Black Hole Fluctuations and Negative Noise Kernel

by

Qingdi Wang

B.Sc., Wuhan University, 2008

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

October 2011

Abstract

In 2007, based on stochastic gravity, Bei-Lok Hu and Albert Roura claimed that black hole fluctuations grow and eventually become important when a black hole has evaporated for a sufficiently long period of time but well before the Planckian regime is reached. In this thesis, we investigate their derivation of the black hole fluctuations and propose two analogue examples to better understand the mechanism of large fluctuations. Our analysis clearly shows the classical nature of the large black hole fluctuations. On the other hand, to test the validity of stochastic gravity, we calculate the centerpiece of stochastic gravity, the noise kernel, first for a perfect reflecting mirror and then for a more realistic mirror which becomes transparent at high frequencies. We find that one of the noise kernel components which corresponds to the fluctuations of energy flux is negative and thus seems to give "imaginary" fluctuations. We also perform calculations of both the fluctuations of forces acting on mirrors and of the fluctuations of energy flux radiated by them and divergent results are obtained. We try to give interpretations of these negative and divergent values. They need further investigation in the future.

Table of Contents

Abstract . . . ii

Table of Contents . . . iii

Acknowledgements . . . v

1 Introduction . . . 1

2 Stochastic gravity . . . 5
- 2.1 From quantum field theory in curved spacetime to semiclassical gravity . . . 5
- 2.2 From semiclassical gravity to stochastic gravity . . . 6

3 Black hole fluctuations . . . 9
- 3.1 Bekenstein's theory . . . 9
- 3.2 Hu and Roura's theory . . . 12
- 3.3 Do the fluctuations really become important before the Planckian regime is reached? . . . 14
  - 3.3.1 Details about the increasing fluctuation . . . 14
  - 3.3.2 Analogue with Brownian motion . . . 15
  - 3.3.3 Analogue with upside down harmonic oscillator . . . 16
  - 3.3.4 Conclusions . . . 18

4 Negative noise kernel in moving mirror models . . . 20
- 4.1 Noise kernel of scalar fields in flat spacetime . . . 20
- 4.2 Noise kernel in perfectly reflecting moving mirror models . . . 23
  - 4.2.1 Moving mirrors and black hole . . . 23
  - 4.2.2 Negative noise kernel in perfectly reflecting moving mirror models . . . 27
- 4.3 Noise kernel in a more realistic moving mirror model . . . 30
- 4.4 Fluctuations of the force acting on the mirror and of the energy flux radiated by the mirror . . . 36

5 Discussions and conclusions 39
5.1 Mathematical aspects of the negative noise kernel 39
5.2 Physical aspects of negative noise kernel 41
5.3 The infinite fluctuations of force acting on the mirror 43

Bibliography 46

Acknowledgements

I would like to thank my supervisor, Dr. William G. Unruh, who has supported me throughout my thesis with his remarkably keen insight and knowledge. I would like to thank Dr. Kristin Schleich for reviewing and carefully modifying my thesis and all her valuable comments. I would also like to thank my roommate Zhen Zhu, who supported me during my busy writing up by cooking for me.

Chapter 1

Introduction

General relativity(hereafter "GR") is one of the two most remarkable pillars in the 20th century physics. In the classical GR, a black hole is predicted to form in a highly dense region with strong gravitational field, which can absorb any matter including photons. Therefore, it was believed to be "dark" in the content of Einstein's theory of GR. However, based on the quantum field theory in curved spacetime, Hawking published an astonishing result in 1974 that a black hole is not completely black but emitting thermal radiation with temperature inversely proportional to the mass of the black hole [1, 2]. Specifically, for a Schwarzschild black hole with mass M, the temperature of the thermal radiation is

$$T = \frac{\hbar c^3}{8\pi k G M}, \tag{1.1}$$

where k is Boltzman's constant. When a black hole loses thermal energy, its mass decreases, therefore resulting in a higher temperature state. This suggests that as a black hole emits thermal radiation, it loses its energy at a gradually faster rate as its mass decreases. Stefan's law provides the order of magnitude estimate of the power emitted by a Schwarzschild black hole (in Planck units):

$$\frac{dM}{dt} = -\sigma A T^4 \sim -M^2(1/M)^4 = -\frac{1}{M^2}. \tag{1.2}$$

Integrating the above equation, we find that a black hole should radiate all of its mass in a finite time τ

$$\tau \sim M^3. \tag{1.3}$$

However, there are some intrinsic limitations to the above description of Hawking radiation because it is based on quantum field theory in curved spacetime, which assumes that quantum fields propagate in a fixed classical background. People realized that, gravitational field, as any other fundamental fields, should also interact with quantum matter and therefore be described by the general framework of quantum field theory. In quantum theory, states of a system are represented by vectors in a Hilbert space, $\mathcal{H}$, and observables are represented by self-adjoint linear maps on $\mathcal{H}$. Outcomes of measurements on an observable for identical quantum states do not give a same value unless the system is coincidently in an eigenstate of that observable. Since observables in general relativity, i.e. spacetime metrics, always give the same value for identical states, general relativity is a purely classical theory. Therefore, it is generally believed that general relativity must be only a classical limit of an underlying fundamental theory of gravity—the so called quantum theory of gravity. To estimate the scale at which the classical theory of GR still holds, one can uniquely combine three fundamental constants of nature, Planck's constant $\hbar$, speed of light in vacuum c, and the gravitational constant G, to form a quantity with the dimension of length, namely the Planck length $l_p \equiv (G\hbar/c^3)^{1/2}$. This quantity sets a minimal scale above which the classical GR still holds. An equivalent quantity to this length scale is the Planck mass, which is defined to be $m_p \equiv (\hbar c/G)^{1/2}$. Henceforce, by the time a black hole has evaporated down to the Planck mass, the approximation of treating gravity classically would not be expected to be still valid, and we have to wait until the complete theory of quantum gravity to figure out what will occur at that stage.

In addition, in quantum field theory in curved spacetime, the gravitational field is supposed to be fixed. This contradicts the principles of classical GR itself. Even in classical GR the spacetime metric g_{ab} is tightly related to the matter distribution in spacetime by Einstein's equation:

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}, \tag{1.4}$$

where T_{ab} is the stress energy tensor of the matter field. In particular, for the Hawking radiation, it is clear that the quantum field carries energy away from a Schwarzschild black hole and the background metric should be affected by the quantum field. This is the so-called back reaction effect of quantum field on black hole metric. To address the problem people proposed semiclassical gravity, which is a better approximation from the quantum field theory in curved spacetime. Unlike its predecessor, in semiclassical gravity the background metric was not fixed but affected by the quantum field propagating on it according to the similclassical Einstein equation in which the source is given by the expectation value of the stress-energy tensor of the quantum fields. By applying this semiclassical theory to the black hole back reaction problem, Bardeen and Massar [3, 4] reached results similar to Eq.(1.2). This mass losing rate, as showed in Eq.(1.2) is generally believed to be valid until the black hole has evaporated down to the Planck mass.

In the 1990s, a new approximation—stochastic gravity [5], was developed. Stochastic gravity includes not only the expectation value but the fluctuations of the quantum field in a new stochastic Einstein-Langevin equation. The centerpiece of stochastic gravity theory is the stress energy bitensor and its expectation value known as the noise kernel. The mathematical properties of this quantity and its physical content in relation to the fluctuations of quantum fields in curved spacetimes are the central issues of this new theory. In 2007, Bei-Lok Hu and Albert Roura [6] proposed a systematic quantum study of the spacetime fluctuations induced by quantum fields in an evaporating black hole by using the stochastic gravity program. They claimed that those fluctuations grow and eventually become important when the black hole has evaporated for a sufficiently long periods of time but well before the Planckian regime is reached. Their result for the growth of the fluctuations of the size of the black hole horizon agrees with the result obtained by Bekenstein in [7].

In this thesis, we first review the stochastic gravity program. Then we investigate the black hole fluctuation theory proposed by Bekenstein in 1984 which is based on an energy conservation argument and the theory proposed by Bei-Lok Hu and Albert Roura in 2007 which is based on the stochastic gravity program [6]. B.L.Hu and Roura have commented that the large fluctuations are mainly driven by classical behavior. To better understand the mechanism producing large fluctuations before the Plankian regime is reached, we proposed two analogue examples, the Brownian motion and the upside down harmonic oscillator. Our analysis clearly shows that the large fluctuations are mainly classical effects which are due to the spread of the classical mass distribution of the black hole. Thus they might not be that important before the Plankian regime is reached. The fluctuations would become important until the black hole has evaporated down to the Plank mass when the classical behavior of the gravitational field breaks down.

On the other hand, to test the validity of stochastic gravity, we also calculate the centerpiece of stochastic gravity, the noise kernel, in moving mirror models. A moving mirror model is a simple example of quantum particle creation which is similar with particle production by black holes. In our calculation, we find that the renormalized noise kernel component N_{0101} is negative and thus seems to give "imaginary" fluctuations. We give an interpretation of this negative value; its physical significance needs further investigation in the future. We also perform the calculations of fluctuations of force acting on the mirror and the fluctuations of energy flux radiated by the mirror. In the calculations, we get divergent fluctuations, which are also unphysical.

This thesis is organized as follows. In Chapter 2, we give a general picture of the stochastic gravity. In Chapter 3, we analyze the black hole fluctuation theory proposed by Bekenstein and B.L.Hu and clearly show the classical nature of the fluctuations by two analogue examples. In Chapter 4, we calculate the noise kernel first for a perfect reflecting mirror and then for a more realistic mirror which becomes transparent at high frequencies. Then we perform the calculations of fluctuations of force acting on the mirror and the fluctuations of energy flux radiated by the mirror. The negative noise kernel component N_{0101} we get implies the energy flux would undergo "imaginary" fluctuations, which are really unphysical. In Chapter 5, we present careful analyses of the "imaginary" fluctuations and the divergent results we get in the calculation of fluctuations of force acting on the mirror in Chapter 4 and try to find out where the problems are.

Chapter 2

Stochastic gravity

2.1 From quantum field theory in curved spacetime to semiclassical gravity

Fundamentally, the black hole fluctuation problem should be investigated in quantum gravity. The goal of quantum gravity is to unify quantum mechanics with general relativity in a self-consistent manner, or more precisely, to formulate a self-consistent theory which reduces to ordinary quantum mechanics in the limit of weak gravity and to general relativity in the limit of large actions. Unfortunately, there is presently no satisfactory quantum theory of gravity. However, we can still use some approximate theories to investigate the black hole fluctuation problem. Note that before trying to combine quantum mechanics with general relativity, we have successfully combined quantum mechanics with special relativity—the quantum field theory in Minkowski spacetime, which is just an approximation when the gravitational field is weak enough to be neglected. Naturally, when the gravitational field is strong, we need a new approximation; a first step is to use quantum field theory in curved spacetime. Quantum field theory in curved spacetime is now a well defined theory both for free fields and interacting fields. In this theory, spacetime is fixed and quantum fields are test fields which propagate in such a spacetime. The key difference between the field theory in Minkowski spacetime and one in curved spacetime is that in the later case, spacetime is dynamical and it is not always possible to define a physically meaningful vacuum state for the quantum field. Even when it is possible to define such a state at some initial times, it may differ from the vacuum state at later times and thus spontaneous creation of particles may occur. Though the quantum field theory in curved spacetime is only an rough approximation to the quantum theory of gravity, it has been widely used. Applications of this in cosmology, such as particle production in expanding Friedmann-Robertson-Walker models, and in black hole physics, such as Hawking radiation, are well known.

Thus we have the first step of an approximation—quantum field theory in curved spacetime. The second step is very natural. Notice that in the first step, we let the background spacetime metric to be fixed. We now go a step further by requiring it to be changed by the back reaction of the quantum fields propagating on it. The back reaction effect would become important when the quantum fields are strong. In semiclassical gravity, matter is represented by quantum matter fields that propagate according to the theory of quantum fields in curved spacetime. The spacetime in which the fields propagate is classical but dynamical. The curvature of the spacetime is given by the semiclassical Einstein equations, which relate the curvature of the spacetime, i.e. the Einstein tensor G_{ab}, to the expectation value of energy momentum tensor T_{ab}:

$$G_{ab} = 8\pi \langle T_{ab} \rangle \,, \tag{2.1}$$

where we have used the Planck units that $\hbar = c = G = k = 1$. Since the gravitational field is coupling to the energy momentum tensor of matter fields, the key object here is the expectation value of the energy momentum tensor of the quantum field in a given quantum state. However, since it is quadratic in field operators, which are only well defined as a distribution on spacetime, it involves ill defined quantities which will diverge. Therefore, to define a physically meaningful quantity, a regularization and renormalization procedure is required. The divergences associated to the expectation value of the energy momentum tensor are also present in Minkowski spacetime and we can use several regularization procedures to obtain finite values. But in a curved background the regularization procedure is more complicated because it is required to preserve general covariance. One regularization procedure which is specially adapted to the curved background is the so called point-splitting method. One of the most important applications of semiclassical gravity are to calculate the Hawking evaporation of black holes. We will review it in Chapter 3.

2.2 From semiclassical gravity to stochastic gravity

Since we use the expectation value of the energy momentum tensor as the matter source of Einstein equation, the semiclassical equation (2.1) will break down if quantum fluctuations are important. For example, if M is a huge mass, and the matter field is in the superposition

$$\frac{1}{\sqrt{2}} \left(|M \;\; at \;\; A\rangle + |M \;\; at \;\; B\rangle \right) , \tag{2.2}$$

where A and B are widely separated, then according to equation (2.1), the gravitational field will behave like half of the matter is at A and the other half is at B. But we would never observe the metric sourced by such a distribution. Instead, it has probability 1/2 of corresponding to the gravitational field of matter at A and has probability 1/2 of corresponding to the gravitational field of matter at B. So naturally, a better approximation to describe the gravitational field is in a probabilistic way. In other words, the semiclassical equations should be substituted by some Langevin-type equations with a stochastic source which describes the quantum fluctuations. A significant step in this direction was made by Bei-Lok Hu. The rest of this section mainly follows the review article written by Bei-Lok Hu and Enric Verdagure [5].

