On Monotone Linear Relations and the Sum Problem in Banach Spaces

by

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Abstract

We study monotone operators in general Banach spaces. Properties and characterizations of monotone linear relations are presented. We focus on the "sum problem" which is the most famous open problem in Monotone Operator Theory, and we provide a powerful sufficient condition for the sum problem. We work on classical types of maximally monotone operators and provide affirmative answers to several open problems posed by Phelps and by Simons. Borwein-Wiersma decomposition and Asplund decomposition of maximally monotone operators are also studied.

Preface

My thesis is primarily based on the following twelve papers:

> [6–8] by Heinz H. Bauschke, Jonathan M. Borwein, Xiangfu Wang and Liangjin Yao;
>
> [14–18] by Heinz H. Bauschke, Xianfu Wang and Liangjin Yao;
>
> [88] by Xianfu Wang and Liangjin Yao;
>
> and
>
> [89–91] by Liangjin Yao.

Specifically, the relationship between the above papers and my thesis is as follows:

> Chapter 3 is mainly based on the work in [15, 17, 18, 89]; Chapter 4 is mainly based on the work in [88]; Chapter 5 is mainly based on the work in [90, 91]; Chapter 6 is all based on the work in [6, 7]; Chapter 7 is mainly based on the work in [15, 17]; Chapter 8 is all based on the work in [8]; and Chapter 9 is mainly based on the work in [18].

For every multi-authored paper, each author contributed equally.

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S–saturated		p. 182
$S^{\perp}$	$\big\{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\big\}$	p. 5
U_X	the open unit ball of X	p. 8
X	a real Banach space	p. 5
$\mathbb{I}_C$	the indicator mapping of C	p. 7
Id	identity mapping	p. 8
Φ_A		p. 143
$\operatorname{bdry} C$	the boundary of C	p. 7
$\operatorname{conv} C$	the convex hull of C	p. 7
$\mathrm{d}(\cdot, C)$	the distance function to a set C	p. 7
$\dim F$	the dimension of F	p. 7
$\operatorname{dom} A$	the domain of A	p. 5
$\operatorname{dom} f$	$f^{-1}(\mathbb{R})$	p. 7
$\ell^2(\mathbb{N})$		p. 30
$\operatorname{gra} A$	the graph of A	p. 5
$\operatorname{int} C$	the interior of C	p. 7
$\overline{C^*}^{\mathrm{w}^*}$	the weak* closure of C^*	p. 7
$\overline{C^{**}}^{\mathrm{w}^*}$	the weak* closure of C^{**}	p. 7
$\overline{C}$	the norm closure of C	p. 7
$\overline{C}^{\mathrm{w}}$	the weak closure of C	p. 7
$\overline{f}$	the lower semicontinuous hull of f	p. 7
$\partial_\varepsilon f$	the ε–subdifferential operator of f	p. 8
$\operatorname{ran} A$	the range of A	p. 5
Sgn		p. 8

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Chapter 1

Introduction

My thesis mainly focuses on monotone operators, which have proved to be a key class of objects in modern Optimization and Analysis. We start with linear relations, which are becoming a centre of attention in Monotone Operator Theory.

In Chapter 3, we gather some basic properties about monotone linear relations, and conditions for them to be maximally monotone. We construct maximally monotone unbounded linear operators. We give some characterizations of the maximal monotonicity of linear operators and we also provide a brief proof of the Brezis-Browder Theorem. In Chapter 4, we focus on finding explicit maximally monotone linear subspace extensions of monotone linear relations, which generalize Crouzeix and Anaya's recent work.

The most important open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximally monotone operators provided that Rockafellar's constraint qualification holds. This is called the "sum problem". The sum problem has an affirmative answer in reflexive spaces, but is still unsolved in general Banach spaces. In Chapter 5, we obtain a powerful sufficient condition for the sum problem to have an affirmative solution, which generalizes other well-known results for this problem obtained by different researchers in recent years. We also prove the case of the sum of a maximally monotone linear relation and the subdifferential operator.

In Chapter 6, we study classical types of maximally monotone operators: dense type, negative-infimum type, Fitzpatrick-Phelps type, etc. We show that every maximally monotone operator of Fitzpatrick-Phelps type must be of dense type. We establish that for a maximally monotone linear relation, being of dense type, negative-infimum type, or Fitzpatrick-Phelps type is equivalent to the adjoint being monotone. The above results provide affirmative answers to two open problems: one posed by Phelps and Simons, and the other by Simons.

The Fitzpatrick function is a very important tool in Monotone Operator Theory. In Chapter 7, we study the properties of the partial inf-convolution of the Fitzpatrick functions associated with maximally monotone operators.

In Chapter 8, we construct some maximally monotone operators that are not of type (D). Using these operators, we show that the partial inf-convolution of two BC-functions will not always be a BC-function, which provides a negative answer to a question posed by Simons.

There are two well known decompositions of maximally monotone operators: Asplund Decomposition and Borwein-Wiersma Decomposition. In Chapter 9, we show that Borwein-Wiersma decomposability implies Asplund decomposability. We present characterizations of Borwein-Wiersma decomposability of maximally monotone linear relations in general Banach spaces and provide a more explicit decomposition in Hilbert spaces.

In this thesis, we solve the following open problems.

(1) Simons posed the following question in [74, page 199] concerning [72, Theorem 41.6] (See Corollary 5.3.6 or [16].):

> *Let* $A : \operatorname{dom} A \to X^*$ *be linear and maximally monotone, let* C *be a nonempty closed convex subset of* X*, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$.
>
> *Is* $A + N_C$ *necessarily maximally monotone?*

(2) Simons posed the following question in [74, Problem 47.6] (See Theorem 6.2.1 or [6].):

> *Let* $A : \operatorname{dom} A \to X^*$ *be linear and maximally monotone. Assume that* A *is of type (FP).*
>
> *Is* A *necessarily of type (NI)?*

(3) Simons posed the following question in [73, Problem 18, page 406] (See Corollary 6.2.2 or [7].):

> *Let* $A : X \rightrightarrows X^*$ *be maximally monotone such that* A *is of type (FP).*
>
> *Is* A *necessarily of type (D)?*

(4) Phelps and Simons posed the following question in [63, Section 9, item 2] (See Corollary 6.3.3 or [6].):

> *Let* $A : \operatorname{dom} A \to X^*$ *be linear and maximally monotone. Assume that* A^* *is monotone.*
>
> *Is* A *necessarily of type (D)*?

(5) Simons posed the following question in [74, Problem 22.12] (See Example 8.3.1(iii)&(v) or [8].):

Let $F_1, F_2 : X \times X^* \to]-\infty, +\infty]$ be proper lower semicontinuous and convex functions. Assume that F_1, F_2 are BC–functions and that

$$\bigcup_{\lambda>0} \lambda \left[P_{X^*} \operatorname{dom} F_1 - P_{X^*} \operatorname{dom} F_2\right] \textit{ is a closed subspace of } X^*.$$

Is $F_1 \Box_1 F_2$ necessarily a BC–function?

The answers are yes, yes, yes, yes and no, respectively.

Chapter 2

Notation and examples

In this chapter, we fix some notation and give some examples. Throughout this thesis, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. The *inverse operator* $A^{-1}\colon X^* \rightrightarrows X$ is given by $\operatorname{gra} A^{-1} = \big\{(x^*,x) \in X^* \times X \mid x^* \in Ax\big\}$; the *domain* of A is $\operatorname{dom} A = \big\{x \in X \mid Ax \neq \varnothing\big\}$, and its *range* is $\operatorname{ran} A = A(X)$. If Z is a real Banach space with dual Z^* and a set $S \subseteq Z$, we define $S^\perp$ by $S^\perp = \big\{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\big\}$. Given a subset D of Z^*, we define $D_\perp$ [63] by $D_\perp = \big\{z \in Z \mid \langle z, d^*\rangle = 0, \quad \forall d^* \in D\big\}$. The *adjoint* of A, written A^*, is defined by

$$\begin{aligned}\operatorname{gra} A^* &= \big\{(x^{**},x^*) \in X^{**} \times X^* \mid (x^*,-x^{**}) \in (\operatorname{gra} A)^\perp\big\}\\ &= \big\{(x^{**},x^*) \in X^{**} \times X^* \mid \langle x^*, a\rangle = \langle a^*, x^{**}\rangle,\ \forall (a,a^*) \in \operatorname{gra} A\big\}.\end{aligned}$$

See Example 2.1.2, Example 2.1.4, Section 3.3 and Cross' book [38] for more information about linear relations.

The *Fitzpatrick function* of A (see [45]) is given by

$$F_A \colon (x,x^*) \in X \times X^* \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big). \tag{2.1}$$

See Chapter 7 for more properties of the Fitzpatrick functions.

Recall that A is *monotone* if

$$\big(\forall (x,x^*) \in \operatorname{gra} A\big)\big(\forall (y,y^*) \in \operatorname{gra} A\big) \quad \langle x-y, x^*-y^*\rangle \geq 0, \tag{2.2}$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). We say $(x,x^*) \in X \times X^*$ is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is of type *Fitzpatrick-Phelps-Veronas (FPV)* if for every open convex set $U \subseteq X$ such that $U \cap \operatorname{dom} A \neq \varnothing$, the implication

$$\begin{aligned}&x \in U \text{ and } (x,x^*) \text{ is monotonically related to } \operatorname{gra} A \cap (U \times X^*)\\ &\Rightarrow (x,x^*) \in \operatorname{gra} A\end{aligned}$$

holds. We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., [22–24], the books [9, 26, 33, 34, 48, 61, 68, 72, 74, 92, 93] and the references therein. We also adopt the standard notation used in these books: Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, $\operatorname{bdry} C$ is the *boundary* of C, $\operatorname{conv} C$ is the *convex hull* of C, and $\overline{C}$ and $\overline{C}^{\mathrm{w}}$ are respectively the *norm closure* of C and *weak closure* of C. For the set $C^* \subseteq X^*$, $\overline{C^*}^{\mathrm{w}^*}$ is the weak* closure of C^*. If $C^{**} \subseteq X^{**}$, $\overline{C^{**}}^{\mathrm{w}^*}$ is the weak* closure of C^{**} in X^{**} with the topology induced by X^*. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

The *indicator mapping* $\mathbb{I}_C \colon X \to X^*$ is defined by

$$\mathbb{I}_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{2.4}$$

The *distance function* to the set C, written as $\mathrm{d}(\cdot, C)$, is defined by $x \mapsto \inf_{c \in C} \|x - c\|$. The *support function* of C, written as σ_C, is defined by $\sigma_C(x^*) = \sup_{c \in C} \langle c, x^* \rangle$. If $D \subseteq X$, we set $C - D = \{x - y \mid x \in C, y \in D\}$. For every $x \in X$, the *normal cone operator* of C at x is defined by $N_C(x) = \big\{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$ (see Example 2.1.5 for more information). For $x, y \in X$, we set $[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. Let $\dim F$ stand for the dimension of a subspace F of X. Given $f \colon X \to \left]-\infty, +\infty\right]$, we set $\operatorname{dom} f = f^{-1}(\mathbb{R})$ and $f^* \colon X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X}(\langle x, x^* \rangle - f(x))$ is the *Fenchel conjugate* of f. The *lower semicontinuous hull* of f is denoted by $\overline{f}$. If f is convex and $\operatorname{dom} f \neq$

$\varnothing$, then $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y)\}$ is the *subdifferential operator* of f. Note that $N_C = \partial\iota_C$ For $\varepsilon \geq 0$, the *ε–subdifferential* of f is defined by $\partial_\varepsilon f\colon X \rightrightarrows X^*\colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y) + \varepsilon\}$. We have $\partial f = \partial_0 f$.

Let $g\colon X \to \left]-\infty, +\infty\right]$. The *inf-convolution* of f and g, $f\Box g$, is defined by

$$f\Box g : x \mapsto \inf_{y\in X} \left[f(y) + g(x-y)\right].$$

Let J be the duality map, i.e., the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$. By [61, Example 2.26],

$$Jx = \{x^* \in X^* \mid \langle x^*, x\rangle = \|x^*\| \cdot \|x\|,\ \text{with } \|x^*\| = \|x\|\}. \tag{2.5}$$

Let Id be the *identity mapping* from X to X. Let Y be a real Banach space. We also set $P_X : X \times Y \to X\colon (x, y) \mapsto x$, and $P_Y : X \times Y \to Y\colon (x, y) \mapsto y$. Let $f\colon X \to \left]-\infty, +\infty\right]$ and $g\colon Y \to \left]-\infty, +\infty\right]$. We define $(f \oplus g)$ on $X \times Y$ by $(f \oplus g)(x, y) = f(x) + g(y)$ for every $(x, y) \in X \times Y$.

The *open unit ball* in X is denoted by $U_X = \{x \in X \mid \|x\| < 1\}$, the *closed unit ball* in X is denoted by $B_X = \{x \in X \mid \|x\| \leq 1\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let Sgn be defined by

$$\mathrm{Sgn}\colon \mathbb{R} \rightrightarrows \mathbb{R}\colon \xi \mapsto \begin{cases} 1, & \text{if } \xi > 0;\\ \left[-1, 1\right], & \text{if } \xi = 0;\\ -1, & \text{if } \xi < 0. \end{cases}$$

Throughout, we shall identify X with its canonical image in the bidual space X^{**}. Furthermore, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are likewise paired via $\langle (x, x^*), (y^*, y^{**}) \rangle = \langle x, y^* \rangle + \langle x^*, y^{**} \rangle$, where $(x, x^*) \in X \times X^*$ and $(y^*, y^{**}) \in X^* \times X^{**}$. Unless mentioned otherwise, the norm on $X \times X^*$, written as $\| \cdot \|_1$, is defined by $\|(x, x^*)\|_1 = \|x\| + \|x^*\|$ for every $(x, x^*) \in X \times X^*$.

2.1 Some examples

Now we give some examples of linear relations and their adjoints. See Example 2.1.1, Example 2.1.2 and Example 2.1.4.

Example 2.1.1 *Figure 2.1 is the graph of the linear operator:*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Example 2.1.2 (Borwein) (See [21, Example 3.1].) *Let* $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be defined by*

$$Ax = \begin{cases} Bx + V, & \text{if } x \in S; \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $B \in \mathbb{R}^{n \times n}$, S *and* V *are subspaces of* $\mathbb{R}^n$. *Then*

$$A^*x = \begin{cases} B^T x + S^\perp, & \text{if } x \in V^\perp; \\ \varnothing, & \text{otherwise.} \end{cases}$$

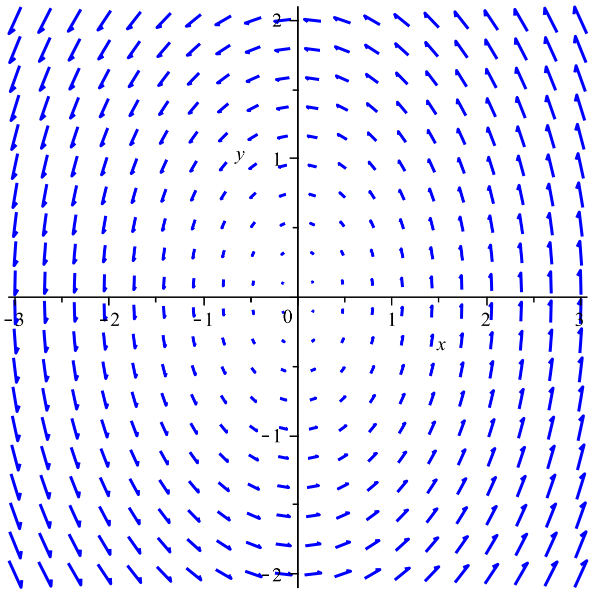

Figure 2.1: field plot of the linear operator A

That is,

$$\operatorname{gra} A = \operatorname{span}\{(s_1, Bs_1), \ldots, (s_p, Bs_p), (0, v_1), \ldots, (0, v_q)\}$$
$$\operatorname{gra} A^* = \operatorname{span}\{(v_1', B^\top v_1'), \ldots, (v_{p'}', B^\top v_{p'}), (0, s_1'), \ldots, (0, s_{q'}')\}$$

where, $(s_1, \ldots, s_p), (v_1, \ldots, v_q)$ *are respectively the bases of* S *and* V *and* $(v_1', \ldots, v_{p'}'), (s_1', \ldots, s_{q'})$ *are respectively the bases of* $V^\perp$ *and* $S^\perp$

Remark 2.1.3 *In Example 2.1.2, take* $S = \mathbb{R}^n$ *and* $V = 0$*, then* $A = B$ *and* $A^* = B^T = A^T$*.*

Let's go to an explicit example of a monotone linear relation.

Example 2.1.4 *Let $A : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ be defined by*

$$Ax = \begin{cases} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} x + \operatorname{span} e_1, & \textit{if } x \in \operatorname{span}\{e_2\}; \\ \varnothing, & \textit{otherwise}, \end{cases}$$

where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. Then

$$A^*x = \begin{cases} \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} x + \operatorname{span}\{e_1, e_3\}, & \textit{if } x \in \operatorname{span}\{e_2, e_3\}; \\ \varnothing, & \textit{otherwise}, \end{cases}$$

and

$$\begin{aligned} \operatorname{gra} A &= \operatorname{span}\{(0, e_1), (e_2, e_1 + 2e_2 + e_3)\} \\ \operatorname{gra} A^* &= \operatorname{span}\{(0, e_1), (0, e_3), (e_2, e_1 + 2e_2 + e_3), (e_3, -e_1 + e_2 + 2e_3)\}. \end{aligned}$$

The following is the explicit formula for the normal cone operator in $\ell^1(\mathbb{N})$.

Example 2.1.5 (Rockafellar) *Suppose that*

$$X = \ell^1(\mathbb{N}), \textit{ with norm } \|(x_n)_{n\in\mathbb{N}}\| = \sum_{n\in\mathbb{N}} |x_n|, \textit{ so that}$$

$X^* = \ell^\infty(\mathbb{N})$ with $\|(x_n^*)_{n\in\mathbb{N}}\|_* = \sup_{n\in\mathbb{N}} |x_n^*|$. The normal cone operator N_{B_X} is maximally monotone; furthermore, for every $x \in \ell^1(\mathbb{N})$,

$$N_{B_X}(x) = \begin{cases} \{0\}, & \text{if } \|x\| < 1; \\ \mathbb{R}_+ \cdot \big(\operatorname{Sgn}(x_n)\big)_{n\in\mathbb{N}}, & \text{if } \|x\| = 1; \\ \varnothing, & \text{if } \|x\| > 1. \end{cases}$$

Proof. By Fact 5.1.2, N_{B_X} is maximally monotone. We now turn to the formula for the normal cone operator. Clearly, $N_{B_X}(x) = \{0\}$ if $\|x\| < 1$, and $N_{B_X}(x) = \varnothing$ if $\|x\| > 1$. Now we suppose $\|x\| = 1$. Assume $x^* \in \ell_\infty(\mathbb{N})$. Then

$$\begin{aligned} & x^* \in N_{B_X}(x) \Leftrightarrow \langle x^*, y - x\rangle \leq 0,\ \forall y \in B_X \Leftrightarrow \|x^*\|_* \leq \langle x^*, x\rangle \\ & \Leftrightarrow \|x^*\|_* = \langle x^*, x\rangle. \end{aligned} \tag{2.6}$$

Clearly,

$$\langle K\big(\operatorname{Sgn}(x_n)\big)_{n=1}^\infty, x\rangle = K\|x\| = K = \|K\big(\operatorname{Sgn}(x_n)\big)_{n=1}^\infty\|_*, \quad \forall K \geq 0.$$

Thus, by (2.6),

$$\big\{\big(K \cdot \operatorname{Sgn}(x_n)\big)_{n=1}^\infty \mid K \geq 0\big\} \subseteq N_{B_X}(x).$$

Let $x^* \in N_{B_X}(x)$. Assume $x^* = (x_n^*)_{n=1}^\infty$. If $x^* = 0$, then $x^* \in \big\{\big(K \cdot \operatorname{Sgn}(x_n)\big)_{n=1}^\infty \mid K \geq 0\big\}$. Now assume $K := \|x^*\|_* \neq 0$. Thus, $|x_n^*| \leq K, \quad \forall n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Now we consider two cases:

Case 1: $x_n = 0$. Clearly, $x_n^* \in K\,[-1,1] = K\,\mathrm{Sgn}(0)$.

Case 2: $x_n \neq 0$. We can suppose $x_n > 0$. By (2.6), we have

$$\begin{aligned} K = x_n^* x_n + \sum_{i \neq n} x_i^* x_i \leq x_n^* x_n + \sum_{i \neq n} \sup_{j \in \mathbb{N}} |x_j^*| \cdot |x_i| = x_n^* x_n + K(1 - x_n) \\ \leq K x_n + K(1 - x_n) = K. \end{aligned}$$

Hence $x_n^* x_n + K(1 - x_n) = K x_n + K(1 - x_n)$. Thus, $x_n^* = K$. Then $x^* \in \big(K \cdot \mathrm{Sgn}(x_n)\big)_{n=1}^{\infty}$. That is,

$$N_{B_X}(x) \subseteq \big\{ \big(K \cdot \mathrm{Sgn}(x_n)\big)_{n=1}^{\infty} \mid K \geq 0 \big\}.$$

Hence $N_{B_X}(x) = \big\{ \big(K \cdot \mathrm{Sgn}(x_n)\big)_{n=1}^{\infty} \mid K \geq 0 \big\}$. ■

Chapter 3

Linear relations

This chapter is mainly based on [15, 17, 18] by Bauschke, Wang and Yao, and my work in [89]. We give some background material on linear relations, present some sufficient conditions for a linear relation to be monotone, and construct some examples of maximally monotone linear relations. Furthermore, we provide a brief proof of the Brezis-Browder Theorem on the characterization of the maximal monotonicity of linear relations. Recently, linear relations have become an interesting topic and are comprehensively studied in Monotone Operator Theory: see [3–5, 14–19, 28–32, 63, 75, 80, 83, 87, 89, 91].

3.1 Properties of linear relations

In this section, we gather some basic properties about monotone linear relations, and conditions for them to be maximally monotone. These results are used frequently in the sequel. We start with properties for general linear relations. If $A\colon X \rightrightarrows X^*$ is a linear relation that is at most single-valued, then we will identify A with the corresponding linear operator from $\operatorname{dom} A$ to X^* and (abusing notation slightly) also write $A\colon \operatorname{dom} A \to X^*$. An analogous comment applies conversely to a linear single-valued operator A

with domain $\operatorname{dom} A$, which we will identify with the corresponding at most single-valued linear relation from X to X^*.

Fact 3.1.1 (See [58, Proposition 2.6.6(c)] or [69, Theorem 4.7 and Theorem 3.12]). *Let C be a subspace of X, and D be a subspace of X^*. Then*

$$(C^{\perp})_{\perp} = \overline{C} = \overline{C}^{\mathrm{w}} \quad \textit{and} \quad (D_{\perp})^{\perp} = \overline{D}^{\mathrm{w}^*}.$$

Fact 3.1.2 (Attouch-Brezis) (See [2, Theorem 1.1] or [74, Remark 15.2]).

Let $f, g : X \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous convex functions. Assume that

$$\bigcup_{\lambda>0} \lambda\left[\operatorname{dom} f - \operatorname{dom} g\right] \quad \textit{is a closed subspace of } X.$$

Then

$$(f+g)^*(z^*) = \min_{y^* \in X^*} \{f^*(y^*) + g^*(z^* - y^*)\}, \quad \forall z^* \in X^*. \tag{3.1}$$

The following result appeared in Cross' book [38]. We give new proofs. The proof of Proposition 3.1.3(ix) was borrowed from [18, Remark 2.2].

Proposition 3.1.3 *Let $A : X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(i) *$A0$ is a linear subspace of X^*.*

(ii) *$Ax = x^* + A0, \quad \forall x^* \in Ax$.*

(iii) *$(\forall (\alpha, \beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\})$ $(\forall x, y \in \operatorname{dom} A)$ $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$.*

(iv) $(A^*)^{-1} = (A^{-1})^*$.

(v) $(\forall x \in \operatorname{dom} A^*)(\forall y \in \operatorname{dom} A)$ $\langle A^*x, y\rangle = \langle x, Ay\rangle$ *is a singleton.*

(vi) *If X is reflexive and* $\operatorname{gra} A$ *is closed, then* $A^{**} = A$.

(vii) $(\operatorname{dom} A)^{\perp} = A^*0$ *and* $\overline{\operatorname{dom} A} = (A^*0)_{\perp}$.

(viii) *If* $\operatorname{gra} A$ *is closed, then* $(\operatorname{dom} A^*)_{\perp} = A0$ *and* $\overline{\operatorname{dom} A^*}^{\mathrm{w}^*} = (A0)^{\perp}$.

(ix) *If* $\operatorname{dom} A$ *is closed, then* $\operatorname{dom} A^* = (\bar{A}0)^{\perp}$ *and thus* $\operatorname{dom} A^*$ *is (weak*) closed, where $\bar{A}$ is the linear relation whose graph is the closure of the graph of A.*

(x) *If* $k \in \mathbb{R} \smallsetminus \{0\}$, *then* $(kA)^* = kA^*$.

Proof. (i): Since $\operatorname{gra} A$ is a linear subspace, $\{0\} \times A0 = \operatorname{gra} A \cap \{0\} \times X^*$ is a linear subspace and hence $A0$ is a linear subspace.

(ii): Let $x \in \operatorname{dom} A$ and $x^* \in Ax$. Then $(x, x^* + A0) = (x, x^*) + (0, A0) \subseteq \operatorname{gra} A$ and hence $x^* + A0 \subseteq Ax$. On the other hand, let $y^* \in Ax$. We have $(0, y^* - x^*) = (x, y^*) - (x, x^*) \in \operatorname{gra} A$. Then $y^* - x^* \in A0$ and thus $y^* \in x^* + A0$. Hence $Ax \subseteq x^* + A0$ and thus $Ax = x^* + A0$.

(iii): Let $(\alpha, \beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\}$ and $\{x, y\} \subseteq \operatorname{dom} A$. We can suppose $\alpha \neq 0$ and $\beta \neq 0$. Take $x^* \in Ax$ and $y^* \in Ay$. Since $\operatorname{gra} A$ is a linear subspace, $\alpha x^* + \beta y^* \in A(\alpha x + \beta y)$. By (ii), $A(\alpha x + \beta y) = \alpha x^* + \beta y^* + A0 = \alpha x^* + A0 + \beta y^* + A0 = \alpha(x^* + \frac{1}{\alpha}A0) + \beta(y^* + \frac{1}{\beta}A0) = \alpha Ax + \beta Ay$.

(iv): We have $(x^*, x^{**}) \in \operatorname{gra}(A^*)^{-1} \Leftrightarrow (x^{**}, x^*) \in \operatorname{gra} A^* \Leftrightarrow (x^*, -x^{**}) \in (\operatorname{gra} A)^{\perp} \Leftrightarrow (x^{**}, -x^*) \in (\operatorname{gra} A^{-1})^{\perp} \Leftrightarrow (x^*, x^{**}) \in \operatorname{gra}(A^{-1})^*$.

(v): Let $x \in \operatorname{dom} A^*$ and $y \in \operatorname{dom} A$. Take $x^* \in A^*x$ and $y^* \in Ay$. We have $\langle x^*, y\rangle = \langle y^*, x\rangle, \forall x^* \in A^*x, y^* \in Ay$. Hence $\langle A^*x, y\rangle$ and $\langle Ay, x\rangle$ are singleton and equal.

(vi): We have $(x, x^*) \in \operatorname{gra} A^{**} \Leftrightarrow (x^*, -x) \in (\operatorname{gra} A^*)^{\perp} = ((\operatorname{gra} -A^{-1})^{\perp})^{\perp} = \operatorname{gra} -A^{-1} \Leftrightarrow (x, x^*) \in \operatorname{gra} A$.

(vii): Clearly, $(\operatorname{dom} A)^{\perp} \subseteq A^*0$. Let $x^* \in A^*0$. We have $\langle x^*, y\rangle + \langle 0, Ay\rangle = 0, \quad \forall y \in \operatorname{dom} A$. Then we have $x^* \in (\operatorname{dom} A)^{\perp}$ and thus $A^*0 \subseteq (\operatorname{dom} A)^{\perp}$. Hence $(\operatorname{dom} A)^{\perp} = A^*0$. By Fact 3.1.1, $\overline{\operatorname{dom} A} = (A^*0)_{\perp}$.

(viii): By Fact 3.1.1,

$$
\begin{aligned}
x^* \in A0 &\Leftrightarrow (0, x^*) \in \operatorname{gra} A = \Big[(\operatorname{gra} A)^{\perp}\Big]_{\perp} = \big[\operatorname{gra} -(A^*)^{-1}\big]_{\perp} \\
&\Leftrightarrow \langle x^*, y^{**}\rangle = 0, \quad \forall y^{**} \in \operatorname{dom} A^* \Leftrightarrow x^* \in (\operatorname{dom} A^*)_{\perp}.
\end{aligned}
$$

Hence $(\operatorname{dom} A^*)_{\perp} = A0$. Take $Y = X^*$, by Fact 3.1.1 again, $\overline{\operatorname{dom} A^*}^{\mathrm{w}^*} = (A0)^{\perp}$.

(ix): Let $\bar{A}$ be the linear relation whose graph is the closure of the graph of A. Then $\operatorname{dom} A = \operatorname{dom} \bar{A}$ and $A^* = \bar{A}^*$. Then by Fact 3.1.2,

$$
\begin{aligned}
\iota_{X^* \times (\bar{A}0)^{\perp}} = \iota^*_{\{0\} \times \bar{A}0} &= \big(\iota_{\operatorname{gra} \bar{A}} + \iota_{\{0\} \times X^*}\big)^* = \iota_{\operatorname{gra}(-\bar{A}^*)^{-1}} \,\square\, \iota_{X^* \times \{0\}} \\
&= \iota_{X^* \times \operatorname{dom} \bar{A}^*}.
\end{aligned}
$$

It is clear that $\operatorname{dom} A^* = \operatorname{dom} \bar{A}^* = (\bar{A}0)^{\perp}$ is closed.

(x): Let $k \in \mathbb{R} \smallsetminus \{0\}$. Then $(x^{**}, x^*) \in \operatorname{gra}(kA)^* \Leftrightarrow (x^*, -x^{**}) \in (\operatorname{gra} kA)^{\perp} \Leftrightarrow (x^*, -kx^{**}) \in (\operatorname{gra} A)^{\perp} \Leftrightarrow (\frac{1}{k}x^*, -x^{**}) \in (\operatorname{gra} A)^{\perp} \Leftrightarrow (x^{**}, \frac{1}{k}x^*) \in \operatorname{gra} A^*$. Hence $(kA)^* = kA^*$. $\blacksquare$

3.2 Properties of monotone linear relations

Proposition 3.2.1, Proposition 3.2.2 and Proposition 3.2.7 were established in reflexive spaces by Bauschke, Wang and Yao in [15, Proposition 2.2]. Here, we adapt the proofs to a general Banach space.

Proposition 3.2.1 *Let $A\colon X \rightrightarrows X^*$ be a linear relation. Then the following hold.*

(i) *Suppose A is monotone. Then $\operatorname{dom} A \subseteq (A0)_\perp$ and $A0 \subseteq (\operatorname{dom} A)^\perp$; consequently, if $\operatorname{gra} A$ is closed, then $\operatorname{dom} A \subseteq \overline{\operatorname{dom} A^*}^{\mathrm{w}^*} \cap X$ and $A0 \subseteq A^*0$.*

(ii) $(\forall x \in \operatorname{dom} A)(\forall z \in (A0)_\perp)$ $\langle z, Ax\rangle$ *is single-valued.*

(iii) $(\forall z \in (A0)_\perp)$ $\operatorname{dom} A \to \mathbb{R}\colon y \mapsto \langle z, Ay\rangle$ *is linear.*

(iv) *A is monotone $\Leftrightarrow$ $(\forall x \in \operatorname{dom} A)$ $\langle x, Ax\rangle$ is single-valued and $\langle x, Ax\rangle \geq 0$.*

(v) *If $(x, x^*) \in (\operatorname{dom} A) \times X^*$ is monotonically related to $\operatorname{gra} A$ and $x_0^* \in Ax$, then $x^* - x_0^* \in (\operatorname{dom} A)^\perp$.*

Proof. (i): Pick $x \in \operatorname{dom} A$. Then there exists $x^* \in X^*$ such that $(x, x^*) \in \operatorname{gra} A$. By the monotonicity of A and since $(0, A0) \subseteq \operatorname{gra} A$, we have $\langle x, x^*\rangle \geq \sup\langle x, A0\rangle$. Since $A0$ is a linear subspace (Proposition 3.1.3(i)), we obtain $x \perp A0$. This implies $\operatorname{dom} A \subseteq (A0)_\perp$ and $A0 \subseteq (\operatorname{dom} A)^\perp$. If $\operatorname{gra} A$ is closed, then Proposition 3.1.3(viii)&(vii) yield $\operatorname{dom} A \subseteq (A0)_\perp \subseteq (A0)^\perp = \overline{\operatorname{dom} A^*}^{\mathrm{w}^*}$ and $A0 \subseteq A^*0$.

(ii): Take $x \in \operatorname{dom} A$, $x^* \in Ax$, and $z \in (A0)_\perp$. By Proposition 3.1.3(ii), $\langle z, Ax\rangle = \langle z, x^* + A0\rangle = \langle z, x^*\rangle$.

(iii): Take $z \in (A0)_\perp$. By (ii), $(\forall y \in \operatorname{dom} A)$ $\langle z, Ay\rangle$ is single-valued. Now let x, y be in $\operatorname{dom} A$, and let α, β be in $\mathbb{R}$. If $(\alpha, \beta) = (0, 0)$, then $\langle z, A(\alpha x + \beta y)\rangle = \langle z, A0\rangle = 0 = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. And if $(\alpha, \beta) \neq (0, 0)$, then Proposition 3.1.3(iii) yields $\langle z, A(\alpha x + \beta y) = \langle z, \alpha Ax + \beta Ay\rangle = \alpha\langle z, Ax\rangle + \beta\langle z, Ay\rangle$. This verifies linearity.

(iv): "$\Rightarrow$": This follows from (i), (ii), and the fact that $(0, 0) \in \operatorname{gra} A$. "$\Leftarrow$": If x and y belong to $\operatorname{dom} A$, then Proposition 3.1.3(iii) yields $\langle x - y, Ax - Ay\rangle = \langle x - y, A(x - y)\rangle \geq 0$.

(v): Let $(x, x^*) \in (\operatorname{dom} A) \times X^*$ be monotonically related to $\operatorname{gra} A$, and take $x_0^* \in Ax$. For every $(v, v^*) \in \operatorname{gra} A$, we have that $x_0^* + v^* \in A(x + v)$ (by Proposition 3.1.3(iii)); hence, $\langle x - (x + v), x^* - (x_0^* + v^*)\rangle \geq 0$ and thus $\langle v, v^*\rangle \geq \langle v, x^* - x_0^*\rangle$. Now take $\lambda > 0$ and replace (v, v^*) in the last inequality by $(\lambda v, \lambda v^*)$. Then divide by λ and let $\lambda \to 0^+$ to see that $0 \geq \sup\langle \operatorname{dom} A,\ x^* - x_0^*\rangle$. Since $\operatorname{dom} A$ is linear, it follows that $x^* - x_0^* \in (\operatorname{dom} A)^\perp$. ■

We say that a linear relation $A : X \rightrightarrows X^*$ is *skew* if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$, equivalently, if $\langle x, x^*\rangle = 0$, $\forall (x, x^*) \in \operatorname{gra} A$; furthermore, A is *symmetric* if $\operatorname{gra} A \subseteq \operatorname{gra} A^*$; equivalently, if $\langle x, y^*\rangle = \langle y, x^*\rangle$, $\forall (x, x^*), (y, y^*) \in \operatorname{gra} A$.

We define the *symmetric part* and the *skew part* of A via

$$A_+ = \tfrac{1}{2}A + \tfrac{1}{2}A^* \quad \text{and} \quad A_\circ = \tfrac{1}{2}A - \tfrac{1}{2}A^*, \tag{3.2}$$

respectively. It is easy to check that A_+ is symmetric and that $A_\circ$ is skew.

Proposition 3.2.2 *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then the following hold.*

(i) *If A is maximally monotone, then $(\operatorname{dom} A)^\perp = A0$ and hence $\overline{\operatorname{dom} A} = (A0)_\perp$.*

(ii) *If $\operatorname{dom} A$ is closed, then: A is maximally monotone $\Leftrightarrow (\operatorname{dom} A)^\perp = A0$.*

(iii) *If A is maximally monotone, then $\overline{\operatorname{dom} A^*}^{\mathrm{w}^*} \cap X = \overline{\operatorname{dom} A} = (A0)_\perp$ and $A0 = A^*0 = A_+0 = A_\circ 0 = (\operatorname{dom} A)^\perp$.*

(iv) *If A is maximally monotone and $\operatorname{dom} A$ is closed, then $\operatorname{dom} A^* \cap X = \operatorname{dom} A$.*

(v) *If A is maximally monotone and $\operatorname{dom} A \subseteq \operatorname{dom} A^*$, then $A = A_+ + A_\circ$, $A_+ = A - A_\circ$, and $A_\circ = A - A_+$.*

Proof. (i): Since $A + N_{\operatorname{dom} A} = A + (\operatorname{dom} A)^\perp$ is a monotone extension of A and A is maximally monotone, we must have $A + (\operatorname{dom} A)^\perp = A$. Then $A0 + (\operatorname{dom} A)^\perp = A0$. As $0 \in A0$, $(\operatorname{dom} A)^\perp \subseteq A0$. The reverse inclusion follows from Proposition 3.2.1(i). Then we have $(\operatorname{dom} A)^\perp = A0$. By Fact 3.1.1, $\overline{\operatorname{dom} A} = (A0)_\perp$.

(ii): "$\Rightarrow$": This follows directly from (i). "$\Leftarrow$": By our assumptions and Fact 3.1.1, $\operatorname{dom} A = (A0)_\perp$. Let (x, x^*) be monotonically related to $\operatorname{gra} A$. We have $\inf [\langle x - 0, x^* - A0\rangle] \geq 0$. Then we have $x \in (A0)_\perp$ and hence $x \in \operatorname{dom} A$. Then by Proposition 3.2.1(v) and Proposition 3.1.3(ii), $x^* \in Ax$. Hence A is maximally monotone.

(iii): By (i) and Proposition 3.1.3(vii), $A0 = (\operatorname{dom} A)^\perp = A^*0$ and thus $A_+0 = A_\circ 0 = A0 = (\operatorname{dom} A)^\perp$. Then by Proposition 3.1.3(viii) and (i),

$\overline{\operatorname{dom} A^*}^{\mathrm{w}^*} \cap X = (A0)_\perp = \overline{\operatorname{dom} A}$.

(iv): Apply (iii) and Proposition 3.1.3(ix) directly.

(v): We show only the proof of $A = A_+ + A_\circ$ as the other two proofs are analogous. Clearly, $\operatorname{dom} A_+ = \operatorname{dom} A_\circ = \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A$. Let $x \in \operatorname{dom} A$, and $x^* \in Ax$ and $y^* \in A^*x$. We write $x^* = \frac{x^*+y^*}{2} + \frac{x^*-y^*}{2} \in (A_+ + A_\circ)x$. Then, by (iii) and Proposition 3.1.3(ii), $Ax = x^* + A0 = x^* + (A_+ + A_\circ)0 = (A_+ + A_\circ)x$. Therefore, $A = A_+ + A_\circ$. ■

Corollary 3.2.3 below first appeared in [63, Corollary 2.6 and Proposition 3.2(h)] by Phelps and Simons. Voisei and Zălinescu showed that the maximality part also holds in locally convex spaces [87, Proposition 23].

Corollary 3.2.3 *Let $A\colon X \to X^*$ be monotone and linear. Then A is maximally monotone and continuous.*

Proof. By Proposition 3.2.2(ii), A is maximally monotone and thus $\operatorname{gra} A$ is closed. By the Closed Graph Theorem, A is continuous. ■

Proposition 3.2.2(ii) provides a characterization of maximal monotonicity for certain monotone linear relations. More can be said in finite-dimensional spaces. We require the following lemma, where $\dim F$ stands for the dimension of a subspace F of X. Lemma 3.2.4 and Proposition 3.2.5 were established by Bauschke, Wang and Yao in [18].

Lemma 3.2.4 *Suppose that X is finite-dimensional and let $A\colon X \rightrightarrows X^*$ be a linear relation. Then* $\dim(\operatorname{gra} A) = \dim(\operatorname{dom} A) + \dim A0$.

Proof. We shall construct a basis of $\operatorname{gra} A$. By Proposition 3.1.3(i), $A0$ is a linear subspace. Let $\{x_1^*, \ldots, x_k^*\}$ be a basis of $A0$, and let $\{x_{k+1}, \ldots, x_l\}$ be a basis of $\operatorname{dom} A$. From Proposition 3.1.3(ii), it is easy to show $\{(0, x_1^*), \ldots, (0, x_k^*), (x_{k+1}, x_{k+1}^*), \ldots, (x_l, x_l^*)\}$ is a basis of $\operatorname{gra} A$, where $x_i^* \in Ax_i,\ i \in \{k+1, \ldots, l\}$. Thus $\dim(\operatorname{gra} A) = l = \dim(\operatorname{dom} A) + \dim A0$. ■

Lemma 3.2.4 allows us to get a satisfactory characterization of maximal monotonicities of linear relations in finite-dimensional spaces.

Proposition 3.2.5 *Suppose that X is finite-dimensional, set $n = \dim X$, and let $A\colon X \rightrightarrows X^*$ be a monotone linear relation. Then A is maximally monotone if and only if $\dim \operatorname{gra} A = n$.*

Proof. Since linear subspaces of X are closed, we see from Proposition 3.2.2(ii) that

$$A \text{ is maximally monotone} \Leftrightarrow \operatorname{dom} A = (A0)^{\perp}. \tag{3.3}$$

Assume first that A is maximally monotone. Then $\operatorname{dom} A = (A0)^{\perp}$. By Lemma 3.2.4, $\dim(\operatorname{gra} A) = \dim(\operatorname{dom} A) + \dim(A0) = \dim((A0)^{\perp}) + \dim(A0) = n$. Conversely, let $\dim(\operatorname{gra} A) = n$. By Lemma 3.2.4, we have that $\dim(\operatorname{dom} A) = n - \dim(A0)$. As $\dim((A0)^{\perp}) = n - \dim(A0)$ and $\operatorname{dom} A \subseteq (A0)^{\perp}$ by Proposition 3.2.1(i), we have that $\operatorname{dom} A = (A0)^{\perp}$. By (3.3), A is maximally monotone. ■

Next, we obtain a key criteria on concerning maximally monotone linear relations, which I will frequently use to construct maximally monotone linear subspace extensions in Chapter 4.

Corollary 3.2.6 *Let $A\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a monotone linear relation. The following are equivalent:*

(i) *A is maximally monotone.*

(ii) $\dim \operatorname{gra} A = n$.

(iii) $\operatorname{dom} A = (A0)^{\perp}$.

For a monotone linear relation $A\colon X \rightrightarrows X^*$ it will be convenient to define (as in, e.g., [5])

$$(\forall x \in X) \quad q_A(x) = \begin{cases} \frac{1}{2}\langle x, Ax\rangle, & \text{if } x \in \operatorname{dom} A; \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition 3.2.7 *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation, let x and y be in* $\operatorname{dom} A$*, and let $\lambda \in \mathbb{R}$. Then q_A is single-valued and*

$$\begin{aligned} &\lambda q_A(x) + (1-\lambda) q_A(y) - q_A(\lambda x + (1-\lambda) y) = \lambda(1-\lambda) q_A(x-y) \\ &= \tfrac{1}{2}\lambda(1-\lambda)\langle x-y, Ax-Ay\rangle. \end{aligned} \tag{3.4}$$

Moreover, q_A is convex.

Proof. Proposition 3.2.1(iv) shows that q_A is single-valued on $\operatorname{dom} A$ and that $q_A \geq 0$. Combining with Proposition 3.2.1(i)&(iii), we obtain (3.4). Then by (3.4), q_A is convex. ■

Fact 3.2.8 (Simons) (See [74, Lemma 19.7 and Section 22].) *Let $A : X \rightrightarrows X^*$ be a monotone operator with convex graph such that* $\operatorname{gra} A \neq \varnothing$*. Then the function*

$$g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra} A}(x, x^*) \tag{3.5}$$

is proper and convex.

Proof. It is clear that g is proper because $\operatorname{gra} A \neq \varnothing$. To see that g is convex, let (a, a^*) and (b, b^*) be in $\operatorname{gra} A$, and let $\lambda \in \,]0,1[$. Set $\mu = 1-\lambda \in \,]0,1[$ and observe that $\lambda(a, a^*) + \mu(b, b^*) = (\lambda a + \mu b, \lambda a^* + \mu b^*) \in \operatorname{gra} A$ by convexity of $\operatorname{gra} A$. Since A is monotone, it follows that

$$\begin{aligned}
&\lambda g(a, a^*) + \mu g(b, b^*) - g\big(\lambda(a, a^*) + \mu(b, b^*)\big) \\
&= \lambda\langle a, a^*\rangle + \mu\langle b, b^*\rangle - \langle \lambda a + \mu b, \lambda a^* + \mu b^*\rangle \\
&= \lambda\mu\langle a - b, a^* - b^*\rangle \\
&\geq 0.
\end{aligned}$$

Therefore, g is convex. ■

Phelps and Simons proved Fact 3.2.9 in the unbounded linear case in [63, Proposition 3.2(a)], but their proof can also be adapted to a linear relation. For readers' convenience, we write down their proof.

Fact 3.2.9 (Phelps-Simons) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then $(x, x^*) \in X \times X^*$ is monotonically related to* $\operatorname{gra} A$ *if and only if*

$$\langle x, x^*\rangle \geq 0 \text{ and } [\langle y^*, x\rangle + \langle x^*, y\rangle]^2 \leq 4\langle x^*, x\rangle\langle y^*, y\rangle, \quad \forall (y, y^*) \in \operatorname{gra} A.$$

Proof. We have

$(x, x^*) \in X \times X^*$ is monotonically related to $\operatorname{gra} A$

$$\Leftrightarrow \lambda^2\langle y, y^*\rangle - \lambda\left[\langle y^*, x\rangle + \langle x^*, y\rangle\right] + \langle x, x^*\rangle = \langle \lambda y^* - x^*, \lambda y - x\rangle \geq 0,$$
$$\forall \lambda \in \mathbb{R}, \forall (y, y^*) \in \operatorname{gra} A$$
$$\Leftrightarrow \langle x, x^*\rangle \geq 0 \text{ and } \left[\langle y^*, x\rangle + \langle x^*, y\rangle\right]^2 \leq 4\langle x^*, x\rangle\langle y^*, y\rangle,$$
$$\forall (y, y^*) \in \operatorname{gra} A \text{ (by [63, Lemma 2.1]).}$$

■

The proof of Proposition 3.2.10(iii) was borrowed from [30, Theorem 2]. Results very similar to Proposition 3.2.10(i)&(ii) are established in [89, Proposition 18.9].

Proposition 3.2.10 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then*

(i) *A_+ is monotone, and $\overline{q_A} + \iota_{\operatorname{dom} A_+} = q_{A_+}$ and thus q_{A_+} is convex.*

(ii) $\operatorname{gra} A_+ \subseteq \operatorname{gra} \partial\overline{q_A}$. *If A_+ is maximally monotone, then $A_+ = \partial\overline{q_A}$.*

(iii) *If A is maximally monotone, then $A^*|_X$ is monotone.*

(iv) *If A is maximally monotone and* $\operatorname{dom} A$ *is closed, then $A^*|_X$ is maximally monotone.*

Proof. Let $x \in \operatorname{dom} A_+$.

(i): Since A is monotone, by Proposition 3.1.3(v) and Proposition 3.2.1(iv), $q_{A_+} = q_A|_{\operatorname{dom} A_+}$ and A_+ is monotone. Then by Proposition 3.2.7, q_{A_+} is convex. Let $y \in \operatorname{dom} A$. Then by Proposition 3.1.3(v) again,

$$0 \leq \tfrac{1}{2}\langle Ax - Ay, x - y\rangle = \tfrac{1}{2}\langle Ay, y\rangle + \tfrac{1}{2}\langle Ax, x\rangle - \langle A_+x, y\rangle, \tag{3.6}$$

we have $q_A(y) \geq \langle A_+x, y\rangle - q_A(x)$. Take lower semicontinuous hull at y and then deduce that $\overline{q_A}(y) \geq \langle A_+x, y\rangle - q_A(x)$. For $y = x$, we have $\overline{q_A}(x) \geq q_A(x)$. On the other hand, $\overline{q_A}(x) \leq q_A(x)$. Altogether, $\overline{q_A}(x) = q_A(x) = q_{A_+}(x)$. Thus (i) holds.

(ii): Let $y \in \operatorname{dom} A$. By (3.6) and (i),

$$q_A(y) \geq q_A(x) + \langle A_+x, y-x\rangle = \overline{q_A}(x) + \langle A_+x, y-x\rangle. \tag{3.7}$$

Since $\operatorname{dom}\overline{q_A} \subseteq \overline{\operatorname{dom} q_A} = \overline{\operatorname{dom} A}$, by (3.7), $\overline{q_A}(z) \geq \overline{q_A}(x) + \langle A_+x, z - x\rangle, \quad \forall z \in \operatorname{dom}\overline{q_A}$. Hence $A_+x \subseteq \partial\overline{q_A}(x)$. If A_+ is maximally monotone, then $A_+ = \partial\overline{q_A}$. Thus (ii) holds.

(iii): Suppose to the contrary that $A^*|_X$ is not monotone. By Proposition 3.2.1(iv), there exists $(x_0, x_0^*) \in \operatorname{gra} A^*$ with $x_0 \in X$ such that $\langle x_0, x_0^*\rangle < 0$. Now we have

$$\begin{aligned}
&\langle -x_0 - y, x_0^* - y^*\rangle = \langle -x_0, x_0^*\rangle + \langle y, y^*\rangle + \langle x_0, y^*\rangle + \langle -y, x_0^*\rangle \\
&= \langle -x_0, x_0^*\rangle + \langle y, y^*\rangle > 0, \quad \forall (y, y^*) \in \operatorname{gra} A.
\end{aligned} \tag{3.8}$$

Thus, $(-x_0, x_0^*)$ is monotonically related to $\operatorname{gra} A$. By the maximal monotonicity of A, $(-x_0, x_0^*) \in \operatorname{gra} A$. Then $\langle -x_0 - (-x_0), x_0^* - x_0^*\rangle = 0$, which contradicts (3.8). Hence $A^*|_X$ is monotone.

