On a Polar Factorization Theorem

by

Pierre Millien

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Abstract

We study the link between two different factorization theorems and their proofs : Brenier's Theorem which states that for any $u \in L^p(\Omega)$, where Ω is a bounded domain in R^d and $1 \leq p \leq \infty$, u can be written as

$$u = \nabla\phi \circ s$$

where ϕ is a convex function, and s a measure preserving transformation, and on the other hand Ghoussoub and Moameni's theorem which states that for any $u \in L^\infty(\Omega)$,

$$u(x) = \nabla_1 H(S(x), x),$$

where H is a convex concave anti-symmetric function, and S is a measure preserving involution.

In a second time we prove that Ghoussoub and Moameni's theorem is true in L^2, and find the decomposition for particular example : $u(x) = |x - 1/2|$.

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Chapter 1

Introduction

This thesis tackles the issue of factorization in analysis. We study two polar decomposition theorems, one by Y. Brenier ([1]) which states that for any $u \in L^p(\Omega)$, with $1 \leq p \leq \infty$, u can be written as

$$u = \nabla\phi \circ s$$

where ϕ is a convex function, and s a measure preserving transformation and a more recent one by N. Ghoussoub and A. Moameni ([5]) which states that for any $u \in L^\infty(\Omega)$,

$$u(x) = \nabla_1 H(S(x), x),$$

where H is a convex concave anti-symmetric function, and S is a measure preserving involution.

Brenier's first approach to the polar factorization was a "projection approach", working in the space $L^2(\Omega)$. His idea was to do the Hilbert projection of u onto the subspace $\mathcal{S}$ of all the measure preserving transformations, then via geometrical arguments, he wanted to deduce the factorization. But this approach failed for technical reasons. So he invented a clever proof based on Monge-Kantorovich's mass transportation theory, which is briefly shown in chapter 1.

Three years later, Gangbo ([3]) invented a new proof for Brenier's theorem using a completely different approach. He proved the result for $u \in L^\infty(\Omega)$ using only elementary convex analysis tools and a variational resolution, and he deduced the result for $u \in L^p(\Omega)$ via a density.

Using a variational method similar to Gangbo's, Ghoussoub and Moameni proved a new version of the polar factorization, the self dual polar factorization in $L^\infty(\Omega)$.

The goal of this thesis is to study the link between the new self dual polar factorization theorem and Brenier's geometrical approach, and to extend the self dual polar factorization theorem to $L^2(\Omega)$ using a density argument similar to Gangbo's. In the last chapter is presented an explicit computation of the factorization for a simple function.

Chapter 2

The mass transportation problem.

We first review a simplified case of the Monge-Kantorovich mass transportation problem, and explain how it can lead to Brenier's polar factorization.

2.1 Monge Kantorovich problem

Let $X \subset R^d$, and $Y \subset R^d$. Consider (X, λ) and (Y, ν) two probabilised spaces. Now consider the class $\mathcal{S}$ of all transformations $s : X \to Y$ which rearrange λ into ν, i.e. such that $s\#\lambda = \nu$ which means that

$$\forall h \in \mathcal{C}^0(Y, R^d), \int_X h(s(x))d\lambda(x) = \int_Y h(y)d\nu(y).$$

Now we introduce a cost function $c : X \times Y \to [0, \infty)$, and a total cost functional :

$$I(s) := \int_X c(x, s(x))d\lambda.$$

The problem is to find

$$\min_{s \in \mathcal{S}} I(s).$$

It turns out that, for the cost function $c(x, y) = \frac{1}{2}|x - y|^2$, there is an s^* such that $s^*\#\lambda = \nu$ which minimize I, and s^* is such that $s^* = D\phi^*$ for some convex function ϕ^*. More over, s^* is unique.

The idea of the proof is to introduce a "relaxed" version of the problem, called the Monge-Kantorovich problem, and solve it via a dual variational principle. To do so, we need to introduce a new class :

$\mathcal{M} := \{\text{Radon probability measures } \pi \text{ on } X \times Y \mid \pi[A \times Y] = \lambda[A],\ \pi[X \times B] = \nu[B]\}.$

We now have a relaxed cost functional :

$$J[\pi] := \int_{X \times Y} c(x, y) d\pi(x, y),$$

and we want to find π^* such that

$$J[\pi^*] = \min_{\pi \in \mathcal{M}} J[\pi].$$

Under appropriate assumptions on the cost c, a compactness argument gives the existence of one optimal measure. Such a measure need not be generated by a one to one mapping $s \in \mathcal{S}$.

In order to get a better characterization of the optimal plan, Kantorovich, inspired by linear programming, introduced a dual problem by defining :

$$\mathcal{L} := \{(u, v) \mid u : X \to R,\ v : Y \to R, u(x) + v(y) \leq c(x, y)\ (x, y \in R^d)\}.$$

A new functional :

$$K[u, v] := \int_X u(x) d\lambda(x) + \int_Y v(y) d\nu(y),$$

and consequently the dual problem is to find an optimal pair $(u^*, v^*) \in \mathcal{L}$ such that

$$K[u^*, v^*] = \max_{(u,v) \in \mathcal{L}} K[u, v].$$

So rather than constructing an $s^* \in \mathcal{S}$ satisfying a non linear constraint, we now have to find the optimal pair (u^*, v^*), which turns out to be a lot easier, and leads to some precise characterization of the optimal plans. For the complete proof, see [8].

