Induced Maps in Galois Cohomology

by

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Abstract

Galois cohomology is an important tool in algebra that can be used to classify isomorphism classes of algebraic objects over a field. In this thesis, we show that many objects of interest in algebra can be described in cohomological terms. Some objects that we discuss include quadratic forms, Pfister forms, G-crossed product algebras, and tuples of central simple algebras. We also provide cohomological interpretation to some induced maps that naturally occur in short exact sequences.

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Dedication

I would like to dedicate this thesis to my beloved sister, who loves mathematics just as much as I do.

Chapter 1

Introduction

In order to facilitate the study of certain mathematical objects over a field, it is common to try and establish a systematic way of organizing them into families. An example of such is the complete classification of connected compact Lie groups over $\mathbb{C}$ by Cartan and Weil. However, over non-algebraically closed fields, complications arise, and a complete classification is more complicated to establish. Over $\mathbb{C}$, a real circle and a real line with its origin removed are isomorphic, while over $\mathbb{R}$, they are not. Thus, to classify objects over such fields, one needs to rely on a theory called Galois cohomology.

Galois cohomology is often used to classify isomorphism classes of algebraic objects over a field. More precisely, given an algebraic structure A over a field k and $G = \mathrm{Aut}_k(A)$, then $\mathrm{H}^1(k, G)$ is in bijective correspondence with k-isomorphism classes of twisted forms of A, that is, classes of algebraic objects A' that become isomorphic to A over a separable closure of k (cf. [4]). For example, in [6], T.A. Springer had shown that $\mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ classifies isomorphism classes of quadratic forms of dimension n over k. For further examples of algebraic objects that can be classified via Galois cohomology, refer to [3].

In this thesis, we show that many objects of interest in algebra can be naturally described by Galois cohomology sets or characterized as images of certain induced maps in Galois cohomology. An example of the classification is to show that elements of $\mathrm{H}^1(k, \mathrm{GL}_n / \mu_d)$ where $d \mid n$ can be identified with central simple algebras over k of degree n and exponent dividing d (cf. Lemma 2.6 [1]).

The structure of this thesis is as follows. In Chapter 2, we introduce the notations and some background material in group and Galois cohomology. In particular, we list some known examples of the first Galois cohomology sets that can be described via isomorphism classes of algebraic objects. In chapters 3 and4, we will discuss some applications of Galois cohomology in quadratic form theory. In particular, we describe quadratic forms of discriminant 1, Pfister forms and scaled Pfister forms in cohomological terms.

Chapter 5 is used to discuss some applications of Galois cohomology in the theory of central simple algebras. We give a cohomological description of l-tuples $(A_1, \ldots, A_l)$ of central simple k-algebras whose Brauer classes satisfy a given system of linear equations of the form $n_1[A_1] + \ldots + n_r[A_r] = 0$ in the Brauer group $\mathrm{Br}(k)$. We also describe G-crossed product algebras as the image of certain induced maps in Galois cohomology. Finally, in chapter 6, we give a cohomological interpretation of the trace form in alternative and Jordan algebras.

Chapter 2

Preliminaries

2.1 Notation

We will use the following notation throughout the thesis:

k - a base field of characteristic 0 containing a primitive 4-th root of unity

k_{sep} - a separable closure of k

Γ - the absolute Galois group of k

K - a finite Galois extension of k

Γ_K - the relative Galois group of K over k

μ_p - the group of p-th roots of unity

$\langle a_1, \ldots, a_n \rangle$ - the diagonal quadratic form $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} a_i x_i^2$.

2.2 A long exact sequence in group cohomology

Let Γ be a group. Suppose that A, B and C are Γ-groups where A is a central subgroup of B and they satisfy the following short exact sequence of Γ-groups:

$$0 \to A \to B \to C \to 0$$

Then, we obtain the following long exact sequence of pointed sets:

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to \mathrm{H}^1(\Gamma, A) \to \mathrm{H}^1(\Gamma, B) \to \mathrm{H}^1(\Gamma, C) \to \mathrm{H}^2(\Gamma, A)$$

2.3 An action of C^G on $\mathrm{H}^1(G, A)$

C^G has a right action on $\mathrm{H}^1(G, A)$ given as follow. Let $c \in C$, lift c to $b \in B$, then the image of $a_s \in \mathrm{H}^1(G, A)$ under c is the cocycle $b^{-1} a_s {}^s b$. We have the following proposition:

Proposition 2.3.1. *(Proposition 39 p.52 [5]) Two elements of $\mathrm{H}^1(G, A)$ have the same image in $\mathrm{H}^1(G, B)$ if and only if they are in the same C^G-orbit.*

2.4 An action of $\mathrm{H}^1(G, A)$ on $\mathrm{H}^1(G, B)$

$\mathrm{H}^1(G, A)$ has a left action on $\mathrm{H}^1(G, B)$ given as follow. Let $\alpha \in \mathrm{H}^1(G, A)$ and $\beta \in \mathrm{H}^1(G, B)$, then $\alpha \cdot \beta = \iota_\star(\alpha)\beta$. Similar to above, in this context, we have the following proposition:

Proposition 2.4.1. *(Proposition 42 p.54 [5]) Two elements of $\mathrm{H}^1(G, B)$ have the same image in $\mathrm{H}^1(G, C)$ if and only if they are in the same orbit under the action of $\mathrm{H}^1(G, A)$.*

2.5 $\mathrm{H}^1(k, G)$

Given a field k and an algebraic group G defined over k_{sep}, we define $\mathrm{H}^1(k, G)$ to be the group cohomology set $\mathrm{H}^1(\Gamma, G)$.

