Black Holes in Spacetime Dimension Other than Four

by

Kory Stevens

B.Sc., The University of Alberta, 2002
M.Sc., The University of Alberta, 2004

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in

The Faculty of Graduate Studies

(Physics)

The University Of British Columbia

(Vancouver)

October, 2011

Abstract

Recently, models of spacetime with extra dimensions has driven interest into higher dimensional black holes. In this thesis, we examine general relativity in spacetimes that are not four dimensional. The thesis is divided into three parts. The first part focuses on the higher dimensional Kerr-de Sitter metrics; we examine the separation of variables for the Hamilton-Jacobi and Klein-Gordon equations that occurs for particles and fields in those metrics. In the second part we consider lower dimensional geons, and give a proof that there cannot be any asymptotically flat geons in a three dimensional spacetime. Finally, we examine charged higher dimensional black holes, and consider the possibility of a higher dimensional generalization of the Kerr-Newman metric. We examine some approaches to find that solution, and demonstrate the form that the electromagnetic potential must have in such a spacetime.

Preface

A version of chapter 3 has been published. M. Vasudevan and K. A. Stevens, *Integrability of particle motion and scalar field propagation in Kerr-(anti) de Sitter black hole spacetimes in all dimensions*, Physical Review D72 (2005) 124008, [gr-qc/0507096]. I performed the calculations in collaboration with then PhD student Muraari Vasudevan and edited the paper.

A version of chapter 4 has been published. K. A. Stevens, K. Schleich, and D. M. Witt, *Non-existence of Asymptotically Flat Geons in 2+1 Gravity*, Classical and Quantum Gravity 26 (2009) 075012, [arXiv:0809.3022]. I performed the proof of theorem 3 and wrote and edited the paper in collaboration with Dr. Schleich and Dr. Witt.

Table of Contents

List of Tables

List of Figures

Acknowledgements

I am very grateful to my supervisors, Kristin Schleich and Don Witt, for providing me with guidance over the course of my PhD. I would like to thank the professors of the courses that I have taken at UBC that I very much enjoyed attending. I would also like to thank the officemates I have had for discussions about many topics.

Chapter 1

Introduction

Spacetimes with extra dimensions have been the objects of much study in the past few decades. For example, superstring theory is naturally formulated on a background with greater than four spacetime dimensions (see [1]). The extra dimensions of superstring theory are usually compactified, so rather than being a size on the scale of usual spatial dimensions, they are of a very small size, possibly on the order of the Planck length. Another possibility, explored by Rubakov and Shaposhnikov is that the universe could be a four dimensional brane embedded in a higher dimensional spacetime, with most of the particles of the theory localized to existing only on the brane [2] (see also [3] for a recent review). In these models, the extra dimensions do not need to be quite as small to remain hidden from experiments. In particular, these models offer a possible solution to the hierarchy problem, that being the question of why the energy scale of gravity $M_{\mathrm{Pl}} \sim 10^{19}$ GeV is so much larger than the electroweak scale at $M_{\mathrm{ew}} \sim 10^{2}$ GeV. Arkani-Hamed, Dinopoulos, and Dvali [4] have demonstrated how to do this by considering the universe to actually have $4{+}d$ dimensions with the extra dimensions d being compact with size R, but R is allowed to be much larger than the Planck scale. In such a universe, suppose we have a gravitating particle of mass M. At distances $r \ll R$, the Newtonian gravitational potential will be given by Gauss's law as

$$V(r \ll R) \sim \frac{M}{M_{\mathrm{Pl}(4+d)}^{d+2}} \frac{1}{r^{d+1}}\,, \tag{1.1}$$

with $M_{\mathrm{Pl}(4+d)}$ being the $4{+}d$-dimensional Planck mass. At much larger scales $r \gg R$, the extra dimensions will no longer cause a drop in the field, and the potential will appear as

$$V(r \gg R) \sim \frac{M}{M_{\mathrm{Pl}(4+d)}^{d+2} R^{d}} \frac{1}{r}\,. \tag{1.2}$$

So we can see that for large r, the four dimensional gravitational scale is given as

$$M^2_{\mathrm{Pl}(4)} = M^{d+2}_{\mathrm{Pl}(4+d)} R^d \,, \tag{1.3}$$

which tells us that if we actually exist on a brane in a higher dimensional spacetime, it is possible for a very large apparent gravitational scale to result from a smaller fundamental one. There has even been suggestions that these models could result in particle colliders creating small scale black holes [5].

In this model, at small distances $r \sim R$, we will begin to see deviations from Newton's graviational law. There has been experimental research examining the validity of the inverse square law down to sub-millimeter scales, putting limits on the possible size of these extra dimensions as no larger than 44μm [6–8].

In this thesis, we will be considering black holes on spacetimes with a number of dimensions other than four. With extra dimensions, compacting the size of those dimensions would complicate the analysis and lead to very difficult equations, so we will typically be working with spaces where the extra dimensions are of infinite size.

About 25 years ago, Myers and Perry found a generalization of the Kerr metric [9] to higher dimensional spacetimes [10]. These metrics are vacuum solutions to the Einstein equations, and they are asymptotically flat and have a well-defined mass and angular momentum. Some time after the Myers-Perry solutions were found, Gibbons, Lu, Page, and Pope discovered a generalization of those metrics to spacetimes with a cosmological constant [11]. Spacetimes with cosmological constants have seen a rise in interest since Maldacena proposed the AdS/CFT correspondence principle, which asserts a correspondence between string theory on an asymptotically locally anti-de Sitter spacetime and an appropriate conformal field theory [12].

Chandrasekhar once pointed out that the Kerr Metric had many properties that have endowed it with an "aura of the miraculous" [13]. In particular, the separability of variables in the Hamilton-Jacobi equation for a particle free-falling in the gravitational field, discovered by Carter [14], and the fact that the Kerr metric is of Petrov type D [15], which is the same Petrov type as the non-rotating, Schwarzschild case. Later properties that were added to this list were the separability of the d'Alembertian and the massive Klein-Gordon equations [16, 17], and the separability of the massless higher spin wave equations [18]. Also discovered was separability of the Dirac equation [19], and later separability of the equilibrium equations for a stationary cosmic string near a Kerr black hole [20].

The property of separability of these equations in the Kerr metric is due to the existence of a rank two Killing tensor in that spacetime. A *Killing tensor* of rank i is a tensor which satisfies

$$\nabla_{(\lambda} K_{\mu_1 \mu_2 \ldots \mu_i)} = 0 \,. \tag{1.4}$$

In the case of a rank one Killing tensor, we have a vector ξ which satisfies $\nabla_{(\lambda}\xi_{\mu)} = 0$. The vector ξ is called a *Killing vector* and it directly represents a symmetry of the metric, specifically that $\mathcal{L}_\xi g = 0$ with $\mathcal{L}$ being the Lie derivative. For higher rank Killing tensors there is no simple geometric interpretation as there is in the rank 1 case, but these tensors still represent a symmetry of the spacetime (see [21] for a discussion). The symmetry is manifested by the existence of a conserved quantity K, defined by

$$K = K_{\mu_1 \mu_2 \ldots \mu_i} P^{\mu_1} P^{\mu_2} \ldots P^{\mu_i} \,. \tag{1.5}$$

This object K will be conserved throughout geodesic motion for a particle freely moving in the spacetime. Note that the identity $\nabla_\lambda g_{\mu\nu} = 0$ guarantees that g will always be a Killing tensor, with a corresponding conserved quantity $-m^2$, the mass of the particle.

To see the importance of a Killing tensor, consider the conserved quantities for a particle in motion in the Kerr geometry. The metric is stationary and axisymmetric, so we have the conserved quantities E and L_z, the energy and z-component of the angular momentum. Another constant of motion is the mass of the particle, but this only gives us three conserved quantities for a four dimensional problem. Carter's discovery of the Killing tensor came with another constant of motion, which allowed for the reduction of the second order equations of motion to first order equations. Those first order equations combined with initial data give a complete solution to the geodesic equation.

More recently, it has been demonstrated that the five dimensional version of the Myers-Perry metrics has many of the "miraculous" features that the Kerr metric has. De Smet noted that this metric is of Petrov type $\underline{22}$, the same as that of a non-rotating black hole, the Tangherlini-Schwarzschild case [22, 23]. Frolov and Stojković demonstrated the separability of the massless scalar field equation [24], and the separability of the Hamilton-Jacobi equation [25]. Later Frolov and this author demonstrated separability of the equations for a stationary cosmic string in the case of a five dimensional Myers-Perry spacetime [26].

In this thesis, we will show a similar "miracle" that occurs for the Kerr-de Sitter metrics found by Gibbons, Lu, Page, and Pope, however in a special case where there are only two independent angular momentum parameters. In the five dimensional case, this assumption carries no loss of generality, but for higher dimensional cases, this is a restriction on the metric to make the calculations easier. This calculation was published in [27] and will be the focus of chapter 3. Chapter 2 will contain an review of the Kerr-de Sitter metrics.

Around fifty years ago, Wheeler and collaborators first introduced the concept of geons [28–30], the idea being that it could be possible for topological structures to exist within a spacetime. These structures could carry charge, and angular momentum, resembling particles when viewed from a large scale. One problem was that geons could produce both electric and magnetic charges from them, however, Sorkin showed that one could remove the magnetic charges by using non-orientable handles [31]. Sorkin also generalized Wheeler's geon into a topological geon [32], allowing for geons to exist on spaces that have the topology of some compact manifold with a single puncture, representing the point at asymptotic infinity. These topological geons did not have the difficulties of instability that that original geons had, they were made stable by the existence of their nontrivial topology. There has also been much research into the quantum mechanics of such objects, such as [33–35], in particular the study of four-dimensional geons motivated the result that all smooth 3-manifolds are allowed as physically reasonable solutions to the Einstein equations [36].

In this thesis we will give a proof that there cannot be any geons in an asymptotically flat three dimensional spacetime, assuming certain energy conditions are met (this proof was also published [37]). The importance of three dimensional gravity was demonstrated by Witten, when twenty years ago he pointed out that three dimensional gravity is exactly soluble [38] (see also [39]). This result is related to the fact that three dimensional classical gravity is "trivial" in a sense, specifically there are no propagating degrees of freedom in three dimensional gravity, so no gravity waves exist. This occurs because the three dimensional Riemann tensor is entirely determined by the Ricci tensor. The proof of nonexistence of geons will be the focus of chapter 3 of this thesis.

One can also generalize vacuum black hole solutions by adding a Maxwell field to the system. The resulting system describes a black hole with charge in addition to the usual mass and angular momentum parameters. The first charged black hole solution was that of Reissner and Nordström [40, 41]. This solution was generalized to include angular momentum shortly after the Kerr solution was published, and it is referred to as the Kerr-Newman metric [42]. This charged, rotating black hole is in many ways the complete unique description of black holes in a four dimensional spacetime [43–45], that is, stationary and asymptotically flat regular electrovac solutions must be equivalent to the Kerr-Newman solution. This is known as the celebrated “No Hair Theorem” for black holes (see [46] for a review). In higher dimensions, the solution space is not as restrictive; there is the black ring solution, examined by Emparan and Reall [47]. This solution is distinct from the higher dimensional Myers-Perry solutions by the fact that the event horizon is of topology $S^2 \times S^1$ rather than S^3, in particular there exist black holes and black rings with the same mass and angular momentum, so simply knowing the mass and angular momentum is not enough to specify the system completely. Non-rotating higher dimensional charged solutions have also been found by Myers and Perry [10], however, there has been no complete solution for a charged rotating black hole metric in an electrovac spacetime. In chapter 4 we will examine some of the attempts to find this solution and prove a new theorem about the form of the electromagnetic potential that such a solution must have.

Throughout the body of this thesis, we will be setting the speed of light and the N-dimensional gravitational constant equal to one. We will use the same curvature sign conventions as Misner, Thorne and Wheeler [48]. The metric we will use will have signature $(-,+,+,+)$, or its generalization to higher and lower dimensions. We will denote N-dimensional spacetime indices with greek letters $(\mu,\nu,... = 0,...,N-1)$, and spatial indices with latin letters from the early part of the alphabet $(a,b,... = 1,...,N-1)$, and for these indices we will obey the convention to sum any repeated indices. We will also make use of indices $(i,j,k,...)$ for miscellaneous purposes, and will explicitly write out sums for these indices when needed.

Chapter 2

Kerr-de Sitter Spacetimes

In this chapter we will review some of the properties of the Kerr-de Sitter metrics [11]. The metrics follow a similar construction to the Myers-Perry metrics [10], in that the spacetime dimensions are divided up into a series of planes of rotation that the black hole rotates in, and each plane of rotation has a spin parameter a_i. The coordinate ϕ_i is the azimuthal angular coordinate associated with the i^{th} plane, the coordinate r is a radial coordinate, and t is the time coordinate. Finally there are n angle cosines μ_i, one for each plane of rotation, and they obey the restriction

$$\sum_{i=1}^{n} \mu_i^2 = 1\,. \tag{2.1}$$

The metrics for N spacetime dimensions split into two cases, depending on whether N is even or odd. When the dimensionality of spacetime is odd, so $N = 2n + 1$, the spatial part divides up evenly into the n planes of rotation, each taking up two spatial dimensions. When the dimensionality of spacetime is even $N = 2n$, the spatial part has one extra dimension after being divided up into the black holes rotational planes (similar to the z-axis in 3+1-dimensional space), and there is an angle cosine μ_n for that extra axis. For notational unity, define a parameter ι which is 0 if N is odd and 1 if N is even.

The ($2n$+1-ι)-dimensional metric is given in the form

$$ds^2 = d\bar{s}^2 + \frac{\alpha}{U}\,(k_\mu\, dx^\mu)^2\,, \tag{2.2}$$

similar to the Kerr-Schild form, but instead of the flat metric $d\bar{s}^2$ is the de Sitter metric,

$$d\bar{s}^2 = -W\,(1-\lambda r^2)\,dt^2 + F\,dr^2 + \sum_{i=1}^{n} \frac{r^2+a_i^2}{1+\lambda a_i^2}\,d\mu_i^2 + \sum_{i=1}^{n-\iota} \frac{r^2+a_i^2}{1+\lambda a_i^2}\,\mu_i^2\,d\phi_i^2$$

$$+\frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\,\mu_i\,d\mu_i}{1+\lambda a_i^2}\right)^2 . \tag{2.3}$$

The functions W, F and, U are given by

$$W=\sum_{i=1}^{n}\frac{\mu_i^2}{1+\lambda a_i^2}\,, \qquad F=\frac{r^2}{1-\lambda\,r^2}\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\,, \tag{2.4}$$

$$U=r^{\iota}\prod_{j=1}^{n-\iota}(r^2+a_j^2)\,\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\,, \tag{2.5}$$

and the null one-form k_μ is

$$k_\mu dx^\mu = W dt + F dr - \sum_{i=1}^{n-\iota}\frac{a_i\mu_i^2}{1+\lambda a_i^2}d\phi_i\,. \tag{2.6}$$

Note that if there is an odd number of spatial dimensions, the final spin parameter a_n is zero, and there is no coordinate ϕ_n, because there is no n^{th} plane of rotation, it is just a single axis. Finally, the parameter α is proportional to the mass of the black hole.

The metric solves the vacuum Einstein equations with cosmological constant

$$R_{\mu\nu}=(N-1)\,\lambda\,g_{\mu\nu}\,, \tag{2.7}$$

note that the constant λ is not the usual cosmological constant, but is scaled by the dimensions (the usual cosmological constant Λ is defined so that the vacuum Einstein equations read $R_{\mu\nu}=\frac{2}{N-2}\Lambda\,g_{\mu\nu}$).