2.1.1 Mass transportation and polar factorization

Let's consider (Ω, λ), where Ω is a bounded subset of R^d and λ is the Lebesgue measure.

Theorem 2.1.1. *(Brenier [1]) Let $u : \Omega \to \Omega$ be a non degenerate L^2 vector field. There is a unique pair $(\nabla\psi, s)$ such that :*

$$\begin{cases} \psi : \Omega \to R^d \text{ is a convex function.} \\ s : \Omega \to \Omega \text{ is measure preserving } (i.e\ s\#\lambda = \lambda) \\ u = \nabla\psi \circ s \end{cases}$$

Moreover, s is the unique orthogonal L^2 projection of u onto $\mathcal{S}$, the set of all measure preserving mapping $\Omega \to \Omega$.

The idea of the proof is the following : We look for a map s which minimizes

$$\int_\Omega |u(x) - \sigma(x)|^2 d\lambda$$

among all $\sigma \in \mathcal{S}$. We introduce the image measure $\pi = (u \times \sigma)\#\lambda$, then we have to find :

$$\min_\sigma \{ \int_{X\times Y} |x-y|^2 d\pi(x,y);\ \pi = (u\times\sigma)\#\lambda,\ \sigma\#\lambda = \lambda \}.$$

Then the idea is that the Monge Kantorovich theory gives a convex ϕ such that $(\nabla\phi \circ u)\#\lambda = \lambda$, and $\pi = (u \times \nabla\phi \circ u)\#\lambda$ is concentrated on the graph of $\nabla\phi$, and is optimal. Then setting $s := \nabla\phi \circ u$ and $\psi = \phi^*$, we get $\nabla\psi \circ s = \nabla\phi^* \circ \nabla\phi \circ u = u$, hence the polar decomposition.

2.2 Polar factorization via a variational approach

2.2.1 Gangbo's approach

Another proof of Brenier's polar factorizations theorem was developed by W. Gangbo in [3] The idea is to solve a minimization problem whose Euler-Lagrange equation turns out to be $u = \nabla\phi \circ s$. The proof does not rely on any of the Monge-Kantorovich tools, and just uses convex analysis. The proposition is proved for mappings $u \in L^\infty$, and the L^p version is deduced by an approximation argument. The idea is the following : let R be such that $u(\Omega) \subset B_R$. Then if we introduce

$$E_R = \{(\phi, \psi) \,|\phi \in C(B_R) \cap L^\infty(B_R),\ \psi \in C(\Omega) \cap L^\infty(\Omega),\ \phi(y) + \psi(z) \geq yz\ \forall (y, z) \in B_R \times \Omega\}$$

and

$$\mathcal{S} = \{s : \Omega \to \Omega \ |s \text{ is a measure preserving mapping}\}.$$

The variational problems are the following :

$$i_\infty = \inf\{I(\phi, \psi) \ |(\phi, \psi) \in E_R\}$$

where

$$I(\phi, \psi) = \int_\Omega [\phi(u(x)) + \psi(x)]dx,$$

and the dual problem is to find

$$\sup\{\int_\Omega u(x).s(x)dx \ |s \in \mathcal{S}\}.$$

2.2.2 A self-dual polar factorization

Using a similar variational approach, Ghoussoub and Moameni recently proved the following :

Theorem 2.2.1. *(Ghoussoub, Moameni,[5]) Let Ω be an open bounded set in R^N, such that $mea(\partial\Omega) = 0$, and let $u \in L^\infty(\Omega, R^N)$ be a non degenerate vector field. Then there exists a globally Lipschitz skew-adjoint saddle* *function H, and an idempotent measure preserving mapping s such that*

$$u(x) = \nabla_1 H(s(x), x) \quad a.e. \quad x \in \Omega,$$

and

$$\int_\Omega (u(x).s(x))dx = \sup_{f \in S} (\int_\Omega (u(x).f(x))dx)$$

If we define $L_H(x,p) = \sup_{y\in\Omega}\{(y.p) - H(y,x)\}$, *we have that* $\int_\Omega L_H(x, u(x))dx = \inf_{\tilde{H}\in\mathcal{H}}\{\int_\Omega L_{\tilde{H}}(x, u(x))dx\}$ *and* $s = \nabla_2 L_H(x, u(x))$.

The question now is to understand what is the link between the self-dual polar decomposition, Monge-Kantorovich's theory, and the Brenier's polar factorization.

Chapter 3

Polar factorization and Hilbert projections

3.1 Abstract polar factorization

We saw that regardless of the approach, in each case, to find the measure preserving mapping (or the self-dual measure preserving mapping), one has to find s such that

$$\int_{\Omega}(u(x).s(x))dx = \sup_{\sigma\in\mathcal{S}}(\int_{\Omega}(u(x).\sigma(x))dx),$$

or equivalently, find

$$\min_{\sigma\in\mathcal{S}}\{\int_{\Omega}|u(x)-\sigma(x)|^2 d\lambda\}.$$

It turns out that the study of the projection problem on a closed set S can lead to an abstract polar factorization theorem , which states that if S is a closed semi group of real Hilbert space H (understand here $H = L^2(\Omega)$, and the group law is the composition of functions) then for "almost every" $u \in H$ (in the sense of Baire) there exists a unique projection of u on S and there is an element in K such that $u = k \circ s$, where K is the polar cone of S defined as follow :

Definition 3.1.1. The polar cone K_S of S is :

$$K = \{u \in H; ((u, e - s)) \geq 0, \forall s \in S\}$$

K is the set of all $u \in H$ for which the identity map e is a Hilbert projection of u on S.

