Extending Erdős-Kac and Selberg–Sathe to Beurling Primes with Controlled Integer Counting Functions

by

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B.S., Western Washington University, 2007

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

April 2013

Abstract

In this thesis we extend two important theorems in analytic prime number theory to a the setting of Beurling primes, namely The Erdős–Kac theorem and a theorem of Saith and Selberg. The Erdős–Kac theorem asserts that the number of prime factors that divide an integer n is, in some sense, normally distributed with mean $\log\log n$ and variance $\sqrt{\log\log n}$. Saith proved and Selberg substantially refined a formula for the counting function of products of k primes with some uniformity on k. A set of Beurling primes is any infinite multiset $\{p_i \mid i \in \mathbb{N}\} \subset \mathbb{R}_{>1}$ such that $p_i \leq p_{i+1}$ for all i and $\lim_{i\to\infty} p_i = \infty$. The set of Beurling primes has a corresponding multiset of Beurling integers formed by all finite products of Beurling primes. We assume that the Beurling integer counting function is approximately linear with varying conditions on the error term in order to prove the stated results. An interesting example of a set of Beurling primes is the set of norms of prime ideals of the ring of integers of a number field. Recently, Granville and Soundararajan have developed a particularly simple proof of the Erdős–Kac theorem which we follow in this thesis. For extending the theorem of Selberg and Sathe much more analytic machinery is needed.

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Acknowledgements

I would like to adamantly thank my advisor Dr. Greg Martin for all of his help and direction in completing this thesis and in making me a better mathematician. I would also like to thank Dr. Lior Silberman for his helpful suggestions and corrections. Finally I want to thank Robert Fraser and Michael Forbes for their assistance in formating.

Dedication

For my wife Haley and my parents Dave and Addie for all of their support throughout my education.

Chapter 1

Introduction and Summary

1.1 What is a Beurling Prime?

When proving results in analytic number theory one does not always use the entire algebraic structure of the integers. Instead the analytic and asymptotic information is much more useful in proving classical results such as the prime number theorem or its generalizations. It turns out that many properties of the integers can be established in the more general context of a countably generated multiplicative semigroup of the positive reals satisfying appropriate analytic assumptions. Beurling noticed that one can prove many classical results in this generalized context.

Fix a multiset $\mathcal{P}$ of real numbers $\{p_i : i \in \mathbb{N}\}$ such that $1 < p_1 \leq p_2 \leq \cdots$ and such that $\lim_{i\to\infty} p_i = \infty$. The elements of $\mathcal{P}$ will be called "Beurling primes". The associated set of "Beurling integers" is the multiset $\mathcal{B}$ of all finite products of Beurling primes. Note that $1 \in \mathcal{B}$, as it is the empty product. Let the set of natural primes be denoted by $\mathbb{P} := \{2, 3, 5, 7, 11, ...\}$. The fact that $\mathcal{B}$ can be a multiset introduces an interesting dynamic and can cause some confusion. For instance one can choose the set of Beurling primes to be all the natural primes with and additional 5 included which is, in some way, distinct from the other 5 already in $\mathbb{P}$. In this case there will actually be $n + 1$ ways to write 5^n as an element of $\mathcal{B}$.

Beurling showed that if one assumes that the integer counting function is roughly like that of the natural numbers then the prime number theorem holds, where naturally the error term depends on the choice of $\mathcal{B}$ as it will for all results in this thesis. More precisely,

Theorem 1.1 (Beurling, 1937)**.** *Given a a set of Beurling primes, $\mathcal{P}$, and the corresponding set of Beurling integers, $\mathcal{B}$, if the integer counting function, $N_{\mathcal{B}}(x) := \sum_{\substack{n\leq x\\ n\in\mathcal{B}}} 1$, has the asymptotic formula*

$$N_{\mathcal{B}}(x) = Ax + O(x(\log x)^{\lambda})$$

for some $A > 0$ and $\lambda < -3/2$ then one gets the following asymptotic formula for $\pi_{\mathcal{B}}(x) := \sum_{\substack{n\leq x\\ n\in\mathcal{P}}} 1$, the Beurling prime counting function:

$$\pi_{\mathcal{B}}(x) \sim \frac{x}{\log x}. \tag{1.1.1}$$

Furthermore this bound is optimal: exists a Beurling primes system with $\lambda = -3/2$ and where 1.1.1 does *not hold.*

There are several, possibly, surprising things about this result. First it is interesting that having twice the number of integers does not seem to effect the number of primes at all; therefore the additional integers introduced must overwhelmingly come from composite numbers. We will make this heuristic more exact in Chapter 4 when we discus the asymptotic formula for the counting function of Beurling integers with exactly k prime factors. Also see that adding a single very small element to $\mathcal{P}$ will greatly increase the number of integers. The next thing that one may find surprising is the way in which the results can be proved, since the methods are very similar, for the most part, to those in the natural prime case.

In addition to this theorem one may also ask for what set of Beurling primes do we get the prime number theorem with a "good" error term. If a set of Beurling primes has a power savings in the error term then using essentially the same methods as in the natural prime case one can prove a prime number theorem with an error term of $O(xe^{-c\theta\sqrt{\log x}})$. In particular define the Beurling zeta function to be $\zeta_{\mathcal{B}}(s) = \sum_{n\in\mathcal{B}} n^{-s}$, then one can show that the Beurling zeta function extends analytically past $\sigma = 1$, one can then create a similarly shaped zero-free region and perform the same integration techniques as when $\mathcal{P} = \mathbb{P}$ to get such a prime number theorem.

1.2 Examples of Sets of Beurling Primes

A simple example of a set of Beurling primes is the natural primes with some small set of elements either added or removed. Note that we could add primes into our system with multiplicity. In the case where we remove a finite set of primes, $\mathcal{T} \subset \mathbb{P}$, the prime number theorem follows trivially from the classical prime number theorem and one can explicitly calculate the integer counting function

$$N_{\mathcal{B}}(x) = \sum_{\substack{n\in\mathbb{N} \\ p\nmid n\ \forall\ p\in\mathcal{T}}} 1 = Ax + O(1) \tag{1.2.1}$$

where $A = \prod_{p\in T}\left(1 - \frac{1}{p}\right)$. To see this note that Ax is an exact formula when x is a multiple of $N = \prod_{p\in\mathcal{T}} p$ and can only be off by a finite amount in $[0, N]$ and the error is periodic with period N. Note that in (1.2.1) the error term depends on $\mathcal{B}$, as will all error terms in this thesis.

The next example provides a way to prove the prime ideal theorem [10]. Let K be a number field such that $n = [K, \mathbb{Q}]$ and let $\mathcal{O}_K$ be its ring of integers. Let $\mathcal{P}$ be the set of norms of prime ideals of $\mathcal{O}_K$. Then if one can show that the total number of ideals with norm less then x is $Ax + O(x^c)$, where $A > 0$ is called the ideal density and $c = 1 - 1/n$. By the unique factorization of prime ideals and the multiplicative property of the norm we then apply the Beurling prime number theorem 1.1 to get the prime ideal theorem.

1.3 Summary of Results

In chapter 2 we extend a major result in probabilistic number theory, the Erdős–Kac theorem, to the case of Beurling primes whenever Mertens's theorem (Corollary 2.1), Chebychev's bound, and a condition on the analog to the Euler ϕ function hold for our set of Beurling primes. Chebychev's bound refers to the property that $\pi(x) \asymp x/\log x$, which Hall [7] and Diamond [2] showed holds precisely when $N_{\mathcal{B}}(x) = Ax + O(x(\log x)^{\gamma})$ where $\gamma < -1$. We also recount from the work of others when exactly a set of Beurling primes has these qualities. Define $\omega(n)$ to be the number of prime factors of $n \in \mathcal{B}$. The Erdős–Kac theorem states that the quantity $\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}$ is, in some sense, normally distributed when n is a considered random variable, uniformly distributed among the integers. Formally we calculate all the moments of the distribution and find that they are, in the limit, the moments of the normal distribution. Probability theory tells us that the normal distribution is determined by its moments which is how we justify the above statement of the theorem.

Theorem 1.2. *Let $\mathcal{P}$ be a set of Beurling primes with corresponding set of integers $\mathcal{B}$ such that the integer counting function has the asymptotic $N_{\mathcal{B}}(x) = Ax + O(x^{\theta})$ where $A > 0$ and $\theta < 1$. Set the notation $C_k := \Gamma(k+1)/2^{k/2}\Gamma(k/2+1)$. Assume that $k \leq \log\log\log\log x$. When k is even*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}}} (\omega_{\mathcal{B}}(n) - \log\log x)^k = A \cdot C_k x(\log\log z)^{k/2}\left(1 + O_A(k^3/(\log\log z))\right) + O(8^k \pi(z)^k).$$

When k is odd

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}}} (\omega_{\mathcal{B}}(n) - \log\log x)^k \ll C_k x(\log\log z)^{k/2}\frac{k^3}{\log\log z} + 8^k\pi(z)^k$$

In particular, it follows simply that $(\omega_{\mathcal{B}}(n) - \log\log n)/\sqrt{\log\log n}$ is normally distributed among natural numbers in the following sense. The asymptotic density

$$\lim_{x\to\infty}\left(\frac{1}{N_{\mathcal{B}}(x)} \cdot \#\left\{n \leq x \;:\; a \leq \frac{\omega_{\mathcal{B}}(n) - \log\log n}{\sqrt{\log\log n}} \leq b\right\}\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}dt.$$

There are many corollaries to the Erdős–Kac theorem including the Hardy–Ramanujan theorem [8] and the Erdős multiplication table theorem [4] among others. These theorems come for free when we prove the Erdős–Kac theorem but are interesting and definitely worth mentioning for their simple interpretation and significance.