At first, let us consider the stress energy tenser operator T_{ab}. Naively, one would replace the classical field $\phi(x)$ in the expression of the classical stress energy tensor with the quantum operator $\phi(x)$(for simplicity, we use the same notation for a classic field and a quantum field operator). But this procedure involves taking the product of two distributions at the same spacetime point, which is ill defined and a regularization procedure is required [8]. There are several regularization methods, and the one we would use in this thesis is the point splitting or point separation regularization method [9, 10]. In this method, one introduces a spacetime point y in the neighborhood of the point x and then uses as the regulator the vector tangent at point x of the geodesic joining x and y. Once the regularization procedure has been introduced, a regularized and renormalized stress energy operator T^R_{ab} may be defined as

$$T^R_{ab} = T_{ab} + F^C_{ab}\mathbf{1}, \tag{2.3}$$

which differs from the regularized T_{ab} by the identity operator times some tensor counterterms F^C_{ab}. These terms depend on the regulator and are local functionals of the metric [11].

Next we consider the tensor operator $t_{ab} = T_{ab} - \langle T_{ab} \rangle$, and define the noise kernel, which is a four index bi-tensor, as the expectation value of the anticommutator of the operator t_{ab}:

$$N_{abcd}(x, y) = \frac{1}{2} \langle \{t_{ab}(x), t_{cd}(y)\} \rangle \,. \tag{2.4}$$

This expectation value is taken in the background metric g_{ab} which is assumed to be a solution of the semiclassical equation (2.1). Bi-tensor means it is a tensor with respect to the first two indices at the point x and a tensor with respect to the last two indices at the point y. Since the noise kernel is defined in terms of the unrenormalized stress energy tensor operators on a given backgound metric g_{ab}, a regulator is implicitly assumed on the right hand side of Eq.(2.4). However, for a linear quantum field, one important property of the symmetric bi-tensor, $N_{abcd}(x,y) = N_{cdab}(y,x)$, is that it is finite because the regularized operator T_{ab} differs from the renormalized operator T^R_{ab} by the identity operator times some tensor counterterms (see Eq.(2.3)) and thus the counterterms are canceled by the subtraction of $\langle T_{ab}\rangle$.

In addition, since the operators T_{ab} are selfadjoint, $N_{abcd}(x,y)$, the expectation value of an anticommutator, is real and positive semi-definite. A proof is given in [12]. This property allows us to define a classical Gaussian stochastic tensor field $\xi_{ab}(x)$:

$$\langle \xi_{ab}(x)\rangle_s = 0, \quad \langle \xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y), \tag{2.5}$$

Where s means statistical average over Gaussian distribution. This stochastic tensor is symmetric $\xi_{ab} = \xi_{ba}$ and divergenceless, $\nabla^a \xi_{ab} = 0$, as a consequence of the fact that the energy momentum tensor operator is divergenceless. Note that this stochastic tensor captures only partially the quantum nature of the fluctuations of the stress energy operator as it assumes that cumulants of higher order are zero. We want to modify the semiclassical Einstein equation (2.1) by introducing a linear correction to the metric g_{ab}, such as $g_{ab} + h_{ab}$, to account consistently for the fluctuation of the energy momentum tensor. The simplest way to do this is to add the stochastic tensor ξ_{ab} directly to the right hand side of equation (2.1):

$$G^{(1)}_{ab}[g+h] = 8\pi \left\langle T^{(1)}_{ab}[g+h]\right\rangle + 8\pi \xi_{ab}[g], \tag{2.6}$$

Since this source is divergenceless with respect to the original metric tensor g_{ab}, it is consistent to put it on the right hand side of Einstein equation. The superscript (1) in this stochastic equation must be regarded as a linear equation for metric perturbation h_{ab} which behaves as a stochastic field tensor. That's because G_{ab} is not linear in the metric tensor, i.e. $G_{ab}(g+h) \neq G_{ab}(g) + G_{ab}(h)$. Therefore the above equation is only a dynamical equation for the metric perturbation h_{ab} to linear order. This equation is the so called Einstein-Langevin equation. In this sense, the Einstein-Langevin equation is a first order extension to the semiclassical Einstein equation of semiclassical gravity and the lowest level representation of stochastic gravity.

B.L.Hu and Roura [6] have applied the stochastic gravity program to the black hole fluctuation problem. They concluded that the black hole fluctuations become important well before the Plankian regime is reached. However, in next chapter, we will carefully analyze their work and show that their argument exaggerates the importance of quantum fluctuations.

Chapter 3

Black hole fluctuations

Black holes emit Hawking radiation with a temperature inversely proportional to their mass. When considering the back reaction of the emitted particles on the spacetime dynamics, one expects that the mass of the black hole decreases as the hole emits particles, resulting in higher and higher temperatures. This picture was indeed obtained from semiclassical gravity calculations and was believed to be valid at least before Planck scale is reached. However, semiclassical gravity neglects the fluctuations of the quantum fields propagating on the background metric. In 1984, Bekenstein [7] investigated the effect of fluctuations based on an energy conservation argument. After the development of stochastic gravity, B.L.Hu and Albert Roura [6] applied it to the fluctuation problem in 2007. In this chapter, we review their theory of black hole fluctuations and propose two analogue examples to better understand the mechanism. Our analyses show that the large fluctuations before Planckian regime is reached might not be that important.

3.1 Bekenstein's theory

Only in this section, we are following Bekenstein's covention and using geometrized units, i.e. $c = G = k = 1$. Define $E(\Delta t)$ is the emitted energy by a Schwatzchild black hole from time t to $t + \Delta t$. The most probable energy radiated by the hole in time Δt is [7]

$$\overline{E}(\Delta t) = \frac{\alpha \hbar \Delta t}{M^2}, \quad \alpha = \frac{9}{10 \times 8^4 \pi}. \tag{3.1}$$

Then based on energy conservation, we have

$$M(t + \Delta t) = M(t) - E(\Delta t) \tag{3.2}$$

Averaging the above equation first with respect to the energy distribution at a certain mass and then with respect to the mass distribution yields

$$\langle M(t + \Delta t) - M(t) \rangle = - \left\langle \overline{E}(\Delta t) \right\rangle \tag{3.3}$$

$$\frac{d\langle M\rangle}{dt} = -\alpha\hbar\left\langle M^{-2}\right\rangle, \tag{3.4}$$

where we have used the Hawking luminosity (3.1). Squaring (3.2), we have

$$M^2(t+\Delta t) = M^2(t) - 2M(t)E(\Delta t) + E^2(\Delta t). \tag{3.5}$$

Averaging the above equation and squaring (3.3) we get

$$\begin{aligned}\left\langle M^2(t+\Delta t)\right\rangle &= \left\langle M^2(t)\right\rangle - 2\left\langle M(t)\overline{E}(\Delta t)\right\rangle + \left\langle \overline{E^2}(\Delta t)\right\rangle \\ &= \left\langle M^2(t)\right\rangle - 2\alpha\hbar\left\langle \frac{1}{M(t)}\right\rangle \Delta t + \left\langle \overline{E^2}(\Delta t)\right\rangle \end{aligned}\tag{3.6}$$

$$\begin{aligned}\langle M(t+\Delta t)\rangle^2 &= \langle M(t)\rangle^2 - 2\langle M(t)\rangle\left\langle \bar{E}(\Delta t)\right\rangle + \left\langle \overline{E}(\Delta t)\right\rangle^2 \\ &= \langle M(t)\rangle^2 - 2\alpha\hbar\langle M(t)\rangle\left\langle \frac{1}{M^2(t)}\right\rangle \Delta t + \left\langle \overline{E}(\Delta t)\right\rangle^2 \end{aligned}\tag{3.7}$$

Subtracting (3.7) from (3.6) yields

$$\begin{aligned}\sigma_M(t+\Delta t) - \sigma_M(t) &= -2\alpha\hbar(\left\langle \frac{1}{M(t)}\right\rangle - \langle M(t)\rangle\left\langle \frac{1}{M^2(t)}\right\rangle)\Delta t \\ &+ \langle \sigma_E\rangle + \left\langle \overline{E}^2\right\rangle - \left\langle \overline{E}\right\rangle^2, \end{aligned}\tag{3.8}$$

where $\sigma_M = \left\langle M^2\right\rangle - \langle M\rangle^2$ is the variance of mass distribution, and [7]

$$\sigma_E = \overline{E^2} - \overline{E}^2 = \frac{\beta\hbar^2\Delta t}{M^3}, \quad \beta = \frac{9}{10\times 8^5\pi^2} \tag{3.9}$$

is the variance of energy distribution for a fixed mass. Since the last two terms $\left\langle \overline{E}^2\right\rangle - \left\langle \overline{E}\right\rangle^2 = \alpha^2\hbar^2(\left\langle \frac{1}{M^4}\right\rangle - \left\langle \frac{1}{M^2}\right\rangle)^2\Delta t^2$ go as Δt^2, after dividing both side by Δt and let $\Delta t \to 0$, these terms would disappear. Therefore we get the differential equation

$$\frac{d\sigma_M}{dt} = -2\alpha\hbar(\left\langle \frac{1}{M}\right\rangle - \langle M\rangle\left\langle \frac{1}{M^2}\right\rangle) + \beta\hbar^2\left\langle \frac{1}{M^3}\right\rangle. \tag{3.10}$$

Suppose that the k^{th} central moment $\left\langle (M - \langle M \rangle)^k \right\rangle$ of M is small when $k \geq 3$. Then we can use the moment expansion

$$
\begin{aligned}
&\left\langle \frac{1}{M} \right\rangle \\
&= \left\langle \frac{1}{\langle M \rangle (1 + \frac{M - \langle M \rangle}{\langle M \rangle})} \right\rangle \\
&= \frac{1}{\langle M \rangle} \left\langle 1 - \frac{M - \langle M \rangle}{\langle M \rangle} + \frac{(M - \langle M \rangle)^2}{\langle M \rangle^2} + o(\frac{(M - \langle M \rangle)^2}{\langle M \rangle^2}) \right\rangle \\
&\approx \frac{\left\langle M^2 \right\rangle}{\langle M \rangle^3}.
\end{aligned} \tag{3.11}
$$

Similar derivations give

$$\left\langle \frac{1}{M^2} \right\rangle \approx -\frac{2}{\langle M \rangle^2} + \frac{3 \left\langle M^2 \right\rangle}{\langle M \rangle^4} \tag{3.12}$$

$$\left\langle \frac{1}{M^3} \right\rangle \approx -\frac{5}{\langle M \rangle^3} + \frac{6 \left\langle M^2 \right\rangle}{\langle M \rangle^5} \tag{3.13}$$

Using the above expansions, (3.10) becomes

$$\frac{d\sigma_M}{dt} = \frac{\hbar}{\langle M \rangle^3}(4\alpha\sigma_M + \beta\hbar(1 + \frac{6\sigma_M}{\langle M \rangle^2})). \tag{3.14}$$

Combining the above equation with (3.4) we get

$$\frac{d\sigma_M}{d \langle M \rangle} = -\frac{\langle M \rangle}{3\sigma_M + \langle M \rangle^2}(4\sigma_M + \frac{\beta\hbar}{\alpha}(1 + \frac{6\sigma_M}{\langle M \rangle^2})). \tag{3.15}$$

When $\sigma_M \ll \langle M \rangle^2$, the above differential equation for σ_M have the approximate solution

$$\sigma_M \approx \frac{\beta\hbar}{4\alpha}(\frac{M_0^4}{\langle M \rangle^4} - 1), \tag{3.16}$$

where M_0 is the initial mass of black hole. As pointed out by Bekenstein, this solution shows that the fluctuations σ_M can grow as large as $\langle M \rangle^2$ for $\langle M \rangle = M_c \sim \hbar^{1/6} M_0^{2/3}$. This critical mass does not have to be small. For example, for $M_0 = 10^{15}g$ the mass distribution becomes broad at $\langle M \rangle \sim 10^8 g$, which is large compared to the Plank mass $\hbar^{\frac{1}{2}} \sim 10^{-5}g$. We can see from (3.4) that the rate of change of $\langle M \rangle$ will then deviate from $-\alpha\hbar \langle M \rangle^{-2}$.

3.2 Hu and Roura's theory

In this section we use the Planck units which are obtained by setting $\hbar = 1$ in geometrized units. Bardeen [3] and Massar [4] have studied the black hole evaporation due to the back reaction of the Hawking radiation emitted by the black hole on the spacetime geometry . For a general spherically symmetric metric there always exists a system of coordinates in which the metric takes the form

$$ds^2 = -e^{2\psi(v,r)}(1-\frac{2m(v,r)}{r})dv^2+2e^{\psi(v,r)}dvdr+r^2(d\theta^2+\sin^2\theta d\varphi^2). \quad (3.17)$$

Spherical symmetry implies that the components $T_{\theta r}$,$T_{\theta v}$,$T_{\varphi r}$ and $T_{\varphi v}$ vanish and the remaining components are independent of the angular coordinates. So the semiclassical Einstein equations associated with the metric in (3.17) become

$$\frac{\partial m}{\partial v} = 4\pi r^2 T_v^r, \quad (3.18)$$

$$\frac{\partial m}{\partial r} = -4\pi r^2 T_v^v, \quad (3.19)$$

$$\frac{\partial \psi}{\partial r} = 4\pi r^2 T_{rr}, \quad (3.20)$$

where $T_{\mu\nu}$ in the above equations are the expectation values of the energy momentum tensor of a scalar field propagating on it. Adopting a useful adiabatic approximation in the regime where the mass of the black hole is much larger than the Planck mass, we have

$$T_v^r = \frac{L_H}{4\pi r^2}, \quad (3.21)$$

where $L_H = \frac{B}{M^2}$(B is a dimensionless parameter that depends on the number of massless fields and their spins; it has been estimated to be of order 10^{-4} [13]). They also use the v component of the stress-energy conservation equation

$$\frac{\partial(r^2 T_v^r)}{\partial r} + r^2\frac{\partial T_v^v}{\partial v} = 0 \quad (3.22)$$

to relate the T_v^r components on the horizon and far from it. Finally they get the equation governing the evolution of the size of the black hole

$$\frac{dM}{dv} = -\frac{B}{M^2}, \quad (3.23)$$

where $M = m(v, \infty)$. Hu and Roura [6] used the stochastic gravity program to investigate the black hole fluctuation problem. As we introduced in Chapter 2, the difference between stochastic gravity and semiclassical gravity is that the former one has a stochastic source added to the right side of semiclassical Einstein equation(2.1) to get the so called Einstein-Langevin equation(2.6). Here they added a stochastic source to the right side of equation(3.18) and got

$$\frac{\partial(m+\eta)}{\partial v} = -\frac{B}{(m+\eta)^2} + 4\pi r^2 \xi_v^r + O(L_H^2), \tag{3.24}$$

where $\eta(v,r)$ is the perturbation of $m(v,r)$. Neglecting terms of order L_H^2 or higher, the above equation reduces to linear order in η:

$$\frac{\partial \eta}{\partial v} = \frac{2B}{m^3}\eta + 4\pi r^2 \xi_v^r. \tag{3.25}$$

In the above equation, they assumed that the change in time of $\eta(v,r)$ was sufficiently slow so that the adiabatic approximation employed in the previous section to obtain the mean evolution of $m(v,r)$ could also be applied to the perturbed quantity $m(v,r) + \eta(v,r)$.

