Computing Motivic Donaldson–Thomas Invariants

by

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Abstract

This thesis develops a method (*dimensional reduction*) to compute motivic Donaldson–Thomas invariants. The method is then employed to compute these invariants in several different cases.

Given a moduli scheme with a symmetric obstruction theory a Donaldson–Thomas type invariant can be defined by integrating Behrend's function over the scheme. Motivic Donaldson–Thomas theory aims to provide a more refined invariant associated to each such moduli space - a *virtual motive*.

From the modern point of view motivic Donaldson–Thomas invariants should be defined for a three dimensional Calabi–Yau category. These categories often arise in a geometric context as the derived category of representations of a *quiver with potential*.

Provided the potential has a linear factor we are able to reduce the problem of computing the corresponding virtual motives to a much simpler one. This includes geometric examples coming from local curves which we compute explicitly.

Preface

This thesis is a compendium of three papers. The Chapters 2, 3, and 4 are essentially three separate articles.

Chapter 2 is a version of independent work published in [38];

Chapter 3 provides an application of the reduction theorem proven in Chapter 2 to the resolved conifold singularity. It is joint work with Prof. Sergey Mozgovoy, Prof. Kentaro Nagao and Prof. Balazs Szendrői [39], my individual contribution to this project was Section 3.3.2;

Chapter 4 generalizes the results of Chapter 3 and comprises joint work with Prof. Kentaro Nagao [40]. Sections 4.5 and 4.9 are my main individual contribution to the work;

[38] A. Morrison, *Motivic invariants of quivers via dimensional reduction*, Selecta Mathematica, published online January 11th 2012.
[39] A. Morrison, S. Mozgovoy, K. Nagao, B. Szendrői. *Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex.* submitted to Advances in Math., July 2011.
[40] A. Morrison, K. Nagao. *Motivic Donaldson–Thomas invariants of small crepant resolutions*, submitted to Algebra & Number Theory, November 2011.

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Dedication

For Rachel and Coen Sanchez.

Chapter 1

Introduction

This thesis contains work collected from three research papers [38], [39], [40] found in Chapters 2, 3 and 4 respectively. Each of the chapters begins with an introduction to the results it contains.

Chapter 2 - contains the dimensional reduction theorem used throughout the thesis in the computation of motivic Donaldson–Thomas invariants. The first six sections introduce all the basic material on quivers, moduli spaces, motives and vanishing cycles. The last section contains the proof of our dimensional reduction theorem.

Chapter 3 - is our first application of the reduction theorem to the resolved conifold. Section 3.2 contains the preliminaries. Section 3.3 computes the reduced series. Then section 3.4 computes the motivic DT invariants (for any choice of stability parameter). The final section identifies the DT/PT moduli spaces for the resolved conifold, giving a motivic version of the DT/PT correspondence.

Chapter 4 - is a generalization of the results in Chapter 3 to a collection of local toric Calabi-Yau threefolds. Sections 4.2-4.4 contain preliminaries on root systems, non-commutative crepant resolutions and motivic DT invariants. Section 4.5 computes the reduced series in a special case (modulo the linear algebra calculations of 4.9). Then in section 4.6 this is generalized using mutations of the quiver and potential. Section 4.7 is almost identical to 3.4 and computes the motivic Donaldson–Thomas invariants (for any choice of stability parameter). Section 4.8 gives another motivic DT/PT type correspondence.

Chapter 5 - is our conclusion. It recapitulates the main results of the thesis with some added analysis. We mention some related work in theoretical physics and end with proposals for future research.

Chapter 2

Dimensional Reduction

This chapter contains a version of independent work published in [38]. It provides a reduction formula for the motivic Donaldson–Thomas invariants associated to a quiver with potential. The method is valid provided the superpotential has a linear factor, it allows us to compute virtual motives in terms of ordinary motivic classes of simpler quiver varieties.

2.1 Introduction

In [60] Thomas defines integer valued invariants associated to compact moduli spaces of stable sheaves on a Calabi–Yau threefold. These numbers contain geometric information about the underlying manifold. In particular they provide a virtual count for the number of curves in each homology class, conjecturally equivalent [37] to the invariants of Gromov–Witten.

Recently there have been several extensions of the integer valued Donaldson–Thomas invariant [33], [32], [4], [27]. In [33] Kontsevich and Soibelman propose to work in a general three dimensional Calabi–Yau category. For a choice of stability condition they get moduli spaces of objects in this category each of which has a motivic DT invariant. The Euler numbers of these motives specialize to the classical invariant.

Quivers with potential provide concrete examples of such CY3 categories. Although quiver categories are of independent interest in representation theory [54] they often contain geometric content, in particular the derived category of many local CY3 folds can be realized in this way.

We provide a reduction theorem for motivic DT invariants of a general class of quivers with potential. The theorem expresses the motivic DT invariants in terms of the ordinary motivic classes of certain reduced quiver representations.

In this way the theorem can be seen as a generalization of [4] where the quiver with potential models the Hilbert scheme of $\mathbb{C}^3$. Or from another point of view, in the cohomological Hall algebra, the reduction statement of [32] section 4.8 is an analog of our motivic result.

The theorem is used here to compute the motivic DT invariants for crepant resolutions of toric singularities. We finish by mentioning some joint work with co-authors on the wall crossing behavior of these motives.

Let Q be a finite quiver with vertex set Q_0, that is a finite directed graph. The representations of Q are indexed by a dimension vector $\alpha \in \mathbb{N}^{Q_0}$. At each vertex i we have a vector space V_i of dimension α_i, and for each arrow $a : i \to j$ we have a linear map $M_a : V_i \to V_j$. Such representations are considered unique up to the linear action of $\mathrm{GL}(V_i)$ on each vector space V_i. We produce a fine moduli space of framed quiver representations by taking a G.I.T quotient. The framing comes from the extra data of a vector $f \in \mathbb{N}^{Q_0} \setminus \{0\}$, and the stability condition is determined by a linearization of the group action χ;

$$N^Q(\alpha, f) = \prod_{a:i\to j} \mathrm{Hom}(V_i, V_j) \times \prod_{i\in Q_0} (V_i)^{f_i} /\!\!/_{\chi} \prod_{i\in Q_0} \mathrm{GL}(V_i).$$

It is a smooth quasi-projective variety (see Section 2.3).

The path algebra $\mathbb{C}Q$ of the quiver Q is the vector space over $\mathbb{C}$ with basis given by the paths in the quiver, the multiplication is then defined by concatenation of paths. An element $W \in \mathbb{C}Q$ of the path algebra is called a potential if it is the sum of cycles, i.e. paths that form loops. We are specifically interested in potentials that have a linear factor;

$$W = L \cdot R$$

here L is linear, that is a sum of length one paths.

The potential defines a function on the moduli space of representations of Q. Given a representation $(M_a)_{a:i\to j}$ insert the matrix M_a in place of each occurrence of a in W then take the trace;

$$\operatorname{Tr} W_{\alpha,f} : N^Q(\alpha, f) \to \mathbb{C}$$

the definition is invariant under the action of the gauge group and so gives a well defined regular function on the moduli space of framed quiver representations. For a specific pair of vectors $\alpha, f \in \mathbb{N}^{Q_0}$ we define the DT moduli space of Q, W to be the scheme theoretic degeneracy locus of $\operatorname{Tr} W_{\alpha,f}$;

$$\mathrm{DT}^{Q,W}(\alpha, f) = \{d \operatorname{Tr} W_{\alpha,f} = 0\} \subset N^{Q,W}(\alpha, f).$$

Behrend, Bryan and Szendrői [4] define a virtual motive associated to each such a locus, essentially given by the motivic Milnor fibre of the map $\operatorname{Tr} W_{\alpha,f}$. Using the same definition we have a motivic DT invariant;

$$[\mathrm{DT}^{Q,W}(\alpha, f)]_{\mathrm{vir}} \in K_0(Var_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}]$$

taking values in the Grothendieck ring of varieties, adjoined with a formal inverse square root of the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$. The conditions described in Section 2.5 allow us to neglect the usual $\hat{\mu}$-action on these invariants.

When the potential has a linear factor $W = L \cdot R$, there exists a reduced space of quiver representations;

$$R^{Q,W}(\alpha) = \{(M_a)_{a:i\to j} \mid R((M_a)_{a:i\to j}) = 0\} \subset \prod_{a:i\to j} \operatorname{Hom}(V_i, V_j).$$

Here the space is a variety and also we have no G.I.T quotient. Our result expresses the virtual motives of the DT moduli spaces in terms of the ordinary motives of the reduced spaces. It is best phrased in the context of a motivic quantum torus.

Let $\mathcal{M}^{St}_{\mathbb{C}}$ be the Grothendieck ring of varieties where we formally invert the general linear groups and a square root of the Lefschetz motive. Now consider the formal power series over this ring, as a $\mathcal{M}_{\mathbb{C}}^{St}$-module it is defined

$$\mathcal{T}_Q := \left\{ \sum_{\alpha \in \mathbb{N}^{Q_0}} m_\alpha \mathbf{t}^\alpha \mid m_\alpha \in \mathcal{M}_{\mathbb{C}}^{St} \right\}.$$

The Euler form $\langle , \rangle_Q$, a pairing associated to the adjacency matrix of Q, defines a non-commutative product $*$ on this set given by,

$$\mathbf{t}^\alpha * \mathbf{t}^\beta = \mathbb{L}^{\frac{1}{2}\left(\langle \alpha,\beta\rangle_Q - \langle \beta,\alpha\rangle_Q\right)} \mathbf{t}^{\alpha+\beta}.$$

$\mathcal{T}_Q$ is the motivic quantum torus of the quiver Q. The virtual motives of the DT moduli spaces give an element;

$$\mathrm{Z}^{Q,W,f}(\mathbf{t}) := \sum_{\alpha \in \mathbb{N}^{Q_0}} [\mathrm{DT}^{Q,W}(\alpha, f)]_{\mathrm{vir}} \mathbf{t}^\alpha \in \mathcal{T}_Q.$$

Similarly the reduced spaces can be assembled into a series;

$$\mathrm{R}^{Q,W}(\mathbf{t}) := \sum_{\alpha \in \mathbb{N}^{Q_0}} \frac{[R^{Q,W}(\alpha)]}{[\mathrm{GL}(\alpha)]} \mathbb{L}^{\frac{1}{2}\langle \alpha,\alpha\rangle_Q} \mathbf{t}^\alpha \in \mathcal{T}_Q$$

here $\mathrm{GL}(\alpha) = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}$. In this context our main result reads simply,

$$\boxed{\mathrm{R}^{Q,W}(\mathbb{L}^{\frac{f}{2}}\mathbf{t}) = \mathrm{Z}^{Q,W,f}(\mathbf{t}) * \mathrm{R}^{Q,W}(\mathbb{L}^{-\frac{f}{2}}\mathbf{t})}$$

where $(\mathbb{L}^\beta \mathbf{t})^\alpha = \mathbb{L}^{\beta\cdot\alpha}\mathbf{t}^\alpha$. This describes all the motivic DT invariants in terms of the ordinary motivic classes of the reduced spaces.

2.2 Quiver with Potential

Let Q be a finite quiver with vertex set Q_0 and arrows $a : i \to j$. A path in the quiver is a sequence of arrows $a_1 \cdot a_2 \cdots a_n$ so that the head of the arrow a_i coincides with the tail of the arrow a_{i+1}, each path has a length given by the number of arrows of which it is composed.

The path algebra of the quiver $\mathbb{C}Q$ is the $\mathbb{C}$ vector space with basis the set of all paths and with multiplication given by concatenation of paths. An element $W \in \mathbb{C}Q$ is called a potential if all the monomials in W are cycles. Then

$$\tilde{W} \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$$

is the class of W up to equivalence of cyclically permuting terms in any monomial. Ginzburg [20], provides a detailed account of such algebras and potentials. Our main interest is in their representations. For each dimension vector $\alpha \in \mathbb{N}^{Q_0}$ we define a linear space of representations of Q,

$$M^Q(\alpha) = \prod_{a:i\to j} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j})$$

the group of automorphisms $\mathrm{GL}(\alpha) = \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i}(\mathbb{C})$ acts on this space by change of basis. The potential W defines a function on the representations of Q. So that given a representation $(M_a)_{a:i\to j}$, we define $W((M_a)_{a:i\to j})$ to be the endomorphism obtained by inserting M_a in the place of each occurrence of a in W. Taking the trace of this linear map gives a complex number $\mathrm{Tr}\, W((M_a)_{a:i\to j})$ well defined under cyclic reordering of each monomial and invariant under the action of $\mathrm{GL}(\alpha)$. In particular this means that the class $\tilde{W} \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ defines a $\mathrm{GL}(\alpha)$ equivariant Chern–Simons functional,

$$\mathrm{Tr}\, \tilde{W}_\alpha : M^Q(\alpha) \to \mathbb{C}.$$

The potentials we consider in this paper satisfy the following special condition.

Definition 2.1. A potential $\tilde{W} \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ has a linear factor if for some lift $W \in \mathbb{C}Q$ there exists a factorization

$$W = L \cdot R \in \mathbb{C}Q$$

where L is a linear combination of arrows, so that for each pair of vertices $i, j \in Q_0$ there is at most one such arrow $a : i \to j$ between them. Moreover we have that no arrow in L occurs in R.

For such a potential we define

$$\begin{aligned} M_1^{Q,W}(\alpha) &:= \operatorname{Tr}\tilde{W}_\alpha^{-1}(1), \\ M_0^{Q,W}(\alpha) &:= \operatorname{Tr}\tilde{W}_\alpha^{-1}(0), \\ R^{Q,W}(\alpha) &:= \{(M_a)_{a:i\to j} \in M^Q(\alpha) \mid R((M_a)_{a:i\to j}) = 0\} \end{aligned}$$

the first two being fibres of the Chern–Simons functional and the latter the zero locus of the reduced equations $R = 0$. The corresponding moduli stacks are defined,

$$\begin{aligned} \mathfrak{M}_1^{Q,W}(\alpha) &:= [M_1^{Q,W}(\alpha)/\operatorname{GL}(\alpha)], \\ \mathfrak{M}_0^{Q,W}(\alpha) &:= [M_0^{Q,W}(\alpha)/\operatorname{GL}(\alpha)], \\ \mathfrak{R}^{Q,W}(\alpha) &:= [R^{Q,W}(\alpha)/\operatorname{GL}(\alpha)]. \end{aligned}$$

There exists a partial order on $\mathbb{N}^{Q_0}$ given by

$$\alpha \geq \beta \text{ if } \alpha_i \geq \beta_i \text{ for all } i \in Q_0$$

and we say that

$$\alpha > \beta \text{ if } \alpha \geq \beta \text{ and } \alpha \neq \beta.$$

As a final piece of notation we introduce two pairings on the semi-group $\mathbb{N}^{Q_0}$ to be used later. Let $\alpha, \beta \in \mathbb{Z}^{Q_0}$ then

$$\begin{aligned} \alpha \cdot \beta &= \sum_i \alpha_i \beta_i \\ \langle \alpha, \beta \rangle_Q &= \sum_i \alpha_i \beta_i - \sum_{a:i\to j} \alpha_i \beta_j. \end{aligned}$$

2.3 Stability for Quiver Representations

We fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$. The stability condition depends upon a choice of framing vector $f \in \mathbb{N}^{Q_0} \setminus \{0\}$, introducing such will produce a fine moduli space of framed quiver representations. Let

$$U^Q(\alpha, f) = \{(M_a, m_l) \mid \underline{\dim}\mathbb{C}\langle M_a\rangle\{m_l\} = \alpha\} \subset M^Q(\alpha) \times \prod_{i\in Q_0}(\mathbb{C}^{\alpha_i})^{f_i}$$

where $\mathbb{C}\langle M_a\rangle\{m_l\}$, is defined to be the span of the collection of vectors m_l under the successive action of the matrices M_a. Now the group $\mathrm{GL}(\alpha)$ acts freely on the $U^Q(\alpha, f)$ giving a fine moduli space of quiver representations.

Proposition 2.1. *[30, King] There exists a character $\chi : \mathrm{GL}(\alpha) \to \mathbb{C}^*$ such that the open subset $U^Q(\alpha, f)$ is precisely the subset of stable points for the action of $\mathrm{GL}(\alpha)$ linearized by χ. In particular the action of $\mathrm{GL}(\alpha)$ on $U^Q(\alpha, f)$ is free, and the quotient*

$$N^Q(\alpha, f) := M^Q(\alpha) \times \prod_{i\in Q_0}(\mathbb{C}^{\alpha_i})^{f_i} /\!\!/_{\chi}\, \mathrm{GL}(\alpha) = U^Q(\alpha, f)/\, \mathrm{GL}(\alpha)$$

is a smooth quasi-projective G.I.T. quotient.

The Chern–Simons functional can be defined on $U^Q(\alpha, f)$ as the pullback of the one on $M^Q(\alpha)$ via the natural projection. This functional is $\mathrm{GL}(\alpha)$ equivariant and consequently descends to a regular function on the smooth quotient

$$\mathrm{Tr}\, \tilde{W}_{\alpha,f} : N^Q(\alpha, f) \to \mathbb{C}.$$

Our main object of interest is the scheme theoretic degeneracy locus of this functional, which we define;

$$\mathrm{DT}^{Q,W}(\alpha, f) := \{d\, \mathrm{Tr}\, \tilde{W}_{\alpha,f} = 0\} \subset N^Q(\alpha, f).$$

Example 2.1. Let Q be the quiver with one vertex $\{\star\}$ and three arrows $\{x, y, z\}$, potential $W = x(yz - zy)$ then it is well known [4, Prop 2.1],

$$\mathrm{DT}^{Q,W}(n, 1) = \mathrm{Hilb}^n(\mathbb{C}^3).$$

Remark 2.1. Details of more general quiver stability conditions can be found in section 3.2.3.

2.4 Grothendieck Rings

Here is an account of Grothendieck rings following Bridgeland [8] (c.f. Toën [59]). We are interested in spaces that are varieties or stacks over $\mathbb{C}$. Denote by

$$Var_{\mathbb{C}} \subset St_{\mathbb{C}}$$

the inclusion of the category of varieties in the category of Artin stacks of finite type. Our first definition is the Grothendieck ring of varieties.

Definition 2.2. The Grothendieck ring of varieties $K_0(Var_{\mathbb{C}})$ is the free abelian group on isomorphism classes of varieties over $\mathbb{C}$, modulo the scissor relations

$$[X] = [Z] + [X \setminus Z]$$

for Z a subvariety of X. Multiplication in $K_0(Var_{\mathbb{C}})$ is given by fibre product

$$[X] \cdot [Y] = [X \times_{\mathbb{C}} Y].$$

For a variety X we will call $[X]$ the motivic class of X. For example consider a Zariski fibration $\pi : E \to B$ with fibre F, then stratifying the base by trivializing neighborhoods and using the scissor relations we get

$$[E] = [F] \cdot [B] \in K_0(Var_{\mathbb{C}}).$$

We can use this fact to give a nice formula for the motivic class of the general linear group.

Lemma 2.2. *We have*

$$[\mathrm{GL}_n] = \prod_{k=0}^{n-1} (\mathbb{L}^n - \mathbb{L}^k) \in K_0(Var_{\mathbb{C}})$$

where $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$.

Proof. The map

$$\pi : \mathrm{GL}_n \to \mathbb{C}^n \setminus \{0\},$$

sending a matrix to its first column, is a Zariski fibration with fibre equal to

$$\mathrm{GL}_{n-1} \times \mathbb{C}^{n-1}$$

so by what was said above

$$[\mathrm{GL}_n] = (\mathbb{L}^n - 1)\mathbb{L}^{n-1}[\mathrm{GL}_{n-1}],$$

and the result follows by induction. □

Recall that the Grassmannian can be defined as the free quotient of the general linear group by the subgroup of matrices fixing a hyperplane of dimension k, so as an immediate corollary of the above lemma we also have a formula for the motivic class of the Grassmannian

$$[Gr(k,n)] = \frac{[\mathrm{GL}_n]}{[\mathrm{GL}_k][\mathrm{GL}_{n-k}][\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})]} \in K_0(Var_{\mathbb{C}}).$$

To define the Grothendieck ring of stacks we need the notion of geometric bijection, we say that a representable morphism

$$f : X \to Y$$

is a geometric bijection if it induces an isomorphism

$$f : X(\mathbb{C}) \to Y(\mathbb{C})$$

between the groupoids of $\mathbb{C}$-valued points. In our definition we shall also need to consider stacks with affine stabilizers, that is stacks X locally of finite type over $\mathbb{C}$ such that for all $x \in X(\mathbb{C})$, $\mathrm{Isom}_{\mathbb{C}}(x,x)$ is an affine algebraic group.

Definition 2.3. The Grothendieck ring of stacks $K_0(St_{\mathbb{C}})$ is the free abelian group of isomorphism classes of finite type Artin stacks of over $\mathbb{C}$ with affine stabilizers, modulo the relations

1. $[X \sqcup Y] = [X] + [Y]$.

2. $[X] = [Y]$ for every geometric bijection $f : X \to Y$.

3. $[X] = [Y]$ for every pair of Zariski fibrations with the same base and fibre.

Again multiplication comes from the fibre product.

The inclusion of the category of varieties in the category of stacks gives an obvious homomorphism of rings

$$K_0(Var_{\mathbb{C}}) \xrightarrow{\varphi} K_0(St_{\mathbb{C}}).$$

The following lemma shows that the image of the general linear group is invertible.

Lemma 2.3. *Every principal* GL_n *bundle* $\pi : Y \to X$ *is a Zariski fibration. Hence*

$$[Y] = [X] \cdot [\mathrm{GL}_n] \in K_0(St_{\mathbb{C}})$$

and in particular taking Y *to be a point we have that* $[\mathrm{GL}_n]$ *is invertible with inverse* $[\mathrm{BGL}_n] \in K_0(St_{\mathbb{C}})$.

Proof. The fibration $\pi : Y \to X$ is a principal GL_n bundle if its pullback to any scheme S is

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{f} & X. \end{array}$$

So $f^*\pi : f^*Y \to S$ is a principal GL_n bundle, therefore locally trivial in the étale topology. As the group GL_n is special [56] this bundle is a Zariski fibration. □

This lemma provides an extension of the map φ above. We have

$$K_0(Var_{\mathbb{C}})[[\mathrm{GL}_n]^{-1} : n \geq 1] \xrightarrow{\tilde{\varphi}} K_0(St_{\mathbb{C}}).$$

Kresch proves [34] that stacks with affine stabilizers are in geometric bijection with quotients Y/GL_n where Y is a variety and n is sufficiently large.

This shows that the map $\tilde{\varphi}$ above is surjective. In fact $\tilde{\varphi}$ is actually an isomorphism of rings.

Lemma 2.4. *[8, c.f. Bridgeland, Lemma 3.9] The ring homomorphism* $\tilde{\varphi} : K_0(Var_{\mathbb{C}})[[\mathrm{GL}_n]^{-1} : n \geq 1] \to K_0(St_{\mathbb{C}})$ *is an isomorphism.*

Since we need them for our definition of virtual motives we hereby define two extended rings of motivic classes

$$\mathcal{M}_{\mathbb{C}} := K_0(Var_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}] \quad \text{and} \quad \mathcal{M}_{\mathbb{C}}^{St} := K_0(St_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}].$$

The topological Euler characteristic of classes in $K_0(Var_{\mathbb{C}})$ extends to $\mathcal{M}_{\mathbb{C}}$ by letting $\chi(\mathbb{L}^{-\frac{1}{2}}) = -1$, giving a ring homomorphism

$$\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}.$$

2.5 Virtual Motives

Let $f : M \to \mathbb{C}$ be a regular function on a smooth quasi-projective variety with Z the scheme theoretic degeneracy locus of f. Using arc spaces Denef and Loeser [14] define the motivic vanishing cycle

$$[\varphi_f]_Z \in K_0^{\hat{\mu}}(Var_Z).$$

This motivic class is relative over the the degeneracy locus Z and has an action of the profinite group $\hat{\mu} = \lim_d \mu_d$ of all roots of unity. From this one can recover the cohomology with its monodromy action of the Milnor fibre at each point $z \in Z$ (see [14] Theorem 3.5.5.). This relative motive can be pushed forward to a point giving a coarser class

$$[\varphi_f] \in K_0^{\hat{\mu}}(Var_{\mathbb{C}})$$

sometimes described as the integral over Z and this class can be used directly to give a definition of the virtual motive in the following case.

Definition 2.4. [4] Let $f : M \to \mathbb{C}$ be a regular function on the a smooth quasi-projective variety, with $Z = \{df = 0\} \subset M$ the scheme theoretic degeneracy locus of f. The virtual motive of Z is defined,

$$[Z]_{\text{vir}} = -\mathbb{L}^{-\frac{\dim M}{2}}[\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

This really depends not only on the scheme Z but on the function f and space M. However taking the Euler characteristic will give a number intrinsic to Z. In particular if Z is a moduli space of stable sheaves on a Calabi–Yau threefold then $\chi([Z]_{\text{vir}})$ will equal the Donaldson-Thomas invariant, by Behrend's description of the constructible function ν in terms of the Milnor fibre [2].

In the case where f is equivariant with respect to a $\mathbb{C}^*$-action satisfying the properties below, the virtual motive lives in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of motivic classes with trivial monodromy action. Moreover, it is given by an easy formula.

Definition 2.5. (Property ($\star$).) Let $f : M \to \mathbb{C}$ be a regular function on a smooth quasi-projective variety M. Then f satisfies Property ($\star$) if there exists a $\mathbb{C}^*$ acting on M, so that for all $m \in M$ and $t \in \mathbb{C}^*$ we have

$$f(t \cdot m) = t \cdot f(m)$$

and secondly we have that for all $m \in M$ the limit

$$\lim_{t \to 0} t \cdot m$$

exists.

Property ($\star$) means that the map $f : M \to \mathbb{C}$ is a trivial fibration over $\mathbb{C}^*$. This is easy to see, letting $M_1 = f^{-1}(1)$ and $M_0 = f^{-1}(0)$ the trivialization is

$$\mathbb{C}^* \times M_1 \xrightarrow{t \cdot m} M \setminus M_0$$

sending a point m in the fibre over 1 to the point $t \cdot m \in M \setminus M_0$.

Proposition 2.5. *[4, Behrend, Bryan, Szendrői] Let $f : M \to \mathbb{C}$ be a regular function on a smooth quasi-projective variety M with $\mathbb{C}^*$-action satisfying Property ($\star$). The motivic class of vanishing cycles $[\varphi_f]$ can be expressed as the motivic difference of the general and central fibres*

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)] \in K_0(Var_{\mathbb{C}})$$

in particular

$$[Z]_{vir} = \mathbb{L}^{-\frac{\dim M}{2}}([f^{-1}(0)] - [f^{-1}(1)]) \in \mathcal{M}_{\mathbb{C}}.$$

In [4] the above theorem is stated under the additional assumption that the $\mathbb{C}^*$-action be circle-compact this implies that in addition to Property ($\star$) the $\mathbb{C}^*$-fixed locus be compact in the analytic topology. As they themselves mention [4, Section 1.7] this requirement is stronger than needed and Property ($\star$) will suffice.

Remark 2.2. More details on equivariant motivic classes and vanishing cycles can be found in the excellent expository article [14].

2.6 Motivic Quantum Torus

The reduction theorem is most easily expressed as a multiplicative relation in a certain quantum torus. Kontsevich and Soibelman see this arising naturally in their work on motivic DT invariants of a three dimensional Calabi–Yau categories [33] and again in the quiver context [32].

Definition 2.6. Let Q be a finite quiver with pairing $\langle , \rangle_Q$. The algebra $\mathcal{T}_Q$, is an $\mathcal{M}_{\mathbb{C}}^{St}$-module

$$\mathcal{T}_Q := \prod_{\alpha \in \mathbb{N}^{Q_0}} \mathcal{M}_{\mathbb{C}}^{St} \mathbf{t}^\alpha,$$

with a non-commutative product,

$$\mathbf{t}^\alpha * \mathbf{t}^\beta = \mathbb{L}^{\frac{1}{2}\left(\langle \alpha, \beta \rangle_Q - \langle \beta, \alpha \rangle_Q\right)} \mathbf{t}^{\alpha+\beta}.$$

Roughly speaking this is a non-commutative ring of power series with motivic classes as coefficients. All the motivic classes we are interested in can be neatly packaged in two such series. The DT series,

$$\mathrm{Z}^{Q,W,f}(\mathbf{t}) := \sum_{\alpha\in\mathbb{N}^{Q_0}} [\mathrm{DT}^{Q,W}(\alpha, f)]_{\mathrm{vir}}\mathbf{t}^{\alpha} \in \mathcal{T}_Q,$$

and the reduced series,

$$\mathrm{R}^{Q,W}(\mathbf{t}) := \sum_{\alpha\in\mathbb{N}^{Q_0}} \frac{[R^{Q,W}(\alpha)]}{[\mathrm{GL}(\alpha)]}\mathbb{L}^{\frac{1}{2}\langle\alpha,\alpha\rangle_Q}\mathbf{t}^{\alpha} \in \mathcal{T}_Q.$$

Remark 2.3. Notice that in the special case when the quiver Q is symmetric the algebra $\mathcal{T}_Q$ is commutative.

2.7 Main Theorem

Here we extend the results of [4, §2.7] to a general quiver and potential with a linear factor.

First we work without stability and consider the Chern–Simons functional

$$\mathrm{Tr}\tilde{W}_{\alpha} : M^Q(\alpha) \to \mathbb{C}.$$

One immediate consequence of the linearity of the potential is that the above map satisfies Property ($\star$), hence is a trivial fibration over $\mathbb{C}^*$. To see this just consider $\mathbb{C}^*$ acting diagonally on the left of the matrices appearing in the linear factor L, one of which will be non-zero if the trace map is. Recall the definitions of Section 2.2

$$\begin{aligned}
M_1^{Q,W}(\alpha) &:= \mathrm{Tr}\,\tilde{W}_{\alpha}^{-1}(1),\\
M_0^{Q,W}(\alpha) &:= \mathrm{Tr}\,\tilde{W}_{\alpha}^{-1}(0),\\
R^{Q,W}(\alpha) &:= \{(M_a) \in M^Q(\alpha) \mid R(M_a) = 0\},
\end{aligned}$$

the first space here is the general fibre, the second the central fibre and the space $R^{Q,W}(\alpha)$ we call the reduced space, since it is the locus of the reduced equation $R = 0$. The corresponding stacks defined in Section 2.2 were denoted $\mathfrak{M}_1^{Q,W}(\alpha), \mathfrak{M}_0^{Q,W}(\alpha)$ and $\mathfrak{R}^{Q,W}(\alpha)$. The following proposition expresses the difference of the general and central fibres in terms of the reduced space.

Proposition 2.6. *Let Q be a finite quiver with potential W. Suppose W has a linear factor. For any dimension vector $\alpha \in \mathbb{N}^{Q_0}$ let $\mathfrak{M}_1^{Q,W}(\alpha)$, $\mathfrak{M}_1^{Q,W}(\alpha)$, $\mathfrak{R}^{Q,W}(\alpha)$ be the stacks defined above. Then*

$$[\mathfrak{M}_0^{Q,W}(\alpha)] - [\mathfrak{M}_1^{Q,W}(\alpha)] = [\mathfrak{R}^{Q,W}(\alpha)]$$

in the Grothendieck ring of stacks $K_0(St_{\mathbb{C}})$.

Proof. The idea is to use the triviality of the fibration and the linearity of the potential to obtain two simple relations between the motivic classes.

Fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$. Consider the map $\operatorname{Tr} \tilde{W}_\alpha : M^Q(\alpha) \to \mathbb{C}$, we stratify the base $\mathbb{C} = \mathbb{C}^* \sqcup \{0\}$. The fibre over $\{0\}$ we call $M_0^{Q,W}(\alpha)$, the central fibre. Since the map is a $\mathbb{C}^*$-equivariant family with Property $(\star)$, then over $\mathbb{C}^*$ it is a trivial fibration, with general fibre $M_1^{Q,W}(\alpha)$. This decomposition of $M^Q(\alpha)$ into two pieces gives our first motivic relation,

$$[M^Q(\alpha)] = [M_0^{Q,W}(\alpha)] + (\mathbb{L} - 1)[M_1^{Q,W}(\alpha)]. \tag{2.1}$$

Now split the arrows of the quiver into two sets, if and only if they occur in the linear factor of the potential;

$$A := \{a : i \to j \mid a \in L\} \text{ and } B := \{b : i \to j \mid b \notin L\}$$

Let $m = (M_a)_{a:i\to j}$ be a Q representation of dimension vector α, using the splitting we decompose m into parts

$$(m_A, m_B) \in \bigoplus_{a:i\to j\in A} \operatorname{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \bigoplus_{b:i\to j\in B} \operatorname{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}).$$

Then

$$L(m) = L(m_A) \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}),$$

and since $W = L \cdot R$ is a sum of cycles

$$R(m) = R(m_B) \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_j}, \mathbb{C}^{\alpha_i}).$$

The trace map gives a non-degenerate pairing between these spaces

$$\mathrm{Tr} \quad : \quad \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_j}, \mathbb{C}^{\alpha_i}) \to \mathbb{C}$$

defined as follows. Let

$$X \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \text{ and } Y \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_j}, \mathbb{C}^{\alpha_i})$$

take the product of these linear maps to get

$$X \cdot Y \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_j}, \mathbb{C}^{\alpha_j})$$

and then take the trace of this endomorphism

$$\mathrm{Tr} : (X, Y) \mapsto \mathrm{Tr}(X \cdot Y).$$

We use this pairing to compute $M_0^{Q,W}(\alpha)$ in order to get a second relation.

Let $m = (m_A, m_B) \in M_0^{Q,W}(\alpha)$, so that $\mathrm{Tr}(L(m_A) \cdot R(m_B)) = 0$. There are two cases to consider firstly when $R(m_B) = 0$ and secondly when $R(m_B) \neq 0$. By definition the locus where $R = 0$ is equal to $R^{Q,W}(\alpha)$. On the compliment of this set consider the projection

$$\begin{aligned} \pi \quad &: \quad M^Q(\alpha) \setminus R^{Q,W}(\alpha) \to \{m_B \mid R(m_B) \neq 0\} \\ &: \quad (m_A, m_B) \mapsto m_B. \end{aligned}$$

This map is a trivial vector bundle. For fixed m_B the condition $\mathrm{Tr}(L(m_A) \cdot R(m_B)) = 0$ is a single linear condition on the matrices m_A in the fibre, and so in the second case the locus of $m \in M_0^{Q,W}(\alpha)$ such that $R \neq 0$, is a vector bundle of rank one lower. Both cases considered we have a formula for the class of $M_0^{Q,W}(\alpha)$,

$$[M_0^{Q,W}(\alpha)] = [R^{Q,W}(\alpha)] + ([M^Q(\alpha)] - [R^{Q,W}(\alpha)])\mathbb{L}^{-1}. \tag{2.2}$$

Finally relations (2.1),(2.2) together imply

$$[M_0^{Q,W}(\alpha)] - [M_1^{Q,W}(\alpha)] = [R^{Q,W}(\alpha)]$$

Dividing by the motivic classes of the automorphism groups $\mathrm{GL}(\alpha)$ gives the corresponding result for stacks.

□

In Section 2.3 we showed that for a choice of framing vector $f \in \mathbb{N}^{Q_0} \setminus \{0\}$ there exists a fine moduli space of quiver representations with Chern–Simons functional

$$\mathrm{Tr}\tilde{W}_{\alpha,f} : N^Q(\alpha, f) \to \mathbb{C}.$$

Again we can use the $\mathbb{C}^*$-action given by acting diagonally on the left of all matrices occurring in the linear factor L to produce a $\mathbb{C}^*$-action satisfying Property ($\star$).

