Essays on Real Estate
Finance and Economics

Strategies for Developments under Uncertainty

by

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Abstract

This thesis contains three essays on real estate finance and economics. Both chapter 2 and chapter 3 study a real estate developer’s timing strategies under uncertainty and chapter 4 studies the ratio of housing price-to-income.

Chapter 2 analyzes the effects of leverage on the timing of developments because most real estate developers depend heavily on leverage. In particular, it investigates the leverage effects on real estate developments that involve conversion of land from agricultural to urban use. Leverage matters for a developer who wants to optimize the timing of developments.

Chapter 3 studies the effects of heterogeneity of real assets on the timing of developments because each property is unique and its value is likely to move together with values of neighboring properties. Heterogeneity matters for a developer who builds multiple properties in a project.

Chapter 4 analyzes the price-to-income ratio which has been used widely as a measure for housing bubble. A cross-city comparison of price-to-income ratios can overestimate bubble in cities of high amenities.
Preface

Chapter 4 is coauthored with Professor Sanghoon Lee. I was responsible for the empirical analysis and the literature review.
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Chapter 1

Introduction

Real estate developers make strategic decisions under uncertainty. As developers can delay a project that incurs sunk cost, they take account of positive opportunity cost of a development option. Development options are widely investigated in the literature on real estate development. Most previous studies, however, propose a pure equity developer, an agent we are not likely to meet in the real estate market. In addition, they consider one underlying asset associated with revenue, even though developers build multiple assets of non-homogeneity.

In chapter 2 we analyze the effects of leverage on the timing of real estate developments that involve conversion of land from agricultural to urban use. We assume that, at the time the land conversion occurs, a developer who maximizes the equity value will obtain a defaultable construction loan at fair market value. Moreover, we abstract from any sources of market friction. We show that, under uncertainty, a leveraged developer is likely to exercise the land conversion option earlier than an unleveraged developer is. Seeking to lower the uncertainty premium, such a leveraged developer has an incentive to lower the threshold level of the land conversion. In urban growth models with irreversible development, the lower threshold level expands equilibrium city size.

In chapter 3 we build a real options model with multiple underlying assets because most real estate developers build multiple real properties of heterogeneity in a project. In the real estate market, each property is unique and its value is spatially correlated to values of neighboring properties. Product diversification strategies can reduce the value of a development option. Under short-sale constraints for the assets, a development option reaches its maximum value with the perfectly positively correlated assets; the value of such an option increases with the spatial autocorrelation coefficient. In addition, we examine the effects of both asset- and project-level uncertainty on timing decisions for real estate development.

While previous chapters study a developer's timing strategies, chapter 4 studies a new
topic: price-to-income ratio. In chapter 4, we argue that a cross-city comparison of price-to-income ratios overestimates the bubble in high QOL (quality of life) cities. The theory is based on Roback (1982). In high QOL cities people are willing to pay high housing rents and to get paid lower wages. This would lead to higher price-to-income ratios even if there were no bubble. We test the theory using data. The challenge is that expected price growth rate may be correlated with QOL across cities. We use our model to disentangle these two effects. Our empirical results show that the QOL bias is significant.
Chapter 2

The Leveraged City

2.1 Introduction

We examine the effects of leverage on the timing of real estate developments that involve conversion of land from agricultural to urban use. A developer determines when to convert undeveloped real estate. When converting such real estate, most developers depend heavily on leverage; real estate development is a “highly leveraged business” (Bookout, 2000). Allen (1995) documents that a range of leverage ratios in real estate is around 60-80%. We study the effects of a defaultable construction loan on levels of reservation rent and price that trigger the conversion.

We show that the timing strategy for a real estate developer can depend on leverage. In particular, a leveraged developer is likely to exercise the land conversion option earlier than an unleveraged developer is. In an uncertain environment, waiting for information is beneficial for such developers. Nevertheless, waiting is less valuable for a leveraged developer than it is for a non-leveraged developer because, by exercising a default option, the leveraged developer can avoid downside outcomes.

Investment timing strategies are investigated in the literature. In this chapter, nonetheless, a strategy for real estate development is not determined by conflict of interest between existing shareholders in Myers (1977) because a debt-equity relationship is formed at the timing of investment. In real estate development, most developers set up a project company funded with equity and debt. Unlike an investor in Modigliani and Miller (1958), moreover, an equity-oriented real estate developer pays construction cost with loans and controls timing strategies. The model of this chapter also holds for other investments, in which an equity holder makes investment decisions.

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1Developers finance development costs with equity, debt, or both. In chapter 2, nonetheless, a developer is an equity investor who finances a construction loan. For details, see section 2.4.

2Financial leverage is a basic concept covered by most real estate textbooks, such as Geltner et al. (2006), and Brueggeman and Fisher (2001). Geltner et al. (2006) argue that developers who “borrow to the hilt” must be quite “creative to make the necessary profits and avoid losing their equity by default.”
In urban growth models, the boundary of an urban area is endogenously determined by land developers. Excepting Riddiough (1997, hereafter Riddiough) and Marseguerra and Cortelezzi (2009, hereafter Marseguerra and Cortelezzi), previous studies have proposed a pure equity developer, an individual we are not likely to meet in the real estate market. In an uncertain business environment, for example, such a developer delays irreversible investment, requiring an uncertainty (or option) premium. Therefore equilibrium city size shrinks because uncertainty increases the threshold level of the land conversion (Capozza and Helsley, 1990, hereafter Capozza and Helsley).

This chapter attempts to incorporate leverage in an optimal timing framework in urban growth models with irreversible investment. In order to incorporate leverage, we first assume that a developer maximizes the value of equity. With limited liability, the developer controls investment decisions including timing decisions. Second, we assume that such a developer obtains a defaultable construction loan. In a perfectly competitive capital market, in addition, a lender originates a commercial loan at fair market value when the land conversion occurs. We observe the developer’s debt to construction cost ratio at the timing of development and the debt-to-equity ratio changes under uncertainty.

Leverage matters to a developer who wants to optimize the timing of the land conversion. Leverage erodes the value of the uncertainty premium, which is interpreted as a markup in Dixit et al. (1999, hereafter Dixit et al.). Seeking to lower the uncertainty premium, the leveraged developer has an incentive to lower the threshold level of the land conversion. In urban growth models, as a consequence, the lower threshold level expands equilibrium city size. Development activities increase in response to improved accessibility to debt credit. In the leveraged city, housing supply is likely to contribute to real estate market dynamics.

By introducing leverage, our analysis extends the literature on real options in real estate development in a standard dynamic programming framework. First, we show that a defaultable loan tends to reduce the uncertainty premium. We allow two coexisting options for real estate developers. Nonetheless, we abstract from tax shields in Marseguerra and Cortelezzi and from a land acquisition loan in Riddiough. Second, we propose that the value of an individual property can be determined by the financial condition of a single borrower. To show the value relevance of financial conditions, however, previous studies assume that some borrowers have the same financial status (Stein, 1995; Kiyotaki and Moore, 1997 and Ortalo-Magne and Rady, 2006). In addition, we contribute a novel
approach to the corporate finance literature. We propose simple closed-form expressions for equity and debt values with a normal diffusion process\(^3\). And we provide a closed-form solution for the problem of leveraged investments.

Several studies consider the developer’s financial conditions. Nevertheless, most of them look at financing constraints (Ambrose and Peek, 2008 and Mayer and Somerville, 1996), rather than capital structure. Note that the goal of the paper is not to determine the developer’s financial mix, but rather to examine the effects of risky debt on the timing of a development. In addition, without any sources of market friction, we cannot endogenize capital structure (Modigliani and Miller, 1958). Investigating borrower’s investment incentives explicitly, nonetheless, chapter 2 also differs from studies on capital structure in real estate (Gau and Wang, 1990). With an optimal stopping framework as well, we study the compound option, whose value is sophisticationally analyzed in Geske (1979). Moreover, the leveraged developer’s incentive problem is distinct from that of Riddiough, because the developer has no pre-existing capital structure before launching a stand-alone project.

The remainder of chapter 2 is structured as follows. Section 2.2 delivers a brief summary of the real options literature both in real estate development and in loan default. Section 2.3 presents a simple model of a pure equity developer, which can be found in the literature. The reader familiar with the literature may skip sections 2.2 and 2.3. Section 2.4 presents the key theoretical model of a developer who obtains a defaultable development loan at the time the land conversion occurs. Section 2.5 applies our model to a durable model of cities. Section 2.6 concludes.

2.2 Literature Survey

This section provides a brief summary of the theoretical real options literature in real estate. We restrict our attention to two options, the development option for land conversion and the default option on debt in the real estate market.

2.2.1 Real Options in Real Estate Development

In an uncertain business environment, developers can delay development, which incurs irreversible expenditures.

\(^3\)Investment projects such as real estate developments may produce negative cash flows. For details, see Capozza and Schwann (1990), Capozza and Sick (1994), and Capozza and Li (1994).
The real estate literature claims that, under uncertainty, developers delay land conversion; development triggers incur the uncertainty premium. Even though developers face multiple uncertainty in the production process, Capozza and Helsley, Capozza and Li (1994, hereafter Capozza and Li), and Capozza and Sick (1994, hereafter Capozza and Sick) suggest a simple assumption of uncertain (bid land) rent that follows an exogenous Arithmetic Brownian motion. Capozza and Helsley show in particular that the land value at the boundary of an urban area adds an additional uncertainty premium. Consequently, equilibrium city size shrinks in urban growth models of irreversible investment. With an optimal stopping framework, this chapter’s position is close to that of Capozza and Helsley in that we examine the implications of uncertainty on the city. Presenting a more general form of Capozza and Helsley’s uncertainty premium, we will show that uncertainty is less of a deterrent to land conversion.

We consider a leveraged developer who determines both the timing of development and the timing of default, as Riddiough and Marseguerra and Cortelezzi also do. However, our developer is distinguished from Riddiough’s developer, because ours finances the development cost with debt, whereas Riddiough’s developer financed a debt for the land acquisition before the development is initiated. Riddiough proposes Myers’s (1977) underinvestment incentive for the developer who has land financing-in-place, but finances no debt at the stage of the development.

We argue that a developer has an overinvestment incentive with a newly issued loan, as Marseguerra and Cortelezzi do. However, our developer is distinguished from Marseguerra and Cortelezzi’s developer, because ours enjoys no interest tax advantage. Their developer resembles Mauer and Sarkar’s (2005) investor, who moves the exercise of the option forward in order to “speed up the realization of interest tax shields earned on debt financing.” Nonetheless, tax-exempt loans are available for real estate developers.\(^4\) With no tax advantage, our leveraged developer has an incentive such as that of Jensen and Meckling’s (1976) overinvestment.

To highlight leverage effects, we will deliberately abstract from factors that developers may take into consideration. Developers often determine development density, which a local authority may control (Capozza and Li). They may be under flexible zoning controls that permit mixed-use development on a single site (Childs et al., 1996 and Geltner et al.\(^4\))

\(^4\) In the US, Miles et al. (2000) report that between 1999 to 2003 the Treasury expected to subsidize around ten billion dollars for the tax-exempt bonds including mortgage revenue bonds.
Developers look at uncertainty based on development cost (Clarke and Reed, 1988) and development lag (Bar-Ilan and Strange, 1996). Developers are heterogeneous (Novy-Marx, 2007) and take into account market structures (Grenadier, 1996; and Wang and Zhou, 2006). Over a long horizon, moreover, developers may develop a property repeatedly (Williams, 1997), or they may abandon it (Williams, 1991).

### 2.2.2 Default Options in Construction Loans

In an uncertain environment, borrowers can delay defaulting on construction loans as they can benefit from delaying such decisions. In the real estate literature, previous studies focus on default on mortgage loans.

Borrowers and lenders can design mortgage contracts that are embedded with various option provisions. With respect to options, most residential mortgage borrowers have options for early termination events such as default and prepayment. In countries such as Canada and the US, nonetheless, residential mortgage borrowers can also purchase a closed-end mortgage that has no prepayment option. In particular, we are looking at the commercial real estate finance market, in which most borrowers are likely to have no prepayment option (Titman and Torous, 1989).

In a dynamic environment, uncertainty creates value in waiting and a sophisticated borrower delays default (Epperson et al., 1985; Kau and Kim, 1994; and Kau et al., 1993). As a borrower, the developer in chapter 2 holds a default option, whose value is incorporated into the equity value. Consequently, the value of an equity investment as a call option is duplicated with a portfolio involving a put option on default.

The mortgage literature suggests with respect to a strategic default decision that a borrower is likely to default when the value of an underlying asset reaches the value of the loan. The default decision, nonetheless, can be distorted by transaction costs (Ambrose et al., 1997; and Capozza et al., 1997), which may vary by types of borrowers or across regions (Deng et al., 2000). After controlling for cost, however, most studies seem to share

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5In the real estate market, empirical evidence is increasing in favor of real options theories. Quigg (1993) proposes that the value of a development option is embedded in the value of land. Holland et al. (2000) and Cunningham (2006) find that the relationship between uncertainty and investment is negative in the commercial market and in the residential market, respectively. The growth control, moreover, can be a deterrent to exercising an option in Seattle (Cunningham, 2007). Controlling for non-diversifiable risks, Bulan et al. (2009) claim that market competition is likely to erode the option value that arises from both idiosyncratic and market risks. In the Canadian market, Somerville (2001) suggests that, rather than starting or completing construction, obtaining a construction permit is close to exercising an option.

---

7
in the consensus that a strategic decision is based on a rational approach to an analysis of the loan default. To create the simplest possible model, furthermore, we assume there are no default-related costs for either the developer or the lender in the theoretical model.

2.3 A Model of the Development Option

Before examining leverage effects, we will heuristically propose a simple framework for the real options model in real estate development; the reader familiar with real options theories may skip this chapter. Note that, in section 2.3, the developer obtains no leverage for (re)development.

2.3.1 A Framework

A developer who owns an underdeveloped building (vacant land) can convert the old building (land) to a new building, while keeping monopolistic power over the location. For simplicity, we suppose that a zoning regulation keeps the developer from changing the current density of one unit, from transferring the current developable right, and from developing a mixed-use building. The developer’s problem is to determine the optimal timing of the development.

Assumptions

The net cash flows from the new and pre-existing buildings are determined according to the net rents represented by the matrix \( \mathbf{R} = (R_1, R_2)^T \), where \( T \) denotes the transpose; the elements \( R_1 \) and \( R_2 \) are the net rent from the new building and the net rent from the pre-existing building (or agricultural land), respectively. The matrix \( \mathbf{R} \) is assumed to conform to the following normal process.

---

6 We can find empirical studies that support the rational approach in both commercial and residential mortgage markets. Titman and Torous (1989) show that the rational approach has sound predictive power; rates for commercial mortgages increase with the volatility of the value of underlying assets. Vandell et al. (1993) document negative relationships between loan-to-value (LTV) ratio and conditional probability of survival to the average loan. The rational approach seems robust in consideration of prepayment and transaction costs (Ciochetti et al., 2002; and Crawford and Rosenblatt, 1995). In the residential mortgage market, moreover, Deng et al. (2000) show with Federal Home Loan Mortgage Corporation data that mortgages with more than 90 per cent initial LTV are 4 to 5 times more likely to default than are those with 80 to 90 per cent LTV. In addition, Ambrose et al. (2001) support the rational approach with Federal Housing Association (FHA) data.

7 The time and building arguments are suppressed if they are clear from the context.
\[ d\mathbf{R} = \mathbf{g}dt + \mathbf{M}dB, \quad (2.1) \]

where the matrix of growth rates \( \mathbf{g} \) is a matrix whose elements are standard Brownian motions. The variance-covariance matrix of changes in rent, \( \Sigma = \mathbf{M}(\mathbf{R})\mathbf{M}^T(\mathbf{R}) \), is symmetric and nonnegative definite. Note that the elements in \( \mathbf{R} \) follow arithmetic Brownian motions.

Throughout chapter 2 we assume that both the developer and the lender are risk-neutral, share information on the realized net rent flows up to time \( t \), and have the same expectations about the distribution of future rents according to equation (2.1). The riskless rate of interest \( r \) is fixed over time and known.

The price of a building in a competitive market equals the expected present value of future rent:

\[ P_{it} \equiv \mathbb{E}_t\left[ \int_t^\infty \{ R_{is}\text{e}^{-r(s-t)} \} ds | R_{it} \right] = \frac{1}{r}(R_{it} + \frac{g}{r}) \quad i = 1, 2, \quad (2.2) \]

where \( \mathbb{E}_t \) is conditional on \( R_{it} \).

The Value of the Development Option

The value of a pre-existing building with a development option would be higher than the value of a durable building would be, as a pre-existing building can be converted to a new building.\(^8\) The value of a pre-existing building with an option is

\[ P_{c2t} = \sup_{\mathbf{t}} \mathbb{E}_t\left\{ \int_t^{\tilde{t}} R_{2s}\text{e}^{-r(s-t)} ds + \int_{\tilde{t}}^\infty R_{1s}\text{e}^{-r(s-t)} ds - Ce^{-r(\tilde{t}-t)} \right\}, \quad (2.3) \]

where \( C \) is the fixed construction cost, and \( \tilde{t} \) is an optimal timing for development adapted to the rent flows up to \( t \). The first term in (2.3) is the present value of the net cash flows from the existing building up to time \( \tilde{t} \). The second term is the present value of cash flows from the new durable building after time \( \tilde{t} \). The third term is the discounted development cost.

Equation (2.3) is converted to

\[ P_{c2t} = P_{2t} + W(\mathbf{R}), \quad (2.4) \]

\(^8\)If \( \tilde{t} \) converges on \( \infty \), the value of development option \( W(\mathbf{R}) \) in (2.4) vanishes and \( P_{c2t}(t) = P_{2t} \).
where \( W(\mathcal{R}) \) is the value of the development option.

The developer’s problem conditioned on the level of rents at the time of development
\[
\mathcal{R} \equiv (R_{1t}, R_{2t})^T
\]
is
\[
W(\mathcal{R}) = \sup_{\bar{t}} \mathbb{E}_t[\int_{\bar{t}}^\infty (R_{1s} - R_{2s})e^{-r(s-\bar{t})}ds - C]e^{-(\bar{t}-t)}|\mathcal{R}, (2.5)
\]
where \( \bar{t} \) is the (unknown) first-hitting time from \( \mathcal{R} \) to \( \mathcal{R} \).

According to both the Bellman equation and Itô’s lemma, the solution for \( W(\mathcal{R}) \) in
\[(2.5)\]
evolves according to the second-order linear ordinary differential equation,
\[
\frac{\sigma_{11}}{2} W_{R_1R_1}(\mathcal{R}) + \frac{\sigma_{22}}{2} W_{R_2R_2}(\mathcal{R}) + \sigma_{12} W_{R_1R_2}(\mathcal{R}) + g_1 W_{R_1}(\mathcal{R}) + g_2 W_{R_2}(\mathcal{R}) - r W(\mathcal{R}) = 0,
\]
where \( W_R \) and \( W_{RR} \) are the first and second derivatives of \( W(\mathcal{R}) \), respectively, and \( \sigma_{ij} \)
is the \( ij \)th element of \( \Sigma \) in the continuation region \( B = \{\mathcal{R}|V(\mathcal{R}) < W(\mathcal{R})\} \), where
\( V(\mathcal{R}) = P_1(R_{1t}) - P_2(R_{2t}) - C \).

The solution has the smooth-pasting conditions,
\[
W_{R_i}(\bar{\mathcal{R}}) = \frac{1}{r}, \quad \text{for } i = 1, 2, \quad (2.6b)
\]
where \( \bar{\mathcal{R}} \) is a point on the boundary of the continuation region. The solution in \((2.6a)\) also
requires the value-matching and non-negative conditions,
\[
W(\bar{\mathcal{R}}) = \frac{1}{r}(\bar{R}_1 + \frac{g_1}{r}) - \frac{1}{r}(\bar{R}_2 + \frac{g_2}{r}) - C, \quad (2.6c)
\]
\[
W(\mathcal{R}) \geq 0. \quad (2.6d)
\]

Equation \((2.6c)\) indicates that, at time \( \bar{t} \), the value of the development option equals the
value of the development. Equation \((2.6d)\) is the condition for the value of the development
option.