In Chapter 3 we provide an interesting examples of a set of Beurling primes which produces extremely large gaps in the corresponding integers. We do this by removing select elements of $\mathbb{P}$ which have a lot of leverage in removing integers approximately exponentially larger.

Next, in chapter 4, we extend a theorem proved by Sathe and greatly simplified by Selberg which gives a formula for counting integers with k prime factors with some uniformity in k. We first derive a formula for a fixed k using the prime number theorem and illustrate, drawing inspiration from Beurling primes, why this formula cannot be uniform, despite some conflicting evidence. Next we use Perron's formula on the partial summation of $d_z(n)$, the generalized devisor function, and find bounds on $\zeta(s)$ to calculate the integral with integrand $\zeta(s)^z$ which Perron's formula gives us. The rest of the chapter relates d_z back to the function which we care about, σ_k, using some arithmetic lemmas. In the end we prove the following theorem and give some corollaries and examples.

Theorem 1.3. *Let $\mathcal{P}$ be a set of Beurling primes with corresponding set of integers $\mathcal{B}$ such that the integer counting function has the asymptotic $N_{\mathcal{B}}(x) = Ax + O(x^\theta)$ where $A > 0$ and $\theta < 1$. Suppose that $R < p_1$ and that*

$$F(s, z) := \prod_{p \in \mathcal{P}} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z.$$

Set $G(z) = A^z \cdot F(1, z)/\Gamma(z + 1)$. Then

$$\sigma_k(x) \;=\; G\left(\frac{k-1}{\log\log x}\right) \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \left(1 \;+\; O_R\left(\frac{k}{(\log\log x)^2}\right)\right)$$

uniformly for $1 \le k \le R \log\log x$.

Chapter 2

Extending the Erdős–Kac Theorem

The goal of this chapter is to prove Theorem 1.2. We start by proving a statement about the average number of prime factors of Beurling integers in Corollary 2.2. To do this we first need a generalization of Mertens's theorem on an asymptotic formula for the sum of reciprocals of prime numbers up to x, given in Corollary 2.1, which follows easily from previously known results.

To prove Theorem 1.2 we build upon the simple proof that was recently discovered by Soundararajan and Granville in [6]. To do this we need to first replicate some arithmetic properties of the integers to the case of Beurling primes with controlled integer counting functions. We do this in Lemma 2.3 and in Theorem 2.4. Finally we prove Theorem 1.2, which follows relatively simply from 2.4.

2.1 Preliminaries

In a recent paper [14] by Paul Pollack he expands on a result of Olofsson, in [13], which states that if

$$N_{\mathcal{B}}(x) \sim Ax$$

for some $A > 0$ then a generalization of Mertens's theorem holds, namely

$$\prod_{\substack{p \le x \\ p \in \mathcal{P}}} \left(1 - \frac{1}{p}\right)^{-1} \sim Ae^{\gamma} \log x.$$

From this result one can derive a generalization of another theorem of Mertens.

Corollary 2.1. *Let $\mathcal{P}$ be a set of Beurling primes and let $\mathcal{B}$ be the corresponding set of Beurling integers. If $N_{\mathcal{B}}(x) \sim Ax$ for some positive A, then*

$$\sum_{\substack{p \le x \\ p \in \mathcal{P}}} \frac{1}{p} = \log\log x + \log A + \gamma - \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{kp^k} + o(1).$$

Proof. Assume that $N_{\mathcal{B}}(x) \sim Ax$ for some $A > 0$. Then by theorem A in [14] we have that

$$\prod_{\substack{p\le x\\ p\in\mathcal{P}}}\left(1-\frac{1}{p}\right)^{-1}=Ae^{\gamma}(\log x)(1+o(1)).$$

Taking logs on both sides and adding zero on the left hand side yields

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\frac{1}{p}+\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\left(\log\left[\left(1-\frac{1}{p}\right)^{-1}\right]-\frac{1}{p}\right)=\log A+\gamma+\log\log x+o(1).$$

Since the Taylor expansion of $\log\left[\left(1-\frac{1}{x}\right)^{-1}\right]-1/x$ is $\sum_{k=2}^{\infty}\frac{1}{kp^k}$ one can see that

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\left(\log\left[\left(1-\frac{1}{p}\right)^{-1}\right]-\frac{1}{p}\right)=\sum_{p\in\mathcal{P}}\sum_{k=2}^{\infty}\frac{1}{kp^k}+O\left(\sum_{\substack{p>x\\ p\in\mathcal{P}}}p^{-2}\right).$$

The series in the error term is $O(x^{-1})$ since $\pi_{\mathcal{B}}(x)\ll N_{\mathcal{B}}(x)\ll_A x$ and the above double sum converges since, by the integral comparison test, we see that

$$\sum_{p\in\mathcal{P}}\sum_{k=2}^{\infty}\frac{1}{kp^k}=\sum_{k=2}^{\infty}\frac{1}{k}\sum_{p\in\mathcal{P}}\frac{1}{p^k}\ll\sum_{k=2}^{\infty}\frac{1}{k}\int_{x=1}^{\infty}x^{-k}=\sum_{k=2}^{\infty}\frac{1}{k(k+1)}\ll 1.$$

Thus, we conclude that

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\frac{1}{p}=\log\log x+\log A+\gamma-\sum_{p\in\mathcal{P}}\sum_{k=2}^{\infty}\frac{1}{kp^k}+o(1).$$

$\square$

From Corollary 2.1 we easily get an average value for $\omega_{\mathcal{B}}$, the function which counts the number of prime factors of $n\in\mathcal{B}$. This is well defined since elements of $\mathcal{B}$ are defined by a product of elements of $\mathcal{P}$. Consider the examples $\mathcal{P}=\mathbb{P}\cup\{6\}$. Then $6\in\mathcal{B}$ has index 2 in the multi set $\mathcal{B}$, that is there are two ways to write 6 is a product of elements of $\mathcal{P}$. In this case, if we write all elements as their defining product over $\mathcal{P}$, then we get $\omega_{\mathcal{B}}(2\cdot 3)=2$ and $\omega_{\mathcal{B}}(6)=1$.

Corollary 2.2. *If $N_{\mathcal{B}}(x)=Ax+O(x(\log x)^{\gamma})$, where $\gamma<-1$ then*

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\omega_{\mathcal{B}}(n)=Ax\log\log x+x\left(\log A+\gamma-\sum_{p\in\mathcal{P}}\sum_{k=2}^{\infty}\frac{1}{kp^k}\right)+o(x).$$

Proof. By switching the order of summation we can write

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\omega_{\mathcal{B}}(n)=\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\sum_{\substack{p|n\\ p\in\mathcal{P}}}1=\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\sum_{\substack{n\le x\\ p|n\\ n\in\mathcal{B}}}1=\sum_{\substack{p\le x\\ p\in\mathcal{P}}}N_{\mathcal{B}}\left(\frac{x}{p}\right).$$

By our assumption on the growth of $N_{\mathcal{B}}(x)$ we see that the above yields our desired main term with some additional error terms. If one uses the trivial error term $N_{\mathcal{B}}(x)=Ax+O(1)$ for when $x\le 2$ we determine that the above is

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\omega_{\mathcal{B}}(n)=Ax\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\frac{1}{p}+O\left(\sum_{\substack{p\le x/2\\ p\in\mathcal{P}}}\frac{x/p}{\log^{\gamma}(x/p)}\right)+O\left(\sum_{\substack{x/2\le p\le x\\ p\in\mathcal{P}}}1\right).$$

By Chebychev's bound for Beurling primes we see that the second error term is $O(x/\log x)$. If $x>p_1^2$, then for the first error term we make a further split in the summation at $\sqrt{x}$ to see that

$$\sum_{\substack{p\le\sqrt{x}\\ d\in\mathcal{P}}}\frac{x/p}{\log^{a}(x/p)}\ll\frac{x}{\log x}\sum_{\substack{p\le\sqrt{x}\\ d\in\mathcal{P}}}\frac{1}{p}\ll\frac{x\log\log x}{\log x}$$

and

$$\sum_{\substack{\sqrt{x}<p\le x/2\\ d\in\mathcal{P}}}\frac{x/p}{\log^{a}(x/p)}\ll\sqrt{x}\sum_{2\le t\le\sqrt{x}}\frac{1}{\log t}\ll\sqrt{x}\,\mathrm{li}(\sqrt{x})\ll\frac{x}{\log x}.$$

Corollary 2.1 yields the desired result. □

From these results we can see that the average value for $\omega_{\mathcal{B}}(n)$ for $n\le x$ is $\log\log x$. We can say much more than this. In fact we can prove a generalization a theorem by Erdős and Kac [5], that $\omega_{\mathcal{B}}(n)$ is normally distributed with mean $\log\log n$ and variance $\sqrt{\log\log n}$. Branching off from there, using only the first two moments of the distribution of $\omega_{\mathcal{B}}$, we also get a generalization of a result by Hardy and Ramanujan which say that almost all integers n have about $\log\log n$ prime factors. Furthermore Erdős' multiplication table theorem - that is, as n goes to infinity, almost no numbers up to n^2 are in the $n\times n$ multiplication table - is also provable in the setting of Beurling primes. Now we prove the Erdős – Kac theorem following the techniques introduced by Granville and Soundararajan in [6]. To this end let us start with some technical lemmas.

