ p^{(0)}_{0,0} = 2\mu,
\end{aligned}$$

And with respect to the transition probabilities, the generating functions are as follows,

$$\begin{aligned}
h(x,y) &= xy - \mu y^2 - \mu x - \lambda x^2,\\
h_1(x,y) &= 2\mu y + \lambda xy - x,\\
h_2(x,y) &= \mu y + \mu + \lambda x - y,\\
h_0(x,y) &= 2\mu + \lambda y - 1.
\end{aligned}$$

3. The Pre-Emptive Priority Queueing System

In this example we consider the $M/M/1$ pre-emptive queueing model. In this queue we have two classes of customers with different priority of being served. Higher and lower priority customers arrivals occur according to a Poisson process with rates of λ_1 and λ_2 respectively. The service rule is FIFO (first in first served), and it is preemptive.

That is, if a lower priority customer is being served, his service is interrupted upon arrival of a higher priority customer to system, and the lower priority customer is waiting at the head of the queue until the service of the higher priority customer is completed and his service continues again. There is one server with exponential service time in this model, which may have different service rates, μ_1, and μ_2 when serving respectively the higher priority customer or lower one. Without loss of generality we assume that $\lambda_1+\lambda_2+\mu_1+\mu_2 = 1$.

The transition probabilities of the random walk of the quarter plane in which the x-axis corresponds to the length of the queue of higher priority customers, and y-axis corresponds to the queue length of lower priority customers are given by,

$$\begin{aligned}
&p_{1,0} = \lambda_1,\ p_{0,1} = \lambda_2,\ p_{-1,0} = \mu_1,\ p_{0,0} = \mu_2\\
&p^{(1)}_{1,0} = \lambda_1,\ p^{(1)}_{0,1} = \lambda_2,\ p^{(1)}_{-1,0} = \mu_1,\ p^{(1)}_{0,0} = \mu_2,\\
&p^{(2)}_{1,0} = \lambda_1,\ p^{(2)}_{0,1} = \lambda_2,\ p^{(2)}_{0,0} = \mu_1,\ p^{(2)}_{0,-1} = \mu_2,\\
&p^{(0)}_{1,0} = \lambda_1,\ p^{(0)}_{0,0} = \mu_1 + \mu_2,\ p^{(0)}_{0,1} = \lambda_2.
\end{aligned}$$

According to the transition probabilities above, we have the following generating functions for the pre-emptive priority queueing system,

$$\begin{aligned}
&h(x, y) = xy - [\lambda_1 x_2 + (\mu_2 + \lambda_2 y)x + \mu_1]y,\\
&h_1(x, y) = [\lambda_1 x^2 + (\mu_2 + \lambda_2 y)x + \mu_1] - x,\\
&h_2(x, y) = [\lambda_1 xy + \lambda_2 y^2 + \mu_1 y + \lambda_2] - y,\\
&h_0(x, y) = [\lambda_1 x + \lambda_2 y + \mu_1 + \mu_2] - 1.
\end{aligned}$$

4. The Restricted Jackson Network

In this queueing model, we have two queues, q_i, $i = 1, 2$. Each of them has an external arrival stream which is a Poisson process with rate of λ_i. The service times in both queues are exponential, and departures from the servers while busy occur at rate of μ_i, $i = 1, 2$. Each customer who completes his service in the q_i has two options, either he gets out of the system with probability of $r_{i,0}$, or he joins the other queue with probability of r_{ij}. Also without loss of generality we assume that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 = 1$.

For this queueing model, the transition probabilities are given by,

$$
\begin{aligned}
&p_{1,0} = \lambda_1,\ p_{0,1} = \lambda_2,\ p_{-1,1} = r_{12}\mu_1,\ p_{-1,0} = r_{10}\mu_1,\ p_{0,-1} = r_{20}\mu_2,\\
&p_{1,-1} = r_{21}\mu_2,\\
&p^{(1)}_{1,0} = \lambda_1,\ p^{(1)}_{0,1} = \lambda_2,\ p^{(1)}_{-1,1} = r_{12}\mu_1,\ p^{(1)}_{-1,0} = r_{10}\mu_1,\ p^{(1)}_{0,0} = \mu_2,\\
&p^{(2)}_{1,0} = \lambda_1,\ p^{(2)}_{0,1} = \lambda_2,\ p^{(2)}_{0,0} = \mu_1,\ p^{(2)}_{0,-1} = r_{20}\mu_2,\ p^{(2)}_{1,-1} = r_{21}\mu_2,\\
&p^{(0)}_{1,0} = \lambda_1,\ p^{(0)}_{0,1} = \lambda_2,\ p^{(0)}_{0,0} = \mu_1 + \mu_2.
\end{aligned}
$$

According to transition probabilities the generating functions are,

$$
\begin{aligned}
h(x,y) =& xy - (\lambda_1 x^2 y + r_{21}\mu_2 x^2 + \lambda_2 x y^2 + r_{20}\mu_2 x + r_{12}\mu_1 y^2 + \\
& r_{10}\mu_1 y),\\
h_1(x,y) =& (r_{12}\mu_1 + \lambda_2 x)y + (r_{10}\mu_1 + \mu_2 x + \lambda_1 x^2) - x,\\
h_2(x,y) =& (r_{21}\mu_2 + \lambda_1 y)x + (r_{20}\mu_2 + \mu_1 y + \lambda_2 y^2) - y,\\
h_0(x,y) =& \lambda_1 x + \lambda_2 y + \mu_1 + \mu_2 - 1.
\end{aligned}
$$

5. The Classical Tandem Queueing Model

In this model, there are two servers in tandem. Customers are arriving to the first server with rate of λ_1, and being served with rate of μ_1. After completion of their service in the first server, they go to the second server to receive service with rate of μ_2, and after completion of the their second service with server two, they leave the system. Without loss of generality we assume that $\lambda + \mu_1 + \mu_2 = 1$.

The transition probabilities for the classical Tandem queue are given by,

$$
\begin{aligned}
&p_{1,0} = \lambda,\ p_{-1,1} = \mu_1,\ p_{0,-1} = \mu_2,\\
&p^{(1)}_{1,0} = \lambda,\ p^{(1)}_{-1,1} = \mu_1,\ p^{(1)}_{0,0} = \mu_2,\\
&p^{(2)}_{1,0} = \lambda,\ p^{(2)}_{0,0} = \mu_1,\ p^{(2)}_{0,-1} = \mu_2,\\
&p^{(0)}_{1,0} = \lambda,\ p^{(0)}_{0,0} = \mu_1 + \mu_2.
\end{aligned}
$$

Using the above transition probabilities, one can find the generating function as follows,

$$\begin{aligned}
h(x,y) &= xy - (\mu_1 y^2 + \lambda x^2 y + \mu_2 x),\\
h_1(x,y) &= (\lambda x^2 + \mu_2 x + \mu_1 y) - x,\\
h_2(x,y) &= (\lambda xy + \mu_2 + \mu_1 y) - y,\\
h_0(x,y) &= \lambda x + \mu_1 + \mu_2 - 1.
\end{aligned}$$

6. Two Demands Queueing System

In this queueing model, there is one stream of customers with arrival rate λ. The server of q_i, $i = 1, 2$ has service rate of μ_i, with an exponential service time independent of arrival rate λ. The stability conditions in this model are $\lambda < \mu_1$, and $\lambda < \mu_2$. Without loss of generality we assume that $\lambda + \mu_1 + \mu_2 = 1$. The transition probabilities for this queue are

$$\begin{aligned}
&p_{1,1} = \lambda,\ p_{-1,0} = \mu_1,\ p_{0,-1} = \mu_2,\\
&p^{(1)}_{1,1} = \lambda,\ p^{(1)}_{-1,0} = \mu_1,\ p^{(1)}_{0,0} = \mu_2,\\
&p^{(2)}_{1,1} = \lambda,\ p^{(2)}_{0,0} = \mu_1,\ p^{(2)}_{0,-1} = \mu_2),\\
&p^{(0)}_{1,1} = \lambda,\ p^{(0)}_{0,0} = \mu_1 + \mu_2.
\end{aligned}$$

Knowing the transition probabilities for this queue, one can get the following generating functions,

$$\begin{aligned}
h(x,y) &= xy - (\mu_1 y + \lambda x^2 y^2 + \mu_2 x),\\
h_1(x,y) &= \lambda x^2 y + \mu_2 x + \mu_1 - x,\\
h_1(x,y) &= \lambda y^2 x + \mu_2 + \mu_1 y - y,\\
h_0(x,y) &= \lambda xy + \mu_1 + \mu_2 - 1.
\end{aligned}$$

2.5 Stability Condition

In this section we will derive the stability conditions of some well-known queueing examples by using the theorem (2.1) by Fayolle, Iasnogorodski, and Malyshev.

1. The Symmetric Join the Shortest Queue Model

Considering the transition probabilities for the Symmetric JSQ model, we have

$$\begin{cases} M_x = \lambda - \mu, \qquad M_y = -\mu, \\ M_x^{(1)} = -2\mu, \qquad M_y^{(1)} = \lambda, \\ M_x^{(2)} = \lambda, \qquad M_y^{(2)} = -\lambda. \end{cases}$$

So by using the theorem (2.1), we have the following situations,

$$(i) \begin{cases} (1)\ \lambda < \mu, \quad -\mu < 0, \\ (2)\ \lambda(\lambda - \mu) - 2\mu^2 < 0, \\ \Rightarrow (\lambda + \mu)(\lambda - 2\mu) < 0 \Rightarrow \lambda < 2\mu, \\ (3)\ (\lambda - \mu)\mu - \lambda\mu < 0, \\ \Rightarrow -\mu^2 < 0, \end{cases}$$

(ii) $\lambda < \mu_1, \qquad -\mu \geq 0,$ (not possible)

(iii) $\mu \leq \lambda, \qquad -\mu_1 < 0,$

$\Rightarrow (\mu + \lambda)(\lambda - 2\mu) < 0 \Rightarrow \lambda < 2\mu.$

Therefore, this queue is ergodic and stable when

$$\lambda < \mu.$$

2. Symmetric Join the Shorter Queue with Coupled processor

From theorem (2.1) we have

$$\begin{cases} M_x = \lambda - \mu, \qquad M_y = -\mu, \\ M_x^{(1)} = -2\mu, \qquad M_y^{(1)} = \lambda, \\ M_x^{(2)} = \lambda, \qquad M_y^{(2)} = -\mu - \mu^*. \end{cases}$$

Hence, the stability conditions are as follows,

$$(i)\begin{cases}(1)\ \lambda<\mu, \quad -\mu<0,\\ (2)\ \lambda(\lambda-\mu)-2\mu^2<0,\\ \Rightarrow(\lambda+\mu)(\lambda-2\mu)<0\Rightarrow\lambda<2\mu,\\ (3)\ (\lambda-\mu)(\mu+\mu^*)-\lambda\mu<0,\\ \Rightarrow(\lambda-\mu)\mu^*-\mu^2<0,\end{cases}$$

(ii) $\lambda<\mu_1, \qquad -\mu\geq 0,$ (not possible)

(iii) $\mu\leq\lambda, \qquad -\mu_1<0,$

$\Rightarrow(\mu+\lambda)(\lambda-2\mu)<0\Rightarrow\lambda<2\mu.$

Therefore, considering the conditions above, this queue is ergodic if

$$\lambda<2\mu.$$

3. The Pre-Emptive Priority Queueing System

For this queueing model we have

$$\begin{cases}M_x=\lambda_1-\mu_1, \qquad M_y=\lambda_2,\\ M_x^{(1)}=\lambda_1-\mu_1, \qquad M_y^{(1)}=\lambda_2,\\ M_x^{(2)}=\lambda_1, \qquad M_y^{(2)}=\lambda_2-\mu_2.\end{cases}$$

Hence, the stability conditions are as follows,

$$(i)\begin{cases}(1)\ \lambda<\mu_1, \quad \lambda_2<0, \quad \text{(not possible)}\\ (2)\ (\lambda_1-\mu_1)\lambda_2-\lambda_2(\lambda_1-\mu_1)<0,\\ (3)\ \lambda_2\lambda_1-(\lambda_1-\mu_1)(\lambda_2-\mu_2)<0,\end{cases}$$

(ii) $\lambda<\mu_1, \quad \lambda_2\geq 0, \quad \lambda_2\lambda_1-(\lambda_1-\mu_1)(\lambda_2-\mu_2)<0,$

$$\Rightarrow\lambda_1\mu_2+\lambda_2\mu_1-\mu_1\mu_2<0\Rightarrow\frac{\lambda_1}{\mu_1}+\frac{\lambda_2}{\mu_2}<1,$$

(iii) $\lambda\geq\mu_1, \quad \lambda_2<0.$ (not possible)

Therefore, this queue is ergodic if, and only if $\rho=\rho_1+\rho_2<1$,

where,

$$\rho_1 = \frac{\lambda_1}{\mu_1}, \quad \text{and} \quad \rho_2 = \frac{\lambda_2}{\mu_2}.$$

4. The Restricted Jackson Network

In this queue matrix M is

$$\begin{cases} M_x = \lambda_1 + r_{21}\mu_2 - r_{12}\mu_1 - r_{10}\mu_1, \\ M_y = \lambda_2 + r_{12}\mu_1 - r_{21}\mu_2 - r_{20}\mu_2, \\ M_x^{(1)} = \lambda_1 - (r_{10}\mu_1 + r_{12}\mu_1), \quad M_y^{(1)} = \lambda_2 + r_{12}\mu_1, \\ M_x^{(2)} = \lambda_1 + r_{21}\mu_2, \quad M_y^{(2)} = \lambda_2 - r_{21}\mu_2. \end{cases}$$

With respect to the matrix M above, and also considering theorem (2.2) we have the stability condition for the restricted Jackson network as follows,

$$\rho_1 < 1, \quad \text{and} \quad \rho_2 < 1.$$

5. The Classical Tandem Queueing Model

For Tandem queue is matrix M is

$$\begin{cases} M_x = \lambda - \mu_1, \qquad M_y = \mu_1 - \mu_2 \\ M_x^{(1)} = \lambda - \mu_1, \qquad M_y^{(1)} = \mu_1, \\ M_x^{(2)} = \lambda, \qquad M_y^{(2)} = -\mu_2. \end{cases}$$

Therefore, by applying theorem (2.1) this queue is ergodic if, and only if $\lambda < \mu_1, \quad \text{and} \quad \lambda < \mu_2$.

6. Two Demands Queueing System

With respect to the transition probabilities for Two demands queueing system we have

$$\begin{cases} M_x = \lambda - \mu_1, \qquad M_y = \lambda - \mu_2 \\ M_x^{(1)} = \lambda - \mu_1, \qquad M_y^{(1)} = \lambda, \\ M_x^{(2)} = \lambda, \qquad M_y^{(2)} = \lambda - \mu_2. \end{cases}$$

Hence, by applying theorem (2.1) this queue is ergodic if, and only if

$$\lambda < min\ \{\mu_1, \mu_2\}.$$

2.6 Riemann Surfaces

Here we provide a summary of algebraic functions and Riemann surfaces which we need for our later discussions. An important class of complex variable functions are algebraic functions and their integrals. An analytic function $y = y(x)$ is called an algebraic function if it satisfies the following equation,

$$a_0(x)y^n + a_1(x)y^{n-1} + ... + a_n(x) = 0, \quad a_0(x) \neq 0, \tag{2.21}$$

where $a_i(x)$, $i = 0, 1, 2, ..., n$ is a polynomial in x with complex coefficients. Furthermore, a rational function of x and y is of the form,

$$R(x, y) = \frac{b_0(x)y^m + b_1(x)y^{m-1} + ... + b_m(x)}{c_0(x)y^k + c_1(x)y^{k-1} + ... + c_k(x)}, \tag{2.22}$$

where b_i, $i = 1, 2, ..., m$, and c_i, $i = 1, 2, ..., k$, are polynomials in x with complex coefficients, and the denominator is not identically zero.

The region on which an algebraic function is defined and single valued is a Riemann surface. The simplest algebraic functions are of the form

$$a_0(x)y + a_1(x) = 0, \tag{2.23}$$

where $a_0(x)$ and $a_1(x)$ are polynomials in x with complex coefficients. Hence, $y = \dfrac{-a_1(x)}{a_0(x)}$ is a rational function of x.

In this thesis we are dealing with algebraic functions of the form

$$h(x, y) = a_0(x)y^2 + a_1(x)y + a_2(x), \tag{2.24}$$

where $a_i(x)$, for $i = 0, 1, 2$ are polynomials in x, and $a_0(x) \neq 0$. In this case we change the variable by

$$\zeta = 2a_0(x)y + a_1(x), \tag{2.25}$$

to get

$$\zeta^2 - p(x) = 0, \tag{2.26}$$

where $p(x) = a_1^2(x) - 4a_0(x)a_2(x)$.