In closing this section, note that rents from new and pre-existing buildings on a given
site cannot be uncorrelated. The analytic solution of equations \((2.6a) - (2.6d)\) are unknown.
Most previous studies consider a case of vacant land development in which
\( R_{2s} = A \geq 0 \) agricultural land rent, \( g_2 = 0 \), \( \sigma_2 = 0 \), and \( \sigma_{12} = 0 \) in \((2.1)\).
2.3.2 The Development Option on Vacant Land

In this section, we suppose that, at the boundary of the urban area, the developer converts land from agricultural to urban use. In the durable city model, the urban boundary is determined by a developer who converts agricultural land at distance $z$ from the Central Business District (CBD); see section 2.5 for details.

After the development is set in motion, the net rents evolve according to a stochastic process

$$dR = g dt + \sigma dB,$$  
(2.7)

where $g$ is the constant drift and $\sigma$ is the constant standard deviation of the rent process.

The present value of the net rents from a durable building in (2.2) is converted to

$$P = \frac{1}{r} \left( R + \frac{g}{r} \right),$$  
(2.8)

with (2.7) and $R \geq -\frac{g}{r}$ for $P \geq 0$.

Before the development occurs, the developer’s problem in (2.5) is converted to

$$W(R_t) = \sup_{\{\bar{t}\}} \mathbb{E}_t\{ \int_{\bar{t}}^{\infty} (R_s - A) e^{-r(s-t)} ds - C \} e^{-r(\bar{t}-t)} | R_t \},$$  
(2.9)

where $R$ and $\bar{R}$ are levels of current rent and of rent at $\bar{t}$, respectively, and

$$\bar{t} = \inf_{\{\bar{t}\}} \{ t + \tau \geq t | R_{t+\tau} \geq \bar{R} \equiv R_{\bar{t}} \}.$$

Agricultural land rent $A$ in (2.9) is constant and can be regarded as the opportunity cost. Note that, from (2.9), the developer has no land acquisition loan whose effects on development are analyzed in Riddiough.

To analyze the development option, we can convert equation (2.6a) to

$$\frac{\sigma^2}{2} W_{RR}(R) + g W_R(R) - r W(R) = 0, \quad R < \bar{R}.$$  
(2.10a)

The associated homogeneous equation for (2.10a) is $W(R) = W_1 e^{\alpha R} + W_2 e^{\beta R}, R < \bar{R}$, where $W_1$ and $W_2$ are to be determined, and $\alpha = (-g + \sqrt{g^2 + 2\sigma^2 r})/\sigma^2 > 0$ and $\beta = \ldots$
\[-(g + \sqrt{g^2 + 2 \sigma^2 r})/\sigma^2 < 0\]  

The boundary conditions in (2.6b) - (2.6d) are

\[
\begin{align*}
W_R(\bar{R}) &= \frac{1}{r}, \\
W(\bar{R}) &= \frac{1}{r}(\bar{R} + \frac{g}{r}) - A - C, \\
\lim_{R \to -\infty} W(R) &= 0.
\end{align*}
\]

The analytic solution for equations (2.10a) - (2.10d):

\[
W(R) = \frac{1}{r}(\bar{R} + \frac{g}{r} - A - rC)e^{-\alpha(\bar{R}-R)}, \quad R < \bar{R},
\]

where the reservation rent, at which development occurs, is

\[
\bar{R} = A + rC + \frac{r - \alpha g}{\alpha r},
\]

where \( r - \alpha g > 0 \). The first and second terms on the right-hand side in (2.11b) are agricultural rent and rent on development costs, respectively. The third term is an increment due to uncertainty.

The optimal development timing in (2.11b) produces the reservation price of developed land

\[
P = \frac{A}{r} + C + \frac{g}{r^2} + \frac{r - \alpha g}{\alpha r^2},
\]

with equation (2.8). The first two terms on the right-hand side in (2.12) are the value of agricultural rents and development costs; the value of agricultural rents is the value of alternate use. The third and fourth terms are the growth premium and the uncertainty premium, respectively. See Capozza and Helsley for details.

### 2.3.3 The Optimal Development Rules

This section explains the optimal development rules; smooth-pasting, value-matching and initial conditions.

9The dynamic behavior of (2.10a) is studied by transforming \( D^n = W_n \); we can rewrite (2.10a) to \( LW(R) = 0 \) where a linear operator \( L \equiv D^2 + \frac{2g}{r}D - \frac{r}{g} \) according to the superposition principle. Based on the Wronskian, a unique solution also exists.

10The stochastic discount rate is lower than the risk-free rate: \( \lim_{\sigma \to 0} \alpha = r/g \)
First, at the optional timing, the slope (derivative) of the value of the development option equals the slope (derivative) of the net value of the development; the developer equalizes both the marginal cost and the marginal benefit. We can covert (2.11a) to

\[ W(R) = \frac{1}{\alpha r} e^{-\alpha R}, \quad (2.13) \]

with (2.11b). The slope of the value of the development option is \( W'(\bar{R}) = \frac{1}{\bar{r}} \).\(^{11}\)

Dixit et al. propose such a condition with a “trade-off between larger versus later net benefits” for a monopoly who adopts price-setting strategies.\(^{12}\) Suppose that the current level of the project value is \( \tilde{P} \) in (2.8). Note that the project is to be completed at \( \tilde{P} > \hat{P} \). Consider an arbitrage trigger level of the project value \( \bar{P} > P_0 \), where \( P_0 \) is the initial value; levels of rent can be represented by levels of the project value according to (2.8). At \( \bar{P} > P_0 \), technically, we derive the first unknown hitting time \( \bar{P} \); with the known distribution of first-hitting times under a Brownian motion, we can derive the expected value of the land conversion project. The right-hand side of (2.11a) is the net present value of the project:

\[ \Pi(\tilde{P}, P_0) \equiv D(\tilde{P}, P_0)(\tilde{P} - C), \quad (2.14) \]

where \( C \equiv \frac{\alpha}{\bar{r}} + C \). Following Dixit et al., we denote \( D(\tilde{P}, P_0) \equiv e^{-\alpha (\tilde{P} - P_0)} \) as the discount factor.

The optimal threshold \( \hat{P} \) is the value of \( \tilde{P} \) which maximizes equation (2.14). The first-order condition is

\[ \hat{P} + \frac{D(\hat{P}, P_0)}{D_{\hat{P}}(\hat{P}, P_0)} = C, \quad (2.15) \]

where \( D_{\hat{P}}(< 0) \) is the partial derivative of \( D(\hat{P}, P_0) \) with respect to its first argument. The discount factor \( D(\hat{P}, P_0) \) is similar to quantity for the price-setting monopoly.

Figure 2.1 is a revised version of figure 1 in Dixit et al. The left-hand side of (2.15) represents the changes in \( \hat{P} \) as a result of an increase in \( D \), which is analogous to the

\(^{11}\)For the optimization of a one-dimensional stopping problem, the smooth-pasting condition is the necessary condition (Brekke and Oksendal, 1991).

\(^{12}\)Shackleton and Sodal (2005) suggest, moreover, that the smooth-pasting condition implies that both option and payoff positions have one equal expected rate of return by making the derivatives of the option and payoff functions equal. We are restricted from showing this, because we solve the developer’s problem using a dynamic programming approach, rather than a contingent claims approach.
marginal revenue function. The optimal reservation price $\bar{P}$ is found at the point where the marginal cost $C$ equals the marginal benefit from the hasty exercise of the option: an increase in $D$. If the developer increases $D$ by one unit, the benefit is increased by $\bar{P} + \frac{D(\bar{P}, P_0)}{D_p(P, P_0)}$, not by $\bar{P}$. Based on the Marshallian rule of $\bar{P} = C$, the developer starts the conversion project at $D^N$, the optimal timing for development when the developer follows a Net Present Value rule.

Second, the developer incurs no loss and realizes no gain upon a conversion decision. Whereas the developer gives up the development opportunity, the developer obtains the net value of the land conversion. Equation (2.13) leads to

$$W(\bar{R}) = \frac{1}{\alpha r} > 0.$$  

The difference between $\bar{P}$ and $C$ is called the markup $\frac{1}{\alpha r}$, which incorporates the value of waiting. In figure 2.1, the developer delays the project from $D^N$ to $\bar{D}$ under uncertainty. The positive net present value distinguishes the real options model from a Marshallian investment model.

Third, the initial condition states that the development option has little value if the rent is low enough.

### 2.4 The Development Option in the Presence of a Defaultable Construction Loan

Most developers obtain development financing, even though the developer invests pure (100%) equity in previous sections. In this section, therefore, we suppose that a developer will obtain development financing in the form of a single, homogeneous class of commercial debt at the timing of development. A developer is likely to set up a stand-alone project company (Somerville, 2002). The developer purchases land, which can be financed with debt (Riddiough, 1997), even though land is not debt-financed in our model.

13 In the commercial real estate finance market, most development loans are bullets; a borrower of a bullet loan pays interest only until the maturity period, and pays the principal back at maturity. Furthermore, commercial borrowers are likely to lose any prepayment option (Titman and Torous, 1989), although recently mortgage investors in the secondary mortgage market often charge a yield maintenance penalty or defeasance.

14 In practice, there are two types of development financing: construction financing and permanent financing. In the theoretical model, we regard permanent financing as construction financing; immediately after construction, construction loans are converted to permanent loans through long-term mortgages. In
We solve a developer’s timing decision backward. After the development, a leveraged developer makes loan payment $m$ per unit of time in order to pin down the value of the construction (permanent) loan. Lenders tend to determine the loan amount according to the projected net operating income (NOI), along with other underwriting criteria such as debt coverage ratio (DCR) and LTV. We assume that the developer will be able to roll over the loan at maturity and pay no loan underwriting or origination costs including discount points even though prepayment is often prohibited in the commercial real estate finance market.

After development, the value of the equity $S(R_t)$ is represented by the optimal stopping problem,

$$S(R_t) = \sup_{\{t_D\}} E_t[\int_t^{t_D} (R_s - m)e^{-r(s-t)} ds|R_t], \quad (2.16)$$

where $t_D$ is an optimal timing of default.

The value of the construction loan $M(R_t)$ is represented by

$$M(R_t) = E_t[\int_t^{t_D} me^{-r(s-t)} ds + \int_{t_D}^{\infty} R_s e^{-r(s-t)} ds|R_t], \quad (2.17)$$

where $t_D$ is strategically determined in (2.16).

Before the development occurs, the developer’s problem is to determine the optimal timing for development with debt financed at the time of the land conversion. The value of the development option for a leveraged developer is represented by the optimal stopping problem

$$F(R_t) = \sup_{\{t^*\}} E_t[\int_t^{t^*} (R_s - A - m)e^{-r(s-t^*)} ds - (C - B)]e^{-r(t^*-t)}|R_t], \quad (2.18)$$

subject to $B = M(R^*)$ in (2.17), where $B \leq C$ is given. Note that $t_D$ is derived in (2.16).

We take no account of development lag in Bar-Ilan and Strange (1996). As construction cost is partially financed with debt at time

$$t^* = \inf_{\{\tau\}} \{t + \tau \geq t \mid R_{t+\tau} \geq R^*\},$$

addition, “developers traditionally obtain a commitment for a permanent loan first and then use that commitment to obtain a construction loan.” (Frej and Peiser, 2003: 20) Alternatively, developers can use other methods such as a standby loan commitment or unit sales.
when $R$ reaches $R^*$ for the first time, even though the developer obtains a debt commitment in advance.

### 2.4.1 Project Valuations

This section provides the market values of equity, debt, and project after the development. With the default option, the value of equity as a call option includes the default option value.

**Equity Valuation: Real Options Approach**

The market value of the developer’s equity in (2.16) evolves according to the ordinary differential equation,

$$\frac{\sigma^2}{2} S_{RR}(R) + g S_R(R) - r S(R) + R - m = 0, \quad R > R_D,$$

which is derived from (2.7) with both the Bellman equation and Itô’s lemma. $R_D$ is the level of rent at time $t_D$ given by

$$t_D = \inf \{ t + \tau \geq t \mid R_{t+\tau} \leq R_D \}.$$

The developer’s net cash flow is $R - m$ per unit of time. The associated inhomogeneous equation for (2.19a) is $S(R) = S_1 e^{\alpha R} + S_2 e^{\beta R} + S_p(R)$, where $S_1$ and $S_2$ and a particular (steady state) solution $S_p(R)$ are to be determined. The equity value in (2.19a) includes the default decision that the developer may make, and thus equation (2.19a) satisfies the boundary conditions:

$$S_R(R_D) = 0,$$  \hspace{1cm} (2.19b)

$$S(R_D) = 0,$$  \hspace{1cm} (2.19c)

$$\lim_{R \to \infty} S(R) = \frac{1}{r} (R + \frac{g}{r}) - \frac{m}{r}.$$  \hspace{1cm} (2.19d)

Equation (2.19b) indicates that, at the optimal default timing, continuing and defaulting are indifferent. Equation (2.19c) recognizes that the developer makes the strategic default decision. Equation (2.19d) is a no-bubbles condition as in Mella-Barral and Perraudin (1997). Note that the developer suffers no default-related costs in (2.19c), and the
developer is not personally liable for the debt. In addition, the equity investor enjoys no
tax benefits highlighted by both Marseguerra and Cortelezzi and Mauer and Sarkar (2005).

The present value of the developer’s equity is the solution to equations (2.19a) - (2.19d):

\[ S(R) = \frac{1}{r} (R + \frac{g}{r}) - \frac{m}{r} - \frac{1}{r \beta} e^{-\beta(R_D - R)}, \quad R > R_D, \]  

(2.20a)

where the optimal default timing is

\[ R_D = \frac{1}{\beta} + m - \frac{g}{r}. \]  

(2.20b)

**Proposition 1.** A leveraged developer delays the default with \( \sigma \), but moves the default forward with \( m \).

See appendix A for the derivation.

Under uncertainty, the borrower delays the default decision to avoid adverse outcomes, as the first term on the right-hand side of (2.20b) is negative.\(^{15}\) The second represents a coupon payment. The last term indicates that the developer delays exercising the option with the high drift. The mortgage default literature documents that, with high volatility or a low LTV ratio, mortgage borrowers postpone default. As the discount rate decreases, the developer delays default.

The equity value is replicated by a portfolio involving a long position in the project, a short position in the riskless debt, and a long position in the put option on default, respectively, on the right side of (2.20a). The developer holds the default option like a put option, whose value is

\[ \Omega \equiv -\frac{1}{\beta r} e^{-\beta(R_D - R)} > 0, \quad R > R_D, \]  

(2.21)

which follows the log-normal diffusion.\(^{16}\)

\(^{15}\)In a static environment, the value of equity capital in (2.20a) is \( S(R) = \frac{1}{r} (R + \frac{g}{r}) - \frac{m}{r} \), as \( \lim_{\sigma \to 0} \frac{1}{\beta} = 0 \). Without uncertainty, the developer will develop the project when \( R > R_D = m - \frac{g}{r} \) in (2.20b). Note that we do not allow the abandonment option for the pure equity developer. However, the abandonment option on a new durable building is embedded implicitly in the model. The value of the non-leveraged equity with an abandonment option in (2.20a) is \( S(R) = -\frac{1}{\beta r} e^{-\beta(R_D - R)} + \frac{1}{r} (R + \frac{2}{r}) \) where \( R > R_D = \frac{1}{\beta} - \frac{g}{r} \), at which point the pure equity developer abandons the building.

\(^{16}\)To stimulate intuition, notice that since \( R_D - R = g (t_D - t) + \sigma db \) in (2.7) we can rewrite \( \Omega = -\frac{1}{\beta r} e^{\beta (t_D - t + \frac{1}{r} \sigma db)} \). Ceteris paribus, the developer is likely to default with high \( \sigma \) or low \( g \). In a stable and growing real estate market, moreover, the developer is less likely to default since \( R_D - R = g E(t_D - t) \).
Proposition 2. The value of the default option increases with $\sigma$, $m$, but decreases with $g$ and $R$.

See appendix A for the derivation.

The default option value increases with $\sigma$ and increases with $m$ [(Epperson et al., 1985; Kau and Kim, 1994)], as the developer defaults at higher levels of rent in (2.20b); the value of a put option increases with the strike price. Note that the option value decreases with the rent, because the value of a put option has a negative relationship with the underlying asset price.

Proposition 2 with (2.20a) leads to

Proposition 3. The equity value increases with $\sigma$ and $R$, but decreases with $m$.

See appendix A for the derivation.

The value of the equity as an option increases with $\sigma$. The equity value increases with levels of rent; the value of the call option increases with the underlying asset price. Note, nonetheless, that, as $R$ increases, the value of the equity increases less than $1/r$ due to the default option. In addition, a marginal change in the equity value with respect to $R$ decreases with $m$: $\frac{\partial^2 S}{\partial R \partial m} < 0$. The equity value decreases as the loan payment increases; the equity value decreases also by less than $1/r$.

Defaultable Construction Loan Valuation

The market value of the debt in (2.17) follows the differential equation:

$$\frac{\sigma^2}{2} M_{RR}(R) + gM_R(R) - rM(R) + m = 0, \quad R > R_D, \quad (2.22a)$$

as the lender obtains $m$ per unit of time until the rent drops to $R_D$. The general solution for equation (2.22a) is $M(R) = M_1e^{\alpha R} + M_2e^{\beta R} + M_p(R)$ where $M_1$, $M_2$, and $M_p(R)$ are to be determined. The loan value must satisfy the initial and boundary conditions,

$$M(R_D) = \frac{1}{r}(R_D + \frac{g}{r}), \quad (2.22b)$$

$$\lim_{R \to \infty} M(R) = \frac{m}{r}. \quad (2.22c)$$

In the deterministic case of $\lim_{\sigma \to 0} \Omega = 0$, the value of the default option disappears. Equation (2.21) may be interpreted as the discounted present value of $\frac{-1}{\beta} > 0$; at each period until the optimal default timing, the borrower can choose to terminate the loan contract.
The frictionless default condition in (2.22b) implies that, when the developer defaults, the lender will take over the default property at no cost. In (2.22c), the lender receives $m$ forever if the rent is large enough.

The analytic solution for equations (2.22a) - (2.22c) is

$$M(R) = \frac{m}{r} + \frac{1}{\beta r} e^{-\beta (R_D - R)}, \quad R > R_D. \quad (2.23)$$

**Corollary 1.** *The value of the construction loan increases with $m$, $R$ and $g$, but decreases with $\sigma$.*

See appendix A for the derivation.

The right-hand side of equation (2.23) is a general form of risky (or defaultable) debt, which is replicated by a long position in the safe asset and a short position in the default option, respectively. Note that the market value of the defaultable debt is lower than the market value of the non-defaultable loan. The value of the debt decreases with the default potential.

The market value of the loan increases with payments, but it increases by less than $1/r$ according to the value of the default option that the lender shorts. Note that the market value of the defaultable debt has a positive relationship with rents and growth rates; as rents increase, the developer is less likely to default. If the loan is non-defaultable, its value is independent of $R$ and $g$. As the lender shorts the option, the value of the debt decreases with volatility.