Lemma 2.7. *There exists a $\mathbb{C}^*$-action on the moduli space $N^Q(\alpha, f)$ satisfying Property ($\star$).*

Proof. Let

$$(m_A, m_B, v) \in \bigoplus_{a:i\to j\in A} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \bigoplus_{\beta:i\to j\in B} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \bigoplus_{i\in Q_0} \mathbb{C}^{\alpha_i}$$

as above we consider the $\mathbb{C}^*$-action given by acting on the left of the matrices in the linear factor

$$t : (m_A, m_B, v) \mapsto (t \cdot m_A, m_B, v).$$

This action is $\mathrm{GL}(\alpha)$ equivariant and descends to a $\mathbb{C}^*$-action on

$$N^Q(\alpha, f) = \prod_{a:i\to j\in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \prod_{i\in Q_0} \mathbb{C}^{\alpha_i} /\!\!/_\chi \mathrm{GL}(\alpha).$$

Moreover by the linearity of the potential the function $\mathrm{Tr}\, \tilde{W}_{\alpha,f}$ is $\mathbb{C}^*$-equivariant. All that remains is to check that the limits exist as $t \to 0$ (c.f. Lemma 2.4 [4]). We consider the affine quotient

$$N_0^Q(\alpha, f) = \prod_{a:i\to j\in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j}) \times \prod_{i\in Q_0} \mathbb{C}^{\alpha_i} /\!\!/_0 \mathrm{GL}(\alpha).$$

given by the GIT quotient at zero stability. Now $N^Q(\alpha, f)$ is projective over $N_0^Q(\alpha, f)$. For any $m = (m_A, m_B, v) \in N_0^Q(\alpha, f)$ the limit $\lim_{t\to 0} t \cdot m$ exists, indeed it is given by the image of $(0, m_B, v)$ in the affine quotient. Since limits exist in $N_0^Q(\alpha, f)$ and $N^Q(\alpha, f)$ is projective over $N_0^Q(\alpha, f)$ the result follows. □

So by Proposition 2.5 the virtual motive of $\mathrm{DT}^{Q,W}(\alpha, f)$ is the difference of the general and central fibres of $\mathrm{Tr}\, \tilde{W}_{\alpha,f}$

$$[\mathrm{DT}^{Q,W}(\alpha, f)]_{\mathrm{vir}} = \mathbb{L}^{-\frac{\dim N^Q(\alpha,f)}{2}}([\mathrm{Tr}\, \tilde{W}_{\alpha,f}^{-1}(0)] - [\mathrm{Tr}\, \tilde{W}_{\alpha,f}^{-1}(1)]).$$

In Proposition 2.6 we ignored stability and expressed the difference of the general and central fibres as equal to the reduced space. Now on adding the stability condition we will get a recursion relation for the virtual motives in terms of the reduced spaces.

Theorem 2.8. *Let Q be a finite quiver with potential W. Suppose that W has a linear factor. For any dimension vector $\alpha \in \mathbb{N}^{Q_0}$ and framing $f \in \mathbb{N}^{Q_0} \setminus \{0\}$, the virtual motives of the moduli spaces $\mathrm{DT}^{Q,W}(\alpha, f)$ satisfy* *a recursion,*

$$[\mathfrak{R}^{Q,W}(\alpha)]\mathbb{L}^{\frac{f\cdot\alpha}{2}} = \sum_{\beta\leq\alpha}\Big(\mathbb{L}^{-\langle\alpha-\beta,\beta\rangle_Q-1/2\langle\beta,\beta\rangle_Q}\cdot[\mathrm{DT}^{Q,W}(\beta,f)]_{vir}$$

$$\cdot[\mathfrak{R}^{Q,W}(\alpha-\beta)]\mathbb{L}^{-\frac{f\cdot(\alpha-\beta)}{2}}\Big).$$

in the ring $\mathcal{M}^{St}_{\mathbb{C}}$. Or equivalently rephrased in the language of Section 2.6

$$\mathrm{R}^{Q,W}(\mathbb{L}^{\frac{f}{2}}\mathbf{t}) = \mathrm{Z}^{Q,W,f}(\mathbf{t}) * \mathrm{R}^{Q,W}(\mathbb{L}^{-\frac{f}{2}}\mathbf{t}).$$

Proof. Fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ and a framing vector $f \in \mathbb{N}^{Q_0}\setminus\{0\}$. First define all the objects without stability

$$\begin{aligned} X^Q(\alpha,f) &:= M^Q(\alpha)\times\prod_{i\in Q_0}(\mathbb{C}^{\alpha_i})^{f_i},\\ Y^{Q,W}(\alpha,f) &:= M_0^{Q,W}(\alpha)\times\prod_{i\in Q_0}(\mathbb{C}^{\alpha_i})^{f_i},\\ Z^{Q,W}(\alpha,f) &:= M_1^{Q,W}(\alpha)\times\prod_{i\in Q_0}(\mathbb{C}^{\alpha_i})^{f_i}. \end{aligned}$$

As shown in Proposition 2.6 the two spaces above are related to the reduced space,

$$[R^{Q,W}(\alpha)]\cdot\mathbb{L}^{f\cdot\alpha} = [Y^{Q,W}(\alpha,f)] - [Z^{Q,W}(\alpha,f)].$$

The stability condition introduced in Section 2.3 depends upon the span of the vectors $\{m_l\}$ under the matrices $(M_a)_{a:i\to j}$. This vector space was earlier defined as

$$\mathbb{C}\langle M_a\rangle\{m_l\}.$$

For a given dimension vector $\beta\in\mathbb{N}^{Q_0}$ set,

$$\begin{aligned} X^Q(\alpha,\beta,f) &:= \{(M_a,m_l)\mid \underline{\dim}\,\mathbb{C}\langle M_a\rangle\{m_l\}=\beta\}\subset X^Q(\alpha,f),\\ Y^{Q,W}(\alpha,\beta,f) &:= X^Q(\alpha,\beta,f)\cap Y^{Q,W}(\alpha,f)\subset Y^{Q,W}(\alpha,f),\\ Z^{Q,W}(\alpha,\beta,f) &:= X^Q(\alpha,\beta,f)\cap Z^{Q,W}(\alpha,f)\subset Z^{Q,W}(\alpha,f). \end{aligned}$$

Since the stability condition required the vectors $\{m_l\}$ generate the entire representation, then in the notation of Section 2.3

$$U^Q(\alpha, f) = X^Q(\alpha, \alpha, f) \text{ and } N^Q(\alpha, f) = X^Q(\alpha, \alpha, f)/\operatorname{GL}(\alpha).$$

So the virtual motive of $\mathrm{DT}^{Q,W}(\alpha, f)$ is the difference of the general and central fibres

$$[\mathrm{DT}^{Q,W}(\alpha, f)]_{\mathrm{vir}} = \mathbb{L}^{-\frac{\dim N^Q(\alpha,f)}{2}} \left(\frac{[Y^{Q,W}(\alpha, \alpha, f)]}{[\mathrm{GL}(\alpha)]} - \frac{[Z^{Q,W}(\alpha, \alpha, f)]}{[\mathrm{GL}(\alpha)]} \right).$$

The dimension of $N^Q(\alpha, f)$ is

$$\begin{aligned} \dim N^Q(\alpha, f) &= \dim M^Q(\alpha) - \dim \mathrm{GL}(\alpha) + \dim \prod_{i \in Q_0} (\mathbb{C}^{\alpha_i})^{f_i} \\ &= -\langle \alpha, \alpha \rangle_Q + \alpha \cdot f. \end{aligned}$$

The remaining task is to compute the difference $[Y^{Q,W}(\alpha, \alpha, f)] - [Z^{Q,W}(\alpha, \alpha, f)]$. Let us start with $Y^{Q,W}(\alpha, \beta, f)$ and $Z^{Q,W}(\alpha, \beta, f)$.

Let $Gr(\beta, \alpha)$ be the Grassmannian of β dimensional subspaces in α dimensional space, that is $Gr(\beta, \alpha) = \prod_{i \in Q_0} Gr(\beta_i, \alpha_i)$. An element of $Y^{Q,W}(\alpha, \beta, f)$ defines a subspace $\mathbb{C}\langle M_a \rangle \{m_l\}$ with dimension vector $\beta \in \mathbb{N}^{Q_0}$, the associated map

$$Y^{Q,W}(\alpha, \beta, f) \to Gr(\beta, \alpha)$$

is a Zariski fibration. To compute the motivic class of the fibre we fix a basis so that

$$M_a = \begin{pmatrix} M'_a & M^*_a \\ 0 & M''_a \end{pmatrix} \text{ with } \begin{cases} M'_a & \in & \operatorname{Hom}(\mathbb{C}^{\beta_i}, \mathbb{C}^{\beta_j}) \\ M^*_a & \in & \operatorname{Hom}(\mathbb{C}^{\beta_i}, \mathbb{C}^{\alpha_j - \beta_j}) \\ M''_a & \in & \operatorname{Hom}(\mathbb{C}^{\alpha_i - \beta_i}, \mathbb{C}^{\alpha_j - \beta_j}) \end{cases}$$

when $a : i \to j$, and vectors

$$m_l = \begin{pmatrix} m'_l \\ 0 \end{pmatrix} \text{ with } \begin{cases} m'_l & \in & \operatorname{Hom}(1, \mathbb{C}^{\beta_i}) \\ 0 & \in & \operatorname{Hom}(1, \mathbb{C}^{\alpha_i - \beta_i}) \end{cases}$$

when m_l is at vertex $i \in Q_0$. The image of the vectors m'_l under the matrices M'_a is now the entire β-dimensional subspace. The Chern–Simons functional also splits with respect to this basis,

$$\operatorname{Tr} \tilde{W}_\alpha(M_a) = \operatorname{Tr} \tilde{W}_\beta(M'_a) + \operatorname{Tr} \tilde{W}_{\alpha-\beta}(M''_a) = 0,$$

in particular there is no restriction on the M^*_a, and they factor out an affine space of dimension $-\langle \alpha - \beta, \beta \rangle_Q + (\alpha - \beta) \cdot \beta$. The two cases to consider are

$$\{\operatorname{Tr} \tilde{W}_\beta(M'_a) = \operatorname{Tr} \tilde{W}_{\alpha-\beta}(M''_a) = 0\},$$

and

$$\{\operatorname{Tr} \tilde{W}_\beta(M'_a) = - \operatorname{Tr} \tilde{W}_{\alpha-\beta}(M''_a) \neq 0\}.$$

In the first case we get an element of $Y^{Q,W}(\beta, \beta, f)$ and an element of $M_0^{Q,W}(\alpha - \beta)$. The other stratum is a trivial $\mathbb{C}^*$-bundle, by the nonzero value of the Chern–Simons functional. Looking at the fibre over 1 gives an element of $Z^{Q,W}(\beta, \beta, f)$ and an element of $M_1^{Q,W}(\alpha - \beta)$. The total motivic class of the fibre is then,

$$\begin{aligned}
&[Y^{Q,W}(\beta, \beta, f)] \cdot \mathbb{L}^{-\langle \alpha-\beta,\beta\rangle_Q + (\alpha-\beta)\cdot\beta} \cdot [M_0^{Q,W}(\alpha - \beta, f)] \\
&\quad + (\mathbb{L} - 1)[Z^{Q,W}(\beta, \beta, f)] \cdot \mathbb{L}^{-\langle \alpha-\beta,\beta\rangle_Q + \langle \alpha-\beta,\beta\rangle} \cdot [M_1^{Q,W}(\alpha - \beta, f)].
\end{aligned}$$

The space $Z^{Q,W}(\alpha, \beta, f)$ also fibres over the Grassmannian $Gr(\beta, \alpha)$, the motivic class of the fibre is computed similarly, as

$$\begin{aligned}
&[Y^{Q,W}(\beta, \beta, f)] \cdot \mathbb{L}^{-\langle \alpha-\beta,\beta\rangle_Q + (\alpha-\beta)\cdot\beta} \cdot [M_1^{Q,W}(\alpha - \beta, f)] \\
&\quad + (\mathbb{L} - 2)[Z^{Q,W}(\beta, \beta, f)] \cdot \mathbb{L}^{-\langle \alpha-\beta,\beta\rangle_Q + (\alpha-\beta)\cdot\beta} \cdot [M_1^{Q,W}(\alpha - \beta, f)] \\
&\quad + [Z^{Q,W}(\beta, \beta, f)] \cdot \mathbb{L}^{-\langle \alpha-\beta,\beta\rangle_Q + (\alpha-\beta)\cdot\beta} \cdot [M_0^{Q,W}(\alpha - \beta, f)].
\end{aligned}$$

We are now ready to deduce the recursion. Stratifying $Y^{Q,W}(\alpha, f)$ and $Y^{Q,W}(\alpha, f)$ by the dimension of $\mathbb{C}\langle M_a \rangle \{m_l\}$ we have

$$Y^{Q,W}(\alpha, f) = \coprod_{\beta \leq \alpha} Y^{Q,W}(\alpha, \beta, f) \text{ and } Z^{Q,W}(\alpha, f) = \coprod_{\beta \leq \alpha} Z^{Q,W}(\alpha, \beta, f).$$

As mentioned the motivic difference of these two spaces was equal to the class of the reduced space (Proposition 2.6)

$$[R^{Q,W}(\alpha)]\mathbb{L}^{f\cdot\alpha} = \sum_{\beta\le\alpha}[Y^{Q,W}(\alpha,\beta,f)] - [Z^{Q,W}(\alpha,\beta,f)].$$

Substituting in our formulas for $Y^{Q,W}(\alpha,\beta,f)$ and $Z^{Q,W}(\alpha,\beta,f)$ above gives

$$\begin{aligned}
[R^{Q,W}(\alpha)]\mathbb{L}^{f\cdot\alpha} &= \sum_{\beta\le\alpha}[Gr(\beta,\alpha)]\mathbb{L}^{-\langle\alpha-\beta,\beta\rangle_Q+(\alpha-\beta)\cdot\beta} \\
&\quad \cdot\big([Y^{Q,W}(\beta,\beta,f)] - [Z^{Q,W}(\beta,\beta,f)]\big) \\
&\quad \cdot\Big([M_0^{Q,W}(\alpha-\beta,f)] - [M_1^{Q,W}(\alpha-\beta,f)]\Big) \\
&= \sum_{\beta\le\alpha}\frac{[\mathrm{GL}(\alpha)]}{[\mathrm{GL}(\beta)][\mathrm{GL}(\alpha-\beta)]}\mathbb{L}^{-\langle\alpha-\beta,\beta\rangle_Q} \\
&\quad \cdot\big([Y^{Q,W}(\beta,\beta,f)] - [Z^{Q,W}(\beta,\beta,f)]\big)\cdot[R^{Q,W}(\alpha-\beta)] \\
&= \sum_{\beta\le\alpha}\frac{[\mathrm{GL}(\alpha)]}{[\mathrm{GL}(\alpha-\beta)]}\mathbb{L}^{-\langle\alpha-\beta,\beta\rangle_Q-1/2\langle\beta,\beta\rangle_Q+1/2f\cdot\beta} \\
&\quad \cdot[\mathrm{DT}^{Q,W}(\beta,f)]_{\mathrm{vir}}\cdot[R^{Q,W}(\alpha-\beta)].
\end{aligned}$$

This recursion formula is a relation in the ring $\mathcal{M}_{\mathbb{C}}$. Dividing out by the motivic classes of the automorphism groups $\mathrm{GL}(\alpha)$ and $\mathrm{GL}(\alpha-\beta)$ gives the corresponding result in $\mathcal{M}_{\mathbb{C}}^{St}$ the Grothendieck ring of stacks

$$[\mathfrak{R}^{Q,W}(\alpha)]\mathbb{L}^{\frac{f\cdot\alpha}{2}} = \sum_{\beta\le\alpha}\Big(\mathbb{L}^{-\langle\alpha-\beta,\beta\rangle_Q-1/2\langle\beta,\beta\rangle_Q}\cdot[\mathrm{DT}^{Q,W}(\beta,f)]_{\mathrm{vir}} \cdot[\mathfrak{R}^{Q,W}(\alpha-\beta)]\mathbb{L}^{-\frac{f\cdot(\alpha-\beta)}{2}}\Big).$$

□

Remark 2.4. As seen in the proof of Theorem 2.8 our result is essentially a relation in the ring $\mathcal{M}_{\mathbb{C}}$. On the last line however we divide by the motivic classes of some automorphism groups to get a corresponding statement in $\mathcal{M}_{\mathbb{C}}^{St}$ the Grothendieck ring of stacks. We make this remark as it is not know whether or not the homomorphism

$$\mathcal{M}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}[(\mathbb{L}^n - 1)^{-1} : n \geq 1] \cong \mathcal{M}_{\mathbb{C}}^{St}$$

is injective.

Chapter 3

The Conifold

This chapter is joint work with Prof. Sergey Mozgozoy, Prof. Kentaro Nagao and Prof. Balazs Szendrői [39]. We compute the motivic Donaldson–Thomas theory of the resolved conifold, in all chambers of the space of stability conditions of the corresponding quiver. The answer is a product formula whose terms depend on the position of the stability vector, generalizing known results for the corresponding numerical invariants. Our formulae imply in particular a motivic form of the DT/PT correspondence for the resolved conifold. The answer for the motivic PT series is in full agreement with the prediction of the refined topological vertex formalism.

3.1 Introduction

A *Donaldson-Thomas* (DT) *invariant* of a Calabi-Yau 3-fold Y is a counting invariant of coherent sheaves on Y, introduced in [60] as a holomorphic analogue of the Casson invariant of a real 3-manifold. A component of the moduli space of (say stable) coherent sheaves on Y carries a symmetric obstruction theory and a virtual fundamental cycle [6, 7]. A DT invariant of a compact Y is then defined as the integral of the constant function 1 over the virtual fundamental cycle of the moduli space.

It is known that the moduli space of coherent sheaves on Y can be locally described as the critical locus of a function, the *holomorphic Chern–Simons functional* (see [27]). Behrend provided a description of DT invariants in terms of the Euler characteristic of the *Milnor fiber* of the CS functional [2]. Inspired by this result, the proposal of [33, 4] was to study the *motivic Milnor fiber* of the CS functional as a motivic refinement of the DT invariant. Such a refinement had been expected in string theory [26, 17].

The purpose of this chapter is to show how the ideas of Szendrői [58] and Nagao and Nakajima [46] can be used to study the motivic refinement of DT theory and related enumerative theories associated to the local conifold $Y = \mathcal{O}_{\mathbb{P}^1}(-1,-1)$, the threefold total space over $\mathbb{P}^1$ of the rank two bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In [58], it was realized that a counting problem closely related to the original DT counting on Y can be formulated algebraically, in terms of counting representations of a certain quiver with potential (see below), the so-called conifold quiver. It was also conjectured there that the algebraic and geometric counting problems are related by wall crossing. The paper [46] realized this, by

- describing the natural chamber structure on the space of stability parameters of the conifold quiver,
- finding chambers which correspond to geometric DT and stable pair (PT), as well as algebraic non-commutative DT invariants, and
- computing the generating function of Donaldson-Thomas type invariants for each chamber.

In this chapter, we consider motivic refinements of these formulae. The motivic refinement is given by the motivic class of vanishing cycles of the conifold potential. This virtual motive "motivates" DT theory and its variants (PT, NCDT) in the sense that its Euler characteristic specialization is the corresponding enumerative invariant of the moduli space.

The main result of this chapter is the computation of the generating series of these virtual motives in all chambers of the space of stability conditions. We constantly use the torus action on Y, together with a result of [4]. We use the factorization property of [33, 32, 44, 51]. We also need one explicit evaluation, Theorem 3.8; we give two proofs of that result, one using an explicit calculation of the generating series of motives of a certain space of matrices, another relying on a further "dimensional reduction" to a problem on a tame quiver.

At large volume, our result agrees (up to a subtlety involving the Hilbert scheme of points) with the refined topological vertex formulae of [26], also discussed in [17] in this context.

The motives considered here exist globally over the moduli spaces. Thus our point of view is slightly different from that of [33], whose general framework involves building motivic invariants from local data. The results here are fully compatible with theirs, but the proofs do not depend on the partially conjectural setup of [33], in particular their integration map.

As well as a motivic refinement, there is also a "categorification" given by the mixed Hodge module of vanishing cycles of the potential; compare [32]. Our results can also be interpreted as computing the generating series of E-polynomials of this categorification.

3.1.1 Main result

Let $J = J(Q, W)$ be the non-commutative crepant resolution of the conifold, a quiver algebra with relations coming from the Klebanov–Witten potential W (see Section 3.3.1 for details). Let $\widetilde{J} = J(\widetilde{Q}, W)$ be the framed algebra given by adding the new vertex ∞ to the quiver of J. In [46], the authors introduce a notion of ζ-(semi)stability of $\widetilde{J}$-modules $\widetilde{V}$ with $\dim \widetilde{V}_\infty \leq 1$ for a stability parameter $\zeta = (\zeta_0, \zeta_1) \in \mathbb{R}^2$.

Let $\alpha \in \mathbb{N}^2$ and let $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ be the moduli space of ζ-stable $\widetilde{J}$-modules $\widetilde{V}$, with $\underline{\dim}\, \widetilde{V} = (\alpha, 1)$. We want to compute the motivic generating series

$$Z_\zeta(y_0, y_1) = \sum_{\alpha \in \mathbb{N}^2} \Big[\mathfrak{M}_\zeta\big(\widetilde{J}, \alpha\big)\Big]_{\mathrm{vir}} \cdot y_0^{\alpha_0} y_1^{\alpha_1} \in \mathcal{M}_{\mathbb{C}}[[y_0, y_1]].$$

Here $[\bullet]_{\mathrm{vir}}$ denotes the *virtual motive* (see Section 3.2.1), an element of a suitable ring of motives $\mathcal{M}_{\mathbb{C}}$.

As proved in [46], the stability parameter space $\mathbb{R}^2$ is a countable union of chambers, within which the moduli spaces and therefore the generating series Z_ζ remain unchanged. The chambers are separated by a set of walls, defined by a set of positive roots

$$\Delta_+ = \Delta_+^{\mathrm{re}} \sqcup \Delta_+^{\mathrm{im}},$$

where

$$\Delta_+^{\mathrm{re}} = \{(i, i-1) \mid i \geq 1\} \cup \{(i-1, i) \mid i \geq 1\},$$
$$\Delta_+^{\mathrm{im}} = \{(i, i) \mid i \geq 1\}.$$

To each element $\alpha = (\alpha_0, \alpha_1) \in \Delta_+$, we associate a finite product as follows: for real roots $\alpha \in \Delta_+^{\mathrm{re}}$, put

$$Z_\alpha(-y_0, y_1) = \prod_{j=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + \frac{1}{2} + j} y_0^{\alpha_0} y_1^{\alpha_1}\right),$$

whereas for imaginary roots $\alpha \in \Delta_+^{\mathrm{im}}$, put

$$Z_\alpha(-y_0, y_1) = \prod_{j=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + 1 + j} y_0^{\alpha_0} y_1^{\alpha_1}\right)^{-1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + 2 + j} y_0^{\alpha_0} y_1^{\alpha_1}\right)^{-1}.$$

Our main result is the following product formula:

Theorem 3.1. *For $\zeta \in \mathbb{R}^2$ not orthogonal to any root,*

$$Z_\zeta(y_0, y_1) = \prod_{\substack{\alpha \in \Delta_+ \\ \zeta \cdot \alpha < 0}} Z_\alpha(y_0, y_1).$$

By [2, 4], the specialization $Z_\zeta(y_0, y_1)|_{\mathbb{L}^{\frac{1}{2}} \to 1}$ is the DT-type series at the generic stability parameter ζ, computed in special cases in [3, 37, 58, 62] and in general in [13, 46]. Previously, all these results have been obtained by torus localization; we obtain new proofs of all these formulae. Since [46] identifies the DT and PT chambers for Y, we in particular get motivic results for these two chambers.

Corollary 3.2. *The refined DT and PT series of the resolved conifold are given by the formulae*

$$Z_{\mathrm{PT}}(-s, T) = \prod_{m \geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{-\frac{m}{2} + \frac{1}{2} + j} s^m T\right)$$

and

$$Z_{\mathrm{DT}}(-s,T) = Z_{\mathrm{PT}}(-s,T) \cdot \prod_{m\geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+1+j} s^m\right)^{-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+2+j} s^m\right)^{-1},$$

written in the geometric variables s, T, with s representing the point class and T representing the curve class as usual.

Thus in particular we compute the first instance of a motivic DT partition function for the original geometric problem of rank-1 invariants of ideal sheaves of points and curves [37] where a curve is present. Corollary 3.2 also proves the motivic version of the DT/PT wall crossing formula. These results are compared to the expectation from the refined topological vertex [26] in Section 3.5.3.

3.2 Preliminaries

3.2.1 Motives

We are working in a version of the ring of motivic weights: let $\mathcal{M}_{\mathbb{C}}$ denote the K-group of the category of effective Chow motives over $\mathbb{C}$, extended by $\mathbb{L}^{-\frac{1}{2}}$, where $\mathbb{L}$ is the Lefschetz motive. It has a natural structure of a λ-ring [19, 23] (see Section 3.2.2 for the definition of a λ-ring) with σ-operations defined by $\sigma_n([X]) = [X^n/S_n]$ and $\sigma_n(\mathbb{L}^{\frac{1}{2}}) = \mathbb{L}^{\frac{n}{2}}$. There is a dimensional completion [5]

$$\widetilde{\mathcal{M}}_{\mathbb{C}} = \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-1}]\!],$$

which is also a λ-ring. Note that in this latter ring, the elements $(1-\mathbb{L}^n)$, and therefore the motives of general linear groups, are invertible. The rings $\mathcal{M}_{\mathbb{C}} \subset \widetilde{\mathcal{M}}_{\mathbb{C}}$ sit in larger rings $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \subset \widetilde{\mathcal{M}}_{\mathbb{C}}^{\hat{\mu}}$ of equivariant motives, where $\hat{\mu}$ is the group of all roots of unity [35].

The map that sends a smooth projective variety X to its E-polynomial

$$E(X,u,v) = \sum_{p,q\geq 0} (-1)^{p+q} \dim H^{p,q}(X,\mathbb{C}) u^p v^q$$

can be extended to the ring homomorphism $E : \widetilde{\mathcal{M}}_{\mathbb{C}} \to \mathbb{Q}[u,v][\![(uv)^{-\frac{1}{2}}]\!]$. This map is a λ-ring homomorphism, where the λ-ring structure on $\mathbb{Q}[u,v][\![(uv)^{-\frac{1}{2}}]\!]$ is given by Adams operations (see Section 3.2.2)

$$\psi_n(f(u,v)) = f(u^n, v^n).$$

The map $E : \mathcal{M}_{\mathbb{C}} \to \mathbb{Q}[u,v,(uv)^{-\frac{1}{2}}]$ can be further specialized to the Euler number $e : \mathcal{M}_{\mathbb{C}} \to \mathbb{Q}$ by $u \mapsto 1, v \mapsto 1, (uv)^{-\frac{1}{2}} \mapsto 1$.

Remark 3.1. Note that the Euler number specialization of $\mathbb{L}^{\frac{1}{2}}$ is $\mathbb{L}^{\frac{1}{2}} \mapsto 1$. This differs from the conventions of [4], where the specialization is $\mathbb{L}^{\frac{1}{2}} \mapsto -1$. This difference results from the fact that [4] uses the λ-ring structure on $\mathcal{M}_{\mathbb{C}}$ with $\sigma_n(-\mathbb{L}^{\frac{1}{2}}) = (-\mathbb{L}^{\frac{1}{2}})^n$ [4, Remark 1.7].

Let $f\colon X \to \mathbb{C}$ be a regular function on a smooth variety X. Using arc spaces, Denef and Loeser [14, 35] define the motivic nearby cycle $[\psi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ and the motivic vanishing cycle

$$[\varphi_f] = [\psi_f] - [f^{-1}(0)] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

of f. Note that if $f = 0$, then $[\varphi_0] = -[X]$. The following result was proved in [4, Prop. 1.11].

Theorem 3.3. *Let $f : X \to \mathbb{C}$ be a regular function on a smooth variety X. Assume that X admits a $\mathbb{C}^*$-action such that f is $\mathbb{C}^*$-equivariant i.e. $f(tx) = tf(x)$ for $t \in \mathbb{C}^*$, $x \in X$, and such that there exist limits* $\lim_{t\to 0} tx$ *for all $x \in X$. Then*

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

Following [4], we define the *virtual motive* of $\operatorname{crit}(f)$ to be

$$[\operatorname{crit}(f)]_{\mathrm{vir}} = -(-\mathbb{L}^{\frac{1}{2}})^{-\dim X}[\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

Thus for a smooth variety X with $f = 0$,

$$[X]_{\mathrm{vir}} = [\operatorname{crit}(0_X)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{-\dim X} \cdot [X].$$

Remark 3.2. The ring $\mathcal{M}_{\mathbb{C}}$ is known to be a homomorphic image of the naive motivic ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}]$. Some of the works cited above work in this ring; the quoted constructions and results carry over to $\mathcal{M}_{\mathbb{C}}$ under this ring homomorphism. We prefer to work in $\mathcal{M}_{\mathbb{C}}$ since that is known to be a λ-ring.

3.2.2 λ-Rings and power structures

Let

$$\Lambda = \varprojlim \mathbb{Z}[\![x_1, \dots, x_n]\!]^{S_n}$$

be the ring of symmetric functions [36]. It is well-known that Λ is generated as an algebra over $\mathbb{Z}$ by elementary symmetric functions

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}$$

as well as by complete symmetric functions

$$h_n = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \dots x_{i_n}.$$

Moreover $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$ is generated over $\mathbb{Q}$ by power sums $p_n = \sum x_i^n$.

A $\mathbb{Q}$-algebra R is called a λ-ring if it is endowed with a map $\circ : \Lambda \times R \to R$ called plethysm, such that $(- \circ r) : \Lambda \to R$ is a ring homomorphism for any $r \in R$ and the maps $\psi_n = (p_n \circ -)$, called Adams operations, are ring homomorphisms satisfying $\psi_1 = \mathrm{Id}_R$ and $\psi_m \psi_n = \psi_{mn}$ for $m, n \ge 1$. Note that plethysm is uniquely determined by Adams operations. It is also uniquely determined by maps $\lambda_n = (e_n \circ -) : R \to R$ called λ-operations and by maps $\sigma_n = (h_n \circ -) : R \to R$ called σ-operations.

Given a λ-ring R, we endow the ring $A = R[\![x_1, \dots, x_m]\!]$ with a λ-ring structure by the rule

$$\psi_n(rx^\alpha) = \psi_n(r) x^{n\alpha}, \qquad r \in R, \alpha \in \mathbb{N}^m.$$

Let $A_+ \subset A$ be an ideal generated by $x_1, \dots, x_m$. We define a map Exp :

$A_+ \to 1 + A_+$, called plethystic exponential, by the rule [19, 43]

$$\operatorname{Exp}(f) = \sum_{n\geq 0} \sigma_n(f) = \exp\Bigg(\sum_{n\geq 1} \frac{1}{n}\psi_n(f)\Bigg).$$

This map has an inverse $\operatorname{Log} : 1 + A_+ \to A_+$, called plethystic logarithm,

$$\operatorname{Log}(f) = \sum_{n\geq 1} \frac{\mu(n)}{n}\psi_n \log(f),$$

where μ is a Möbius function.

We define a power structure map $\operatorname{Pow} : (1 + A_+) \times A \to 1 + A_+$ by the rule [43]

$$\operatorname{Pow}(f, g) = \operatorname{Exp}(g \operatorname{Log}(f)).$$

In the case when R is a ring of motives, the power structure map has the following geometric interpretation [22]. Let

$$f = 1 + \sum_{\alpha>0} [A_\alpha] x^\alpha,$$

where A_α are algebraic varieties. Then

$$\operatorname{Pow}(f, [X]) = \sum_{k:\mathbb{N}^m\to\mathbb{N}} \Bigg[\Big(F_{|k|}X \times \prod_{\alpha\in\mathbb{N}^m} A_\alpha^{k(\alpha)}\Big)/ \prod_{\alpha\in\mathbb{N}^m} S_{k(\alpha)}\Bigg] x^{\sum k(\alpha)\alpha},$$

where the sum runs over maps $k : \mathbb{N}^m \to \mathbb{N}$ with finite support, $|k| = \sum_{\alpha\in\mathbb{N}^m} k(\alpha)$, the configuration space $F_n X$ is given by

$$F_n X = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\},$$

and the product of symmetric groups $\prod_{\alpha\in\mathbb{N}^m} S_{k(\alpha)}$ acts on both factors in the obvious way. The quotient in square brackets parametrizes elements in

$$\bigcup_{\psi:X\to\mathbb{N}^m} \prod_{x\in X} A_{\psi(x)}$$

with $\psi : X \to \mathbb{N}^m$ satisfying $\#\{x \in X \mid \psi(x) = \alpha\} = k(\alpha)$ for any $\alpha \in$

$\mathbb{N}^m \backslash \{0\}$ (see [41]). Therefore we can also write

$$\mathrm{Pow}(f, [X]) = \sum_{\psi : X \to \mathbb{N}^m} \prod_{x \in X} [A_{\psi(x)}] x^{\psi(x)},$$

where the sum runs over maps $\psi : X \to \mathbb{N}^m$ with finite support.

3.2.3 Quivers and moduli spaces

Let Q be a quiver, with vertex set Q_0 and edge set Q_1. For an arrow $a \in Q_1$, we denote by $s(a) \in Q_0$ (resp. $t(a) \in Q_0$) the vertex at which a starts (resp. ends). We define the Euler-Ringel form χ on $\mathbb{Z}^{Q_0}$ by the rule

$$\chi(\alpha, \beta) = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{s(a)} \beta_{t(a)}, \qquad \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

We define the skew-symmetric bilinear form $\langle \bullet, \bullet \rangle$ of the quiver Q to be

$$\langle \alpha, \beta \rangle = \chi(\alpha, \beta) - \chi(\beta, \alpha), \qquad \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

Given a Q-representation M, we define its dimension vector $\underline{\dim}\, M \in \mathbb{N}^{Q_0}$ by $\underline{\dim}\, M = (\dim M_i)_{i \in Q_0}$. Let $\alpha \in \mathbb{N}^{Q_0}$ be a dimension vector and let $V_i = \mathbb{C}^{\alpha_i}$, $i \in Q_0$. We define

$$R(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$$

and

$$G_\alpha = \prod_{i \in Q_0} \mathrm{GL}(V_i).$$

Note that G_α naturally acts on $R(Q, \alpha)$ and the quotient stack

$$\mathfrak{M}(Q, \alpha) = [R(Q, \alpha)/G_\alpha]$$

gives the moduli stack of representations of Q with dimension vector α.

Let W be a potential on Q, a finite linear combination of cyclic paths in Q. Denote by $J = J_{Q,W}$ the Jacobian algebra, the quotient of the path algebra $\mathbb{C}Q$ by the two-sided ideal generated by formal partial derivatives of the potential W. Let

$$f_\alpha : R(Q,\alpha) \to \mathbb{C}$$

be the G_α-invariant function defined by taking the trace of the map associated to the potential W. As it is now well known [55, Proposition 3.8], a point in the critical locus $\mathrm{crit}(f_\alpha)$ corresponds to a J-module. The quotient stack

$$\mathfrak{M}(J,\alpha) = \left[\mathrm{crit}(f_\alpha)/G_\alpha\right]$$

gives the moduli stack of J-modules with dimension vector α.

Definition 3.1. A *central charge* is a group homomorphism $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$ such that

$$Z(\alpha) \in \mathbb{H}_+ = \{re^{i\pi\varphi} \mid r > 0, 0 < \varphi \leq 1\}$$

for any $\alpha \in \mathbb{N}^{Q_0}\backslash\{0\}$. Given $\alpha \in \mathbb{N}^{Q_0}\backslash\{0\}$, the number $\varphi(\alpha) = \varphi \in (0,1]$ such that $Z(\alpha) = re^{i\pi\varphi}$, for some $r > 0$, is called the phase of α.

Definition 3.2. For any nonzero Q-representation (resp. J-module) V, we define $\varphi(V) = \varphi(\underline{\dim}\, V)$. A Q-representation (resp. J-module) V is said to be Z-(semi)stable if for any proper nonzero Q-subrepresentation (resp. J-submodule) $U \subset V$ we have

$$\varphi(U)(\leq)\varphi(V).$$

Definition 3.3. Given $\zeta \in \mathbb{R}^{Q_0}$, define the central charge $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$ by the rule

$$Z(\alpha) = -\zeta \cdot \alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i \in Q_0} \alpha_i$. We say that a Q-representation (resp. J-module) is ζ-(semi)stable if it is Z-(semi)stable.