We define the GCD the using the multiplicative structure of $\mathcal{B}$, that is the GCD of n and m denoted, (n,m) is just the product of all elements of $\mathcal{P}$ which are in the prime factorization of both n and m. One has to be a bit careful here because there may be multiple ways to write any integer as a product of Beurling primes. For instance if a set of Beurling primes contains 2 and $\sqrt{2}$ then it would be the case that $(\sqrt{2}^2, 2) = 1$. We also define $\phi_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$, using the product formula, as follows

$$\phi_{\mathcal{B}}(n) = n \cdot \prod_{\substack{p|n \\ p\in\mathcal{P}}} \left(1 - \frac{1}{p}\right), \text{ and } \tau_{\mathcal{B}}(n) = \prod_{\substack{p^\alpha || n \\ p\in\mathcal{P}}} 1 + \alpha = \sum_{\substack{d|n \\ d\in\mathcal{B}}} 1$$

Lemma 2.3. *Let $\mathcal{P}$ be a set of Beurling primes such that $N_{\mathcal{B}}(x) = Ax + O(x^\theta)$ for some $A > 0$ and $\theta < 1$. For r and d in $\mathcal{B}$*

$$\sum_{\substack{n\leq x \\ (n,r)=d \\ n\in\mathcal{B}}} 1 = \frac{Ax}{d} \frac{\phi_{\mathcal{B}}(r/d)}{r/d} + O\left(\left(\frac{x}{r}\right)^\theta g_\theta(r/d)\right)$$

where $g_s(n) = \prod_{p|n}(1 + p^s)$.

Much of the theory of Arithmetic functions is similar in the context of Beurling primes and evaluating the sum in Lemma 2.3 for the natural primes is a special case of this lemma.

Proof. Let $d \in \mathcal{B}$ be given and let r be given such that $d|r$. Then

$$\sum_{\substack{n\leq x \\ n\in\mathcal{B} \\ (n,r)=d}} 1 = \sum_{\substack{n\leq x \\ n\in\mathcal{B} \\ d|n \\ (n/d,r/d)=1}} 1 = \sum_{\substack{k\leq x/d \\ k\in\mathcal{B} \\ (k,r/d)=1}} 1.$$

Now we use the fact that $(\mu \star 1)(n) = e(n)$, where e is the identity in the ring of arithmetic functions under Dirichlet convolution. We use this identity to detect when k and $\frac{r}{d}$ are coprime to get

$$\sum_{\substack{n\leq x \\ n\in\mathcal{B} \\ (n,r)=d}} 1 = \sum_{\substack{k\leq \frac{x}{d} \\ k\in\mathcal{B}}} \sum_{\substack{c|(k,\frac{r}{d}) \\ c\in\mathcal{B}}} \mu_{\mathcal{B}}(c).$$

Now we switch the order of summation which yields that the above is

$$\sum_{\substack{n\leq x \\ n\in\mathcal{B} \\ (n,r)=d}} 1 = \sum_{\substack{c|\frac{r}{d} \\ c\in\mathcal{B}}} \sum_{\substack{k\leq\frac{x}{d} \\ c|k \\ k\in\mathcal{B}}} \mu_{\mathcal{B}}(c) = \sum_{\substack{c|\frac{r}{d} \\ c\in\mathcal{B}}} \sum_{\substack{k\leq\frac{x}{cd} \\ k\in\mathcal{B}}} \mu_{\mathcal{B}}(c) = \sum_{\substack{c|\frac{r}{d} \\ c\in\mathcal{B}}} \left[\mu(c)\frac{Ax}{cd} + O\left(|\mu(c)| \cdot \left(\frac{x}{cd}\right)^\theta\right)\right].$$

Finally, note that $\phi(n) = \mu(n) \star n$ as seen by examining the product formula for the functions involved. Therefore

$$\frac{Ax}{d}\frac{1}{r/d}\sum_{\substack{c|\frac{r}{d}\\ c\in\mathcal{B}}}\mu_{\mathcal{B}}(c)\frac{r/d}{c}+O\Bigg(\left(\frac{x}{r}\right)^{\theta}\sum_{\substack{c|\frac{r}{d}\\ c\text{ square free}\\ c\in\mathcal{B}}}\left(\frac{r}{cd}\right)^{\theta}\Bigg)=\frac{Ax}{d}\frac{\phi_{\mathcal{B}}(r/d)}{r/d}+O\left(\left(\frac{x}{r}\right)^{\theta}g_{\theta}(r/d)\right).$$

□

2.2 The Main Technical Work

Toward the goal of proving Erdős–Kac theorem, in the style of Granville and Soundararajan we introduce a sort of "balanced prime counting function". For a given $p\in\mathcal{P}$ define the function, $f:\mathcal{B}\to\mathbb{Q}$, as

$$f_p(n)=\begin{cases}1-1/p & p|n\\ -1/p & \text{otherwise.}\end{cases}$$

For example, a sort of balanced prime divisor counting function for $n\in\mathcal{B}$ is

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}}}f_p(n)=\omega_{\mathcal{B}}(n)-\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{p}.$$

Furthermore one can generalize this function for any $r\in\mathcal{B}$. If $r=\prod_{i=1}^{s}p_i^{\alpha_i}$ then set $f_r(n)=\prod_{i=1}^{s}f_{p_i}(n)^{\alpha_i}$. There is not an obvious reason why this "balancing" would be helpful, but it is. In other proofs of the Erdős–Kac theorem the expression

$$\sum_{n\le x}(\omega(n)-\log\log x)^k$$

is expanded and, carefully, one can cancel out the large terms and bound the terms that remain. The balanced prime divisor counting function takes care of all of the cancellation automatically. This makes the proof shorter and easier.

Theorem 2.4. *Let x be large, set $z=x^{1/k}$ and let $k<\log\log\log\log(z)$. Define notation for the moments of the normal distribution to be $C_k:=\Gamma(k+1)/[2^{k/2}\cdot\Gamma(k/2+1)]$. If k is even then*

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\Bigg(\sum_{\substack{p\le z\\ p\in\mathcal{P}}}f_p(n)\Bigg)^k=A\cdot C_kx(\log\log x)^{k/2}\left(1+O_A((k^3/\log\log z))\right)+O(8^k\pi(z)^k).$$

If k is odd then

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left(\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right)^k \ll C_k x(\log\log x)^{k/2}(k^3/\log\log x)) + O(8^k\pi(z)^k).$$

Proof. By reordering the summation we get that

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left(\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right)^k = \sum_{\substack{p_1,\dots p_k\le z\\ p_i\in\mathcal{P}}}\sum_{\substack{n\le x\\ n\in\mathcal{B}}} f_{p_1\cdots p_k}(n). \tag{2.2.1}$$

If $R=\prod_{i=1}^s p_i^{\alpha_i}$ then set $r=\prod_{i=1}^s p_i$. Let $n\in\mathcal{B}$ be given. If $d=(n,r)$ then

$$f_R(n)=\prod_{p^\alpha\|R} f_p(n)^\alpha = \prod_{\substack{p^\alpha\|R\\ p\mid n}}\left(1-\frac{1}{p}\right)^\alpha \prod_{\substack{p^\alpha\|R\\ p\nmid n}}\left(\frac{-1}{p}\right)^\alpha = \prod_{\substack{p^\alpha\|R\\ p\mid d}}\left(1-\frac{1}{p}\right)^\alpha \prod_{\substack{p^\alpha\|R\\ p\nmid d}}\left(\frac{-1}{p}\right)^\alpha = f_R(d).$$

From this fact we can sort the sum, $\sum_{n\le x} f_R(n)$, by what the value of (r,n) takes, which will bring us back to one of our lemmas.

$$\sum_{n\le x} f_R(n) = \sum_{d\mid r} f_R(d)\sum_{\substack{n\le x\\ (n,r)=d\\ n\in\mathcal{B}}} 1$$

By Lemma 2.3, we know the formula for the inner sum. Therefore we get that the above is

$$\sum_{n\le x} f_R(n) = \sum_{d\mid r}\left(\frac{Ax}{d}\frac{\phi_{\mathcal{B}}(r/d)}{r/d} + O\left(\left(\frac{x}{r}\right)^\theta g_\theta(r/d)\right)\right) f_R(d).$$