We will also need the vector form of the one-form k, it is given as

$$k^\mu\,\partial_\mu = -\frac{1}{1-\lambda r^2}\,\partial_t+\partial_r-\sum_{i=1}^{n-\iota}\frac{a_i}{r^2+a_i^2}\,\partial_{\phi_i}\,. \tag{2.8}$$

For our purposes, it will be convenient to switch to Boyer-Lindquist coordinates, which are characterized by the metric not having any cross terms involving the differential dr. The Boyer-Lindquist coordinates are obtained by the following coordinate transformation from the (t,ϕ_i) coordinates to the (τ,φ_i) coordinates:

$$dt = d\tau + \frac{\alpha}{(1-\lambda r^2)(V-\alpha)}dr\,,$$

$$d\phi_i = d\varphi_i - \lambda a_i d\tau + \frac{\alpha\, a_i}{(r^2 + a_i^2)(V - \alpha)} dr\,, \tag{2.9}$$

where the function V is given by

$$V = \frac{U}{F} = r^{\iota-2}(1 - \lambda r^2) \prod_{i=1}^{n-\iota} (r^2 + a_i^2)\,. \tag{2.10}$$

In these coordinates, the metric takes the form

$$\begin{aligned} ds^2 = &-W(1-\lambda r^2)\, d\tau^2 + \frac{U}{V-\alpha} dr^2 + \frac{\alpha}{U}\left(d\tau - \sum_{i=1}^{n-\iota} \frac{a_i \mu_i^2\, d\varphi_i}{1+\lambda a_i^2}\right)^2 \\ &+ \sum_{i=1}^{n-\iota} \frac{r^2 + a_i^2}{1+\lambda a_i^2} \mu_i^2\, (d\varphi_i - \lambda\, a_i\, d\tau)^2 + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1+\lambda a_i^2}\, d\mu_i^2 \\ &+ \frac{\lambda}{W(1-\lambda r^2)} \left(\sum_{i=1}^{n} \frac{(r^2 + a_i^2)\mu_i\, d\mu_i}{1+\lambda a_i^2} \right)^2, \end{aligned} \tag{2.11}$$

2.1 Inverting the Metric

In later calculations we will need the inverse of the Kerr-de Sitter metric, so we will spend this next section finding those terms. To do so we will make use of the fact that the metric is in Kerr-Schild form as in equation (2.2), so the inverse metric can be expressed as

$$g^{\mu\nu} = \bar{\eta}^{\mu\nu} - \frac{\alpha}{U} k^\mu k^\nu\,, \tag{2.12}$$

with $\bar{\eta}^{\mu\nu}$ being the inverse of the de Sitter metric. Note that the metric is block diagonal, the $\{r,\, \tau,\, \phi_i\}$-sector is decoupled from the $\{\mu_i\}$-sector, so each of these sectors can be inverted separately, and since k has no μ components, we can consider (2.12) to hold in the $\{r,\, \tau,\, \phi_i\}$-sector as well. The coordinate change (2.9) can be used on the inverse metric to obtain the following components of $g^{\mu\nu}$ in Boyer-Lindquist coordinates

$$g^{\tau r} = g^{\varphi_i r} = 0\,,$$

$$g^{rr} = \frac{V - \alpha}{U}\,,$$

$$g^{\tau\tau} = Q - \frac{\alpha^2}{U(1-\lambda r^2)^2(V-\alpha)}\,,$$

$$g^{\tau\varphi_i} = \lambda a_i Q - \frac{\alpha^2 a_i(1+\lambda a_i^2)}{U(1-\lambda r^2)^2(V-\alpha)(r^2+a_i^2)} - \frac{\alpha}{U}\frac{a_i}{(1-\lambda r^2)(r^2+a_i^2)}\,,$$

$$g^{\varphi_i\varphi_j} = \frac{(1+\lambda a_i^2)}{(r^2+a_i^2)\mu_i^2}\delta^{ij} + \lambda^2 a_i a_j Q + \frac{Q^{ij}}{U}$$

$$+\frac{\alpha^2 a_i a_j(1+\lambda a_i^2)(1+\lambda a_j^2)}{U(1-\lambda r^2)^2(V-\alpha)(r^2+a_i^2)(r^2+a_j^2)}\,, \tag{2.13}$$

and the objects Q and Q^{ij} are defined to be

$$Q = -\frac{1}{W(1-\lambda r^2)} - \frac{\alpha}{U}\frac{1}{(1-\lambda r^2)^2}\,, \tag{2.14}$$

$$Q^{ij} = \frac{-\alpha^2\lambda a_i a_j[(1+\lambda a_j^2)(r^2+a_i^2)+(1+\lambda a_i^2)(r^2+a_j^2)]}{(1-\lambda r^2)^2(V-\alpha)(r^2+a_i^2)(r^2+a_j^2)} - \alpha\frac{a_i a_j}{(r^2+a_i^2)(r^2+a_j^2)}$$

$$-\frac{\alpha\lambda a_i a_j}{(1-\lambda r^2)}\left[\frac{1}{(r^2+a_i^2)}+\frac{1}{(r^2+a_j^2)}\right] - \frac{\alpha^2 a_i a_j[(1+\lambda a_i^2)+(1+\lambda a_j^2)]}{(1-\lambda r^2)^2(V-\alpha)(r^2+a_i^2)(r^2+a_j^2)}\,. \tag{2.15}$$

Inverting the μ sector of the metric is much more difficult, since the μ's do not actually represent independent coordinates, but are restricted by the relation (2.1), it is necessary to substitute in that restriction before inverting the metric. Unfortunately this has the effect of complicating the natural symmetry of the metric, making is very difficult to invert. In a pervious paper [49], the calculation was simplified by assuming that all spin parameters of the black hole were identical, $a_i = a$, thus allowing easy inversion of the rest of the metric. This assumption meant that the calculation only worked for an odd number of spacetime dimensions however, since in an even number of dimensions there is an extra spin parameter a_n which is always zero.

For our purposes, we will work in a somewhat more general case by assuming that the spin parameters can take on at most two possible values, a and b. In an even number of spacetime dimensions, at least one of the values must be zero, so this we will assume $b = 0$ there, and in an odd number of spacetime dimensions the parameters a and b can take on any values. Specifically, we will be assuming

$$a_i = a \quad \text{for} \quad i = 1, ..., p \quad , \qquad a_{j+p} = b \quad \text{for} \quad j = 1, ..., q\,, \tag{2.16}$$

with $p+q=n$.

To deal with the restriction (2.1), we will first break the μ_i coordinates up into the two groups corresponding to the separate rotation parameters by defining a new coordinate θ as follows:

$$\mu_i = \sigma_i \sin\theta \quad \text{for} \quad i=1,...,p\,, \qquad \mu_{j+p} = \nu_j \cos\theta \quad \text{for} \quad j=1,...,q\,, \tag{2.17}$$

and σ_i and ν_i must satisfy the constraints

$$\sum_{i=1}^{p} \sigma_i^2 = 1\,, \qquad \sum_{j=1}^{q} \nu_j^2 = 1\,. \tag{2.18}$$

Next it is sensible to introduce spherical coordinates α_i and β_i to satisfy the constraints (2.18),

$$\sigma_i = \left(\prod_{k=1}^{p-i} \sin\alpha_k\right) \cos\alpha_{p-i+1}\,,$$

$$\nu_i = \left(\prod_{k=1}^{q-i} \sin\beta_k\right) \cos\beta_{q-i+1}\,, \tag{2.19}$$

with the definition that $\alpha_p = 0$ and $\beta_q = 0$ and the understanding that for any product with a smaller upper bound than a lower one, the product is automatically 1.

With these coordinates, the μ part of the metric (2.11) can be written as

$$ds_\mu^2 = \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{r^2+a^2}{\Sigma_a} \sin^2\theta \sum_{i=1}^{p-1} \left(\prod_{k=1}^{i-1} \sin^2\alpha_k\right) d\alpha_i^2$$

$$+ \frac{r^2+b^2}{\Sigma_b} \cos^2\theta \sum_{i=1}^{q-1} \left(\prod_{k=1}^{i-1} \sin^2\beta_k\right) d\beta_i^2\,, \tag{2.20}$$

where we have defined

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,,$$

$$\Delta_\theta = 1 + \lambda a^2 \cos^2\theta + \lambda b^2 \sin^2\theta\,,$$

$$\Sigma_a = 1 + \lambda a^2\,,$$

$$\Sigma_b = 1 + \lambda b^2 \,. \tag{2.21}$$

We can see that this metric is diagonal, so the inverse of the metric is given by

$$\begin{aligned}
g^{\theta\theta} &= \frac{\Delta_\theta}{\rho^2}\,, \\
g^{\alpha_i\alpha_j} &= \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\alpha_k\right)}\delta_{ij}\,, \qquad i,j=1,...,p\,, \\
g^{\beta_i\beta_j} &= \frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\frac{1}{\left(\prod_{k=1}^{i-1}\sin^2\beta_k\right)}\delta_{ij}\,, \qquad i,j=1,...,q\,.
\end{aligned} \tag{2.22}$$

This finishes the inverting of the Kerr-de Sitter metrics, lastly we will note a few identities that will be useful in the coming calculations

$$\begin{aligned}
U &= \rho^2 Z\,, \\
W &= \frac{\Delta_\theta}{\Sigma_a\Sigma_b}\,, \\
Q &= \frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Sigma_r} - \frac{\alpha}{\rho^2 Z\Sigma_r^2(V-\alpha)}\,,
\end{aligned} \tag{2.23}$$

where we have defined

$$\begin{aligned}
Z &= r^\iota(r^2+a^2)^{p-1}(r^2+b^2)^{q-1-\iota}\,, \\
\Sigma_r &= 1-\lambda r^2\,.
\end{aligned} \tag{2.24}$$

Chapter 3

Separation of Variables in Kerr-de Sitter Spacetimes

3.1 Introduction

In this chapter we will study the Hamilton-Jacobi and massive Klein-Gordon equations in the Kerr-de Sitter black hole backgrounds of the previous chapter. We will demonstrate how a complete separation can be carried out in the case when there are two sets of equal black hole rotation parameters. From this, we will derive first order equations of motion for particles and examine some of their properties. Finally, we will explicitly construct a Killing tensor that exists in these metrics and is related to the separability. The results in this chapter were also published in [27].

3.2 The Hamilton-Jacobi Equation

The Hamilton-Jacobi equation for a particle undergoing motion in a curved background with metric $g_{\mu\nu}$ is given by

$$-\frac{\partial S}{\partial l} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}\,, \tag{3.1}$$

where S is the action associated with the particle, l is an affine parameter along the worldline, and H is the Hamiltonian.

We will be considering the motion of a particle in the Kerr-de Sitter metrics and we will use the assumption of only having two different spin parameters. We will see how this assumption is made to enlarge the isometry group of the spacetime. Planes 1 through p will have spin parameter a and planes $p+1$ through $p+q$ will have spin parameter b.

We will attempt an additive separation of variables as follows, let us assume S is given by

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{p} \Phi_i \varphi_i + \sum_{i=1}^{q} \Psi_i \varphi_{p+i} + S_r(r) + S_\theta(\theta) + \sum_{i=1}^{p-1} S_{\alpha_i}(\alpha_i) + \sum_{i=1}^{q-1} S_{\beta_i}(\beta_i)\,. \tag{3.2}$$

Since τ and φ_i are cyclic coordinates, we have used their conserved conjugate momenta E and Φ_i or Ψ_i. If the number of spacetime dimensions is even, then φ_{p+q} is not a coordinate, so we will adopt the convention that $\Psi_q = 0$ in that case.

Using the form of the inverse metric (2.13), (2.22) we can substitute (3.2) into (3.1) to get

$$\begin{aligned}
-\rho^2 m^2 &= \left[\frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} - \frac{\alpha}{Z\Sigma_r} - \frac{\alpha^2}{Z\Sigma_r^2}\right] E^2 \\
&+2\left[\frac{a\Sigma_a\Sigma_b}{\Delta_\theta} - \frac{a\Sigma_a\Sigma_b}{\Sigma_r} - \frac{\alpha\lambda a}{Z\Sigma_r^2} - \frac{\alpha^2 a\Sigma_a - \alpha a\Sigma_r(V-\alpha)}{Z\Sigma_r^2(V-\alpha)(r^2+a^2)}\right]\sum_{i=1}^{p}(-E)\Phi_i \\
&+2\left[\frac{b\Sigma_a\Sigma_b}{\Delta_\theta} - \frac{b\Sigma_a\Sigma_b}{\Sigma_r} - \frac{\alpha\lambda b}{Z\Sigma_r^2} - \frac{\alpha^2 b\Sigma_b - \alpha b\Sigma_r(V-\alpha)}{Z\Sigma_r^2(V-\alpha)(r^2+b^2)}\right]\sum_{i=1}^{q}(-E)\Psi_i \\
&+\sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2 a^2\left(\frac{\Sigma_a\Sigma_b}{\Delta_\theta} - \frac{Z\Sigma_a\Sigma_b - \alpha\lambda}{Z\lambda\Sigma_r}\right) + \frac{\alpha^2 a^2\Sigma_a^2}{Z(V-\alpha)(r^2+a^2)^2} + \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j \\
&+\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2 b^2\left(\frac{\Sigma_a\Sigma_b}{\Delta_\theta} - \frac{Z\Sigma_a\Sigma_b - \alpha\lambda}{Z\lambda\Sigma_r}\right)\right. \\
&\left. + \frac{\alpha^2 b^2\Sigma_a^2}{Z(V-\alpha)(r^2+b^2)^2} + \frac{Q^{(i+p)(j+p)}}{Z}\right]\Psi_i\Psi_j \\
&+2\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\Delta_\theta} - \frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} - \frac{\alpha}{Z\Sigma_r}\right)\right. \\
&\left. + \frac{\alpha^2 ab\Sigma_a\Sigma_b}{Z(V-\alpha)(r^2+a^2)(r^2+b^2)} + \frac{Q^{i(j+p)}}{Z}\right]\Phi_i\Psi_j\,. \\
&+\Delta_\theta\left[\frac{dS_\theta(\theta)}{d\theta}\right]^2 + \frac{V-\alpha}{Z}\left[\frac{dS_r(r)}{dr}\right]^2
\end{aligned}$$

$$+\frac{\rho^2\Sigma_a}{(r^2+a^2)\sin^2\theta}\sum_{i=1}^{p}\frac{\Phi_i^2}{\sigma_i^2}+\sum_{i=1}^{p-1}\frac{\rho^2\Sigma_a}{(r^2+a^2)\sin^2\theta\prod_{k=1}^{i-1}\sin^2\alpha_k}\left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2$$

$$+\frac{\rho^2\Sigma_b}{(r^2+b^2)\cos^2\theta}\sum_{i=1}^{p}\frac{\Psi_i^2}{\nu_i^2}+\sum_{i=1}^{q-1}\frac{\rho^2\Sigma_b}{(r^2+b^2)\cos^2\theta\prod_{k=1}^{i-1}\sin^2\beta_k}\left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2 . \quad (3.3)$$

Note that the quantities σ_i and ν_i are not coordinates, but functions of the angular coordinates α_i and β_i defined by (2.19). We can see that all of the dependence on the α_i coordinates is contained in the second last line, and all of the dependence on the β_i coordinates is contained in the final line. This tells us we can separate out the α and β parts of those terms as constants

$$J_1^2=\sum_{i=1}^{p}\left[\frac{\Phi_i^2}{\sigma_i^2}+\frac{1}{\prod_{k=1}^{i-1}\sin^2\alpha_k}\left(\frac{dS_{\alpha_i}}{d\alpha_i}\right)^2\right],$$

$$L_1^2=\sum_{i=1}^{q}\left[\frac{\Psi_i^2}{\nu_i^2}+\frac{1}{\prod_{k=1}^{i-1}\sin^2\beta_k}\left(\frac{dS_{\beta_i}}{d\beta_i}\right)^2\right], \quad (3.4)$$

with J_1^2 and L_1^2 being separation constants, and we may now replace the final two lines of (3.3) with $\frac{\rho^2\Sigma_a}{(r^2+a^2)\sin^2\theta}J_1^2+\frac{\rho^2\Sigma_b}{(r^2+b^2)\cos^2\theta}L_1^2$.

To separate the rest of (3.3), we can isolate all the θ dependence away from the r dependence. Recall that the functions Z, V, and Q^{ij} only depend on r and have no other coordinate dependence. After r and θ have been separated, one arrives at the equations

$$K=m^2r^2-\left[\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r}+\frac{\alpha}{Z\Sigma_r}+\frac{\alpha^2}{Z\Sigma_r^2}\right]E^2$$

$$+2\left[\frac{a\Sigma_a\Sigma_b}{\Sigma_r}+\frac{\alpha\lambda a}{Z\Sigma_r^2}+\frac{\alpha^2a\Sigma_a}{Z\Sigma_r^2(V-\alpha)(r^2+a^2)}+\frac{\alpha a}{Z\Sigma_r(r^2+a^2)}\right]\sum_{i=1}^{p}(-E)\Phi_i$$

$$+2\left[\frac{b\Sigma_a\Sigma_b}{\Sigma_r}+\frac{\alpha\lambda b}{Z\Sigma_r^2}+\frac{\alpha^2b\Sigma_b}{Z\Sigma_r^2(V-\alpha)(r^2+b^2)}+\frac{\alpha b}{Z\Sigma_r(r^2+b^2)}\right]\sum_{i=1}^{q}(-E)\Psi_i$$

$$+\sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r}+\frac{\alpha}{Z\Sigma_r}\right)-\frac{\alpha^2a^2\Sigma_a^2}{Z(V-\alpha)(r^2+a^2)^2}-\frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j$$

$$+\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2 b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r}+\frac{\alpha}{Z\Sigma_r}\right)-\frac{\alpha^2 b^2\Sigma_a^2}{Z(V-\alpha)(r^2+b^2)^2}-\frac{Q^{(i+p)(j+p)}}{Z}\right]\Psi_i\Psi_j$$

$$+2\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r}+\frac{\alpha}{Z\Sigma_r}\right)\right.$$

$$\left.-\frac{\alpha^2 ab\Sigma_a\Sigma_b}{Z(V-\alpha)(r^2+a^2)(r^2+b^2)}-\frac{Q^{i(j+p)}}{Z}\right]\Phi_i\Psi_j$$

$$+\frac{V-\alpha}{Z}\left[\frac{dS_r(r)}{dr}\right]^2+\frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2+\frac{\Sigma_b(r^2+a^2)}{r^2+b^2}L_1^2\,, \tag{3.5}$$

and

$$-K=m^2a^2\cos^2\theta+m^2b^2\sin^2\theta+\Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2+\Sigma_a\cot^2\theta J_1^2+\Sigma_b\tan^2\theta L_1^2$$

$$+\frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2-2\sum_{i=1}^{p}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i-2\sum_{i=1}^{q}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i+\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j$$

$$+\sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j+2\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j\,, \tag{3.6}$$

with K being the separation constant.

Now the Hamilton-Jacobi equation (3.3) has been separated into its r, θ, α_i, and β_i sectors, and all that remains is to separate out the individual α_i and β_i coordinates in (3.4). The metric parameters α, λ, a and b do not appear in that equation, so the result is exactly what you would get for the spherical coordinates in flat spacetime.

It is possible to separate the functions $S_{\alpha_i}(\alpha_i)$ from eachother, one at a time getting the following inductive form

$$\left(\frac{dS_{\alpha_k}}{d\alpha_k}\right)^2=J_k^2-\frac{J_{k+1}^2}{\sin^2\alpha_k}-\frac{\Phi_{p-k+1}^2}{\cos^2\alpha_k}\qquad\text{for}\quad k=1,...,p-2\,,$$

$$\left(\frac{dS_{\alpha_{p-1}}}{d\alpha_{p-1}}\right)^2=J_{p-1}^2-\frac{\Phi_1^2}{\sin^2\alpha_{p-1}}-\frac{\Phi_2^2}{\cos^2\alpha_{p-1}}\,, \tag{3.7}$$

and similarly for β,

$$\left(\frac{dS_{\beta_k}}{d\beta_k}\right)^2 = L_k^2 - \frac{L_{k+1}^2}{\sin^2\beta_k} - \frac{\Psi_{q-k+1}^2}{\cos^2\beta_k} \qquad \text{for} \quad k = 1, ..., q-2\,,$$

$$\left(\frac{dS_{\beta_{q-1}}}{d\beta_{q-1}}\right)^2 = L_{q-1}^2 - \frac{\Psi_1^2}{\sin^2\beta_{q-1}} - \frac{\Psi_2^2}{\cos^2\beta_{q-1}}\,. \tag{3.8}$$

This completes the separation of the Hamilton-Jacobi equation for a particle moving in the Kerr-de Sitter spacetime with two unequal rotation parameters.