Remark 3.3. Let the central charge Z be as in Definition 3.3. Define the slope function $\mu : \mathbb{N}^{Q_0}\backslash\{0\} \to \mathbb{R}$ by $\mu(\alpha) = \frac{\zeta \cdot \alpha}{|\alpha|}$. If $l \subset \mathbb{H} = \mathbb{H}_+ \cup \{0\}$ is a ray such that $Z(\alpha) \in l$ then $l = \mathbb{R}_{\geq 0}(-\mu(\alpha), 1)$. This implies that $\varphi(\alpha) < \varphi(\beta)$ if and only if $\mu(\alpha) < \mu(\beta)$.

We say that $\zeta \in \mathbb{R}^{Q_0}$ is α-generic if for any $0 < \beta < \alpha$ we have $\varphi(\beta) \neq \varphi(\alpha)$. This condition implies that any ζ-semistable Q-representation (resp. J-module) is automatically ζ-stable.

Let $R_\zeta(Q,\alpha)$ denote the open subset of $R(Q,\alpha)$ consisting of ζ-semistable representations. Let $f_{\zeta,\alpha}$ denote the restriction of f_α to $R_\zeta(Q,\alpha)$. The quotient stacks

$$\mathfrak{M}_\zeta(Q,\alpha) = \left[R_\zeta(Q,\alpha)/G_\alpha\right], \qquad \mathfrak{M}_\zeta(J,\alpha) = \left[\mathrm{crit}(f_{\zeta,\alpha})/G_\alpha\right] \tag{3.1}$$

give the moduli stacks of Q-representations and J-modules with dimension vector α.

3.2.4 Motivic DT invariants

Let (Q,W) be a quiver with a potential and let $J = J_{Q,W}$ be its Jacobian algebra. Recall that the degeneracy locus of the function $f_\alpha : R(Q,\alpha) \to \mathbb{C}$ defines the locus of J-modules, so that the quotient stack

$$\mathfrak{M}(J,\alpha) = [\mathrm{crit}(f_\alpha)/G_\alpha]$$

is the stack of J-modules with dimension vector α. We define motivic Donaldson-Thomas invariants

$$[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}} = \frac{[\mathrm{crit}(f_\alpha)]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}},$$

where $[G_\alpha]_{\mathrm{vir}}$ refers to the virtual motive of the pair $(G_\alpha, 0)$.

Definition 3.4. A subset $I \subset Q_1$ is called a cut of (Q,W) if in the associated grading g_I on Q given by

$$g_I(a) = \begin{cases} 1 & a \in I, \\ 0 & a \notin I, \end{cases}$$

the potential W is homogeneous of degree 1.

Throughout this section we assume that (Q,W) admits a cut. Then the space $R(Q,\alpha)$ admits a $\mathbb{C}^*$-action satisfying the conditions of Theorem 3.3 for the function $f_\alpha : R(Q,\alpha) \to \mathbb{C}$. This implies

$$[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{-\dim R(Q,\alpha)} \frac{[f_\alpha^{-1}(0)] - [f_\alpha^{-1}(1)]}{[G_\alpha]_{\mathrm{vir}}}$$
$$= (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{[f_\alpha^{-1}(0)] - [f_\alpha^{-1}(1)]}{[G_\alpha]}. \quad (3.2)$$

Generally, for an arbitrary stability parameter ζ, we define

$$[\mathfrak{M}_\zeta(J,\alpha)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{[f_{\zeta,\alpha}^{-1}(0)] - [f_{\zeta,\alpha}^{-1}(1)]}{[G_\alpha]}, \quad (3.3)$$

where, as before, $f_{\zeta,\alpha}$ denote the restriction of $f_\alpha : R(Q,\alpha) \to \mathbb{C}$ to $R_\zeta(Q,\alpha)$.

Lemma 3.4. *Let $\alpha \in \mathbb{N}^{Q_0}$ be such that $\alpha_i = 1$ for some $i \in Q_0$ (this will be the case for framed representations studied later) and let $\zeta \in \mathbb{R}^{Q_0}$ be α-generic. Then*

$$[\mathfrak{M}_\zeta(J,\alpha)]_{\mathrm{vir}} = \frac{[\mathrm{crit}(f_{\zeta,\alpha})]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}}.$$

Proof. Let

$$M_\zeta(Q,\alpha) = R_\zeta(Q,\alpha)/G_\alpha$$

be the smooth moduli space of ζ-semistable Q-representations having dimension vector α, and let $f'_{\zeta,\alpha} : M_\zeta(Q,\alpha) \to \mathbb{C}$ be the map induced by $f_{\zeta,\alpha} : R_\zeta(Q,\alpha) \to \mathbb{C}$. Note that $R_\zeta(Q,\alpha) \to M_\zeta(Q,\alpha)$ is a principal bundle with the structure group $PG_\alpha = G_\alpha/\mathbb{C}^*$. The group PG_α is a product of general linear groups (here we use our assumption that there exists $i \in Q_0$ with $\alpha_i = 1$). Therefore $R_\zeta(Q,\alpha) \to M_\zeta(Q,\alpha)$ is locally trivial in Zariski topology. This implies

$$\frac{[\mathrm{crit}(f_{\zeta,\alpha})]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}} = \frac{[\mathrm{crit}(f'_{\zeta,\alpha})]_{\mathrm{vir}}}{[\mathrm{GL}_1]_{\mathrm{vir}}}.$$

As (Q,W) admits a cut, the space $M_\zeta(Q,\alpha)$ admits a $\mathbb{C}^*$-action satisfying the conditions of Theorem 3.3 for the function $f'_{\zeta,\alpha} : M_\zeta(Q,\alpha) \to \mathbb{C}$ (one uses the fact that $M_\zeta(Q,\alpha)$ is projective over $R(Q,\alpha)/\!\!/G_\alpha)$. This implies

$$\frac{[\mathrm{crit}(f'_{\zeta,\alpha})]_{\mathrm{vir}}}{[\mathrm{GL}_1]_{\mathrm{vir}}} = \frac{-(-\mathbb{L}^{\frac{1}{2}})^{-\dim M_\zeta(Q,\alpha)}[\varphi_{f'_{\zeta,\alpha}}]}{(-\mathbb{L}^{\frac{1}{2}})^{-1}(\mathbb{L}-1)}$$
$$= \frac{(-\mathbb{L}^{\frac{1}{2}})^{\dim G_\alpha - \dim R(Q,\alpha)}}{\mathbb{L}-1}([f'^{-1}_{\zeta,\alpha}(0)] - [f'^{-1}_{\zeta,\alpha}(1)])$$
$$= (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)}\frac{[f^{-1}_{\zeta,\alpha}(0)] - [f^{-1}_{\zeta,\alpha}(1)]}{[G_\alpha]}. \quad (3.4)$$

□

3.2.5 Twisted algebra and central charge

Definition 3.5. The *twisted motivic algebra* associated to the quiver Q is the associative $\widetilde{\mathcal{M}}_{\mathbb{C}}$-algebra

$$\mathcal{T}_Q = \prod_{\alpha \in \mathbb{N}^{Q_0}} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y^\alpha$$

generated by formal variables y^α that satisfy the relation

$$y^\alpha \cdot y^\beta = (-\mathbb{L}^{\frac{1}{2}})^{\langle \alpha,\beta \rangle} y^{\alpha+\beta},$$

with $\langle \bullet, \bullet \rangle$ the skew-symmetric form of the quiver Q.

Note that if the quiver Q is symmetric, i.e. its skew-symmetric form is identically zero, then $\mathcal{T}_Q$ is commutative.

Remark 3.4. This algebra, which is all we are going to need, is (a completion of) the "positive half" of the motivic quantum torus of Kontsevich–Soibelman [33].

A ray in the upper half plane $\mathbb{H} = \mathbb{H}_+ \cup \{0\}$ is a half line which has the origin as its end. For a ray $l \subset \mathbb{H}$ and a central charge Z, we put

$$\mathcal{T}_{Z,l} = \prod_{\alpha \in Z^{-1}(l) \cap \mathbb{N}^{Q_0}} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y^\alpha,$$

a subalgebra of the twisted algebra $\mathcal{T}_Q$.

Lemma 3.5 (Kontsevich–Soibelman [31, Theorem 6])**.** *For any element $A = \sum_{\alpha\in\mathbb{N}^{Q_0}} A_\alpha y^\alpha \in \mathcal{T}_Q$ with $A_0 = 1$, there is a unique factorization*

$$A = \prod_{l\subset\mathbb{H}}^{\curvearrowright} A_{Z,l} \tag{3.5}$$

with $A_{Z,l} \in \mathcal{T}_{Z,l}$, where the product is taken in the clockwise order over all rays.

Proof. For a positive real number r, we put

$$\mathcal{T}_Q^{(r)} = \prod_{\alpha\in\mathbb{N}^{Q_0},|\alpha|<r} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y^\alpha, \quad \mathcal{T}_{Z,l}^{(r)} = \prod_{\alpha\in Z^{-1}(l)\cap\mathbb{N}^{Q_0},|\alpha|<r} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y^\alpha$$

which we consider as factor algebras of $\mathcal{T}_Q$ and $\mathcal{T}_{Z,l}$ respectively. Let $A^{(r)} \in \mathcal{T}_Q^{(r)}$ denote the image of A under the canonical projection $\mathcal{T}_Q \twoheadrightarrow \mathcal{T}_Q^{(r)}$. It is enough to show that for any r there is a unique factorization

$$A^{(r)} = \prod_{l\subset\mathbb{H}}^{\curvearrowright} A_{Z,l}^{(r)}$$

with $A_{Z,l}^{(r)} \in \mathcal{T}_{Z,l}^{(r)}$. Note that the set of rays $l \in \mathbb{H}$ such that

$$\{\alpha \in Z^{-1}(l) \mid |\alpha| < r\} \neq \emptyset$$

is finite. We order this set $(l_1, \ldots, l_N)$ so that

$$\arg Z(l_1) < \cdots < \arg Z(l_N).$$

First we put $A_{Z,l_1}^{(r)}$ to be the summand of $A^{(r)}$ contained in $\mathcal{T}_{Z,l_1}^{(r)}$. For $1 < i \leq N$ we define $A_{Z,l_i}^{(r)}$ to be the summand of

$$A^{(r)} \cdot \big(A_{Z,l_1}^{(r)}\big)^{-1} \cdot \cdots \cdot \big(A_{Z,l_{i-1}}^{(r)}\big)^{-1}$$

contained in $\mathcal{T}_{Z,l_i}^{(r)}$. Uniqueness is clear from the construction. □

3.2.6 Generating series of motivic DT invariants

Let (Q, W) be a quiver with a potential admitting a cut, and let $J = J_{Q,W}$ be its Jacobian algebra.

Definition 3.6. We define the generating series of the motivic Donaldson-Thomas invariants of (Q, W) by

$$A_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} [\mathfrak{M}(J,\alpha)]_{\mathrm{vir}} \cdot y^\alpha = \sum_{\alpha\in\mathbb{N}^{Q_0}} \frac{[\mathrm{crit}(f_\alpha)]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}} \cdot y^\alpha \in \mathcal{T}_Q,$$

the subscript referring to the fact that we think of this series as the universal series.

Given a cut I of (Q, W), we define a new quiver $Q_I = (Q_0, Q_1\backslash I)$. Let $J_{W,I}$ be the quotient of $\mathbb{C}Q_I$ by the ideal

$$(\partial_I W) = (\partial W/\partial a, a \in I).$$

Proposition 3.6. *If (Q, W) admits a cut I, then*

$$A_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)+2d_I(\alpha)} \frac{[R(J_{W,I},\alpha)]}{[G_\alpha]} y^\alpha,$$

where $d_I(\alpha) = \sum_{(a:i\to j)\in I} \alpha_i\alpha_j$ for any $\alpha \in \mathbb{Z}^{Q_0}$.

Proof. Let $f = f_\alpha : R(Q, \alpha) \to \mathbb{C}$. According to (3.2) we have

$$[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{[f^{-1}(0)] - [f^{-1}(1)]}{G_\alpha}.$$

It is proved in [51, Theorem 4.1] and [38, Prop. 7.1] that

$$[f^{-1}(1)] - [f^{-1}(0)] = -\mathbb{L}^{d_I(\alpha)}[R(J_{W,I},\alpha)].$$

Therefore

$$[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)+2d_I(\alpha)} \frac{[R(J_{W,I},\alpha)]}{[G_\alpha]}.$$

□

Let $\zeta \in \mathbb{R}^{Q_0}$ be some stability parameter and let $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$ be the central charge determined by ζ as in Definition 3.3.

Definition 3.7. Let $l = \mathbb{R}_{\geq 0}(-\mu, 1) \subset \mathbb{H}$ be a ray (see Remark 3.3). We put

$$A_{Z,l} = A_{\zeta,\mu} = \sum_{\substack{\alpha \in \mathbb{N}^{Q_0} \\ Z(\alpha) \in l}} [\mathfrak{M}_\zeta(J, \alpha)]_{\mathrm{vir}} \cdot y^\alpha \in \mathcal{T}_Q.$$

The Harder-Narashimhan filtrations provide a filtration on $R(Q, \alpha)$. This filtration induces the following *factorization property.*

Theorem 3.7. *Assume that* (Q, W) *has a cut. Then we have*

$$A_U = \prod_{l}^{\curvearrowright} A_{Z,l},$$

where the product is taken in the clockwise order over all rays.

Proof. This is originally a result of Kontsevich–Soibelman [33], though their proof depends on a conjectural integral identity. Assuming the existence of a cut, Theorem 3.3 leads to a simplified proof, written out in [51] and [44]. □

3.3 The Universal DT Series of the Conifold Quiver

3.3.1 Motivic DT invariants for the conifold quiver

Let (Q, W) be the conifold quiver with potential. Recall that Q has vertices $0, 1$ and arrows $a_i : 0 \to 1$, $b_i : 1 \to 0$ for $i = 1, 2$. The potential is given by

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

We have

$$\chi(\alpha, \alpha) = \alpha_0^2 + \alpha_1^2 - 4\alpha_0\alpha_1.$$

Let $J_W = \mathbb{C}Q/\partial W$ be the Jacobian algebra of (Q, W). Then $I = \{a_1\}$ is easily seen to be a cut for (Q, W). Let $Q_I = (Q_0, Q_1 \backslash I)$ be the quiver defined by the cut, and $J_{W,I}$ the quotient of $\mathbb{C}Q_I$ by the ideal

$$(\partial_I W) = (\partial W/\partial a, a \in I).$$

It follows from Proposition 3.6 that the coefficients of the universal Donaldson-Thomas series $A_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} A_\alpha y^\alpha$ are given by

$$A_\alpha = (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)+2\alpha_0\alpha_1}\frac{[R(J_{W,I},\alpha)]}{[G_\alpha]} = (-\mathbb{L}^{\frac{1}{2}})^{(\alpha_0-\alpha_1)^2}\frac{[R(J_{W,I},\alpha)]}{[G_\alpha]}.$$

The goal of this section is to prove the following result.

Theorem 3.8. *We have*

$$A_U(y_0,y_1) = \operatorname{Exp}\left(\frac{(\mathbb{L}+\mathbb{L}^2)y_0y_1 - \mathbb{L}^{\frac{1}{2}}(y_0+y_1)}{\mathbb{L}-1}\sum_{n\geq 0}(y_0y_1)^n\right). \qquad (3.6)$$

Equivalently,

$$A_U(y_0,y_1) = \prod_{\alpha\in\Delta_+} A^\alpha(y_0,y_1), \qquad (3.7)$$

where for roots $\alpha\in\Delta_+$, we put

$$A^\alpha(y_0,y_1) = \begin{cases} \operatorname{Exp}\left(\frac{-\mathbb{L}^{-\frac{1}{2}}}{1-\mathbb{L}^{-1}}y^\alpha\right) = \prod_{j\geq 0}\left(1-\mathbb{L}^{-j-\frac{1}{2}}y^\alpha\right) & \alpha\in\Delta_+^{\mathrm{re}},\\ \operatorname{Exp}\left(\frac{1+\mathbb{L}}{1-\mathbb{L}^{-1}}y^\alpha\right) = \prod_{j\geq 0}\left(1-\mathbb{L}^{-j}y^\alpha\right)^{-1}\left(1-\mathbb{L}^{-j+1}y^\alpha\right)^{-1} & \alpha\in\Delta_+^{\mathrm{im}}.\end{cases}$$

The equivalence of the exponential and product forms (3.6)–(3.7) follows from formal manipulations. In the following two subsections, we give two proofs of Theorem 3.8. The first one develops the method of [18] (c.f. [12]). The second proof uses another "dimensional reduction" to reduce the problem to that of representations of the tame quiver of affine type $A_1^{(1)}$.

3.3.2 First proof

The goal is to compute the generating function of motives of moduli of representations of $J_{W,I}$-modules. Up to a group action, these moduli spaces are given concretely as spaces of triples of matrices:

$$R(J_{W,I},\alpha) = \{(A_2,B_1,B_2) \in \mathrm{Hom}(V_0,V_1)\times\mathrm{Hom}(V_1,V_0)^{\times 2} \mid B_1A_2B_2 = B_2A_2B_1\}.$$

In this section, for simplicity, we will denote this space by $R(\alpha)$. The proof begins by reducing the problem to two simpler ones via a stratification of $R(\alpha)$. For $(A_2,B_1,B_2) \in R(\alpha)$, consider the linear map

$$A_2 \oplus B_2 : V_0 \oplus V_1 \to V_0 \oplus V_1.$$

For any such endomorphism, the vector space $V = V_0 \oplus V_1$ has a second decomposition $V = V^I \oplus V^N$ on which $A_2 \oplus B_2$ decomposes into an invertible map and a nilpotent map (c.f. [18, Lemma 1]). Namely,

$$A_2^I \oplus B_2^I : V^I \to V^I \text{ and } A_2^N \oplus B_2^N : V^N \to V^N.$$

Define V_i^I (resp. V_i^N) to be the intersection of V_i and V^I (resp. V^N), so that we have decompositions $V_0 = V_0^I \oplus V_0^N$ and $V_1 = V_1^I \oplus V_1^N$, with

$$A_2 = A_2^I \oplus A_2^N \in \mathrm{Hom}(V_0^I, V_1^I) \oplus \mathrm{Hom}(V_0^N, V_1^N),$$

$$B_2 = B_2^I \oplus B_2^N \in \mathrm{Hom}(V_1^I, V_0^I) \oplus \mathrm{Hom}(V_1^N, V_0^N).$$

Notice that A_2^I, B_2^I are invertible, in particular we have

$$\dim(V_0^I) = \dim(V_1^I) = \frac{1}{2}\dim(V^I).$$

A little bit of linear algebra shows that a matrix B_1 satisfying $B_1A_2B_2 = B_2A_2B_1$ has a similar block decomposition with respect to the splitting $V = V^I \oplus V^N$;

$$B_1 = B_1^I \oplus B_1^N \in \mathrm{Hom}(V_1^I, V_0^I) \oplus \mathrm{Hom}(V_1^N, V_0^N).$$

The relation $B_1 A_2 B_2 = B_2 A_2 B_1$ now becomes two independent sets of equations, $B_1^I A_2^I B_2^I = B_2^I A_2^I B_1^I$ and $B_1^N A_2^N B_2^N = B_2^N A_2^N B_1^N$. Define

$$R_a^I = \{(A_2, B_1, B_2) \in R((a,a)) \mid A_2 \oplus B_2 \text{ is invertible}\};$$

$$R_\alpha^N = \{(A_2, B_1, B_2) \in R(\alpha) \mid A_2 \oplus B_2 \text{ is nilpotent}\}.$$

Over the stratum of $R(\alpha)$ where $\dim(V^I) = 2a$, we have a Zariski locally trivial fibre bundle

$$\begin{array}{ccc} R_a^I \times R_\alpha^N & \longrightarrow & \{(A_2, B_1, B_2) \in R(\alpha) \mid \dim(V_I) = 2a\} \\ & & \downarrow \\ & & \mathcal{M}(a, \alpha) \end{array}$$

where $\mathcal{M}(a,\alpha)$ is the space of direct sum decompositions $V_0 \cong V_0^I \oplus V_0^N$, $V_1 \cong V_1^I \oplus V_1^N$. Hence stratifying $R(\alpha)$ by $\dim(V^I)$ gives the following relation in the Grothendieck ring of varieties:

$$[R(\alpha)] = \sum_{a=0}^{\min(\alpha_0,\alpha_1)} [R_a^I] \cdot [R^N_{(\alpha_0 - a, \alpha_1 - a)}] \cdot \frac{[\mathrm{GL}_{\alpha_0}]}{[\mathrm{GL}_a][\mathrm{GL}_{\alpha_0 - a}]} \cdot \frac{[\mathrm{GL}_{\alpha_1}]}{[\mathrm{GL}_a][\mathrm{GL}_{\alpha_1 - a}]}.$$

We collect the above motives into two generating series

$$I(y) = \sum_{a \geq 0} \frac{[R_a^I]}{[\mathrm{GL}_a]^2} y^a$$

and

$$N(y_0, y_1) = \sum_{\alpha \in N^Q} \frac{[R_\alpha^N]}{[\mathrm{GL}_{\alpha_0}][\mathrm{GL}_{\alpha_1}]} (-\mathbb{L}^{1/2})^{(\alpha_0 - \alpha_1)^2} y_0^{\alpha_0} y_1^{\alpha_1}.$$

Multiplying the above relation by $(-\mathbb{L}^{1/2})^{(\alpha_0 - \alpha_1)^2} t_0^{\alpha_0} t_1^{\alpha_1}$ and summing gives an equality of power series

$$A_U = I(y_0 y_1) \cdot N(y_0, y_1).$$

It remains to compute $I(y)$ and $N(y_0, y_1)$.

First consider $I(y)$. If π is a partition of a we will write $\pi \vdash a$, and denote its length $l(\pi)$, and size $|\pi|$. Then we have two spaces

$$R_a^I = \{(A_2, B_1, B_2) \in \mathrm{Iso}\big(V_0^I, V_1^I\big) \times \mathrm{Hom}\big(V_1^I, V_0^I\big) \times \mathrm{Iso}\big(V_1^I, V_0^I\big) \mid B_1 A_2 B_2 = B_2 A_2 B_1\}$$

and

$$C_a^I = \{(C_1, C_2) \in \mathrm{End}(V_1^I) \times \mathrm{GL}(V_1^I) \mid C_2^{-1} C_1 C_2 = C_1\},$$

together with a map $\beta : R_a^I \to C_a^I$ given by

$$\beta(A_2, B_1, B_2) = (A_2 B_1, A_2 B_2).$$

The map β is a $\mathrm{GL}(V_0^I)$-torsor associated to a global gauge fixing, $g \cdot (A_2, B_1, B_2) = (A_2 g^{-1}, gB_1, gB_2)$. Since the general linear group is a special group, the map β is a locally trivial $\mathrm{GL}(V_0^I)$ bundle in the Zariski topology. The base of the fibration C_a^I is a commuting variety whose motivic class is known [12] to equal

$$[\mathrm{GL}_a] \sum_{\pi \vdash a} \mathbb{L}^{l(\pi)}.$$

Therefore

$$\begin{aligned} I(y) &= \sum_{a \geq 0} \frac{[R_a^I]}{[\mathrm{GL}_a]^2} y^a = \sum_{a \geq 0} \frac{[C_a^I]}{[\mathrm{GL}_a]} y^a = \sum_{\pi} \mathbb{L}^{l(\pi)} y^{|\pi|} \\ &= \prod_{i=1}^{\infty} \frac{1}{1 - \mathbb{L} y^i} = \mathrm{Exp}\left(\mathbb{L} \sum_{n \geq 1} y^n\right). \end{aligned}$$

All that remains is to compute the series $N(y_0, y_1)$. Given now that the matrix $A_2 \oplus B_2$ is nilpotent, there exists a basis of V^N, $\{a_{s_1}^{i_1}, b_{s_2}^{i_2}, c_{s_3}^{i_3}, d_{s_4}^{i_4}\}$,

$1 \leq i_j \leq k_j$, $1 \leq s_j \leq r_i^j$, such that

$$V_0^N \ni a^{i_1}_{r_i^1} \overset{A_2}{\mapsto} a^{i_1}_{r_i^1-1} \overset{B_2}{\mapsto} a^{i_1}_{r_i^1-2} \overset{A_2}{\mapsto} \cdots \overset{A_2}{\mapsto} a^{i_1}_1 \in V_1^N \setminus 0 \text{ and } B_2(a^{i_1}_1) = 0,$$
$$V_1^N \ni b^{i_2}_{r_i^2} \overset{B_2}{\mapsto} b^{i_2}_{r_i^2-1} \overset{A_2}{\mapsto} b^{i_2}_{r_i^2-2} \overset{B_2}{\mapsto} \cdots \overset{B_2}{\mapsto} b^{i_2}_1 \in V_0^N \setminus 0 \text{ and } A_2(b^{i_2}_1) = 0,$$
$$V_0^N \ni c^{i_3}_{r_i^3} \overset{A_2}{\mapsto} c^{i_3}_{r_i^3-1} \overset{B_2}{\mapsto} c^{i_3}_{r_i^3-2} \overset{A_2}{\mapsto} \cdots \overset{B_2}{\mapsto} c^{i_3}_1 \in V_0^N \setminus 0 \text{ and } A_2(c^{i_3}_1) = 0,$$
$$V_1^N \ni d^{i_4}_{r_i^4} \overset{B_2}{\mapsto} d^{i_4}_{r_i^4-1} \overset{A_2}{\mapsto} d^{i_4}_{r_i^4-2} \overset{B_2}{\mapsto} \cdots \overset{A_2}{\mapsto} d^{i_4}_1 \in V_1^N \setminus 0 \text{ and } B_2(d^{i_4}_1) = 0.$$

As the numbers r_i^1, r_i^2 are always even and r_i^3, r_i^4 always odd, it is combinatorially convenient to define $\hat{r}_i^1 = r_i^1/2, \hat{r}_i^2 = r_i^2/2, \hat{r}_i^3 = (r_i^3+1)/2, \hat{r}_i^4 = (r_i^4+1)/2$. Also after reordering we may assume that $\hat{r}_i^j \geq \hat{r}_{i+1}^j$. Up to a choice of the above basis, the matrix $A_2 \oplus B_2$ is determined by four partitions

$$\pi_1 : |\pi_1| = \hat{r}_1^1 + \hat{r}_2^1 + \hat{r}_3^1 + \ldots + \hat{r}_{k_1}^1$$
$$\pi_2 : |\pi_2| = \hat{r}_1^2 + \hat{r}_2^2 + \hat{r}_3^2 + \ldots + \hat{r}_{k_2}^2$$
$$\pi_3 : |\pi_3| = \hat{r}_1^3 + \hat{r}_2^3 + \hat{r}_3^3 + \ldots + \hat{r}_{k_3}^3$$
$$\pi_4 : |\pi_4| = \hat{r}_1^4 + \hat{r}_2^4 + \hat{r}_3^4 + \ldots + \hat{r}_{k_4}^4$$

with

$$\alpha_0 = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4| - l(\pi_4)$$
$$\alpha_1 = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4| - l(\pi_3).$$

With respect to the above basis, denote the normal form of $A_2 \oplus B_2$ by $A_2^{\{\pi_j\}} \oplus B_2^{\{\pi_j\}}$. The space R_α^N can be stratified by this data, giving

$$[R_\alpha^N] = \sum_{\pi_1,\pi_2,\pi_3,\pi_4} [R(\pi_1,\pi_2,\pi_3,\pi_4)],$$

where $R(\pi_1,\pi_2,\pi_3,\pi_4)$ is the stratum of R_α^N, where $A_2 \oplus B_2$ has normal form $A_2^{\{\pi_j\}} \oplus B_2^{\{\pi_j\}}$. The space $R(\pi_1,\pi_2,\pi_3,\pi_4)$ is a vector bundle

$$p : R(\pi_1,\pi_2,\pi_3,\pi_4) \to \{(A_2,B_2) \mid A_2 \oplus B_2 \sim A_2^{\{\pi_j\}} \oplus B_2^{\{\pi_j\}}\}$$

over the space of all matrices with this normal form, with fibre the linear space of matrices

$$\{B_1 \mid B_1 A_2^{\{\pi_j\}} B_2^{\{\pi_j\}} = B_2^{\{\pi_j\}} A_2^{\{\pi_j\}} B_1\}.$$

We compute the fibre and base by a linear algebra calculation to deduce

$$[R(\pi_1,\pi_2,\pi_3,\pi_4)] = [\mathrm{GL}_{\alpha_0}]\cdot[\mathrm{GL}_{\alpha_1}] f(\pi_1)f(\pi_2)g(\pi_3)g(\pi_4)(-\mathbb{L}^{1/2})^{-(l(\pi_3)-l(\pi_4))^2},$$

where we are given

$$f(\pi) = \prod_{i\geq 1} \mathbb{L}^{b_i^2}/[\mathrm{GL}_{b_i}] \text{ for } \pi = (1^{b_1}2^{b_2}3^{b_3}\cdots),$$

and

$$g(\pi) = \prod_{i\geq 1} (-\mathbb{L}^{1/2})^{b_i^2}/[\mathrm{GL}_{b_i}] \text{ for } \pi = (1^{b_1}2^{b_2}3^{b_3}\cdots).$$

Substituting this into the generating series gives

$$\begin{aligned} N(y_0,y_1) &= \sum_{\alpha_0,\alpha_1\geq 0} \sum_{\substack{\pi_1,\pi_2,\pi_3,\pi_4 \\ \alpha_0=|\pi_1|+|\pi_2|+|\pi_3|+|\pi_4|-l(\pi_4) \\ \alpha_1=|\pi_1|+|\pi_2|+|\pi_3|+|\pi_4|-l(\pi_3)}} f(\pi_1)f(\pi_2)g(\pi_3)g(\pi_4)y_0^{\alpha_0}y_1^{\alpha_1} \\ &= \sum_{\pi_1} f(\pi_1)(y_0y_1)^{|\pi_1|} \sum_{\pi_2} f(\pi_2)(y_0y_1)^{|\pi_2|} \sum_{\pi_3} g(\pi_3)(y_0y_1)^{|\pi_3|}y_1^{-l(\pi_3)} \\ &\quad \cdot \sum_{\pi_4} g(\pi_4)(y_0y_1)^{|\pi_4|}y_0^{-l(\pi_4)}. \end{aligned}$$

The series for f and g have well known formulas [36]

$$\sum_{\pi} f(\pi)y^{|\pi|} = \prod_{i,j=1}^{\infty}(1-\mathbb{L}^{1-j}y^i)^{-1} = \mathrm{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1}\sum_{n\geq 1} y^n\right),$$

and

$$\sum_{\pi} g(\pi)y^{|\pi|}a^{-l(\pi)} = \prod_{i,j=1}^{\infty}(1+(-\mathbb{L}^{1/2})^{-2j+1}y^i a^{-1}) = \mathrm{Exp}\left(\frac{\mathbb{L}^{1/2}}{1-\mathbb{L}}\sum_{n\geq 1} y^n a^{-1}\right).$$

Hence

$$N(y_0, y_1) = \operatorname{Exp}\left(\frac{2\mathbb{L}}{\mathbb{L}-1}\sum_{n\geq 1}(y_0y_1)^n\right)\cdot \operatorname{Exp}\left(\frac{-\mathbb{L}^{1/2}}{\mathbb{L}-1}\sum_{n\geq 1} y_0^n y_1^{n-1} + y_0^{n-1}y_1^n\right).$$

Multiplying the series I and N gives

$$A_U(y_0, y_1) = \operatorname{Exp}\left(\frac{(\mathbb{L}+\mathbb{L}^2)y_0y_1 - \mathbb{L}^{1/2}(y_0+y_1)}{\mathbb{L}-1}\sum_{n\geq 0}(y_0y_1)^n\right).$$

3.3.3 Second proof: another dimensional reduction

Recall that representations of the cut algebra $J_{W,I}$ are given by triples (A_2, B_1, B_2), where $A_2 : V_0 \to V_1$, $B_1, B_2 : V_1 \to V_0$ are linear maps satisfying

$$B_1A_2B_2 = B_2A_2B_1. \tag{3.8}$$

The pair (A_2, B_2) gives a representation of the quiver

$$C^2 = (0, 1; a : 0 \to 1, b : 1 \to 0).$$

Given the dimension vector $\alpha \in \mathbb{N}^2$, let $R(J_{W,I}, \alpha)$ be the space of representations of $J_{W,I}$ having dimension vector α. Let $R(C^2, \alpha)$ be the space of representations of C^2 having dimension vector α. There is a forgetful map

$$g : R(J_{W,I}, \alpha) \to R(C^2, \alpha), \qquad (A_2, B_1, B_2) \to (A_2, B_2).$$

Its fibers are linear vector spaces. This map is equivariant with respect to the natural action of $G_\alpha = \mathrm{GL}_{\alpha_0} \times \mathrm{GL}_{\alpha_1}$ on both sides. Given a C^2-representation $M = (M_0, M_1; M_a, M_b)$, let $\rho(M)$ be the dimension of the fiber of g over M. Let $M^0 = (M_0; M_bM_a)$ and $M^1 = (M_1; M_aM_b)$ be representations of the Jordan quiver C^1 (one vertex and one loop). Then it follows from (3.8) that

$$\rho(M) = \dim \operatorname{Hom}_{C^1}(M^1, M^0).$$

More generally, for any two representations of C^2

$$M = (M_0, M_1; M_a, M_b), \quad N = (N_0, N_1; N_a, N_b)$$

we define

$$\rho(M, N) = \dim \operatorname{Hom}_{C^1}(M^1, N^0).$$

If M is some representation of C^2 having dimension vector α, then the contribution of its G_α-orbit (i.e. isomorphism class) to $[R(C^2, \alpha)]/[G_\alpha]$ is $1/[\operatorname{Aut} M]$. The contribution of the preimage of its G_α-orbit to $[R(J_{W,I}, \alpha)]/[G_\alpha]$ is $\mathbb{L}^{\rho(M)}/[\operatorname{Aut} M]$.

Let $M = \oplus M_i^{n_i}$ be a decomposition of a C^2-representation M into the sum of indecomposable representations. Then by [41, Theorem 1.1]

$$[\operatorname{Aut} M] = [\operatorname{End}(M)] \cdot \prod_i (\mathbb{L}^{-1})_{n_i},$$

where $(q)_n = (q; q)_n = \prod_{k=1}^n (1 - q^k)$ is the q-Pochhammer symbol. Thus the contribution of the preimage of the G_α-orbit of M to $[R(J_{W,I}, \alpha)]/[G_\alpha]$ is

$$\frac{\mathbb{L}^{\rho(M)}}{[\operatorname{Aut} M]} = \frac{\mathbb{L}^{\rho(M,M)-h(M,M)}}{\prod_i (\mathbb{L}^{-1})_{n_i}} = \frac{\prod_{i,j} \mathbb{L}^{n_i n_j (\rho(M_i,M_j)-h(M_i,M_j))}}{\prod_i (\mathbb{L}^{-1})_{n_i}}, \tag{3.9}$$

where $h(M, N) = \dim \operatorname{Hom}(M, N)$ for any C^2-representations M, N. We will compute the numbers $h(M, N)$ and $\rho(M, N)$ for indecomposable representations M, N of C^2. As is well known, the indecomposable representations of C^2 are the following:

1. Representations I_n of dimension $(n, n-1)$, $n \geq 1$.
2. Representations P_n of dimension $(n-1, n)$, $n \geq 1$.
3. Representations $R_{t,n} = (Id_n, J_{t,n})$, $n \geq 1$, $t \in \mathbb{C}$, of dimension (n, n). There are also representations $R_{\infty,n} = (J_{0,n}, Id_n)$, $n \geq 1$, of dimension (n, n). Here $J_{t,n}$ denotes the Jordan block of size n with value t on the diagonal.

Remark 3.5. Define duality on representations of C^2 by

$$D(M_0, M_1; M_{12}, M_{21}) = (M_0^\vee, M_1^\vee; M_{21}^\vee, M_{12}^\vee).$$

Then $\mathrm{Hom}(DM, DN) = \mathrm{Hom}(N, M)^\vee$ and

$$D(I_n) = I_n, \quad D(P_n) = P_n, \quad D(R_{t,n}) = R_{t^{-1},n}.$$

Define equivalence (cyclic shift) by

$$C(M_0, M_1; M_{12}, M_{21}) = (M_1, M_0; M_{21}, M_{12}).$$

Then $\mathrm{Hom}(CM, CN) = \mathrm{Hom}(M, N)$ and

$$C(I_n) = P_n, \quad C(P_n) = I_n, \quad C(R_{t,n}) = R_{t^{-1},n}.$$

The proofs of the following two propositions are easy exercises.