Define $\mathrm{Id}_s(n) := n^s$. To estimate the error term over the sum of divisors of r see that when r is square free g_θ takes the simpler form $g_\theta(r/d)=\sigma_\theta(r/d)=1\star\mathrm{Id}_\theta(n)$. Furthermore for $d\mid r$ square free our $|f_R(d)|$ has the simple form $|f_R(d)|=\frac{1}{r/d}\frac{\phi(d)}{d}=\frac{\phi(d)}{r}$. Therefore

$$\begin{aligned}
\left(\frac{x}{r}\right)^\theta\sum_{d\mid r} g_\theta(r)|f_R(d)| &= \left(\frac{x}{r}\right)^\theta\sum_{d\mid r}\frac{\phi(d)}{r}\sum_{c\mid(r/d)}(r/cd)^\theta\\
&=\frac{x^\theta}{r}\sum_{d\mid r}\frac{\phi(d)}{d^\theta}\sum_{c\mid(r/d)}\frac{1}{c^\theta}\\
&\ll\frac{x^\theta}{r}\sum_{d\mid r} d^{1-\theta}\tau(r/d)
\end{aligned} \tag{2.2.2}$$

Since $d|r$ is square free and $\omega_{\mathcal{B}}(r) \leq k$ the quantity $\tau(d) \leq 2^k$ so $\sum_{d|r} \tau(r/d) d^{1-\theta} \ll 4^k r^{1-\theta}$. Therefore the quantity in the last line of (2.2.2) is $\ll \left(\frac{x}{r}\right)^\theta 4^k$.

Therefore we conclude that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{B}}} f_R(x) = \frac{Ax}{r} \sum_{d|r} f_R(d) \phi_{\mathcal{B}}(r/d) + O\left(\left(\frac{x}{r}\right)^\theta 4^k \right). \tag{2.2.3}$$

We define the function $G(R) := \prod_{p^\alpha || R} \left[\frac{1}{p}\left(1 - \frac{1}{p}\right)^\alpha + \left(-\frac{1}{p}\right)^\alpha \left(1 - \frac{1}{p}\right) \right]$ and claim that the following summation formula holds, $G(R) = \frac{1}{r} \sum_{d|r} f_R(d) \phi_{\mathcal{B}}(r/d)$. To see that this is true, observe that if one expands the product each term will simply be a product of some terms of the form $\frac{1}{p}\left(1 - \frac{1}{p}\right)^\alpha$ and the rest of the terms of the form $\left(-\frac{1}{p}\right)^\alpha \left(1 - \frac{1}{p}\right)$, which yields that

$$G(R) = \sum_{d|r} \prod_{p|d} \frac{1}{p}\left(1 - \frac{1}{p}\right)^\alpha \cdot \prod_{p \nmid d} \left(\frac{-1}{p}\right)^\alpha \left(1 - \frac{1}{p}\right).$$

To be pedantic and separate everything out neatly, we get that

$$G(R) = \sum_{d|r} \prod_{p|d} \frac{1}{p} \prod_{p|d} \left(1 - \frac{1}{p}\right)^\alpha \cdot \prod_{p|r/d} \left(\frac{-1}{p}\right)^\alpha \prod_{p \nmid d} \left(1 - \frac{1}{p}\right).$$

The first product yields $1/d$. The second and third product combine to give $f_R(d)$ and the final product is the product formula for $\frac{\phi_{\mathcal{B}}(r/d)}{r/d}$. We conclude that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{B}}} f_R(n) = Ax \cdot G(R) + O\left(\left(\frac{x}{r}\right)^\theta 4^k \right).$$

Applying the above to the right hand side of (2.2.1) gives us

$$\sum_{\substack{p_1, \ldots p_k \leq z \\ p_i \in \mathcal{P}}} \sum_{\substack{n \leq x \\ n \in \mathcal{B}}} f_{p_1 \cdots p_k}(n) = \sum_{\substack{p_1, \ldots p_k \leq z \\ p_i \in \mathcal{P}}} Ax \cdot G(p_1 \cdots p_k) + O\left(\left(\frac{x}{p_1 \cdots p_k}\right)^\theta 4^k \right)$$

Summing the error term over all k-tuples of primes less than z yields

$$\sum_{\substack{p_1, \ldots p_k \leq z \\ p_i \in \mathcal{P}}} \sum_{\substack{n \leq x \\ n \in \mathcal{B}}} f_{p_1 \cdots p_k}(n) = Ax \cdot \sum_{\substack{p_1, \ldots p_k \leq z \\ p_i \in \mathcal{P}}} G(p_1 \cdots p_k) + O(8^k \pi(z)^k) \tag{2.2.4}$$

since $\sum_{p \leq z} p^{-\theta} \ll z^{-\theta} \pi(z)$. Note that $G(R) = 0$ if and only if R has a prime divisor, p, such that $p||R$.

Thus we can eliminate many of the terms from the sum

$$\sum_{\substack{p_1,\ldots p_k \le z \\ p_i \in \mathcal{P}}} G(p_1 \cdots p_k) = \sum_{\substack{p_1,\ldots p_k \le z \\ p_i \in \mathcal{P} \\ p_1 \cdots p_k \text{square-full}}} G(p_1 \cdots p_k).$$

For $p_1 \cdots p_k = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$ to be square-full we must have $\alpha_i \ge 2$ for all i, thus we see that $s \le k/2$. We rewrite the above sum over square-full numbers according to how many primes appear in each numbers factorization. This shows us that the above is

$$= \sum_{s \le k/2} \sum_{\substack{q_1 < \ldots < q_s \le z \\ q_i \in \mathcal{P}}} \sum_{\substack{\alpha_1,\ldots,\alpha_s \ge 2 \\ \sum \alpha_i = k}} \frac{k!}{\alpha_1! \cdots \alpha_s!} G(q_1^{\alpha_1} \cdots q_s^{\alpha_s}). \tag{2.2.5}$$

Here the factorial term comes when we restrict the q_i's to an increasing sequence. When k is even then the terms when $\alpha_i = 2$ for all i are the main contributors. When $s = k/2$, we get the terms

$$\sum_{\substack{q_1 < \cdots < q_{k_2} \le z \\ q_i \in \mathcal{P}}} \frac{k!}{2^{k/2}} G(q_1^2 \cdots q_{k/2}^2).$$

When $R = \prod_{i=1}^s q_i^2$, then we evaluate $G(R)$ to be

$$G(R) = \prod_{i=1}^{s} \left(\frac{1}{q_i} \left(1 - \frac{1}{q_i}\right)^2 + \left(\frac{-1}{q_i}\right)^2 \left(1 - \frac{1}{q_i}\right) \right) = \prod_{i=1}^{s} \frac{1}{q_i} \left(1 - \frac{1}{q_i}\right).$$

If we remove the increasing condition on the above sum and pull out all constants we get our Gaussian moments, C_k. Thus the terms in (2.2.5) when $s = k/2$ can be seen to be

$$\frac{k!}{2^{k/2}(k/2)!} \sum_{\substack{q_1,\ldots,q_s \le z \\ q_i \in \mathcal{P} \\ q_i \text{ distinct}}} \prod_{i=1}^{k/2} \frac{1}{q_i} \left(1 - \frac{1}{q_i}\right).$$

We can bound the above sum by above if we ignore distinctness and by below if we overcompensate for distinctness by ignoring the largest summands. Let π_n be the Beurling prime which comes n^{th} closest to maximize $\frac{1}{t}\left(1 - \frac{1}{t}\right)$. Note that $t = 2$ maximizes this function so when n is small π_n should be relatively close to 2. To be more specific on how to acheive a lower bound, take $q_1, \ldots, q_j$ as given, then the sum over just the $q_{j+1} < z$ is clealy

$$\ge \sum_{\substack{p \le z \\ p \in \mathcal{P} \\ p \ne \pi_i, 1 \le i \le j}} \frac{1}{p} - \frac{1}{p^2}$$

Applying this argument for all $1 \leq j \leq k/2$, and applying the trivial upper bound we get the pair of inequalities

$$\left(\sum_{\substack{p\leq z \\ p\in\mathcal{P} \\ p\neq \pi_i, 1\leq i\leq k/2}} \frac{1}{p} - \frac{1}{p^2} \right)^{k/2} \leq \sum_{\substack{q_1,\ldots,q_s\leq z \\ q_i\in\mathcal{P} \\ q_i \text{ distinct}}} \prod_{i=1}^{k/2} \frac{1}{q_i}\left(1-\frac{1}{q_i}\right) \leq \left(\sum_{\substack{p\leq z \\ p\in\mathcal{P}}} \frac{1}{p} - \frac{1}{p^2} \right)^{k/2}.$$

The inner sum of the lower bound can differ from the inner sum of the upper bound by at most $k/8$. So, by Corollary 2.1 we get that the term from (2.2.5) with $s = k/2$ contributes

$$\frac{k!}{2^{k/2}(k/2)!} (\log\log z + O_A(k))^{k/2} = \frac{k!}{2^{k/2}(k/2)!} (\log\log z)^{k/2} \left(1 + O\left(k^3/(\log\log z)\right)\right). \tag{2.2.6}$$

To deal with the terms when $s < k/2$ use the trivial estimate that

$$0 \leq G(q_1^{\alpha_1} \cdots q_s^{\alpha_s}) \leq 1/(q_1^{\alpha_1} \cdots q_s^{\alpha_s}).$$