**The Value of the Project and Origination of a Defaultable Loan**

According to (2.8), (2.20a), and (2.23) in the previous sections, we conclude that after development, the value of the land conversion project equals the sum of the value of equity and the value of debt; the Modigliani and Miller (1958) theorem holds:

$$P(R) = S(R) + M(R). \quad (2.24)$$

With a defaultable loan, our approach is distinguished from that of Mella-Barral and Perraudin (1997) and Merton (1974), because we derive the value of equity and debt using a normal diffusion process.
The value of the debt in a competitive capital market equals the expected present value of future payments. At the optimal time of development, the lender originates the loan at the fair price:

\[ B = M(R^*) = \frac{m}{r} + \frac{1}{\beta r} e^{-\beta(R_D - R^*)} = \frac{m}{r} - \Omega(R^*). \quad R^* > R_D. \quad (2.25) \]

Upon loan origination, the lender devalues the construction loan because of the developer’s risk-shifting problem observed in Jensen and Meckling (1976).\(^{17}\) Note, however, that the lender evaluates the problem and will surrender the ex ante option value. While the face value of the defaultable debt \( B \), once originated, remains invariant over time, the market value of the development loan \( M(R) \) fluctuates and the LTV ratio, a measure of financial leverage, depends on time.

We can show that there is an upper bound of the amount of the construction loan that the developer can borrow. The maximum payment available to the developer, from (2.23), is

\[ \tilde{m}(R) = R - \frac{1}{\beta} + \frac{r}{g}, \]

with the sufficient condition: \( \frac{\partial M(R)}{\partial m} < 0 \).

In addition, the interest rate, the yield paid by the defaultable debt \( i \), equals \( m/B \),

\[ i = \frac{m}{B} = r - \frac{1}{\beta} e^{-\beta(R_D - R^*)}/B \]

(2.26)

with (2.25). If we define the credit spread by \( i - r \), then we know that the lender will charge wider credit spreads with high potential loss in the second term on the right-hand side of (2.26); the credit spread increases with \( \sigma \), and \( m \) (Merton, 1974), but decreases with \( g \) and \( R \) (see appendix A for details). Moreover, the interest rate \( i \), from (2.26), is convex with respect to \( m \), which is supported theoretically and empirically by Cannaday and Yang (1996), and Hendershott and Shilling (1989), respectively.

\(^{17}\) At the time of loan origination \( t^* \), \( \frac{\partial S}{\partial B} = \frac{\partial S}{\partial m} = -1 \) in (2.20a), whereas \( \frac{\partial M}{\partial B} = \frac{\partial M}{\partial m} = 1 \) in (2.23); one extra dollar of the loan reduces the developer’s equity value by \$1. Overall, financial structure is irrelevant to the value of a real asset: \( \frac{\partial S}{\partial m} = -\frac{1}{r}(1 - e^{-\beta(R_D - R^*)}) \) in proposition 3, while \( \frac{\partial m}{\partial B} = r/(1 - e^{-\beta(R_D - R^*)}) \) in (2.25).
2.4.2 The Development Option for a Leveraged Developer

This section shows that, for a developer with a defaultable construction loan outstanding, the timing of the development is a decreasing function of leverage, and, at the optimal time, the option value of the uncertainty premium decreases with leverage.

**Leveraged Optimal Development Timing**

The value of the leveraged option $F(R)$ in (2.18) evolves according to

$$\frac{\sigma^2}{2} F_{RR}(R) + g F_R(R) - r F(R) = 0, \quad R < R^*. \tag{2.27a}$$

The option value must satisfy the conditions,

$$F_R(R^*) = S_R(R^*), \tag{2.27b}$$

$$F(R^*) = S(R^*) - \frac{A}{r} - (C - B), \tag{2.27c}$$

$$\lim_{R \to -\infty} F(R) = 0. \tag{2.27d}$$

Equation (2.27b) recognizes the condition for equity value maximization. Note that due to the default option a marginal change in the equity value on the right-hand side in (2.27b) is lower than a marginal change in the project value on the right-hand side in (2.10b). Equation (2.27c) indicates that the proceeds of the loan issue are used for the development project with no land acquisition financing. At time $t^*$, the option value equals the value of the net equity investment in light of the opportunity cost. Note that $C$ is independent of $m$. When the rent is low enough, the developer will delay the project in (2.27d). The analytic solution for equations (2.27a) - (2.27d) along with (2.25) is

$$F(R) = \frac{1}{r} \left[ R^* + \frac{g}{r} - m - A - r(C - B) - \frac{1}{\beta} e^{-\beta(R^* - R)} \right] e^{-\alpha(R^* - R)}, \quad R < R^* \tag{2.28a}$$

where

$$R^* = A + rC + \frac{r\Theta(R^*; m) - ag}{\alpha r}, \tag{2.28b}$$

where $\Theta(R^*; m, \sigma, g, r) \equiv 1 - e^{-\beta(R^* - R^*{D})}$ with $0 < \Theta < 1$ as the splitting rule for the leveraged developer. As we expected, the equity value maximizer takes no account of the value of waiting for the lender and, ex ante, the lender is indifferent to the developer’s strategic behavior.\(^\text{18}\)

\(^\text{18}\)At the ex post stage, the lender may regret her decision because the developer launched an immature project.
For a unique solution of (2.28b), see appendix A.

**Proposition 4.** Compared with a non-leveraged developer, a leveraged developer exercises the development option earlier.

See appendix [A](#) for the derivation.

With leverage, we propose a reduced increment due to uncertainty in (2.28b). Moreover the reservation rent decreases with leverage; $R^* < \bar{R}$ in (2.11b) and the reservation price also decreases with leverage. Even though $R^*$ is slightly smaller than $\bar{R}$, the optimal development timing can be significantly different in a dynamic environment. Note that the value of the conversion option decreases, as $m$ (or $M$) increases and as $R^*$ decreases (see appendix [A](#)). The leveraged developer is more likely to default with $m$ and is less likely to default with $R^*$.

Under no uncertainty, we expect no leverage effects, as $\lim_{\sigma \to 0} R^* = \lim_{\sigma \to 0} \bar{R} = A + rC$. Even under uncertainty, a non-defaultable loan also has no impact on the development timing in (2.28b): $\Theta = 1$. This result is similar to that of Geske (1979), who shows that the value of a (fixed-maturity) call option on stock does not vary with a debt that matures at infinity.$^{19}$

Compared with a non-leveraged developer, a leveraged developer exercises the development option at the lower trigger levels of equilibrium rent and price. We can show this with the reservation price for the urban land

$$P^* = \frac{A}{r} + C + \frac{g}{r^2} + \frac{r \Theta (R^*; m) - \alpha g}{\alpha r^2},$$

(2.29) with (2.28b). Equation (2.29) implies that, at the fringe of the urban area, the value of a real property can be determined by leverage; the uncertainty terms in (2.11b) and (2.12) are discounted according to $\Theta$. The developer with defaultable loan financing speeds up the option exercise as leverage truncates loss in bad states of the world.

One caveat pertaining to the model is that since the development trigger in (2.28b) and the default trigger in (2.20b) are integrated, the analytical solution is insufficient to infer the developer’s behavior with respect to other variables such as $\sigma$.

$^{19}$Without leverage, our model is similar to Dixit (1989)’s model of two correlated decisions of entry and exit simultaneously. Nonetheless, our model is distinct from Dixit (1989)’s, because the developer finances the debt and has no reentry option, but does have a default option. To see the abandonment option in our model, we know $\Theta(\sigma, g, r, 0, R^*) \equiv 1 - e^{-\beta (R_D - R^*)}$, where $R_D = \frac{1}{\beta} - \frac{2}{r}$ when $m$ goes to zero.
Optimal Development Rules with Leverage

This section explains the optimal development rules for a developer with a defaultable loan outstanding. The first rule of the smooth-pasting condition for such a developer must be revised with leverage; we can covert (2.28a) to

$$F(R : m) = \frac{\Theta(R^* : m)}{\alpha r} e^{-\alpha(R^* - R)}.$$  \hspace{1cm} (2.30)

The slope of the value of the development option decreases to

$$F'(R^*) = \frac{\Theta(R^* - m)}{r} < \frac{1}{r};$$

note that the slope is a function of $m$.

Similarly to (2.14), the right-hand side of (2.28a) is converted to

$$\Pi^M(\tilde{P}, P_0) \equiv D(\tilde{P}, P_0)[\tilde{P} - M(\tilde{P}) - (C - B)]$$  \hspace{1cm} (2.31)

with (2.23), where $\Pi^M$ is the net present value of the equity investment because the underlying asset of the conversion option is the value of the equity investment. The project payoff fluctuates under uncertainty and the equity payoff $\Pi^M(\tilde{P})$ does as well. Nonetheless, the equity maximizer takes no account of information on $M(\tilde{P})$; the lender is better of waiting, as shown in corollary 1.

The leveraged optimal threshold $P^*$ is the value of $\tilde{P}$, which maximizes equation (2.31). The first-order condition, with (2.25), is

$$\tilde{P} + \frac{\Theta(\tilde{P} : m)D(\tilde{P}, P_0)}{D_{\tilde{P}}(\tilde{P}, P_0)} = C.$$  \hspace{1cm} (2.32)

The optimal reservation price $P^*$ is found at the point where the marginal benefit equals the marginal cost. Note that the marginal benefit from the hasty exercise of the conversion option increases with leverage; see figure [2.1] which shows that the leveraged marginal benefit $\tilde{P} + \frac{\Theta D}{D_{\tilde{P}}}$ from an increase in $D$ is higher than the non-leveraged marginal benefit $\tilde{P} + \frac{D}{D_{\tilde{P}}}$ from an increase in $D$. The leveraged marginal benefit has a positive relationship with leverage due to the premium related to the financial risks.

To see the mixed effects of uncertainty risks due to fluctuations of the underlying asset and the financial risks of leverage, we convert (2.31), with (2.23), to

$$\Pi^M(\tilde{R}, R_0) = D(\tilde{R}, R_0)[\frac{1}{r}(\tilde{R} + \frac{g}{r}) - C + \Omega(\tilde{R} : m) - \Omega(R^* : m)],$$  \hspace{1cm} (2.33)
where \( \tilde{R} < R^* \). We know \( \Omega(\tilde{R} : m) > \Omega(R^* : m) \) because the value of an American put option that a leveraged developer owns decreases with the underlying asset price.

In the short run, when the developer does not react to an increase in the loan payment, equation (2.33) yields to

\[
\frac{\partial \Pi^M(\tilde{R}, R_0)}{\partial m} = \frac{D(\tilde{R}, R_0)}{r} \left[ e^{-\beta(R_D - \tilde{R})} - e^{-\beta(R_D - R^*)} \right] > 0,
\]

which implies that the value of the equity investment increases with greater financial risks.

In the long run, when the developer reacts to an increase in the loan payment, equation (2.33) yields to

\[
\frac{\partial \Pi^M(\tilde{R}, R_0)}{\partial m} = \frac{D(\tilde{R}, R_0)}{r} \left[ e^{-\beta(R_D - \tilde{R})} - e^{-\beta(R_D - R^*)} \right] + \frac{D(\tilde{R}, R_0)}{r} e^{-\beta(R_D - R^*)} \frac{\partial R^*}{\partial m}. \tag{2.34}
\]

The second term of the right-hand side of (2.34) implies that the value of the equity investment is negatively affected by the early exercise of the conversion option. With the defaultable loan, at time \( R^* \), the value of the equity investment decreases:

\[
\frac{\partial \Pi^M(R^*, R_0)}{\partial m} = \frac{D(R^*, R_0)}{r} e^{-\beta(R_D - R^*)} \frac{\partial R^*}{\partial m} < 0.
\]

Our model bears a strong similarity to the analysis of Geske (1979), who insists that, with risky debt, the value of a call option on the equity value (or the value of a compound option) is less than the value of a call option on the value of a firm because the underlying asset is equity-based, rather than firm-based.\(^{20}\)

Second, the uncertainty premium at \( t^* \) for a leveraged developer is lower than it is at \( \bar{t} \) for an unleveraged developer. At \( R^* \),

\[
F(R^* : m) = \frac{\Theta(R^* ; m)}{\alpha r} \tag{2.35}
\]

Our model is more general than that of Capozza and Helsley. With the non-defaultable construction loan or with no construction loan, we conclude that the markup of \( F(R^*) = F(\tilde{R}) = \frac{1}{\alpha r} \) in (2.35); with a non-defaultable loan or no loan, a (leveraged) developer who has no default option exercises the development option at \( R^* = \tilde{R} \) in (2.11b). With

\(^{20}\)Geske (1979) and recently Elettra and Rossella (2003) argue that the value of an option on equity (as an option) decreases with an increase in debt, which implies a decrease in equity. Within the real options framework, we show that the option value decreases with \( m \) due to the accelerated development timing.
defaultable debt, however, the developer accelerates the exercise of the land conversion option and seeks to lower the uncertainty premium in (2.35). In figure 2.1 at the optimal time to commence development, the uncertainty premium for a leveraged developer is lower than it is for an unleveraged developer. Leverage moves the discount factor closer to $D^N$.

Lastly, the initial condition is similar with leverage.

### 2.4.3 Comparative Statics

In urban growth models with irreversible development, an unleveraged developer delays land conversion with the uncertainty premium. We propose, however, that, compared with what an unleveraged developer does, a leveraged developer hastens development, even though the leveraged developer delays development relative to the certainty case. As mentioned before, our model is limited with respect to the analytical comparative statics.

For a numerical analysis, we adopt the following set of base-case parameter values for our simulation results. The cost of exercising development option $C$ to build the new building is 300 and the agricultural land rent is 10. We assume that the drift of rent $g$ and the standard deviation of rent $\sigma$ are 1 and 4, respectively. The annualized risk-free interest rate $r$ is 0.03. Finally, we assume initially that the construction loan has a fixed promised loan payment $m$ of 9. Table 2.1 summarizes the results for numerical comparative statics in this section.

#### Leverage

The optimal development timing for an unleveraged developer $\bar{R}$ in (2.11b) is 25.66, whereas that for a leveraged developer $R^*$ in (2.28b) decreases to 25.65; while $\bar{P}$ is 1,966.66, $P^*$ is 1,966.39. The Marshallian development triggers are $R^M = 19.00$ and $P^M = 1744.44$.

Even though we showed the effects of leverage on reservation rent and reservation price, we can show proposition 4 numerically with figure 2.2, which shows optimal trigger prices with varying debt payment $m$, which corresponds to the loan-to-construction-cost ratio. It is interesting to compare a leveraged developer’s decision with that of an unleveraged developer. In figure 2.2, the straight and dashed lines represent $P^*$ and $\bar{P}$, respectively, with respect to the loan-to-construction-cost ratio. The straight line monotonically decreases

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21. All the parameter values except $m$ are based on Capozza and Sick. In addition, we need an assumption of an initial rent which is less than optimal. When $m = 9$, the value of a loan at time $t^*$ is 299.95, and thus the ratio of loan to construction cost is 99.98% for the purpose of highlighting leverage effects.
with the ratio, whereas the dashed line is invariant to the ratio. Note that two lines do not intersect at $m = 0$ (no debt) due to no abandonment, which we mentioned before. With the numerical exercise, this section conforms to proposition 4.

**Uncertainty**

Uncertainty delays development and this is true with leverage. For $\bar{R}$ in (2.11b), it is easy to show that the reservation rent increases with uncertainty $\frac{\partial \bar{R}}{\partial \sigma} = -\frac{1}{\alpha} \alpha > 0$. For $R^*$ in (2.28b), nonetheless, it is not a straightforward matter to show the effects of uncertainty with an analytic analysis; for details, see appendix A.

In figure 2.3, again, the straight and dashed lines indicate $P^*$ and $\bar{P}$, respectively. Both $P^*$ and $\bar{P}$ monotonically increase with $\sigma$, as uncertainty causes the uncertainty premium to increase. Consequently, we confirm that uncertainty also delays a leveraged investment by the numerical analysis: $\frac{\partial P^*}{\partial \sigma} > 0$ and $\frac{\partial \bar{P}}{\partial \sigma} = \frac{1}{r} \frac{\partial \bar{R}}{\partial \sigma} > 0$.

**Expected Growth in Rent and the Risk-free Rate of Interest**

As the expected growth increases, the unleveraged reservation rent in (2.11b) is undetermined, whereas the unleveraged reservation price in (2.12) increases; $\frac{\partial \bar{R}}{\partial g} = -\frac{1}{\alpha^2} \alpha_g - \frac{1}{r} > 0$ and $\frac{\partial \bar{P}}{\partial g} = -\frac{1}{\alpha^2} \alpha_g > 0$. As the risk-free interest rate increases, the unleveraged reservation rent increases, whereas the unleveraged reservation price decreases; $\frac{\partial \bar{R}}{\partial r} = c - \frac{1}{\alpha^2} \alpha_r + \frac{2}{r} > 0$ and $\frac{\partial \bar{P}}{\partial r} = -A \frac{\alpha_r}{\alpha^2} - \frac{1}{\alpha^2} \alpha_r < 0$.

With leverage, we can show numerically that the comparative statics for the leveraged reservation rent and the leveraged reservation price in (2.28b) and (2.29) is the same as it is for the unleveraged reservation triggers; $\frac{\partial P^*}{\partial g} > 0$, $\frac{\partial R^*}{\partial r} > 0$ and $\frac{\partial P^*}{\partial r} < 0$.

**2.5 Leverage, Uncertainty and the City**

This section discusses the implications of our model for (1) city size, (2) land price and rent, and (3) leapfrogging in urban growth models of cities.

**2.5.1 The City Boundary**

A key implication of our analysis is that the boundary of a city, where development occurs, expands with leverage. A leveraged developer has an incentive to start a land conversion project at low trigger levels.
To introduce the location, we assume that a monocentric-city on homogeneous long strip land has a CBD. Land is differentiated with distance $z$ from the CBD. The CBD rent is $R_C$ and the land bid rent is a decreasing function of $z$; $R(z) = R_C - z$. At time $t$, the developer converts agricultural land to urban land at the boundary of the city $z^*$. In an urban area with a durable structure, bid rents are an increasing function of the distance from $z^*$

$$R(z) - R^* = z^* - z,$$

where $R(z)$ is the land bid rent at location $z$ at time $t$ and $R^* \equiv R(z^*)$. Equation (2.36) is the same as in Capozza and Helsley and Capozza and Li.

The boundary of an urban area is endogenously determined by a land developer who considers price dynamics (Mayer and Somerville, 2000). And the boundary is endogenously determined by

$$\frac{\partial z^*}{\partial m} = -\frac{\partial R^*}{\partial m} > 0,$$

and

$$\frac{\partial z^*}{\partial \sigma} = -\frac{\partial R^*}{\partial \sigma} < 0.$$

We conclude that, in a highly leveraged city, the equilibrium city boundary expands. Figure 2.4 shows that the city boundary expands to $z^*$ with leverage from $\overline{z}$. Even in such a leveraged city, moreover, the equilibrium city size decreases with uncertainty. In addition, our model converges on Capozza and Helsley’s model if the developer obtains non-defaultable debt or no debt, because with non-defaultable debt, $\frac{\partial z^*}{\partial m} = 0$, and with no debt, $z^* = R_C - R^* = R_C - \overline{R}$.

At the urban fringe, construction activities increase as developers are more accessible to financial credit; accessibility to debt credit with which to finance development can be a contributing factor with respect to market dynamics. We can argue that housing supply has procyclical effects on market dynamics in the leveraged city. This line of reasoning is similar to that taken in previous studies, such as Stein (1995) and Lamont and Stein (1999) in the housing market and Kiyotaki and Moore (1997) in the land market, which argue that borrowers’ financing capacity can amplify asset-price dynamics.
2.5.2 The Structure of Land Rents and Prices

The equilibrium land rent function at time $t$ is

$$R(z) = \begin{cases} A + rC + \frac{r\Theta - \alpha g}{\alpha r} + z^* - z, & z \leq z^*, \\ A, & z > z^*, \end{cases}$$

from (2.36) and (2.28b). In the urban area, the last term of (2.37) is the location rent. The price of urban land, from (2.8), is

$$P^u(z) = A + C + \frac{g}{r^2} + \frac{r\Theta - \alpha g}{\alpha r^2} + \frac{(z^* - z)}{r}, \quad z \leq z^*,$$

where the last term of (2.38) is the location value.

