The Isoperimetric Inequality in $\mathbb{R}^n$

by

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Abstract

This thesis presents a complete proof of the isoperimetric inequality for a smooth surface in Euclidean space. The proof uses the Brunn-Minkowski Inequality, the formulae for the first variations of area and Alexandrov's theorem.

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Chapter 1

Introduction

The question of how to minimize the surface area for a given volume was posed as early as 200 BCE by Archimedes and Zenodorus in Ancient Greece [9, p.159]. However, it would take over 2000 years before the isoperimetric inequality was proven rigorously.

In its simplest form, the isoperimetric inequality states that for a region in the plane of area A enclosed by a curve of length L,

$$4\pi A \leq L^2$$

and equality holds if and only if the curve is a circle. The history of the two-dimensional problem is very entertainingly laid out by Victor Blasjo in his article "The Evolution of the Isoperimetric Problem" [2]. Possibly the Babylonian and certainly Greek and Arab mathematicians were interested in this problem, motivated by astronomy. Zenodorus more or less proved that the circle has a greater area than any polygon with the same perimeter.

The problem was further explored by Jakob Steiner (1796-1863). Steiner developed five proofs of the isoperimetric theorem, in each case showing that one can increase the area of any figure that is not a circle while preserving the perimeter. However, his proofs had one major flaw: he assumed the existence of a solution. Weierstrass was the first to finally prove the existence of a solution in 1879, and thus the proof of the two-dimensional isoperimetric inequality was at last complete.

The isoperimetric inequality in $\mathbb{R}^n$ states that in order to minimize the surface area of a domain given a fixed volume, the domain must be a sphere. Precisely:

Theorem 1. *Let D be a domain in $\mathbb{R}^n$. The volume V of the region and the volume A of its boundary satisfy*

$$n^n \omega_n V^{n-1} \leq A^n$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Moreover, equality holds if and only if $D = B_r^n(a)$, a ball of radius r about some point a in $\mathbb{R}^n$

Burago and Zalgaller summarize the development of the higher dimensional problem in [3]. The first complete proof of the three-dimensional case was given by Schwartz in 1884. His method was expanded on by Schmidt to proof the higher-dimensional case in 1939.

This thesis presents a complete, self-contained proof of the isoperimetric inequality for smooth domains in $\mathbb{R}^n$ in all dimensions, which is difficult to find in one place in the existing literature. In order to prove the inequality, we will first use the Brunn-Minkowski Inequality to show that a sphere will minimize the surface area. Then we must show that the sphere is in fact the only surface that will minimize the surface area. We will use the formulae for the first variations of area to show that if a surface S has minimum area among all surfaces bounding the same volume then the mean curvature of S must be constant. Finally, we will appeal to Alexandrov's Theorem, specifically, that any surface with constant mean curvature must be a sphere. While the isoperimetric inequality can be proved for an arbitrary domain $D \subset \mathbb{R}^n$, we will only prove it for domains bounded by smooth surfaces.

Chapter 2

Brunn-Minkowski Inequality

The Brunn-Minkowski inequality is an inequality relating the areas of compact subsets of $\mathbb{R}^n$. In this chapter we prove the Brunn-Minkowski Inequality in arbitrary dimensions. We then use the Brunn-Minkowski inequality to give a short proof that sphere minimizes volume among all surfaces enclosing a given volume.

Theorem 2 (The Brunn-Minkowski Inequality)**.** *Let A and B be two bounded open subsets of Euclidean space $\mathbb{R}^n$. Then*

$$(volA)^{1/n} + (volB)^{1/n} \leq (vol(A+B))^{1/n}.$$

The following proof is a generalization to n-dimensions of the three-dimensional proof presented in Chapter 7 of Montiel and Ros [8].

Proof. For any two A and B subsets of $\mathbb{R}^n$, we will define their *sum* $A+B$ as the set

$$A + B = \{a + b | a \in A, b \in B\}.$$

We will first prove this inequality in the case where A and B are hyperrectangles. Then we will consider the case when A and B are the sums of hyperrectangles. Finally, we will generalize to any bounded open $A, B \subset \mathbb{R}^n$.

Consider the hyperrectangles $A = I_1 \times ... \times I_n$ and $B = J_1 \times ... \times J_n$, where I_i and J_i are bounded open intervals in $\mathbb{R}$ with lengths a_i and b_i respectively, for each $i = 1, ..., n$. Then

$$\begin{aligned}\frac{(volA)^{1/n} + (volB)^{1/n}}{(vol(A+B))^{1/n}} &= \frac{(\prod_{i=1}^n a_i)^{1/n} + (\prod_{i=1}^n b_i)^{1/n}}{\prod_{i=1}^n (a_i + b_i)^{1/n}} \\ &= (\prod_{i=1}^n \frac{a_i}{a_i + b_i})^{1/n} + (\prod_{i=1}^n \frac{b_i}{a_i + b_i})^{1/n}\end{aligned}$$

But the right hand side is the sum of the geometric means of $\{\frac{a_i}{a_i+b_i}\}_{i=1}^n$ and $\{\frac{b_i}{a_i+b_i}\}_{i=1}^n$. Since the geometric mean is always less than or equal to the

arithmetic mean,

$$\frac{(volA)^{1/n} + (volB)^{1/n}}{(vol(A+B))^{1/n}} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{a_i}{a_i + b_i} + \frac{1}{n}\sum_{i=1}^{n} \frac{b_i}{a_i + b_i} = 1.$$

This proves the inequality in the case of hyperrectangles.