The next problem in stochastic gravity is to determine the stochastic source—the noise kernel ξ_v^r in the above equation. Unfortunately, the calculation is highly nontrivial even if we compute it on the Schwarzschild spacetime. To proceed, they further assumed that the fluctuations of the radiated energy flux far from the horizon are exactly correlated with the fluctuations of the negative energy flux crossing the horizon. They argued that since the generation of Hawking radiation is especially sensitive to what happens near the horizon, they could consider $\eta(v) = \eta(v, 2M(v))$. (3.25) becomes

$$\frac{d\eta(v)}{dv} = \frac{2B}{M^3(v)}\eta(v) + \xi(v), \tag{3.26}$$

where $\xi(v) \equiv (4\pi r^2 \xi_v^r)(v, r \approx 6M(v))$. Then they followed the similar derivation of Bekenstein and derive directly an equation for $\left\langle \eta^2(v) \right\rangle_\xi$ ($\langle\rangle_\xi$ is statistical average over stochastic distribution of ξ):

$$\frac{d}{dv}\left\langle \eta^2(v) \right\rangle_\xi = \frac{4B}{M^3(v)}\left\langle \eta^2(v) \right\rangle_\xi + 2\left\langle \eta(v)\xi(v) \right\rangle_\xi. \tag{3.27}$$

by multiplying (3.26) by $2\eta(v)$ and then taking the expectation value. For simplicity, they considered quantities smeared over a time of order M and introduce the Markovian approximation $\langle \xi(v)\xi(v') \rangle_\xi \approx (\epsilon_0/M^3(v))\delta(v-v')$, where ϵ_0 has been estimated to lie between $0.1B$ and B [14]. For this case $\langle \eta(v)\xi(v) \rangle_\xi = \frac{\epsilon_0}{2M^3(v)}$, and the above equation becomes

$$\frac{d}{dv}\left\langle \eta^2(v) \right\rangle_\xi = \frac{4B}{M^3(v)}\left\langle \eta^2(v) \right\rangle_\xi + \frac{\epsilon_0}{M^3(v)}. \tag{3.28}$$

Changing from the v coordinate to the mass function $M(v)$ for the background solution, (3.28) becomes

$$\frac{d}{dM}\left\langle \eta^2(M) \right\rangle_\xi = -\frac{4}{M}\left\langle \eta^2(M) \right\rangle_\xi - \frac{\epsilon_0/B}{M}. \tag{3.29}$$

The solution of this equation is given by

$$\left\langle \eta^2(M) \right\rangle_\xi = \left\langle \eta^2(M_0) \right\rangle_\xi (\frac{M_0}{M})^4 + \frac{\epsilon_0}{4B}[(\frac{M_0}{M})^2 - 1]. \tag{3.30}$$

So they found that the spherically symmetric fluctuations of the horizon size of an evaporating black hole become important at late times, and even comparable to its mean value when $M \sim M_0^{2/3}$. They claimed that for a sufficiently massive black hole, the fluctuations become significant well before the Planckian regime is reached. More specifically, for a solar mass black hole they become comparable to the mean value when the black hole radius is of the order of 10 nm, whereas for a supermassive black hole with $M \sim 10^7 M_\odot$, that happens when the radius reaches a size of the order of 1 mm.

3.3 Do the fluctuations really become important before the Planckian regime is reached?

3.3.1 Details about the increasing fluctuation

Let us first investigate the physics in detail by carefully analyzing Bekenstein's results (3.16). In Bekenstein's derivation, it is clear that the generation of fluctuations comes from the fluctuations of emitted energy σ_E. If $\sigma_E = 0$, Eq.(3.15) for the evolution of σ_M becomes

$$\frac{d\sigma_M}{d\langle M \rangle} = -\frac{4\langle M \rangle \sigma_M}{3\sigma_M + \langle M \rangle^2}. \tag{3.31}$$

It has a solution $\sigma_M \equiv 0$ for initial condition $\sigma_M(M = M_0) = 0$; there will be no fluctuation at all if there is no fluctuation of emitted energy. In the beginning, $\sigma_M \ll \frac{\beta\hbar}{\alpha}$, so the second term of Eq.(3.15) is dominant and it becomes

$$\frac{d\sigma_M}{d\langle M\rangle} = -\frac{\beta\hbar}{\alpha}\frac{1}{\langle M\rangle} < 0. \tag{3.32}$$

So σ_M will slowly increase as $\frac{\beta\hbar}{\alpha}\ln\frac{M}{M_0}$ as the mass of black hole decreases due to evaporation. The increasing mass fluctuations of this part is mainly due to the fluctuations of the emitted energy. After a certain time(approximately at $t = \frac{(1-e^{-3/4})M_0^3}{3\alpha\hbar}$) when $\langle M\rangle < \frac{M_0}{e^{1/4}} \approx 0.78M_0$, the first term become dominant and (3.15) becomes

$$\frac{d\sigma_M}{d\langle M\rangle} \approx -\frac{4\sigma_M}{\langle M\rangle}. \tag{3.33}$$

Then σ_M will increase as fast as $\frac{1}{\langle M\rangle^4}$, i.e. the fluctuation is accelerated. The factor σ_M in the right hand side of the above equation clearly shows that the increasing mass fluctuations of this part is mainly due to the spread of the mass distribution.

The results obtained by Hu and Roura are based on the stochastic gravity program. They are actually in agreement with Bekenstein's result which is based on energy conservation. They mentioned in [6] that this is mainly a classical effect. In the beginning, the small fluctuations of the quantum field induce the stochastic but classical fluctuations of background metric, and the spread of the background metric results in the acceleration of the black hole fluctuation. This mechanism can be better understood by the following two analogues with Brownian motion and upside down harmonic oscillator which are stated in 3.3.2 and 3.3.3 respectively.

3.3.2 Analogue with Brownian motion

The fluctuations start to become significant only after a long time, more precisely a time on the order of evaporation time when the mass of the black hole has decreased substantially. Thus for the first stage of black hole evaporation, we can assume σ_M is small enough that $\left\langle\frac{1}{M^2}\right\rangle \approx \frac{1}{\langle M\rangle^2}$ so the evolution of the mean mass of the black hole obey the equation

$$\frac{d\langle M\rangle}{dt} = -\frac{\alpha\hbar}{\langle M\rangle^2}. \tag{3.34}$$

It has the solution

$$\langle M(t)\rangle = (M_0^3 - 3\alpha\hbar t)^{\frac{1}{3}}. \tag{3.35}$$

Substituting the above solution into (3.16), we get

$$\begin{aligned}\sigma_M &\approx \frac{\beta\hbar}{4\alpha}\left(\frac{M_0^4}{(M_0^3 - 3\alpha\hbar t)^{\frac{4}{3}}} - 1\right) \\ &= \frac{\beta\hbar}{4\alpha}\left(\frac{1}{(1 - \frac{3\alpha\hbar t}{M_0^3})^{\frac{4}{3}}} - 1\right) \\ &\approx \frac{\beta\hbar^2}{M_0^3}t. \end{aligned} \tag{3.36}$$

Note that the above formula for σ_M has the same form as the mean square displacement $\overline{(\Delta x)^2}$ in the Brownian motion, which is

$$\overline{(\Delta x)^2} = 2Dt, \tag{3.37}$$

where D is diffusion constant. So the behavior of mass distribution is exactly the same as the position distribution of Brownian motion.

3.3.3 Analogue with upside down harmonic oscillator

Now let us consider the upside down harmonic oscillator, i.e. a particle moving in the potential [15]

$$V(x) = -\frac{1}{2}kx^2, \quad k > 0. \tag{3.38}$$

We assume that at $t = 0$, the particle is described by a wave function which is centered and peaked at $x = 0$. The evolution of this particle is governed by the Schröedinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}kx^2\psi. \tag{3.39}$$

Suppose the solution has the form

$$\psi(x,t) = A(t)e^{-B(t)x^2}. \tag{3.40}$$

Then $A(t)$ and $B(t)$ obey the following equations

$$i\hbar\frac{dA}{dt} = \frac{\hbar^2}{m}AB \tag{3.41}$$

$$i\hbar\frac{dB}{dt} = \frac{1}{2}k + \frac{2\hbar^2}{m}B^2. \tag{3.42}$$

It is useful to introduce the parameters

$$a^2 = \frac{\hbar}{\sqrt{mk}}, \tag{3.43}$$

$$\omega^2 = \frac{k}{m}. \tag{3.44}$$

In terms of these parameters, the solutions are

$$A = (2\pi)^{-\frac{1}{4}} \frac{(\sin 2\phi)^{1/4}}{\sqrt{a\cos(\phi - i\omega t)}}, \tag{3.45}$$

$$B = \frac{1}{2a^2}\tan(\phi - i\omega t), \tag{3.46}$$

where ϕ is a real constant of integration which is related to the width of the wave packet at $t = 0$. The behavior of $\psi(x,t)$ for large t is given by

$$\psi(x,t) \sim \frac{1}{\sqrt{a}}(\frac{2\sin 2\phi}{\pi})^{\frac{1}{4}} e^{[-\frac{1}{2}(\omega t + i\phi)]} e^{(-e^{-2\omega t}\frac{x^2\sin 2\phi}{a^2} + \frac{ix^2}{2a^2})}. \tag{3.47}$$

The average value of x is still 0 and the average value of x^2 is given by

$$\left\langle x^2 \right\rangle \sim \frac{1}{4}a^2 \frac{e^{2\omega t}}{\sin 2\phi}. \tag{3.48}$$

The ϕ dependence of $\left\langle x^2 \right\rangle$ is easily understood: $\left\langle x^2 \right\rangle$ is minimized when $\phi = \pi/4$. If $\phi < \pi/4$, the initial probability distribution for x is broad and thus spreads quickly. If $\phi > \pi/4$, the initial wave function is narrowly peaked at x, but the spread at momentum is large and therefore this large spread of momentum results in a rapid spreading of the x distribution.

Next we consider the commutator of the operators x and p, where $p = -i\hbar\frac{\partial}{\partial x}$. Note that

$$xp\psi = \sqrt{mk}x^2\psi + O(e^{-2\omega t}), \tag{3.49}$$

$$px\psi = \sqrt{mk}x^2\psi - i\hbar\psi + O(e^{-2\omega t}). \tag{3.50}$$

The commutator contribution $-i\hbar\psi$ will be small compared to the other term if $\hbar \ll \sqrt{mk}x^2$, i.e. $a^2 \ll x^2$. From (3.48) we know that at large times $x^2 \gg a^2$, the commutator $[x,p]$ becomes negligible and therefore the particle moves classically. However, the wave function is definitely not sharply peaked at one particular classical trajectory but instead by a classical probability distribution:

$$\begin{aligned} f(x,p,t) &= |\psi(x,t)|^2\,\delta(p - \sqrt{mk}x) \\ &= \sqrt{\frac{2\sin 2\phi}{\pi a^2}} e^{-\omega t} e^{-2e^{-2\omega t}x^2\sin 2\phi/a^2}\delta(p - \sqrt{mk}x). \end{aligned} \tag{3.51}$$

This model gives a very good example that a system which initially behaves quantum mechanically but later classical effect dominates.

Now let's compare the behavior of the black hole mass M with the position x of the upside down harmonic oscillator. We have already concluded in 3.3.1 that, in the beginning the fluctuations, σ_M, of black hole mass increase very slowly as $\frac{\beta\hbar}{\alpha}\ln\frac{M}{M_0}$. This part of the fluctuations are mainly due to Eq.(3.9)—the fluctuations of emitted energy. The relative fluctuation $\frac{\sigma_E}{\overline{E}^2}\sim M$, i.e. it has maximum value in the first stage of black hole evaporation. This can be easily understood as follows: initially the black hole has very large mass and thus very low temperature. The radiation is so weak that the time at which a black hole emits a particle is highly random, hence the quantum effects dominate. However, as we conclude 3.3.1, after a certain time (approximately at $t=\frac{(1-e^{-3/4})M_0^3}{3\alpha\hbar}$) when σ_M grows large enough that the first term of Eq.(3.15) becomes dominant and the fluctuation will accelerate due to the spread of mass. After that time, the increasing of the fluctuations due to quantum fluctuations of emitted energy becomes negligible and the classical effect dominates.

3.3.4 Conclusions

Now we can conclude that the fluctuations do become large before the Planckian regime is reached, but this fact (that the fluctuations become large) might not be that important until the mass of black hole evaporates to the Planck mass.