Proposition 3.9. *We have*

1. $h(R_{s,m}, R_{t,n}) = \min\{m, n\}$ *if* $s = t$ *or* $s, t \in \{0, \infty\}$. *It is* 0 *otherwise.*
2. $h(I_m, R_{t,n}) = h(R_{t,n}, P_m) = \begin{cases} \min\{m-1, n\} & t = 0; \\ \min\{m, n\} & t = \infty; \\ 0 & t \in \mathbb{C}^*. \end{cases}$
3. $h(R_{t,n}, I_m) = h(P_m, R_{t,n}) = \begin{cases} \min\{m-1, n\} & t = \infty; \\ \min\{m, n\} & t = 0; \\ 0 & t \in \mathbb{C}^*. \end{cases}$
4. $h(I_m, I_n) = h(P_m, P_n) = \min\{m, n\}$.
5. $h(I_m, P_n) = h(P_n, I_m) = \min\{m, n\} - 1$.

Proposition 3.10. *We have*

1. $\rho(R_{s,m}, R_{t,n}) = \min\{m, n\}$ *if* $s = t$ *or* $s, t \in \{0, \infty\}$. *It is* 0 *otherwise.*

2. $\rho(I_m, R_{t,n}) = \rho(R_{t,n}, P_m) = \begin{cases} \min\{m-1, n\} & t = 0, \infty; \\ 0 & t \in \mathbb{C}^*. \end{cases}$

3. $\rho(R_{t,n}, I_m) = \rho(P_m, R_{t,n}) = \begin{cases} \min\{m, n\} & t = 0, \infty; \\ 0 & t \in \mathbb{C}^*. \end{cases}$

4. $\rho(I_m, I_n) = \rho(P_n, P_m) = \min\{m-1, n\}$.

5. $\rho(I_m, P_n) = \min\{m, n\} - 1$.

6. $\rho(P_n, I_m) = \min\{m, n\}$.

Corollary 3.11. *For any C^2-representations M, N, let*

$$d(M, N) = \rho(M, N) - h(M, N).$$

If M, N are indecomposable, then

$$d(M,N) + d(N,M) = \begin{cases} 1 & M = I_m, N = P_n; \\ -1 - \delta_{m,n} & M = I_m, N = I_n \text{ or } M = P_m, N = P_n; \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 3.8. We can decompose any C^2-representation as

$$M = I \oplus P \oplus \bigoplus_{t \in \mathbb{P}^1} R_t = \bigoplus_{i \geq 1} I_i^{m_i} \oplus \bigoplus_{i \geq 1} P_i^{n_i} \oplus \bigoplus_{t \in \mathbb{P}^1} \bigoplus_{i \geq 1} R_{t,i}^{r_i(t)}.$$

With the representation M we associate partitions $\mu, \eta \in \mathcal{P}$ and $\lambda(t) \in \mathcal{P}$, for $t \in \mathbb{P}^1$, in the following way:

$$\mu_k = \sum_{i \geq k} m_i, \qquad \eta_k = \sum_{i \geq k} n_i, \qquad \lambda_k(t) = \sum_{i \geq k} r_i(t).$$

Applying Corollary 3.11, we obtain

$$
\begin{aligned}
\rho(M,M) - h(M,M) &= \sum_{i,j\geq 1} m_i n_j - \sum_{i\geq j\geq 1}(m_i m_j + n_i n_j)\\
&= -\frac{1}{2}\left(\Big(\sum_{i\geq 1}(m_i - n_i)\Big)^2 + \sum_{i\geq 1} m_i^2 + \sum_{i\geq 1} n_i^2\right), \quad (3.10)
\end{aligned}
$$

an expression that we are going to denote by $d(\mu,\eta)$. The dimension vectors of the summands of M are given by

$$
\begin{aligned}
\underline{\dim}\, I &= \left(\sum_{i\geq 1} i m_i, \sum_{i\geq 1}(i-1)m_i\right) = (|\mu|, |\mu| - \mu_1),\\
\underline{\dim}\, P &= \left(\sum_{i\geq 1}(i-1)n_i, \sum_{i\geq 1} i n_i\right) = (|\eta| - \eta_1, |\eta|),\\
\underline{\dim}\, R_t &= \left(\sum_{i\geq 1} i r_i(t), \sum_{i\geq 1} i r_i(t)\right) = (|\lambda(t)|, |\lambda(t)|).
\end{aligned}
$$

Applying equation (3.9) we obtain

$$
\begin{aligned}
A_U &= \sum_{\alpha\in\mathbb{N}^2} (-\mathbb{L}^{\frac{1}{2}})^{(\alpha_0-\alpha_1)^2} \frac{[R(J_{W,I},\alpha)]}{[G_\alpha]} y^\alpha\\
&= \sum_{\mu,\eta\in\mathcal{P}} (-\mathbb{L}^{\frac{1}{2}})^{(\mu_1-\eta_1)^2} \frac{y_0^{|\mu|+|\eta|-\eta_1} y_1^{|\mu|+|\eta|-\mu_1} \mathbb{L}^{d(\mu,\eta)}}{\prod_{i\geq 1}(\mathbb{L}^{-1})_{\mu_i-\mu_{i+1}}(\mathbb{L}^{-1})_{\eta_i-\eta_{i+1}}} \sum_{\lambda:\mathbb{P}^1\to\mathcal{P}} \prod_{t\in\mathbb{P}^1} f_{\lambda(t)},
\end{aligned}
\quad (3.11)
$$

where

$$
f_\lambda = \frac{(y_0 y_1)^{|\lambda|}}{\prod_{i\geq 1}(\mathbb{L}^{-1})_{\lambda_i-\lambda_{i+1}}}.
$$

By Hua formula's (see [24] or [43, Theorem 6]) applied to the quiver with one loop, we obtain

$$
f = \sum_{\lambda\in\mathcal{P}} f_\lambda = \operatorname{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1}\sum_{n\geq 1}(y_0 y_1)^n\right).
$$

Therefore, using the geometric description of power structures (Section 3.2.2), we obtain

$$\sum_{\lambda:\mathbb{P}^1\to\mathcal{P}}\prod_{t\in\mathbb{P}^1} f_{\lambda(t)} = \operatorname{Pow}(f,[\mathbb{P}^1]) = \operatorname{Exp}\left(\frac{(\mathbb{L}+1)\mathbb{L}}{\mathbb{L}-1}\sum_{n\ge1}(y_0y_1)^n\right).$$

On the other hand it follows from (3.10) that

$$(\mu_1-\eta_1)^2+2d(\mu,\eta) = -\sum_{i\ge1} m_i^2 - \sum_{i\ge1} n_i^2$$

and therefore

$$A_U = \operatorname{Exp}\left(\frac{\mathbb{L}+\mathbb{L}^2}{\mathbb{L}-1}\sum_{n\ge1}(y_0y_1)^n\right)\sum_{\mu,\eta\in\mathcal{P}}\frac{y_0^{|\mu|+|\eta|-\eta_1}y_1^{|\mu|+|\eta|-\mu_1}(-\mathbb{L}^{\frac12})^{-\sum_i(m_i^2+n_i^2)}}{\prod_{i\ge1}(\mathbb{L}^{-1})_{m_i}(\mathbb{L}^{-1})_{n_i}}, \tag{3.12}$$

where we denote $m_i=\mu_i-\mu_{i+1}$, $n_i=\eta_i-\eta_{i+1}$. Define

$$H(x,q^{\frac12}) = \sum_{n\ge0}\frac{(-q^{\frac12})^{-n^2}x^n}{(q^{-1})_n} = \sum_{n\ge1}\frac{(xq^{\frac12})^n}{(q)_n} = \operatorname{Exp}\left(\frac{xq^{\frac12}}{1-q}\right),$$

where the last equality follows from the Heine formula [28, 42]. Then the sum in (3.12) can be written in the form

$$\begin{aligned}
&\sum_{(m_i)_{i\ge1},(n_i)_{i\ge1}}\prod_{i\ge1}\frac{(y_0^iy_1^{i-1})^{m_i}(y_0^{i-1}y_1^i)^{n_i}(-\mathbb{L}^{\frac12})^{-m_i^2-n_i^2}}{(\mathbb{L}^{-1})_{m_i}(\mathbb{L}^{-1})_{n_i}}\\
&\qquad=\prod_{i\ge1}\left(\sum_{m\ge0}\frac{(y_0^iy_1^{i-1})^m(-\mathbb{L}^{\frac12})^{-m^2}}{(\mathbb{L}^{-1})_m}\sum_{n\ge0}\frac{(y_0^{i-1}y_1^i)^n(-\mathbb{L}^{\frac12})^{-n^2}}{(\mathbb{L}^{-1})_n}\right)\\
&=\prod_{i\ge1}H(y_0^iy_1^{i-1},\mathbb{L}^{\frac12})H(y_0^{i-1}y_1^i,\mathbb{L}^{\frac12}) = \operatorname{Exp}\left(\frac{\mathbb{L}^{\frac12}}{1-\mathbb{L}}\sum_{i\ge1}(y_0^iy_1^{i-1}+y_0^{i-1}y_1^i)\right).
\end{aligned}$$

The second proof of Theorem 3.8 is complete. □

3.3.4 Decomposing the universal series

In this section, we decompose the product from Theorem 3.8. We will say that a stability parameter ζ is generic, if for any stable J-module V, we have $\zeta \cdot \underline{\dim} V \neq 0$. For generic stability parameter ζ, let $\mathfrak{M}^{+}_{\zeta}(J,\alpha)$ (resp. $\mathfrak{M}^{-}_{\zeta}(J,\alpha)$) denote the moduli stacks of J-modules V such that $\underline{\dim} V = \alpha$ and such that all the HN factors F of V with respect to the stability parameter ζ satisfy $\zeta \cdot \underline{\dim} F > 0$ (resp. < 0). We put

$$A^{\pm}_{\zeta} = \sum_{\alpha \in \mathbb{N}_{Q_0}} [\mathfrak{M}^{\pm}_{\zeta}(J,\alpha)]_{\mathrm{vir}} \cdot y^{\alpha}.$$

Lemma 3.12. *The generating series $A^{\pm}_{\zeta}$ are given by*

$$A^{\pm}_{\zeta} = \prod_{\substack{\alpha \in \Delta_+ \\ \pm\zeta\cdot\alpha < 0}} A^{\alpha},$$

where $A^{\alpha} = A^{\alpha}(y_0, y_1)$ were defined in Theorem 3.8. We have

$$A_U = A^{+}_{\zeta} A^{-}_{\zeta}.$$

Proof. By Theorem 3.7, we have a factorization in $\mathcal{T}_Q$ (note that $\mathcal{T}_Q$ is commutative and we don't need to take the ordered product)

$$A_U = \prod_{\mu \in \mathbb{R}} A_{\zeta,\mu},$$

where $A_{\zeta,\mu}$ were defined in Definition 3.7. Similarly we have $A^{\pm}_{\zeta} = \prod_{\pm\mu>0} A_{\zeta,\mu}$.

By Theorem 3.8, we have

$$A_U = \prod_{\alpha \in \Delta_+} A^{\alpha},$$

where A^{α} contain only powers $y^{k\alpha}$, $k \geq 0$. By the uniqueness of the factorizations from Lemma 3.5, we obtain

$$A_{\zeta,\mu} = \prod_{\substack{\alpha \in \Delta_+ \\ \mu(\alpha)=\mu}} A^{\alpha}$$

and the statement of the lemma follows. ☐

3.4 Motivic DT with Framing

3.4.1 Framed quiver

Let Q be a quiver with a distinguished vertex $0 \in Q_0$ and let W be a potential. We denote by $\widetilde{Q}$ the corresponding framed quiver, the new quiver obtained from Q by adding a new vertex ∞ and a single new arrow $\infty \to 0$. Let $\widetilde{J} = J_{\widetilde{Q},W}$ be the Jacobian algebra corresponding to the quiver with potential $(\widetilde{Q}, W)$, where we view W as a potential for $\widetilde{Q}$ in the obvious way. Any $\widetilde{Q}$-representation (resp. $\widetilde{J}$-module) $\widetilde{V}$ can be written as a triple $(V, \widetilde{V}_\infty, s)$, where V is a Q-representation (resp. J-module), $\widetilde{V}_\infty$ is a vector space, and $s : \widetilde{V}_\infty \to V_0$ is a linear map. We will always do this identification without mentioning.

The twisted motivic algebra $\mathcal{T}_Q$ of the original quiver sits as a subalgebra inside the algebra $\mathcal{T}_{\widetilde{Q}}$ associated to the framed quiver $\widetilde{Q}$. Note that in $\mathcal{T}_{\widetilde{Q}}$ we have

$$y_\infty \cdot y^{(\alpha,0)} = (-\mathbb{L}^{\frac{1}{2}})^{-\alpha_0} \cdot y^{(\alpha,1)} = \mathbb{L}^{-\alpha_0} \cdot y^{(\alpha,0)} \cdot y_\infty, \tag{3.13}$$

where we put

$$y_\infty = y^{(0,1)}.$$

In particular, $\mathcal{T}_{\widetilde{Q}}$ is never commutative.

3.4.2 Stability for framed representations

Let $\zeta \in \mathbb{R}^{Q_0}$ be a vector, which we will refer to as the stability parameter.

Definition 3.8. A $\widetilde{Q}$-representation (resp. $\widetilde{J}$-module) $\widetilde{V}$ with $\dim \widetilde{V}_\infty = 1$ is said to be ζ-(semi)stable, if it is (semi)stable with respect to $(\zeta, \zeta_\infty) \in \mathbb{R}^{\widetilde{Q}}$ (see Definition 3.2), where $\zeta_\infty = -\zeta \cdot \underline{\dim}\, V$. Equivalently, the following conditions should be satisfied:

- for any $\widetilde{Q}$-subrepresentation (resp. $\widetilde{J}$-submodule) $0 \neq \widetilde{V}' \subset \widetilde{V}$ with $\widetilde{V}'_\infty = 0$, we have
$$\zeta \cdot \underline{\dim}\, V' \,(\leq)\, 0;$$

- for any $\widetilde{Q}$-quotient representation (resp. $\widetilde{J}$-quotient module) $\widetilde{V} \twoheadrightarrow \widetilde{V}'' \neq 0$ with $\widetilde{V}''_\infty = 0$, we have

$$\zeta \cdot \underline{\dim}\, V'' \,(\geq)\, 0.$$

As in Section 3.3.4, a stability parameter $\zeta \in \mathbb{R}^{Q_0}$ is said to be *generic*, if for any stable J-module V we have $\zeta \cdot \underline{\dim}\, V \neq 0$.

3.4.3 Motivic DT invariants with framing

For a stability parameter $\zeta \in \mathbb{R}^{Q_0}$ and a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, let as before $\zeta_\infty = -\zeta \cdot \alpha$, $\widetilde{\alpha} = (\alpha, 1)$, and let

$$\mathfrak{M}_\zeta(\widetilde{Q}, \alpha) = [R_{(\zeta,\zeta_\infty)}(\widetilde{Q}, \widetilde{\alpha})/G_\alpha], \qquad \mathfrak{M}_\zeta(\widetilde{J}, \alpha) = [R_{(\zeta,\zeta_\infty)}(\widetilde{J}, \widetilde{\alpha})/G_\alpha]$$

denote the moduli stack of ζ-stable $\widetilde{Q}$-representations (resp. $\widetilde{J}$-modules) with dimension vector $\widetilde{\alpha}$. The corresponding stacks for the trivial stability $\zeta = 0$ will be denoted by $\mathfrak{M}(\widetilde{Q}, \alpha)$ and $\mathfrak{M}(\widetilde{J}, \alpha)$.

Remark 3.6. Note that the stack $\mathfrak{M}_\zeta(\widetilde{Q}, \alpha)$ is slightly different from the stack $\mathfrak{M}_{(\zeta,\zeta_\infty)}(\widetilde{Q}, \widetilde{\alpha})$ which was defined in (3.1) to be $[R_{(\zeta,\zeta_\infty)}(\widetilde{Q}, \widetilde{\alpha})/G_{\widetilde{\alpha}}]$. The same applies to the stacks of $\widetilde{J}$-modules.

Definition 3.9. Let

$$\widetilde{A}_U = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}\big(\widetilde{J}, \alpha\big)\right]_{\text{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}},$$

where $\left[\mathfrak{M}\big(\widetilde{J}, \alpha\big)\right]_{\text{vir}}$ is defined similarly to (3.2). For any stability parameter $\zeta \in \mathbb{R}^{Q_0}$ let

$$\widetilde{A}_\zeta = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}_\zeta\big(\widetilde{J}, \alpha\big)\right]_{\text{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}},$$

where $\left[\mathfrak{M}_\zeta\big(\widetilde{J}, \widetilde{\alpha}\big)\right]_{\text{vir}}$ is defined similarly to (3.3). Let also, as in the Introduction,

$$Z_\zeta = \sum_{\alpha \in \mathbb{N}^{Q_0}} \left[\mathfrak{M}_\zeta\big(\widetilde{J}, \alpha\big)\right]_{\text{vir}} \cdot y^{\alpha} \in \mathcal{T}_Q.$$

3.4.4 Relating the universal and framed series

In this subsection we assume that Q is a symmetric quiver and therefore $\mathcal{T}_Q$ is commutative. The following theorem relates results of the previous section on the universal series to the framed invariants of this section.

Theorem 3.13. *For generic stability parameter ζ, we have*

$$Z_\zeta = \frac{A_\zeta^-(-\mathbb{L}^{\frac{1}{2}}y_0, y_1, \dots)}{A_\zeta^-(-\mathbb{L}^{-\frac{1}{2}}y_0, y_1, \dots)}, \tag{3.14}$$

where A_ζ^- were defined in Section 3.3.4.

This result is [45, Corollary 4.17]. In the rest of this subsection, we provide an alternative approach to this theorem. The main difference is that here we study just two stability parameters, while the result in [45] was obtained by studying an infinite sequence of parameters between these two.

Proposition 3.14. *Let $\widetilde{V}$ be a $\widetilde{Q}$-representation (resp. a $\widetilde{J}$-module) with $\dim \widetilde{V}_\infty = 1$. Then there exists the unique filtration*

$$0 = \widetilde{U}^0 \subset \widetilde{U}^1 \subset \widetilde{U}^2 \subset \widetilde{U}^3 = \widetilde{V}$$

such that with $\widetilde{V}^i = \widetilde{U}^i / \widetilde{U}^{i-1}$ we have

1. *$\widetilde{V}^1_\infty = 0$ and all the HN factors F of V^1 with respect to the stability parameter ζ satisfy $\zeta \cdot \underline{\dim} F > 0$,*

2. *$\widetilde{V}^2_\infty = 1$ and $\widetilde{V}^2$ is ζ-semistable,*

3. *$\widetilde{V}^3_\infty = 0$ and all the HN factors F of V^3 with respect to the stability parameter ζ satisfy $\zeta \cdot \underline{\dim} F < 0$.*

Proof. We will work only with $\widetilde{Q}$-representations. We take sufficiently small $\varepsilon > 0$ and define the central charge

$$Z_{\zeta,\varepsilon}(\widetilde{\alpha}) = -\zeta \cdot \alpha + (\varepsilon|\alpha| + \alpha_\infty)\sqrt{-1}, \qquad \widetilde{\alpha} = (\alpha, \alpha_\infty).$$

Let $\widetilde{W}$ be a $\widetilde{Q}$-representation with $\dim \widetilde{W}_\infty = 1$. For any submodule $\widetilde{W}' = W'$ of $\widetilde{W}$ with $\widetilde{W}'_\infty = 0$, we have

$$\zeta \cdot \underline{\dim}\, W' \gtrless 0 \iff \arg Z_{\zeta,\varepsilon}(\underline{\dim}\, \widetilde{W}') \gtrless \arg Z_{\zeta,\varepsilon}(\underline{\dim}\, \widetilde{W}).$$

Hence $\widetilde{W}$ is $Z_{\zeta,\varepsilon}$-stable if and only if it is ζ-stable. Then, the Harder-Narashimhan filtration for $Z_{\zeta,\varepsilon}$-stability is the required filtration. □

The filtration from Proposition 3.14 induces the following factorization in the same way as Theorem 3.7:

Proposition 3.15. *We have*

$$\widetilde{A}_U = A_\zeta^+ \cdot \widetilde{A}_\zeta \cdot A_\zeta^-$$

in the motivic algebra $\mathcal{T}_{\widetilde{Q}}$.

Proposition 3.16.

$$\widetilde{A}_U = A_U \cdot y_\infty.$$

Proof. Any $\widetilde{Q}$-module (resp. $\widetilde{J}$-module) $\widetilde{V}$ with $\dim \widetilde{V}_\infty = 1$ and $\underline{\dim}\, V = \alpha$ has a unique filtration

$$0 \subset V \subset \widetilde{V}$$

with

$$\widetilde{V}/V \simeq S_\infty,$$

the simple module concentrated at the vertex ∞. Thus the factorization follows. □

Proof of Theorem 3.13. We have

$$\begin{aligned}
\widetilde{A}_\zeta &= (A_\zeta^+)^{-1} \cdot \widetilde{A}_U \cdot (A_\zeta^-)^{-1} && \text{(Proposition 3.15)} \\
&= (A_\zeta^+)^{-1} \cdot (A_\zeta^+ \cdot A_\zeta^- \cdot y_\infty) \cdot (A_\zeta^-)^{-1} && \text{(Prop. 3.16 and Lemma 3.12)} \\
&= y_\infty \cdot \frac{A_\zeta^-(\mathbb{L}y_0, y_1, \dots)}{A_\zeta^-(y_0, y_1, \dots)}. && \text{(Equation (3.13))}
\end{aligned} \tag{3.15}$$

It follows from (3.13) that $y_\infty \cdot Z_\zeta(-\mathbb{L}^{\frac{1}{2}} y_0, \dots) = \widetilde{A}_\zeta$. Combining this with (3.15) we get the statement of Theorem 4.21. □

3.4.5 Application to the conifold

The following theorem is the main result of this section, announced as Theorem 3.1. Let (Q, W) be the conifold quiver with potential.

Theorem 3.17. *For generic* $\zeta \in \mathbb{R}^2$*,*

$$Z_\zeta(y_0, y_1) = \prod_{\substack{\alpha \in \Delta_+ \\ \zeta \cdot \alpha < 0}} Z_\alpha(y_0, y_1), \tag{3.16}$$

with

$$Z_\alpha(-y_0, y_1) = \begin{cases} \prod_{j=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + \frac{1}{2} + j} y^\alpha\right) & \alpha \in \Delta_+^{\mathrm{re}} \\ \prod_{j=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + 1 + j} y^\alpha\right)^{-1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + 2 + j} y^\alpha\right)^{-1} & \alpha \in \Delta_+^{\mathrm{im}} \end{cases}$$

Proof. Substituting the result of Lemma 3.12 into Theorem 3.13, we get the product form (3.16), with

$$Z_\alpha(y_0, y_1) = A^\alpha(-\mathbb{L}^{\frac{1}{2}} y_0, y_1) / A^\alpha(-\mathbb{L}^{-\frac{1}{2}} y_0, y_1).$$

Now use the expression for A^α from Theorem 3.8. □

3.5 DT/PT Series

3.5.1 Chambers and the moduli spaces for the conifold

Let (Q, W) be the conifold quiver with potential. In the space $\mathbb{R}^2$ of stability parameters, consider the lines

$$\begin{aligned} L_+(m) &= \{(\zeta_0, \zeta_1) \mid m\,\zeta_0 + (m-1)\zeta_1 = 0\} \quad (m \geq 1), \\ L_\infty &= \{(\zeta_0, \zeta_1) \mid \zeta_0 + \zeta_1 = 0\}, \\ L_-(m) &= \{(\zeta_0, \zeta_1) \mid m\,\zeta_0 + (m+1)\zeta_1 = 0\} \quad (m \geq 0). \end{aligned}$$

It is immediately seen that these are exactly the lines orthogonal to the roots in Δ_+ with respect to the standard inner product. Let $L \subset \mathbb{R}^2$ denote the union of this countable set of lines. The complement of L in $\mathbb{R}^2$ is a countable union of open cones. Denote by Y^+ the flop of Y along the embedded rational curve.

Theorem 3.18. [46, Lemma 3.1 and Propositions 2.10-2.13] *The set of generic parameters in $\mathbb{R}^2$ is the complement of the union L of the lines defined above.*

1. *For ζ with $\zeta_0 < 0$ and $\zeta_1 < 0$, the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the NCDT moduli spaces, the moduli spaces of cyclic J-modules from [58].*

2. *For ζ near the line L_∞ with $\zeta_0 < \zeta_1$ and $\zeta_0 + \zeta_1 < 0$, the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the commutative DT moduli spaces of Y from [37], the moduli spaces of subschemes on Y with support in dimension at most 1.*

3. *For ζ near the line L_∞ with $\zeta_0 < \zeta_1$ and $\zeta_0 + \zeta_1 > 0$, the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the PT moduli spaces of Y introduced in [53]; these are moduli spaces of stable rank-1 coherent systems.*

4. *For ζ near the line L_∞ with $\zeta_0 > \zeta_1$ and $\zeta_0 + \zeta_1 < 0$, the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the commutative DT moduli spaces of the flop Y^+.*

5. *For ζ near the line L_∞ with $\zeta_0 < \zeta_1$ and $\zeta_0 + \zeta_1 > 0$, the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the PT moduli spaces of the flop Y^+.*

6. *For ζ with $\zeta_0 > 0$ and $\zeta_1 > 0$, the moduli space $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ consists of a point for $\alpha = 0$ and is otherwise empty.*

Remark 3.7. Note that "near" in the above statements means sufficiently near depending on the dimension vector $(\alpha, 1)$.

3.5.2 Motivic PT and DT invariants

Proposition 3.19. *The refined partition functions of the resolved conifold Y for the DT and PT chambers are given by*

$$Z_{\mathrm{PT}}(-y_0, y_1) = \prod_{m\geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+\frac{1}{2}+j} y_0^m y_1^{m-1}\right) \tag{3.17}$$

and

$$Z_{\mathrm{DT}}(-y_0, y_1) = Z_{\mathrm{PT}}(-y_0, y_1) \cdot \prod_{m\geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+1+j} y_0^m y_1^m\right)^{-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+2+j} y_0^m y_1^m\right)^{-1}. \tag{3.18}$$

Proof. Let $\zeta = (-1+\varepsilon, 1)$, $0 < \varepsilon \ll 1$, be some stability corresponding to PT moduli spaces. Then

$$\{\alpha \in \Delta_+ \mid \zeta \cdot \alpha < 0\} = \{(m, m-1) \mid m \geq 1\}.$$

Applying Theorem 3.17 we obtain

$$Z_{PT}(-y_0, y_1) = \prod_{\zeta\cdot\alpha<0} Z_\alpha(-y_0, y_1) = \prod_{m\geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{-\frac{m}{2}+\frac{1}{2}+j} y_0^m y_1^{m-1}\right).$$

The proof of the second formula is similar. □

Let us re-write these formulae in the perhaps more familiar large radius parameters $T = y_1^{-1}$, $s = y_0 y_1$, corresponding to the cohomology class of a point and a curve on the geometry Y. We obtain

$$Z_{\mathrm{PT}}(-s,T) = \prod_{m\geq 1}\prod_{j=0}^{m-1}\left(1-\mathbb{L}^{-\frac{m}{2}+\frac{1}{2}+j}s^m T\right) \tag{3.19}$$

and

$$Z_{\mathrm{DT}}(-s,T) = Z_{\mathrm{PT}}(-s,T)\cdot\prod_{m\geq 1}\prod_{j=0}^{m-1}\left(1-\mathbb{L}^{-\frac{m}{2}+1+j}s^m\right)^{-1}\left(1-\mathbb{L}^{-\frac{m}{2}+2+j}s^m\right)^{-1}. \tag{3.20}$$

The specializations at $\mathbb{L}^{\frac{1}{2}} = 1$ are the PT and DT series of the resolved conifold respectively, given by the standard expressions

$$\overline{Z}_{\mathrm{PT}}(-s,T) = \prod_{m\geq 1}(1-Ts^m)^m = \mathrm{Exp}\left(\frac{-T}{(s^{\frac{1}{2}}-s^{-\frac{1}{2}})^2}\right)$$

and

$$\overline{Z}_{\mathrm{DT}}(-s,T) = M(s)^2\prod_{m\geq 1}(1-Ts^m)^m,$$

with $M(s) = \prod_{m\geq 1}(1-s^m)^{-m}$ the MacMahon function, and $\overline{}$ denoting generating series of numerical (as opposed to motivic) invariants.

Wall crossing at the special wall L_∞ is the PT/DT wall crossing of [53]. On the PT side, the coefficient of the T^0 term is just 1, since if there is no curve present, the only possible PT pair consists of the structure sheaf of Y (with zero map). On the DT side, the moduli space with zero curve class is the moduli space of ideal sheaves of point clusters on Y, in other words the Hilbert scheme of points of Y. Hence the ratio of the T^0 terms gives the generating function of virtual motives of the Hilbert scheme of points of Y [4]:

$$\sum_{n=0}^{\infty}[Y^{[n]}]_{\mathrm{vir}}(-s)^n = \prod_{m\geq 1}\prod_{j=0}^{m-1}\left(1-\mathbb{L}^{1+j-\frac{m}{2}}s^m\right)^{-1}\left(1-\mathbb{L}^{2+j-\frac{m}{2}}s^m\right)^{-1}.$$

At $\mathbb{L}^{\frac{1}{2}} = 1$, we obtain the MNOP result $M(s)^2$. Note in particular that, as proved in [4] but contrary to the speculations of [17], the motivic refinement is not a square, though both products are combinatorial refinements of the usual MacMahon series.

Remark 3.8. Note that our results in fact imply a full factorization

$$Z_{\mathrm{DT}}(s,T) = \left(\sum_{n=0}^{\infty} [Y^{[n]}]_{\mathrm{vir}} s^n\right) Z_{\mathrm{PT}}(s,T), \tag{3.21}$$

with the middle sum being a product of refined MacMahon series as in [4]. This is a motivic analogue of the factorization

$$\overline{Z}_{X,\mathrm{DT}}(s,T) = M(s)^{e(X)} \overline{Z}_{X,\mathrm{PT}}(s,T) \tag{3.22}$$

conjectured for a quasi-projective Calabi–Yau threefold X in [37, 53], proved in [9] following earlier proofs of a version of this statement in [57, 61]. In general, the only definition we have of the motivic $Z_{X,\mathrm{DT}}$ and $Z_{X,\mathrm{PT}}$ is through the partially conjectural setup of [33]. Assuming that relevant parts of [33] are put on a firm footing, including the integration ring homomorphism from the motivic Hall algebra to the motivic quantum torus, it seems likely that the proof of [9] can be adapted to prove the motivic version (3.21) in general.

3.5.3 Connection with the refined topological vertex

The standard way to compute the unrefined PT series of the resolved conifold Y is via the topological vertex [1]. From the toric combinatorics, we obtain the formula

$$\overline{Z}_{\mathrm{PT}}^{\mathrm{vertex}}(-s,T) = \sum_{\lambda} \overline{C}_{\lambda\emptyset\emptyset}(s) \overline{C}_{\lambda^t\emptyset\emptyset}(s)(-T)^{|\lambda|},$$

see e.g. [26, (63)]. On the right hand side, the sum runs over all partitions; for a partition λ, λ^t denotes the conjugate partition, and $\overline{C}_{\lambda\mu\nu}(s)$ is the topological vertex expression of [1]. In the case when $\mu = \nu = \emptyset$, $\overline{C}_{\lambda\emptyset\emptyset}(s)$ can be expressed as a simple Schur function, and then Cauchy's identity immediately gives

$$\overline{Z}_{\rm PT}^{\rm vertex}(-s,T) = \prod_{m\geq 1}(1-Ts^m)^m = \overline{Z}_{\rm PT}(-s,T).$$

In mathematical terms [37], this equality (or rather its DT version) expresses torus localization, the combined expression $M(s)\overline{C}_{\lambda\mu\nu}(s)$ being the generating function of 3-dimensional partitions with given 2-dimensional asymptotics along the coordinate axes.

The refined PT partition function as computed by the refined topological vertex is [26, (67)]

$$Z_{\rm PT}^{\rm vertex}(t,q,T) = \sum_{\lambda} C_{\lambda\emptyset\emptyset}(q,t)C_{\lambda^t\emptyset\emptyset}(t,q)(-T)^{|\lambda|},$$

where $C_{\lambda\mu\nu}(q,t)$ is now the refined topological vertex expression. Using the Cauchy identity again, this sum reduces to [26, (67)]

$$\begin{aligned} Z_{\rm PT}^{\rm vertex}(t,q,T) &= \prod_{i,j\geq 1}\left(1-Tq^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right) \\ &= \exp\left(\sum_{n\geq 1}\frac{-T^n}{n(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})}\right) \\ &= \operatorname{Exp}\left(\frac{-T}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(t^{\frac{1}{2}}-t^{-\frac{1}{2}})}\right). \end{aligned} \tag{3.23}$$

Proposition 3.20. *We have*

$$Z_{\rm PT}^{\rm vertex}(t,q,T) = Z_{PT}(-s,T),$$

when we make the change of variables $q=\mathbb{L}^{\frac{1}{2}}s$, $t=\mathbb{L}^{-\frac{1}{2}}s$, *with* $(qt)^{\frac{1}{2}}=s$.

Proof. This is immediate when we compare (3.19) with (3.23). □

Thus we obtain a proof of the "motivic=refined" correspondence [17] in this example. Note however that the situation is very different from the unrefined story: here we are not giving a full mathematical interpretation of the refined topological vertex expression $C_{\lambda\mu\nu}(q,t)$; indeed, it remains a very interesting problem to find one. We are only checking that the results agree in the particular case of the resolved conifold.

3.5.4 Connection to the cohomological Hall algebra

An alternative to considering the refined motivic invariants is to consider the mixed Hodge modules of vanishing cycles [16, 32]. The description as the vanishing locus of the trace of the potential endows the moduli spaces $\mathfrak{M}_\zeta\big(\widetilde{J}, \widetilde{\alpha}\big)$ with the mixed Hodge modules of vanishing cycles of the trace function. Recently, the cohomologies of the moduli spaces with coefficients in these mixed Hodge modules have been organized into an algebra in [32], the (critical) cohomological Hall algebra. Replacing $\mathbb{L}$ by q in all our formulae, we obtain generating series of E-polynomials of these mixed Hodge modules, the analogues of the formulae of [16] in our situation.

Chapter 4

Local Toric Examples

This chapter is joint work with Prof. Kentaro Nagao [40]. Here we generalize the work of the previous chapter to compute the motivic Donaldson–Thomas theory of small crepant resolutions of a toric Calabi–Yau 3-folds.

4.1 Introduction

As mentioned this Chapter is a continuation of [39]. We recall that a *Donaldson–Thomas* (DT) *invariant* of a Calabi–Yau 3-fold Y is a counting invariant of coherent sheaves on Y, introduced in [60] as a holomorphic analogue of the Casson invariant of a real 3-manifold. A component of the moduli space of stable coherent sheaves on Y carries a symmetric obstruction theory and a virtual fundamental cycle [6, 7]. A DT invariant of a compact Y is then defined as the integral of the constant function 1 over the virtual fundamental cycle of the moduli space.

It is known that the moduli space of coherent sheaves on Y can be locally described as the critical locus of a function, the *holomorphic Chern–Simons functional* (see [27]). Behrend provided a description of DT invariants in terms of the Euler characteristic of the *Milnor fiber* of the CS functional [2]. Inspired by this result, the proposal of [33, 4] was to study the *motivic Milnor fiber* of the CS functional as a motivic refinement of the DT invariant. Such a refinement had been expected in string theory [26, 17].

On the other hand, in [58], it was proposed to study counting invariants for the non-commutative crepant resolution (NCCR) of the conifold, which are called non-commutative Donaldson–Thomas (ncDT) invariants. It was also conjectured there that ncDT and DT invariants are related by wall crossing. The paper [46] realized this, by

- describing the chamber structure on the space of stability parameters for the NCCR,
- finding chambers which correspond to geometric DT and stable pair (PT), as well as ncDT invariants, and
- computing the generating function of DT type invariants for each chamber.