The price of urban land is unaffected by leverage. With (2.36), we can re-express (2.38) into

$$P^u(z) = R_C + \frac{g}{r^2} - \frac{z}{r}, \quad z \leq z^*,$$

which is independent of $m$ or $\sigma$, which also has no impact on the price of urban land. Figure 2.4 shows that the trigger prices decrease and thus the boundary of a city expands with leverage. Consequently, we can argue that the value of urban land on the fringe of the city is determined by a marginal developer, whose financial conditions can be different from those of other landowners.

The price of agricultural land, from (2.28a), is

$$P^a(z) = A + \Theta(R^* : m)e^{-\alpha(z - z^*)}, \quad z > z^*,$$

which is a function of both $m$ and $\sigma$. In the short run, when the developer does not react to an increase in $m$, the effect of leverage on agricultural land prices is negative. In the long run, however, the effect of $m$ on land rent and price is ambiguous and the effect of $\sigma$ on land rent and price is also ambiguous (see appendix A for details).

2.5.3 Leapfrogging

The effects of leverage on the reservation rent and the reservation price are surprising. For a given city size, both the reservation rent and the reservation price at the urban boundary are lower with leverage.
At the outskirts of a city, landowners may operate under unique financial conditions. With heterogeneous landowners (Novy-Marx, 2007), in terms of leverage, we can address “leapfrogging”: discontinuous development of land near the boundary of an urban area. Suppose that two landowners near the city boundary operate under differing financial conditions. The two landowners are contemplating how to determine the timing of the land conversion. When converting land, the landowner located nearer to the city boundary will nonetheless obtain no leverage, whereas the landowner located farther away from the city boundary will finance all development costs with debt for any number of reasons. Although the location farther away from the city boundary offers lower rent, a leveraged landowner who has lower development triggers will start a land conversion project earlier than an unleveraged developer who is located nearer to the city boundary will.

2.5.4 Discussion

The essential point that entails our results is that the developer in our study finances a defaultable construction loan. Taking no account of information available to the lender, the developer with a non-defaultable development loan outstanding will, in some states of the world, pass up future opportunities that make the project more valuable. As a result, leverage matters to the developer who determines the timing of the land conversion project, even though the value of the completed project is irrelevant to leverage.

When the developer maximizes the total value of the project, the developer waits until the value of the project reaches the value at which we observe no agency problem. The social developer, who maximizes the total value, if such a developer exists, will not accelerate the land conversion with leverage. The difference between \( \bar{P} \) in (2.12) and \( P^* \) in (2.29) can be considered the social cost from the perspective of the developer and the lender. Nonetheless, consumers from whom our model abstracts may obtain benefits.

Other developer’ incentives, such as governance on and control over the project, on which we are silent in section 2.4, would make the developer consider the total benefits of waiting to commence development. Moreover, we may force the developer to consider the total value of the project. For example, the developer and the lender can negotiate the optimal timing, even though the contract will not be attractive to a developer who can finance debt at fair market value on the capital market.

\[22\] We often observe vacant raw land inside the urban boundary; according to our model of the durable city, we can not explain another type of leapfrogging: improved but vacant land ready for the development.
2.6 Conclusion

The primary goal of this chapter is to incorporate leverage in a real options model in real estate development. To see the effects of leverage on development timing, we assume that a developer maximizes the equity value rather than the total value of a project and finances a defaultable construction loan. Note that, in the model, we have no sources of market friction.

We show that such a developer is likely to move the development decision forward with leverage, because the leveraged developer does not consider the lender’s benefit of waiting. As the developer takes on more highly leveraged positions, the developer starts the land conversion project at lower reservation triggers. In durable city models in which the urban boundary is endogenously determined, the equilibrium size of the city expands because a leveraged developer who converts land from agricultural to urban use has an incentive to overinvest. In the leveraged city, the housing supply is likely to contribute to real estate market dynamics.

By considering both the development option and the default option that are available to real estate developers, we show that the timing strategy for a land conversion project can depend on the developer’s financial condition. Moreover, we divide the value of an investment whose underlying asset follows a normal diffusion process into the value of the equity and the value of the debt. We then examine the leveraged developer’s decision on the timing of exercising the development option. By doing so, we provide a more general solution than Capozza and Helsley do.
The optimal threshold for an unleveraged developer $\bar{P}$ and the corresponding optimal discount factor is $\bar{D}(\bar{P}, P_0)$. Note that $\bar{P} > C$ and the markup is $\frac{1}{\alpha r}$. With a simple NPV rule, the optimal timing is $D^N$. Under uncertainty, an project is delayed as $\tilde{D}(\bar{P}, P_0) < D^N$. For a leveraged developer, the optimal threshold is $P^*$ and the corresponding optimal discount factor is $D^*(P^*, P_0)$. The optimal threshold decreases with leverage as $D^*(P^*, P_0) > \bar{D}(\bar{P}, P_0)$ and also the markup decreases with leverage; $\frac{\Theta(P^*, m)}{\alpha r} < \frac{1}{\alpha r}$. 
For the figure, following Capozza and Sick, we assume $C = 300$, $A = 10$, $g = 1$, $\sigma = 4$ and $r = 0.03$. The straight and dashed lines represent $P^*$ and $\bar{P}$, respectively. The straight line monotonically decreases with the loan-to-construction-cost ratio, whereas the dashed line is invariant with respect to the loan-to-construction-cost ratio. A leveraged developer brings forward land conversion.
For the figure, following Capozza and Sick we assume $C = 300$, $A = 10$, $g = 1$ and $r = 0.03$. The straight and dashed lines represent $P^*$ and $\bar{P}$, respectively. Both lines increase with $\sigma$. Under uncertainty, a developer delays land conversion with or without leverage.
Without consideration of the developer’s financial condition, the city boundary that is determined by the developer is $\bar{z}$. Nonetheless, the city boundary expands to $z^*$, because a leveraged developer has the lower triggers.
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Chapter 3

Heterogeneous Assets and Development Options

3.1 Introduction

The traditional approach to real options in real estate development assumes that development options have one underlying asset associated with revenue.\(^{23}\) Even though a developer builds multiple heterogeneous real properties, a development option is a function of revenue determined by a single stochastic process (Capozza and Li, 1994).\(^{24,25}\) In the real estate market, nonetheless, each property is unique and its value is spatially correlated to values of neighboring properties.\(^{26}\) In addition, mixed use buildings can be found in most downtown neighborhoods. In one building, for instance, residential units can be built together with retail or office units such as the University Village near the University of British Columbia area. Cash flows from residential units are different from those from other units, even though they tend to move together according to market conditions.

We build a real options model with heterogeneous assets because most real estate developers build interrelated multiple assets, which are currently either in place or under consideration. Throughout chapter 3, an ‘asset’ can be regarded as a commercial (or residential) unit that generates rent flows. Introducing Markowitz (1952)’s theory in a

\(^{23}\)Uncertainty can originate from other factors such as development cost (Clarke and Reed 1988 and Wang and Zhou 2006). Nonetheless, this chapter focuses on uncertainty of future revenue (net cash flow).

\(^{24}\)If each asset is treated as an individual project, a developer of chapter 3 has an option to develop multiple interrelated projects simultaneously.

\(^{25}\)When a development opportunity generates multiple properties, the development option can be determined by a single stochastic process as long as properties are homogeneous in the sense that their values perfectly covary with a correlation coefficient of positive one. For example, when a widget factory in Dixit and Pindyck (1994) produces two widgets at half price rather than one widget at full price, the value of an investment option can be invariant with respect to the number of products.

\(^{26}\)Even though we perfectly control for location, values of identical properties on a location will be less than perfectly correlated because a two dimensional Cartesian \(\{x,y\}\) coordinate system fails to control for the height of buildings.
real options framework, we address a diversification issue in real estate development. In a development project, in addition, we analyze the effects of both asset- and project-level uncertainty on development timing. In the theoretical model, note that a developer builds multiple assets with the exercise of a development option.

Under short-sale constraints for the assets, a developer is less likely to delay development with heterogeneous assets than a developer is with homogeneous assets. In an uncertain business environment, a sophisticated developer delays development because the aggregate uncertainty of a project assigns the positive option value for waiting (Capozza and Helsley, and Dixit and Pindyck, 1994). We show that when developers build non-identical assets whose values covary with a correlation coefficient of less than one, a strategy of product diversification can reduce levels of aggregate risk that a developer faces. Note that the heterogeneity of the assets is distinguished from Novy-Marx (2007)’s supply-side heterogeneity, which attenuates the effect of competition in real options theory (Grenadier, 2002).

By analyzing interrelated assets, we show that the value of a development option can be inaccurate without consideration of spatial correlation. The interrelationship between assets is referred to as spatial autocorrelation in the real estate market. The spatial relationship can be represented by Tobler’s (1970: 236) “first law of geography: everything is related to everything else, but near things are more related than distant things.” The farther real assets are, the less their values covary. Dubin (1988) argues that, without consideration of spatial relationship, estimated values of real estate can be inaccurate. Under short-sale constraints for the assets, in particular, the value of a development option increases because underlying assets have a higher spatial correlation coefficient. In an uncertain business environment, a developer is less likely to delay a development decision, as assets are less far away (or are more spatially correlated).

Our model is close to that of Childs et al. (1998) who analyze two projects in a European option framework. In an American option framework, however, our model considers a development option with a portfolio of multiple underlying assets[27] A development option can be composed of underlying assets that are distributed across geographic regions and that can be of different property types. Childs et al. (1996) and Geltner et al. (1996) also examine a development option with two underlying assets. With a real property under

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[27]Note that by exercising one development option, our developer builds multiple properties at the same time; we do not consider a portfolio of multiple development options
flexible zoning controls, their developer has the option to diversify through mixed uses. However, we analyze single-use real estate development with multiple assets; in our model, each property has no option for alternative use, but properties can be under either the same or different zoning controls.

By introducing multiple uncertainty, our analysis contributes to the literature. First, we derive the value of an option whose underlying assets are heterogeneous, but interrelated. With $n$ stochastic processes, we provide a close-form value of a development option in a standard framework of dynamic programming. Second, we analyze both the uncertainty of an individual asset and the uncertainty of a project. The effects of the uncertainty of an asset on development timing can be different from those of the aggregate uncertainty of a project. Third, we analyze interrelated assets (or projects) in an American option framework. In real estate development, in particular, we allow non-homogenous projects in Somerville (1999).

In order to build a simplest possible model, we abstract from some industry-specific characteristics. First, cash flows are independently determined by the number of assets of a project; implementing a portfolio strategy incurs no cost for a real estate developer. Second, a developer of this chapter has a static portfolio strategy whose adjustment over time is restricted as in Markowitz (1952). Third, under multi-dimensional uncertainty, the cash flows of each asset follow a normal diffusion process. Unlike our model, previous models in the literature were limited to the analysis of a real option with only two underlying assets, whose values follow log-normal diffusion processes.

The remainder of chapter 3 is structured as follows. Section 3.2 delivers a traditional real options model with a single stochastic process. The reader familiar with the literature may skip this section. Section 3.3 builds a real options model with $n$-dimensional stochastic processes. In order to investigate our model further, we consider the case when each property has equal weight in development and also we provide a simple numerical example. Section 3.4 concludes.

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28Under multiple-use zoning, Childs et al. (1996)'s developer starts single-use development with no conversion option, even though the choice of land use is flexible. Geltner et al. (1996)'s developer has a convert (redevelopment) option with which a conversion of a property into alternative use is flexible. In addition, Capozza and Li (1994) also deal with two underlying assets by examining a developer who converts one use (land) to another (building), even though they abstract from uncertainty on one underlying asset: land. In particular, the development plan for Capozza and Li (1994)'s developer is close to that of our developer.
3.2 The Development Option with A Single Property

Before examining $n$-dimensional heterogeneous diffusion processes, this section, which may be skipped by the reader who familiar with a real options model, proposes a traditional real options model with a single diffusion process.

By converting a vacant plot into an urban area, a developer constructs one unit of durable structure, a real asset. A developer maximizes the value of a project and builds a single use property. The developer is not allowed to transfer the developable right. Development is instantaneously completed with no uncertainty at the construction stage. Right after development, an asset generates the net revenue (or net rental rate) $p_t$ at time $t$ in an uncertain environment.

**Assumption 1.** The rent follows the normal diffusion process with a drift.

$$ dp_t = g dt + \sigma db_t, $$

where the drift $g$ and the instantaneous conditional standard deviation $\sigma$ per unit time. Both $g$ and $\sigma$ are independent of time $t$. $b$ denotes a standard Brownian motion and $db$ is the increment to $b$. As we have a single arithmetic Brownian motion, $\sigma$ represents the aggregate (total) uncertainty of a development project. In an urban context, previous studies such as Capozza and Helsley, and Capozza and Li suggest the rent process in (3.1), which allows negative net cash flows. For details, see Capozza and Schwann (1990) and Capozza and Sick.

Throughout this chapter, we assume that all the developers are risk-neutral, share information on the realized cash flows up to time $t$, and have the same expectations about the distribution of future rent flows according to (3.1). The known riskless rate of interest $r$ is fixed over time.

In a perfectly competitive market, the price of a real asset equals the expected present value of the future rent flows:

$$ v_t = \mathbb{E}_t \left\{ \int_0^\infty [p_t + gs + \sigma b_s] e^{-rs} ds | p_t \right\} $$

$$ = \frac{1}{r} (p_t + \frac{g}{r}), \quad (3.2) $$

where $\mathbb{E}_t$ is expectation conditional on the information available at time $t$. 

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The developer's problem is to determine the optimal timing of development. Before development, the option value of waiting for development is

\[ f(p_t) = \sup_{\{i\}} E_t \left[ \int_t^\infty p_s e^{-r(s-t)} ds - k \right] e^{-r(t-t')} [p_t, p_t], \]  

(3.3)

where the optimal development time is \( \hat{t} \) and development cost is \( k \). The first term on the right-hand side of (3.3) is the present value of the rent flows and the second term is discounted development cost, which includes the value of vacant land or finished land ready for development.

According to both the Bellman equation and Itô’s lemma, the solution for \( f(p) \) evolves according to the second-order linear ordinary differential equation:

\[ \frac{\sigma^2}{2} \frac{\partial^2 f(p)}{\partial p^2} + g \frac{\partial f(p)}{\partial p} - rf(p) = 0. \]  

(3.4a)

The general solution for equation (3.4a) is

\[ f(p) = c_1 e^{\alpha p} + c_2 e^{\beta p}, \]  

where \( \alpha = (-g + \sqrt{g^2 + 2\sigma^2 r}) / \sigma^2 > 0 \), and \( \beta < 0 \). Both \( c_1 \) and \( c_2 \) are to be determined according to the boundary conditions:

\[ \lim_{p \to -\infty} f(p) = 0, \]  

(3.4b)

\[ f(\hat{p}) = \frac{1}{r} (\hat{p} + \frac{g}{r}) - k, \]  

(3.4c)

\[ f_p(\hat{p}) = \frac{1}{r}. \]  

(3.4d)

Equation (3.4b) indicates that the option has no value when \( p \) is low enough and, according to the initial condition, the solution for the development option is

\[ f(p) = c_1 e^{\alpha p} . \]  

Equation (3.4c) is the value-matching condition that, at (unknown) optimal time \( \hat{p} \equiv p_t \), the value of an option equals the net present value of a project. Equation (3.4d) is the value-pasting condition that, at \( \hat{p} \), the marginal cost of losing a development option equals the marginal benefit.

The analytic solution for the equations (3.4a) - (3.4d) is

\[ f(p) = \begin{cases} \frac{1}{\alpha} e^{-\alpha(\hat{p}-p)} & \text{for } p \leq \hat{p} \\ \frac{1}{r} (p + \frac{g}{r}) - k & \text{for } p > \hat{p}, \end{cases} \]  

(3.5a)

\footnote{The time subscript is suppressed if they are clear from the context.}
where the reservation trigger is
\[ \hat{p} = r - \frac{\alpha g}{\alpha r} + rk. \] (3.5b)

The first term on the right-hand side of equation (3.5b) is an increment due to uncertainty, and the second term is rent on the development cost.

**Proposition 5.** A developer delays the development decision with the aggregate risk of a development project \( \sigma \).

The proof is
\[ \frac{\partial \hat{p}}{\partial \sigma} = -\frac{1}{\alpha^2} \frac{\partial \alpha}{\partial \sigma} = \frac{\sigma}{\sqrt{g^2 + 2r\sigma^2}}. \]

Uncertainty delays real estate development (Capozza and Helsley; and Capozza and Li), which is a general implication of a real options model (Dixit and Pindyck, 1994). Cunningham (2006) and Bulan et al. (2009) find empirical evidence of proposition 5 by controlling for lot size and project size, respectively. As proposition 5 considers a development project with one underlying asset, it takes no account of heterogeneity of real properties.

The reservation value of a development project is
\[ \hat{v} = \frac{1}{\alpha r} + k, \] (3.6)
with equations (3.2) and (3.5b). The first term on the right-hand side of (3.6) is the value of a development option at time \( \hat{p} \) and the second term is development cost. Under uncertainty, the reservation value increases by the option value in (3.6), whereas, under no uncertainty, the reservation value equals \( k \).

### 3.3 The Development Option with Heterogeneous Multiple Properties

As most developers build heterogeneous, but interrelated multiple assets in a single project, this section builds a real options model with \( n \) interrelated stochastic processes associated with revenue. Developer’s strategies will be determined by the relationship between assets and by the uncertainty of an individual property. Mixed-use development can be used as a hedging strategy (Childs et al., 1996). Note that each property is of single use and a project that builds \( n \) properties at the same time has properties of different use.
A developer builds predetermined \( n \geq 1 \) real assets, each of which is unique and has its own process of cash flows.\(^{30}\) \( n \) stochastic processes are represented by the matrix

\[
dP_t = \mathcal{G} dt + \Sigma d\mathcal{B}_t, \tag{3.7}
\]

where the matrix of growth rates \( \mathcal{G}_{n \times 1} = [g_1 \cdots g_n]^T \), where \( T \) denotes the transpose. The diagonal matrix \( \Sigma_{n \times n} \) has \( i \)th diagonal entries of \( \sigma_i \) per unit time for unit \( i \) and all the non-diagonal entries are zero. Note that \( \sigma_i \) represents the uncertainty of an individual asset \( i \), rather than the aggregate uncertainty of a project \( \sigma \) in the previous section.

The \( n \)-dimensional Brownian motion \( \mathcal{B}_t_{n \times 1} = [b_{1,t} \cdots b_{n,t}]^T \) is a vector of Brownian motions. The increment of a Brownian motion in continuous time is \( db_{i,t} = \epsilon_{i,t} \sqrt{dt} \) for any \( i \), where \( \epsilon_{i,t} \) is a random variable that follows a normal distribution with the first two moments \((0, 1)\). Consequently, \( E(db_{i,t}) = 0 \), and \( V(db_{i,t}) = E([db_{i,t}]^2) = dt \) for any \( i \), where \( V \) is conditional variance.

**Assumption 2.** Property values are spatially correlated and the spatial correlation decreases as the (socio-economic) distance between assets increases or as the degree of heterogeneity between assets increases.

The correlation between assets in the real estate market can be represented by spatial (auto)correlation in \cite{Tobler1970, Dubin1988}. Physical distance matters in the real estate market; due to social and economic interactions of residents, the socio-economic distance is also important. Real estate values are interrelated by location attributes such as neighborhood characteristics, accessibility and proximity externalists, and by socio-economic attributes of neighborhood residents such as household income, education levels, race, ethnicity, and religious affiliation in a neighborhood. Moreover, the more heterogeneous real assets are, the less correlated their values are. In a local real estate market in which properties share the same location amenities and have similar neighborhood characteristics, for example, the value of a residential house is less correlated with the value of a commercial building than with the value of another residential house.