First of all, the fluctuations become large only after a long period of the order of evaporation time when the black hole has emitted almost all of its mass. Secondly, and more importantly, the large fluctuations are mainly "classical fluctuations" but not "quantum fluctuations". We have seen from 3.3.2 that the behavior of the mass distribution is exactly the same as the position distribution of Brownian motion. The mean square displacement $\overline{(\Delta x)^2}$ of Brownian motion grows as $2Dt$ and it will also become large and even go to infinity after a sufficient period of time. Do the fluctuations in Brownian motion become important at that time? Obviously not. The fluctuations of Brownian motion have equal importance at all times. Thus similarly, one may not expect that the fluctuations of the black hole become important before the Planck regime is reached.

On the other hand, from the detailed analogy with the upside down harmonic oscillator in the 3.3.3, we conclude that the large fluctuations are mainly due to classical effects which come from the spread of "classical" mass distribution, which is just "classical" stochastic behavior but not "quantum" stochastic behavior. Notice that even in stochastic gravity, the gravitational field also behaves classically. Therefore, the large fluctuations at later times but well before Planck regime are mainly normal classical behaviors and might not be important. The fluctuations will not become important until the Planck regime is reached when the classical behavior of gravitational field breaks down.

Chapter 4

Negative noise kernel in moving mirror models

The source in semiclassical gravity is the expectation value of the stress energy tensor of a quantum field. The central issues are how to define the vacuum, how to control divergences by regularization and renormalization, and its physical content in relation to the dynamics of the background metric. As a revised theory from semiclassical gravity, the central piece in stochastic gravity is the stress energy bi-tensor and its expectation value known as the noise kernel. The central issues in this theory are how to control the divergences of the noise kernel by regularization and renormalization and how they induce the metric fluctuations. The calculation of noise kernel in curved spacetime is highly nontrivial even in the simplest Schwarzschild spacetime. To better understand the properties of noise kernels, we calculate the noise kernel in moving mirror models, in which the spacetimes are flat but nontrivial. This model has been proven to be a very good example of particle creation and is very similar to Hawking radiation of black hole. In our calculation, we find that the noise kernel component N_{0101} is negative. Negative value of N_{0101} implies that the fluctuation of energy flux T_{tx} is "imaginary" and can induce "imaginary" fluctuations of the background metric, which is unphysical. Careful analysis of the meaning of this negative value is given in Chapter 5. We also calculate the fluctuations of the force acting on the mirror and of the energy flux radiated by the mirror in the last section.

4.1 Noise kernel of scalar fields in flat spacetime

Before we start to calculate the noise kernel for moving mirror models, we first construct the general formalism of noise kernel calculation in this section.

A stress energy tensor is not well defined since it is built from the product of a pair of field operators evaluated at a single point. It can only be defined as an operator valued distribution on spacetime. To resolve this problem, regularizations and renormalizations are required. In 1960s, DeWitt [16] introduced the covariant point separation scheme. It was brought to more popular use in the 1970s in the context of quantum field theory in curved spacetime as a method for obtaining a finite quantum stress tensor. In this scheme, one introduces an artificial separation of the single spacetime point x into a pair of closely separated points x and x'. Then the divergent terms involving field products such as $\phi(x)^2$ become $\phi(x)\phi(x')$, whose expectation value is finite and thus well defined for $x \neq x'$. Once the divergences present are identified, they may be regularized and removed to get a well-defined, finite stress tensor at a single spacetime point.

Similarly, the stress energy bi-tensor and the noise kernel are also not well defined because they are built from the product of four field operators evaluated at two points or even a single point. Therefore, we need to regularize the stress energy bi-tenor and noise kernel to obtain a finite expression too. This has been carried out by Phillips and Hu [17] using the "modified" point separation scheme and a general expression related to the quantum field's Green function was obtained. In the rest section we will show how their method works and derive the general expression for the noise kernel in flat spacetime.

Consider a massless real scalar field $\phi(x)$ in flat spacetime. The stress energy tensor is

$$T_{ab}(x) = \phi_{,a}(x)\phi_{,b}(x) - \frac{1}{2}\eta_{ab}\eta^{cd}\phi_{,c}(x)\phi_{,d}(x). \tag{4.1}$$

Next we perform the point separation technique. In curved spacetime, one needs to parallel transport the vector components back to a common point. However, in flat spacetime with flat coordinates, that is trivial. The calculations in this thesis are only performed on flat spacetime with flat coordinates; thus it is convenient to introduce a point separated differential operator $\mathcal{T}_{ab}$ without the parallel displacement factors:

$$\mathcal{T}_{ab} = \frac{1}{2}(\partial_{a'}\partial_b + \partial_a\partial_{b'}) - \frac{1}{2}\eta_{ab}(\eta^{cd'}\partial_c\partial_{d'}), \tag{4.2}$$

where prime $'$ means the corresponding differential operator acts on the point x'. We can define the expectation value of the stress energy tensor of the field by the following point separating method:

$$\begin{aligned}
\langle T_{ab}(x)\rangle &= \frac{1}{2}\lim_{x'\to x}\left\langle \mathcal{T}_{ab}\{\phi(x),\phi(x')\}\right\rangle \\
&= \frac{1}{2}\lim_{x'\to x}\mathcal{T}_{ab}(D(x,x')+D(x',x)) \\
&= \lim_{x'\to x}\left\langle T_{ab}(x,x')\right\rangle . \qquad (4.3)
\end{aligned}$$

Here we have defined two point separated stress energy tensor operator

$$T_{ab}(x,x') = \frac{1}{2}\mathcal{T}_{ab}\{\phi(x),\phi(x')\} \qquad (4.4)$$

and the Wightman function for a massless scalar field

$$D(x,x') = \langle 0|\,\phi(x)\phi(x')\,|0\rangle . \qquad (4.5)$$

The separation of points is an artificial construct, so neither point should be favored and thus the product of field configurations is taken to be a symmetrized operator product:

$$\phi(x)\phi(y) \to \frac{1}{2}\{\phi(x),\phi(y)\} = \frac{1}{2}(\phi(x)\phi(y)+\phi(y)\phi(x)). \qquad (4.6)$$

Through the same way, for flat spacetime with flat coordinates, we can define the noise kernel as follows:

$$\begin{aligned}
N_{abcd}(x,y) &= \frac{1}{2}\left\langle\{T_{ab}(x)-\langle T_{ab}(x)\rangle\,, T_{cd}(y)-\langle T_{cd}(y)\rangle\}\right\rangle \\
&= \frac{1}{2}\left\langle\{T_{ab}(x),T_{cd}(y)\}\right\rangle - \langle T_{ab}(x)\rangle\langle T_{cd}(y)\rangle \\
&= \lim_{\substack{x'\to x\\ y'\to y}}\frac{1}{2}\left\langle\{T_{ab}(x,x'),T_{cd}(y,y')\}\right\rangle \\
&\qquad -\left\langle T_{ab}(x,x')\right\rangle\left\langle T_{cd}(y,y')\right\rangle \\
&= \lim_{\substack{x'\to x\\ y'\to y}}\frac{1}{2}\left\langle\{\frac{1}{2}\mathcal{T}_{ab}\{\phi(x),\phi(x')\},\frac{1}{2}\mathcal{T}_{cd}\{\phi(y),\phi(y')\}\}\right\rangle \\
&\qquad -\left\langle\frac{1}{2}\mathcal{T}_{ab}\{\phi(x),\phi(x')\}\right\rangle\left\langle\frac{1}{2}\mathcal{T}_{cd}\{\phi(y),\phi(y')\}\right\rangle \\
&= \lim_{\substack{x'\to x\\ y'\to y}}\mathcal{T}_{ab}\mathcal{T}_{cd}D(x,x',y,y'), \qquad (4.7)
\end{aligned}$$

where $D(x, x', y, y')$ is

$$\begin{aligned} D(x, x', y, y') &= \frac{1}{8}[\left\langle\{\{\phi(x), \phi(x')\}, \{\phi(y), \phi(y')\}\}\right\rangle \\ &- 2\left\langle\{\phi(x), \phi(x')\}\right\rangle \left\langle\{\phi(y), \phi(y')\}\right\rangle]. \end{aligned} \tag{4.8}$$

We can use Wick's theorem [18], which is a method of reducing arbitrary products of creation and annihilation operators to sums of products of pairs of these operators, to simplify $D(x, x', y, y')$. In the case we are considering, Wick's theorem for the four point function is

$$\begin{aligned} \left\langle\phi(x)\phi(x')\phi(y)\phi(y')\right\rangle &= D(x, x')D(y, y') \\ &+ D(x, y)D(x', y') + D(x, y')D(x', y). \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), we get

$$\begin{aligned} D(x, x', y, y') &= \frac{1}{2}[D(x, y)D(x', y') + D(x, y')D(x', y) \\ &+ D(y, x)D(y', x') + D(y, x')D(y', x)]. \end{aligned} \tag{4.10}$$

Now we can easily see that when $(x', y') \to (x, y)$,

$$D(x, x', y, y') \to D(x, y)^2 + D(y, x)^2. \tag{4.11}$$

Therefore, the noise kernel defined via Eq.(4.7) is well defined for $x \neq y$.

4.2 Noise kernel in perfectly reflecting moving mirror models

After defining the regularized stress energy bi-tensor and noise kernel, we calculate the noise kernel for perfectly reflecting moving mirrors in this section. In next section we propose a more realistic mirror model in which the mirror becomes transparent at high frequencies and calculate the noise kernel in this model. We obtain the same negative value of N_{0101} for both models.

4.2.1 Moving mirrors and black hole

We first review the perfectly reflecting moving mirror models. The vacuum of a quantum field in flat spacetime can be non-trivially modified by a single reflecting boundary (mirror) and particle creation can occur. Fulling and

Davies [19, 20] have shown that for a particular mirror trajectory, the mirror acts precisely as a black body. It is thus a good analogue of a black hole.

Consider a mirror moving along the trajectory

$$\begin{aligned} x &= x(t), \quad |\dot{x}(t)| < 1, \\ x(t) &= 0, \quad t < 0, \end{aligned} \tag{4.12}$$

in a two dimensional Minkowski spacetime. A massless scalar field ϕ, satisfying the field equation

$$\Box\phi = \frac{\partial^2 \phi}{\partial u \partial v} = 0 \tag{4.13}$$

with reflecting boundary condition

$$\phi(t, x(t)) = 0 \tag{4.14}$$

has a set of mode solutions

$$\phi_\omega(u,v) = i(4\pi\omega)^{-\frac{1}{2}}(e^{-i\omega v} - e^{-i\omega h(u)}), \quad \omega > 0, \tag{4.15}$$

where $u = t - x,\ v = t + x$ are null coordinates. $h(u) = 2\tau_u - u$ and τ_u is determined implicitly by the trajectory (4.12) through

$$\tau_u - x(\tau_u) = u. \tag{4.16}$$

Now, in the spacetime region to the right of the mirror, the quantum field ϕ can be expanded in terms of mode solutions (4.15) as

$$\phi(t,x) = \int_0^{+\infty} (a_\omega \phi_\omega + a_\omega^\dagger \phi_\omega^*) d\omega. \tag{4.17}$$

Then the Wightman function for the field ϕ in the moving mirror model with modes (4.15) is

$$\begin{aligned} D(x,y) &= \langle \phi(x)\phi(y) \rangle \\ &= \langle 0| \int_0^{+\infty} \int_0^{+\infty} (a_\omega \phi_\omega(x) + a_\omega^\dagger \phi_\omega^*(x)) \\ &\quad (a_{\omega'} \phi_{\omega'}(y) + a_{\omega'}^\dagger \phi_{\omega'}^*(y)) d\omega d\omega' \, |0\rangle \\ &= \int_0^{+\infty} \int_0^{+\infty} \delta(\omega - \omega') \phi_\omega(x) \phi_{\omega'}^*(y) d\omega d\omega' \\ &= \int_0^{+\infty} \phi_\omega(x) \phi_\omega^*(y) d\omega \\ &= \frac{1}{4\pi} \int_0^{+\infty} \frac{1}{\omega} [e^{-i\omega v} - e^{-i\omega h(u)}][e^{i\omega s} - e^{i\omega h(r)}] d\omega, \end{aligned} \tag{4.18}$$

where (u, v) and (r, s) are null coordinates corresponding to two dimensional spacetime points x and y respectively. Applying the differential operator $\partial_{a'}\partial_b$ on the above equation, we have

$$\begin{aligned}
&\partial_{a'}\partial_b D(x, y) \\
&= \frac{1}{4\pi}\int_0^{+\infty} \omega[e^{-i\omega v} - (-1)^b h'(u)e^{-i\omega h(u)}][e^{i\omega s} - (-1)^a h'(r)e^{i\omega h(r)}]d\omega \\
&= \frac{1}{4\pi}[-\frac{1}{(v-s)^2} + \frac{(-1)^a h'(r)}{(v-h(r))^2} + \frac{(-1)^b h'(u)}{(s-h(u))^2} - \frac{(-1)^{(a+b)}h'(u)h'(r)}{h(u)-h(r)}],
\end{aligned} \tag{4.19}$$

Substituting (4.2) and (4.19) into (4.4), we get the components of point separated stress energy tensor

$$\begin{aligned}
&\langle T_{00}(x, y)\rangle = \langle T_{11}(x, y)\rangle \\
&= -\frac{h^2(u) - 2h(u)h(r) + h^2(r) + (s-v)^2 h'(u)h'(r)}{4\pi(s-v)^2\left[h(u)-h(r)\right]^2}
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
&\langle T_{01}(x, y)\rangle = \langle T_{10}(x, y)\rangle \\
&= -\frac{h^2(u) - 2h(u)h(r) + h^2(r) - (s-v)^2 h'(u)h'(r)}{4\pi(s-v)^2\left[h(u)-h(r)\right]^2}.
\end{aligned} \tag{4.21}$$