Now we will consider the case $A = \bigcup_{i=1}^{l} A_i$ and $B = \bigcup_{i=1}^{m} B_i$, where each of these unions is a disjoint union and each A_i, B_i is a hyperrectangle in the same form as in case one. We will use induction on $n + m$ to prove the inequality in this case.

The base case, $n + m = 2$, is true by case one. Now suppose that $n + m \geq 3$ and suppose that the inequality holds whenever the total number of hyperrectangles is less than $n + m$. Without loss of generality, we may assume that $n \geq 2$. Notice that we can find an $(n-1)$-dimensional hyperplane separating $A_1 = I_1 \times ... \times I_n$ and $A_2 = J_1 \times ... \times J_n$. Since A_1 and A_2 are disjoint, there must be some $i \in 1, \ldots, n$ such that I_i and J_i are also disjoint. Then if $p \in \mathbb{R}$ is any point in between I_i and J_i, the $(n-1)$-dimensional hyperplane P orthogonal to the ith coordinate axis that contains p separates A_1 and A_2.

The hyperplane P divides $\mathbb{R}^n$ into two open half-spaces, P^+ and P^-. Define $A^+ = A \cap P^+$ and $A^- = A \cap P^-$. Then

$$A^+ = \bigcup_{i=1}^{l^+} A_i^+ \text{ and } A^- = \bigcup_{i=1}^{l^-} A_i^-$$

are also sums of hyperrectangles. Furthermore, $l^+, l^- < l$ since P separates A_1 and A_2. We can now find another hyperplane Q parallel to P such that

$$\frac{volA^+}{volA} = \frac{volB^+}{volB},$$

where B^+ and B^- are in the half-spaces Q^+ and Q^-. To find such a plane, let Q be a hyperplane parallel to P such that B is entirely on one side of Q. Then displace Q in the ith direction towards the other side of B. The value $\frac{volB^+}{volB}$ will vary continuously from 0 to 1 as Q is displaced so at some point the two values above will be equal. We will also have

$$\frac{volA^-}{volA} = \frac{volB^-}{volB}$$

since $volA^+ + volA^- = volA$ and $volB^+ + volB^- = volB$. We can also consider the disjoint finite unions of hyperrectangles

$$B^+ = \bigcup_{i=1}^{m^+} B_i^+ \text{ and } B^- = \bigcup_{i=1}^{m^-} B_i^-.$$

Note that m^+ and m^- may be equal to m.

Now we may apply to inductive hypothesis to $A^\pm$ and $B^\pm$ to get

$$vol(A^\pm + B^\pm) \geq [(volA^\pm)^{1/n} + (volB^\pm)^{1/n})]^n.$$

Note that since $A^\pm \subset P^\pm$ and $B^\pm \subset Q^\pm$, then

$$A^\pm + B^\pm \subset P^\pm + Q^\pm = (P+Q)^\pm$$

. Also, $A^+ + B^+$ and $A^- + B^-$ are two disjoint subsets of $A + B$, so

$$\begin{aligned}
vol(A+B) &\geq vol(A^+ + B^+) + vol(A^- + B^-) \\
&\geq [(volA^+)^{1/n} + (volB^+)^{1/n})]^n + [(volA^-)^{1/n} + (volB^-)^{1/n})]^n \\
&= volA^+[1 + (\frac{volB^+}{volA^+})^{1/n}]^n + volA^-[1 + (\frac{volB^-}{volA^-})^{1/n}]^n \\
&= volA^+[1 + (\frac{volB}{volA})^{1/n}]^n + volA^-[1 + (\frac{volB}{volA})^{1/n}]^n \\
&= volA[1 + (\frac{volB}{volA})^{1/n}]^n \\
&= [(volA)^{1/n} + (volB)^{1/n}].
\end{aligned}$$

This completes the proof of the second case

Now let A and B be any two bounded open subsets in $\mathbb{R}^n$. Then we can approximate A and B by sets of form considered in case two. Specifically, there exist $\{A_i\}_{i=1}^\infty \subset A$ and $\{B_i\}_{i=1}^\infty \subset B$, where each A_i and B_i is a sum of hyperrectangles, such that

$$\lim_{i\to\infty} volA_i = volA \text{ and } \lim_{i\to\infty} volB_i = volB.$$

Then since $A_i + B_i \subset A + B$ for each i,

$$(vol(A+B))^{1/n} \geq (vol(A_i + B_i))^{1/n} \geq (volA_i)^{1/n} + (volB_i)^{1/n}$$

by case two. Finally, we can take limits, since all of these sets are bounded, to get

$$vol(A+B)^{1/n} \geq (volA)^{1/n} + (volB)^{1/n}.$$

□

We will now introduce some notation. $B_r^n(a)$ will denote the open ball in $\mathbb{R}^n$ centred at a with radius r and $B_r^n := B_r^n(0)$. $S_r^{n-1}(a)$ will denote the $(n-1)$-dimensional sphere in $\mathbb{R}^n$ with radius r centred at a and $S_r^n := S_r^n(0)$. Finally, we will define $\omega_n := volB_1^n$.