\(^{30}\)In order to determine \( n \) endogenously, Capozza and Li assume perfectly divisible housing supply with a concave production function.
The spatial correlation in chapter 3 is that, in an uncertain environment, if cash flows from an individual asset change, then those from nearby assets are likely to change. Brownian motions in (3.7) have a relationship of 

\[ E_t [db_i, \tau db_j, \tau] = \rho_{ij} \tau \] 

for \( i \neq j \) and \( \tau \geq t \), where \( i, j = 1, \ldots, n \); the instantaneous coefficient of spatial correlation between processes is \( \rho_{ij} = \rho_{ji} \in (-1, 1) \). Consequently,

\[
(d\mathcal{B})(d\mathcal{B})^T = \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1n} \\
\rho_{21} & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \vdots & \cdots & 1
\end{pmatrix} dt
\]

is symmetric and positive definite for all \( t \) almost surely.\(^{31}\) The correlation coefficient \( \rho_{ij} \) should be weakly less-than-perfect for any \( i \neq j \). Both real properties and their inter-relationship are unique.

Note that Capozza and Li have \( g = g_i, \sigma = \sigma_i, b_t = b_{i,t} \) and \( \rho = \rho_{ij} = 1 \) for any \( i \) and \( j \). In addition, Childs et al. (1996) and Geltner et al. (1996) has only two interrelated underlying assets. With a single asset in a project, note that equation (3.7) turns to equation (3.1). By equations (3.2) and (3.1), moreover, \( dv_i = \frac{g_i}{r} dt + \sigma_i \sigma_{ij}^r db_i \) for any \( i \in [1, n] \).

As \( dv_i \) has an error term correlated with an error term of other assets, the values of properties are correlated.\(^{32}\)

The total rents from a development project of \( n \) assets is

\[ \mathcal{P} = \mathcal{W}^T \mathcal{P}, \quad (3.8) \]

where \( \mathcal{W}^T 1 = 1 \), where \( 1 \) is a column vector of ones. For simplicity, we assume \( \mathcal{P} \) is independent of \( n \); the spatial relationship will matter, even though \( \mathcal{P} \) is a function of \( n \). We allow non-homogeneous projects (developers) in Somerville (1999), as the project size is different according to a unique strategy \( \mathcal{W} \). Note that the indivisible housing units are

\(^{31}\)Our assumption on \( \mathcal{B} \) is consistent with that of other finance literature such as Dixit and Pindyck (1994). Note, nonetheless, that our correlation coefficient is different from that of Wang and Zhou (2006) who suggest a correlation between revenue and development cost. However, mathematicians often define \( E_t [db_i, \tau db_j, \tau] = 0 \) for \( i \neq j \) when they assume multiple independent Brownian motions.

\(^{32}\)E[\( dv_i dv_j \)] = \( \frac{\sigma_i \sigma_{ij}^r}{r} dt \), according to the box algebra that will be discussed later. Furthermore, \( dv_i \) is a special case of general multi-variable stochastic calculus, in which a stochastic variable has multiple error terms, for example, \( dv_i = \frac{g_i}{r} dt + \sum_{j=1}^n \frac{\sigma_{ij}^r}{r} db_j \) for any \( i \)
large enough for generating cash flows (Colwell and Sirmans, 1978); each entry of $W$ is a point in real number $\mathbb{R}$.\footnote{We can endogenously determine $W$ when a developer’s goal is not to postpone a project. The proof is available upon request.}

According to Itô’s lemma, the total differential of (3.8) is
\[ d\mathbb{P}(P) = (\nabla\mathbb{P})^T dP + \frac{1}{2}(d\mathbb{P})^T (\nabla^2\mathbb{P})dP, \]
because the literature on real options ignores the terms of order 3 or higher and assumes no time derivative. Furthermore, $\nabla\mathbb{P} = W_{n \times 1}$ and $\nabla^2\mathbb{P} = 0_{n \times n}$.\footnote{The gradient with respect to $P$ of $\mathbb{P}$ is $\nabla\mathbb{P} \equiv \begin{bmatrix} \frac{\partial \mathbb{P}(P)}{\partial p_1} & \cdots & \frac{\partial \mathbb{P}(P)}{\partial p_n} \end{bmatrix}^T$ and the Hessian with respect to $P$ of $\mathbb{P}$ is $\nabla^2 \mathbb{P}(P) \equiv \begin{bmatrix} \frac{\partial^2 \mathbb{P}(P)}{\partial p_1^2} & \cdots & \frac{\partial^2 \mathbb{P}(P)}{\partial p_1 \partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathbb{P}(P)}{\partial p_n \partial p_1} & \cdots & \frac{\partial^2 \mathbb{P}(P)}{\partial p_n^2} \end{bmatrix}$.} As a result, a portfolio of arithmetic Brownian motions in (3.8) evolves according to the process with $n$-dimensional Brownian motions:
\[ d\mathbb{P} = W^T Gdt + W^T MB \] (3.9)

with (3.7). Equation (3.9) yields
\[ (d\mathbb{P})^2 = S^2 dt, \] (3.10)
where the aggregate risk of a development project with multiple assets
\[ S \equiv \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \rho_{ij} \sigma_i \sigma_j} \] (3.11)
is the standard deviation of a project as we ignores the terms $dt$ of order 3/2 or higher, which go faster to zero than $dt$ does. See appendix B for derivation; equation (3.11) implies $\sum_{i=1}^n \sum_{j=1}^n w_i w_j \rho_{ij} \sigma_i \sigma_j > 0$. Note that $S$ represents the aggregate uncertainty of a project $\sigma$ in the previous section.

In a perfectly competitive market, the value of a portfolio of real assets is
\[
V = E_t \left\{ \int_0^\infty [W^T P + (W^T G)s + W^T MB_s]e^{-rs} ds \mid \mathbb{P}_t \right\} = \frac{1}{r}(P + G),
\] (3.12)
where $G \equiv W^T G$.

The developer’s problem is to determine the optimal timing of the development. Before development, the option value of waiting for development is

$$F(P_t) = \sup_{t^*} \mathbb{E}_t \left[ \int_{t^*}^{\infty} P_s e^{-r(s-t^*)} ds - k \right] e^{-r(t^* - t)|P_t|},$$

where $P_s$ is the level of cash flows at $s$ and $k$ is the development cost. The first term on the right-hand side of (3.13) is the present value of the portfolio cash flows. In order to compare a project with one underlying asset and a project with multiple underlying assets, the development cost on the right-hand side of (3.13) equals that in (3.4).

According to both the Bellman equation and Itô’s lemma, the solution for $F(P)$ in (3.13) evolves according to the second-order linear ordinary differential equation

$$\frac{S^2}{2} \frac{\partial^2 F(P)}{\partial P^2} + G \frac{\partial F(P)}{\partial P} - rF(P) = 0,$$

whose general solution is $F(P) = C_1 e^{AP} + C_2 e^{B P}$, where $A = (-G + \sqrt{G^2 + 2Sr^2})/S^2 > 0$ and $B < 0$. We also have the boundary conditions such as

$$\lim_{P \to -\infty} F(P) = 0,$$

$$F(P^*) = \frac{1}{r} (P + \frac{G}{r}) - k,$$

$$F(P^*) = \frac{1}{r}.$$

Equation (3.14b) indicates that the option is of no value when the portfolio cash flows are low enough. Equation (3.14b) is the value-matching condition that, at (unknown) time $P^* \equiv P_{t^*}$, the value of a development option equals the net present value of portfolio investment. Equation (3.14d) is the smooth-pasting condition that, at time of the development, the marginal cost of losing a development option equals the marginal benefit of initiating a development project with multiple assets.

The analytic solution for equations (3.14) is

$$F(P) = \begin{cases} \frac{1}{A^r} e^{-A(P^* - P)} & \text{for } P \leq P^* \\ \frac{1}{r} \left( P + \frac{G}{r} \right) - k & \text{for } P > P^* \end{cases}$$

where the reservation trigger is

$$P^* = \frac{r - AG}{Ar} + rk.$$
The first term on the right-hand side of (3.15b) is an increment due to the aggregate uncertainty of a project, and the second term is rent on development cost. Note that the structure of the reservation trigger in (3.15b) is similar to that in (3.5b). Unlike $\hat{p}$ in (3.5b), $P^*$ in (3.15b) allows the heterogeneity of the assets; $P^*$ incorporates the correlation between assets and uncertainty of an individual asset.

**Proposition 6.** i) When short-sale constraints bind, a developer delays development with $\rho_{ij}$. 

ii) When short-sale constraints does not bind, a) if a developer holds a same position in assets $i$, $j$, then a developer delays development with $\rho_{ij}$; and b) if a developer holds a different position in assets $i$, $j$, then a developer does not delay development with $\rho_{ij}$.

\[
\frac{\partial P^*}{\partial \rho_{ij}} = \begin{cases} 
\frac{\partial P^*}{\partial A} \frac{\partial A}{\partial S} \frac{\partial S}{\partial \rho_{ij}} > 0, & \text{if } w_i w_j \sigma_i \sigma_j > 0 \\
\frac{\partial P^*}{\partial A} \frac{\partial A}{\partial S} \frac{\partial S}{\partial \rho_{ij}} \leq 0, & \text{otherwise}
\end{cases}
\]

See appendix B for derivation.

Proposition 6 is one of the key implications. A developer can decrease the uncertainty premium by implementing strategies of geographic and economic diversification or of property type & size diversification. As expected, developer’s timing strategy is a function of the correlation coefficient. Under short-sale constraints, a developer delays development with $\rho_{ij}$, since the aggregate uncertainty of a project increases with $\rho_{ij}$. However, the reservation trigger does not always increase with $\rho_{ij}$. Under no short-sale constraints, $S$ decreases as $\rho_{ij}$ increases if a developer holds a different position between asset $i$ and asset $j$: $w_i w_j < 0$. Nonetheless, when a developer holds the same position in both asset $i$ and asset $j$, a diversification strategy brings forward the development. Note that our heterogeneity is distinguished from Novy-Marx (2007)’s heterogeneity, which reinforces the value of an option.

The effects of the uncertainty of an asset on development timing can be different from those of the aggregate uncertainty of a project.

**Proposition 7.** If $\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j > 0$, then a developer delays development with $\sigma_i$. If $\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j \leq 0$, then a developer does not delay development with $\sigma_i$.

---

35 The real estate literature traditionally emphasizes diversification issues and, within the real estate asset classes, see Seiler et al. (1999) for an overview of the literature.
\[
\frac{\partial P^*}{\partial \sigma_i} = \begin{cases} 
\frac{\partial P^*}{\partial \sigma_i} > 0, & \text{if } \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j > 0 \\
\frac{\partial P^*}{\partial \sigma_i} \leq 0, & \text{otherwise}
\end{cases}
\]

See appendix B for derivation.

Uncertainty of an asset delays development when \(\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j > 0\). Nonetheless, it does not always delay development, since proposition 7 indicates that, when \(\sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j < 0\), a developer brings development forward with uncertainty of an asset. In addition, even though a developer has short-sale constraints, a developer can bring forward development with uncertainty when values of spatial correlation coefficients are negative. Plazzi et al. (2010) provide empirical evidence that the expected growth in office rent is (significantly) negative over one year or three years when rent to price ratios are higher in the U.S. commercial real estate market from the mid 1990s to the early 2000s.

With equations (3.12), (3.15b), we have the reservation price for development

\[
V^* = \frac{1}{K_P} + k. \tag{3.16}
\]

The first term on the right-hand side of (3.16) is the value of a development option at time \(P^*\) and the second term is development cost. Note that the option value for a project with multiple assets is determined by \(g_i, \sigma_i\) and \(\rho_{ij}\) for any \(i, j\).

### 3.3.2 A Special Case: Equally Weighted Assets under Short-Sale Constraints

In the previous section, a developer has portfolio strategies that allow short-sale of real properties in the real estate market. Unlike investors in the financial market, however, most real estate developers are likely to face short-sale constraints. Therefore, this section imposes short-sale constraints for the assets. In order to investigate implications of the model, furthermore, we assume that a developer builds \(n\) assets which are identical except for the location. Note that assets can be distributed across markets.

The drift in (3.9) for equally weighted portfolios is

\[
\bar{G} = (\sum_{i=1}^{n} g_i)/n, \tag{3.17}
\]

and the aggregate risk of a project in (3.11) for equally weighted portfolios turns to
\[ \tilde{S} = \sqrt{\frac{1}{n} \sigma_{ii} + \frac{n - 1}{n} \sigma_{ij}} \]  

(3.18)

per unit time where \( \sigma_{ii} \equiv \sum_{i=1}^{n} \frac{1}{n} \sigma_{i}^{2} \) is the average variance of an individual asset and \( \sigma_{ij} \equiv \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \rho_{ij} \sigma_{i} \sigma_{j} \) is the average covariance of assets. See appendix B for derivation.

Our model is more general than that of Capozza and Helsley and Capozza and Schwann (1990), which present a case of a single underlying asset. Even though Capozza and Li (1994) allow multiple assets, they pay no attention to the heterogeneity of the assets and their (spatial) correlation. In order to see this, we can show that, with \( g_{i} = g \) and \( \sigma_{i} = \sigma \) for any \( i \) and \( \rho_{ij} = 1 \) for any \( i, j \), \( \tilde{G} \) in (3.17) and \( \tilde{S} \) in (3.18) equal \( g \) and \( \sigma \sqrt{\frac{1}{n} + \frac{n-1}{n}} \rho = \sigma \), respectively. Consequently, \( V^* \) in (3.16) for a project with multiple assets of homogeneity equals \( \tilde{v} \) in (3.6) for a project with a single assets. With a single asset, furthermore, \( V^* \) equals \( \tilde{v} \); this is an assumption of Capozza and Helsley and Capozza and Schwann (1990).

Equation (3.18) shows that the aggregate project risk can be broken down into two pieces. By increasing the number of assets, we can reduce the first term on the right-hand side of (3.18), which is called the non-market component of a project’s standard deviation. Even by increasing the number of assets, we can not eliminate the second term on the right-hand side of (3.18).

**Corollary 2.** The aggregate uncertainty of a development project decreases as a developer builds more assets of non-homogeneity.

\[ \lim_{n \to \infty} \tilde{S} = \sqrt{\sigma_{ij}} \]

Even with numerous assets, the non-diversifiable risk matters for a developer. In order to illustrate corollary 2 we assume that a developer builds identical properties in one small market and thus we can adopt the base-case parameter values: \( g_{i} = 1 \) and \( \sigma_{i} = 4 \) for any \( i \) and also \( k = 300 \) and \( r = 0.03 \) according to Capozza and Sick. Consequently, \( \tilde{G} \) in (3.17) and \( \tilde{S} \) in (3.18) equal 1 and \( \sqrt{\frac{16}{n} + \frac{8(n-1)}{n}} \), respectively.

Even though a developer delays development under uncertainty, a developer is less likely to delay development with multiple assets due to the spatial relationship. Figure 3.1 shows the reservation triggers \( P^* \) in (3.15b) with respect to the number of properties. The maximum reservation trigger is 15.66 with a single asset and it decreases with \( n \). Under
no uncertainty, note that the reservation trigger is \( rk = 9 \), while, under uncertainty, the reservation trigger \( \hat{\rho} \) in (3.5b) with a single asset is 15.66. In figure 3.1, the reservation trigger is invariant with respect to \( n \) with \( \rho_{ij} = 1 \) for any \( i, j \). This is a traditional approach to the value of a development option in the literature. With \( \rho_{ij} \neq 1 \), nevertheless, we argue that the reservation trigger decreases and becomes flatter as \( n \) increases. The reservation trigger is higher than 9.\(^{36}\) Note that when \( \rho_{ij} = 0 \) for any \( i, j \), the reservation trigger fast approaches to 9 as non-market component of a project’s standard deviation disappears with \( n \).

The deterrent effect of the aggregate uncertainty is smaller with the spatial correlation than without it. The value of a development option has a positive relationship with the aggregate risk \( \bar{S} \), which is positively related with a correlation coefficient. The value of a development option increases with correlation.

**Corollary 3.** When short-sale constraints bind, the traditional approach in real estate development that ignores the (spatial) correlation overestimates the value of a development option.

\[
\frac{\partial \bar{S}}{\partial \rho_{ij}} = \frac{1}{2S} \left( \frac{2\sigma_i \sigma_j}{n^2} \right) > 0.
\]

In order to illustrate corollary 3, we keep assumptions for figure 3.1. With \( n = 2 \), nonetheless, figure 3.2 shows \( P^* \) with respect to \( \rho_{12} \) and \( P^* \) reaches its maximum value with \( \rho_{12} = 1 \).

The uncertainty of an individual asset does not always delay development. When the sum of covariances between \( p_i \) and \( p_{j \neq i} \) has a negative value that is significantly larger than the variance of \( p_i \), a developer can bring forward development with \( \sigma_i \). When all the spatial correlation coefficients are positive, nevertheless, the uncertainty of an individual asset delays development.

**Corollary 4.** With equally weighted portfolios of \( n \) assets, if \( \sigma_i + \sum_{j=1, j \neq i}^n \rho_{ij} \sigma_j > 0 \), then a developer delays development with \( \sigma_i \) and otherwise, a developer does not delay development.

\(^{36}\)Note that we assume that a developer incurs no diversification cost as in Markowitz (1952); when \( p \) or \( k \) is determined by \( n \), the effects of diversification strategies will diminish.
\[
\frac{\partial S}{\partial \sigma_i} = \frac{1}{S} \left( \sigma_i + \sum_{j=1,j\neq i}^n \rho_{ij} \sigma_j \right)\frac{n^2}{S}.
\]

In order to illustrate a case of \(\frac{\partial S}{\partial \sigma_i} < 0\) in corollary 4, we keep most assumptions for figure 3.1 except for \(n = 2\) and \(\rho_{12} = -0.5\). Figure 3.3 shows \(P^*\) with respect to \(\sigma_1\). The aggregate uncertainty of a project does not always increase with the uncertainty of an individual asset; with a negative spatial correlation coefficient, \(\frac{\partial S}{\partial \sigma_i}\) can be negative when \(\sigma_1 < 2\).

### 3.4 Conclusion

We show that heterogeneity of real properties revises the value of an option for waiting in the literature on real estate development. Under short-sale constraints for the assets, for example, uncertainty can have smaller deterrent effects for a developer with a diversification strategy than those for a developer without it. By providing a closed-form value of an option with \(n\) interrelated stochastic processes, we analyze the effects of spatial correlation on development timing decisions. In addition, we show that the effects of uncertainty of an asset on development can be different from those of uncertainty of a project.

Our results seem similar to development strategies in practice. Introducing the term of plattage in the land market, Colwell and Sirmans (1978) argue that a land owner is better off with an additional value, which is created by subdividing a large parcel into two or more small lots. Although the value function of a portfolio of heterogeneous assets is independent of the number of assets, the option value of our model decreases with respect to the number of assets in a project. As a consequence, our developer is better off with an incentive to bring forward development with the lower project risk.

In the real estate market, we look at a diversification strategy of real estate developers who can be of heterogeneity in Novy-Marx (2007). For empirical evidence relevant to this chapter, real estate market-wide (or economy-wide) models may be inappropriate because the capital asset pricing model (CAPM) assumes a homogeneous investor. We argue that given expected return, a portfolio of mixed assets hastens investment with a decrease in the

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\[\text{We abstract from Colwell and Sirmans (1993)’s land, whose parcels (assets) are too small to be developed. With assembly cost, the relationship between value of undeveloped land and parcel size can be different even though the diversification effects remain.}\]
variance for the whole project. Therefore, we suggest an individual project-level approach such as [Henriksson (1984)] who tests the validity of using the CAPM for mutual funds.