3.3 Equations of Motion

Using the equations from the last section, the action (3.2) can be written in the form

$$S = \frac{1}{2}m^2 l - E\tau + \sum_{i=1}^{p}\Phi_i\varphi_i + \sum_{i=1}^{q}\Psi_i\varphi_{p+i} + \int^r \sqrt{R(r')}dr' + \int^\theta \sqrt{\Theta(\theta')}d\theta'$$
$$+\sum_{i=1}^{p-1}\int^{\alpha_i}\sqrt{A_i(\alpha_i')}d\alpha_i' + \sum_{i=1}^{q-1}\int^{\beta_i}\sqrt{B_i(\beta_i')}d\beta_i'\,, \tag{3.9}$$

with $R(r)$ being given by (3.5)

$$\frac{V-\alpha}{Z}R(r) = -m^2r^2 + \left[\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r} + \frac{\alpha^2}{Z\Sigma_r^2}\right]E^2$$
$$-2\left[\frac{a\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda a}{Z\Sigma_r^2} + \frac{\alpha^2 a\Sigma_a}{Z\Sigma_r^2(V-\alpha)(r^2+a^2)} + \frac{\alpha a}{Z\Sigma_r(r^2+a^2)}\right]\sum_{i=1}^{p}(-E)\Phi_i$$
$$-2\left[\frac{b\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda b}{Z\Sigma_r^2} + \frac{\alpha^2 b\Sigma_b}{Z\Sigma_r^2(V-\alpha)(r^2+b^2)} + \frac{\alpha b}{Z\Sigma_r(r^2+b^2)}\right]\sum_{i=1}^{q}(-E)\Psi_i$$
$$-\sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2a^2\Sigma_a^2}{Z(V-\alpha)(r^2+a^2)^2} - \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j$$
$$-\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2b^2\Sigma_a^2}{Z(V-\alpha)(r^2+b^2)^2} - \frac{Q^{(i+p)(j+p)}}{Z}\right]\Psi_i\Psi_j$$

$$-2\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r}+\frac{\alpha}{Z\Sigma_r}\right)\right.$$

$$\left.-\frac{\alpha^2 ab\Sigma_a\Sigma_b}{Z(V-\alpha)(r^2+a^2)(r^2+b^2)}-\frac{Q^{i(j+p)}}{Z}\right]\Phi_i\Psi_j$$

$$-\frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2-\frac{\Sigma_b(r^2+a^2)}{r^2+b^2}L_1^2-K\,, \tag{3.10}$$

$\Theta(\theta)$ being given by (3.6)

$$\Delta_\theta\Theta(\theta)=-m^2a^2\cos^2\theta-m^2b^2\sin^2\theta-\Sigma_a\cot^2\theta J_1^2-\Sigma_b\tan^2\theta L_1^2$$

$$-\frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2+2\sum_{i=1}^{p}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i+2\sum_{i=1}^{q}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i-\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j$$

$$-\sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j-2\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j-K\,, \tag{3.11}$$

and $A_i(\alpha_i)$ and $B_i(\beta_i)$ are given by (3.7) and (3.8)

$$A_i(\alpha_i)=J_i^2-\frac{J_{i+1}^2}{\sin^2\alpha_i}-\frac{\Phi_{p-i+1}^2}{\cos^2\alpha_i}\qquad\text{for}\quad i=1,...,p-2\,,$$

$$A_{p-1}(\alpha_{p-1})=J_{p-1}^2-\frac{\Phi_1^2}{\sin^2\alpha_{p-1}}-\frac{\Phi_2^2}{\cos^2\alpha_{p-1}}\,,$$

$$B_i(\beta_i)=L_i^2-\frac{L_{i+1}^2}{\sin^2\beta_i}-\frac{\Psi_{q-i+1}^2}{\cos^2\beta_i}\qquad\text{for}\quad i=1,...,q-2\,,$$

$$B_{q-1}(\beta_{q-1})=L_{q-1}^2-\frac{\Psi_1^2}{\sin^2\beta_{q-1}}-\frac{\Psi_2^2}{\cos^2\beta_{q-1}}\,. \tag{3.12}$$

To obtain the equations of motion from this action, we differentiate S with respect to the parameters m^2, K, E, J_i^2, L_i^2, Φ_i, Ψ_i and set these derivatives equal to constants. However, these new constants can be set to zero by freedom of choice of the origin of coordinates or by changing constants of integration. The equations that one obtains are of the form

$$\frac{\partial S}{\partial m^2}=0\Rightarrow l=\int\frac{Zr^2}{V-\alpha}\frac{dr}{\sqrt{R(r)}}+\int\frac{(a^2\cos^2\theta+b^2\sin^2\theta)d\theta}{\Delta_\theta\sqrt{\Theta(\theta)}}\,,$$

$$
\begin{aligned}
\frac{\partial S}{\partial K} = 0 \Rightarrow \int \frac{d\theta}{\Delta_\theta \sqrt{\Theta(\theta)}} &= \int \frac{Z}{V-\alpha} \frac{dr}{\sqrt{R(r)}}\,, \\
\frac{\partial S}{\partial J_1^2} = 0 \Rightarrow \int \frac{d\alpha_1}{\sqrt{A_1(\alpha_1)}} &= \int \frac{Z}{V-\alpha} \frac{\Sigma_a (r^2+b^2)}{r^2+a^2} \frac{dr}{\sqrt{R(r)}} + \int \frac{\Sigma_a \cot^2\theta d\theta}{\Delta_\theta \sqrt{\Theta(\theta)}}\,, \\
\frac{\partial S}{\partial J_i^2} = 0 \Rightarrow \int \frac{d\alpha_i}{\sqrt{A_i(\alpha_i)}} &= \int \frac{1}{\sin^2 \alpha_{i-1}} \frac{d\alpha_{i-1}}{\sqrt{A_{i-1}}}\,, \qquad i = 2, ..., p-2\,, \\
\frac{\partial S}{\partial L_1^2} = 0 \Rightarrow \int \frac{d\beta_1}{\sqrt{B_1(\beta_1)}} &= \int \frac{Z}{V-\alpha} \frac{\Sigma_b (r^2+a^2)}{r^2+b^2} \frac{dr}{\sqrt{R(r)}} + \int \frac{\Sigma_b \tan^2\theta d\theta}{\Delta_\theta \sqrt{\Theta(\theta)}}\,, \\
\frac{\partial S}{\partial L_i^2} = 0 \Rightarrow \int \frac{d\beta_i}{\sqrt{B_i(\beta_i)}} &= \int \frac{1}{\sin^2 \beta_{i-1}} \frac{d\beta_{i-1}}{\sqrt{B_{i-1}(\beta_{i-1})}}\,, \qquad i = 2, ..., q-2\,.
\end{aligned}
\tag{3.13}
$$

There are n more equations one can obtain for the azimuthal coordinates φ_i, obtained by differentiating S with respect to the angular momenta Φ_i and Ψ_i. Also there is an equation involving the time coordinate τ that can be found by differentiating S with respect to E. These additional equations are lengthy and not very illuminating however, so we will not list them, but they can be found very simply by following this procedure.

Rather than having equations of motion in integral form, it is possible to write them in differential form by differentiating (3.13) with respect to the affine parameter l

$$
\begin{aligned}
\rho^2 \frac{dr}{dl} &= \frac{V-\alpha}{Z} \sqrt{R(r)}\,, \\
\rho^2 \frac{d\theta}{dl} &= \Delta_\theta \sqrt{\Theta(\theta)}\,, \\
\frac{(r^2+a^2)}{\Sigma_a} \frac{d\alpha_k}{dl} &= \frac{\sqrt{A_k(\alpha_k)}}{\sin^2\theta \prod_{i=1}^{k-1} \sin^2 \alpha_i}\,, \qquad k = 1, ..., p-1\,, \\
\frac{(r^2+b^2)}{\Sigma_b} \frac{d\beta_k}{dl} &= \frac{\sqrt{B_k(\beta_k)}}{\cos^2\theta \prod_{i=1}^{k-1} \sin^2 \beta_i}\,, \qquad k = 1, ..., q-1\,.
\end{aligned}
\tag{3.14}
$$

Naturally, there are similar differential equations involving τ and φ_i that one can obtain from the integral equations of motion that we did not list.

3.3.1 The Radial Equation

Next we will analyse the radial equation of motion by considering motion in the black hole exterior. The allowed regions of particle motion necessarily have a positive value for the quantity $R(r)$, owing to equation (3.14).

When $\lambda = 0$, at large distances the dominant contribution to $R(r)$ is $E^2 - m^2$, so we can say that for $E^2 > m^2$ we can have unbounded orbits, but if $E^2 < m^2$ only bound orbits will be possible. If λ is nonzero then the dominant term at large r is $\frac{m^2}{\lambda r^2}$, so if $\lambda > 0$ unbound orbits will be possible, while if $\lambda < 0$ then there will only be bound orbits.

One can further analyse radial motion by decomposing $R(r)$ as a quadratic in E as

$$R(r) = \gamma E^2 - 2\zeta E + \kappa, \tag{3.15}$$

where

$$\gamma = \frac{Z}{V-\alpha}\left[\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r} + \frac{\alpha^2}{Z\Sigma_r^2}\right],$$

$$\zeta = \frac{-Z}{V-\alpha}\left[\frac{a\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda a}{Z\Sigma_r^2} + \frac{\alpha^2 a\Sigma_a}{Z\Sigma_r^2(V-\alpha)(r^2+a^2)} + \frac{\alpha a}{Z\Sigma_r(r^2+a^2)}\right]\sum_{i=1}^{p}\Phi_i$$
$$-\frac{Z}{V-\alpha}\left[\frac{b\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda b}{Z\Sigma_r^2} + \frac{\alpha^2 b\Sigma_b}{Z\Sigma_r^2(V-\alpha)(r^2+b^2)} + \frac{\alpha b}{Z\Sigma_r(r^2+b^2)}\right]\sum_{j=1}^{q}\Psi_j\,,$$

$$\kappa = \Bigg\{-\sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2a^2\Sigma_a^2}{Z(V-\alpha)(r^2+a^2)^2} - \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j$$
$$-\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2b^2\Sigma_a^2}{Z(V-\alpha)(r^2+b^2)^2} - \frac{Q^{(i+p)(j+p)}}{Z}\right]\Psi_i\Psi_j$$
$$-2\sum_{i=1}^{p}\sum_{j=1}^{q}\Bigg[\lambda^2ab\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right)$$
$$-\frac{\alpha^2ab\Sigma_a\Sigma_b}{Z(V-\alpha)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+p)}}{Z}\Bigg]\Phi_i\Psi_j$$
$$-\frac{\Sigma_a(r^2+b^2)}{r^2+a^2}J_1^2 - \frac{\Sigma_b(r^2+a^2)}{r^2+a^2}L_1^2 + K - m^2r^2\Bigg\}\frac{Z}{V-\alpha}\,. \tag{3.16}$$

Now we can see that the turning points for radial motion will occur at $E = V_\pm$, where

$$V_\pm = \frac{\zeta \pm \sqrt{\zeta^2 - \gamma\kappa}}{\gamma}\,. \tag{3.17}$$

3.3.2 The Angular Equation

One can also examine the angular parts of the equations of motion (3.14) to study motion of the angular coordinates α_i and β_i. Consider a particle which has a motion such that α_i is a constant for each value of i, this means that the motion is described by the equations

$$A_i(\alpha_i = \alpha_{i0}) = \frac{dA_i}{d\alpha_i}(\alpha_i = \alpha_{i0}) = 0\,, \qquad i = 1, ..., p-1\,, \tag{3.18}$$

with α_{i0} being that constant value of α_i.

These equations can be solved to find

$$\frac{J_{i+1}^2}{\sin^4 \alpha_{i0}} = \frac{\Phi_{p-i-1}^2}{\cos^4 \alpha_{i0}}\,,$$

$$J_i^2 = \frac{J_{i+1}^2}{\sin^2 \alpha_{i0}} + \frac{\Phi_{p-i+1}^2}{\cos^2 \alpha_{i0}}\,, \qquad i = 1, ..., p-1\,. \tag{3.19}$$

We can see that if $\alpha_{i0} = 0$ then $J_{i+1} = 0$, and if $\alpha_{i0} = \pi/2$ then $\Phi_{p-i+1} = 0$.

Examining A_i in the general case the subspace $\alpha_i = 0$ can only be reached if $J_{i+1} = 0$, and $\alpha_i = \pi/2$ can only be reached if $\Phi_{p-i+1} = 0$. The motion will be completely in the subspace $\alpha_i = 0$ only if $J_i^2 = \Phi_{p-i+1}^2$, and the motion will be completely in the subspace $\alpha_i = \pi/2$ only if $J_i^2 = J_{i+1}^2$.

Naturally, equivalent results hold for the β_i equations as well.

3.4 The Klein-Gordon Equation

Consider a scalar field χ with mass m in a gravitational background with metric g with dynamics governed by the action

$$S[\chi] = -\frac{1}{2}\int \left((\nabla\chi)^2 + \eta R \chi^2 + m^2\chi^2\right)\sqrt{-g}\; d^N x\,, \tag{3.20}$$

with a curvature-dependent coupling with constant η. In the Kerr-de Sitter metrics we are interested in, $R = N(N-1)\lambda$ is a constant, so we may simply replace m^2 with $m^2 - \eta R$ to remove the term proportional to η. Variation of this action leads to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\chi) = m^2\chi\,. \tag{3.21}$$

Carter [16] has noted that separation of the Klein-Gordon equation is often connected with separation of the Hamilton-Jacobi equation. Since we have seen that the Hamilton-Jacobi equation undergoes separation in the case that there are only two distinct spin parameters, it is natural to assume that again for the analysis here.

First we must calculate the determinant of the Kerr-de Sitter metric. In Boyer-Lindquist coordinates, and using the θ, α_i and β_i coordinates from (2.17)-(2.19), the determinant can be expressed as

$$g = -\frac{RTAB\rho^4}{\Sigma_a^{2p}\Sigma_b^{2q-2\iota}}\,, \tag{3.22}$$

where

$$\begin{aligned} R &= r^2(r^2+a^2)^{2p-2}(r^2+b^2)^{2q-2-\iota}\,,\\ T &= \sin^{4p-2}\theta\cos^{4q-2-2\iota}\theta\,,\\ A &= \prod_{i=1}^{p-1}\sin^{4p-4i-2}\alpha_i\cos^2\alpha_i\,,\\ B &= \prod_{i=1}^{q-1}\sin^{4q-4i-2}\beta_i\cos^2\beta_i\cos^{-2\iota}\beta_1\,. \end{aligned} \tag{3.23}$$

Note that R and T are functions of r and θ only, while A and B depend only on the variables α_i and β_i.

Using this and the form of the inverse metric calculated earlier, the Klein-Gordon equation becomes:

$$m^2\chi = \frac{1}{\rho^2\sqrt{R}}\partial_r\left(\sqrt{R}\frac{V-\alpha}{Z}\partial_r\chi\right) + \frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\sum_{i=1}^{p}\frac{1}{\sigma_i^2}\partial^2_{\varphi_i}\chi$$

$$+\frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\sum_{i=1}^{q}\frac{1}{\nu_i^2}\partial^2_{\varphi_{i+p}}\chi + \frac{1}{\rho^2\sqrt{T}}\partial_\theta\left(\sqrt{T}\Delta_\theta\partial_\theta\chi\right)$$

$$
\begin{aligned}
&+\frac{\Sigma_a}{(r^2+a^2)\sin^2\theta}\left[\sum_{i=1}^{p-1}\frac{1}{\sqrt{A}}\partial_{\alpha_i}\left(\frac{\sqrt{A}}{\prod_{k=1}^{i-1}\sin^2\alpha_k}\partial_{\alpha_i}\chi\right)\right] \\
&+\frac{\Sigma_b}{(r^2+b^2)\cos^2\theta}\left[\sum_{i=1}^{q-1}\frac{1}{\sqrt{B}}\partial_{\beta_i}\left(\frac{\sqrt{B}}{\prod_{k=1}^{i-1}\sin^2\beta_k}\partial_{\beta_i}\chi\right)\right] \\
&+\left[\frac{\Sigma_a\Sigma_b}{\lambda\rho^2\Delta_\theta}-\frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Sigma_r}-\frac{\alpha}{\rho^2 Z\Sigma_r}-\frac{\alpha^2}{\rho^2 Z\Sigma_r^2}\right]\partial_\tau^2\chi \\
&+2\left[\frac{a\Sigma_a\Sigma_b}{\rho^2\Delta_\theta}-\frac{a\Sigma_a\Sigma_b}{\rho^2\Sigma_r}-\frac{\alpha\lambda a}{\rho^2 Z\Sigma_r^2}-\frac{\alpha^2 a\Sigma_a}{\rho^2 Z\Sigma_r^2(V-\alpha)(r^2+a^2)}\right. \\
&\qquad\left.-\frac{\alpha a}{\rho^2 Z\Sigma_r(r^2+a^2)}\right]\sum_{i=1}^{p}\partial_\tau\partial_{\varphi_i}\chi \\
&+2\left[\frac{b\Sigma_a\Sigma_b}{\rho^2\Delta_\theta}-\frac{b\Sigma_a\Sigma_b}{\rho^2\Sigma_r}-\frac{\alpha\lambda b}{\rho^2 Z\Sigma_r^2}-\frac{\alpha^2 b\Sigma_b}{\rho^2 Z\Sigma_r^2(V-\alpha)(r^2+b^2)}\right. \\
&\qquad\left.-\frac{\alpha b}{\rho^2 Z\Sigma_r(r^2+b^2)}\right]\sum_{i=1}^{q}\partial_\tau\partial_{\varphi_{p+i}}\chi \\
&+\sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2a^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta}-\frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Sigma_r}-\frac{\alpha}{\rho^2 Z\Sigma_r}\right)\right. \\
&\qquad\left.+\frac{\alpha^2a^2\Sigma_a^2}{\rho^2 Z(V-\alpha)(r^2+a^2)^2}+\frac{Q^{ij}}{\rho^2 Z}\right]\partial_{\varphi_i}\partial_{\varphi_j}\chi \\
&+\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2b^2\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta}-\frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Sigma_r}-\frac{\alpha}{\rho^2 Z\Sigma_r}\right)\right. \\
&\qquad\left.+\frac{\alpha^2b^2\Sigma_a^2}{\rho^2 Z(V-\alpha)(r^2+b^2)^2}+\frac{Q^{(i+p)(j+p)}}{\rho^2 Z}\right]\partial_{\varphi_{i+p}}\partial_{\varphi_{j+p}}\chi \\
&+2\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\lambda^2ab\left(\frac{\Sigma_a\Sigma_b}{\rho^2\Delta_\theta}-\frac{\Sigma_a\Sigma_b}{\rho^2\lambda\Sigma_r}-\frac{\alpha}{\rho^2 Z\Sigma_r}\right)\right. \\
&\qquad\left.+\frac{\alpha^2ab\Sigma_a\Sigma_b}{\rho^2 Z(V-\alpha)(r^2+a^2)(r^2+b^2)}+\frac{Q^{i(j+p)}}{\rho^2 Z}\right]\partial_{\varphi_i}\partial_{\varphi_{j+p}}\chi\,.
\end{aligned}
\tag{3.24}
$$

To separate this equation, we assume a multiplicative separation of χ:

$$\chi = \chi_r(r)\chi_\theta(\theta)e^{-iE\tau}e^{i\sum_i^p \Phi_i\varphi_i}e^{i\sum_i^q \Psi_i\varphi_{p+i}}\left(\prod_{i=1}^{m-1}\chi_{\alpha_i}(\alpha_i)\right)\left(\prod_{i=1}^{p-1}\chi_{\beta_i}(\beta_i)\right), \tag{3.25}$$

and we will again use the convention that $\Psi_p = 0$ in the case of an even number of spacetime dimensions.