It is clear that when $y \to x$, the above expression for $T_{ab}(x)$ will diverge. To regularize this quantity, we evaluate the above expression on two separate spacetime points (t, x) and $(t + \epsilon, x)$ and then expand it as a power series in ϵ. Changing to null coordinates, the two spacetime points are (u, v) and $(u + \epsilon, v + \epsilon)$ and the results are

$$\begin{aligned}
&\langle T_{00}(x)\rangle = \langle T_{11}(x)\rangle \\
&= -\frac{1}{2\pi\epsilon^2} + \frac{3h''(u)^2 - 2p'(u)h'''(u)}{48\pi h'(u)^2} + O(\epsilon), \\
&\langle T_{01}(x)\rangle = \langle T_{10}(x)\rangle \\
&= -\frac{3h''(u)^2 - 2h'(u)h'''(u)}{48\pi h'(u)^2} + O(\epsilon)
\end{aligned} \tag{4.22}$$

In the above equations, the various derivatives of h are evaluated at $u \equiv t-x$. If we perform exactly the same point separation calculations on Minkowski spacetime, i.e. if the mirror did not exist, we would find that

$$\langle T_{00}(x)\rangle_M = \langle T_{11}(x)\rangle_M = -\frac{1}{2\pi\epsilon^2} \tag{4.23}$$

$$\langle T_{01}(x)\rangle_M = \langle T_{10}(x)\rangle_M = 0. \tag{4.24}$$

The above equations give a strong suggestion that the divergent leading term $\frac{1}{2\pi\epsilon^2}$ in Eq.(4.22) corresponds to the infinite vacuum energy and thus should be discarded. Therefore, the final results are

$$\begin{aligned} \langle T_{00}(u)\rangle &= \langle T_{11}(u)\rangle = -\langle T_{01}(u)\rangle = -\langle T_{10}(u)\rangle \\ &= \frac{3h''(u)^2 - 2h'(u)h'''(u)}{48\pi h'(u)^2}. \end{aligned} \tag{4.25}$$

We now evaluate (4.25) for a particular mirror trajectory with the asymptotic form

$$x(t) \to -t - Ae^{-2\kappa t} + B \quad as\ t \to \infty, \tag{4.26}$$

where $A, B, \kappa > 0$ are constants. From (4.16) and (4.26) one obtains

$$h(u) \equiv 2\tau(u) - u \to B - Ae^{-\kappa(u+B)} \quad as\ u \to \infty. \tag{4.27}$$

Substituting (4.27) into (4.25) yields

$$\langle T_{00}\rangle = \frac{\kappa^2}{48\pi}, \quad t \to \infty. \tag{4.28}$$

Using the Bogolubov transformation, Fulling and Davies [20] also showed that the energy flux of the mirror with trajectory (4.26) is thermal. In other words, the moving mirror acts precisely as a black body with temperature

$$T = \frac{\kappa}{2\pi}, \tag{4.29}$$

which is exactly the same as Hawking radiation of a black hole whose spectrum is also thermal with temperature

$$T = \frac{1}{8\pi M} \tag{4.30}$$

if we identify $\kappa = \frac{M}{4}$.

Another interesting moving mirror system is one with a hyperbolic trajectory which recedes with constant acceleration:

$$\begin{aligned} x(t) &= B - (B^2 + t^2)^{\frac{1}{2}}, \quad && t \geq 0 \\ &= 0, && t < 0. \end{aligned} \tag{4.31}$$

From (4.16) and (4.26) one obtains

$$h(u) = \frac{Bu}{B+u}. \tag{4.32}$$

Substituting (4.32) into (4.25) yields

$$\langle T_{00}\rangle = \langle T_{11}\rangle = \langle T_{01}\rangle = 0, \tag{4.33}$$

which implies there is no energy at all radiated during intervals when the mirror acceleration is constant. However, the calculation of the Bogolubov transformation coefficients

$$\alpha_{\omega'\omega} = i(B/\pi)e^{-i(\omega+\omega')B}K_1\left[2iB(\omega\omega')^{\frac{1}{2}}\right] \tag{4.34}$$

$$\beta_{\omega'\omega} = (B/\pi)e^{i(\omega-\omega')B}K_1\left[2B(\omega\omega')^{\frac{1}{2}}\right], \tag{4.35}$$

where K_1 is a modified Bessel function, clearly shows that the mirror emits particles since $\beta_{\omega'\omega}$ has non-zero off-diagonal elements. This example beautifully illustrates the loose connection between particles and energy in quantum field theory. The presence of quanta need not imply the presence of energy [20].

4.2.2 Negative noise kernel in perfectly reflecting moving mirror models

In this subsection, we calculate the noise kernel for moving mirror models introduced in the last subsection. In two dimensional spacetime, the differential operator $\mathcal{T}_{ab}\mathcal{T}_{cd}$ becomes

$$\begin{aligned}
\mathcal{T}_{ab}\mathcal{T}_{cd} & \\
&= \frac{1}{4}\left[(\partial_{a'}\partial_b + \partial_a\partial_{b'}) - \eta_{ab}(-\partial_t\partial_{t'} + \partial_x\partial_{x'})\right] \\
&\quad \times \left[(\partial_{c'}\partial_d + \partial_c\partial_{d'}) - \eta_{cd}(-\partial_t\partial_{t'} + \partial_x\partial_{x'})\right] \\
&= \frac{1}{4}[\partial_{a'}\partial_b\partial_{c'}\partial_d + \partial_{a'}\partial_b\partial_c\partial_{d'} + \partial_a\partial_{b'}\partial_{c'}\partial_d + \partial_a\partial_{b'}\partial_c\partial_{d'} \\
&\quad - \eta_{cd}(-\partial_{a'}\partial_b\partial_t\partial_{t'} + \partial_{a'}\partial_b\partial_x\partial_{x'} - \partial_a\partial_{b'}\partial_t\partial_{t'} + \partial_a\partial_{b'}\partial_x\partial_{x'}) \\
&\quad - \eta_{ab}(-\partial_t\partial_{t'}\partial_{c'}\partial_d - \partial_t\partial_{t'}\partial_c\partial_{d'} + \partial_x\partial_{x'}\partial_{c'}\partial_d + \partial_x\partial_{x'}\partial_c\partial_{d'}) \\
&\quad + \eta_{cd}\eta_{ab}(\partial_t\partial_{t'}\partial_t\partial_{t'} - \partial_t\partial_{t'}\partial_x\partial_{x'} - \partial_x\partial_{x'}\partial_t\partial_t + \partial_x\partial_{x'}\partial_x\partial_{x'})]
\end{aligned} \tag{4.36}$$

Applying the differential operator $\partial_{a'}\partial_b\partial_{c'}\partial_d$ on $D(x,y)D(x',y')$ of Eq.(4.18) we get

$$\begin{aligned}
&\partial_a\partial_b\partial_c\partial_d D(x,y)D(x',y')\\
&\quad=\frac{1}{16\pi^2}\partial_a\partial_b\partial_c\partial_d\int_0^{+\infty}\frac{1}{\omega}\left[e^{-i\omega v}-e^{-i\omega h(u)}\right]\left[e^{i\omega s}-e^{i\omega h(r)}\right]d\omega\\
&\qquad\qquad\cdot\int_0^{+\infty}\frac{1}{\omega}\left[e^{-i\omega q}-e^{-i\omega h(p)}\right]\left[e^{i\omega n}-e^{i\omega h(m)}\right]d\omega\\
&\quad=\frac{1}{16\pi^2}\int_0^{+\infty}\omega\left[e^{-i\omega v}-(-1)^b h'(u)e^{-i\omega h(u)}\right]\left[e^{i\omega s}-(-1)^d h'(r)e^{i\omega h(r)}\right]d\omega\\
&\qquad\cdot\int_0^{+\infty}\omega\left[e^{-i\omega q}-(-1)^a h'(p)e^{-i\omega h(p)}\right]\left[e^{i\omega n}-(-1)^c h'(m)e^{i\omega h(m)}\right]d\omega\\
&\quad=\frac{1}{16\pi^2}\left[-\frac{1}{(v-s)^2}+\frac{(-1)^d h'(r)}{[v-h(r)]^2}+\frac{(-1)^b h'(u)}{[s-h(u)]^2}-\frac{(-1)^{b+d}h'(r)h'(u)}{[h(u)-h(r)]^2}\right]\\
&\qquad\cdot\left[-\frac{1}{(q-n)^2}+\frac{(-1)^c h'(m)}{[q-h(m)]^2}+\frac{(-1)^a h'(p)}{[n-h(p)]^2}-\frac{(-1)^{a+c}h'(p)h'(m)}{[h(p)-h(m)]^2}\right]
\end{aligned}\tag{4.37}$$

Combining (4.37) and (4.36) together and applying them to (4.7), we obtain

$$\begin{aligned}
&N_{0000}(x,y)=N_{1111}(x,y)\\
&\quad=\frac{1}{8\pi^2}\left(\frac{1}{(s-v)^4}+\frac{h'^2(r)}{(v-h(r))^4}+h'(u)^2\left(\frac{1}{(s-h(u))^4}+\frac{h'(r)^2}{(h(u)-h(r))^4}\right)\right)
\end{aligned}\tag{4.38}$$

$$\begin{aligned}
&N_{0101}(x,y)=N_{1010}(x,y)\\
&\quad=\frac{1}{8\pi^2}\left(\frac{1}{(s-v)^4}-\frac{h'^2(r)}{(v-h(r))^4}+h'(u)^2\left(-\frac{1}{(s-h(u))^4}+\frac{h'(r)^2}{(h(u)-h(r))^4}\right)\right)
\end{aligned}\tag{4.39}$$

For the particular constant acceleration trajectory (4.31), the results are

$$\begin{aligned}
&N_{0000}(x,y)=N_{1111}(x,y)\\
&\quad=\frac{1}{8\pi^2}\left(\frac{1}{(s-v)^4}+\frac{1}{(u-r))^4}+B^4\left(\frac{1}{(B(s-u)+su)^4}+\frac{1}{(B(v-r)+vr)^4}\right)\right)
\end{aligned}\tag{4.40}$$

$$N_{0101}(x,y) = N_{1010}(x,y)$$
$$= \frac{1}{8\pi^2}\Big(\frac{1}{(s-v)^4} + \frac{1}{(u-r))^4} - B^4\left(\frac{1}{(B(s-u)+su)^4} + \frac{1}{(B(v-r)+vr)^4}\right)\Big) \tag{4.41}$$

It is clear that when $y \to x$, the above expression for the noise kernel will diverge. To make sense of this quantity, we follow the same logic we used in the previous subsection and evaluate the above expression on two separate spacetime points (t,x) and $(t+\epsilon,x)$ then expand it as a power series in ϵ. Changing to null coordinates, the two spacetime points are (u,v) and $(u+\epsilon, v+\epsilon)$. The components of noise kernel are

$$\begin{aligned}
&N_{0000}(x) = N_{1111}(x)\\
&= \frac{1}{4\pi^2}\epsilon^{-4} + \frac{-3h''(u)^2 + 2h'(u)h^{(3)}(u)}{48\pi^2 h'(u)^2}\epsilon^{-2}\\
&+ \frac{3h''(u)^3 - 4h'(u)h''(u)h^{(3)}(u) + h'(u)^2h^{(4)}(u)}{48\pi^2 h'(u)^3}\epsilon^{-1}\\
&+ \{\frac{h'(u)^2}{4\pi^2\,[v-h(u)]^4} - \frac{5h''(u)^4}{128\pi^2 h'(u)^4} + \frac{h''(u)^2h^{(3)}(u)}{12\pi^2 h'(u)^3}\\
&- \frac{5h^{(3)}(u)^2}{288\pi^2 h'(u)^2} - \frac{h''(u)h^{(4)}(u)}{32\pi^2 h'(u)^2} + \frac{h^{(5)}(u)}{160\pi^2 h'(u)}\} + O(\epsilon)
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
&N_{0101}(x) = N_{1010}(x)\\
&= \frac{1}{4\pi^2}\epsilon^{-4} + \frac{-3h''(u)^2 + 2h'(u)h^{(3)}(u)}{48\pi^2 h'(u)^2}\epsilon^{-2}\\
&+ \frac{3h''(u)^3 - 4h'(u)h''(u)h^{(3)}(u) + h'(u)^2h^{(4)}(u)}{48\pi^2 h'(u)^3}\epsilon^{-1}\\
&+ \{-\frac{h'(u)^2}{4\pi^2\,[v-h(u)]^4} - \frac{5h''(u)^4}{128\pi^2 h'(u)^4} + \frac{h''(u)^2h^{(3)}(u)}{12\pi^2 h'(u)^3}\\
&- \frac{5h^{(3)}(u)^2}{288\pi^2 h'(u)^2} - \frac{h''(u)h^{(4)}(u)}{32\pi^2 h'(u)^2} + \frac{h^{(5)}(u)}{160\pi^2 h'(u)}\} + O(\epsilon)
\end{aligned} \tag{4.43}$$

In particular, for a mirror at rest, we have $h(u) = u$ and thus

$$N_{0000}(x) = N_{1111}(x) \;=\; \frac{1}{4\pi^2\epsilon^4} + \frac{1}{4\pi^2(v-u)^4} \tag{4.44}$$

$$N_{0101}(x) = N_{1010}(x) \;=\; \frac{1}{4\pi^2\epsilon^4} - \frac{1}{4\pi^2(v-u)^4} \tag{4.45}$$

For the particular constant acceleration trajectory (4.31), the results are

$$\begin{aligned} &N_{0000}(x) = N_{1111}(x) \\ &= \frac{1}{4\pi^2\epsilon^4} + \frac{B^4}{4\pi^2\left[B(u-v)-uv\right]^4} + \frac{B^4(u+v)\epsilon}{4\pi^2\left[B(u-v)-uv\right]^5} + O(\epsilon^2) \end{aligned} \tag{4.46}$$

$$\begin{aligned} &N_{0101}(x) = N_{1010}(x) \\ &= \frac{1}{4\pi^2\epsilon^4} - \frac{B^4}{4\pi^2\left[B(u-v)-uv\right]^4} - \frac{B^4(u+v)\epsilon}{4\pi^2\left[B(u-v)-uv\right]^5} + O(\epsilon^2). \end{aligned} \tag{4.47}$$

If we perform exactly the same point separation calculations on Minkowski spacetime, i.e. if the mirror did not exist, we would find that

$$N_{0000} = N_{1111} = N_{0101} = \frac{1}{4\pi^2\epsilon^4}. \tag{4.48}$$

Like the previous case, the calculation of the expectation value of stress energy tensor, the above equation shows that the divergent leading term $\frac{1}{4\pi^2\epsilon^4}$ comes from the infinite vacuum energy and thus should be discarded. Therefore we get the finite expression of noise kernels:

$$N_{0000}(x) = N_{1111}(x) = \frac{B^4}{4\pi^2\left[B(u-v)-uv\right]^4} \tag{4.49}$$

$$N_{0101}(x) = N_{1010}(x) = -\frac{B^4}{4\pi^2\left[B(u-v)-uv\right]^4} \tag{4.50}$$

Note that

$$N_{0101}(x) = \left\langle (T_{01}(x) - \langle T_{01}(x)\rangle)^2 \right\rangle \tag{4.51}$$

is the variance of T_{01} and should never be negative! The unphysical results are discussed in Chapter 5.