Here are some basic known examples of Galois cohomology:

1. $\mathrm{H}^1(k, \mathrm{GL}_n(k_{sep})) = \{1\}$ (Hilbert's Theorem 90).

2. $\mathrm{H}^1(k, \mu_d) \cong k^\times/(k^\times)^d$, where given $a \in k^\times/(k^\times)^d$, the corresponding cocycle $\gamma : \Gamma \to \mu_d$ is defined by $\gamma(s) = b^{-1}s(b)$ for all $s \in \Gamma$, and some $b \in k_{sep}^\times$ such that $b^d = a$.

3. $\mathrm{H}^1(k, S_n) \cong \{$isomorphism classes of étale algebra of dimension n over $k\}$, where an *étale k-algebra L* of dimension n is a direct sum of n finite separable field extensions of k.

4. $\mathrm{H}^1(k, G) \cong$ {isomorphism classes of G-Galois algebras of dimension n over k} for a finite group G, where a *G-Galois k-algebra L* is an étale algebra over k endowed with an action by G such that $L^G = k$ (cf. p.287-288 [3]).

5. $\mathrm{H}^1(k, \mathrm{O}_n(k_{sep})) \cong$ {isomorphism classes of quadratic forms of dimension n over k}, where a *quadratic form* of dimension n is a homogeneous polynomial of degree 2 in n variables.

6. $\mathrm{H}^1(k, \mathrm{PGL}_n(k_{sep})) \cong$ {isomorphism classes of central simple algebras of degree n over k}, where an associative algebra A over k is a *central simple algebra* if $\mathrm{Z}(A) = k$, and A contains no nontrivial proper two sided ideals.

7. Let $G_1, \ldots, G_n$ be n algebraic groups over k_{sep}. Then,

$$\mathrm{H}^1(k, \prod_{i=1}^{n} G_i) \cong \prod_{i=1}^{n} \mathrm{H}^1(k, G_i).$$

2.6 The Brauer group $\mathrm{H}^2(k, \mathbb{G}_m)$

One can identify elements of $\mathrm{H}^2(k, \mathbb{G}_m)$ with Brauer equivalence classes of central simple algebras over k. Recall that from the short exact sequence:

$$1 \to \mathbb{G}_m(k_{sep}) \to \mathrm{GL}_n(k_{sep}) \to \mathrm{PGL}_n(k_{sep}) \to 0,$$

the connecting map $\delta : \mathrm{H}^1(k, \mathrm{PGL}_n(k_{sep})) \to \mathrm{H}^2(k, \mathbb{G}_m(k_{sep}))$ maps a central simple algebra A to its equivalence class in the Brauer group.

Chapter 3

Quadratic forms and Pfister forms

Consider the following embedding of algebraic groups:

$$\begin{aligned}\iota : &\mu_2^n \longrightarrow \mathrm{O}_n(k_{sep})\\ &(\varepsilon_1, \dots, \varepsilon_n) \mapsto \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)\end{aligned}$$

Recall that we may identify $H^1(k, \mu_2^n)$ with $(k^\times/(k^\times)^2)^n$. We observe that:

Lemma 3.0.1. *The induced map $\iota_\star : \mathrm{H}^1(k, \mu_2^n) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ is precisely the map which sends $(a_1, \dots, a_n)$ to $\langle a_1, \dots, a_n \rangle$, where $a_i \in k^\times/(k^\times)^2$ for $1 \le i \le n$.*

Proof. Let $\alpha : \Gamma \to \mu_2^n$ be a cocycle in $\mathrm{Z}^1(k, \mu_2^n)$. Since $(a_i)_{i=1}^n \in (k^\times/(k^\times)^2)^n$ is an n-tuple representing α, one has that for all $s \in \Gamma$,

$$\alpha(s) = (b_1^{-1}s(b_1), \dots, b_n^{-1}s(b_n)), \text{ for some } b_i^2 = a_i \text{ and } b_i \in k_{sep}^\times.$$

Fix one such n-tuple $(b_1, \dots, b_n) \in (k_{sep}^\times)^n$ where $b_i^2 = a_i$. Denote by K the compositum of Galois extensions $k(b_i)$ in k_{sep} for $1 \le i \le n$. Then, K is also a finite Galois extension over k. Let $\{e_i\}_{i=1}^n$ be a basis of k^n, and $\{e_i^\star\}_{i=1}^n$ the corresponding dual basis. Fix a tensor $x = \sum_{i=1}^n e_i^\star \otimes e_i^\star \in ((k^n)^\star)^{\otimes 2}$, which represents the split quadratic form over k^n. Since $\iota_\star(\alpha) \in \mathrm{Z}^1(k, \mathrm{O}^n(k_{sep})) \subseteq \mathrm{Z}^1(k, \mathrm{GL}_n(k_{sep}))$, it follows from $\mathrm{H}^1(k, \mathrm{GL}_n(k_{sep}))$ being trivial that $\iota_\star(\alpha)$ is cohomologous to 0 in $\mathrm{Z}^1(\Gamma_K, \mathrm{GL}_n(K))$. Thus, there exists $f \in \mathrm{GL}(K^n)$ such that for all $s \in \Gamma_K$, $\iota_\star(\alpha)(s) = f^{-1}s(f)$. Suppose that there exists $h \in \mathrm{GL}(K^n)$ such that $f^{-1}s(f) = h^{-1}s(h)$. Then, $hf^{-1} = s(hf^{-1})$ for all $s \in \Gamma_K$, and hence, $hf^{-1} \in \mathrm{GL}(k^n)$. Therefore, $hf^{-1}(x)$ defines an isometric quadratic form to x on k^n, which further implies that $h(x)$ defines an isometric quadratic form to $f(x)$ on K^n. So, it suffices to consider f being the matrix $\mathrm{diag}(b_1, \ldots, b_n) \in \mathrm{GL}(K^n)$. Then, f satisfies the equation $\iota_\star(\alpha)(s) = f^{-1}s(f)$ trivially. Thus,

$$f(x) = \sum_{i=1}^{n} b_i e_i^\star \otimes b_i e_i^\star = \sum_{i=1}^{n} a_i e_i^\star \otimes e_i^\star.$$