This approach is developed in [1]. The problem for our case is that S is not a group in both cases. The set of all measure preserving transformation is a semi group but not a group (some of its elements are not invertible), and the set of all self dual measure preserving transformation is not a semi group, but all its elements are invertible.

Since Brenier was still able to prove that the polar cone of the set of all measure preserving transformation is exactly the set $\{\nabla\psi,\ \psi \in W^{1,2}(\Omega),\ \psi\ convex\}$, we tried to see what would still hold in the case where

$$S = \{\text{Self dual measure preserving transformations}\},$$

and find its polar cone.

3.2 The self dual case

All the theorems that still hold need the assumption that S is closed. It is the case indeed :

Proposition 3.2.1. *Let S be the set of all self dual measure preserving transformation, i.e. $\forall s \in S$:*

$$\forall f \in L^1(\omega),\ f \circ s \in L^1(\omega),\ and\ \int_\Omega f \circ s(x)dx = \int_\Omega f(x)dx,$$

and $s \circ s = e$.
Then S is closed.

Proof. Let $(s_n)_{n \in N} \in S^N$, $s_n \longrightarrow s$. s is still measure preserving (see [1]). s is also idempotent : $\forall H \in L^1 \cap C^0$, H antisymmetric, the dominated convergence theorem and the continuity of H give :

$$\int_\Omega H(s_n(x), x)dx \longrightarrow \int_\Omega (s(x), x)dx,$$

which gives us

$$\int_\Omega H(s(x), x)dx = 0.$$

Now for any $H \in L^1$, there exists a sequence H_n of continuous antisymmetric functions, converging to H. We get :

$$\int_\Omega H_n(s(x), x)dx \longrightarrow \int_\Omega H(s(x), x)dx,$$

which gives

$$\int_\Omega H(s(x), x)dx = 0.$$

Then, considering $H(x, y) = |s(x) - y| - |s(y) - x|$ and $\int_\Omega H(s(x), x)dx = \int_\Omega |s^2(x) - x|dx = 0.$ □

The first theorem that still holds is the one giving the existence of the projection.

Theorem 3.2.2. *(Edelstein) Let S be a closed bounded subset of a real Hilbert space H. Then, the set of all $u \in H$ for which there is a unique Hilbert projection $\pi(u)$ on S contains a dense countable intersection of open sets $H \backslash N$, defined by : $H \backslash N = u \in H; \forall \epsilon > 0, \exists \delta > 0$ such that $\|s_1 - s_2\| \leq \epsilon,\ s_1, s_2 \in S$, whenever $|s_i - u| \leq dist(u, S) = \delta, i = 1, 2$. Moreover, π is continuous from $H \backslash N$ into H.*

Theorem 3.2.3. *Let S be a closed bounded subset of a sphere centred at the origin in a real Hilbert space H. Then, the projector operator $\pi : H \longrightarrow S$ can be characterized as the gradient of the Lipschitz continuous convex function*

$$J(u) = \sup_{s \in S}((u, s)).$$

More precisely, one has $\partial J(u) = \pi(u)$, for all $u \in H \backslash N$

The proof can be found in [1].

Then we managed to determine what S's polar cone is :

Proposition 3.2.4. *Let* $\tilde{K} = \{u \in L^2(\Omega); u = \nabla_1 H(x,x)\}$ *for some* $H \in \mathcal{H}\}$ *Then* $\tilde{K}$ *is* S*'s polar cone, i.e.* $\tilde{K} = K_S$.

In the proof, we will use the following lemma, due to Krauss ([6]):

Lemma 3.2.5. *For each monotone operator* $A \subset H \times H$*, such that* $D(A) \neq \emptyset$*, there exists a lower closed skew symmetric saddle function* $L_A : H \times H \longrightarrow R$ *with* $coD(A) \subset DomL_A \subset \overline{co}D(A)$ *such that* T_{L_A}*, defined by* $f \in T_{L_A}x := [f, -f] \in \partial L_A(x,x)$*, is a maximal monotone extension of* A.

Proof. First step : $\tilde{K} \subset K_S$. Let $u \in \tilde{K}$, $u = \nabla_1 H(x,x)$ with $H \in \mathcal{H}$. For all $s \in S$, $H(s(x),x) \geq H(x,x) + \nabla_1 H(x,x).(s(x) - x)$

We get that :

$$\int_\Omega \nabla_1 H(x,x).(x - s(x))dx \geq \int_\Omega H(s(x),x) - H(x,x)dx$$

$$\int_\Omega \nabla_1 H(x,x).(x - s(x))dx \geq 0,$$

Because since s is measure preserving on Ω, $(s \times Id)$ is also measure preserving on Ω^2