We are now ready to use the Brunn-Minkowski Inequality to prove the inequality part of the isoperimetric inequality.

Corollary 1. *Let D be a domain with smooth boundary surface in $\mathbb{R}^n$ with volume V and surface area A. Then $A^n \geq n^n\omega_n V^{n-1}$.*

Proof. Let E be the boundary of D, ∂D and let $D_r = D + B_r^n$. Then by the Brunn-Minkowski Inequality,

$$volD_r \geq ((volD)^{1/n} + (volB_r^n)^{1/n})^n = (V^{1/n} + \omega_n^{1/n} r)^n$$

Applying the binomial theorem gives $volD_r \geq V + nV^{\frac{n-1}{n}}\omega_n^{1/n} r$. So,

$$\frac{volD_r - volD}{r} \geq n\omega_n^{1/n} V^{\frac{n-1}{n}}$$

$volD_r - volD$ is the volume of the part of $E + B_r^n$ lying outside of D. Now we will get a similar inequality for the part of $E + B_r^n$ lying inside of D. Let $D_{-r} = D \setminus (E + B_r^n)$. Then

$$volD \geq \left((volD_{-r})^{1/n} + \omega_n^{1/n} r\right)^n \geq volD_{-r} + n(volD_{-r})^{\frac{n-1}{n}}\omega_n^{1/n} r.$$

Rearranging gives

$$\frac{volD - volD_{-r}}{r} \geq n(vol(D_{-r})^{\frac{n-1}{n}}\omega_n^{1/n}.$$

Putting these two inequalities together, we get

$$\frac{vol(E + B_r^n)}{r} \geq n\omega_n^{1/n}(V^{\frac{n-1}{n}} + (volD_{-r})^{\frac{n-1}{n}}).$$

We will now let $r \to 0$ and since E is smooth,

$$\lim_{r\to 0} \frac{vol(E + B_r^n)}{r} = 2volE = 2A \geq 2nw_n^{1/n} V^{\frac{n-1}{n}}$$

□

Note that if D is the ball B_r^n, then $V = \omega_n r^n$ and $A = n\omega_n r^{n-1}$. Then $A^n = n^n\omega_n^n r^{n(n-1)} = n^n\omega_n(\omega_n r^n)^{n-1} = n^n\omega_n V^{n-1}$. Thus, in the case of the sphere, we have equality in the above inequality. Now we want to show that the sphere is the unique surface that gives equality in the isoperimetric inequality.

Chapter 3

Variation Formulae

To prove that the sphere is the unique surface that minimizes the surface area of a fixed volume, we will use the formulae for first variation of area and volume. Let $D \subset \mathbb{R}^n$ be bounded by a smooth surface S. Let $\varphi : S \to \mathbb{R}$ be a smooth real-valued function on S and let S_t denote the surface obtained by displacing each point $p \in S$ by the vector $t\varphi(p)N(p)$, where N is the unit exterior normal field to S. Let D_t denote the region enclosed by S_t. If $A(t)$ is the area of $S(t)$, $V(t)$ is the volume of D_t, and H is the mean curvature of S with respect to the normal field N, then the formulae of first variation are:

$$A'(0) = -\int_S \varphi H \, dA$$

$$V'(0) = \int_S \varphi \, dA$$

Proof. The first equation is the standard formula for the first variation of volume (see for example Kühnel [7, §3D], or Sakai [11, p.71]), which we will now derive. Without loss of generality, we may assume that S can be parametrized by a single coordinate chart as $f : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$. Then S_t is parametrized by

$$f_t(u_1, \ldots, u_{n-1}) := f(u_1, \ldots, u_{n-1}) + t\varphi(u_1, \ldots, u_{n-1})N(u_1, \ldots, u_{n-1}).$$

We have

$$\frac{\partial f_t}{\partial t} = \varphi N \tag{3.1}$$

and

$$\frac{\partial f_t}{\partial u_i} = \frac{\partial f}{\partial u_i} + t\frac{\partial \varphi}{\partial u_i} \cdot N + t \cdot \varphi \cdot \frac{\partial N}{\partial u_i}$$

The first fundamental form of S_t relative to this parametrization is given by

$$\begin{aligned} g_{ij}(t) &= \left\langle \frac{\partial f_t}{\partial u_i}, \frac{\partial f_t}{\partial u_j} \right\rangle \\ &= g_{ij} + 2t\varphi \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial N}{\partial u_j} \right\rangle + t^2 \left(\varphi^2 \left\langle \frac{\partial N}{\partial u_i}, \frac{\partial N}{\partial u_j} \right\rangle + \frac{\partial \varphi}{\partial u_i} \frac{\partial h}{\partial u_j} \right) \\ &= g_{ij} - 2t\varphi h_{ij} + t^2 \left(\varphi^2 \left\langle \frac{\partial N}{\partial u_i}, \frac{\partial N}{\partial u_j} \right\rangle + \frac{\partial \varphi}{\partial u_i} \frac{\partial h}{\partial u_j} \right) \end{aligned}$$

where h_{ij} denotes the second fundamental form of S (see [7, §3B,§3D]. In particular, notice that

$$\left. \frac{d}{dt} \right|_{t=0} g_{ij}(t) = -2\varphi h_{ij}. \tag{3.2}$$

The volume of S_t is

$$A(S_t) = \int_{S_t} dA_t = \int_U \sqrt{\det g_{ij}(t)}\, du_1 \dots du_{n-1}.$$