So, $\iota_\star(\alpha)$ corresponds to the quadratic space $(K^n, f(x))$, where $f(x)$ is precisely the quadratic form represented by $\langle a_1, \ldots, a_n\rangle$. □

Corollary 3.0.2. *The map* $\iota_\star : \mathrm{H}^1(k, \mu_2^n) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ *defined in Lemma 3.0.1 is surjective.*

Proof. Let q be a quadratic form corresponding to a cohomology class α in $\mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$. Since every quadratic form is diagonalizable, there exists $(a_1, \ldots, a_n) \in k^n$ such that q is equivalent to a diagonal quadratic form $q' = \mathrm{diag}(a_1, \ldots, a_n)$. Then, q and q' are isometric, which means that q' also represents α. Thus, if $\pi : k^\times \to k^\times/(k^\times)^2$ is the projection map, then $(\pi(a_i))_{i=1}^n \in (k^\times/(k^\times)^2)^n \cong (\mathrm{H}^1(k, \mu_2))^n$ is such that $\iota_\star((\pi(a_i))_{i=1}^n) = \alpha$. □

Lemma 3.0.3. *Let* $\phi : \mu_2^m \to \mu_2^n$ *be the map defined by* $\phi((\varepsilon_j)_{j=1}^m) = (\prod_{j=1}^{m} \varepsilon_j^{e_{i,j}})_{i=1}^n$. *Then, the induced map* $\phi_\star : \mathrm{H}^1(k, \mu_2^m) \to \mathrm{H}^1(k, \mu_2^n)$ *is precisely the map* $\phi_\star((a_j)_{j=1}^m) = (\prod_{j=1}^{m} a_j^{e_{i,j}})_{i=1}^n$, *where* $a_i \in k^\times/(k^\times)^2$ *for* $1 \leq i \leq m$.

Proof. Let $\alpha : \Gamma \to \mu_2^m$ be a cocycle in $\mathrm{Z}^1(k, \mu_2^m)$. Since $(a_j)_{j=1}^m \in (k^\times/(k^\times)^2)^m$ corresponds to the cocycle α, one has that for all $s \in \Gamma$,

$$\alpha(s) = (b_j^{-1}s(b_j))_{j=1}^m, \text{ for some } b_j^2 = a_j \text{ and } b_j \in k_{sep}^\times.$$

Hence, the resulting cocycle $\phi_\star(\alpha) : \Gamma \to \mu_2^n$ is the map

$$\phi_\star(\alpha)(s) = (\prod_{j=1}^{m}(b_j^{-1}s(b_j))^{e_{i,j}})_{i=1}^n = (\prod_{j=1}^{m}((b_j^{e_{i,j}})^{-1}s(b_j^{e_{i,j}}))_{i=1}^n \text{ for all } s \in \Gamma.$$

Thus, an n-tuple of elements in $k^\times/(k^\times)^2$ which corresponds to $\phi_\star(\alpha)$ is

$$(\prod_{j=1}^{m}(b_j^{e_{i,j}})^2)_{i=1}^n = (\prod_{j=1}^{m} a_j^{e_{i,j}})_{i=1}^n,$$

as desired. $\square$

Corollary 3.0.4. *Let $\{\chi_i\}_{i=1}^{2^n}$ be the set of all distinct maps of the form $\chi_i : \mu_2^n \to \mu_2$ such that $\chi_i((\varepsilon_j)_{j=1}^n) = \prod_{j=1}^{n} \varepsilon_j^{d_{i,j}}$ for $d_{i,j} \in \{0,1\}$. Consider an embedding map $\lambda : \mu_2^n \to \mathrm{O}_{2^n}(k_{sep})$ via $\lambda((\varepsilon_j)_{j=1}^n) = \mathrm{diag}(\chi_1((\varepsilon_j)_{j=1}^n), \dots, \chi_{2^n}((\varepsilon_j)_{j=1}^n)$. Then, the induced map $\lambda_\star : \mathrm{H}^1(k, \mu_2^n) \to \mathrm{H}^1(k, \mathrm{O}_{2^n}(k_{sep}))$ is defined by:*

$$\lambda_\star((a_j)_{j=1}^n) = \langle \chi_1((a_j)_{j=1}^n), \dots, \chi_{2^n}((a_j)_{j=1}^n\rangle,$$

where $a_i \in k^\times/(k^\times)^2$ for $1 \le i \le n$.