From this we get that these terms are bounded above by

$$\sum_{s<k/2} \frac{k!}{s!} \left(\sum_{\substack{q\leq z \\ q\in\mathcal{P}}} \frac{1}{q} \right)^s \sum_{\substack{\alpha_1,\ldots,\alpha_s\geq 2 \\ \sum \alpha_i = k}} \frac{1}{\alpha_1!\cdots\alpha_s!}.$$

The number of ways of writing $k = \sum_{\substack{1\leq i\leq s \\ \alpha_i\geq 2}} \alpha_i$ is the same as the number of ways of writing $k - s = \sum_{\substack{1\leq i\leq s \\ \alpha_i\geq 1}} \alpha_i$. Picture lining $k - s$ objects. Then we can separate them into s groups, of size at least 1, by placing $s - 1$ partitions between any combination of their $k - s - 1$ gaps. The number of ways to write $k - s = \sum_{\substack{1\leq i\leq s \\ \alpha_i\geq 1}} \alpha_i$ is therefore $\binom{k-s-1}{s-1}$. Now using Corollary 2.1 we bound the terms when $s < k/2$ above by

$$< \sum_{s<k/2} \frac{k!}{s!2^s} \binom{k-s-1}{s-1} (\log\log z + O_A(1))^s. \tag{2.2.7}$$

Now recall equations (2.2.4), (2.2.6) and (2.2.7). Stringing these together we get that

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left(\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right)^k = \sum_{\substack{p_1,\dots p_k\le z\\ p_i\in\mathcal{P}}}\sum_{\substack{n\le x\\ n\in\mathcal{B}}} f_{p_1\cdots p_k}(n)$$

$$=A\,x \sum_{\substack{p_i\le z\\ 1\le i\le k\\ p_i\in\mathcal{P}}} G(p_1\cdots p_k) + O(8^k\pi_{\mathcal{B}}(z)^k)$$

$$=A\,C_k x(\log\log z + O_A(1+k^3/\log\log z))^{k/2}$$
$$+\,O\left(x\sum_{s<k/2}\frac{k!}{2^s s!}\binom{k-s-1}{s-1}(\log\log z+O_A(1))^s + 8^k\pi_{\mathcal{B}}(z)^k\right)$$

Note that we can bound the sum $\sum_{s<k/2}\frac{k!}{2^s s!}\binom{k-s-1}{s-1}(\log\log z)^s$ by a convergent geometric sum since the ratio of consecutive terms are $\frac{2(s+1)s}{(k-s)\log\log z}$ which is much smaller than 1. Hence

$$\sum_{s<k/2}\frac{k!}{2^s s!}\binom{k-s-1}{s-1}(\log\log z)^s \ll \frac{k!}{2^{k/2}(k/2)!}(\log\log z)^{\lceil k/2\rceil-1}.$$

□

Now we want to say something about the sum $\sum_{n\le x}(\omega_{\mathcal{B}}(n)-\log\log x)^k$ with some uniformity over k.

Recall Theorem 1.2.

Let $\mathcal{P}$ be a set of Beurling primes with corresponding set of integers $\mathcal{B}$ such that the integer counting function has the asymptotic $N_{\mathcal{B}}(x) = Ax + O(x^\theta)$ where $A>0$ and $\theta<1$. Set $C_k := \Gamma(k+1)/2^{k/2}\Gamma(k/2+1)$. Assume that $k\le \log\log\log\log x$. When k is even

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}(\omega_{\mathcal{B}}(n)-\log\log x)^k = A\cdot C_k x(\log\log z)^{k/2}\left(1+O_A(k^3/(\log\log z))\right) + O(8^k\pi(z)^k).$$

When k is odd

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}(\omega_{\mathcal{B}}(n)-\log\log x)^k \ll C_k x(\log\log z)^{k/2}\frac{k^3}{\log\log z} + 8^k\pi(z)^k$$

Proof of Theorem 1.2. Set $z = x^{1/k}$ to see that

$$\omega_{\mathcal{B}}(n) - \log\log x = \sum_{p \le z} f_p(n) + \sum_{\substack{p|n \\ p>z}} 1 + \left(\sum_{p \le z} 1/p - \log\log x\right).$$

If $n \le x$ then we know that n can have, at most, $k-1$ prime factors larger than z, thus $\sum_{\substack{p|n \\ p>z}} 1 \ll k$. To handle the third sum recall Corollary 2.1 to see that

$$\sum_{p \le z} 1/p - \log\log x = \log\log x^{1/k} - \log\log x + O(1) = \log \frac{\log x^{1/k}}{\log x} + O(1) = \log(1/k) + O(1) \ll \log k.$$

Therefore, we conclude that

$$\omega_{\mathcal{B}}(n) - \log\log x = \sum_{p \le z} f_p(n) + O(k).$$

Using the above result and the binomial theorem, then for some positive constant c, say the explicit constant from the error term in the above equation, we get

$$(\omega_{\mathcal{B}}(n) - \log\log x)^k = \left(\sum_{p \le z} f_p(n)\right)^k + O\left(\sum_{\ell=0}^{k-1} (ck)^{k-\ell} \binom{k}{\ell} \left|\sum_{p \le z} f_p(n)\right|^{\ell}\right).$$

This should look familiar. Now we will apply Theorem 2.4 to evaluate this sum.

$$\sum_{n \le x} (\omega_{\mathcal{B}}(n) - \log\log x)^k = \sum_{n \le x} \left(\sum_{p \le z} f_p(n)\right)^k + O\left(\sum_{n \le x} \sum_{\ell=0}^{k-1} (ck)^{k-\ell} \binom{k}{\ell} \left|\sum_{p \le z} f_p(n)\right|^{\ell}\right) \tag{2.2.8}$$

Directly from the statement of Theorem 2.4 we can correctly conclude, if $k < \log\log\log\log z$ and is even, then the main term is

$$\sum_{n \le x} \left(\sum_{p \le z} f_p(n)\right)^k = A \cdot C_k x (\log\log z)^{k/2} \left(1 + O_A(k^3/\log\log z)\right) + O(8^k \pi(z)^k).$$

We will also use Theorem 2.4 to bound the error tern in (2.2.8).

If ℓ is even then, just as above, Theorem 2.4 bounds the sum in the error term. If ℓ is odd then the error term of 2.2.8 is bounded using the Cauchy–Schwarz inequality and Theorem 2.4 and is seen to be

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left|\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right|^{\ell}$$

$$\le\left(\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left(\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right)^{\ell-1}\right)^{1/2}\left(\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\left(\sum_{\substack{p\le z\\ p\in\mathcal{P}}} f_p(n)\right)^{\ell+1}\right)^{1/2}$$

$$\ll\sqrt{C_{\ell-1}C_{\ell+1}}\,x(\log\log z)^{\ell}.$$

$\square$

In particular Theorem 1.2 proves that

$$\lim_{x\to\infty}\frac{1}{N_{\mathcal{B}}(x)}\sum_{n\le x}\left(\frac{\omega_{\mathcal{B}}(n)-\log\log x}{\sqrt{\log\log x}}\right)^k=\begin{cases}C_k & k\text{ is odd}\\ 0 & k\text{ is even}\end{cases}$$

which coincide with the moments of the normal distribution. Now we will show that the difference

$$\sum_{n\le x}\left(\frac{\omega_{\mathcal{B}}(n)-\log\log x}{\sqrt{\log\log x}}\right)^k-\sum_{n\le x}\left(\frac{\omega_{\mathcal{B}}(n)-\log\log n}{\sqrt{\log\log n}}\right)^k=o(x). \tag{2.2.9}$$

This will be enough to prove that $\frac{\omega_{\mathcal{B}}(n)-\log\log n}{\sqrt{\log\log n}}$ is asymptotically normally distributed since the normal distribution is determined by its moments. In other words the asymptotic density

$$\lim_{x\to\infty}\left(\frac{1}{N_{\mathcal{B}}(x)}\cdot\#\left\{n\le x\ :\ a\le\frac{\omega_{\mathcal{B}}(n)-\log\log n}{\sqrt{\log\log n}}\le b\right\}\right)=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}dt.$$

To prove (2.2.9) see that the triangle inequality for the L_k norm gives

$$\left|\left(\sum_{1<n\le x}|\omega_{\mathcal{B}}(n)-\log\log x|^k\right)^{1/k}-\left(\sum_{1<n\le x}|\omega_{\mathcal{B}}(n)-\log\log n|^k\right)^{1/k}\right|$$
$$\le\left(\sum_{1<n\le x}(\log\log x-\log\log n)^k\right)^{1/k}$$

To calculate a bound for the sum $\sum_{1<n\leq x}(\log\log x - \log\log n)^k$ we split the sum at $x/\log x$. For values of n between $x/\log x$ and x we see that

$$\log\log x - \log\log n \ll \frac{\log\log x}{\log x}.$$

Therefore

$$\sum_{x/\log x<n\leq x}(\log\log x - \log\log n)^k \ll x\cdot\left(\frac{\log\log x}{\log x}\right)^k = o(x).$$

For values of n between 1 and $x/\log x$ we use the trivial bound

$$\sum_{1<n\leq x/\log x}(\log\log x - \log\log n)^k \ll \frac{x(\log\log x)^k}{(\log x)^{k-1}} = o(x).$$

Chapter 3

A Set of Beurling Primes that Induces Large Gaps in its Set of Integers

Recall the standard set of primes of the natural numbers is being denoted by $\mathbb{P}$. In this short chapter $\mathcal{P}$ will be an infinite subset of $\mathbb{P}$ and $\mathcal{B}$ will be the comprised of the elements of $\mathbb{N}$ that have a prime factorization containing primes only found in $\mathcal{P}$. That is $\mathcal{B}$ is the smallest semigroup containing all elements of $\mathcal{P}$. If we define $\pi_{\mathcal{B}}$ as the counting function of $\mathcal{P}$ and $N_{\mathcal{B}}$ as the counting function $\mathcal{B}$ then we show the existence of such a $\mathcal{P}$ as to make very large gaps between consecutive elements of $\mathcal{B}$. To be precise, our constructed $\mathcal{B}$ has the property that for every $\gamma < 1$ there is a $x \in \mathbb{R}$ such that there is a x^{γ} sized gap of integers inside of $(x, 2x)$.