(iv): By Proposition 3.1.3(ix), $\operatorname{dom} A^*|_X = (A0)_\perp$ and thus $\operatorname{dom} A^*|_X$ is closed. By Fact 3.1.1 and Proposition 3.2.2(i), $(\operatorname{dom} A^*|_X)^\perp = ((A0)_\perp)^\perp = \overline{A0}^{\mathrm{w}*} = A0$. Then by Proposition 3.2.2(iii), $(\operatorname{dom} A^*|_X)^\perp = A^*0$. Apply (iii) and Proposition 3.2.2(ii), $A^*|_X$ is maximally monotone. ■

Proposition 3.2.11 *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then A is symmetric $\Leftrightarrow$ $A = A^*|_X$.*

Proof. "$\Rightarrow$": Assume that A is symmetric, i.e., $\operatorname{gra} A \subseteq \operatorname{gra} A^*$. Since A is maximally monotone, by Proposition 3.2.10(iii), $A = A^*|_X$.

"$\Leftarrow$": Obvious. ■

Fact 3.2.12 (Phelps-Simons) (See [63, Theorem 2.5 and Lemma 4.4].) *Let $A : \operatorname{dom} A \to X^*$ be monotone and linear. The following hold.*

(i) *If A is maximally monotone, then* $\operatorname{dom} A$ *is dense (and hence A^* is at most single-valued).*

(ii) *Assume that A is skew such that* $\operatorname{dom} A$ *is dense. Then* $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ *and $A^*|_{\operatorname{dom} A} = -A$.*

Fact 3.2.13 (Brezis-Browder) (See [30, Theorem 2].) *Assume X is reflexive. Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Then the following are equivalent.*

(i) *A is maximally monotone.*

(ii) *A^* is maximally monotone.*

(iii) *A^* is monotone.*

In Theorem 3.2.15, established in [89, Theorem 18.5], we provide a new and simpler proof to show the hard part (iii)$\Rightarrow$(i) in Fact 3.2.13. We first need the following fact.

Fact 3.2.14 (Simons-Zălinescu) (See [77, Theorem 1.2] or [72, Theorem 10.6].)

Assume X is reflexive. Let $A\colon X \rightrightarrows X^*$ be monotone. Then A is maximally monotone if and only if

$$\operatorname{gra} A + \operatorname{gra}(-J) = X \times X^*.$$

Now we come to the hard part (iii)⇒(i) in Theorem 3.2.13. The proof was inspired by that of [93, Theorem 32.L].

Theorem 3.2.15 *Assume X is reflexive. Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation with closed graph. Suppose A^* is monotone. Then A is maximally monotone.*

Proof. By Fact 3.2.14, it suffices to show that $X \times X^* \subseteq \operatorname{gra} A + \operatorname{gra}(-J)$. For this, let $(x, x^*) \in X \times X^*$ and we define $g : X \times X^* \to \left]-\infty, +\infty\right]$ by

$$(y, y^*) \mapsto \tfrac{1}{2}\|y^*\|^2 + \tfrac{1}{2}\|y\|^2 + \langle y^*, y\rangle + \iota_{\operatorname{gra} A}(y - x, y^* - x^*).$$

We have $f : (y, y^*) \mapsto \langle y^*, y\rangle + \iota_{\operatorname{gra} A}(y-x, y^*-x^*) = \langle y^*, y\rangle + \iota_{\operatorname{gra} A+(x,x^*)}(y, y^*)$. By Fact 3.2.8 and the assumption that $\operatorname{gra} A$ is closed, f is proper lower semicontinuous and convex. Hence g is lower semicontinuous convex and coercive. According to [92, Theorem 2.5.1(ii)], g has minimizers. Suppose that (z, z^*) is a minimizer of g. Then $(z - x, z^* - x^*) \in \operatorname{gra} A$, hence,

$$(x, x^*) \in \operatorname{gra} A + (z, z^*). \tag{3.9}$$

On the other hand, since (z, z^*) is a minimizer of g, $(0,0) \in \partial g(z, z^*)$. By a result of Rockafellar (see [37, Theorem 2.9.8] and [92, Theorem 3.2.4(ii)] or [60, Theorem 1.93 and Proposition 1.107(ii)]), there exist $(z_0^*, z_0) \in \partial(\iota_{\operatorname{gra} A}(\cdot - x, \cdot - x^*))(z, z^*) = \partial\iota_{\operatorname{gra} A}(z - x, z^* - x^*) = (\operatorname{gra} A)^{\perp}$, and $(v, v^*) \in X \times X^*$ with $v^* \in Jz, z^* \in Jv$ such that

$$(0,0) = (z^*, z) + (v^*, v) + (z_0^*, z_0).$$

Then

$$\big(-(z+v), z^* + v^*\big) \in \operatorname{gra} A^*.$$

Since A^* is monotone,

$$\langle z^* + v^*, z + v\rangle = \langle z^*, z\rangle + \langle z^*, v\rangle + \langle v^*, z\rangle + \langle v^*, v\rangle \leq 0. \tag{3.10}$$

Note that since $\langle z^*, v\rangle = \|z^*\|^2 = \|v\|^2$, $\langle v^*, z\rangle = \|v^*\|^2 = \|z\|^2$, by (3.10), we have

$$\tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}\|z^*\|^2 + \langle z^*, z\rangle + \tfrac{1}{2}\|v^*\|^2 + \tfrac{1}{2}\|v\|^2 + \langle v, v^*\rangle \leq 0.$$

Hence $z^* \in -Jz$. By (3.9), $(x, x^*) \in \operatorname{gra} A + \operatorname{gra}(-J)$. ■

Remark 3.2.16 *Haraux provides a very simple proof of Theorem 3.2.15 in Hilbert spaces in [51, Theorem 10], but the proof could not be adapted to reflexive Banach spaces (The proof is based on the application of Minty's Theorem).*

3.3 An unbounded skew operator on $\ell^2(\mathbb{N})$

In this section, we construct a maximally monotone and skew operator S on $\ell^2(\mathbb{N})$ such that $-S^*$ is not maximally monotone. This answers Svaiter's question raised in [80]. We also show its domain is a proper subset of the domain of its adjoint S^*, i.e., $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$. Throughout this section, H denotes a Hilbert space. Section 3.3 is all based on the work in [17] by Bauschke, Wang and Yao .

Let $\ell^2(\mathbb{N})$ denote the Hilbert space of real square-summable sequences $(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, \ldots)$ with $\sum_{i\geq 1} x_i^2 < +\infty$.

Example 3.3.1 *Let $H = \ell^2(\mathbb{N})$, and $S : \operatorname{dom} S \to \ell^2(\mathbb{N})$ be given by*

$$Sy = \frac{\left(\sum_{i<n} y_i - \sum_{i>n} y_i\right)_{n\in\mathbb{N}}}{2} = \left(\sum_{i<n} y_i + \tfrac{1}{2} y_n\right)_{n\in\mathbb{N}},$$
$$\forall y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S, \tag{3.11}$$

where $\operatorname{dom} S = \left\{y = (y_n) \in \ell^2(\mathbb{N}) \mid \sum_{i\geq 1} y_i = 0, \left(\sum_{i\leq n} y_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\right\}$ *and $\sum_{i<1} y_i$ is understood to mean 0. In matrix form,*

$$S = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 0 & -1 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 1 & 0 & -1 & \cdots & -1 & -1 & \cdots \\ 1 & 1 & 1 & 1 & 0 & \cdots & -1 & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \end{pmatrix},$$

or

$$S = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & \frac{1}{2} & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \frac{1}{2} & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \end{pmatrix}.$$

Using the second matrix, it is easy to see that S is injective.

Proposition 3.3.2 *Let S be defined as in Example 3.3.1. Then S is skew.*

Proof. Let $x = (x_n)_{n\in\mathbb{N}} \in \operatorname{dom} S$. Then $\left(\sum_{i\le n} x_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Thus,

$$\begin{aligned}\ell^2(\mathbb{N}) \ni \left(\sum_{i\le n} x_i\right)_{n\in\mathbb{N}} - \tfrac{1}{2}x &= \left(\sum_{i\le n} x_i\right)_{n\in\mathbb{N}} - \tfrac{1}{2}(x_n)_{n\in\mathbb{N}} \\ &= \left(\sum_{i<n} x_i + \tfrac{1}{2}x_n\right)_{n\in\mathbb{N}} = Sx.\end{aligned}$$

Hence S is well defined. Clearly, S is linear on $\operatorname{dom} S$. Now we show S is skew.

Let $y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S$, and $s = \sum_{i\ge 1} y_i$. Then $\left(\sum_{i\le n} y_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Hence $\left(\sum_{i<n} y_i\right)_{n\in\mathbb{N}} = \left(\sum_{i\le n} y_i\right)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Since $s = 0$,

$$\begin{aligned}\ell^2(\mathbb{N}) \ni -\left(\sum_{i<n} y_i\right)_{n\in\mathbb{N}} &= 0 - \left(\sum_{i<n} y_i\right)_{n\in\mathbb{N}} = \left(\sum_{i\ge 1} y_i - \sum_{i<n} y_i\right)_{n\in\mathbb{N}} \\ &= \left(\sum_{i\ge n} y_i\right)_{n\in\mathbb{N}},\end{aligned}$$

$$\Big(\sum_{i\geq n+1} y_i\Big)_{n\in\mathbb{N}} = 0 - \Big(\sum_{i\leq n} y_i\Big)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}). \tag{3.12}$$

Thus, by (3.12),

$$-2\langle Sy, y\rangle = \Big\langle \Big(\sum_{i>n} y_i - \sum_{i<n} y_i\Big)_{n\in\mathbb{N}}, y\Big\rangle = \Big\langle \Big(\sum_{i\geq n+1} y_i + \sum_{i\geq n} y_i\Big)_{n\in\mathbb{N}}, y\Big\rangle \tag{3.13}$$

$$\begin{aligned}
&= \Big\langle \Big(\sum_{i\geq 1} y_i, \sum_{i\geq 2} y_i, \ldots\Big) + \Big(\sum_{i\geq 2} y_i, \sum_{i\geq 3} y_i, \ldots\Big), y\Big\rangle \\
&= \langle (s, s-y_1, s-(y_1+y_2), \ldots) + (s-y_1, s-(y_1+y_2), \ldots), (y_1, y_2, \ldots)\rangle \\
&= [sy_1 + (s-y_1)y_2 + (s-(y_1+y_2))y_3 + \cdots] + \\
&\qquad [(s-y_1)y_1 + (s-(y_1+y_2))y_2 + (s-(y_1+y_2+y_3))y_3 + \cdots] \\
&= \lim_n [sy_1 + (s-y_1)y_2 + \cdots + (s-(y_1+\cdots+y_{n-1}))y_n] + \\
&\qquad \lim_n [(s-y_1)y_1 + (s-(y_1+y_2))y_2 + \cdots + (s-(y_1+\cdots+y_n))y_n] \\
&= \lim_n [s(y_1+\cdots+y_n) - y_1y_2 - (y_1+y_2)y_3 - \cdots - (y_1+\cdots+y_{n-1})y_n] + \\
&\qquad [s(y_1+\cdots+y_n) - (y_1^2+\cdots+y_n^2) - y_1y_2 - \cdots - (y_1+\cdots+y_{n-1})y_n] \\
&= \lim_n [2s(y_1+\cdots+y_n) - (y_1+\cdots+y_n)^2] = 2s^2 - s^2 = s^2 = 0.
\end{aligned}$$

Hence S is skew. ■

Proposition 3.3.3 *Let S be defined as in Example 3.3.1. Then S is a maximally monotone operator. In particular,* $\operatorname{gra} S$ *is closed.*

Proof. By Proposition 3.3.2, S is skew. Let $(x, x^*) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ be monotonically related to $\operatorname{gra} S$. Write $x = (x_n)_{n\in\mathbb{N}}$ and $x^* = (x_n^*)_{n\in\mathbb{N}}$. By

Fact 3.2.9, we have

$$\langle Sy, x\rangle + \langle x^*, y\rangle = 0, \quad \forall y \in \operatorname{dom} S. \tag{3.14}$$

Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$: the nth entry is 1 and the others are 0. Then let $y = -e_1 + e_n$. Thus $y \in \operatorname{dom} S$ and $Sy = (-\frac{1}{2}, -1, \ldots, -1, -\frac{1}{2}, 0, \ldots)$. Then by (3.14),

$$-x_1^* + x_n^* - \tfrac{1}{2}x_1 - \tfrac{1}{2}x_n - \sum_{i=2}^{n-1} x_i = 0 \Rightarrow x_n^* = x_1^* - \tfrac{1}{2}x_1 + \sum_{i=1}^{n-1} x_i + \tfrac{1}{2}x_n. \tag{3.15}$$

Since $x^* \in \ell^2(\mathbb{N})$ and $x \in \ell^2(\mathbb{N})$, we have $x_n^* \to 0, x_n \to 0$. Thus by (3.15),

$$-\sum_{i\geq 1} x_i = x_1^* - \tfrac{1}{2}x_1. \tag{3.16}$$

Next we show $-\sum_{i\geq 1} x_i = x_1^* - \frac{1}{2}x_1 = 0$. Let $s = \sum_{i\geq 1} x_i$. Then by (3.15) and (3.16),

$$\begin{aligned}
2x^* = 2(x_n^*)_{n\in\mathbb{N}} &= 2\Big(-\sum_{i\geq 1} x_i + \sum_{i<n} x_i + \tfrac{1}{2}x_n \Big)_{n\in\mathbb{N}} \\
&= \Big(-2\sum_{i\geq 1} x_i + 2\sum_{i<n} x_i + x_n \Big)_{n\in\mathbb{N}} \\
&= \Big(-2\sum_{i\geq n} x_i + x_n \Big)_{n\in\mathbb{N}} = \Big(-\sum_{i\geq n} x_i - \sum_{i\geq n} x_i + x_n \Big)_{n\in\mathbb{N}} \\
&= \Big(-\sum_{i\geq n} x_i - \sum_{i\geq n+1} x_i \Big)_{n\in\mathbb{N}}.
\end{aligned} \tag{3.17}$$

On the other hand, by (3.15) and (3.16),

$$\ell^2(\mathbb{N}) \ni x^* - \tfrac{1}{2}x = \Big(-\sum_{i\geq 1} x_i + \sum_{i<n} x_i + \tfrac{1}{2}x_n\Big)_{n\in\mathbb{N}} - (\tfrac{1}{2}x_n)_{n\in\mathbb{N}} = \Big(-\sum_{i\geq n} x_i\Big)_{n\in\mathbb{N}}.$$

Then by (3.17),

$$2x^* = \Big(-\sum_{i\geq n} x_i\Big)_{n\in\mathbb{N}} + \Big(-\sum_{i\geq n+1} x_i\Big)_{n\in\mathbb{N}}.$$

Then by Fact 3.2.9, similar to the proof in (3.13) in Proposition 3.3.2, we have

$$\begin{aligned}
0 \geq -2\langle x^*, x\rangle &= \langle \Big(\sum_{i\geq n} x_i\Big)_{n\in\mathbb{N}} + \Big(\sum_{i\geq n+1} x_i\Big)_{n\in\mathbb{N}}, x\rangle \\
&= \langle \Big(\sum_{i\geq 1} x_i, \sum_{i\geq 2} x_i, \ldots\Big) + \Big(\sum_{i\geq 2} x_i, \sum_{i\geq 3} x_i, \ldots\Big), x\rangle \\
&= 2s^2 - s^2 = s^2.
\end{aligned}$$

Hence $s = 0$, i.e., $x_1^* = \frac{1}{2}x_1$ by (3.16). By (3.15), $x^* = \Big(\sum_{i<n} x_i + \frac{1}{2}x_n\Big)_{n\in\mathbb{N}}$. Thus

$$\ell^2(\mathbb{N}) \ni x^* + \tfrac{1}{2}x = \Big(\sum_{i<n} x_i + \tfrac{1}{2}x_n\Big)_{n\in\mathbb{N}} + \big(\tfrac{1}{2}x_n\big)_{n\in\mathbb{N}} = \Big(\sum_{i\leq n} x_i\Big)_{n\in\mathbb{N}}.$$

Hence $x \in \operatorname{dom} S$ and $x^* = Sx$. Thus, S is maximally monotone. Hence $\operatorname{gra} S$ is closed. ■

Remark 3.3.4 *Let S be as in Example 3.3.1. Since $e_1 = (1, 0, 0, \ldots, 0, \ldots) \notin \operatorname{dom} S$, the operator S is unbounded.*

Proposition 3.3.5 *Let S be defined as in Example 3.3.1. Then*

$$S^*y = \left(\sum_{i>n} y_i + \tfrac{1}{2}y_n\right)_{n\in\mathbb{N}}, \quad \forall y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S^*, \tag{3.18}$$

where $\operatorname{dom} S^* = \big\{y = (y_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \mid \sum_{i\geq 1} y_i \in \mathbb{R}, \left(\sum_{i>n} y_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\big\}$. *In matrix form,*

$$S^* = \begin{pmatrix} \frac{1}{2} & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 0 & \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 0 & 0 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \cdots & 1 & 1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \end{pmatrix}.$$

Moreover, $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$, $S^* = -S$ *on* $\operatorname{dom} S$, *and* S^* *is not skew.*

Proof. Let $y = (y_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ with $\left(\sum_{i>n} y_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, and $y^* = \left(\sum_{i>n} y_i + \frac{1}{2}y_n\right)_{n\in\mathbb{N}}$. Now we show $(y, y^*) \in \operatorname{gra} S^*$. Let $s = \sum_{i\geq 1} y_i$ and $x \in \operatorname{dom} S$. Then we have

$$\begin{aligned}
&\langle y, Sx\rangle + \langle y^*, -x\rangle = \left\langle y, \tfrac{1}{2}x + \left(\sum_{i<n} x_i\right)_{n\in\mathbb{N}}\right\rangle + \left\langle \tfrac{1}{2}y + \left(\sum_{i>n} y_i\right)_{n\in\mathbb{N}}, -x\right\rangle \\
&= \left\langle y, \left(\sum_{i<n} x_i\right)_{n\in\mathbb{N}}\right\rangle + \left\langle \left(\sum_{i>n} y_i\right)_{n\in\mathbb{N}}, -x\right\rangle \\
&= \lim_n \left[y_2x_1 + y_3(x_1 + x_2) + \cdots + y_n(x_1 + \cdots + x_{n-1})\right] \\
&\quad - \lim_n \left[x_1(s - y_1) + x_2(s - y_1 - y_2) + \cdots + x_n(s - y_1 - \cdots - y_n)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_n \left[x_1(y_2 + \cdots + y_n) + x_2(y_3 + \cdots + y_n) + \cdots + x_{n-1}y_n\right] \\
&\quad - \lim_n \left[x_1(s - y_1) + x_2(s - y_1 - y_2) + \cdots + x_n(s - y_1 - \cdots - y_n)\right] \\
&= \lim_n \left[x_1(y_1 + y_2 + \cdots + y_n - s) + x_2(y_1 + y_2 + \cdots + y_n - s) + \cdots \right. \\
&\quad \left. + x_n(y_1 + y_2 + \cdots + y_n - s)\right] \\
&= \lim_n \left[(x_1 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n - s)\right] \\
&= 0.
\end{aligned}
$$

Hence $(y, y^*) \in \operatorname{gra} S^*$.

On the other hand, let $(a, a^*) \in \operatorname{gra} S^*$ with $a = (a_n)_{n\in\mathbb{N}}$ and $a^* = (a^*_n)_{n\in\mathbb{N}}$. Now we show

$$
\left(\sum_{i>n} a_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \text{ and } a^* = \left(\sum_{i>n} a_i + \tfrac{1}{2}a_n\right)_{n\in\mathbb{N}}. \tag{3.19}
$$

Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$: the nth entry is 1 and the others are 0. Then let $y = -e_1 + e_n$. Thus $y \in \operatorname{dom} S$ and $Sy = (-\frac{1}{2}, -1, \ldots, -1, -\frac{1}{2}, 0, \ldots)$. Then,

$$
0 = \langle a^*, y\rangle + \langle -Sy, a\rangle = -a^*_1 + a^*_n + \tfrac{1}{2}a_1 + \tfrac{1}{2}a_n + \sum_{i=2}^{n-1} a_i
$$

$$
\Rightarrow a^*_n = a^*_1 - \tfrac{1}{2}a_1 - \sum_{i=2}^{n-1} a_i - \tfrac{1}{2}a_n. \tag{3.20}
$$

Since $a^* \in \ell^2(\mathbb{N})$ and $a \in \ell^2(\mathbb{N})$, $a^*_n \to 0, a_n \to 0$. Thus by (3.20),

$$
a^*_1 = \tfrac{1}{2}a_1 + \sum_{i>1} a_i, \tag{3.21}
$$

from which we see that $\sum_{i\geq 1} a_i \in \mathbb{R}$. Combining (3.20) and (3.21), we have

$$a_n^* = \sum_{i>n} a_i + \tfrac{1}{2} a_n$$

Thus, (3.19) holds. Hence (3.18) holds.

Now for $x \in \operatorname{dom} S$, since $\sum_{i\geq 1} x_i = 0$, we have

$$\begin{aligned} S^* x &= \Big(\tfrac{1}{2}x_n + \sum_{i>n} x_i\Big)_{n\in\mathbb{N}} = \Big(-\tfrac{1}{2}x_n + \sum_{i\geq n} x_i\Big)_{n\in\mathbb{N}} \\ &= \Big(-\tfrac{1}{2}x_n - \sum_{i<n} x_i\Big)_{n\in\mathbb{N}} = -Sx. \end{aligned}$$

We note that S^* is not skew since for $e_1 = (1, 0, \ldots)$, $\langle S^* e_1, e_1\rangle = \langle 1/2 e_1, e_1\rangle = 1/2$. As $e_1 = (1, 0, 0, \ldots, 0, \ldots) \in \operatorname{dom} S^*$ but $e_1 \notin \operatorname{dom} S$, we have $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$. ■

Proposition 3.3.6 *Let S be defined as in Example 3.3.1. Then*

$$\langle S^* y, y\rangle = \tfrac{1}{2} s^2, \quad \forall y \in \operatorname{dom} S^* \textit{ with } \quad s = \sum_{i\geq 1} y_i. \tag{3.22}$$

Proof. Let $y = (y_n)_{n\in\mathbb{N}} \in \operatorname{dom} S^*$, and $s = \sum_{i\geq 1} y_i$. By Proposition 3.3.5, we have $s \in \mathbb{R}$ and

$$\begin{aligned} &\langle S^* y, y\rangle = \langle \Big(\sum_{i>n} y_i + \tfrac{1}{2} y_n\Big)_{n\in\mathbb{N}}, y\rangle = \langle \Big(\sum_{i\geq n} y_i - \tfrac{1}{2} y_n\Big)_{n\in\mathbb{N}}, y\rangle \\ &= \lim_n [s y_1 + (s - y_1) y_2 + \cdots + (s - y_1 - y_2 - \cdots - y_{n-1}) y_n \\ &\quad - \tfrac{1}{2}(y_1^2 + y_2^2 + \cdots + y_n^2)] \\ &= \lim_n [s(y_1 + \cdots + y_n) - y_1 y_2 - (y_1 + y_2) y_3 - \cdots \end{aligned}$$

$$
\begin{aligned}
&\quad - (y_1 + y_2 + \cdots + y_{n-1})y_n] - \tfrac{1}{2}\left[y_1^2 + y_2^2 + \cdots + y_n^2\right] \\
&= \lim_n \left[s(y_1 + \cdots + y_n)\right] \\
&\quad - \lim_n [y_1y_2 + (y_1 + y_2)y_3 + \cdots + (y_1 + y_2 + \cdots + y_{n-1})y_n \\
&\quad + \tfrac{1}{2}(y_1^2 + y_2^2 + \cdots + y_n^2)] \\
&= s^2 - \lim_n \tfrac{1}{2}\left[y_1 + y_2 + \cdots + y_n\right]^2 \\
&= s^2 - \tfrac{1}{2}s^2 \\
&= \tfrac{1}{2}s^2.
\end{aligned}
$$

Hence (3.22) holds. ■

Proposition 3.3.7 *Let S be defined as in Example 3.3.1. Then $-S$ is not maximally monotone.*

Proof. By Proposition 3.3.2, $-S$ is skew. Let $e_1 = (1, 0, 0, \ldots, 0, \ldots)$. Then $e_1 \notin \operatorname{dom} S = \operatorname{dom}(-S)$. Thus, $(e_1, \frac{1}{2}e_1) \notin \operatorname{gra}(-S)$. We have for every $y \in \operatorname{dom} S$,

$$\langle e_1, \tfrac{1}{2}e_1\rangle \geq 0 \text{ and } \langle e_1, -Sy\rangle + \langle y, \tfrac{1}{2}e_1\rangle = -\tfrac{1}{2}y_1 + \tfrac{1}{2}y_1 = 0.$$

By Fact 3.2.9, $(e_1, \frac{1}{2}e_1)$ is monotonically related to $\operatorname{gra}(-S)$. Hence $-S$ is not maximally monotone. ■

Suppose that $X = \ell^2(\mathbb{N})$. We proceed to show that for every maximally monotone and skew operator S, the operator $-S$ has a unique maximally monotone extension, namely $S^*|_X$.

Theorem 3.3.8 *Let $S : \operatorname{dom} S \to X^*$ be a maximally monotone skew operator. Then $-S$ has a unique maximally monotone extension: $S^*|_X$.*

Proof. By Fact 3.2.12, $\operatorname{gra}(-S) \subseteq \operatorname{gra} S^*|_X$. Assume T is a maximally monotone extension of $-S$. Let $(x, x^*) \in \operatorname{gra} T$. Then (x, x^*) is monotonically related to $\operatorname{gra}(-S)$. By Fact 3.2.9,

$$\langle x^*, y\rangle + \langle -x, Sy\rangle = \langle x^*, y\rangle + \langle x, -Sy\rangle = 0, \quad \forall y \in \operatorname{dom} S.$$

Thus $(x, x^*) \in \operatorname{gra} S^*|_X$. Since $(x, x^*) \in \operatorname{gra} T$ is arbitrary, we have $\operatorname{gra} T \subseteq \operatorname{gra} S^*|_X$. By Fact 3.2.10(iii), $S^*|_X$ is monotone. Hence $T = S^*|_X$. ■

Remark 3.3.9 *Note that [87, Proposition 17] also implies that $-S$ has a unique maximally monotone extension, where S is as in Theorem 3.3.8.*

Remark 3.3.10 *Define the* right and left shift operators $R, L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ *by*

$$Rx = (0, x_1, x_2, \ldots), \quad Lx = (x_2, x_3, \ldots), \quad \forall\ x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N}).$$

One can verify that in Example 3.3.1

$$S = (\operatorname{Id} - R)^{-1} - \frac{\operatorname{Id}}{2}, \quad S^* = (\operatorname{Id} - L)^{-1} - \frac{\operatorname{Id}}{2}.$$

The maximally monotone operators $(\operatorname{Id} - R)^{-1}$ and $(\operatorname{Id} - L)^{-1}$ have been utilized by Phelps and Simons, see [63, Example 7.4].

Example 3.3.11 ($S+S^*$ fails to be maximally monotone) *Let S be defined as in Example 3.3.1. Then neither S nor S^* has full domain. By Fact 3.2.12,* $\forall x \in \operatorname{dom}(S+S^*) = \operatorname{dom} S$, *we have*

$$(S+S^*)x = 0.$$

Thus $S+S^*$ has a proper monotone extension from $\operatorname{dom}(S+S^*)$ *to the* 0 *map on* $\ell^2(\mathbb{N})$. *Consequently, $S+S^*$ is not maximally monotone. This supplies a different example for showing that the constraint qualification in the sum problem of maximal monotone operators cannot be substantially weakened, see [63, Example 7.4].*

Svaiter introduced $S^{\vdash}$ in [80], which is defined by

$$\operatorname{gra} S^{\vdash} = \big\{(x,x^*) \in X \times X^* \mid (x^*,x) \in (\operatorname{gra} S)^{\perp}\big\}.$$

Hence $S^{\vdash} = -S^*|_X$.

Definition 3.3.12 *Let $S : X \rightrightarrows X^*$ be skew. We say S is* maximally skew (*termed* "maximal self-cancelling" in [80]) *if no proper enlargement (in the sense of graph inclusion) of S is skew. We say T is* a maximally skew extension *of S if T is maximally skew and* $\operatorname{gra} T \supseteq \operatorname{gra} S$.

Lemma 3.3.13 *Let $S : X \rightrightarrows X^*$ be a maximally monotone skew operator. Then both S and $-S$ are maximally skew.*

Proof. Clearly, S is maximally skew. Now we show $-S$ is maximally skew. Let T be a skew operator such that $\operatorname{gra}(-S) \subseteq \operatorname{gra} T$. Thus, $\operatorname{gra} S \subseteq$

$\operatorname{gra}(-T)$. Since $-T$ is monotone and S is maximally monotone, $\operatorname{gra} S = \operatorname{gra}(-T)$. Then $-S = T$. Hence $-S$ is maximally skew. ■

Fact 3.3.14 (Svaiter) (See [80].) *Let $S : X \rightrightarrows X^*$ be maximally skew. Then either $-S^*|_X$(i.e., $S^{\vdash}$) or $S^*|_X$(i.e., $-S^{\vdash}$) is maximally monotone.*

In [80], Svaiter asked whether or not $-S^*|_X$(i.e., $S^{\vdash}$) is maximally monotone if S is maximally skew. Now we can give a negative answer, even though S is maximally monotone and skew.

Theorem 3.3.15 *Let S be defined as in Example 3.3.1. Then S is maximally skew, but $-S^*$ is not monotone, so not maximally monotone.*

Proof. Let $e_1 = (1, 0, 0, \ldots, 0, \ldots)$. By Proposition 3.3.5, $(e_1, -\frac{1}{2}e_1) \in \operatorname{gra}(-S^*)$, but $\langle e_1, -\frac{1}{2}e_1\rangle = -\frac{1}{2} < 0$. Hence $-S^*$ is not monotone. ■

By Theorem 3.3.15, $-S^*|_X$(i.e., $S^{\vdash}$) is not always maximally monotone. Can one improve Svaiter's result to: "If S is maximally skew, then $S^*|_X$ (i.e., $-S^{\vdash}$) is always maximally monotone"?

Theorem 3.3.16 *There exists a maximally skew operator T on $\ell^2(\mathbb{N})$ such that T^* is not maximally monotone. Consequently, Svaiter's result is optimal.*

Proof. Let $T = -S$, where S be defined as in Example 3.3.1. By Lemma 3.3.13, T is maximally skew. Then by Theorem 3.3.15 and Proposition 3.1.3(x), $T^* = (-S)^* = -S^*$ is not maximally monotone. Hence Svaiter's result cannot be further improved. ■

3.4 The inverse Volterra operator on $L^2[0,1]$

Section 3.4 is all based on the work in [17] by Bauschke, Wang and Yao .

Let V be the Volterra integral operator. In this section, we systematically study $T = V^{-1}$ and its skew part $S = \frac{1}{2}(T - T^*)$. It turns out that T is neither skew nor symmetric and that its skew part S admits two maximally monotone and skew extensions T_1, T_2 (in fact, anti-self-adjoint) even though $\operatorname{dom} S$ is a dense linear subspace of $L^2[0,1]$. This will give another simpler example of Phelps-Simons' showing that the constraint qualification for the sum of monotone operators cannot be significantly weakened, see [78, Theorem 5.5] or [83].

Definition 3.4.1 ([15]) *Let $A : H \rightrightarrows H$ be a linear relation. We say that A is anti-self-adjoint if $A^* = -A$.*

To study the Volterra operator and its inverse, we shall frequently need the following generalized integration-by-parts formula, see [79, Theorem 6.90].

Fact 3.4.2 (Generalized integration by parts) *Assume that x, y are absolutely continuous functions on the interval $[a,b]$. Then*

$$\int_a^b xy' + \int_a^b x'y = x(b)y(b) - x(a)y(a).$$

Fact 3.2.13 allows us to claim the following proposition.

Proposition 3.4.3 *Let $A : H \rightrightarrows H$ be a linear relation. If $A^* = -A$, then both A and $-A$ are maximally monotone and skew.*

Proof. Since $A = -A^*$, we have that $\operatorname{dom} A = \operatorname{dom} A^*$ and that A has closed graph. Now $\forall x \in \operatorname{dom} A$, by Proposition 3.1.3(v),

$$\langle Ax, x\rangle = \langle x, A^*x\rangle = -\langle x, Ax\rangle \quad \Rightarrow \quad \langle Ax, x\rangle = 0.$$

Hence A and $-A$ are skew. As $A^* = -A$ is monotone, Fact 3.2.13 shows that A is maximally monotone.

Now $-A = A^* = -(-A)^*$ and $-A$ is a linear relation. Similar arguments show that $-A$ is maximally monotone. ■

Example 3.4.4 (Volterra operator) *(See [5, Example 3.3].) Set* $H = L^2[0,1]$. *The* Volterra integration operator *[52, Problem 148] is defined by*

$$V\colon H \to H\colon x \mapsto Vx, \quad \text{where} \quad Vx\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_0^t x, \tag{3.23}$$

and its adjoint is given by

$$t \mapsto (V^*x)(t) = \int_t^1 x, \quad \forall x \in X.$$

Then

(i) *Both* V *and* V^* *are maximally monotone since they are monotone, continuous and linear.*

(ii) *Both ranges*

$$\begin{aligned}\operatorname{ran} V = \{x \in L^2[0,1] : \quad & x \text{ is absolutely continuous}, x(0) = 0, \\ & x' \in L^2[0,1]\},\end{aligned} \tag{3.24}$$

and

$$\operatorname{ran} V^* = \{x \in L^2[0,1] : \quad x \textit{ is absolutely continuous}, x(1) = 0,$$
$$x' \in L^2[0,1]\}, \tag{3.25}$$

are dense in $L^2[0,1]$, and both V and V^* are one-to-one.

(iii) $\operatorname{ran} V \cap \operatorname{ran} V^* = \{Vx \ \mid x \in e^{\perp}\}$, *where $e \equiv 1 \in L^2[0,1]$.*

(iv) *Define $V_+ x = \frac{1}{2}(V + V^*)(x) = \frac{1}{2}\langle e, x\rangle e$. Then V_+ is self-adjoint and*

$$\operatorname{ran} V_+ = \operatorname{span}\{e\}.$$

(v) *Define $V_\circ x = \frac{1}{2}(V - V^*)(x) : t \mapsto \frac{1}{2}[\int_0^t x - \int_t^1 x] \quad \forall x \in L^2[0,1]$, $t \in [0,1]$. Then $V_\circ$ is anti-self-adjoint and*

$$\operatorname{ran} V_\circ = \{x \in L^2[0,1] : \ x \textit{ is absolutely continuous on } [0,1], x' \in L^2[0,1],$$
$$x(0) = -x(1)\}.$$

Proof. (i) By Fact 3.4.2,

$$\langle x, Vx\rangle = \int_0^1 x(t) \int_0^t x(s) ds dt = \frac{1}{2}\bigg(\int_0^1 x(s) ds\bigg)^2 \geq 0,$$

so V is monotone.

As $\operatorname{dom} V = L^2[0,1]$ and V is continuous, $\operatorname{dom} V^* = L^2[0,1]$. Let $x, y \in$

$L^2[0,1]$. We have

$$\begin{aligned}\langle Vx, y\rangle &= \int_0^1 \int_0^t x(s)ds y(t)dt = \int_0^1 x(t)dt \int_0^1 y(s)ds - \int_0^1 \int_0^t y(s)dsx(t)dt \\ &= \int_0^1 \left(\int_0^1 y(s)ds - \int_0^t y(s)ds\right)x(t)dt = \int_0^1 \int_t^1 y(s)dsx(t)dt \\ &= \langle V^* y, x\rangle,\end{aligned}$$

thus $(V^*y)(t) = \int_t^1 y(s)ds \ \forall t \in [0,1]$.

(ii) To show (3.24), if $z \in \operatorname{ran} V$, then

$$z(t) = \int_0^t x \quad \text{for some } x \in L^2[0,1],$$

and hence $z(0) = 0$, z is absolutely continuous, and $z' = x \in L^2[0,1]$. On the other hand, if $z(0) = 0$, z is absolutely continuous, $z' \in L^2[0,1]$, then $z = Vz'$.

To show (3.25), if $z \in \operatorname{ran} V^*$, then

$$z(t) = \int_t^1 x \quad \text{for some } x \in L^2[0,1],$$

and hence $z(1) = 0$, z is a absolutely continuous, and $z' = -x \in L^2[0,1]$. On the other hand, if $z(1) = 0$, z is absolutely continuous, $z' \in L^2[0,1]$, then $z = V^*(-z')$.

(iii) follows from (ii) (or see [5]).

(iv) is clear.

(v) If x is absolutely continuous, $x(0) = -x(1)$, $x' \in L^2[0,1]$, we have

$$V_\circ x'(t) = \tfrac{1}{2}\Big(\int_0^t x' - \int_t^1 x'\Big) = \tfrac{1}{2}\Big(x(t) - x(0) - x(1) + x(t)\Big) = x(t).$$

This shows that $x \in \operatorname{ran} V_\circ$. Conversely, if $x \in \operatorname{ran} V_\circ$, i.e.,

$$x(t) = \frac{1}{2}\int_0^t y - \frac{1}{2}\int_t^1 y \quad \text{for some } y \in L^2[0,1],$$

then x is absolutely continuous, $x' = y \in L^2[0,1]$ and $x(0) = -x(1) = -\frac{1}{2}\int_0^1 y$. ■

Theorem 3.4.5 (Inverse Volterra operator) *Let $H = L^2[0,1]$, and V be the Volterra integration operator. We let $T = V^{-1}$ and $D = \operatorname{dom} T \cap \operatorname{dom} T^*$. Then the following hold.*

(i) *$T : \operatorname{dom} T \to X$ is given by $Tx = x'$ with*

$$\begin{aligned}&\operatorname{dom} T\\ &= \{x \in L^2[0,1] : \quad x \textit{ is absolutely continuous}, x(0) = 0, x' \in L^2[0,1]\},\end{aligned}$$

and $T^* : \operatorname{dom} T^* \to L^2[0,1]$ is given by $T^*x = -x'$ with

$$\begin{aligned}&\operatorname{dom} T^*\\ &= \{x \in L^2[0,1] : \quad x \textit{ is absolutely continuous}, x(1) = 0, x' \in L^2[0,1]\}.\end{aligned}$$

Both T and T^* are maximally monotone linear operators.

(ii) *T is neither skew nor symmetric.*

(iii) *The linear subspace*

$$D = \big\{x \in L^2[0,1] : \quad x \text{ is absolutely continuous}, x(0) = x(1) = 0, \\ x' \in L^2[0,1]\big\}$$

is dense in $L^2[0,1]$. Moreover, T and T^* are skew on D.

Proof. (i): T and T^* are maximally monotone because $T = V^{-1}$, and $T^* = (V^{-1})^* = (V^*)^{-1}$ and Example 3.4.4(i). By Example 3.4.4(ii), $T : L^2[0,1] \to L^2[0,1]$ has

$$\begin{aligned}
&\operatorname{dom} T \\
&= \{x \in L^2[0,1] : \quad x \text{ is absolutely continuous}, x(0) = 0, x' \in L^2[0,1]\} \\
&\operatorname{dom} T^* \\
&= \{x \in L^2[0,1] : \quad x \text{ is absolutely continuous}, x(1) = 0, x' \in L^2[0,1]\}
\end{aligned}$$

$Tx = x'$, $\forall x \in \operatorname{dom} T$, $T^*y = -y'$ and $\forall y \in \operatorname{dom} T^*$.

Note that by Fact 3.4.2,

$$\langle Tx, x\rangle = \int_0^1 x'x = \frac{1}{2}x^2(1) - \frac{1}{2}x^2(0) = \frac{1}{2}x(1)^2 \quad \forall x \in \operatorname{dom} T, \tag{3.26}$$

$$\langle T^*x, x\rangle = \int_0^1 -x'x = -(\frac{1}{2}x(1)^2 - \frac{1}{2}x(0)^2) = \frac{1}{2}x(0)^2 \quad \forall x \in \operatorname{dom} T^*. \tag{3.27}$$

(ii): Letting $x(t)=t, y(t)=t^2$ we have

$$\langle Tx,x\rangle = \int_0^1 t = \tfrac{1}{2}, \quad \langle x,Ty\rangle = \int_0^1 2t^2 = \tfrac{2}{3} \neq \tfrac{1}{3} = \int_0^1 t^2 = \langle Tx,y\rangle$$

$$\Rightarrow \langle Tx,x\rangle \neq 0, \langle Tx,y\rangle \neq \langle x,Ty\rangle.$$

(iii): By (i), $D=\operatorname{dom} T \cap \operatorname{dom} T^*$ is clearly a linear subspace. For $x\in D$, $x(0)=x(1)=0$, from (3.26) and (3.27),

$$\langle Tx,x\rangle = \tfrac{1}{2}x(1)^2 = 0, \quad \langle T^*x,x\rangle = \tfrac{1}{2}x(0)^2 = 0.$$

Hence both T and T^* are skew on D. The fact that D is dense in $L^2[0,1]$ follows from [79, Theorem 6.111]. ■

Our proof of (ii), (iii) in the following theorem follows the ideas of [69, Example 13.4].

Theorem 3.4.6 (The skew part of the inverse Volterra operator) *Let H $= L^2[0,1]$, and T be defined as in Theorem 3.4.5. Let $S=\frac{T-T^*}{2}$.*

(i) *$Sx = x'$ ($\forall x \in \operatorname{dom} S$) and $\operatorname{gra} S = \{(Vx,x) \mid x\in e^{\perp}\}$, where $e\equiv 1 \in L^2[0,1]$. In particular,*

$$\begin{aligned}\operatorname{dom} S &= \{x\in L^2[0,1]: \ x \textit{ is absolutely continuous}, x(0)=x(1)=0,\\ &\qquad x'\in L^2[0,1]\},\\ \operatorname{ran} S &= \{y\in L^2[0,1] : \langle e,y\rangle = 0\} = e^{\perp}.\end{aligned}$$

Moreover, $\operatorname{dom} S$ *is dense, and*

$$S^{-1} = V|_{e^\perp}, \quad (-S)^{-1} = V^*|_{e^\perp}, \tag{3.28}$$

consequently, S is skew, and neither S nor $-S$ is maximally monotone.

(ii) *The adjoint of S has* $\operatorname{gra} S^* = \{(V^*x^* + le, x^*) \mid x^* \in L^2[0,1],\ l \in \mathbb{R}\}$. *More precisely,*

$$\begin{aligned} S^*x &= -x' \quad \forall x \in \operatorname{dom} S^*, \ \textit{with} \\ \operatorname{dom} S^* &= \{x \in L^2[0,1] : \ x \textit{ is absolutely continuous on } [0,1], \\ &\quad x' \in L^2[0,1]\}, \\ \operatorname{ran} S^* &= L^2[0,1]. \end{aligned}$$

Neither S^* nor $-S^*$ is monotone. Moreover, $S^{**} = S$.

(iii) *Let* $T_1 : \operatorname{dom} T_1 \to L^2[0,1]$ *be defined by*

$$\begin{aligned} T_1 x &= x', \quad \forall x \in \operatorname{dom} T_1, \ \textit{with} \\ \operatorname{dom} T_1 &= \{x \in L^2[0,1] : \quad x \textit{ is absolutely continuous}, x(0) = x(1), \\ &\quad x' \in L^2[0,1]\}. \end{aligned}$$

Then $T_1^* = -T_1$,

$$\operatorname{ran} T_1 = e^\perp. \tag{3.29}$$

Hence T_1 is skew, and a maximally monotone extension of S; and $-T_1$ is skew and a maximally monotone extension of $-S$.

Proof. (i): By Theorem 3.4.5(iii), we get $\operatorname{dom} S$ directly. Now $(\forall x \in \operatorname{dom} S = \operatorname{dom} T \cap \operatorname{dom} T^*)$ $Tx = x'$ and $T^*x = -x'$, so $Sx = x'$. Then Example 3.4.4(iii) implies $\operatorname{gra} S = \{(Vx, x) \mid x \in e^{\perp}\}$. Hence

$$\operatorname{gra} S^{-1} = \{(x, Vx) \mid x \in e^{\perp}\}. \tag{3.30}$$

Theorem 3.4.5(iii) implies $\operatorname{dom} S$ is dense. Furthermore, $\operatorname{gra}(-S) = \{(Vx, -x) \mid x \in e^{\perp}\}$, so

$$\operatorname{gra}(-S)^{-1} = \{(x, -Vx) \mid x \in e^{\perp}\}. \tag{3.31}$$

Since

$$V^*x(t) = \int_t^1 x - 0 = \int_t^1 x - \int_0^1 x = -\int_0^t x = -Vx(t) \quad \forall t \in [0,1], \forall x \in e^{\perp}$$

we have $-Vx = V^*x, \forall x \in e^{\perp}$. Then by (3.31),

$$\operatorname{gra}(-S)^{-1} = \{(x, V^*x) \mid x \in e^{\perp}\}. \tag{3.32}$$

Hence, (3.30) and (3.32) together establish (3.28). As both V, V^* are maximally monotone with full domain, we conclude that $S^{-1}, (-S)^{-1}$ are not maximally monotone, thus $S, -S$ are not maximally monotone.

(ii): By (i), we have

$$\begin{aligned}&(x, x^*) \in \operatorname{gra} S^* \Leftrightarrow \langle -x, y\rangle + \langle x^*, Vy\rangle = 0, \quad \forall y \in e^{\perp}\\ &\Leftrightarrow \langle -x + V^*x^*, y\rangle = 0, \quad \forall y \in e^{\perp} \Leftrightarrow x - V^*x^* \in \operatorname{span}\{e\}.\end{aligned}$$

Equivalently, $x = V^*x^* + ke$ for some $k \in \mathbb{R}$. This means that x is absolutely continuous, $x^* = -x' \in L^2[0,1]$

On the other hand, if x is absolutely continuous and $x' \in L^2[0,1]$, observe that

$$x(t) = \int_t^1 -x' + x(1)e,$$

so that $x - V^*(-x') \in \operatorname{span}\{e\}$ and $(x, -x') \in \operatorname{gra} S^*$. It follows that

$$\operatorname{dom} S^* = \{x \in L^2[0,1] : \ x \text{ is absolutely continuous on } [0,1], x' \in L^2[0,1]\},$$

$$\operatorname{ran} S^* = L^2[0,1], \qquad \text{and}$$

$$S^*x = -x', \ \forall x \in \operatorname{dom} S^*.$$

Since

$$\langle S^*x, x\rangle = -\int_0^1 x'x = -\left(\frac{1}{2}x(1)^2 - \frac{1}{2}x(0)^2\right),$$

we conclude that neither S^* nor $-S^*$ is monotone.

Now we show $S^{**} = S$. V has closed graph $\Rightarrow V|_{e^\perp}$ has closed graph $\Rightarrow$ S^{-1} has closed graph $\Rightarrow S$ has closed graph $\Rightarrow \operatorname{gra} S = \operatorname{gra} S^{**} \Rightarrow S^{**} = S$.

(iii): To show (3.29), suppose that x is absolutely continuous and that $x(0) = x(1)$. Then

$$\int_0^1 x' = x(1) - x(0) = 0 \quad \Rightarrow T_1 x = x' \in e^\perp.$$

Conversely, if $x \in L^2[0,1]$ satisfies $\langle e, x\rangle = 0$, we define $t \mapsto z(t) = \int_0^t x$, then z is absolutely continuous, $z(0) = z(1)$, $T_1 z = x$. Hence $\operatorname{ran} T_1 = e^\perp$.

T_1 is skew, because for every $x \in \operatorname{dom} T_1$, we have

$$\langle T_1 x, x\rangle = \int_0^1 x'x = \tfrac{1}{2}x(1)^2 - \tfrac{1}{2}x(0)^2 = 0.$$

Moreover, $T_1^* = -T_1$: indeed, as T_1 is skew, by Fact 3.2.12, $\operatorname{gra}(-T_1) \subseteq \operatorname{gra} T_1^*$. To show that $T_1^* = -T_1$, take $z \in \operatorname{dom} T_1^*, \varphi = T_1^* z$. Put $\Phi(t) = \int_0^t \varphi$. We have $\forall y \in \operatorname{dom} T_1$,

$$\int_0^1 y'z = \langle T_1 y, z\rangle = \langle T_1^* z, y\rangle = \langle \varphi, y\rangle = \int_0^1 y\varphi = \int_0^1 y\Phi' \tag{3.33}$$

$$= [\Phi(1)y(1) - \Phi(0)y(0)] - \int_0^1 \Phi y'. \tag{3.34}$$

Using $y = e \in \operatorname{dom} T_1$ gives $\Phi(1) - \Phi(0) = 0$, from which $\Phi(1) = \Phi(0) = 0$. It follows from (3.33)–(3.34) that $\int_0^1 y'(z + \Phi) = 0 \ \forall y \in \operatorname{dom} T_1$. Since $\operatorname{ran} T_1 = e^\perp$, $z + \Phi \in \operatorname{span}\{e\}$, say $z + \Phi = ke$ for some constant $k \in \mathbb{R}$. Then z is absolutely continuous, $z(0) = z(1)$ since $\Phi(0) = \Phi(1) = 0$, and $T_1^* z = \varphi = \Phi' = -z'$. This implies that $\operatorname{dom} T_1^* \subseteq \operatorname{dom} T_1$. Then by Fact 3.2.12, $T_1^* = -T_1$. It remains to apply Proposition 3.4.3. ∎

Remark 3.4.7 *Let S be defined in Theorem 3.4.6. Now we give a new proof to show that $S^{**} = S$ in Theorem 3.4.6 (ii). Applying similar arguments as [42, Example 8.22], one can indeed show that S has a closed graph, so $S^{**} = S$. Or, by [63, Proposition 3.2(e)], S has a closed graph, then $S^{**} = S$.*

Fact 3.4.8 *Let H be a Hilbert space and $A : H \rightrightarrows H$. Then $(-A)^{-1} =$*

$A^{-1} \circ (-\operatorname{Id})$. *If A is a linear relation, then*

$$(-A)^{-1} = -A^{-1}.$$

Proof. This follows from the definition of the set-valued inverse. Indeed, $x \in (-A)^{-1}(x^*) \Leftrightarrow (x, x^*) \in \operatorname{gra}(-A) \Leftrightarrow (x, -x^*) \in \operatorname{gra} A \Leftrightarrow x \in A^{-1}(-x^*)$. When A is a linear relation, $x \in (-A)^{-1}(x^*) \Leftrightarrow (x, -x^*) \in \operatorname{gra} A \Leftrightarrow (-x, x^*) \in \operatorname{gra} A \Leftrightarrow -x \in A^{-1}x^* \Leftrightarrow x \in -A^{-1}(x^*)$. ■

Theorem 3.4.9 (The inverse of the skew part of Volterra operator) *Let $H = L^2[0,1]$, and V be the Volterra integration operator, and $V_\circ : L^2[0,1] \to L^2[0,1]$ be given by*

$$V_\circ = \frac{V - V^*}{2}.$$

Define $T_2 : \operatorname{dom} T_2 \to L^2[0,1]$ by $T_2 = V_\circ^{-1}$. Then

(i) $T_2 x = x', \quad \forall x \in \operatorname{dom} T_2$ *where*

$$\operatorname{dom} T_2 = \{x \in H : \ x \text{ is absolutely continuous on } [0,1], \\ x' \in H, x(0) = -x(1)\}. \tag{3.35}$$

(ii) $T_2^* = -T_2$, *and both $T_2, -T_2$ are maximally monotone and skew.*

Proof. (i): Since

$$V_\circ x(t) = \tfrac{1}{2}\left(\int_0^t x - \int_t^1 x\right),$$

$V_\circ$ is a one-to-one map. Then

$$V_\circ^{-1}\left(\frac{1}{2}(\int_0^t x - \int_t^1 x)\right) = x(t) = \left(\frac{1}{2}(\int_0^t x - \int_t^1 x)\right)',$$

which implies $T_2 x = V_\circ^{-1} x = x'$ for $x \in \operatorname{ran} V_\circ$. As $\operatorname{dom} T_2 = \operatorname{ran} V_\circ$, by Example 3.4.4(v), $\operatorname{ran} V_\circ$ can be written as (3.35).