Proof. Observe that $\lambda = \iota \circ \phi$, where $\iota : \mu_2^{2^n} \to \mathrm{O}_n(k_{sep})$ is the embedding defined in Lemma 3.0.1, and $\phi : \mu_2^n \to \mu_2^{2^n}$ defined in Lemma 3.0.3 with $e_{i,j} = d_{i,j}$. Hence, it follows from the aforementioned Lemmas that $\lambda_\star = \iota_\star \circ \phi_\star$ is the desired map. $\square$

3.1 Pfister forms

A 2^n-dimensional quadratic form η is *an n-fold Pfister form* if $\eta = \bigotimes_{i=1}^{n} \langle 1, a_i \rangle$. Using above results, one arrives at the following characterization:

Corollary 3.1.1. *A 2^n-dimensional quadratic form η is an n-fold Pfister form if and only if $\eta \in \lambda_\star(\mathrm{H}^1(k, \mu_2^n))$, where $\lambda_\star$ is the map defined in Corollary 3.0.4.*

Proof. The statement follows from the observation that an n-fold Pfister form $\ll a_1, \dots, a_n \gg$ can be expressed as:

$$\ll a_1, \dots, a_n \gg = \bigotimes_{i=1}^{n} \langle 1, a_i \rangle = \langle \chi_1((a_j)_{j=1}^n), \dots, \chi_{2^n}((a_j)_{j=1}^n\rangle,$$

where $\{\chi_i\}_{i=1}^{2^n}$ are precisely the maps defined in Corollary 3.0.4. □

3.2 Scaled Pfister forms

A 2^n-dimensional quadratic form γ is *an n-fold scaled Pfister form* if $\gamma = a_0\eta$ for some n-fold Pfister form η. Similar to above, one obtains the following:

Corollary 3.2.1. *Let $\psi : \mu_2^{n+1} \to \mathrm{O}_n(k_{sep})$ be the map defined by:*

$$\psi((\varepsilon_j)_{j=0}^n) = \varepsilon_0 \operatorname{diag}(\chi_1((\varepsilon_j)_{j=1}^n), \dots, \chi_{2^n}((\varepsilon_j)_{j=1}^n)),$$

where $\{\chi_i\}_{i=1}^{2^n}$ are the maps defined in Corollary 3.0.4. Then, a 2^n-dimensional quadratic form η is an n-fold scaled Pfister form if and only if $\eta \in \psi_\star(\mathrm{H}^1(k, \mu_2^{n+1}))$.

Proof. Observe that $\psi = \iota \circ \phi$, where $\iota : \mu_2^{2^n} \to \mathrm{O}_n(k_{sep})$ is the embedding defined in Lemma 3.0.1, and $\phi : \mu_2^{n+1} \to \mu_2^{2^n}$ defined in Lemma 3.0.3 with $e_{i,j} = d_{i,j}$ for $1 \geq j$, and $e_{i,0} = 1$ for all i. Thus, by Lemma 3.0.1 and Lemma 3.0.3, $\psi_\star = \iota_\star \circ \phi_\star$ is the map such that given $(a_j)_{j=0}^n \in (k^\times/(k^\times)^2)^{n+1}$, one has that

$$\psi_\star((a_j)_{j=0}^n) = \langle a_0\chi_1((a_j)_{j=1}^n), \dots, a_0\chi_{2^n}((a_j)_{j=1}^n\rangle,$$

which is precisely the scaled Pfister form $\langle a_0\rangle \otimes \bigotimes_{i=1}^n \langle 1, a_i\rangle$. □

Chapter 4

Galois cohomology of special orthogonal groups

In order to study $\mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$, we will rely on the long exact sequence in cohomology to relate $\mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$ to $\mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$, since the latter is better understood. Then, by giving an interpretation to the induced maps between them, we will give a cohomological intepretation to this Galois cohomology set.

Recall the following discriminant map:

$$\mathrm{disc} : \mathrm{H}^1(k, \mathrm{O}_n(k_{sep})) \to k_{sep}$$
$$\langle a_1, \ldots, a_n \rangle \mapsto \prod_{j=1}^{n} a_j$$

We observe that:

Lemma 4.0.2. *Let* $\det : \mathrm{O}_n(k_{sep}) \to \mu_2$ *be the determinant map. Then,* $\det_\star : \mathrm{H}^1(k, \mathrm{O}_n(k_{sep})) \to \mathrm{H}^1(k, \mu_2)$ *is the discriminant map.*

Proof. Denote by ξ the inclusion map of $\mathrm{SO}_n(k_{sep})$ into $\mathrm{O}_n(k_{sep})$, and $\iota : \mu_2^n \to \mathrm{O}_n(k_{sep})$ the map defined in Lemma 3.0.1. Furthermore, let $\phi : \mu_2^{n-1} \to \mu_2^n$ be the embedding such that $\phi((\varepsilon_i)_{i=1}^{n-1}) = (\varepsilon_i)_{i=1}^{n}$, where $\varepsilon_n = \prod_{i=1}^{n-1} \varepsilon_i$, and $\lambda : \mu_2^n \to \mu_2$ the map which maps $(\varepsilon_i)_{i=1}^{n}$ to $\prod_{i=1}^{n} \varepsilon_i$. Also, since $\mathrm{Im}(\iota\phi) \subseteq \mathrm{SO}_n(k_{sep})$, denote by $\tau : \mu_2^{n-1} \to \mathrm{SO}_n(k_{sep})$ the composition of ϕ and ι. Then, observe that one has the following commutative diagram where the two horizontal sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_2^{n-1} & \xrightarrow{\phi} & \mu_2^n & \xrightarrow{\lambda} & \mu_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \mathrm{SO}_n(k_{sep}) & \xrightarrow{\xi} & \mathrm{O}_n(k_{sep}) & \xrightarrow{\det} & \mu_2 & \longrightarrow & 0 \end{array}$$

So, one obtains the following commutative diagram of long exact sequences in cohomology:

$$\begin{array}{ccccccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\delta'} & (k^\times/(k^\times)^2)^{n-1} & \xrightarrow{\phi_\star} & (k^\times/(k^\times)^2)^n & \xrightarrow{\lambda_\star} & k^\times/(k^\times)^2 \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \tau_\star} & & \downarrow{\scriptstyle \iota_\star} & & \downarrow{\scriptstyle \cong} \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\delta} & \mathrm{H}^1(k, \mathrm{SO}_n) & \xrightarrow{\xi_\star} & \mathrm{H}^1(k, \mathrm{O}_n) & \xrightarrow{\det_\star} & k^\times/(k^\times)^2 \end{array} \tag{4.1}$$