The Klein-Gordon equation now separates. The r and θ portions are given as

$$\begin{aligned}
K = {} & \frac{1}{\chi_r\sqrt{R}}\frac{d}{dr}\left(\sqrt{R}\frac{V-\alpha}{Z}\frac{d\chi_r}{dr}\right) + \left[\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r} + \frac{\alpha^2}{Z\Sigma_r^2}\right]E^2 \\
& + \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\lambda^2 a^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2 a^2\Sigma_a^2}{Z(V-\alpha)(r^2+a^2)^2} - \frac{Q^{ij}}{Z}\right]\Phi_i\Phi_j \\
& + \sum_{i=1}^{q}\sum_{j=1}^{q}\left[\lambda^2 b^2\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right) - \frac{\alpha^2 b^2\Sigma_a^2}{Z(V-\alpha)(r^2+b^2)^2} - \frac{Q^{(i+p)(j+p)}}{Z}\right]\Psi_i\Psi_j \\
& + 2\sum_{i=1}^{p}\sum_{j=1}^{q}\left[\lambda^2 ab\left(\frac{\Sigma_a\Sigma_b}{\lambda\Sigma_r} + \frac{\alpha}{Z\Sigma_r}\right)\right. \\
& \left. - \frac{\alpha^2 ab\Sigma_a\Sigma_b}{Z(V-\alpha)(r^2+a^2)(r^2+b^2)} - \frac{Q^{i(j+p)}}{Z}\right]\Phi_i\Psi_j \\
& - 2\left[\frac{a\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda a}{Z\Sigma_r^2} + \frac{\alpha^2 a\Sigma_a}{Z\Sigma_r^2(V-\alpha)(r^2+a^2)} + \frac{\alpha a}{Z\Sigma_r(r^2+a^2)}\right]\sum_{i=1}^{p}E\Phi_i \\
& - 2\left[\frac{b\Sigma_a\Sigma_b}{\Sigma_r} + \frac{\alpha\lambda b}{Z\Sigma_r^2} + \frac{\alpha^2 b\Sigma_b}{Z\Sigma_r^2(V-\alpha)(r^2+b^2)} + \frac{\alpha b}{Z\Sigma_r(r^2+b^2)}\right]\sum_{i=1}^{q}E\Psi_i \\
& - \Sigma_a\frac{r^2+b^2}{r^2+a^2}\sum_{i=1}^{q}L_1\Phi_i^2 - \Sigma_b\frac{r^2+a^2}{r^2+b^2}\sum_{i=1}^{q}M_1\Psi_i^2 - m^2 r^2,
\end{aligned} \tag{3.26}$$

and

$$-K = \frac{1}{\chi_\theta\sqrt{T}}\frac{d}{d\theta}\left(\sqrt{T}\Delta_\theta\frac{d\chi_\theta}{d\theta}\right) - \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2 - m^2(a^2\cos^2\theta + b^2\sin^2\theta)$$

$$+L_1\cot^2\theta+M_1\tan^2\theta-2\lambda a^2\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{p}\sum_{j=1}^{p}\Phi_i\Phi_j-2\lambda b^2\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{q}\sum_{j=1}^{q}\Psi_i\Psi_j$$

$$-4\lambda ab\frac{\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{p}\sum_{j=1}^{q}\Phi_i\Psi_j+2\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{i=1}^{p}E\Phi_i+2\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}\sum_{j=1}^{q}E\Phi_j\,. \tag{3.27}$$

K, L_1, and M_1 are separation constants. The α and β parts of the equations are given by a series of equations

$$L_k=\frac{L_{k+1}}{\sin^2\alpha_k}-\frac{\Phi^2_{p-k+1}}{\cos^2\alpha_k}+\frac{1}{\chi_{\alpha_k}\cos\alpha_k\sin^{2p-2k-1}\alpha_k}\frac{d}{d\alpha_k}\left(\cos\alpha_k\sin\alpha_k\frac{d\chi_{\alpha_k}}{d\alpha_k}\right)\,, \tag{3.28}$$

for $k=1,...,p-1$ and

$$M_k=\frac{M_{k+1}}{\sin^2\beta_k}-\frac{\Psi^2_{q-k+1}}{\cos^2\beta_k}+\frac{1}{\chi_{\beta_k}\cos\beta_k\sin^{2q-2k-1}\beta_k}\frac{d}{d\beta_k}\left(\cos\beta_k\sin\beta_k\frac{d\chi_{\beta_k}}{d\beta_k}\right)\,, \tag{3.29}$$

for $k=1,...,q-1$.

This completes the separation of the Klein-Gordon equation for a scalar field acting in the Kerr-(anti) de Sitter spacetime with two unequal spin parameters.

3.5 Symmetry of the Metric

Now we would like to say something about the underlying symmetries of these spacetimes that allowed for the separation of variables of the Hamilton-Jacobi and Klein-Gordon equations. In general relativity, symmetries of metrics are represented by Killing vector fields that exist in the spacetime. A vector field ξ is said to be a Killing vector if it obeys the Killing equation

$$\mathcal{L}_\xi\, g=0\,, \tag{3.30}$$

which is the statement that the metric g is unchanged under an infinitesimal translation in the direction of ξ. In component notation, equation (3.30) can be expressed as

$$\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu=0\,, \tag{3.31}$$

so the derivative of a Killing vector is antisymmetric.

Killing vectors are also connected with conserved quantities in geodesic motion, specifically if ξ is a Killing vector and p is the momentum vector tangent the worldline of a particle in geodesic motion, then the quantity $\xi \cdot p$ is conserved. Specifically

$$p^\mu \nabla_\mu (p^\nu \xi_\nu) = 0\,, \tag{3.32}$$

which is easily shown to be true from the fact that $\nabla\xi$ is antisymmetric and the geodesic equation $p^\mu \nabla_\mu p^\nu = 0$. In the Kerr-de Sitter spacetimes there are Killing vectors associated with coordinates φ_i and τ, and correspond to conserved quatities Φ_i, Ψ_i, and E. When some of the rotation parameters are equal there is additional symmetry associated with "rotating" the individual planes into eachother. One can construct these Killing vectors explicitly by selecting two of the planes that have equal rotation parameters with labels i and j. These planes have angle cosines μ_i and μ_j and azimuthal coordinates φ_i and φ_j. Introduce Cartesian-like coordinates on the planes as

$$x_i = r\mu_i \cos\varphi_i\,, \qquad y_i = r\mu_i \sin\varphi_i$$

$$x_j = r\mu_j \cos\varphi_j\,, \qquad y_j = r\mu_j \sin\varphi_j\,. \tag{3.33}$$

Next define rotation generators on the planes as

$$L_{uv} = u\partial_v - v\partial_u\,, \tag{3.34}$$

where u and v can be any of x_i, y_i, x_j, y_j. If $u = x_i$ and $v = y_i$ then the vector $L_{x_i y_i}$ is the same as ∂_{φ_i}. The other combinations for L_{uv} will not be Killing vectors except in the case that the chosen planes have spin parameter equal to zero, however, the combinations

$$\xi_{ij} = L_{x_i x_j} + L_{y_i y_j}\,, \qquad \zeta_{ij} = L_{x_i y_j} + L_{x_j y_i}\,, \tag{3.35}$$

are Killing vectors whenever $a_i = a_j$.

One can also generalize the concept of a Killing vector to construct Killing tensors. One defines a Killing tensor $K_{\mu_1\mu_2\ldots\mu_n}$ as a symmetric rank n tensor which has the feature that the totally symmetric part of ∇K vanishes,

$$\nabla_{(\nu} K_{\mu_1\mu_2\ldots\mu_n)} = 0\,. \tag{3.36}$$

Then we can see that

$$p^\mu \nabla_\mu (K_{\mu_1\mu_2\ldots\mu_n} p^{\mu_1} p^{\mu_2} \ldots p^{\mu_n}) = 0\,, \tag{3.37}$$

when p obeys the geodesic equation, so that $K_{\mu_1\mu_2...\mu_n}p^{\mu_1}p^{\mu_2}...p^{\mu_n}$ is a conserved quantity.

We can derive a Killing tensor for the Kerr-de Sitter spacetimes by considering our separation constant to be a constant of the motion which is quadratic in the momenta. Recall equation (3.6),

$$-K = m^2a^2\cos^2\theta + m^2b^2\sin^2\theta + \Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2 + \Sigma_a\cot^2\theta J_1^2 + \Sigma_b\tan^2\theta L_1^2$$
$$+\frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}E^2 - 2\sum_{i=1}^{p}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}E\Phi_i - 2\sum_{i=1}^{q}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}E\Psi_i + \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Phi_j$$
$$+\sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\Psi_i\Psi_j + 2\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}\Phi_i\Psi_j\,, \tag{3.38}$$

from this we may read off the form of the Killing tensor from $K = K^{\mu\nu}p_\mu p_\nu = K^{\mu\nu}\partial_\mu S\partial_\nu S$ as

$$K^{\mu\nu} = -g^{\mu\nu}\left(a^2\cos^2\theta + b^2\sin^2\theta\right) - \frac{\Sigma_a\Sigma_b}{\lambda\Delta_\theta}\delta^\mu_\tau\delta^\nu_\tau - \Sigma_a\cot^2\theta J_1^{\mu\nu} - \Sigma_b\tan^2\theta L_1^{\mu\nu}$$
$$-\sum_{i=1}^{p}\frac{a\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_\tau\delta^\nu_{\varphi_i} + \delta^\mu_{\varphi_i}\delta^\nu_\tau) - \sum_{i=1}^{q}\frac{b\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_\tau\delta^\nu_{\varphi_{i+p}} + \delta^\mu_{\varphi_{i+p}}\delta^\nu_\tau)$$
$$-\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\lambda^2a^2\Sigma_a\Sigma_b}{\Delta_\theta}\delta^\mu_{\varphi_i}\delta^\nu_{\varphi_j} - \sum_{i=1}^{q}\sum_{j=1}^{q}\frac{\lambda^2b^2\Sigma_a\Sigma_b}{\Delta_\theta}\delta^\mu_{\varphi_{i+p}}\delta^\nu_{\varphi_{j+p}}$$
$$-\sum_{i=1}^{p}\sum_{j=1}^{q}\frac{\lambda^2ab\Sigma_a\Sigma_b}{\Delta_\theta}(\delta^\mu_{\varphi_i}\delta^\nu_{\varphi_{j+p}} + \delta^\mu_{\varphi_{j+p}}\delta^\nu_{\varphi_i}) - \Delta_\theta\delta^\mu_\theta\delta^\nu_\theta\,. \tag{3.39}$$

where $J_1^{\mu\nu}$ and $L_1^{\mu\nu}$ are reducible Killing tensors, which means that they can be expressed as linear combinations of products of Killing vectors. In the case of even spacetime dimension, there is no coordinate φ_{p+q}, so one has to set $\delta^\mu_{\varphi_{p+q}} = 0$.

To see that J_1 and L_1 are reducible explicitly, we first examine the series of separation of the α_i and β_i coordinates (3.7)-(3.8) which gave us a series of Killing tensors,

$$J_k^{\mu\nu} = \frac{J_{k+1}^{\mu\nu}}{\sin^2\alpha_k} + \frac{\delta^\mu_{\varphi_{p-k+1}}\delta^\nu_{\varphi_{p-k+1}}}{\cos^2\alpha_k} + \delta^\mu_{\alpha_k}\delta^\nu_{\alpha_k} \qquad \text{for} \quad k = 1, ..., p-2\,,$$

$$J^{\mu\nu}_{p-1} = \delta^{\mu}_{\alpha_{p-1}}\delta^{\nu}_{\alpha_{p-1}} + \frac{\delta^{\mu}_{\varphi_1}\delta^{\nu}_{\varphi_1}}{\sin^2\alpha_{p-1}} + \frac{\delta^{\mu}_{\varphi_2}\delta^{\nu}_{\varphi_2}}{\cos^2\alpha_{p-1}}\,, \tag{3.40}$$

and

$$L^{\mu\nu}_{k} = \frac{L^{\mu\nu}_{k+1}}{\sin^2\beta_k} + \frac{\delta^{\mu}_{\varphi_{p+q-k+1}}\delta^{\nu}_{\varphi_{p+q-k+1}}}{\cos^2\beta_k} + \delta^{\mu}_{\beta_k}\delta^{\nu}_{\beta_k} \qquad \text{for} \quad k = 1, ..., q-2\,,$$

$$J^{\mu\nu}_{q-1} = \delta^{\mu}_{\beta_{q-1}}\delta^{\nu}_{\beta_{q-1}} + \frac{\delta^{\mu}_{\varphi_{p+1}}\delta^{\nu}_{\varphi_{p+1}}}{\sin^2\beta_{q-1}} + \frac{\delta^{\mu}_{\varphi_{p+2}}\delta^{\nu}_{\varphi_{p+2}}}{\cos^2\beta_{q-1}}\,. \tag{3.41}$$

To see that these J_k and L_k are reducible, an explicit calculation confirms that one can write them in the form

$$J^{\mu\nu}_{p-k} = \sum_{i=1}^{k+1}\delta^{\mu}_{\varphi_i}\delta^{\nu}_{\varphi_i} - \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}\delta^{(\mu}_{\varphi_i}\delta^{\nu)}_{\varphi_j}$$
$$+\sum_{i=1}^{k+1}\sum_{j=1}^{i-1}\xi^{\mu}_{ij}\xi^{\nu}_{ij} + \sum_{i=1}^{k+1}\sum_{j=1}^{i-1}\zeta^{\mu}_{ij}\zeta^{\nu}_{ij}\,, \qquad k = 1, ..., p-1\,, \tag{3.42}$$

and

$$L^{\mu\nu}_{q-k} = \sum_{i=p+1}^{p+k+1}\delta^{\mu}_{\varphi_i}\delta^{\nu}_{\varphi_i} - \sum_{i=p+1}^{p+k+1}\sum_{j=p+1}^{i-1}\delta^{(\mu}_{\varphi_i}\delta^{\nu)}_{\varphi_j}$$
$$+\sum_{i=p+1}^{p+k+1}\sum_{j=p+1}^{i-1}\xi^{\mu}_{ij}\xi^{\nu}_{ij} + \sum_{i=p+1}^{p+k+1}\sum_{j=p+1}^{i-1}\zeta^{\mu}_{ij}\zeta^{\nu}_{ij}\,, \qquad k = 1, ..., q-1\,, \tag{3.43}$$

with ξ_{ij} and ζ_{ij} defined in (3.35). One can confirm through calculation that the tensors $K^{\mu\nu}$, $J^{\mu\nu}_k$, and $L^{\mu\nu}_k$ all obey (3.36), and so they are all Killing tensors.

Chapter 4

Geons

4.1 Introduction

Geons were introduced about fifty years ago, by Wheeler and collaborators [29, 30], the idea being that it could be possible for topological structures to exist within a spacetime. These structures could carry charge, and angular momentum, resembling particles when viewed from a large scale. There has been much research into the quantum mechanics of such objects, such as [33–35], in particular the study of four-dimensional geons motivated the result that all smooth 3-manifolds are allowed as physically reasonable solutions to the Einstein equations [36].

An intuitive example of a geon is to imagine a "handle" on an otherwise simple space, as figure 4.1. One can see that this space is topologically equivalent to a torus missing a single point.

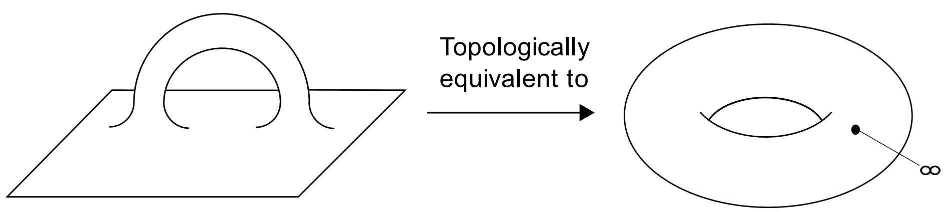

Figure 4.1: A "handle" on an otherwise simple space.

A more explicit example of a geon is the AdS soliton, first examined by Horowitz and Myers [50], the metric in N dimensions is given by

$$ds^2 = -r^2 dt^2 + \frac{1}{V(r)} dr^2 + V(r) d\phi^2 + r^2 \sum_{i=1}^{N-3} dy_i^2 \,, \tag{4.1}$$

with $V(r) = |\lambda| r^2 \left(1 - \frac{r_0^{N-1}}{r^{N-1}}\right)$ and $\lambda < 0$. The coordinate r varies from r_0

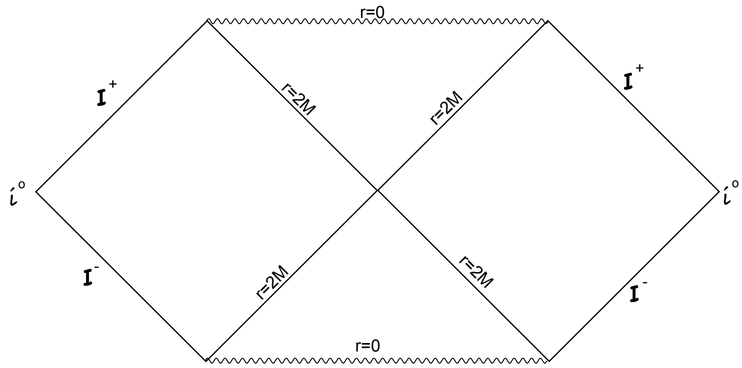

Figure 4.2: Maximal analytic extension of Schwarzschild spacetime Each point except the points labelled i_0 represents a two-sphere. The surface $r = 2M$ is the event horizon, and $r = 0$ is the location of the singularity. Note that there are two asymptotic regions.

to infinity, and regularity near $r = r_0$ demands that ϕ have period $\frac{4\pi}{|\lambda|(N-1)r_0}$, but the y_i can have arbitrary periods.