A two-dimensional manifold $\mathbb{M}$ is a topological space, if every point of that is surrounded by a neighbourhood, which is homeomorphic to an open disk in the complex plane $\mathbb{C}$. A pair $\{U, \varphi\}$, which is formed by neighbourhood $U \subseteq \mathbb{M}$ and its associated homeomorphism φ is called a chart. The mapping, $\Phi : U \to \mathbb{C}$, defines a system of local coordinated in U. A collection of charts $\{(U_i, \varphi_i),\ i \in I\}$, where for some index set I, $\{U_i,\ i \in I\}$ is an open covering of $\mathbb{M}$, is called Atlas $\mathbb{A}$.

A connected two-dimensional manifold $\mathbb{M}$ is Riemann surface S, if there exists an atlas $\mathbb{A}_{\mathbb{S}}$ with the following property:

For any pair $\{U, \varphi\}$, $\{V, \varphi\}$ of charts in $\mathbb{A}_{\mathbb{S}}$, such that $U \cap V \neq \phi$, the mapping $\varphi\ o\ \psi^{-1}$ is holomorphic in $\psi(U \cap V) \subset \mathbb{C}$.

The classical notion of holomoorphic functions can be generalized to the case of Riemann surfaces. Let S be a Riemann surface, $\mathbb{A}_{\mathbb{S}}$ its atlas, and $Y \subset S$ an open connected set of S. A function $f :\ Y \to \mathbb{C}$ is holomorphic in Y, if, for any chart $\{U, \varphi\}$ in $\mathbb{A}_{\mathbb{S}}$, the mapping $f\ o\ \varphi^{-1}\ :\ \varphi(U) \to \mathbb{C}$ is holomorphic in the normal sense in the open set $\varphi(U) \subset\ \mathbb{C}$.

Now let S and T be two Riemann surfaces. The mapping $f\ :\ S \to T$ is holomorphic if, for any pair of charts $\{U, \varphi\}$, $\{V, \varphi\}$ belonging to $\mathbb{A}_{\mathbb{S}}$ and $\mathbb{A}_{\mathbb{T}}$ respectively, with $f(U)\ \subset\ V$, the mapping $\psi\ o\ f\ \varphi^{-1}$ is holomorphic in $\Phi(U)\ \subset \mathbb{C}$. The uniqueness theorem remains valid: if f_1 and f_2 are two holomorphic functions from S to T, which are equal on an infinite compact set of S, then they must be equal everywhere.

2.6.1 Genus 1 or Genus 0

The Riemann surfaces which are generated by fundamental form and generating functions are divided into 2 cases, genus 1 or genus 0. In fact, Riemann surface has genus 0 if, and only if the discriminant, $D(x) = b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0$, of the generating function equation, $Q(x) = a(x)y^2 + b(x)y + c(x) = 0$, has multiple zeros. That is, whenever the number of zeros of the discriminant is 1 or 2 the Riemann surface is of genus 0 which creates sphere, and when the number of discriminant's zeros is 3 or 4 the Riemann surface is of genus 1 which creates torus. For further studies one can refer to [19]. The following lemma gives the explicit classification of genus 1 and 0.

Lemma 2.1. *(Fayolle, Iasnogorodski, Malyshev) For all non singular random walks, a Riemann surface has genus 0 if, and only if one the following holds,*

$$M_x = M_y = 0,$$

$$p_{1,0} = p_{1,1} = p_{0,1} = 0,$$

$$p_{1,0} = p_{1,-1} = p_{0,-1} = 0,$$

$$p_{-1,0} = p_{-1,-1} = p_{0,-1} = 0,$$

$$p_{0,1} = p_{-1,0} = p_{-1,1} = 0.$$

2.7 Analysis of the Kernel

The equation, $h(x, y) = 0$, is called kernel equation. It is usually a quadratic equation in terms of x and y. If we want to solve the kernel equation for y, we see that for each value of x, there are two values for y. The solutions of this equation gives us a Riemann surface. Whenever the delta of $D(x)$ or $D(y)$ has three or four distinct zeros, then the Riemann surface will be of genus 1, which yields a torus. However, if $D(x)$ or $D(y)$ has one, or two distinct roots, then the Riemann surface will be of genus 0, which yields a sphere. We denote the roots of $D(x)$ by x_1, x_2, x_3, and x_4. There is an important lemma by Fayolle, Iasnogorodski, and Malyshev [19], which indicates the relationship between x_i, $i = 1, 2, 3, 4$, under different conditions. Here we state the lemma.

Lemma 2.2. *(Fayolle, Iasnogorodski, and Malyshev) A non-singular random walk with $M_y \neq 0$, $y(x)$ has two real branch points x_1 and x_2 inside the unit circle, and two real branch points, x_3 and x_4, outside of the unit circle. The following classifications hold for the pair (x_3, x_4).*

1. *If $p_{1,0} > 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 < x_4 < \infty$;*
2. *If $p_{1,0} = 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 < x_4 = \infty$;*
3. *If $p_{1,0} < 2\sqrt{p_{1,1}p_{1,-1}}$, then $1 < x_3 \leq -x_4 < \infty$; Also for the pair (x_1, x_2) we have*
4. *If $p_{-1,0} > 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $0 < x_1 < x_2 < 1$;*
5. *If $p_{-1,0} = 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $x_1 = 0$ and $0 < x_2 < 1$;*
6. *If $p_{-1,0} < 2\sqrt{p_{-1,1}p_{-1,-1}}$, then $0 < -x_1 \leq x_2 < 1$.*

Proof of this lemma can be found in [19].

Lemma 2.3. *(Fayolle, Iasnogorodski, and Malyshev) For non-singular random walks with $M_y = 0$, one of the branch points of $y(x)$ is 1. Furthermore, if $M_x < 0$, two branch points have a modulus greater than 1, and the remaining ones have a modulus less than 1, and if $M_x > 0$, two branch points have a modulus less than 1, and the other ones have a modulus greater than 1.*

2.7.1 Queueing Examples

In this section we provide some queueing examples to see how the above lemma works for some queueing models. Since in all queueing examples we will study $M_y \neq 0$, then we have

$$0 \leq x_1 < x_2 < 1 < x_3 < x_4, \quad \text{or} \quad 0 \leq x_1 < x_2 < 1 < x_3 < x_4 = \infty.$$

1. The symmetric JSQ Model and the JSQ Model with Coupled Processors

Referring to the transition diagrams for the Symmetric JSQ model, and JSQ model with coupled processors, we have

$$p_{1,0} = p_{1,1} = p_{-1,0} = p_{-1,-1} = 0, \quad p_{1,-1} = \lambda, \quad p_{-1,1} = \mu.$$

Therefore, according to lemma (2.2) and (2.3), we obtain

$$0 = x_1 < x_2 < 1 < x_3 < x_4 = \infty.$$

2. The Pre-Emptive Priority Queueing System

The Pre-emptive priority queueing model is singular, and so its corresponding Riemann surface is of genus 0. Since we are dealing with the non-singular random walks, we can not discuss this here.

3. The Restricted Jackson Network

From transition diagrams for these two queueing model we have

$p_{1,0} = \lambda_1$, $p_{1,-1} = r_{21}\mu_2$, $p_{-1,0} = r_{10}\mu_1$, $p_{-1,1} = r_{12}\mu_1$, $p_{1,1} = p_{-1,-1} = 0$.

Therefore, according to the lemmas (2.2) and (2.3), we have

$$0 < x_1 < x_2 < 1 < x_3 < x_4.$$

4. The Classical Tandem Queues

According to the transition probabilities for Tandem queue, we obtain

$$p_{1,1} = p_{1,-1} = p_{-1,0} = p_{-1,-1} = 0, \quad p_{1,0} = \lambda, \quad p_{-1,1} = \mu_1.$$

Hence, referring to the two lemmas (2.2) and (2.3), we have

$$0 = x_1 < x_2 < 1 < x_3 < x_4.$$

5. The Queueing Systems with Two Demands

From transition probabilities for Two demand queueing model, we obtain

$$p_{1,0} = p_{1,-1} = p_{-1,0} = p_{-1,1} = p_{-1,-1} = 0, \quad p_{1,1} = \lambda.$$

Therefore, from the two lemmas (2.2) and (2.3) above, we have

$$0 < x_1 < x_2 < 1 < x_3 < x_4 = \infty.$$

Chapter 3

Exact Tail Asymptotic Analysis of a General Random Walk Model Using Kernel Method

3.1 Kernel Method and Generating Functions

The kernel method was first introduced by Knuth [38] and then was developed by Banderier et al [4]. Suppose we have a fundamental form $A(x,y)F(x,y) = B(x,y)G(x) + C(x,y)$, where $F(x,y)$, and $G(x)$ are unknown, and $A(x,y)$, $B(x,y)$, and $C(x,y)$ are known two-variable complex functions. The main idea of the kernel method is to find the solutions of the equation $F(x,y) = 0$. Suppose x and $Y_0(x)$ is the solutions of $F(x,y) = 0$, then obviously, $G(x) = \dfrac{-C(x,Y_0(x))}{B(x,Y_0(x))}$. In kernel method, finding the location of poles of the unknown functions, as well as determination of the dominant singularities of the generating functions are sufficient for the exact tail asymptotics analysis.

In this section we provide the generating functions for a general random walk model in the first and fourth quadrants of the two-dimensional random walk. The generating functions are listed as follows,

(1) generating functions for the first quadrant:

$$
\begin{aligned}
P_n(x) &= \sum_{m\geq 1} \pi_{m,n} x^m, \\
Q_0(y) &= \sum_{n\geq 1} \pi_{0,n} y^n, \\
Q_m(y) &= \sum_{n\geq 1} \pi_{m,n} y^n, \\
P(x,y) &= \sum_{m\geq 1, n\geq 1} \pi_{m,n} x^m y^n,
\end{aligned} \tag{3.1}
$$

(2) generating functions for the fourth quadrant:

$$
\begin{aligned}
P_n^{(-)}(x) &= \sum_{m\geq 1} \pi_{m,n} x^m, \\
Q_0^{(-)}(y) &= \sum_{n\leq -1} \pi_{0,n} y^{-n}, \\
Q_m^{(-)}(y) &= \sum_{n\leq -1} \pi_{m,n} y^{-n}, \\
P^{(-)}(x,y) &= \sum_{m\geq 1, n\leq -1} \pi_{m,n} x^m y^{-n},
\end{aligned} \tag{3.2}
$$

(3) generating functions for the boundary x-axis:

$$
P_0(x) = \sum_{m\geq 1} \pi_{m,0} x^m. \tag{3.3}
$$

Using the generating functions defined above, we can write the balance equations for the first and fourth quadrants of the plane in order to find the fundamental forms of a random walk model. Balance equation describes that the probability of leaving each stationary state is equal to the probability of moving to that state. We categorize the balance equations as follows,
(1) for the boundary on x-axis:

if $m \geq 2$ and $n = 0$,

$$\pi_{m,0} = \sum_{i=-1}^{1} \pi_{m+i,-1}\ p^{(-1)}_{-i,1} + \sum_{i=-1}^{1} \pi_{m+i,1}\ p_{-i,-1} + \sum_{i=-1,1} \pi_{m+i,0}\ p^{(1)}_{-i,0} + \pi_{m,0}\ p_{0,0}, \tag{3.4}$$

if $m = 1$ and $n = 0$,

$$\pi_{1,o} = \sum_{i=1}^{2} \pi_{i,-1}\ p^{(-)}_{-i+1,1} + \sum_{i=1}^{2} \pi_{i,1}\ p_{-i+1,-1} + \pi_{0,-1}\ p^{(2-)}_{1,1} + \pi_{0,1}\ p^{(2)}_{1,-1} + \pi_{0,0}\ p^{(0)}_{1,0} + \pi_{2,0}\ p^{(1)}_{-1,0}, \tag{3.5}$$

if $m = 0$ and $n = 0$,

$$\pi_{0,0} = \pi_{0,-1}\ p^{(-2)}_{0,1} + \pi_{1,-1}\ p^{(-)}_{-1,1} + \pi_{1,0}\ p^{(1)}_{-1,0} + \pi_{1,1}\ p_{-1,-1} + \pi_{0,1}\ p^{(2)}_{0,-1} + \pi_{0,0}\ p^{(0)}_{0,0}, \tag{3.6}$$

(2) for the boundary on y-axis:
if $m = 0$ and $n \geq 2$,

$$\pi_{0,n} = \sum_{i=-1}^{1} \pi_{0,n+i}\ p^{(2)}_{0,-i} + \sum_{i=-1}^{1} \pi_{1,n+i}\ p_{-1,-i}, \tag{3.7}$$

if $m = 0$ and $n = 1$,

$$\pi_{0,1} = \sum_{i=1}^{2} \pi_{1,i}\ p_{-1,-i+1} + \sum_{i=1}^{2} \pi_{0,i}\ p^{(2)}_{0,-i+1} + \pi_{0,0}\ p^{(0)}_{0,1} + \pi_{1,0}\ p^{(1)}_{-1,1}, \tag{3.8}$$

if $m = 0$ and $n = -1$,

$$\pi_{0,-1} = \sum_{i=-2}^{-1} \pi_{1,i}\ p^{(-)}_{-1,-i-1} + \sum_{i=-2}^{-1} \pi_{0,i}\ p^{(2-)}_{0,-i-1} + \pi_{0,0}\ p^{(0)}_{-1,0} + \pi_{1,0}\ p^{(1)}_{-1,-1}, \tag{3.9}$$

if $m = 0$ and $n \leq -2$,

$$\pi_{0,n} = \sum_{i=-1}^{1} \pi_{0,n+i}\ p^{(2-)}_{0,-i} + \sum_{i=-1}^{1} \pi_{1,n+i}\ p^{(-)}_{-1,-i}, \tag{3.10}$$

(3) for the interior of the first quadrant:
if $m \geq 2$ and $n = 1$,

$$\pi_{m,1} = \sum_{i=-1}^{1} \pi_{m+i,0}\ p^{(1)}_{-i,1} + \sum_{i=-1}^{1} \pi_{m+i,2}\ p_{-i,-1} + \sum_{i=-1,1} \pi_{m+i,0}\ p_{-i,0} + \pi_{m,1}\ p_{0,0}, \tag{3.11}$$

if $m \geq 2$ and $n \geq 2$,

$$\pi_{m,n} = \sum_{i=-1}^{1} \pi_{m+i,n-1}\ p_{-i,1} + \sum_{i=-1}^{1} \pi_{m+i,n+1}\ p_{-i,-1} + \sum_{i=-1,1} \pi_{m+i,n}\ p_{-i,0} + \pi_{m,n}\ p_{0,0}, \tag{3.12}$$

if $m = 1$ and $n \geq 2$,

$$\pi_{1,n} = \sum_{i=-1}^{1} \pi_{0,n+i}\ p^{(2)}_{1,-i} + \sum_{i=-1}^{1} \pi_{2,n+i}\ p_{-1,-i} + \sum_{i=-1,1} \pi_{1,n+i}\ p_{0,-i}, \tag{3.13}$$

if $m = 1$ and $n = 1$,

$$\pi_{1,1} = \sum_{i=1}^{2} \pi_{i,0}\ p^{(1)}_{-i+1,1} + \sum_{i=1}^{2} \pi_{0,i}\ p^{(2)}_{1,-i+1} + \sum_{i=1}^{2} \pi_{2,i}\ p_{-1,-i+1} + \pi_{0,0}\ p^{(0)}_{1,1} + \pi_{1,2}\ p_{0,-1}, \tag{3.14}$$

(4) for the interior of the fourth quadrant:
if $m \geq 2$ and $n = -1$,

$$\pi_{m,-1} = \sum_{i=-1}^{1} \pi_{m+i,0}\ p^{(1)}_{-i,-1} + \sum_{i=-1}^{1} \pi_{m+i,-2}\ p^{(-)}_{-i,1} + \sum_{i=-1,1} \pi_{m+i,-1}\ p^{(-)}_{-i,0} + \pi_{m,-1}\ p_{0,0}, \tag{3.15}$$

if $m \geq 2$ and $n \leq -2$,

$$\pi_{m,n} = \sum_{i=-1}^{1} \pi_{m+i,n-1}\ p^{(-)}_{-i,1} + \sum_{i=-1}^{1} \pi_{m+i,n+1}\ p^{(-)}_{-i,-1} + \sum_{i=-1,1} \pi_{m+i,n}\ p^{(-)}_{-i,0} + \pi_{m,n}\ p_{0,0}, \tag{3.16}$$

if $m = 1$ and $n \leq -2$,

$$\pi_{1,n} = \sum_{i=-1}^{1} \pi_{0,n+i}\ p^{(2-)}_{1,-i} + \sum_{i=-1}^{1} \pi_{2,n+i}\ p^{(-)}_{-1,-i} + \sum_{i=-1,1} \pi_{1,n+i}\ p^{(-)}_{0,-i}, \tag{3.17}$$

if $m = 1$ and $n = -1$,

$$\pi_{1,-1} = \sum_{i=1}^{2} \pi_{i,0}\ p^{(1)}_{-i+1,-1} + \sum_{i=-2}^{-1} \pi_{0,i}\ p^{(2-)}_{1,-i-1} + \sum_{i=-2}^{-1} \pi_{2,i}\ p^{(-)}_{-1,-i-1} + \pi_{0,0}\ p^{(0)}_{1,-1} + \pi_{1,-2}\ p^{(-)}_{0,1} + \pi_{1,-1}\ p^{(-)}_{0,0}. \tag{3.18}$$

Now according to the above balance equations one can get the following fundamental forms.