The first variation of volume is

$$\begin{aligned} A'(0) &= \left. \frac{d}{dt} \right|_{t=0} A(S_t) \\ &= \left. \frac{d}{dt} \right|_{t=0} \int_U \sqrt{\det g_{ij}(t)}\, du_1 \dots du_{n-1} \\ &= \int_U \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det g_{ij}(t)}\, du_1 \dots du_{n-1}. \end{aligned}$$

Using the chain rule we have

$$\begin{aligned} \frac{d}{dt} \sqrt{\det g_{ij}(t)} &= \sum_{k,l} \frac{\partial}{\partial g_{kl}} \sqrt{\det g_{ij}}\ \frac{d}{dt} g_{kl}(t) \\ &= \sum_{k,l} \frac{1}{2\sqrt{\det g_{ij}}} \frac{\partial \det g_{ij}}{\partial g_{kl}}\ \frac{d}{dt} g_{kl}(t). \end{aligned}$$

By a formula from linear algebra for the derivative of a determinant,

$$\frac{\partial \det g_{ij}}{\partial g_{kl}} = (\det g_{ij}) g^{kl}$$

where as usual, g^{kl} denotes the kl-entry of the inverse of the matrix (g_{ij}). Therefore,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\sqrt{\det g_{ij}(t)} &= \frac{1}{2}\sum_{k,l} g^{kl}\ \left.\frac{d}{dt}\right|_{t=0} g_{kl}(t)\ \sqrt{\det g_{ij}}\\ &= -\varphi\sum_{k,l} g^{kl}h_{kl}\ \sqrt{\det g_{ij}}\\ &= -\varphi H\ \sqrt{\det g_{ij}}\end{aligned}$$

where in the second equality we used equation (3.2) and in the last equality we used the formula $H = \sum_{k,l} g^{kl}h_{kl}$ for the mean curvature [7, §3B] . Finally we have

$$\begin{aligned}A'(0) &= \int_U \left.\frac{d}{dt}\right|_{t=0}\sqrt{\det g_{ij}(t)}\ du_1\dots du_{n-1}\\ &= -\int_S \varphi\sum_{k,l}\varphi H\ \sqrt{\det g_{ij}}\ du_1\dots du_{n-1}\\ &= -\int_S \varphi H\ dA\end{aligned}$$

which proves the first formula.

On the other hand, we can also write the first variation formula in another way. Since

$$\begin{aligned}\frac{d}{dt}g_{kl}(t) &= \frac{d}{dt}\left\langle\frac{\partial f_t}{\partial u_k},\frac{\partial f_t}{\partial u_l}\right\rangle\\ &= \left\langle\nabla_{\frac{d}{dt}}\frac{\partial f_t}{\partial u_k},\frac{\partial f_t}{\partial u_l}\right\rangle + \left\langle\frac{\partial f_t}{\partial u_k},\nabla_{\frac{d}{dt}}\frac{\partial f_t}{\partial u_l}\right\rangle\end{aligned}$$

we have

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \sqrt{\det g_{ij}(t)} &= \sum_{k,l} g^{kl} \left\langle \nabla_{\frac{d}{dt}} \frac{\partial f_t}{\partial u_k}, \frac{\partial f_t}{\partial u_l} \right\rangle \bigg|_{t=0} \sqrt{\det g_{ij}} \\
&= \sum_{k,l} g^{kl} \left\langle \nabla_{\frac{\partial}{\partial u_k}} \frac{\partial f_t}{\partial t}, \frac{\partial f_t}{\partial u_l} \right\rangle \bigg|_{t=0} \sqrt{\det g_{ij}} \\
&= \sum_{k,l} g^{kl} \left\langle \nabla_{\frac{\partial}{\partial u_k}} \varphi N, \frac{\partial f}{\partial u_l} \right\rangle \sqrt{\det g_{ij}} \\
&= \operatorname{div}_S(\varphi N) \sqrt{\det g_{ij}}
\end{aligned}$$

where in the second to last equality we used equation (3.1), and in the last equality we used the formula for the divergence (see Sakai [11, pp.31,71]). Therefore,

$$\begin{aligned}
A'(0) &= \frac{d}{dt}\bigg|_{t=0} A(S_t) \\
&= \int_U \frac{d}{dt}\bigg|_{t=0} \sqrt{\det g_{ij}(t)}\, du_1 \dots du_{n-1} \\
&= \int_S \operatorname{div}_S(\varphi N)\, dA.
\end{aligned}$$

A similar derivation as above applies to calculate the first variation of volume of D, the region enclosed by S, for the variation D_t of D by the variation field φN, giving us the formula:

$$V'(0) = \frac{d}{dt}\bigg|_{t=0} V(D_t) = \int_D \operatorname{div}_D(\varphi N)\, dV$$

By the divergence theorem (see [11, p.71]) we can rewrite this as

$$\begin{aligned}
V'(0) &= \int_D \operatorname{div}_D(\varphi N)\, dV \\
&= \int_S \langle \varphi N, N \rangle\, dA \\
&= \int_S \varphi\, dA
\end{aligned}$$

and this proves the second formula. □

Chapter 4

Constant Mean Curvature of Surface with Minimum Area

The variation formulae derived in the previous chapter will be used in this chapter to prove that if a surface has minimum area among all surfaces bounding the same volume, then the mean curvature of the surface must be constant. The proof is a generalization to n-dimensions of the proof given by Osserman [10, p.1187].