As we mentioned, a caveat of our model is that a developer has no timing diversification strategies in [Childs et al., (1998)] who makes the comparison between parallel development and sequential development. In order to derive the value of an American-style option, we often use the backward induction algorithm. After building \( n - 1 \) (where \( n \geq 2 \)) assets, for example, an option holder contemplates the optimal time for building last \( n \)th asset. For optimal conditions in real options, nonetheless, the option value and the net present value of investment should be a function of the same process. We relegate this to future research.
For the figure, we assume $k = 300$, $r = 0.03$, $g_i = 1$ and $\sigma_i = 4$ for any $i$. The reservation trigger rent decreases with the number of underlying assets; with $\rho = 1$ the reservation trigger is invariant with respect to $n$. Note that the reservation trigger decreases faster as $\rho$ decreases. Note that under no uncertainty the reservation trigger is $rk = 9$. 
Figure 3.2: Effects of $\rho$ on $P^*$

For the figure, we assume $g_1 = g_2 = 1, \sigma_1 = \sigma_2 = 4, k = 300, r = 0.03$ and $n = 2$, The reservation trigger increases with $\rho_{12}$ and reaches the maximum value at $\rho_{12} = 1$. 
For the figure, we assume \( g_1 = g_2 = 1, \sigma_2 = 4, k = 300, r = 0.03, n = 2 \) and \( \rho_{12} = -0.5 \). When \( \sigma_1 < 2 \) (or \( \sigma_1 + \rho_{12} \sigma_2 < 0 \)), the reservation trigger decreases with \( \sigma_1 \). With \( \sigma_1 > 2 \) (or \( \sigma_1 + \rho_{12} \sigma_2 > 0 \)), the reservation trigger increases with \( \sigma_1 \).
Chapter 4

The Price-to-income Ratio and the Quality of Life
with Professor Sanghoon Lee

4.1 Introduction

The price-to-income ratio has been widely used as a measure for housing market bubble. If a city has a very high price-to-income ratio, this is often considered a sign of the bubble, because people would not pay such prices if they did not expect housing price to go up in the future. For example, Demographia (2010) classifies areas with price-to-income ratios greater than 4.5 as severe bubble areas.

Chapter 4 argues that a cross-city comparison of price-to-income ratios can overestimate the bubble in high quality of life (QOL) cities. Our theory builds on Roback (1982), who argues that workers are willing to pay higher rents and get paid lower wages in order to live in high QOL cities (e.g., mild climates, coastal locations). Because housing rent and wage directly affect housing price and income, this leads to higher price-to-income ratios in these cities. Note that the expectation of future price growth - a necessary condition for the bubble - does not play any role in this theory.

We take the theory to data and estimate this QOL bias. A challenge is that QOL may be correlated with price growth expectation: people may expect price to go up faster in higher amenity cities. Thus, we need to disentangle the QOL effect and the expectation effect in accounting for price-to-income ratios. Our model suggests that the QOL bias can be separately estimated by looking at rent-to-income ratios instead of price-to-income ratios. The intuition is that rent-to-income ratios do not depend on price growth expectation but respond to the QOL effect. Our empirical analysis confirms that the QOL bias is

\footnote{The expectation of future price increase does not necessarily mean the irrational bubble. See discussion below.}
significant.

The common problem among all bubble indicators is that distinguishing between rational and irrational expectation is virtually impossible. For example, if a city is expected to grow quickly in size, people can rationally expect its housing price to go up in the future and this is not considered a bubble. Beside this common problem among all bubble indicators, a few papers pointed out the limitations of price-to-income ratio. For example, Gallin (2006) shows that house price and income are not cointegrated even over a long time span. Himmelberg et al. (2005) argue that price-to-income ratios should not be compared over time without controlling for interest rate change. It is also well-known among economists that one cannot directly compare the price-to-income ratios across cities because different cities may have different levels of fundamental price-to-income ratios (e.g., Himmelberg et al. (2005))). The novelty of chapter 4 is that we provide an explicit theoretical mechanism why the fundamental ratios may differ across cities and test this particular mechanism. By investigating a cross-sectional bias of price-to income with economic variables such as amenities, which are assumed to be stable over time, this chapter is distinguished from Gallin (2006) and Himmelberg et al. (2005), which consider inter-temporal bias.

The remainder of chapter 4 is structured as follows. Section 4.2 presents the model and its theoretical implications. Section 4.3 presents our estimation strategy, data, and results. Section 4.4 concludes.

4.2 Theory

This section provides the model and its theoretical implications. The model is based on the idea of Roback (1982), who shows how wage and housing rent are determined as functions of consumption amenities (i.e., QOL) and production amenities. In order to pin down housing price, we add landlords who decide whether or not to purchase housing.

4.2.1 Model

There are numerous cities indexed by \( s \in S \subset \mathbb{N} \). Cities differ exogenously in their consumption amenities \( A \), production amenities \( O \), and expected future housing price growth \( e \). There are two commodities: a composite good and housing. The composite good is freely tradable with zero transportation cost while housing must be locally provided. The zero transportation cost for the composite good implies that its equilibrium price would
be same across all cities. We normalize this price to 1 and use the composite good as the numeraire.

Workers are homogeneous and freely mobile across cities with zero moving cost. A worker first chooses a city in which to live and then chooses her consumption bundle consisting of composite good $q$ and housing $h$. Their utility function $U(q,h;A)$ is increasing in consumption amenities $A$. Each worker supplies one unit of labor. The decision of a worker can be summarized by the following maximization problem:

$$\max_s V(r_s,w_s;A_s)$$

where

$$V(r_s,w_s;A_s) \equiv \max_{q,h} U(q,h;A_s) \text{ subject to } q + r_s h = w_s,$$

where $r_s$, $w_s$, and $A_s$ are housing rent, wage, and consumption amenities in city $s$. We obtain $\partial V/\partial w > 0$, $\partial V/\partial r < 0$, and $\partial V/\partial A > 0$, using the envelope theorem. In other words, the utility of workers living in city $s$ increases as local wage increases, rent falls, or consumption amenity level increases.

Because workers can move across cities with zero cost, they get the same utility across all cities in equilibrium. This leads to the following equilibrium condition:

$$V(r_s,w_s;A_s) = \bar{u} \text{ for } s \in S,$$

where $\bar{u}$ is the common equilibrium utility level in the economy.

Each city has numerous firms producing composite goods. All firms use identical constant returns to scale technology and thus we can consider one aggregate firm for each city that behaves as a perfectly competitive firm. We assume that the aggregate firm uses labor $n$ and buildings $h$ which come from the same stock of housing as the workers’ housing. Its production function $F$ is increasing in production advantage $O$. The decision of a firm in city $s$ can be summarized by the following maximization problem:

$$\max_{n,h} F(n,h;O_s) - w_s n - r_s h,$$

where $n$ and $h$ are employed labor and housing.

The market for the composite good is perfectly competitive and each firm thus makes zero profit in equilibrium. Because the firms use the same constant returns to scale technology, the zero profit condition is equivalent to unit cost being equal to unit output price:

$$C(r_s, w_s; O_s) = 1 \text{ for } s \in S,$$
where $C$ is the unit cost function. We obtain $\partial C/\partial r > 0$ and $\partial C/\partial w > 0$ because housing and labor are production inputs. We obtain $\partial C/\partial O < 0$ because firms can produce the composite good more efficiently as the local productivity $O$ increases.

Absentee landlords decide whether to purchase housing or not in each city. If they purchase housing, they rent it out to workers and firms. They can also earn capital gain if housing price increases. The cost of owning housing is the interest payment for housing price. Under this simple formulation the following condition holds in equilibrium.

$$i - e_s = \frac{r_s}{p_s} \text{ for } s \in S. \quad (4.3)$$

However, other factors may enter the landlords’ decision such as depreciation, and risk premium. To account for other factors without complicating the model, we assume a capitalization rate function $k$ at which landlords are indifferent between buying housing and not buying housing.

$$k(i, e_s) = \frac{r_s}{p_s} \text{ for } s \in S. \quad (4.4)$$

where we assume $\partial k/\partial i > 0$ and $\partial k/\partial e < 0$. These assumptions mean that landlords are willing to pay higher prices to purchase housing if price appreciation expectation is higher or interest is lower. Note that condition (4.3) is a special case of condition (4.4).

We are ready to define the equilibrium of our model. An equilibrium consists of wage $w_s$, rent $r_s$, and housing price $p_s$ for each city $s \in S$ satisfying the equilibrium conditions (4.1), (4.2), and (4.4).

### 4.2.2 Theoretical Implications

This section derives two theoretical implications: first, price-to-income ratio increases with the expectation of housing price increase in the future; second, price-to-income ratio increases with local consumption amenities. These two results jointly imply that the expectation effect and the consumption amenity effect are not distinguishable simply by looking at the price-to-income ratios across cities.

---

39 We could allow workers and firms to make the purchase decision. However, we chose to have landlords separate from workers and firms, to preserve the very popular Roback model for workers and firms. This makes our model easier to follow.

40 We do not explicitly model housing supply to keep our model simple. We can do this because housing supply does not affect housing prices, rents, or wages due to the free mobility assumption. See Lee and Li (2010) for an example where housing supply is explicitly modeled into the Roback model.
We begin with the expectation effect. Suppose that consumption amenities $A$ and production advantage $O$ are constant across cities. We take the total derivatives of equilibrium conditions (4.1), (4.2), and (4.4) and characterize the expectation elasticity of the price-to-income ratio.

\[
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log e} = -\frac{ep}{r} \frac{\partial k}{\partial e}.
\]

Because $\partial k/\partial e$ is negative, we obtain the following result.

**Proposition 8.** Suppose that $A$ and $O$ are constant across all cities. The price-to-income ratio is higher in cities with higher expectation on future price growth.

\[
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log e} > 0
\]

Now we turn to the consumption amenity effect. Suppose that $e$ and $O$ are constant across cities. We take the total derivatives of the equilibrium conditions (4.1), (4.2), and (4.4), and calculate the consumption amenity elasticity of the price-to-income ratio.

\[
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log A} = \frac{A \frac{\partial V}{\partial A} \left( r \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial w} \right)}{rw \left( - \frac{\partial V}{\partial r} \frac{\partial C}{\partial w} + \frac{\partial C}{\partial r} \frac{\partial V}{\partial w} \right)}.
\]

Because $\partial V/\partial A > 0$, $\partial V/\partial w > 0$, $\partial V/\partial r < 0$, $\partial C/\partial r > 0$, and $\partial C/\partial w > 0$, we obtain the following result.

**Proposition 9.** Suppose that $e$ and $O$ are constant across all cities. The price-to-income ratio is higher in cities with higher consumption amenities.

\[
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log A} > 0
\]

Proposition 8 shows that the price-to-income ratio is higher in cities with higher expectation on future growth. This is consistent with the common interpretation of price-to-income ratio as a bubble indicator. However, Proposition 9 shows that the price-to-income ratio may be higher in a city because the city has great QOL. Thus, price-to-income ratio, when used as a bubble indicator, can overestimate the housing market bubble in high QOL cities.
4.3 Empirical Analysis

This section tests if the consumption amenity bias is statistically significant. We need to do more than show a positive correlation between price-to-income ratios and QOL scores across cities because expectation on future price increase may be correlated with QOL. People may expect price to go up faster in high QOL cities. Thus, we need to control for the expectation effect to test the amenity effect we are interested in. We use our model to show that rent-to-income ratios can separate out the amenity effect from the expectation effect. This result is intuitive because rent-to-income ratios depend on QOL but not on expectation. We test whether rent-to-income ratios are positively correlated with QOL scores.

We need consumption amenity scores across cities as well as incomes, housing prices, and housing rents across cities. We use a standard data set (Integrated Public Use Microsample; IPUMS) for incomes, housing prices, and housing rents. However, there are no universally accepted QOL scores for cities. The two most popular ones - Cities Ranked & Rated and Places Rated Almanac - assign scores to various city characteristics (e.g., climate, crime, education) and combine these subscores to provide composite QOL scores. The QOL economics literature also infers QOL scores for cities using revealed preference theory for workers and firms (e.g., Rosen, 1979; Blomquist et al., 1988; Roback, 1988; Gy oursko and Tracy, 1991; Rappaport, 2008; and Albouy, 2008). We run our test using the two popular rankings and the latest rankings from the QOL literature by Albouy (2008). As an additional robustness check, we also run the test using only climate scores from Cities Ranked & Rated and Places Rated Almanac to avoid potential endogeneity problems in city characteristics.

4.3.1 Estimation Strategy

We begin by explaining our estimation strategy. When we allow expectation \( e \) to depend on consumption amenities \( A \), we obtain the following equation:

\[
\frac{d \log \left( \frac{p}{w} \right)}{d \log A} = \frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log A} + \frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log e} \frac{d \log e}{d \log A},
\]

(4.5)

where the first term on the right hand side is the QOL bias and the second term is the expectation effect correlated with QOL.
It turns out that the expectation effect on price-to-income ratio is identical to that on the price-to-rent ratio.

\[
\begin{align*}
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log e} &= -\frac{e p}{r} \frac{\partial k}{\partial e} = \frac{\partial \log \left( \frac{p}{e} \right)}{\partial \log e} \\
\end{align*}
\]

which we obtain by taking total derivatives of the equilibrium conditions \((4.1), (4.2),\) and \((4.4).\) The intuition is that expectation affects both price-to-income ratio and price-to-rent ratio in the same way; expectation affects price that is common across the two rates, but does not affect either current wage or rent.

By plugging equation \((4.6)\) into equation \((4.5)\), we obtain the following:

\[
\begin{align*}
\frac{\partial \log \left( \frac{p}{w} \right)}{\partial \log A} &= \frac{d \log \left( \frac{p}{w} \right)}{d \log A} - \frac{\partial \log \left( \frac{p}{e} \right)}{\partial \log e} \frac{\partial \log e}{\partial \log A} \\
&= \frac{d \log \left( \frac{p}{w} \right)}{d \log A} - \frac{\partial \log \left( \frac{p}{e} \right)}{\partial \log A} = \frac{\partial \log \frac{r}{w}}{\partial \log A}.
\end{align*}
\]

Equation \((4.7)\) suggests that we can estimate the QOL bias by regressing \(\log \left( \frac{r}{w} \right)\) on \(\log A.\)

### 4.3.2 Data

*Cities Ranked & Rated* \((\text{Sperling and Sander, 2007})\) provides QOL scores for 373 US metropolitan areas.\(^{41}\) For each metropolitan area, they determine composite QOL score by aggregating scores for ten categories: economy & jobs, cost of living, climate, education, health & health care, crime, transportation, leisure, arts & culture, and one subjective category. See table \(C.1\) for more details on how they calculate these scores.

The composite scores from *Cities Ranked & Rated* do not correspond directly to our consumption amenities \(A.\) The consumption amenities in our model correspond to QOL without considering income and housing cost, while the QOL composite scores in the book include these factors. Thus, following *Albouy (2008)*, we regenerate the consumption amenities score using their subcategory scores. We remove the economy & jobs and cost of living categories and generate QOL scores using the other categories. Although the book does not provide the weights used to produce their composite scores, we can back out these weights by regressing their logged composite score on logged score for each category. (see table \(C.2\) To get a feel for the new QOL scores we generated, we report them for the top

\(^{41}\)Scores are reported at primary metropolitan statistical area (PMSA) levels.
20 cities and bottom 10 cities in table 4.1. The differences between the QOL scores in the book and the new QOL scores are clear in cities such as San Francisco. According to our definition, San Francisco has a high consumption amenity score and is ranked the second. However, because San Francisco has a high housing price, the QOL score in the book is not as high and this city is ranked the 73rd.

*Places Rated Almanac* (Savageau, 2007) provides QOL scores for 379 metropolitan areas. It rates a city in nine categories: housing, jobs, ambience, crime, transportation, education, health care, recreation and climate. Again, we generate new QOL scores using the seven categories excluding housing and job categories. Unlike the *Cities Ranked & Rated*, the *Places Rated Almanac* puts equal weight on each category when calculating their composite score and we use the same weighting scheme. We report the new QOL scores for the top 20 cities and bottom 10 cities in table 4.1.

The QOL literature, seminated by Rosen (1979) and Roback (1982), infers QOL scores for cities from wages and rents. The QOL scores are identified because workers are willing to get paid lower wages and pay higher rents, combined with firms’ indifference condition across cities. We use the latest QOL scores in the literature, by Albouy (2008). One minor technical problem is that his QOL scores are negative for some low QOL cities and we cannot take the log of these negative scores. We rescale his QOL scores by adding 0.5, as reported in table 4.2.

Housing price, housing rent, and income data come from Census 2000 Integrated Public Use Microdata Series (IPUMS) 5 percent sample by Ruggles et al. (2010). Because housing quality and human capital distribution vary across cities, we run hedonic and Mincer regressions to control for these heterogeneities, following the QOL literature. One difference is that we run quantile regressions as we are interested in median values; price-to-income ratio is defined as the ratio of median housing price to median income. As we have the median values, we have a representative housing unit and a representative laborer in each city. Nonetheless, note that home ownership structure may vary across cities.

For housing price and rent we use IPUMS data. To control for quality differences we run hedonic regressions and calculate estimated price for a representative house. For housing price we regress housing values on housing characteristics and metropolitan area dummy variables using owner-occupied units. For housing rent we regress annual gross rents on housing characteristics and metropolitan area dummy variables using rented units. Both regressions are weighted by census-household weights. The housing characteristics dummy
variables include the number of rooms (9 dummies), the number of bedrooms (4), age (8), the number of units in the structure (5), plumbing facilities (2), kitchen or cooking facilities (2), house acreage (2), and acreage of property (7), in addition to 283 metropolitan area dummies. Using the coefficient estimates, we generate housing prices and rents across cities for a benchmark house for each city, one with four rooms, two bedrooms and complete plumbing and kitchen facilities, with one unit in the structure, built in the 1970s, and on a property size less than 1 acre.

For personal income we use IPUMS data. To control for heterogeneity among workers we run a Mincer regression and calculate wage for a representative worker across cities. We regress total personal income on personal characteristics and metropolitan area dummies, using workers who are aged 25 to 55 and worked more than 30 hours per week and more than 26 weeks per year. We weight the regression by census-person weights. The personal characteristics dummies include educational attainment (12 dummies), industry (9), occupation (9), sex (2), marital status (4), veteran status (2), race (5), citizenship (2), and English fluency (2). With the estimates we generate median income for a benchmark worker who is a 40 year old male, has 4 years of college education, is married but spouse can be absent, can speak English, is not a veteran, or a minority, and has a professional job (in 0-99 occupation group) in the service industry (800-800 industry group).

Table 4.3 presents a summary statistics for these variables. Average house price is $77,269, with a minimum of $36,404 in McAllen-Edinburg-Pharr-Mission, TX and a maximum of $309,905 in San Jose, CA. Average rent per month is $549, with the minimum of $346 in Johnstown, PA and the maximum of $1,201 in San Jose, CA. Average individual income is $46,795, with the minimum of $36,295 in McAllen-Edinburg-Pharr-Mission, TX and the maximum of $71,846 in Stamford, CT. Mean price-to-income ratio is 1.62, with a minimum of 1.00 in Beaumont-Port Arthur-Orange,TX, and a maximum of 4.97 in Santa Cruz, CA. Mean price-to-annual rent ratio is 11.43, and the mean annual rent-to-income ratio is 0.14.

4.3.3 Estimation Result

We run our test using each of the three sets of QOL indices: Cities Ranked & Rated, Places Rated Almanac, and Albouy (2008). We also run the test using only climate scores for cities from Cities Ranked & Rated and Places Rated Almanac. As mentioned above, we
use climate scores as additional robustness checks, because other city characteristics used to generate QOL scores (e.g., education, crime, healthcare, and transportation) may have endogeneity problems. The results are presented in table 4.5.