For the first quadrant, fundamental form is

$$P(x,y)h(x,y) = P_0(x)h_1(x,y) + Q_0(y)h_2(x,y) + \pi_{0,0}h_0(x,y) + P_{-1}(x)A(x) + \pi_{0,-1}B(x), \tag{3.19}$$

where

$$\begin{aligned}
h(x,y) &= 1 - \sum_{i=-1}^{1}\sum_{j=-1}^{1} p_{i,j}x^i y^j, \\
h_1(x,y) &= \sum_{i=-1}^{1}\sum_{j=0}^{1} p_{i,j}^{(1)} x^i y^j - 1, \\
h_2(x,y) &= \sum_{i=0}^{1}\sum_{j=-1}^{1} p_{i,j}^{(2)} x^i y^j - 1, \\
h_0(x,y) &= \sum_{i=0}^{1}\sum_{j=0}^{1} p_{i,j}^{(0)} x^i y^j - 1, \\
A(x) &= \sum_{i=-1}^{1} p_{i,1}^{(-)} x^i, \\
B(x) &= p_{0,1}^{(-)} + p_{1,1}^{(2-)} x. \qquad (3.20)
\end{aligned}$$

For the fourth quadrant, fundamental form is

$$\begin{aligned}
&P^{(-)}(x,y)h^{(-)}(x,y) = P_0(x)h_1^{(-)}(x,y) + Q_0^{(-)}(y)h_2^{(-)}(x,y) + \\
&\pi_{0,0}h_0^{(-)}(x,y) + P_{-1}(x)A^{(-)}(x) + \pi_{0,-1}B^{(-)}x, \qquad (3.21)
\end{aligned}$$

where

$$
\begin{aligned}
h^{(-)}(x,y) &= 1 - \sum_{i=-1}^{1}\sum_{j=-1}^{1} p_{i,j}^{(-)} x^i y^{-j},\\
h_1(x,y) &= \sum_{i=-1}^{1} p_{i,-1}^{(1)} x^i y,\\
h_2^{(-)}(x,y) &= \sum_{i=0}^{1}\sum_{j=-1}^{1} p_{i,j}^{(2-)} x^i y^{-j} - 1,\\
h_0^{(-)}(x,y) &= \sum_{i=0}^{1} p_{i,-1}^{(0)} x^i y,\\
A^{(-)}(x) &= -A(x) = -\sum_{i=-1}^{1} p_{i,1}^{(-)} x^i,\\
B^{(-)}(x) &= -B(x) = -p_{0,1}^{(-)} + p_{1,1}^{(2-)} x. \qquad (3.22)
\end{aligned}
$$

Lemma 3.1. *For $|x| \leq 1$ and $|y| \leq 1$, we have the following two expressions for generating functions as follows,*

$P(x,y)h(x,y)+P^{(-)}(x,y)h^{(-)}(x,y) = P_0(x)H_1(x,y)+Q_0(y)h_2(x,y)+$

$Q_0^{(-)}(y)h_2^{(-)}(x,y) + \pi_{0,0}H_0(x,y)$,
where
$H_1(x,y) = h_1(x,y) + h_1^{(-)}(x,y)$, *and*
$H_0(x,y) = h_0(x,y) + h_0^{(-)}(x,y)$.

Proof. This follows from equations (3.19) and (3.21). □

3.2 Key Kernel

For the first quadrant of the random walk plane, the kernel equation, $h(x,y) = 0$, gives an algebraic curve which turns out to be a Riemann surface. The zeros of the kernel equation are called branches of the fundamental form. In this section we analyse the kernel equation. For the first quadrant, the kernel equation becomes a quadratic equation in terms of y, which is

$$h(x, y) = a(x)y^2 + b(x)y + c(x) = 0, \tag{3.23}$$

where

$$\begin{aligned} a(x) &= -p_{1,1}x^2 - p_{0,1}x - p_{-1,1}, \\ b(x) &= -p_{1,0}x^2 + x - p_{-1,0}, \\ c(x) &= -p_{1,-1}x^2 - p_{0,-1}x - p_{-1,-1}. \end{aligned} \tag{3.24}$$

Therefore, the branches of the fundamental form are,

$$Y_{\pm}(x) = \frac{-b(x) \pm \sqrt{D(x)}}{2a(x)}, \tag{3.25}$$

where

$$D(x) = b^2(x) - 4a(x)c(x) \tag{3.26}$$

Among the zeros, $Y_{\pm}(x)$, the one with smaller modulus is denoted by $Y_0(x)$ and the one with greater modulus is denoted by $Y_1(x)$. Therefore,

$$Y_0(x) = \begin{cases} Y_-(x) & \text{if } |Y_-(x)| \leq |Y_+(x)|, \\ Y_+(x) & \text{if } |Y_+(x)| \leq |Y_-(x)|. \end{cases} \tag{3.27}$$

The zeros of the equation $D(x) = b^2(x) - 4a(x)c(x) = 0$ are called branch points. In our case there are at most four branch points denoted by x_1, x_2, x_3, and x_4. As mentioned earlier, when the number of branch points are one or two, the corresponding Riemann surface of kernel equation is sphere, and when the number of branch points are three ot four, the corresponding Riemann surface will be torus.

We also can express $h(x, y)$ in quadratic form of x as follows,

$$h(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = 0, \tag{3.28}$$

where

$$\begin{aligned}\tilde{a}(y) &= -p_{1,1}y^2 - p_{1,0}y - p_{1,-1},\\ \tilde{b}(y) &= -p_{0,1}y^2 + y - p_{0,-1},\\ \tilde{c}(y) &= -p_{-1,1}y^2 - p_{-1,0}y - p_{-1,-1}.\end{aligned} \tag{3.29}$$

Hence, the solutions of kernel equation are

$$X_{\pm}(y) = \frac{-\tilde{b}(y) \pm \sqrt{\tilde{D}(y)}}{2\tilde{a}(y)}, \tag{3.30}$$

where

$$\tilde{D}(y) = \left(\tilde{b}(y)\right)^2 - 4\tilde{a}(y)\tilde{c}(y). \tag{3.31}$$

There are at most four branch points for $D(y)$ denoted by y_1, y_2, y_3, and y_4. Similar to (3.27), Among the roots $X_{\pm}(y)$, the one with smaller modulus is denoted by $X_0(y)$ and the one with greater modulus is denoted by $X_1(y)$. Therefore,

$$X_0(y) = \begin{cases} X_-(y) & \text{if } |X_-(y)| \le |X_+(y)|, \\ X_+(y) & \text{if } |X_+(y)| \le |X_-(y)|. \end{cases} \tag{3.32}$$

If we write the kernel equation for the fourth quadrant, we get

$$h^{(-)}(x,y) = a^{(-)}(x)y^2 + b^{(-)}(x)y + c^{(-)}(x) = 0, \tag{3.33}$$

where

$$\begin{aligned}a^{(-)}(x) &= -p^{(-)}_{1,1}x^2 - p^{(-)}_{0,1}x - p^{(-)}_{-1,1},\\ b^{(-)}(x) &= -p^{(-)}_{1,0}x^2 + x - p^{(-)}_{-1,0},\\ c^{(-)}(x) &= -p^{(-)}_{1,-1}x^2 - p^{(-)}_{0,-1}x - p^{(-)}_{-1,-1}.\end{aligned} \tag{3.34}$$

The solutions for kernel equation are now,

$$Y^{(-)}_{\pm}(x) = \frac{-b^{(-)}(x) \pm \sqrt{D^{(-)}(x)}}{2a^{(-)}(x)}, \tag{3.35}$$

where

$$D^{(-)}(x) = (b^{(-)})^2(x) - 4a^{(-)}(x)c^{(-)}(x). \tag{3.36}$$

We can also write the kernel equation of the fourth quadrant as follows,

$$h(x,y) = \tilde{a}^{(-)}(y)x^2 + \tilde{b}^{(-)}(y)x + \tilde{c}^{(-)}(y) = 0, \tag{3.37}$$

where

$$\begin{aligned}
\tilde{a}^{(-)}(y) &= -p_{1,1}^{(-)}y^2 - p_{1,0}^{(-)}y - p_{1,-1}^{(-)},\\
\tilde{b}^{(-)}(y) &= -p_{0,1}^{(-)}y^2 + y - p_{0,-1}^{(-)},\\
\tilde{c}^{(-)}(y) &= -p_{-1,1}^{(-)}y^2 - p_{-1,0}^{(-)}y - p_{-1,-1}^{(-)}.
\end{aligned}$$

Hence, the solutions of kernel equation are

$$X_{\pm}^{(-)}(y) = \frac{-\tilde{b}^{(-)}(y) \pm \sqrt{\tilde{D}^{(-)}(y)}}{2\tilde{a}^{(-)}(y)}, \tag{3.38}$$

where

$$\tilde{D}^{(-)}(y) = (\tilde{b}^{(-)})^2(y) - 4\tilde{a}^{(-)}(y)\tilde{c}^{(-)}(y).$$

In this thesis, we define $[x_3, x_4] = [-\infty, x_4] \cup [x_3, \infty]$ when $x_4 < -1$, similarly we define $[y_3, y_4] = [-\infty, y_4] \cup [y_3, \infty]$ when $y_4 < -1$. Since all the generating functions are defined in the complex plane, to ensure the continuity or to avoid the transition from one branch to another, we consider the following cut planes

$$\begin{aligned}
\tilde{\mathbb{C}}_x &= \mathbb{C}_x - [x_3, x_4],\\
\tilde{\tilde{\mathbb{C}}}_x &= \mathbb{C}_x - [x_1, x_2] \cup [x_3, x_4],\\
\tilde{\mathbb{C}}_y &= \mathbb{C}_y - [y_3, y_4],\\
\tilde{\tilde{\mathbb{C}}}_y &= \mathbb{C}_y - [y_1, y_2] \cup [y_3, y_4].
\end{aligned} \tag{3.39}$$

Lemma 3.2. *The function $Y_0(x)$ and $Y_1(x)$ are meromorphic in cut plane $\tilde{C}_x$, Moreover,* *(i) $Y_0(x)$ has one zero and no poles. Hence, $Y_0(x)$ is analytic in $\tilde{C}_x$.*

(ii) $Y_1(x)$ has two poles and no zeros.

(iii) $|Y_0(x)| < |Y_1(x)|$ on the whole cut plane $\tilde{\tilde{C}}_x$, and $|Y_0(x)| = |Y_1(x)|$ only on cuts.

(iv) If $|x| = 1$, then $|Y_0(x)| \leq 1$. Moreover, $Y_0(1) = 1$.

Proof. Since we are in the cut plane $\tilde{\tilde{C}}_x$, which excludes the case of $b(x) = 0$, we have

$$Y_0(x) = \begin{cases} \dfrac{2c(x)}{-b(x) + \sqrt{D_1(x)}}, & \text{if } -b(x) > 0, \\ \dfrac{2c(x)}{-b(x) - \sqrt{D_1(x)}}, & \text{if } -b(x) < 0, \end{cases}$$

$$Y_0(x)Y_1(x) = \frac{c(x)}{a(x)},$$

hence, it is clear that (i) $Y_0(x)$ has one zero and no poles, and (ii) $Y_1(x)$ has two poles and no zeros. (iii) is true according to the definition of $Y_0(x)$ and $Y_1(x)$. (iv) It is proved in lemma 2.3.4. of Fayolle, Iasnogorodski, and Malyshev [19]. □

3.3 Asymptotic Analysis of $P_0(x)$, $Q_0(y)$ and $Q_0^{(-)}(y)$

In this section we find the singularities of the generating functions, $P_0(x)$, $Q_0(y)$ and $Q_0^{(-)}(y)$ in order to find the asymptotic behaviour of them. Let

$$\begin{aligned} f(x) &= -Q_0(Y_0(x))h_2(x, Y_0(x)), \\ f^{(-)}(x) &= -Q_0^{(-)}(Y_0^{(-)}(x))h_2^{(-)}(x, Y_0^{(-)}(x)), \\ g_0(x) &= -\pi_{0,0}Q_0(x), \\ g_1(x) &= h_1(x, Y_0(x)) + h_1^{(-)}(x, Y^{(-)}(x)). \end{aligned} \tag{3.40}$$

From lemma 3.1. and equations (3.40) we have,

$$P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{g_1(x)}, \tag{3.41}$$

Let x^* be the zero of $g_1(x)$. Hence,

$$g_1(x^*) = h_1(x^*, Y_0(x^*)) + h_1^{(-)}(x^*, Y^{(-)}(x^*)) = 0.$$

Now for analysing the singularity for $Q_0(y)$ we need the following notations. Let

$$\begin{aligned}
l_1(y) =& - P_0(X_0(y))h_1(X_0(y), y),\\
l_2(y) =& - \pi_{0,0}\ h_0(X_0(y), y) - P_{-1}(X_0(y))\ A(X_0(y)) -\\
&\pi_{0,-1}\ B(X_0(y)),\\
g_2(y) =& h_2(X_0(y), y).
\end{aligned} \tag{3.42}$$

Hence, from equations (3.19) and (3.42) we have,

$$Q_0(y) = \frac{l_1(y) + l_2(y)}{g_2(y)} \tag{3.43}$$

Now let y^* be the zero of $g_2(y)$. Hence,

$$g_2(y^*) = h_2\bigg(X_0(y^*), y^*\bigg) = 0.$$

Also for asymptotic analysing of the $Q_0^{(-)}$ we need the following notations. Let

$$\begin{aligned}
l_1^{(-)}(y) =& - P_0(X_0^{(-)}(y))h_1^{(-)}(X_0^{(-)}(y), y),\\
l_2^{(-)}(y) =& - \pi_{0,0}\ h_0^{(-)}(X_0^{(-)}(y), y) - P_{-1}(X_0^{(-)}(y))\ A^{(-)}(X_0^{(-)}(y)) -\\
&\pi_{0,-1}\ B^{(-)}(X_0(y)),\\
g_2^{(-)}(y) =& h_2^{(-)}(X_0^{(-)}(y), y).
\end{aligned} \tag{3.44}$$

Therefore, from equations (3.21) and (3.44) we have,

$$Q_0(y) = \frac{l_1^{(-)}(y) + l_2^{(-)}(y)}{g_2^{(-)}(y)} \tag{3.45}$$

In this equation let $y^*_{(-)}$ be the zero of $g_2^{(-)}(y)$. Therefore,

$$g_2^{(-)}(y) = h_2^{(-)}\left(X_0^{(-)}(y^*_{(-)}), y^*_{(-)}\right) = 0.$$

The following theorem is proved by H. Li and Y. Zhao in [50].

Theorem 3.1. *Let z be a pole of $P_0(x)$ with the smallest modulus in the disk $|x| < x_3$. Then one of the following three cases must hold:*

1. *z is a zero of $h_1(x, Y_0(x))$;*
2. *$y^* = Y_0(z)$ is a zero of $h_2(X_0(y), y)$ and $|y^*| > 1$; or*
3. *$z^* = X_0(y^*)$ is a zero of $h_1(x, Y_0(x))$ and $|z^*| > 1$.*

Using theorem 3.1, it is easily shown that x^*, $\tilde{x_1}$, $\tilde{x_1}^{(-)}$, x_3, and $x_3^{(-)}$ are singularities of $P_0(x)$, where
$\tilde{x_1} = X_1(y^*)$, and $\tilde{x_1}^{(-)} = X_1^{(-)}(y^*_{(-)})$.
The singularity with smallest modulus greater than 1 will be called the dominant singularity for $P_0(y)$, and consequently the radius of convergence of $P_0(y)$ can be extended to this point by analytic continuation [19].
Similarly, according to (3.42) and (3.43), y^*, $\tilde{y_1}$, y_3 are singularities of $Q_0(y)$, where $\tilde{y_1} = Y_1(x^*)$, and according to (3.44) and (3.45), $y^*_{(-)}$, $\tilde{y_1}^{(-)}$, and $y_3^{(-)}$ are singularities of $Q_0^{(-)}(x)$, where $\tilde{y_1}^{(-)} = Y_1^{(-)}(x^*)$.