Lemma 1. *Let S be a closed hypersurface in $\mathbb{R}^n$. If S has minimum volume among all hypersurfaces bounding the same volume V, then the mean curvature of S must be constant.*

Proof. Suppose first that there exists a function $\varphi : S \to \mathbb{R}$ so that $V'(0) = 0$ and $A'(0) \neq 0$. Then, applying a transformation that multiplies all distances by $\left(\frac{V}{V(t)}\right)^{1/n}$, transforms S_t into a surface $\hat{S}_t$ bounding the same volume V, but having surface area $\hat{A}(t)$ which for sufficiently small t is either strictly greater than or strictly less that the original surface area A, depending on the sign of t. So if S has minimum surface area among all surfaces bounding the volume V, we must have that if $V'(0) = 0$ then $A'(0) = 0$, i.e. if

$$\int_S \varphi \, dA = 0 \text{ then } \int_S \varphi H \, dA = 0$$

Suppose the constant mean curvature H is not constant on S. Then there are two points x, $y \in S$ such that H takes different values at x and y. Choose a smooth function $\varphi : S \to \mathbb{R}$ such that φ is zero except on small neighbourhoods of x and y, and having opposite signs on these neighbourhoods, and so that $\int_S \varphi \, dA = 0$ but $\int_S \varphi H \, dA > 0$. Such a φ decreases the mean curvature where H is bigger and increases it where it is smaller, so it will preserve the volume while decreasing the surface area. This contradicts the assumption that S has minimal surface area. Thus H must be constant on S. $\square$

Chapter 5

The Alexandrov Theorem

Alexandrov's theorem states that the only closed hypersurfaces with constant mean curvature embedded in Euclidean space are the round spheres. Combined with Lemma 1 from the previous chapter, this theorem implies that *if* a hypersurface minimizes the surface area of a domain of fixed volume, then this hypersurface must be a sphere. Together with the Brunn-Minkowski Inequality, which showed that a sphere will indeed minimize surface area, this concludes the proof of the Isoperimetric Inequality.

Theorem 3. *Any compact embedded hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ is a sphere.*

5.0.1 Proof

Since Alexandrov's original proof of the theorem in 1958, other simplified proofs of the result have been given. Notably, in 1976-77, Reilly obtained a purely analytical proof. This section, however, presents the original proof of the Alexandrov theorem, as described by Alías and Malacarne [1]. Besides the result itself, this proof is of interest as the origin of Alexandrov's reflection method. Alexandrov's reflection method has proven to be an important method for showing symmetries of solutions to certain classes of second order nonlinear elliptic partial differential equations. Since Alexandrov's original proof, the method has been used to prove other major results in geometry and partial differential equations.

Before beginning the proof, we give some background on uniformly elliptic linear operators and Hopf's maximum principle. We will then use the uniformly elliptic quasi-linear mean curvature operator to develop a maximum principle for constant mean curvature hypersurfaces.

For this step, we will use Hopf's Maximum Principle, which applies to uniformly elliptic linear operators. To define uniformly elliptic, we will begin with a domain $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{C}^2(\Omega)$ with Euclidean gradient

$Du = (u_1, ..., u_n) \in \mathbb{R}^n$. A second order partial differential equation operator L has the form

$$L[u](x) = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij}(x) + \sum_{k=1}^{n} b_k(x)u_k(x),$$

where the coefficients $a_{ij} = a_{ji}$ and b_k are continuous functions. Then L is said to be *elliptic at* $x \in \Omega$ if the symmetric matrix $[a_{ij}(x)]$ is positive definite. L is *elliptic in* Ω if it is elliptic at each point in Ω. Finally, L is *uniformly elliptic in* Ω if the function Λ/λ is bounded in Ω, where $\Lambda(x) > 0$ and $\lambda(x) > 0$ are the maximum and minimum eigenvalues of the positive definite matrix $[a_{ij}(x)]$.

Hopf's Maximum Principle is a generalization of the property that harmonic functions do not attain their maximum at interior points unless they are constant. In this last example, harmonic functions are solutions of $\Delta[u] = 0$, where $\Delta[u] = \Sigma_i u_{ii}$ is the Laplacian, a uniformly elliptic operator.

Theorem 4 (Hopf's maximum principle)**.** *a) (Interior point) Suppose that u satisfies the inequality $L[u] \geq 0$, with L uniformly elliptic in Ω. If u achieves its maximum at an interior point of Ω, then u is constant in Ω.*

b) (Boundary point) Let u satisfy $L[u] \geq 0$ with L uniformly elliptic in a domain Ω with smooth boundary $\partial\Omega$. If u achieves its maximum at a boundary point where Du exists, then any outward directional derivative of u at this point is positive, unless u is constant in Ω.

Before applying Hopf's Maximum Principle, we need some more definitions. Let M be an orientable connected hypersurface with non-empty smooth boundary in $\mathbb{R}^{n+1}$. We can choose a globally-defined unit normal vector field N along M and we may assume that M is oriented by N. The shape operator $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ of the hypersurface with respect to N is a self-adjoint linear operator on T_pM for each $p \in M$. Its eigenvalues $\kappa_1(p), ..., \kappa_n(p)$ are the *principal curvatures* of the hypersurface at p. The *mean curvature* of the hypersurface is defined by $nH(p) = tr(A) = \sum_i \kappa_i(p)$.