Second step : $K_S \subset \tilde{K}$. Let $u \in K_S$, then u is monotone. Let us consider a pair $(x_1, x_2) \in \Omega$. When R is small enough, then $B(x_i, R) \subset \Omega, \forall i \in \{1,2\}$. Then we define s_r, which is a Lebesgue measure preserving involution :

$$s_R(x) = \begin{cases} x - x_2 + x_1 & \text{if } x \in B(x_2, R) \\ x - x_1 + x_2 & \text{if } x \in B(x_2, R) \\ x & \text{otherwise} \end{cases}$$

Since $u \in K_S$ we have $\int_\Omega u(x).(x - s_r(x))dx \geq 0$, which is :

$$(x_1 - x_2).\left[\int_{B_R} u(x_1 + Ry)dy - \int_{B_R} u(x_2 + Ry)dy\right] \geq 0.$$

Since u is Lebesgue integrable, almost every point $x \in \Omega$ is a Lebesgue point, which means

$$u(x) = \lim_{R \to 0} \mid B \mid^{-1} \int_B u(x + Ry)dy.$$

This leads to :

$$(x_1 - x_2).(u(x_1) - u(x_2)) \geq 0, \text{for a.e. } x_1, x_2 \in \Omega$$

We now have that every $u \in K_S$ is monotone. We now use Krauss' theorem, and the graph $\{(x, u(x)), x \in \Omega \subset (x, \nabla_1 L(x, x)), x \in \Omega\}$, for some skew symmetric saddle function L, which gives $u \in \tilde{K}$. □

We did not succeed in proving the self dual polar factorization in L^2 via the abstract polar factorization theory, but we found out some similarities between the polar cone of S and the $\nabla_1 H$ that appears in Ghoussoub and Moameni's theorem. Even though this technique did not work, we still managed to prove that the self dual polar factorization holds in L^2, using an approximation argument, similar to the one that Gangbo used to prove the polar factorization in L^2 using the L^∞ result.

Chapter 4

The self dual polar factorization in L^2

Ghoussoub and Moameni proved the polar factorization theorem (theorem 2.1) for $u \in L^\infty(\Omega)$.

We want to prove this result in L^2. The following proof is the work of Abbas Moameni and me. I would like to thank him for all his help.

Let $u \in L^2(\Omega)$ be a non degenerate vector field. Define $u_n : \Omega \longrightarrow R$ by

$$u_n(x) = r_n(u(x)),$$

where r_n is a diffeomorphism from R^d onto B_n such that $\|r_n(y)\| \leq \|y\|$, for all $y \in R^d$ and $r_n(y) \longrightarrow y$ uniformly on any compact subset of R^d.

We have that for each $n \in N$, $u_n \in L^\infty(\Omega,\ R^d)$, and u_n is non degenerate. As $\sup\{\|u_n(x)\| | x \in \Omega\} \leq n$, we have, using the self dual polar factorization theorem, that there exists an measure preserving involution s_n, a continuous Lipschitz (the constant depends on n) saddle function H_n such that

$$u_n(x) = \nabla_1 H_n(s_n(x), x) \quad a.e. \quad x \in \Omega,$$

and

$$\int_\Omega (u_n(x).s_n(x))dx = \sup_{f\in S}(\int_\Omega (u_n(x).f(x))dx).$$

Let us show that $(H_n)_{n\in N}$ is bounded in $L^2(\Omega)$.

Since H_n is convex in the first variable, we get :

$H_n(y,x) - H_n(S_nx,x) \geq (y - S_nx.\nabla_1 H_n(S_nx,x))$

$H_n(y,x) - H_n(S_nx,x) \geq (y - S_nx.u_n(x))$

$H_n(S_nx,x) \geq |u_n(x)||S_nx - x|$

We also have that $H_n(S_nx,S_nx) - H_n(x,S_nx) \geq (S_nx - x.u_n(x))$

$H_n(x,S_nx) \leq (S_nx - x.u_n(x))$

$H_n(S_nx,x) \geq -|S_nx - x||u_n(x)|$

We get that :

$$-|S_nx - x||u_n(x)| \leq H_n(S_nx,x) \leq |u_n(x)||S_nx - x|.$$

Now, we have the following :

$H_n(y,x) - H_n(S_nx,x) \geq (y - S_nx.u_n(x))$

$H_n(y,x) \geq -|y - S_nx||u_n(x)| + H_n(S_nx,x)$

$H_n(y,x) \geq -|y - S_nx||u_n(x)| - |u_n(x)||S_nx - x|$

$H_n(x,y) \leq |y - S_nx||u_n(x)| + |u_n(x)||S_nx - x|$

Similarly we have

$H_n(x,y) - H_n(S_ny,y) \geq (x - S_ny.u_n(y))$

$H_n(x,y) \geq -|x - S_ny||u_n(y)| + H_n(S_ny,y)$

$H_n(x,y) \geq -|x - S_ny||u_n(y)| - |u_n(y)||S_ny - y|$

So we get that :

$$-|x-S_ny||u_n(y)|-|u_n(y)||S_ny-y| \leq H_n(x,y) \leq |y-S_nx||u_n(x)|+|u_n(x)||S_nx-x|,$$

which implies that H_n is a bounded sequence in $L^2(\Omega\times\Omega)$.