3.4 Tauberian-Like Theorem

In this section we recall the Tauberian-like theorem which is an important tool for our calculation of asymptotic analysis of generating functions, $P_0(x)$, $Q_0(y)$ and $Q_0^{(-)}(y)$.

Theorem 3.2. *Assume that $C(z)$ is an analytic function defined on $\triangle(\phi, \epsilon) = \{z : z \leq (1+\epsilon), \epsilon \geq 0, < \phi < \frac{\pi}{2}\}$, except at $z = 1$. Suppose $C(z) = \sum_{n\geq 0} c_n z^n$, and a is the dominant singularity of $C(z)$*

on the convergence circle, and $\lim_{z\to a}(1-\dfrac{z}{a})^h\ C(z)=b$. *then*

$$c_n \sim \frac{b\ n^{Re(h)-1}}{Re(a)^n\Gamma\Big(Re(h)\Big)} \qquad as\ n\to\infty. \tag{3.46}$$

3.5 Exact Tail Asymptotic for $P_0(x)$

In this section we will find the exact tail asymptotic behaviour of the marginal stationary distribution along x-axis, with applying the Tauberian-like theorem to the generating function $P_0(x)$. For this purpose we need the following equations. Since $Y_0(x)$ and $Y_0^{(-)}(x)$ can be written as

$$\begin{aligned}
Y_0(x) &= a(x)+b(x)\sqrt{1-\frac{x}{x_3}},\\
Y_0^{(-)}(x) &= a^{(-)}(x)+b^{(-)}(x)\sqrt{1-\frac{x}{x_3^{(-)}}},
\end{aligned} \tag{3.47}$$

we can write the following equations,

$$\begin{aligned}
g_1(x) &= a_1(x)+b_1(x)\sqrt{1-\frac{x}{x_3}}+b_2(x)\sqrt{1-\frac{x}{x_3^{(-)}}},\\
Y_0(x_3)-Y_0(x) &= (1-\frac{x}{x_3})a^*(x)-b(x)\sqrt{1-\frac{x}{x_3}},\\
Y_0^{(-)}(x_3^{(-)})-Y_0^{(-)}(x) &= (1-\frac{x}{x_3^{(-)}})a^{(-)*}(x)-b^{(-)}(x)\sqrt{1-\frac{x}{x_3^{(-)}}},\\
g_1(x)-g_1(x_3) &= (1-\frac{x}{x_3})a_1^*(x)+b_1(x)\sqrt{1-\frac{x}{x_3}},\\
g_1(x)-g_1(x_3^{(-)}) &= (1-\frac{x}{x_3^{(-)}})a_1^{(-)*}(x)+b_1(x)\sqrt{1-\frac{x}{x_3^{(-)}}}.
\end{aligned} \tag{3.48}$$

To simplify our later calculations let

$$\begin{aligned}
\alpha(x) &= \left(\sqrt{1-\frac{x}{x_3}}\, a^*(x) - b(x)\right),\\
\alpha^{(-)}(x) &= \left(\sqrt{1-\frac{x}{x_3^{(-)}}}\, a^{(-)*}(x) - b^{(-)}(x)\right),\\
\beta(x) &= \left(\sqrt{1-\frac{x}{x_3}}\, a_1^*(x) - b_1(x)\right),\\
\beta^{(-)}(x) &= \left(\sqrt{1-\frac{x}{x_3^{(-)}}}\, a_1^{(-)*}(x) - b_1^{(-)}(x)\right),
\end{aligned} \tag{3.49}$$

Hence, according to (3.49) we can rewrite (3.48) as

$$\begin{aligned}
Y_0(x_3) - Y_0(x) &= \sqrt{1-\frac{x}{x_3}}\, \alpha(x),\\
Y_0^{(-)}(x_3^{(-)}) - Y_0^{(-)}(x) &= \sqrt{1-\frac{x}{x_3^{(-)}}}\, \alpha^{(-)}(x),\\
g_1(x) - g_1(x_3) &= \sqrt{1-\frac{x}{x_3}}\, \beta(x),\\
g_1(x) - g_1(x_3^{(-)}) &= \sqrt{1-\frac{x}{x_3^{(-)}}}\, \beta^{(-)}(x).
\end{aligned} \tag{3.50}$$

According to (3.40), (3.41), and (3.50), we divide the asymptotic behaviour of $\pi_{m,0}$ into five cases: (1) exact geometric decay rate, (2) exact geometric decay rate with factor $m^{-1/2}$, (3) exact geometric decay rate with factor $m^{1/2}$, (4) exact geometric decay rate with factor m, and (5) exact geometric decay rate with factor $m^{-3/2}$. Next, we will show the details on the five cases above.

Case 1. If $x_{dom} = x^*$, or $x_{dom} = \tilde{x}_1$, or $x_{dom} = \tilde{x_1}^{(-)}$, or $x_{dom} = \tilde{x}_1 = \tilde{x_1}^{(-)}$, or $x_{dom} = \tilde{x}_1 = x_3^{(-)}$, or $x_{dom} = \tilde{x_1}^{(-)} = x_3$, or $x_{dom} = x^* = \tilde{x}_1 = x_3$, or $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3^{(-)}$, or $x_{dom} = \tilde{x}_1 = \tilde{x_1}^{(-)} = x_3$, or $x_{dom} = \tilde{x}_1 = \tilde{x_1}^{(-)} = x_3^{(-)}$, or $x_{dom} = x^* = \tilde{x}_1 = x_3 = x_3^{(-)}$, or $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3 = x_3^{(-)}$, or $x_{dom} = x^* = \tilde{x}_1 = \tilde{x_1}^{(-)} = x_3 =$

$x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{L}{(x_{dom})^m}, \tag{3.51}$$

where L is a constant, and x_{dom} is dominant singularity of $P_0(x)$. We will prove it in the following conditions.

Condition 1.1. If $x_{dom} = x^*$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{(x - x_{dom})\ g_1^*(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \frac{f(x^*) + f^{(-)}(x^*) + g_0(x^*)}{x^*\ g_1^*(x^*)} = L_1,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_1}{(x^*)^m}.$$

Condition 1.2. If $x_{dom} = \tilde{x}_1$, then

$$P_0(x) = \frac{f(x)\dfrac{(1 - \dfrac{Y_0(x)}{y^*})}{(1 - \dfrac{Y_0(x)}{y^*})} + f^{(-)}(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \frac{y^* f(\tilde{x}_1)(1 - \dfrac{Y_0(\tilde{x}_1)}{y^*})}{\tilde{x}_1\ Y_0'(\tilde{x}_1)\ g_1(\tilde{x}_1)} = L_2,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_2}{(\tilde{x}_1)^m}.$$

Condition 1.3. If $x_{dom} = \tilde{x}_1{}^{(-)}$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x)\dfrac{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})} + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}}(1-\frac{x}{x_{dom}})P_0(x) = \frac{y^*_{(-)}\ f^{(-)}(\tilde{x_1}^{(-)})(1-\dfrac{Y_0(\tilde{x_1}^{(-)})}{y^*})}{\tilde{x_1}^{(-)}\ \dfrac{dY_0^{(-)}(x)}{dx}|_{x=\tilde{x_1}^{(-)}}\ g_1(\tilde{x_1}^{(-)})}$$
$$= L_3,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_3}{(\tilde{x_1}^{(-)})^m}.$$

Condition 1.4. If $x_{dom} = \tilde{x}_1 = \tilde{x_1}^{(-)}$, then

$$P_0(x) = \frac{f(x)\dfrac{(1-\dfrac{Y_0(x)}{y^*})}{(1-\dfrac{Y_0(x)}{y^*})} + f^{(-)}(x)\dfrac{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})} + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}}(1-\frac{x}{x_{dom}})P_0(x) =$$

$$\left.\frac{\dfrac{y^*}{x\ \dfrac{dY_0(x)}{dx}}f(x)(1-\dfrac{Y_0(x)}{y^*}) + \dfrac{y^*_{(-)}}{x\ \dfrac{dY_0^{(-)}(x)}{dx}}f^{(-)}(x)(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{g_1(x)}\right|_{x=\tilde{x}_1}$$

$$= L_4,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_4}{(\tilde{x}_1)^m}.$$

Condition 1.5. If $x_{dom} = \tilde{x}_1 = x_3^{(-)}$, then

$$P_0(x) = \frac{f(x)\dfrac{(1-\dfrac{Y_0(x)}{y^*})}{(1-\dfrac{Y_0(x)}{y^*})} + f^{(-)}(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}}(1-\frac{x}{x_{dom}})P_0(x) = \frac{y^*\ f(x)(1-\dfrac{Y_0(x)}{y^*})}{x\ \dfrac{dY_0(x)}{dx} g_1(x)}\Bigg|_{x=\tilde{x}_1} = L_5,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_5}{(\tilde{x}_1)^m}.$$

Condition 1.6. If $x_{dom} = \tilde{x_1}^{(-)} = x_3$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x)\dfrac{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})} + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}}(1-\frac{x}{x_{dom}})P_0(x) = \frac{y^*_{(-)}\ f^{(-)}(x)(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{x\ \dfrac{dY_0^{(-)}(x)}{dx}\ g_1(x)}\Bigg|_{x=\tilde{x_1}^{(-)}}$$
$$= L_6,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_6}{(x_3)^m}.$$

Condition 1.7. If $x_{dom} = x^* = \tilde{x}_1 = x_3$,

$$P_0(x) = \frac{\dfrac{y^*(1 - \dfrac{Y_0}{y^*})\ f(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + f^{(-)}(x) + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left. \frac{y^*\ f(x)(1 - \dfrac{Y_0(x)}{y^*})}{\alpha(x)\ \beta(x)} \right|_{x=x^*} = L_7,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_7}{(x_*)^m}.$$

Condition 1.8. If $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3^{(-)}$,

$$P_0(x) = \frac{\dfrac{y^*_{(-)}(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})\ f^{(-)}(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha^{(-)}(x)} + f(x) + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta^{(-)}(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left. \frac{y^*_{(-)}\ f(x)(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}})}{\alpha^{(-)}(x)\ \beta^{(-)}(x)} \right|_{x=x^*} = L_8,$$

Therefore, applying theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_8}{(x_*)^m}.$$

Condition 1.9. If $x_{dom} = \tilde{x_1} = \tilde{x_1}^{(-)} = x_3$, then

$$P_0(x) = \frac{f^{(-)}(x)\dfrac{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)} + \dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left.\frac{y^*\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{x\ \dfrac{dY_0^{(-)}(x)}{dx}\ g_1(x)}\right|_{x=\tilde{x_1}} = L_9,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_9}{(x_3)^m}.$$

Condition 1.10. If $x_{dom} = \tilde{x_1} = \tilde{x_1}^{(-)} = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\left(1 - \dfrac{Y_0(x)}{y^*}\right)} + \dfrac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha^{(-)}(x)} + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left.\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{x\ \dfrac{dY_0(x)}{dx}\ g_1(x)}\right|_{x=\tilde{x_1}} = L_{10},$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_{10}}{(\tilde{x_1})^m}.$$

Condition 1.11. If $x_{dom} = x^* = \tilde{x_1} = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + f^{(-)}(x) + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left.\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\alpha(x)\ \beta(x)}\right|_{x=x^*} = L_{11},$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_{11}}{(x^*)^m}.$$

Condition 1.12. If $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha^{(-)}(x)} + f(x) + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) = \left.\frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\alpha^{(-)}(x)\ \beta(x)}\right|_{x=x^*}$$
$$= L_{12},$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_{12}}{(x^*)^m}.$$

Condition 1.13. If $x_{dom} = x^* = \tilde{x_1} = \tilde{x_1}^{(-)} = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + \dfrac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha^{(-)}(x)}}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)} + \frac{g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} (1 - \frac{x}{x_{dom}}) P_0(x) =$$

$$\left(\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\alpha(x)\ \beta(x)} + \frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\alpha^{(-)}(x)\ \beta(x)} \right)\Bigg|_{x=x^*} = L_{13},$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{L_{13}}{(x^*)^m}.$$

Case 2. If $x_{dom} = x^* = x_3$, or $x_{dom} = x^* = x_3^{(-)}$, or $x_{dom} = \tilde{x_1} = x_3$, or $x_{dom} = \tilde{x_1}^{(-)} = x_3^{(-)}$, or $x_{dom} = x^* = x_3 = x_3^{(-)}$, or $x_{dom} = \tilde{x_1} = x_3 = x_3^{(-)}$, or $x_{dom} = \tilde{x_1}^{(-)} = x_3 = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{u\ m^{-1/2}}{\sqrt{\pi}(x_{dom})^m}. \tag{3.52}$$

where u is a constant, and x_{dom} is dominant singularity for $P_0(x)$. The proofs are given in the following conditions.

Condition 2.1. If $x_{dom} = x^* = x_3$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{\sqrt{1 - \frac{x}{x_{dom}}}\ \beta(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\ P_0(x) = \left. \frac{f(x) + f^{(-)}(x) + g_0(x)}{\beta(x)} \right|_{x=x^*} = u_1,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_1\ m^{-1/2}}{\sqrt{\pi}(x^*)^m}.$$

Condition 2.2. If $x_{dom} = x^* = x_3^{(-)}$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{\sqrt{1 - \frac{x}{x_{dom}}}\ \beta^{(-)}(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\ P_0(x) = \left. \frac{f(x) + f^{(-)}(x) + g_0(x)}{\beta^{(-)}(x)} \right|_{x=x^*} = u_2,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_2\ m^{-1/2}}{\sqrt{\pi}(x^*)^m}.$$

Condition 2.3. If $x_{dom} = \tilde{x_1} = x_3$, then

$$P_0(x) = \frac{\dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \frac{x}{x_{dom}}}\ \alpha(x)} + f^{(-)}(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\; P_0(x) = \frac{y^*\; f(x)\left(1-\dfrac{Y_0(x)}{y^*}\right)}{\alpha(x)\; g_1(x)}\Bigg|_{x=\tilde{x_1}} = u_3,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_3\; m^{-1/2}}{\sqrt{\pi}(x_3)^m}.$$

Condition 2.4. If $x_{dom} = \tilde{x_1}^{(-)} = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*_{(-)}\; f^{(-)}(x)\left(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1-\dfrac{x}{x_{dom}}}\; \alpha^{(-)}(x)} + f(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\; P_0(x) = \frac{y^*_{(-)}\; f^{(-)}(x)\left(1-\dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\alpha^{(-)}(x)\; g_1(x)}\Bigg|_{x=\tilde{x_1}^{(-)}}$$
$$= u_4,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_4\; m^{-1/2}}{\sqrt{\pi}(x_3^{(-)})^m}.$$

Condition 2.5. If $x_{dom} = x_* = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{\sqrt{1-\dfrac{x}{x_{dom}}}\; \beta(x)},$$

$$\Rightarrow \lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\; P_0(x) = \frac{f(x) + f^{(-)}(x) + g_0(x)}{\beta(x)}\Bigg|_{x=x_*} = u_5,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_5\ m^{-1/2}}{\sqrt{\pi}(x_*)^m}.$$

Condition 2.6. If $x_{dom} = \tilde{x_1} = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + f^{(-)}(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\ P_0(x) = \left.\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\alpha(x)\ g_1(x)}\right|_{x=\tilde{x_1}} = u_6,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_6\ m^{-1/2}}{\sqrt{\pi}(x_3)^m}.$$

Condition 2.7. If $x_{dom} = \tilde{x_1}^{(-)} = x_3 = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha^{(-)}(x)} + f(x) + g_0(x)}{g_1(x)},$$

$$\Rightarrow \lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}}\ P_0(x) = \left.\frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\alpha^{(-)}(x)\ g_1(x)}\right|_{x=\tilde{x_1}^{(-)}}$$
$$= u_7,$$

Therefore, by theorem 3.2 we get

$$\pi_{m,0} \sim \frac{u_7\ m^{-1/2}}{\sqrt{\pi}(x_3)^m}.$$

Case 3. If $x_{dom} = x^* = \tilde{x_1} = x_3^{(-)}$, or $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3$, or $x_{dom} = x^* = \tilde{x_1} = \tilde{x_1}^{(-)} = x_3$, or $x_{dom} = x^* = \tilde{x_1} = \tilde{x}_1^{(-)} = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{n\ m^{1/2}}{\frac{\sqrt{\pi}}{2}(x_{dom})^m}. \tag{3.53}$$

where n is a constant, and x_{dom} is dominant singularity for $P_0(x)$. The proofs are given in the following conditions.