(ii): Since $\operatorname{dom} V = \operatorname{dom} V^* = L^2[0,1]$, $V_\circ$ is skew on $L^2[0,1]$, so maximally monotone. Then $T_2 = V_\circ^{-1}$ is maximally monotone.

Since $V_\circ$ is skew and $\operatorname{dom} V_\circ = L^2[0,1]$, we have $V_\circ^* = -V_\circ$, by Fact 3.4.8,

$$T_2^* = (V_\circ^{-1})^* = (V_\circ^*)^{-1} = (-V_\circ)^{-1} = -V_\circ^{-1} = -T_2.$$

By Proposition 3.4.3, we have both T_2 and $-T_2$ are maximally monotone and skew. ■

Remark 3.4.10 *Note that while $V_\circ$ is continuous on $L^2[0,1]$, the operator S given in Theorem 3.4.6 is discontinuous.*

Combining Theorem 3.4.5, Theorem 3.4.6 and Theorem 3.4.9, we can summarize the relationships among the differentiation operators encountered in this section.

Corollary 3.4.11 *Let T be defined in Theorem 3.4.5 and S, T_1 be defined in Theorem 3.4.6 and T_2 be defined in Theorem 3.4.9. Then the domain of the skew operator S is dense in $L^2[0,1]$. Neither S nor $-S$ is maximally monotone. Neither S^* nor $-S^*$ is monotone.*

The linear operators S, T, T_1, T_2 satisfy:

$$\operatorname{gra} S \subsetneqq \operatorname{gra} T \subsetneqq \operatorname{gra}(-S^*),$$

$$\operatorname{gra} S \subsetneqq \operatorname{gra} T_1 \subsetneqq \operatorname{gra}(-S^*),$$

$$\operatorname{gra} S \subsetneqq \operatorname{gra} T_2 \subsetneqq \operatorname{gra}(-S^*).$$

While S is skew, T, T_1, T_2 are maximally monotone and T_1, T_2 are skew. Also,

$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}(T^*) \subsetneqq \operatorname{gra} S^*,$$

$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}(-T_1) \subsetneqq \operatorname{gra} S^*,$$

$$\operatorname{gra}(-S) \subsetneqq \operatorname{gra}(-T_2) \subsetneqq \operatorname{gra} S^*.$$

While $-S$ is skew, $T^*, -T_1, -T_2$ are maximally monotone and $-T_1, -T_2$ are skew.

Remark 3.4.12 *(i): Note that while T_1, T_2 are maximally monotone, $-T_1$, $-T_2$ are also maximally monotone. This is in stark contrast with the maximally monotone skew operator given in Proposition 3.3.3 and Proposition 3.3.7 such that its negative is not maximally monotone.*

(ii): Even though the skew operator S in Theorem 3.4.6 has $\operatorname{dom} S$ *dense in $L^2[0,1]$, it still admits two distinct maximally monotone and skew extensions T_1, T_2.*

Example 3.4.13 ($T + T^*$ fails to be maximally monotone) *Let T be defined as in Theorem 3.4.5, and T_1, T_2 be respectively defined in Theo-*

rem 3.4.6 and Theorem 3.4.9. Now $\forall x \in \operatorname{dom} T \cap \operatorname{dom} T^*$*, we have*

$$Tx + T^*x = x' - x' = 0.$$

Thus $T + T^*$ *has a proper monotone extension from* $\operatorname{dom} T \cap \operatorname{dom} T^* \subsetneqq L^2[0,1]$ *to the* 0 *map on* $L^2[0,1]$*. Consequently,* $T + T^*$ *is not maximally monotone. Note that* $\operatorname{dom} T \cap \operatorname{dom} T^*$ *is dense in* $L^2[0,1]$ *and that* $\operatorname{dom} T - \operatorname{dom} T^*$ *is a dense subspace of* $L^2[0,1]$*. This supplies a simpler example for showing that the constraint qualification in the sum problem of maximally monotone operators cannot be substantially weakened, see [63, Example 7.4]. Similarly, by Theorems 3.4.6 and Theorem 3.4.9,* $T_i^* = -T_i$*, we conclude that* $T_i + T_i^* = 0$ *on* $\operatorname{dom} T_i$*, a dense subset of* $L^2[0,1]$*; thus,* $T_i + T_i^*$ *fails to be maximally monotone while both* T_i, T_i^* *are maximally monotone.*

3.5 Discussion

The Brezis-Browder Theorem (see Fact 3.2.13) is a very important characterization of maximal monotonicities of monotone relations. The original proof [30] is based on the application of Zorn's Lemma by constructing a series of finite-dimensional subspaces, which is complicated. In Theorem 3.2.15, we establish the Brezis-Browder Theorem by considering the fact that a lower semicontinuous, convex and coercive function on a reflexive space has at least one minimizer. In [75], Simons generalized the Brezis-Browder Theorem to SSDB spaces. The Brezis-Browder Theorem and Corollary 3.2.6 are essential tools for the construction of maximally monotone linear subspace extensions of a monotone linear relation, which will be discussed in detail in Chapter 4.

There will be an interesting question for the future work on the Brezis-Browder Theorem in a general Banach space:

> *Let* $A\colon X \rightrightarrows X^*$ *be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Assume* $A^*|_X$ *is monotone.*
> *Is* A *necessarily maximally monotone?*

In Sections 3.3 and 3.4, some explicit monotone linear relations were constructed in Hilbert spaces, which gave a negative answer to a question raised by Svaiter [80] and which showed that the constraint qualification in the sum problem for maximally monotone operators cannot be weakened (see [63, Example 7.4]). In particular, these two sections will provide concrete examples for the characterization of decomposable monotone linear relations discussed in Chapter 9.

Chapter 4

Maximally monotone extensions of monotone linear relations

This chapter is based on [88] by Wang and Yao. We consider the linear relation $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:

$$\operatorname{gra} G = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid Ax + Bx^* = 0\} \quad \text{where} \tag{4.1}$$

$$A, B \in \mathbb{R}^{p \times n}, \tag{4.2}$$

$$\operatorname{rank}(A\ B) = p. \tag{4.3}$$

Our main concern is to find explicit extensions of G that are maximally monotone linear relations. Recently, finding constructive maximally monotone extensions, instead of using Zorn's lemma, has been a very active topic [11, 13, 39–41]. In [39], Crouzeix and Ocaña-Anaya gave an algorithm for finding maximally monotone linear subspace extensions of G, but it is not clear what the maximally monotone extensions are analytically. In this chapter, we provide some maximally monotone extensions of G with closed analytical forms. Along the way, we also give a new proof of Crouzeix and Ocaña-Anaya's characterizations on monotonicity and maximal monotonicity of G. Our key tool is the Brezis-Browder characterization of maximally monotone linear relations.

In this chapter, we use the following notation. Counting multiplicities, let

$$\lambda_1, \lambda_2, \ldots, \lambda_k \text{ be all positive eigenvalues of } (AB^\intercal + BA^\intercal) \text{ and} \tag{4.4}$$

$$\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_p \text{ be nonpositive eigenvalues of } (AB^\intercal + BA^\intercal). \tag{4.5}$$

Moreover, let v_i be an eigenvector of eigenvalue λ_i of $(AB^\intercal + BA^\intercal)$ satisfying $\|v_i\| = 1$, and $\langle v_i, v_j \rangle = 0$ for $1 \leq i \neq j \leq q$. It will be convenient to put

$$\mathrm{Id}_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_p \end{pmatrix}, \quad V = [v_1\, v_2\, \ldots\, v_p]\,. \tag{4.6}$$

4.1 Auxiliary results on linear relations

In this section, we collect some facts and preliminary results which will be used in the sequel.

We first provide a result about subspaces on which a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e, an $n \times n$ matrix, is monotone. For $M \in \mathbb{R}^{n\times n}$, define three subspaces of $\mathbb{R}^n$, namely, the positive eigenspace, null eigenspace and negative eigenspace associated with $M + M^\intercal$ by

$$\mathbf{V}_+(M) = \operatorname{span}\left\{\begin{array}{ll} w_1, \ldots, w_s: & w_i \text{ is an eigenvector associated with} \\ & \text{a positive eigenvalue } \alpha_i \text{ of } M + M^\intercal \\ & \langle w_i, w_j \rangle = 0 \ \forall\ i \neq j, \|w_i\| = 1, \\ & i, j = 1, \ldots, s. \end{array}\right\}$$

$$\mathbf{V}_{\mathbf{0}}(M) = \operatorname{span}\left\{\begin{array}{ll} w_{s+1}, \ldots, w_l: & w_i \text{ is an eigenvector associated with} \\ & \text{the 0 eigenvalue of } M + M^\intercal \\ & \langle w_i, w_j \rangle = 0 \ \forall\ i \neq j, \|w_i\| = 1, \\ & i, j = s+1, \ldots, l. \end{array}\right\}$$

$$\mathbf{V}_-(M) = \operatorname{span}\left\{\begin{array}{ll} w_{l+1}, \ldots, w_n: & w_i \text{ is an eigenvector associated with} \\ & \text{a negative eigenvalue } \alpha_i \text{ of } M + M^\intercal \\ & \langle w_i, w_j \rangle = 0 \ \forall\ i \neq j, \|w_i\| = 1, \\ & i, j = l+1, \ldots, n. \end{array}\right\}$$

which is possible since a symmetric matrix always has a complete orthonormal set of eigenvectors, [59, pages 547–549].

Proposition 4.1.1 *Let M be an $n \times n$ matrix. Then*

(i) *M is strictly monotone on $\mathbf{V}_+(M)$. Moreover, $M + M^\intercal : \mathbf{V}_+(M) \to \mathbf{V}_+(M)$ is a bijection.*

(ii) *M is monotone on $\mathbf{V}_+(M) + \mathbf{V}_{\mathbf{0}}(M)$.*

(iii) *$-M$ is strictly monotone on $\mathbf{V}_-(M)$. Moreover, $-(M+M^\intercal) : \mathbf{V}_-(M) \to$*

$\mathbf{V}_{-}(M)$ *is a bijection.*

(iv) $-M$ *is monotone on* $\mathbf{V}_{-}(M) + \mathbf{V_0}(M)$.

(v) *For every* $x \in \mathbf{V_0}(M)$, $(M + M^{\intercal})x = 0$ *and* $\langle x, Mx\rangle = 0$.

In particular, the orthogonal decomposition holds: $\mathbb{R}^n = \mathbf{V}_{+}(M) \oplus \mathbf{V_0}(M) \oplus \mathbf{V}_{-}(M)$.

Proof. (i): Let $x \in \mathbf{V}_{+}(M)$. Then $x = \sum_{i=1}^{s} l_i w_i$ for some $(l_1, \ldots, l_s) \in \mathbb{R}^s$. Since $\{w_1, \cdots, w_s\}$ is a set of orthonormal vectors, they are linearly independent so that

$$x \neq 0 \quad \Leftrightarrow \quad (l_1, \ldots, l_s) \neq 0.$$

Note that $\alpha_i > 0$ when $i = 1, \ldots, s$ and $\langle w_i, w_j\rangle = 0$ for $i \neq j$. We have

$$\begin{aligned} 2\langle x, Mx\rangle = \langle x, (M + M^{\intercal})x\rangle &= \langle\sum_{i=1}^{s} l_i w_i, (M + M^{\intercal})(\sum_{i=1}^{s} l_i w_i)\rangle \\ &= \langle\sum_{i=1}^{s} l_i w_i, \sum_{i=1}^{s} l_i \alpha_i w_i\rangle = \sum_{i=1}^{s} \alpha_i l_i^2 > 0 \end{aligned}$$

if $x \neq 0$.

For every $x \in \mathbf{V}_{+}(M)$ with $x = \sum_{i=1}^{s} l_i w_i$, we have

$$(M + M^{\intercal})x = \sum_{i=1}^{s} l_i (M + M^{\intercal}) w_i = \sum_{i=1}^{s} \alpha_i l_i w_i \in \mathbf{V}_{+}(M).$$

As $\alpha_i > 0$ for $i = 1, \ldots, s$ and $\{w_1, \ldots, w_s\}$ is an orthonormal basis of $\mathbf{V}_{+}(M)$, we conclude that $M + M^{\intercal} : \mathbf{V}_{+}(M) \to \mathbf{V}_{+}(M)$ is a bijection.

(ii): Let $x \in \mathbf{V}_+(M)+\mathbf{V_0}(M)$. Then $x = \sum_{i=1}^{l} l_i w_i$ for some $(l_1, \ldots, l_l) \in \mathbb{R}^l$. Note that $\alpha_i \geq 0$ when $i = 1, \ldots, l$ and $\langle w_i, w_j \rangle = 0$ for $i \neq j$. We have

$$\begin{aligned} 2\langle x, Mx \rangle = \langle x, (M + M^\intercal)x \rangle &= \langle \sum_{i=1}^{l} l_i w_i, (M + M^\intercal)(\sum_{i=1}^{l} l_i w_i) \rangle \\ &= \langle \sum_{i=1}^{l} l_i w_i, \sum_{i=1}^{l} l_i \alpha_i w_i \rangle = \sum_{i=1}^{l} \alpha_i l_i^2 \geq 0. \end{aligned}$$

The proofs for (iii), (iv) are similar to (i), (ii). (v): Obvious. ■

Corollary 4.1.2 *The following hold:*

(i)

$$\operatorname{gra} T = \{(B^\intercal u, A^\intercal u) \mid u \in \mathbf{V}_+(BA^\intercal)\}$$

is strictly monotone.

(ii)

$$\operatorname{gra} T = \{(B^\intercal u, A^\intercal u) \mid u \in \mathbf{V}_+(BA^\intercal) + \mathbf{V_0}(BA^\intercal)\}$$

is monotone.

(iii)

$$\operatorname{gra} T = \{(B^\intercal u, -A^\intercal u) \mid u \in \mathbf{V}_-(BA^\intercal)\}$$

is strictly monotone.

(iv)

$$\operatorname{gra} T = \{(B^\intercal u, -A^\intercal u) \mid u \in \mathbf{V}_-(BA^\intercal) + \mathbf{V_0}(BA^\intercal))\}$$

is monotone.

Proof. As $\langle B^\intercal u, A^\intercal u\rangle = \langle u, BA^\intercal u\rangle\ \forall u \in \mathbb{R}^n$, the result follows from Proposition 4.1.1 by letting $M = BA^\intercal$. ■

Lemma 4.1.3 *For every subspace $S \subseteq \mathbb{R}^p$, the following hold.*

$$\dim\{(B^\intercal u, A^\intercal u) \mid u \in S\} = \dim S. \tag{4.7}$$

$$\dim\{(B^\intercal u, -A^\intercal u) \mid u \in S\} = \dim S. \tag{4.8}$$

Proof. See [59, page 208, Exercise 4.4.9]. ■

The following fact is straightforward from the definition of V.

Fact 4.1.4 *We have*

$$(AB^\intercal + BA^\intercal)V = V\,\mathrm{Id}_\lambda\,.$$

Some basic properties of G are:

Lemma 4.1.5 (i) $\operatorname{gra} G = \ker(A\ \ B)$.

(ii) $G0 = \ker B, G^{-1}(0) = \ker A$.

(iii) $\operatorname{dom} G = P_X(\ker(A\ \ B))$ *and* $\operatorname{ran} G = P_{X^*}(\ker(A\ \ B))$.

(iv) $\operatorname{ran}(G + \mathrm{Id}) = P_{X^*}(\ker(A - B\ \ B)) = P_X(\ker(A\ \ B - A))$, *and*

$$\operatorname{dom} G = P_X(\ker(A - B\ \ B)), \quad \operatorname{ran} G = P_{X^*}(\ker(A\ \ (B - A)).$$

(v) $\dim \operatorname{gra} G = 2n - p$.

Proof. (i), (ii), (iii) follow from the definition of G. Since

$$Ax+Bx^* = 0 \quad \Leftrightarrow \quad (A-B)x+B(x+x^*) = 0 \quad \Leftrightarrow \quad A(x+x^*)+(B-A)x^* = 0,$$

(iv) holds.

(v): We have

$$2n = \dim \ker(A\ B) + \dim \operatorname{ran} \begin{pmatrix} A^\intercal \\ B^\intercal \end{pmatrix} = \dim \operatorname{gra} G + p.$$

Hence $\dim \operatorname{gra} G = 2n - p$. ■

The following result summarizes the monotonicities of G^* and G.

Lemma 4.1.6 *The following hold.*

(i) $\operatorname{gra} G^* = \{(B^\intercal u, -A^\intercal u) \mid u \in \mathbb{R}^p\}$.

(ii) G^* *is monotone* $\Leftrightarrow$ *the matrix* $A^\intercal B+B^\intercal A \in \mathbb{R}^{p\times p}$ *is negative-semidefinite.*

(iii) *Assume* G *is monotone. Then* $n \leq p$. *Moreover,* G *is maximally monotone if and only if* $\dim \operatorname{gra} G = n = p$.

Proof. (i): By Lemma 4.1.5(i), we have

$$(x, x^*) \in \operatorname{gra} G^* \Leftrightarrow (x^*, -x) \in \operatorname{gra} G^\perp = \operatorname{ran} \begin{pmatrix} A^\intercal \\ B^\intercal \end{pmatrix} = \{(A^\intercal u, B^\intercal u) \mid u \in \mathbb{R}^p\}.$$

Thus $\operatorname{gra} G^* = \{(B^\intercal u, -A^\intercal u) \mid u \in \mathbb{R}^p\}$.

(ii): Since $\operatorname{gra} G^*$ is a linear subspace, by (i),

$$
\begin{aligned}
&G^* \text{ is monotone} \Leftrightarrow \langle B^\intercal u, -A^\intercal u\rangle \geq 0, \quad \forall u \in \mathbb{R}^p\\
&\Leftrightarrow \langle u, -BA^\intercal u\rangle \geq 0, \quad \forall u \in \mathbb{R}^p\\
&\Leftrightarrow \langle u, BA^\intercal u\rangle \leq 0, \quad \forall u \in \mathbb{R}^p \Leftrightarrow \langle u, (A^\intercal B + B^\intercal A)u\rangle \leq 0, \quad \forall u \in \mathbb{R}^p\\
&\Leftrightarrow (A^\intercal B + B^\intercal A) \text{ is negative semidefinite.}
\end{aligned}
$$

(iii): By Fact 3.2.6 and Lemma 4.1.5(v), $2n-p = \dim \operatorname{gra} G \leq n \Rightarrow n \leq p$. By Fact 3.2.6 and Lemma 4.1.5(v) again, G is maximally monotone $\Leftrightarrow$ $2n-p = \dim \operatorname{gra} G = n \Leftrightarrow \dim \operatorname{gra} G = p = n$. ■

4.1.1 One linear relation: two equivalent formulations

The linear relation G given by (4.1)–(4.3):

$$\operatorname{gra} G = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid \ Ax + Bx^* = 0\} \tag{4.9}$$

is *an intersection of p linear hyperplanes.* It can be equivalently described as *a span of $q = 2n-p$ points in $\mathbb{R}^n \times \mathbb{R}^n$.* Indeed, for (4.9) we can use Gaussian elimination to reduce $(A\ B)$ to row echelon form. Then back substitute to solve for the basic variables in terms of the free variables, see [59, page 61]. The row-echelon form gives

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = h_1 y_1 + \cdots + h_{2n-p} y_{2n-p} = \begin{pmatrix} C \\ D \end{pmatrix} y$$

where $y \in \mathbb{R}^{2n-p}$ and

$$\begin{pmatrix} C \\ D \end{pmatrix} = (h_1, \ldots, h_{2n-p})$$

with C, D being $n \times (2n-p)$ matrices. Therefore,

$$\operatorname{gra} G = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} \middle| \; y \in \mathbb{R}^{2n-p} \right\} = \operatorname{ran} \begin{pmatrix} C \\ D \end{pmatrix} \tag{4.10}$$

which is a span of $2n-p$ points in $\mathbb{R}^n \times \mathbb{R}^n$. The two formulations (4.9) and (4.10) coincide when $p = q = n$, $\mathrm{Id} = -B = C$ and $D = A$ in which $\mathrm{Id} \in \mathbb{R}^{n \times n}$.

4.2 Explicit maximally monotone extensions of monotone linear relations

In this section, we give explicit maximally monotone linear subspace extensions of G by using $\mathbf{V}_+(AB^\intercal)$ or V_g. A characterization of all maximally monotone extensions of G is also given. We also provide a new proof for Crouzeix and Ocaña-Anaya's characterizations of the monotonicity and the maximal monotonicity of G. We shall use notations given in (4.1)–(4.6), in particular, G is in the form of (4.9).

Lemma 4.2.1 *Let $M \in \mathbb{R}^{p \times p}$, and linear relations $\widetilde{G}$ and $\widehat{G}$ be defined by*

$$\operatorname{gra} \widetilde{G} = \{(x, x^*) \mid M^\intercal A x + M^\intercal B x^* = 0\}$$
$$\operatorname{gra} \widehat{G} = \{(B^\intercal u, -A^\intercal u) \mid \; u \in \operatorname{ran} M\}.$$

Then $(\widetilde{G})^* = \widehat{G}$.

Proof. Let $(y, y^*) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we have

$$
\begin{aligned}
&(y, y^*) \in \operatorname{gra}(\widetilde{G})^* \\
&\Leftrightarrow (y^*, -y) \in (\operatorname{gra}\widetilde{G})^\perp = \left(\ker \begin{pmatrix} M^\intercal A & M^\intercal B \end{pmatrix}\right)^\perp = \operatorname{ran} \begin{pmatrix} A^\intercal M \\ B^\intercal M \end{pmatrix} \\
&\Leftrightarrow (y, y^*) \in \operatorname{gra}\widehat{G}.
\end{aligned}
$$

Hence $(\widetilde{G})^* = \widehat{G}$. ■

Lemma 4.2.2 *Define linear relations* $\widetilde{G}$ *and* $\widehat{G}$ *by*

$$
\begin{aligned}
\operatorname{gra}\widetilde{G} &= \{(x, x^*) \mid V_g A x + V_g B x^* = 0\} \\
\operatorname{gra}\widehat{G} &= \{(B^\intercal u, -A^\intercal u) \mid \ u \in \mathbf{V}_-(BA^\intercal) + \mathbf{V_0}(BA^\intercal)\},
\end{aligned}
$$

where V_g *is* $(p-k) \times p$ *matrix defined by*

$$
V_g = \begin{pmatrix} v_{k+1}^\intercal \\ v_{k+2}^\intercal \\ \vdots \\ v_p^\intercal \end{pmatrix}.
$$

Then

(i) $\widehat{G}$ *is monotone.*

(ii) $(\widehat{G})^* = \widetilde{G}$.

(iii) $\operatorname{gra}\widetilde{G} = \operatorname{gra} G + \left\{ \begin{pmatrix} B^{\intercal} \\ A^{\intercal} \end{pmatrix} u \;\middle|\; u \in \mathbf{V}_{+}(BA^{\intercal}) \right\}.$

Proof. (i): Apply Corollary 4.1.2(iv).

(ii): Notations are as in (4.6). Define the $p \times p$ matrix N by

$$N = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{Id} \end{pmatrix}$$

in which $\mathrm{Id} \in \mathbb{R}^{(p-k)\times(p-k)}$. Then we have

$$N^{\intercal}V^{\intercal} = \left((v_1 \ \cdots v_k \ \ V_g^{\intercal}) \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{Id} \end{pmatrix} \right)^{\intercal} = \begin{pmatrix} 0 \\ V_g \end{pmatrix}. \tag{4.11}$$

Then we have

$$\begin{aligned} V_g Ax + V_g Bx^* = 0 \Leftrightarrow \begin{pmatrix} 0 \\ V_g Ax + V_g Bx^* \end{pmatrix} = 0 \\ \Leftrightarrow N^{\intercal}V^{\intercal}Ax + N^{\intercal}V^{\intercal}Bx^* = 0, \quad \forall (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n. \end{aligned}$$

Hence

$$\operatorname{gra}\widetilde{G} = \{(x, x^*) \mid N^{\intercal}V^{\intercal}Ax + N^{\intercal}V^{\intercal}Bx^* = 0\}.$$

Thus by Lemma 4.2.1 with $M = VN$,

$$\operatorname{gra}(\widetilde{G})^* = \{(B^{\intercal}u, -A^{\intercal}u) \mid u \in \operatorname{ran} VN = \operatorname{ran} \begin{pmatrix} 0 & V_g^{\intercal} \end{pmatrix}$$

$$= \mathbf{V}_{-}(BA^{\intercal}) + \mathbf{V}_{\mathbf{0}}(BA^{\intercal})\} = \operatorname{gra} \widehat{G}.$$

Hence $(\widehat{G})^{*} = (\widetilde{G})^{**} = \widetilde{G}$.

(iii): Let J be defined by

$$\operatorname{gra} J = \operatorname{gra} G + \left\{ \begin{pmatrix} B^{\intercal} \\ A^{\intercal} \end{pmatrix} u \,\middle|\, u \in \mathbf{V}_{+}(BA^{\intercal}) \right\}.$$

Then we have

$$(\operatorname{gra} J)^{\perp} = (\operatorname{gra} G)^{\perp} \cap \left\{ \begin{pmatrix} B^{\intercal} \\ A^{\intercal} \end{pmatrix} u \,\middle|\, u \in \mathbf{V}_{+}(BA^{\intercal}) \right\}^{\perp}.$$

By Lemma 4.1.5(i),

$$\operatorname{gra} G^{\perp} = \left\{ \begin{pmatrix} A^{\intercal} \\ B^{\intercal} \end{pmatrix} w \,\middle|\, w \in \mathbb{R}^{p} \right\}$$

Then

$$\begin{pmatrix} A^{\intercal} \\ B^{\intercal} \end{pmatrix} w \in \left\{ \begin{pmatrix} B^{\intercal} \\ A^{\intercal} \end{pmatrix} u \,\middle|\, u \in \mathbf{V}_{+}(BA^{\intercal}) \right\}^{\perp}$$

if and only if

$$\langle (A^{\intercal}w, B^{\intercal}w), (B^{\intercal}u, A^{\intercal}u) \rangle = 0 \quad \forall\, u \in \mathbf{V}_{+}(BA^{\intercal}),$$

that is,

$$\langle A^\intercal w, B^\intercal u\rangle + \langle B^\intercal w, A^\intercal u\rangle = \langle w, (AB^\intercal + BA^\intercal)u\rangle = 0 \quad \forall u \in \mathbf{V}_+(AB^\intercal). \tag{4.12}$$

Because $AB^\intercal + BA^\intercal : \mathbf{V}_+(AB^\intercal) \mapsto \mathbf{V}_+(AB^\intercal)$ is onto by Proposition 4.1.1(i), we obtain that (4.12) holds if and only if $w \in \mathbf{V}_-(AB^\intercal) + \mathbf{V}_\mathbf{0}(AB^\intercal)$. Hence

$$(\operatorname{gra} J)^\perp = \{(A^\intercal w, B^\intercal w) \mid \ w \in \mathbf{V}_-(BA^\intercal) + \mathbf{V}_\mathbf{0}(BA^\intercal)\},$$

from which $\operatorname{gra} J^* = \operatorname{gra} \widehat{G}$. Then by (i),

$$\operatorname{gra} \widetilde{G} = \operatorname{gra}(\widehat{G})^* = \operatorname{gra} J^{**} = \operatorname{gra} J.$$

■

We are ready to apply the Brezis-Browder Theorem, namely Fact 3.2.13, to improve Crouzeix and Ocaña-Anaya's characterizations of monotonicity and maximal monotonicity of G and provide a different proof.

Theorem 4.2.3 *Let $\widehat{G}, \widetilde{G}$ be defined in Lemma 4.2.2. The following are equivalent:*

(i) *G is monotone;*

(ii) *$\widetilde{G}$ is monotone;*

(iii) *$\widetilde{G}$ is maximally monotone;*

(iv) *$\widehat{G}$ is maximally monotone;*

(v) $\dim \mathbf{V}_+(BA^\intercal) = p - n$, *equivalently,* $AB^\intercal + BA^\intercal$ *has exactly* $p - n$ *positive eigenvalues (counting multiplicity).*

Proof. (i)⇔(ii): Lemma 4.2.2(iii) and Corollary 4.1.2(i).

(ii)⇔(iii)⇔(iv): Note that $\widetilde{G} = \big(\widehat{G}\big)^*$ and $\widehat{G}$ is always a monotone linear relation by Corollary 4.1.2(iv). It suffices to combine Lemma 4.2.2 and Fact 3.2.13.

(i)⇒(v): Assume that G is monotone. Then $\widetilde{G}$ is monotone by Lemma 4.2.2(iii) and Corollary 4.1.2(i). By Lemma 4.2.2(ii), Corollary 4.1.2(iv) and Fact 3.2.13, $\widehat{G}$ is maximally monotone, so that $\dim(\operatorname{gra}\widehat{G}) = p-k = n$ by Fact 3.2.6 and Lemma 4.1.3, thus $k = p-n$. Note that for each eigenvalue of a symmetric matrix, its geometric multiplicity is the same as its algebraic multiplicity [59, page 512].

(v)⇒(i): Assume that $k = p - n$. Then $\dim(\operatorname{gra}\widehat{G}) = p - k = n$ by Lemma 4.1.3, so that $\widehat{G}$ is maximally monotone by Fact 3.2.6(i)(ii). By Lemma 4.2.2(ii) and Fact 3.2.13, $\widetilde{G}$ is monotone, which implies that G is monotone. ■

Corollary 4.2.4 *Assume that G is monotone. Then*

$$\begin{aligned}\operatorname{gra}\widetilde{G} &= \operatorname{gra} G + \left\{ \begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} u \;\middle|\; u \in \mathbf{V}_+(BA^\intercal\rangle \right\}\\ &= \{(x, x^*) \mid V_g Ax + V_g Bx^* = 0\}\end{aligned}$$

is a maximally monotone extension of G, where

$$V_g = \begin{pmatrix} v_{p-n+1}^{\intercal} \\ v_{p-n+2}^{\intercal} \\ \vdots \\ v_p^{\intercal} \end{pmatrix}.$$

Proof. Combine Theorem 4.2.3 and Lemma 4.2.2(iii) directly. ∎

Note that Corollary 4.2.4 gives both types of maximally monotone extensions of G, namely, type (4.9) and type (4.10). A remark is in order to compare our extension with the one by Crouzeix and Ocaña-Anaya.

Remark 4.2.5 *(i). Crouzeix and Ocaña-Anaya [39] defines the union of monotone extension of G as*

$$S = \operatorname{gra} G + \left\{ \begin{pmatrix} B^{\intercal} \\ A^{\intercal} \end{pmatrix} u \,\middle|\, u \in K \right\},$$

where $K = \{u \in \mathbb{R}^n \mid \langle u, (AB^{\intercal} + BA^{\intercal})u\rangle \geq 0\}$. Although this is the set monotonically related to G, it is not monotone in general as long as $(AB^{\intercal}+BA^{\intercal})$ has both positive eigenvalues and negative eigenvalues. Indeed, let (α_1, u_1) and (α_2, u_2) be eigen-pairs of $(AB^{\intercal} + BA^{\intercal})$ with $\alpha_1 > 0$ and $\alpha_2 < 0$. We have

$$\langle u_1, (AB^{\intercal}+BA^{\intercal})u_1\rangle = \alpha_1\|u_1\|^2 > 0, \quad \langle u_2, (AB^{\intercal}+BA^{\intercal})u_2\rangle = \alpha_2\|u_2\|^2 < 0.$$

Choose $\epsilon > 0$ sufficiently small so that

$$\langle u_1 + \epsilon u_2, (AB^\intercal + BA^\intercal)(u_1 + \epsilon u_2)\rangle > 0.$$

Then

$$\begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} u_1, \begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} (u_1 + \epsilon u_2) \in S.$$

However,

$$\begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} (u_1 + \epsilon u_2) - \begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} u_1 = \epsilon \begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} u_2$$

has

$$\langle \epsilon B^\intercal u_2, \epsilon A^\intercal u_2\rangle = \epsilon^2 \langle u_2, BA^\intercal u_2\rangle = \epsilon^2 \frac{\langle u_2, (AB^\intercal + BA^\intercal)u_2\rangle}{2} < 0.$$

Therefore S is not monotone. By using $\mathbf{V}_+(BA^\intercal) \subseteq K$, we have obtained a maximally monotone extension of G .

(ii). Crouzeix and Ocaña-Anaya [39] find a maximally monotone linear subspace extension of G algorithmically by using $\tilde{u}_k \in \operatorname{gra} \widetilde{G_k} \setminus \operatorname{gra} G_k$ and constructing $\operatorname{gra} G_{k+1} = \operatorname{gra} G_k + \mathbb{R}\tilde{u}_k$ where

$$\tilde{u}_k = \begin{pmatrix} B_k^\intercal \\ A_k^\intercal \end{pmatrix} u_k, \quad \langle u_k, (A_k B_k^\intercal + B_k A_k^\intercal) u_k\rangle \geq 0.$$

This recursion is done until $\dim \operatorname{gra} G_k = n$. In particular, each u_k may be chosen as an eigenvector associated with a positive eigenvalue of $A_k B_k^\intercal + B_k A_k^\intercal$, which is possible since $p > n$ when G_k is not maximally monotone.

Their construction uses both formulations, namely, (4.9) *and* (4.10)*. No computation indications are given on the passage from one formulation to the other one.*

The following result extends the characterization of maximally monotone linear relations given by Crouzeix and Ocaña-Anaya [39].

Theorem 4.2.6 *Let $\widehat{G}, \widetilde{G}$ be defined in Lemma 4.2.2. The following are equivalent:*

(i) *G is maximally monotone;*

(ii) *$p = n$ and G is monotone;*

(iii) *$p = n$ and $AB^\intercal + BA^\intercal$ is negative semidefinite.*

(iv) *$p = n$ and $\widehat{G}$ is maximally monotone.*

Proof. (i)⇒(ii): Apply Lemma 4.1.6(iii).

(ii)⇒(iii): Apply Theorem 4.2.3(i)(v) directly .

(iii)⇒(i): Assume that $p = n$ and $(AB^\intercal + BA^\intercal)$ is negative semidefinite. Then $k = 0$ and $\widetilde{G} = G$. It follows that $\dim(\operatorname{gra}\widehat{G}) = p - k = n$ by Lemma 4.1.3, so that $\widehat{G}$ is maximally monotone by Corollary 4.1.2(iv) and Fact 3.2.6(i)(ii). Since $\left(\widehat{G}\right)^* = \widetilde{G}$ by Lemma 4.2.2(ii), Fact 3.2.13 gives that $\widetilde{G} = G$ is maximally monotone.

(iii)⇒(iv): Assume that $p = n$ and $(AB^\intercal + BA^\intercal)$ is negative semidefinite. We have $k = 0$ and $\dim(\operatorname{gra}\widehat{G}) = p - k = n - 0 = n$. Hence (iv) holds by Corollary 4.1.2(iv) and Fact 3.2.6(i)(ii).

(iv)⇒(iii): Assume that $\widehat{G}$ is maximally monotone and $p = n$. We have $\dim(\operatorname{gra}\widehat{G}) = p - k = n - k = n$ so that $k = 0$. Hence $(AB^\intercal + BA^\intercal)$ is negative semidefinite. ■

Corollary 4.2.4 supplies only one maximally monotone linear subspace extension of G. Can we find all of them? Surprisingly, we may give a characterization of all the maximally monotone linear subspace extensions of G when it is given in the form of (4.9).

Theorem 4.2.7 *Let G be monotone. Then $\widetilde{G}$ is a maximally monotone extension of G if and only if there exists $N \in \mathbb{R}^{p\times p}$ with rank of n such that $N^\intercal \operatorname{Id}_\lambda N$ is negative semidefinite and*

$$\operatorname{gra}\widetilde{G} = \{(x, x^*) \mid N^\intercal V^\intercal Ax + N^\intercal V^\intercal Bx^* = 0\}. \tag{4.13}$$

Proof. "⇒": By Lemma 4.1.6(i), we have

$$\operatorname{gra} G^* = \{(B^\intercal u, -A^\intercal u) \mid u \in \mathbb{R}^p\}. \tag{4.14}$$

Since $\operatorname{gra} G \subseteq \operatorname{gra}\widetilde{G}$ and thus $\operatorname{gra}(\widetilde{G})^*$ is a subspace of $\operatorname{gra} G^*$.

Thus by (4.14), there exists a subspace F of $\mathbb{R}^p$ such that

$$\operatorname{gra}(\widetilde{G})^* = \{(B^\intercal u, -A^\intercal u) \mid u \in F\}. \tag{4.15}$$

By Fact 3.2.13, Fact 3.2.6 and Lemma 4.1.3, we have

$$\dim F = n. \tag{4.16}$$

Thus, there exists $N \in \mathbb{R}^{p\times p}$ with rank n such that $\operatorname{ran} VN = F$ and

$$\operatorname{gra}(\widetilde{G})^* = \{(B^\intercal VNy, -A^\intercal VNy) \mid y \in \mathbb{R}^p\}. \tag{4.17}$$

As $\widetilde{G}$ is maximally monotone, $(\widetilde{G})^*$ is maximally monotone by Fact 3.2.13, so

$$N^\intercal V^\intercal (BA^\intercal + AB^\intercal)VN \text{ is negative semidefinite.}$$

Using Fact 4.1.4, we have

$$N^\intercal \operatorname{Id}_\lambda N = N^\intercal V^\intercal V \operatorname{Id}_\lambda N = N^\intercal V^\intercal (AB^\intercal + BA^\intercal)VN \tag{4.18}$$

which is negative semidefinite. (4.13) follows from (4.17) by Lemma 4.2.1 using $M = VN$.

"$\Leftarrow$": By Lemma 4.2.1, we have

$$\operatorname{gra}(\widetilde{G})^* = \{(B^\intercal VNu, -A^\intercal VNu) \mid u \in \mathbb{R}^p\}. \tag{4.19}$$

Observe that $(\widetilde{G})^*$ is monotone because $N^\intercal V^\intercal (AB^\intercal + BA^\intercal)VN = N^\intercal \operatorname{Id}_\lambda N$ is negative semidefinite by Fact 4.1.4 and the assumption. As $\operatorname{rank}(VN) = n$, it follows from (4.19) and Lemma 4.1.3 that $\dim \operatorname{gra}(\widetilde{G})^* = n$. Therefore $(\widetilde{G})^*$ is maximally monotone by Fact 3.2.6. Applying Fact 3.2.13 for $T = (\widetilde{G})^*$ yields that $\widetilde{G} = (\widetilde{G})^{**}$ is maximally monotone. ■

From the above proof, we see that to find a maximally monotone extension of G one essentially need to find subspace $F \subseteq \mathbb{R}^p$ such that $\dim F = n$ and $AB^\intercal + BA^\intercal$ is negative semidefinite on F. If $F = \operatorname{ran} M$ and $M \in \mathbb{R}^{p\times p}$

with $\operatorname{rank} M = n$, one can let $N = V^\intercal M$. The maximally monotone linear subspace extension of G is

$$\widetilde{G} = \{(x, x^*) \mid M^\intercal Ax + M^\intercal Bx^* = 0\}.$$

In Corollary 4.2.4, one can choose $M = \begin{pmatrix} \underbrace{0\ 0\ \cdots 0}_{n} & v_{p-n+1} & \cdots & v_p \end{pmatrix}$.

Corollary 4.2.8 *Let G be monotone. Then $\widetilde{G}$ is a maximally monotone extension of G if and only if there exists $M \in \mathbb{R}^{p\times p}$ with rank of n such that $M^\intercal(AB^\intercal + BA^\intercal)M$ is negative semidefinite and*

$$\operatorname{gra}\widetilde{G} = \{(x, x^*) \mid M^\intercal Ax + M^\intercal Bx^* = 0\}. \tag{4.20}$$

Note that G may have different representations in terms of A, B. The maximally monotone extension of $\widetilde{G}$ given in Theorem 4.2.7 and Corollary 4.2.4 relies on A, B matrices and N. This might lead to different maximally monotone extensions, see Section 4.5.

Remark 4.2.9 *A referee for the paper [88] pointed out that there is a shorter way to see Theorem 4.2.7. Consider the maximally monotone linear subspace extension of G of type:*

$$\operatorname{gra}\widetilde{G} = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid \widetilde{A}x + \widetilde{B}x^* = 0\} \supseteq \operatorname{gra} G$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{R}^{n\times n}$. With the nonsingular $p \times p$ matrix V given as in (4.6), *an equivalent formulation of G is*

$$\operatorname{gra} G = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid V^\intercal A x + V^\intercal B x^* = 0\}.$$

As $\widetilde{G}$ is maximally monotone, the $n \times 2n$ matrix has $\operatorname{rank}(\widetilde{A}, \widetilde{B}) = n$ *and the matrix*

$$\widetilde{A}\widetilde{B}^\intercal + \widetilde{B}\widetilde{A}^\intercal \in \mathbb{R}^{n\times n}$$

is negative semidefinite. Since $\operatorname{gra}\widetilde{G} \supseteq \operatorname{gra} G$*, we have*

$$\operatorname{ran}\begin{pmatrix} \widetilde{A}^\intercal \\ \widetilde{B}^\intercal \end{pmatrix} = (\operatorname{gra}\widetilde{G})^\perp \subseteq (\operatorname{gra} G)^\perp = \operatorname{ran}\begin{pmatrix} (V^\intercal A)^\intercal \\ (V^\intercal B)^\intercal \end{pmatrix}.$$

Therefore, there exists a $p \times n$ matrix N with $\operatorname{rank} N = n$ *such that*

$$\begin{pmatrix} \widetilde{A}^\intercal \\ \widetilde{B}^\intercal \end{pmatrix} = \begin{pmatrix} (V^\intercal A)^\intercal \\ (V^\intercal B)^\intercal \end{pmatrix} N = \begin{pmatrix} (V^\intercal A)^\intercal N \\ (V^\intercal B)^\intercal N \end{pmatrix}$$

from which $\widetilde{A} = N^\intercal V^\intercal A, \widetilde{B} = N^\intercal V^\intercal B$. Then the $n \times n$ matrix

$$\begin{aligned} \widetilde{A}\widetilde{B}^\intercal + \widetilde{B}\widetilde{A}^\intercal &= N^\intercal V^\intercal A (N^\intercal V^\intercal B)^\intercal + N^\intercal V^\intercal B (N^\intercal V^\intercal A)^\intercal && (4.21)\\ &= N^\intercal V^\intercal (AB^\intercal + BA^\intercal) V N && (4.22)\\ &= N^\intercal \operatorname{Id}_\lambda N. && (4.23) \end{aligned}$$

Therefore, all maximally monotone linear subspace extensions of G can be *obtained by using*

$$\operatorname{gra}\widetilde{G} = \{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid N^\intercal V^\intercal A x + N^\intercal V^\intercal B x^* = 0\}$$

in which the $p \times n$ *matrix* N *satisfies* $\operatorname{rank} N = n$ *and* $N^\intercal \operatorname{Id}_\lambda N$ *is negative semidefinite.*

4.3 Minty parameterizations

Although G is set-valued in general, when G is monotone it has an elegant Minty parametrization in terms of A, B, which is what we are going to show in this section.

Lemma 4.3.1 *The linear relation* G *is monotone if and only if*

$$\|y\|^2 - \|y^*\|^2 \geq 0, \text{ whenever} \tag{4.24}$$

$$(A + B)y + (B - A)y^* = 0. \tag{4.25}$$

Consequently, if G *is monotone then the* $p \times n$ *matrix* $B - A$ *must have full column rank, namely* n*.*

Proof. Define the $2n \times 2n$ matrix

$$P = \begin{pmatrix} 0 & \operatorname{Id} \\ \operatorname{Id} & 0 \end{pmatrix}$$

where $\mathrm{Id} \in \mathbb{R}^{n\times n}$. It is easy to see that G is monotone if and only if

$$\left\langle (x, x^*), P \begin{pmatrix} x \\ x^* \end{pmatrix} \right\rangle \geq 0,$$

whenever $Ax + Bx^* = 0$. Define the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

and put

$$\begin{pmatrix} x \\ x^* \end{pmatrix} = Q \begin{pmatrix} y \\ y^* \end{pmatrix}.$$

Then G is monotone if and only if

$$\|y\|^2 - \|y^*\|^2 \geq 0, \text{ whenever} \tag{4.26}$$

$$(A + B)y + (B - A)y^* = 0. \tag{4.27}$$

If $(B - A)$ does not have full column rank, then there exists $y^* \neq 0$ such that $(B - A)y^* = 0$. Then $(0, y^*)$ satisfies (4.27) but (4.26) fails. Therefore, $B - A$ has to be full column rank. ∎

Theorem 4.3.2 (Minty parametrization) *Assume that G is a monotone operator. Then $(x, x^*) \in \operatorname{gra} G$ if and only if*

$$x = \frac{1}{2}[\mathrm{Id} + (B - A)^\dagger (B + A)]y \tag{4.28}$$

$$x^* = \frac{1}{2}[\mathrm{Id} - (B - A)^\dagger (B + A)]y \tag{4.29}$$

for $y = x + x^* \in \operatorname{ran}(\operatorname{Id} + G)$. *Here the Moore-Penrose inverse* $(B-A)^\dagger = [(B-A)^\intercal(B-A)]^{-1}(B-A)^\intercal$. *In particular, when* G *is maximally monotone, we have*

$$\operatorname{gra} G = \{((B-A)^{-1}By, -(B-A)^{-1}Ay) \mid y \in \mathbb{R}^n\}.$$

Proof. As $(B-A)$ is full column rank, $(B-A)^\intercal(B-A)$ is invertible. It follows from (4.25) that $(B-A)^\intercal(A+B)y + (B-A)^\intercal(B-A)y^* = 0$ so that

$$y^* = -((B-A)^\intercal(B-A))^{-1}(B-A)^\intercal(A+B)y = -(B-A)^\dagger(A+B)y.$$

Then

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(y - y^*) = \frac{1}{\sqrt{2}}[\operatorname{Id} + (B-A)^\dagger(B+A)]y \\ x^* &= \frac{1}{\sqrt{2}}(y + y^*) = \frac{1}{\sqrt{2}}[\operatorname{Id} - (B-A)^\dagger(B+A)]y \end{aligned}$$

where $y = \frac{x+x^*}{\sqrt{2}}$ with $(x, x^*) \in \operatorname{gra} G$. Since $\operatorname{ran}(\operatorname{Id} + G)$ is a subspace, we have

$$\begin{aligned} x &= \frac{1}{2}[\operatorname{Id} + (B-A)^\dagger(B+A)]\tilde{y} \\ x^* &= \frac{1}{2}[\operatorname{Id} - (B-A)^\dagger(B+A)]\tilde{y} \end{aligned}$$

with $\tilde{y} = x + x^* \in \operatorname{ran}(\operatorname{Id} + G)$.

If G is maximally monotone, then $p = n$ by Theorem 4.2.6 and hence $B - A$ is invertible, thus $(B-A)^\dagger = (B-A)^{-1}$. Moreover, $\operatorname{ran}(G + \operatorname{Id}) = \mathbb{R}^n$.

Then (4.28) and (4.29) imply that

$$x = \frac{1}{2}(B-A)^{-1}[B-A+(B+A)]y = (B-A)^{-1}By \tag{4.30}$$

$$x^* = \frac{1}{2}(B-A)^{-1}[(B-A)-(B+A)]y = -(B-A)^{-1}Ay \tag{4.31}$$

for $y \in \mathbb{R}^n$. ■

Remark 4.3.3 *See Lemma 4.1.5 for* $\operatorname{ran}(G+\operatorname{Id})$. *Note that as* G *is a monotone linear relation, the mapping*

$$z \mapsto ((G+\operatorname{Id})^{-1}, \operatorname{Id}-(G+\operatorname{Id})^{-1})(z)$$

is bijective and linear from $\operatorname{ran}(G+\operatorname{Id})$ *to* $\operatorname{gra} G$, *therefore* $\dim(\operatorname{ran}(G+\operatorname{Id})) = \dim(\operatorname{gra} G)$.

Corollary 4.3.4 *Let* G *be a monotone operator. Then* $\widetilde{G}$ *defined in Corollary 4.2.4, the maximally monotone extension of* G, *has its Minty parametrization given by*

$$\operatorname{gra}\widetilde{G} = \{((V_gB - V_gA)^{-1}V_gBy, -(V_gB - V_gA)^{-1}V_gAy) \mid y \in \mathbb{R}^n\}$$

where V_g *is given as in Corollary 4.2.4.*

Proof. Since $\operatorname{rank}(V_g) = n$ and $\operatorname{rank}(A\ B) = p$, by Lemma 4.1.3(4.7), $\operatorname{rank}(V_gA\ V_gB) = n$. Then we can apply Corollary 4.2.4 and Theorem 4.3.2 directly. ■

Corollary 4.3.5 *When G is maximally monotone,*

$$\operatorname{dom} G = (B-A)^{-1}(\operatorname{ran} B), \quad \operatorname{ran} G = (B-A)^{-1}(\operatorname{ran} A).$$

Recall that $T : \mathbb{R}^n \to \mathbb{R}^n$ is *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \quad \forall\, x, y \in \operatorname{dom} T.$$

In terms of matrices, we have

Corollary 4.3.6 *Suppose that $p = n$, $AB^{\intercal} + BA^{\intercal}$ is negative semidefinite. Then $(B-A)^{-1}B$ and $-(B-A)^{-1}A$ are firmly nonexpansive.*

Proof. By Theorem 4.2.6, G is maximally monotone. Theorem 4.3.2 gives that

$$(B-A)^{-1}B = (\operatorname{Id} + G)^{-1}, \quad -(B-A)^{-1}A = (\operatorname{Id} + G^{-1})^{-1}.$$

Being resolvents of monotone operators G, G^{-1}, they are firmly nonexpansive, see [9, 43] or [13, Fact 2.5]. ■

4.4 Maximally monotone extensions with the same domain or the same range

How do we find maximally monotone linear subspace extensions of G if it is given in the form of (4.10)? The purpose of this section is to find maximally monotone linear subspace extensions of G which keep either $\operatorname{dom} G$ or $\operatorname{ran} G$

unchanged. For a closed convex set $S \subseteq \mathbb{R}^n$, let N_S denote its normal cone mapping.