4.3 Noise kernel in a more realistic moving mirror model

A realistic mirror cannot reflect lights at all frequencies. In this subsection, we propose a more realistic mirror model which becomes transparent at high frequencies. Consider a mirror with internal degree of freedom q coupled with a real scalar field ϕ. The action is

$$\begin{aligned} S =&\frac{1}{2}\int\int ((\partial_t\phi)^2 - (\partial_x\phi)^2)dtdx + \frac{1}{2}\int ((\frac{dq}{d\tau})^2 - \Omega^2 q^2)d\tau \\ &+ \epsilon_0 \int (\frac{d}{d\tau}\phi(\tau, x(\tau)))q(\tau)d\tau, \end{aligned} \tag{4.52}$$

where ϵ_0 is the coupling constant. The last term is integrated along a trajectory $x = x(\tau)$ which is parametrized by the proper time τ of the mirror trajectory. For simplicity, we consider a mirror at rest whose trajectory is $x(\tau) = 0,\ \tau = t$. The Lagrangian is

$$L = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)\right)dx + \frac{1}{2}\left(\left(\frac{dq}{dt}\right)^2 - \Omega^2 q^2\right) + \epsilon q(t)\frac{\partial\phi}{\partial t}(t,0), \tag{4.53}$$

Using the Euler-Lagrangian equations, we obtain the equation of motion

$$\ddot{\phi} - \phi'' = -\epsilon_0\dot{q}\delta(x) \tag{4.54}$$

$$\ddot{q} + \Omega^2 q = \epsilon_0\dot{\phi}(t,0). \tag{4.55}$$

It is easy to check that the solution of Eq.(4.54) is of the following form

$$\phi = \phi_0 - \frac{\epsilon_0}{2}q(t-|x|), \tag{4.56}$$

where ϕ_0 is the solution of the homogeneous equation

$$\ddot{\phi} - \phi'' = 0. \tag{4.57}$$

Substituting (4.56) into (4.55) we obtain

$$\ddot{q} + \Omega^2 q = \epsilon_0\partial_t\phi(t,0) = -\epsilon_0\partial_t\phi_0(t,0) - \frac{\epsilon_0^2}{2}\dot{q} \tag{4.58}$$

To solve this equation, we use the Fourier expansion of $q(t)$ and $\phi_0(t,x)$ as follows:

$$q(t) = \int_0^{+\infty}\left[g(\omega)e^{-i\omega t} + g(\omega)^* e^{-i\omega t}\right]d\omega, \tag{4.59}$$

$$\begin{aligned}\phi_0(t,x) = &\int_0^{+\infty}\left(\alpha(-\omega)e^{-i\omega(t+x)} + \alpha(-\omega)^* e^{i\omega(t+x)}\right)d\omega \\ &+\int_0^{+\infty}\left(\alpha(\omega)e^{-i\omega(t-x)} + \alpha(\omega)^* e^{i\omega(t-x)}\right)d\omega,\end{aligned} \tag{4.60}$$

where $g(\omega)$ and $\alpha(\omega)$ are complex valued coefficients. Using the above expansion in (4.58), we get

$$\left(-\omega^2 - \frac{i}{2}\epsilon_0^2\omega + \Omega^2\right)g(\omega) + \epsilon_0\left[-i\omega(\alpha(\omega) + \alpha(-\omega))\right] = 0 \tag{4.61}$$

$$\left(-\omega^2 + \frac{i}{2}\epsilon_0^2\omega + \Omega^2\right)g(\omega)^* + \epsilon_0\left[i\omega(\alpha(\omega)^* + \alpha(-\omega))^*\right] = 0. \tag{4.62}$$

Thus we obtain the classical general solutions of $q(t)$ and $\phi(t,x)$ as follows:

$$\begin{aligned} q(t) = \int_0^{+\infty} [&\frac{-i\omega\epsilon_0(\alpha(\omega)+\alpha(-\omega))}{-\omega^2-\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{-i\omega t} \\ &+\frac{i\omega\epsilon_0(\alpha(\omega)^*+\alpha(-\omega)^*)}{-\omega^2+\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{+i\omega t}]d\omega \end{aligned} \tag{4.63}$$

$$\begin{aligned} \phi = \phi_0 - \frac{\epsilon_0}{2}\int_0^{+\infty} [&\frac{-i\omega\epsilon_0(\alpha(\omega)+\alpha(-\omega))}{-\omega^2-\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{-i\omega(t-|x|)} \\ &+\frac{i\omega\epsilon_0(\alpha(\omega)^*+\alpha(-\omega)^*)}{-\omega^2+\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{+i\omega(t-|x|)}]d\omega \end{aligned} \tag{4.64}$$

We now quantize q and ϕ by replacing $\alpha(\omega)$ and $\alpha^*(\omega)$ with operators a_ω and $a_\omega^\dagger$ with an normalization factor. The normalization factor can be determined by $(\phi_0(\omega), \phi_0(\omega')) = \delta(\omega-\omega')$, where the inner product $(,)$ is defined by

$$(\phi_1,\phi_2) = -i\int \{\phi_1(x)\partial_t\phi_2^*(x) - [\partial_t\phi_1(x)]\phi_2^*(x)\}dx, \tag{4.65}$$

and

$$\phi_0(\omega) = \alpha(\omega)e^{-i\omega(t-x)} + \alpha(\omega)^* e^{i\omega(t-x)}. \tag{4.66}$$

After normalization, we obtain

$$\begin{aligned} q(t) = \int_0^{+\infty} -i\epsilon_0\sqrt{\frac{\omega}{4\pi}}[&\frac{(a_\omega+a_{-\omega})}{-\omega^2-\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{-i\omega t} \\ &-\frac{(a_\omega^\dagger+a_{-\omega}^\dagger)}{-\omega^2+\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{+i\omega t}]d\omega \end{aligned} \tag{4.67}$$

$$\begin{aligned} \phi = &\int_0^{+\infty}\frac{1}{\sqrt{4\pi\omega}}[a_\omega e^{-i\omega(t-x)} + a_{-\omega}e^{-i\omega(t+x)} \\ &\qquad + a_\omega^\dagger e^{i\omega(t-x)} + a_{-\omega}^\dagger e^{i\omega(t+x)}]d\omega \\ &-\frac{\epsilon_0^2}{2}\int_0^{+\infty}\sqrt{\frac{\omega}{4\pi}}[\frac{-i(a_\omega+a_{-\omega})}{-\omega^2-\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{-i\omega(t-|x|)} \\ &\qquad +\frac{i(a_\omega^\dagger+a_{-\omega}^\dagger)}{-\omega^2+\frac{i}{2}\epsilon_0^2\omega+\Omega^2}e^{+i\omega(t-|x|)}]d\omega. \end{aligned} \tag{4.68}$$

To calculate the noise kernel for this model, we follow the procedures of point separation regularization. First, we calculate the two point Wightman function

$$
\begin{aligned}
&\langle \phi(x)\phi(y)\rangle \\
&= \left\langle \phi_0(t,x)\phi_0(t',x')\right\rangle - \frac{\epsilon_0}{2}\left\langle \phi_0(t,x)q(t'-|x'|)\right\rangle \\
&\quad - \frac{\epsilon_0}{2}\left\langle q(t-|x|)\phi_0(t',x')\right\rangle + \frac{\epsilon_0^2}{4}\left\langle q(t-|x|)q(t'-|x'|)\right\rangle
\end{aligned}
\tag{4.69}
$$

The four terms in the above expressions are

$$
\left\langle \phi_0(t,x)\phi_0(t',x')\right\rangle = \int_0^{+\infty}\frac{d\omega}{4\pi\omega}\left(e^{-i\omega[(t-t')-(x-x')]} + e^{-i\omega[(t-t')+(x-x')]}\right)
\tag{4.70}
$$

$$
\begin{aligned}
\left\langle \phi_0(t,x)q(t'-|x'|)\right\rangle = \epsilon_0 \int_0^{+\infty} &\frac{i}{4\pi(-\omega^2 + \frac{i}{2}\epsilon_0^2\omega + \Omega^2)} \\
&\cdot\left(e^{-i\omega[(t-t')-(x-|x'|)]} + e^{-i\omega[(t-t')+(x+|x'|)]}\right) d\omega
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
\left\langle q(t-|x|)\phi_0(t',x')\right\rangle = \epsilon_0 \int_0^{+\infty} &\frac{-i}{4\pi(-\omega^2 - \frac{i}{2}\epsilon_0^2\omega + \Omega^2)} \\
&\cdot\left(e^{-i\omega[(t-t')-(|x|-x')]} + e^{-i\omega[(t-t')-(|x|+x')]}\right) d\omega
\end{aligned}
\tag{4.72}
$$

$$
\begin{aligned}
&\left\langle q(t-|x|)q(t'-|x'|)\right\rangle \\
&\quad = 2\epsilon_0^2 \int_0^{+\infty}\frac{\omega}{4\pi|-\omega^2 + \frac{i}{2}\epsilon_0^2\omega + \Omega^2|^2} e^{-i\omega[(t-t')-(|x|-|x'|)]} d\omega
\end{aligned}
\tag{4.73}
$$

Combining above equations, we get

$$
\begin{aligned}
&\langle \phi(x)\phi(y)\rangle \\
&= \int_0^{\infty}\{\frac{1}{4\pi\omega}\left[e^{-i\omega[(t-t')-(x-x')]} + e^{-i\omega[(t-t')+(x-x')]}\right] \\
&\qquad - \frac{i\epsilon_0^2}{8\pi(-\omega^2 + \frac{i}{2}\epsilon^2\omega + \Omega^2)}\left[e^{-i\omega[(t-t')-(x-|x'|)]} + e^{-i\omega[(t-t')+(x+|x'|)]}\right] \\
&\qquad - \frac{-i\epsilon_0^2}{8\pi(-\omega^2 - \frac{i}{2}\epsilon_0^2\omega + \Omega^2)}\left[e^{-i\omega[(t-t')-(|x|-x')]} + e^{-i\omega[(t-t')-(|x|+x')]}\right] \\
&\qquad + \frac{\omega\epsilon_0^4}{8\pi|-\omega^2 + \frac{i}{2}\epsilon_0^2\omega + \Omega^2|^2} e^{-i\omega[(t-t')-(|x|-|x'|)]}\} d\omega
\end{aligned}
\tag{4.74}
$$

Applying the operator $\partial_{a'}\partial_b\partial_{c'}\partial_d$ on $\langle\phi(x)\phi(y)\rangle\,\langle\phi(x')\phi(y')\rangle$ yields

$$
\begin{aligned}
&\partial_{a'}\partial_b\partial_{c'}\partial_d \langle\phi(x)\phi(y)\rangle \left\langle\phi(x')\phi(y')\right\rangle \\
&= \partial_b\partial_d \langle\phi(x)\phi(y)\rangle\, \partial_{a'}\partial_{c'} \left\langle\phi(x')\phi(y')\right\rangle \\
&= \frac{1}{16\pi^2}\{-\frac{(-1)^{b+d}}{[(t-x)-(t'-x')]^2} - \frac{1}{[(t+x)-(t'+x')]^2} \\
&\quad + 2(-\mathrm{sgn}(x))^b(-\mathrm{sgn}(x'))^d F[(t-|x|)-(t'-|x'|)] \\
&\quad - (-1)^b(-\mathrm{sgn}(x'))^d F[(t-x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x'))^d F[(t+x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x))^b(-1)^d F[(t-|x|) \\
&\quad - (t'-x')] - (-\mathrm{sgn}(x))^b F[(t-|x|)-(t'+x')] \\
&\quad - i[(-1)^b(-\mathrm{sgn}(x'))^d G[(t-x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x'))^d G[(t+x)-(t'-|x'|)] \\
&\quad + (-\mathrm{sgn}(x))^b(-1)^d G[(t-|x|)-(t'-x')] \\
&\quad - (-\mathrm{sgn}(x))^b G[(t-|x|)-(t'+x')]]\} \\
&\times \{-\frac{(-1)^{a+c}}{[(t''-x'')-(t'''-x''')]^2} - \frac{1}{[(t''+x'')-(t'''+x''')]^2} \\
&\quad + 2(-\mathrm{sgn}(x''))^a(-\mathrm{sgn}(x'''))^c F[(t''-|x''|)-(t'''-|x'''|)] \\
&\quad - (-1)^a(-\mathrm{sgn}(x'''))^c F[(t''-x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x'''))^c F[(t''+x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x''))^a(-1)^c F[(t''-|x''|)-(t'''-x''')] \\
&\quad - (-\mathrm{sgn}(x''))^a F[(t''-|x''|)-(t'''+x''')] \\
&\quad - i[(-1)^a(-\mathrm{sgn}(x'''))^c G[(t''-x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x'''))^c G[(t''+x'')-(t'''-|x'''|)] \\
&\quad + (-\mathrm{sgn}(x''))^a(-1)^c G[(t''-|x''|)-(t'''-x''')] \\
&\quad - (-\mathrm{sgn}(x''))^a G[(t''-|x''|)-(t'''+x''')]]\},
\end{aligned}
\tag{4.75}
$$

where (t,x), (t',x'), (t'',x'') and (t''',x''') are the coordinates of spactime points x, y, x', y' respectively and the functions F, G are determined by the following integrals:

$$F(\chi) = \frac{\epsilon_0^4}{4}\int_0^{+\infty} \frac{\omega^3}{(-\omega^2+\Omega^2)^2+\frac{\epsilon_0^4}{4}\omega^2} e^{-i\omega\chi} d\omega \tag{4.76}$$

$$G(\chi) = \frac{\epsilon_0^2}{2}\int_0^{+\infty} \frac{\omega^2(-\omega^2+\Omega^2)}{(-\omega^2+\Omega^2)^2+\frac{\epsilon_0^4}{4}\omega^2} e^{-i\omega\chi} d\omega. \tag{4.77}$$