The only result that the results in this chapter rely heavily on is the fact that the counting function for integers less than x with prime factors all less than y, denoted by $\psi(x, y)$ has the property that if $y = \log x$ then $\psi(x, y) = x^{o(1)}$. As a source for counting integers with small prime factors take [12], Section 7.1. Define $C[x, y]$ as the function which counts integers less than x with at least 1 prime factor larger than y. Clearly we see that if $x > 1$ and $y \leq 2$ then $C[x, y] = [x]$. Again it is simple to see that if $y \geq x$ then we get that $C[x, y] = 0$. It is plain to see that $C[x, y] + \psi(x, y) = [x]$.

Proposition 3.1. *For any $\gamma \in (0, 1)$ there exists a large $N_{\gamma} \in \mathbb{N}$ such that there exist x^{γ} consecutive integers in $[x, 2x]$ with a prime factor larger than $\log x$ whenever $x > N_{\gamma}$.*

Proof. Let $\gamma \in (0, 1)$ be given and set $\kappa = (1 - \gamma)/2$, so that $\gamma < \kappa < 1$. Without loss of generality $\log x$ is an integer since, if not, a larger value of x we be sufficient. Therefore $\log(x + x^{\gamma}) < \log 2 + \log x < \log x + 1$. Now, consider the difference

$$C[x + x^{\kappa}, \log x] - C[x, \log x] = C[x + x^{\kappa}, \log(x + x^{\kappa})] - C[x, \log x] = x^{\kappa} - x^{o(1)}.$$

Therefore in the interval $[x, x + x^{\kappa}]$ we have x^{κ} integers with a prime factor at least as big as $\log x$ with at most $x^{o(1)}$ exceptions. By the pigeonhole principle, we must have at least $x^{\kappa - o(1)}$ consecutive integers with a prime factor larger than $\log x$. For x sufficiently large (say, larger than N_{γ}) we have that $\kappa - o(1) > \gamma$. This means that for such an x we have at least x^{γ} consecutive integers between x and $2x$ all with a prime factor larger than $\log x$.

$\square$

A corollary to this proposition is the main result of the chapter.

Corollary 3.2. *There exists a set of Beurling primes $\mathcal{P} \subset \mathbb{P}$ such that $\pi_{\mathcal{B}}(x) \sim \pi(x)$ and for every $\gamma \in (0,1)$ there exist an x such that there is a x^γ size gap between two consecutive Beurling integers between x and $2x$.*

Proof. Define the sequence $\gamma_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$. Let N_γ be chosen to be as in Proposition 3.1. Choose $x_1 > N_{\gamma_1}$ and inductively Choose

$$x_n > \left(e^{2x_{n-1}},\ N_{\gamma_n}\right).$$

For each $n \in \mathbb{N}$ find, according to Proposition 3.1, the $x_n^{\gamma_n}$ consecutive integers that all have a prime factor larger than $\log(x_n)$ and call the set of such integers I_n. Now, define $Q(n)$ to be the largest prime factor of n and define

$$\mathcal{R}_n = \{Q(k)\ :\ k \in I_n\}.$$

Note that if $i < j$ are integers, then for $a \in \mathcal{R}_j$ we have that $a > \log x_j > \log(e^{2x_{j-1}}) \geq 2x_i$. Also, for $b \in \mathcal{R}_i$ we have $b \leq 2x_i$. Hence for $i \neq j$ we have that $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$. Furthermore note that the size of $\mathcal{R}_n$ is less than or equal to $x_n^{\gamma_n}$, and all of the elements of $\mathcal{R}_n$ are within the interval $[\log x_n, 2x_n]$.

Now we choose the set of Beurling primes which will satisfy the statement of the corollary

$$\mathcal{P}_{wow} := \mathbb{P} \setminus \bigcup_{m=1}^{\infty} \mathcal{R}_m.$$

By the above discussion we see that for all $n \in \mathbb{N}$ the set $I_n \cap \mathcal{B}_{wow} = \emptyset$. Therefore for all $n \in \mathbb{N}$ we have $x_n^{\gamma_n}$ sized gaps at each x_n. Furthermore it is easy to see that $\pi_{\mathcal{P}_{wow}}(x) \leq \pi(x)$. But also

$$\pi_{\mathcal{P}_{wow}}(x) \geq \pi(x) - \sum_{x_n \leq x} x_n^{\gamma_n}.$$

Call $m := \max\{i \in \mathbb{N} \mid\ x_m \leq x\}$. Then

$$\sum_{i=1}^{m} x_i^{\gamma_i} \leq x_m^{\gamma_m} + \sum_{i=1}^{m-1} x_i \ll x_m^{\gamma_m} + (m-1)x_{m-1}.$$

Since $m - 1 \leq x_{m-1} \leq \log x_m \leq \log x$ and also $m \ll \log_\star x \ll \log\log x$. Therefore the above is $\sum_{i=1}^{m} x_i^{\gamma_i} \ll (\log x)^2$. Hence we see that $\pi(x) \sim \pi_{\mathcal{P}_{wow}}(x)$. $\square$

Chapter 4

Extending a Theorem of Selberg and Sathe

4.1 The Non-Uniform Case

We would like to expand the prime number theorem for Beurling primes (Theorem 1.1) to counting products of k elements of $\mathcal{P}$. Consider the counting function of Beurling integers with exactly k prime factors less than x, denoted by $\sigma_{\mathcal{B},k}(x)$. Then we have the classical result due to Landau [9] – for the set of natural primes, $\mathbb{P}$ – that for a fixed k the following asymptotic relationship holds

$$\sigma_{k,\mathbb{P}}(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}. \tag{4.1.1}$$

It is tempting to think that this result is uniform in k since counting the integers less then x according to how many prime in their factorization shows us that $\sum_{k\in\mathbb{N}} \sigma_{k,\mathbb{P}}(x) = \lfloor x \rfloor$, while Taylor series tell us that $\sum_{k\in\mathbb{N}} \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} = x$. It turns out that this formula is actually not uniform. Now, using the inspiration of Beurling primes we can show that (4.1.1) is not uniform over all k.

Consider the set of Beurling primes $\mathcal{P} = \mathbb{P} \setminus \{3\}$. Then for a fixed k Theorem 4.1 shows that that (4.1.1) will hold, yet clearly

$$\sum_{k\geq 1} \sigma_{\mathcal{B},k}(x) = \sum_{\substack{n\leq x\\ n\in\mathcal{B}}} 1 = 2x/3 + O(1).$$

So it seams that there cannot be such a wide range of uniformity. In the natural prime case a theorem of Sathe [15] [16] [17] [18], which was greatly simplified by Selberg [19], gives a formula for $\sigma_k(x)$ which is uniform for all $k < R\log\log x$ where $R < 2$. In the setting of Beurling primes we will prove an analogous statement in theorem 1.3. First, we prove the non-uniform case.

Theorem 4.1. *For a fixed positive integer k and a fixed set of Beurling primes with integer counting functions $N_{\mathcal{B}}(x) = Ax + O(x/log^{\gamma}x)$ with $A > 0$ and $\gamma > 3/2$, then the counting function for integers with exactly k prime factors has asymptotic formula*

$$\sigma_{\mathcal{B},k}(x) \sim \frac{x(\log\log x)^{k-1}}{\log x}.$$

Beurling proved in [1] that it is possible that the prime number theorem will not hold for a set of Beurling primes if we have an error term in the integer counting function which has $\gamma = 3/2$. Since the prime number theorem is a base case of this theorem it is also true that $3/2$ is best possible for Theorem 4.1.