Proposition 4.4.1 *Assume that $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a monotone linear relation. Then*

(i) $T_1 = T + N_{\operatorname{dom} T}$, *i.e.,*

$$x \mapsto T_1 x = \begin{cases} Tx + (\operatorname{dom} T)^{\perp} & \text{if } x \in \operatorname{dom} T \\ \emptyset & \text{otherwise} \end{cases}$$

is maximally monotone. In particular, $\operatorname{dom} T_1 = \operatorname{dom} T$.

(ii) $T_2 = (T^{-1} + N_{\operatorname{ran} T})^{-1}$ *is a maximally monotone extension of T and* $\operatorname{ran} T_2 = \operatorname{ran} T$.

Proof. (i): Since $0 \in T0 \subseteq (\operatorname{dom} T)^{\perp}$ by [15, Proposition 2.2(i)], we have $T_1 0 = T0 + (\operatorname{dom} T)^{\perp} = (\operatorname{dom} T)^{\perp}$ so that $\operatorname{dom} T_1 = \operatorname{dom} T = (T_1 0)^{\perp}$. Hence T_1 is maximally monotone by Fact 3.2.6.

(ii): Apply (i) to T^{-1} to see that $T^{-1} + N_{\operatorname{ran} T}$ is a maximally monotone extension of T^{-1} with $\operatorname{dom}(T^{-1} + N_{\operatorname{ran} T}) = \operatorname{ran} T$. Therefore, T_2 is a maximally monotone extension of T with $\operatorname{ran} T_2 = \operatorname{ran} T$. ■

Define linear relations $E_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ $(i = 1, 2)$ by

$$\operatorname{gra} E_1 = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} + \begin{pmatrix} 0 \\ (\operatorname{ran} C)^{\perp} \end{pmatrix} \,\middle|\, y \in \mathbb{R}^{2n-p} \right\}, \tag{4.32}$$

$$\operatorname{gra} E_2 = \left\{ \begin{pmatrix} Cy \\ Dy \end{pmatrix} + \begin{pmatrix} (\operatorname{ran} D)^{\perp} \\ 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R}^{2n-p} \right\}. \tag{4.33}$$

Theorem 4.4.2 (i) *E_1 is a maximally monotone extension of G with* $\operatorname{dom} E_1 = \operatorname{dom} G$. *Moreover,*

$$\operatorname{gra} E_1 = \operatorname{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ (\operatorname{ran} C)^{\perp} \end{pmatrix} = \operatorname{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ \ker C^{\intercal} \end{pmatrix}. \tag{4.34}$$

(ii) *E_2 is a maximally monotone extension of G with* $\operatorname{ran} E_2 = \operatorname{ran} G$. *Moreover,*

$$\operatorname{gra} E_2 = \operatorname{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} (\operatorname{ran} D)^{\perp} \\ 0 \end{pmatrix} = \operatorname{ran} \begin{pmatrix} C \\ D \end{pmatrix} + \begin{pmatrix} \ker D^{\intercal} \\ 0 \end{pmatrix}. \tag{4.35}$$

Proof. (i): Note that $\operatorname{dom} G = \operatorname{ran} C$. The maximal monotonicity follows from Proposition 4.4.1. (4.34) follows from (4.32) and the fact that $(\operatorname{ran} C)^{\perp} = \ker C^{\intercal}$ [59, page 405].

(ii): Apply (i) to G^{-1}, i.e.,

$$\operatorname{gra} G^{-1} = \left\{ \begin{pmatrix} Dy \\ Cy \end{pmatrix} \,\middle|\, y \in \mathbb{R}^{2n-p} \right\} \tag{4.36}$$

and followed by taking the set-valued inverse. ■

Apparently, both extensions E_1, E_2 rely on $\operatorname{gra} G$, $\operatorname{dom} G, \operatorname{ran} G$, not on the A, B. In this sense, E_1, E_2 are intrinsic maximally monotone linear subspace extensions.

Remark 4.4.3 *Theorem 4.4.2 is much easier to use than Corollary 4.2.8 when G is written in the form of* (4.10). *Indeed, it is not hard to check that*

$$\operatorname{gra}(E_1^*) = \{(B^\intercal u, -A^\intercal u) \mid B^\intercal u \in \operatorname{dom} G, u \in \mathbb{R}^p\}. \tag{4.37}$$

$$\operatorname{gra}(E_2^*) = \{B^\intercal u, -A^\intercal u) \mid A^\intercal u \in \operatorname{ran} G, u \in \mathbb{R}^p\}. \tag{4.38}$$

According to Fact 3.2.13, E_i^* is maximally monotone and $\dim E_i^* = n$*. This implies that*

$$\dim\{u \in \mathbb{R}^p \mid B^\intercal u \in \operatorname{dom} G\} = n, \qquad \dim\{u \in \mathbb{R}^p \mid A^\intercal u \in \operatorname{ran} G\} = n.$$

Let $M_i \in \mathbb{R}^{p\times p}$ with $\operatorname{rank} M_i = n$ *and*

$$\{u \in \mathbb{R}^p \mid B^\intercal u \in \operatorname{dom} G\} = \operatorname{ran} M_1, \tag{4.39}$$

$$\{u \in \mathbb{R}^p \mid A^\intercal u \in \operatorname{ran} G\} = \operatorname{ran} M_2. \tag{4.40}$$

Corollary 4.2.8 shows that

$$\operatorname{gra} E_i = \{(x, x^*) \mid M_i^\intercal Ax + M_i^\intercal Bx^* = 0\}.$$

However, finding M_i from (4.39) *and* (4.40) *may not be as easy as it seems.*

Remark 4.4.4 *Unfortunately, we do not know how to determine* **all** *maximally monotone linear subspace extensions of G if it is given in the form of*

(4.10).

4.5 Examples

In the final section, we illustrate our maximally monotone extensions by considering three examples. In particular, they show that maximally monotone extensions $\widetilde{G}$ rely on the representation of G in terms of A, B and choices of N we shall use. However, the maximally monotone extensions E_i are intrinsic, depending only on $\operatorname{gra} G$.

Example 4.5.1 *Consider*

$$\operatorname{gra} G = \left\{ (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ C \end{pmatrix} x^* = 0 \right\}$$

where $C \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, and $\mathrm{Id} \in \mathbb{R}^{n\times n}$. Clearly,

$$\operatorname{gra} G = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

We have

(i) *For every $\alpha \in [-1, 1]$, $\widetilde{G}_\alpha$ defined by*

$$\operatorname{gra} \widetilde{G}_\alpha = \begin{cases} \{(0, \mathbb{R}^n)\}, & \text{if } \alpha = 1; \\ \left\{ (x, \frac{1+\alpha}{1-\alpha} C^{-1} x) \mid x \in \mathbb{R}^n \right\}, & \text{otherwise} \end{cases}$$

is a maximally monotone linear extension of G.

(ii) $E_1 = \widetilde{G}_1$ *and* $E_2 = \widetilde{G}_{-1}$.

Proof. (i): To find $\widetilde{G}_\alpha$, we need eigenvectors of

$$\mathbf{A} = \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix} (0 \ C^\intercal) + \begin{pmatrix} 0 \\ C \end{pmatrix} (\mathrm{Id} \ \ 0) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.$$

Counting multiplicity, the positive definite matrix C has eigen-pairs (λ_i, w_i) $(i = 1, \ldots, n)$ such that $\lambda_i > 0, \|w_i\| = 1$ and $\langle w_i, w_j \rangle = 0$ for $i \neq j$. As such, the matrix $\mathbf{A}$ has $2n$ eigen-pairs, namely

$$\left(\lambda_i, \begin{pmatrix} w_i \\ w_i \end{pmatrix} \right)$$

and

$$\left(-\lambda_i, \begin{pmatrix} w_i \\ -w_i \end{pmatrix} \right)$$

with $i = 1, \ldots, n$. Put $W = (w_1 \ \cdots \ w_n) \in \mathbb{R}^{n \times n}$ and write

$$V = \begin{pmatrix} W & W \\ W & -W \end{pmatrix}.$$

Then

$$W^\intercal C W = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

In Theorem 4.2.7, take

$$N_\alpha = \begin{pmatrix} \mathbf{0} & \alpha \operatorname{Id} \\ \mathbf{0} & \operatorname{Id} \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$$

where $\operatorname{Id} \in \mathbb{R}^{n \times n}$. We have $\operatorname{rank} N_\alpha = n$,

$$N_\alpha^\intercal \operatorname{Id}_\lambda N_\alpha = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\alpha^2 - 1) W^\intercal C W \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\alpha^2 - 1) D \end{pmatrix}$$

being negative semidefinite, and

$$V N_\alpha = \begin{pmatrix} 0 & (1+\alpha) W \\ 0 & (\alpha - 1) W \end{pmatrix}.$$

Then by Theorem 4.2.7, we have a maximally monotone linear extension $\widetilde{G}_\alpha$ given by

$$\begin{aligned} \operatorname{gra} \widetilde{G}_\alpha &= \left\{ (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{pmatrix} 0 \\ (1+\alpha) W^\intercal x + (\alpha - 1) W^\intercal C x^* \end{pmatrix} = 0 \right\} \\ &= \big\{ (x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid (1+\alpha) x + (\alpha - 1) C x^* = 0 \big\} \\ &= \begin{cases} \{(0, \mathbb{R}^n)\}, & \text{if } \alpha = 1; \\ \Big\{ (x, \frac{1+\alpha}{1-\alpha} C^{-1} x) \mid x \in \mathbb{R}^n \Big\}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence we get the desired result.

(ii): It is immediate from Theorem 4.4.2 and (i). ∎

Example 4.5.2 *Consider*

$$\operatorname{gra} G = \left\{ (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\middle|\, \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = 0 \right\}.$$

Then

(i) *the linear operators $\widetilde{G}_i : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ for $i = 1, 2$ given by*

$$\widetilde{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1+\sqrt{2}}{2-\sqrt{2}} \end{pmatrix}, \quad \widetilde{G}_2 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{2}}{10} \end{pmatrix}$$

are two maximally monotone extensions of G.

(ii)

$$E_1(x_1, 0) = (x_1, \mathbb{R}) \quad \forall x_1 \in \mathbb{R}.$$

(iii)

$$E_2(x_1, y) = (x_1, 0) \quad \forall x_1, y \in \mathbb{R}.$$

Proof. We have

$$\operatorname{gra} G = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{R} \right\}$$

is monotone. Since $\dim G = 1$, G is not maximally monotone by Fact 3.2.6.

The matrix

$$AB^\intercal + BA^\intercal = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

has a positive eigenvalue $-1+\sqrt{2}$ with an eigenvector

$$u = \begin{pmatrix} 0 \\ 1 \\ 1-\sqrt{2} \end{pmatrix} \quad \text{so that} \quad \begin{pmatrix} B^\intercal \\ A^\intercal \end{pmatrix} u = \begin{pmatrix} 0 \\ 2-\sqrt{2} \\ 0 \\ -1+\sqrt{2} \end{pmatrix}.$$

Then by Corollary 4.2.4,

$$\begin{aligned} \operatorname{gra} \widetilde{G}_1 &= \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_1 \\ 0 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\} + \left\{ \begin{pmatrix} 0 \\ 2-\sqrt{2} \\ 0 \\ -1+\sqrt{2} \end{pmatrix} x_2 \middle| \, x_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ (2-\sqrt{2})x_2 \\ x_1 \\ (-1+\sqrt{2})x_2 \end{pmatrix} \middle| \, x_1, x_2 \in \mathbb{R} \right\}. \end{aligned}$$

Therefore,

$$\widetilde{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1+\sqrt{2}}{2-\sqrt{2}} \end{pmatrix}$$

is a maximally monotone extension of G.

Now we have

$$\mathrm{Id}_\lambda = \begin{pmatrix} -1+\sqrt{2} & 0 & 0 \\ 0 & -1-\sqrt{2} & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{-1+\sqrt{2}} & -\frac{1}{-1-\sqrt{2}} & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4.41}$$

Take

$$N = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{4.42}$$

We have $\operatorname{rank} N = 2$ and

$$N^\intercal \,\mathrm{Id}_\lambda\, N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -7-3\sqrt{2} & 1+\sqrt{2} \\ 0 & 1+\sqrt{2} & -4 \end{pmatrix} \tag{4.43}$$

is negative semidefinite. By Theorem 4.2.7, with V, N given in (4.41) and (4.42), we use the NullSpace command in Maple to solve

$$(VN)^\intercal Ax + (VN)^\intercal Bx^* = 0,$$

and get

$$\operatorname{gra}\widetilde{G}_2 = \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-2\sqrt{2}\\5\sqrt{2}\\0\\1\end{pmatrix}\right\}.$$

Thus $\widetilde{G}_2 = \begin{pmatrix}1 & -2\sqrt{2}\\0 & 5\sqrt{2}\end{pmatrix}^{-1} = \begin{pmatrix}1 & \frac{2}{5}\\0 & \frac{\sqrt{2}}{10}\end{pmatrix}$ is another maximally monotone extension of G.

On the other hand,

$$\operatorname{gra}E_1 = \left\{\begin{pmatrix}x_1\\0\\x_1\\0\end{pmatrix}\middle|\ x_1 \in \mathbb{R}\right\} + \begin{pmatrix}0\\0\\0\\\mathbb{R}\end{pmatrix} = \left\{\begin{pmatrix}x_1\\0\\x_1\\\mathbb{R}\end{pmatrix}\middle|\ x_1 \in \mathbb{R}\right\}$$

gives

$$E_1(x_1, 0) = (x_1, \mathbb{R}) \quad \forall x_1 \in \mathbb{R}.$$

We have

$$\operatorname{gra}E_2 = \left\{\begin{pmatrix}x_1\\\mathbb{R}\\x_1\\0\end{pmatrix}\middle|\ x_1 \in \mathbb{R}\right\},$$

which gives $E_2(x_1, y) = (x_1, 0) \quad \forall x_1, y \in \mathbb{R}$. ■

In [11], the authors use autoconjugates to find maximally monotone extensions of monotone operators. In general, it is not clear whether the maximally monotone extensions of a linear relation is still a linear relation. As both monotone operators in Examples 4.5.2 and 4.5.1 are subsets of $\{(x,x) \mid x \in \mathbb{R}^n\}$, [11, Example 5.10] shows that the maximally monotone extension obtained by autoconjugates must be Id, which is different from the ones given here.

Example 4.5.3 *Consider* $\operatorname{gra} G = \{(x,x^*) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid Ax + Bx^* = 0\}$ *where*

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 5 \\ 1 & 7 \\ 0 & 2 \end{pmatrix}, \textit{thus}\,(A\ B) = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 0 & 1 & 7 \\ 3 & 1 & 0 & 2 \end{pmatrix}. \tag{4.44}$$

Then the linear operators $\widetilde{G}_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *for* $i = 1, 2$ *given by*

$$\widetilde{G}_1 = \begin{pmatrix} \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} & \frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}, \quad \widetilde{G}_2 = \begin{pmatrix} \frac{33}{4} - \frac{\sqrt{201}}{6} & \frac{13}{4} - \frac{\sqrt{201}}{6} \\ -\frac{29}{20} + \frac{\sqrt{201}}{30} & -\frac{9}{20} + \frac{\sqrt{201}}{30} \end{pmatrix}$$

are two maximally monotone linear extensions of G.

Moreover,

$$\operatorname{gra} E_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} x_2 \,\middle|\, x_1, x_2 \in \mathbb{R} \right\}$$

and

$$\operatorname{gra} E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} x_2 \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}.$$

Proof. We have $\operatorname{rank}(A\ B) = 3$ *and*

$$\mathrm{Id}_\lambda = \begin{pmatrix} 13+\sqrt{201} & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 13-\sqrt{201} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{20}{1+\sqrt{201}} & 0 & \frac{20}{1-\sqrt{201}} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{4.45}$$

and

$$V_g = \begin{pmatrix} 0 & -1 & 1 \\ \frac{20}{1-\sqrt{201}} & 1 & 1 \end{pmatrix}. \tag{4.46}$$

Clearly, here $p = 3, n = 2$ *and* $AB^\intercal + BA^\intercal$ *has exactly* $p - n = 3 - 2 = 1$ *positive eigenvalue. By Theorem 4.2.3(i)(v),* G *is monotone. Since* $AB^\intercal + BA^\intercal$ *is not negative semidefinite, by Theorem 4.2.6(i)(iii),* G *is not maximally monotone.*

With V_g *given in* (4.46) *and* A, B *in* (4.44)*, use the NullSpace command* *in maple to solve* $V_g Ax + V_g Bx^* = 0$ *and obtain* $\widetilde{G}_1$ *defined by*

$$\operatorname{gra}\widetilde{G}_1 = \operatorname{span}\left\{\begin{pmatrix} -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \\ \frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} \\ 0 \\ 1 \end{pmatrix}\right\}.$$

By Corollary 4.2.4, $\widetilde{G}_1$ *is a maximally monotone linear subspace extension of* G*. Then*

$$\widetilde{G}_1 = \begin{pmatrix} -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ \frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{-117+17\sqrt{201}}{2(-1+\sqrt{201})} & \frac{-107+7\sqrt{201}}{2(-1+\sqrt{201})} \\ -\frac{-23+3\sqrt{201}}{2(-1+\sqrt{201})} & -\frac{-21+\sqrt{201}}{2(-1+\sqrt{201})} \end{pmatrix}.$$

Let N *be defined by*

$$N = \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.47}$$

Then $\operatorname{rank} N = 2$ *and*

$$N^\intercal \operatorname{Id}_\lambda N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & \frac{338-24\sqrt{201}}{25} \end{pmatrix}$$

is negative semidefinite.

With N *in* (4.47)*,* A, B *in* (4.44) *and* V *in* (4.45)*, use the NullSpace command in maple to solve* $(VN)^\intercal Ax + (VN)^\intercal Bx^* = 0$*. By Theorem 4.2.7,* *we get a maximally monotone linear extension of* G, $\widetilde{G}_2$, *defined by*

$$\widetilde{G}_2 = \begin{pmatrix} -\frac{9}{20} + \frac{\sqrt{201}}{30} & -\frac{13}{4} + \frac{\sqrt{201}}{6} \\ \frac{29}{20} - \frac{\sqrt{201}}{30} & \frac{33}{4} - \frac{\sqrt{201}}{6} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{33}{4} - \frac{\sqrt{201}}{6} & \frac{13}{4} - \frac{\sqrt{201}}{6} \\ -\frac{29}{20} + \frac{\sqrt{201}}{30} & -\frac{9}{20} + \frac{\sqrt{201}}{30} \end{pmatrix}.$$

To find E_1 *and* E_2, *using the LinearSolve command in Maple, we get* $\operatorname{gra} G = \operatorname{ran}\begin{pmatrix} C \\ D \end{pmatrix}$, *where*

$$C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad D = \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

It follows from Theorem 4.4.2 that

$$\operatorname{gra} E_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} x_2 \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}$$

and

$$\operatorname{gra} E_2 = \left\{ \begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} x_2 \;\middle|\; x_1, x_2 \in \mathbb{R} \right\}.$$

■

4.6 Discussion

A direction for future work in this chapter is to write computer code to find the maximally monotone subspace extension of G, and to generalize the results into a Hilbert space by applying the Brezis-Browder Theorem.

Chapter 5

The sum problem

Let A and B be maximally monotone operators from X to X^*. Clearly, the *sum operator*

$$A+B\colon X\rightrightarrows X^*\colon x\mapsto Ax+Bx=\big\{a^*+b^*\mid a^*\in Ax \text{ and } b^*\in Bx\big\}$$

is monotone. Rockafellar established the following very important result in 1970.

Theorem 5.0.1 (Rockafellar's sum theorem) (See [66, Theorem 1].) *Suppose that X is reflexive. Let $A,B: X\rightrightarrows X^*$ be maximally monotone. Assume that A and B satisfy the classical* constraint qualification
$\operatorname{dom} A\cap\operatorname{int}\operatorname{dom} B\neq\varnothing$ *Then $A+B$ is maximally monotone.*

The most famous open problem concerns the maximal monotonicity of the sum of two maximally monotone operators in general Banach spaces, which is called the "sum problem". See Simons' monograph [74] and [22–24, 86, 90] for a comprehensive account of some recent developments. In this chapter, we prove the maximal monotonicity of $A+B$ provided that $\operatorname{dom} A\cap\operatorname{int}\operatorname{dom} B\neq\varnothing$, $A+N_{\overline{\operatorname{dom} B}}$ is of type (FPV), and $\operatorname{dom} A\cap\overline{\operatorname{dom} B}\subseteq\operatorname{dom} B$.

We also show the maximal monotonicity of $A+B$ when A is a maximally monotone linear relation and B is a subdifferential operator satisfying $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$.

This chapter is mainly based on my work in [90, 91].

5.1 Basic properties

Fact 5.1.1 (Rockafellar) (See [65, Theorem 3], [74, Corollary 10.3 and Theorem 18.1], or [92, Theorem 2.8.7(iii)].) *Let $f, g : X \to \left]-\infty, +\infty\right]$ be proper convex functions. Assume that there exists a point $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that g is continuous at x_0. Then for every $z^* \in X^*$, there exists $y^* \in X^*$ such that*

$$(f+g)^*(z^*) = f^*(y^*) + g^*(z^* - y^*). \tag{5.1}$$

Furthermore, $\partial(f+g) = \partial f + \partial g$.

Fact 5.1.2 (Rockafellar) (See [67, Theorem A], [92, Theorem 3.2.8], [74, Theorem 18.7] or [54, Theorem 2.1]) *Let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.*

Fact 5.1.3 (See [61, Theorem 2.28].) *Let $A : X \rightrightarrows X^*$ be monotone such that $\operatorname{int} \operatorname{dom} A \neq \varnothing$. Assume that $x \in \operatorname{int} \operatorname{dom} A$. Then A is locally bounded at x, i.e., there exist $\delta > 0$ and $K > 0$ such that*

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \operatorname{dom} A.$$

Fact 5.1.4 (See [61, Proposition 3.3 and Proposition 1.11].) *Let $f : X \to$*

$]-\infty,+\infty]$ *be a lower semicontinuous convex function and* $\operatorname{int}\operatorname{dom} f \neq \varnothing$*. Then* f *is continuous on* $\operatorname{int}\operatorname{dom} f$ *and* $\partial f(x) \neq \varnothing$ *for every* $x \in \operatorname{int}\operatorname{dom} f$*.*

Fact 5.1.5 (Fitzpatrick) (See [45, Corollary 3.9].) *Let* $A\colon X \rightrightarrows X^*$ *be maximally monotone, and set*

$$F_A\colon X \times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \sup_{(a,a^*)\in\operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big). \tag{5.2}$$

Then for every $(x,x^*) \in X \times X^*$*, the inequality* $\langle x,x^*\rangle \leq F_A(x,x^*)$ *is true, and equality holds if and only if* $(x,x^*) \in \operatorname{gra} A$*.*

Fact 5.1.6 (Fitzpatrick) (See [45, Theorem 3.4].) *Let* $A\colon X \rightrightarrows X^*$ *be monotone. Then* $\operatorname{conv}\operatorname{dom} A \subseteq P_X(\operatorname{dom} F_A)$*.*

Fact 5.1.7 (See [84, Theorem 3.4 and Corollary 5.6] or [74, Theorem 24.1(b)].) *Let* $A, B : X \rightrightarrows X^*$ *be maximally monotone operators. Assume*

$$\bigcup_{\lambda>0} \lambda\left[P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)\right] \text{ is a closed subspace of } X.$$

If

$$F_{A+B} \geq \langle \cdot,\, \cdot\rangle \text{ on } \quad X \times X^*, \tag{5.3}$$

then $A + B$ *is maximally monotone.*

Fact 5.1.8 (Simons) (See [74, Theorem 27.1 and Theorem 27.3].) *Let* $A : X \rightrightarrows X^*$ *be maximally monotone with* $\operatorname{int}\operatorname{dom} A \neq \varnothing$*. Then* $\operatorname{int}\operatorname{dom} A = \operatorname{int}\left[P_X \operatorname{dom} F_A\right]$, $\overline{\operatorname{dom} A} = \overline{P_X\left[\operatorname{dom} F_A\right]}$*, and* $\overline{\operatorname{dom} A}$ *is convex.*

Fact 5.1.9 (Simons) (See [70, Lemma 2.2].) *Let $f : X \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous and convex. Let $x \in X$ and $\lambda \in \mathbb{R}$ be such that* $\inf f < \lambda < f(x) \leq +\infty$, *and set*

$$K := \sup_{a \in X, a \neq x} \frac{\lambda - f(a)}{\|x - a\|}.$$

Then $K \in \left]0, +\infty\right[$ and for every $\varepsilon \in \left]0, 1\right[$, there exists $(y, y^*) \in \operatorname{gra} \partial f$ such that

$$\langle y - x, y^* \rangle \leq -(1 - \varepsilon) K \|y - x\| < 0. \tag{5.4}$$

Fact 5.1.10 (Simons) (See [74, Theorem 48.6(a)].) *Let $f : X \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex. Let $(x, x^*) \in X \times X^*$ be such that $(x, x^*) \notin \operatorname{gra} \partial f$ and let $\alpha > 0$. Then for every $\varepsilon > 0$, there exists $(y, y^*) \in \operatorname{gra} \partial f$ with $y \neq x$ and $y^* \neq x^*$ such that*

$$\left| \frac{\|x - y\|}{\|x^* - y^*\|} - \alpha \right| < \varepsilon \tag{5.5}$$

and

$$\left| \frac{\langle x - y, x^* - y^* \rangle}{\|x - y\| \cdot \|x^* - y^*\|} + 1 \right| < \varepsilon. \tag{5.6}$$

Fact 5.1.11 (Simons) (See [74, Corollary 28.2].) *Let $A \colon X \rightrightarrows X^*$ be maximally monotone. Then*

$$\overline{\operatorname{span}(P_X \operatorname{dom} F_A)} = \overline{\operatorname{span}\left[\operatorname{dom} A\right]}. \tag{5.7}$$

Now we cite some results on maximally monotone operators of type (FPV).

Fact 5.1.12 (Fitzpatrick-Phelps and Verona-Verona) (See [47, Corollary 3.4], [81, Theorem 3] or [74, Theorem 48.4(d)].) *Let $f : X \to]-\infty, +\infty]$ be proper, lower semicontinuous, and convex. Then ∂f is of type (FPV).*

Fact 5.1.13 (Simons) (See [74, Theorem 44.2].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone of type (FPV). Then*

$$\overline{\operatorname{dom} A} = \overline{\operatorname{conv} \operatorname{dom} A} = \overline{P_X \operatorname{dom} F_A}.$$

Fact 5.1.14 (Simons) (See [74, Theorem 46.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then A is of type (FPV).*

Fact 5.1.15 (Simons and Verona-Verona) (See [74, Thereom 44.1] or [81].) *Let $A : X \rightrightarrows X^*$ be maximally monotone. Suppose that for every closed convex subset C of X with $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$, the operator $A + N_C$ is maximally monotone. Then A is of type (FPV).*

The following statement first appeared in [72, Theorem 41.5]. However, on [74, page 199], concerns were raised about the validity of the proof of [72, Theorem 41.5]. In [85], Voisei recently provided a result that generalizes and confirms [72, Theorem 41.5] and hence the following fact.

Fact 5.1.16 (Voisei) *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV) with convex domain, let C be a nonempty closed convex subset of X,* *and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then* $A + N_C$ *is maximally monotone.*

Corollary 5.1.17 *Let* $A : X \rightrightarrows X^*$ *be maximally monotone of type (FPV) with convex domain, let* C *be a nonempty closed convex subset of* X*, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then* $A + N_C$ *is of type* (FPV).

Proof. By Fact 5.1.16, $A+N_C$ is maximally monotone. Let D be a nonempty closed convex subset of X, and suppose that $\operatorname{dom}(A+N_C) \cap \operatorname{int} D \neq \varnothing$. Let $x_1 \in \operatorname{dom} A \cap \operatorname{int} C$ and $x_2 \in \operatorname{dom}(A + N_C) \cap \operatorname{int} D$. Thus, there exists $\delta > 0$ such that $x_1 + \delta U_X \subseteq C$ and $x_2 + \delta U_X \subseteq D$. Then for small enough $\lambda \in \,]0, 1[$, we have $x_2 + \lambda(x_1 - x_2) + \frac{1}{2}\delta U_X \subseteq D$. Clearly, $x_2 + \lambda(x_1 - x_2) + \lambda\delta U_X \subseteq C$. Thus $x_2 + \lambda(x_1 - x_2) + \frac{\lambda\delta}{2} U_X \subseteq C \cap D$. Since $\operatorname{dom} A$ is convex, $x_2 + \lambda(x_1 - x_2) \in \operatorname{dom} A$ and $x_2 + \lambda(x_1 - x_2) \in \operatorname{dom} A \cap \operatorname{int}(C \cap D)$. By Fact 5.1.1 , $A + N_C + N_D = A + N_{C\cap D}$. Then, by Fact 5.1.16 (applied to A and $C \cap D$), $A + N_C + N_D = A + N_{C\cap D}$ is maximally monotone. By Fact 5.1.15, $A + N_C$ is of type (FPV). ■

Corollary 5.1.18 *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, let* C *be a nonempty closed convex subset of* X*, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then* $A + N_C$ *is of type* (FPV).

Proof. Apply Fact 5.1.14 and Corollary 5.1.17. ■

The following Lemma 5.1.19 is from [16, Lemma 2.5].

Lemma 5.1.19 *Let* C *be a nonempty closed convex subset of* X *such that* $\operatorname{int} C \neq \varnothing$. *Let* $c_0 \in \operatorname{int} C$ *and suppose that* $z \in X \smallsetminus C$. *Then there exists* $\lambda \in \,]0, 1[$ *such that* $\lambda c_0 + (1 - \lambda) z \in \operatorname{bdry} C$.

Proof. Let $\lambda = \inf\big\{t \in [0,1] \mid tc_0 + (1-t)z \in C\big\}$. Since C is closed,

$$\lambda = \min\big\{t \in [0,1] \mid tc_0 + (1-t)z \in C\big\}. \tag{5.8}$$

Because $z \notin C$, $\lambda > 0$. We now show that $\lambda c_0 + (1-\lambda)z \in \operatorname{bdry} C$. Assume to the contrary that $\lambda c_0 + (1-\lambda)z \in \operatorname{int} C$. Then there exists $\delta \in \left]0, \lambda\right[$ such that $\lambda c_0 + (1-\lambda)z - \delta(c_0 - z) \in C$. Hence $(\lambda-\delta)c_0 + (1-\lambda+\delta)z \in C$, which contradicts (5.8). Therefore, $\lambda c_0 + (1-\lambda)z \in \operatorname{bdry} C$. Since $c_0 \notin \operatorname{bdry} C$, we also have $\lambda < 1$. ■

The proof of the next result follows closely the proof of [74, Theorem 53.1]. Lemma 5.1.20 was established by Bauschke, Wang and Yao in [19, Lemma 2.10].

Lemma 5.1.20 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation, and let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous and convex function. Suppose that $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$, $(z, z^*) \in X \times X^*$ is monotonically related to $\operatorname{gra}(A + \partial f)$ and $z \in \operatorname{dom} A$. Then $z \in \operatorname{dom} \partial f$.*

Proof. Let $c_0 \in X$ and $y^* \in X^*$ be such that

$$c_0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \quad \text{and} \quad (z, y^*) \in \operatorname{gra} A. \tag{5.9}$$

Take $c_0^* \in Ac_0$, and set

$$M := \max\big\{\|y^*\|, \|c_0^*\|\big\}, \tag{5.10}$$

$D := [c_0, z]$, and $h := f + \iota_D$. By (5.9), Fact 5.1.4 and Fact 5.1.1, $\partial h = \partial f + \partial \iota_D$. Set $g\colon X \to\ ]-\infty, +\infty] : x \mapsto h(x+z) - \langle z^*, x\rangle$. It remains to show that

$$0 \in \operatorname{dom} \partial g. \tag{5.11}$$

If $\inf g = g(0)$, then (5.11) holds. Now suppose that $\inf g < g(0)$. Let $\lambda \in \mathbb{R}$ be such that $\inf g < \lambda < g(0)$, and set

$$K_\lambda := \sup_{g(x)<\lambda} \frac{\lambda - g(x)}{\|x\|}. \tag{5.12}$$

We claim that

$$K_\lambda \leq M.$$

By Fact 5.1.9, we have $K_\lambda \in\ ]0, \infty[$ and $\forall \varepsilon \in\ ]0,1[$, by $\operatorname{gra} \partial g = \operatorname{gra} \partial h - (z, z^*)$ there exists $(x, x^*) \in \operatorname{gra} \partial h$ such that

$$\langle x - z, x^* - z^*\rangle \leq -(1-\varepsilon) K_\lambda \|x - z\| < 0. \tag{5.13}$$

Since $\partial h = \partial f + \partial \iota_D$, there exists $t \in [0,1]$ with $x_1^* \in \partial f(x)$ and $x_2^* \in \partial \iota_D(x)$ such that $x = tc_0 + (1-t)z$ and $x^* = x_1^* + x_2^*$. Then $\langle x - z, x_2^*\rangle \geq 0$. Thus, by (5.13),

$$\langle x - z, x_1^* - z^*\rangle \leq \langle x - z, x_1^* + x_2^* - z^*\rangle \leq -(1-\varepsilon) K_\lambda \|x - z\| < 0. \tag{5.14}$$

As $x = tc_0 + (1-t)z$ and A is a linear relation, we have $(x, tc_0^* + (1-t)y^*) \in \operatorname{gra} A$. Since (z, z^*) is monotonically related to $\operatorname{gra}(A + \partial f)$, by (5.10),

$$\langle x - z, x_1^* - z^* \rangle \geq -\langle x - z, tc_0^* + (1-t)y^* \rangle \geq -M\|x - z\|. \tag{5.15}$$

Combining (5.15) and (5.14), we obtain

$$-M\|x - z\| \leq -(1-\varepsilon)K_\lambda \|x - z\| < 0. \tag{5.16}$$

Hence, $(1-\varepsilon)K_\lambda \leq M$. Letting $\varepsilon \downarrow 0$, we deduce that $K_\lambda \leq M$. Then, by (5.12) and letting $\lambda \uparrow g(0)$, we get

$$g(y) + M\|y\| \geq g(0), \quad \forall y \in X. \tag{5.17}$$

In view of [74, Example 7.1], we conclude that $0 \in \operatorname{dom} \partial g$. Hence (5.11) holds and thus $z \in \operatorname{dom} \partial f$. ■

5.2 Maximality of the sum of a (FPV) operator and a full domain operator

The following result plays a key role in the proof of Theorem 5.2.4. The first half of its proof follows along the lines of the proof of [74, Theorem 44.2].

Proposition 5.2.1 *Let $A, B : X \rightrightarrows X^*$ be maximally monotone with* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. *Assume that $A + N_{\overline{\operatorname{dom} B}}$ is maximally monotone of type (FPV), and* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. *Then* $\overline{P_X [\operatorname{dom} F_{A+B}]} =$

$\overline{\operatorname{dom} A \cap \operatorname{dom} B}$.

Proof. By Fact 5.1.6, $\overline{\operatorname{dom} A \cap \operatorname{dom} B} = \overline{\operatorname{dom}(A+B)} \subseteq \overline{P_X\left[\operatorname{dom} F_{A+B}\right]}$. It suffices to show that

$$P_X\left[\operatorname{dom} F_{A+B}\right] \subseteq \overline{\operatorname{dom} A \cap \operatorname{dom} B}. \tag{5.18}$$

After translating the graphs if necessary, we can and do assume that $0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B$ and $(0,0) \in \operatorname{gra} B$.

To show (5.18), we take $z \in P_X\left[\operatorname{dom} F_{A+B}\right]$ and we assume to the contrary that

$$z \notin \overline{\operatorname{dom} A \cap \operatorname{dom} B}. \tag{5.19}$$

Thus $\alpha = \mathrm{d}(z, \overline{\operatorname{dom} A \cap \operatorname{dom} B}) > 0$. Now take $y_0^* \in X^*$ such that

$$\|y_0^*\| = 1 \quad \text{and} \quad \langle z, y_0^* \rangle \geq \tfrac{2}{3}\|z\|. \tag{5.20}$$

Set

$$U_n = [0, z] + \tfrac{\alpha}{4n} U_X, \quad \forall n \in \mathbb{N}. \tag{5.21}$$

Since $0 \in N_{\overline{\operatorname{dom} B}}(x), \forall x \in \operatorname{dom} B$, $\operatorname{gra} B \subseteq \operatorname{gra}(B + N_{\overline{\operatorname{dom} B}})$. Since B is maximally monotone and $B + N_{\overline{\operatorname{dom} B}}$ is a monotone extension of B, we must have $B = B + N_{\overline{\operatorname{dom} B}}$. Thus

$$A + B = A + N_{\overline{\operatorname{dom} B}} + B. \tag{5.22}$$

Since $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$ by assumption, we obtain

$$\operatorname{dom} A \cap \operatorname{dom} B \subseteq \operatorname{dom}(A + N_{\overline{\operatorname{dom} B}}) = \operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} A \cap \operatorname{dom} B.$$

Hence

$$\operatorname{dom} A \cap \operatorname{dom} B = \operatorname{dom}(A + N_{\overline{\operatorname{dom} B}}). \tag{5.23}$$

By (5.19) and (5.23), $z \notin \operatorname{dom}(A + N_{\overline{\operatorname{dom} B}})$ and thus $(z, ny_0^*) \notin \operatorname{gra}(A + N_{\overline{\operatorname{dom} B}}), \forall n \in \mathbb{N}$. For every $n \in \mathbb{N}$, since $z \in U_n$ and since $A + N_{\overline{\operatorname{dom} B}}$ is of type (FPV) by assumption, we deduce the existence of $(z_n, z_n^*) \in \operatorname{gra}(A + N_{\overline{\operatorname{dom} B}})$ such that $z_n \in U_n$ and

$$\langle z - z_n, z_n^* \rangle > n\langle z - z_n, y_0^* \rangle, \quad \forall n \in \mathbb{N}. \tag{5.24}$$

Hence, using (5.21), there exists $\lambda_n \in [0, 1]$ such that

$$\|z - z_n - \lambda_n z\| = \|z_n - (1 - \lambda_n)z\| < \tfrac{1}{4}\alpha, \quad \forall n \in \mathbb{N}. \tag{5.25}$$

By the triangle inequality, we have $\|z - z_n\| < \lambda_n\|z\| + \frac{1}{4}\alpha$ for every $n \in \mathbb{N}$. From the definition of α and (5.23), it follows that $\alpha \leq \|z - z_n\|$ and hence that $\alpha < \lambda_n\|z\| + \frac{1}{4}\alpha$. Thus,

$$\tfrac{3}{4}\alpha < \lambda_n\|z\|, \quad \forall n \in \mathbb{N}. \tag{5.26}$$

By (5.25) and (5.20),

$$\langle z - z_n - \lambda_n z, y_0^* \rangle \geq -\|z_n - (1-\lambda_n)z\| > -\tfrac{1}{4}\alpha, \quad \forall n \in \mathbb{N}. \tag{5.27}$$

By (5.27), (5.20) and (5.26),

$$\langle z - z_n, y_0^* \rangle > \lambda_n \langle z, y_0^* \rangle - \tfrac{1}{4}\alpha > \tfrac{2}{3}\tfrac{3}{4}\alpha - \tfrac{1}{4}\alpha = \tfrac{1}{4}\alpha, \quad \forall n \in \mathbb{N}. \tag{5.28}$$

Then, by (5.24) and (5.28),

$$\langle z - z_n, z_n^* \rangle > \tfrac{1}{4} n\alpha, \quad \forall n \in \mathbb{N}. \tag{5.29}$$

By (5.21), there exist $t_n \in [0,1]$ and $b_n \in \frac{\alpha}{4n} U_X$ such that $z_n = t_n z + b_n$. Since $t_n \in [0,1]$, there exists a convergent subsequence of $(t_n)_{n\in\mathbb{N}}$, which, for convenience, we still denote by $(t_n)_{n\in\mathbb{N}}$. Then $t_n \to \beta$, where $\beta \in [0,1]$. Since $b_n \to 0$, we have

$$z_n \to \beta z. \tag{5.30}$$

By (5.23), $z_n \in \operatorname{dom} A \cap \operatorname{dom} B$; thus, $\|z_n - z\| \geq \alpha$ and $\beta \in [0,1[$. In view of (5.22) and (5.29), we have, for every $z^* \in X^*$,

$$\begin{aligned}
&F_{A+B}(z,z^*) = F_{A+N_{\overline{\operatorname{dom} B}}+B}(z,z^*)\\
&\geq \sup_{\{n\in\mathbb{N}, y^*\in X^*\}} \left[\langle z_n, z^*\rangle + \langle z - z_n, z_n^*\rangle + \langle z - z_n, y^*\rangle - \iota_{\operatorname{gra} B}(z_n, y^*)\right]\\
&\geq \sup_{\{n\in\mathbb{N}, y^*\in X^*\}} \left[\langle z_n, z^*\rangle + \tfrac{1}{4}n\alpha + \langle z - z_n, y^*\rangle - \iota_{\operatorname{gra} B}(z_n, y^*)\right].
\end{aligned} \tag{5.31}$$

We now claim that

$$F_{A+B}(z,z^*) = \infty. \tag{5.32}$$

We consider two cases.

Case 1: $\beta = 0$.

By (5.30) and Fact 5.1.3 (applied to $0 \in \operatorname{int} \operatorname{dom} B$), there exist $N \in \mathbb{N}$ and $K > 0$ such that

$$Bz_n \neq \varnothing \quad \text{and} \quad \sup_{y^* \in Bz_n} \|y^*\| \leq K, \quad \forall n \geq N. \tag{5.33}$$

Then, by (5.31),

$$\begin{aligned}
F_{A+B}(z,z^*) &\geq \sup_{\{n\geq N, y^*\in X^*\}} \left[\langle z_n, z^*\rangle + \tfrac{1}{4}n\alpha + \langle z - z_n, y^*\rangle - \iota_{\operatorname{gra} B}(z_n, y^*)\right] \\
&\geq \sup_{\{n\geq N, y^*\in Bz_n\}} \left[-\|z_n\|\cdot\|z^*\| + \tfrac{1}{4}n\alpha - \|z - z_n\|\cdot\|y^*\|\right] \\
&\geq \sup_{\{n\geq N\}} \left[-\|z_n\|\cdot\|z^*\| + \tfrac{1}{4}n\alpha - K\|z - z_n\|\right] \quad \text{(by (5.33))} \\
&= \infty \quad \text{(by (5.30))}.
\end{aligned}$$

Thus (5.32) holds.

Case 2: $\beta \neq 0$.

Take $v_n^* \in Bz_n$. We consider two subcases.

Subcase 2.1: $(v_n^*)_{n\in\mathbb{N}}$ is bounded. By (5.31),

$$F_{A+B}(z,z^*) \geq \sup_{\{n\in\mathbb{N}\}} \left[\langle z_n, z^*\rangle + \tfrac{1}{4}n\alpha + \langle z - z_n, v_n^*\rangle\right]$$

$$\geq \sup_{\{n\in\mathbb{N}\}} \left[-\|z_n\|\cdot\|z^*\| + \tfrac{1}{4}n\alpha - \|z-z_n\|\cdot\|v_n^*\|\right]$$

$$= \infty \quad \text{(by (5.30) and the boundedness of } (v_n^*)_{n\in\mathbb{N}}).$$

Hence (5.32) holds.

Subcase 2.2: $(v_n^*)_{n\in\mathbb{N}}$ is unbounded.

We first show

$$\limsup_{n\to\infty} \langle z - z_n, v_n^*\rangle \geq 0. \tag{5.34}$$

Since $(v_n^*)_{n\in\mathbb{N}}$ is unbounded and after passing to a subsequence if necessary, we assume that $\|v_n^*\| \neq 0, \forall n \in \mathbb{N}$ and that $\|v_n^*\| \to +\infty$. By $0 \in \operatorname{int}\operatorname{dom} B$ and Fact 5.1.3, there exist $\delta > 0$ and $M > 0$ such that

$$By \neq \varnothing \quad \text{and} \quad \sup_{y^*\in By} \|y^*\| \leq M, \quad \forall y \in \delta B_X. \tag{5.35}$$

Then we have

$$\begin{aligned}
&\langle z_n - y, v_n^* - y^*\rangle \geq 0, \quad \forall y \in \delta U_X, y^* \in By, n \in \mathbb{N}\\
&\Rightarrow \langle z_n, v_n^*\rangle - \langle y, v_n^*\rangle + \langle z_n - y, -y^*\rangle \geq 0, \quad \forall y \in \delta U_X, y^* \in By, n \in \mathbb{N}\\
&\Rightarrow \langle z_n, v_n^*\rangle - \langle y, v_n^*\rangle \geq \langle z_n - y, y^*\rangle, \quad \forall y \in \delta U_X, y^* \in By, n \in \mathbb{N}\\
&\Rightarrow \langle z_n, v_n^*\rangle - \langle y, v_n^*\rangle \geq -(\|z_n\| + \delta)M, \quad \forall y \in \delta U_X, n \in \mathbb{N} \quad \text{(by (5.86))}\\
&\Rightarrow \langle z_n, v_n^*\rangle \geq \langle y, v_n^*\rangle - (\|z_n\| + \delta)M, \quad \forall y \in \delta U_X, n \in \mathbb{N}\\
&\Rightarrow \langle z_n, v_n^*\rangle \geq \delta\|v_n^*\| - (\|z_n\| + \delta)M, \quad \forall n \in \mathbb{N}\\
&\Rightarrow \langle z_n, \tfrac{v_n^*}{\|v_n^*\|}\rangle \geq \delta - \tfrac{(\|z_n\|+\delta)M}{\|v_n^*\|}, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{5.36}$$

By the Banach-Alaoglu Theorem (see [69, Theorem 3.15]), there exist a weak* convergent *subnet* $(v^*_\gamma)_{\gamma\in\Gamma}$ of $(v^*_n)_{n\in\mathbb{N}}$, say

$$\frac{v^*_\gamma}{\|v^*_\gamma\|} \overset{\mathrm{w}^*}{\rightharpoonup} w^* \in X^*. \tag{5.37}$$

Using (5.30) and taking the limit in (5.36) along the subnet, we obtain

$$\langle \beta z, w^* \rangle \geq \delta. \tag{5.38}$$

Since $\beta > 0$, we have

$$\langle z, w^* \rangle \geq \tfrac{\delta}{\beta} > 0. \tag{5.39}$$

Now we assume to the contrary that

$$\limsup_{n\to\infty} \langle z - z_n, v^*_n \rangle < -\varepsilon,$$

for some $\varepsilon > 0$.

Then, for all n sufficiently large,

$$\langle z - z_n, v^*_n \rangle < -\tfrac{\varepsilon}{2},$$

and so

$$\langle z - z_n, \tfrac{v^*_n}{\|v^*_n\|} \rangle < -\tfrac{\varepsilon}{2\|v^*_n\|}. \tag{5.40}$$

Then by (5.30) and (5.37), taking the limit in (5.40) along the subnet again, we see that

$$\langle z - \beta z, w^* \rangle \leq 0.$$

Since $\beta < 1$, we deduce $\langle z, w^* \rangle \leq 0$ which contradicts (5.39). Hence (5.34) holds. By (5.31),

$$\begin{aligned}
F_{A+B}(z, z^*) &\geq \sup_{\{n \in \mathbb{N}\}} \left[\langle z_n, z^* \rangle + \tfrac{1}{4} n\alpha + \langle z - z_n, v_n^* \rangle \right] \\
&\geq \sup_{\{n \in \mathbb{N}\}} \left[-\|z_n\| \cdot \|z^*\| + \tfrac{1}{4} n\alpha + \langle z - z_n, v_n^* \rangle \right] \\
&\geq \limsup_{n \to \infty} \left[-\|z_n\| \cdot \|z^*\| + \tfrac{1}{4} n\alpha + \langle z - z_n, v_n^* \rangle \right] \\
&= \infty \quad \text{(by (5.30) and (5.34))}.
\end{aligned}$$

Hence

$$F_{A+B}(z, z^*) = \infty. \tag{5.41}$$

Therefore, we have proved (5.32) in all cases. However, (5.32) contradicts our original choice that $z \in P_X[\operatorname{dom} F_{A+B}]$. Hence $P_X[\operatorname{dom} F_{A+B}] \subseteq \overline{\operatorname{dom} A \cap \operatorname{dom} B}$ and thus (5.18) holds. Thus we have $\overline{P_X[\operatorname{dom} F_{A+B}]} = \overline{\operatorname{dom} A \cap \operatorname{dom} B}$. ■

Corollary 5.2.2 *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV) with convex domain, and $B : X \rightrightarrows X^*$ be maximally monotone with* $\operatorname{dom} A \cap$

$\operatorname{int}\operatorname{dom} B \neq \varnothing$. *Assume that* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. *Then*

$$\overline{P_X\left[\operatorname{dom} F_{A+B}\right]} = \overline{\operatorname{dom} A \cap \operatorname{dom} B}.$$

Proof. Combine Fact 5.1.8, Corollary 5.1.17 and Proposition 5.2.1. ■

Corollary 5.2.3 *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $B : X \rightrightarrows X^*$ *be maximally monotone with* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \varnothing$. *Assume that* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. *Then* $\overline{P_X\left[\operatorname{dom} F_{A+B}\right]} = \overline{\operatorname{dom} A \cap \operatorname{dom} B}$.

Proof. Combine Fact 5.1.8, Corollary 5.1.18 and Proposition 5.2.1. Alternatively, combine Fact 5.1.14 and Corollary 5.2.2. ■

We are now ready for our main result in this section.