Following the same procedure used in the last subsection, we first combine (4.75) and (4.36) together and apply them to (4.7) to obtain the finite expression of noise kernel components for spacetime points $x \neq y$. Then we evaluate them on two separate spacetime points (t, x) and $(t+\epsilon, x)$ and expand them as power series in ϵ. The results are, for space coordinate $x > 0$, i.e. on the right side of the mirror,

$$\begin{aligned}
&N_{0000} = N_{1111} \\
&= \frac{1}{4\pi^2\epsilon^4} + \frac{1}{8\pi^2}[F(-2x)^2 + F(2x)^2 \\
&- 2iF(-2x)G(-2x) - G(-2x)^2 + 2iF(2x)G(2x) - G(2x)^2] + O(\epsilon_0) \\
&= \frac{1}{4\pi^2\epsilon^4} + \frac{1}{4\pi^2}((\mathrm{Re}F(2x) - \mathrm{Im}G(2x))^2 - (\mathrm{Im}F(2x) + \mathrm{Re}G(2x))^2) + O(\epsilon_0),
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
&N_{0101} = \frac{1}{4\pi^2\epsilon^4} - \frac{1}{8\pi^2}[F(-2x)^2 + F(2x)^2 \\
&- 2iF(-2x)G(-2x) - G(-2x)^2 + 2iF(2x)G(2x) - G(2x)^2] + O(\epsilon_0) \\
&= \frac{1}{4\pi^2\epsilon^4} - \frac{1}{4\pi^2}((\mathrm{Re}F(2x) - \mathrm{Im}G(2x))^2 - (\mathrm{Im}F(2x) + \mathrm{Re}G(2x))^2) + O(\epsilon_0);
\end{aligned} \tag{4.79}$$

and for space coordinate $x < 0$, i.e. on the left side of the mirror,

$$\begin{aligned}
&N_{0000} = N_{1111} \\
&= \frac{1}{4\pi^2\epsilon^4} + \frac{1}{8\pi^2}[F(-2x)^2 + F(2x)^2 \\
&+ 2iF(-2x)G(-2x) - G(-2x)^2 - 2iF(2x)G(2x) - G(2x)^2] + O(\epsilon_0) \\
&= \frac{1}{4\pi^2\epsilon^4} + \frac{1}{4\pi^2}((\mathrm{Re}F(2x) + \mathrm{Im}G(2x))^2 - (\mathrm{Im}F(2x) - \mathrm{Re}G(2x))^2) + O(\epsilon_0),
\end{aligned} \tag{4.80}$$

$$N_{0101} = \frac{1}{4\pi^2\epsilon^4} - \frac{1}{8\pi^2}[F(-2x)^2 + F(2x)^2$$
$$+ 2iF(-2x)G(-2x) - G(-2x)^2 - 2iF(2x)G(2x) - G(2x)^2] + O(\epsilon_0)$$
$$= \frac{1}{4\pi^2\epsilon^4} - \frac{1}{4\pi^2}((\mathrm{Re}F(2x) + \mathrm{Im}G(2x))^2 - (\mathrm{Im}F(2x) - \mathrm{Re}G(2x))^2) + O(\epsilon_0). \tag{4.81}$$

Again, the divergent leading term $\frac{1}{4\pi^2\epsilon^4}$ would be discarded. Actually, the validity of discarding this term is questionable. Discussions about it will be presented in Chapter 5.

4.4 Fluctuations of the force acting on the mirror and of the energy flux radiated by the mirror

To better understand the behavior of the noise kernel, in this section we study the fluctuations of force acting on the mirror and of the energy flux radiated by the mirror. It is, after all, these forces and energies which are important for the motion of the mirror. Additionally, they can really be measured.

Suppose space coordinate $x > 0$. Let spacetime points $x^+ = (t, x)$, $x^- = (t, -x)$, i.e. x^+ and x^- are symmetrically located on the two sides of the mirror with a distance x. Then the fluctuation of force acting on the mirror is

$$\lim_{x\to 0^+} \{\left\langle (T_{11}(x^+) - T_{11}(x^-))^2 \right\rangle - \left\langle (T_{11}(x^+) - T_{11}(x^-)) \right\rangle^2\}$$
$$= \lim_{x\to 0^+} \{\left\langle T_{11}(x^+)^2 \right\rangle + \left\langle T_{11}(x^-)^2 \right\rangle - \left\langle T_{11}(x^+)T_{11}(x^-) \right\rangle - \left\langle T_{11}(x^-)T_{11}(x^+) \right\rangle$$
$$- \left\langle T_{11}(x^+) \right\rangle^2 - \left\langle T_{11}(x^-) \right\rangle^2 + \left\langle T_{11}(x^+) \right\rangle \left\langle T_{11}(x^-) \right\rangle + \left\langle T_{11}(x^-) \right\rangle \left\langle T_{11}(x^+) \right\rangle\}$$
$$= \lim_{x\to 0^+} \{N_{1111}(x^+) + N_{1111}(x^-) - N_{1111}(x^+, x^-) - N_{1111}(x^-, x^+)\}. \tag{4.82}$$

If there is no mirror, the fluctuation is just zero. For a perfectly reflecting mirror, there is no correlation between fields on the two sides of the mirror, i.e. $N_{1111}(x^+, x^-) = 0$. Thus substituting (4.45) into (4.82), we get the fluctuation of force

$$\sigma_F = \lim_{x\to 0^+} \{N_{1111}(x^+) + N_{1111}(x^-)\}$$
$$= \lim_{x\to 0^+} \frac{1}{32\pi^2 x^4} \to \infty. \tag{4.83}$$

This result is unphysical since it shows that although the expectation value of force acting on the mirror is zero, these forces undergo infinite fluctuations that the position of the mirror would be totally unstable. It is very natural to suspect that the cause of this unphysical result is using the arbitrarily high frequency modes. Thus one expect that for the more realistic mirror which becomes transparent at high frequencies, we can probably get a finite result. Now let us perform the calculation for the mirror we proposed in 4.3. Following similar procedures, we get

$$\begin{aligned}
&N_{1111}(x^+,x^-) = N_{1111}(x^-,x^+)\\
&= \frac{1}{64\pi^2}\cdot\frac{1}{x^4}\cdot\left[1+2x^2F(-2x)+2x^2F(2x)-2ix^2G(-2x)+2ix^2G(2x)\right]^2\\
&\quad+\frac{1}{16\pi^2}\left[-F(-2x)+F(2x)+iG(-2x)+iG(2x)\right]^2\\
&= \frac{1}{64\pi^2}\left(\frac{1}{x^4}+\frac{8(\mathrm{Re}F(2x)-\mathrm{Im}G(2x))}{x^2}\right)\\
&+\frac{1}{4\pi^2}((\mathrm{Re}F(2x)-\mathrm{Im}G(2x))^2-(\mathrm{Im}F(2x)+\mathrm{Re}G(2x))^2)
\end{aligned} \tag{4.84}$$

Similarly to what we do before, if we perform exactly the same point separation calculations on Minkowski spacetime, i.e. if the mirror did not exist, we would find that

$$N_{1111}(x^+,x^-) = N_{1111}(x^-,x^+) = \frac{1}{64\pi^2x^4}. \tag{4.85}$$

Thus this divergent leading term should also be discarded. Note that we have already found the expression for $N_{1111}(x^+)$ in 4.3:

$$\begin{aligned}
&N_{1111}(x^+) = N_{1111}(x^-)\\
&= \frac{1}{8\pi^2}[F(-2x)^2+F(2x)^2\\
&\quad-2iF(-2x)G(-2x)-G(-2x)^2+2iF(2x)G(2x)-G(2x)^2]+O(\epsilon_0)\\
&= \frac{1}{4\pi^2}((\mathrm{Re}F(2x)-\mathrm{Im}G(2x))^2-(\mathrm{Im}F(2x)+\mathrm{Re}G(2x))^2)
\end{aligned} \tag{4.86}$$

Substituting (4.86) and (4.84) into (4.82) and discarding the divergent leading term $\frac{1}{64\pi^2 x^4}$, we get the fluctuation of force acting on the mirror

$$\begin{aligned}\sigma_F(x) &= -\frac{1}{4\pi^2 x^2}(\mathrm{Re}F(2x) - \mathrm{Im}G(2x))] \\ &= -\frac{1}{4\pi^2 x^2}[\frac{\epsilon_0^4}{4}\int_0^{+\infty} \frac{\omega^3}{(-\omega^2+\Omega^2)^2 + \frac{\epsilon_0^4}{4}\omega^2}\cos(2\omega x)d\omega \\ &+ \frac{\epsilon_0^2}{2}\int_0^{+\infty} \frac{\omega^2(-\omega^2+\Omega^2)}{(-\omega^2+\Omega^2)^2 + \frac{\epsilon_0^4}{4}\omega^2}\sin(2\omega x)d\omega]\end{aligned} \tag{4.87}$$

The first integral in the above equation goes as $\ln x$, and the second one goes as $-1/x$. Therefore, for this model we still get a divergent fluctuation which goes as $\frac{1}{x^3}$ as $x \to 0$. Although this divergence becomes weaker, it is still unphysical. The effect of high frequencies can not be completely removed in this model. Discussions about this unphysical result are given in Chapter 5. In addition, following exactly the same procedure, we get the fluctuation of energy radiated by the mirror:

$$\begin{aligned}\sigma_E(x) &= -\frac{1}{4\pi^2 x^2}(\mathrm{Re}F(2x) - \mathrm{Im}G(2x))] \\ &+ \frac{1}{\pi^2}[-(\mathrm{Re}F - \mathrm{Im}G)^2 + (\mathrm{Im}F + \mathrm{Re}G)^2].\end{aligned} \tag{4.88}$$

Chapter 5

Discussions and conclusions

In last chapter, we calculate the noise kernel for a prefect reflecting mirror model and a more realistic model in which the mirror becomes transparent at high frequencies. Both models give the same negative value for the noise kernel component N_{0101}, which is contradictory to the fact that noise kernel is always positive semi-definite. This is unphysical, since the negative value implies that the quantum field induces "imaginary" fluctuations of the metric tensor, which is a real quantity. The negative noise kernel implies that the method we used in our calculations is pathological. To find out where the problem is, we examine our calculations from a mathematical aspect and then from a physical perspective. In addition, we discuss the cause of the unphysical results of fluctuations of force acting on the mirror in the end of this chapter.

5.1 Mathematical aspects of the negative noise kernel

By carefully examining each step of our calculation, we find that the problem may come from the last line of (4.37) where we evaluate the integral of the form

$$R(\delta) = \int_0^{+\infty} \omega e^{-i\omega\delta} d\omega. \tag{5.1}$$

The above integral does not converge. However, we can always obtain a value by analytic continuation. The function $R(\delta)$ is well defined in the lower half complex plane, so we can move the point a a little bit downwards and then move it back by taking the limit to make sense the above integral:

$$\begin{aligned} R(\delta) &= \lim_{\eta\to 0^+} \int_0^{+\infty} \omega e^{-i\omega(\delta - i\eta)} d\omega \\ &= \lim_{b\to 0^+} \left(-\frac{1}{(\delta - i\eta)^2}\right) \\ &= -\frac{1}{\delta^2}. \end{aligned} \tag{5.2}$$

This kind of analytic continuation is widely used in field theory, but its validity is rarely discussed. Actually the negative value we encountered here has the same mechanism with the negative energy density in Casimir effect [21].