Up to a subsequence, $H_n \longrightarrow_{weakly} H$. Now, H is still an anti-symmetric function almost everywhere:

Let $(x_0, y_0) \in \Omega^2$. Let's consider

$$f_\epsilon(x,y) = \frac{1}{|B((x_0,y_0),\epsilon)|} 1_{B((x_0,y_0),\epsilon)}(x,y)$$

$$g_\epsilon(x,y) = \frac{1}{|B((y_0,x_0),\epsilon)|} 1_{B((y_0,x_0),\epsilon)}(x,y).$$

We have, for ϵ small enough, $\forall n \in N$,

$$\int_{\Omega^2} f_\epsilon(x,y) H_n(x,y) dxdy + \int_{\Omega^2} g_\epsilon(x,y) H_n(x,y) dxdy = 0.$$

Taking the limit when $n \longrightarrow \infty$ we have that

$$\forall \epsilon > 0, \int_{\Omega^2} f_\epsilon(x,y) H(x,y) dxdy = -\int_{\Omega^2} g_\epsilon(x,y) H(x,y) dxdy.$$

Now, using Lebesgue's differentiation theorem, we get, when $\epsilon \longrightarrow 0$:

$$H(x_0,y_0) = -H(y_0,x_0), \ a.e.$$

Since $H_n \longrightarrow_{weakly} H$ in L^2, there exists a sequence $(\tilde{H}_n)_{n \in N}$ of convex combinations $\tilde{H}_i = \sum_{finite} \alpha_{i,n} H_n, \ \alpha_{i,n} \geq 0, \sum_n \alpha_{i,n} = 1$ which converges to H strongly. So up to a subsequence, we have $\tilde{H}_n(x,y) \longrightarrow H(x,y)$ for almost every $x \in \Omega$ and every $y \in \Omega$. For every $y \in \Omega$, let's denote by G_y the set $G_y \subset \Omega$, of all the points x in Ω such that $\tilde{H}_n(x,y) \longrightarrow H(x,y)$. We have $|G_y| = |\Omega|$.

Now, let's fix $y_0 \in \Omega$. Let

$$\forall x: \ \tilde{H}(x,y_0) := \inf\{\sum \lambda_i H(x_i,y_0), \ \sum_i \lambda_i = 1, \ x_i \in G_{y_0}, \ x = \sum \lambda_i x_i\}.$$

$\tilde{H}(.,y_0)$ is a convex function and $\tilde{H}(x,y_0) \leq H(x,y_0)$. But $\tilde{H}(.,y_0) = H(.,y_0)$ almost everywhere. Indeed, for any $x \in G_{y_0}$, for any convex combination of points $(x_i)_i$ in G_{y_0} such that $\sum \lambda_i x_i = x$, we have $H(x,y_0) \leq \sum \lambda_i H(x_i,y_0)$ so, going to the infinitum we get $H(x,y_0) \leq \tilde{H}(x,y_0)$, $\forall x \in$

G_{y_0}.

Since for every y almost everywhere $\tilde{H}(.,y) = H(.,y)$ and almost everywhere in Ω^2, $H(x,y) = -H(y,x)$, we have, almost everywhere in Ω^2, $\tilde{H}(x,y) = -\tilde{H}(y,x)$.

For every $y \in \Omega$, $x \mapsto \tilde{H}(x,y)$ is convex. Since $\tilde{H}(.,y)$ is in $L^2(\Omega)$, it is finite for almost every x. By convexity it is finite every where on the interior of Ω, so it is also continuous on the interior of Ω.

Putting together the two last points, we get that

$$\tilde{H}(x,y) = -\tilde{H}(y,x)$$

for every $x, y \in \Omega$.

Let's consider $g_n(x) = L_n(x, u_n(x)) = (s_n.u_n) - H_n(s_n(x), x)$. Since $g_n \in L^2(\Omega)$, $g_n \longrightarrow_{weakly} g$ in L^2.
We have that

$$L_n(x, u_n(x)) \longrightarrow_{weakly} g(x),$$

which implies

$$\int_\Omega L_n(x, u_n(x))dx \longrightarrow \int_\Omega g(x)dx.$$

Now for every x, y we have : $H_n(y,x) + L_n(x, u_n(x)) \geq (y.u_n)$, which gives when taking the integral over Ω :

$$\int_\Omega L_n(x, u_n(x))dx + \int_\Omega H_n(y,x)dx \geq \int_\Omega (y.u_n(x))dx$$

$$\int_\Omega L_n(x, u_n(x))dx \geq \int_\Omega (y.u_n(x)) - H_n(y,x)dx$$

Now let us introduce, for any $y_0 \in \Omega$, $\epsilon > 0$:

$$f_{\epsilon,y_0}(y) := \frac{1}{|B(y_0,\epsilon)|} 1_{B(y_0,\epsilon)}.$$

Now, for any $y_0 \in \Omega$, we multiply the previous inequality by f_{ϵ,y_0} and integrate on Ω with respect to y :

$$\int_\Omega \int_\Omega L_n(x,u_n(x))dx f_{\epsilon,y_0}(y)dy \geq \int_\Omega \int_\Omega ((y.u_n(x)) - H_n(y,x)) f_{\epsilon,y_0}(y)dxdy.$$

Now we can take the limit because H_n is weakly convergent, and get :

$$\int_\Omega g(x)dx \geq \int_\Omega \int_\Omega ((y.u(x)) - H(y,x)) f_{\epsilon,y_0}(y)dxdy.$$