Theorem 5.2.4 *Let* $A, B : X \rightrightarrows X^*$ *be maximally monotone with* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \varnothing$. *Assume that* $A + N_{\overline{\operatorname{dom} B}}$ *is maximally monotone of type (FPV), and that* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. *Then* $A + B$ *is maximally monotone.*

Proof. After translating the graphs if necessary, we can and do assume that $0 \in \operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B$ and that $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} B$. By Fact 5.1.5, $\operatorname{dom} A \subseteq P_X(\operatorname{dom} F_A)$ and $\operatorname{dom} B \subseteq P_X(\operatorname{dom} F_B)$. Hence,

$$\bigcup_{\lambda>0} \lambda\big(P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)\big) = X. \tag{5.42}$$

Thus, by Fact 5.1.7, it suffices to show that

$$F_{A+B}(z,z^*) \geq \langle z, z^*\rangle, \quad \forall (z,z^*) \in X \times X^*. \tag{5.43}$$

Take $(z,z^*) \in X \times X^*$. Then

$$\begin{aligned} F_{A+B}(z,z^*) = \sup_{\{x,x^*,y^*\}} & [\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle + \langle z-x, y^*\rangle \\ & - \iota_{\operatorname{gra} A}(x,x^*) - \iota_{\operatorname{gra} B}(x,y^*)]. \end{aligned} \tag{5.44}$$

Assume to the contrary that

$$F_{A+B}(z,z^*) < \langle z, z^*\rangle. \tag{5.45}$$

Then $(z,z^*) \in \operatorname{dom} F_{A+B}$ and, by Proposition 5.2.1,

$$z \in \overline{\operatorname{dom} A \cap \operatorname{dom} B} = \overline{P_X\left[\operatorname{dom} F_{A+B}\right]}. \tag{5.46}$$

Next, we show that

$$F_{A+B}(\lambda z, \lambda z^*) \geq \lambda^2 \langle z, z^*\rangle, \quad \forall \lambda \in \left]0,1\right[. \tag{5.47}$$

Let $\lambda \in \left]0,1\right[$. By (5.46) and Fact 5.1.8, $z \in \overline{P_X \operatorname{dom} F_B}$. By Fact 5.1.8 again and $0 \in \operatorname{int} \operatorname{dom} B$, $0 \in \operatorname{int} \overline{P_X \operatorname{dom} F_B}$. Then, by [92, Theorem 1.1.2(ii)], we have

$$\lambda z \in \operatorname{int} \overline{P_X \operatorname{dom} F_B} = \operatorname{int}\left[P_X \operatorname{dom} F_B\right]. \tag{5.48}$$

Combining (5.48) and Fact 5.1.8, we see that $\lambda z \in \operatorname{int}\operatorname{dom} B$.

We consider two cases.

Case 1: $\lambda z \in \operatorname{dom} A$.

By (5.44),

$$\begin{aligned} F_{A+B}(\lambda z, \lambda z^*) \geq \sup_{\{x^*, y^*\}} &\big[\langle \lambda z, \lambda z^*\rangle + \langle \lambda z, x^*\rangle - \langle \lambda z, x^*\rangle + \langle \lambda z - \lambda z, y^*\rangle \\ &- \iota_{\operatorname{gra} A}(\lambda z, x^*) - \iota_{\operatorname{gra} B}(\lambda z, y^*)\big] \\ = &\langle \lambda z, \lambda z^*\rangle. \end{aligned}$$

Hence (5.47) holds.

Case 2: $\lambda z \notin \operatorname{dom} A$.

Using $0 \in \operatorname{dom} A \cap \operatorname{dom} B$ and the convexity of $\overline{\operatorname{dom} A \cap \operatorname{dom} B}$ (which follows from (5.46)), we obtain $\lambda z \in \overline{\operatorname{dom} A \cap \operatorname{dom} B} \subseteq \overline{\operatorname{dom} A \cap \overline{\operatorname{dom} B}}$. Set

$$U_n = \lambda z + \tfrac{1}{n} U_X, \quad \forall n \in \mathbb{N}. \tag{5.49}$$

Then $U_n \cap \operatorname{dom}(A + N_{\overline{\operatorname{dom} B}}) \neq \varnothing$. Since $(\lambda z, \lambda z^*) \notin \operatorname{gra}(A + N_{\overline{\operatorname{dom} B}})$, $\lambda z \in U_n$, and $A + N_{\overline{\operatorname{dom} B}}$ is of type (FPV), there exists $(b_n, b_n^*) \in \operatorname{gra}(A + N_{\overline{\operatorname{dom} B}})$ such that $b_n \in U_n$ and

$$\langle \lambda z, b_n^*\rangle + \langle b_n, \lambda z^*\rangle - \langle b_n, b_n^*\rangle > \lambda^2 \langle z, z^*\rangle, \quad \forall n \in \mathbb{N}. \tag{5.50}$$

Since $\lambda z \in \operatorname{int}\operatorname{dom} B$ and $b_n \to \lambda z$, by Fact 5.1.3, there exist $N \in \mathbb{N}$ and $M > 0$ such that

$$b_n \in \operatorname{int}\operatorname{dom} B \quad \text{and} \quad \sup_{v^* \in Bb_n} \|v^*\| \leq M, \quad \forall n \geq N. \tag{5.51}$$

Hence $N_{\overline{\operatorname{dom} B}}(b_n) = \{0\}$ and thus $(b_n, b_n^*) \in \operatorname{gra} A$ for every $n \geq N$. Thus by (5.44), (5.50) and (5.51),

$$\begin{aligned}
&F_{A+B}(\lambda z, \lambda z^*) \\
&\geq \sup_{\{v^* \in Bb_n\}} \left[\langle b_n, \lambda z^*\rangle + \langle \lambda z, b_n^*\rangle - \langle b_n, b_n^*\rangle + \langle \lambda z - b_n, v^*\rangle\right], \quad \forall n \geq N \\
&\geq \sup_{\{v^* \in Bb_n\}} \left[\lambda^2\langle z, z^*\rangle + \langle \lambda z - b_n, v^*\rangle\right], \quad \forall n \geq N \quad (\text{by } (5.50)) \\
&\geq \sup \left[\lambda^2\langle z, z^*\rangle - M\|\lambda z - b_n\|\right], \quad \forall n \geq N \quad (\text{by } (5.51)) \\
&\geq \lambda^2\langle z, z^*\rangle \quad (\text{by } b_n \to \lambda z).
\end{aligned} \tag{5.52}$$

Hence $F_{A+B}(\lambda z, \lambda z^*) \geq \lambda^2\langle z, z^*\rangle$.

We have established that (5.47) holds in both cases. Since $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} B$, we obtain $(\forall (x, x^*) \in \operatorname{gra}(A+B))\ \langle x, x^*\rangle \geq 0$. Thus, $F_{A+B}(0,0) = 0$. Now define

$$f\colon\ [0,1] \to \mathbb{R}\colon t \to F_{A+B}(tz, tz^*).$$

Then f is continuous on $[0,1]$ by [92, Proposition 2.1.6]. From (5.47), we obtain

$$F_{A+B}(z, z^*) = \lim_{\lambda \to 1^-} F_{A+B}(\lambda z, \lambda z^*) \geq \lim_{\lambda \to 1^-} \langle \lambda z, \lambda z^*\rangle = \langle z, z^*\rangle, \tag{5.53}$$

which contradicts (5.45). Hence

$$F_{A+B}(z, z^*) \geq \langle z, z^* \rangle. \tag{5.54}$$

Therefore, (5.43) holds, and $A + B$ is maximally monotone. ■

Theorem 5.2.4 allows us to deduce both new and previously known sum theorems.

Corollary 5.2.5 *Let $f : X \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex, and let $B : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. Assume that $\operatorname{dom} \partial f \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. Then $\partial f + B$ is maximally monotone.*

Proof. By Fact 5.1.8 and Fact 5.1.1, $\partial f + N_{\overline{\operatorname{dom} B}} = \partial(f + \iota_{\overline{\operatorname{dom} B}})$. Then by Fact 5.1.12, $\partial f + N_{\overline{\operatorname{dom} B}}$ is type of (FPV). Now apply Theorem 5.2.4. ■

Corollary 5.2.6 *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone with full domain. Then $A + B$ is maximally monotone.*

Proof. Since $A + N_{\overline{\operatorname{dom} B}} = A + N_X = A$ and thus $A + N_{\overline{\operatorname{dom} B}}$ is maximally monotone of type (FPV), the conclusion follows from Theorem 5.2.4. ■

Corollary 5.2.7 (Verona-Verona) (See [82, Corollary 2.9(a)] or [74, Theorem 53.1].) *Let $f : X \to \,]-\infty, +\infty]$ be proper, lower semicontinuous, and convex, and let $B : X \rightrightarrows X^*$ be maximally monotone with full domain. Then $\partial f + B$ is maximally monotone.*

Proof. Clear from Corollary 5.2.5. Alternatively, combine Fact 5.1.12 and Corollary 5.2.6. ■

Corollary 5.2.8 (Heisler) (See [62, Remark, page 17].) *Let $A, B : X \rightrightarrows X^*$ be maximally monotone with full domain. Then $A + B$ is maximally monotone.*

Proof. Let C be a nonempty closed convex subset of X. By Corollary 5.2.7, $N_C + A$ is maximally monotone. Thus, A is of type (FPV) by Fact 5.1.15. The conclusion now follows from Corollary 5.2.6. ■

Corollary 5.2.9 *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV) with convex domain, and let $B : X \rightrightarrows X^*$ be maximally monotone with* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$*. Assume that* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$*. Then $A + B$ is maximally monotone.*

Proof. Combine Fact 5.1.8, Corollary 5.1.17 and Theorem 5.2.4. ■

Corollary 5.2.10 (Voisei) (See [85].) *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV) with convex domain, let C be a nonempty closed convex subset of X, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$*. Then $A + N_C$ is maximally monotone.*

Proof. Apply Corollary 5.2.9. ■

Corollary 5.2.11 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximally monotone with* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$*. Assume that* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$*. Then $A + B$ is maximally monotone.*

Proof. Combine Fact 5.1.14 and Corollary 5.2.9. ■

Corollary 5.2.12 (See [16, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, let C be a nonempty closed convex subset of X, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then $A + N_C$ is maximally monotone.*

Proof. Apply Corollary 5.2.11. ■

Corollary 5.2.13 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximally monotone with full domain. Then $A + B$ is maximally monotone.*

Proof. Apply Corollary 5.2.11. ■

Example 5.2.14 *Suppose that $X = L^1[0, 1]$, let*

$$D = \big\{x \in X \mid x \textit{ is absolutely continuous}, x(0) = 0, x' \in X^*\big\},$$

and set

$$A \colon X \rightrightarrows X^* \colon x \mapsto \begin{cases} \{x'\}, & \textit{if } x \in D; \\ \varnothing, & \textit{otherwise.} \end{cases}$$

By Phelps and Simons' [63, Example 4.3], A is an at most single-valued maximally monotone linear relation with proper dense domain, and A is neither symmetric nor skew. Now let J be the duality mapping, *i.e., $J = \partial\frac{1}{2}\|\cdot\|^2$. Then Corollary 5.2.13 implies that $A + J$ is maximally monotone. To the best of our knowledge, the maximal monotonicity of $A + J$ cannot be deduced from any previously known result.*

Remark 5.2.15 *In [19], it was shown that the sum problem has an affirmative solution when A is a linear relation, B is the subdifferential operator of a proper lower semicontinuous sublinear function, and Rockafellar's constraint qualification holds. When the domain of the subdifferential operator is closed, then that result can be deduced from Theorem 5.2.4. However, it is possible that the domain of the subdifferential operator of a proper lower semicontinuous sublinear function does not have to be closed. For an example, see [3, Example 5.4]: Set $C = \{(x,y) \in \mathbb{R}^2 \mid 0 < 1/x \leq y\}$ and $f = \iota_C^*$ given by*

$$f(x,y) := \begin{cases} -2\sqrt{xy}, & \text{if } x \leq 0 \text{ and } y \leq 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then f is not subdifferentiable at any point in the boundary of its domain, except at the origin. Thus, in the general case, we do not know whether or not it is possible to deduce the result in [19] from Theorem 5.2.4.

5.3 Maximality of the sum of a linear relation and a subdifferential operator

Theorem 5.3.1 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $f : X \to \,]-\infty, +\infty]$ be a proper lower semicontinuous convex function with $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. Then $A + \partial f$ is maximally monotone.*

Proof. After translating the graphs if necessary, we can and do assume that $0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f$ and that $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} \partial f$. By Fact 5.1.5 and

Fact 5.1.2, $\operatorname{dom} A \subseteq P_X(\operatorname{dom} F_A)$ and $\operatorname{dom} \partial f \subseteq P_X(\operatorname{dom} F_{\partial f})$. Hence,

$$\bigcup_{\lambda>0} \lambda\big(P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_{\partial f})\big) = X. \tag{5.55}$$

Thus, by Fact 5.1.2 and Fact 5.1.7, it suffices to show that

$$F_{A+\partial f}(z,z^*) \geq \langle z, z^*\rangle, \quad \forall (z,z^*) \in X \times X^*. \tag{5.56}$$

Take $(z,z^*) \in X \times X^*$. Then

$$\begin{aligned} F_{A+\partial f}(z,z^*) = \sup_{\{x,x^*,y^*\}} &[\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle + \langle z-x, y^*\rangle \\ &- \iota_{\operatorname{gra} A}(x,x^*) - \iota_{\operatorname{gra} \partial f}(x,y^*)]. \end{aligned} \tag{5.57}$$

Assume to the contrary that

$$F_{A+\partial f}(z,z^*) + \lambda < \langle z, z^*\rangle, \tag{5.58}$$

where $\lambda > 0$.

Thus by (5.58),

$$(z,z^*) \text{ is monotonically related to } \operatorname{gra}(A+\partial f). \tag{5.59}$$

We claim that

$$z \notin \operatorname{dom} A. \tag{5.60}$$

Indeed, if $z \in \operatorname{dom} A$, apply (5.59) and Lemma 5.1.20 to get $z \in \operatorname{dom} \partial f$. Thus $z \in \operatorname{dom} A \cap \operatorname{dom} \partial f$ and hence $F_{A+\partial f}(z, z^*) \geq \langle z, z^* \rangle$ which contradicts (5.58). This establishes (5.60).

By (5.58) and the assumption that $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} \partial f$, we have

$$\begin{aligned}
&\sup\left[\langle 0, z^*\rangle + \langle z, A0\rangle - \langle 0, A0\rangle + \langle z, \partial f(0)\rangle\right] \\
&= \sup_{a^* \in A0, b^* \in \partial f(0)} \left[\langle z, a^*\rangle + \langle z, b^*\rangle\right] < \langle z, z^*\rangle.
\end{aligned}$$

Thus, because $A0$ is a linear subspace,

$$z \in X \cap (A0)^{\perp}. \tag{5.61}$$

Then, by Proposition 3.2.2(i), we have

$$z \in \overline{\operatorname{dom} A}. \tag{5.62}$$

Combine (5.60) and (5.62),

$$z \in \overline{\operatorname{dom} A} \backslash \operatorname{dom} A. \tag{5.63}$$

Set

$$U_n = z + \tfrac{1}{n} U_X, \quad \forall n \in \mathbb{N}. \tag{5.64}$$

By (5.63), $(z, z^*) \notin \operatorname{gra} A$ and $U_n \cap \operatorname{dom} A \neq \varnothing$. Since $z \in U_n$ and A is of type (FPV) by Fact 5.1.14, there exists $(a_n, a_n^*) \in \operatorname{gra} A$ with $a_n \in U_n, n \in \mathbb{N}$

such that

$$\langle z, a_n^* \rangle + \langle a_n, z^* \rangle - \langle a_n, a_n^* \rangle > \langle z, z^* \rangle. \tag{5.65}$$

Then we have

$$a_n \to z. \tag{5.66}$$

Now we claim that

$$z \in \overline{\operatorname{dom} \partial f}. \tag{5.67}$$

Suppose to the contrary that $z \notin \overline{\operatorname{dom} \partial f}$. By the Brøndsted-Rockafellar Theorem (see [61, Theorem 3.17] or [92, Theorem 3.1.2]), $\overline{\operatorname{dom} \partial f} = \overline{\operatorname{dom} f}$. Since $0 \in \operatorname{int} \operatorname{dom} \partial f \subseteq \operatorname{int} \operatorname{dom} f \subseteq \operatorname{int} \overline{\operatorname{dom} f}$, then by Lemma 5.1.19, there exists $\delta \in \,]0,1[$ such that

$$\delta z \in \operatorname{bdry} \overline{\operatorname{dom} f}. \tag{5.68}$$

Set $g_n : X \to \,]-\infty, +\infty]$ by

$$g_n = f + \iota_{[0,a_n]}, \quad n \in \mathbb{N} \tag{5.69}$$

Since $z \notin \overline{\operatorname{dom} f}$, $z \notin \operatorname{dom} f \cap [0, a_n] = \operatorname{dom} g_n$. Thus $(z, z^*) \notin \operatorname{gra} \partial g_n$. Then by Fact 5.1.10, there exist $\beta_n \in [0,1]$ and $x_n^* \in \partial g_n(\beta_n a_n)$ with $x_n^* \neq z^*$

and $\beta_n a_n \neq z$ such that

$$\frac{\|z-\beta_n a_n\|}{\|z^*-x_n^*\|} \geq n \tag{5.70}$$

$$\frac{\langle z-\beta_n a_n, z^*-x_n^*\rangle}{\|z-\beta_n a_n\|\cdot\|z^*-x_n^*\|} < -\tfrac{3}{4}. \tag{5.71}$$

By (5.66), $\|z-\beta_n a_n\|$ is bounded. Then by (5.70), we have

$$x_n^* \to z^*. \tag{5.72}$$

Since $0 \in \operatorname{int}\operatorname{dom} f$, f is continuous at 0 by Fact 5.1.4. Then by $0 \in \operatorname{dom} f \cap \operatorname{dom} \iota_{[0,a_n]}$ and Fact 5.1.1, we have that there exist $w_n^* \in \partial f(\beta_n a_n)$ and $v_n^* \in \partial\iota_{[0,a_n]}(\beta_n a_n)$ such that $x_n^* = w_n^* + v_n^*$. Then by (5.72),

$$w_n^* + v_n^* \to z^*. \tag{5.73}$$

Since $\beta_n \in [0,1]$, there exists a convergent subsequence of $(\beta_n)_{n\in\mathbb{N}}$, which, for convenience, we still denote by $(\beta_n)_{n\in\mathbb{N}}$. Then $\beta_n \to \beta$, where $\beta \in [0,1]$. Then by (5.66),

$$\beta_n a_n \to \beta z. \tag{5.74}$$

We claim that

$$\beta \leq \delta < 1. \tag{5.75}$$

In fact, suppose to the contrary that $\beta > \delta$. By (5.74), $\beta z \in \overline{\operatorname{dom} f}$. Then by $0 \in \operatorname{int}\operatorname{dom} f$ and [92, Theorem 1.1.2(ii)], $\delta z = \frac{\delta}{\beta}\beta z \in \operatorname{int}\overline{\operatorname{dom} f}$, which contradicts (5.68).

We can and do suppose that $\beta_n < 1$ for every $n \in \mathbb{N}$. Then by $v_n^* \in \partial\iota_{[0,a_n]}(\beta_n a_n)$, we have

$$\langle v_n^*, a_n - \beta_n a_n\rangle \leq 0. \tag{5.76}$$

Dividing by $(1-\beta_n)$ on both sides of the above inequality, we have

$$\langle v_n^*, a_n\rangle \leq 0. \tag{5.77}$$

Since $(0,0) \in \operatorname{gra} A$, $\langle a_n, a_n^*\rangle \geq 0, \forall n \in \mathbb{N}$. Then by (5.65), we have

$$\begin{aligned}\langle z, \beta_n a_n^*\rangle + \langle \beta_n a_n, z^*\rangle - \beta_n^2\langle a_n, a_n^*\rangle \quad &\geq \langle \beta_n z, a_n^*\rangle + \langle \beta_n a_n, z^*\rangle - \beta_n\langle a_n, a_n^*\rangle \\ \geq \beta_n\langle z, z^*\rangle. \end{aligned}\tag{5.78}$$

Then by (5.78),

$$\langle z - \beta_n a_n, \beta_n a_n^*\rangle \geq \langle \beta_n z - \beta_n a_n, z^*\rangle. \tag{5.79}$$

Since $\operatorname{gra} A$ is a linear subspace and $(a_n, a_n^*) \in \operatorname{gra} A$, $(\beta_n a_n, \beta_n a_n^*) \in \operatorname{gra} A$. By (5.58), we have

$$\begin{aligned}\lambda <&\langle z - \beta_n a_n, z^* - w_n^* - \beta_n a_n^*\rangle \\ &= \langle z - \beta_n a_n, z^* - w_n^*\rangle + \langle z - \beta_n a_n, -\beta_n a_n^*\rangle\end{aligned}$$

$$< -\tfrac{3}{4}\|z-\beta_n a_n\|\cdot\|z^*-w_n^*-v_n^*\| + \langle z-\beta_n a_n, v_n^*\rangle$$
$$+ \langle z-\beta_n a_n, -\beta_n a_n^*\rangle \quad \text{(by (5.71))}$$
$$\leq -\tfrac{3}{4}\|z-\beta_n a_n\|\cdot\|z^*-w_n^*-v_n^*\| + \langle z-\beta_n a_n, v_n^*\rangle$$
$$- \langle \beta_n z-\beta_n a_n, z^*\rangle \quad \text{(by (5.79))}.$$

Then

$$\lambda < \langle z-\beta_n a_n, v_n^*\rangle - \langle \beta_n z-\beta_n a_n, z^*\rangle. \tag{5.80}$$

Now we consider two cases:

Case 1: $(w_n^*)_{n\in\mathbb{N}}$ is bounded.

By (5.73), $(v_n^*)_{n\in\mathbb{N}}$ is bounded. By the Banach-Alaoglu Theorem (see [69, Theorem 3.15]), there exist a weak* convergent *subnet* $(v_\gamma^*)_{\gamma\in\Gamma}$ of $(v_n^*)_{n\in\mathbb{N}}$, say

$$v_\gamma^* \overset{\mathrm{w}^*}{\rightharpoonup} v_\infty^* \in X^*. \tag{5.81}$$

Combine (5.66), (5.74) and (5.81), and pass the limit along the subnet of (5.80) to get that

$$\lambda \leq \langle z-\beta z, v_\infty^*\rangle. \tag{5.82}$$

By (5.75), divide by $(1-\beta)$ on both sides of (5.82) to get

$$\langle z, v_\infty^* \rangle \geq \tfrac{\lambda}{1-\beta} > 0. \tag{5.83}$$

On the other hand, by (5.66) and (5.81), taking the limit along the subnet of (5.77) we get that

$$\langle v_\infty^*, z \rangle \leq 0, \tag{5.84}$$

which contradicts (5.83).

Case 2: $(w_n^*)_{n\in\mathbb{N}}$ is unbounded.

Since $(w_n^*)_{n\in\mathbb{N}}$ is unbounded and after passing to a subsequence if necessary, we assume that $\|w_n^*\| \neq 0, \forall n \in \mathbb{N}$ and that $\|w_n^*\| \to +\infty$.

By the Banach-Alaoglu Theorem again, there exist a weak* convergent *subnet* $(w_\nu^*)_{\nu\in I}$ of $(w_n^*)_{n\in\mathbb{N}}$, say

$$\frac{w_\nu^*}{\|w_\nu^*\|} \overset{\mathrm{w}^*}{\rightharpoonup} w_\infty^* \in X^*. \tag{5.85}$$

By $0 \in \operatorname{int}\operatorname{dom}\partial f$ and Fact 5.1.3, there exist $\rho > 0$ and $M > 0$ such that

$$\partial f(y) \neq \varnothing \quad \text{and} \quad \sup_{y^*\in\partial f(y)} \|y^*\| \leq M, \quad \forall y \in \rho U_X. \tag{5.86}$$

Then by $w_n^* \in \partial f(\beta_n a_n)$, we have

$$\langle \beta_n a_n - y, w_n^* - y^* \rangle \geq 0, \quad \forall y \in \rho U_X, y^* \in \partial f(y)$$

$$\Rightarrow \langle \beta_n a_n, w_n^*\rangle - \langle y, w_n^*\rangle + \langle \beta_n a_n - y, -y^*\rangle \geq 0, \quad \forall y \in \rho U_X, y^* \in \partial f(y)$$

$$\Rightarrow \langle \beta_n a_n, w_n^*\rangle - \langle y, w_n^*\rangle \geq \langle \beta_n a_n - y, y^*\rangle, \quad \forall y \in \rho U_X, y^* \in \partial f(y)$$

$$\Rightarrow \langle \beta_n a_n, w_n^*\rangle - \langle y, w_n^*\rangle \geq -(\|\beta_n a_n\| + \rho)M, \quad \forall y \in \rho U_X \quad \text{(by (5.86))}$$

$$\Rightarrow \langle \beta_n a_n, w_n^*\rangle \geq \langle y, w_n^*\rangle - (\|\beta_n a_n\| + \rho)M, \quad \forall y \in \rho U_X$$

$$\Rightarrow \langle \beta_n a_n, w_n^*\rangle \geq \rho\|w_n^*\| - (\|\beta_n a_n\| + \rho)M$$

$$\Rightarrow \langle \beta_n a_n, \tfrac{w_n^*}{\|w_n^*\|}\rangle \geq \rho - \frac{(\|\beta_n a_n\| + \rho)M}{\|w_n^*\|}, \quad \forall n \in \mathbb{N}. \tag{5.87}$$

Combining (5.74) and (5.85), taking the limit in (5.87) along the subnet, we obtain

$$\langle \beta z, w_\infty^*\rangle \geq \rho. \tag{5.88}$$

Then we have $\beta \neq 0$ and thus $\beta > 0$. Then by (5.88),

$$\langle z, w_\infty^* \geq \tfrac{\rho}{\beta} > 0. \tag{5.89}$$

By (5.73) and $\frac{z^*}{\|w_n^*\|} \to 0$, we have

$$\frac{w_n^*}{\|w_n^*\|} + \frac{v_n^*}{\|w_n^*\|} \to 0. \tag{5.90}$$

By(5.85), taking the weak* limit in (5.90) along the subnet, we obtain

$$\frac{v_\nu^*}{\|w_\nu^*\|} \overset{\mathrm{w}^*}{\rightharpoonup} -w_\infty^*. \tag{5.91}$$

Dividing by $\|w_n^*\|$ on the both sides of (5.80), we get that

$$\frac{\lambda}{\|w_n^*\|} < \langle z - \beta_n a_n, \frac{v_n^*}{\|w_n^*\|}\rangle - \frac{\langle \beta_n z - \beta_n a_n, z^*\rangle}{\|w_n^*\|}. \tag{5.92}$$

Combining (5.74), (5.66) and (5.91), taking the limit in (5.92) along the subnet, we obtain

$$\langle z - \beta z, -w_\infty^*\rangle \geq 0. \tag{5.93}$$

By (5.75) and (5.93),

$$\langle z, -w_\infty^*\rangle \geq 0, \tag{5.94}$$

which contradicts (5.89).

Altogether $z \in \overline{\operatorname{dom} \partial f} = \overline{\operatorname{dom} f}$.

Next, we show that

$$F_{A+\partial f}(tz, tz^*) \geq t^2\langle z, z^*\rangle, \quad \forall t \in \left]0,1\right[. \tag{5.95}$$

Let $t \in \left]0,1\right[$. By $0 \in \operatorname{int} \operatorname{dom} f$ and [92, Theorem 1.1.2(ii)], we have

$$tz \in \operatorname{int} \operatorname{dom} f. \tag{5.96}$$

By Fact 5.1.4,

$$tz \in \operatorname{int} \operatorname{dom} \partial f. \tag{5.97}$$

Set

$$H_n = tz + \tfrac{1}{n}U_X, \quad \forall n \in \mathbb{N}. \tag{5.98}$$

Since $\operatorname{dom} A$ is a linear subspace, $tz \in \overline{\operatorname{dom} A} \backslash \operatorname{dom} A$ by (5.63). Then $H_n \cap \operatorname{dom} A \neq \varnothing$. Since $(tz, tz^*) \notin \operatorname{gra} A$ and $tz \in H_n$, A is of type (FPV) by Fact 5.1.14, there exists $(b_n, b_n^*) \in \operatorname{gra} A$ such that $b_n \in H_n$ and

$$\langle tz, b_n^* \rangle + \langle b_n, tz^* \rangle - \langle b_n, b_n^* \rangle > t^2 \langle z, z^* \rangle, \quad \forall n \in \mathbb{N}. \tag{5.99}$$

Since $tz \in \operatorname{int} \operatorname{dom} \partial f$ and $b_n \to tz$, by Fact 5.1.3, there exist $N \in \mathbb{N}$ and $K > 0$ such that

$$b_n \in \operatorname{int} \operatorname{dom} \partial f \quad \text{and} \quad \sup_{v^* \in \partial f(b_n)} \|v^*\| \leq K, \quad \forall n \geq N. \tag{5.100}$$

Hence

$$\begin{aligned}
&F_{A+\partial f}(tz, tz^*) \\
&\geq \sup_{\{c^* \in \partial f(b_n)\}} \left[\langle b_n, tz^* \rangle + \langle tz, b_n^* \rangle - \langle b_n, b_n^* \rangle + \langle tz - b_n, c^* \rangle \right], \quad \forall n \geq N \\
&\geq \sup_{\{c^* \in \partial f(b_n)\}} \left[t^2 \langle z, z^* \rangle + \langle tz - b_n, c^* \rangle \right], \quad \forall n \geq N \quad \text{(by (5.99))} \\
&\geq \sup \left[t^2 \langle z, z^* \rangle - K \|tz - b_n\| \right], \quad \forall n \geq N \quad \text{(by (5.100))} \\
&\geq t^2 \langle z, z^* \rangle \quad (\text{by } b_n \to tz).
\end{aligned} \tag{5.101}$$

Hence $F_{A+\partial f}(tz, tz^*) \geq t^2 \langle z, z^* \rangle$.

We have established (5.95). Since $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} \partial f$, we obtain $(\forall (d,d^*) \in \operatorname{gra}(A+\partial f))\ \langle d, d^*\rangle \geq 0$. Thus, $F_{A+\partial f}(0,0) = 0$. Now define

$$j\colon\ [0,1] \to \mathbb{R}\colon t \to F_{A+\partial f}(tz, tz^*).$$

Then j is continuous on $[0,1]$ by (5.58) and [92, Proposition 2.1.6]. From (5.95), we obtain

$$F_{A+\partial f}(z,z^*) = \lim_{t\to 1^-} F_{A+\partial f}(tz,tz^*) \geq \lim_{t\to 1^-} \langle tz, tz^*\rangle = \langle z, z^*\rangle, \tag{5.102}$$

which contradicts (5.58). Hence

$$F_{A+\partial f}(z,z^*) \geq \langle z, z^*\rangle. \tag{5.103}$$

Therefore, (5.56) holds, and $A+\partial f$ is maximally monotone. ■

Remark 5.3.2 *In Theorem 5.3.1, when* $\operatorname{int}\operatorname{dom} A \cap \operatorname{dom} \partial f \neq \varnothing$*, we have* $\operatorname{dom} A = X$ *since* $\operatorname{dom} A$ *is a linear subspace. Therefore, we can obtain the maximal monotonicity of* $A+\partial f$ *from the Verona-Verona result (see [82, Corollary 2.9(a)], [74, Theorem 53.1] or [90, Corollary 3.7]).*

Corollary 5.3.3 *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and* $f : X \to \,]-\infty,+\infty]$ *be a proper lower semicontinuous convex function with* $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} \partial f \neq \varnothing$*. Then* $A+\partial f$ *is of type* (FPV)*.*

Proof. By Theorem 5.3.1, $A+\partial f$ is maximally monotone. Let C be a nonempty closed convex subset of X, and suppose that $\operatorname{dom}(A+\partial f)\cap \operatorname{int} C \neq \varnothing$. Let $x_1 \in \operatorname{dom} A \cap \operatorname{int}\operatorname{dom}\partial f$ and $x_2 \in \operatorname{dom}(A+\partial f) \cap \operatorname{int} C$. Thus, there exists $\delta > 0$ such that $x_1 + \delta U_X \subseteq \operatorname{dom} f$ and $x_2 + \delta U_X \subseteq C$. Then for small enough $\lambda \in]0, 1[$, we have $x_2 + \lambda(x_1 - x_2) + \frac{1}{2}\delta U_X \subseteq C$. Clearly, $x_2+\lambda(x_1-x_2)+\lambda\delta U_X \subseteq \operatorname{dom} f$. Thus $x_2+\lambda(x_1-x_2)+\frac{\lambda\delta}{2}U_X \subseteq \operatorname{dom} f \cap C = \operatorname{dom}(f + \iota_C)$. By Fact 5.1.4, $x_2 + \lambda(x_1 - x_2) + \frac{\lambda\delta}{2}U_X \subseteq \operatorname{dom} \partial(f + \iota_C)$. Since $\operatorname{dom} A$ is convex, $x_2 + \lambda(x_1 - x_2) \in \operatorname{dom} A$ and $x_2 + \lambda(x_1 - x_2) \in \operatorname{dom} A \cap \operatorname{int}[\operatorname{dom} \partial(f + \iota_C)]$. By Fact 5.1.1 , $\partial f + N_C = \partial(f + \iota_C)$. Then, by Theorem 5.3.1 (applied to A and $f + \iota_C$), $A + \partial f + N_C = A + \partial(f + \iota_C)$ is maximally monotone. By Fact 5.1.15, $A + \partial f$ is of type (FPV). ■

Corollary 5.3.4 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function with* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. *Then*

$$\overline{\operatorname{dom}(A + \partial f)} = \overline{\operatorname{conv} \operatorname{dom}(A + \partial f)} = \overline{P_X \operatorname{dom} F_{A+\partial f}}.$$

Proof. Combine Corollary 5.3.3 and Fact 5.1.13. ■

Now by Corollary 5.3.3, we can deduce Fact 5.1.14 that is used in the proof of Theorem 5.3.1.

Corollary 5.3.5 (Simons) (See [74, Theorem 46.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then A is of type (FPV).*

Proof. Let $f = \iota_X$. Then by Corollary 5.3.3, we have that $A = A + \partial f$ is of type (FPV). ■

Corollary 5.3.6 (See [16, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, let C be a nonempty closed convex subset of X,* *and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then* $A + N_C$ *is maximally monotone.*

Corollary 5.3.7 (See [19, Theorem 3.1].) *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, let* $f : X \to \left]-\infty, +\infty\right]$ *be a proper lower semicontinuous sublinear function, and suppose that* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. *Then* $A + \partial f$ *is maximally monotone.*

5.4 An example and comments

Example 5.4.1 *Suppose that* $X = L^1[0,1]$ *with norm* $\|\cdot\|_1$, *let*

$$D = \big\{x \in X \mid x \text{ is absolutely continuous}, x(0) = 0, x' \in X^*\big\},$$

and set

$$A\colon X \rightrightarrows X^* : x \mapsto \begin{cases} \{x'\}, & \text{if } x \in D; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Define $f : X \to \left]-\infty, +\infty\right]$ *by*

$$f(x) = \begin{cases} \frac{1}{1-\|x\|_1^2}, & \text{if} \|x\| < 1; \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.104}$$

Clearly, X *is a nonreflexive Banach space. By Phelps and Simons' [63, Example 4.3],* A *is an at most single-valued maximally monotone linear relation with proper dense domain, and* A *is neither symmetric nor skew. Since* $g(t) = \frac{1}{1-t^2}$ *is convex and increasing on* $[0,1[$ *(by* $g''(t) = 2(1-t^2)^{-2} + 8t^2(1-t^2)^{-3} \geq 0, \forall t \in [0,1[$*),* f *is convex. Clearly,* f *is proper lower* *semicontinuous, and by Fact 5.1.4, we have*

$$\operatorname{dom} f = U_X = \operatorname{int}\operatorname{dom} f = \operatorname{dom}\partial f = \operatorname{int}[\operatorname{dom}\partial f]. \tag{5.105}$$

Since $0 \in \operatorname{dom} A \cap \operatorname{int}[\operatorname{dom}\partial f]$, *Theorem 5.3.1 implies that* $A + \partial f$ *is maximally monotone. To the best of our knowledge, the maximal monotonicity of* $A + \partial f$ *cannot be deduced from any previously known result.*

Remark 5.4.2 *To the best of our knowledge, the results in [19, 82, 84, 86, 90] cannot establish the maximal monotonicity in Example 5.4.1.*

(1) Verona and Verona (see [82, Corollary 2.9(a)] or [74, Theorem 53.1] or [90, Corollary 3.7]) showed the following: "Let $f : X \to\ ]-\infty, +\infty]$ *be proper, lower semicontinuous, and convex, let* $A : X \rightrightarrows X^*$ *be maximally monotone, and suppose that* $\operatorname{dom} A = X$. *Then* $\partial f + A$ *is maximally monotone." The* $\operatorname{dom} A$ *in Example 5.4.1 is proper dense, hence* $A+\partial f$ *in Example 5.4.1 cannot be deduced from the Verona -Verona result.*

(2) In [84, Theorem 5.10(η)], Voisei showed that the sum problem has an affirmative solution when $\operatorname{dom} A \cap \operatorname{dom} B$ *is closed,* $\overline{\operatorname{dom} A}$ *is convex and Rockafellar's constraint qualification holds. In Example 5.4.1,* $\operatorname{dom} A \cap \operatorname{dom}\partial f$ *is not closed by* (5.105). *Hence we cannot apply for [84, Theorem 5.10(η)].*

(3) In [86, Corollary 4], Voisei and Zălinescu showed that the sum problem has an affirmative solution when ${}^{ic}(\operatorname{dom} A) \neq \varnothing, {}^{ic}(\operatorname{dom} B) \neq \varnothing$ *and* $0 \in {}^{ic}[\operatorname{dom} A - \operatorname{dom} B]$. *Since the* $\operatorname{dom} A$ *in Example 5.4.1 is a proper*

dense linear subspace, ${}^{ic}(\operatorname{dom} A) = \varnothing$*. Thus we cannot apply for [86, Corollary 4]. (Given a set* $C \subseteq X$*, we define* ${}^{ic}C$ *by*

$$ {}^{ic}C = \begin{cases} {}^{i}C, & \text{if } \operatorname{aff} C \text{ is closed;} \\ \varnothing, & \text{otherwise,} \end{cases} $$

where ${}^{i}C$ *[92] is the* intrinsic core *or* relative algebraic interior *of* C*, defined by* ${}^{i}C = \{a \in C \mid \forall x \in \operatorname{aff}(C - C), \exists \delta > 0, \forall \lambda \in [0, \delta] : a + \lambda x \in C\}$*.)*

(4) In [19], it was shown that the sum problem has an affirmative solution when A *is a linear relation,* B *is the subdifferential operator of a proper lower semicontinuous sublinear function, and Rockafellar's constraint qualification holds. Clearly,* f *in Example 5.4.1 is not sublinear. Then we cannot apply for it. Theorem 5.3.1 truly generalizes [19].*

(5) In [90, Corollary 3.11], it was shown that the sum problem has an affirmative solution when A *is a linear relation,* B *is a maximally monotone operator satisfying Rockafellar's constraint qualification and* $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$*. In Example 5.4.1, since* $\operatorname{dom} A$ *is a linear subspace, we can take* $x_0 \in \operatorname{dom} A$ *with* $\|x_0\| = 1$*. Thus, by* (5.105)*, we have that*

$$ x_0 \in \operatorname{dom} A \cap \overline{U_X} = \operatorname{dom} A \cap \overline{\operatorname{dom} \partial f} \quad \text{but} \quad x_0 \notin U_X = \operatorname{dom} \partial f. \tag{5.106} $$

Thus $\operatorname{dom} A \cap \overline{\operatorname{dom} \partial f} \nsubseteq \operatorname{dom} \partial f$ *and thus we cannot apply [90, Corollary 3.11] either.*

5.5 Discussion

As we can see, Fact 5.1.7 plays an important role in the proof of Theorem 5.2.4 and Theorem 5.3.1. Theorem 5.2.4 presents a powerful sufficient condition for the sum problem. The following question posed by Simons in [72, Problem 41.4] remains open:

> *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV), let C be a nonempty closed convex subset of X, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$.
>
> *Is $A + N_C$ necessarily maximally monotone?*

If the above result holds, by Theorem 5.2.4, we can get the following result:

Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. Assume that $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. Then $A + B$ is maximally monotone.

Chapter 6

Classical types of maximally monotone operators

This chapter is based on the work by Bauschke, Borwein, Wang and Yao in [6, 7]. We study three classical types of maximally monotone operators: *dense type*, *negative-infimum type*, and *Fitzpatrick-Phelps type*.

We show that every maximally monotone operator of Fitzpatrick-Phelps type must be of dense type. This provides affirmative answers to two questions posed by Stephen Simons and it implies that various important notions of monotonicity coincide.

Moreover, we prove that for a maximally monotone linear relation, the monotonicities of dense type, of negative-infimum type, and of Fitzpatrick-Phelps type are the same and equivalent to monotonicity of the adjoint. This result also provides an affirmative answer to one problem posed by Phelps and Simons.

6.1 Introduction and auxiliary results

We now recall the three fundamental types of monotonicity.

Definition 6.1.1 *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then three key types of monotone operators are defined as follows.*

(i) *A is* of dense type or type (D) *(1971, [49], [62] and [76, Theorem 9.5]) if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with*

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exists a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ weak×strong converges to (x^{**}, x^*).*

(ii) *A is* of type negative infimum (NI) *(1996, [71]) if*

$$\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle a, x^* \rangle + \langle a^*, x^{**} \rangle - \langle a, a^* \rangle\big) \geq \langle x^{**}, x^* \rangle,$$

for every $(x^{**}, x^*) \in X^{**} \times X^*$.

(iii) *A is* of type Fitzpatrick-Phelps (FP) *(1992, [46]) if whenever U is an open convex subset of X^* such that $U \cap \operatorname{ran} A \neq \varnothing$, $x^* \in U$, and $(x, x^*) \in X \times X^*$ is monotonically related to $\operatorname{gra} A \cap (X \times U)$ it must follow that $(x, x^*) \in \operatorname{gra} A$.*

All three of these properties are known to hold for the subgradient of a closed convex function and for every maximally monotone operator on a reflexive space [26, 72, 74]. These and other relationships known amongst these and other monotonicity notions are described in [26, Chapter 9].

Now we introduce some notation. Let $F : X \times X^* \to \,]-\infty, +\infty]$. We say F is a *representative* of a maximally monotone operator $A : X \rightrightarrows X^*$ if

F is lower semicontinuous and convex with $F \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$ and

$$\operatorname{gra} A = \{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\}.$$

Let $(z, z^*) \in X \times X^*$. Then $F_{(z,z^*)} : X \times X^* \to \left]-\infty, +\infty\right]$ [55, 57, 74] is defined by (for every $(x, x^*) \in X \times X^*$)

$$\begin{aligned} F_{(z,z^*)}(x, x^*) &= F(z + x, z^* + x^*) - \big(\langle x, z^* \rangle + \langle z, x^* \rangle + \langle z, z^* \rangle\big) \\ &= F(z + x, z^* + x^*) - \langle z + x, z^* + x^* \rangle + \langle x, x^* \rangle. \end{aligned} \tag{6.1}$$

We recall the following basic fact regarding the second dual ball:

Fact 6.1.2 (Goldstine) (See [58, Theorem 2.6.26] or [44, Theorem 3.27].) *The weak*-closure of B_X in X^{**} is $B_{X^{**}}$.*

Fact 6.1.3 (Borwein) (See [20, Theorem 1] or [92, Theorem 3.1.1].) *Let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous and convex function. Let $\varepsilon > 0$ and $\beta \geq 0$ (where $\frac{1}{0} = \infty$). Assume that $x_0 \in \operatorname{dom} f$ and $x_0^* \in \partial_\varepsilon f(x_0)$. There exist $x_\varepsilon \in X, x_\varepsilon^* \in X^*$ such that*

$$\begin{aligned} &\|x_\varepsilon - x_0\| + \beta\,|\langle x_\varepsilon - x_0, x_0^* \rangle| \leq \sqrt{\varepsilon}, \quad x_\varepsilon^* \in \partial f(x_\varepsilon), \\ &\|x_\varepsilon^* - x_0^*\| \leq \sqrt{\varepsilon}(1 + \beta\|x_0^*\|), \quad |\langle x_\varepsilon - x_0, x_\varepsilon^* \rangle| \leq \varepsilon + \frac{\sqrt{\varepsilon}}{\beta}. \end{aligned}$$

Fact 6.1.4 (Simons) (See [73, Theorem 17] or [74, Theorem 37.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone and of type (D). Then A is of type (FP).*

Fact 6.1.5 (Simons / Marques Alves and Svaiter) (See [71, Lemma 15] or [74, Theorem 36.3(a)], and [56, Theorem 4.4].) *Let $A : X \rightrightarrows X^*$ be maximally monotone, and let $F : X \times X^* \to \,]-\infty, +\infty]$ be a representative of A. Then the following are equivalent.*

(i) *A is type of (D).*

(ii) *A is of type (NI).*

(iii) *For every $(x_0, x_0^*) \in X \times X^*$,*

$$\inf_{(x,x^*)\in X\times X^*} \Big[F_{(x_0,x_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\Big] = 0.$$

6.2 Every maximally monotone operator of Fitzpatrick-Phelps type is actually of dense type

In Theorem 6.2.1 of this section (see also [7]), we provide an affirmative answer to the following question, posed by S. Simons [73, Problem 18, page 406]:

> *Let $A : X \rightrightarrows X^*$ be maximally monotone such that A is of type (FP).*
> *Is A necessarily of type (D)?*

In consequence, in Corollary 6.2.2 we record that the three notions in Definition 6.1.1 actually coincide.

Simons posed another question in [74, Problem 47.6]:

Let $A : \operatorname{dom} A \to X^*$ be linear and maximally monotone. Assume that A is of type (FP).

Is A necessarily of type (NI)?

By Fact 6.1.5, [74, Problem 47.6] is a special case of [73, Problem 18, page 406].

Let $A : X \rightrightarrows X^*$ be monotone. For convenience, we defined Φ_A on $X^{**} \times X^*$ by

$$\Phi_A \colon (x^{**}, x^*) \mapsto \sup_{(a,a^*) \in \operatorname{gra} A} \big(\langle x^{**}, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle\big).$$

Then we have $\Phi_A|_{X \times X^*} = F_A$. The next theorem is our first main result in Chapter 6. In conjunction with the corollary that follows, it provides the affirmative answer promised to Simons's problem posed in [73, Problem 18, page 406].

Theorem 6.2.1 *Let $A : X \rightrightarrows X^*$ be maximally monotone such that A is of type (FP). Then A is of type (NI).*

Proof. After translating the graph if necessary, we can and do suppose that $(0, 0) \in \operatorname{gra} A$. Let $(x_0^{**}, x_0^*) \in X^{**} \times X^*$. We must show that

$$\Phi_A(x_0^{**}, x_0^*) \geq \langle x_0^{**}, x_0^* \rangle \tag{6.2}$$

and we consider two cases.

Case 1: $x_0^{**} \in X$.

Then (6.2) follows directly from Fact 5.1.5.

Case 2: $x_0^{**} \in X^{**} \smallsetminus X$.

By Fact 6.1.2, there exists a bounded net $(x_\alpha)_{\alpha \in I}$ in X that weak* converges to x_0^{**}. Thus, we have

$$M = \sup_{\alpha \in I} \|x_\alpha\| < +\infty \tag{6.3}$$

and

$$\langle x_\alpha, x_0^* \rangle \to \langle x_0^{**}, x_0^* \rangle. \tag{6.4}$$

Now we consider two subcases.

Subcase 2.1: There exists $\alpha \in I$, such that $(x_\alpha, x_0^*) \in \operatorname{gra} A$.

By definition,

$$\Phi_A(x_0^{**}, x_0^*) \geq \langle x_\alpha, x_0^* \rangle + \langle x_0^{**}, x_0^* \rangle - \langle x_\alpha, x_0^* \rangle = \langle x_0^{**}, x_0^* \rangle.$$

Hence (6.2) holds.

Subcase 2.2: We have

$$(x_\alpha, x_0^*) \notin \operatorname{gra} A, \quad \forall \alpha \in I. \tag{6.5}$$

Set

$$U_\varepsilon = [0, x_0^*] + \varepsilon U_{X^*}, \tag{6.6}$$

where $\varepsilon > 0$. Observe that U_ε is open and convex. Since $(0,0) \in \operatorname{gra} A$, we have, by the definition of U_ε, $0 \in \operatorname{ran} A \cap U_\varepsilon$ and $x_0^* \in U_\varepsilon$. In view of (6.5) and because A is of type (FP), there exists a net $(a_{\alpha,\varepsilon}, a^*_{\alpha,\varepsilon})$ in $\operatorname{gra} A$ such that $a^*_{\alpha,\varepsilon} \in U_\varepsilon$ and

$$\langle a_{\alpha,\varepsilon}, x_0^*\rangle + \langle x_\alpha, a^*_{\alpha,\varepsilon}\rangle - \langle a_{\alpha,\varepsilon}, a^*_{\alpha,\varepsilon}\rangle > \langle x_\alpha, x_0^*\rangle, \quad \forall \alpha \in I. \tag{6.7}$$

Now fix $\alpha \in I$. By (6.7),

$$\langle a_{\alpha,\varepsilon}, x_0^*\rangle + \langle x_0^{**}, a^*_{\alpha,\varepsilon}\rangle - \langle a_{\alpha,\varepsilon}, a^*_{\alpha,\varepsilon}\rangle > \langle x_0^{**} - x_\alpha, a^*_{\alpha,\varepsilon}\rangle + \langle x_\alpha, x_0^*\rangle.$$

Hence,

$$\Phi_A(x_0^{**}, x_0^*) > \langle x_0^{**} - x_\alpha, a^*_{\alpha,\varepsilon}\rangle + \langle x_\alpha, x_0^*\rangle. \tag{6.8}$$

Since $a^*_{\alpha,\varepsilon} \in U_\varepsilon$, there exist

$$t_{\alpha,\varepsilon} \in [0,1] \text{ and } b^*_{\alpha,\varepsilon} \in U_{X^*} \tag{6.9}$$

such that

$$a^*_{\alpha,\varepsilon} = t_{\alpha,\varepsilon} x_0^* + \varepsilon b^*_{\alpha,\varepsilon}. \tag{6.10}$$

Using (6.8), (6.10), and (6.3), we deduce that

$$\begin{aligned}
\Phi_A(x_0^{**}, x_0^*) &> \langle x_0^{**} - x_\alpha, t_{\alpha,\varepsilon} x_0^* + \varepsilon b^*_{\alpha,\varepsilon}\rangle + \langle x_\alpha, x_0^*\rangle \\
&= t_{\alpha,\varepsilon}\langle x_0^{**} - x_\alpha, x_0^*\rangle + \varepsilon\langle x_0^{**} - x_\alpha, b^*_{\alpha,\varepsilon}\rangle + \langle x_\alpha, x_0^*\rangle \\
&\geq t_{\alpha,\varepsilon}\langle x_0^{**} - x_\alpha, x_0^*\rangle - \varepsilon\|x_0^{**} - x_\alpha\| + \langle x_\alpha, x_0^*\rangle
\end{aligned}$$

$$\geq t_{\alpha,\varepsilon}\langle x_0^{**} - x_\alpha, x_0^*\rangle - \varepsilon(\|x_0^{**}\| + M) + \langle x_\alpha, x_0^*\rangle. \tag{6.11}$$

In view of (6.9) and since $\alpha \in I$ was chosen arbitrarily, we take the limit in (6.11) and obtain with the help of (6.4) that

$$\Phi_A(x_0^{**}, x_0^*) \geq -\varepsilon(\|x_0^{**}\| + M) + \langle x_0^{**}, x_0^*\rangle. \tag{6.12}$$

Next, letting $\varepsilon \to 0$ in (6.12), we have

$$\Phi_A(x_0^{**}, x_0^*) \geq \langle x_0^{**}, x_0^*\rangle. \tag{6.13}$$

Therefore, (6.2) holds in all cases. ■

We now obtain the promised corollary:

Corollary 6.2.2 *Let $A\colon X \rightrightarrows X^*$ be maximally monotone. Then the following are equivalent.*

(i) *A is of type (D).*

(ii) *A is of type (NI).*

(iii) *A is of type (FP).*

Proof. "(i)⇒(iii)": Fact 6.1.4. "(iii)⇒(ii)": Theorem 6.2.1 . "(ii)⇒(i)": Fact 6.1.5. ■

Remark 6.2.3 *Let $A\colon X \rightrightarrows X^*$ be maximally monotone. Corollary 6.2.2 establishes the equivalences of the key types (D), (NI), and (FP), which as* *noted all hold when X is reflexive or $A = \partial f$, where $f\colon X \to \,]-\infty, +\infty]$ is convex, lower semicontinuous, and proper (see [26, 72, 74]).*

Furthermore, these notions are also equivalent to type (ED), see [76]. For a nonlinear operator they also coincide with the uniqueness *of maximal extensions to X^{**} (see [56]). In [26, p. 454] there is a discussion of this result and of the linear case.*

Finally, when A is a linear relation, it has recently been established that all these notions coincide with the monotonicity of the adjoint *multifunction A^* (see Section 6.3).*

6.3 The adjoint of a maximally monotone linear relation

In this section, we provide tools to give an affirmative answer to a question posed by Phelps and Simons. Phelps and Simons posed the following question in [63, Section 9, item 2]:

> *Let $A : \operatorname{dom} A \to X^*$ be linear and maximally monotone. Assume that A^* is monotone.*
> *Is A necessarily of type (D)?*

Theorem 6.3.1 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then A is of type (NI) if and only if A^* is monotone.*

Proof.