For the conifold, the dimension of the fiber of the crepant resolution is less than 2 (we say that the resolution is small). This condition plays an important role in many places of the paper. Affine toric Calabi–Yau 3-folds which have small crepant resolutions are classified as follows:

1. $\mathcal{X} = \mathcal{X}_{N_0,N_1} := \{XY - Z^{N_0}W^{N_1}\}$ for $N_0 > 0$ and $N_1 \geq 0$, or
2. $\mathcal{X} = \mathcal{X}_{(\mathbb{Z}/2\mathbb{Z})^2} := \mathbb{C}^3/(\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2$ acts on $\mathbb{C}^3$ with weights $(1,0)$, $(0,1)$ and $(1,1)$.

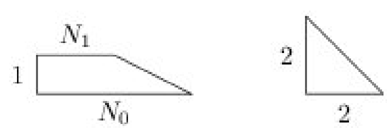

Figure 4.1: Polygons for $\mathcal{X}_{N_0,N_1}$ and $\mathcal{X}_{(\mathbb{Z}/2\mathbb{Z})^2}$

In [47], counting invariants for non-commutative and commutative crepant resolutions of $\{XY - Z^{N_0}W^{N_1}\}$ were studied. First, we provided descriptions of NCCRs of $\{XY - Z^{N_0}W^{N_1}\}$ in terms of a quiver with potential. Given N_0 and N_1, the quivers with potential are not unique. However it was also shown that any such quivers with potential are related by a sequence of mutations. Finally, generalizations of the results in [46] are given.

In [39], we provided motivic refinements of formulae in [46]. For the proof, we needed one explicit evaluation of the "universal" series ([39, §2]) and a wall-crossing argument ([39, §3]).

In this chapter, we will show similar formulae for $\{XY - Z^{N_0}W^{N_1}\}$, that is, motivic refinements of the formulae in [47]. The wall-crossing argument works without modifications (§4.7, 4.8), while the evaluation part is more involved (Theorem 4.1). Our strategy is as follows:

- First, in §4.5, we evaluate the universal series for a specific NCCR using a generalization of the calculation [39, §2.2].
- Then, in §4.6, we evaluate the universal series for a general NCCR. In [51], Nagao has provided a formula which describes how the universal series changes under mutation (§4.7, 4.8). Although we assume that the quiver has no loops and 2-cycles in [51], we can apply a parallel argument in our setting as well.

Since any two NCCRs are related by a sequence of mutations, the evaluation is done.

4.1.1 Main result

Let Γ be the quadrilateral (or the triangle in case $N_1 = 0$) as in Figure 4.1 and σ be a partition Γ, that is, a division of Γ into N-tuples of triangles with area 1/2. We will associate σ with a quiver with potential $(Q_\sigma, \omega_\sigma)$. The set of vertices of the quiver Q_σ is $\hat{I} := \mathbb{Z}/N\mathbb{Z}$, which is identified with $\{0, \ldots, N-1\}$. A vertex has a loop if and only if it is in the subset $\hat{I}_r \subset \hat{I}$ (see (4.1) for the definition). It is shown in [47, §1] that the Jacobian algebra $J_\sigma := J(Q_\sigma, \omega_\sigma)$ is an NCCR of

$$\mathcal{X} := \operatorname{Spec}\left(\mathbb{C}[X, Y, Z, W]/(XY - Z^{N_0}W^{N_1})\right).$$

Let Δ be the set of roots of type $\hat{A}_N$ and $\Delta_{\sigma,+}$ (resp. $\Delta^{\mathrm{re}}_{\sigma,+}$, $\Delta^{\mathrm{im}}_{\sigma,+}$) denote the set of positive (resp. positive real, positive imaginary) roots. [1]

For $\alpha \in \mathbb{N}^{\hat{I}}$, let $\mathfrak{M}(J_\sigma, \alpha)$ be the moduli stack of J_σ-modules V with $\underline{\dim}\, V = \alpha$. We define the generating series of the motivic DT invariants of

[1]From the view point of the root system, a choice of a partition σ corresponds to a choice of a set of simple roots.

(Q_σ, W_σ) by

$$A_U^\sigma(y) = A_U^\sigma(y_0, \ldots, y_{N-1}) := \sum_{\alpha \in \mathbb{N}^{Q_0}} [\mathfrak{M}(J_\sigma, \alpha)]_{\mathrm{vir}} \cdot y^\alpha \in \mathcal{M}_{\mathbb{C}}[[y_0, \ldots, y_{N-1}]].^2$$

Here $y^\alpha := \prod (y_i)^{\alpha_i}$ and $[\bullet]_{\mathrm{vir}}$ denotes the *virtual motive* (see Section 4.4.1), an element of a suitable ring of motives $\mathcal{M}_{\mathbb{C}}$. The subscript referring to the fact that we think of this series as the universal series.

To each root $\alpha \in \Delta_{\sigma,+}$, we associate an infinite product as follows:

- for a real root $\alpha \in \Delta_{\sigma,+}^{\mathrm{re}}$ such that $\sum_{k \notin \hat{I}_r} \alpha_k$ is odd, put

$$\begin{aligned} A^\alpha(y) &:= \mathrm{Exp}\left(\frac{-\mathbb{L}^{-1/2}}{1 - \mathbb{L}^{-1}} y^\alpha \right) \\ &= \prod_{j \geq 0} \left(1 - \mathbb{L}^{-j-1/2} y^\alpha \right) \end{aligned}$$

- for a real root $\alpha \in \Delta_{\sigma,+}^{\mathrm{re}}$ such that $\sum_{k \notin \hat{I}_r} \alpha_k$ is even, put

$$\begin{aligned} A^\alpha(y) &:= \mathrm{Exp}\left(\frac{1}{1 - \mathbb{L}^{-1}} y^\alpha \right) \\ &= \prod_{j \geq 0} \left(1 - \mathbb{L}^{-j} y^\alpha \right)^{-1} \end{aligned}$$

- for an imaginary root $\alpha \in \Delta_{\sigma,+}^{\mathrm{im}}$, put

$$\begin{aligned} A^\alpha(y) &:= \mathrm{Exp}\left(\frac{N - 1 + \mathbb{L}}{1 - \mathbb{L}^{-1}} y^\alpha \right) \\ &= \prod_{j \geq 0} \left(1 - \mathbb{L}^{-j} y^\alpha \right)^{1-N} \cdot \left(1 - \mathbb{L}^{-j+1} y^\alpha \right)^{-1}. \end{aligned}$$

The main result of this paper is the following formula:

[2] For the wall-crossing of motivic DT theory, a twisted product on y_α's twisted by the Euler form plays a crucial role. In this case, the twisted product coincides with the usual commutative product since the Euler form is trivial.

Theorem 4.1.

$$A_U^\sigma(y) = \prod_{\alpha \in \Delta_{\sigma,+}} A^\alpha(y).$$

This is proved in §4.5 and §4.6.2.

4.1.2 Corollaries

Let $\widetilde{J}_\sigma = J(\widetilde{Q}_\sigma, W_\sigma)$ be the framed algebra given by adding the new vertex ∞ and the new arrow from ∞ to 0 to the quiver of J_σ. In [46], the authors introduce a notion of ζ-(semi)stability of $\widetilde{J}$-modules $\widetilde{V}$ with $\dim \widetilde{V}_\infty \leq 1$ for a stability parameter $\zeta \in \mathbb{R}^{\hat{I}}$.

For $\alpha \in \mathbb{N}^{\hat{I}}$, let $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ be the moduli space of ζ-stable $\widetilde{J}$-modules $\widetilde{V}$ with $\underline{\dim}\, \widetilde{V} = (\alpha, 1)$. We want to compute the motivic generating series

$$Z_\zeta(y) = Z_\zeta(y_0, \ldots, y_{N-1}) := \sum_{\alpha \in \mathbb{N}^{\hat{I}}} \Big[\mathfrak{M}_\zeta\big(\widetilde{J}, \alpha\big)\Big]_{\mathrm{vir}} \cdot y^\alpha \in \mathcal{M}_{\mathbb{C}}[[y_0, \ldots, y_{N-1}]].$$

To each root $\alpha \in \Delta_{\sigma,+}$, we put

$$Z_\alpha(y_0, \ldots, y_{N-1}) := \frac{A^\alpha(-\mathbb{L}^{1/2} y_0, y_1, \ldots, y_{N-1})}{A^\alpha(-\mathbb{L}^{-1/2} y_0, y_1, \ldots, y_{N-1})}.$$

They are given as follows:

- for a real root $\alpha \in \Delta^{\mathrm{re}}_{\sigma,+}$ such that $\sum_{k \notin \hat{I}_r} \alpha_k$ is odd, we have

$$Z_\alpha(-y_0, \ldots, y_{N-1}) = \prod_{i=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + \frac{1}{2} + i} y^\alpha\right),$$

- for a real root $\alpha \in \Delta^{\mathrm{re}}_{\sigma,+}$ such that $\sum_{k \notin \hat{I}_r} \alpha_k$ is even, we have

$$Z_\alpha(-y_0, \ldots, y_{N-1}) = \prod_{i=0}^{\alpha_0 - 1} \left(1 - \mathbb{L}^{-\frac{\alpha_0}{2} + 1 + i} y^\alpha\right)^{-1},$$

- for an imaginary root $\alpha \in \Delta^{\mathrm{im}}_{\sigma,+}$, we have

$$Z_\alpha(-y_0,\dots,y_{N-1}) = \prod_{i=0}^{\alpha_0-1} \left(1-\mathbb{L}^{-\frac{\alpha_0}{2}+1+i}y^\alpha\right)^{1-N}\cdot\left(1-\mathbb{L}^{-\frac{\alpha_0}{2}+2+j}y^\alpha\right)^{-1}$$

Applying the same argument as [39, §3], we get the following formula (§4.7):

Corollary 4.2. *For $\zeta \in \mathbb{R}^{\tilde{I}}$ not orthogonal to any root, we have*

$$Z_\zeta(y) = \prod_{\substack{\alpha\in\Delta_{\sigma,+}\\ \zeta\cdot\alpha<0}} Z_\alpha(y_0,\dots,y_{N-1}).$$

By [2, 4], the specialization $Z_\zeta(y)|_{\mathbb{L}^{\frac{1}{2}}\to 1}$ is the DT-type series at the generic stability parameter ζ, computed in [47].

Let $\mathcal{Y}_\sigma \to \mathcal{X}$ be the crepant resolution corresponding to σ. The non-commutative crepant resolution J_σ is derived equivalent to $\mathcal{Y}_\sigma$. In [46, §3], we find a stability parameter ζ_{DT} (resp. ζ_{PT}) such that the moduli space coincides with the Hilbert scheme (resp. the stable pair moduli space) for $\mathcal{Y}_\sigma$.

Let $Z^\sigma_{\mathrm{DT}}(s,T_1,\dots,T_{N-1})$ (resp. $Z^\sigma_{\mathrm{DT}}(s,T_1,\dots,T_{N-1})$) be the generating function of DT (resp. PT) invariants of $\mathcal{Y}_\sigma$. Here s is the variable for the homology class of a point and T_i is the variable for the homology class of the i-th component C_i of the exceptional curve. The variable change induced by the derived equivalence is given as follows:

$$s := y_0\cdot y_1\cdot\cdots\cdot y_{N-1}, \quad T_i = y_i.$$

For $1\le a\le b\le N-1$, we put

$$C_{[a,b]} := [C_a]+\cdots+[C_b] \in H_2(\mathcal{Y}_\sigma,\mathbb{Z}),$$

where C_i is a component of the exceptional curve and let

$$T_{[a,b]} = T_a\cdot\cdots\cdot T_b$$

be the corresponding monomial. Let $c(a,b)$ denote the number of $(-1,-1)$-curves in $\{C_i \mid a \le i \le b\}$. We define infinite products as follows:

- If $c(a,b)$ is odd, we put

$$Z_{[a,b]} = Z_{[a,b]}(s, T_{[a,b]}) := \prod_{n=1}^{\infty} \left(\prod_{i=0}^{n-1} \left(1 - \mathbb{L}^{-\frac{n}{2}+\frac{1}{2}+i} \cdot (-s)^n \cdot T_{[a,b]} \right) \right).$$

- If $c(a,b)$ is even, we put

$$Z_{[a,b]} = Z_{[a,b]}(s, T_{[a,b]}) := \prod_{n=1}^{\infty} \left(\prod_{i=0}^{n-1} \left(1 - \mathbb{L}^{-\frac{n}{2}+1+i} \cdot (-s)^n \cdot T_{[a,b]} \right)^{-1} \right).$$

- For imaginary roots, we put

$$Z_{\mathrm{im}} = Z_{\mathrm{im}}(s) := \prod_{n=1}^{\infty} \left(\prod_{i=0}^{n-1} \left(1 - \mathbb{L}^{-\frac{n}{2}+1+i}(-s)^n \right)^{1-N} \left(1 - \mathbb{L}^{-\frac{n}{2}+2+i}(-s)^n \right)^{-1} \right).$$

Corollary 4.3. *(1) The refined DT and PT series of $\mathcal{Y}_\sigma$ are given by the formulae :*

$$Z_{\mathrm{DT}}(s, T_1, \ldots, T_{N-1}) = Z_{\mathrm{im}}(s) \cdot \prod_{1 \le a \le b \le N-1} Z_{[a,b]}(s, T_{[a,b]})$$

and

$$Z_{\mathrm{PT}}(s, T_1, \ldots, T_{N-1}) = \prod_{1 \le a \le b \le N-1} Z_{[a,b]}(s, T_{[a,b]})$$

(2) The generating function of virtual motives of the Hilbert scheme of points on $\mathcal{Y}_\sigma$ is given by the formula :

$$Z_{0\text{-dim}}(s) := \sum_{n=0}^{\infty} \left[(\mathcal{Y}_\sigma)^{[n]} \right]_{\mathrm{vir}} \cdot s^n = Z_{\mathrm{im}}.$$

(3) The refined version of DT-PT correspondence for $\mathcal{Y}_\sigma$ holds :

$$Z_{\mathrm{DT}}(s, T_1, \ldots, T_{N-1}) = Z_{0\text{-dim}}(s) \times Z_{\mathrm{PT}}(s, T_1, \ldots, T_{N-1}).$$

Remark. The formula in (2) is a direct consequence of the formula for Z_{DT} in (1), since the polynomial in the $T_{[a,b]}$ variables does not contribute.

4.2 Root Systems

Let $N_0 > 0$ and $N_1 \geq 0$ be integers such that $N_0 \geq N_1$ and set $N = N_0 + N_1$. We set

$$\begin{aligned} I &= \{1, \ldots, N-1\}, \\ \hat{I} &= \{0, 1, \ldots, N-1\}, \\ \tilde{I} &= \left\{\frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}\right\}, \\ \tilde{\mathbb{Z}} &= \left\{n + \frac{1}{2} \,\middle|\, n \in \mathbb{Z}\right\}. \end{aligned}$$

For $l \in \mathbb{Z}$ and $j \in \tilde{\mathbb{Z}}$, let $\underline{l} \in \hat{I}$ and $\underline{j} \in \tilde{I}$ be the elements such that $l - \underline{l} \equiv j - \underline{j} \equiv 0$ modulo N.

Let $\mathbb{Z}^{\hat{I}}$ be the free Abelian group with basis $\{\alpha_i \mid i \in \hat{I}\}$, where α_i is called a simple root. We put

$$\begin{aligned} \Delta_+^{\mathrm{fin}} &:= \{\alpha_{[a,b]} := \alpha_a + \cdots \alpha_b \mid 1 \leq a \leq b \leq N-1\} \\ \Delta_+^{\mathrm{re},+} &:= \{\alpha_{[a,b]} + n \cdot \delta \mid \alpha_{[a,b]} \in \Delta_+^{\mathrm{fin}}, n \in \mathbb{Z}_{\geq 0}\} \\ \Delta_+^{\mathrm{re},-} &:= \{-\alpha_{[a,b]} + n \cdot \delta \mid \alpha_{[a,b]} \in \Delta_+^{\mathrm{fin}}, n \in \mathbb{Z}_{>0}\} \end{aligned}$$

and

$$\Delta_+^{\mathrm{re}} := \Delta_+^{\mathrm{re},+} \sqcup \Delta_+^{\mathrm{re},-}, \quad \Delta_+^{\mathrm{im}} := \{n \cdot \delta \mid n \in \mathbb{Z}_{>0}\}$$

where $\delta := \alpha_0 + \cdots + \alpha_{N-1}$ is the (positive minimal) imaginary root.

For $k \in \hat{I}$, the simple reflection at k is the group homomorphism given by

$$\begin{aligned} \mathbb{Z}^{\hat{I}} &\to \mathbb{Z}^{\hat{I}} \\ \alpha_i &\mapsto \alpha_i - C_{ik} \cdot \alpha_k \end{aligned}$$

where C is the Cartan matrix of type $\hat{A}_N$. This gives a self-bijection of

$\Delta_+^{\mathrm{re},+} \backslash \{\alpha_k\}$.

4.3 Non-Commutative Crepant Resolutions

4.3.1 Quivers with potential

We denote by Γ the quadrilateral (or the triangle in case $N_1 = 0$) with vertices $(0,0)$, $(0,1)$, $(N_0, 0)$ and $(N_1, 1)$. Note that the affine toric Calabi–Yau 3-fold corresponding to Γ is $\mathcal{X} = \{XY - Z^{N_0} W^{N_1}\}$.

A partition σ of Γ is a pair of functions $\sigma_x \colon \tilde{I} \to \tilde{\mathbb{Z}}$ and $\sigma_y \colon \tilde{I} \to \{0,1\}$ such that

- $\sigma(i) := (\sigma_x(i), \sigma_y(i))$ gives a bijection between $\tilde{I}$ and the following set:

$$\left\{ \left(\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \dots, \left(N_0 - \frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{3}{2}, 1\right), \dots, \left(N_1 - \frac{1}{2}, 1\right) \right\},$$

- if $i < j$ and $\sigma_y(i) = \sigma_y(j)$ then $\sigma_x(i) > \sigma_x(j)$.

Giving a partition σ of Γ is equivalent to dividing Γ into N-tuples of triangles $\{T_i\}_{i \in \tilde{I}}$ with area $1/2$ so that T_i has $(\sigma_x(i) \pm 1/2, \sigma_y(i))$ as its vertices. Let Γ_σ be the corresponding diagram, Δ_σ be the fan and $f_\sigma \colon \mathcal{Y}_\sigma \to \mathcal{X}$ be the crepant resolution of $\mathcal{X}$. We put

$$\hat{I}_r := \left\{ k \in \hat{I} \,\middle|\, \sigma_y(k - \frac{1}{2}) = \sigma_y(k + \frac{1}{2}) \right\}. \tag{4.1}$$

Example 4.1. Let us consider as an example the case $N_0 = 4$, $N_1 = 2$ and

$$(\sigma(i))_{i \in \tilde{I}} = \left(\left(\frac{7}{2}, 0\right), \left(\frac{3}{2}, 1\right), \left(\frac{5}{2}, 0\right), \left(\frac{3}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, 0\right) \right).$$

We show the corresponding diagram Γ_σ in Figure 4.2.

Let S be the union of an infinite number of rhombi with edge length 1 as in Figure 4.3 which is located so that the centers of the rhombi are on a line parallel to the x-axis in $\mathbb{R}^2$ and H be the union of infinite number of

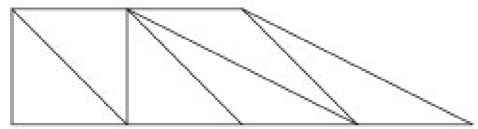

Figure 4.2: Γ_σ

hexagons with edge length 1 as in Figure 4.4 which is located so that the centers of the hexagons are in a line parallel to the x-axis in $\mathbb{R}^2$. We make

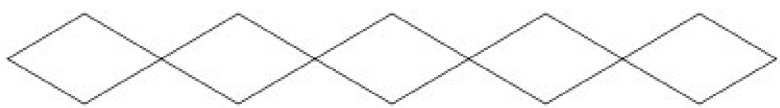

Figure 4.3: S

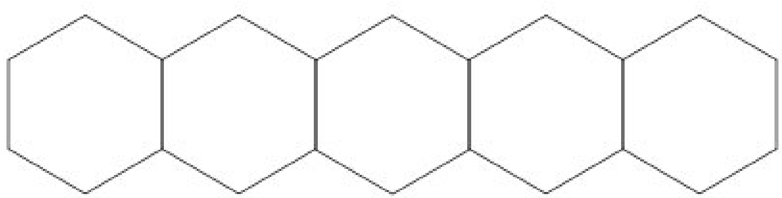

Figure 4.4: H

the sequence $\tau = \tau_\sigma : \mathbb{Z} \to \{S, H\}$ which maps l to S (resp. H) if l modulo N is not in $\hat{I}_r$ (resp. is in $\hat{I}_r$) and cover the whole plane $\mathbb{R}^2$ by arranging S's and H's according to this sequence (see Figure 4.5). We regard this as a graph on the 2-dimensional torus $\mathbb{R}^2/\Lambda$, where Λ is the lattice generated by $(\sqrt{3}, 0)$ and $(N_0 - N_1, (N_0 - N_1)\sqrt{3} + N_1)$. We can color the vertices of this graph black or white so that each edge connects a black vertex and a white one. Let P_σ denote this bipartite graph on the torus. For each edge

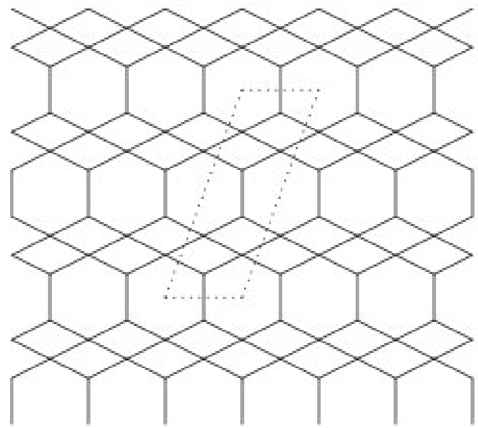

Figure 4.5: P_σ in case Example 4.1

$h^\vee$ in P_σ, we make its dual edge h directed so that we see the black end of $h^\vee$ on our right hand side when we cross $h^\vee$ along h in the given direction. Let Q_σ denote the resulting quiver. The set of vertices of the quiver Q_σ is $\hat{I}$, which is identified with $\mathbb{Z}/N\mathbb{Z}$. The set of edges of the quiver Q_σ is given by

$$H := \left(\coprod_{i \in \tilde{I}} h_i^+\right) \sqcup \left(\coprod_{i \in \tilde{I}} h_i^-\right) \sqcup \left(\coprod_{k \in \hat{I}_r} r_k\right).$$

Here h_i^+ (resp. h_i^-) is an edge from $i - \frac{1}{2}$ to $i + \frac{1}{2}$ (resp. from $i + \frac{1}{2}$ to $i - \frac{1}{2}$), r_k is an edge from k to itself.

For each vertex q of P_σ, let ω_q be the potential[3] which is the composition of all arrows in Q_σ corresponding to edges in P_σ with q as their ends. We define

$$\omega_\sigma := \sum_{q\,:\,\text{black}} \omega_q - \sum_{q\,:\,\text{white}} \omega_q.$$

The relations of the Jacobian algebra are as follows:

- $h_i^+ \circ r_{i-\frac{1}{2}} = r_{i+\frac{1}{2}} \circ h_i^+$ and $r_{i-\frac{1}{2}} \circ h_i^- = h_i^- \circ r_{i+\frac{1}{2}}$ for $i \in \tilde{I}$ such that

[3]A potential of a quiver Q is an element in $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, i.e. a linear combination of equivalence classes of cyclic paths in Q where two paths are equivalent if they coincide after a cyclic rotation.

$i-\frac{1}{2}, i+\frac{1}{2} \in \hat{I}_r$.

- $h_i^+ \circ r_{i-\frac{1}{2}} = h_{i+1}^- \circ h_{i+1}^+ \circ h_i^+$ and $r_{i-\frac{1}{2}} \circ h_i^- = h_i^- \circ h_{i+1}^- \circ h_{i+1}^+$ for $i \in \tilde{I}$ such that $i-\frac{1}{2} \in \hat{I}_r$, $i+\frac{1}{2} \notin \hat{I}_r$.

- $h_i^+ \circ h_{i-1}^+ \circ h_{i-1}^- = r_{i+\frac{1}{2}} \circ h_i^+$ and $h_{i-1}^+ \circ h_{i-1}^- \circ h_i^- = h_i^- \circ r_{i+\frac{1}{2}}$ for $i \in \tilde{I}$ such that $i-\frac{1}{2} \notin \hat{I}_r$, $i+\frac{1}{2} \in \hat{I}_r$.

- $h_i^+ \circ h_{i-1}^+ \circ h_{i-1}^- = h_{i+1}^- \circ h_{i+1}^+ \circ h_i^+$ and $h_{i-1}^+ \circ h_{i-1}^- \circ h_i^- = h_i^- \circ h_{i+1}^- \circ h_{i+1}^+$ for $i \in \tilde{I}$ such that $i-\frac{1}{2}, i+\frac{1}{2} \notin \hat{I}_r$.

- $h_{i-\frac{1}{2}}^+ \circ h_{i-\frac{1}{2}}^- = h_{i+\frac{1}{2}}^- \circ h_{i+\frac{1}{2}}^+$ for $k \in \hat{I}_r$.

Remark 4.1. Note these quivers were previously considered in **??**. Another detailed account of their definition can be found there.

4.3.2 NCCR and derived equivalence

Let $\pi\colon \mathcal{Y}_\sigma \to \mathcal{X}$ be the crepant resolution corresponding to σ.

Theorem 4.4. *[47, Theorem 1.15 and Theorem 1.20]*

$$D^b(\mathrm{mod} J_\sigma) \simeq D^b(\mathrm{Coh}\mathcal{Y}_\sigma)$$

The equivalence is given by an explicit tilting vector bundle which is a direct sum of line bundles [47, Theorem 1.10]. In particular, the following map is compatible with the derived equivalence:

$$\begin{array}{ccccc} H^0(Y_\sigma,\mathbb{Z}) & \oplus & H^2(Y_\sigma,\mathbb{Z}) & \to & \mathbb{Z}^I \\ [\mathrm{pt}] & & & \mapsto & \delta \\ & & [C_i] & \mapsto & \alpha_i \end{array}$$

where α_i is the i-th fundamental vector and $\delta := \alpha_0 + \alpha_1 + \cdots \alpha_{N-1}$.

4.3.3 Mutation and derived equivalence

Derksen–Weyman–Zelevinsky's mutation ([15]) of a quiver with a potential induces a derived equivalence of the derived categories of Ginzburg's dgas

([29]). Moreover, the relation between the module categories of Jacobian algebras has a description in terms of torsion pair and tilting, which plays a crucial role for the wall-crossing formulae ([33, 48]). In this paper, we can not apply [15] and [29] since we have loops and oriented 2-cycles in the quiver. In this subsection, we see derived equivalences and descriptions of module categories using the explicit computations given in [47, §3].

Let k be an edge of the partition σ which is a diagonal of a parallelogram. Note that such k corresponds to a vertex without loops. Let σ' denote the partition which is obtained by a "flip" of the edge k.

Let P_i be the indecomposable projective J_σ-module associated to a vertex i. Note that as a vector space P_i is the space of linear combinations of path ending at the vertex i. We define

$$P'_k := \mathrm{coker}(P_k \to P_{k-1} \oplus P_{k+1}).$$

and put $P'_i = P_i$ for $i \neq k$. Here the map $P_k \to P_{k\pm1}$ above is induced by the arrow from k to $k \pm 1$.

Theorem 4.5. *[47, Proposition 3.1]*

(1)

$$\mathrm{End}(\oplus P'_i)^{\mathrm{op}} \simeq J_{\sigma'}.$$

(2)

$$\Phi_k := \mathbf{R}\mathrm{Hom}(\oplus P'_i, \bullet) \colon D^b(\mathrm{mod} J_\sigma) \to D^b(\mathrm{mod} J_{\sigma'})$$

provides an equivalence.

For a J_σ-module $V = \oplus_{i \in \hat{I}} V_i$, we have

$$\left(H^j_{\mathrm{mod} J_{\sigma'}}(\Phi_k(V))\right)_i = \begin{cases} V_i & i \neq k, j = 0, \\ \ker\left(V_{k-1} \oplus V_{k+1} \to V_k\right) & i = k, j = 0, \\ \mathrm{coker}\left(V_{k-1} \oplus V_{k+1} \to V_k\right) & i = k, j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As for dimension vectors, the simple reflection is compatible with the derived equivalence.

By the description above, we have

$$\begin{aligned}
\mathrm{mod}J_\sigma \cap \Phi_k^{-1}(\mathrm{mod}J_{\sigma'}) &= \{V \in \mathrm{mod}J_\sigma \mid \mathrm{coker}\,(V_{k-1} \oplus V_{k+1} \to V_k) = 0\} \\
&= \{V \in \mathrm{mod}J_\sigma \mid \mathrm{Hom}(V, s_k) = 0\} \\
&=: (\mathrm{mod}J_\sigma)^k, \\
\mathrm{mod}J_\sigma \cap \Phi_k^{-1}(\mathrm{mod}J_{\sigma'})[1] &= \{V \in \mathrm{mod}J_\sigma \mid V_i = 0 \ (i \neq k)\} \\
&=: \mathcal{S}_k.
\end{aligned}$$

In other-words, $((\mathrm{mod}J_\sigma)^k, \mathcal{S}_k)$ is a torsion pair of $\mathrm{mod}J_\sigma$ and $\Phi_k^{-1}(\mathrm{mod}J_{\sigma'})$ is obtained from $\mathrm{mod}J_\sigma$ by tilting with respect to this torsion pair (see [48, §3.1]). Then we have

$$\begin{aligned}
\mathrm{mod}J_{\sigma'} \cap \Phi_k(\mathrm{mod}J_\sigma) &= \{V \in \mathrm{mod}J_{\sigma'} \mid \mathrm{Hom}(s'_k, V) = 0\} \\
&=: (\mathrm{mod}J_{\sigma'})_k.
\end{aligned}$$

In summary, we have the following:

Proposition 4.6. *The equivalence Φ_k induces an equivalence of* $(\mathrm{mod}J_\sigma)^k$ *and* $(\mathrm{mod}J_{\sigma'})_k$.

In the proof of [47, Proposition 3.1], the author provides the isomorphism in Proposition 4.5 (1) explicitly. For $V \in \mathrm{mod}J_\sigma \cap \Phi_k^{-1}(\mathrm{mod}J_{\sigma'})$, the map

$$\left(H^0_{\mathrm{mod}J_{\sigma'}}(\Phi_k(V))\right)_{k-1} \to \left(H^0_{\mathrm{mod}J_{\sigma'}}(\Phi_k(V))\right)_k$$

is induced by the following morphism:

$$R_{k-1} \oplus R_{k-1,k+1} \colon V_{k-1} \to V_{k-1} \oplus V_{k+1}$$

where

$$R_{k-1} := \begin{cases} r_{k-1} & k-1 \in \hat{I}_r, \\ h^+_{k-\frac{3}{2}} \circ h^-_{k-\frac{3}{2}} & k-1 \notin \hat{I}_r \end{cases}$$

and

$$R_{k-1,k+1} := h^-_{k+\frac{1}{2}} \circ h^+_{k+\frac{1}{2}} \circ h^+_{k-\frac{1}{2}}.$$

4.3.4 Cut and mutation

Let (Q, W) be a quiver with potential. To each subset $C \subset Q_1$ we associate a grading g_C on Q by

$$g_C(a) = \begin{cases} 1 & a \in C, \\ 0 & a \in C. \end{cases}$$

A subset $C \subset Q_1$ is called a cut if W is homogeneous of degree 1 with respect to g_C. Denote by Q_C, the subquiver of Q with the vertex set Q_0 and the arrow set $Q_1 \backslash C$. We define the truncated Jacobian algebra by

$$J(Q, W)_C := J(Q, W)/\langle C \rangle$$

Let k be a vertex of Q_σ without loops and C be a cut of (Q_σ, w_σ) such that $g_C(h^+_{k+\frac{1}{2}}) = 1$[4]. We define a cut C' of $(Q_{\sigma'}, w_{\sigma'})$ by the following conditions:

- $g_{C'}(h^+_{k-\frac{1}{2}}) = 1$, and
- $g_{C'}(h^\pm_i) = g_C(h^\pm_i)$ if $i \neq k - \frac{1}{2}, k + \frac{1}{2}$.

Proposition 4.7. *(See [51, Proposition 4.12]) The equivalence* Φ_k *induces an equivalence of* $(\mathrm{mod} J_{\sigma,C})_k$ *and* $(\mathrm{mod} J_{\sigma',C'})^k$*.*

Proof. It is enough to show that if $h^+_{k+\frac{1}{2}}$ vanishes on V then $h^+_{k-\frac{1}{2}}$ vanished on $\Phi_k(V)$.

Since $g_C(h^\pm_{k-\frac{1}{2}}) = 0$, we have

- $g_C(r_{k-1}) = 1$ if $k - 1 \in \hat{I}_r$, and
- $g_C(h^+_{k-\frac{3}{2}}) = 1$ or $g_C(h^-_{k-\frac{3}{2}}) = 1$ if $k - 1 \notin \hat{I}_r$

[4] We can construct a cut of (Q_σ, w_σ) as follows: First, by coupling h^+_i and h^-_i for each i, we group the arrows in Q_σ into $N + |\hat{I}_r|$ groups. Note that $N + |\hat{I}_r|$ is even. These groups have the natural cyclic order and we label each of them by odd or even. Choose (any) one arrow from each odd (or even) labelled group, then we get a cut.

and so R_{k-1} vanishes. Since $g_C(h^+_{k+\frac{1}{2}}) = 1$, we see that $R_{k-1,k+1}$ vanishes. □

4.4 Motivic Donaldson–Thomas Invariants

4.4.1 Motives

We are working in a version of the ring of motivic weights: let $\mathcal{M}_\mathbb{C}$ denote the K-group of the category of effective Chow motives over $\mathbb{C}$, extended by $\mathbb{L}^{-\frac{1}{2}}$, where $\mathbb{L}$ is the Lefschetz motive. It has a natural structure of a λ-ring [19, 23] with σ-operations defined by $\sigma_n([X]) = [X^n/S_n]$ and $\sigma_n(\mathbb{L}^{\frac{1}{2}}) = \mathbb{L}^{\frac{n}{2}}$. We put

$$\widetilde{\mathcal{M}}_\mathbb{C} = \mathcal{M}_\mathbb{C}[\![\mathbb{L}^{-1}]\!],$$

which is also a λ-ring. Note that in this latter ring, the elements $(1 - \mathbb{L}^n)$, and therefore the motives of general linear groups, are invertible. The rings $\mathcal{M}_\mathbb{C} \subset \widetilde{\mathcal{M}}_\mathbb{C}$ sit in larger rings $\mathcal{M}^{\hat{\mu}}_\mathbb{C} \subset \widetilde{\mathcal{M}}^{\hat{\mu}}_\mathbb{C}$ of equivariant motives, where $\hat{\mu}$ is the group of all roots of unity [35].

Let $f\colon X \to \mathbb{C}$ be a regular function on a smooth variety X. Using arc spaces, Denef and Loeser [14, 35] define the motivic nearby cycle $[\psi_f] \in \mathcal{M}^{\hat{\mu}}_\mathbb{C}$ and the motivic vanishing cycle

$$[\varphi_f] := [\psi_f] - [f^{-1}(0)] \in \mathcal{M}^{\hat{\mu}}_\mathbb{C}$$

of f. Note that if $f = 0$, then $[\varphi_0] = -[X]$. The following result was proved in [4, Prop. 1.11].