Given a cohomology class $\alpha \in \mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$, one may represent $\zeta_\star(\alpha)$ by a diagonal quadratic form $\langle a_1, \ldots, a_n \rangle$ for some $a_j \in k^\times/(k^\times)^2$. It follows from the definition of $\iota_\star$ that $(a_j)_{j=1}^n$ is an n-tuple in $(k^\times/(k^\times)^2)^n$ such that $\iota_\star((a_j)_{j=1}^n) = \alpha$. By Lemma 3.0.3, $\lambda_\star((a_j)_{j=1}^n) = \prod_{j=1}^n a_j$. Thus,

$$\det{}_\star(\alpha) = \det{}_\star \iota_\star((a_j)_{j=1}^n) = \lambda_\star((a_j)_{j=1}^n) = \prod_{j=1}^n a_j,$$

which is precisely the discriminant map. □

Lemma 4.0.3. *The map $\xi_\star : \mathrm{H}^1(k, \mathrm{SO}_n(k_{sep})) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ in Diagram 4.1 is injective.*

Proof. Suppose that $\xi_\star$ is not injective. So, there exist two distinct cohomology classes α and β in $\mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$ such that $\xi_\star(\alpha) = \xi_\star(\beta)$. By Proposition 2.3.1, α and β belong to the same $\mathbb{Z}/2\mathbb{Z}$-orbit, that is, $s(\alpha) = \beta$, where s denotes the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. Since $\iota_\star$ is surjective, there exists an element $x \in (k^\times/(k^\times)^2)^n$ such that $\iota_\star(x) = \xi_\star(\alpha)$. From diagram 4.1, one has that

$$\lambda_\star(x) = \det{}_\star(\iota_\star(x)) = \det{}_\star(\xi_\star(\alpha)) = 0.$$

So, it follows from the sequence being exact that there exists an element $x_0 \in (k^\times/(k^\times)^2)^{n-1}$ such that $\phi_\star(x_0) = x$. Observe that

$$\xi_\star(\tau_\star(x_0)) = \iota_\star(\phi_\star(x_0)) = \iota_\star(x) = \xi_\star(\alpha) = \xi_\star(\beta).$$

Thus, by Proposition 2.3.1, $\tau_\star(x_0)$ belongs to the same $\mathbb{Z}/2\mathbb{Z}$-orbit as α and β. Since a $\mathbb{Z}/2\mathbb{Z}$-orbit can only have at most two distinct elements, without loss of generality, one may assume that $\tau_\star(x_0) = \alpha$. By Lemma 3.0.3, $\phi_\star$ is a group homomorphism such that $\phi((a_i)_{i=1}^{n-1}) = (a_i)_{i=1}^{n}$, where $a_n = \prod_{i=1}^{n-1} a_i$. It is clear from its definition that $\phi_\star$ is injective. Hence, by Proposition 2.3.1, the action of $\mathbb{Z}/2\mathbb{Z}$ on $(k^\times/(k^\times)^2)^{n-1}$ is trivial. So, since $\tau_\star$ respects the actions of $\mathbb{Z}/2\mathbb{Z}$ on $(k^\times/(k^\times)^2)^{n-1}$ and $\mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$, one has that

$$\alpha = \tau_\star(x_0) = \tau_\star(s(x_0)) = s(\tau_\star(x_0)) = s(\alpha) = \beta,$$

which contradicts the assumption that α and β are distinct. Thus, one concludes that $\xi_\star$ is an injective map, as desired. □

By applying the above two lemmas, one obtains the following result:

Theorem 4.0.4. *$H^1(k, \mathrm{SO}_n(k_{sep}))$ is naturally in bijection with the set of classes of quadratic k-forms with discriminant* 1*.*

Proof. By Lemma 4.0.3, since $\xi_\star$ is injective, one may identify $\mathrm{H}^1(k, \mathrm{SO}_n(k_{sep}))$ with the kernel of map $\det_\star$, which is the discriminant map by Lemma 4.0.2. Thus, the statement of the theorem follows. □

Chapter 5

Central simple algebras

5.1 Central simple algebras

Consider a subgroup $C \subseteq \mathbb{G}_m^l$. Via the embedding of each element of $\mathbb{G}_m^l$ as an l-tuple of scalar matrices, one obtains an embedding of C in $\prod_{i=1}^{l} \mathrm{GL}_{n_i}$. Let $G = (\prod_{i=1}^{l} \mathrm{GL}_{n_i})/C$.

Lemma 5.1.1. *The induced map* $\pi_\star : \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i})$ *is injective.*

Proof. Consider the short exact sequence:

$$1 \to \mathbb{G}_m^l/C \to G \to \prod_{i=1}^{l} \mathrm{PGL}_{n_i} \to 1,$$

which induces the following exact sequence in cohomology:

$$\ldots \to \mathrm{H}^1(k, \mathbb{G}_m^l/C) \to \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i})$$

Since $\mathbb{G}_m^l/C \cong \mathbb{G}_m^l$, we have that $\mathrm{H}^1(k, \mathbb{G}_m^l/C) \cong \mathrm{H}^1(k, \mathbb{G}_m^l) = \{1\}$. So, the action of $\mathrm{H}^1(k, \mathbb{G}_m^l/C)$ is trivial on $\mathrm{H}^1(k, G)$. Hence, it follows from Proposition 2.4.1 that $\pi_\star : \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i})$ is injective. □

Let $(\chi_1, \ldots, \chi_l)$ be a set of generators for the group of characters of $\mathbb{G}_m^l$ which vanish on C. Consider the map $\pi^C : \mathbb{G}_m^l \to \mathbb{G}_m^l/C$ where $\pi^C(g) = (\chi_1(g), \ldots, \chi_l(g))$.