Condition 3.1 If we have $x_{dom} = x_* = \tilde{x_1} = x_3^{(-)}$, then

$$P_0(x) = \frac{\dfrac{f(x)\left(1-\dfrac{Y_0(x)}{y^*}\right)}{\left(1-\dfrac{Y_0(x)}{y^*}\right)} + f^{(-)}(x) + g_0(x)}{\sqrt{1-\dfrac{x}{x_{dom}}\ \beta^{(-)}(x)}},$$

$$\lim_{x\to x_{dom}} \left(1-\frac{x}{x_{dom}}\right)^{3/2} P_0(x) = \left.\frac{y^*\ f(x)\left(1-\dfrac{Y_0(x)}{y^*}\right)}{x\ \dfrac{dY_0(x)}{dx}\ \beta^{(-)}(x)}\right|_{x=x^*} = n_1,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{n_1\ m^{1/2}}{\frac{\sqrt{\pi}}{2}(x^*)^m}.$$

Condition 3.2. If $x_{dom} = x^* = \tilde{x_1}^{(-)} = x_3$, then we have

$$P_0(x) = \frac{\dfrac{f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)} + f(x) + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\begin{aligned}\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^{3/2} P_0(x) &= \left.\frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{x\ \dfrac{dY_0^{(-)}(x)}{dx}\ \beta(x)}\right|_{x=x^*} \\ &= n_2,\end{aligned}$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{n_2\ m^{1/2}}{\dfrac{\sqrt{\pi}}{2}(x^*)^m}.$$

Condition 3.3. If we have $x_{dom} = x^* = \tilde{x_1} = \tilde{x_1}^{(-)} = x_3$, then

$$P_0(x) = \frac{\dfrac{f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)} + \dfrac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \alpha(x)} + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}}\ \beta(x)},$$

$$\begin{aligned}\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^{3/2} P_0(x) &= \left.\frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{x\ \dfrac{dY_0^{(-)}(x)}{dx}\ \beta(x)}\right|_{x=x^*} \\ &= n_3,\end{aligned}$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{n_3 \ m^{1/2}}{\frac{\sqrt{\pi}}{2}(x^*)^m}.$$

Condition 3.4. If $x_{dom} = x^* = \tilde{x}_1 = \tilde{x}_1^{(-)} = x_3^{(-)}$, then we have

$$P_0(x) = \frac{\dfrac{f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\left(1 - \dfrac{Y_0(x)}{y^*}\right)} + \dfrac{y^*_{(-)} \ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\sqrt{1 - \dfrac{x}{x_{dom}}} \ \alpha^{(-)}(x)} + g_0(x)}{\sqrt{1 - \dfrac{x}{x_{dom}}} \ \beta^{(-)}(x)},$$

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^{3/2} P_0(x) = \left. \frac{y^* \ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{x \ \dfrac{dY_0(x)}{dx} \ \beta^{(-)}(x)} \right|_{x=x^*} = n_4,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{n_4 \ m^{1/2}}{\frac{\sqrt{\pi}}{2}(x^*)^m}.$$

Case 4. If $x_{dom} = x^* = \tilde{x}_1$, or $x_{dom} = x^* = \tilde{x}_1{}^{(-)}$, or $x_{dom} = x^* = \tilde{x}_1 = \tilde{x}_1{}^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{r \ m}{(x_{dom})^m}, \tag{3.54}$$

where r is a constant, and x_{dom} is dominant singularity of $P_0(x)$. We will give the proofs in the following cases.

Condition 4.1. If we have $x_{dom} = x^* = \tilde{x_1}$, then

$$P_0(x) = \frac{\dfrac{f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\left(1 - \dfrac{Y_0(x)}{y^*}\right)} + f^{(-)}(x) + g_0(x)}{\left(x - x_{dom}\right)\ g_1^*(x)},$$

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 P_0(x) = \left.\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{x^2\ \dfrac{dY_0(x)}{dx}\ g_1^*(x)}\right|_{x=x^*} = r_1,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{r_1\ m}{(x^*)^m}.$$

Condition 4.2. If we have $x_{dom} = x^* = \tilde{x_1}^{(-)}$, then

$$P_0(x) = \frac{\dfrac{f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)} + f(x) + g_0(x)}{\left(x - x_{dom}\right)\ g_1^*(x)},$$

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 P_0(x) = \left.\frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{x^2\ \dfrac{dY_0^{(-)}(x)}{dx}\ g_1^*(x)}\right|_{x=x^*}$$
$$= r_2,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{r_2\ m}{(x^*)^m}.$$

Condition 4.3. If $x_{dom} = x^* = \tilde{x_1} = \tilde{x_1}^{(-)}$, then we have

$$P_0(x) = \frac{\dfrac{f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{\left(1 - \dfrac{Y_0(x)}{y^*}\right)} + \dfrac{f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)} + g_0(x)}{\left(x - x_{dom}\right)\ g_1^*(x)},$$

$$\lim_{x \to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 P_0(x) =$$

$$\left(\frac{y^*\ f(x)\left(1 - \dfrac{Y_0(x)}{y^*}\right)}{x^2\ \dfrac{dY_0(x)}{dx}\ g_1^*(x)} + \frac{y^*_{(-)}\ f^{(-)}(x)\left(1 - \dfrac{Y_0^{(-)}(x)}{y^*_{(-)}}\right)}{x^2\ \dfrac{dY_0^{(-)}(x)}{dx}\ g_1^*(x)}\right)\Bigg|_{x=x^*} = r_3,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{r_3\ m}{(x^*)^m}.$$

Case 5. If $x_{dom} = x_3$, or $x_{dom} = x_3^{(-)}$, or $x_{dom} = x_3 = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{s\ m^{-3/2}}{\sqrt{\pi}(x_{dom})^m}, \tag{3.55}$$

where s is a constant, and x_{dom} is the dominant singularity of $P_0(x)$. The proofs are given in the following conditions, but first we need to define the following functions in order to simplify the later calculations. Let

$$L(x) = -\ h_2(x, Y_0(x))\frac{dQ_0(Y_0(x))}{dx} + \left(-\ Q_0(Y_0(x))\frac{dh_2(x,y)}{dy} - \pi_{0,0}\frac{dh_0(x,y)}{dy}\right)\Bigg|_{y=Y_0(x)},$$

$$
\begin{aligned}
L^{(-)}(x) = & - h_2^{(-)}(x, Y_0^{(-)}(x)) \frac{dQ_0^{(-)}(Y_0^{(-)}(x))}{dx} + \\
& \left(- Q_0^{(-)}(Y_0^{(-)}(x)) \frac{dh_2^{(-)}(x,y)}{dy} - \pi_{0,0} \frac{dh_0^{(-)}(x,y)}{dy} \right) \Bigg|_{y=Y_0^{(-)}(x)},
\end{aligned}
$$

$$
\begin{aligned}
N(x) = & - \left(Q_0(Y_0(x)) \frac{dh_2(x,y)}{dx} + \pi_{0,0} \frac{dh_0(x,y)}{dx} \right) \Bigg|_{y=Y_0(x)} + \\
& - \left(Q_0^{(-)}(Y_0^{(-)}(x)) \frac{dh_2^{(-)}(x,y)}{dx} + \pi_{0,0} \frac{dh_0^{(-)}(x,y)}{dx} \right) \Bigg|_{y=Y_0^{(-)}(x)},
\end{aligned}
$$

$$
\begin{aligned}
O(x) &= \frac{dh_1(x,y)}{dy} \Bigg|_{y=Y_0(x)}, \\
O^{(-)}(x) &= \frac{dh_1^{(-)}(x,y)}{dy} \Bigg|_{y=Y_0^{(-)}(x)}, \\
Z(x) &= \frac{dh_1(x,y)}{dx} \Bigg|_{y=Y_0(x)} + \frac{dh_1^{(-)}(x,y)}{dx} \Bigg|_{y=Y_0^{(-)}(x)}, \\
M(x) &= L(x) g_1(x) - O(x) \Big(f(x) + f^{(-)}(x) + g_0(x) \Big), \\
M^{(-)}(x) &= L^{(-)}(x) g_1(x) - O^{(-)}(x) \Big(f(x) + f^{(-)}(x) + g_0(x) \Big), \\
U(x) &= N(x) g_1(x) - Z(x) \Big(f(x) + f^{(-)}(x) + g_0(x) \Big), \\
K(x) &= \frac{da(x)}{dx} + \frac{db(x)}{dx} \sqrt{1 - \frac{x}{x_3}}, \\
K^{(-)}(x) &= \frac{da^{(-)}(x)}{dx} + \frac{db^{(-)}(x)}{dx} \sqrt{1 - \frac{x}{x_3^{(-)}}}.
\end{aligned} \tag{3.56}
$$

Condition 5.1. If we have $x_{dom} = x_3$, we write,

$$\frac{dP_0(x)}{dx} = \frac{\left(K(x) - \frac{b(x)}{2x_3\sqrt{1-\frac{x}{x_3}}}\right) M(x)}{\left(g_1(x)\right)^2} + \frac{\left(K^{(-)}(x) - \frac{b^{(-)}(x)}{2x_3^{(-)}\sqrt{1-\frac{x}{x_3^{(-)}}}}\right) M^{(-)}(x) + U(x)}{\left(g_1(x)\right)^2}, \quad (3.57)$$

$$\lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\,\frac{d\left(P_0(x)\right)}{dx} = \left.\frac{-b(x)\ M(x)}{2x\left(g_1(x)\right)^2}\right|_{x=x_3} = s_1,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{s_1\ m^{-3/2}}{\sqrt{\pi}(x_3)^m}.$$

Condition 5.2. If $x_{dom} = x_3^{(-)}$, according to (3.57) we have,

$$\lim_{x\to x_{dom}} \sqrt{1-\frac{x}{x_{dom}}}\,\frac{d\left(P_0(x)\right)}{dx} = \left.\frac{-b^{(-)}(x)\ M^{(-)}(x)}{2x\left(g_1(x)\right)^2}\right|_{x=x_3^{(-)}} = s_2,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{s_2\ m^{-3/2}}{\sqrt{\pi}(x_3^{(-)})^m}.$$

bf Condition 5.3. If we have $x_{dom} = x_3 = x_3^{(-)}$, according to (3.57) we have,

$$\lim_{x \to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}} \frac{d\Big(P_0(x)\Big)}{dx} = \frac{-b(x)\ M(x) - b^{(-)}(x)\ M^{(-)}(x)}{2x\ \Big(g_1(x)\Big)^2}\Bigg|_{x=x_3} = s_3,$$

Therefore, using theorem 3.2 we get

$$\pi_{m,0} \sim \frac{s_3\ m^{-3/2}}{\sqrt{\pi}(x_3)^m}.$$

3.6 Exact Tail Asymptotic for $Q_0(y)$

In this section we find the exact tail asymptotic behaviour of the marginal stationary distribution along y-axis with applying theorem 3.2 to the generating function $Q_0(y)$. From (3.42) and (3.43) we have

$$Q_0(y) = \frac{l_1(y) + l_2(y)}{g_2(y)}.$$

Since $X_0(y)$ can be written as

$$X_0(y) = c(y) + d(y)\sqrt{1 - \frac{y}{y_3}}, \tag{3.58}$$

we can write the following equations

$$\begin{aligned} g_2(y) &= c_1(y) + d_1(y)\sqrt{1 - \frac{y}{y_3}}, \\ X_0(y_3) - X_0(y) &= (1 - \frac{y}{y_3})c^*(y) - d(y)\sqrt{1 - \frac{y}{y_3}}, \\ g_2(y_3) - g_2(y) &= c_1^*(y)(1 - \frac{y}{y_3}) - d_1(y)\sqrt{1 - \frac{y}{y_3}}. \end{aligned} \tag{3.59}$$

Hence, according to (3.42), (3.43), and (3.59), we characterize the exact tail asymptotic behaviour of $\pi_{0,n}$ into four cases: (1) exact geometric decay rate, (2) exact geometric decay rate with factor $n^{-1/2}$, (3) exact geometric decay rate with factor $n^{-3/2}$, and (4) exact geometric decay rate with factor n. In the following we will show the details on the cases above.

Case 1. If the conditions, $|y_{dom}| = y^* < \min\{\tilde{y}_1, y_3\}$, or $|y_{dom}| = y^* = \tilde{y}_1 = y_3$, or $|y_{dom}| = \tilde{y}_1 < \min\{y^*, y_3\}$ hold, then

$$\pi_{0,n} \sim \frac{c}{(y_{dom})^n}, \tag{3.60}$$

where c is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$. We will give the proof in the following conditions.

Condition 1.1. If $|y_{dom}| = y^* < \min\{\tilde{y}_1, y_3\}$, then we have

$$Q_0(y) = \frac{l_1(y) + l_2(y)}{(y - y_{dom})g_2^*(y)},$$

$$\Rightarrow \lim_{y\to y_{dom}} \left(1 - \frac{y}{y_{dom}}\right)Q_0(y) = \left.\frac{l_1(y) + l_2(y)}{g_2^*(y)}\right|_{y=y^*} = c_1.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{c_1}{(y^*)^n}.$$

Condition 1.2. If $|y_{dom}| = y^* = \tilde{y}_1 = y_3$, then

$$Q_0(y) = \frac{\dfrac{x^*\ l_1(y)\left(1 - \dfrac{X_0(y)}{x^*}\right)}{\sqrt{1 - \dfrac{y}{y_{dom}}}\left(\sqrt{1 - \dfrac{y}{y_{dom}}}\ c^*(y) - d(y)\right)} + l_2(y)}{\sqrt{1 - \dfrac{y}{y_{dom}}}\left(\sqrt{1 - \dfrac{y}{y_{dom}}}\ c_1^*(y) - d_1(y)\right)},$$

$$\Rightarrow \lim_{y\to y_{dom}} \left(1 - \frac{y}{y_{dom}}\right)Q_0(y) =$$

$$\left.\frac{x^* \ l_1(y)\left(1-\dfrac{X_0(y)}{x^*}\right)}{\left(\sqrt{1-\dfrac{y}{y_{dom}}}\ c^*(y)-d(y)\right)\left(\sqrt{1-\dfrac{y}{y_{dom}}}\ c_1^*(y)-d_1(y)\right)}\right|_{y=y^*}=c_2.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{c_2}{(y^*)^n}.$$

Condition 1.3. If $|y_{dom}| = \tilde{y_1} < \min\{y^*, y_3\}$, then

$$Q_0(y) = \frac{\dfrac{l_1(y)\left(1-\dfrac{X_0(y)}{x^*}\right)}{\left(1-\dfrac{X_0(y)}{x^*}\right)} + l_2(y)}{g_2(y)},$$

$$\Rightarrow \lim_{y\to y_{dom}}\left(1-\frac{y}{y_{dom}}\right)Q_0(y) = \left.\frac{x^* \ l_1(y)\left(1-\dfrac{X_0(y)}{x^*}\right)}{y\ \dfrac{dX_0(y)}{dy}\ g_2(y)}\right|_{y=\tilde{y_1}} = c_3.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{c_3}{(\tilde{y_1})^n}.$$

Case 2. If the conditions, $|y_{dom}| = y^* = y_3 < \tilde{y_1}$, or $|y_{dom}| = \tilde{y_1} = y_3 < y^*$ hold, then

$$\pi_{0,n} \sim \frac{t\ n^{-1/2}}{\sqrt{\pi}(y_{dom})^n}, \tag{3.61}$$

where t is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$. The proofs are given in the following conditions.