Proof. To avoid cumbersome notation we drop the subscript $\mathcal{B}$, but keep in mind hat we are always working with Beurling integers and primes. Consider a slightly different function, $\pi_k(x)$, which counts Beurling integers less than x which are products of k distinct Beurling primes. I claim that $\pi_k(x)$ has the asymptotic formula $\pi_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$. We see that

$$\begin{aligned}
0 \leq\ \sigma_k(x) - \pi_k(x) &\ll \sum_{1\leq i\leq k-1} \#\Big\{p_1^{\alpha_1}\cdots p_i^{\alpha_i} \leq x \mid p_j \in \mathcal{B} \mid \sum_{\substack{1\leq j\leq i\\ \alpha_j\geq 1}} \alpha_j = k\Big\} \\
&\ll \sum_{1\leq i\leq k-1} \binom{k-1}{i-1} \#\Big\{p_1\cdots p_i \leq x \mid p_j \in \mathcal{B}\Big\} \\
&\ll \pi_{k-1}(x) \sum_{1\leq i\leq k-1} \binom{k-1}{i-1} \ll_k \pi_{k-1}(x)
\end{aligned} \tag{4.1.2}$$

because, as discussed in the proof of Theorem 2.4, the number of ways to sum i positive integers to k is $\binom{k-1}{i-1}$. The case for when $k = 1$ is taken care of by the prime number theorem for Beurling primes. Now assume the induction hypothesis. Every product of $k+1$ distinct primes has exactly $k+1$ options for a prime p_0 to omit so

$$\sum_{\substack{p_0\leq x\\ p_0\in\mathcal{P}}} \pi_k(x/p) = \sum_{\substack{p_0\leq x\\ p_0\in\mathcal{P}}} \sum_{\substack{p_1\cdots p_k\leq x/p\\ p_i\in\mathcal{P}}} 1 = (k+1)\pi_{k+1}(x) + O(\pi_k(x)) \tag{4.1.3}$$

since $\sum_{\substack{p_0\leq x\\ p_0\in\mathcal{P}}} \pi_k(x/p)$ also count products of $k+1$ prime factors with at most 1 repeated factor. Writing $\sum_{\substack{p_0\leq x\\ p_0\in\mathcal{P}}} \pi_k(x/p)$ as a Riemann–Stieltjes integral and applying partial summation we get that

$$
\begin{aligned}
\sum_{\substack{p_0\le x\\ p_0\in\mathcal{P}}} \pi_k(x/p) &\sim \frac{x}{(k-1)!}\cdot\sum_{\substack{p\le x\\ p\in\mathcal{P}}} \frac{(\log\log x/p)^{k-1}}{p\log(x/p)} \\
&\sim \frac{x}{(k-1)!}\cdot\sum_{\substack{n\le x\\ n\in\mathcal{B}}} \frac{(\log\log x/n)^{k-1}}{n\log(x/n)}\mathbb{I}_{\mathcal{P}}(n) \\
&= \frac{x}{(k-1)!}\cdot\pi(x)\cdot\frac{(\log\log x)^{k-1}}{x\log x} - \frac{x}{(k-1)!}\int_{p_1}^{x}\pi(y)\frac{d}{dy}\left[\frac{(\log\log x/y)^{k-1}}{y\log x/y}\right].
\end{aligned}
$$

We see that the first term in the above is $\ll_k x(\log\log x)^{k-1}/(\log x)^2$ so we suspect that this should go into the error term. To calculate the above integral we apply the prime number theorem to get the rather complicated expression

$$
\begin{aligned}
&\int_{p_1}^{x}\pi(y)\frac{d}{dy}\left[\frac{(\log\log x/y)^{k-1}}{y\log x/y}\right] \\
&\qquad \sim \int_{p_1}^{x}\frac{-(\log\log x/y)^{k-1}}{y\log y\log x/y} + \frac{(\log\log x/y)^{k-1}}{y\log y(\log x/y)^2} - \frac{(k-1)(\log\log x/y)^{k-2}}{y\log y(\log x/y)^2}dy.
\end{aligned}
$$

Now combining the smaller integrals into an error term shows us that

$$
\frac{(k-1)!(k+1)\pi_k(x)}{x} = (1+o(1))\int_{p_1}^{x}\frac{(\log\log x/y)^{k-1}}{y\log y\log x/y}dy + O\left(\int_{p_1}^{x}\frac{(\log\log x/y)^{k-1}}{y\log y(\log x/y)^2}dy + \frac{(\log\log x)^{k-2}}{\log x}\right) \tag{4.1.4}
$$

Set $I=\int_{p_1}^{x}\frac{-(\log\log x/y)^{k-1}}{y\log y\log x/y}dy$. To see that the error term is genuinely smaller than I split the integral up at $\sqrt{x}$ and get the bound

$$
\int_{p_1}^{\sqrt{x}}\frac{-(\log\log x/y)^{k-1}}{y\log y(\log x/y)^2}dy \ll \frac{I}{\log x}
$$

for the integral over small values of y. For the integral for large values of y use the substitution $v=\log\log x/y$ and repeated integration by parts to get that

$$\int_{\sqrt{x}}^{x} \frac{-(\log\log x/y)^{k-1}}{y\log y(\log x/y)^2}dy \ll \frac{1}{\log x}\int_{p_1}^{x} \frac{-(\log\log x/y)^{k-1}}{y(\log x/y)^2}dy$$

$$= -\int_{\log\log\sqrt{x}}^{\log\log x} \frac{v^{k-1}}{e^v}dv$$

$$\ll \frac{(\log\log x)^{k-1}}{(\log x)^2}.$$

So the above calculations the error term in (4.1.4) yields

$$\frac{(k-1)!(k+1)\pi_k(x)}{x} \sim \int_{p_1}^{x} \frac{(\log\log x/y)^{k-1}}{y\log y\log x/y}dy. \tag{4.1.5}$$

All that is required to prove Theorem 4.1 is to evaluate the above integral. Start by noting that

$$\int_{x/e}^{x} \frac{\pi_k(x/u)}{\log u} \ll \int_{x/e}^{x} \frac{du}{u} \ll \frac{x}{\log x}$$

so we can replace the upper bound of the integral in (4.1.5) with x/e. Now make the substitution $t = \log y$ and then use partial fraction decomposition to see that

$$\frac{(k-1)!(k+1)\pi_k(x)}{x} \sim \int_{1}^{\log x-1} \frac{(\log(\log x-t))^{k-1}}{t(\log x-t)}dt$$

$$= \frac{1}{\log x}\int_{1}^{\log x-1} \frac{(\log(\log x-t))^{k-1}}{s-t}dt + \frac{1}{\log x}\int_{1}^{\log x-1} \frac{(\log(\log x-t))^{k-1}}{t}dt.$$

A simple calculation shows that the first integral

$$\int_{1}^{\log x-1} \frac{(\log(\log x-t))^{k-1}}{\log x-t}dt = \frac{1}{k}\left[-\log(\log x-t)^k\right]_1^{\log x-1} = \frac{1}{k}\log(\log x-1)^k \sim \frac{1}{k}\log\log x.$$

To get a grasp on the size of the second integral make the further split at $\frac{\log x}{2}$ and see that

$$\int_{(\log x)/2}^{\log x-1} \frac{(\log(\log x - t))^{k-1}}{t} dt \ll (\log\log x)^{k-1} \cdot \int_{(\log x)/2}^{\log x-1} \frac{dt}{t} \ll (\log\log x)^{k-1}$$

and using the binomial theorem and the Taylor expansion of $\log(1-x/c)$ we get that

$$\begin{aligned}\int_1^{(\log x)/2} \frac{(\log(\log x - t))^{k-1}}{t} dt &= \int_1^{(\log x)/2} \frac{(\log\log x + \log(1-t/\log x))^{k-1}}{t} dt \\ &= \int_1^{(\log x)/2} \frac{(\log\log x)^{k-1} + O((\log\log x)^{k-2})}{t} dt \\ &= (\log\log x)^k + O((\log\log x)^{k-1}).\end{aligned}$$

Therefore we combine the calculations of these integrals to conclude that

$$\frac{(k-1)!(k+1)\sigma_k(x)}{x} \sim \int_1^{\log x-1} \frac{(\log(\log x - t))^{k-1}}{t(\log x - t)} dt \sim \left(1+\frac{1}{k}\right) \frac{(\log\log x)^k}{\log x}.$$

Rearrange terms gives us $\pi_k(x) \sim \frac{(x\log\log x)^k}{k!\log x}$. Because $\sigma_1(x) = \pi_1(x)$ induction and (4.1.2) proves that $\sigma_k(x) \sim \frac{(x\log\log x)^k}{k!\log x}$ □

Similar to the case with natural primes we will be able to prove a result with more uniformity. In the case of natural primes, this is called the Selberg–Sathe formula. In fact, we will use similar methods of proof for our more general result, but first we must show that these methods are actually valid and just as powerful when we change to using Beurling primes. In order to use the same methods we require that our zeta function be analytic to the left of the line $\sigma = 1$. To achieve an extended range of analyticity we require that our integer counting function has the asymptotic approximation

$$N_{\mathcal{B}}(x) = Ax + O(x^\theta) \tag{4.1.6}$$

where $A > 0$ and $0 \le \theta < 1$. We will show that this asymptotic property will imply analyticity to the left of $\sigma = 1$ in Section 4.3.