"$\Rightarrow$": Suppose to the contrary that there exists $(a_0^{**}, a_0^*) \in \operatorname{gra} A^*$ such that $\langle a_0^{**}, a_0^* \rangle < 0$. Then we have

$$\begin{aligned}\sup_{(a,a^*)\in\operatorname{gra} A}\big(\langle a, -a_0^*\rangle + \langle a_0^{**}, a^*\rangle - \langle a, a^*\rangle\big) &= \sup_{(a,a^*)\in\operatorname{gra} A}\{-\langle a, a^*\rangle\} \\ &= 0 < \langle -a_0^{**}, a_0^*\rangle,\end{aligned}$$

which contradicts that A is of type (NI). Hence A^* is monotone.

"$\Leftarrow$": Define

$$F : X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^*\rangle.$$

Since A is maximally monotone, Fact 3.2.8 implies that F is proper lower semicontinuous and convex, and a representative of A. Let $(v_0, v_0^*) \in X \times X^*$. Recalling (6.1), note that

$$F_{(v_0,v_0^*)} \colon (x, x^*) \mapsto \iota_{\operatorname{gra} A}(v_0 + x, v_0^* + x^*) + \langle x, x^*\rangle \tag{6.14}$$

is proper lower semicontinuous and convex. By Fact 5.1.1, there exists $(y^{**}, y^*) \in X^{**} \times X^*$ such that

$$\begin{aligned}K &:= \inf_{(x,x^*)\in X\times X^*}\Big[F_{(v_0,v_0^*)}(x, x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\Big] \\ &= -\big(F_{(v_0,v_0^*)} + \tfrac{1}{2}\|\cdot\|^2 + \tfrac{1}{2}\|\cdot\|^2\big)^*(0,0) \\ &= -F^*_{(v_0,v_0^*)}(y^*, y^{**}) - \tfrac{1}{2}\|y^{**}\|^2 - \tfrac{1}{2}\|y^*\|^2. \end{aligned} \tag{6.15}$$

Since $(x,x^*)\mapsto F_{(v_0,v_0^*)}(x,x^*)+\frac{1}{2}\|x\|^2+\frac{1}{2}\|x^*\|^2$ is coercive, there exist $M>0$ and a sequence $(a_n,a_n^*)_{n\in\mathbb{N}}$ in $X\times X^*$ such that, $\forall n\in\mathbb{N}$,

$$\|a_n\|+\|a_n^*\|\leq M \tag{6.16}$$

and

$$\begin{aligned}
&F_{(v_0,v_0^*)}(a_n,a_n^*)+\tfrac{1}{2}\|a_n\|^2+\tfrac{1}{2}\|a_n^*\|^2\\
&<K+\tfrac{1}{n^2}=-F^*_{(v_0,v_0^*)}(y^*,y^{**})-\tfrac{1}{2}\|y^{**}\|^2-\tfrac{1}{2}\|y^*\|^2+\tfrac{1}{n^2}\quad\text{(by (6.15))}\\
&\Rightarrow F_{(v_0,v_0^*)}(a_n,a_n^*)+\tfrac{1}{2}\|a_n\|^2+\tfrac{1}{2}\|a_n^*\|^2+F^*_{(v_0,v_0^*)}(y^*,y^{**})+\tfrac{1}{2}\|y^{**}\|^2\\
&\quad+\tfrac{1}{2}\|y^*\|^2<\tfrac{1}{n^2} &(6.17)\\
&\Rightarrow F_{(v_0,v_0^*)}(a_n,a_n^*)+F^*_{(v_0,v_0^*)}(y^*,y^{**})+\langle a_n,-y^*\rangle+\langle a_n^*,-y^{**}\rangle<\tfrac{1}{n^2} &(6.18)\\
&\Rightarrow (y^*,y^{**})\in\partial_{\frac{1}{n^2}}F_{(v_0,v_0^*)}(a_n,a_n^*)\quad\text{(by [92, Theorem 2.4.2(ii)]).} &(6.19)
\end{aligned}$$

Set $\beta=\frac{1}{\max\{\|y^*\|,\|y^{**}\|\}+1}$. Then by Fact 6.1.3, there exist sequences $(\widetilde{a_n},\widetilde{a_n^*})_{n\in\mathbb{N}}$ in $X\times X^*$ and $(y_n^*,y_n^{**})_{n\in\mathbb{N}}$ in $X^*\times X^{**}$ such that, $\forall n\in\mathbb{N}$,

$$\begin{aligned}
&\|a_n-\widetilde{a_n}\|+\|a_n^*-\widetilde{a_n^*}\|+\beta\left|\langle\widetilde{a_n}-a_n,y^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y^{**}\rangle\right|\leq\tfrac{1}{n} &(6.20)\\
&\max\{\|y_n^*-y^*\|,\|y_n^{**}-y^{**}\|\}\leq\tfrac{2}{n} &(6.21)\\
&\left|\langle\widetilde{a_n}-a_n,y_n^*\rangle+\langle\widetilde{a_n^*}-a_n^*,y_n^{**}\rangle\right|\leq\tfrac{1}{n^2}+\tfrac{1}{n\beta} &(6.22)\\
&(y_n^*,y_n^{**})\in\partial F_{(v_0,v_0^*)}(\widetilde{a_n},\widetilde{a_n^*}). &(6.23)
\end{aligned}$$

Then, $\forall n\in\mathbb{N}$, we have

$$\langle\widetilde{a_n},y_n^*\rangle+\langle\widetilde{a_n^*},y_n^{**}\rangle-\langle a_n,y^*\rangle-\langle a_n^*,y^{**}\rangle$$

$$
\begin{aligned}
&= \langle \widetilde{a_n} - a_n, y_n^*\rangle + \langle a_n, y_n^* - y^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y_n^{**}\rangle + \langle a_n^*, y_n^{**} - y^{**}\rangle \\
&\leq \left|\langle \widetilde{a_n} - a_n, y_n^*\rangle + \langle \widetilde{a_n^*} - a_n^*, y_n^{**}\rangle\right| + |\langle a_n, y_n^* - y^*\rangle| + |\langle a_n^*, y_n^{**} - y^{**}\rangle| \\
&\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \|a_n\| \cdot \|y_n^* - y^*\| + \|a_n^*\| \cdot \|y_n^{**} - y^{**}\| \quad \text{(by (6.22))} \\
&\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + (\|a_n\| + \|a_n^*\|) \cdot \max\{\|y_n^* - y^*\|, \|y_n^{**} - y^{**}\|\} \\
&\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \tfrac{2}{n}M \quad \text{(by (6.16) and (6.21))}. \qquad (6.24)
\end{aligned}
$$

By (6.20), $\forall n \in \mathbb{N}$, we have

$$
\big|\|a_n\| - \|\widetilde{a_n}\|\big| + \big|\|a_n^*\| - \|\widetilde{a_n^*}\|\big| \leq \tfrac{1}{n}. \qquad (6.25)
$$

Thus by (6.16), $\forall n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\big|\|a_n\|^2 - \|\widetilde{a_n}\|^2\big| + \left|\|a_n^*\|^2 - \|\widetilde{a_n^*}\|^2\right| \\
&= \big|\|a_n\| - \|\widetilde{a_n}\|\big|\big(\|a_n\| + \|\widetilde{a_n}\|\big) + \left|\|a_n^*\| - \|\widetilde{a_n^*}\|\right| \left(\|a_n^*\| + \|\widetilde{a_n^*}\|\right) \\
&\leq \tfrac{1}{n}\left(2\|a_n\| + \tfrac{1}{n}\right) + \tfrac{1}{n}\left(2\|a_n^*\| + \tfrac{1}{n}\right) \quad \text{(by (6.25))} \\
&\leq \tfrac{1}{n}(2M + \tfrac{2}{n}) = \tfrac{2}{n}M + \tfrac{2}{n^2}. \qquad (6.26)
\end{aligned}
$$

Similarly, by (6.21), for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\big|\|y_n^*\|^2 - \|y^*\|^2\big| &\leq \tfrac{4}{n}\|y^*\| + \tfrac{4}{n^2} \leq \tfrac{4}{n\beta} + \tfrac{4}{n^2}, \\
\big|\|y_n^{**}\|^2 - \|y^{**}\|^2\big| &\leq \tfrac{4}{n}\|y^{**}\| + \tfrac{4}{n^2} \leq \tfrac{4}{n\beta} + \tfrac{4}{n^2}. \qquad (6.27)
\end{aligned}
$$

Thus, $\forall n \in \mathbb{N}$,

$$
F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 + \tfrac{1}{2}\|y_n^{**}\|^2
$$

$$
\begin{aligned}
&= \big[F_{(v_0,v_0^*)}(\widetilde{a_n},\widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*,y_n^{**}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 \\
&\quad + \tfrac{1}{2}\|y_n^{**}\|^2\big] \\
&\quad - \big[F_{(v_0,v_0^*)}(a_n,a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*,y^{**}) + \tfrac{1}{2}\|y^{**}\|^2 \\
&\quad + \tfrac{1}{2}\|y^*\|^2\big] \\
&\quad + \big[F_{(v_0,v_0^*)}(a_n,a_n^*) + \tfrac{1}{2}\|a_n\|^2 + \tfrac{1}{2}\|a_n^*\|^2 + F^*_{(v_0,v_0^*)}(y^*,y^{**}) + \tfrac{1}{2}\|y^{**}\|^2 \\
&\quad + \tfrac{1}{2}\|y^*\|^2\big] \\
&< \Big[F_{(v_0,v_0^*)}(\widetilde{a_n},\widetilde{a_n^*}) + F^*_{(v_0,v_0^*)}(y_n^*,y_n^{**}) - F_{(v_0,v_0^*)}(a_n,a_n^*) - F^*_{(v_0,v_0^*)}(y^*,y^{**})\Big] \\
&\quad + \tfrac{1}{2}\Big[\|\widetilde{a_n}\|^2 + \|\widetilde{a_n^*}\|^2 - \|a_n\|^2 - \|a_n^*\|^2\Big] \\
&\quad + \tfrac{1}{2}\big[\|y_n^*\|^2 + \|y_n^{**}\|^2 - \|y^{**}\|^2 - \|y^*\|^2\big] + \tfrac{1}{n^2} \quad \text{(by (6.17))} \\
&\leq \Big[\langle\widetilde{a_n},y_n^*\rangle + \langle\widetilde{a_n^*},y_n^{**}\rangle - \langle a_n,y^*\rangle - \langle a_n^*,y^{**}\rangle\Big] \quad \text{(by (6.23))} \\
&\quad + \tfrac{1}{2}\Big(\big|\|\widetilde{a_n}\|^2 - \|a_n\|^2\big| + \Big|\|\widetilde{a_n^*}\|^2 - \|a_n^*\|^2\Big|\Big) \\
&\quad + \tfrac{1}{2}\big(\big|\|y_n^*\|^2 - \|y^*\|^2\big| + \big|\|y_n^{**}\|^2 - \|y^{**}\|^2\big|\big) + \tfrac{1}{n^2} \\
&\leq \tfrac{1}{n^2} + \tfrac{1}{n\beta} + \tfrac{2}{n}M + \tfrac{1}{n}M + \tfrac{1}{n^2} + \tfrac{4}{n\beta} + \tfrac{4}{n^2} + \tfrac{1}{n^2} \text{ (by (6.24), (6.26) and (6.27))} \\
&= \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M. \qquad (6.28)
\end{aligned}
$$

By (6.23), (6.14), and [92, Theorem 3.2.4(vi)&(ii)], there exists a sequence $(z_n^*, z_n^{**})_{n\in\mathbb{N}}$ in $(\operatorname{gra} A)^\perp$ and such that

$$(y_n^*, y_n^{**}) = (\widetilde{a_n^*}, \widetilde{a_n}) + (z_n^*, z_n^{**}), \quad \forall n \in \mathbb{N}. \tag{6.29}$$

Since A^* is monotone and $(z_n^{**}, z_n^*) \in \operatorname{gra}(-A^*)$, it follows from (6.29) that, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
&\langle y_n^*, y_n^{**}\rangle - \langle y_n^*, \widetilde{a_n}\rangle - \langle y_n^{**}, \widetilde{a_n^*}\rangle + \langle \widetilde{a_n^*}, \widetilde{a_n}\rangle = \langle y_n^* - \widetilde{a_n^*}, y_n^{**} - \widetilde{a_n}\rangle \qquad (6.30)\\
&\quad = \langle z_n^*, z_n^{**}\rangle \leq 0\\
&\Rightarrow \langle y_n^*, y_n^{**}\rangle \leq \langle y_n^*, \widetilde{a_n}\rangle + \langle y_n^{**}, \widetilde{a_n^*}\rangle - \langle \widetilde{a_n^*}, \widetilde{a_n}\rangle.
\end{aligned}$$

Then by (6.14) and (6.23), we have $\langle \widetilde{a_n^*}, \widetilde{a_n}\rangle = F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*})$ and, $\forall n \in \mathbb{N}$,

$$\langle y_n^*, y_n^{**}\rangle \leq \langle y_n^*, \widetilde{a_n}\rangle + \langle y_n^{**}, \widetilde{a_n^*}\rangle - F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) = F^*_{(v_0,v_0^*)}(y_n^*, y_n^{**}). \qquad (6.31)$$

By (6.28) and (6.31), $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}
&F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \langle y_n^*, y_n^{**}\rangle + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 + \tfrac{1}{2}\|y_n^*\|^2 + \tfrac{1}{2}\|y_n^{**}\|^2\\
&\quad < \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M\\
&\Rightarrow F_{(v_0,v_0^*)}(\widetilde{a_n}, \widetilde{a_n^*}) + \tfrac{1}{2}\|\widetilde{a_n}\|^2 + \tfrac{1}{2}\|\widetilde{a_n^*}\|^2 < \tfrac{7}{n^2} + \tfrac{5}{n\beta} + \tfrac{3}{n}M. \qquad (6.32)
\end{aligned}$$

Thus by (6.32),

$$\inf_{(x,x^*)\in X\times X^*}\left[F_{(v_0,v_0^*)}(x,x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] \leq 0. \qquad (6.33)$$

By (6.14),

$$\inf_{(x,x^*)\in X\times X^*}\left[F_{(v_0,v_0^*)}(x,x^*) + \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2\right] \geq 0. \qquad (6.34)$$

Combining (6.33) with (6.34), we obtain

$$\inf_{(x,x^*)\in X\times X^*}\Big[F_{(v_0,v_0^*)}(x,x^*)+\tfrac{1}{2}\|x\|^2+\tfrac{1}{2}\|x^*\|^2\Big]=0. \tag{6.35}$$

Thus by Fact 6.1.5, A is of type (NI). ■

Remark 6.3.2 *The proof of the necessary part of Theorem 6.3.1 follows closely that of [30, Theorem 2]. The proof of the sufficient part of Theorem 6.3.1 was partially inspired by that of [93, Theorem 32.L] and that of [54, Theorem 2.1].*

Combining Corollary 6.2.2 and Theorem 6.3.1, we get the following result.

Corollary 6.3.3 *Let $A: X \rightrightarrows X^*$ be a maximally monotone linear relation. Then the following are equivalent.*

(i) *A is of type (D).*

(ii) *A is of type (NI).*

(iii) *A is of type (FP).*

(iv) *A^* is monotone.*

Remark 6.3.4 *When A is linear and continuous, Corollary 6.3.3 is due to Bauschke and Borwein [4, Theorem 4.1]. Phelps and Simons in [63, Theorem 6.7] considered the case when A is linear but possibly discontinuous; they arrived at some of the implications of Corollary 6.3.3 in that case.*

Corollary 6.3.3(iv)⇒(i) gives an affirmative answer to a problem posed by Phelps and Simons in [63, Section 9, item 2] on the converse of [63, Theorem 6.7(c)⇒(f)].

It is interesting to compare Corollary 6.3.3 with the following related result by Brezis and Browder. Suppose that X *is reflexive and let* $A\colon X \rightrightarrows X^*$ *be a monotone linear relation with closed graph. Then* A *is maximally monotone if and only if* A^* *is (maximally) monotone; see [28–30] and also the recent works [70, 89].*

We conclude with an application of Corollary 6.3.3 to an operator studied previously by Phelps and Simons [63].

Example 6.3.5 *Suppose that* $X = L^1[0,1]$ *so that* $X^* = L^\infty[0,1]$*, let*

$$D = \big\{x \in X \mid x \text{ is absolutely continuous}, x(0) = 0, x' \in X^*\big\},$$

and set

$$A\colon X \rightrightarrows X^*\colon x \mapsto \begin{cases} \{x'\}, & \text{if } x \in D; \\ \varnothing, & \text{otherwise.} \end{cases}$$

By [63, Example 4.3], A *is an at most single-valued maximally monotone linear relation with proper dense domain, and* A *is neither symmetric nor skew. Moreover,*

$$\operatorname{dom} A^* = \{z \in X^{**} \mid z \text{ is absolutely continuous}, z(1) = 0, z' \in X^*\} \subseteq X$$

$A^*z = -z', \forall z \in \operatorname{dom} A^*$*, and* A^* *is monotone. Therefore, Corollary 6.3.3* *implies that A is of type (D), of type (NI), and of type (FP).*

6.4 Discussion

Our first main result (Theorem 6.2.1) in this chapter is obtained by applying Goldstine's Theorem (see Fact 6.1.2). Simons, Marques Alves and Svaiter's characterization of type (D) operators and Borwein's generalization of the Brøndsted-Rockafellar theorem are the main tools for obtaining the other main result (Theorem 6.3.1). Corollary 6.3.3 motivates the following question:

> *Let $A : X \rightrightarrows X^*$ be a monotone linear relation with closed graph. Assume that A^* is monotone.*
> *Is A necessarily of type (D)?*

Chapter 7

Properties of monotone operators and the partial inf convolution of Fitzpatrick functions

Chapter 7 is mainly based on the work in [15, 17] by Bauschke, Wang and Yao.

Let $F_1, F_2 \colon X \times X^* \to \,]-\infty, +\infty]$. Then the *partial inf-convolution* on the second variable $F_1 \Box_2 F_2$, is the function defined on $X \times X^*$ by

$$F_1 \Box_2 F_2 \colon (x, x^*) \mapsto \inf_{y^* \in X^*} F_1(x, x^* - y^*) + F_2(x, y^*).$$

In this chapter, we study the properties of $F_A \Box_2 F_B$ for two maximally monotone operators A and B. We also consider the connection between $F_A \Box_2 F_B$ and F_{A+B}. Then we provide a new proof of the following result due to Voisei [83]: *Let $A, B \colon X \rightrightarrows X^*$ be maximally monotone linear relations, and suppose that $[\operatorname{dom} A - \operatorname{dom} B]$ is closed. Then $A + B$ is maximally monotone.*

7.1 Auxiliary results

The next result was first established in [5, Proposition 2.2(v)] by Bauschke, Borwein and Wang in a Hilbert space. Now we generalize it to a general Banach space.

Proposition 7.1.1 *Let $A\colon X \to X^*$ be linear and monotone. Then*

$$F_A(x,x^*) = 2q^*_{A_+}(\tfrac{1}{2}x^*+\tfrac{1}{2}A^*x) = \tfrac{1}{2}q^*_{A_+}(x^*+A^*x), \quad \forall (x,x^*) \in X\times X. \tag{7.1}$$

If $\operatorname{ran} A_+$ *is closed, then* $\operatorname{dom} q^*_{A_+} = \operatorname{ran} A_+$.

Proof. By Proposition 3.1.3(ix), $\operatorname{dom} A^* \cap X = X$. Hence for every $(x,x^*) \in X \times X^*$,

$$\begin{aligned} F_A(x,x^*) &= \sup_{y\in X} \left[\langle x, Ay\rangle + \langle y, x^*\rangle - \langle y, Ay\rangle\right] \\ &= 2\sup_{y\in X} \left[\langle y, \tfrac{1}{2}x^* + \tfrac{1}{2}A^*x\rangle - q_{A_+}(y)\right] \\ &= 2q^*_{A_+}(\tfrac{1}{2}x^* + \tfrac{1}{2}A^*x) \\ &= \tfrac{1}{2}q^*_{A_+}(x^* + A^*x). \end{aligned} \tag{7.2}$$

By [92, Proposition 2.4.4(iv) and Theorem 2.3.3],

$$\operatorname{ran} \partial q_{A_+} \subseteq \operatorname{dom} \partial q^*_{A_+}. \tag{7.3}$$

By Proposition 3.2.10, $\operatorname{ran} \partial q_{A_+} = \operatorname{ran} A_+$. Then by (7.3),

$$\operatorname{ran} A_+ \subseteq \operatorname{dom} \partial q^*_{A_+} \subseteq \operatorname{dom} q^*_{A_+} \tag{7.4}$$

Then by the Brøndsted-Rockafellar Theorem (see [92, Theorem 3.1.2]),

$$\operatorname{ran} A_+ \subseteq \operatorname{dom} \partial q^*_{A_+} \subseteq \operatorname{dom} q^*_{A_+} \subseteq \overline{\operatorname{ran} A_+}.$$

By the assumption that $\operatorname{ran} A_+$ is closed, we have $\operatorname{ran} A_+ = \operatorname{dom} \partial q^*_{A_+} = \operatorname{dom} q^*_{A_+}$. ■

Now we give a direct proof of the following result.

Fact 7.1.2 (Bartz-Bauschke-Borwein-Reich-Wang) (See [3, Corollary 5.9].) *Let C be a closed convex nonempty set of X. Then $F_{N_C} = \iota_C \oplus \iota^*_C$.*

Proof. Let $(x, x^*) \in X \times X^*$. Then we have

$$\begin{aligned} F_{N_C}(x, x^*) &= \sup_{(c,c^*)\in \operatorname{gra} N_C} [\langle x, c^*\rangle + \langle c, x^*\rangle - \langle c, c^*\rangle] \\ &= \sup_{(c,c^*)\in \operatorname{gra} N_C, k\geq 0} [\langle x, kc^*\rangle + \langle c, x^*\rangle - \langle c, kc^*\rangle] \\ &= \sup_{(c,c^*)\in \operatorname{gra} N_C, k\geq 0} [k(\langle x, c^*\rangle - \langle c, c^*\rangle) + \langle c, x^*\rangle] \end{aligned} \tag{7.5}$$

By (7.5),

$$\begin{aligned} &(x, x^*) \in \operatorname{dom} F_{N_C} \Rightarrow \sup_{(c,c^*)\in \operatorname{gra} N_C} [\langle x, c^*\rangle - \langle c, c^*\rangle] \leq 0 \\ &\Leftrightarrow \inf_{(c,c^*)\in \operatorname{gra} N_C} [-\langle x, c^*\rangle + \langle c, c^*\rangle] \geq 0 \\ &\Leftrightarrow \inf_{(c,c^*)\in \operatorname{gra} N_C} [\langle c - x, c^* - 0\rangle] \geq 0 \\ &\Leftrightarrow (x, 0) \in \operatorname{gra} N_C \quad \text{(by Fact 5.1.2)} \end{aligned}$$

$$\Leftrightarrow x \in C. \tag{7.6}$$

Now assume $x \in C$. By (7.5),

$$F_{N_C}(x, x^*) = \iota_C^*(x^*). \tag{7.7}$$

Combine (7.6) and (7.7), $F_{N_C} = \iota_C \oplus \iota_C^*$. ■

Following Penot [64], if $F\colon X \times X^* \to \,]-\infty, +\infty]$, we set

$$F^{\intercal}\colon X^* \times X\colon (x^*, x) \mapsto F(x, x^*). \tag{7.8}$$

Fact 7.1.3 (Fitzpatrick) (See [45, Proposition 4.2 and Theorem 4.3].) *Let $A : X \rightrightarrows X^*$ be a monotone operator. Then $F_A^{*\intercal} = \langle\cdot,\cdot\rangle$ on* $\operatorname{gra} A$ *and*

$$\big\{x \in X \mid \exists x^* \in X^* \textit{ such that } F_A^*(x^*, x) = \langle x, x^*\rangle\big\} \subseteq \overline{\operatorname{conv}(\operatorname{dom} A)}.$$

Fact 7.1.4 (See [92, Theorem 2.4.14].) *Let $f : X \to \,]-\infty, +\infty]$ be a sublinear function. Then the following hold.*

(i) $\partial f(x) = \{x^* \in \partial f(0) \mid \langle x^*, x\rangle = f(x)\}, \quad \forall x \in \operatorname{dom} f.$

(ii) *If f is lower semicontinuous, then $f = \sup\langle\cdot, \partial f(0)\rangle$.*

Fact 7.1.5 (Simons and Zălinescu) (See [78, Theorem 4.2].)

Let Y be a Banach space and $F_1, F_2\colon X \times Y \to \,]-\infty, +\infty]$ be proper, lower semicontinuous, and convex. Assume that for every $(x, y) \in X \times Y$,

$$(F_1 \Box_2 F_2)(x, y) > -\infty$$

and that $\bigcup_{\lambda>0} \lambda\left[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\right]$ *is a closed subspace of* X. *Then for every* $(x^*, y^*) \in X \times X^*$,

$$(F_1 \Box_2 F_2)^*(x^*, y^*) = \min_{w^* \in X^*} \left[F_1^*(x^* - w^*, y^*) + F_2^*(w^*, y^*)\right].$$

The following result was first established in [21, Theorem 7.4]. Now we give a new proof.

Fact 7.1.6 (Borwein) *Let* $A, B : X \rightrightarrows X^*$ *be linear relations such that* $\operatorname{gra} A$ *and* $\operatorname{gra} B$ *are closed. Assume that* $\operatorname{dom} A - \operatorname{dom} B$ *is closed. Then*

$$(A+B)^* = A^* + B^*.$$

Proof. We have

$$\iota_{\operatorname{gra}(A+B)} = \iota_{\operatorname{gra} A} \Box_2 \iota_{\operatorname{gra} B}. \tag{7.9}$$

Let $(x^{**}, x^*) \in X^{**} \times X^*$. Since $\operatorname{gra} A$ and $\operatorname{gra} B$ are closed convex, $\iota_{\operatorname{gra} A}$ and $\iota_{\operatorname{gra} B}$ are proper lower semicontinuous and convex. Then by Fact 7.1.5 and (7.9), there exists $y^* \in X^*$ such that

$$\begin{aligned}
\iota_{\operatorname{gra}(A+B)^*}(x^{**}, x^*) &= \iota_{\left(\operatorname{gra}(A+B)\right)^\perp}(-x^*, x^{**}) \\
&= \iota^*_{\operatorname{gra}(A+B)}(-x^*, x^{**}) \quad \text{(since } \operatorname{gra}(A+B) \text{ is a subspace)} \\
&= \iota^*_{\operatorname{gra} A}(y^*, x^{**}) + \iota^*_{\operatorname{gra} B}(-x^* - y^*, x^{**}) \\
&= \iota_{(\operatorname{gra} A)^\perp}(y^*, x^{**}) + \iota_{(\operatorname{gra} B)^\perp}(-x^* - y^*, x^{**}) \\
&= \iota_{\operatorname{gra} A^*}(x^{**}, -y^*) + \iota_{\operatorname{gra} B^*}(x^{**}, x^* + y^*)
\end{aligned}$$

$$= \iota_{\operatorname{gra}(A^*+B^*)}(x^{**}, x^*). \tag{7.10}$$

Thus $\operatorname{gra}(A+B)^* = \operatorname{gra}(A^*+B^*)$ and hence $(A+B)^* = A^*+B^*$. ■

Lemma 7.1.7 *Let $A, B\colon X \rightrightarrows X^*$ be maximally monotone, and suppose that $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace of X. Set*

$$E = \big\{x \in X \mid \exists x^* \in X^* \text{ such that } F_A^*(x^*, x) = \langle x, x^*\rangle\big\}$$

and

$$F = \big\{x \in X \mid \exists x^* \in X^* \text{ such that } F_B^*(x^*, x) = \langle x, x^*\rangle\big\}.$$

Then

$$\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[E - F]\,.$$

Moreover, if A and B are of type (FPV), then we have

$$\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B]\,.$$

Proof. Using Fact 7.1.3, we see that

$$\begin{aligned}
&\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \subseteq \bigcup_{\lambda>0} \lambda\,[E - F] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{conv}(\operatorname{dom} A)} - \overline{\operatorname{conv}(\operatorname{dom} B)}\right] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{conv}(\operatorname{dom} A) - \operatorname{conv}(\operatorname{dom} B)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\lambda>0} \lambda\overline{[\operatorname{conv}(\operatorname{dom} A - \operatorname{dom} B)]} \\
&\subseteq \overline{\bigcup_{\lambda>0} \lambda\,[\operatorname{conv}(\operatorname{dom} A - \operatorname{dom} B)]} \\
&= \bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \quad \text{(using the assumption).}
\end{aligned}
$$

Hence $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[E - F]$.

Now assume that A, B are of type (FPV). Then by Fact 5.1.6 and Fact 5.1.13, we have

$$
\begin{aligned}
&\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \subseteq \bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{dom} A} - \overline{\operatorname{dom} B}\right] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{dom} A - \operatorname{dom} B}\right] \subseteq \overline{\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]} \\
&= \bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \quad \text{(using the assumption).}
\end{aligned}
$$

■

Corollary 7.1.8 *Let $A, B\colon X \rightrightarrows X^*$ be maximally monotone linear relations, and suppose that $[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace. Then*

$$
\begin{aligned}
\bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B] &= [\operatorname{dom} A - \operatorname{dom} B] \\
&= \bigcup_{\lambda>0} \lambda\left[P_X \operatorname{dom} F_A^{*\intercal} - P_X \operatorname{dom} F_B^{*\intercal}\right].
\end{aligned}
$$

Proof. Apply directly Fact 5.1.14 and Lemma 7.1.7. ■

Corollary 7.1.9 *Let $A\colon X \rightrightarrows X^*$ be maximally monotone linear relations and $C \subseteq X$ be a closed convex set. Assume that $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - C]$ is a closed subspace. Then*

$$\begin{aligned}\bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_{N_C}] &= \bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - C]\\ &= \bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_A^{*\intercal} - P_X \operatorname{dom} F_{N_C}^{*\intercal}\right].\end{aligned}$$

Proof. Let $B = N_C$. Then apply directly Fact 5.1.14, Fact 5.1.12 and Lemma 7.1.7. ∎

Fact 7.1.10 (See [74, Lemma 23.9] or [10, Proposition 4.2].) *Let $A, B\colon X \rightrightarrows X^*$ be monotone operators and* $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$. *Then $F_{A+B} \leq F_A \Box_2 F_B$.*

Proof. Let $(x, x^*) \in X \times X^*$ and $y^* \in X^*$. Then we have

$$\begin{aligned}
&F_A(x,y^*) + F_B(x,x^*-y^*) = \sup_{(a,a^*)\in\operatorname{gra} A} [\langle a,y^*\rangle + \langle x,a^*\rangle - \langle a,a^*\rangle]\\
&\quad + \sup_{(b,b^*)\in\operatorname{gra} B} [\langle b,x^*-y^*\rangle + \langle x,b^*\rangle - \langle b,b^*\rangle]\\
&= \sup_{(a,a^*)\in\operatorname{gra} A,(b,b^*)\in\operatorname{gra} B} \big[\langle a,y^*\rangle + \langle x,a^*\rangle - \langle a,a^*\rangle + \langle b,x^*-y^*\rangle + \langle x,b^*\rangle\\
&\quad - \langle b,b^*\rangle\big]\\
&\geq \sup_{(a,a^*)\in\operatorname{gra} A,(a,b^*)\in\operatorname{gra} B} \big[\langle a,y^*\rangle + \langle x,a^*\rangle - \langle a,a^*\rangle + \langle a,x^*-y^*\rangle + \langle x,b^*\rangle\\
&\quad - \langle a,b^*\rangle\big]\\
&= \sup_{(a,a^*)\in\operatorname{gra} A,(a,b^*)\in\operatorname{gra} B} [\langle a,x^*\rangle + \langle x,a^*+b^*\rangle - \langle a,a^*+b^*\rangle]\\
&= F_{A+B}(x,x^*). \qquad (7.11)
\end{aligned}$$

Then $\inf_{y^*\in X^*}\left[F_A(x,y^*)+F_B(x,x^*-y^*)\right]\geq F_{A+B}(x,x^*)$ and thus $F_A\Box_2 F_B(x,x^*)\geq F_{A+B}(x,x^*)$. ■

We now discover more properties of $F_A\Box_2 F_B$.

Proposition 7.1.11 was first established by Bauschke, Wang and Yao in [15, Proposition 5.9] when X is a reflexive space. We now provide a nonreflexive version.

Proposition 7.1.11 *Let $A,B\colon X\rightrightarrows X^*$ be maximally monotone and suppose that $\bigcup_{\lambda>0}\lambda\left[\operatorname{dom}A-\operatorname{dom}B\right]$ is a closed subspace of X. Then $F_A\Box_2 F_B$ is proper, norm×weak* lower semicontinuous and convex, and the partial infimal convolution is exact everywhere.*

Proof. Define $F_1,F_2\colon X\times X^*\to\,]-\infty,+\infty]$ by

$$F_1:(x,x^*)\mapsto F_A^*(x^*,x),\quad F_2:(x,x^*)\mapsto F_B^*(x^*,x).$$

Since F_A,F_B is norm-weak* lower semicontinuous,

$$F_1^*(x^*,x)=F_A(x,x^*),\quad F_2^*(x^*,x)=F_B(x,x^*),\quad \forall (x,x^*)\in X\times X^*. \tag{7.12}$$

Take $(x,x^*)\in X\times X^*$. By Fact 5.1.5,

$$\big(F_1\Box_2 F_2\big)(x,x^*)\geq\langle x,x^*\rangle>-\infty.$$

In view of Lemma 7.1.7,

$$\bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\right] = \bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B\right] \quad \text{is a closed subspace.}$$

By Fact 7.1.5 and (7.12),

$$\begin{aligned}\big(F_1 \Box_2 F_2\big)^*(x^*, x) &= \min_{y^* \in X^*} \left[F_1^*(x^* - y^*, x) + F_2^*(y^*, x)\right] \\ &= \min_{y^* \in X^*} \left[F_A(x, x^* - y^*) + F_B(x, y^*)\right] = \big(F_A \Box_2 F_B\big)(x, x^*).\end{aligned}$$

Hence $F_A \Box_2 F_B$ is proper, norm$\times$weak* lower semicontinuous and convex, and the partial infimal convolution is exact. ■

Proposition 7.1.12 (See [15, Proposition 5.5].) *Let X be reflexive and $A\colon X \rightrightarrows X^*$ be a monotone linear relation with nonempty closed graph. Then $F_A^*\colon (x^*, x) \mapsto \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^*\rangle$.*

Proof. Define $g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra} A}(x, x^*)$. Thus by Fact 3.2.8 and the assumption, g is proper, lower semicontinuous and convex.

By definition of F_A, $F_A(x, x^*) = g^*(x^*, x)$ (for every $(x, x^*) \in X \times X^*$). Therefore, by [92, Theorem 2.3.3] we have $F_A^{*\intercal} = g$. ■

The next new result provides a sufficient but not necessary condition for the maximality of the sum of two maximally monotone operators.

Proposition 7.1.13 *Let $A, B\colon X \rightrightarrows X^*$ be maximally monotone and suppose that $\bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B\right]$ is a closed subspace of X. Assume that $F_A \Box_2 F_B = F_{A+B}$. Then $A + B$ is maximally monotone.*

Proof. We first show

$$F_{A+B} \geq \langle \cdot, \cdot \rangle. \tag{7.13}$$

Let $(x, x^*) \in X \times X^*$ and $y^* \in X^*$. Then by Fact 5.1.5, we have

$$F_A(x, y^*) + F_B(x, x^* - y^*) \geq \langle x, y^* \rangle + \langle x, x^* - y^* \rangle = \langle x, x^* \rangle.$$

Then

$$F_A \Box_2 F_B(x, x^*) = \inf_{y^* \in X^*} \left[F_A(x, y^*) + F_B(x, x^* - y^*) \right] \geq \langle x, x^* \rangle. \tag{7.14}$$

By (7.14) and the assumption that $F_A \Box_2 F_B = F_{A+B}$, we have (7.13) holds. Combining (7.13) and Fact 5.1.7, $A + B$ is maximally monotone. ■

Let $A, B \colon X \rightrightarrows X^*$ be maximally monotone such that $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$. By Fact 7.1.10 $F_A \Box_2 F_B \geq F_{A+B}$. It naturally raises a question: Does the equality always hold under the Rockafellar's constraint qualification: $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$ (which was also asked by the referee of [90])? The equality has a far-reaching meaning. If this were true, then Proposition 7.1.13 would directly solve the sum problem in the affirmative. However, in general, it cannot hold. The easiest example probably is [10, Example 4.7] by Bauschke, McLaren and Sendov on two projection operators in one dimensional space. Now we give another counterexample on a maximally monotone linear relation and the subdifferential of a proper lower semicontinuous sublinear function, which thus implies that we cannot approach the maximality of the sum of a linear relation A and the subdifferential of a proper lower semicontinuous sublinear function f by showing $F_A \Box_2 F_{\partial f} = F_{A+\partial f}$.

Example 7.1.14 *Let X be a Hilbert space, B_X be the closed unit ball of X and* Id *be the identity mapping from X to X. Let $f : x \in X \to \|x\|$. Then we have*

$$F_{\partial f} \Box_2 F_{\mathrm{Id}}(x, x^*) = \|x\| + \begin{cases} 0, & \textit{if} \|x + x^*\| \leq 1; \\ \frac{1}{4}\|x + x^*\|^2 - \frac{1}{2}\|x + x^*\| + \frac{1}{4}, & \textit{if} \|x + x^*\| > 1. \end{cases} \tag{7.15}$$

We also have $F_{\partial f + \mathrm{Id}} \neq F_{\partial f} \Box_2 F_{\mathrm{Id}}$ when $X = \mathbb{R}$.

Proof. By [10, Example 3.10 and Example 3.3], we have

$$F_{\mathrm{Id}}(x, x^*) = \frac{1}{4}\|x + x^*\|^2 \tag{7.16}$$

$$F_{\partial f}(x, x^*) = \|x\| + \iota_{B_X}(x^*), \quad \forall (x, x^*) \in X \times X. \tag{7.17}$$

Note that

$$\partial f(x) = \begin{cases} B_X, & \text{if } x = 0; \\ \{\frac{x}{\|x\|}\}, & \text{otherwise.} \end{cases} \tag{7.18}$$

$$N_{B_X}(x) = \begin{cases} 0, & \text{if } \|x\| < 1; \\ [0, \infty[\cdot x, & \text{if } \|x\| = 1; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{7.19}$$

Indeed, clearly $\partial f(0) = B_X$. Assume $x \neq 0$. By Fact 7.1.4(i),

$$x^* \in \partial f(x) \Leftrightarrow x^* \in B_X, \langle x^*, x\rangle = \|x\| \Leftrightarrow \|x^*\| = 1, \langle x^*, x\rangle = \|x\| \cdot \|x^*\|$$
$$\Leftrightarrow x^* = \tfrac{x}{\|x\|}.$$

Hence (7.18) holds. Similarly, (7.19) holds.

Then by (7.16) and (7.17),

$$(F_{\partial f}\Box_2 F_{\mathrm{Id}})(x, x^*) = \inf_{y^*}\left[\|x\| + \iota_{B_X}(y^*) + \tfrac{1}{4}\|x + x^* - y^*\|^2\right]$$
$$= \|x\| + \tfrac{1}{4}\|x + x^*\|^2 + \tfrac{1}{2}\inf_{y^*}\left[\langle x + x^*, y^*\rangle + \iota_{B_X}(y^*) + \tfrac{1}{2}\|y^*\|^2\right]. \qquad (7.20)$$

We consider two cases:

Case 1: $\|x + x^*\| \leq 1$. Then we directly obtain that

$$\inf_{y^*}\left[\langle x + x^*, y^*\rangle + \iota_{B_X}(y^*) + \tfrac{1}{2}\|y^*\|^2\right] = -\tfrac{1}{2}\|x + x^*\|^2.$$

And thus, $F_{\partial f}\Box_2 F_{\mathrm{Id}}(x, x^*) = \|x\|$.

Case 2: $\|x+x^*\| > 1$. Since $K : y^* \in X \to \langle x+x^*, y^*\rangle + \iota_{B_X}(y^*) + \frac{1}{2}\|y^*\|^2$ is convex, y_0^* is a minimizer of K if and only if $0 \in x + x^* + y_0^* + N_{B_X}(y_0^*)$. Since $\|x + x^*\| > 1$, by (7.19), $\|y_0^*\| = 1$. Thus by (7.19) again, there exists $\rho > 0$ such that $0 = x + x^* + y_0^* + \rho y_0^*$. Then we have $\rho + 1 = \|x + x^*\|$ and $y_0^* = -\frac{x+x^*}{\|x+x^*\|}$. Thus $\inf K = K(y_0^*) = -\|x + x^*\| + \frac{1}{2}$. Then $F_{\partial f}\Box_2 F_{\mathrm{Id}}(x, x^*) = \|x\| + \frac{1}{4}\|x + x^*\|^2 - \frac{1}{2}\|x + x^*\| + \frac{1}{4}$. Hence (7.15) holds.

In order to show $F_{\partial f+\mathrm{Id}} \neq F_{\partial f}\Box_2 F_{\mathrm{Id}}$, we consider the case when $X=\mathbb{R}$. Now we consider the point $(-1,4)$. Then by (7.15),

$$(F_{\partial f}\Box_2 F_{\mathrm{Id}})(-1,4) = 1+1 = 2. \tag{7.21}$$

On the other hand,

$$
\begin{aligned}
&F_{\partial f+\mathrm{Id}}(-1,4) = \sup_{x\in\mathbb{R}} \left[\langle x,4\rangle + \langle -1, x+\partial f(x)\rangle - \langle x, \partial f(x)+x\rangle\right]\\
&= \sup_{x\in\mathbb{R}} \left[\langle x,3\rangle + \langle -1, \partial f(x)\rangle - \langle x, \partial f(x)+x\rangle\right]\\
&= \sup_{x\in\mathbb{R}} \left[\langle x,3\rangle + \langle -1, \partial f(x)\rangle - |x| - |x|^2\right] \quad \text{(by Fact 7.1.4(i))}\\
&= \max\Big\{ \sup_{x>0} \left[\langle x,3\rangle + \langle -1, \partial f(x)\rangle - |x| - |x|^2\right],\\
&\sup_{x=0} \left[\langle x,3\rangle + \langle -1, \partial f(x)\rangle - |x| - |x|^2\right],\\
&\sup_{x<0} \left[\langle x,3\rangle + \langle -1, \partial f(x)\rangle - |x| - |x|^2\right] \Big\}\\
&= \max\Big\{ \sup_{x>0} \left[\langle x,3\rangle - 1 - |x| - |x|^2\right], 1, \sup_{x<0} \left[\langle x,3\rangle + 1 - |x| - |x|^2\right] \Big\}\\
&\quad \text{(by (7.18))}\\
&= \max\Big\{ \sup_{x>0} \left[\langle x,3\rangle - 1 - x - |x|^2\right], 1, 1\Big\}\\
&= \max\Big\{ \sup_{x>0} \left[2x - 1 - |x|^2\right], 1\Big\}\\
&= \max\{0,1\} = 1 \neq 2 = F_{\partial f}\Box_2 F_{\mathrm{Id}}(-1,4) \quad \text{(by (7.21))}.
\end{aligned}
$$

Hence $F_{\partial f+\mathrm{Id}} \neq F_{\partial f}\Box_2 F_{\mathrm{Id}}$. ■

7.2 Fitzpatrick function of the sum of two linear relations

Section 7.2 is mainly based on the work in [15, 17] by Bauschke, Wang and Yao.

Theorem 7.2.1 was first proved in [15, Theorem 5.10] by Bauschke, Wang and Yao in a reflexive space. Now we generalize it to a general Banach space.

Theorem 7.2.1 (Fitzpatrick function of the sum) *Let* $A, B\colon X \rightrightarrows X^*$ *be maximally monotone linear relations, and suppose that* $[\operatorname{dom} A - \operatorname{dom} B]$ *is closed. Then* $F_{A+B} = F_A \Box_2 F_B$.

Proof. Let $(z, z^*) \in X \times X^*$. By Fact 7.1.10, it suffices to show that

$$F_{A+B}(z, z^*) \geq (F_A \Box_2 F_B)(z, z^*). \tag{7.22}$$

If $(z, z^*) \notin \operatorname{dom} F_{A+B}$, then (7.22) clearly holds.

Now assume that $(z, z^*) \in \operatorname{dom} F_{A+B}$. Then

$$\begin{aligned}
&F_{A+B}(z, z^*)\\
&= \sup_{\{x,x^*,y^*\}} [\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle + \langle z - x, y^*\rangle - \iota_{\operatorname{gra} A}(x, x^*)\\
&\quad - \iota_{\operatorname{gra} B}(x, y^*)].
\end{aligned} \tag{7.23}$$

Let $Y = X^*$ and define $F, K : X \times X^* \times Y \to\ ]-\infty, +\infty]$ respectively by

$$F :(x, x^*, y^*) \in X \times X^* \times Y \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra} A}(x, x^*)$$

$$K : (x, x^*, y^*) \in X \times X^* \times Y \mapsto \langle x, y^* \rangle + \iota_{\operatorname{gra} B}(x, y^*)$$

Then by (7.23),

$$F_{A+B}(z, z^*) = (F + K)^*(z^*, z, z). \tag{7.24}$$

By Fact 3.2.8 and the assumptions, F and K are proper lower semicontinuous and convex, and

$$\operatorname{dom} F - \operatorname{dom} K = [\operatorname{dom} A - \operatorname{dom} B] \times X^* \times Y \quad \text{is a closed subspace.}$$

Thus by Fact 3.1.2 and (7.24), there exist $(z_0^*, z_0^{**}, z_1^{**}) \in X^* \times X^{**} \times Y^*$ such that

$$\begin{aligned}
F_{A+B}(z, z^*) &= F^*(z^* - z_0^*, z - z_0^{**}, z - z_1^{**}) + K^*(z_0^*, z_0^{**}, z_1^{**}) \\
&= F^*(z^* - z_0^*, z, 0) + K^*(z_0^*, 0, z) \quad (\text{by } (z, z^*) \in \operatorname{dom} F_{A+B}) \\
&= F_A(z, z^* - z_0^*) + F_B(z, z_0^*) \\
&\geq (F_A \Box_2 F_B)(z, z^*).
\end{aligned}$$

Thus (7.22) holds and hence $F_{A+B} = F_A \Box_2 F_B$. ■

The following result was first established by Voisei in [83]. Simons gave another proof in [74, Theorem 46.3]. Now we give a new approach for showing this result.

Theorem 7.2.2 *Let $A, B \colon X \rightrightarrows X^*$ be maximally monotone linear rela-*

tions, and suppose that $[\operatorname{dom} A - \operatorname{dom} B]$ *is closed. Then* $A+B$ *is maximally monotone.*

Proof. Combining Theorem 7.2.1, Corollary 7.1.8, and Proposition 7.1.13, we have $A+B$ is maximally monotone. ■

The following examples show that the constraint on the domain in Theorem 7.2.1 cannot be weakened. The rest of this section is all based on the work in [17] by Bauschke, Wang and Yao.

Let S be defined in Example 3.3.1, i.e.,

$$S\colon \operatorname{dom} S \to \ell^2(\mathbb{N})\colon y \mapsto \left(\tfrac{1}{2}y_n + \sum_{i<n} y_i\right)_{n\in\mathbb{N}}, \tag{7.25}$$

with

$$\operatorname{dom} S = \left\{ y = (y_n) \in \ell^2(\mathbb{N}) \;\middle|\; \sum_{i\geq 1} y_i = 0, \left(\sum_{i\leq n} y_i\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \right\}.$$

We explicitly compute the Fitzpatrick functions F_{S+S^*}, F_S, F_{S^*}, and show that $F_{S+S^*} \neq F_S \Box_2 F_{S^*}$ even though S, S^* are linear maximally monotone with $\operatorname{dom} S - \operatorname{dom} S^*$ being a dense linear subspace in $\ell^2(\mathbb{N})$.