Condition 2.1. If $|y_{dom}| = y^* = y_3 < \tilde{y_1}$, then

$$Q_0(y) = \frac{l_1(y) + l_2(y)}{\sqrt{1-\dfrac{y}{y_{dom}}}\left(\sqrt{1-\dfrac{y}{y_{dom}}}\ c_1^*(y) - d_1(y)\right)},$$

$$\Rightarrow \lim_{y \to y_{dom}} \sqrt{1 - \frac{y}{y_{dom}}} Q_0(y) = \frac{l_1(y) + l_2(y)}{\left(\sqrt{1 - \frac{y}{y_{dom}}}\ c_1^*(y) - d_1(y) \right)} \Bigg|_{y=y^*}$$
$$= t_1.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{t_1\ n^{-1/2}}{\sqrt{\pi}(y^*)^n}.$$

Condition 2.2. If $|y_{dom}| = \tilde{y}_1 = y_3 < y^*$, then

$$Q_0(y) = \frac{\dfrac{x^*\ l_1(y)\left(1 - \dfrac{X_0(y)}{x^*}\right)}{\sqrt{1 - \dfrac{y}{y_{dom}}}\left(\sqrt{1 - \dfrac{y}{y_{dom}}}\ c^*(y) - d(y)\right)} + l_2(y)}{g_2(y)},$$

$$\Rightarrow \lim_{y \to y_{dom}} \sqrt{1 - \frac{y}{y_{dom}}} Q_0(y) = \frac{x^*\ l_1(y)\left(1 - \dfrac{X_0(y)}{x^*}\right)}{\left(\sqrt{1 - \dfrac{y}{y_{dom}}}\ c^*(y) - d(y)\right)\ g_2(y)} \Bigg|_{y=\tilde{y}_1}$$
$$= t_2.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{t_2\ n^{-1/2}}{\sqrt{\pi}(y_3)^n}.$$

Case 3. If the condition $|y_{dom}| = y_3 < \min\{\tilde{y}_1, y^*\}$ holds, then

$$\pi_{0,n} \sim \frac{u\ n^{-3/2}}{\sqrt{\pi}(y_{dom})^n}, \tag{3.62}$$

where u is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$. The proof is given in the following condition, but first we need the following assigned notations in order to simplify the calculations.

In (3.19) let $C(x) = P_{-1}(x)A(x) + \pi_{0,-1}B(x)$. Therefore, according to (3.42) we have

$$Q_0(y) = \frac{l_1(y) - \pi_{0,0}h_0(X_0(y), y) - C(X_0(y))}{g_2(y)}. \tag{3.63}$$

Also lets define

$$\begin{aligned}
D(y) =& - h_1(X_0(y), y)\frac{dP_0(X_0(y))}{dx} - \frac{dC(X_0(y))}{dx} - \\
&\left(P_0(X_0(y))\frac{dh_1(x,y)}{dx} + \pi_{0,0}\frac{dh_0(x,y)}{dx}\right)\Bigg|_{x=X_0(y)},\\
E(y) =& -\left(P_0(X_0(y))\frac{dh_1(x,y)}{dy} + \pi_{0,0}\frac{dh_0(x,y)}{dy}\right)\Bigg|_{x=X_0(y)},\\
I(y) =& \frac{dh_2(x,y)}{dx}\Bigg|_{x=X_0(y)},\\
J(y) =& \frac{dh_2(x,y)}{dy}\Bigg|_{x=X_0(y)},\\
V(y) =& D(y)g_2(y) - I(y)\Big(l_1(y) + l_2(y)\Big),\\
W(y) =& E(y)g_2(y) - J(y)\Big(l_1(y) + l_2(y)\Big),\\
R(y) =& \frac{dc(y)}{dy} + \frac{d\ d(y)}{dy}\sqrt{1 - \frac{y}{y_3}}.
\end{aligned} \tag{3.64}$$

Condition 3.1. If $|y_{dom}| = y_3 < \min\{\tilde{y}_1, y^*\}$, then

$$\frac{dQ_0(y)}{dy} = \frac{\left(R(y) - \frac{d(y)}{2y_3\sqrt{1-\frac{y}{y_3}}}\right)V(y) + W(y)}{\Big(g_2(y)\Big)^2},$$

$$\Rightarrow \lim_{y\to y_{dom}} \sqrt{1 - \frac{y}{y_{dom}}}\ \frac{dQ_0(y)}{dy} = \frac{-d(y)V(y)}{2y_3\Big(g_2(y)\Big)^2}\Bigg|_{y=y_3} = u_1.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{u_1 \ n^{-3/2}}{\sqrt{\pi}(y_3)^n}.$$

Case 4. If the condition $|y_{dom}| = y^* = \tilde{y}_1 < y_3$ holds, then

$$\pi_{0,n} \sim \frac{v \ n}{(y_{dom})^n}, \tag{3.65}$$

where v is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$. The proof is given below.

Condition 4.1. If $|y_{dom}| = y^* = \tilde{y}_1 < y_3$, then

$$Q_0(y) = \frac{\dfrac{l_1\left(1 - \dfrac{X_0(y)}{x^*}\right)}{\left(1 - \dfrac{X_0(y)}{x^*}\right)} + l_2(y)}{(y - y_{dom}) \ g_2^*(y)},$$

$$\Rightarrow \lim_{y \to y_{dom}} \left(1 - \frac{y}{y_{dom}}\right)^2 Q_0(y) = \left.\frac{x^* \ l_1(y)\left(1 - \dfrac{X_0(y)}{x^*}\right)}{y^2 \ \dfrac{dX_0(y)}{dy} \ g_2^*(y)}\right|_{y=y^*} = v_1.$$

Therefore, using theorem 3.2 we get

$$\pi_{0,n} \sim \frac{v_1 \ n}{(y^*)^n}.$$

Because of the symmetry, the exact tail asymptotic behaviour of the marginal stationary distribution along $-y$-axis is similar to the asymptotics behaviour of $\pi_{0,n}$. Therefore, we are not going through the details of the analysis of $Q_0^{(-)}(y)$.

Chapter 4

Exact Tail Asymptotic Behaviour of the Joint Stationary Distributions of Generalized-JSQ with Kernel Method

4.1 Generalized-JSQ Model and Kernel Method

In this section we describe the Generalized-JSQ model with two servers. We will model this queue as a two-dimensional random walk in the first and fourth quadrant of the random walk plane, and for each quadrant we will write a specific fundamental form.

In the Generalized-JSQ model we have three arrival streams. The first stream is a Poisson process with rate of λ_1, which is dedicated to server 1. That is, this stream only goes to server 1. The second stream is a Poisson process with rate of λ_2, which is dedicated to server 2, and the third stream is a Poisson process with rate of λ, which is called smart stream. That is, this stream determines the shorter queue, and joins that queue. We have exponential distributions for service times, where the rate of service in server 1 is μ_1, and in server 2 is μ_2.

Since G-JSQ is a two-dimensional random walk model, and we assume that random walks are considered to be without jumps, that is, the maximum step in any direction is 1. Therefore, the transition probabilities for the Generalized-JSQ model are as follows,

$$\bar{P}_{(a,b),(a+i,b+j)} = \begin{cases} p_{i,j} & \text{if } a \geq 1 \text{ and } b \geq 1,\ -1 \leq i,j \leq 1, \\ p^{0}_{i,j} & \text{if } (a,b) = (0,0),\ -1 \leq i,j \leq 1, \\ p^{(1)}_{i,j} & \text{if } a \geq 1 \text{ and } b = 0,\ -1 \leq i,j \leq 1, \\ p^{(2)}_{i,j} & \text{if } a = 0 \text{ and } b \geq 1,\ -1 \leq i,j \leq 1, \\ p^{(2-)}_{i,j} & \text{if } a = 0 \text{ and } b \leq -1,\ -1 \leq i,j \leq 1, \\ p^{(-)}_{i,j} & \text{if } a \geq 1 \text{ and } b \leq -1,\ -1 \leq i,j \leq 1, \end{cases}$$

where $p_{i,j}$, $p^{(0)}_{i,j}$, $p^{(1)}_{i,j}$, $p^{(2)}_{i,j}$, $p^{(-)}_{i,j}$, and $p^{(2-)}_{i,j}$ are non-negative real numbers having the following property, $\sum_{i,j=0\pm 1} p_{i,j} = 1$, $\sum_{i,j=0,\pm 1} p^{(1)}_{i,j} = 1$, $\sum_{i=0,1,j=0,\pm 1} p^{(0)}_{i,j} = 1$, $\sum_{i=0,1,j=0,\pm 1} p^{(2)}_{i,j} = 1$, $\sum_{i=0,1,j=0,\pm 1} p^{(2-)}_{i,j} = 1$, and $\sum_{i,j=0,\pm 1} p^{(-)}_{i,j} = 1$.

By modelling the Generalized-JSQ into the first and fourth quadrant of the random walk plane, where x-axis corresponds to $\min\{Q_1, Q_2\}$ and y-axis corresponds to $(Q_1 - Q_2)$, we have the following probabilities,

$$\begin{aligned} & p_{1,-1} = \lambda_1 + \lambda,\ p_{0,1} = \lambda_2,\ p_{-1,1} = \mu_1,\ p_{0,-1} = \mu_2, \\ & p^{(1)}_{0,1} = \lambda_2 + \lambda/2,\ p^{(1)}_{-1,1} = \mu_1,\ p^{(1)}_{0,-1} = \lambda_1 + \lambda/2,\ p^{(1)}_{-1,-1} = \mu_2, \\ & p^{(2)}_{0,1} = \lambda_2,\ p^{(2)}_{1,-1} = \lambda_1 + \lambda,\ p^{(2)}_{0,-1} = \mu_2,\ p^{(2)}_{0,0} = \mu_1, \\ & p^{(0)}_{0,1} = \lambda_2 + \lambda/2,\ p^{(0)}_{0,-1} = \lambda_1 + \lambda/2,\ p^{(0)}_{0,0} = \mu_2 + \mu_2, \\ & p^{(-)}_{1,-1} = \lambda_1 + \lambda,\ p^{(-)}_{0,1} = \lambda_1,\ p^{(-)}_{-1,1} = \mu_2,\ p^{(-)}_{0,-1} = \mu_1, \\ & p^{(2-)}_{1,-1} = \lambda_2 + \lambda,\ p^{(2-)}_{0,1} = \lambda_1,\ p^{(2-)}_{0,-1} = \mu_1,\ p^{(2-)}_{0,0} = \mu_2. \end{aligned} \tag{4.1}$$

From the transition probabilities, one can write the balance equations as follows,
(1) for interior of the first quadrant:
if $n = 1$, then

$$\pi_{m,1} = \mu_2 \pi_{m,2} + (\lambda_1 + \lambda)\pi_{m-1,2} + (\lambda_2 + \lambda/2)\pi_{m,0} + \mu_1 \pi_{m+1,0},$$

if $n \geq 2$, then

$$\pi_{m,n} = (\lambda_1 + \lambda)\pi_{m-1,n+1} + \mu_2\pi_{m,n+1} + \mu_1\pi_{m+1,n-1} + \lambda_2\pi_{m,n-1},$$

(2) for the boundary on x-axis:
if $n = 0$, then

$$\pi_{m,0} = (\lambda_1 + \lambda)\pi_{m-1,1} + \mu_2\pi_{m,1} + \mu_1\pi_{m,-1} + (\lambda_2 + \lambda)\pi_{m-1,-1},$$

(3) for interior of the fourth quadrant:
if $n = -1$, then

$$\pi_{m,-1} = (\lambda_1 + \lambda/2)\pi_{m,0} + (\lambda_2 + \lambda)\pi_{m-1,-2} + \mu_2\pi_{m+1,0} + \mu_1\pi_{m,-2},$$

if $n \leq -2$, then

$$\pi_{m,n} = \lambda_1\pi_{m,n+1} + (\lambda_2 + \lambda)\pi_{m-1,n-1} + \mu_2\pi_{m+1,n+1} + \mu_1\pi_{m,n-1}. \tag{4.2}$$

Hence, by using the generating functions (3.1), (3.2), (3.3), and balance equations (4.2), we can derive two fundamental forms including one equation for the first quadrant , and one equation for the fourth quadrant as follows,
(1) for the first quadrant,

$$\begin{aligned} P(x,y)h(x,y) =& P_0(x)h_1(x,y) + Q_0(y)h_2(x,y) + \pi_{0,0}h_0(x,y) + \\ & \pi_{0,-1}B(x) + p_{-1}(x)A(x), \end{aligned} \tag{4.3}$$

(2) and for the fourth quadrant,

$$\begin{aligned} P^{(-)}(x,y)h^{(-)}(x,y) =& P_0(x)h_1^{(-)}(x,y) + Q_0^{(-)}(y)h_2^{(-)}(x,y) + \\ & \pi_{0,0}h_0^{(-)}(x,y) - \pi_{0,-1}B(x) + \mu_1) - \\ & p_{-1}(x)A(x), \end{aligned}$$

where,

$$
\begin{aligned}
h(x,y) &= \Big(1-\mu_2 y^{-1} - x(\lambda_1+\lambda)y^{-1} - \mu_1 x^{-1}y - \lambda_2 y\Big),\\
h^{(-)}(x,y) &= \Big(1-\mu_1 y^{-1} - (\lambda_2+\lambda)xy^{-1} - \mu_2 x^{-1}y - \lambda_1 y\Big),\\
h_1(x,y) &= \Big(\mu_1 x^{-1}y + (\lambda_2+\lambda/2)y - 1\Big),\\
h_1^{(-)}(x,y) &= \Big(\mu_2 x^{-1}y + (\lambda_1+\lambda/2)y\Big),\\
h_2(x,y) &= \Big((\lambda_1+\lambda)xy^{-1} + \lambda_2 y + \mu_2 y^{-1} + \mu_1 - 1\Big),\\
h_2^{(-)}(x,y) &= \Big((\lambda_2+\lambda)xy^{-1} + \lambda_1 y + \mu_1 y^{-1} + \mu_2 - 1\Big),\\
h_0(x,y) &= \Big((\lambda_2+\lambda/2)y + \mu_1 + \mu_2 - 1\Big),\\
h_0^{(-)}(x,y) &= \Big((\lambda_1+\lambda/2)y\Big),\\
A(x) &= \Big((\lambda_2+\lambda)x + \mu_1\Big),\\
B(x) &= \Big((\lambda_2+\lambda)x + \mu_1\Big). \qquad (4.4)
\end{aligned}
$$

Therefore, according to lemma 3.1. the fundamental form for the both first and fourth quadrant is

$$
\begin{aligned}
&P(x,y)h(x,y) + P^{(-)}(x,y)h^{(-)}(x,y) = P_0(x)H_1(x,y) +\\
&Q_0(y)h_2(x,y) + Q_0^{(-)}(y)h_2^{(-)}(x,y) + \pi_{0,0}H_0(x,y), \qquad (4.5)
\end{aligned}
$$

where,

$$
\begin{aligned}
H_1(x,y) &= h_1(x,y) + h_1^{(-)}(x,y),\\
H_0(x,y) &= h_0(x,y) + h_0^{(-)}(x,y). \qquad (4.6)
\end{aligned}
$$

4.2 Branch Points and Branches

In this section we introduce the kernel equation for the Generalized-JSQ model, and provide details on properties of branches and branch points of the kernel equation. The polynomial $h(x,y)$ is called kernel. Therefore, according to (3.23), for the generalized-JSQ model, kernel equation is

$$h(x,y) = a(x)y^2 + b(x)y + c(x) = 0,$$

where

$$\begin{aligned} a(x) &= \Big(-\lambda_2 x - \mu_1 \Big), \\ b(x) &= x, \\ c(x) &= \Big(-(\lambda_1 + \lambda)x^2 - \mu_2 x \Big). \end{aligned} \tag{4.7}$$

from (3.25), the branches of the kernel equation, $h(x,y) = 0$, are

$$Y_{\pm}(x) = \frac{-b(x) \pm \sqrt{D_1(x)}}{2a(x)},$$

where

$$D_1(x) = b(x)^2 - 4a(x)c(x).$$

Furthermore, as stated in (3.28), kernel equation for the first quadrant can be written as,

$$h(x,y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) = 0,$$

where

$$\begin{aligned} \tilde{a}(y) &= \Big(-\lambda_1 - \lambda \Big), \\ \tilde{b}(y) &= \Big(-\lambda_2 y^2 + y - \mu_2 \Big), \\ \tilde{c}(y) &= \Big(-\mu_1 y^2 \Big). \end{aligned} \tag{4.8}$$

from (3.30), the solutions for x are,

$$X_{\pm}(y) = \frac{-\tilde{b}(y) \pm \sqrt{D_2(y)}}{2\tilde{a}(y)},$$

where

$$D_2(y) = \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y),$$

Similarly as stated in (3.33), kernel equation of the fourth quadrant is

$$h^{(-)}(x, y) = a^{(-)}(x)y^2 + b^{(-)}(x)y + c^{(-)}(x) = 0,$$

where

$$\begin{aligned}
a^{(-)}(x) &= (-\lambda_1 x - \mu_2), \\
b^{(-)}(x) &= x, \\
c^{(-)}(x) &= (-(\lambda_2 + \lambda)x^2 - \mu_1 x),
\end{aligned} \tag{4.9}$$

and from (3.37), the kernel equation for the fourth quadrant can be written as

$$h^{(-)}(x, y) = \tilde{a}^{(-)}(y)x^2 + \tilde{b}^{(-)}(y)x + \tilde{c}^{(-)}(y) = 0,$$

where

$$\begin{aligned}
\tilde{a}^{(-)}(y) &= (-\lambda_2 - \lambda), \\
\tilde{b}^{(-)}(y) &= (-\lambda_1 y^2 + y - \mu_1), \\
\tilde{c}^{(-)}(y) &= (-\mu_2 y^2).
\end{aligned} \tag{4.10}$$

4.3 Asymptotic Analysis of $P_0(x)$ of the Generalized-JSQ

In this section we make $P_0(x)$ of the G-JSQ in terms of other generating functions from the fundamental forms. For simplification and reducing the unknown generating functions, we make the kernel polynomial zero by substituting the branches of kernel polynomial of the Generalized-JSQ into the fundamental form of the first and fourth quadrant, and consequently get the simpler expression for $P_0(x)$. Therefore, recalling (4.6) we have

$$P_0(x)\Bigg(h_1(x,Y_0(x))+h_1^{(-)}(x,Y_0^{(-)}(x))\Bigg)+Q_0(Y_0(x))h_2(x,Y_0(x))+$$
$$Q_0^{(-)}(Y_0^{(-)}(x))h_2^{(-)}(x,Y_0^{(-)}(x))+$$
$$\pi_{0,0}\Bigg(h_0(x,Y_0(x))+h_0^{(-)}(x,Y_0^{(-)}(x))\Bigg)=0, \tag{4.11}$$

where $h_1(x,Y_0(x))$, $h_2(x,Y_0(x))$, $h_1^{(-)}(x,Y_0^{(-)}(x))$, $h_2^{(-)}(x,Y_0^{(-)}(x))$, $h_0(x,Y_0(x))$, and $h_0^{(-)}(x,Y_0(x))$ can be found from (4.5).
In order to analyse the asymptotic behaviour of $P_0(x)$, we need to get information about the dominant singularity of $P_0(x)$. The following lemma gives the solutions of the equation $\Bigg(h_1(x,Y_0(x))+h_1^{(-)}(x,Y_0^{(-)}(x))\Bigg)=0$, which is one of the singularities of $P_0(x)$.