Lemma 5.1.2. $(\gamma_1, \ldots, \gamma_l)$ *is in the kernel of* $\pi_\star^C : \mathrm{H}^2(k, \mathbb{G}_m^l) \to \mathrm{H}^2(k, \mathbb{G}_m^l/C)$ *if and only if for every character* $\chi = (a_1, \ldots, a_l) \in \mathbb{Z}^l$ *of* $\mathbb{G}_m^l$ *that vanishes on* C, $\gamma_1^{\otimes a_1} \otimes \ldots \otimes \gamma_l^{\otimes a_l} = 0$ *in* $\mathrm{H}^2(k, \mathbb{G}_m)$.

Proof. Let $\gamma = (\gamma_1, \ldots, \gamma_l) \in \ker(\pi_\star^C)$. We have the following short exact sequence of abelian groups:

$$1 \to C \to \mathbb{G}_m^l \to \mathbb{G}_m^l/C \to 1,$$

which induces the following exact sequence in cohomology:

$$\ldots \to \mathrm{H}^2(k, C) \to \mathrm{H}^2(k, \mathbb{G}_m^l) \to \mathrm{H}^2(k, \mathbb{G}_m^l/C) \to \ldots$$

So, $\ker(\pi_\star^C) = \mathrm{Im}(\iota_\star)$. Then, since ι is the inclusion map, as a cocycle, $\gamma \in \mathrm{Z}^2(k, C)$. So, $\chi_i \circ \gamma = 0$ for all i. Let χ be a character of $\mathbb{G}_m^l$ that vanishes on C, then $\chi = \sum_{i=1}^{l} d_i(\chi_i)$. Clearly, $\chi \circ \gamma = 0$. Hence, $\chi_\star : \mathrm{H}^2(k, \mathbb{G}_m^l) \to \mathrm{H}^2(k, \mathbb{G}_m)$ maps γ to the trivial cocycle in $\mathrm{H}^2(k, \mathbb{G}_m)$. On the other hand, $\chi(\gamma) = \bigotimes_{i=1}^{l} \gamma_i^{\otimes a_i}$. Therefore, $\gamma_1^{\otimes a_1} \otimes \ldots \otimes \gamma_l^{\otimes a_l} = 0$ in $\mathrm{H}^2(k, \mathbb{G}_m)$ for every $\chi = (a_1, \ldots, a_l)$ that vanishes on C.

Conversely, let $\gamma \in \mathrm{H}^2(k, \mathbb{G}_m^l)$ such that $\gamma_1^{\otimes a_1} \otimes \ldots \otimes \gamma_l^{\otimes a_l} = 0$ in $\mathrm{H}^2(k, \mathbb{G}_m)$ for every $\chi = (a_1, \ldots, a_l)$ that vanishes on C. Then, $\mathrm{Im}(\gamma) \subseteq C$, since otherwise, there exists a character χ such that $\chi \circ \gamma \neq 0$. Hence, $\gamma \in \mathrm{Im}(\iota_\star) = \ker(\pi_\star^C)$, as desired.

□

Using the above lemmas, we arrive at the following theorem:

Theorem 5.1.3. $\mathrm{H}^1(k, G)$ *are in one-to-one correspondence with* k*-isomorphism classes of* l*-tuples* $(A_1, \ldots, A_l)$ *of central simple algebras over* k*, such that each* A_i *is of degree* n_i*, and for every character* $\chi = (a_1, \ldots, a_l) \in \mathbb{Z}^l$ *of* $\mathbb{G}_m^l$ *that vanishes on* C, $A_1^{\otimes a_1} \otimes \ldots \otimes A_l^{\otimes a_l} = 0$ *in* $\mathrm{H}^2(k, \mathbb{G}_m)$.

Proof. Consider the following commutative diagram where the two rows are exact:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m^l & \xrightarrow{\iota} & \prod_{i=1}^{l} \mathrm{GL}_{n_i} & \xrightarrow{\pi} & \prod_{i=1}^{l} \mathrm{PGL}_{n_i} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi^C} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \cong} & & \\
1 & \longrightarrow & \mathbb{G}_m^l/C & \xrightarrow{\iota} & G & \xrightarrow{\pi} & \prod_{i=1}^{l} \mathrm{PGL}_{n_i} & \longrightarrow & 1
\end{array}$$

which induces the following commutative diagram in Galois cohomology:

$$\begin{array}{ccccc}
1 = \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{GL}_{n_i}) & \longrightarrow & \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i}) & \xrightarrow{\delta} & \mathrm{H}^2(k, \mathbb{G}_m^l) \\
\downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \pi_\star^C} \\
\mathrm{H}^1(k, G) & \xrightarrow{\pi_\star} & \mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i}) & \xrightarrow{\delta_C} & \mathrm{H}^2(k, \mathbb{G}_m^l/C)
\end{array}$$