This space is asymptotically locally anti-de Sitter in the sense defined in the appendix, and it has been shown to have minimal energy under small metric perturbations [51, 52]. If one takes a spatial slice of this spacetime and conformally completes it, the space has a topology of $T^{N-3} \times \mathbb{R}^2$, with T^n the n-torus, so we can see that spatial slices of the AdS soliton are topologically different than that of flat space.

As another example, one can consider the $\mathbb{RP}^3$ geon, discussed in [53]. This is a spacetime that one gets by starting with the maximal extension of the Schwarzschild spacetime [54] (see figure 4.2) and identifying points from the "left" and "right" spaces with each other. In order to do this consistently across the middle without destroying the manifold structure, the sphere that usually represents the middle is replaced with the projective space $\mathbb{RP}^2$ (see figure 4.3).

The resulting space has spatial slices that are topologically $\mathbb{RP}^3 - \{p\}$, with p representing the point at infinity, so the spatial slices are topologically nontrivial. This idea will be central to the precise definition of a geon, which we will go over now.

A globally hyperbolic spacetime M is said to be an N-dimensional *geon* if its Cauchy surface V can be expressed as $V = \Pi - \{p\}$ with Π being any

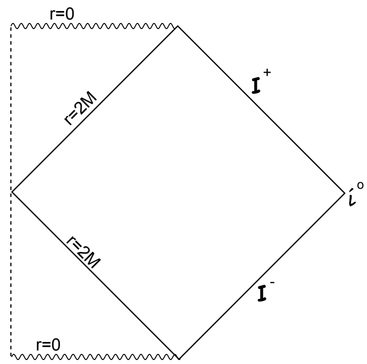

Figure 4.3: The $\mathbb{RP}^3$ geon
This space is obtained by identifying opposite sides of Schwarzschild space. Each point represents a two-sphere, except for the dashed line where each point is $\mathbb{RP}^2$, and the point i_0 which is a single point.

compact manifold without boundary, other than the sphere S^{N-1}, and $\{p\}$ denotes a singleton set. The intention of this definition is that p corresponds to the "point at infinity" removed from the Cauchy slice, and if what is left has a more complicated topology than a sphere, then we say the spacetime is a geon.

An asymptotically flat geon is one that satisfies the definition of an asymptotically flat spacetime, while an asymptotically AdS geon is one that satisfies the definition of an asymptotically AdS spacetime.

4.2 Nonexistence of Asymptotically Flat Three Dimensional Geons

In this section we will prove that any asymptotically flat, three dimensional spacetime that obeys the Einstein equations cannot contain geons, assuming reasonable conditions on the stress-energy of the spacetime. The proof will be done through a series of theorems, first demonstrating that in any three dimensional spacetime, the domain of outer communication must have a simple topology, so any geons will necessarily be hidden behind an event horizon. We then show that any three dimensional spacetime which has an event horizon must have a negative cosmological constant, and so cannot be asymptotically flat. This proof was also a published work [37]. We will make use of some of the terminology discussed in the appendix. We must begin with a discussion of energy conditions, which are assumptions about the stress energy tensor for a spacetime.

We will say that a spacetime obeys the *null energy condition* (NEC) if $T_{\mu\nu}W^\mu W^\nu \geq 0$ for all null vectors W^μ, the *weak energy condition* (WEC) if instead $T_{\mu\nu}W^\mu W^\nu \geq 0$ for timelike vectors W^μ, and the *dominant energy condition* (DEC) if whenever W^μ is a future-directed, nonspacelike vector, then $-T^\mu_{\ \nu}W^\nu$ is also a future-directed, nonspacelike vector [55]. These energy conditions can be physically interpreted as how local observers view flow of energy in spacetime. For an observer with momentum W^μ, the momentum of local energy is given by the vector $-T^\mu_{\ \nu}W^\nu$, so the DEC is the statement that the local energy as viewed by a timelike observer flows in a timelike direction. Note also that the DEC implies the WEC. We can interpret the WEC by noting that the quantity $T_{\mu\nu}W^\mu W^\nu$ is the local energy density as viewed by an observer with momentum W^μ, so the WEC is the statement that timelike observers see positive energy density. The NEC is less physically clear, but is mathematically practical. First note that a violation of the NEC would mean there exists a null vector W^μ such that $T_{\mu\nu}W^\mu W^\nu < 0$, but then by continuity this must also hold in some neighbourhood of W^μ, since there must be a timelike vector in this neighbourhood we can see that a violation of the NEC implies a violation of the WEC, so the NEC is a weaker restriction than the WEC. Further, when W^μ is null, the Einstein equations tell us $T_{\mu\nu}W^\mu W^\nu = R_{\mu\nu}W^\mu W^\nu$, so the NEC can be easily interpreted as a statement on the geometry of solutions.

We begin by considering the region of space which is external to the event horizons. We will define the *domain of outer communications* (DOC) as $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ for an asymptotically flat (AF) spacetime with null infinity $\mathcal{I}^+$ and $\mathcal{I}^-$ and for an asymptotically locally anti-de Sitter (ALADS) spacetime the DOC will be defined as $I^-(\mathcal{I}) \cap I^+(\mathcal{I})$ for $\mathcal{I}$ the boundary of the spacetime. Intuitively, we can see that the DOC is the region of spacetime which can receive messages from infinity and can send messages to infinity.

To begin the proof, we will need to make use of the principle of topological censorship (PTC). This principle roughly says that any nontrivial topological structures in a spacetime must always be hidden behind an event horizon. Letting $\mathcal{I}$ represent the boundary of M in the Penrose compactified spacetime, we may state the PTC as follows:

The Principle of Topological Censorship: Every nonspacelike curve whose initial and final endpoints belong on $\mathcal{I}$ is homotopic to a curve that lies entirely within $\mathcal{I}$.

In [56] it has been shown that if a spacetime M with boundary $\mathcal{I}$ and DOC D obeys the following conditions:

(1)$D' = D \cup \mathcal{I}$ is globally hyperbolic

(2)$\mathcal{I}$ admits a compact spacelike cut

(3)The NEC holds in D

Then the PTC holds on D.

In that paper, the authors actually used a weaker condition than the NEC, it was sufficient for the NEC to hold on average for null geodesics in D, however, for our purposes we will be using the stronger DEC anyway.

There is a similar theorem for AdS spacetimes, demonstrated in [57]. In this case, it was needed for a locally AdS spacetime to be globally hyperbolic, however anti-de Sitter space is not generically globally hyperbolic, so the definition must be modified somewhat. We say that a ALADS spacetime M with boundary $\mathcal{I}$ is globally hyperbolic if the spacetime-with-boundary $M \cup \mathcal{I}$ is globally hyperbolic.

In both cases, there is a corollary to the theorem which is a statement about the fundamental homotopy group:

If the PTC holds for D', then the group homomorphism $i_* : \pi_1(\mathcal{I}) \to \pi_1(D')$ is surjective.

We will use this result in our first theorem.

Theorem 1. *Let M be a 2+1-dimensional spacetime which is a globally hyperbolic AF or ALADS spacetime with boundary, and the stress-energy for M obeys the NEC. Let D be the DOC of M and V be a Cauchy surface in D. Then V is either the disk B^2 or an annulus $\mathbb{R} \times S^1$.*

Proof. Consider the DOC in the Penrose compactified spacetime $D' = D \cup \mathcal{I}$, let V' be the Cauchy surface V extended into this spacetime, so V' has a boundary at infinity $\partial V' = \sigma$. Because D' satisfies the conditions needed to prove topological censorship, then the homomorphism of fundamental groups $i_* : \pi_1(\sigma) \to \pi_1(V')$ induced by inclusion must be surjective. In other words, the sequence $\pi_(\sigma) \to \pi_1(V') \to 1$ is exact.

Because V' is two dimensional, the only possible topology on σ is S^1, the only closed connected 1-dimensional manifold. As we know that $\pi_1(S^1) = \mathbb{Z}$ we then have $\mathbb{Z} \to \pi_1(V') \to 1$ for our exact sequence. This exact sequence tells us that $\pi_1(V') = \mathbb{Z}/\ker i_*$. As i_* is a homomorphism from $\mathbb{Z}$, its kernel must be a subgroup of $\mathbb{Z}$. The only subgroups of $\mathbb{Z}$ are of the form $s\mathbb{Z}$ where s is some nonnegative integer, so we must have $\pi_1(V') = \mathbb{Z}/s\mathbb{Z} = \mathbb{Z}_s$.

Classification of 2-manifolds tells us that if we have a 2-manifold with boundary V' such that $\pi_1(V') = \mathbb{Z}_s$ and $\partial V' = \sigma$, then we must have $s = 1$ and V' is a disk $\mathbb{R}^2$ or we have $s = 0$ and V' is an annulus $\mathbb{R} \times S^1$. □

We can see that theorem 1 gives us information about the spatial part of the DOC, which is the part that can be reached from infinity. This theorem still allows a Cauchy slice of the full spacetime (which may include parts outside of D) to have a more complicated topology. Our next theorem will demonstrate that 3-dimensional geons always come with an event horizon.

Theorem 2. *Let M be a $2+1$-dimensional spacetime which is a globally hyperbolic AF or ALADS geon spacetime with boundary, and the stress-energy for M obeys the NEC. Then M contains an event horizon.*

Proof. Since M is a geon, it must have a Cauchy surface $\Pi - \{p\}$. If the spacetime has no horizon, then we would have that the DOC of M would be the same as all of M. According to theorem 1, the Cauchy surface of M must then be either $\mathbb{R}^2$ or $\mathbb{R} \times S^1$. These spaces are topologically equivalent to $S^2 - \{p\}$ and $\mathbb{R}^2 - \{p\}$ respectively. We can see that neither of these has nontrivial topology, and so are not the Cauchy surface of a geon. Thus, it must be the case that a geon spacetime has a horizon. □

Theorem 2 tells us that if the spacetime has any non-trivial topology, it must be hidden behind an event horizon. The next theorem will demonstrate that any spacetime which has an event horizon must have a negative cosmological constant. The proof will also require use of the DEC.

Theorem 3. *Let M be a globally hyperbolic $2+1$-dimensional spacetime with possible event horizons. Suppose M obeys the following conditions 1), 2), 3), and either 4a) or 4b):*

1)The Einstein equations with cosmological constant $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$,

2) $T_{\mu\nu}$ obeys the DEC.

3) The spacetime contains a trapped surface.

4a) The metric induced on spatial sections of the manifold are analytic in the region near the event horizon.

4b) The spacetime admits a maximal slicing.

Then the spacetime cannot be asymptotically flat.

Proof. Suppose V is a Cauchy surface in M. Since M has a trapped surface, it must have a trapped region, consider the intersection of that trapped region with the surface V. Since the trapped region does not extend all the way out to infinity (or else there would be no event horizon), it must have an outer boundary. Let $\mathcal{U}$ represent the outer boundary of the trapped region which lies in V. We can see that $\mathcal{U}$ must be the outermost trapped surface, so it is a marginal trapped surface. Let l be the future-directed geodesic null vector orthogonal to $\mathcal{U}$ with zero expansion. Let n be the other future-directed null vector which is orthogonal to $\mathcal{U}$. We may normalize l and n such that $l \cdot n = -1$. Now we wish to add one more vector m so that the basis (l, n, m) is complete. The remaining spatial vector m, which must be parallel to the tangent vector to $\mathcal{U}$, can be normalized so that $m \cdot m = 1$. The metric tensor then decomposes as

$$g_{\mu\nu} = m_\mu m_\nu - l_\mu n_\nu - l_\nu n_\mu \,. \tag{4.2}$$

Note that since $\mathcal{U}$ is a marginally trapped surface, the expansion of outgoing null geodesics $\theta = m_\mu m^\nu \nabla_\nu l^\mu$ is zero in that surface. Next, consider deforming $\mathcal{U}$ outwards a distance w along $-n$, where n has been chosen so that the direction of positive w is outward toward infinity. Since l is hypersurface orthogonal, we may use the scaling freedom $l \to e^\chi l$, $n \to e^{-\chi} n$ to ensure that l obeys

$$m^\nu n^\mu \nabla_\mu l_\nu = m^\nu n^\mu \nabla_\nu l_\mu \,, \tag{4.3}$$

which in the language of [58] is $\tau = \overline{\alpha} + \beta$. Moving outward a distance w along $-n$, The expansion θ of l changes according to

$$\frac{d\theta}{dw} = -n^\rho \nabla_\rho (m_\mu m^\nu \nabla_\nu l^\mu) \,. \tag{4.4}$$

From (4.2) we can see that

$$\nabla_\nu m^\mu = l^\mu m_\sigma \nabla_\nu n^\sigma + n^\mu m_\sigma \nabla_\nu l^\sigma \,. \tag{4.5}$$

Expanding out (4.4) and using the fact that l is geodesic, we can write this as

$$\frac{d\theta}{dw} = -R^\mu{}_{\sigma\rho\nu} l^\sigma n^\rho m^\nu m_\mu - n^\rho m^\nu m_\mu \nabla_\nu \nabla_\rho l^\mu - p^\mu p_\mu - p^\nu n^\mu \nabla_\nu l_\mu \,, \tag{4.6}$$

where we have defined $p^\mu = m^\mu m^\nu n^\sigma \nabla_\nu l_\sigma$, and this p is a spacelike vector. One can calculate the divergence of p in the surface $\mathcal{U}$ using (4.3), it is given by

$$m^\mu m^\nu \nabla_\nu p_\mu = m^\nu m_\mu n^\rho \nabla_\nu \nabla_\rho l^\mu + p^\nu n^\rho \nabla_\nu l_\rho$$

$$+m^\nu m_\rho \nabla_\nu n^\rho m^\tau m_\mu \nabla_\tau l^\mu + m^\nu m_\mu \nabla_\nu l^\mu n^\sigma n^\lambda \nabla_\lambda l_\sigma \,. \tag{4.7}$$

Using this expression, and using (4.2), we can see that

$$\frac{d\theta}{dw} = -R^\mu{}_{\sigma\rho\nu} l^\sigma n^\rho l^\nu n_\mu + R_{\mu\nu} l^\mu n^\nu - p^\mu p_\mu - m^\mu m^\nu \nabla_\nu p_\mu$$

$$+m^\tau m_\mu \nabla_\tau l^\mu \left(m^\nu m_\rho \nabla_\nu n^\rho + n^\sigma n^\lambda \nabla_\lambda l_\sigma\right) \,. \tag{4.8}$$

In the final term, we may replace $m^\tau m_\mu \nabla_\tau l^\mu$ with θ. Consider now defining $l' = e^\phi l$ and $n' = e^\phi n$, with ϕ a function on the surface $\mathcal{U}$, but ϕ does not vary as we move out along w, $\frac{d\phi}{dw} = 0$. Defining $p'^\mu = p^\mu - m^\mu m^\nu \nabla_\nu \phi$, we can express the last equation as

$$\frac{d\theta'}{dw} = -R_{\mu\nu\sigma\lambda} n^\mu l^\nu n^\sigma l^\lambda + R_{\mu\nu} n^\mu l^\nu - m^\mu m^\nu \nabla_\nu p_\mu - m^\mu m^\nu \nabla_\mu \nabla_\nu \phi$$

$$-p'^\mu p'_\mu + \theta'(m^\nu m_\rho \nabla_\nu n'^\rho + n'^\sigma n'^\lambda \nabla_\lambda l'_\sigma) \,, \tag{4.9}$$

where $\theta' = m_\mu m^\nu \nabla_\nu l'^\mu = e^\phi \theta$. It is possible to choose the function ϕ so that the sum of the terms on the right is a constant on the surface $\mathcal{U}$. The sign of this constant is determined by the value of the integral of these terms on the surface. The terms $m^\mu m^\nu \nabla_\nu p_\mu$ and $m^\mu m^\nu \nabla_\mu \nabla_\nu \phi$ are total divergences in the surface $\mathcal{U}$, so they integrate to zero. The remaining terms determine the overall sign

$$\mathrm{sgn}\left(\frac{d\theta'}{dw}\right) = \mathrm{sgn}\left(\int_{\mathcal{U}} (-R_{\mu\nu\lambda\kappa} n^\mu l^\nu n^\lambda l^\kappa + R_{\mu\nu} n^\mu l^\nu - p'^\mu p'_\mu + \theta' \gamma) dA\right) \,, \tag{4.10}$$

where $\gamma = e^\phi(m^\nu m_\rho \nabla_\nu n'^\rho + n'^\sigma n'^\lambda \nabla_\lambda l'_\sigma)$. Since $\theta' = e^\phi \theta$ and $\frac{d\phi}{dw} = 0$, it must be the case that $\frac{d\theta'}{dw}$ and $\frac{d\theta}{dw}$ have the same sign. We can simplify our expression for $\frac{d\theta}{dw}$ using the relation between the Riemann tensor and the Ricci tensor in three dimensions,

$$R_{\mu\nu\lambda\kappa} = g_{\mu\lambda} R_{\nu\kappa} + g_{\nu\kappa} R_{\mu\lambda} - g_{\mu\kappa} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\kappa} + \frac{1}{2}(g_{\mu\kappa} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\kappa}) R \,, \tag{4.11}$$

so we then have

$$\text{sgn}\left(\frac{d\theta}{dw}\right) = \text{sgn}\left(\int_{\mathcal{U}} (\theta\gamma - R_{\mu\nu} l^\mu n^\nu - R/2 - p'^\mu p'_\mu) dA\right) . \tag{4.12}$$

We can replace the terms involving the Ricci tensor by using the Einstein equations contracted with l and n,

$$R_{\mu\nu} n^\mu l^\nu + R/2 = 8\pi T_{\mu\nu} n^\mu l^\nu + \Lambda , \tag{4.13}$$

giving us

$$\text{sgn}\left(\frac{d\theta}{dw}\right) = \text{sgn}\left(\int_{\mathcal{U}} (\theta\gamma - p'^\mu p'_\mu - 8\pi T_{\mu\nu} n^\mu l^\nu - \Lambda) dA\right) . \tag{4.14}$$

Let us now denote

$$\mathcal{A} = \int_{\mathcal{U}} dA \quad \text{and} \quad Q = \int_{\mathcal{U}} (8\pi T_{\mu\nu} n^\mu l^\nu + p'^\mu p'_\mu) dA , \tag{4.15}$$

$\mathcal{A} \geq 0$ is the area of the trapped surface, and the DEC along with the fact that p' is spacelike guarantees that $Q \geq 0$. We can then write

$$\text{sgn}\left(\frac{d\theta}{dw}\right) = \text{sgn}\left(\int_{\mathcal{U}} \theta\gamma dA - Q - \Lambda\mathcal{A}\right) . \tag{4.16}$$

We know that θ is zero on the surface $\mathcal{U}$, and w is aligned such that increasing w is outward, toward infinity. We know that $\mathcal{U}$ is the outermost trapped surface, so it cannot be the case that $\frac{d\theta}{dw}$ is negative. Since on $\mathcal{U}$ we have $\theta = 0$, we must have $Q + \Lambda\mathcal{A} \leq 0$ in order to guarantee that $\text{sgn}\left(\frac{d\theta}{dw}\right) \geq 0$. As we know that Q and $\mathcal{A}$ are each positive, that can only leave us with

$$\Lambda \leq 0 . \tag{4.17}$$

So far, we have made use of our conditions *1) 2) 3)* to demonstrate that Λ must be nonpositive. The rest of the proof makes use of conditions *4a)* and *4b)* to rule out the possibility that the spacetime can be asymptotically flat. We will begin with a proof making use of condition *4a)*.