A *quasi-linear partial differential operator* Q has the form

$$Q[u] = \sum_{i,j=1}^{n} a_{ij}(Du)u_{ij} + b(Du)$$

where the coefficients $a_{ij} = a_{ji}$ and b are functions in $C^1(\mathbb{R}^n)$. Q is said to be *elliptic with respect to a function u at $x \in \Omega$* if the symmetric matrix $[a_{ij}(Du)(x)]$ is positive definite; it is *uniformly elliptic with respect to u* if Λ/λ is bounded in Ω, where $\lambda(x)$ and $\Lambda(x)$ are the minimum and maximum eigenvalues of $[a_{ij}(Du)(x)]$.

For our purposes, we need a specific quasi-linear operator, associated with the mean curvature H.

Locally, around the point p, one can write the hypersurface M as a graph of a C^2 function $u(x_1, \ldots, x_n)$ over a neighbourhood Ω of the origin in the tangent plane to M at p. Then M can locally be parametrized by:

$$f : \Omega \to \mathbb{R}^{n+1}$$

where

$$f(x_1, \ldots, x_n) = (x_1, \ldots x_n, u(x_1, \ldots, x_n)).$$

We have

$$f_i = (0, \ldots, 1, \ldots, 0, u_i),$$

where here f_i denotes the partial derivative of f with respect to x_i, and the "1" is in the i-th slot. Also,

$$f_{ij} = (0, \ldots, 0, u_{ij}).$$

The local representation of the induced metric on S relative to the given parametrization f is

$$g_{ij} = \langle f_i, f_j \rangle = \delta_{ij} + u_i u_j.$$

Since the hypersurface M is locally the level surface $x_{n+1} - u(x_1, \ldots x_n) = 0$, we see that the upward unit normal to S is

$$\nu = \frac{1}{\sqrt{1 + u_1^2 + \cdots u_n^2}}(-u_1, \ldots, -u_n, 1).$$

Set $W^2 = 1 + u_1^2 + \cdots u_n^2 = 1 + |Du|^2$. Then,

$$\nu = \frac{1}{W}(-u_1, \ldots, -u_n, 1).$$

The local representation of the second fundamental form of S relative to the given parametrization is (see [7, p.66])

$$h_{ij} = \langle f_{ij}, \nu \rangle = \frac{u_{ij}}{W}.$$

By inspection one can check that the inverse of the matrix (g_{ij}) is given by

$$g^{ij} = \frac{1}{W^2}(W^2\delta_{ij} - u_i u_j).$$

The mean curvature of M is (see [7, p.71])

$$H = \frac{1}{n}\sum_{i,j=1}^{n} g^{ij}h_{ij} = \frac{1}{nW^3}\sum_{i,j=1}^{n}(W^2\delta_{ij} - u_i u_j)u_{ij}.$$

Therefore, the condition that the mean curvature is a constant H is given by

$$\sum_{i,j=1}^{n}(W^2\delta_{ij} - u_i u_j)u_{ij} = nW^3 H.$$

Define the differential operator

$$\mathcal{H}[u] = \sum_{i,j=1}^{n}(W^2\delta_{ij} - u_i u_j)u_{ij}.$$

This is a second order quasilinear operator, as defined above, with $a^{ij} = a^{ji}$. Here

$$a^{ij} = W^2\delta_{ij} - u_i u_j.$$

By inspection, one can check that this $n \times n$ matrix (a^{ij}) has eigenvalue 1 with eigenvector parallel to the Euclidean gradient of u, $Du = (u_1, \ldots, u_n)$, and eigenvalue W^2 with multiplicity $n-1$ (with eigenvectors orthogonal to the gradient of u). Thus,

$$\lambda(x, u, Du) = 1 \quad \text{and} \quad \Lambda(x, u, Du) = W^2$$

Since M is the graph of u over Ω, Du is bounded on Ω, and so and so $\Lambda/\lambda = 1 + |Du|^2$ is bounded on Ω. Therefore, $\mathcal{H}$ is uniformly elliptic on Ω.

Theorem 5 (The Maximum Principle for Constant Mean Curvature)**.** *a) (Interior Point) Let M and M' be oriented hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvatures H and H' respectively, satisfying $H \leq H'$. If M and M' have the same normal vectors at a tangency point $p \in M \bigcap M'$, then M cannot remain above M' in a neighbourhood of p, unless the hypersurfaces coincide locally.*

b) (Boundary Point) Let M and M' be oriented hypersurfaces with boundaries ∂M and $\partial M'$ in $\mathbb{R}^{n+1}$ with constant mean curvatures satisfying $H \leq H'$. Assume that M and M' as well as their boundaries are both tangent at $p \in \partial M \cap \partial M'$, with the same normal vectors at the tangency point. Then M cannot remain above M' in a neighbourhood of p unless the hypersurfaces coincide locally.

Proof. Without loss of generality, we may assume that $p = \vec{0}$ is the origin in $\mathbb{R}^{n+1}$. We will write M and M' near $\vec{0}$ as the graphs of u and u', respectively, with normal vectors pointing upwards. Then $u(\vec{0}) = 0 = u'(\vec{0})$ and $Du(\vec{0}) = \vec{0} = Du'(\vec{0})$.