By Lemma 5.1.1, the map $\pi_\star$ is injective. Thus, we can identify each element in $\mathrm{H}^1(k, G)$ with its image in $\mathrm{H}^1(k, \prod_{i=1}^{l} \mathrm{PGL}_{n_i}) \cong \prod_{i=1}^{l} \mathrm{H}^1(k, \mathrm{PGL}_{n_i})$, which is a k-isomorphism class of l-tuples $(A_1, \ldots, A_l)$ where each A_i is a central simple algebra of degree n_i. Since the bottom row is exact and the diagram is commutative, $\mathrm{Im}(\pi_\star) = \ker(\delta_C) = \ker(\pi_\star^C \circ \delta)$. Moreover, it follows from the top row being exact that $\ker \delta = \{1\}$, and hence, $\delta(\mathrm{Im}(\pi_\star)) = \ker(\pi_\star^C)$. Observe that δ maps an l-tuple of central simple algebras $(A_1, \ldots, A_l)$ to the l-tuple $([A_1], \ldots, [A_l])$ of its class in the Brauer group. By Lemma 5.1.2, $([A_1], \ldots, [A_l]) \in \ker(\pi_\star^C)$ if and only if for every character $\chi = (a_1, \ldots, a_l) \in \mathbb{Z}^l$ of $\mathbb{G}_m^l$ that vanishes on C, $[A_1^{\otimes a_1} \otimes \ldots \otimes A_l^{\otimes a_l}] = 0$. Thus, the statement of the proposition follows. □

5.2 *G*-crossed product algebras

A *G-crossed product algebra* A over k is a central simple k-algebra that contains a maximal G-Galois sub-algebra L.

Let G be a finite group of order n. Denote by $T \cong \mathbb{G}^m/\Delta$ the split maximal torus consisting of diagonal matrices up to scalar multiples in PGL_n. Since S_n acting on T by permutation, the embedding of G into S_n induces an embedding of $T \rtimes G$ into $T \rtimes S_n$, which is a subgroup of PGL_n. Thus, we have an embedding ι of $T \rtimes G$ into PGL_n.

Proposition 5.2.1. $\mathrm{H}^1(k, T \rtimes G)$ *classifies isomorphism classes of G-crossed product algebras over k.*

Proof. Consider pairs of the form (A, L), where A is a central simple algebra of degree n over k, L is a faithful left G-module (where G acts by permutation), and L is also an n-dimensional commutative semisimple subalgebra of A. A morphism f of two such pairs (A_1, L_1) and (A_2, L_2) is a k-algebra homomorphism $f : A_1 \to A_2$, such that $f(L_1) \subseteq L_2$ and $f\mid_{L_1}$ is G-equivariant. Let the split object of these pairs be (A_0, L_0) where $A_0 = \mathrm{M}_n(k)$ and $L_0 = \bigoplus_{g \in G} kg$, embedded into A_0 as the diagonal matrices. We first show that $\mathrm{Aut}((A_0 \otimes k_{sep}, L_0 \otimes k_{sep})) \cong T \rtimes G$. Clearly, $T \rtimes G \subseteq \mathrm{Aut}((A_0 \otimes k_{sep}, L_0 \otimes k_{sep}))$ since every elements of $T \rtimes G$ is invertible, and G acts on T by permutation. Conversely, let $f \in \mathrm{Aut}((A_0 \otimes k_{sep}, L_0 \otimes k_{sep}))$. Then, $f\mid_{L_0 \otimes k_{sep}}$ is a G-equivariant k-algebra automorphism. Observe that if $f(1e \otimes 1) = \sum_{g \in G} 1g \otimes a_g$, then f being G-equivariant implies that for any $g' \in G$,

$$f(1g' \otimes 1) = g' \cdot f(1e \otimes 1) = g' \cdot \left(\sum_{g \in G} 1g \otimes a_g \right) = \sum_{g \in G} 1(g'g) \otimes a_g.$$

So, f is determined uniquely by $f(1e \otimes 1)$. Thus, since $L_0 \otimes k_{sep}$ is embedded in $A_) \otimes k_{sep} \cong \mathrm{M}_n(k_{sep})$ as the diagonal matrices, f is represented by an element of $T \rtimes G$. Since elements of $T \rtimes G$ are invertible, f can be extended uniquely to an automorphism of $A_0 \otimes k_{sep}$ by the Skolem-Noether theorem. Thus, $\mathrm{Aut}((A_0 \otimes k_{sep}, L_0 \otimes k_{sep}) \cong T \rtimes G$.

By Galois descent, $\mathrm{H}^1(k, T \rtimes G)$ classifies isomorphism classes of twisted forms of (A_0, L_0) over k. Observe that A_0 is also the split object for the central simple algebras over k, and as a subalgebra of A_0, elements of L_0 are invertible. So, $L_0 \cong k^n$, which is the split G-Galois algebra. Hence, twisted forms of (A_0, L_0) are precisely the G-crossed product algebras over k. □

Corollary 5.2.2. *Let A be a G-cross product algebra over k, and $\alpha \in \mathrm{Z}^1(k, \mathrm{PGL}_n)$ that corresponds to A. Then, $\alpha \in \iota_\star(\mathrm{H}^1(k, T \rtimes G))$.*

Proof. The proposition follows from the previous lemma and the observation that since $\iota^\star$ is induced by the inclusion map, it maps the class of a pair (A, L) to the class of A in $\mathrm{H}^1(k, \mathrm{PGL}_n(k_{sep}))$. □

Chapter 6

Trace forms

6.1 Trace

Recall that a k-algebra A is *strictly power-associative* if for all $x \in A$ and $a, b \in \mathbb{Z}$, $x^a x^b = x^{a+b}$.