Theorem 4.8. *Let $f : X \to \mathbb{C}$ be a regular function on a smooth variety X. Assume that X admits a $\mathbb{C}^*$-action such that f is $\mathbb{C}^*$-equivariant i.e. $f(tx) = tf(x)$ for $t \in \mathbb{C}^*$, $x \in X$, and such that there exist limits* $\lim_{t\to 0} tx$ *for all $x \in X$. Then*

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)] \in \mathcal{M}_\mathbb{C} \subset \mathcal{M}^{\hat{\mu}}_\mathbb{C}.$$

Following [4], we define the *virtual motive* of $\mathrm{crit}(f)$ to be

$$[\mathrm{crit}(f)]_{\mathrm{vir}} := -(-\mathbb{L}^{\frac{1}{2}})^{-\dim X}[\varphi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

For a smooth variety X, we put

$$[X]_{\mathrm{vir}} := [\mathrm{crit}(0_X)]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{-\dim X} \cdot [X].$$

4.4.2 Quivers and moduli spaces

Let Q be a quiver, with vertex set Q_0 and edge set Q_1. For an arrow $a \in Q_1$, we denote by $s(a) \in Q_0$ (resp. $t(a) \in Q_0$) the vertex at which a starts (resp. ends). We define the Euler–Ringel form χ on $\mathbb{Z}^{Q_0}$ by the rule

$$\chi(\alpha, \beta) = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{s(a)} \beta_{t(a)}, \qquad \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

Given a Q-representation M, we define its dimension vector $\underline{\dim}\, M \in \mathbb{N}^{Q_0}$ by $\underline{\dim}\, M = (\dim M_i)_{i \in Q_0}$. Let $\alpha \in \mathbb{N}^{Q_0}$ be a dimension vector and let $V_i = \mathbb{C}^{\alpha_i}$, $i \in Q_0$. We define

$$R(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$$

and

$$G_\alpha := \prod_{i \in Q_0} \mathrm{GL}(V_i).$$

Note that G_α naturally acts on $R(Q, \alpha)$ and the quotient stack

$$\mathfrak{M}(Q, \alpha) := [R(Q, \alpha)/G_\alpha]$$

gives the moduli stack of representations of Q with dimension vector α.

Let W be a potential on Q, a finite linear combination of cyclic paths in Q. Denote by $J = J_{Q,W}$ the Jacobian algebra, the quotient of the path algebra $\mathbb{C}Q$ by the two-sided ideal generated by formal partial derivatives of the potential W. Let

$$f_\alpha : R(Q,\alpha) \to \mathbb{C}$$

be the G_α-invariant function defined by taking the trace of the map associated to the potential W. As it is now well known [55, Proposition 3.8], a point in the critical locus $\mathrm{crit}(f_\alpha)$ corresponds to a J-module. The quotient stack

$$\mathfrak{M}(J,\alpha) := \left[\mathrm{crit}(f_\alpha)/G_\alpha\right]$$

gives the moduli stack of J-modules with dimension vector α.

Definition 4.1. A *central charge* is a group homomorphism $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$ such that

$$Z(\alpha) \in \mathbb{H}_+ = \{re^{i\pi\varphi} \mid r > 0, 0 < \varphi \leq 1\}$$

for any $\alpha \in \mathbb{N}^{Q_0}\backslash\{0\}$. Given $\alpha \in \mathbb{N}^{Q_0}\backslash\{0\}$, the number $\varphi(\alpha) = \varphi \in (0,1]$ such that $Z(\alpha) = re^{i\pi\varphi}$, for some $r > 0$, is called the phase of α.

Definition 4.2. For any nonzero Q-representation (resp. J-module) V, we define $\varphi(V) = \varphi(\underline{\dim}\, V)$. A Q-representation (resp. J-module) V is said to be Z-(semi)stable if for any proper nonzero Q-subrepresentation (resp. J-submodule) $U \subset V$ we have

$$\varphi(U)(\leq)\varphi(V).$$

Definition 4.3. Given $\zeta \in \mathbb{R}^{Q_0}$, define the central charge $Z : \mathbb{Z}^{Q_0} \to \mathbb{C}$ by the rule

$$Z(\alpha) = -\zeta \cdot \alpha + i|\alpha|,$$

where $|\alpha| = \sum_{i \in Q_0} \alpha_i$. We say that a Q-representation (resp. J-module) is ζ-(semi)stable if it is Z-(semi)stable.

Remark 4.2. Let the central charge Z be as in Definition 4.3. Define the slope function $\mu : \mathbb{N}^{Q_0}\backslash\{0\} \to \mathbb{R}$ by $\mu(\alpha) = \frac{\zeta\cdot\alpha}{|\alpha|}$. If $l \subset \mathbb{H} = \mathbb{H}_+ \cup \{0\}$ is a ray such that $Z(\alpha) \in l$ then $l = \mathbb{R}_{\geq 0}(-\mu(\alpha), 1)$. This implies that $\varphi(\alpha) < \varphi(\beta)$ if and only if $\mu(\alpha) < \mu(\beta)$.

We say that $\zeta \in \mathbb{R}^{Q_0}$ is α-generic if for any $0 < \beta < \alpha$ we have $\varphi(\beta) \neq \varphi(\alpha)$. This condition implies that any ζ-semistable Q-representation (resp. J-module) is automatically ζ-stable.

Let $R_\zeta(Q, \alpha)$ denote the open subset of $R(Q, \alpha)$ consisting of ζ-semistable representations. Let $f_{\zeta,\alpha}$ denote the restriction of f_α to $R_\zeta(Q, \alpha)$. The quotient stacks

$$\mathfrak{M}_\zeta(Q, \alpha) := \left[R_\zeta(Q, \alpha)/G_\alpha\right], \qquad \mathfrak{M}_\zeta(J, \alpha) := \left[\operatorname{crit}(f_{\zeta,\alpha})/G_\alpha\right] \tag{4.2}$$

give the moduli stacks of ζ-semistable Q-representations and J-modules with dimension vector α.

4.4.3 Motivic DT invariants

Let (Q, W) be a quiver with a potential and let $J = J_{Q,W}$ be its Jacobian algebra. Recall that the degeneracy locus of the function $f_\alpha : R(Q, \alpha) \to \mathbb{C}$ defines the locus of J-modules, so that the quotient stack

$$\mathfrak{M}(J, \alpha) := [\operatorname{crit}(f_\alpha)/G_\alpha]$$

is the stack of J-modules with dimension vector α. We define motivic Donaldson–Thomas invariants by

$$[\mathfrak{M}(J, \alpha)]_{\text{vir}} := \frac{[\operatorname{crit}(f_\alpha)]_{\text{vir}}}{[G_\alpha]_{\text{vir}}}.$$

For a stability parameter ζ, we define

$$[\mathfrak{M}_\zeta(J, \alpha)]_{\text{vir}} = \frac{[\operatorname{crit}(f_{\zeta,\alpha})]_{\text{vir}}}{[G_\alpha]_{\text{vir}}}. \tag{4.3}$$

where, as before, $f_{\zeta,\alpha}$ denote the restriction of $f_\alpha : R(Q, \alpha) \to \mathbb{C}$ to $R_\zeta(Q, \alpha)$.

4.4.4 Generating series of motivic DT invariants

Let (Q, W) be a quiver with a potential admitting a cut, and let $J = J_{Q,W}$ be its Jacobian algebra.

Definition 4.4. We define the generating series of the motivic Donaldson–Thomas invariants of (Q, W) by

$$A_U(y) = \sum_{\alpha \in \mathbb{N}^{Q_0}} [\mathfrak{M}(J, \alpha)]_{\mathrm{vir}} \cdot y^\alpha = \sum_{\alpha \in \mathbb{N}^{Q_0}} \frac{[\mathrm{crit}(f_\alpha)]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}} \cdot y^\alpha \in \mathcal{T}_Q,$$

the subscript referring to the fact that we think of this series as the universal series.

Given a cut C of (Q, W), we define a new quiver $Q_C = (Q_0, Q_1 \backslash C)$. Let J_C be the quotient of $\mathbb{C}Q_C$ by the ideal

$$(\partial_C W) = (\partial W / \partial a, a \in C).$$

Proposition 4.9. *[39, Proposition 1.14] If (Q, W) admits a cut C, then*

$$A_U(y) = \sum_{\alpha \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)+2d_I(\alpha)} \frac{[R(J_C, \alpha)]}{[G_\alpha]} y^\alpha,$$

where $d_C(\alpha) = \sum_{(a:i \to j) \in C} \alpha_i \alpha_j$ for any $\alpha \in \mathbb{Z}^{Q_0}$.

The quiver with potential (Q_σ, w_σ) introduced in §4.3 admits a cut (see §4.3.4) and Proposition 4.9 can be applied. In the next section we use this to compute the universal series in a specific case.

4.5 The Universal DT Series: Special Case

Let us fix an integer N' with $0 \leq N' \leq N$. Throughout this section we fix σ to be the unique partition defined such that

$$\hat{I}_r = \{0, 1, 2, 3, \ldots N' - 1\}.$$

In other words the partition such that the quiver with potential (Q_σ, w_σ) has loops at the first N' vertices only. The aim of this section is to prove Theorem 4.1 for this quiver with potential.

We define three fixed subsets of the vertices

$$\begin{aligned} I_1 &:= \{0, 1, \ldots, N'-1\} \subset \mathbb{Z}/N, \\ I_2 &:= \{N', N'+2, N'+4, \ldots, N-2\} \subset \mathbb{Z}/N, \\ I_3 &:= \{N'+1, N'+3, N'+5, \ldots, N-1\} \subset \mathbb{Z}/N. \end{aligned}$$

Then there exists a cut C given by the collection of arrows

$$C = \left\{ h_i^- \mid i - \frac{1}{2} \notin I_2 \right\}.$$

By Proposition 4.9 the coefficients of the universal DT series $A_U^\sigma(y) = \sum_{\alpha \in \mathbb{N}^Q} A_\alpha y^\alpha$ are given by

$$A_\alpha = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{\chi(\alpha,\alpha)+2d_C(\alpha)} \frac{[R(J_{\sigma,C},\alpha)]}{[G_\alpha]} y^\alpha$$

where $d_C(\alpha) = \sum_{(a:i\to j)\in C} \alpha_i \alpha_j$. To begin we find a simple expression for the term $\chi(\alpha,\alpha) + 2d_C(\alpha)$ in the exponent. We know by definition that

$$\begin{aligned} \chi(\alpha,\alpha) &= \sum_{i \in I_1 \cup I_2 \cup I_3} \alpha_i^2 - \sum_{i \in I_1} \alpha_i^2 - \sum_{i \in I_1 \cup I_2 \cup I_3} \alpha_i \alpha_{i+1} - \sum_{i \in I_1 \cup I_2 \cup I_3} \alpha_{i+1}\alpha_i, \\ d_I(\alpha) &= \sum_{i \in I_1} \alpha_i \alpha_{i+1} + \sum_{i \in I_3} \alpha_{i+1} \alpha_i, \end{aligned}$$

so it follows

$$\begin{aligned} \chi(\alpha,\alpha) + 2d_C(\alpha) &= \sum_{i \in I_2 \cup I_3} \alpha_i^2 - 2 \cdot \sum_{i \in I_2} \alpha_i \alpha_{i+1}, \\ &= \sum_{i \in I_2} (\alpha_{i+1} - \alpha_i)^2. \end{aligned}$$

Our next goal is to factorize $A_U^\sigma(y)$ into two simpler series. This proceeds by analyzing the motivic classes $[R(J_{\sigma,C},\alpha)]$.

Given a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ and a representation of a $J_{\sigma,C}$-module

$$V = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} V_i$$

we focus on a specific element

$$H := h^+_{1/2} + h^+_{3/2} + \cdots + h^+_{N-1/2} \in \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \mathrm{Hom}(V_i, V_{i+1}).$$

This map H acts as an endomorphism of the vector space V. Given any such linear map

$$H : V \to V$$

there exists a unique splitting $V = V^I \oplus V^N$ with maps

$$\begin{array}{lcll} H^I & : & V^I \to V^I & \text{invertible} \\ H^N & : & V^N \to V^N & \text{nilpotent} \end{array}$$

so that

$$H = H^I \oplus H^N.$$

Moreover in our case the above splitting respects the grading by $i \in I_1 \cup I_2 \cup I_3$. To be explicit we have that

$$V^I = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} V_i^I$$

where $V_i^I := V_i \cap V^I$ (similarly $V^N = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} V_i^N$ with $V_i^N := V_i \cap V^N$). One immediate consequence of this is that

$$\dim\left(V_i^I\right) = \dim\left(V_{i+1}^I\right) \text{ for all } i \in I_1 \cup I_2 \cup I_3,$$

indeed this is clear since the block form of H^I demands that it map V_i^I to V_{i+1}^I via an isomorphism. We are now ready to decompose the computation of $A_U^\sigma(y)$ into two simpler subproblems.

Definition 4.5. (Invertible series) We define

$$R^I(a) := \{r \in R(J_{\sigma,C}, \alpha) \mid H \text{ is invertible, } \alpha_i = a\ \forall i\}$$

and the series

$$I^\sigma(y) := \sum_{a \geq 0} \frac{[R^I(a)]}{[\mathrm{GL}(a)]^N} y^a.$$

Definition 4.6. (Nilpotent series) We define

$$R^N(\alpha) := \{r \in R(J_{\sigma,C}, \alpha) \mid H \text{ is nilpotent}\}$$

and the series

$$N^\sigma(y) := \sum_{\alpha \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{1/2})^{\sum_{i \in I_2} (\alpha_{i+1} - \alpha_i)^2} \frac{[R^N(\alpha)]}{[G_\alpha]} y^\alpha.$$

The following lemma shows that the series $A_U^\sigma(y)$ factorizes into the product of the two just defined.

Lemma 4.10. *We have*

$$A_U^\sigma(y) = I^\sigma(y_0 \cdots y_{N-1}) \cdot N^\sigma(y).$$

Proof. This formula follows directly from a stratification of the variety $R(J_{\sigma,C}, \alpha)$ by the dimension of V_i^I.

Fix $\alpha \in \mathbb{N}^{Q_0}$, we stratify $R(J_{\sigma,C}, \alpha)$ by $\dim(V_i^I) = a$. Let

$$\underline{a} := (a, a, \dots, a) \in \mathbb{N}^{Q_0},$$

and

$$\alpha' \text{ such that } \alpha = \underline{a} + \alpha' \in \mathbb{N}^{Q_0}.$$

There is a Zariski locally trivial fibration

$$\begin{array}{rcl} R^I(a) \times R^N(\alpha') & \to & \{r \in R(J_{\sigma,C}, \alpha) \mid \dim(V_i^I) = a \text{ for } H \in r\} \\ & & \downarrow \\ & & \mathcal{M}(a, \alpha). \end{array}$$

Here $\mathcal{M}(a, \alpha)$ is the space parameterizing splittings $V_i = V_i^I \oplus V_i^N$. To see this one checks that the arrows r_i, $h^-_{i+1/2}$ in the representation also preserve the splitting, so the entire representation splits into $V^I \oplus V^N$. This follows easily from the relations and some basic linear algebra.

Splittings of the vector space $V_i = V_i^I \oplus V_i^N$ are parameterized by

$$\mathrm{GL}(\alpha_i) / \left(\mathrm{GL}(a) \times \mathrm{GL}(\alpha'_i)\right)$$

and hence the motivic class of the base is

$$[\mathcal{M}(a, \alpha)] = \frac{[G_\alpha]}{[\mathrm{GL}(a)]^N \cdot [G_{\alpha'}]}.$$

Summing over each stratum with $\dim(V_i^I) = a$ we get

$$[R(J_{\sigma,C}, \alpha)] = [G_\alpha] \cdot \sum_{a=0}^{\min_i\{\alpha_i\}} \frac{[R^I(a)]}{[\mathrm{GL}(a)]^N} \cdot \frac{[R^N(\alpha')]}{[G_{\alpha'}]}.$$

Multiplying both sides of this expression by $(-\mathbb{L}^{1/2})^{\sum_{i \in I_2} (\alpha_{i+1} - \alpha_i)^2} y^\alpha$ and summing gives

$$A_U^\sigma(y) = \left(\sum_{a \geq 0} \frac{[R^I(a)]}{[\mathrm{GL}(a)]^N} \prod_{i=0}^{N-1} y_i^a \right) \cdot \left(\sum_{\alpha' \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{1/2})^{\sum_{i \in I_2} (\alpha'_{i+1} - \alpha'_i)^2} \frac{[R^N(\alpha')]}{[G_{\alpha'}]} y^{\alpha'} \right)$$

proving the result. □

In the next two sections we compute formulas for $I^\sigma(y)$ and $N^\sigma(y)$.

4.5.1 Step One: The invertible case $I^\sigma(y)$

Proposition 4.11. *We have*

$$I^\sigma(y) = \mathrm{Exp}\left(\mathbb{L}\frac{y}{1-y}\right).$$

Proof. A $J_{\sigma,C}$-module $r \in R(J_{\sigma,C},\alpha)$ is given by a vector space

$$V = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} V_i$$

of dimension $\alpha \in \mathbb{N}^{Q_0}$ and a collection of linear maps

$$\begin{array}{lcll} r_i & : & V_i \to V_i & \text{for } i \in I_1 \\ h^-_{i+1/2} & : & V_{i+1} \to V_i & \text{for } i \in I_2 \\ h^+_{i+1/2} & : & V_i \to V_{i+1} & \text{for } i \in I_1 \cup I_2 \cup I_3 \end{array}$$

satisfying the relations coming from cyclic differentiation of the potential

$$\begin{array}{lcll} r_i h^+_{i-1/2} & = & h^+_{i-1/2} r_{i-1} & \text{for } i \in [1, N'-1] \cap I_1 \\ r_0 h^+_{N-1/2} & = & h^+_{N-1/2} h^+_{N-3/2} h^-_{N-3/2} & \\ h^-_{N'+1/2} h^+_{N'+1/2} h^+_{N'-1/2} & = & h^+_{N'-1/2} r_{N'-1} & \\ h^-_{i+3/2} h^+_{i+3/2} h^+_{i+1/2} & = & h^+_{i+1/2} h^+_{i-1/2} h^-_{i-1/2} & \text{for } i = [N'+1, N-3] \cap I_3. \end{array}$$

assuming moreover that $r \in R^I(a)$ then

$$h^+_{i+1/2} : V_i \to V_{i+1} \text{ is invertible } \forall i \in I_1 \cup I_2 \cup I_3.$$

This allows us to express $R^I(a)$ as a $\prod_{i=1}^{N-1} \mathrm{GL}(V_i)$ torsor over a commuting variety

$$\begin{array}{lcll} \pi & : & R^I(a) & \to & C(a) \\ & : & (r_i, h^+_{i+1/2}, h^-_{i+1/2}) & \mapsto & (r_0, h^+_{N-1/2} h^+_{N-3/2} \cdots h^+_{3/2} h^+_{1/2}) \end{array}$$

where

$$C(a) = \{(A,B) \in \mathrm{End}(V_0) \times \mathrm{GL}(V_0) \mid AB = BA\}.$$

The free action of $\prod_{i=1}^{N-1} \mathrm{GL}(V_i)$ on $R^I(a)$ is given by

$$\begin{array}{rcll}(g_1,\ldots,g_{N-1}) & : & r_i \mapsto g_i r_i g_i^{-1} & \text{for } i \in [1, N'-1] \\ & : & h^+_{1/2} \mapsto g_1 h^+_{1/2} & \\ & : & h^+_{N-1/2} \mapsto h^+_{N-1/2} g_{N-1}^{-1} & \\ & : & h^+_{i+1/2} \mapsto g_{i+1} h^+_{i+1/2} g_i^{-1} & \text{for } i \in [1, N-2] \\ & : & h^-_{i+1/2} \mapsto g_i h^-_{i+1/2} g_{i+1}^{-1} & \text{for } i \in I_2.\end{array}$$

As $\mathrm{GL}(a)$ is a special group the torsor splits in the Zariski topology, motivically we have

$$[R^I(a)] = [\mathrm{GL}(a)]^{N-1} \cdot [C(a)].$$

Thus

$$I^\sigma(y) = \sum_{a \geq 0} \frac{[C(a)]}{[\mathrm{GL}(a)]} y^a.$$

The generating series for the commuting variety is obtained in [12] giving the result. □

4.5.2 Step Two: The nilpotent case $N^\sigma(y)$

This section is the final step in the calculation. Here we compute $N^\sigma(y)$ and obtain the formula of $A^\sigma_U(y)$.

We fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$. As before a $J_{\sigma,C}$-module is given by a vector space

$$V = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} V_i$$

of dimension α and a collection of linear maps

$$\begin{array}{rcll} r_i & : & V_i \to V_i & \text{for } i \in I_1 \\ h^-_{i+1/2} & : & V_{i+1} \to V_i & \text{for } i \in I_2 \\ h^+_{i+1/2} & : & V_i \to V_{i+1} & \text{for } i \in I_1 \cup I_2 \cup I_3 \end{array}$$

satisfying the relations of the potential (see Proposition 4.11). Throughout this section we insist that the map

$$H = h^+_{1/2} + h^+_{3/2} + \cdots h^+_{N-1/2} \in \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \mathrm{Hom}(V_i, V_{i+1})$$

is nilpotent. In fact $R^N(\alpha)$ is exactly the collection of all such representations (see Definition 4.6). In particular if we let $|\alpha| := \dim(V)$ then we know that $H^{|\alpha|} = 0$. This gives a filtration of the vector space,

$$V = V^{|\alpha|} \supset V^{|\alpha|-1} \supset \cdots \supset V^1 \supset V^0 = \{0\}$$

where

$$V^j = \{v \in V \mid H^j(v) = 0\}.$$

Moreover the filtration respects the grading by $i \in I_1 \cup I_2 \cup I_3$, by which we mean that

$$V^j = \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \left(V^j \cap V_i\right)$$

where V_i is the summand at the ith vertex of the quiver. By considering the vector space V as a representation of the nilpotent matrix H we can identify V with a $\mathbb{C}[x]$-module supported at the origin. Modules for a principal ideal domain have a simple structure. In particular we have

$$V \cong \bigoplus_{j=1}^{d} \left(\mathbb{C}[x]/(x^j)\right)^{\oplus b_j}$$

as a $\mathbb{C}[x]$-module. The next proposition provides a more refined version of this statement where each factor in this decomposition is generated by a vector from a vector space V_i.

Proposition 4.12. *For each $i \in I_1 \cup I_2 \cup I_3$ there exists collection of integers b^i_j so that*

$$V \cong \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \bigoplus_{j=1}^{d} \left(\mathbb{C}[x]/(x^j)\right)^{\oplus b^i_j}$$

where the factor $\left(\mathbb{C}[x]/(x^j)\right)^{\oplus b^i_j}$ is generated as a $\mathbb{C}[x]$-module by vectors in

V_i. Moreover the numbers b^i_j are uniquely determined by the above conditions.

Proof. We will argue by induction on d, the largest integer such that $b_d \neq 0$. As such we can assume that for each $j \leq d-1$ the factor $\mathbb{C}[x]/(x^j)$ is generated by a vector in some V_i. Now let $e_1, \dots, e_{b_d}$ be a generating set for the factor $\left(\mathbb{C}[x]/(x^d)\right)^{\oplus b_d}$, and define $W := \operatorname{span}\{e_1, \dots, e_{b_d}\}$. We consider the projection operators

$$p_i : V \to V_i / V_i \cap V^{d-1}$$

and set $W_i := p_i(W)$ and $b^i_d = \dim W_i$. We claim that $p_0 \oplus \cdots \oplus p_{N-1} : W \to W_0 \oplus \cdots \oplus W_{N-1}$ is an isomorphism. The map is clearly onto and an injection since any vector in the kernel must lie in V^{d-1}. Now consider a lifting of the vector space $V_i \supset W'_i \twoheadrightarrow W_i \subset V_i/V_i \cap V^{d-1}$ then

$$W'_i \oplus HW'_i \oplus \cdots \oplus H^{d-1}W'_i \subset V$$

is a submodule of V isomorphic to $\left(\mathbb{C}[x]/(x^d)\right)^{\oplus b^i_d}$. Summing over all i we have that $\left(\mathbb{C}[x]/(x^d)\right)^{\sum_i b^i_d}$ is a submodule of V, hence it follows that $\sum_i \dim W_i = \sum_i b^i_d \leq b_d = \dim W$ and so for dimension reasons we get

$$V \cong \left(\bigoplus_{i=0}^{N-1} \left(\mathbb{C}[x]/(x^d)\right)^{\oplus b^i_d} \right) \bigoplus \left(\bigoplus_{j=1}^{d-1} \left(\mathbb{C}[x]/(x^j)\right)^{\oplus b_j} \right).$$

Here each factor $\left(\mathbb{C}[x]/(x^d)\right)^{\oplus b^i_d}$ is generated by vectors in V_i, so by our inductive hypothesis the entire module is generated by vectors in V_i.

Finally we prove the uniqueness statement. Assume we have two distinct such decompositions

$$V \cong \bigoplus_{i=0}^{N-1} \bigoplus_{j=1}^{d} (\mathbb{C}[x]/(x^j))^{\oplus b^i_j} \cong \bigoplus_{i=0}^{N-1} \bigoplus_{j=1}^{d} (\mathbb{C}[x]/(x^j))^{\oplus c^i_j}.$$

By restricting to subrepresentations if necessary we can assume that $b^i_d \neq c^i_d$

for some i. However in this case

$$b_d^i = \dim\left(\ker(H^d : V_i \to V_{i+d})/V_i \cap V^{d-1}\right) = c_d^i$$

is a contradiction. This proves the last part of the lemma. □

Next we organize this data in the way most helpful to our cause.

Definition 4.7. Let $0 \leq a, b \leq N-1$. We define

$$|b-a| = \min\{r \in \{0, 1, \ldots, N-1\} \mid b = a + r \mod N\}.$$

Intuitively this is the distance from a to b in the cyclic direction $i \to i+1$ corresponding to the map H.

Definition 4.8. Suppose we have a decomposition of V as a $\mathbb{C}[x]$-module as in Proposition 4.12 . Define $V^{a,b}$ to be the vector subspace corresponding to summand

$$\bigoplus_{l \geq 1} \left(\mathbb{C}[x]/(x^{N(l-1)+|b-a|+1})\right)^{b^a_{N(l-1)+|b-a|+1}}.$$

and relabel the integers

$$b_l^{a,b} := b^a_{N(l-1)+|b-a|+1},$$

to define partitions

$$\pi^{[a,b]} := (1^{b_1^{a,b}} 2^{b_2^{a,b}} 3^{b_3^{a,b}} \cdots).$$

Notice that the above definition depends on the choice of the decomposition in Proposition 4.12. However all such vector spaces are isomorphism abstractly as $\mathbb{C}[x]$-modules. We can think of these vector spaces as being generated by the nilpotent vectors that start at the ath vertex and are annihilated at the $b+1$th vertex under the action of the map H.

The next lemma makes explicit how to recover the dimension vector of a representation from the datum of the N^2 partitions $\{\pi^{[a,b]} \mid 0 \leq a, b \leq N-1\}$.

Lemma 4.13. *Given a representation $r \in R^N(\alpha)$ so that the endomorphism H has type $\{\pi^{[a,b]}\}$ the dimension vector of the representation r is given by*

$$\alpha_i = \sum_{a,b} |\pi^{[a,b]}| - \sum_{a,b: i \notin [a,b]} l(\pi^{[a,b]})$$

where $|\pi^{[a,b]}|$ and $l(\pi^{[a,b]})$ are the size and length of the partition $\pi^{[a,b]}$.

Proof. This is clear since

$$V = \bigoplus_{a,b} V^{a,b}$$

and

$$\dim\left(V^{a,b} \cap V_i\right) = \begin{cases} |\pi^{[a,b]}| & \text{if } i \in [a,b] \\ |\pi^{[a,b]}| - l(\pi^{[a,b]}) & \text{if } i \notin [a,b]. \end{cases}$$

□

We can use this to give a simple reformulation of the term $\chi(\alpha,\alpha) + 2d_C(\alpha)$ appearing in the series N^σ.

Corollary 4.14. *We have*

$$\chi(\alpha,\alpha) + 2d_C(\alpha) = \sum_{i \in I_2} \left(\sum_{b \neq i} l(\pi^{[i+1,b]}) - \sum_{c \neq i+1} l(\pi^{[c,i]}) \right)^2.$$

Proof. In our initial analysis of these terms we saw that

$$\chi(\alpha,\alpha) + 2d_C(\alpha) = \sum_{i \in I_2} (\alpha_{i+1} - \alpha_i)^2$$

and now by lemma 4.13 we have

$$\alpha_{i+1} - \alpha_i = \sum_{b \neq i} l(\pi^{[i+1,b]}) - \sum_{c \neq i+1} l(\pi^{[c,i]}).$$

□

The above classification has been for the purpose of breaking the variety $R^N(\alpha)$ down into simpler pieces.

Definition 4.9. Given N^2 partitions $\{\pi^{[a,b]} \mid 0 \leq a,b \leq N-1\}$ and a dimension vector α as in lemma 4.13 we define

$$R(\{\pi^{[a,b]}\}) = \{r \in R^N(\alpha) \mid H \text{ has type } \{\pi^{[a,b]}\}\}.$$

This provides a stratification of $R^N(\alpha)$ into strata where the normal form of H has a fixed type. We will proceed to compute the motivic classes of each of these stratum.

A representation in $R(\{\pi^{[a,b]}\})$ is given explicitly by a vector space $V = \bigoplus_{i\in I_1\cup I_2\cup I_3} V_i$ and a collection of linear maps corresponding to the arrows r_i with $i \in I_1$,$h^-_{i+1/2}$ with $i \in I_2$ and $h^+_{i+1/2}$ with $i \in I_1 \cup I_2 \cup I_3$. In addition the linear maps satisfy relations

$$\begin{array}{lcll} r_i h^+_{i-1/2} & = & h^+_{i-1/2} r_{i-1} & \text{for } i \in [1, N'-1] \cap I_1 \\ r_0 h^+_{N-1/2} & = & h^+_{N-1/2} h^+_{N-3/2} h^-_{N-3/2} & \\ h^-_{N'+1/2} h^+_{N'+1/2} h^+_{N'-1/2} & = & h^+_{N'-1/2} r_{N'-1} & \\ h^-_{i+3/2} h^+_{i+3/2} h^+_{i+1/2} & = & h^+_{i+1/2} h^+_{i-1/2} h^-_{i-1/2} & \text{for } i = [N'+1, N-3] \cap I_3. \end{array}$$

and we require that the map

$$H = h^+_{1/2} + h^+_{3/2} + \cdots h^+_{N-1/2} \in \bigoplus_{i\in I_1\cup I_2\cup I_3} \mathrm{Hom}(V_i, V_{i+1})$$

has a type given by the partitions $\{\pi^{[a,b]} \mid 0 \leq a, b \leq N-1\}$. The linear map H contains all the information of the maps $h^+_{i+1/2}$. For brevity we make the following definition packaging all the remaining linear maps into one.

Definition 4.10. Given a representation as above, we define the linear map

$$L := r_0 + r_1 + \cdots r_{N'-1} + h^-_{N'+1/2} + \cdots h^-_{N-3/2} \in \bigoplus_{i\in I_1} \mathrm{Hom}(V_i, V_i) \bigoplus_{i\in I_2} \mathrm{Hom}(V_{i+1}, V_i).$$

From now on in order to compute the motivic class of $R(\{\pi^{[a,b]}\})$ we will work with a choice of coordinates. Let

$$v^{a,b}_l(k) \in V_a$$

be such that $v_l^{a,b}(k)$ generates the kth summand of $\mathbb{C}[x]/(x^{N(l-1)+|b-a|+1})^{\oplus b_l^{a,b}}$ in the decomposition of Proposition 4.12. Then we have that

$$\mathcal{B} := \{H^p v_l^{a,b}(k) \mid 1 \leq k \leq b_l^{a,b}, 0 \leq a, b \leq N-1, 0 \leq p \leq N(l-1)+|b-a|+1\}$$

forms a basis of V.

Definition 4.11. We define $H(\pi^{[a,b]})$ to be the matrix representation of the map H with respect to the basis $\mathcal{B}$. Also define

$$\begin{aligned} F(\{\pi^{[a,b]}\}) &:= \{L \mid (L, H(\pi^{[a,b]})) \in R(\{\pi^{[a,b]}\})\} \\ N(\{\pi^{[a,b]}\}) &:= \{H \mid H \text{ has type } \{\pi^{[a,b]}\}\}. \end{aligned}$$

Then $R(\{\pi^{[a,b]}\})$ has a decomposition as a vector bundle.

Lemma 4.15. $R(\{\pi^{[a,b]}\})$ *has the structure of a vector bundle*

$$\begin{array}{ccc} F(\{\pi^{[a,b]}\}) & \to & R(\{\pi^{[a,b]}\}) \\ & & \downarrow \\ & & N(\{\pi^{[a,b]}\}). \end{array}$$

In particular we have that

$$[R(\{\pi^{[a,b]}\})] = [F(\{\pi^{[a,b]}\})] \cdot [N(\{\pi^{[a,b]}\})]$$

in the Grothendieck ring of varieties.

Proof. The projection map

$$\begin{array}{ccccc} p & : & R(\{\pi^{[a,b]}\}) & \to & N(\{\pi^{[a,b]}\}) \\ & : & (L, H) & \mapsto & H. \end{array}$$

defines the bundle structure with zero section $H \mapsto (0, H)$. The fibre is the linear space of all such L. $\square$

Here the base of the vector bundle is the space of all matrices of type

$\{\pi^{[a,b]}\}$ these are all conjugate to $H(\pi^{[a,b]})$, therefore we have a torsor,

$$\begin{array}{rcccl} \pi & : & G_\alpha & \to & N(\pi^{[a,b]}) \\ & : & P & \mapsto & PH(\pi^{[a,b]})P^{-1}. \end{array}$$

This is a torsor for the group $S'(\{\pi^{[a,b]}\}) := \mathrm{Stab}_{G_\alpha}(H(\pi^{a,b}))$. This group is given as the group of units in an algebra.

Definition 4.12. We identify $S'(\{\pi^{[a,b]}\})$ with the group of multiplicative units in the following algebra

$$S(\{\pi^{[a,b]}\}) := \left\{ N \in \prod_{i=0}^{N-1} \mathrm{End}(\alpha_i) \middle| \; NH(\pi^{[a,b]}) = H(\pi^{[a,b]})N \right\}.$$

Since $S'(\{\pi^{[a,b]}\})$ is the group of units of an algebra it is a special group and so the above torsor splits in the Zariski topology. The next lemma gives a formula of the motivic class of the group $S'(\{\pi^{[a,b,]}\})$ and via the splitting of the above torsor we deduce a formula for the class of $N(\{\pi^{[a,b]}\})$. Before stating the lemma we create some notation.