Since $H = \tilde{H}$ almost everywhere :

$$\int_\Omega g(x)dx \geq \int_\Omega \int_\Omega ((y.u(x)) - \tilde{H}(y,x)) f_{\epsilon,y_0}(y)dxdy.$$

Now, $\tilde{H}$ is continuous so we can use Lebesgue differentiation theorem by taking the limit when $\epsilon \longrightarrow 0$. We get , for every $y_0 \in \Omega$,

$$\int_\Omega g(x)dx \geq \int_\Omega (y_0.u(x)) - \tilde{H}(y_0,x)dx.$$

Now, taking the supremum over y_0 :

$$\int_\Omega g(x)dx \geq \sup_{y\in\Omega} \int_\Omega (y.u(x)) - \tilde{H}(y,x)dx,$$

which is :

$$\int_\Omega g(x)dx \geq \int_\Omega L_{\tilde{H}}(x,u(x))dx.$$

Now, for any $\hat{H} \in \mathcal{H}$, we have :

$$\int_\Omega L_n(x,u_n(x))dx \leq \int_\Omega L_{\hat{H}}(x,u_n(x))dx,$$

taking the limit we have :

$$\int_\Omega L_{\tilde{H}}(x,u(x))dx \leq \int_\Omega g(x)dx \leq \int_\Omega L_{\hat{H}}(x,u(x))dx.$$

So we found an optimal $\tilde{H}$, anti symmetric convex concave. Now if we take

$$S(x) \in \partial_2 L_{\tilde{H}}(x, u(x)),$$

we have that S is self dual measure preserving and that

$$u(x) = \nabla_1 H(S(x), x).$$

(See [5]).

Chapter 5

Case study

In this section we study a particular case. $\Omega = [0,1]$ and $u(x) = |x - 1/2|$. The following computations are due to Bernard Maurey from university Paris VII.

5.1 Finding S

We find s by maximizing $\int_\Omega u(x).S(x)dx$. We are looking for S of the following type : $S(x) = \alpha - x$ if $x \in [0, \alpha]$ and $S(x) = x$ for $x \in [\alpha, 1]$. We find $\alpha = \sqrt{2}/2$.

5.2 Finding $H(x, Sx)$

Let's set $\alpha = \sqrt{2}/2$. When $0 \leq x\alpha$, $Sx = \alpha - x$. Set

$$f(x) = H(x, \alpha - x)$$

such that

$$f'(x) = \nabla_1 H(x, \alpha - x) - \nabla_2 H(x, \alpha - x) = u(Sx) + u(x) = u(\alpha - x) + u(x).$$

When $0 \leq x \leq \beta := \alpha - 1/2$, we have $u(\alpha - x) = \alpha - x - 1/2$ and $u(x) = 1/2 - x$ so $f'(x) = \alpha - 2x$. When $\beta \leq x \leq 1/2$, we also have $\beta \leq \alpha - x \leq 1/2$, $f'(x) = 1/2 - (\alpha - x) + 1/2 - x = 1 - \alpha$, and to finish when $1/2 \leq x \leq \alpha$,

$f'(x) = x - 1/2 + (1/2 - (\alpha - x)) = 2x - \alpha$. Since $f(\alpha/2) = H(\alpha/2, \alpha/2) = 0$, we can deduce

$$f(x) = (1-\alpha)(x - \alpha/2) = (1-\alpha)x - \beta/2$$

when $\beta < x < 1/2$, so $f(\beta) = (1-\alpha)\beta - \beta/2 = \beta/2 - 1/2 + \alpha/2 = \alpha - 3/4$, and $f(1/2) = (1-\alpha)(1/2 - \beta/2) = -\alpha + 3/4$. then when $0 \leq x \leq \beta$,

$$f(x) = f(\beta) + \int_{\beta}^{x} (\alpha - 2t)dt = -x^2 + \alpha x + \alpha/2 - 1/2.$$

We have $f(0) = -\alpha\beta$. When $1/2 \leq x \leq \alpha$,

$$f(x) = f(1/2) + \int_{1/2}^{x} (2t - \alpha)dt = x^2 - \alpha x + \alpha\beta.$$

When $\alpha \leq x \leq 1$, we have $Sx = x$ and $f(x) = H(x, x) = 0$.

5.3 Convexity inequalities

We are now going to give lower and upper estimates on H obtained by concavity - convexity. When $0 < y_0 < \beta$, we have $1/2 < x_0 > \alpha$, $H(x_0, y_0) = f(x_0) = x_0^2 - \alpha x_0 + \alpha\beta$ and for any x we get the lower estimate on H by finding the tangent in x at x_0 to the convex function $H(., y_0)$ by setting :

$$C_0(x, y_0) = H(x_0, y_0) + (x - x_0)\nabla_1 H(x_0, y_0) = H(x_0, y_0) + (x - x_0)u(y_0).$$

We get that

$$C_0(x, y_0) = x_0^2 - \alpha x_0 + \alpha\beta + (x - x_0)(1/2 - y_0) = (\alpha - y_0)^2 - \alpha(\alpha - y_0) + (x + y_0 - \alpha)(1/2 - y_0) + \alpha\beta,$$

$$C_0(x, y_0) = (x + y_0 - \alpha)(1/2) - xy_0 + \alpha\beta = -xy_0 + (x + y_0)/2 - \beta.$$