Let A be a strictly power-associative unital k-algebra of dimension n, and $(e_i)_{i=1}^n$ a basis of A over k. Consider a generic element $x = \sum x_i e_i \in A \otimes k(x_1, \ldots, x_n)$. There exists a monic polynomial

$$P_{A,x}(t) = t^m - s_1(x)t^{m-1} + \ldots + (-1)^m s_m(x),$$

of least degree such that $P_{A,x}(x) = 0$, and each s_i is a homogeneous polynomial in terms of $x_1, \ldots, x_n$ with coefficients in k. Then, $s_1(x) = T_A$ is the generic trace of the algebra A over k. Define the following bilinear trace map:

$$\begin{aligned} \mathrm{Tr}_{A/k} : A \times A &\to k \\ (x, y) &\mapsto T_A(xy). \end{aligned}$$

If A is an alternative algebra or a Jordan algebra, then $\mathrm{Tr}_{A/k}$ is an associative, symmetric bilinear form (cf. Corollary 4 p.227 [2]). In particular, $\mathrm{Tr}_{A/k}$ is non-singular if and only if A is separable (cf. Lemma 32.4 [3]). Moreover, if $\sigma \in \mathrm{Aut}(A)$, then $\sigma(T_A(x)) = T_A(\sigma(x))$, that is, the trace form $\mathrm{Tr}_{A/k}$ is invariant under the k-automorphism group of A (cf. Theorem 1 p.224 [2]). Hence, there exists an embedding $\iota : \mathrm{Aut}_k(A) \to \mathrm{O}_n(k_{sep}, \mathrm{Tr}_{A/k})$.

Theorem 6.1.1. *Let A be an alternative or Jordan n-dimensional k-algebra. Then, the induced map $\iota_\star : \mathrm{H}^1(k, \mathrm{Aut}_k(A)) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}, \mathrm{Tr}_{A/k}))$ maps* *a twisted form A' of A to its trace form $\mathrm{Tr}_{A'/k}$.*

Proof. Let A' be a twisted form of A in $\mathrm{H}^1(k, \mathrm{Aut}_k(A))$, and K a finite Galois extension of k such that $A' \in \mathrm{H}^1(K/k, \mathrm{Aut}_k(A))$. Let $V = k^n$, and $x \in V^\star \otimes V^\star \otimes V$ the structure constant of A. Then, (V, x) corresponds to the trivial cocycle in $\mathrm{H}^1(K/k, \mathrm{Aut}_k(A))$. Suppose that (V, x') is the associated element to A' in $E(K/k, (V, x))$. So, there exists a K-isomorphism $f : V_K \to V_K$ such that $f(x_K) = x'_K$, and the cocycle responding to A' is $\gamma : \Gamma_K \to \mathrm{Aut}_k(A)$ such that $\gamma(s) = f^{-1}s(f)$ for $s \in \Gamma_K$. Consequently, $\iota_\star(\gamma) = \iota \circ \gamma : \Gamma_K \to \mathrm{O}_n(k_{sep}, \mathrm{Tr}_{A/k})$. With V defined as above, and $y \in V^\star \otimes V^\star$ representing the trace form $\mathrm{Tr}_{A/k}$, one obtains that the quadratic space corresponding to $\iota_\star(\gamma)$ is $(V, f(y_K))$. Denote by y' the tensor corresponding to $\mathrm{Tr}_{A'/k}$. It remains to be shown that $(V_K, y_K) \cong_K (V_K, y'_K)$ via f and $(V, f(y)) \cong_k (V, y')$ via the identity map. Clearly, f is a K-isomorphism of V_K, and for any $a, b \in (V_K, \mathrm{Tr}_{A',k} \otimes K)$, one has that

$$\begin{aligned} y'_K(a,b) &= T_{A'}(x'_K(a,b))) = T_{A'}(f(x_K)(a,b)) \\ &= T_A(x_K(f^{-1}(a), f^{-1}(b))) = y_K(f^{-1}(a), f^{-1}(b)). \end{aligned}$$

Hence, $(V_K, y_K) \cong_K (V_K, y'_K)$ via f. Since $f(x_K) = x'_K$, f is a k-algebra homomorphism. So, $f(y_K) = y'_K$ implies that $f(y) = y'$. Thus, $(V, f(y)) \cong_k (V, y')$ via the identity map. Thus, the associated quadratic form to $\iota_\star(\gamma)$ is $\mathrm{Tr}_{A',k}$, as desired. □

Remark 6.1.2. We can apply the above theorem to the following two important classes of algebras: étale algebras and G-Galois algebras. Recall that étale algebras of dimension n and G-Galois algebras over k are classified by $\mathrm{H}^1(k, S_n(k_{sep}))$ and $\mathrm{H}^1(k, G)$ for a finite group G, respectively. So, let $\iota : S_n(k_{sep}) \to \mathrm{O}_n(k_{sep})$ be the embedding of permutation matrices into the orthogonal group, and $\rho : G \to S_n$ the Cayley representation map. Then, $\iota_\star : \mathrm{H}^1(k, S_n(k_{sep})) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ is the map which sends an étale k-algebra L to its trace form $\mathrm{Tr}_{L/k}$. Similarly, the induced map $(\iota \circ \rho)_\star : \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \mathrm{O}_n(k_{sep}))$ is the map sending a G-Galois algebra L of dimension n to its trace form $\mathrm{Tr}_{L/k}$.

Chapter 7

Conclusion

In this thesis, we show that many objects of interest in algebra can be naturally described by Galois cohomology sets or characterized as images of certain induced maps. This is an attempt to establish a connection between Galois cohomology and abstract algebra, allowing us to view common algebraic objects from a different perspective. In many cases, this alternate view leads to new approaches to existing problems in the area.

A limitation of this work is that the treatment for each case is specific to the objects being considered. There is no reason to expect that an induced map has a natural interpretation because the same cohomological set may have various reasonable algebraic meanings. For example, $\mathrm{H}^1(k, S_n)$ can be thought of as isomorphism classes of étale algebras or of S_n-Galois algebras. Thus, as in this thesis, working with individual cases is necessary in order to understand the connection between these two areas of mathematics.

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