We can rule out the case that the spacetime is asymptotically flat by showing that the inequality in (4.17) must be a strict one. We can do this by using analyticity of the metric near the event horizon, note that θ is obtained from the metric by an analytic calculation, so it must also be analytic. Also recall that because $\mathcal{U}$ was the outermost trapped surface, θ must become positive as w increases. We will show a contradiction to this by using equation (4.16) to demonstrate that all of the coefficients in the Taylor expansion of $\theta(w)$ must vanish. Specifically, we will show that for all nonnegative integers m

$$\lim_{w \to 0^+} \frac{\theta}{w^m} = 0 \,. \tag{4.18}$$

This condition is clearly true for $n = 0$, as $w = 0$ is the surface $\mathcal{U}$, now suppose there was a violation of (4.18), then there is a smallest value that violates it. Let us denote the smallest m that violates (4.18) as k. Positivity of θ outside of $\mathcal{U}$ guarantees that

$$\lim_{w \to 0^+} \frac{\theta}{w^k} > 0 \,, \tag{4.19}$$

but, since k must be positive, this is equivalent to

$$\lim_{w \to 0^+} \frac{d\theta/dw}{w^{k-1}} > 0 \,. \tag{4.20}$$

Making use of (4.16) and the fact that the signature function is continuous on positive values, we can then say

$$\lim_{w \to 0^+} \left(\int_{\mathcal{U}} \frac{\theta}{w^{k-1}} \gamma \, dA - \frac{Q}{w^{k-1}} \right) > 0 \,, \tag{4.21}$$

where we have assumed $\Lambda = 0$. Since $Q \geq 0$ and (4.18) is satisfied for $m = k-1$, the left side of this equation must be nonpositive. This proves that there is no smallest violation to (4.18), which then shows that all derivatives of θ must vanish at $w = 0$. Since θ is an analytic function near $\mathcal{U}$, the vanishing of all of its derivatives implies that θ is zero somewhere outside of $\mathcal{U}$, and so a trapped surface exists somewhere outside of $\mathcal{U}$, which is impossible. Thus we must have $\Lambda \neq 0$ if the metric is analytic in a region near $\mathcal{U}$.

Note that in the vacuum case $T_{\mu\nu} = 0$, they metric will be analytic, also if there is matter near the horizon such that Q as defined by (4.15) is nonzero, then the inequality on (4.17) becomes a strict one and we do not need the analyticity requirement.

Finally, we will show how to deal with the asymptotically flat case by making use of condition *4b)* instead. First, recall the Gauss-Bonnet theorem on a two-dimensional spatial surface M, which relates the geometry of M to its Euler characteristic χ,

$$\int_M R\,dA + 2\int_{\partial M} K\,ds = 4\pi\chi\,, \tag{4.22}$$

where R is the Ricci scalar on M and $K = y^\mu y^\nu \nabla_\mu z_\nu$ with y being the tangent unit vector on the boundary and z being the normal unit vector on the boundary. Consider that we have a flat spatial slice with a boundary only at infinity, $R = 0$ and $\chi = 1$ for a flat spatial slice, and therefore it must be the case that

$$\int_\infty K\,ds = 2\pi\,, \tag{4.23}$$

and since this equation only depends on the local geometry at ∂M, it must hold for any asymptotically flat spatial slice.

Now consider our case, with a trapped region, consider a spatial slice and denote the part of the spatial slice exterior to the trapped region as D. If D is an annulus, it will have $\chi = 0$, otherwise it will have a negative Euler characteristic. For such a spatial slice, the Einstein equations read

$${}_2R = 16\pi\rho + K_{ab}K^{ab} - K^a{}_a K^b{}_b\,, \tag{4.24}$$

with ${}_2R$ being the Ricci scalar on the slice, K_ab the extrinsic curvature, and $\rho = T_{\mu\nu}\xi^\mu\xi^\nu$ being the energy density measured using the vector ξ as the normal vector to the spatial slicing. The DEC guarantees that $\rho \geq 0$, and if we have a maximal slicing then $K^a{}_a = 0$, so we are left with ${}_2R \geq 0$.

Applying the Gauss-Bonnet theorem to our surface D, which has a boundary at infinity and at our marginally trapped surface $\mathcal{U}$, we have

$$\int_D {}_2R\,dA + 2\int_\infty K\,ds + 2\int_{\mathcal{U}} K\,ds = 4\pi\chi \leq 0\,. \tag{4.25}$$

As we have seen, ${}_2R$ must be nonnegative, and applying (4.23) since D is asymptotically flat, we are left with

$$\int_{\mathcal{U}} K\,ds \leq -2\pi\,. \tag{4.26}$$

However, as $\mathcal{U}$ is a marginally trapped surface, it must have $K = 0$, so this is a contradiction.

As one final note, it is possible to have a space with an asymptotically flat region that does not obey (4.23), if the asymptotic region has a deficit angle. This could happen if the spacetime were to asymptote as a cone, for example. In this case, we would still have $\int_\infty K\,ds = \delta > 0$ and so we would instead obtain

$$\int_{\mathcal{U}} K\,ds \leq -\delta\,, \tag{4.27}$$

which is still in contradiction to $\mathcal{U}$ being a trapped surface. □

Finally, we may prove our final theorem, which states that three-dimensional geons cannot be asymptotically flat.

Theorem 4. *Let M be a globally hyperbolic AF or ALADS spacetime-with-boundary that obeys the Einstein equations with cosmological constant. If the stress-energy tensor obeys the DEC and is analytic, then if M is a geon the spacetime cannot be asymptotically flat.*

Proof. Because the DEC implies the NEC, we have satisfied the conditions of theorem 2, so the topology must be hidden behind a horizon. We will now show that there is no topology behind a horizon in the AF case.

To do this, we must make use of theorem 3, so we must show that M contains a trapped surface. Consider a Cauchy surface in M, denoted V. Since M is a geon, we must have that V is a compact manifold with a point removed $V = \Pi - \{p\}$, and Π is something other than the two-sphere S^2. So, Π must be multiply connected, and as such has a universal covering space which is a multiple cover of Π. Denote this universal covering space as $\hat{\Pi}$.

We have seen that V can be expressed as a punctured manifold, $\Pi - \{p\}$, similarly we can construct a space we will call $\hat{V}$ by puncturing the corresponding points out of $\hat{\Pi}$. This $\hat{V}$ is a covering space of V, but it is not a universal cover because it will not be simply connected in general, due to the punctured points. If we give $\hat{V}$ the geometry of V as its covering space, then we can see that $\hat{V}$ has multiple asymptotic regions which are either all AF or all AADS. We can also construct $\hat{M}$ a the corresponding covering space of M.

The geometries of $\hat{V}$ and $\hat{M}$ are locally the same as V and M, and $\hat{V}$ and $\hat{M}$ have more than one asymptotic region. If one takes a large circle in one asymptotic region and considers null geodesics travelling from that circle to that asymptotic region, the expansion of those null geodesics is positive relative to the corresponding asymptotic region. However, the expansion is negative relative to a different asymptotic region, and there is therefore a trapped surface in the spacetime given by $\hat{M}$. This means that the spacetime $\hat{M}$ satisfies the conditions of theorem 3, and therefore $\hat{M}$ cannot be AF. Since $\hat{M}$ has the same local geometry as M, it must also be the case that M cannot be AF. □

It is worth noting that there are 3-dimensional spacetimes which do have event horizons, such as the BTZ black hole [59]. This solution represents a rotating black hole in a three dimensional spacetime with a negative cosmological constant, consistent with the results we have proven here. Even in such three dimensional black hole spacetimes though, there can only be one event horizon per asymptotic region, as the principle of topological censorship still holds.

Chapter 5

Charged Higher Dimensional Black Holes

5.1 Introduction

In this chapter we will discuss solutions to the Einstein-Maxwell equations, that is, geometries which obey the Einstein equations with stress-energy generated by an electromagnetic field, and the field simultaneously satisfies Maxwells equations. We will specifically be looking for solutions which physically represent isolated black holes in an asymptotically flat spacetime. We will review attempts that others have made to approach this problem, and prove a new theorem demonstrating the form of the electromagnetic potential in a higher dimensional rotating black hole spacetime.

Some solutions to Einstein-Maxwell equations have been known for some time, specifically, in a 3+1-dimensional spacetime the solution which represents an isolated black hole is the Kerr-Newman solution, derived in [42]. The metric and electromagnetic potential are given as

$$ds^2 = -dt^2 + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{2Mr - Q^2}{\rho^2}(dt + a\sin^2\theta d\phi)^2 \,,$$

$$A_\mu dx^\mu = \frac{Qr}{\rho^2}(dt + a\sin^2\theta d\phi) \,, \tag{5.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \,, \qquad \Delta = r^2 + a^2 - 2Mr + Q^2 \,. \tag{5.2}$$

This solution describes a rotating black hole with mass M, angular momentum Ma and charge Q.

Higher dimensional black hole solutions have also been discovered, the first of which was examined by Tangherlini [60], who considered a higher dimensional version of the Schwarzschild metric. The metric can be expressed as

$$ds^2 = -\left(1 - \frac{\alpha}{r^{N-3}}\right) dt^2 + \left(1 - \frac{\alpha}{r^{N-3}}\right)^{-1} dr^2 + r^2 d\Omega_{N-2}^2 , \qquad (5.3)$$

where $d\Omega_k^2$ is the line element on the k-sphere and the parameter α is proportional to the mass of the black hole (the proportionality constant depending on the number of dimensions N).

This metric was generalized by Myers and Perry [10] to a rotating black hole in N-dimensions, the 5-dimensional one can be expressed as

$$\begin{aligned} ds^2 = -dt^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + (r^2 + b^2)\cos^2\theta d\psi^2 + \frac{\rho^2 r^2}{\Delta} dr^2 \\ +\rho^2 d\theta^2 + \frac{\alpha}{\rho^2}\left[dt + a\sin^2\theta d\phi + b\cos^2\theta d\psi\right]^2 , \end{aligned} \qquad (5.4)$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta \qquad \Delta = (r^2 + a^2)(r^2 + b^2) - \alpha r^2 . \qquad (5.5)$$

The black hole represented by this metric has two angular momenta, proportional to the parameters a and b, and a mass proportional to α.

Myers and Perry also derived the metric for a charged black hole in N dimensions. The metric and corresponding electromagnetic potential are given by

$$\begin{aligned} ds^2 = -\left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right) dt^2 + \left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right)^{-1} dr^2 + r^2 d\Omega_{N-2}^2 , \\ A_\mu dx^\mu = \frac{Q}{(N-3)r^{N-3}} dt . \end{aligned} \qquad (5.6)$$

These metrics give us an interesting picture of charged and rotating black holes in varying spacetime dimension, however there is a part of the picture that is missing, there has been no complete solution for a charged, rotating, higher dimensional black hole. Approaches to the solution to this problem will be the topic of this chapter.

Black Holes	3+1 Dimensional	Higher Dimensional
Static	Schwarzschild	Tangherlini
Rotating	Kerr	Myers-Perry
Charged	Reissner-Nordström	Myers-Perry
Rotating + Charged	Kerr-Newman	unknown

Table 5.1: Vacuum and electromagnetic black hole solutions

5.2 Complex Coordinate Substitution

In [61], it was shown that one can "derive" the Kerr metric by means of a complex coordinate substitution on the Schwarzschild metric, we will go over that derivation here.

Begin with the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (5.7)$$

and transform coordinates from t to the null coordinate u defined by

$$u = t - r - 2M\ln(r - 2M)\,, \quad (5.8)$$

to give us the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \quad (5.9)$$

The inverse metric can be expressed as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \overline{m}^\nu + m^\nu \overline{m}^\mu\,, \quad (5.10)$$

with the null tetrad vectors being

$$l^\mu \partial_\mu = \partial_r\,, \qquad n^\mu \partial_\mu = \partial_u - \frac{1}{2}\left(1 - \frac{2M}{r}\right)\partial_r\,,$$

$$m^\mu \partial_\mu = \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right)\,, \quad \overline{m}^\mu \partial_\mu = \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)\,. \quad (5.11)$$

At the point we perform assume the coordinates r and u can take on complex values, and rewrite the null tetrad as

$$l^\mu \partial_\mu = \partial_r\,, \qquad n^\mu \partial_\mu = \partial_u - \frac{1}{2}\left[1 - M\left(\frac{1}{r} + \frac{1}{\overline{r}}\right)\right]\partial_r\,,$$

$$m^\mu \partial_\mu = \frac{1}{\overline{r}\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right) \quad \overline{m}^\mu \partial_\mu = \frac{1}{r\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right)\,. \quad (5.12)$$

with $\bar{r}$ being the complex conjugate of r. Note that the prescription is to keep l and n real, while keeping m and $\overline{m}$ complex conjugates.

Now we may perform the coordinate transformation that changes the Schwarzschild metric into the Kerr metric. The transformation is given by

$$r' = r + ia\cos\theta\,, \quad u' = u - ia\cos\theta\,, \quad \theta' = \theta\,, \quad \phi' = \phi\,. \tag{5.13}$$

If we apply this transformation to the null vectors and then restrict r' and u' to be real valued, we obtain the tetrad

$$l'^{\mu}\partial_{\mu} = \partial_{r'}\,, \qquad n'^{\mu}\partial_{\mu} = \partial_{u'} - \frac{1}{2}\left(1 - \frac{2Mr'}{r'^2 + a^2\cos^2\theta}\right)\partial_{r'}\,,$$

$$m'^{\mu}\partial_{\mu} = \frac{1}{(r' + ia\cos\theta)\sqrt{2}}\left[ia\sin\theta(\partial_{u'} - \partial_{r'}) + \partial_{\theta'} + \frac{i}{\sin\theta'}\partial_{\phi'}\right]\,, \tag{5.14}$$

and $\overline{m}'$ as the complex conjugate of m'.

At this point, the metric $g'^{\mu\nu} = -l'^{\mu}n'^{\nu} - l'^{\nu}n'^{\mu} + m'^{\mu}\overline{m}'^{\nu} + m'^{\nu}\overline{m}'^{\mu}$ is the Kerr metric, with a coordinate transformation applied. The explicit coordinate transformation is given by

$$du' = -dt - \frac{r'^2 + a^2}{r'^2 + a^2 - 2Mr'}dr'\,, \qquad d\phi' = d\Phi + \frac{a}{r'^2 + a^2 - 2Mr'}dr'\,, \tag{5.15}$$

Giving us the Kerr metric in coordinates which have been named (t, r', θ', Φ). So we can see that it is possible to "derive" the Kerr metric by a complex coordinate transformation of the Schwarzschild metric. This is not a derivation in a strict sense, but is rather a trick that arrives at the correct answer. The transformation (5.13) applies angular momentum onto the static Schwarzschild spacetime to give us the Kerr spacetime.

One can perform a similar trick on the Reissner-Nordström metric to arrive at the Kerr-Newman metric. The Reissner-Nordström metric with the null coordinate u can be expressed as

$$g^{\mu\nu} = -l^{\mu}n^{\nu} - l^{\nu}n^{\mu} + m^{\mu}\overline{m}^{\nu} + m^{\nu}\overline{m}^{\mu}\,, \tag{5.16}$$

where

$$l^{\mu}\partial_{\mu} = \partial_r\,, \qquad n^{\mu}\partial_{\mu} = \partial_u - \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\partial_r\,,$$

$$m^\mu \partial_\mu = \frac{1}{r\sqrt{2}} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right) , \quad \overline{m}^\mu \partial_\mu = \frac{1}{r\sqrt{2}} \left(\partial_\theta - \frac{i}{\sin\theta} \partial_\phi \right) . \tag{5.17}$$

Now we once again allow r and u to take on complex values, and we rewrite the tetrad vectors as

$$l^\mu \partial_\mu = \partial_r \, , \qquad n^\mu \partial_\mu = \partial_u - \frac{1}{2} \left[1 - M \left(\frac{1}{r} + \frac{1}{\overline{r}} \right) + \frac{Q^2}{r\overline{r}} \right] \partial_r \, ,$$

$$m^\mu \partial_\mu = \frac{1}{\overline{r}\sqrt{2}} \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right) , \quad \overline{m}^\mu \partial_\mu = \frac{1}{r\sqrt{2}} \left(\partial_\theta - \frac{i}{\sin\theta} \partial_\phi \right) , \tag{5.18}$$

again keeping l and n real with m and $\overline{m}$ complex conjugates. Note that we could have replaced the term $\frac{Q^2}{r^2}$ in n with $\frac{Q^2}{2}(\frac{1}{r^2} + \frac{1}{\overline{r}^2})$ instead of $\frac{Q^2}{r\overline{r}}$, however this would not have worked out to give the correct answer in the end.