Lemma 4.1. *Suppose the following two inequalities hold,*

(1) $\mu_1-\mu_2-2\mu_1\mu_2+2\lambda_2\mu_1^2+2\lambda_2\mu_2^2+2\mu_1\mu_2^2+4\mu_1^2\mu_2-3\mu_1^2+2\mu_1^3+\mu_2^2+4\lambda_2\mu_1\mu_2<0$,

(2) $\mu_1-\mu_2-2\mu_1\mu_2+2\lambda_2\mu_1^2+2\lambda_2\mu_2^2+2\lambda\mu_1^2+2\lambda\mu_2^2+2\mu_1\mu_2^2+4\mu_1^2\mu_2-3\mu_1^2+2\mu_1^3+\mu_2^2+4\lambda_2\mu_1\mu_2+4\lambda\mu_1\mu_2>0$,

then $x^*=(\frac{1}{\rho})^2=(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2+\lambda})^2$ *is the solution of* $h_1(x,Y_0(x))+h_1^{(-)}(x,Y_0^{(-)}(x))=0$.

Proof. If the first inequality holds, we have $Y_0(\rho^{-2})=\rho^{-1}$. Also from the second inequality we have $Y_0^{(-)}(\rho^{-2})=\rho^{-1}$. Therefore,

$$h_1(\rho^{-2},Y_0(\rho^{-2}))+h_1^{(-)}(\rho^{-2},Y_0^{(-)}(\rho^{-2}))=0.$$

This completes the proof. □

From now on let x^* be the solution of $h_1(x,Y_0(x))+h_1^{(-)}(x,Y_0^{(-)}(x))=0$, with the smallest modulus greater than 1. Hence, $h_1(x^*,Y_0(x^*))+h_1^{(-)}(x^*,Y_0^{(-)}(x^*))=0$, for $|x^*|>1$.

4.4 Asymptotic Analysis of $Q_0(y)$ of the Generalized-JSQ

Recall (4.3), which gives the fundamental form for the first quadrant. Now with replacing x by $X_0(y)$ in (4.3) we get,

$$Q_0(y) = \frac{-P_0(X_0)h_1(X_0,y) - \pi_{0,0}h_0(X_0,y)}{h_2(X_0,y)} - \frac{(P_{-1}(X_0) + \pi_{0,-1})((\lambda_2+\lambda)X_0 + \mu_1)}{h_2(X_0,y)}. \tag{4.12}$$

Now for finding the solution of $h_2(X_0,y) = 0$ with the smallest modulus greater than 1, we define the function,

$$f(y) = \tilde{a}(y)h_2(X_0,y)h_2(X_1,y).$$

Hence, we have the following lemma.

Lemma 4.2. *$y = \frac{\mu_2}{\lambda_2}$ is the number with the smallest modulus greater than 1 which makes $f(y) = 0$.*

Proof. Equation $f(y) = \tilde{a}(y)h_2(X_0,y)h_2(X_1,y) = 0$ implies,

$$-(\lambda_1+\lambda)\mu_1 y(y-1)(\lambda_2 y - \mu_2) = 0.$$

Therefore, $y = \frac{\mu_2}{\lambda_2}$ is the zero of $f(y)$ with the smallest modulus greater than 1. □

Lemma 4.3. *$y = \frac{\mu_2}{\lambda_2} = y^*$ is the zero of $h_2(X_0(y),y)$.*

Proof. From (4.11) one can get

$$D_2(\frac{\mu_2}{\lambda_2}) = \frac{\mu_2^2(2\mu_1+\mu_2+\lambda_2-1)^2}{\lambda_2^2},$$

since we assumed the stability conditions for the system, ($\mu_1 > \lambda_1 + \lambda$), which implies the inequality, $2\mu_1 + \mu_2 + \lambda_2 > 1$, from (4.11) one can get

$$X_+(\frac{\mu_2}{\lambda_2}) = \frac{\mu_2}{\lambda_2}, \text{ and } \quad X_-(\frac{\mu_2}{\lambda_2}) = \frac{\mu_2\mu_1}{\lambda_2(\lambda_1+\lambda)}.$$

With the above stability condition it is straight forward to see that

$$X_0(\frac{\mu_2}{\lambda_2}) = \frac{\mu_2}{\lambda_2},$$

Therefore,

$$h_2\left(X_0(\frac{\mu_2}{\lambda_2}), \frac{\mu_2}{\lambda_2}\right) = (\lambda_1+\lambda)\frac{\mu_2}{\lambda_2} + \lambda_2(\frac{\mu_2}{\lambda_2})^2 + \mu_2 + (\mu_1 - 1)\frac{\mu_2}{\lambda_2} = 0.$$

From now on we denote the solution of $h_2(X_0(y), y) = 0$ with the smallest modulus greater than 1 by y^*. □

4.5 Asymptotic Analysis of $Q_0^{(-)}(y)$ of the Generalized-JSQ

In this section, in order to analyse the asymptotic behaviour of $Q_0^{(-)}(y)$ of the Generalized-JSQ, we replace x by $X_0^{(-)}(y)$ in (4.4) to get,

$$\begin{aligned} Q_0^{(-)}(y) =& \frac{-P_0(X_0^{(-)}(y))h_1^{(-)}(X_0^{(-)}(y), y) - \pi_{0,0}h_0^{(-)}(X_0^{(-)}(y), y)}{h_2^{(-)}(X_0^{(-)}(y), y)} + \\ & \frac{-(P_{-1}(X_0^{(-)}(y)) + \pi_{0,-1})((\lambda_2+\lambda)X_0^{(-)}(y) + \mu_1)}{h_2^{(-)}(X_0^{(-)}(y), y)}. \end{aligned}$$

Lemma 4.4. *Let $g(y) = \tilde{a}^{(-)}(y)h_2^{(-)}(X_0^{(-)}(y), y)h_2^{(-)}(X_1^{(-)}(y), y)$, then $y = \frac{\mu_1}{\lambda_1}$ is a solution of $g(y) = 0$, with the smallest modulus greater than 1.*

Proof. If $g(y) = \tilde{a}^{(-)}(y)h_2^{(-)}(X_0^{(-)}(y), y)h_2^{(-)}(X_1^{(-)}(y), y) = 0$, then

$$-(\lambda_2+\lambda)\mu_2 y(y-1)(\lambda_1 y - \mu_1) = 0.$$

Therefore, $y = \frac{\mu_1}{\lambda_1}$ is a solution of $g(y) = 0$, with the smallest modulus greater than 1. □

Lemma 4.5. *$y = \dfrac{\mu_1}{\lambda_1}$ is a solution of $h_2^{(-)}(X_0^{(-)}(y), y) = 0$.*

Proof. From (4.15) we can get

$$D_2^{(-)}(\frac{\mu_1}{\lambda_1}) = \frac{\mu_1^2(\mu_2 - \lambda_2 - \lambda)^2}{\lambda_1^2}.$$

By using (4.15) and assuming the stability condition, which is ($\mu_2 > \lambda_2 + \lambda$), we can get $X_+^{(-)}(\frac{\mu_1}{\lambda_1}) = \frac{\mu_1}{\lambda_1}$, and $X_-^{(-)}(\frac{\mu_1}{\lambda_1}) = \frac{\mu_1\mu_2}{\lambda_1(\lambda_2+\lambda)}$. Therefore,

$$h_2^{(-)}(X_0^{(-)}(\frac{\mu_1}{\lambda_1}), \frac{\mu_1}{\lambda_1}) = (\lambda_2 + \lambda)\frac{\mu_1}{\lambda_1} + \lambda_1(\frac{\mu_1}{\lambda_1})^2 + \mu_1 + (\mu_2 - 1)\frac{\mu_1}{\lambda_1} = 0.$$

From now on we denote the solution of $h_2^{(-)}(X_0^{(-)}(y), y) = 0$ with the smallest modulus greater than 1 by $y^*_{(-)}$. □

4.6 Exact Tail Asymptotic for $P_0(x)$ of the Generalized-JSQ

In this section according to the results of the section 3.5 of this thesis, we find the exact tail asymptotics behaviour of $\pi_{m,0}$ of the generalized-JSQ by applying the theorem 3.2 to $P_0(x)$. If the inequalities of lemma 4.1. and two stability conditions, $\mu_i > \lambda_i + \lambda$ for $i = 1, 2$ hold, then the results of the section 3.5, indicate that the dominant singularity of $P_0(x)$ can be either $(\frac{1}{\rho})^2 = (\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2+\lambda})^2$, or $X_1(\frac{1}{\rho_2}) = X_1(\frac{\mu_2}{\lambda_2})$, or $X_1^{(-)}(\frac{1}{\rho_1}) = X_1^{(-)}(\frac{\mu_1}{\lambda_1})$, or x_3, the third branch point of $h(x, y)$, or $x_3^{(-)}$, the third branch point of $h^{(-)}(x, y)$. Therefore, we can characterize the exact tail asymptotic behaviour of $P_0(x)$ of the Generalized-JSQ into 5 cases: (1) exact geometric decay rate, (2) exact geometric decay rate with factor $m^{-1/2}$, (3) exact geometric decay rate with factor $m^{1/2}$, (4) exact geometric decay rate with factor m, and (5) exact geometric decay rate with factor $m^{-3/2}$. Next, we will show the details on the five cases above.

Case 1. If $x_{dom} = (\frac{1}{\rho})^2$, or $x_{dom} = X_1(\frac{1}{\rho_2})$, or $x_{dom} = X_1^{(-)}(\frac{1}{\rho_1})$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1})$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = x_3^{(-)}$, or $x_{dom} = X_1^{(-)}(\frac{1}{\rho_1}) = x_3$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = x_3$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1^{(-)}(\frac{1}{\rho_1}) = x_3^{(-)}$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1}) = x_3$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1}) = x_3^{(-)}$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = x_3 = x_3^{(-)}$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1^{(-)}(\frac{1}{\rho_1}) = x_3 = x_3^{(-)}$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1}) = x_3 = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{L}{(x_{dom})^m},$$

where ($\lim_{x\to x_{dom}}(1 - \frac{x}{x_{dom}})P_0(x) = L$) is a constant, and x_{dom} is dominant singularity of $P_0(x)$, which can be obtained from case 1 of section 3.5.

Case 2. If $x_{dom} = (\frac{1}{\rho})^2 = x_3$, or $x_{dom} = (\frac{1}{\rho})^2 = x_3^{(-)}$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = x_3$, or $x_{dom} = X_1^{(-)}(\frac{1}{\rho_1}) = x_3^{(-)}$, or $x_{dom} = (\frac{1}{\rho})^2 = x_3 = x_3^{(-)}$, or $x_{dom} = X_1(\frac{1}{\rho_2}) = x_3 = x_3^{(-)}$, or $x_{dom} = X_1^{(-)}(\frac{1}{\rho_1}) = x_3 = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{u\ m^{-1/2}}{\sqrt{\pi}(x_{dom})^m},$$

where ($\lim_{x\to x_{dom}}\sqrt{1 - \frac{x}{x_{dom}}}\ P_0(x) = u$) is a constant, and x_{dom} is dominant singularity for $P_0(x)$, which can be obtained from case 2 of section 3.5.

Case 3. If $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = x_3^{(-)}$, or $x_{dom} = (\frac{1}{\rho})^2 =$

$X_1^{(-)}(\frac{1}{\rho_1}) = x_3$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1}) = x_3$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1}) = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{n\ m^{1/2}}{\frac{\sqrt{\pi}}{2}(x_{dom})^m},$$

where $(\lim_{x\to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^{3/2} P_0(x) = n)$ is a constant, and x_{dom} is dominant singularity for $P_0(x)$, which can be obtained from case 3 of section 3.5.

Case 4. If $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2})$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1^{(-)}(\frac{1}{\rho_1})$, or $x_{dom} = (\frac{1}{\rho})^2 = X_1(\frac{1}{\rho_2}) = X_1^{(-)}(\frac{1}{\rho_1})$ holds, then

$$\pi_{m,0} \sim \frac{r\ m}{(x_{dom})^m},$$

where $(\lim_{x\to x_{dom}} \left(1 - \frac{x}{x_{dom}}\right)^2 P_0(x) = r)$ is a constant, and x_{dom} is dominant singularity of $P_0(x)$, which can be obtained from case 4 of section 3.5.

Case 5. If $x_{dom} = x_3$, or $x_{dom} = x_3^{(-)}$, or $x_{dom} = x_3 = x_3^{(-)}$ holds, then

$$\pi_{m,0} \sim \frac{s\ m^{-3/2}}{\sqrt{\pi}(x_{dom})^m},$$

where $(\lim_{x\to x_{dom}} \sqrt{1 - \frac{x}{x_{dom}}} \frac{d\Big(P_0(x)\Big)}{dx} = s)$ is a constant, and x_{dom} is the dominant singularity of $P_0(x)$, which can be obtained from case 5 of section 3.5.

4.7 Exact Tail Asymptotic for $Q_0(y)$ of the Generalized-JSQ

In this section we determine the exact tail asymptotic behaviour of $\pi_{0,n}$ for Generalized-JSQ with applying the theorem 3.2 to the generating function $Q_0(y)$. If the inequalities of lemma 4.1 and two stability conditions, $\mu_i > \lambda_i + \lambda$ for $i = 1, 2$, hold, then due to the results of the section 3.6, the dominant singularity of $Q_0(x)$ is either $\dfrac{1}{\rho_2} = \dfrac{\mu_2}{\lambda_2}$, or $Y_1(\dfrac{1}{\rho^2}) = Y_1((\dfrac{\mu_1+\mu_2}{\lambda_1+\lambda_2+\lambda})^2)$, or y_3, the third branch point of $h(x, y)$.

Hence, according to section 3.6, we can categorize the exact tail asymptotic behaviour for the $Q_0(y)$ of the generalized JSQ into 4 cases: (1) exact geometric decay rate, (2) exact geometric decay rate with factor $n^{-1/2}$, (3) exact geometric decay rate with factor $n^{-3/2}$, and (4) exact geometric decay rate with factor n. In the following we will show the details on the cases above.