Definition 4.13. The following linear spaces have dimension

$$\begin{array}{rcl} T(\{\pi^{[a,b]}\}) & := & \dim F(\{\pi^{[a,b]}\}) \\ B(\{\pi^{[a,b]}\}) & := & \dim S(\{\pi^{[a,b]}\}). \end{array}$$

Lemma 4.16. *We have*

$$[S'(\{\pi^{[a,b]}\})] = [S(\{\pi^{[a,b]}\})] \cdot \prod_{0 \le a,b \le N-1} \frac{1}{f\left(\pi^{[a,b]}\right)}$$

where

$$f\left(\pi^{[a,b]}\right) := \prod_{l \ge 1} \frac{[\mathrm{End}(b_l^{a,b})]}{[\mathrm{GL}(b_l^{a,b})]}.$$

So as a consequence

$$[R(\{\pi^{a,b}\})] = [G_\alpha] \cdot \mathbb{L}^{T(\{\pi^{[a,b]}\}) - B(\{\pi^{[a,b]}\})} \cdot \prod_{0 \le a,b \le N-1} f\left(\pi^{[a,b]}\right).$$

Proof. Let

$$W_l^{a,b} := \mathrm{span}_{\mathbb{C}}\{v_l^{a,b}(k) \mid 1 \leq k \leq b_l^{a,b}\}$$

be the span of the basis elements $v_l^{a,b}(k)$ for $1 \leq k \leq b_l^{a,b}$. We have both inclusion and projection

$$W_l^{a,b} \hookrightarrow V \twoheadrightarrow W_l^{a,b}.$$

This gives a map of algebras

$$\begin{aligned} \pi \quad &: \quad S(\{\pi^{[a,b]}\}) \to \prod_{a,b,l} \mathrm{End}(W_l^{a,b}) \\ &: \quad N \mapsto \oplus_{a,b,l} N|_{W_l^{a,b}}. \end{aligned}$$

This splits as a trivial vector bundle, whose rank is the dimension of the total space minus the dimension of the base. Since we have that the group $S'(\{\pi^{[a,b]}\})$ is the group of units in $S(\{\pi^{[a,b]}\})$, we can identify $S'(\{\pi^{[a,b]}\})$ as the inverse image of the units on the right hand side. This is a trivial vector bundle of rank equal to $\dim S(\{\pi^{[a,b]}\}) - \dim \prod_{a,b,l} \mathrm{End}(W_l^{a,b})$. We have an isomorphism of varieties

$$S'(\{\pi^{[a,b]}\}) \equiv \frac{S(\{\pi^{[a,b]}\})}{\prod_{a,b,l} \mathrm{End}(W_l^{a,b})} \times \prod_{a,b,l} \mathrm{GL}(W_l^{a,b})$$

so motivically we have

$$[S'(\{\pi^{[a,b]}\})] = [S(\{\pi^{[a,b]}\})] \cdot \prod_{0 \leq a,b \leq N-1} \frac{1}{f\left(\pi^{[a,b]}\right)}.$$

In lemma 4.15 we saw that

$$[R(\{\pi^{[a,b]}\})] = [F(\{\pi^{[a,b]}\})] \cdot [N(\{\pi^{[a,b]}\})]$$

Now we know that $N(\{\pi^{[a,b]}\})$ is a torsor for the group $S'(\{\pi^{[a,b]}\})$ whose motive we have just computed we deduce

$$\begin{aligned}
[R(\{\pi^{[a,b]}\})] &= [F(\{\pi^{[a,b]}\})] \cdot \frac{[G_\alpha]}{[S'(\{\pi^{[a,b]}\})]} \\
&= [F(\{\pi^{[a,b]}\})] \cdot \frac{[G_\alpha]}{[S(\{\pi^{[a,b]}\})]} \cdot \prod_{0\le a,b\le N-1} f\left(\pi^{[a,b]}\right) \\
&= [G_\alpha] \cdot \mathbb{L}^{T(\{\pi^{[a,b]}\})-B(\{\pi^{[a,b]}\})} \cdot \prod_{0\le a,b\le N-1} f\left(\pi^{[a,b]}\right).
\end{aligned}$$

□

The next proposition computes the difference $T(\{\pi^{[a,b]}\}) - B(\{\pi^{[a,b]}\})$. Its proof is found in Section 4.9.

Proposition 4.17. *We have* $T(\{\pi^{[a,b]}\}) - B(\{\pi^{[a,b]}\})$ *equals to*

$$-\frac{1}{2}\sum_{i\in I_2}\left(\sum_{b\neq i} l(\pi^{[i+1,b]}) - \sum_{c\neq i+1} l(\pi^{[c,i]})\right)^2$$
$$-\frac{1}{2}\sum_{a\in I_3, b\notin I_2}\sum_{i\ge 1}(b_i^{a,b})^2 - \frac{1}{2}\sum_{a\notin I_3, b\in I_2}\sum_{i\ge 1}(b_i^{a,b})^2.$$

Proof. The proof is a linear algebra calculation. See Section 4.9. □

As a corollary we deduce the formula for $N^\sigma(y)$.

Proposition 4.18. *Let*

$$\begin{aligned}
S &= \{[a,b] \mid a\in I_3, b\notin I_2 \text{ or } a\notin I_3, b\in I_2\}, \\
y_{[a,b]} &= y_a \cdot y_{a+1}\cdots y_b, \\
y' &= y_0\cdot y_1 \cdots y_{N-1},
\end{aligned}$$

then we have

$$N^\sigma(y) = \mathrm{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1}\frac{1}{1-y'}\left(\sum_{[a,b]\notin S} y_{[a,b]} - \mathbb{L}^{-\frac{1}{2}}\sum_{[a,b]\in S} y_{[a,b]}\right)\right).$$

Proof. Recall our initial definition of $N^\sigma(y)$

$$N^\sigma(y) = \sum_{\alpha \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{1/2})^{\chi(\alpha,\alpha)+2d_C(\alpha)} \frac{[R^N(\alpha)]}{[G_\alpha]} y^\alpha$$

In Proposition 4.12 we saw that it was possible to stratify each of the varieties $R^N(\alpha)$ by the type $\{\pi^{[a,b]}\}$ of the cycle H. This gives

$$N^\sigma(y) = \sum_{\alpha \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{1/2})^{\chi(\alpha,\alpha)+2d_C(\alpha)} [G_\alpha]^{-1} \left(\sum_{\{\pi^{[a,b]}\} \vdash \alpha} [R(\{\pi^{[a,b]}\})] \right) y^\alpha.$$

The motivic class of $R(\{\pi^{[a,b]}\})$ was computed in lemma 4.16 substituting this into the above formula gives

$$\sum_{\alpha \in \mathbb{N}^{Q_0}} (-\mathbb{L}^{1/2})^{\chi(\alpha,\alpha)+2d_C(\alpha)} \left(\sum_{\{\pi^{[a,b]}\} \vdash \alpha} \mathbb{L}^{T(\{\pi^{[a,b]}\})-B(\{\pi^{[a,b]}\})} \cdot \prod_{0 \le a,b \le N-1} f\left(\pi^{[a,b]}\right) \right) y^\alpha.$$

Lemma 4.13 showed how the dimension vector depended on the partitions we had

$$\alpha_i = \sum_{0 \le a,b \le N-1} |\pi^{[a,b]}| - \sum_{[a,b] \not\ni i} l(\pi^{[a,b]})$$

and an immediate corollary was that

$$\chi(\alpha,\alpha) + 2d_C(\alpha) = \sum_{i \in I_2} \left(\sum_{b \neq i} l(\pi^{[i+1,b]}) - \sum_{c \neq i+1} l(\pi^{[c,i]}) \right)^2.$$

Combining this with the formula for the difference $T(\{\pi^{[a,b]}\}) - B(\{\pi^{[a,b]}\})$ (Proposition 4.17) gives

$$\begin{aligned} N^\sigma(y) &= \sum_{\{\pi^{[a,b]}\}} \left(\prod_{[a,b] \notin S} f\left(\pi^{[a,b]}\right) \right) \cdot \left(\prod_{[a,b] \in S} f\left(\pi^{[a,b]}\right) \prod_{l \ge 1} (-\mathbb{L}^{\frac{1}{2}})^{-(b_l^{a,b})^2} \right) \\ &\quad \cdot \prod_{i=0}^{N-1} y_i^{\sum_{0 \le a,b \le N-1} |\pi^{[a,b]}| - \sum_{[a,b] \not\ni i} l(\pi^{[a,b]})}. \end{aligned}$$

To simplify notation set

$$g(\pi) \quad := \quad f(\pi) \cdot \prod_{l \geq 1} (-\mathbb{L}^{\frac{1}{2}})^{-b_l^2} \quad \text{for} \quad \pi = (1^{b_1} 2^{b_2} 3^{b_3} \cdots)$$

then rearranging the products and summations gives

$$\begin{aligned} N^{\sigma}(y) \quad = \quad & \prod_{[a,b] \notin S} \sum_{\pi^{[a,b]}} f\left(\pi^{[a,b]}\right) \cdot y'^{\left|\pi^{[a,b]}\right| - l\left(\pi^{[a,b]}\right)} \cdot y_{[a,b]}^{l(\pi^{[a,b]})} \\ & \cdot \prod_{[a,b] \in S} \sum_{\pi^{[a,b]}} g\left(\pi^{[a,b]}\right) \cdot y'^{\left|\pi^{[a,b]}\right| - l\left(\pi^{[a,b]}\right)} \cdot y_{[a,b]}^{l(\pi^{[a,b]})}. \end{aligned}$$

Both of these series are know to have product expansions [36]

$$\begin{aligned} f(t,a) \quad &= \quad \textstyle\sum_{\pi} f(\pi)\, a^{l(\pi)} t^{|\pi| - l(\pi)} \quad = \quad \operatorname{Exp}\left(\frac{1}{1-\mathbb{L}^{-1}} \cdot \frac{a}{1-t}\right) \\ g(t,a) \quad &= \quad \textstyle\sum_{\pi} g(\pi)\, a^{l(\pi)} t^{|\pi| - l(\pi)} \quad = \quad \operatorname{Exp}\left(\frac{(-\mathbb{L}^{\frac{1}{2}})^{-1}}{1-\mathbb{L}^{-1}} \cdot \frac{a}{1-t}\right). \end{aligned}$$

Now N^{σ} is a product of such series and multiplying together the corresponding exponential generating series gives the desired result

$$N^{\sigma}(y) = \operatorname{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1} \frac{1}{1-y'} \left(\sum_{[a,b] \notin S} y_{[a,b]} - \mathbb{L}^{-\frac{1}{2}} \sum_{[a,b] \in S} y_{[a,b]}\right)\right).$$

□

Now we have computed I^{σ}, N^{σ} and so by lemma 4.10

$$A_U^{\sigma}(y) = \operatorname{Exp}\left(\mathbb{L}\frac{y'}{1-y'} + \frac{\mathbb{L}}{\mathbb{L}-1} \frac{1}{1-y'} \left(\sum_{[a,b] \notin S} y_{[a,b]} - \mathbb{L}^{-1/2} \sum_{[a,b] \in S} y_{[a,b]}\right)\right)$$

or to reformulate this as a product over the set of roots

$$\operatorname{Exp}\left(\frac{1}{1-\mathbb{L}^{-1}}\left((\mathbb{L}+N-1)\sum_{\alpha\in\Delta^{im}_{\sigma,+}} y^\alpha + \sum_{\substack{\alpha\in\Delta^{re}_{\sigma,+}\\ \sum_{I_2\cup I_3}\alpha_i \text{ is even}}} y^\alpha - \mathbb{L}^{-\frac{1}{2}} \sum_{\substack{\alpha\in\Delta^{re}_{\sigma,+}\\ \sum_{I_2\cup I_3}\alpha_i \text{ is odd}}} y^\alpha\right)\right).$$

Thus proving Theorem 4.1

$$A_U^\sigma(y) = \prod_{\alpha\in\Delta_{\sigma,+}} A^\alpha(y)$$

for the special case of the partition σ.

4.6 The Universal DT Series: General Case

In this section we will prove Theorem 4.1 for any partition σ.

4.6.1 Mutation and the root system

Recall that the simple reflection provides a bijection between $\Delta_{\sigma,+}\backslash\{\alpha_k\}$ and $\Delta_{\sigma',+}\backslash\{\alpha'_k\}$ (see §4.3.3). The simple root α_k maps to $-\alpha'_k$.

For $\alpha \in \Delta^{\mathrm{re}}_+$, let x_α be a simple module with $\underline{\dim}\alpha$. By [47, Proposition 2.14], $\sum_{i\notin\hat{I}_r}\alpha_i$ is odd (resp. even) if and only if $\operatorname{ext}^1(x,x) = 0$ (resp. $= 1$). In particular, the parity of $\sum_{i\notin\hat{I}_r}\alpha_i$ is preserved by the simple reflection.

4.6.2 Wall-crossing formula

Theorem 4.19. *[51, Theorem 4.9]*

$$A_U^{\sigma'}(\mathbf{y}) = \frac{A_U^\sigma(\mathbf{y})}{\mathbb{E}(y_k)} \times \mathbb{E}(y_k^{-1})$$

Proof. Step 1 : By the observation in §4.3.3, we have the following factorization:

$$A_U^\sigma = \mathbb{E}(y_k) \times A_k^\sigma$$

where

$$\mathbb{E}(y) := \sum_{n\geq 0} \frac{[\mathrm{pf}]}{[\mathrm{GL}_n]_{\mathrm{vir}}} \cdot y^n, \quad y_k := y_{\alpha_k}$$

and A^{σ}_k is the generating series of virtual motives of moduli stacks of objects in $(\mathrm{mod} J_\sigma)_k$. We also have

$$A^{\sigma'}_U = A^{\sigma',k} \times \mathbb{E}(y_k^{-1})$$

where $A^{\sigma',k}$ is the generating series of virtual motives of moduli stacks of objects in $(\mathrm{mod} J_{\sigma'})^k$.
Step 2 : By Proposition 4.7, we have $A^{\sigma}_k = A^{\sigma',k}$ (see [51, Theorem4.7]). □

Now Theorem 4.1 for any σ follows from the result in §4.5 combined with Theorem 4.19 and the remark in §4.6.1.

Remark 4.3. These mutations were first used in [47] for the case of the classical invariants.

4.6.3 Factorization of the universal series

We will say that a stability parameter ζ is generic, if for any stable J_σ-module V, we have $\zeta \cdot \underline{\dim} V \neq 0$. For generic stability parameter ζ, let $\mathfrak{M}^+_\zeta(J_\sigma, \alpha)$ (resp. $\mathfrak{M}^-_\zeta(J_\sigma, \alpha)$) denote the moduli stacks of J_σ-modules V such that $\underline{\dim} V = \alpha$ and such that all the HN factors F of V with respect to the stability parameter ζ satisfy $\zeta \cdot \underline{\dim} F > 0$ (resp. < 0). Let $[\mathfrak{M}^\pm_\zeta(J_\sigma, \alpha)]_{\mathrm{vir}}$ denote the virtual motive of the moduli stack defined in the same way as (4.3). We put

$$A^\pm_\zeta(y) = \sum_{\alpha \in \mathbb{N}^{\hat{I}}} [\mathfrak{M}^\pm_\zeta(J, \alpha)]_{\mathrm{vir}} \cdot y^\alpha.$$

Lemma 4.20. *[39, Lemma 2.6] The generating series $A^\pm_\zeta$ are given by*

$$A^\pm_\zeta(y) = \prod_{\alpha \in \Delta_{\sigma,+}, \pm\zeta\cdot\alpha < 0} A^\alpha(y).$$

4.7 Motivic DT with Framing

We denote by $\tilde{Q}_\sigma$ the new quiver obtained from Q_σ by adding a new vertex ∞ and a single new arrow $\infty \to 0$. Let $\tilde{J}_\sigma = J_{\tilde{Q}_\sigma, w_\sigma}$ be the Jacobian algebra corresponding to the quiver with potential $(\tilde{Q}_\sigma, w_\sigma)$, where we view w_σ as a potential for $\tilde{Q}_\sigma$ in the obvious way.

Let $\zeta \in \mathbb{R}^{\hat{I}}$ be a vector, which we will refer to as the stability parameter. A $\tilde{J}_\sigma$-representation $\widetilde{V}$ with $\dim \widetilde{V}_\infty = 1$ is said to be ζ-(semi)stable, if it is (semi)stable with respect to $(\zeta, \zeta_\infty) \in \mathbb{R}^{\hat{I} \sqcup \{\infty\}}$ (see Definition 4.2), where $\zeta_\infty = -\zeta \cdot \underline{\dim}\, V$. As in §4.4.2, a stability parameter $\zeta \in \mathbb{R}^{Q_0}$ is said to be *generic*, if for any stable J-module V we have $\zeta \cdot \underline{\dim}\, V \neq 0$.

For a stability parameter $\zeta \in \mathbb{R}^{Q_0}$ and a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{\hat{I}}$, let $\mathfrak{M}_\zeta(\widetilde{J}_\sigma, \alpha)$ denote the moduli stack of ζ-semistable $\widetilde{J}_\sigma$-representations with dimension vector $(\alpha, 1)$. As in the introduction, we define the generating function:

$$Z_\zeta(y_0, \ldots, y_{N-1}) = Z_\zeta(y) := \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^{\hat{I}}} \left[\mathfrak{M}_\zeta\big(\widetilde{J}_\sigma, \alpha\big)\right]_{\mathrm{vir}} \cdot y^\alpha.$$

Theorem 4.21. *[39, Proposition 4.6] For a generic stability parameter ζ, we have*

$$Z_\zeta(y) = \frac{A_\zeta^-(-\mathbb{L}^{\frac{1}{2}} y_0, y_1, \ldots, y_{N-1})}{A_\zeta^-(-\mathbb{L}^{-\frac{1}{2}} y_0, y_1, \ldots, y_{N-1})}, \tag{4.4}$$

where A_ζ^- were defined in §4.6.3.

Combined with Lemma 4.20, we get the formula in Corollary 4.2.

Remark 4.4. If we cross the wall W_α, we get (or lose) a factor $Z_\alpha(y)$ in the generating function. This is compatible with the result in [50].

4.8 DT/PT Series

4.8.1 Chambers in the moduli spaces

For a root $\alpha \in \Lambda$, let W_α denote the hyperplane in the space $\mathbb{R}^{\hat{I}}$ of stability parameters which is orthogonal to α. We put

$$W = W_\delta \cup \bigcup_{\alpha \in \Delta^{\mathrm{re}}_{\sigma,+}} W_\alpha.$$

A connected component of the complement of W in $\mathbb{R}^{\hat{I}}$ is called a chamber.

Theorem 4.22. [47, Proposition 2.10], [46, Proposition 3.10, 3.11] *The set of generic parameters in $\mathbb{R}^{\hat{I}}$ is the compliment of W.*

1. *For ζ with $\zeta_i < 0$ ($\forall i$), the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the NCDT moduli spaces, the moduli spaces of cyclic J-modules from [58].*

2. *For ζ in the same chamber as $(1 - N + \varepsilon, 1, 1, \ldots, 1)$ ($0 < \varepsilon \ll 1$), the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the DT moduli spaces of Y_σ from [37], the moduli spaces of subschemes on Y_σ with support in dimension at most 1.*

3. *For ζ in the same chamber as $(1 - N - \varepsilon, 1, 1, \ldots, 1)$ ($0 < \varepsilon \ll 1$), the moduli spaces $\mathfrak{M}_\zeta(\widetilde{J}, \alpha)$ are the PT moduli spaces of Y_σ introduced in [53]; these are moduli spaces of stable rank-1 coherent systems.*

Remark 4.5. In the above statements ε depends on the dimension vector $(\alpha, 1)$.

4.8.2 Motivic PT and DT invariants

Let

$$\zeta_{\mathrm{DT}} = (1 - N - \varepsilon, 1, 1, \ldots, 1), \quad \zeta_{\mathrm{PT}} = (1 - N + \varepsilon, 1, 1, \ldots, 1)$$

($0 < \varepsilon \ll 1$) be stability parameters corresponding to DT and PT moduli spaces. Then we have

$$\{\alpha \in \Delta_{\sigma,+} \mid \zeta_{\mathrm{DT}} \cdot \alpha < 0\} = \Delta_+^{\mathrm{re},+},$$
$$\{\alpha \in \Delta_{\sigma,+} \mid \zeta_{\mathrm{PT}} \cdot \alpha < 0\} = \Delta_+^{\mathrm{re},+} \sqcup \Delta_+^{\mathrm{im}}.$$

As we mentioned in the introduction the variable change induced by the derived equivalence is given by

$$s := y_0 \cdot y_1 \cdot \cdots \cdot y_{N-1}, \quad T_i = y_i.$$

Here s is the variable for the homology class of a point and T_i is the variable for the homology class of C_i. Then we get the formulae in Corollary 4.3.

4.8.3 Connection with the refined topological vertex

As studied in [49], we can apply the vertex operator method [52] to get a product expansion of the refined topological vertex for $\mathcal{Y}_\sigma$. Then we see that the PT generating function can be described by the refined topological vertices normalized by the refined MacMahon functions. [5]

4.9 Linear Algebra Computation

Throughout this computation we will work with a fixed choice of basis $\mathcal{B}$. In §4.5.2 we chose a basis

$$\mathcal{B} = \{H^p v_l^{a,b}(k) \mid 1 \le k \le b_l^{a,b}, 0 \le a, b \le N-1, 0 \le p \le N(l-1)+|b-a|+1\}$$

and defined linear spaces $F(\{\pi^{[a,b]}\}$ equal to

$$\left\{ L \in \bigoplus_{i \in I_1} \mathrm{Hom}(V_i, V_i) \oplus \bigoplus_{i \in I_2} \mathrm{Hom}(V_{i+1}, V_i) \;\middle|\; (L, H(\pi^{[a,b]})) \in R(\{\pi^{[a,b]}\}) \right\}$$

[5] Unfortunately, the DT generating function does not coincide with the refined topological vertex. See [39, §4.3] for detail.

and $S(\{\pi^{[a,b]}\})$ equal to

$$\left\{ N \in \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \mathrm{Hom}(V_i, V_i) \,\middle|\, \left[N, H(\pi^{[a,b]}) \right] = 0 \right\}$$

with dimensions $T(\{\pi^{[a,b]}\}) = \dim F(\{\pi^{[a,b]}\})$ and $B(\{\pi^{[a,b]}\}) = \dim S(\{\pi^{[a,b]}\})$. Our goal is to prove Proposition 4.17, that is to show that the difference $T(\{\pi^{[a,b]}\}) - B(\{\pi^{[a,b]}\})$ is equal to

$$-\frac{1}{2} \sum_{i \in I_2} \left(\sum_{b \neq i} l(\pi^{[i+1,b]}) - \sum_{c \neq i+1} l(\pi^{[c,i]}) \right)^2$$
$$-\frac{1}{2} \sum_{a \in I_3, b \notin I_2} \sum_{i \geq 1} (b_i^{a,b})^2 - \frac{1}{2} \sum_{a \notin I_3, b \in I_2} \sum_{i \geq 1} (b_i^{a,b})^2.$$

For some early examples it becomes clear that the dimension of $F(\{\pi^{[a,b]}\})$ and $S(\{\pi^{[a,b]}\})$ are determined by solving a set of linearly independent equations. We will see that these dimensions are quadratic polynomials in the number of parts $b_l^{a,b}$ of the partitions $\{\pi^{[a,b]}\}$. An initial means of simplifying the calculation is to break the spaces $F(\{\pi^{[a,b]}\})$ and $S(\{\pi^{[a,b]}\})$ down into simpler spaces. One easy observation is that not only are the spaces $F(\{\pi^{[a,b]}\})$ and $S(\{\pi^{[a,b]}\})$ linear but they come with a natural vector space structure, the origin corresponding to the zero matrix in both cases. This means that as vector spaces we have decompositions

$$F(\{\pi^{[a,b]}\}) = \bigoplus_{0 \leq a,b,c,d \leq N-1} F(\pi^{[a,b]}, \pi^{[c,d]})$$
$$S(\{\pi^{[a,b]}\}) = \bigoplus_{0 \leq a,b,c,d \leq N-1} S(\pi^{[a,b]}, \pi^{[c,d]})$$

whose summands are given by the following definition.

Definition 4.14. We define

$$\begin{aligned} F(\pi^{[a,b]}, \pi^{[c,d]}) &= F(\{\pi^{[a,b]}\}) \cap \bigoplus_{i \in I_1 \cup I_2} \mathrm{Hom}(V^{a,b}, V^{c,d}) \\ S(\pi^{[a,b]}, \pi^{[c,d]}) &= S(\pi^{[a,b]}, \pi^{[c,d]}) \cap \bigoplus_{i \in I_1 \cup I_2 \cup I_3} \mathrm{Hom}(V^{a,b}, V^{c,d}). \end{aligned}$$

These subspaces are essentially given by the block matrices for the decomposition $V = \bigoplus_{0 \leq a,b \leq N-1} V^{a,b}$.

Definition 4.15. We define

$$\begin{aligned} T(\pi^{[a,b]}, \pi^{[c,d]}) &= \dim F(\pi^{[a,b]}, \pi^{[c,d]}) \\ B(\pi^{[a,b]}, \pi^{[c,d]}) &= \dim S(\pi^{[a,b]}, \pi^{[c,d]}). \end{aligned}$$

Both $T(\pi^{[a,b]}, \pi^{[c,d]})$ and $B(\pi^{[a,b]}, \pi^{[c,d]})$ can be written as quadratic expressions in the number of parts of $\pi^{[a,b]}$ and $\pi^{[c,d]}$. To do this we introduce a quadratic form on the space of all partitions and a combinatorial operation that removes a box from each column of the the partition.

Definition 4.16. We define

$$\begin{aligned} M &: \mathcal{P} \otimes \mathcal{P} \to \mathbb{Z}_{\geq 0} \\ &: (1^{b_1} 2^{b_2} 3^{b_3} \cdots) \otimes (1^{c_1} 2^{c_2} 3^{c_3} \cdots) \mapsto \sum_{i \geq 1} \left(\sum_{j \geq i} b_j \right) \left(\sum_{j \geq i} c_j \right) \\ ' &: \mathcal{P} \to \mathcal{P} \\ &: \pi = (1^{b_1} 2^{b_2} 3^{b_3} \cdots) \mapsto \pi' = (1^{b_2} 2^{b_3} 3^{b_4} \cdots). \end{aligned}$$

Let us begin with the easier case. We compute dimensions $B(\pi^{[a,b]}, \pi^{[c,d]})$ of the spaces $S(\pi^{[a,b]}, \pi^{[c,d]})$.

Lemma 4.23. *Let $N \in S(\pi^{[a,b]}, \pi^{[c,d]})$ then the matrix N is uniquely determined by its value on the vectors $v_l^{a,b}(k)$. Moreover the only restriction on the image of such a vector is that it lie in the linear subspace*

$$N(v_l^{a,b}) \in V_a \cap V^{c,d} \cap V^{N \cdot (l-1) + |b-a| + 1}.$$

Proof. To define the linear map N on the space $V^{a,b}$ it suffices to define its value at each of the basis vectors

$$\{H^r v_l^{a,b}(k) \mid 0 \leq r \leq N \cdot (l-1) + |b-a|, 1 \leq k \leq b_l^{a,b}\}.$$

However for $N \in S(\pi^{[a,b]}, \pi^{[c,d]})$ we have

$$N(H^r v_l^{a,b}(k)) = H^r(N v_l^{a,b}(k)),$$

therefore the value of N at each $H^r v_l^{a,b}(k)$ is determined by $N v_l^{a,b}(k)$. This proves the first part of the lemma. Now we know that the matrix N maps the vector space at the ath vertex to itself $V_a \to V_a$, also since $N \in S(\pi^{[a,b]}, \pi^{[c,d]})$ we insist that its image be in $V^{c,d}$. The only additional condition on the image of the vector $v_l^{a,b}(k)$ is

$$H^{N\cdot(l-1)+|b-a|+1}(N v_l^{a,b}(k)) = N(H^{N\cdot(l-1)+|b-a|+1} v_l^{a,b}(k)) = 0.$$

Combining these three conditions above we have,

$$N(v_l^{a,b}(k)) \in V_a \cap V^{c,d} \cap V^{N\cdot(l-1)+|b-a|+1}.$$

□

Corollary 4.24. *We have*

$$B(\pi^{[a,b]}, \pi^{[c,d]}) = \begin{cases} M(\pi^{[a,b]}, \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| \leq |b-a| \\ M((\pi^{[a,b]})', \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| > |b-a| \\ M(\pi^{[a,b]}, (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| \leq |b-a| \\ M((\pi^{[a,b]})', (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| > |b-a|. \end{cases}$$

Proof. Let $N \in S(\pi^{[a,b]}, \pi^{[c,d]})$. Each vector $v_l^{a,b}(k)$ with $1 \leq k \leq b_l^{a,b}$ can take any value in the vector space $V_a \cap V^{c,d} \cap V^{N\cdot(l-1)+|b-a|+1}$ and so the dimension of $S(\pi^{a,b}, \pi^{[c,d]})$ is given by

$$B(\pi^{[a,b]}, \pi^{[c,d]}) = \sum_{l \geq 0} b_l^{a,b} \cdot \dim\left(V_a \cap V^{c,d} \cap V^{N\cdot(l-1)+|b-a|+1}\right).$$

Counting the number of basis vectors of $V^{c,d}$ that lie in V_a we see there are four possibilities for $\dim\left(V_a \cap V^{c,d} \cap V^{N\cdot(l-1)+|b-a|+1}\right)$:

$$\begin{array}{ll} \sum_{i=1}^{l} ib_i^{c,d} + l\sum_{i\geq l} b_i^{c,d} & \text{if } a \in [c,d] \text{ and } |d-a| \leq |b-a| \\ \sum_{i=1}^{l-1} ib_i^{c,d} + (l-1)\sum_{i\geq l} b_i^{c,d} & \text{if } a \in [c,d] \text{ and } |d-a| > |b-a| \\ \sum_{i=1}^{l} ib_{i+1}^{c,d} + l\sum_{i\geq l} b_{i+1}^{c,d} & \text{if } a \notin [c,d] \text{ and } |d-a| \leq |b-a| \\ \sum_{i=1}^{l-1} ib_{i+1}^{c,d} + (l-1)\sum_{i\geq l} b_{i+1}^{c,d} & \text{if } a \notin [c,d] \text{ and } |d-a| > |b-a|. \end{array}$$

Consider the first case $a \in [c,d]$ and $|d-a| \leq |b-a|$ then

$$\begin{aligned} B(\pi^{[a,b]}, \pi^{[c,d]}) &= \sum_{l\geq 1} b_l^{a,b} \cdot \left(\sum_{i=1}^{l} ib_i^{c,d} + l\sum_{i\geq l} b_i^{c,d} \right) \\ &= \sum_{i\geq 1} \left(\sum_{l\geq i} b_l^{a,b} \right) \cdot \left(\sum_{l\geq i} b_l^{c,d} \right) \\ &= M(\pi^{[a,b]}, \pi^{[c,d]}). \end{aligned}$$

The other three cases are identical. The relabeling of the partitions in these cases is encoded by the operation $\pi \mapsto \pi'$. □

Now we turn to computing the dimensions $T(\pi^{[a,b]}, \pi^{[c,d]})$ of the spaces $F(\pi^{[a,b]}, \pi^{[c,d]})$. This will be more intricate.

Lemma 4.25. *Suppose $a \in I_1 \cup I_3$ and $L \in F(\pi^{[a,b]}, \pi^{[c,d]})$ then the map L is uniquely determined by its value on the vectors $v_l^{a,b}(k)$. Moreover the only restriction on the image of such a vector is that it lie in a linear subspace*

$$Lv_l^{a,b}(k) \in \begin{cases} V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \text{if } a \in I_1 \text{ and } b \notin I_2 \\ V_a \cap V^{N\cdot(l-1)+|b-a|} \cap V^{c,d} & \text{if } a \in I_1 \text{ and } b \in I_2 \\ V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+2} \cap V^{c,d} & \text{if } a \in I_3 \text{ and } b \notin I_2 \\ V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \text{if } a \in I_3 \text{ and } b \in I_2. \end{cases}$$

Proof. To define the linear map L on the space $V^{a,b}$ it suffices to define its value at each of the basis vectors

$$\{H^r v_l^{a,b}(k) \mid 0 \leq r \leq N \cdot (l-1) + |b-a|, 1 \leq k \leq b_l^{a,b}\}.$$

However for $L \in F(\pi^{[a,b]}, \pi^{[c,d]})$ we know that the pair $(L, H(\pi^{[a,b]})) \in R(\{\pi^{[a,b]}\})$ satisfy the relations coming from the potential:

$$\begin{array}{lcll} r_i h^+_{i-1/2} & = & h^+_{i-1/2} r_{i-1} & \text{for } i \in [1, N'-1] \cap I_1 \\ r_0 h^+_{N-1/2} & = & h^+_{N-1/2} h^+_{N-3/2} h^-_{N-3/2} & \\ h^-_{N'+1/2} h^+_{N'+1/2} h^+_{N'-1/2} & = & h^+_{N'-1/2} r_{N'-1} & \\ h^-_{i+3/2} h^+_{i+3/2} h^+_{i+1/2} & = & h^+_{i+1/2} h^+_{i-1/2} h^-_{i-1/2} & \text{for } i = [N'+1, N-3] \cap I_3. \end{array}$$

As in Lemma 4.23 once the value of L is determined for $v_l^{a,b}(k)$ it is uniquely determined for all $H^r v_l^{a,b}(k)$ by the condition that the above relations be satisfied for the pair $(L, H(\pi^{[a,b]})$. To be precise if $a \in I_1$ we have

$$L : H^r(v_l^{a,b}(k)) \mapsto \begin{cases} H^r L(v_l^{a,b}(k)) & \text{if } a + r \in I_1 \\ 0 & \text{if } a + r \in I_2 \\ H^{r-1} L(v_l^{a,b}(k)) & \text{if } a + r \in I_3 \end{cases}$$

and if $a \in I_3$ then

$$L : H^r(v_l^{a,b}(k)) \mapsto \begin{cases} H^{r+1} L(v_l^{a,b}(k)) & \text{if } a + r \in I_1 \\ 0 & \text{if } a + r \in I_2 \\ H^r L(v_l^{a,b}(k)) & \text{if } a + r \in I_3. \end{cases}$$

Since $L \in F(\pi^{[a,b]}, \pi^{[c,d]})$ by definition its image must lie in the space $V^{c,d}$, also if $a \in I_1$ then $L : V_a \to V_a$ and if $a \in I_3$ then $L : V_a \to V_{a-1}$. The only further condition on the image of a vector $v_l^{a,b}(k)$ is that its image be killed by a high enough power of H. It is given that $H^{N \cdot (l-1)+|b-a|+1} v_l^{a,b}(k) = 0$ so then $H^t(L v_l^{a,b}(k)) = 0$ where the exponent t is read off for the defining relations on L above. In the separate cases

$$Lv_l^{a,b}(k) \in \begin{cases} V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \text{if } a \in I_1 \text{ and } b \notin I_2 \\ V_a \cap V^{N\cdot(l-1)+|b-a|} \cap V^{c,d} & \text{if } a \in I_1 \text{ and } b \in I_2 \\ V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+2} \cap V^{c,d} & \text{if } a \in I_3 \text{ and } b \notin I_2 \\ V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \text{if } a \in I_3 \text{ and } b \in I_2. \end{cases}$$

proving the result. □

We have a result similar to Lemma 4.25 when $a \in I_2$.

Lemma 4.26. *Suppose $a \in I_2$ and $L \in F(\pi^{[a,b]}, \pi^{[c,d]})$ then the map L is uniquely determined by its value on the vectors $Hv_l^{a,b}(k)$. Moreover the only restriction on the image of such a vector is that it lie in a linear subspace*

$$L(Hv_l^{a,b}(k)) \in \begin{cases} V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \textit{if } b \notin I_2 \\ V_a \cap V^{N\cdot(l-1)+|b-a|} \cap V^{c,d} & \textit{if } b \in I_2 \end{cases}$$

Proof. Again we know that to define the linear map L on the space $V^{a,b}$ it suffices to define its value at each of the basis vectors

$$\{H^r v_l^{a,b}(k) \mid 0 \leq r \leq N \cdot (l-1) + |b-a|, 1 \leq k \leq b_l^{a,b}\}.$$

Since by definition if $a \in I_2$ then $Lv_l^{a,b}(k) = 0$ the map is already trivially determined on these vectors and their image does not suffice to determine the map in general. However if we consider the vectors $Hv_l^{a,b}(k)$ then once the value of L is determined for $Hv_l^{a,b}(k)$ it is uniquely determined for all $H^r v_l^{a,b}(k)$ by the condition that the relations (see Lemma 4.25) be satisfied by the pair $(L, H(\pi^{[a,b]})$. To be precise if $a \in I_2$ we have

$$L : H^r(v_l^{a,b}(k)) \mapsto \begin{cases} H^r L(Hv_l^{a,b}(k)) & \text{if } a+r \in I_1 \\ 0 & \text{if } a+r \in I_2 \\ H^{r-1} L(Hv_l^{a,b}(k)) & \text{if } a+r \in I_3. \end{cases}$$

By definition we know that the image of L lies in $V^{c,d}$ and also that for $a \in I_2$ we have $L : V_{a+1} \to V_a$. As before the only remaining condition on the image of $v_l^{a,b}(k)$ is that it be killed by a high enough power of H. From the definition of L above we see that

$$L(Hv_l^{a,b}(k)) \in \begin{cases} V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d} & \text{if } b \notin I_2 \\ V_a \cap V^{N\cdot(l-1)+|b-a|} \cap V^{c,d} & \text{if } b \in I_2 \end{cases}$$

proving the result. □

The following notation collects the dimensions of all the vector spaces encountered in the last two Lemmas.