In Casimir effect, one calculates the quantity $\frac{1}{2}\left\langle(\dot{\phi}^2+\phi'^2)\right\rangle$, which is the expectation value of the square of Hermitian operators $\dot{\phi}$ and ϕ'. This quantity is divergent, and regularization methods must be used to obtain a finite value. One usual way is to use the zeta function regularization [22]. Here we use the same method as that used in previous chapter, the point separation regularization, to show that the negative noise kernel and negative energy density are generated by the same mechanism. For simplicity, we calculate the Casimir effect of a massless scalar field in a two dimensional spacetime. Consider two perfectly reflecting boundaries at points $x=0$ and $x=L$. The normalized mode solutions for a massless field ϕ is

$$\phi_n(x) = \frac{1}{\sqrt{4n\pi}}(e^{-i\frac{n\pi}{L}v} - e^{-i\frac{n\pi}{L}u}), \quad n = 1, 2, 3, ... \tag{5.3}$$

The two point Wightman function is

$$\begin{aligned} D(x,y) &= \sum_{n=1}^{+\infty} \phi_n(x)\phi_n^*(y) \\ &= \sum_{n=1}^{+\infty} \frac{1}{4n\pi}[e^{-i\frac{n\pi}{L}(v-s)} - e^{-i\frac{n\pi}{L}(v-r)} \\ &- e^{-i\frac{n\pi}{L}(u-s)} + e^{-i\frac{n\pi}{L}(u-r)}]. \end{aligned} \tag{5.4}$$

Applying the differential operator $\partial_{a'}\partial_b$ on $D(x,y)$ we get

$$\begin{aligned} \partial_{a'}\partial_b D(x,y) = \sum_{n=1}^{+\infty} \frac{n\pi}{4L^2}[e^{-i\frac{n\pi}{L}(v-s)} - (-1)^a e^{-i\frac{n\pi}{L}(v-r)} \\ - (-1)^b e^{-i\frac{n\pi}{L}(u-s)} + (-1)^{a+b} e^{-i\frac{n\pi}{L}(u-r)}]. \end{aligned} \tag{5.5}$$

Here we have a summation of the form $R(\delta) = \sum_{n=1}^{+\infty} ne^{-in\delta}$, which is basically the discretized version of the integral (5.1). It also does not converge. We now use the same technique of analytic continuation to make it converge as follows:

$$
\begin{aligned}
R(\delta) &= i\frac{d}{d\delta}\sum_{n=1}^{+\infty} e^{-in\delta} \\
&= i\frac{d}{d\delta}\lim_{\eta\to 0^+}\sum_{n=1}^{+\infty} e^{-in(\delta - i\eta)} \\
&= \frac{e^{-i\delta}}{(1-e^{-i\delta})^2}
\end{aligned}
\tag{5.6}
$$

Combining (4.2),(4.4),(5.5) and (5.6) together and then expanding the point separated stress energy tensor as a power series of geodesic distance ϵ, we get:

$$\langle T_{tt}\rangle = -\frac{1}{2\pi\epsilon^2} - \frac{\pi}{24L^2} - \frac{\pi^3\epsilon^2}{480L^4} + O(\epsilon^2). \tag{5.7}$$

Taking the limit $\epsilon \to 0$ and discarding the divergent leading term, we obtain the negative energy density

$$\langle T_{tt}\rangle = -\frac{\pi}{24L^2}. \tag{5.8}$$

Note that $\langle T_{tt}\rangle = \frac{1}{2}\left\langle(\dot{\phi}^2 + \phi'^2)\right\rangle$ is the sum of squares of Hermitian operators, $\dot{\phi}^2$ and ϕ'^2, and on the constant acceleration trajectory (4.31) we have $\langle T_{ab}\rangle = 0$. Thus the noise kernel component $N_{0000} = \left\langle T_{tt}^2\right\rangle = \frac{1}{4}\left\langle(\dot{\phi}^2 + \phi'^2)^2\right\rangle$, which contains terms of the fourth power of a Hermitian operator such as $\dot{\phi}^4$, is positive as we expected. But the component $N_{0101} = \left\langle T_{tx}^2\right\rangle = \left\langle(\dot{\phi}\phi')^2\right\rangle$ only contains a term consisting of the square of the Hermitian operator $\dot{\phi}\phi'$, thus we should not be surprised with it now having a negative value if we accept $\left\langle(\dot{\phi}^2 + \phi'^2)\right\rangle$ is negative. In other words, the negative noise kernel we obtained in last chapter has the same mechanism as the negative energy density in the Casimir effect. The latter is widely believed to be true by many physicists.

5.2 Physical aspects of negative noise kernel

Mathematically possible results are not always physically significant. The negative noise kernel component N_{0101} is mathematically similar to the negative energy in the Casimir effect. In the Casimir effect, one can naturally interpret the negative value as a proof of negative energy density and predict the attraction force between two electrically neutral conducting surfaces. Although this force is extremely weak, experiments with more and more accuracy have been undertaken that measure the attractive force [23–25]. However, the physical significance of the negative $\langle N_{0101} \rangle$ is unclear. it is the expectation value of $\left\langle (T_01 - \langle T_01 \rangle)^2 \right\rangle$. If it is negative, $T_01 - \langle T_01 \rangle$ must be imaginary. What does the "imaginary" mean? Will it induce the "imaginary" fluctuations of the gravitational field? What are the physical effects of it?

It is obvious that the negative noise kernel contradicts all physical laws we know so far. Thus more likely, the problem may come from our calculations. In particular, regularizing the noise kernel by throwing away the infinity is probably the wrong thing to do. Although it does work in regularizing the stress energy tensor, it would not work for regularizing the noise kernel. In fact, the noise, N_{0101}, is confusing. This is the noise of energy flux, but right at the mirror, the ingoing flux equals to the outgoing flux for all modes, i.e. at the mirror we would expect it to not only have zero expectation value, but also have zero noise. However, our calculations show that N_{0101} is not zero but negative infinite (4.45) after dropping the divergent leading term $\frac{1}{4\pi^2\epsilon^4}$. We note that if we do not drop that infinite term, it may be canceled by the second term which is divergent on the mirror and thus we get zero noise. Unfortunately, this would result in another problem, that the noise becomes infinite when you go away from the mirror.

L.H.Ford et. al. got the negative power spectrum in [26], which is similar with the negative noise kernel we get here. They argued that because the negative value is always associated with coordinate space correlation functions which are singular at coincident points, and thus cannot represent the expectation value of a meaningful squared quantity. However, the correlation functions at distinct points are meaningful and can be integrated over spacetime to get a physically meaningful quantity. Now we are becoming more and more convinced that one cannot subtract that divergent leading term of the noise kernel—it is an essential part of the kernel, i.e. we think that the stochastic gravity approach has its own points, but that regularizing the noise by throwing away the infinity is probably not the correct way to do. One needs a different argument to handle the infinities or turn them into finite quantities.

5.3 The infinite fluctuations of force acting on the mirror

In the last section of last chapter, we find that the fluctuations of force acting on a perfectly reflecting mirror go to infinity as $1/x^4$ (Eq.(4.83)). We suspect that the divergent result comes from the arbitrarily high frequency modes and we would get a finite result for the mirror which becomes transparent at high frequencies. However, the result is that the fluctuations go as $\frac{1}{x^3}$ (Eq.(4.87)). Though it is a much weaker divergence compared with perfect reflecting case but is still divergent.

By carefully examining our calculations, we find that the problem still comes from the arbitrarily high frequency modes. Let's go back to the modes expansion (4.68). The first term of this equation is the usual modes expansion for a free scalar field propagating on Minkowski spacetime, the second term representing the modes reflected by the mirror. One observes that the reflecting modes approaches to zero at high frequencies, i.e. the high frequency modes freely propagate on the spacetime. If one calculate the expectation value of stress energy momentum, the high frequency effect can be removed directly by renormalization. However, it is not this case in the calculations of the noise. Observing Eq.(4.75)

$$
\begin{aligned}
&\partial_{a'}\partial_b\partial_{c'}\partial_d \left\langle \phi(x)\phi(y)\right\rangle \left\langle \phi(x')\phi(y')\right\rangle \\
&= \partial_b\partial_d \left\langle \phi(x)\phi(y)\right\rangle \partial_{a'}\partial_{c'} \left\langle \phi(x')\phi(y')\right\rangle \\
&= \frac{1}{16\pi^2}\{-\frac{(-1)^{b+d}}{[(t-x)-(t'-x')]^2} - \frac{1}{[(t+x)-(t'+x')]^2} \\
&\quad + 2(-\mathrm{sgn}(x))^b(-\mathrm{sgn}(x'))^d F[(t-|x|)-(t'-|x'|)] \\
&\quad - (-1)^b(-\mathrm{sgn}(x'))^d F[(t-x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x'))^d F[(t+x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x))^b(-1)^d F[(t-|x|) \\
&\quad - (t'-x')] - (-\mathrm{sgn}(x))^b F[(t-|x|)-(t'+x')] \\
&\quad - i[(-1)^b(-\mathrm{sgn}(x'))^d G[(t-x)-(t'-|x'|)] \\
&\quad - (-\mathrm{sgn}(x'))^d G[(t+x)-(t'-|x'|)] \\
&\quad + (-\mathrm{sgn}(x))^b(-1)^d G[(t-|x|)-(t'-x')] \\
&\quad - (-\mathrm{sgn}(x))^b G[(t-|x|)-(t'+x')]]\} \\
&\times \{-\frac{(-1)^{a+c}}{[(t''-x'')-(t'''-x''')]^2} - \frac{1}{[(t''+x'')-(t'''+x''')]^2} \\
&\quad + 2(-\mathrm{sgn}(x''))^a(-\mathrm{sgn}(x'''))^c F[(t''-|x''|)-(t'''-|x'''|)] \\
&\quad - (-1)^a(-\mathrm{sgn}(x'''))^c F[(t''-x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x'''))^c F[(t''+x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x''))^a(-1)^c F[(t''-|x''|)-(t'''-x''')] \\
&\quad - (-\mathrm{sgn}(x''))^a F[(t''-|x''|)-(t'''+x''')] \\
&\quad - i[(-1)^a(-\mathrm{sgn}(x'''))^c G[(t''-x'')-(t'''-|x'''|)] \\
&\quad - (-\mathrm{sgn}(x'''))^c G[(t''+x'')-(t'''-|x'''|)] \\
&\quad + (-\mathrm{sgn}(x''))^a(-1)^c G[(t''-|x''|)-(t'''-x''')] \\
&\quad - (-\mathrm{sgn}(x''))^a G[(t''-|x''|)-(t'''+x''')]]\},
\end{aligned}
$$

we find that there are additional cross terms relating to the arbitrarily high frequencies come out. More specifically, the above equation is a product of two terms, the first line of each term $-\frac{(-1)^{b+d}}{[(t-x)-(t'-x')]^2} - \frac{1}{[(t+x)-(t'+x')]^2}$ and $(-1)^{a+c}[(t''-x'')-(t'''-x''')]^2 - \frac{1}{[(t''+x'')-(t'''+x''')]^2}$ correspond to the arbitrarily high frequency modes and the rest of each term correspond to the reflected low frequency modes. When you expand the product, besides the product of the first line of each term(the product of high frequency modes with high frequency modes), additional cross terms which are products of high frequency modes and low frequency modes come out. It is these terms contribute to the $\frac{1}{x^3}$ divergence. This example clearly shows that the arbitrarily high frequency divergence cannot be renormalized in the noise calculations as before when we calculate the expectation value of stress energy tensor.

The above analysis gives a suggestion that including arbitrarily high frequency modes is inappropriate even in flat spacetime. In principle, the arbitrarily high frequency modes are questionable since it is based on physics down to Plank scale. It eventually runs into the whole problem of quantum gravity. Our unphysical results of noise calculations even in flat spacetime give a signal that the stochastic gravity program may not be a good approximation in the sense that the noise is strongly related to quantum gravity effects. We are still do not know what is happening in the neighborhood of the mirror and further investigation is needed in the future.

Bibliography

[1] S.W. Hawking. Black hole explosions. *Nature*, 248:30–31, 1974.

[2] S.W. Hawking. Particle Creation by Black Holes. *Commun.Math.Phys.*, 43:199–220, 1975.

[3] James M. Bardeen. Black Holes Do Evaporate Thermally. *Phys.Rev.Lett.*, 46:382–385, 1981.

[4] S. Massar. The Semiclassical back reaction to black hole evaporation. *Phys.Rev.*, D52:5857–5864, 1995.

[5] B.L. Hu and E. Verdaguer. Stochastic Gravity: Theory and Applications. *Living Rev.Rel.*, 11:3, 2008.

[6] B.L. Hu and Albert Roura. Metric fluctuations of an evaporating black hole from back reaction of stress tensor fluctuations. *Phys.Rev.*, D76:124018, 2007.

[7] J.D.Bekenstein. *148-159, Quantum Theory of Gravity, edited by S.M.Christensen.* Adam Hilger, Bristol, 1984.

[8] Robert M. Wald. *394, General Relativity.* The University of Chicago Press, Chicago and London, 1984.

[9] S.M. Christensen. Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background: The Covariant Point Separation Method. *Phys.Rev.*, D14:2490–2501, 1976.

[10] S.M. Christensen. Regularization, Renormalization, and Covariant Geodesic Point Separation. *Phys.Rev.*, D17:946–963, 1978.

[11] Rosario Martin and Enric Verdaguer. Stochastic semiclassical gravity. *Phys.Rev.*, D60:084008, 1999.

[12] B.L. Hu and E. Verdaguer. Stochastic gravity: A Primer with applications. *Class.Quant.Grav.*, 20:R1–R42, 2003.

[13] Don N. Page. Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole. *Phys.Rev.*, D13:198–206, 1976.

[14] C.H. Wu and L.H. Ford. Fluctuations of the Hawking flux. *Phys.Rev.*, D60:104013, 1999.

[15] Alan H. Guth and So-Young Pi. The Quantum Mechanics of the Scalar Field in the New Inflationary Universe. *Phys.Rev.*, D32:1899–1920, 1985.

[16] Bryce S. DeWitt. Quantum Field Theory in Curved Space-Time. *Phys.Rept.*, 19:295–357, 1975.

[17] Nicholas G. Phillips and B.L. Hu. Noise kernel in stochastic gravity and stress energy bitensor of quantum fields in curved space-times. *Phys.Rev.*, D63:104001, 2001.

[18] Michael E. Peskin and Daniel V. Schroeder. *88-90, An Introduction to quantum field theory.* Westview Press.

[19] S.A. Fulling and P.C.W. Davies. Radiation from a Moving Mirror in Two Dimensional Space-Time: Conformal Anomaly. *Proc.Roy.Soc.Lond.*, A348:393–414, 1976.

[20] P.C.W. Davies and S.A. Fulling. Radiation from Moving Mirrors and from Black Holes. *Proc.Roy.Soc.Lond.*, A356:237–257, 1977.

[21] H.B.G. Casimir. On the Attraction Between Two Perfectly Conducting Plates. *Indag.Math.*, 10:261–263, 1948.

[22] S.W. Hawking. Zeta function regularization of path integrals in curved spacetime. *Communications in Mathematical Physics*, 55:133–148, 1977.

[23] M.J. Sparnaay. Measurements of attractive forces between flat plates. *Physica*, 24:751–764, 1958.

[24] G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso. Measurement of the Casimir force between parallel metallic surfaces. *Phys.Rev.Lett.*, 88:041804, 2002.

[25] P. Zuurbier, S. de Man, G. Gruca, K. Heeck, and D. Iannuzzi. Measurement of the Casimir force with a ferrule-top sensor. *New J.Phys.*, 13:023027, 2011.

[26] Jen-Tsung Hsiang, Chun-Hsien Wu, and L.H. Ford. Negative Power Spectra in Quantum Field Theory. *Phys.Lett.*, A375:2296–2299, 2011.