Performing the same coordinate transformation as before, with $r' = r + ia\cos\theta$ and $u' = u - ia\cos\theta$, and then restricting r' and u' to take on real values, we obtain the new tetrad

$$l'^\mu \partial_\mu = \partial_{r'} \, , \qquad n'^\mu \partial_\mu = \partial_{u'} - \frac{1}{2} \left(1 - \frac{2Mr' - Q^2}{r'^2 + a^2 \cos^2\theta} \right) \partial'_r \, ,$$

$$m'^\mu \partial_\mu = \frac{1}{(r' + ia\cos\theta)\sqrt{2}} \left[ia\sin\theta(\partial_{u'} - \partial_{r'}) + \partial_{\theta'} + \frac{i}{\sin\theta'} \partial_{\phi'} \right] , \tag{5.19}$$

and again $\overline{m}'$ as the complex conjugate of m'.

With these vectors, the metric $g'^{\mu\nu} = -l'^\mu n'^\nu - l'^\nu n'^\mu + m'^\mu \overline{m}'^\nu + m'^\nu \overline{m}'^\mu$ is the same as the Kerr-Newman metric, with a coordinate transformation applied.

Given how successful the transformation (5.13) was in generating rotating black hole solutions out of nonrotating ones, we might expect something similar will happen even when spacetime is not four dimensional. In this next section we will analyse that calculation.

5.3 Higher Dimensional Complex Coordinate Substitution

One can perform the complex coordinate substitution on higher dimensional metrics as well. This calculation was performed by [62], and we will examine that calculation here.

We being with the higher dimensional charged metric and electromagnetic potential

$$ds^2 = -\left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right) dt^2 + \left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right)^{-1} dr^2 + r^2 d\Omega^2_{N-2} \, ,$$

$$A_\mu dx^\mu = \frac{Q}{(N-3)r^{N-3}} dt \, , \tag{5.20}$$

and transform into the null coordinates

$$du = dt - \left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right)^{-1} dr \, , \tag{5.21}$$

to get the metric

$$ds^2 = -\left(1 - \frac{\alpha}{r^{N-3}} - \frac{Q^2}{r^{2N-6}}\right) du^2 - 2dudr + r^2 d\Omega^2_{N-2} \, . \tag{5.22}$$

To introduce our vieltrad for the inverse metric, we must have a suitable coordinatization of $d\Omega^2_{N-2}$. We will introduce $N-2$ coordinates, denoted θ_i so that we can write the spherical part of the metric as

$$d\Omega^2_{N-2} = d\theta_1^2 + \sin^2\theta_1 (d\theta_2^2 + \sin^2\theta_2 (d\theta_3^2 + ...(d\theta^2_{N-2}))) \, . \tag{5.23}$$

At this point we can express the inverse metric as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + \sum_{i=1}^{k} \left(m_i^\mu \overline{m}_i^\nu + m_i^\nu \overline{m}_i^\mu\right) \, , \tag{5.24}$$

where

$$l^\mu \partial_\mu = \partial_r \, , \qquad n^\mu \partial_\mu = \partial_u - \frac{1}{2}\left(1 - \frac{\alpha}{r^{N-3}} + \frac{Q^2}{r^{2N-6}}\right) \partial_r \, ,$$

$$m_1^\mu \partial_\mu = \frac{1}{r\sqrt{2}} \left(\partial_{\theta_1} + \frac{i}{\sin\theta_1} \partial_{\theta_2} \right) ,$$
$$m_2^\mu \partial_\mu = \frac{1}{r\sin\theta_1 \sin\theta_2 \sqrt{2}} \left(\partial_{\theta_3} + \frac{i}{\sin\theta_3} \partial_{\theta_4} \right) .$$

$$m_k^\mu \partial_\mu = \frac{1}{r\sin\theta_1 \sin\theta_2 ... \sin\theta_{N-4} \sqrt{2}} \left(\partial_{\theta_{N-3}} + \frac{i}{\sin\theta_{N-3}} \partial_{\theta_{N-2}} \right) , \qquad (5.25)$$

we have introduced $\frac{N-2}{2}$ vectors m_i.

At this point the coordinate change (5.13) is introduced and the new metric is interpreted to have real valued coordinates. After doing a coordinate change, the metric we arrive at is

$$ds^2 = -dt^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$
$$+ \frac{\alpha r^{N-3} - Q^2}{r^{2N-8}\rho^2} (dt + a\sin^2\theta d\phi)^2 + r^2 \cos^2\theta d\Omega_{N-4}^2 ,$$
$$A_\mu dx^\mu = \frac{Q}{(N-3) r^{N-5} \rho^2} (dt + a\sin^2\theta d\phi) , \qquad (5.26)$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta , \qquad \Delta = r^2 + a^2 - \frac{\alpha}{r^{N-5}} + \frac{Q^2}{r^{2N-8}} . \qquad (5.27)$$

The metric produced by this calculation does not satisfy the Einstein equations however. Given the metric, the vector potential A does satisfy Maxwells equations, and in any of the special cases of $N = 4$, $Q = 0$, or $a = 0$, then this metric does produce the correct answer one would expect.

It is worth noting that the process of introducing the vectors m is a well-defined process only if N is even, however, the final answer can be interpreted in odd N as well as even, and it does produce the correct result for odd N in the special case of $Q = 0$.

One has to wonder why the complex coordinate transformation method yields the correct results in a four dimensional spacetime, but does not work in higher dimensions. Xu [62] suggests that it may be because the electromagnetic stress-energy tensor,

$$T_{\mu\nu} = F_\mu{}^\sigma F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\lambda} F^{\sigma\lambda} , \qquad (5.28)$$

has zero trace in four dimensions, but not in higher dimensions. This means that the Ricci scalar is nonzero for higher dimensional charged solutions, which complicates the situation compared to the four dimensional case.

5.4 Approximate Solutions

Another possible way to derive the metric which represents a charged rotating black hole in higher dimensions would be to begin with an approximate solution. There has been some research [63–65] into examining black hole solutions with slow rotation, ignoring terms proportional to a^2 in the metric and electromagnetic tensor. The metric and electromagnetic potential are given by

$$ds^2 = -\left(1 - \frac{\alpha}{r^{N-3}} + \frac{Q^2}{r^{2N-6}}\right) dt^2 + \left(1 - \frac{\alpha}{r^{N-3}} + \frac{Q^2}{r^{2N-6}}\right)^{-1} dr^2$$

$$+2a \sin^2\theta \frac{\alpha r^{N-3} - Q^2}{r^{2N-6}} dt d\phi + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega^2_{N-4}) \,,$$

$$A_\mu dx^\mu = -\frac{Q}{(N-3)r^{N-3}}(dt + a \sin^2\theta d\phi) \,. \tag{5.29}$$

This metric and electromagnetic potential does satisfy the Einstein and Maxwell equations, provided that terms of order a^2 are discarded. Note that this metric can be arrived at by taking the metric (5.26) to linear order in a.

We can see that the metric (5.29) does correctly represent the spacetime near a slowly rotating black hole, where the angular momentum per unit mass is much smaller than the characteristic length of the event horizon.

One can also consider a case with very weak charge, so that the term Q does not show up in the metric. The is essentially solving the Maxwell equations on a background of a five-dimensional rotating black hole. The electromagnetic potential one arrives at is the same as the one given in (5.26), suggesting that this electromagnetic potential is possibly the correct one for a rotating charged black hole.

We can make this last statement more rigourous by considering a spacetime metric where the charge is “slowly turned on”, and showing that the electromagnetic potential A does not change during this process.

Theorem 5. *Let (M, g) be a spacetime with a metric which depends analytically on a parameter Q, and let g_0 be the metric one obtains by setting $Q = 0$. If A is linear in Q and satisfies the Maxwell equations $\Box A = 0$ for the metric g, then A satisfies the Maxwell equations $\Box_0 A = 0$ for the metric g_0.*

Proof. If g is analytic in Q, then it must be given exactly by its Taylor expansion, which means that the linear operator $\Box$ must also be given by its Taylor expansion. Acting on an arbitrary field φ, $\Box$ must appear as

$$\Box\varphi = \sum_{i=0}^{\infty} Q^i \Box_i \varphi \,. \tag{5.30}$$

Note that $\Box_0$ is the operator associated with g_0. If A is linear in Q, we can write $A = QB$, where the field B does not depend on Q. We then have

$$\Box A = \sum_{i=0}^{\infty} Q^{i+1} \Box_i B \,, \tag{5.31}$$

if $\Box A = 0$ is to hold for all values of Q, then order by order we must have $\Box_i B = 0$, specifically this must mean that $\Box_0 A = 0$. □

We would also like to note that the condition that A be linear in Q can be justified by observing that the electric charge is typically defined as an integral of F over some Gaussian surface enclosing the space of interest. Due to this, Q will be linear in F, and since F is linear in A it should be the case that A is linear in Q.

Theorem 5 tells us that, if we were to find a solution that represents a charged and rotating black hole, the electromagnetic potential A must be given by the same expression that solves the test charge case,

$$A_\mu dx^\mu = \frac{Q}{(N-3) r^{N-5} \rho^2} (dt + a \sin^2\theta d\phi) \,. \tag{5.32}$$

So we can see that the electromagnetic potential in (5.26) is the correct expression, it is the metric that does not work out.

The existence of a solution that would represent a charged rotating black hole in five dimensions can be argued for, especially if one considers the five dimensional Einstein-Maxwell equations to be representing something "physical". In this case the idea that one could simply throw some charge and angular momentum into a physical black hole is not surprising. From a mathematical perspective we have already seen approximate solutions, and the only question is if these approximate solutions are actually approximating a full solution. This is a subject called "linearization stability" for differential equations, and was analysed for the Einstein equations by [66]. In that paper they considered a metric g that satisfies the vacuum Einstein equations

$$\mathbf{R}(g) = 0\,, \tag{5.33}$$

with $\mathbf{R}(g)$ being the Ricci tensor of g, and you have a tensor h that satisfies the linearized Einstein equations

$$\partial(\mathbf{R}(g)) \cdot h = 0\,, \tag{5.34}$$

with $\partial(\mathbf{R}(g))$ the derivative of the Ricci tensor viewed as a function of the metric. In this case, one asks if there exists a metric $g(\lambda)$, which depends on a parameter λ, such that $g(\lambda)$ solves the Einstein equations and $g(0) = g$ and $\partial_\lambda g(\lambda)|_{\lambda=0} = h$. If this "full solution" $g(\lambda)$ exists then we say that the background metric g is linearization stable. [66] showed conditions for metrics under which they would have this feature of linearization stability.

For our purposes we would like two generalizations on the work of [66], first we need that the proof works in five dimensions, instead of four, and second we need to consider metrics which solve the Einstein-Maxwell equations rather than the vacuum Einstein equations. The first requirement is simple enough, one can work through all the results of their paper in any number of dimensions greater than four and it all holds, however the removal of vacuum equations represents a more difficult complication. More work is needed in this area.

Chapter 6

Discussion

We will now review the main results of this thesis.

We began by demonstrating separation of variables of the Hamilton-Jacobi and Klein-Gordon equations for particles and scalar fields on higher dimensional black hole spacetimes, specifically the Kerr-de Sitter spacetimes derived in [11]. We have showed that complete separation of variables was possible in the special case of two unequal spin parameters. The occurrence of such separation is connected to the existence of a Killing tensor in the spacetime, which represents an underlying symmetry. The assumption of equal spin parameters was used to eliminate many of the off-diagonal components that would exist in the metric otherwise. This assumption greatly enlarged the symmetry group of the spacetime, giving rise to additional Killing vectors outlined in section 3.5. In [67] it is demonstrated that the higher dimensional Myers-Perry metrics always posses a Killing tensor, however one extra conserved quantity, along with the already present rotational and time-symmetric Killing vectors, is not enough to ensure separation of variables in the case that the number of spacetime dimensions is larger than 5. We analysed the equations of motion that resulted from the Hamilton-Jacobi equations, and examined the behaviour in some special cases. There is the possibility of further research in this area if one can relax the assumption of two unequal spin parameters, however, complete separation is only likely if additional hidden symmetries exist, and that might not be the case.

Next we examined geons, which are gravitational structures held together by their nontrivial topology. We proved a theorem that states that an asymptotically flat three dimensional spacetime which obeys the dominant energy condition cannot be a geon. This result occurs because of the principle of topological censorship guarantees that null curves that travel from “infinity” and back to “infinity” must be continuously deformable to a curve that stays at “infinity”. This means that any geon structure must be hidden behind an event horizon. The next part of the proof showed that three dimensional spacetimes with event horizons cannot be asymptotically flat. An additional assumption of analyticity of the metric or the existence of a maximal slicing of the spacetime was needed to prove this result. The principle of topological censorship combined with the nonexistence of event horizons in an asymptotically flat space was then used to guarantee the nonexistence of asymptotically flat three dimensional geons.

Finally, we considered charged higher dimensional rotating black holes. This is important as it completes a picture of the black hole solutions. The Kerr solution describes a rotating black hole in four dimensions, and the Reissner-Nordström solution describes a charged static black hole in four dimensions. The combination of these two, the Kerr-Newman solution, describes a rotating charged black hole in a four dimensional spacetime, and forms a complete description of four dimensional electrovac stationary black holes. In higher dimensions, the Myers-Perry solution describes a rotating black hole, and Myers and Perry also discovered the higher dimensional charged static solution. One would like to "combine" these into a charged and rotating solution. This has proven difficult, however, we did an overview of the literature examining slowly rotating and weakly charged solutions. We have also proven the form that the electromagnetic potential A most likely takes in the general case, if a solution exists. There is still much to do in this area, the full solution for a charged, rotating black hole in Maxwell-Einstein theory would be very interesting to find.

Bibliography

[1] J. H. Schwarz, *Superstring Theory*, *Phys. Rept.* **89** (1982) 223–322.

[2] V. A. Rubakov and M. E. Shaposhnikov, *Do We Live Inside a Domain Wall?*, *Phys. Lett.* **B125** (1983) 136–138.

[3] K. K. Roy Maartens, *Brane-world gravity*, *Living Reviews in Relativity* **13** (2010), no. 5.

[4] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B429** (1998) 263–272, [hep-ph/9803315].

[5] D. M. Eardley and S. B. Giddings, *Classical black hole production in high-energy collisions*, *Phys. Rev.* **D66** (2002) 044011, [gr-qc/0201034].

[6] **EOT-WASH Group** Collaboration, E. G. Adelberger, *Sub-millimeter tests of the gravitational inverse square law*, hep-ex/0202008.

[7] C. D. Hoyle *et al.*, *Sub-millimeter tests of the gravitational inverse-square law*, *Phys. Rev.* **D70** (2004) 042004, [hep-ph/0405262].

[8] D. J. Kapner *et al.*, *Tests of the gravitational inverse-square law below the dark-energy length scale*, *Phys. Rev. Lett.* **98** (2007) 021101, [hep-ph/0611184].

[9] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (1963) 237–238.

[10] R. C. Myers and M. J. Perry, *Black Holes in Higher Dimensional Space-Times*, *Ann. Phys.* **172** (1986) 304.

[11] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope, *Rotating black holes in higher dimensions with a cosmological constant*, *Phys. Rev. Lett.* **93** (2004) 171102, [hep-th/0409155].

[12] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[13] S. Chandrasekhar, *The Mathematical Theory of Black Holes.* Oxford University Press, 1983.

[14] B. Carter, *Global structure of the Kerr family of gravitational fields*, *Phys. Rev.* **174** (1968) 1559–1571.

[15] W. Kinnersley, *Type D Vacuum Metrics*, *J. Math. Phys.* **10** (1969) 1195–1203.

[16] B. Carter, *Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations*, *Commun. Math. Phys.* **10** (1968) 280.

[17] D. R. Brill, P. L. Chrzanowski, C. Martin Pereira, E. D. Fackerell, and J. R. Ipser, *Solution of the scalar wave equation in a kerr background by separation of variables*, *Phys. Rev.* **D5** (1972) 1913–1915.

[18] S. A. Teukolsky, *Rotating black holes - separable wave equations for gravitational and electromagnetic perturbations*, *Phys. Rev. Lett.* **29** (1972) 1114–1118.

[19] S. Chandrasekhar, *The Solution of Dirac's Equation in Kerr Geometry*, *Proc. Roy. Soc. Lond.* **A349** (1976) 571–575.

[20] V. P. Frolov, V. Skarzhinsky, A. Zelnikov, and O. Heinrich, *Equilibrium Configurations of a Cosmic String Near a Rotating Black Hole*, *Phys. Lett.* **B224** (1989) 255.

[21] M. Walker and R. Penrose, *On quadratic first integrals of the geodesic equations for type [22] spacetimes*, *Commun. Math. Phys.* **18** (1970) 265–274.

[22] P.-J. De Smet, *Black holes on cylinders are not algebraically special*, *Class. Quant. Grav.* **19** (2002) 4877–4896, [hep-th/0206106].

[23] P.-J. De Smet, *The Petrov type of the five-dimensional Myers-Perry metric*, *Gen. Rel. Grav.* **36** (2004) 1501–1504, [gr-qc/0312021].

[24] V. P. Frolov and D. Stojkovic, *Quantum radiation from a 5-dimensional rotating black hole*, *Phys. Rev.* **D67** (2003) 084004, [gr-qc/0211055].

[25] V. P. Frolov and D. Stojkovic, *Particle and light motion in a space-time of a five- dimensional rotating black hole*, *Phys. Rev.* **D68** (2003) 064011, [gr-qc/0301016].

[26] V. P. Frolov and K. A. Stevens, *Stationary strings near a higher-dimensional rotating black hole*, *Phys. Rev.* **D70** (2004) 044035, [gr-qc/0404035].

[27] M. Vasudevan and K. A. Stevens, *Integrability of particle motion and scalar field propagation in Kerr-(anti) de Sitter black hole spacetimes in all dimensions*, *Phys. Rev.* **D72** (2005) 124008, [gr-qc/0507096].

[28] J. A. Wheeler, *Geons*, *Phys. Rev.* **97** (1955) 511–536.