Case 1. If the conditions, $|y_{dom}| = \dfrac{1}{\rho_2} < \min\ \{Y_1(\dfrac{1}{\rho^2}), y_3\}$, or $|y_{dom}| = \dfrac{1}{\rho_2} = Y_1(\dfrac{1}{\rho^2}) = y_3$, or $|y_{dom}| = Y_1(\dfrac{1}{\rho^2}) < \min\ \{\dfrac{1}{\rho_2}, y_3\}$ hold, then

$$\pi_{0,n} \sim \frac{c}{(y_{dom})^n},$$

where $(\lim\limits_{y\to y_{dom}} \left(1 - \dfrac{y}{y_{dom}}\right) Q_0(y) = c)$ is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$, which can be obtained from case 1 of section 3.6.

Case 2. If the conditions, $|y_{dom}| = \dfrac{1}{\rho_2} = y_3 < Y_1(\dfrac{1}{\rho^2})$, or $|y_{dom}| = Y_1(\dfrac{1}{\rho^2}) = y_3 < \dfrac{1}{\rho_2}$ hold, then

$$\pi_{0,n} \sim \frac{t\ n^{-1/2}}{\sqrt{\pi}(y_{dom})^n},$$

where $(\lim_{y\to y_{dom}} \sqrt{1-\frac{y}{y_{dom}}}Q_0(y) = t)$ is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$, which can be obtained from case 2 of section 3.6.

Case 3. If the condition $|y_{dom}| = y_3 < \min\{Y_1(\frac{1}{\rho^2}), \frac{1}{\rho_2}\}$ holds, then

$$\pi_{0,n} \sim \frac{u\ n^{-3/2}}{\sqrt{\pi}(y_{dom})^n},$$

where $(\lim_{y\to y_{dom}} \sqrt{1-\frac{y}{y_{dom}}}\frac{dQ_0(y)}{dy} = u)$ is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$, which can be obtained from case 3 of section 3.6.

Case 4. If the condition $|y_{dom}| = \frac{1}{\rho_2} = Y_1(\frac{1}{\rho^2}) < y_3$ holds, then

$$\pi_{0,n} \sim \frac{v\ n}{(y_{dom})^n},$$

where $(\lim_{y\to y_{dom}} \left(1-\frac{y}{y_{dom}}\right)^2 Q_0(y) = v)$ is a constant, and y_{dom} is the dominant singularity of $Q_0(y)$, which can be obtained from case 4 of section 3.6.

Because of the symmetry, the exact tail asymptotics behaviour of $\pi_{0,-n}$ for $n = 1, 2, 3, ...$ is similar to $\pi_{0,n}$ for $n = 1, 2, 3, ...$, and so we are not going through the details of the analysis of $Q_0^{(-)}(y)$.

Chapter 5

Conclusion

In this thesis, I extended the idea of a random walk model in the first quadrant by Fayolle, Malyshev, and Iasnogorodski [19], to find the general formulation for the generating functions, and fundamental form for a general random walk model in the first quadrant and fourth quadrant of the plane. In addition, I extended the results of the kernel method in the random walk plane by Li and Zhao [50], to investigate the exact tail asymptotics behaviour of a general random walk model in the first and fourth quadrant of the plane.

Moreover, I found the exact tail asymptotics behaviour of the Generalized-JSQ, which is a two-dimensional queueing model in the first and fourth quadrant of the plane by using the kernel method. This model was already studied by Li, Miyazawa, and Zhao [45], and Zhao and Grassmann [75]. But the advantage of the kernel method is that it is a simpler and faster method compared with other methods.

Although many studies have been done in the area of random walks, this research field has still a lot of problems to be solved and discovered. Here I give some suggestions for future investigations in this area:

(1) Studying the exact tail asymptotics behaviour of the generating functions in the three-dimensional random walk model using the kernel method, which has a large application in the manufacturing, communication, and healthcare industry. As an example one can work on parallel queueing problem with three servers.

(2) Studying the exact tail asymptotics behaviour of the two-dimensional queueing problems modelled in the first and fourth quadrants of the random walk plane with jump steps bounded by two instead of jump steps bounded by one.

Bibliography

[1] Abate, J. and Whitt, W. (1997) Asymptotics for M/G/1 low-priority waiting-time tail probabilities, Queueing Systems, 25, 173-233.

[2] Adan, I., Foley, R.D. and McDonald, D.R. (2009) Exact asymptotics for the stationary distribution of a Markov chain: a production model, Queueing Systems, 62, 311-344.

[3] IJBF Adan, J. Wessels, and WHM Zijm, Analysis of the Symmetric Shortest Queue Problem, Stochastic Models, 6:691-713, 1990.

[4] Banderier, C., Bousquet-Mlou, M., Denise, A., Flajolet, P., Gardy, D. and Gouyou- Beauchamps, D. (2002). Generating functions for generating trees, (Formal Power Series and Algebraic Combinatorics, Barcelona, 1999), Discrete Math. 246:1-3, 29-55.

[5] B. Blaszczyszyn, A. Frey and V. Schmidt, Light Traffic Approximation for Markov-Modulated Mulriserver Queues, Stochastic Models, Vol. 11, p. 423-445, 1995.

[6] Amarjit Budhiraja, Paul Dupuis, and Vasileios Matoulas, Large Deviations for Infinite Dimensional Stochastic Dynamical Systems, Annals of Probability, Vol. 6, No. 4, 1390-1420, 2008.

[7] J.P.C. Blanc, A note on waiting times in systems with queues in parallel, Journal of Applied Probability. Vol. 24, p. 540-546, 1987.

[8] Bender, E. (1974) Asymptotic methods in enumeration, SIAM Review, 16, 485-513.

[9] Borovkov, A.A. and Mogul'skii, A.A. (2001) Large deviations for Markov chains in the positive quadrant, Russian Math. Surveys, 56, 803-916.

[10] Bousquet-Melou, M. (2005) Walks in the quarter plane: Kreweras algebraic model, Annals of Applied Probability, 15, 1451-1491.

[11] J. W. Cohen, Analysis of a Two-Dimensional Algebraic Nearest-Neighbour Random Walk (Queue with Paired Services), Technical Report BS-R9437, CWI, Amsterdam, 1994.

[12] Cohen, J. W. and Boxma, O. J. (1983) Boundary Value Problems in Queueing System Analysis, North-Holland, Amsterdam.

[13] D. R. Cox and H. D. Miller, The Theory of Stochastic Processes, John Wiley and Sons Inc., New York 1965, Chapter 2 and 5.

[14] Dai, J. and Miyazawa, M. (2010) Reflecting Brownian motion in two dimensions: Exact asymptotics for the stationary distribution, submitted.

[15] Robert D. Foley and David R. McDonald, Large Deviation of a Modified Jackson Network: Stability and Rough Asymptotics, The Annals of Applied Probability. Vol. 15, No. 1B, 519C541, 2005.

[16] Foley, R.D. and McDonald, R.D. (2001). Join-the-shortest-queue: Stability and exact asymptotics. Ann. Appl. Probab. 11, 569-607.

[17] Foley, R.D. and McDonald, D.R. (2001) Join the shortest queue: stability and exact asymptotics, Annals of Applied Probability, 11, 569-607.

[18] Foley, R.D. and McDonald, R.D. (2005) Bridges and networks: exact asymptotics, Annals of Applied Probability, 15, 542-586.

[19] G. Fayolle, R. Iasnogorodski and V. Malyshev, Random Walks in the Quarter Plane, Springer, Berlin, 1999.

[20] Fayolle, G. and Iasnogorodski, R. (1979) Two coupled processors: the reduction to a Riemann- Hilbert problem, Z. Wahrscheinlichkeitsth, 47, 325-351.

[21] Fayolle, G., King, P.J.B. and Mitrani, I. (1982) The solution of certain two-dimensional Markov models, Adv. Appl. Prob., 14, 295-308.

[22] Flajolet, P. and Odlyzko, A. (1990) Singularity analysis of generating functions, SIAM J. Disc. Math., 3, 216-240.

[23] Flajolet, F. and Sedgewick, R. (2009) Analytic Combinatorics, Cambridge University Press.

[24] Flatto, L. and McKean, H.P. (1977) Two queues in parallel, Comm. Pure Appl. Math., 30, 255-263.

[25] Flatto, L. and Hahn, S. (1984) Two parallel queues created by arrivals with two demands I, SIAM J. Appl. Math., 44, 1041-1053.

[26] Flatto, L. (1985) Two parallel queues created by arrivals with two demands II, SIAM J. Appl. Math., 45, 861-878.

[27] Foss, S., and Chernova, N. (1991). On the ergodicity of multichannel not fully accessible communication systems. Problems of Information Transmission, 27, 94-99.

[28] Foss, S., and Chernova, N. (1998). On the stability of partially accessible queue with state-dependent routing. Queueing Systems, 29, 55-73.

[29] Guillemin, F. and Leeuwarden, J. (2009) Rare event asymptotics for a random walk in the quarter plane, submitted.

[30] D. P. Gaver, Diffusion Approximation and Models for Certain Congestion Problems, Journal of Applied Probability, Vol. 5, p. 607-1968.

[31] Haque, L. (2003) Tail Behaviour for Stationary Distributions for Two-Dimensional Stochastic Mod- els, Ph.D. Thesis, Carleton University, Ottawa, ON, Canada.

[32] Haque, L., Liu, L. and Zhao, Y.Q. (2005) Sucient conditions for a geometric tail in a QBD process with countably many levels and phases, Stochastic Models, 21(1), 77-99.

[33] He, Q., Li, H. and Zhao, Y.Q. (2009) Light-tailed behaviour in QBD process with countably many phases, Stochastic Models, 25, 50-75.

[34] Halfin, S. (1985). The shortest queue problem. J. Appl. Probab. 22, 865-878.

[35] G. Hooghiemstra, M. Keane and S. Van de Ree, Power Series for Stationary Distributions of Coupled Processor Models, SIAM J. Appl. Math. Vol. 48, 1159-1166, 1988.

[36] Khanchi, Aziz (2008) State of a network when one node overloads, Ph.D. Thesis, University of Ottawa.

[37] Khanchi, Aziz (2009) Asymptotic hitting distribution for a reflected random walk in the positive quardrant, submitted.

[38] D. E. Knuth, The art of computer programming, vol. 1: Fundamental Algorithms, Addison-Wesley, 1973, Third edition, 1997.

[39] Kobayashi, M. and Miyazawa, M. (2010) Tail asymptotics of the stationary distribution of a two dimensional re-acting random walk with unbounded upward jumps, submitted.

[40] Kobayashi, M. and Miyazawa, M. (2011) Tail asymptotics of the stationary distribution of a two dimensional reflecting random walk with unbounded upward jumps, submitted.

[41] Kobayashi, M., Miyazawa, M. and Zhao, Y.Q. (2010) Tail asymptotics of the occupation measure for a Markov additive process with an M/G/1-type background process, accepted by Stochastic Models.

[42] Kroese, D.P., Scheinhardt, W.R.W. and Taylor, P.G. (2004) Spectral properties of the tandem Jackson network, seen as a quasi-birth-and-death process, Annals of Applied Probability, 14(4), 2057-2089.

[43] Kurkova, I.A. and Suhov, Y.M. (2003) Malyshev's theory and JS-queues. Asymptotics of stationary probabilities, The Annals of Applied Probability, 13, 1313-1354.

[44] Kurkova, I.A. (2001). A load-balanced network with two servers. Queueing Systems, 37, 379-389

[45] Li, L., Miyazawa, M. and Zhao, Y. (2007) Geometric decay in a QBD process with countable background states with applications to a join-the-shortest-queue model, Stochastic Models, 23, 413-438.

[46] Li, H. and Zhao, Y.Q. (2005) A retrial queue with a constant retrial rate, server break downs and impatient customers, Stochastic Models, 21, 531-550.

[47] Li, H. and Zhao, Y.Q. (2009) Exact tail asymptotics in a priority queue-characterizations of the preemptive model, Queueing Systems, 63, 355-381.

[48] Li, H. and Zhao, Y.Q. (2011) Exact tail asymptotics in a priority queue—characterizations of the non-preemptive model, Queueing Systems, accepted.

[49] Li, H. and Zhao, Y.Q. (2011) Tail asymptotics for a generalized two demand queueing model-A kernel method, Queueing Systems, accepted.

[50] Li, H., Yiqiang Q. Zhao, A Kernel Method for Exact Tail Asymptotics-Random Walks in the Quater Plane, submitted.

[51] Lieshout, P. and Mandjes, M. (2008) Asymptotic analysis of Levy-driven tandem queues, Queueing Systems, 60, 203-226.

[52] Liu, L., Miyazawa, M. and Zhao, Y.Q. (2008) Geometric decay in level-expanding QBD models, Annals of Operations Research, 160, 83-98.

[53] Mishna, M. (2009) Classifying lattice walks restricted to the quarter plane, Journal of Combinatorial Theory, Series A, 116, 460-477.

[54] Miyazawa, M. (2004) The Markov renewal approach to M/G/1 type queues with countably many background states, Queueing Systems, 46, 177-196.

[55] Miyazawa, M. (2007) Doubly QBD process and a solution to the tail decay rate problem, in Proceedings of the Second Asia-Pacific Symposium on Queueing Theory and Network Applications, Kobe, Japan.

[56] Miyazawa, M. (2009) Two sided DQBD process and solutions to the tail decay rate problem and their applications to the generalized join shortest queue, in Advances in Queueing Theory and Network Applications, edited by W. Yue, Y. Takahashi and H. Takaki, 3-33, Springer, New York.

[57] Miyazawa, M. and Zhao, Y.Q. (2004) The stationary tail asymptotics in the G/I/G1 type queue with countably many background states, Adv. in Appl. Probab., 36(4), 1231-1251.

[58] Morrison, J.A. (2007) Processor sharing for two queues with vastly different rates, Queueing Systems, 57, 19-28.

[59] Motyer, Allan J. and Taylor, Peter G. (2006) Decay rates for quasi-birth-and-death process with countably many phases

and tri-diagonal block generators, Advances in Applied Probability, 38, 522-544.

[60] V. A. Malyshev, Random Walks. The Wiener-Hopf Equations in Quadrant of the Plane. Galois Automorphisms. Moscow Univ., Press, 1970. (In Russian)

[61] V. A. Malyshev, Analytic Method in the Theory of Two-Dimensional Random Walks, Sib, Math. J. 13 1314-1327, 1972.

[62] V. A. Malyshev, Asymptotic Behaviour of Stationary Probabilities for Two-Dimensional Positive Random Walks, Sib. Math. J. 14 156-169, 1973.

[63] Mitzenmacher, M.D. (1996). The power of two choices in randomized load balancing. PhD thesis. University of California at Berkeley.

[64] Raschel, K. (2010) Green functions and Martin compactification for killed random walks related to SU(3), Elect. Comm. in Probab., 15, 176-190.

[65] Sheldon M. Ross, Introduction to Probability Models (8th Edition), 2003.

[66] Sharifnia, A. (1997). Instability of the join-the-shortest-queue and FCFS policies in queueing systems and their stabilization. Operations Research, 45, 309-314.

[67] Suhov, Yu.M., and Vvedenskaya, N.D. (2002). Fast Jackson networks with dynamic routing. Probl. Inform. Transmission, 38, 136-159.

[68] Takahashi, Y., Fujimoto, K. and Makimoto, N. (2001) Geometric decay of the steady-state probabilities in a quasi-birth-and-death process with a countable number of phases, Stochastic Models, 17(1), 1-24.

[69] Tandra, R., Hemachandra, N. and Manjunath, D. (2004). Job minimum cost queue for multiclass customers. Stability and Performance bounds. Probability in the Engineering and informational Sciences, 18, 445-472.

[70] Tang, J. and Zhao, Y.Q. (2008) Stationary tail asymptotics of a tandem queue with feedback, Annals of Operations Research, 160, 173-189.

[71] Vvedenskaya, N.D. and Suhov, Yu.M. (2004). Functional equations in asymptotic problems of queueing theory. Journal of Mathematical Sciences (N.Y.), 120, 1255-1276. 22 ABRAMOV.

[72] Vvedenskaya, N.D., Dobrushin, R.L., and Karpelevich, F.I. (1996). A queueing system with selection the shortest of two queues: An asymptotic approach. Probl. Inform. Transmission, 32, 15-27.

[73] Winston, W. (1977). Optimality of the shortest line discipline. J. Appl. Probab. 14, 181-189.

[74] Wright, P. (1992) Two parallel processors with coupled inputs, Adv. Appl. Prob., 24, 986-1007.

[75] Yiqiang Zhao, Winfried K. Grassmann. 1995. Queueing Analysis of a Jockeying Model. Operations Research 43, 520-529.