Lemma 7.2.3 *Let* X *be a reflexive space and* $S : \operatorname{dom} S \to X^*$ *be a maximally monotone skew linear operator. Then*

$$F_S = \iota_{\operatorname{gra}(-S^*)}$$

and

$$F_{S^*}^{*\intercal} = F_{S^*} = \iota_{\operatorname{gra} S^*} + \langle \cdot, \cdot \rangle.$$

Proof. By Proposition 7.1.12,

$$F_S^* = (\iota_{\operatorname{gra} S})^{\intercal}.$$

Then

$$\begin{aligned} F_S &= \left(F_S^{*\intercal}\right)^{*\intercal} = \left(\iota_{\operatorname{gra} S}\right)^{*\intercal} = \left(\iota_{\operatorname{gra} S}^{\intercal}\right)^{*} = \left(\iota_{\operatorname{gra} S^{-1}}\right)^{*} \\ &= \iota_{(\operatorname{gra} S^{-1})^{\perp}} = \iota_{\operatorname{gra}(-S^*)}. \end{aligned} \tag{7.26}$$

From Fact 3.2.12, $\operatorname{gra}(-S) \subseteq \operatorname{gra} S^*$, we have

$$F_{S^*} \geq F_{(-S)} = \iota_{\operatorname{gra} -(-S)^*} = \iota_{\operatorname{gra} S^*},$$

this shows that $\operatorname{dom} F_{S^*} \subseteq \operatorname{gra} S^*$. By the Brezis-Browder theorem (Fact 3.2.13) and Fact 5.1.5, $F_{S^*}(x, x^*) = \langle x, x^* \rangle \ \forall (x, x^*) \in \operatorname{gra} S^*$. Hence $F_{S^*} = \iota_{\operatorname{gra} S^*} + \langle \cdot, \cdot \rangle$. Again by Proposition 7.1.12, $F_{S^*}^{*\intercal} = \iota_{\operatorname{gra} S^*} + \langle \cdot, \cdot \rangle$. ■

Theorem 7.2.4 *Let $H = \ell^2(\mathbb{N})$ and S be defined as in Example 3.3.1. Then*

$$\begin{aligned} F_{S+S^*}(x, x^*) &= \iota_{H \times \{0\}}(x, x^*) \\ F_S \Box_2 F_{S^*}(x, x^*) &= \begin{cases} \frac{1}{2}s^2, & \text{if } (x, x^*) \in \operatorname{dom} S^* \times \{0\} \text{ with } s = \sum_{i \geq 1} x_i; \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \tag{7.27}$$

Consequently, $F_S \Box_2 F_{S^*} \neq F_{S+S^*}$.

Proof. By Fact 3.2.12,

$$(S+S^*)|_{\operatorname{dom} S} = 0. \tag{7.28}$$

Let $(x,x^*) \in H \times H$. Using (7.28) and Fact 3.2.12, we have

$$F_{S+S^*}(x,x^*) = \sup_{a \in \operatorname{dom} S} \langle x^*, a\rangle = \iota_{(\operatorname{dom} S)^\perp}(x^*) = \iota_{\{0\}}(x^*) = \iota_{H\times\{0\}}(x,x^*). \tag{7.29}$$

Then by Fact 7.1.10, we have

$$(F_S \Box_2 F_{S^*})(x,x^*) = \infty, \quad x^* \neq 0. \tag{7.30}$$

It follows from Lemma 7.2.3 that

$$\begin{aligned}
(F_S \Box_2 F_{S^*})(x,0) &= \inf_{y^* \in H} \{F_S(x,y^*) + F_{S^*}(x,-y^*)\}\\
&= \inf_{y^* \in H} \{\iota_{\operatorname{gra}(-S^*)}(x,y^*) + \iota_{\operatorname{gra} S^*}(x,-y^*) + \langle x, -y^*\rangle\}\\
&= \inf_{y^* \in H} \{\iota_{\operatorname{gra} S^*}(x,-y^*) + \langle x,-y^*\rangle\}. \qquad (7.31)
\end{aligned}$$

Thus, $F_S \Box_2 F_{S^*}(x,0) = \infty$ if $x \notin \operatorname{dom} S^*$. Now suppose $x \in \operatorname{dom} S^*$ and $s = \sum_{i\geq 1} x_i$. Then by (7.31) and Proposition 3.3.6, we have

$$F_S \Box_2 F_{S^*}(x,0) = \langle x, S^* x\rangle = \tfrac{1}{2}s^2.$$

Combine the results above, (7.27) holds. Since $\operatorname{dom} S^* \neq H$, $F_S \Box_2 F_{S^*} \neq F_{S+S^*}$. ■

Let $A : H \rightrightarrows H$ be a maximally monotone linear relation. Then [15, Theorem 7.6] shows that: $A^* = -A$ if and only if $\big(\operatorname{dom} A = \operatorname{dom} A^*$ and $F_A = F_A^{*\intercal}\big)$. Let $A = S^*$ with S defined as in Example 3.3.1. Lemma 7.2.3 shows that $F_A = F_A^{*\intercal}$, but $A^* = S \neq -S^* = -A$ by Proposition 3.3.5. Hence the requirement $\operatorname{dom} A = \operatorname{dom} A^*$ cannot be omitted.

Let V be the Volterra integral operator. In the rest of this section, we systematically study $T = V^{-1}$ and its adjoint T^*.

We compute the Fitzpatrick functions F_T, F_{T^*}, F_{T+T^*}, and we show that $F_T \Box_2 F_{T^*} \neq F_{T+T^*}$. This shows that the constraint qualification for the formula of the Fitzpatrick function of the sum of two maximally monotone operators cannot be significantly weakened either.

To study Fitzpatrick functions of sums of maximally monotone operators, we need:

Lemma 7.2.5 *Let $H = L^2[0,1]$ and V be the Volterra integration operator defined in Example 3.4.4 and $e \equiv 1 \in L^2[0,1]$. Then*

$$q^*_{V_+}(z) = \iota_{\operatorname{span}\{e\}}(z) + \langle z, e\rangle^2, \quad \forall z \in L^2[0,1].$$

Proof. Let $z \in H$. By Example 3.4.4(iv) and Fact 7.1.1, we have

$$q^*_{V_+}(z) = \infty, \quad \text{if } z \notin \operatorname{span}\{e\}.$$

Now suppose that $z = le$ for some $l \in \mathbb{R}$. By Example 3.4.4(iv),

$$\begin{aligned} q^*_{V_+}(z) &= \sup_{x\in H}\{\langle x, z\rangle - q_{V_+}(x)\} = \sup_{x\in H}\{\langle x, le\rangle - \tfrac{1}{4}\langle x, e\rangle^2\} \\ &= l^2 = \langle le, e\rangle^2 = \langle z, e\rangle^2. \end{aligned}$$

Hence $q^*_{V_+}(z) = \iota_{\operatorname{span}\{e\}}(z) + \langle z, e\rangle^2$. ■

Lemma 7.2.6 *Let $H = L^2[0,1]$ and T be defined as in Theorem 3.4.5. We have (for every $(x, y^*) \in H \times H$)*

$$F_T(x, y^*) = F_V(y^*, x) = \iota_{\operatorname{span}\{e\}}(x + V^*y^*) + \tfrac{1}{2}\langle x + V^*y^*, e\rangle^2,$$
$$F_{T^*}(x, y^*) = F_{V^*}(y^*, x) = \iota_{\operatorname{span}\{e\}}(x + Vy^*) + \tfrac{1}{2}\langle x + Vy^*, e\rangle^2. \tag{7.32}$$

Proof. Apply Fact 5.1.5, Fact 7.1.1 and Lemma 7.2.5 to obtain the formula for F_T. Let $(x, y^*) \in H \times H$. By Proposition 3.1.3(iv), Fact 7.1.1 and Lemma 7.2.5 again, we have

$$\begin{aligned} F_{T^*}(x, y^*) = F_{V^*}(y^*, x) = \tfrac{1}{2}q^*_{V^*_+}(x + V^{**}y^*) = \tfrac{1}{2}q^*_{V_+}(x + Vy^*) \\ = \iota_{\operatorname{span}\{e\}}(x + Vy^*) + \tfrac{1}{2}\langle x + Vy^*, e\rangle^2. \end{aligned}$$

■

Remark 7.2.7 *Theorem 7.2.8 below gives another example showing that $F_{T+T^*} \neq F_T \Box_2 F_{T^*}$ while T, T^* are maximally monotone, and $\operatorname{dom} T - \operatorname{dom} T^*$ is a dense subspace in $L^2[0,1]$. This again shows that the assumption that $\operatorname{dom} A - \operatorname{dom} B$ is closed in Theorem 7.2.1 cannot be weakened substantially.*

Theorem 7.2.8 *Let $H = L^2[0,1]$ and T be defined as in Theorem 3.4.5, $e \equiv 1 \in L^2[0,1]$ and set*

$$C = \{x \in L^2[0,1] : \quad x \text{ is absolutely continuous, and } x' \in L^2[0,1]\}.$$

Then

$$F_{T+T^*}(x,x^*) = \iota_{H\times\{0\}}(x,x^*), \quad \forall (x,x^*)\in H\times H$$
$$(F_T\Box_2 F_{T^*})(x,x^*) = \begin{cases} \frac{1}{2}\left[x(1)^2 + x(0)^2\right], & \textit{if } (x,x^*)\in C\times\{0\};\\ \infty, & \textit{otherwise.}\end{cases} \tag{7.33}$$

Consequently, $F_T\Box_2 F_{T^*} \neq F_{T+T^*}$.

Proof. By Theorem 3.4.5(i) and Example 3.4.4(iii),

$$(T+T^*)y = 0, \quad \forall y\in \operatorname{dom} T\cap \operatorname{dom} T^* = \{Vx \ \mid x\in e^{\perp}\}. \tag{7.34}$$

Let $(x,x^*)\in H\times H$. Using Theorem 3.4.5(iii) and (7.34), we see that

$$\begin{aligned} F_{T+T^*}(x,x^*) &= \sup_{y\in\operatorname{dom} T\cap\operatorname{dom} T^*}\langle x^*, y\rangle = \sup_{y\in H}\langle x^*, y\rangle \\ &= \iota_{\{0\}}(x^*) = \iota_{H\times\{0\}}(x,x^*). \end{aligned} \tag{7.35}$$

By Fact 7.1.10, we have

$$\big(F_T\Box_2 F_{T^*}\big)(x,x^*) = \infty, \quad \forall\ x^*\neq 0. \tag{7.36}$$

When $x^*=0$, by Lemma 7.2.6,

$$\begin{aligned} \big(F_T\Box_2 F_{T^*}\big)(x,0) &= \inf_{y^*\in H}\{F_T(x,y^*) + F_{T^*}(x,-y^*)\} \\ &= \inf_{y^*\in H}\{\iota_{\operatorname{span}\{e\}}(x+V^*y^*) + \tfrac{1}{2}\langle x+V^*y^*, e\rangle^2 \\ &\quad + \iota_{\operatorname{span}\{e\}}(x-Vy^*) + \tfrac{1}{2}\langle x-Vy^*, e\rangle^2\}. \end{aligned} \tag{7.37}$$

Observe that

$$\begin{aligned}
&x + V^*y^* \in \operatorname{span}\{e\}, x - Vy^* \in \operatorname{span}\{e\}\\
&\Leftrightarrow x - Vy^* + Vy^* + V^*y^* \in \operatorname{span}\{e\}, x - Vy^* \in \operatorname{span}\{e\}\\
&\Leftrightarrow x - Vy^* \in \operatorname{span}\{e\}, \quad \text{(by Example 3.4.4(iv))}\\
&\Leftrightarrow x \in Vy^* + \operatorname{span}\{e\} \Leftrightarrow x \text{ is absolutely continuous and } y^* = x'.
\end{aligned}$$

Therefore, $(F_T \Box_2 F_{T^*})(x, 0) = \infty$ if $x \notin C$. For $x \in C$, using (7.37) and the fact that $x - Vx' = x(0)e$ and $x + V^*x' = x(1)e$, we obtain

$$\begin{aligned}
&\big(F_T \Box_2 F_{T^*}\big)(x, 0) = \tfrac{1}{2}\langle x + V^*x', e\rangle^2 + \tfrac{1}{2}\langle x - Vx', e\rangle^2\\
&= \tfrac{1}{2}x(1)^2 + \tfrac{1}{2}x(0)^2 = \tfrac{1}{2}\left[x(1)^2 + x(0)^2\right].
\end{aligned}$$

Thus, (7.33) holds. Consequently, $F_T \Box_2 F_{T^*} \neq F_{T+T^*}$. ■

7.3 Fitzpatrick function of the sum of a linear relations and a normal cone operator

The proof of Theorem 7.3.1 partially follows that of [16, Theorem 3.1] by Bauschke, Wang and Yao.

Theorem 7.3.1 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, let C be a nonempty closed convex subset of X, and suppose that* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$. *Then $F_{A+N_C} = F_A \Box_2 F_{N_C}$.*

Proof. Let $(z, z^*) \in X \times X^*$. By Fact 7.1.10, it suffices to show that

$$F_{A+N_C}(z, z^*) \geq (F_A \Box_2 F_{N_C})(z, z^*). \tag{7.38}$$

By Corollary 5.3.4,

$$P_X\left[\operatorname{dom} F_{A+N_C}\right] \subseteq \overline{\left[\operatorname{dom}(A + N_C)\right]} \subseteq C.$$

Thus, (7.38) holds if $z \notin C$. Now assume that $z \in C$. Set

$$g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra} A}(x, x^*). \tag{7.39}$$

By Fact 3.2.8, g is convex. Hence,

$$h = g + \iota_{C \times X^*} \tag{7.40}$$

is convex as well. Let

$$c_0 \in \operatorname{dom} A \cap \operatorname{int} C, \tag{7.41}$$

and let $c_0^* \in Ac_0$. Then $(c_0, c_0^*) \in \operatorname{gra} A \cap (\operatorname{int} C \times X^*) = \operatorname{dom} g \cap \operatorname{int} \operatorname{dom} \iota_{C \times X^*}$. By Fact 5.1.4, $\iota_{C \times X^*}$ is continuous at (c_0, c_0^*). Then,

$$\begin{aligned}
& F_{A+N_C}(z, z^*) \\
&= \sup_{(x,x^*,c^*)} \big[\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle + \langle z - x, c^*\rangle - \iota_{\operatorname{gra} A}(x, x^*) \\
&\quad - \iota_{\operatorname{gra} N_C}(x, c^*)\big] \\
&\geq \sup_{(x,x^*)} \left[\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle - \iota_{\operatorname{gra} A}(x, x^*) - \iota_{C \times X^*}(x, x^*)\right]
\end{aligned}$$

$$= \sup_{(x,x^*)} [\langle x, z^*\rangle + \langle z, x^*\rangle - h(x,x^*)]$$

$$= h^*(z^*, z)$$

$$= g^*(y^*, y^{**}) + \iota^*_{C\times X^*}(z^* - y^*, z - y^{**})$$

$$\text{(by Fact 5.1.1, } \exists (y^*, y^{**}) \in X^* \times X^{**})$$

$$= g^*(y^*, y^{**}) + \iota^*_C(z^* - y^*) + \iota_{\{0\}}(z - y^{**}).$$

We consider two cases:

Case 1: $z \neq y^{**}$. Clearly, $F_{A+N_C}(z,z^*) = +\infty \geq (F_A \Box_2 F_{N_C})(z,z^*)$.

Case 2: $z = y^{**}$. Then

$$F_{A+N_C}(z,z^*) \geq g^*(y^*, y^{**}) + \iota^*_C(z^* - y^*) = F_A(z, y^*) + \iota^*_C(z^* - y^*)$$

$$= F_A(z,y^*) + \iota^*_C(z^* - y^*) + \iota_C(z) \geq F_A \Box_2 (\iota_C + \iota^*_C)$$

$$= (F_A \Box_2 F_{N_C})(z,z^*) \quad \text{(by Fact 7.1.2).}$$

Hence (7.38) holds and thus $F_{A+N_C} = F_A \Box_2 F_{N_C}$. ■

7.4 Discussion

It would be interesting to find out whether Theorem 7.3.1 generalizes to the following:

> Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, let C be a nonempty closed convex subset of X. Assume that $\left[\operatorname{dom} A - \bigcup_{\lambda>0} \lambda C\right]$ is a closed subspace of X.
> Is it necessarily true that $F_{A+N_C} = F_A \Box_2 F_{N_C}$?

Chapter 8

BC–functions and examples of type (D) operators

This chapter is based on the work in [8] by Bauschke, Borwein, Wang and Yao.

We first introduce some notation related to this chapter. Let $F : X \times X^* \to \left]-\infty, +\infty\right]$. We say F is a *BC–function* (BC stands for "bigger conjugate") [74] if F is proper and convex with

$$F^*(x^*, x) \geq F(x, x^*) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*. \tag{8.1}$$

Let Y be a real Banach space, and let $F_1, F_2 \colon X \times Y \to \left]-\infty, +\infty\right]$. Then the function $F_1 \Box_1 F_2$ is defined on $X \times Y$ by

$$F_1 \Box_1 F_2 \colon (x, y) \mapsto \inf_{u \in X} \{F_1(u, y) + F_2(x - u, y)\}.$$

In Example 8.3.1(iii)&(v) of this chapter, we provide a negative answer to the following question posed by S. Simons [74, Problem 22.12]:

Let $F_1, F_2 : X \times X^* \to \left]-\infty, +\infty\right]$ be lower semicontinuous BC–

functions and

$$\bigcup_{\lambda>0} \lambda\left[P_{X^*} \operatorname{dom} F_1 - P_{X^*} \operatorname{dom} F_2\right] \text{ is a closed subspace of } X^*.$$

Is $F_1 \Box_1 F_2$ necessarily a BC–function?

8.1 Auxiliary results

Fact 8.1.1 (Banach and Mazur) (See [44, Theorem 5.17]).) *Every separable Banach space is isometric to a subspace of $C[0,1]$.*

Fact 8.1.2 (Fitzpatrick) (See [45, Corollary 3.9 and Proposition 4.2].) *Let $A\colon X \rightrightarrows X^*$ be maximally monotone. Then F_A is a BC–function and $F_A = \langle\cdot,\cdot\rangle$ on* $\operatorname{gra} A$.

Let Y be a real Banach space. Let $L : X \to Y$ be linear. We say L is an *isomorphism into* Y if L is one to one, continuous and L^{-1} is continuous on $\operatorname{ran} L$. We say L is an *isometry* if $\|Lx\| = \|x\|, \forall x \in X$. The spaces X, Y are called *isometric* if there exists an isometry from X onto Y. Let $A : X \rightrightarrows X^*$ be monotone and S be a subspace of X. We say A is S*–saturated* if

$$Ax + S^{\perp} = Ax, \quad \forall x \in \operatorname{dom} A.$$

Fact 8.1.3 (Simons and Zălinescu) (See [74, Theorem 16.4(b)].)

Let Y be a Banach space and $F_1, F_2\colon X \times Y \to \left]-\infty, +\infty\right]$ be proper, *lower semicontinuous and convex. Assume that for every* $(x,y)\in X\times Y$*,*

$$(F_1\Box_1 F_2)(x,y) > -\infty$$

and that $\bigcup_{\lambda>0}\lambda\left[P_Y \operatorname{dom} F_1 - P_Y \operatorname{dom} F_2\right]$ *is a closed subspace of* Y*. Then for every* $(x^*,y^*)\in X^*\times Y^*$*,*

$$(F_1\Box_1 F_2)^*(x^*,y^*) = \min_{u^*\in Y^*}\left[F_1^*(x^*,u^*) + F_2^*(x^*,y^*-u^*)\right].$$

Fact 8.1.4 (Simons) (See [74, Theorem 28.9].) *Let* Y *be a real Banach space, and* $L: Y\to X$ *be continuous and linear with* $\operatorname{ran} L$ *closed and* $\operatorname{ran} L^* = Y^*$*. Let* $A: X\rightrightarrows X^*$ *be monotone with* $\operatorname{dom} A\subseteq \operatorname{ran} L$ *such that* $\operatorname{gra} A\neq\varnothing$*. Then* A *is maximally monotone if and only if* A *is* $\operatorname{ran} L$*–saturated and* L^*AL *is maximally monotone.*

Fact 8.1.5 (See [58, Theorem 3.1.22(b)] or [44, Exercise 2.39(i), page 59].) *Let* Y *be a real Banach space. Assume that* $L: Y\to X$ *is an isomorphism into* X*. Then* $\operatorname{ran} L^* = Y^*$*.*

Corollary 8.1.6 *Let* Y *be a real Banach space, and* $L: Y\to X$ *be an isomorphism into* X*. Let* $T: Y\rightrightarrows Y^*$ *be monotone. Then* T *is maximally monotone if, and only if* $(L^*)^{-1}TL^{-1}$ *is maximally monotone.*

Proof. Let $A = (L^*)^{-1}TL^{-1}$. Then $\operatorname{dom} A\subseteq\operatorname{ran} L$. Since L is an isomorphism into X, $\operatorname{ran} L$ is closed. By Fact 8.1.5, $\operatorname{ran} L^* = Y^*$. Hence $\operatorname{gra}(L^*)^{-1}TL^{-1}\neq\varnothing$ if and only if $\operatorname{gra} T\neq\varnothing$. Clearly, A is monotone. Since $(0,(\operatorname{ran} L)^{\perp})\subseteq\operatorname{gra}(L^*)^{-1}$, $A = (L^*)^{-1}TL^{-1}$ is $\operatorname{ran} L$–saturated. By

Fact 8.1.4, $A = (L^*)^{-1}TL^{-1}$ is maximally monotone if and only if $L^*AL = T$ is maximally monotone. ■

The following result will allow us for constructing operators that are not of type (D) in different Banach spaces.

Corollary 8.1.7 *Let Y be a real Banach space, and $L : Y \to X$ be an isomorphism into X. Let $T : Y \rightrightarrows Y^*$ be maximally monotone. Assume that T is not of type (D). Then $(L^*)^{-1}TL^{-1}$ is maximally monotone but is not of type (D).*

Proof. By Corollary 8.1.6, $(L^*)^{-1}TL^{-1}$ is maximally monotone. By Fact 6.1.5 or Corollary 6.2.2 , there exists $(y_0^{**}, y_0^*) \in Y^{**} \times Y^*$ such that

$$\sup_{(b,b^*)\in \operatorname{gra} T} \left\{ \langle y_0^{**}, b^* \rangle + \langle y_0^*, b \rangle - \langle b, b^* \rangle \right\} < \langle y_0^{**}, y_0^* \rangle. \tag{8.2}$$

By Fact 8.1.5, there exists $x_0^* \in X^*$ such that $L^*x_0^* = y_0^*$. Let $A = (L^*)^{-1}TL^{-1}$. Then we have

$$\begin{aligned}
&\sup_{(a,a^*)\in \operatorname{gra} A} \left\{ \langle L^{**}y_0^{**}, a^* \rangle + \langle x_0^*, a \rangle - \langle a, a^* \rangle \right\} \\
&= \sup_{(Ly,a^*)\in \operatorname{gra} A} \left\{ \langle y_0^{**}, L^*a^* \rangle + \langle x_0^*, Ly \rangle - \langle Ly, a^* \rangle \right\} \\
&= \sup_{(Ly,a^*)\in \operatorname{gra} A} \left\{ \langle y_0^{**}, L^*a^* \rangle + \langle L^*x_0^*, y \rangle - \langle y, L^*a^* \rangle \right\} \\
&= \sup_{(Ly,a^*)\in \operatorname{gra} A} \left\{ \langle y_0^{**}, L^*a^* \rangle + \langle y_0^*, y \rangle - \langle y, L^*a^* \rangle \right\} \\
&= \sup_{(y,y^*)\in \operatorname{gra} T} \left\{ \langle y_0^{**}, y^* \rangle + \langle y_0^*, y \rangle - \langle y, y^* \rangle \right\} \\
&\quad (\text{by } (Ly, a^*) \in \operatorname{gra} A \Leftrightarrow (y, L^*a^*) \in \operatorname{gra} T)
\end{aligned}$$

$$< \langle y_0^{**}, y_0^* \rangle \quad \text{(by (8.2))}$$

$$= \langle L^{**} y_0^{**}, x_0^* \rangle. \tag{8.3}$$

Thus A is not type (NI) and hence $A = (L^*)^{-1} T L^{-1}$ is not type (D) by Fact 6.1.5. ■

8.2 Main construction

We shall give an abstract framework for constructing non type (D) operators in non-reflexive spaces.

Lemma 8.2.1 *Let $A\colon X \rightrightarrows X^*$ be a skew linear relation. Then*

$$F_A = \iota_{\operatorname{gra}(-A^*) \cap X \times X^*}. \tag{8.4}$$

Proof. Let $(x_0, x_0^*) \in X \times X^*$. We have

$$\begin{aligned} F_A(x_0, x_0^*) &= \sup_{(x,x^*) \in \operatorname{gra} A} \{ \langle (x_0^*, x_0), (x, x^*) \rangle - \langle x, x^* \rangle \} \\ &= \sup_{(x,x^*) \in \operatorname{gra} A} \langle (x_0^*, x_0), (x, x^*) \rangle \\ &= \iota_{(\operatorname{gra} A)^\perp}(x_0^*, x_0) \\ &= \iota_{\operatorname{gra}(-A^*)}(x_0, x_0^*) \\ &= \iota_{\operatorname{gra}(-A^*) \cap X \times X^*}(x_0, x_0^*). \end{aligned}$$

Hence (8.4) holds. ■

The main result in this chapter is Theorem 8.2.2, which our constructed examples are based on.

Theorem 8.2.2 *Let $A : X^* \to X^{**}$ be linear and continuous. Assume that* $\operatorname{ran} A \subseteq X$ *and there exists $e \in X^{**} \backslash X$ such that*

$$\langle Ax^*, x^* \rangle = \langle e, x^* \rangle^2, \quad \forall x^* \in X^*.$$

Let P and S respectively be the symmetric part and antisymmetric part of A. Let $T : X \rightrightarrows X^*$ be defined by

$$\begin{aligned} \operatorname{gra} T &= \{(-Sx^*, x^*) \mid x^* \in X^*, \langle e, x^* \rangle = 0\} \\ &= \{(-Ax^*, x^*) \mid x^* \in X^*, \langle e, x^* \rangle = 0\}. \end{aligned} \tag{8.5}$$

Let $f : X \to \,]-\infty, +\infty]$ be a proper lower semicontinuous and convex function. Set $F = f \oplus f^*$ on $X \times X^*$. Then the following hold.

(i) *T is maximally monotone.*

(ii) $\operatorname{gra} T^* = \{(Sx^* + re, x^*) \mid x^* \in X^*,\ r \in \mathbb{R}\}$.

(iii) *T is not of type (D).*

(iv) *$F_T = \iota_C$, where*

$$C = \{(-Ax^*, x^*) \mid x^* \in X^*\}. \tag{8.6}$$

(v) *If* $\operatorname{dom} T \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$*, then $T + \partial f$ is maximally monotone.*

(vi) *F and F_T are BC–functions on $X \times X^*$.*

(vii) *Moreover,*

$$\bigcup_{\lambda>0} \lambda\big(P_{X^*}(\operatorname{dom} F_T) - P_{X^*}(\operatorname{dom} F)\big) = X^*.$$

Assume that there exists $(v_0, v_0^*) \in X \times X^*$ such that

$$f^*(v_0^*) + f^{**}(v_0 - A^* v_0^*) < \langle v_0, v_0^* \rangle. \tag{8.7}$$

Then $F_T \Box_1 F$ is not a BC–function.

(viii) *Assume that $\left[\operatorname{ran} A - \bigcup_{\lambda>0} \lambda \operatorname{dom} f\right]$ is a closed subspace of X and that $\varnothing \neq \operatorname{dom} f^{**} \circ A^*|_{X^*} \subsetneqq \{e\}_\perp$. Then $T + \partial f$ is not of type (D).*

Proof. (i): Now we claim that

$$Px^* = \langle x^*, e\rangle e, \ \forall x^* \in X^*. \tag{8.8}$$

Since $\langle \cdot, e\rangle e = \partial(\frac{1}{2}\langle \cdot, e\rangle^2)$ and by [63, Theorem 5.1], $\langle \cdot, e\rangle e$ is a symmetric operator on X^*. Clearly, $A - \langle \cdot, e\rangle e$ is skew. Then (8.8) holds.

Let $x^* \in X^*$ with $\langle e, x^* \rangle = 0$. Then we have

$$Sx^* = \langle x^*, e\rangle e + Sx^* = Px^* + Sx^* = Ax^* \in \operatorname{ran} A \subseteq X.$$

Thus (8.5) holds and T is well defined.

We have S is skew and hence T is skew. Let $(z, z^*) \in X \times X^*$ be monotonically related to $\operatorname{gra} T$. By Fact 3.2.9, we have

$$0 = \langle z, x^* \rangle + \langle -Sx^*, z^* \rangle = \langle z + Sz^*, x^* \rangle, \quad \forall x^* \in \{e\}_\perp.$$

Thus by Fact 3.1.1, we have $z + Sz^* \in (\{e\}_\perp)^\perp = \operatorname{span}\{e\}$ and then

$$z = -Sz^* + \kappa e, \ \exists \kappa \in \mathbb{R}. \tag{8.9}$$

By Fact 3.2.9 again,

$$\kappa \langle z^*, e \rangle = \langle -Sz^* + \kappa e, z^* \rangle = \langle z, z^* \rangle \geq 0. \tag{8.10}$$

Then by (8.9) and (8.8),

$$Az^* = Pz^* + Sz^* = Pz^* + \kappa e - z = [\langle z^*, e \rangle + \kappa]\, e - z. \tag{8.11}$$

By the assumptions that $z \in X$, $Az^* \in X$ and $e \notin X$, $[\langle z^*, e \rangle + \kappa] = 0$ by (8.11). Then by (8.10), we have $\langle z^*, e \rangle = \kappa = 0$ and thus $(z, z^*) \in \operatorname{gra} T$ by (8.9). Hence T is maximally monotone.

(ii): Let $(x_0^{**}, x_0^*) \in X^{**} \times X^*$. Then we have

$$\begin{aligned}
&(x_0^{**}, x_0^*) \in \operatorname{gra} T^* \Leftrightarrow \langle x_0^*, Sx^* \rangle + \langle x^*, x_0^{**} \rangle = 0, \quad \forall x^* \in \{e\}_\perp \\
&\Leftrightarrow \langle x^*, x_0^{**} - Sx_0^* \rangle = 0, \quad \forall x^* \in \{e\}_\perp \\
&\Leftrightarrow x_0^{**} - Sx_0^* \in (\{e\}_\perp)^\perp = \operatorname{span}\{e\} \quad \text{(by Fact 3.1.1)} \\
&\Leftrightarrow x_0^{**} - Sx_0^* = re, \quad \exists r \in \mathbb{R}.
\end{aligned}$$

Thus $\operatorname{gra} T^* = \{(Sx^* + re, x^*) \mid x^* \in X^*,\ r \in \mathbb{R}\}$.

(iii): By (ii), T^* is not monotone. Then by Corollary 6.3.3, T is not of type (D).

(iv): By (ii), we have

$$\begin{aligned}
&(z, z^*) \in \operatorname{gra}(-T^*) \cap X \times X^* \\
&\Leftrightarrow (z, z^*) = (-Sz^* - re, z^*), \quad z \in X,\ \exists r \in \mathbb{R},\ z^* \in X^* \\
&\Leftrightarrow (z, z^*) = (-Sz^* - \langle z^*, e\rangle e + [\langle z^*, e\rangle - r]\, e, z^*), \quad \exists r \in \mathbb{R},\ z^* \in X^* \\
&\Leftrightarrow (z, z^*) = (-Az^* + [\langle z^*, e\rangle - r]\, e, z^*), \quad \exists r \in \mathbb{R},\ z^* \in X^* \text{ (by (8.8))} \\
&\Leftrightarrow (z, z^*) = (-Az^*, z^*), \quad \langle z^*, e\rangle = r \\
&\quad \text{(by } z, Az^* \in X \text{ and } e \notin X), \exists r \in \mathbb{R},\ z^* \in X^* \\
&\Leftrightarrow (z, z^*) \in \{(-Ax^*, x^*) \mid x^* \in X^*\} = C.
\end{aligned}$$

Thus by Lemma 8.2.1, we have $F_T = \iota_C$.

(v): Apply (i) and Theorem 5.3.1.

(vi): Clearly, F is a BC–function. By (i) and Fact 8.1.2, we have F_T is a BC–function.

(vii): By (iv), we have

$$\bigcup_{\lambda>0} \lambda\big(P_{X^*}(\operatorname{dom} F_T) - P_{X^*}(\operatorname{dom} F)\big) = X^*. \tag{8.12}$$

Then for every $(x, x^*) \in X \times X^*$ and $u \in X$, by (vi),

$$F_T(x - u, x^*) + F(u, x^*) = F_T(x - u, x^*) + (f \oplus f^*)(u, x^*)$$

$$\geq \langle x-u, x^*\rangle + \langle u, x^*\rangle = \langle x, x^*\rangle.$$

Hence

$$(F_T \Box_1 F)(x, x^*) \geq \langle x, x^*\rangle > -\infty. \tag{8.13}$$

Then by (8.12), (8.13) and Fact 8.1.3,

$$\begin{aligned}
(F_T \Box_1 F)^*(v_0^*, v_0) &= \min_{x^{**}\in X^{**}} F_T^*(v_0^*, x^{**}) + F^*(v_0^*, v_0 - x^{**}) \\
&\leq F_T^*(v_0^*, A^*v_0^*) + F^*(v_0^*, v_0 - A^*v_0^*) \\
&= 0 + F^*(v_0^*, v_0 - A^*v_0^*) \quad \text{(by (iv))} \\
&= (f \oplus f^*)^*(v_0^*, v_0 - A^*v_0^*) = (f^* \oplus f^{**})(v_0^*, v_0 - A^*v_0^*) \\
&= f^*(v_0^*) + f^{**}(v_0 - A^*v_0^*) \\
&< \langle v_0^*, v_0\rangle \quad \text{(by 8.7)}.
\end{aligned} \tag{8.14}$$

Hence $F_A \Box_1 F$ is not a BC–function.

(viii): By the assumption, there exists $x_0^* \in \operatorname{dom} f^{**} \circ A^*|_{X^*}$ such that $\langle e, x_0^*\rangle \neq 0$. Let $\varepsilon_0 = \frac{\langle e, x_0^*\rangle^2}{2}$. By [92, Theorem 2.4.4(iii)]), there exists $y_0^{***} \in \partial_{\varepsilon_0} f^{**}(A^*x_0^*)$. By [92, Theorem 2.4.2(ii)]),

$$f^{**}(A^*x_0^*) + f^{***}(y_0^{***}) \leq \langle A^*x_0^*, y_0^{***}\rangle + \varepsilon_0. \tag{8.15}$$

Then by [74, Lemma 45.15] or the proof of [67, Eq.(2.5) in Proposition 1], there exists $y_0^* \in X^*$ such that

$$f^{**}(A^* x_0^*) + f^*(y_0^*) < \langle A^* x_0^*, y_0^* \rangle + 2\varepsilon_0. \tag{8.16}$$

Let $z_0^* = y_0^* + x_0^*$. Then by (8.16), we have

$$\begin{aligned} f^{**}(A^* x_0^*) + f^*(z_0^* - x_0^*) &< \langle A^* x_0^*, z_0^* - x_0^* \rangle + 2\varepsilon_0 \\ &= \langle A^* x_0^*, z_0^* \rangle - \langle A^* x_0^*, x_0^* \rangle + 2\varepsilon_0 \\ &= \langle A^* x_0^*, z_0^* \rangle - \langle x_0^*, A x_0^* \rangle + 2\varepsilon_0 \\ &= \langle A^* x_0^*, z_0^* \rangle - 2\varepsilon_0 + 2\varepsilon_0 \\ &= \langle A^* x_0^*, z_0^* \rangle. \end{aligned} \tag{8.17}$$

Then for every $(x, x^*) \in X \times X^*$ and $u^* \in X$, by (vi),

$$\begin{aligned} F_T(x, x^* - u^*) + F(x, u^*) &= F_T(x, x^* - u^*) + (f \oplus f^*)(x, u^*) \\ &\geq \langle x, x^* - u^* \rangle + \langle x, u^* \rangle = \langle x, x^* \rangle. \end{aligned}$$

Hence

$$(F_T \Box_2 F)(x, x^*) \geq \langle x, x^* \rangle > -\infty. \tag{8.18}$$

Then by (8.18), (iv) and Fact 7.1.5,

$$(F_T \Box_2 F)^*(z_0^*, A^* x_0^*)$$

$$
\begin{aligned}
&= \min_{y^*\in X^*} F_T^*(y^*, A^*x_0^*) + F^*(z_0^* - y^*, A^*x_0^*)\\
&\leq F_T^*(x_0^*, A^*x_0^*) + F^*(z_0^* - x_0^*, A^*x_0^*)\\
&= 0 + F^*(z^* - x_0^*, A^*x_0^*) \quad \text{(by (iv))}\\
&= (f\oplus f^*)^*(z_0^* - x_0^*, A^*x_0^*)\\
&= f^*(z_0^* - x_0^*) + f^{**}(A^*x_0^*)\\
&< \langle z_0^*, A^*x_0^*\rangle \quad \text{(by (8.17))}. \qquad (8.19)
\end{aligned}
$$

Let $F_0 : X \times X^* \to \,]-\infty, +\infty]$ be defined by

$$(x, x^*) \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra}(T+\partial f)}(x, x^*). \tag{8.20}$$

Clearly, $F_T\Box_2 F \leq F_0$ on $X \times X^*$ and thus $(F_T\Box_2 F)^* \geq F_0^*$ on $X^* \times X^{**}$. By (8.19), $F_0^*(z_0^*, A^*x_0^*) < \langle z_0^*, A^*x_0^*\rangle$. Hence $T + \partial f$ is not of type (NI) and thus $T + \partial f$ is not of type (D) by Fact 6.1.5. ■

8.3 Examples and applications

Example 8.3.1 *Suppose that*

$$X = c_0, \textit{ with norm } \|\cdot\|_\infty \textit{ so that } X^* = \ell^1(\mathbb{N}) \textit{ with norm } \|\cdot\|_1,$$

and $X^{**} = \ell^\infty(\mathbb{N})$ *with norm* $\|\cdot\|_*$. *Let* $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ *with* $\limsup \alpha_n \neq 0$, *and let* $A_\alpha : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ *be defined by*

$$(A_\alpha x^*)_n = \alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^*, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}).$$

Let P_α and S_α respectively be the symmetric part and antisymmetric part of A_α. Let $T_\alpha : c_0 \rightrightarrows X^*$ be defined by

$$
\begin{aligned}
\operatorname{gra} T_\alpha &= \big\{(-S_\alpha x^*, x^*) \big| \, x^* \in X^*, \langle \alpha, x^* \rangle = 0\big\} \\
&= \big\{(-A_\alpha x^*, x^*) \big| \, x^* \in X^*, \langle \alpha, x^* \rangle = 0\big\} \\
&= \Big\{\big((-\sum_{i>n} \alpha_n \alpha_i x_i^* + \sum_{i<n} \alpha_n \alpha_i x_i^*)_{n\in\mathbb{N}}, x^*\big) \Big| \, x^* \in X^*, \langle \alpha, x^* \rangle = 0\Big\}.
\end{aligned}
\tag{8.21}
$$

Then the following hold.

(i) $\langle A_\alpha x^*, x^* \rangle = \langle \alpha, x^* \rangle^2, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. *Hence* (8.21) *is well defined.*

(ii) T_α *is a maximally monotone operator that is not of type (D).*

(iii) $F_{T_\alpha} \Box_1 (\|\cdot\| \oplus \iota_{B_{X^*}})$ *is not a BC–function.*

(iv) $T_\alpha + \partial\|\cdot\|$ *is a maximally monotone operator that is not of type (D).*

(v) *If* $\frac{1}{\sqrt{2}} < \|\alpha\|_* \leq 1$, *then* $F_{T_\alpha} \Box_1 (\frac{1}{2}\|\cdot\|^2 \oplus \frac{1}{2}\|\cdot\|_1^2)$ *is not a BC–function.*

(vi) $T_\alpha + \lambda J$ *is a maximally monotone operator that is not of type (D) for every* $\lambda > 0$.

(vii) *There exists a linear operator* $L : c_0 \to C[0,1]$ *that is an isometry from* c_0 *to a subspace of* $C[0,1]$. *Then for every* $\lambda > 0$, $(L^*)^{-1}(T_\alpha + \partial\|\cdot\|)L^{-1}$ *and* $(L^*)^{-1}(T_\alpha + \lambda J)L^{-1}$ *are maximally monotone operators that are not of type (D).*

(viii) *Let $G : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ be Gossez's operator [50] defined by*

$$\big(G(x^*)\big)_{n\in\mathbb{N}} = \sum_{i>n} x_i^* - \sum_{i<n} x_i^*, \quad \forall (x_n^*)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}).$$

Then $T_e : c_0 \rightrightarrows \ell^1(\mathbb{N})$ as defined by

$$\operatorname{gra} T_e = \{(-G(x^*), x^*) \mid x^* \in \ell^1(\mathbb{N}), \langle x^*, e\rangle = 0\}$$

is a maximally monotone operator that is not of type (D), where $e = (1, 1, \ldots, 1, \ldots)$.

Proof. We have $\alpha \notin c_0$. Since $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, A_α is linear and continuous and $\operatorname{ran} A_\alpha \subseteq c_0 \subseteq \ell^\infty(\mathbb{N})$.

(i): We have

$$\begin{aligned}
\langle A_\alpha x^*, x^*\rangle &= \sum_n x_n^*(\alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^*) = \sum_n \alpha_n^2 {x_n^*}^2 + 2\sum_n\sum_{i>n} \alpha_n\alpha_i x_n^* x_i^* \\
&= \sum_n \alpha_n^2 {x_n^*}^2 + \sum_{n\neq i} \alpha_n\alpha_i x_n^* x_i^* \\
&= (\sum_n \alpha_n x_n^*)^2 = \langle \alpha, x^*\rangle^2, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}). \qquad (8.22)
\end{aligned}$$

Then the proof of Theorem 8.2.2 shows that the symmetric part P_α of A_α is $P_\alpha x^* = \langle \alpha, x^*\rangle\alpha$ (for every $x^* \in \ell^1(\mathbb{N})$). Thus, the skew part S_α of A_α is

$$\begin{aligned}
(S_\alpha x^*)_{n\in\mathbb{N}} &= (A_\alpha x^*)_{n\in\mathbb{N}} - (P_\alpha x^*)_{n\in\mathbb{N}} = \Big(\alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^* - \sum_i \alpha_n\alpha_i x_i^*\Big)_{n\in\mathbb{N}} \\
&= \Big(\sum_{i>n} \alpha_n\alpha_i x_i^* - \sum_{i<n} \alpha_n\alpha_i x_i^*\Big)_{n\in\mathbb{N}}. \qquad (8.23)
\end{aligned}$$

Then by Theorem 8.2.2, (8.21) is well defined.

(ii): Combine Theorem 8.2.2(i)&(iii).

(iii): Let $f = \|\cdot\|$ on $X = c_0$. Then $f^* = \iota_{B_{X^*}}$ by [92, Corollary 2.4.16]. Since $\alpha \neq 0$, there exists $i_0 \in \mathbb{N}$ such that $\alpha_{i_0} \neq 0$. Let $e_{i_0} = (0, \ldots, 0, 1, 0, \ldots)$, i.e., the i_0th component is 1 and the others are 0. Then by (8.23), we have

$$S_\alpha e_{i_0} = \alpha_{i_0}(\alpha_1, \ldots, \alpha_{i_0-1}, 0, -\alpha_{i_0+1}, -\alpha_{i_0+2}, \ldots). \tag{8.24}$$

Then

$$\begin{aligned} A_\alpha^* e_{i_0} &= P_\alpha e_{i_0} - S_\alpha e_{i_0} \\ &= \alpha_{i_0}(0, \ldots, 0, \alpha_{i_0}, 2\alpha_{i_0+1}, 2\alpha_{i_0+2}, \ldots). \end{aligned} \tag{8.25}$$

Now set $v_0^* = e_{i_0}$ and $v_0 = 3\|\alpha\|_*^2 e_{i_0}$. Thus by (8.25),

$$\begin{aligned} v_0 - A_\alpha^* v_0^* &= 3\|\alpha\|_*^2 e_{i_0} - A_\alpha^* e_{i_0} \\ &= (0, \ldots, 0, 3\|\alpha\|_*^2 - \alpha_{i_0}^2, -2\alpha_{i_0}\alpha_{i_0+1}, -2\alpha_{i_0}\alpha_{i_0+2}, \ldots) \end{aligned} \tag{8.26}$$

We have

$$\begin{aligned} f^*(v_0^*) + f^{**}(v_0 - A_\alpha^* e_{i_0}) &= \iota_{B_{X^*}}(e_{i_0}) + \|v_0 - A_\alpha^* e_{i_0}\|_* \\ &= \left\| 3\|\alpha\|_* e_{i_0} - A_\alpha^* e_{i_0} \right\|_* \\ &< 3\|\alpha\|_*^2 \quad \text{(by (8.26))} \\ &= \langle v_0, v_0^* \rangle. \end{aligned}$$

Hence by Theorem 8.2.2 (vii), $F_{T_\alpha}\Box_1(\|\cdot\|\oplus\iota_{B_{X^*}})$ is not a BC–function.

(iv): Let $f=\|\cdot\|$ on X. Since $\operatorname{dom} f^{**}=X^{**}$, $\varnothing\neq\operatorname{dom} f^{**}\circ A_\alpha^*|_{X^*}\nsubseteqq\{e\}_\perp$. Then apply Theorem 8.2.2(v)&(viii) directly.

(v): Let $f=\frac{1}{2}\|\cdot\|^2$ on $X=c_0$. Then $f^*=\frac{1}{2}\|\cdot\|_1^2$ and $f^{**}=\frac{1}{2}\|\cdot\|_*^2$. By $\frac{1}{\sqrt{2}}<\|\alpha\|_*\leq 1$, take $|\alpha_{i_0}|^2>\frac{1}{2}$. Let e_{i_0} be defined as in the proof of (iii). Then set $v_1^*=\frac{1}{2}e_{i_0}$ and $v_1=\big(1+\frac{1}{2}\alpha_{i_0}^2\big)e_{i_0}$.

By (8.25), we have

$$v_1-A_\alpha^*v_1^*=(0,\ldots,0,1,-\alpha_{i_0}\alpha_{i_0+1},-\alpha_{i_0}\alpha_{i_0+2},\ldots)\tag{8.27}$$

Since $|\alpha_{i_0}\alpha_j|\leq\|\alpha\|_*^2\leq 1,\ \forall j\in\mathbb{N}$, then

$$\|v_1-A_\alpha^*v_1^*\|\leq 1.\tag{8.28}$$

We have

$$\begin{aligned}
&f^*(v_1^*)+f^{**}(v_1-A_\alpha^*v_1^*)=\tfrac{1}{2}\|v_1^*\|_1^2+\tfrac{1}{2}\|v_1-A_\alpha^*v_1^*\|_*^2\\
&\leq\tfrac{1}{8}+\tfrac{1}{2}\quad(\text{by (8.28)})\\
&<\tfrac{\alpha_{i_0}^2}{4}+\tfrac{1}{2}\quad(\text{by }\alpha_{i_0}^2>\frac{1}{2})\\
&=\langle v_1^*,v_1\rangle.
\end{aligned}$$

Hence by Theorem 8.2.2(vii), $F_{T_\alpha}\Box_1(\frac{1}{2}\|\cdot\|^2\oplus\frac{1}{2}\|\cdot\|_*^2)$ is not a BC–function.

(vi): Let $\lambda>0$ and $f=\frac{\lambda}{2}\|\cdot\|^2$ on $X=c_0$. Then $f^{**}=\frac{\lambda}{2}\|\cdot\|_*^2$. The rest of the proof is very similar to that of (iv).

(vii) : Since c_0 is separable by [58, Example 1.12.6] or [44, Proposition 1.26(ii)], by Fact 8.1.1, there exists a linear operator $L : c_0 \to C[0,1]$ that is an isometry from c_0 to a subspace of $C[0,1]$. Then combine (iv), (vi) and Corollary 8.1.7.

(viii): Apply (ii) . ■

Remark 8.3.2 *The maximal monotonicity of the operator T_e in Example 8.3.1(viii) was established by Voisei and Zălinescu in [87, Example 19] and then later a direct proof was given by Bueno and Svaiter in [32, Lemma 2.1]. Bueno and Svaiter also proved that T_e is not of type (D) in [32]. Here we give a short and direct proof of the above results. Example 8.3.1(iii)&(v) provide a negative answer to Simons' problem in [74, Problem 22.12].*

8.4 Discussion

The idea of the construction of the operator A in (Theorem 8.2.2) comes from [4, Theorem 5.1] by Bauschke and Borwein. The main tool involved in the main result (Theorem 8.2.2) is Simons and Zălinescu's version of Attouch-Brezis theorem.

Chapter 9

On Borwein-Wiersma decompositions of monotone linear relations

This chapter is mainly based on [18] by Bauschke, Wang and Yao, in which although we worked in a reflexive Banach space in [18], we can adapt most results from a reflexive space to a general Banach space.

It is well known that every square matrix can be decomposed into the sum of a symmetric matrix and an antisymmetric matrix, where the symmetric part is a gradient of a quadratic function. In this chapter, we provide the necessary and sufficient conditions for a maximally monotone linear relation to be Borwein-Wiersma decomposable, i.e., to be the sum of a subdifferential operator and a skew operator. We also show that Borwein-Wiersma decomposability implies Asplund decomposability.

9.1 Decompositions

Definition 9.1.1 (Borwein-Wiersma decomposition [27]) *The set-valued operator $A : X \rightrightarrows X^*$ is* Borwein-Wiersma decomposable *if*

$$A = \partial f + S, \tag{9.1}$$

where $f : X \to]-\infty, +\infty]$ is proper lower semicontinuous and convex, and where $S : X \rightrightarrows X^*$ is skew and at most single-valued. The right side of (9.1) *is a* Borwein-Wiersma decomposition *of A.*

Note that every single-valued linear monotone operator A with full domain is Borwein-Wiersma decomposable, with Borwein-Wiersma decomposition

$$A = A_+ + A_\circ = \nabla q_A + A_\circ. \tag{9.2}$$

Definition 9.1.2 (Asplund irreducibility [1]) *The set-valued operator $A\colon X \rightrightarrows X^*$ is* irreducible *(sometimes termed "acyclic" [27]) if whenever*

$$A = \partial f + S,$$

with $f : X \to]-\infty, +\infty]$ proper lower semicontinuous and convex, and $S : X \rightrightarrows X^*$ monotone, then necessarily $\operatorname{ran}(\partial f)|_{\operatorname{dom} A}$ is a singleton.

As we shall see in Section 9.1, the following decomposition is less restrictive.

Definition 9.1.3 (Asplund decomposition [1]) *The set-valued operator*

$A\colon X \rightrightarrows X^*$ is Asplund decomposable *if*

$$A = \partial f + S, \tag{9.3}$$

where $f\colon X \to\ ]-\infty, +\infty]$ is proper, lower semicontinuous, and convex, and where S is irreducible. The right side of (9.3) *is an* Asplund decomposition *of A.*

The following fact, due to Censor, Iusem and Zenios [36, 53], was previously known in $\mathbb{R}^n$. Here we give a different proof and extend the result to Banach spaces.