In row (A) of table 4.5, we show the QOL elasticity of the price-to-income ratio. We regress log price-to-income ratio on log QOL scores and report the coefficients. The results show a strong correlation between the price-to-income ratio and the amenity scores. For every 1 percent increase in the QOL index, the price-to-income ratio tends to increase by 0.57 (t-statistic: 11.7) percent with Cities Ranked & Rated, by 0.32 (7.7) percent with Places Rated Almanac, by 2.13 (14.9) percent with Albouy (2008), by 0.05 (2.8) percent with Cities Ranked & Rated climate scores, and by 0.08 (5.1) percent with Places Rated Almanac climate scores.

However, as equation (4.5) shows, some of these correlations may be driven by the correlation between QOL and expectation on future housing price growth. Equation (4.7) suggests that the QOL effect can be captured by subtracting the QOL elasticity of price-to-rent ratio from that of the price-to-income ratio or simply by calculating the QOL elasticity of rent-to-income ratio.

In row (B) of table 4.5, we show the QOL elasticity of the price-to-rent ratio. The results show a significant correlation between QOL scores and the price-to-rent ratios. This indicates that people expect housing price to increase more in higher QOL cities. For every 1 percent increase in the QOL index, price-to-rent ratio tends to increase by 0.26 (6.6) percent with Cities Ranked & Rated, by 0.14 (4.6) with Places Rated Almanac, by 0.93 (7.4) with Albouy (2008), by 0.01 (0.9) percent with Cities Ranked & Rated climate scores, and by 0.03 (3.0) percent with Places Rated Almanac climate scores.

In row (C) of table 4.5, we show the QOL elasticity of the rent-to-income ratio that captures the QOL effect on the price-to-income ratio after controlling for the expectation effect. The results show the QOL effect on price-to-income ratio is significant. For every 1 percent increase in the QOL index, rent-to-income ratios tend to increase by 0.31 (13.2) percent with Cities Ranked & Rated, by 0.18 (8.5) percent with Places Rated Almanac, by 1.20 (18.8) percent with Albouy (2008), by 0.04 (4.5) percent with Cities Ranked & Rated climate scores, and by 0.05 (5.9) percent with Places Rated Almanac climate scores.

Table 4.6 has amenity corrected price-to-income ratios with Cities Ranked & Rated. Compared to table 4.4, the price-to-income ratio decreases by 0.18 in Santa Cruz, CA where amenity scores are 36% higher than the mean, while it increases by 0.3 in Beaumont-Port.
Arthur-Orange, TX where amenity scores are 6% lower than the mean.

4.4 Conclusion

Chapter 4 argues that a direct comparison of price-to-income ratios can overestimate the bubble in high QOL cities. It is well-known among economists that price-to-income ratios cannot be compared across cities because fundamental ratios may differ across cities. This chapter provides one explicit mechanism that makes these ratios differ across cities and tests this particular mechanism. Our empirical test shows that the QOL bias is significant.
Table 4.1: Top 20 and Bottom 10 Cities: Cities Ranked & Rated and Places Rated Almanac

<table>
<thead>
<tr>
<th>QOL Rank</th>
<th>Metropolitan Area</th>
<th>QOL Score</th>
<th>QOL Rank</th>
<th>Metropolitan Area</th>
<th>QOL Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Nassau-Suffolk, NY</td>
<td>81</td>
<td>1</td>
<td>San Francisco, CA</td>
<td>87</td>
</tr>
<tr>
<td>2</td>
<td>San Francisco-San Mateo-Redwood City, CA</td>
<td>79</td>
<td>2</td>
<td>Pittsburgh, PA</td>
<td>85</td>
</tr>
<tr>
<td>3</td>
<td>Santa Ana-Anaheim-Irvine, CA</td>
<td>78</td>
<td>3</td>
<td>Seattle-Bellevue-Everett, WA</td>
<td>85</td>
</tr>
<tr>
<td>4</td>
<td>Napa, CA</td>
<td>77</td>
<td>4</td>
<td>San Jose-Sunnyvale, CA</td>
<td>83</td>
</tr>
<tr>
<td>5</td>
<td>San Jose-Sunnyvale-Santa Clara, CA</td>
<td>77</td>
<td>5</td>
<td>Philadelphia, PA</td>
<td>83</td>
</tr>
<tr>
<td>6</td>
<td>Charlottesville, VA</td>
<td>77</td>
<td>6</td>
<td>Newark-Union, NJ-PA</td>
<td>83</td>
</tr>
<tr>
<td>7</td>
<td>Bridgeport-Stamford-Norwalk, CT</td>
<td>77</td>
<td>7</td>
<td>Boston-Quincy, MA</td>
<td>82</td>
</tr>
<tr>
<td>8</td>
<td>Oxnard-Thousand Oaks-Ventura, CA</td>
<td>76</td>
<td>8</td>
<td>New York, NY-NJ</td>
<td>82</td>
</tr>
<tr>
<td>9</td>
<td>Los Angeles-Long Beach-Glendale, CA</td>
<td>75</td>
<td>9</td>
<td>Washington, DC-VA-MD-WV</td>
<td>82</td>
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<tr>
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<td>Boulder, CO</td>
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<td>Portland-Vancouver, OR-WA</td>
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<tr>
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<td>Philadelphia, PA</td>
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<td>San Diego, CA</td>
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<tr>
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<td>Providence, RI-MA</td>
<td>79</td>
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<td>13</td>
<td>San Diego-Carlsbad-San Marcos, CA</td>
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<td>13</td>
<td>Nassau-Suffolk, NY</td>
<td>79</td>
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<td>14</td>
<td>New York-White Plains, NY</td>
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<td>14</td>
<td>Baltimore-Towson, MD</td>
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<td>New Haven-Milford, CT</td>
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<td>Rochester, NY</td>
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<td>16</td>
<td>Winchester, VA-WV</td>
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<td>16</td>
<td>Santa Ana-Anaheim-Irvine, CA</td>
<td>79</td>
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<td>Ann Arbor, MI</td>
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<td>Madison, WI</td>
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<td>18</td>
<td>Santa Barbara-Santa Maria, CA</td>
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<td>Minneapolis-St. Paul, MN-WI</td>
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<td>San Luis Obispo-Paso Robles, CA</td>
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<td>Virginia Beach-Norfolk, VA-NC</td>
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<td>364</td>
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<td>Danville, IL</td>
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<td>371</td>
<td>Morristown, TN</td>
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<td>372</td>
<td>Elkhart-Goshen, IN</td>
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<td>Clarksville, TN-KY</td>
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<td>373</td>
<td>Elizabethtown, KY</td>
<td>21</td>
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<td>368</td>
<td>Decatur, AL</td>
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<td>374</td>
<td>Warner Robins, GA</td>
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</tr>
<tr>
<td>369</td>
<td>Florence, SC</td>
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<td>Sumter, SC</td>
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<tr>
<td>370</td>
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<td>Morristown, TN</td>
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<td>372</td>
<td>Gadsden, AL</td>
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<td>378</td>
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<td>373</td>
<td>Pine Bluff, AR</td>
<td>24</td>
<td>379</td>
<td>Goldsboro, NC</td>
<td>15</td>
</tr>
</tbody>
</table>

* We adjust original data to fit our purpose. See text for details.
### Table 4.2: Top 20 and Bottom 10 Cities (cont.): Albouy (2008)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Metropolitan Area</th>
<th>QOL Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Honolulu, HI</td>
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<tr>
<td>2</td>
<td>Santa Barbara-Santa Maria-Lompoc, CA</td>
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<td>Salinas, CA</td>
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</tr>
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<td>4</td>
<td>Santa Fe, NM</td>
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</tr>
<tr>
<td>5</td>
<td>San Luis Obispo-Atascadero-Paso Robles, CA</td>
<td>0.62</td>
</tr>
<tr>
<td>6</td>
<td>San Francisco-Oakland-San Jose, CA</td>
<td>0.61</td>
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<tr>
<td>7</td>
<td>San Diego, CA</td>
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<tr>
<td>8</td>
<td>Naples, FL</td>
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<tr>
<td>9</td>
<td>Medford-Ashland, OR</td>
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<td>340</td>
<td>Beaumont-Port Arthur, TX</td>
<td>0.41</td>
</tr>
<tr>
<td>341</td>
<td>Kokomo, IN</td>
<td>0.39</td>
</tr>
</tbody>
</table>

* We adjust original data to fit our purpose. See text for details.
<table>
<thead>
<tr>
<th></th>
<th>Obs</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median Home Price</strong></td>
<td>283</td>
<td>$77,269</td>
<td>$35,184</td>
<td>$36,404 McAllen-Edinburg-Pharr-Mission, TX</td>
<td>$309,907 San Jose, CA</td>
</tr>
<tr>
<td><strong>Median Rent</strong></td>
<td>283</td>
<td>$549</td>
<td>$119</td>
<td>$346 Johnstown, PA</td>
<td>$1,201 San Jose, CA</td>
</tr>
<tr>
<td><strong>Average Individual Income</strong></td>
<td>283</td>
<td>$46,794</td>
<td>$5,040</td>
<td>$36,295 McAllen-Edinburg-Pharr-Mission, TX</td>
<td>$71,846 Stamford, CT</td>
</tr>
<tr>
<td><strong>Price-to-income ratio</strong></td>
<td>283</td>
<td>1.62</td>
<td>0.55</td>
<td>1.00 Beaumont-Port Arthur-Orange,TX</td>
<td>4.97 Santa Cruz, CA</td>
</tr>
<tr>
<td><strong>Price-to-annual rent ratio</strong></td>
<td>283</td>
<td>11.43</td>
<td>2.40</td>
<td>7.95 San Antonio, TX</td>
<td>24.35 Santa Cruz, CA</td>
</tr>
<tr>
<td><strong>Annual rent-to-income ratio</strong></td>
<td>283</td>
<td>13.99%</td>
<td>1.90%</td>
<td>10.38% Decatur, AL</td>
<td>21.97% San Jose, CA</td>
</tr>
</tbody>
</table>

Authors' calculations based on the 2000 Census
Table 4.4: Price-to-income Ratios: Top and Bottom 20 Cities

<table>
<thead>
<tr>
<th>Metropolitan Area</th>
<th>Price-to-income Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Santa Cruz, CA</td>
<td>4.97</td>
</tr>
<tr>
<td>San Jose, CA</td>
<td>4.73</td>
</tr>
<tr>
<td>Santa Barbara-Santa Maria-Lompoc, CA</td>
<td>4.24</td>
</tr>
<tr>
<td>Honolulu, HI</td>
<td>3.97</td>
</tr>
<tr>
<td>San Francisco-Oakland-Vallejo, CA</td>
<td>3.87</td>
</tr>
<tr>
<td>Salinas-Sea Side-Monterey, CA</td>
<td>3.72</td>
</tr>
<tr>
<td>Stamford, CT</td>
<td>3.68</td>
</tr>
<tr>
<td>Santa Rosa-Petaluma, CA</td>
<td>3.35</td>
</tr>
<tr>
<td>Los Angeles-Long Beach, CA</td>
<td>3.28</td>
</tr>
<tr>
<td>San Luis Obispo-Atascocita-P Robles, CA</td>
<td>3.05</td>
</tr>
<tr>
<td>Ventura-Oxnard-Simi Valley, CA</td>
<td>3.04</td>
</tr>
<tr>
<td>San Diego, CA</td>
<td>2.99</td>
</tr>
<tr>
<td>Santa Fe, NM</td>
<td>2.80</td>
</tr>
<tr>
<td>Boston, MA-NH</td>
<td>2.68</td>
</tr>
<tr>
<td>New York-Northeastern NJ</td>
<td>2.60</td>
</tr>
<tr>
<td>Seattle-Everett, WA</td>
<td>2.56</td>
</tr>
<tr>
<td>Yolo, CA</td>
<td>2.44</td>
</tr>
<tr>
<td>Barnstable-Yarmouth, MA</td>
<td>2.37</td>
</tr>
<tr>
<td>Eugene-Springfield, OR</td>
<td>2.27</td>
</tr>
<tr>
<td>Danbury, CT</td>
<td>2.27</td>
</tr>
<tr>
<td>San Antonio, TX</td>
<td>1.19</td>
</tr>
<tr>
<td>Lubbock, TX</td>
<td>1.18</td>
</tr>
<tr>
<td>Wichita Falls, TX</td>
<td>1.18</td>
</tr>
<tr>
<td>Dothan, AL</td>
<td>1.18</td>
</tr>
<tr>
<td>Abilene, TX</td>
<td>1.18</td>
</tr>
<tr>
<td>Anniston, AL</td>
<td>1.18</td>
</tr>
<tr>
<td>Wichita, KS</td>
<td>1.17</td>
</tr>
<tr>
<td>Galveston-Texas City, TX</td>
<td>1.15</td>
</tr>
<tr>
<td>Decatur, IL</td>
<td>1.14</td>
</tr>
<tr>
<td>Macon-Warner Robins, GA</td>
<td>1.14</td>
</tr>
<tr>
<td>Jamestown-Dunkirk, NY</td>
<td>1.13</td>
</tr>
<tr>
<td>Gadsden, AL</td>
<td>1.12</td>
</tr>
<tr>
<td>Waco, TX</td>
<td>1.10</td>
</tr>
<tr>
<td>Albany, GA</td>
<td>1.10</td>
</tr>
<tr>
<td>Houston-Brazoria, TX</td>
<td>1.09</td>
</tr>
<tr>
<td>Brownsville-Harlingen-San Benito, TX</td>
<td>1.07</td>
</tr>
<tr>
<td>Flint, MI</td>
<td>1.05</td>
</tr>
<tr>
<td>McAllen-Edinburg-Pharr-Mission, TX</td>
<td>1.00</td>
</tr>
<tr>
<td>Odessa, TX</td>
<td>1.00</td>
</tr>
<tr>
<td>Beaumont-Port Arthur-Orange, TX</td>
<td>1.00</td>
</tr>
</tbody>
</table>

* Quality controlled ratio
Table 4.5: The QOL Elasticities of Price-to-Income Ratio, Price-to-Rent Ratio, and Rent-to-Income Ratio

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>QOL elasticity of price-to-income ratio (A)</td>
<td>0.57 (11.7)</td>
<td>0.32 (7.7)</td>
<td>2.13 (14.9)</td>
<td>0.05 (2.8)</td>
<td>0.08 (5.1)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.34</td>
<td>0.18</td>
<td>0.44</td>
<td>0.33</td>
<td>0.09</td>
</tr>
<tr>
<td>QOL elasticity of price-to-rent ratio (B)</td>
<td>0.26 (6.6)</td>
<td>0.14 (4.6)</td>
<td>0.93 (7.4)</td>
<td>0.01 (0.9)</td>
<td>0.03 (3.0)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.14</td>
<td>0.18</td>
<td>0.16</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>QOL elasticity of rent-to-income ratio (C)</td>
<td>0.31 (13.2)</td>
<td>0.18 (8.5)</td>
<td>1.20 (18.8)</td>
<td>0.04 (4.5)</td>
<td>0.05 (5.9)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.340</td>
<td>0.21</td>
<td>0.56</td>
<td>0.07</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Standard error in parenthesis
Table 4.6: Amenity Corrected Price-to-income Ratios: Top and Bottom 20 Cities (Cities Ranked & Rated)

<table>
<thead>
<tr>
<th>Metropolitan Area</th>
<th>Price-to-income(^*) Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Santa Cruz, CA</td>
<td>4.79</td>
</tr>
<tr>
<td>San Jose, CA</td>
<td>4.44</td>
</tr>
<tr>
<td>Santa Barbara-Santa Maria-Lompoc, CA</td>
<td>4.00</td>
</tr>
<tr>
<td>Honolulu, HI</td>
<td>3.84</td>
</tr>
<tr>
<td>Salinas-Sea Side-Monterey, CA</td>
<td>3.64</td>
</tr>
<tr>
<td>San Francisco-Oakland-Vallejo, CA</td>
<td>3.56</td>
</tr>
<tr>
<td>Santa Rosa-Petaluma, CA 01</td>
<td></td>
</tr>
<tr>
<td>San Luis Obispo-Atascad-P Robles, CA</td>
<td>2.82</td>
</tr>
<tr>
<td>Ventura-Oxnard-Simi Valley, CA</td>
<td>2.76</td>
</tr>
<tr>
<td>San Diego, CA</td>
<td>2.74</td>
</tr>
<tr>
<td>Santa Fe, NM</td>
<td>2.70</td>
</tr>
<tr>
<td>Yolo, CA</td>
<td>2.48</td>
</tr>
<tr>
<td>Boston, MA-NH</td>
<td>2.48</td>
</tr>
<tr>
<td>Seattle-Everett, WA</td>
<td>2.37</td>
</tr>
<tr>
<td>New York-Northeastern, NJ</td>
<td>2.36</td>
</tr>
<tr>
<td>Barnstable-Yarmouth, MA</td>
<td>2.23</td>
</tr>
<tr>
<td>Medford, OR</td>
<td>2.17</td>
</tr>
<tr>
<td>Eugene-Springfield, OR</td>
<td>2.16</td>
</tr>
<tr>
<td>Flagstaff, AZ-UT</td>
<td>2.13</td>
</tr>
<tr>
<td>...</td>
<td></td>
</tr>
<tr>
<td>Pittsburgh, PA</td>
<td>1.25</td>
</tr>
<tr>
<td>Lakeland-Winterhaven, FL</td>
<td>1.25</td>
</tr>
<tr>
<td>Vineland-Milville-Bridgetown, NJ</td>
<td>1.25</td>
</tr>
<tr>
<td>Binghamton, NY</td>
<td>1.24</td>
</tr>
<tr>
<td>Albany, GA</td>
<td>1.23</td>
</tr>
<tr>
<td>Richmond-Petersburg, VA</td>
<td>1.23</td>
</tr>
<tr>
<td>McAllen-Edinburg-Pharr-Mission, TX</td>
<td>1.23</td>
</tr>
<tr>
<td>Brownsville-Harlingen-San Benito, TX</td>
<td>1.22</td>
</tr>
<tr>
<td>Syracuse, NY</td>
<td>1.22</td>
</tr>
<tr>
<td>Wichita, KS</td>
<td>1.20</td>
</tr>
<tr>
<td>Lubbock, TX</td>
<td>1.19</td>
</tr>
<tr>
<td>Atlanta, GA</td>
<td>1.18</td>
</tr>
<tr>
<td>Macon-Warner Robins, GA</td>
<td>1.15</td>
</tr>
<tr>
<td>San Antonio, TX</td>
<td>1.15</td>
</tr>
<tr>
<td>Waco, TX</td>
<td>1.13</td>
</tr>
<tr>
<td>Abilene, TX</td>
<td>1.11</td>
</tr>
<tr>
<td>Odessa, TX</td>
<td>1.08</td>
</tr>
<tr>
<td>Dallas-Fort Worth, TX</td>
<td>1.07</td>
</tr>
<tr>
<td>Houston-Brazoria, TX</td>
<td>1.05</td>
</tr>
<tr>
<td>Beaumont-Port Arthur-Orange, TX</td>
<td>1.03</td>
</tr>
</tbody>
</table>

\(^*\) Quality controlled ratio
Chapter 5

Conclusion

We study a real estate developer’s timing strategies under uncertainty in both chapter 2 and chapter 3. The former shows the effects of leverage on investment decisions and the latter shows the effects of heterogeneity of assets on investment decisions.