[29] C. W. Misner and J. A. Wheeler, *Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space*, *Annals Phys.* **2** (1957) 525–603.

[30] J. A. Wheeler, *On the Nature of quantum geometrodynamics*, *Annals Phys.* **2** (1957) 604–614.

[31] R. Sorkin, *The Quantum Electromagnetic Field in Multiply Connected Space*, *J. Phys.* **A12** (1979) 403–421.

[32] R. D. Sorkin, *Introduction to Topological Geons*, . Print-85-0952 (SYRACUSE).

[33] J. L. Friedman and R. D. Sorkin, *Spin 1/2 From Gravity*, *Phys. Rev. Lett.* **44** (1980) 1100–1103.

[34] J. l. Friedman and D. M. Witt, *Internal Symmetry Groups Of Quantum Geons*, *Phys. Lett.* **B120** (1983) 324–328.

[35] D. M. Witt, *Symmetry Groups Of State Vectors In Canonical Quantum Gravity*, *J. Math. Phys.* **27** (1986) 573–592.

[36] D. M. Witt, *Vacuum Space-Times That Admit No Maximal Slice*, *Phys. Rev. Lett.* **57** (1986) 1386–1389.

[37] K. A. Stevens, K. Schleich, and D. M. Witt, *Non-existence of Asymptotically Flat Geons in 2+1 Gravity*, *Class. Quant. Grav.* **26** (2009) 075012, [arXiv:0809.3022].

[38] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, *Nucl. Phys.* **B311** (1988) 46.

[39] E. Witten, *Three-Dimensional Gravity Revisited*, arXiv:0706.3359.

[40] H. Reissner, *Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie*, *Ann. Phys. (Germany)* **50** (1916) 106–120.

[41] G. Nordstrom, *On the energy of the gravitational field in Einstein's theory*, *Proc. Kon. Ned. Akad. Wet.* **20** (1918) 1238–1245.

[42] E. T. Newman *et al.*, *Metric of a Rotating, Charged Mass*, *J. Math. Phys.* **6** (1965) 918–919.

[43] W. Israel, *Event horizons in static vacuum space-times*, *Phys. Rev.* **164** (1967) 1776–1779.

[44] B. Carter, *Axisymmetric Black Hole Has Only Two Degrees of Freedom*, *Phys. Rev. Lett.* **26** (1971) 331–333.

[45] V. P. Frolov and I. D. Novikov, *Black Holes: Basic Concepts and New Developments.* Kluwer Academic Publishers, 1998.

[46] J. D. Bekenstein, *Black hole hair: Twenty-five years after*, gr-qc/9605059.

[47] R. Emparan and H. S. Reall, *A rotating black ring in five dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101, [hep-th/0110260].

[48] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* W. H. Freeman and Company, 1973.

[49] M. Vasudevan, K. A. Stevens, and D. N. Page, *Separability of the Hamilton-Jacobi and Klein-Gordon equations in Kerr-de Sitter metrics*, *Class. Quant. Grav.* **22** (2005) 339–352, [gr-qc/0405125].

[50] G. T. Horowitz and R. C. Myers, *The AdS/CFT Correspondence and a New Positive Energy Conjecture for General Relativity*, *Phys. Rev.* **D59** (1998) 026005, [hep-th/9808079].

[51] N. R. Constable and R. C. Myers, *Spin-two glueballs, positive energy theorems and the AdS/CFT correspondence*, *JHEP* **10** (1999) 037, [hep-th/9908175].

[52] G. J. Galloway, S. Surya, and E. Woolgar, *A uniqueness theorem for the adS soliton*, *Phys. Rev. Lett.* **88** (2002) 101102, [hep-th/0108170].

[53] J. L. Friedman, K. Schleich, and D. M. Witt, *Topological censorship*, *Phys. Rev. Lett.* **71** (1993) 1486–1489, [gr-qc/9305017].

[54] M. D. Kruskal, *Maximal extension of Schwarzschild metric*, *Phys. Rev.* **119** (1960) 1743–1745.

[55] S. Hawking and G. Ellis, *The large scale structure of space-time.* Cambridge University Press, 1973.

[56] G. J. Galloway, K. Schleich, D. M. Witt, and E. Woolgar, *Topological Censorship and Higher Genus Black Holes*, *Phys. Rev.* **D60** (1999) 104039, [gr-qc/9902061].

[57] G. J. Galloway, K. Schleich, D. Witt, and E. Woolgar, *The AdS/CFT correspondence conjecture and topological censorship*, *Phys. Lett.* **B505** (2001) 255–262, [hep-th/9912119].

[58] E. Newman and R. Penrose, *An Approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566–578.

[59] M. Banados, C. Teitelboim, and J. Zanelli, *The Black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849–1851, [hep-th/9204099].

[60] F. R. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, *Nuovo Cim.* **27** (1963) 636–651.

[61] E. T. Newman and A. I. Janis, *Note on the Kerr spinning particle metric*, *J. Math. Phys.* **6** (1965) 915–917.

[62] D.-Y. Xu, *Exact Solutions of Einstein and Einstein-Maxwell Equations In Higher Dimensional Space-Time*, *Class. Quant. Grav.* **5** (1988) 871–881.

[63] A. N. Aliev, *Charged Slowly Rotating Black Holes in Five Dimensions*, *Mod. Phys. Lett.* **A21** (2006) 751–758, [gr-qc/0505003].

[64] A. N. Aliev, *Rotating black holes in higher dimensional Einstein-Maxwell gravity*, *Phys. Rev.* **D74** (2006) 024011, [hep-th/0604207].

[65] A. N. Aliev, *Higher dimensional rotating charged black holes*, gr-qc/0612169.

[66] A. E. Fischer and J. E. Marsden, *Linearization Stability of the Einstein equations*, *Bulletin of the American Mathematical Society* **79** (1973) 997–1003.

[67] V. P. Frolov and D. Kubiznak, *'Hidden' symmetries of higher dimensional rotating black holes*, *Phys. Rev. Lett.* **98** (2007) 011101, [gr-qc/0605058].

[68] A. Ashtekar and R. O. Hansen, *A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity*, *J. Math. Phys.* **19** (1978) 1542–1566.

[69] R. M. Wald, *General Relativity.* The University of Chicago Press, 1984.

[70] A. Ashtekar and A. Magnon, *Asymptotically anti-de Sitter space-times*, *Class. Quant. Grav.* **1** (1984) L39–L44.

Appendix A

Causality and Asymptotic Spaces

Here we will discuss causality in a technical sense, and define what it means for a spacetime to be asymptotically flat. Asymptotically flat spacetimes are important as they represent isolated gravitating systems, such that far away from them their gravitational influence is negligible. One would like to define an asymptotically flat space as one in which the metric approaches a flat metric as $r \to \infty$. However, it is difficult to be sure that such a definition is coordinate independent. Ashtekar and Hansen [68] demonstrated a formalism where one can add in a "point at infinity" by making use of a conformal transformation. Before we can discuss this definition, however, we must first review the notion of casuality in the context of general relativity.

A.1 Causality

In this section, we will prepare the formalism of how to define causality in general relativity. We will assume spacetime is time-orientiable, that is, it is possible to divide non-spacelike vectors into two categories, labelled future-directed and past-directed, and that it is possible to make this choice in a continuous way. Once we have this division, it is possible to talk about the *chronological future* of a point p, denoted $I^+(p)$, which is the set of points that one can reach from p by following a future-directed, timelike curve. One can also talk about the chronological future of a set of points, by simply taking the union of the chronological futures of each individual point in the set. It is also possible to define the chronological past of a point p, denoted $I^-(p)$, in a similar way, replacing 'future' with 'past'. We will also talk about the *causal future* of a point p, denoted $J^+(p)$, which is the set of points that one can reach from p by following a future-directed, non-spacelike curve. Similarly, it is possible to define the causal future of a set of points and to define the causal past J^-.

One would like to discuss spacetimes that have no closed timelike curves. If spacetime contained a closed timelike curve, then a physical observer in that spacetime would be able to effectively "time travel", to visit their past self. We will say that a spacetime that contains no closed timelike curves satisfies the *chronology condition*. Note that one can also word this condition as saying that there exists no point p with the property that $p \in I^+(p)$. One can also strengthen this condition to require that there be no closed non-spacelike curves, this condition is called the *causality condition*, and can also be rewritten that there is no point p with the property that $p \in J^+(p)$. We will also spend some time discussing spacetimes that obey the *strong causality condition*, which holds at a point p if for each neighbourhood S of p it contains a neighbourhood $Q \subset S$ of p such that non-spacelike curves do not intersect Q more than once.

A *Cauchy surface* in a spacetime is a spacelike surface such that every non-spacelike curve intersects it exactly once. If one were discussing evolution of fields on a spacetime, you would specify all of your initial data on a Cauchy surface, that and the field equations would be sufficient to specify the field everywhere in the future of that surface. This leads us to the idea of a *globally hyperbolic* space, such a space must obey the strong causality condition everywhere and must also have $J^+(p) \cap J^-(q)$ be compact for all points p and q. This is saying that $J^+(p) \cap J^-(q)$ should not contain any points on the edge of the spacetime (at infinity or a singularity). The intuitive reason for this requirement is that then initial data on a Cauchy surface will specify the field at points in the future of that surface, there is no interference from points which are outside of the spacetime.

A.2 Asymptotically Flat Spacetimes

Now we wish to define what it means for a spacetime to be asymptotically flat. This formulation is often useful for examining properties of asymptotic behaviour of fields that are radiating in the spacetime. To consider this analysis, we will examine with an example given in [69]. We will begin with Minkowski spacetime in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{A.1}$$

Suppose we wanted to examine properties of radiation going to infinity in this spacetime, we would need to take limits going to infinity along null directions so it is useful to introduce null coordinates

$$v = t + r\,, \qquad u = t - r\,, \tag{A.2}$$

so the metric becomes

$$ds^2 = -du\,dv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{A.3}$$

At this point, if we wanted to examine properties of radiation, we would consider $v \to \infty$ and look at the lowest nonvanishing term of the field. This is not a great method for doing things though, as it is not coordinate invariant, and so it is not clear how to generalize this process to a curved spacetime. It would be much simpler to make infinity into a definite "place", so that we could simply examine properties of fields at that "place".

The simplest thing to do would be to simply perform a coordinate transformation from v to $V = 1/v$, so that $v \to \infty$ is located instead at $V = 0$. In these coordinates, the metric appears as

$$ds^2 = \frac{1}{V^2}du\,dv + \frac{1}{4}\left(\frac{1}{V} - u\right)^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{A.4}$$

Unsurprisingly, this metric is singular at $V = 0$, making it difficult to do tensor analysis at that "place".

At this point, suppose we consider a new, unphysical metric, which is conformally related to our metric. We will define $g' = \Omega^2 g$, and use $\Omega = V$ as the conformal factor. This unphysical metric looks like

$$ds'^2 = g'_{\mu\nu}dx^\mu dx^\nu = -du\,dV + \frac{1}{4}(1-uV)^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{A.5}$$

which is well behaved at $V = 0$. We can then extend the spacetime to include the points represented by $V = 0$, which could not be included in the original spacetime, Minkowski spacetime being inextendible. In this unphysical metric, we can do tensor analysis at $V = 0$, which in terms of the physical metric, would be located "at infinity". The effect is that we have used a conformal transformation to bring infinity to a finite place.

In fact, it would be even better to use a conformal factor that was well behaved throughout the spacetime. The one we used $V = 1/v$ needlessly diverged near $v = 0$ which is the location of a physical place in the original spacetime. Furthermore, our construction has allowed us access to "future null infinity" $v \to \infty$, we have no access to "past null infinity" $(u \to \infty)$ or "spatial infinity" $(r \to \infty)$. A different choice of conformal factor can solve all of these problems, let us use $g' = \Omega^2 g$ with

$$\Omega^2 = 4(1+v^2)^{-1}(1+u^2)^{-1}\,, \tag{A.6}$$

then g' is a smooth metric everywhere in the original Minkowski spacetime, but we can extend the metric by performing a coordinate transformation from (v, u) to (T, R) defined by

$$T = \tan^{-1} v + \tan^{-1} u\,, \qquad R = \tan^{-1} v - \tan^{-1} u\,. \tag{A.7}$$

Then T and R have the coordinate restrictions

$$-\pi < T + R < \pi\,, \quad -\pi < T - R < \pi\,, \quad R \geq 0\,. \tag{A.8}$$

The metric g' in these coordinates is given by

$$ds'^2 = g'_{\mu\nu} dx^\mu dx^\nu = -dT^2 + dR^2 + \sin^2 R (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{A.9}$$

which is just the metric on $\mathbb{R} \times S^3$ except that we have the coordinate restriction (A.8). Naturally, it is possible to extend this metric to all $T \in \mathbb{R}$ and $R \in [0, \pi]$. So we have seen that Minkowski space is conformal to an open subset of $\mathbb{R} \times S^3$. The subset is a described by a triangle in (T, R) space given by the restrictions (A.8) (see figure A.1). The boundary of this triangle breaks into five important pieces that we will describe. First there is "future timelike infinity", often denoted i^+, which is located at $R = 0$, $T = \pi$, next there is "future null infinity", often denoted $\mathcal{I}^+$, which is the line $T = -R + \pi$ for $R \in [0, \pi]$, i^+ is at the top end of $\mathcal{I}^+$. At the lower end of $\mathcal{I}^+$ is "spacelike infinity", denoted i^0 and located at the point $T = 0, R = \pi$, then the line $T = R - \pi$ for $R \in [0, \pi]$ is "past null infinity", denoted $\mathcal{I}^-$, and finally at the bottom of the triangle is "past timelike infinity", located at $R = 0$, $T = -\pi$, and denoted i^-.

Note that all timelike geodesics begin at i^- and end at i^+, all null geodesics begin at $\mathcal{I}^-$ and end at $\mathcal{I}^+$, and all spacelike geodesics begin and end at i^0.

If one wishes to consider the behaviour of fields in the space, then depending on the asymptotics one wishes to impose, it typically will be sufficient to require that a suitable power of Ω^{-1} times the field be well behaved at one of these infinities.

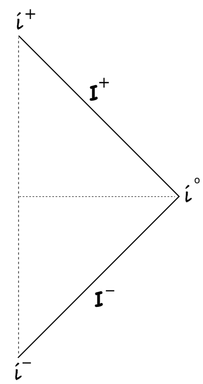

Figure A.1: Minkowski space in T-R coordinates.
The vertical dashed line is $R = 0$, while the horizontal dashed line is $T = 0$. Each point is actually a two sphere, except for the line $R = 0$, the center of spherical coordinates and the point $R = \pi$, $T = 0$, the point at infinity.

This completes the description of the asymptotics of Minkowski space in terms of the conformal space g'. One would like to define an asymptotically flat space as one which can undergo a similar analysis, we expect a space to be mapped into a larger unphysical space, which it is conformal to. If the unphysical space with the conformal factor has the correct behaviour at "infinity", we will then say that to original space was in fact asymptotically flat. However, often we only expect our spaces to be flat at "spatial or null infinity", rather than at "timelike infinity", which would describe early or late times at a fixed location. The reason is that we may wish to consider an isolated gravitating body, which exists at a finite location for all times. Thus, we will not demand any particular restrictions on the properties of i^+ and i^-, but instead focus on features of i^0 and $\mathcal{I}^+$ and $\mathcal{I}^-$. The full conditions were described in [68], and we will discuss them here.

We will say that a spacetime with metric (M, g) is *asymptotically flat* (AF) if it can be conformally included into a larger spacetime M' with a metric g' related to g by a conformal transformation $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with the following features:

(1) g' is smooth everywhere except possibly at a single point i^0.

(2) $\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = M' \backslash M$, where $\overline{S}$ denotes the closure of a set S. So the boundary of M, denoted ∂M, is made up of three disjoint parts, i^0, $\mathcal{I}^+$, and $\mathcal{I}^-$, where $\mathcal{I}^\pm = \partial J^\pm(i^0) \backslash i^0$.

(3) There exists an open neighbourhood, V, of ∂M such that the spacetime V with metric g' is strongly causal.

(4) On $\mathcal{I}^+$ and $\mathcal{I}^-$ we have $\Omega = 0$, $\nabla'_\mu \Omega \neq 0$, with ∇' being the derivative operator associated with g'. We also requite $\Omega(i^0) = 0$ and $\lim_{i^0} \nabla'_\mu \Omega = 0$ and $\lim_{i^0} \nabla'_\mu \nabla'_\nu \Omega = 2g'_{\mu\nu}$.

Let us now discuss these conditions in detail. Condition (1) is used as the definition of the point i^0 as the point at infinity, the metric needs to be well-behaved everywhere except possibly that point, which is intended to be physically inaccessible. Condition (2) guarantees that i^0 is spacelike related to all points in M, that everywhere timelike and null related to i^0 is not in M. Condition (3) guarantees that the distant points in M have the same causality conditions as one would get in a flat spacetime. Condition (4) ensures i^0 actually behaves correctly as a point at infinity, demanding Ω and $\nabla'\Omega$ be zero at i^0 ensures that i^0 is actually infinitely far away from points in M, and the requirement for $\nabla'\nabla'\Omega$ to approach $2g'$ ensures the metric has the correct asymptotic falloff. This condition is discussed in more detail in [68]. The larger space M' is often called the *Penrose compactified spacetime*.

We can also modify these conditions to define what it means for a spacetime to be asymptotically anti-de Sitter. The precise formulation is discussed in [70]. A spacetime (M, g) is called *asymptotically locally anti-de Sitter* (ALADS) if it can be conformally included into a larger spacetime M' with a metric g' related to g by a conformal transformation $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with the following features:

(1) $\mathcal{I} = \partial M$ is topologically $S^n \times \mathbb{R}$ and, on $\mathcal{I}$, $\Omega = 0$

(2) g satisfies $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$ with $\Lambda < 0$ and $\Omega^{-3} T_\mu^{\ \nu}$ admits a smooth limit to $\mathcal{I}$

These conditions are discussed in detail in [70], but it is worth noting in particular that the sign of Λ guarantees that $\mathcal{I}$ is a timelike surface, rather than a null one. This means that there exist spacelike geodesics that go there, removing the need for i^0 in ALADS spacetimes. The definition is "local" in the sense that $\mathcal{I}$ might have a more complicated topology than anti-de Sitter space usually has.