Assume towards contradiction that M is above M' near p. Then $u \geq u'$ in a neighbourhood U of $\vec{0}$ in $\mathbb{R}^{n+1}$, if p is an interior point, or in the upper half-space, if p is a boundary point. By taking a smaller neighbourhood U if necessary, we may assume that Du' is bounded and the mean curvature operator $\mathcal{H}$ is uniformly elliptic with respect to u'. Write a segment from u to u' as $u_t = (1-t)u + tu'$. Then $\frac{d}{dt}u_t = u' - u$ and $\frac{d}{dt}Du_t = Du' - Du$. Also, $\mathcal{H}[u] \leq \mathcal{H}[u']$ since $H \leq H'$ and $n, W \geq 0$.

Then,

$$0 \leq \mathcal{H}[u'] - \mathcal{H}[u] = \sum_{i,j} a_{ij}(Du')(u'_{ij} - u_{ij}) + \sum_{i,j} \{a_{ij}(Du') - a_{ij}(Du)\}u_{ij}.$$

We will now rewrite $\mathcal{H}[u'] - \mathcal{H}[u]$ as a linear second order partial differential equation operator L. First set $w = u' - u$ and set $c_{ij} - a_{ij}(Du')$. We will now transform the second sum in the above inequality:

$$\sum_{i,j} \{a_{ij}(Du') - a_{ij}(Du)\}u_{ij} = \sum_{i,j} \{ \int_0^1 \frac{d}{dt}\{a_{ij}(Du_t)\}dt\}u_{ij}.$$

Then, by the chain rule,

$$\sum_{i,j} \{ \int_0^1 \sum_k \frac{\partial a_{ij}}{\partial u_k}(Du_t)\frac{d}{dt}\{(Du_t)_k\}dt\}u_{ij}.$$

Since $\frac{d}{dt}Du_t = Du' - Du$, this expression becomes

$$\sum_k (\sum_{i,j} \int_0^1 \frac{\partial a_{ij}}{\partial u_k} Du_t dt)(Du' - Du)_k.$$

Now $w_k = (Du' - Du)_k$ and

$$b_k = \sum_{i,j} \{ \int_0^1 \frac{\partial a_{ij}}{\partial u_k}(Du_t)dt \}.$$

This allows us to rewrite the inequality above as

$$0 \leq L[w] = \sum_{i,j} c_{ij} w_{ij} + \sum_k b_k w_k.$$

The coefficients c_{ij} and b_k are continuous so L is a linear second order partial differential equation operator. Furthermore, L is uniformly elliptic since $\mathcal{H}$ is uniformly elliptic with respect to u'. Thus, we can apply Hopf's Maximum Principle to conclude that $u - u' = w \equiv 0$ in our neighbourhood U. Therefore, the hypersurfaces coincide locally. □

Now that we have a maximum principle for constant mean curvature hypersurfaces, we are ready to show that a hypersurface with constant mean curvature has a symmetry hyperplane in each direction of $\mathbb{R}^{n+1}$ and thus that M must be a sphere. We will prove this step using Alexandrov's reflection method.

Proof. Let M be a connected, compact embedded hypersurface with constant mean curvature. Then M is the boundary of a compact domain $\bar{\Omega} \subset \mathbb{R}^{n+1}$. Consider an arbitrary direction in $\mathbb{R}^{n+1}$. Without loss of generality, we may assume that it is the direction of the x_{n+1} axis, that M is contained in the half-space $\{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$ and that M is tangent to the hyperplane $\Pi_0 = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. For each $t > 0$, we define

$$M_t = \{x \in M : x_{n+1} \leq t\}.$$

Now reflect M_t with respect the hyperplane $\Pi_t = \{x \in \mathbb{R}^{n+1} : x_n + 1 = t\}$ to obtain the hypersurface

$$M_t^* = \{(x_1, ..., x_n, 2t - x_{n+1} \in \mathbb{R}^{n+1} : x \in M_t\}.$$

Since M is smooth and has constant mean curvature, for small enough $t > 0$, $M_t^* \subset \bar{\Omega}$, where $\bar{\Omega}$ is the compact domain with M as its boundary. Let $s > 0$ be the greatest positive number such that $M_s^* \subset \bar{\Omega}$. Either there exists a point $p \in M \cap M_s^* - \Pi_s$ or there is no such point.

Consider the first case, where we have a point $p \in M \cup M_s$ and $p \notin \Pi_s$. Note that M and M_s have the same constant mean curvature and the same orientation at p, thus they have the same normal vectors at p and we can apply the maximum principle we proved above. Thus M and M_s coincide in a neighbourhood around p.

Now consider the second case, where $M \cup M_s^* - \Pi_s$ is empty. Recall that s was chosen to be the largest positive number such that $M_s^* \subset \bar{\Omega}$. By this choice of s, there is a point $p \in M \cap \Pi_s$ such that the tangent space to M at p is orthogonal to Π_s. Thus we have a point $p \in \partial M \cap \partial M_s^*$ such that M, M_s, ∂M and ∂M_s^* are all tangent at p, since M and M_s have the same orientation and constant mean curvature. Now we can apply the maximum principle for a boundary point to conclude that M and M_s coincide in a neighbourhood of p.