We have that

$$C_0(x, y) = -xy + (x + y)/2 - \beta, \qquad 0 \leq y \leq \beta.$$

When $1/2 < y_0 < \alpha$, we have $0 < x_0 < \alpha$, $H(x_0, y_0) = f(x_0) = -x_0^2 + \alpha x_0 - \alpha\beta$ and for any x :

$$C_0(x, y_0) = -x_0^2 + \alpha x_0 - \alpha\beta - (x - x_0)(1/2 - y_0).$$

We have that

$$C_0(x, y) = xy - (x + y)/2 + \beta, \qquad 1/2 \le y \le \alpha.$$

When $\beta < y_0 < 1/2$, we have $\beta < x_0 < 1/2$, $H(x_0, y_0) = f(x_0) = (1 - \alpha)x_0 - \beta/2$ and for any x

$$C(x, y_0) = (1 - \alpha)x_0 - \beta/2 + (x - x_0)(1/2 - y_0)$$

$$C(x, y_0) = (1 - \alpha)(\alpha - y_0 - \beta/2) + (x + y_0 - \alpha)(1/2 - y_0)$$

$$C(x, y_0) = -y_0^2 - xy_0 + x/2 + (\alpha + \beta)y_0 - 1/4$$

So we have that

$$C(x, y) = -y^2 - xy + x/2 + (\alpha + \beta)y - 1/4, \qquad \beta \le y \le 1/2.$$

For $y_0 \ge \alpha$, we have $x_0 = y_0$, $f(x_0) = 0$ and for any x

$$C_0(x, y_0) = (x - y_0)(y_0 - 1/2),$$

so

$$C_0(x, y) = (x - y)(y - 1/2), \qquad \alpha \le y \le 1.$$

Let's now find the upper estimates $C_1(x, y)$ concave (actually affine) in y. By construction we find that $C_1(x, y) = -C_0(y, x)$, so we get

$$C_1(x, y) = xy - (x + y)/2 + \beta, \qquad 0 \le x \le \beta,$$

$$C_1(x,y) = x^2 + xy - y/2 - (\alpha+\beta)x + 1/4, \qquad \beta \le x \le 1/2,$$

$$C_1(x,y) = -xy + (x+y)/2 - \beta, \qquad 1/2 \le x \le \alpha,$$

$$C_1(x,y) = x^2 - xy - x/2 + y/2, \qquad \alpha \le x \le 1.$$

It looks like $C_0 \le C_1$, which is necessary if the problem has a solution that uses this transformation S. The solution H has to satisfy

$$C_0 \le H \le C_1.$$

In the square $0 \le x \le \beta$, $1/2 \le y \le \alpha$, we have

$$C_0(x,y) = xy - (x+y)/2 + \beta = C_1(x,y)$$

which shows that we found at least that part of the definition of H.

The functions $f = C_0$ and $f = C_1$ both satisfy the equations

$$\nabla_1 f(x, Sx) = u(Sx), \quad \nabla_2 f(x, Sx) = -u(x).$$

Since $C_0 \le C_1$, it turns out that any function between C_0 and C_1 will still satisfy the equations. We can then try to regularize : if $C_{1,0}$ is the convexified in x of C_1; we have $C_{1,0} \le C_1$ by definition and $C_0 \le C_{1,0}$ because C_0 is convex in x. We could imagine a succession of regularization in x (convexification), and in y (concavification), but it turns out that here the convexified $C_{1,0}$ in x of C_1 appears to be convex - concave. (This has been checked only on a computer for now.) And it satisfies both equations. We will finish by setting

$$H(x,y) = (C_{1,0}(x,y) - C_{1,0}(y,x))/2.$$

We first give what we found for $C_{1,0}$. If

$$\alpha \le y \le 1 \text{ and } 0 < x < x_1(y) := \sqrt{y-\beta},$$

then

$$C_{1,0}(x,y) = -x_1(y)^2 + y/2 + x(2x_1(y) - y - 1/2)$$
$$C_{1,0} = -xy + 2x\sqrt{y-\beta} - x/2 - y/2 + \beta.$$

If

$$\beta \le y \le \alpha \text{ and } x_0(y) := \alpha - \sqrt{2\beta y - \beta^2} \le x \le \alpha,$$

then

$$C_{1,0}(x,y) = -x_0(y)^2 + 1/4 - y/2 + x(2x_0(y) + y - \alpha - \beta)$$

$$C_{1,0}(x,y) = xy - 2x\sqrt{2\beta y - \beta^2} + x/2 - (1/2+2\beta)y + 2\alpha\sqrt{2\beta y - \beta^2} + 1/2 - \alpha;$$

When it is not those two cases we just set

$$C_{1,0} = C_1(x,y).$$

$C_{1,0}$ is the largest convex concave function satisfying our equations.