Definition 4.17. We define integers

$$d_{a,b:c,d}(l) = \begin{cases} \dim(V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d}) & \text{if } a \in I_1 \cup I_2 \text{ and } b \notin I_2 \\ \dim(V_a \cap V^{N\cdot(l-1)+|b-a|} \cap V^{c,d}) & \text{if } a \in I_1 \cup I_2 \text{ and } b \in I_2 \\ \dim(V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+2} \cap V^{c,d}) & \text{if } a \in I_3 \text{ and } b \notin I_2 \\ \dim(V_{a-1} \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d}) & \text{if } a \in I_3 \text{ and } b \in I_2. \end{cases}$$

From Lemmas 4.25 and 4.26 we deduce the dimension of the spaces $F(\pi^{[a,b]}, \pi^{[c,d]})$.

Corollary 4.27. *If $a \in I_1 \cup I_2$ and $b \notin I_2$ then $T(\pi^{[a,b]}, \pi^{[c,d]})$ equals*

$$\begin{cases} M(\pi^{[a,b]}, \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| \le |b-a| \\ M((\pi^{[a,b]})', \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| > |b-a| \\ M(\pi^{[a,b]}, (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| \le |b-a| \\ M((\pi^{[a,b]})', (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| > |b-a|. \end{cases}$$

If $a \in I_1 \cup I_2$ and $b \in I_2$ then $T(\pi^{[a,b]}, \pi^{[c,d]})$ equals

$$\begin{cases} M(\pi^{[a,b]}, \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| \le |b-a|-1 \\ M((\pi^{[a,b]})', \pi^{[c,d]}) & \text{if } a \in [c,d] \text{ and } |d-a| > |b-a|-1 \\ M(\pi^{[a,b]}, (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| \le |b-a|-1 \\ M((\pi^{[a,b]})', (\pi^{[c,d]})') & \text{if } a \notin [c,d] \text{ and } |d-a| > |b-a|-1. \end{cases}$$

If $a \in I_3$ *and* $b \notin I_2$ *then* $T(\pi^{[a,b]}, \pi^{[c,d]})$ *equals*

$$\begin{cases} M(\pi^{[a,b]}, \pi^{[c,d]}) & \text{if } a-1 \in [c,d] \text{ and } |d-(a-1)| \leq |b-a|+1 \\ M((\pi^{[a,b]})', \pi^{[c,d]}) & \text{if } a-1 \in [c,d] \text{ and } |d-(a-1)| > |b-a|+1 \\ M(\pi^{[a,b]}, (\pi^{[c,d]})') & \text{if } a-1 \notin [c,d] \text{ and } |d-(a-1)| \leq |b-a|+1 \\ M((\pi^{[a,b]})', (\pi^{[c,d]})') & \text{if } a-1 \notin [c,d] \text{ and } |d-(a-1)| > |b-a|+1. \end{cases}$$

If $a \in I_3$ *and* $b \in I_2$ *then* $T(\pi^{[a,b]}, \pi^{[c,d]})$ *equals*

$$\begin{cases} M(\pi^{[a,b]}, \pi^{[c,d]}) & \text{if } a-1 \in [c,d] \text{ and } |d-(a-1)| \leq |b-a| \\ M((\pi^{[a,b]})', \pi^{[c,d]}) & \text{if } a-1 \in [c,d] \text{ and } |d-(a-1)| > |b-a| \\ M(\pi^{[a,b]}, (\pi^{[c,d]})') & \text{if } a-1 \notin [c,d] \text{ and } |d-(a-1)| \leq |b-a| \\ M((\pi^{[a,b]})', (\pi^{[c,d]})') & \text{if } a-1 \notin [c,d] \text{ and } |d-(a-1)| > |b-a|. \end{cases}$$

Proof. We know that if $a \in I_1 \cup I_3$ (resp. $a \in I_2$) then the map $L \in F(\pi^{[a,b]}, \pi^{[c,d]})$ is determined by its value at the vectors $v_l^{a,b}(k)$ (resp. $Hv_l^{a,b}(k)$) for $1 \leq k \leq b_l^{a,b}$. In the notation of the previous definition such a vector takes values in a space of dimension $d_{a,b;c,d}(l)$. So in all cases the total dimension of the space $F(\pi^{[a,b]}, \pi^{[c,d]})$ equals

$$T(\pi^{[a,b]}, \pi^{[c,d]}) = \sum_{l \geq 1} b_l^{a,b} \cdot d_{a,b;c,d}(l).$$

In the above definition of $d_{a,b;c,d}(l)$ there are four possible forms depending on the value of a and b. Lets consider the first case when $a \in I_1 \cup I_2$ and $b \notin I_2$. Then we have that $d_{a,b;c,d}(l) = \dim\left(V_a \cap V^{N\cdot(l-1)+|b-a|+1} \cap V^{c,d}\right)$. Counting the number of basis vectors of $V^{c,d}$ that lie in V_a we see there are four possibilities for $\dim\left(V_a \cap V^{c,d} \cap V^{N\cdot(l-1)+|b-a|+1}\right)$:

$$\begin{array}{ll} \sum_{i=1}^{l} i b_i^{c,d} + l \sum_{i \geq l} b_i^{c,d} & \text{if } a \in [c,d] \text{ and } |d-a| \leq |b-a| \\ \sum_{i=1}^{l-1} i b_i^{c,d} + (l-1) \sum_{i \geq l} b_i^{c,d} & \text{if } a \in [c,d] \text{ and } |d-a| > |b-a| \\ \sum_{i=1}^{l} i b_{i+1}^{c,d} + l \sum_{i \geq l} b_{i+1}^{c,d} & \text{if } a \notin [c,d] \text{ and } |d-a| \leq |b-a| \\ \sum_{i=1}^{l-1} i b_{i+1}^{c,d} + (l-1) \sum_{i \geq l} b_{i+1}^{c,d} & \text{if } a \notin [c,d] \text{ and } |d-a| > |b-a|. \end{array}$$

In the first case $a \in [c,d]$ and $|d-a| \leq |b-a|$ and

$$\begin{aligned} T(\pi^{[a,b]},\pi^{[c,d]}) &= \sum_{l\geq 1} b_l^{a,b} \cdot \left(\sum_{i=1}^{l} i b_i^{c,d} + l \sum_{i\geq l} b_i^{c,d} \right) \\ &= \sum_{i\geq 1} \left(\sum_{l\geq i} b_l^{a,b} \right) \cdot \left(\sum_{l\geq i} b_l^{c,d} \right) \\ &= M(\pi^{[a,b]},\pi^{[c,d]}). \end{aligned}$$

In the second case $a \in [c,d]$ and $|d-a| > |b-a|$ and

$$\begin{aligned} T(\pi^{[a,b]},\pi^{[c,d]}) &= \sum_{l\geq 1} b_l^{a,b} \cdot \left(\sum_{i=1}^{l-1} i b_i^{c,d} + (l-1) \sum_{i\geq l} b_i^{c,d} \right) \\ &= \sum_{i\geq 1} \left(\sum_{l\geq i} b_{l+1}^{a,b} \right) \cdot \left(\sum_{l\geq i} b_l^{c,d} \right) \\ &= M((\pi^{[a,b]})',\pi^{[c,d]}). \end{aligned}$$

In the third case $a \notin [c,d]$ and $|d-a| \leq |b-a|$ and

$$\begin{aligned} T(\pi^{[a,b]},\pi^{[c,d]}) &= \sum_{l\geq 1} b_l^{a,b} \cdot \left(\sum_{i=1}^{l} i b_{i+1}^{c,d} + l \sum_{i\geq l} b_{i+1}^{c,d} \right) \\ &= \sum_{i\geq 1} \left(\sum_{l\geq i} b_l^{a,b} \right) \cdot \left(\sum_{l\geq i} b_{l+1}^{c,d} \right) \\ &= M(\pi^{[a,b]},(\pi^{[c,d]})'). \end{aligned}$$

Finally in the fourth case $a \notin [c,d]$ and $|d-a| > |b-a|$ and we have

$$\begin{aligned} T(\pi^{[a,b]}, \pi^{[c,d]}) &= \sum_{l \geq 1} b_l^{a,b} \cdot \left(\sum_{i=1}^{l-1} i b_{i+1}^{c,d} + (l-1) \sum_{i \geq l} b_{i+1}^{c,d} \right) \\ &= \sum_{i \geq 1} \left(\sum_{l \geq i} b_{l+1}^{a,b} \right) \cdot \left(\sum_{l \geq i} b_{l+1}^{c,d} \right) \\ &= M((\pi^{[a,b]})', (\pi^{[c,d]})'). \end{aligned}$$

This completes the situation when $a \in I_1 \cup I_2$ and $b \notin I_2$. In the other situations $a \in I_1 \cup I_2$ and $b \in I_2$, or $a \in I_3$ and $b \notin I_2$, or $a \in I_3$ and $b \in I_2$. All of these cases can be dealt with in a similar manner. $\square$

Now we have computed all the dimensions $T(\pi^{[a,b]}, \pi^{[c,d]})$ and $B(\pi^{[a,b]}, \pi^{[c,d]})$. The next lemma combines corollaries 4.24 and 4.27 to compute their difference. We see that in most cases there is an exact cancellation.

Lemma 4.28. *We have*

$$T(\pi^{[a,b]}, \pi^{[c,d]}) = B(\pi^{[a,b]}, \pi^{[c,d]}) \textit{ aside from the following cases,}$$

Case 1: $a \in I_1 \cup I_2$, $b = d \in I_2$

$$\begin{array}{ll} M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]}) & \text{if } a \in [c,b] \\ M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})') & \text{if } a \notin [c,b]. \end{array}$$

Case 2: $a \in I_3$, $b \notin I_2$, $d = a-1 \in I_2$

$$\begin{array}{ll} M(\pi^{[a,b]}, \pi^{[a,a-1]}) - M((\pi^{[a,b]})', \pi^{[a,a-1]}) & \text{if } a = c \\ M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})') & \text{if } a \neq c. \end{array}$$

Case 3: $a \in I_3$, $b \notin I_2$, $a = c$, $d \neq a-1$,

$$\begin{array}{ll} M(\pi^{[a,b]}, (\pi^{[a,d]})') - M(\pi^{[a,b]}, \pi^{[a,d]}) & \text{if } |d-a| \leq |b-a|, \\ M((\pi^{[a,b]})', (\pi^{[a,d]})') - M((\pi^{[a,b]})', \pi^{[a,d]}) & \text{if } |d-a| > |b-a|. \end{array}$$

Case 4: $a \in I_3,\ b \in I_2,\ d = a-1,$

$$\begin{array}{ll} M(\pi^{[a,b]}, \pi^{[a,a-1]}) - M((\pi^{[a,b]})', \pi^{[a,a-1]}) & \text{if } a = c \text{ and } b \neq a-1 \\ M(\pi^{[a,a-1]}, \pi^{[c,a-1]}) - M(\pi^{[a,a-1]}, (\pi^{[c,a-1]})') & \text{if } a \neq c \text{ and } b = a-1 \\ M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})') & \text{if } a \neq c \text{ and } b \neq a-1. \end{array}$$

Case 5: $a \in I_3,\ b \in I_2,\ a-1 \in [c,d],\ d \neq a-1,\ b = d$

$$M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]}).$$

Case 6: $a \in I_3,\ b \in I_2,\ a-1 \notin [c,d],\ a = c,\ |d-a| < |b-a|$

$$M(\pi^{[a,b]}, (\pi^{[a,d]})') - M(\pi^{[a,b]}, \pi^{[a,d]}).$$

Case 7: $a \in I_3,\ b \in I_2,\ a-1 \notin [c,d]$

$$\begin{array}{ll} M((\pi^{[a,b]})', (\pi^{[a,b]})') - M(\pi^{[a,b]}, \pi^{[a,b]}) & \text{if } a = c \text{ and } b = d \\ M((\pi^{[a,b]})', (\pi^{[a,d]})') - M((\pi^{[a,b]})', \pi^{[a,d]}) & \text{if } a = c \text{ and } |d-a| > |b-a| \\ M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})') & \text{if } a \neq c \text{ and } b = d. \end{array}$$

Proof. Compare corollaries 4.24 and 4.27. □

Our aim throughout this appendix has been to prove Proposition 4.17 and deduce that the difference $\sum_{0 \le a,b,c,d \le N-1} T(\pi^{[a,b]}, \pi^{[c,d]}) - B(\pi^{[a,b]}, \pi^{[c,d]})$ equals

$$-\frac{1}{2}\sum_{i \in I_2}\left(\sum_{b \neq i} l(\pi^{[i+1,b]}) - \sum_{c \neq i+1} l(\pi^{[c,i]})\right)^2 - \frac{1}{2}\sum_{a \in I_3, b \notin I_2}\sum_{i \geq 1}(b_i^{a,b})^2 - \frac{1}{2}\sum_{a \notin I_3, b \in I_2}\sum_{i \geq 1}(b_i^{a,b})^2.$$

So all that remains is to check this sum agrees with the values we computed. First we will transform it into a expression in terms of the $M(\pi^{[a,b]}, \pi^{[c,d]})$.

To do this we need the simple identities

$$
\begin{aligned}
M(\pi^{[a,b]},\pi^{[c,d]}) - M((\pi^{[a,b]})',(\pi^{[c,d]})') &= \sum_{l\geq 1}\left(\sum_{i\geq l} b_i^{a,b}\cdot\sum_{i\geq l} b_i^{c,d} - \sum_{i\geq l} b_{i+1}^{a,b}\cdot\sum_{i\geq l} b_{i+1}^{c,d}\right)\\
&= \sum_{i\geq 1} b_i^{a,b}\cdot\sum_{i\geq 1} b_i^{c,d}\\
&= l(\pi^{[a,b]})\cdot l(\pi^{[c,d]})
\end{aligned}
$$

and

$$
\begin{aligned}
M(\pi^{[a,b]},\pi^{[a,b]}) - M((\pi^{[a,b]})',\pi^{[a,b]}) &= \sum_{l\geq 1}\left(\sum_{i\geq l} b_i^{a,b}\cdot\sum_{i\geq l} b_i^{a,b} - \sum_{i\geq l} b_{i+1}^{a,b}\cdot\sum_{i\geq l} b_i^{a,b}\right)\\
&= \sum_{l\geq 1} b_l^{a,b}\cdot\sum_{i\geq l} b_i^{c,d}\\
&= \frac{1}{2}l(\pi^{[a,b]})^2 + \frac{1}{2}\sum_{l\geq 1}(b_l^{a,b})^2.
\end{aligned}
$$

Using these two identities and some simple algebraic manipulations we can rewrite Proposition 4.17 as the statement that the difference $\sum_{0\leq a,b,c,d\leq N-1} T(\pi^{[a,b]},\pi^{[c,d]}) - B(\pi^{[a,b]},\pi^{[c,d]})$ equals

$$
\begin{aligned}
&\sum_{i\in I_2}\sum_{b\neq i, c\neq i+1} && M(\pi^{[i+1,b]},\pi^{[c,i]}) && - && M((\pi^{[i+1,b]})',(\pi^{[c,i]})')\\
+\ &\sum_{i\in I_2}\sum_{\substack{b<d\\ b,d\neq i}} && M((\pi^{[i+1,b]})',(\pi^{[i+1,d]})') && - && M(\pi^{[i+1,b]},\pi^{[i+1,d]})\\
+\ &\sum_{i\in I_2}\sum_{\substack{a<c\\ a,c\neq i+1}} && M((\pi^{[a,i]})',(\pi^{[c,i]})') && - && M(\pi^{[a,i]},\pi^{[c,i]})\\
+\ &\sum_{a\in I_3, b\in I_2, b\neq a-1} && M((\pi^{[a,b]})',(\pi^{[a,b]})') && - && M(\pi^{[a,b]},\pi^{[a,b]})\\
+\ &\sum_{[a,b]\in S} && M((\pi^{[a,b]})',\pi^{[a,b]}) && - && M(\pi^{[a,b]},\pi^{[a,b]}).
\end{aligned}
$$

We will take a systematic approach, accounting for these terms one by one, all in all we will check nine separate cases.

First let us assess the contribution from terms involving partitions $\pi^{[r,s]}$ with $r,s\in I_1$. Comparing with Lemma 4.28 in all seven cases there is no discrepancy when $a,b\in I_1$ or $c,d\in I_1$ and therefore there is no contribution from these terms in agreement with the above sum.

Secondly we assess the contribution from terms involving partitions $\pi^{[r,s]}$ with $r \in I_1$ and $s \in I_2$. Considering Lemma 4.28 we note the following cases,

Case 1: $a \in I_1 \cup I_2, b \in I_2, b = d$

$$\begin{array}{ll} M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]}) & \text{if } a \in [c,b] \\ M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})') & \text{if } a \notin [c,b]. \end{array}$$

Case 2: $a \in I_3, b \notin I_2, c \in I_1, d = a-1 \in I_2$

$M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{a,b})', (\pi^{[c,a-1]})')$.

Case 4: $a \in I_3, b \in I_2, c \in I_1, d = a-1$

$$\begin{array}{ll} M(\pi^{[a,a-1]}, \pi^{[c,a-1]}) - M(\pi^{[a,a-1]}, (\pi^{[c,a-1]})') & \text{if } b = a-1 \\ M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})') & \text{if } b \neq a-1. \end{array}$$

Case 5: $a \in I_3, b \in I_2, c \in I_2, b = d, a-1 \in [c,b], b \neq a-1$

$M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]})$.

Case 7: $a \in I_3, b \in I_2, c \in I_1, b = d, a-1 \notin [c,b]$

$M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})')$.

The sum total of these cases gives

$$\begin{array}{rccc} & \displaystyle\sum_{a \in I_1, b \in I_2} M((\pi^{[a,b]})', \pi^{[a,b]}) & - & M(\pi^{[a,b]}, \pi^{[a,b]}) \\ + & \displaystyle\sum_{\substack{a<c: a,c \neq b+1 \\ a \in I_1, b \in I_2 \text{ or } c \in I_1, b \in I_2}} M((\pi^{[a,b]})', (\pi^{[c,b]})') & - & M(\pi^{[a,b]}, \pi^{[c,b]}) \\ + & \displaystyle\sum_{b \neq i, c \in I_1, i \in I_2} M(\pi^{[i+1,b]} \pi^{[c,i]}) & - & M((\pi^{[i+1,b]})', (\pi^{[c,i]})') \end{array}$$

this accounts for all the terms involving partitions $\pi^{[a,b]}$ with $a \in I_1$ and $b \in I_2$.

Thirdly we assess the contribution from terms involving partitions $\pi^{[r,s]}$ with $r \in I_1$ and $s \in I_3$. Comparing with lemma 4.28 in all seven cases there is no discrepancy when $a \in I_1$ and $b \in I_3$ or $c \in I_1$ and $d \in I_3$

and therefore there is no contribution from these terms in agreement with Proposition 4.17. We have now observed the correct contributions from all terms involving partitions $\pi^{[r,s]}$ where $r \in I_1$.

Fourthly we will consider contributions from terms involving partitions $\pi^{[r,s]}$ where $r \in I_2$ and $s \in I_1$. As in the first and third cases on comparing with lemma 4.28 in all seven cases there is no discrepancy when $a \in I_2$ and $b \in I_1$ or $c \in I_2$ and $d \in I_1$. Again this is in full agreement with Proposition 4.17.

Fifthly we consider contributions from terms involving partitions $\pi^{[r,s]}$ where $r \in I_2$ and $s \in I_2$. This time on comparing with lemma 4.28 we observe some nontrivial contributions as desired. This case is almost identical to when $r \in I_1$ and $s \in I_2$. In the lemma the following cases contribute

Case 1: $a \in I_1 \cup I_2, b \in I_2, b = d$

$$M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]}) \quad \text{if } a \in [c,b]$$
$$M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})') \quad \text{if } a \notin [c,b].$$

Case 2: $a \in I_3, b \notin I_2, c \in I_2, d = a-1 \in I_2$

$$M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{a,b})', (\pi^{[c,a-1]})').$$

Case 4: $a \in I_3, b \in I_2, c \in I_2, d = a-1$

$$M(\pi^{[a,a-1]}, \pi^{[c,a-1]}) - M(\pi^{[a,a-1]}, (\pi^{[c,a-1]})') \quad \text{if } b = a-1$$
$$M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})') \quad \text{if } b \neq a-1.$$

Case 5: $a \in I_3, b \in I_2, c \in I_2, b = d, a-1 \in [c,b], b \neq a-1$

$$M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]}).$$

Case 7: $a \in I_3, b \in I_2, c \in I_2, b = d, a-1 \notin [c,b]$

$$M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})').$$

The sum total of these cases gives

$$\begin{array}{llcl} & \sum_{a\in I_2,b\in I_2} M((\pi^{[a,b]})',\pi^{[a,b]}) & - & M(\pi^{[a,b]},\pi^{[a,b]}) \\ + & \sum_{\substack{a<c:a,c\neq b+1\\ a,b\in I_2 \text{ or } c,b\in I_2}} M((\pi^{[a,b]})',(\pi^{[c,b]})') & - & M(\pi^{[a,b]},\pi^{[c,b]}) \\ + & \sum_{c\in I_2 i\in I_2 b\neq i} M(\pi^{[i+1,b]}\pi^{[c,i]}) & - & M((\pi^{[i+1,b]})',(\pi^{[c,i]})'). \end{array}$$

These terms again agree with those of Proposition 4.28.

Sixthly we move to consider terms involving partitions $\pi^{[r,s]}$ where $r \in I_2$ and $s \in I_3$. considering the lemma we see that there is no contribution for these terms as desired. We have now observed the correct contributions from all terms involving partitions $\pi^{[r,s]}$ where $r \in I_1 \cup I_2$ only the cases when $r \in I_3$ remain, we may restrict to consider only those differences $T(\pi^{[a,b]},\pi^{[c,d]}) - B(\pi^{[a,b]},\pi^{[c,d]})$ where both $a, c \in I_3$.

Seventhly we consider terms involving partitions $\pi^{[r,s]}$ where $r \in I_3$ and $s \in I_1$. Considering lemma 4.28 we see that the following cases give non-trivial contributions.

Case 2: $a \in I_3, b \notin I_2, d = a-1 \in I_2$

$M(\pi^{[a,b]},\pi^{[a,a-1]}) - M((\pi^{[a,b]})',\pi^{[a,a-1]})$ if $a = c$

$M(\pi^{[a,b]},\pi^{[c,a-1]}) - M((\pi^{[a,b]})',(\pi^{[c,a-1]})')$ if $a \neq c$.

Case 3: $a \in I_3, b \notin I_2, a = c, d \neq a-1$

$M(\pi^{[a,b]},(\pi^{[a,d]})') - M(\pi^{[a,b]},\pi^{[a,d]})$ if $|d-a| \leq |b-a|$,

$M((\pi^{[a,b]})',(\pi^{[a,d]})') - M((\pi^{[a,b]})',\pi^{[a,d]})$ if $|d-a| > |b-a|$.

Case 6: $a \in I_3, b \in I_2, a-1 \notin [c,d], a = c, |d-a| < |b-a|, d \in I_1$

$M(\pi^{[a,b]},(\pi^{[a,d]})') - M(\pi^{[a,b]},\pi^{[a,d]})$.

Case 7: $a \in I_3, b \in I_2, a-1 \notin [c,d], d \in I_1$

$M((\pi^{[a,b]})',(\pi^{[a,d]})') - M((\pi^{[a,b]})',\pi^{[a,d]})$ if $a = c$ and $|d-a| > |b-a|$.

The sum total of these cases gives

$$\begin{array}{llll} & \displaystyle\sum_{a\in I_3, b\in I_1} & M((\pi^{[a,b]})', \pi^{[a,b]}) & - \quad M(\pi^{[a,b]}, \pi^{[a,b]}) \\ + & \displaystyle\sum_{\substack{b<d:b,d\neq a \\ a\in I_2 \text{ and } b\in I_1 \text{ or } d\in I_1}} & M((\pi^{[a+1,b]})', (\pi^{[a+1,d]})') & - \quad M(\pi^{[a+1,b]}, \pi^{[a+1,d]}) \\ + & \displaystyle\sum_{i\in I_2 b\in I_1} & M(\pi^{[i+1,b]}\pi^{[c,i]}) & - \quad M((\pi^{[i+1,b]})', (\pi^{[c,i]})'). \end{array}$$

Eighthly we consider terms involving partitions $\pi^{[r,s]}$ where $r \in I_3$ and $s \in I_3$. Here considering lemma 4.28 we see that the following cases give non-trivial contributions.

Case 2: $a \in I_3, b \notin I_2, d = a-1 \in I_2$

$M(\pi^{[a,b]}, \pi^{[a,a-1]}) - M((\pi^{[a,b]})', \pi^{[a,a-1]})$ if $a = c$

$M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})')$ if $a \neq c$.

Case 3: $a \in I_3, b \notin I_2, a = c, d \neq a-1$

$M(\pi^{[a,b]}, (\pi^{[a,d]})') - M(\pi^{[a,b]}, \pi^{[a,d]})$ if $|d-a| \leq |b-a|$,

$M((\pi^{[a,b]})', (\pi^{[a,d]})') - M((\pi^{[a,b]})', \pi^{[a,d]})$ if $|d-a| > |b-a|$.

Case 6: $a \in I_3, b \in I_2, a-1 \notin [c,d], a = c, |d-a| < |b-a|, d \in I_3$

$M(\pi^{[a,b]}, (\pi^{[a,d]})') - M(\pi^{[a,b]}, \pi^{[a,d]})$.

Case 7: $a \in I_3, b \in I_2, a-1 \notin [c,d], d \in I_3$

$M((\pi^{[a,b]})', (\pi^{[a,d]})') - M((\pi^{[a,b]})', \pi^{[a,d]})$ if $a = c$ and $|d-a| > |b-a|$.

The sum total of these cases gives

$$\begin{array}{llll} & \sum_{a\in I_3, b\in I_3} & M((\pi^{[a,b]})', \pi^{[a,b]}) & - \quad M(\pi^{[a,b]}, \pi^{[a,b]}) \\ + & \sum_{\substack{b<d:b,d\neq a \\ a\in I_2 \text{ and } b\in I_3 \text{ or } d\in I_3}} & M((\pi^{[a+1,b]})', (\pi^{[a+1,d]})') & - \quad M(\pi^{[a+1,b]}, \pi^{[a+1,d]}) \\ + & \sum_{i\in I_2 b\in I_3} & M(\pi^{[i+1,b]}\pi^{[c,i]}) & - \quad M((\pi^{[i+1,b]})', (\pi^{[c,i]})'). \end{array}$$

Now we have accounted for all terms involving partitions other that the case $\pi^{[r,s]}$ with $r \in I_3$ and $s \in I_2$. So we can now restrict to consider terms $T(\pi^{[a,b]}, \pi^{[c,d]}) - B(\pi^{[a,b]}, \pi^{[c,d]})$ with $a, c \in I_3$ and $b, d \in I_2$ as follows.

Ninthly we consider terms involving partitions $\pi^{[r,s]}$ where $r \in I_3$ and $s \in I_2$. Here considering lemma 4.28 we see that the following cases give non-trivial contributions.

Case 4: $a \in I_3, b \in I_2, d = a-1,$

$$\begin{array}{ll} M(\pi^{[a,b]}, \pi^{[a,a-1]}) - M((\pi^{[a,b]})', \pi^{[a,a-1]}) & \text{if } a = c \text{ and } b \neq a-1 \\ M(\pi^{[a,a-1]}, \pi^{[c,a-1]}) - M(\pi^{[a,a-1]}, (\pi^{[c,a-1]})') & \text{if } a \neq c \text{ and } b = a-1 \\ M(\pi^{[a,b]}, \pi^{[c,a-1]}) - M((\pi^{[a,b]})', (\pi^{[c,a-1]})') & \text{if } a \neq c \text{ and } b \neq a-1. \end{array}$$

Case 5: $a \in I_3, b \in I_2, a-1 \in [c,d], d \neq a-1, b = d$

$M((\pi^{[a,b]})', \pi^{[c,b]}) - M(\pi^{[a,b]}, \pi^{[c,b]})$.

Case 6: $a \in I_3, b \in I_2, a-1 \notin [c,d], a = c, |d-a| < |b-a|$

$M(\pi^{[a,b]}, (\pi^{[a,d]})') - M(\pi^{[a,b]}, \pi^{[a,d]})$.

Case 7: $a \in I_3, b \in I_2, a-1 \notin [c,d]$

$$\begin{array}{ll} M((\pi^{[a,b]})', (\pi^{[a,b]})') - M(\pi^{[a,b]}, \pi^{[a,b]}) & \text{if } a = c \text{ and } b = d \\ M((\pi^{[a,b]})', (\pi^{[a,d]})') - M((\pi^{[a,b]})', \pi^{[a,d]}) & \text{if } a = c \text{ and } |d-a| > |b-a| \\ M((\pi^{[a,b]})', (\pi^{[c,b]})') - M(\pi^{[a,b]}, (\pi^{[c,b]})') & \text{if } a \neq c \text{ and } b = d. \end{array}$$

The sum total of these cases gives the remaining terms

$$\begin{aligned}
& \sum_{a\in I_3, b\in I_2, a-1\neq b} && M((\pi^{[a,b]})', (\pi^{[a,b]})') && - && M(\pi^{[a,b]}, \pi^{[a,b]}) \\
+ & \sum_{\substack{b<d: b,d\neq a \\ a\in I_2 \text{ and } b\in I_2 \text{ or } d\in I_2}} && M((\pi^{[a+1,b]})', (\pi^{[a+1,d]})') && - && M(\pi^{[a+1,b]}, \pi^{[a+1,d]}) \\
+ & \sum_{\substack{a<c: a,c\neq b \\ b\in I_2 \text{ and } a\in I_3 \text{ or } c\in I_3}} && M((\pi^{[a,b]})', (\pi^{[c,b]})') && - && M(\pi^{[a,b]}, \pi^{[c,b]}) \\
+ & \sum_{\substack{b\neq i \text{ and } a-1\neq i \\ i\in I_2 \text{ and } b\in I_2 \text{ or } c\in I_3}} && M(\pi^{[i+1,b]}, \pi^{[c,i]}) && - && M((\pi^{[i+1,b]})', (\pi^{[c,i]})').
\end{aligned}$$

This completes the proof of Proposition 4.17.

Chapter 5

Conclusion

5.1 Summary of Results

The central goal of this thesis was to produce a method to compute the motivic Donaldson–Thomas invariants of three dimensional Calabi–Yau categories [33]. Theorem 2.8 now provides a schematic way to determine these invariants.

We have applied this to the derived category of coherent sheaves on various Calabi–Yau threefolds [39] [40]. In particular we have produced formulas for the motivic Donaldson–Thomas invariants of crepant resolutions in all chambers of our space of stability parameters, for example Theorem 3.1 and Corollary 4.2.

One obvious drawback of the method is that the categories we work with must come from a quiver with potential. In this way it is not possible to study derived categories of sheaves on a projective Calabi–Yau threefold. Indeed this remains a problem in the computation of the classical invariants, which have yet to be computed for the quintic threefold, the main known examples are toric [37].

Secondly, from the quiver point of view we have an affine space of stability conditions. In the cases we have studied the classical DT/PT/ncDT moduli spaces are all present in this picture [46],[47]. However when the threefold contains a divisor class (e.g. local surfaces) this is not known to be true.

Hopefully new techniques can be developed in the future to broaden our field of view. Joint work with Jim Bryan is underway in this direction, see future work below.

5.2 Connections to String Theory

In type IIB string theory there exists a collection of dynamical objects know as D-branes. The branes which saturate a certain energy bound are called BPS-states. Mathematically these BPS-branes can be described by the category of coherent sheaves on a Calabi–Yau threefold. Bridgeland defined a stability condition on this category inspired by the notion of Π-stability in the physics literature [10]. From this modern point of view the Donadlson–Thomas invariants should be considered virtual counts of the number of stable BPS-branes.

Recently string theorist have been working to understand a so called refined BPS state count. It is conjectured that these refined counts are given by the motivic Donaldson–Thomas invariants computed in this thesis. One validation of this comes from comparing our results with those of Dimofte and Gukov [17]. Another interesting recent development of physicists is the formulation of a refined topological vertex [26], this provides a formalism to compute refined BPS invariants on toric Calabi–Yau threefolds. We notice that this formalism is in agreement with our motivic calculations (see Section 3.5.3 and Section 4.8.3). Hopefully future work will be fruitful in producing a mathematical version of this refined vertex.

5.3 Future Research

5.3.1 Localization

This is work in progress with Jim Bryan. Our aim is to discover a localization type formula for computing motivic vanishing cycles (and hence motivic Donaldson–Thomas invariants).

Let $f : X \to \mathbb{C}$ be a regular function on a smooth quasi-projective scheme. We assume that f is $\mathbb{C}^*$-equivariant so that $f(t \cdot x) = t \cdot f(x)$ and also that $\lim_{t\to 0} t \cdot x$ exists for all $x \in X$. Then we know (see Theorem 2.5) that the (absolute) motivic vanishing cycle has a trivial monodromy action and equals the difference of the general and central fibers

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)] \in K_0(Var_{\mathbb{C}}).$$

We would like to describe this motivic class as a sum over the fixed points of the $\mathbb{C}^*$-action. We have shown that the only fixed points that have non-trivial contribution are those in the singular locus of the central fibre. As mentioned above a good localization formula would potentially be useful in the calculation of motivic DT invariants. We would like to prove the following:

Conjecture 5.1. *The motivic DT series for the orbifold* $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ *equals*

$$\begin{aligned} Z(y_0, y_1, y_2, y_3) = \prod_{m\geq 1} & \left(\prod_{k=0}^{m-1} (1 - \mathbb{L}^{-\frac{m}{2}+1+k} y^m)^{-3} (1 - \mathbb{L}^{-\frac{m}{2}+2+k} y^m)^{-1} \right) \\ & \cdot \left(\prod_{k=0}^{m-1} \prod_{i=0}^{3} (1 + \mathbb{L}^{-\frac{m}{2}+\frac{1}{2}+k} y^m y_i)^{-1} (1 + \mathbb{L}^{-\frac{m}{2}+\frac{1}{2}+k} y^m y_i^{-1})^{-1} \right) \\ & \cdot \left(\prod_{k=0}^{m-1} \prod_{0\leq i<j\leq 3} (1 - \mathbb{L}^{-\frac{m}{2}+1+k} y^m y_i^{-1} y_j^{-1}) \right) \end{aligned}$$

where $y := y_0 y_1 y_2 y_3$.

I have verified this conjecture with a computer in low degrees.

In [62], B. Young solved the corresponding enumerative problem by using Graber–Pandharipande localization [21]. Then the classical DT invariant equals the number of torus fixed points on the Hilbert scheme. There is a bijection

$$\{\text{piles of 3d - boxes}\} \longleftrightarrow \{\mathbb{C}^*\text{- fixed points of Hilb}(\mathbb{C}^3)\}.$$

Young enumerates piles of colored 3d - boxes using a clever trick involving vertex operator algebras. I am optimistic that finding a motivic localization formula will provide a refined combinatorial generating series which can be handled in a similar way. Through solving this problem we are aiming to develop a mathematical framework for the refined topological vertex.

In physics the topological vertex [1] provides a powerful formalism to compute the A-model topological string partition function. A mathematical counterpart of this was developed via localization [37] [11] giving a way to compute the classical DT invariants of any toric Calabi-Yau threefold solely in terms of its web diagram.

Recently, A. Iqbal, C. Kozçaz and C. Vafa [26] have proposed a refined topological vertex as a means to compute the refined topological string partition function. This vertex is more subtle giving refined BPS state counts and breaking much of the symmetry enjoyed by the usual vertex. I hope that by providing a localization formula for motivic vanishing cycles we will give a mathematical description of this object and provide a calculus formalizing the computation of motivic DT invariants on toric Calabi-Yau threefolds. However such a localization formula could be useful for computing the cohomology of vanishing cycles in other contexts.

5.3.2 Motivic zeta functions

Another interesting avenue for future research comes from the singularities of schemes $\mathcal{M} = \{df = 0\}$ that are the degeneracy locus of a regular function on a smooth variety $f : X \to \mathbb{C}$. The motivic zeta function contains much more information about the singularities of $\mathcal{M}$ than the motivic vanishing cycle. The theory of local zeta functions also has analytic and p-adic versions. The interaction of these theories is very interesting. In particular the geometry of the singularities of f determines much of the number theoretic behavior. It is conjectured (see [25]) that the poles of the local zeta function are given by the roots of the Bernstein-Sato polynomial $b_f(s)$.

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