Fact 9.1.4 (Censor, Iusem and Zenios) *The subdifferential operator of a proper lower semicontinuous convex function $f\colon X \to\ ]-\infty, +\infty]$ is* paramonotone, *i.e., if*

$$x^* \in \partial f(x), \quad y^* \in \partial f(y), \tag{9.4}$$

and

$$\langle x^* - y^*, x - y\rangle = 0, \tag{9.5}$$

then $x^* \in \partial f(y)$ and $y^* \in \partial f(x)$.

Proof. By (9.5),

$$\langle x^*, x\rangle + \langle y^*, y\rangle = \langle x^*, y\rangle + \langle y^*, x\rangle. \tag{9.6}$$

By (9.4),

$$f^*(x^*) + f(x) = \langle x^*, x\rangle, \quad f^*(y^*) + f(y) = \langle y^*, y\rangle.$$

Adding them, followed by using (9.6), yields

$$f^*(x^*) + f(y) + f^*(y^*) + f(x) = \langle x^*, y\rangle + \langle y^*, x\rangle,$$

$$[f^*(x^*) + f(y) - \langle x^*, y\rangle] + [f^*(y^*) + f(x) - \langle y^*, x\rangle] = 0.$$

Since each bracketed term is nonnegative, we must have $f^*(x^*) + f(y) = \langle x^*, y\rangle$ and $f^*(y^*) + f(x) = \langle y^*, x\rangle$. It follows that $x^* \in \partial f(y)$ and that $y^* \in \partial f(x)$. ■

The following result provides a powerful criterion for determining whether a given operator is irreducible and hence Asplund decomposable.

Theorem 9.1.5 *Let $A : X \rightrightarrows X^*$ be monotone and at most single-valued. Suppose that there exists a dense subset D of* $\operatorname{dom} A$ *such that*

$$\langle Ax - Ay, x - y\rangle = 0 \quad \forall x, y \in D.$$

Then A is irreducible and hence Asplund decomposable.

Proof. Let $a \in D$ and $D' := D - \{a\}$. Define $A' : \operatorname{dom} A - \{a\} \to A(\cdot + a)$. Then A is irreducible if and only if A' is irreducible. Now we show A' is irreducible. By assumptions, $0 \in D'$ and

$$\langle A'x - A'y, x - y\rangle = 0 \quad \forall x, y \in D'.$$

Let $A' = \partial f + R$, where f is proper lower semicontinuous and convex, and R is monotone. Since A' is single-valued on $\operatorname{dom} A'$, we have that ∂f and R

are single-valued on $\operatorname{dom} A'$ and that

$$R = A' - \partial f \quad \text{on } \operatorname{dom} A'.$$

By taking $x_0^* \in \partial f(0)$, rewriting $A' = (\partial f - x_0^*) + (x_0^* + R)$, we can and do suppose $\partial f(0) = \{0\}$. For $x, y \in D'$ we have $\langle A'x - A'y, x - y\rangle = 0$. Then for $x, y \in D'$

$$\begin{aligned} 0 \le \langle R(x) - R(y), x - y\rangle &= \langle A'x - A'y, x - y\rangle - \langle \partial f(x) - \partial f(y), x - y\rangle \\ &= -\langle \partial f(x) - \partial f(y), x - y\rangle. \end{aligned}$$

On the other hand, ∂f is monotone, thus,

$$\langle \partial f(x) - \partial f(y), x - y\rangle = 0, \quad \forall x, y \in D'. \tag{9.7}$$

Using $\partial f(0) = \{0\}$,

$$\langle \partial f(x) - 0, x - 0\rangle = 0, \quad \forall x \in D'. \tag{9.8}$$

As ∂f is paramonotone by Fact 9.1.4, $\partial f(x) = \{0\}$ so that $x \in \operatorname{argmin} f$. This implies that $D' \subseteq \operatorname{argmin} f$ since $x \in D'$ was chosen arbitrarily. As f is lower semicontinuous, $\operatorname{argmin} f$ is closed. Using that D' is dense in $\operatorname{dom} A'$, it follows that $\operatorname{dom} A' \subseteq \overline{D'} \subseteq \operatorname{argmin} f$. Since ∂f is single-valued on $\operatorname{dom} A'$, $\partial f(x) = \{0\}$, $\forall x \in \operatorname{dom} A'$. Hence we have A' is irreducible, and so is A. ■

Remark 9.1.6 *In Theorem 9.1.5, the assumption that A be at most single-*

valued is important: indeed, let L be a proper subspace of $\mathbb{R}^n$. Then $\partial\iota_L$ is a linear relation and skew, yet $\partial\iota_L = \partial\iota_L + 0$ is not irreducible.

Theorem 9.1.5 and the definitions of the two decomposabilities now yield the following.

Corollary 9.1.7 *Let $A : X \rightrightarrows X^*$ be maximally monotone such that A is Borwein-Wiersma decomposable. Then A is Asplund decomposable.*

We proceed to give a few sufficient conditions for a maximally monotone linear relation to be Borwein-Wiersma decomposable. The following simple observation will be needed.

Lemma 9.1.8 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that A is Borwein-Wiersma decomposable, say $A = \partial f + S$, where $f \colon X \to \,]-\infty, +\infty]$ is proper, lower semicontinuous, and convex, and where $S \colon X \rightrightarrows X^*$ is at most single-valued and skew. Then the following hold.*

(i) $\partial f + \mathbb{I}_{\operatorname{dom} A} \colon x \mapsto \begin{cases} \partial f(x), & \text{if } x \in \operatorname{dom} A; \\ \varnothing, & \text{otherwise} \end{cases}$ *is a monotone linear relation.*

(ii) $\operatorname{dom} A \subseteq \operatorname{dom} \partial f \subseteq \operatorname{dom} f \subseteq (A0)_\perp$.

(iii) *If A is maximally monotone, then* $\operatorname{dom} A \subseteq \operatorname{dom} \partial f \subseteq \operatorname{dom} f \subseteq \overline{\operatorname{dom} A}$.

(iv) *If A is maximally monotone and* $\operatorname{dom} A$ *is closed, then* $\operatorname{dom} \partial f = \operatorname{dom} A = \operatorname{dom} f$.

Proof. (i): Indeed, on $\operatorname{dom} A$, we see that $\partial f = A - S$ is the difference of two linear relations.

(ii): Clearly $\operatorname{dom} A \subseteq \operatorname{dom} \partial f$. As $S0 = 0$, we have $A0 = \partial f(0)$. Thus, $\forall x^* \in A0$, $x \in X$,

$$\langle x^*, x\rangle \leq f(x) - f(0).$$

Then $\sigma_{A0}(x) \leq f(x) - f(0)$, where σ_{A0} is the support function of $A0$. If $x \notin (A0)_\perp$, then $\sigma_{A0}(x) = +\infty$ since $A0$ is a linear subspace, so $f(x) = +\infty$, $\forall x \notin (A0)_\perp$. Therefore, $\operatorname{dom} f \subseteq (A0)_\perp$. Altogether, (ii) holds.

(iii): Combine (ii) with Proposition 3.2.2(i). (iv): Apply (iii). ■

Theorem 9.1.9 *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation such that* $\operatorname{dom} A \subseteq \operatorname{dom} A^*$. *Then A is Borwein-Wiersma decomposable via*

$$A = \partial\overline{q_A} + S,$$

where S is an arbitrary linear single-valued selection of $A_\circ$. Moreover, $\partial\overline{q_A} = A_+$ *on* $\operatorname{dom} A$.

Proof. From Proposition 3.2.10(i), A_+ is monotone and $q_{A_+} = q_A$, using Proposition 3.2.10(ii), $\operatorname{gra} A_+ \subseteq \operatorname{gra} \partial\overline{q_{A_+}} = \operatorname{gra} \partial\overline{q_A}$. Let $S : \operatorname{dom} A \to X^*$ be a linear selection of $A_\circ$ (the existence of which is guaranteed by a standard Zorn's lemma argument). Then, S is skew. Thus, by Proposition 3.2.2(v), we have $\operatorname{gra} A = \operatorname{gra}(A_+ + S) \subseteq \operatorname{gra}(\partial\overline{q_A} + S)$. Since A is maximally monotone, $A = \partial\overline{q_A} + S$, which is the announced Borwein-Wiersma decomposition. Moreover, $\partial\overline{q_A} = A - S = A_+$ on $\operatorname{dom} A$. ■

Corollary 9.1.10 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation such that A is symmetric. Then A is Borwein-Wiersma decomposable, with decompositions $A = \partial\overline{q_A} + 0$. If X is reflexive, then A^{-1} is Borwein-Wiersma decomposable with $A^{-1} = \partial q_A^* + 0$.*

Proof. Using Proposition 3.2.11, we obtain $A = A^*|_X$. Hence, Theorem 9.1.9 applies; in fact, $A = \partial\overline{q_A}$. If X is reflexive, then we have $A^{-1} = \partial\overline{q_A}^* = \partial q_A^*$ by [92, Theorem 2.4.4(iv) and Theorem 2.3.1(iv)]. From Proposition 3.1.3(iv), we have $A^{-1} = (A^*)^{-1} = (A^{-1})^*$. Then $A^{-1} = \partial\overline{q_{A^{-1}}}$. Hence $A^{-1} = \partial\overline{q_{A^{-1}}} = \partial q_A^*$. ■

Corollary 9.1.11 *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation such that $\operatorname{dom} A$ is closed, and let S be a single-valued linear selection of $A_\circ$. Then $q_A = \overline{q_A}$, $A_+ = \partial q_A$ is maximally monotone, and A and $A^*|_X$ are Borwein-Wiersma decomposable, with decompositions $A = A_+ + S$ and $A^*|_X = A_+ - S$, respectively.*

Proof. Proposition 3.2.2(iv) implies that $\operatorname{dom} A^*|_X = \operatorname{dom} A$. By Proposition 3.2.10(iv), $A^*|_X$ is maximally monotone. In view of Proposition 3.2.2(v), $A = A_+ + A_\circ$ and $A^*|_X = A_+ - A_\circ$. Theorem 9.1.9 yields the Borwein-Wiersma decomposition $A = \partial\overline{q_A} + S$. Hence $\operatorname{dom} A \subseteq \operatorname{dom} \partial\overline{q_A} \subseteq \operatorname{dom} \overline{q_A} \subseteq \overline{\operatorname{dom} A} = \operatorname{dom} A$. In turn, since $\operatorname{dom} A = \operatorname{dom} A_+$ and $q_A = q_{A_+}$, this implies that $\operatorname{dom} A_+ = \operatorname{dom} \partial\overline{q_{A_+}} = \operatorname{dom} \overline{q_{A_+}}$. In view of Proposition 3.2.10(i)&(ii), $q_{A_+} = \overline{q_{A_+}}$ and $\operatorname{gra} A_+ \subseteq \operatorname{gra} \partial\overline{q_{A_+}}$. By Theorem 9.1.9, $A_+ = \partial\overline{q_A}$ on $\operatorname{dom} A$. Since $\operatorname{dom} A = \operatorname{dom} A_+ = \operatorname{dom} \partial\overline{q_A}$ and $q_A = q_{A_+} = \overline{q_{A_+}} = \overline{q_A}$, this implies that $A_+ = \partial q_A = \partial\overline{q_A}$ *everywhere*. Therefore, A_+ is maximally monotone. Then we obtain the Borwein-Wiersma decomposition $A^*|_X = A_+ - S$. ■

Theorem 9.1.12 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation such that A is skew, and let S be a single-valued linear selection of A. Then A is Borwein-Wiersma decomposable via $\partial\iota_{\overline{\operatorname{dom} A}} + S$.*

Proof. Clearly, S is skew. Proposition 3.1.3(ii) and Proposition 3.2.2(iii) imply that $A = A0 + S = (\operatorname{dom} A)^{\perp} + S = \partial\iota_{\overline{\operatorname{dom} A}} + S$, as announced. Alternatively, by [80, Lemma 2.2], $\operatorname{dom} A \subseteq \operatorname{dom} A^*$ and now apply Theorem 9.1.9. ■

Under a mild constraint qualification, the sum of two Borwein-Wiersma decomposable operators is also Borwein-Wiersma decomposable and the decomposition of the sum is the corresponding sum of the decompositions.

Proposition 9.1.13 (sum rule) *Let A_1 and A_2 be maximally monotone linear relations from X to X^*. Suppose that A_1 and A_2 are Borwein-Wiersma decomposable via $A_1 = \partial f_1 + S_1$ and $A_2 = \partial f_2 + S_2$, respectively. Suppose that $\operatorname{dom} A_1 - \operatorname{dom} A_2$ is closed. Then $A_1 + A_2$ is Borwein-Wiersma decomposable via $A_1 + A_2 = \partial(f_1 + f_2) + (S_1 + S_2)$.*

Proof. By Lemma 9.1.8(iii), $\operatorname{dom} A_1 \subseteq \operatorname{dom} f_1 \subseteq \overline{\operatorname{dom} A_1}$ and $\operatorname{dom} A_2 \subseteq \operatorname{dom} f_2 \subseteq \overline{\operatorname{dom} A_2}$. Hence $\operatorname{dom} A_1 - \operatorname{dom} A_2 \subseteq \operatorname{dom} f_1 - \operatorname{dom} f_2 \subseteq \overline{\operatorname{dom} A_1} - \overline{\operatorname{dom} A_2} \subseteq \overline{\operatorname{dom} A_1 - \operatorname{dom} A_2} = \operatorname{dom} A_1 - \operatorname{dom} A_2$. Thus, $\operatorname{dom} f_1 - \operatorname{dom} f_2 = \operatorname{dom} A_1 - \operatorname{dom} A_2$ is a closed subspace of X. By [74, Theorem 18.2], $\partial f_1 + \partial f_2 = \partial(f_1 + f_2)$; furthermore, $S_1 + S_2$ is clearly skew. The result thus follows. ■

9.2 Uniqueness results

The main result in this section (Theorem 9.2.8) states that if a maximally monotone linear relation A is Borwein-Wiersma decomposable, then the subdifferential part of its decomposition is unique on $\operatorname{dom} A$. We start by showing that subdifferential operators that are monotone linear relations are actually symmetric, which is a variant of a well known result from Calculus.

Lemma 9.2.1 *Let $f : X \to \,]-\infty, +\infty]$ be proper, lower semicontinuous, and convex. Suppose that the maximally monotone operator ∂f is a linear relation with closed domain. Then $\partial f = (\partial f)^*$.*

Proof. Set $A = \partial f$ and $Y = \operatorname{dom} f$. Since $\operatorname{dom} A$ is closed, the Brøndsted-Rockafellar Theorem (see [74, Theorem 18.6]) implies that $\operatorname{dom} f = Y = \operatorname{dom} A$. By Proposition 3.2.2(iv), $\operatorname{dom} A^*|_X = \operatorname{dom} A$. Let $x \in Y$ and consider the directional derivative $g = f'(x; \cdot)$, i.e.,

$$g\colon X \to [-\infty, +\infty] : y \mapsto \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}.$$

By [92, Theorem 2.1.14], $\operatorname{dom} g = \bigcup_{r \geq 0} r \cdot (\operatorname{dom} f - x) = Y$. On the other hand, f is lower semicontinuous on X. Thus, since $Y = \operatorname{dom} f$ is a Banach space, $f|_Y$ is continuous by [92, Theorem 2.2.20(b)]. Altogether, in view of [92, Theorem 2.4.9], $g|_Y$ is continuous. Hence g is lower semicontinuous. Using [92, Corollary 2.4.15] and Fact 3.1.3(v), we now deduce that $(\forall y \in Y)$ $g(y) = \sup\langle \partial f(x), y\rangle = \langle Ax, y\rangle = \langle x, A^* y\rangle$. We thus have proved that

$$(\forall x \in Y)(\forall y \in Y) \quad f'(x; y) = \langle Ax, y\rangle = \langle x, A^* y\rangle. \tag{9.9}$$

In particular, $f|_Y$ is *differentiable*. Now fix x, y, z in Y. Then, using (9.9), we see that

$$\begin{aligned}\langle Az, y\rangle &= \lim_{s\downarrow 0} \frac{\langle A(x+sz), y\rangle - \langle Ax, y\rangle}{s} = \lim_{s\downarrow 0} \frac{f'(x+sz; y) - f'(x; y)}{s} \\ &= \lim_{s\downarrow 0}\lim_{t\downarrow 0}\Big(\frac{f(x+sz+ty) - f(x+sz)}{st} - \frac{f(x+ty) - f(x)}{st}\Big).\end{aligned} \tag{9.10}$$

Set $h\colon \mathbb{R} \to \mathbb{R}\colon s \mapsto f(x+sz+ty) - f(x+sz)$. Since $f|_Y$ is differentiable, so is h. For $s > 0$, the Mean Value Theorem thus yields $r_{s,t} \in \,]0, s[$ such that

$$\begin{aligned}&\frac{f(x+sz+ty) - f(x+sz)}{s} - \frac{f(x+ty) - f(x)}{s} \\ &= \frac{h(s)}{s} - \frac{h(0)}{s} = h'(r_{s,t}) \\ &= f'(x + r_{s,t}z + ty; z) - f'(x + r_{s,t}z; z) \\ &= t\langle Ay, z\rangle.\end{aligned} \tag{9.11}$$

Combining (9.10) with (9.11), we deduce that $\langle Az, y\rangle = \langle Ay, z\rangle$. Thus, A is symmetric. The result now follows from Proposition 3.2.11. ■

To improve Lemma 9.2.1, we need the following "shrink and dilate" technique.

Lemma 9.2.2 *Let $A : X \rightrightarrows X^*$ be a monotone linear relation, and let Z be a closed subspace of* $\operatorname{dom} A$*. Set $B = (A + \mathbb{I}_Z) + Z^{\perp}$. Then B is maximally monotone and* $\operatorname{dom} B = Z$.

Proof. Since $Z \subseteq \operatorname{dom} A$ and $B = A + \partial\iota_Z$ it is clear that B is a monotone linear relation with $\operatorname{dom} B = Z$. By Proposition 3.2.2 (i), we have

$$Z^{\perp} \subseteq B0 = A0 + Z^{\perp} \subseteq (\operatorname{dom} A)^{\perp} + Z^{\perp} \subseteq Z^{\perp} + Z^{\perp} = Z^{\perp}.$$

Hence $B0 = Z^{\perp} = (\operatorname{dom} B)^{\perp}$. Therefore, by Proposition 3.2.2(ii), B is maximally monotone. ■

Theorem 9.2.3 *Let $f : X \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous, and convex, and let Y be a linear subspace of X. Suppose that $\partial f + \mathbb{I}_Y$ is a linear relation. Then $\partial f + \mathbb{I}_Y$ is symmetric.*

Proof. Put $A = \partial f + \mathbb{I}_Y$. Assume that $(x, x^*), (y, y^*) \in \operatorname{gra} A$. Set $Z = \operatorname{span}\{x, y\}$. Let $B : X \rightrightarrows X^*$ be defined as in Lemma 9.2.2. Clearly, $\operatorname{gra} B \subseteq \operatorname{gra} \partial(f + \iota_Z)$. In view of the maximal monotonicity of B, we see that $B = \partial(f + \iota_Z)$. Since $\operatorname{dom} B = Z$ is closed, it follows from Lemma 9.2.1 that $B = B^*$. In particular, we obtain that $\langle x^*, y\rangle = \langle y^*, x\rangle$. Hence, $\langle \partial f(x), y\rangle = \langle \partial f(y), x\rangle$ and therefore $\partial f + \mathbb{I}_Y$ is symmetric. ■

Lemma 9.2.4 *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation such that A is Borwein-Wiersma decomposable. Then* $\operatorname{dom} A \subseteq \operatorname{dom} A^*$.

Proof. By hypothesis, there exists a proper lower semicontinuous and convex function $f\colon X \to \left]-\infty, +\infty\right]$ and an at most single-valued skew operator S such that $A = \partial f + S$. Hence $\operatorname{dom} A \subseteq \operatorname{dom} S$, and Theorem 9.2.3 implies that $(A - S) + \mathbb{I}_{\operatorname{dom} A}$ is symmetric. Let x and y be in $\operatorname{dom} A$. Then

$$\langle Ax - 2Sx, y\rangle = \langle Ax - Sx, y\rangle - \langle Sx, y\rangle = \langle Ay - Sy, x\rangle - \langle Sx, y\rangle$$

$$= \langle Ay, x\rangle - \langle Sy, x\rangle - \langle Sx, y\rangle = \langle Ay, x\rangle,$$

which implies that $(A - 2S)x \subseteq A^*x$. Therefore, $\operatorname{dom} A = \operatorname{dom}(A - 2S) \subseteq \operatorname{dom} A^*$. ■

Remark 9.2.5 *We can now derive part of the conclusion of Proposition 9.1.13 differently as follows. Since* $\operatorname{dom} A_1 - \operatorname{dom} A_2$ *is closed, Voisei proved in [83] (see Theorem 7.2.2 or [74, Theorem 46.3]) that* $A_1 + A_2$ *is maximally monotone; moreover, Fact 7.1.6 yields* $(A_1 + A_2)^* = A_1^* + A_2^*$. *Using Lemma 9.2.4, we thus obtain* $\operatorname{dom}(A_1 + A_2) = \operatorname{dom} A_1 \cap \operatorname{dom} A_2 \subseteq \operatorname{dom} A_1^* \cap \operatorname{dom} A_2^* = \operatorname{dom}(A_1^* + A_2^*) = \operatorname{dom}(A_1 + A_2)^*$. *Therefore,* $A_1 + A_2$ *is Borwein-Wiersma decomposable by Theorem 9.1.9.*

Theorem 9.2.6 (characterization of subdifferential operators) *Let* $A : X \rightrightarrows X^*$ *be a monotone linear relation. Then* A *is maximally monotone and symmetric* $\Leftrightarrow$ *there exists a proper lower semicontinuous convex function* $f : X \to\ ]-\infty, +\infty]$ *such that* $A = \partial f$.

Proof. "$\Rightarrow$": Proposition 3.2.10(ii). "$\Leftarrow$": Apply Theorem 9.2.3 with $Y = X$. ■

Remark 9.2.7 *Theorem 9.2.6 generalizes [63, Theorem 5.1] of Phelps and Simons.*

Theorem 9.2.8 (uniqueness of the subdifferential part) *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation such that* A *is Borwein-Wiersma decomposable. Then on* $\operatorname{dom} A$, *the subdifferential part in the de-*

composition is unique and equals to A_+, and the skew part must be a linear selection of $A_\circ$.

Proof. Let f_1 and f_2 be proper lower semicontinuous convex functions from X to $]-\infty, +\infty]$, and let S_1 and S_2 be at most single-valued skew operators from X to X^* such that

$$A = \partial f_1 + S_1 = \partial f_2 + S_2. \tag{9.12}$$

Set $D = \operatorname{dom} A$. Since S_1 and S_2 are single-valued on D, we have $A - S_1 = \partial f_1$ and $A - S_2 = \partial f_2$ on D. Hence $\partial f_1 + \mathbb{I}_D$ and $\partial f_2 + \mathbb{I}_D$ are monotone linear relations with

$$(\partial f_1 + \mathbb{I}_D)(0) = (\partial f_2 + \mathbb{I}_D)(0) = A0. \tag{9.13}$$

By Theorem 9.2.3, $\partial f_1 + \mathbb{I}_D$ and $\partial f_2 + \mathbb{I}_D$ are symmetric, i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \langle \partial f_1(x), y\rangle = \langle \partial f_1(y), x\rangle \quad \text{and} \quad \langle \partial f_2(x), y\rangle = \langle \partial f_2(y), x\rangle.$$

Thus,

$$(\forall x \in D)(\forall y \in D) \quad \langle \partial f_2(x) - \partial f_1(x), y\rangle = \langle \partial f_2(y) - \partial f_1(y), x\rangle. \tag{9.14}$$

On the other hand, by (9.12), $(\forall x \in D)$ $S_1x - S_2x \in \partial f_2(x) - \partial f_1(x)$. Then by Fact 3.2.2(iii), Proposition 3.2.1(ii) and Proposition 3.1.3(v),

$$(\forall x \in D)(\forall y \in D) \quad \langle \partial f_2(x) - \partial f_1(x), y\rangle = \langle S_1x - S_2x, y\rangle \tag{9.15}$$

$$= -\langle S_1 y - S_2 y, x\rangle$$
$$= -\langle \partial f_2(y) - \partial f_1(y), x\rangle.$$

Now fix $x \in D$. Combining (9.14) and (9.15), we get $(\forall y \in D)$ $\langle \partial f_2(x) - \partial f_1(x), y\rangle = 0$. Using Fact 3.2.2(iii), we see that

$$\partial f_2(x) - \partial f_1(x) \subseteq D^{\perp} = (\operatorname{dom} A)^{\perp} = A0.$$

Hence, in view of Lemma 9.1.8(i), (9.13), and Fact 3.1.3(ii),

$$\partial f_1 + \mathbb{I}_D = \partial f_2 + \mathbb{I}_D.$$

By Lemma 9.2.4 and Theorem 9.1.9, we consider the case when $f_2 = \overline{q_A}$ so that $\partial f_2 = A_+$ on D. Hence $\partial f_1 = A_+$ on D and, if $x \in D$, then $S_1 x \in Ax - \partial f_1(x) = Ax - A_+ x = A_\circ x$ by Proposition 3.2.2(v). ■

Remark 9.2.9 *In a Borwein-Wiersma decomposition, the skew part need not be unique: indeed, assume that $X = \mathbb{R}^2$, set $Y := \mathbb{R} \times \{0\}$, and let S be given by $\operatorname{gra} S = \big\{\big((x,0),(0,x)\big) \mid x \in \mathbb{R}\big\}$. Then S is skew and the maximally monotone linear relation $\partial\iota_Y$ has two distinct Borwein-Wiersma decompositions, namely $\partial\iota_Y + 0$ and $\partial\iota_Y + S$.*

Proposition 9.2.10 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Suppose that A is Borwein-Wiersma decomposable, with subdifferential part ∂f, where $f\colon X \to \left]-\infty,+\infty\right]$ is proper, lower semicontinuous and convex. Then there exists a constant $\alpha \in \mathbb{R}$ such that the following hold.*

(i) $f = \overline{q_A} + \alpha$ *on* $\operatorname{dom} A$.

(ii) *If* $\operatorname{dom} A$ *is closed, then* $f = \overline{q_A} + \alpha = q_A + \alpha$ *on* X.

Proof. Let S be a linear single-valued selection of $A_\circ$. By Lemma 9.2.4, $\operatorname{dom} A \subseteq \operatorname{dom} A^*$. In turn, Theorem 9.1.9 yields

$$A = \partial \overline{q_A} + S.$$

Let $\{x, y\} \subset \operatorname{dom} A$. By Theorem 9.2.8, $\partial f + \mathbb{I}_{\operatorname{dom} A} = \partial \overline{q_A} + \mathbb{I}_{\operatorname{dom} A}$. Now set $Z = \operatorname{span}\{x, y\}$, apply Lemma 9.2.2 to the monotone linear relation $\partial f + \mathbb{I}_{\operatorname{dom} A} = \partial \overline{q_A} + \mathbb{I}_{\operatorname{dom} A}$, and let B be as in Lemma 9.2.2. Note that $\operatorname{gra} B = \operatorname{gra}(\partial \overline{q_A} + \partial \iota_Z) \subseteq \operatorname{gra} \partial(\overline{q_A} + \iota_Z)$ and that $\operatorname{gra} B = \operatorname{gra}(\partial f + \partial \iota_Z) \subseteq \operatorname{gra} \partial(f + \iota_Z)$. By the maximal monotonicity of B, we conclude that $B = \partial(\overline{q_A} + \iota_Z) = \partial(f + \iota_Z)$. By [67, Theorem B], there exists $\alpha \in \mathbb{R}$ such that $f + \iota_Z = \overline{q_A} + \iota_Z + \alpha$. Hence $\alpha = f(x) - \overline{q_A}(x) = f(y) - \overline{q_A}(y)$ and repeating this argument with $y \in (\operatorname{dom} A) \smallsetminus \{x\}$, we see that

$$f = \overline{q_A} + \alpha \quad \text{on } \operatorname{dom} A \tag{9.16}$$

and (i) is thus established. Now assume in addition that $\operatorname{dom} A$ is closed. Applying Lemma 9.1.8(iv) with both ∂f and $\partial \overline{q_A}$, we obtain

$$\operatorname{dom} \overline{q_A} = \operatorname{dom} \partial \overline{q_A} = \operatorname{dom} A = \operatorname{dom} \partial f = \operatorname{dom} f.$$

Consequently, (9.16) now yields $f = \overline{q_A} + \alpha$. Finally, Corollary 9.1.11 implies that $q_A = \overline{q_A}$. ■

9.3 Characterizations and examples

The following characterization of the Borwein-Wiersma decomposability of a maximally monotone linear relation is quite pleasing.

Theorem 9.3.1 (Borwein-Wiersma decomposability) *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then the following are equivalent.*

(i) *A is Borwein-Wiersma decomposable.*

(ii) $\operatorname{dom} A \subseteq \operatorname{dom} A^*$.

(iii) $A = A_+ + A_\circ$.

Proof. "(i)⇒(ii)": Lemma 9.2.4. "(i)⇐(ii)": Theorem 9.1.9. "(ii)⇒(iii)": Proposition 3.2.2(v). "(ii)⇐(iii)": This is clear. ■

Corollary 9.3.2 *Assume X is reflexive. Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Then both A and A^* are Borwein-Wiersma decomposable if and only if* $\operatorname{dom} A = \operatorname{dom} A^*$.

Proof. Combine Theorem 9.3.1, Fact 3.2.13, and Fact 3.1.3(vi). ■

We shall now provide two examples of a linear relation S in the Hilbert space to illustrate that the following do occur:

- S is Borwein-Wiersma decomposable, but S^* is not.
- Neither S nor S^* is Borwein-Wiersma decomposable.
- S is not Borwein-Wiersma decomposable, but S^{-1} is.

Example 9.3.3 *Suppose that X is the Hilbert space $\ell^2(\mathbb{N})$, and set*

$$S\colon \operatorname{dom} S \to X\colon y \mapsto \Big(\tfrac{1}{2}y_n + \sum_{i<n} y_i\Big)_{n\in\mathbb{N}}, \tag{9.17}$$

with

$$\operatorname{dom} S = \Big\{ y = (y_n)_{n\in\mathbb{N}} \in X \;\Big|\; \sum_{i\geq 1} y_i = 0, \Big(\sum_{i\leq n} y_i\Big)_{n\in\mathbb{N}} \in X \Big\}.$$

Then

$$S^*\colon \operatorname{dom} S^* \to X\colon y \mapsto \Big(\tfrac{1}{2}y_n + \sum_{i>n} y_i\Big)_{n\in\mathbb{N}} \tag{9.18}$$

where

$$\operatorname{dom} S^* = \Big\{ y = (y_n)_{n\in\mathbb{N}} \in X \;\Big|\; \Big(\sum_{i>n} y_i\Big)_{n\in\mathbb{N}} \in X \Big\}.$$

Then S can be identified with an at most single-valued linear relation such that the following hold. (See [63, Theorem 2.5] and Proposition 3.3.2, Proposition 3.3.3, Proposition 3.3.5, and Theorem 3.3.8.)

(i) *S is maximally monotone and skew.*

(ii) *S^* is maximally monotone but not skew.*

(iii) *$\operatorname{dom} S$ is dense in $\ell^2(\mathbb{N})$, and $\operatorname{dom} S \subsetneqq \operatorname{dom} S^*$.*

(iv) *$S^* = -S$ on $\operatorname{dom} S$.*

In view of Theorem 9.3.1, S is Borwein-Wiersma decomposable while S^* is not. However, both S and S^* are irreducible and Asplund decomposable by

Theorem 9.1.5. Because S^* is irreducible but not skew, we see that the class of irreducible operators is strictly larger than the class of skew operators.

Example 9.3.4 (Inverse Volterra operator) *(See Example 3.4.4 and Theorem 3.4.5.) Suppose that X is the Hilbert space $L^2[0,1]$, and consider the* Volterra integration operator *(see, e.g., [52, Problem 148]), which is defined by*

$$V\colon X \to X\colon x \mapsto Vx, \quad \textit{where} \quad Vx\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_0^t x, \tag{9.19}$$

and set $A = V^{-1}$. Then

$$V^*\colon X \to X\colon x \mapsto V^*x, \quad \textit{where} \quad V^*x\colon [0,1] \to \mathbb{R}\colon t \mapsto \int_t^1 x,$$

and the following hold.

(i) *We have*

$$\begin{aligned}\operatorname{dom} A = \big\{x \in X \;\big|\; &x \textit{ is absolutely continuous, } x(0) = 0,\\ &\textit{and } x' \in X\big\}\end{aligned}$$

and

$$A\colon \operatorname{dom} A \to X\colon x \mapsto x'.$$

(ii) *We have*

$$\operatorname{dom} A^* = \big\{x \in X \;\big|\; x \textit{ is absolutely continuous, } x(1) = 0,$$

$$\text{and } x' \in X\Big\}$$

and

$$A^*\colon\ \operatorname{dom} A^* \to X : x \mapsto -x'.$$

(iii) *Both A and A^* are maximally monotone linear operators.*

(iv) *Neither A nor A^* is symmetric.*

(v) *Neither A nor A^* is skew.*

(vi) $\operatorname{dom} A \not\subseteq \operatorname{dom} A^*$, *and* $\operatorname{dom} A^* \not\subseteq \operatorname{dom} A$.

(vii) $Y = \operatorname{dom} A \cap \operatorname{dom} A^*$ *is dense in* X.

(viii) *Both $A + \mathbb{I}_Y$ and $A^* + \mathbb{I}_Y$ are skew.*

By Theorem 9.1.5, both A and A^* are irreducible and Asplund decomposable. On the other hand, by Theorem 9.3.1, neither A nor A^* is Borwein-Wiersma decomposable. Finally, $A^{-1} = V$ and $(A^*)^{-1} = V^*$ are Borwein-Wiersma decomposable since they are continuous linear operators with full domain.

Remark 9.3.5 (an answer to Borwein and Wiersma's question) *The operators S, S^*, A, and A^* defined in this section are all irreducible and Asplund decomposable, but none of them has full domain. This provides an answer to [27, Question (4) in Section 7]:*

> Can one exhibit an irreducible operator whose domain is not the whole space?

9.4 When X is a Hilbert space

Throughout this short section, we suppose that X is a Hilbert space. Recall (see, e.g., [42, Chapter 5] for basic properties) that if C is a nonempty closed convex subset of X, then the *(nearest point) projector* P_C is well defined and continuous. If Y is a closed subspace of X, then P_Y is linear and $P_Y = P_Y^*$.

Definition 9.4.1 *Let $A\colon X \rightrightarrows X$ be a maximally monotone linear relation. We define Q_A by*

$$Q_A : \operatorname{dom} A \to X : x \mapsto P_{Ax}x.$$

Note that Q_A is monotone and a single-valued selection of A because $(\forall x \in \operatorname{dom} A)$ Ax is a nonempty closed convex subset of X.

Proposition 9.4.2 (linear selection) *Let $A\colon X \rightrightarrows X$ be a maximally monotone linear relation. Then the following hold.*

(i) $(\forall x \in \operatorname{dom} A)$ $Q_A x = P_{(A0)^\perp}(Ax)$, *and* $Q_A x \in Ax$.

(ii) Q_A *is monotone and linear.*

(iii) $A = Q_A + A0$.

Proof. Let $x \in \operatorname{dom} A = \operatorname{dom} Q_A$ and let $x^* \in Ax$. Using Proposition 3.1.3(ii), we see that

$$\begin{aligned} Q_A x = P_{Ax}x = P_{x^*+A0}x = x^* + P_{A0}(x - x^*) = x^* + P_{A0}x - P_{A0}x^* \\ = P_{A0}x + P_{(A0)^\perp}x^* = P_{(A0)^\perp}x^*. \end{aligned}$$

Since $x^* \in Ax$ is arbitrary, we have thus established (i). Now let x and y be in $\operatorname{dom} A$, and let α and β be in $\mathbb{R}$. If $\alpha = \beta = 0$, then, by Proposition 3.1.3(i), we have $Q_A(\alpha x + \beta y) = Q_A 0 = P_{A0}0 = 0 = \alpha Q_A x + \beta Q_A y$. Now assume that $\alpha \neq 0$ or $\beta \neq 0$. By (i) and Proposition 3.1.3(iii), we have

$$\begin{aligned} Q_A(\alpha x + \beta y) &= P_{(A0)^\perp} A(\alpha x + \beta y) = \alpha P_{(A0)^\perp}(Ax) + \beta P_{(A0)^\perp}(Ay) \\ &= \alpha Q_A x + \beta Q_A y. \end{aligned}$$

Hence Q_A is a linear selection of A and (ii) holds. Finally, (iii) follows from Proposition 3.1.3(ii). ■

Example 9.4.3 *Let $A\colon X \rightrightarrows X$ be maximally monotone and skew. Then $A = \partial\iota_{\overline{\operatorname{dom} A}} + Q_A$ is a Borwein-Wiersma decomposition.*

Proof. By Proposition 9.4.2(ii), Q_A is a linear selection of A. Now apply Theorem 9.1.12. ■

Example 9.4.4 *Let $A\colon X \rightrightarrows X$ be a maximally monotone linear relation such that $\operatorname{dom} A$ is closed. Set $B = P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A}$ and $f = q_B + \iota_{\operatorname{dom} A}$. Then the following hold.*

(i) *$B\colon X \to X$ is continuous, linear, and maximally monotone.*

(ii) *$f\colon X \to \left]-\infty, +\infty\right]$ is convex, lower semicontinuous, and proper.*

(iii) *$A = \partial\iota_{\operatorname{dom} A} + B$.*

(iv) *$\partial f + B_\circ$ is a Borwein-Wiersma decomposition of A.*

Proof. (i): By Proposition 9.4.2(ii), Q_A is monotone and a linear selection of A. Hence, $B\colon X \to X$ is linear; moreover, $(\forall x \in X)$ $\langle x, Bx\rangle = \langle x,\ P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A} x\rangle = \langle P_{\operatorname{dom} A} x,\ Q_A P_{\operatorname{dom} A} x\rangle \geq 0$. Altogether, $B\colon X \to X$ is linear and monotone. By Corollary 3.2.3, B is continuous and maximally monotone.

(ii): By (i), q_B is thus convex and continuous; in turn, f is convex, lower semicontinuous, and proper.

(iii): Using Proposition 9.4.2(i) and Proposition 3.2.2(iii), we have $(\forall x \in X)$ $(Q_A P_{\operatorname{dom} A})x \in (A0)^{\perp} = \overline{\operatorname{dom} A} = \operatorname{dom} A$. Hence, $(\forall x \in \operatorname{dom} A)$ $Bx = (P_{\operatorname{dom} A} Q_A P_{\operatorname{dom} A})x = Q_A x \in Ax$. Thus, $B + \mathbb{I}_{\operatorname{dom} A} = Q_A$. In view of Proposition 9.4.2(iii) and Proposition 3.2.2(iii), we now obtain $A = B + \mathbb{I}_{\operatorname{dom} A} + A0 = B + \partial\iota_{\operatorname{dom} A}$.

(iv): It follows from (iii) and (9.2) that $A = B + \partial\iota_{\operatorname{dom} A} = \nabla q_B + \partial\iota_{\operatorname{dom} A} + B_{\circ} = \partial(q_B + \iota_{\operatorname{dom} A}) + B_{\circ} = \partial f + B_{\circ}$. ■

Proposition 9.4.5 *Let $A : X \rightrightarrows X$ be such that* $\operatorname{dom} A$ *is a closed subspace of X. Then A is a maximally monotone linear relation $\Leftrightarrow$ $A = \partial\iota_{\operatorname{dom} A} + B$, where $B : X \to X$ is linear and monotone.*

Proof. "$\Rightarrow$": This is clear from Example 9.4.4(i)&(iii). "$\Leftarrow$": Clearly, A is a linear relation. By Corollary 3.2.3, B is continuous and maximally monotone. Using Rockafellar's sum theorem [66] or Theorem 5.3.1, we conclude that $\partial\iota_{\operatorname{dom} A} + B$ is maximally monotone. ■

9.5 Discussion

The original papers by Asplund [1] and by Borwein and Wiersma [27] concerned the additive decomposition of a maximally monotone operator whose domain has *nonempty interior*. In this chapter, we focused on maximally monotone *linear relations* and we specifically allowed for domains with empty interior. All maximally monotone linear relations on finite-dimensional spaces are Borwein-Wiersma decomposable; however, this fails in infinite-dimensional settings. We presented characterizations of Borwein-Wiersma decomposability of maximally monotone linear relations in general Banach spaces and provided a more explicit decomposition in Hilbert spaces.

The characterization of Asplund decomposability and the corresponding construction of an Asplund decomposition remain interesting unresolved topics for future explorations, even for maximally monotone linear operators whose domains are proper dense subspaces of infinite-dimensional Hilbert spaces.

Chapter 10

Conclusion

Let us conclude by listing our findings of all relevant chapters.

Chapter 3: The Brezis-Browder Theorem (see Fact 3.2.13) is a very important characterization of maximal monotonicities of monotone relations. The original proof [30] is based on the application of Zorn's Lemma by constructing a series of finite-dimensional subspaces, which is complicated. In Theorem 3.2.15, we establish the Brezis-Browder Theorem by considering the fact that a lower semicontinuous, convex and coercive function on a reflexive space has at least one minimizer. In [75], Simons generalized the Brezis-Browder Theorem to SSDB spaces. The Brezis-Browder Theorem and Corollary 3.2.6 are essential tools for the construction of maximally monotone linear subspace extensions of a monotone linear relation.

There will be an interesting question for the future work on the Brezis-Browder Theorem in a general Banach space:

> *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A$ *is closed. Assume $A^*|_X$ is monotone.*
> *Is A necessarily maximally monotone?*

In Sections 3.3 and 3.4, some explicit monotone linear relations were constructed in Hilbert spaces, which gave a negative answer to a question raised by Svaiter [80] and which showed that the constraint qualification in the sum problem for maximally monotone operators cannot be weakened (see [63, Example 7.4]). In particular, these two sections will provide concrete examples for the characterization of decomposable monotone linear relations.

Chapter 4: A direction for future work in this chapter is to write computer code to find the maximally monotone subspace extension of G, and to generalize the results into a Hilbert space by applying the Brezis-Browder Theorem.

Chapter 5: As we can see, Fact 5.1.7 plays an important role in the proof of Theorem 5.2.4 and Theorem 5.3.1. Theorem 5.2.4 presents a powerful sufficient condition for the sum problem. The following question posed by Simons in [72, Problem 41.4] remains open:

> *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV), let C be a nonempty closed convex subset of X, and suppose that $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$.*
>
> *Is $A + N_C$ necessarily maximally monotone?*

If the above result holds, by Theorem 5.2.4, we can get the following result:

Let $A : X \rightrightarrows X^*$ be maximally monotone of type (FPV), and let $B : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. Assume that $\operatorname{dom} A \cap \overline{\operatorname{dom} B} \subseteq \operatorname{dom} B$. Then $A + B$ is maximally monotone.

Chapter 6: Our first main result (Theorem 6.2.1) in this chapter is obtained by applying Goldstine's Theorem (see Fact 6.1.2). Simons, Marques Alves and Svaiter's characterization of type (D) operators and Borwein's generalization of the Brøndsted-Rockafellar theorem are the main tools for obtaining the other main result (Theorem 6.3.1). Corollary 6.3.3 motivates the following question:

> *Let $A : X \rightrightarrows X^*$ be a monotone linear relation with closed graph. Assume that A^* is monotone.*
> *Is A necessarily of type (D)?*

Chapter 7: It would be interesting to find out whether Theorem 7.3.1 generalizes to the following:

> Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, let C be a nonempty closed convex subset of X. Assume that
>
> $$\left[\operatorname{dom} A - \bigcup_{\lambda>0} \lambda C\right] \quad \text{is a closed subspace of } X.$$
>
> Is it necessarily true that $F_{A+N_C} = F_A \Box_2 F_{N_C}$?

Chapter 8: The idea of the construction of the operator A in (Theorem 8.2.2) comes from [4, Theorem 5.1] by Bauschke and Borwein. The main tool involved in the main result (Theorem 8.2.2) is Simons and Zălinescu's version of Attouch-Brezis theorem.

Chapter 9: The original papers by Asplund [1] and by Borwein and Wiersma [27] concerned the additive decomposition of a maximally monotone operator whose domain has *nonempty interior*. In this chapter, we focused on maximally monotone *linear relations* and we specifically allowed for domains with empty interior. All maximally monotone linear relations on finite-dimensional spaces are Borwein-Wiersma decomposable; however, this fails in infinite-dimensional settings. We presented characterizations of Borwein-Wiersma decomposability of maximally monotone linear relations in general Banach spaces and provided a more explicit decomposition in Hilbert spaces.

The characterization of Asplund decomposability and the corresponding construction of an Asplund decomposition remain interesting unresolved topics for future explorations, even for maximally monotone linear operators whose domains are proper dense subspaces of infinite-dimensional Hilbert spaces.

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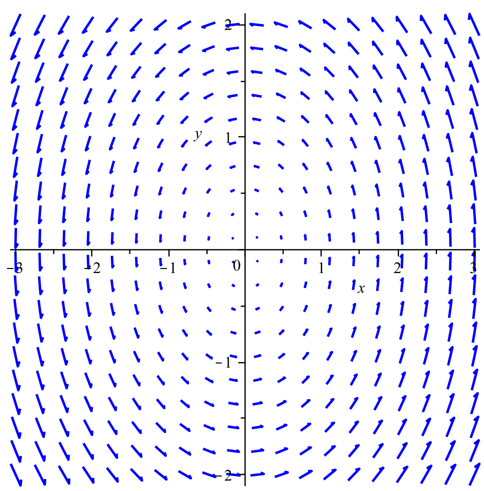
Appendix A

Maple code

The following is the Maple code to plot Figure 2.1.

```
>restart: Loading Student:-LinearAlgebra
with(plots):
>fieldplot((Matrix(2, 2, {(1, 1) = 0, (1, 2) = -1, (2, 1) = 1,
 (2, 2) = 0})).
(Vector(2, {(1) = x, (2) = y})), x = -3 .. 3, y = -2 .. 2,
 thickness = 2, colour = blue)
```



The following is the Maple code used to verify the calculations for Example 4.5.2 on $\widetilde{G_2}$

```
>restart: Loading Student:-LinearAlgebra
>A := Matrix(3, 2, {(1, 1) = −1, (1, 2) = 0, (2, 1) = 0,
(2, 2) = 0, (3, 1) = 0, (3, 2) = −1});
>B := Matrix(3, 2, {(1, 1) = 1, (1, 2) = 0, (2, 1) = 0,
(2, 2) = 1, (3, 1) = 0, (3, 2) = 1})
>T:=A.Transpose(B)+B.Transpose(A)
>Eigenvalues(T)
>Eigenvectors(T)
>Idlam:=[[[−1+sqrt(2),0,0],[0,−1−sqrt(2),0],[0,0,−2]]]
>V := Matrix(3, 3, {(1, 1) = 0, (1, 2) = 0, (1, 3) = 1,
(2, 1) = −1/(sqrt(2)−1), (2, 2) = −1/(−1−sqrt(2)),
 (2, 3) = 0, (3, 1) = 1, (3, 2) = 1, (3, 3) = 0})
>N:= Matrix(3, 3, {(1, 1) = 0, (1, 2) = −1, (1, 3) = 1,
 (2, 1) = 0, (2, 2) = 2, (2, 3) = −1, (3, 1) = 0,
  (3, 2) = 1, (3, 3) = 1})
>M:=Transpose(N).Idlam.N
>evalf(Eigenvalues(M))
>NullSpace(`<|>`(Transpose(N).Transpose(V).A,
 Transpose(N).Transpose(V).B))
>C := Matrix(2, 2, {(1, 1) = 1, (1, 2) = −2*sqrt(2),
 (2, 1) = 0, (2, 2) = 5*sqrt(2)})
>tilde{G_2}:= 1/C
```

The following is the Maple code used to verify the calculations for Example 4.5.3 on $\widetilde{G_1}, \widetilde{G_2}, E_1$ and E_2.

```
>restart: Loading Student:-LinearAlgebravoisei
>A := Matrix([[1, 1], [2, 0], [3, 1]])
>B := Matrix([[1, 5], [1, 7], [0, 2]])
>K := Matrix([[1, 1, 1, 5], [2, 0, 1, 7], [3, 1, 0, 2]])
>Rank(K)
>K1:= A.Transpose(B)+B.Transpose(A)
>Eigenvectors(K1)
>Idlam := Matrix(3, 3, {(1, 1) = 13+sqrt(201), (1, 2) = 0,
 (1, 3) = 0, (2, 1) = 0, (2, 2) = -6, (2, 3) = 0,
 (3, 1) = 0, (3, 2) = 0, (3, 3) = 13-sqrt(201)})
 >V:= Matrix(3, 3, {(1, 1) = 20/(1+sqrt(201)), (1, 2) = 0,
  (1, 3) = 20/(1-sqrt(201)),
  (2, 1) = 1, (2, 2) = -1, (2, 3) = 1, (3, 1) = 1,
   (3, 2) = 1, (3, 3) = 1})
>V_g := Matrix(2, 3, {(1, 1) = 0, (1, 2) = -1, (1, 3) = 1,
 (2, 1) = 20/(1-sqrt(201)),
 (2, 2) = 1, (2, 3) = 1})
>L :=NullSpace(`<|>`(V_g.A, V_g.B))
>C0 := Matrix(2, 2, {(1, 1) = -(-21+sqrt(201))/(-2+2*sqrt(201)),
 (1, 2) = -(-107+7*sqrt(201))/(-2+2*sqrt(201)),
 (2, 1) = (-23+3*sqrt(201))/(-2+2*sqrt(201)),
  (2, 2) = (-117+17*sqrt(201))/(-2+2*sqrt(201))})
```

```
>tilde{G_1}:= 1/C0
>N := Matrix(3, 3, {(1, 1) = 0, (1, 2) = 0, (1, 3) = 1/5,
 (2, 1) = 0,
(2, 2) = 1, (2, 3) = 0, (3, 1) = 0, (3, 2) = 0, (3, 3) = 1})
>M := Transpose(N).Idlam.N
>evalf(Eigenvalues(M))
>NullSpace(`<|>`(Transpose(N).Transpose(V).A,
 Transpose(N).Transpose(V).B))
>C1 := Matrix(2, 2, {(1, 1) = -9/20+(1/30)*sqrt(201),
 (1, 2) = -13/4+(1/6)*sqrt(201),
 (2, 1) = 29/20-(1/30)*sqrt(201),
 (2, 2) = 33/4-(1/6)*sqrt(201)})
>tilde{G_2}:= 1/C1
>vec := Vector(3, {(1) = 0, (2) = 0, (3) = 0})
>LinearSolve(`<|>`(A, B, vec), free = t)
```

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