5.4 Final expressions for H

Here are the equations, assuming $0 \le x \le y \le 1$.

$$(1) \qquad H(x,y) = -y^2/2 + x\sqrt{y-\beta} - x/2 + \beta/2$$

if $\alpha \le y \le 1$ and $0 \le x \le \sqrt{y-\beta}$,

$$(2) \qquad H(x,y) = x^2/2 - y^2/2 - x/2 + y/2$$

if $\alpha \le y \le 1$ and $\sqrt{y-\beta} \le x \le y$,

$$(3) \qquad H(x,y) = xy - x/2 - y/2 + \beta$$

if $\leq x \leq \beta$ and $1/2 \leq y \leq \alpha$,

$$H(x,y) = x^2/2 - (\alpha - y)\sqrt{2\beta x - \beta^2} - y/2 + \alpha/2 - 1/8 \tag{4}$$

if $\beta \leq x \leq \alpha + \beta - \sqrt{1-\alpha}$ and $\alpha - \sqrt{2\beta x - \beta^2} \leq y \leq \beta/2 + (\alpha - x)^2/(2\beta)$,

$$H(x,y) = (\alpha - x)\sqrt{2\beta y - \beta^2} - (\alpha - y)\sqrt{2\beta x - \beta^2} + \alpha(x - y) \tag{5}$$

if $\alpha + \beta - \sqrt{1-\alpha} \leq y \leq \alpha$ and $\alpha - \sqrt{2\beta y - \beta^2} \leq x \leq y$,

$$H(x,y) = -y^2/2 + \beta y + \alpha/2 - 3/8 \tag{6}$$

if $0 \leq x \leq \beta$ and $\beta \leq y \leq 1/2$,

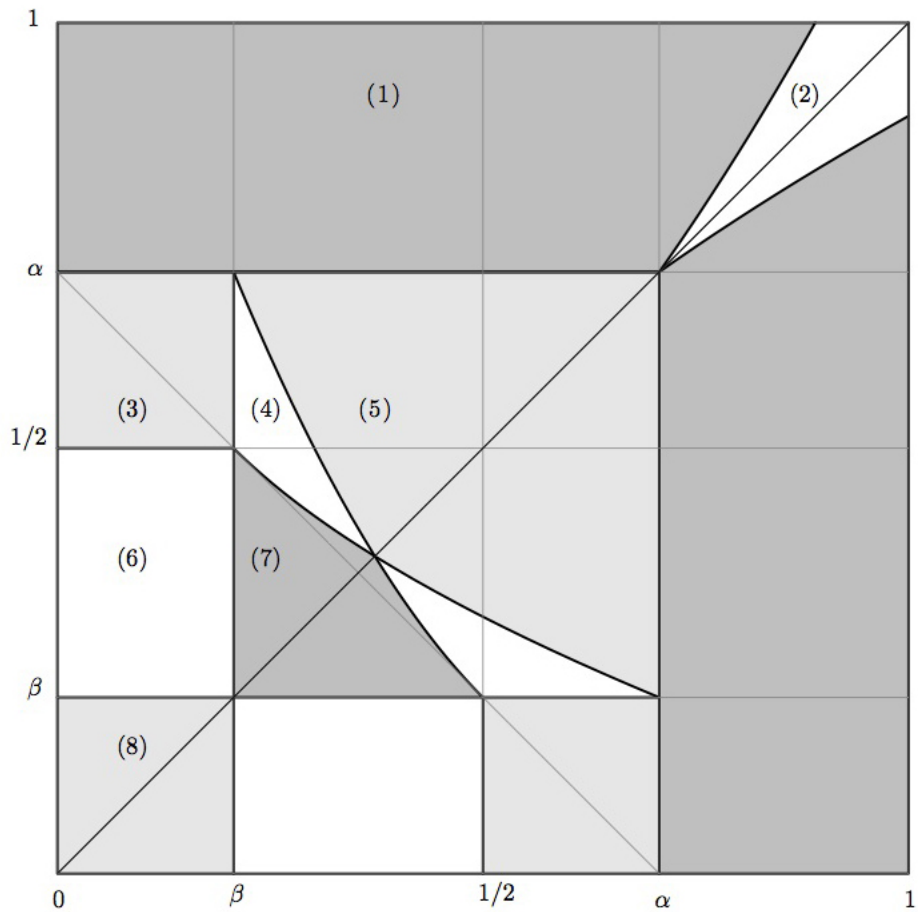
$$H(x,y) = x^2/2 - y^2/2 - \beta x + \beta y \tag{7}$$

if $\beta \leq x \leq \alpha + \beta - \sqrt{1-\alpha}$ and $x \leq y \leq \alpha - \sqrt{2\beta x - \beta^2}$,

$$H(x,y) = 0 \tag{8}$$

if $0 \leq y \leq \beta$ and $0 \leq x \leq y$.

Figure 5.1: Expression for H



Chapter 6

Conclusion

Even though it is still not clear whether the self dual polar factorization can be written as a mass transportation problem, the geometrical approach in L^2 via the projection onto the space of all measure preserving involutions $\mathcal{S}$ shows a lot of similarities with Brenier's theory, for instance the polar cone that we find is $\tilde{K} = \{u \in L^2(\Omega); u = \nabla_1 H(x, x)\}$. This approach does not give a proof of the self dual polar decomposition, but it still gives a new geometrical meaning to it.

The fact that we were able to extend the result to $L^2(\Omega)$ comforts the idea that the measure preserving involution found in the decomposition is the Hilbert projection of u onto $\mathcal{S}$.

The explicit computation of H and s for a simple function shows that even though the theorem is a powerful existence result, constructing the decomposition is a problem, even for simple functions, because the proof of the decomposition is not constructive.

The next step would be to build a numerical scheme to find H and S, in order to better understand the link between u and its decomposition.

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