Under uncertainty, leverage encourages a real estate developer to move forward development decisions. The use of development financing is critical for real estate developers. In order to finance development cost, they obtain construction loans on a non-recourse basis, building an independent project company. Chapter 2 analyzes leveraged strategies for developers as Riddiough and Marseguerra and Cortelezzi do. However, our developer finances no land acquisition loan in Riddiough and enjoys no tax shields in Marseguerra and Cortelezzi. By abstracting from existing capital and interest tax advantage, chapter 2 highlights the leverage effects on real estate developments. Nonetheless, note that the goal of chapter 2 is not to determine the developer’s financial structure, but to examine the effects of a risky construction loan on the timing of real estate investment decisions.

Under uncertainty, heterogeneity of the assets grants a developer diversification benefits. By introducing Markowitz (1952) in a real options framework, we show that a developer is less likely to delay investment with heterogeneous assets, rather than homogeneous assets. In real estate development projects, developers build multiple assets of non-homogeneity whose values are imperfectly correlated. This distinguishes real estate developments from supply in other manufacturing industries. Chapter 3 analyzes the effects of heterogeneity on investment decisions. We show that the heterogeneity revises the value of an option for waiting in the literature in real estate development. Under short-sale constraints, for example, uncertainty can have smaller deterrent effects for a developer with a diversification strategy than those for a developer without it. Nonetheless, we abstract from a dynamic strategy for product mix and a timing diversification strategy.

We study the price-to-income ratio in chapter 4. We argue that when we compare price-to-income ratios cross-sectionally in order for an analysis of housing bubbles, we are likely
to have misleading results. The price-to-income ratio can overestimate bubbles in cities of high amenities because it is likely to increase in local consumption amenities. Our theory is based on [Roback (1982)]. In order to provide empirical evidence, we use two popular ratings for the quality of living across cities and show that there is a strong positive correlation between price-to-income ratios and quality of living indices across cities. Nonetheless, we do not provide any alternative bubble indicators.
Bibliography


Appendix A

Proofs of Chapter 2

Note the signs of partial derivatives of $\alpha$ and $\beta$: $\alpha_\sigma < 0$, $\alpha_r > 0$, $\alpha_g < 0$, $\beta_\sigma > 0$, $\beta_r < 0$, and $\beta_g < 0$.

Proposition 1

\[
\frac{\partial R_D}{\partial \sigma} = -\frac{1}{\beta^2} \beta_\sigma < 0 \text{ as } \beta_\sigma < 0,
\]
\[
\frac{\partial R_D}{\partial m} > 1,
\]
\[
\frac{\partial R_D}{\partial r} = \frac{g}{r^2} > 0,
\]
\[
\frac{\partial R_D}{\partial g} = -\frac{1}{\beta^2} \beta_g - \frac{1}{r}.
\]

The sign of $\frac{\partial R_D}{\partial g}$ is undetermined.

Proposition 2

\[
\frac{\partial \Omega}{\partial \sigma} = e^{-\beta(R_D-R)} \frac{\beta_\sigma}{r^2} + \frac{(R_D-R)e^{-\beta(R_D-R)}}{r\beta} \beta_\sigma + \frac{e^{-\beta(R_D-R)}}{r} \frac{\partial R_D}{\partial \sigma}
\]
\[
= -\beta_\sigma (R_D - R) \Omega > 0,
\]
\[
\frac{\partial \Omega}{\partial m} = -\beta \Omega > 0,
\]
\[
\frac{\partial \Omega}{\partial g} = e^{-\beta(R_D-R)} \frac{\beta_g}{r^2} + \frac{(R_D-R)e^{-\beta(R_D-R)}}{r\beta} \beta_g + \frac{e^{-\beta(R_D-R)}}{r} \frac{\partial R_D}{\partial g}
\]
\[
= \frac{(R_D - R)e^{-\beta(R_D-R)}}{r\beta} \beta_g - \frac{e^{-\beta(R_D-R)}}{r^2} < 0 \text{ as } \beta_g < 0,
\]
\[
\frac{\partial \Omega}{\partial R} = \beta \Omega < 0.
\]
\[
\frac{\partial \Omega}{\partial r} = \frac{e^{-\beta(R_D-R)}}{r^2 \beta} + \frac{(R_D - R)e^{-\beta(R_D-R)}}{r \beta} \beta + \frac{e^{-\beta(R_D-R)}}{r} \frac{\partial R_D}{\partial r} \\
= \frac{\beta g + r e^{-\beta(R_D-R)}}{\beta r} + \frac{(R_D - R)e^{-\beta(R_D-R)}}{r \beta} \beta_r.
\]

Although the sign of \(\frac{\partial \Omega}{\partial r}\) is undetermined, we may observe \(\frac{\partial \Omega}{\partial r} < 0\) if \(\sigma\) is large enough as \(r > \beta g\).

**Proposition 3**

\[
\begin{align*}
\frac{\partial S}{\partial \sigma} & = \frac{\partial \Omega}{\partial \sigma}, \\
\frac{\partial S}{\partial R} & = \frac{1}{r}(1 - e^{-\beta(R_D-R)}) > 0, \\
\frac{\partial S}{\partial m} & = -\frac{1}{r}(1 - e^{-\beta(R_D-R)}) < 0, \\
\frac{\partial S}{\partial g} & = \frac{(R_D - R)e^{-\beta(R_D-R)}}{r \beta} \beta + \frac{(1 - e^{-\beta(R_D-R)})}{r^2}, \\
\frac{\partial S}{\partial r} & = \frac{\beta g + r e^{-\beta(R_D-R)}}{\beta r} + \frac{(R_D - R)e^{-\beta(R_D-R)}}{r \beta} \beta_r - \frac{R - m}{r^2} - \frac{g}{r^3}.
\end{align*}
\]

The signs of \(\frac{\partial S}{\partial g}\) and \(\frac{\partial S}{\partial r}\) are undetermined.

**Corollary 1**

\[
\begin{align*}
\frac{\partial M}{\partial \sigma} & = -\frac{\partial \Omega}{\partial \sigma} < 0, \\
\frac{\partial M}{\partial m} & = \frac{1}{r}(1 - e^{-\beta(R_D-R)}) > 0, \\
\frac{\partial M}{\partial R} & = -\beta \Omega > 0, \\
\frac{\partial M}{\partial g} & = -\frac{\partial \Omega}{\partial g} > 0, \\
\frac{\partial M}{\partial r} & = \frac{\beta g + r e^{-\beta(R_D-R)}}{\beta r} + \frac{(R_D - R)e^{-\beta(R_D-R)}}{r \beta} \beta_r - \frac{m}{r^2}.
\end{align*}
\]

The sign of \(\frac{\partial M}{\partial r}\) is undetermined.
Credit spread

\[
\frac{\partial (i - r)}{\partial \sigma} = \frac{e^{-\beta (RD - R^*)}}{\beta} \frac{\partial R}{\partial \sigma} + \frac{(R_D - R^*) e^{-\beta (RD - R^*)}}{\beta} \beta + e^{-\beta (RD - R^*)} \frac{\partial R_D}{\partial \sigma} \\
= -\beta \sigma (R_D - R^*) r \Omega / M > 0,
\]

\[
\frac{\partial (i - r)}{\partial m} = e^{-\beta (RD - R^*)} / M > 0,
\]

\[
\frac{\partial (i - r)}{\partial g} = \left[ \frac{e^{-\beta (RD - R^*)}}{\beta} \beta g + \frac{(R_D - R^*) e^{-\beta (RD - R^*)}}{\beta} \beta g + e^{-\beta (RD - R^*)} \frac{\partial R_D}{\partial g} / M \right] = \left[ \frac{(R_D - R) e^{-\beta (RD - R^*)}}{\beta} \beta g - e^{-\beta (RD - R^*)} / r \right] / M < 0,
\]

\[
\frac{\partial (i - r)}{\partial R^*} = -e^{-\beta (RD - R^*)} / M < 0.
\]

Unique solution for equation (2.28b)

Define \( f(R) = R - (A + rC + \frac{r - \alpha g}{\alpha}) \) and \( g(R) = -e^{-\beta (RD - R)} / \alpha \). We know \( g(R) < 0 \) and \( \lim_{R \to \infty} g(R) = 0 \), and also \( g'(R) > 0 \) and \( g''(R) < 0 \) for any \( R \). At the lower boundary at \( R = R_D \) in equation (2.25), \( f(R_D) = \frac{1}{\beta} - \frac{g}{r} + m - (A + rC + \frac{1}{\alpha} - \frac{g}{r}) < -\frac{1}{\alpha} = g(R_D) \) as the maximum value of \( m = -\frac{1}{\beta} + rC \).

Proposition 4

Along with the above proof, we can show \( \frac{\partial R^*}{\partial m} < 0 \) explicitly, because \( \frac{\partial f(R)}{\partial m} = 0 \) and \( \frac{\partial g(R)}{\partial m} = \frac{\beta}{\alpha} e^{-\beta (RD - R)} \frac{\partial R_D}{\partial m} = \frac{\beta}{\alpha} e^{-\beta (RD - R)} < 0 \) with \( R > R_D \).

Comparative statics of the option value

From (2.35), \( \frac{\partial F}{\partial m} = \frac{\beta e^{-\beta (RD - R^*)}}{\alpha r} < 0 \) and \( \frac{\partial F}{\partial M} = \frac{\beta e^{-\beta (RD - R^*)}}{\alpha M} = \frac{\beta e^{-\beta (RD - R^*)}}{\alpha (1 - e^{-\beta (RD - R^*)})} < 0 \). Moreover, \( \frac{\partial F}{\partial \sigma} = \frac{-\beta e^{-\beta (RD - R^*)}}{\alpha r} \). But, the signs of \( \frac{\partial F}{\partial \sigma} \), \( \frac{\partial F}{\partial r} \) and \( \frac{\partial F}{\partial g} \) are undetermined.

Undetermined relationship between \( R^* \) and \( \sigma \)

We can define \( \frac{\partial R^*}{\partial \sigma} = \left[ \frac{-1 - e^{-\beta (RD - R^*)} / \alpha^2}{\alpha^2} + \frac{\beta e^{-\beta (RD - R^*)} \partial R_D}{\partial \sigma} \right] + \left[ R_D e^{-\beta (RD - R^*)} \partial R_D / \partial \sigma \right] / \left[ 1 + \frac{\beta e^{-\beta (RD - R^*)} / \alpha^2}{\alpha^2} + \frac{\beta e^{-\beta (RD - R^*)} \partial R_D}{\partial \sigma} \right] > 0 \) as long as both \( -1 < \frac{\beta}{\alpha} e^{-\beta (RD - R^*)} < 0 \) and \( \left| \frac{1 - e^{-\beta (RD - R^*)} / \alpha^2}{\alpha^2} + \frac{\beta e^{-\beta (RD - R^*)} \partial R_D}{\partial \sigma} \right| > \left| \frac{R_D e^{-\beta (RD - R^*)} \partial R_D / \partial \sigma}{\alpha^2} \right| \). Nonetheless, note that the difference between \( \hat{R} \) and \( R^* \) increase with \( \sigma \) because we already know \( \hat{R} - R^* = \frac{e^{-\beta (RD - R^*)}}{\alpha} > 0 \), \( R < R^* \), where \( \hat{R} \) in (2.11b) and \( R^* \) in (2.28b).
Comparative statics of equation (2.40)

The sign of $\frac{\partial P^a(z)}{\partial m}$ is negative in the short run when $\frac{\partial R^*}{\partial m} = 0$; the sign of $\frac{\partial P^a(z)}{\partial m}$ is undetermined in the long run $\frac{\partial R^*}{\partial m} \neq 0$ because $\frac{\partial P^a(z)}{\partial m} = \frac{\beta}{\alpha r} e^{-\beta(R_D-R^*)} e^{-\alpha(R^*-R)} + \frac{1-e^{-\beta(R_D-R^*)}}{r} e^{-\alpha(R^*-R)} \frac{\partial R^*}{\partial m}$.

The sign of $\frac{\partial P^a(z)}{\partial \sigma}$ is also undetermined as $\frac{\partial P^a(z)}{\partial \sigma} = -\frac{\Theta(R^* : m)}{\alpha^2 r} \alpha_{\sigma} + \frac{\beta e^{-\beta(R_D-R^*)} e^{-\alpha(R^*-R)}}{\alpha r} - \frac{\beta e^{-\beta(R_D-R^*)} e^{-\alpha(R^*-R)}}{\alpha r} \frac{\partial R_D}{\partial \sigma} \frac{\beta}{\alpha r} e^{-\beta(R_D-R^*)} e^{-\alpha(R^*-R)} + \frac{1-e^{-\beta(R_D-R^*)}}{r} e^{-\alpha(R^*-R)} \frac{\partial R^*}{\partial \sigma}$. 

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Appendix B

Proofs of Chapter 3

Equation (3.10)

Equation (3.9) yields

\[
(dP)^2 = (W^T G dt + W^T M dB)(W^T G dt + W^T M dB)
\]

\[
= W^T G W^T G(dt)^2 + W^T G W^T M(dt dB)
\]

\[
+ W^T M W^T G(dt dB) + W^T M dB W^T M dB
\]

\[
= W^T M dB W^T M dB
\]

\[
= \left( \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j \right) dt,
\]

where \( \rho_{ij} = 1 \) with \( i = j \) and \( \rho_{ij} \in [-1,1] \) with \( i \neq j \), since \((dt)^2 = 0\), \( dt dB_i = 0\), and \( dB_i dB_i = dt \) for any \( i \) according to the box algebra in Steele (2001: 127) and Dixit and Pindyck (1994: 80).

Proposition 6

\[
\frac{\partial \bar{P}^*}{\partial \rho_{ij}} = \frac{\partial \bar{P}^*}{\partial A} \frac{\partial A}{\partial S} \frac{\partial S}{\partial \rho_{ij}}
\]

\[
= \frac{-1}{A^2} \left\{ \frac{G^2 + r S^2}{\sqrt{G^2 + 2r S^2}} \right\} - \frac{2w_i w_j \sigma_i \sigma_j}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j}}
\]

\[
= \frac{\mathcal{S}(w_i w_j \sigma_i \sigma_j)}{\sqrt{G^2 + 2r S^2}} \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j}
\]

\[
= \frac{w_i w_j \sigma_i \sigma_j}{\sqrt{G^2 + 2r S^2}}
\]
Proposition 7

\[
\frac{\partial \bar{P}^*}{\partial \sigma_i} = \frac{\partial \bar{P}^*}{\partial \sigma_i} - \frac{1}{A} \left\{ \frac{G - \frac{G^2 + rS^2}{\sqrt{G^2 + 2rS^2}}}{S^3} \right\} \frac{2 \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j}{2 \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j}} \right.
\]

\[
= \frac{S \left( \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j \right)}{\sqrt{G^2 + 2rS^2}} \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j}
\]

\[
= \sum_{j=1}^{n} w_i w_j \rho_{ij} \sigma_j \right)
\]

\[
\frac{\sqrt{G^2 + 2rS^2}}{\sqrt{G^2 + 2rS^2}}}
\]

Equation (3.18)

\[
S^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} w_i w_j \rho_{ij} \sigma_i \sigma_j, \text{ where } w_i = 1/n \text{ for any } i
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sigma_i^2 + \frac{n-1}{n} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{n(n-1)} \rho_{ij} \sigma_i \sigma_j
\]

\[
= \frac{1}{n} \sigma_{ii} + \frac{n-1}{n} \sigma_{ij},
\]

where \( \sigma_{ii} \equiv \sum_{i=1}^{n} \frac{1}{n} \sigma_i^2 \) and \( \sigma_{ij} \equiv \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{1}{n(n-1)} \rho_{ij} \sigma_i \sigma_j. \)
## Appendix C

### Tables for Chapter 4

<table>
<thead>
<tr>
<th>Category</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost of Living</td>
<td>Indexes and Taxes: Cost of living index, Buying power index, Income tax rate, Sales tax rate, Property tax rate Housing: Median home price, Home price appreciation, Median rent, Homes owned, Home price ratio Necessities: Food index, Housing index, Utilities index, Transportation index, Healthcare index, Miscellaneous cost index</td>
</tr>
<tr>
<td>Climate</td>
<td>Temperature: Avg January low, Avg July high, Annual days &gt; 90°F, Annual days &lt; 52°F, Annual days snow &gt; 15 inches Precipitation: Annual inches precipitation, Annual days precipitation, Annual inches snowfall, Annual days rain &gt; 0.5 inches, Annual days snow &gt; 1.5 inches</td>
</tr>
<tr>
<td>Education</td>
<td>Achievement: High school degree, 2-year college degree, 4-year college degree, Graduate/professional degree Public Schools: Expenditure per pupil, Student/teacher ratio, Attending public school, State SAT score, State ACT score Higher Education: No 2-year college, No 4-year college/universities, No highly ranked universities</td>
</tr>
<tr>
<td>Health &amp; Hazard &amp; Illness</td>
<td>Health Care: Air quality score, Water quality score, Pollen/algal score, Cancer mortality per capita, Depression days per month, Stress score Physicians per capita, Hospital beds per capita, Cost per doctor visit, No. teaching hospitals, Cost per dental visit</td>
</tr>
<tr>
<td>Crime</td>
<td>Crime: Violent crime rate, Change in violent crime rate, Property crime rate, Change in property crime rate</td>
</tr>
<tr>
<td>Transportation</td>
<td>Commute: Average commute time, Percent commute &gt; 60 minutes, Commute by auto, Commute by mass transit, Work at home, Mass transit miles per capita Intercity Services: Major airport within 60 miles, Size of regional airport, Daily airline activity, Amtrack service Automotive: Insurance: annual premium, Gas: cost per gallon, Daily vehicle miles per capita</td>
</tr>
<tr>
<td>Leisure</td>
<td>Dining &amp; Shopping: Restaurant rating, Outlet mall score, No. Starbucks, No. warehouse club Entertainment: Professional sports rating, College sports rating, Zoo/aquarium rating, Amusement park rating, Botanical garden/arboretum rating Outdoor Activities: Golf-course rating, Ski area rating, Sq. miles inland water, Miles of coastline, National park rating</td>
</tr>
<tr>
<td>Arts &amp; Culture</td>
<td>Media &amp; Libraries: Arts radio rating, No. public libraries, Library volumes per capita Performing arts: Classical music rating, Ballet/dance rating, Professional theater rating, University arts programs rating Museums: Overall museum rating, Art museum rating, Science museum rating, Children’s museum rating</td>
</tr>
<tr>
<td>Subjective Category (Quality of Life)</td>
<td>Physical attractiveness: physical setting and overall appearance, attractiveness and functionality of the downtown core Heritage: Metropolitan area with well-preserved historic districts and public buildings Overall ease of living ease of living incorporates crowdedness, attitude and friendliness of people, and simplicity of infrastructure</td>
</tr>
</tbody>
</table>

### Table C.2: Log-log regressions: *Cities Ranked & Rated*

<table>
<thead>
<tr>
<th>Log (score of variable)</th>
<th>Coefficient (St. Dev.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economy &amp; Job</td>
<td>0.11 (0.02)</td>
</tr>
<tr>
<td>Cost of Living</td>
<td>0.11 (0.02)</td>
</tr>
<tr>
<td>Climate</td>
<td>0.17 (0.01)</td>
</tr>
<tr>
<td>Education</td>
<td>0.12 (0.02)</td>
</tr>
<tr>
<td>Health &amp; Health care</td>
<td>0.09 (0.02)</td>
</tr>
<tr>
<td>Crime</td>
<td>0.09 (0.02)</td>
</tr>
<tr>
<td>Transportation</td>
<td>0.12 (0.01)</td>
</tr>
<tr>
<td>Leisure</td>
<td>0.10 (0.02)</td>
</tr>
<tr>
<td>Arts &amp; Culture</td>
<td>0.12 (0.02)</td>
</tr>
<tr>
<td>Subjective</td>
<td>0.08 (0.02)</td>
</tr>
<tr>
<td>Adjusted R²</td>
<td>0.996</td>
</tr>
<tr>
<td>Obs.</td>
<td>333</td>
</tr>
</tbody>
</table>