In either case we have found a point p with a neighbourhood where M and M_s coincide. Now we will define $\tilde{S}$ to be the connected component of M_s^* that contains p. Also, let S be the part of M_s which is reflected to become $\tilde{S}$, so

$$\tilde{S} = \{(x_1, ..., x_n, 2s - x_{n+1}) \in \mathbb{R}^{n+1} : x \in S\}.$$

Set

$$\mathcal{A} = \{q \in \tilde{S} : M \text{ and } \tilde{S} \text{ coincide in a neighbourhood of } q\}.$$

To see that $\mathcal{A}$ is open, let $q \in \mathcal{A}$. Then there exists a neighbourhood U of q where M and $\tilde{S}$ coincide. Since U is open we can find a neighbourhood around every point of U where M and S coincide. Thus $U \subset \mathcal{A}$. $\mathcal{A}$ is also closed. Take $q \in \mathcal{A}^c$; then there is no neighbourhood of q where M and $\tilde{S}$ coincide. Since M and S are smooth, we can find a neighbourhood U of q where M and $\tilde{S}$ don't coincide at any point of U. Thus $\mathcal{A}^c$ is open and $\mathcal{A}$ is closed. Furthermore, $\mathcal{A}$ is non-empty since $p \in \mathcal{A}$ and M and M_s^* coincide in a neighbourhood of p, as shown above. Since $\tilde{S}$ is connected, $\tilde{S} \subset M$. So we have that $S \cup S$ is a compact, connected hypersurface contained in M. Thus $\tilde{S} \cup S$ coincides with M. Therefore, Π_s is a symmetry hyperplane of M in an arbitrary direction. This concludes the proof of Alexandrov's theorem: a closed hypersurface with constant mean curvature embedded in Euclidean space must be a round sphere. $\square$

Chapter 6

Conclusion

This thesis presented a complete self-contained proof of the isoperimetric inequality for smooth domains in $\mathbb{R}^n$, in all dimensions, which was previously difficult to find in one place in the literature. Problems related to the classical isoperimetric inequality in $\mathbb{R}^n$ remain an active area of current research, as described below. Also the techniques used in the proof and presented in this thesis are important methods in modern research in geometric analysis.

The variational methods developed in chapter 3 are key ideas in the calculus of variations. The variation formulae are central to constant mean curvature and minimal surface theory, which are both active areas of research. The Alexandrov Theorem proved in chapter 5 has been generalized in the last 30 years to the case of constant scalar curvature, and to constant higher order mean curvatures. The corresponding problem to the Alexandrov theorem for the case of non-empty boundary remains an open problem, although results have be obtained in special cases (see [1]). Also, the Alexandrov reflection method itself, which was presented in detail in chapter 5, continues to be an important technique in differential geometry and partial differential equations. A first striking result in differential geometry using Alexandrov's method was R. Schoen's characterization [12] of the catenoid as the only complete connected properly immersed minimal surface in Euclidean space with finite total curvature and two embedded ends. Alexandrov's reflection method has been used to prove numerous other results on constant mean curvature surfaces. In partial differential equations, J. Serrin [13], by an application of the reflection method, proved that a solution to the Poisson equation on a domain with over-determined boundary conditions must be a radial solution on the ball. Also, famous and highly-cited work of Gidas, Ni and Nirenberg [5] proves symmetry of positive solutions to a class of nonlinear second order spherically symmetric elliptic equations, using ideas based on the Alexandrov reflection method.

Problems related to the isoperimetric inequality in $\mathbb{R}^n$ remain an active area of current research. In one direction, it is natural to ask whether the

isoperimetric inequality holds on manifolds. While this is certainly not true in general, the inequality $4\pi A \leq L^2$ is classically known to hold on any disk type surface of non-positive Gauss curvature, and in particular, on any disk type minimal surface in $\mathbb{R}^n$. The inequality can easily be seen not to hold for general surfaces of non-positive Gauss curvature, when one drops the simple connectivity assumption; e.g. on a long cylinder, the perimeter remains fixed, while the area can be made as large as one wants. However, similar examples cannot be constructed in the case of minimal surfaces, and it is conjectured that $4\pi A \leq L^2$ *for any minimal surface in* $\mathbb{R}^n$. Many special cases have been proved, but the general conjecture is an open and much-studied problem (see [4]).

As described in this thesis, the isoperimetric inequality in $\mathbb{R}^n$ is the problem of minimizing the surface area among all domains having given volume, and the solution is that the unique extremal is the domain bounded by a sphere. A physical illustration of this mathematical principle is a single soap bubble which quickly finds the round sphere as the least-area way to enclose the fixed volume of air trapped inside. Similarly, bubble clusters seek the least-area way to enclose and separate *several* regions of prescribed volumes [9, Chapter 13]. In 2000, it was proved that the standard "double bubble", consisting of three spherical caps meeting at 120 degrees, is the least-area way to enclose *two* prescribed volumes in $\mathbb{R}^3$. This problem was referred to as the "Double Bubble Conjecture", and a description of the problem and its eventual solution can be found in Morgan [9, Chapter 14]. The problem in higher dimensions and for more than two regions remains open. Soap bubble cluster theory is just one of many active areas of research in Geometric Measure Theory.

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