4.2 Perron's Formula for Beurling Primes

The first thing that is necessary to prove Theorem 1.3 is Perron's formula and error bounds on the remainder upon subtraction by a finite integral. That is we want to bound R in the following equation:

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}ds=\frac{1}{2\pi i}\int_{\sigma_0-iT_0}^{\sigma_0+iT_0}\alpha(s)\frac{x^s}{s}ds+R(T_0).$$

Perron's formula, in this case, is just a special case of an inverse Mellin transform [11]. To see this, note that the integer counting function is locally continuous (in fact locally constant), and converges in right half plains.

Lemma 4.2. *Let $\{a_n\}_{n\in\mathcal{B}}$ be an arithmetic sequence and let $\alpha(s):=\sum_{n\in\mathcal{B}}a_n\cdot n^{-s}$ be its associated Dirichlet Series. Define $\sigma_a:=\inf\{\sigma\in\mathbb{R}\mid\sum_{n\in\mathcal{B}}|a_n|n^{-\sigma}<\infty\}$, the absolute abscissa of convergence for $\alpha(s)$. If $\sigma_0>\max(0,\sigma_a)$ and $x>0$, then*

$${\sum_{n\le x}}' a_n=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}ds+R$$

where

$$R=\frac{1}{\pi}\sum_{\substack{x/2<n<x\\ n\in\mathcal{B}}}a_n\,\mathrm{si}\left(T\log\frac{x}{n}\right)-\frac{1}{\pi}\sum_{\substack{x/2<n<x\\ n\in\mathcal{B}}}a_n\,\mathrm{si}\left(T\log\frac{n}{x}\right)+O\left(\frac{4^{\sigma_0+x^{\sigma_0}}}{T}\sum_n\frac{|a_n|}{n^{\sigma_0}}\right)\tag{4.2.1}$$

$$\ll\sum_{\substack{x/2<n<2x\\ n\neq x\\ n\in\mathcal{B}}}|a_n|\min\left(1,\frac{x}{T}|x-n|\right)+\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n\in\mathcal{B}}|a_n|n^{-\sigma_0}\tag{4.2.2}$$

Proof. The series $\alpha(s)$ is absolutely convergent on the interval $[\sigma_0-iT,\sigma_0+iT]$ so

$$\frac{1}{2\pi i}\int_{\sigma_0-iT_0}^{\sigma_0+iT_0}\alpha(s)\frac{x^s}{s}ds=\sum_{n\in\mathcal{B}}a_n\frac{1}{2\pi i}\int_{\sigma_0-iT_0}^{\sigma_0+iT_0}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

Therefore it suffices to evaluate $\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}y^s\frac{ds}{s}$ for y in the four intervals $(0,1/2],\ [1/2,1],\ [1,2],\ [2,\infty)$. This is taken care of for us already since it is the same as the integrals that need to be evaluated for the natural prime case. For a reference see [12][section 5.1]. This proves (4.2.1)

Now for the less exact but more user friendly bound note that $\mathrm{si}(x)\ll\min(1,1/x)$. Also, see that $|\log n/x|=|\log(1+(n-x)/x)|\asymp|x-n|$. So, if $x/2\le n\le 2x$, then

$$\operatorname{si}(T|\log n/x|) \ll \min\left(1, \frac{1}{T|\log n/x|}\right) \asymp \min\left(1, \frac{1}{T|x-n|}\right).$$

Applying this bound to 4.2.1 we get that

$$R \ll \sum_{\substack{x/2<n<2x \\ n\neq x \\ n\in\mathcal{B}}} |a_n| \min\left(1, \frac{x}{T}|x-n|\right) + \frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n\in\mathcal{B}} |a_n| n^{-\sigma_0}.$$

□

4.3 The Bound on $\zeta_{\mathcal{B}}$

In our calculations leading up to Theorem 1.3 we will need a bound on $\zeta_{\mathcal{B}}(\sigma+it)$ in the region where $\sigma > 1 - c\theta/(2\log t)$ and $|t| > t_0$ and the region $2 > \sigma > 1 - c\theta/(2\log t)$ and $|t| \le t_0$, where t_0 is chosen to be slightly away from the pole at $s = 1$. We need these bounds to evaluate the integral given to us by Perron's formula in Lemma 4.2. To calculate these bounds we first need to know that $\zeta_{\mathcal{B}}$ is analytic to the left of $\sigma = 1$. In [3] there is a generalization of this for any zeta function of a continuous measure, but this simpler fact can be seen by applying the same methods that one would use to show this for the zeta function on natural primes. Start by separating the tail of the zeta function. For $\sigma > 1$

$$\zeta_{\mathcal{B}}(s) = \sum_{n\in\mathcal{B}} n^{-s} = \sum_{\substack{n\le x \\ n\in\mathcal{B}}} n^{-s} + \sum_{\substack{n>x \\ n\in\mathcal{B}}} n^{-s}.$$

Now we deal with the tail. Define $\epsilon(u) := N_{\mathcal{B}}(u) - Au$, where $N_{\mathcal{B}}(x)$ is as in (4.1.6). Note that $\epsilon(u) \ll u^{\theta}$. Now write the tail of the zeta function as

$$\sum_{\substack{n>x \\ n\in\mathcal{B}}} n^{-s} = \int_x^{\infty} u^{-s} dN_{\mathcal{B}}(u) = \int_x^{\infty} Au^{-s} du - \int_x^{\infty} u^{-s} d\epsilon(u).$$

The first integral can now be evaluated for $\sigma > 1$ and, using integration by parts, the second integral converges for $\sigma > \theta$. Using partial summation on the finite sum $\sum_{\substack{n\le x \\ n\in\mathcal{B}}} n^{-s}$ gives us that for $s \neq 1$

$$\zeta_{\mathcal{B}}(s) = \frac{Ax^{1-s}}{s-1} + \epsilon(x)\cdot x^{-s} + \int_x^{\infty} \epsilon(u)\cdot du^{-s} = \frac{Ax^{1-s}}{s-1} + \epsilon(x)\cdot x^{-s} - s\int_x^{\infty} \epsilon(u)\cdot u^{-s-1} du.$$

Since the integrand in the above is an analytic function of s we see that $\zeta_{\mathcal{B}}(s)$ is analytic in any region not containing $s = 1$ contained in the half plane $\sigma > \theta$, by the uniqueness of analytic continuation. In particular, when $x = 1$ we have

$$\zeta_{\mathcal{B}}(s) = \frac{A}{s-1} - s \int_1^\infty \epsilon(u) \cdot u^{-s-1} du. \tag{4.3.1}$$

It is possible to determine more information about $\zeta_B ints(s)$ to the left of $\sigma = 1$. Landau's methods in [10] showed that $\zeta_{\mathcal{B}}$ has a zero free region to the left of the curve $\sigma = \frac{c\theta}{\log t}$ for some small constant c. Using this zero free region we can determine a bound for $\zeta(s)$ which will be useful in applying Perron's formula in a later calculation.

Lemma 4.3. *For $s \in \{\sigma + it \mid \sigma > 1 - \frac{c(1-\theta)}{2\log t}$ and $|t| \geq t_0\}$ we get the bound*

$$\zeta_{\mathcal{B}}(s) \ll A \log t.$$

For the region $\{\sigma it \mid 2 \geq \sigma > 1 - \frac{c(1-\theta)}{2\log t}$ and $|t| \leq t_0\}$ we have that

$$\frac{\zeta_{\mathcal{B}}(s)^z}{s} = \frac{A^z}{(s-1)^z} + O\left(\frac{1}{|s-1|^{\Re z - 1}}\right).$$

One should note that one could make this lemma a corollary to a more general bound on $\zeta(s)$ but all we require for Theorem 1.3 are these specific bounds. In particular we calculate bounds for $\frac{\zeta'}{\zeta}(s)$ and for $|\log \zeta(s)|$ in the same regions as above.

Proof. Take $s = 1 + r + it$ and assume that $r \geq \frac{1-\theta}{\log t}$. By logarithmic differentiation of $\zeta_{\mathcal{B}}(s)$ with respect to s and the fact that $\pi_{\mathcal{B}}$ is a non-decreasing function we have

$$\begin{aligned} \left|\frac{\zeta_{\mathcal{B}}'}{\zeta_{\mathcal{B}}}(s)\right| &= \left|\int_{1^-}^\infty x^{-s} \cdot \log x \, d\pi_{\mathcal{B}}(x)\right| \\ &\leq \int_{1^-}^\infty x^{-1-r} \cdot \log x \, d\pi_{\mathcal{B}}(x). \end{aligned}$$

Furthermore, by (4.3.1) we see that

$$\left|\frac{\zeta_{\mathcal{B}}'}{\zeta_{\mathcal{B}}}(s)\right| \leq \frac{\zeta_{\mathcal{B}}'}{\zeta_{\mathcal{B}}}(1+r) \ll \frac{1}{r} \leq \frac{\log t}{1-\theta}.$$

In particular the above holds for $s_1 = 1 + \frac{\log t}{1-\theta} + it$. Now we need to prove the same inequality for smaller values of r. Using Jensen's inequality and the Borel–Caratheodory lemma, or (50) in [3] for $\sigma \geq 1 - \frac{1-\theta}{4}$ we have

$$\frac{\zeta'_{\mathcal{B}}}{\zeta_{\mathcal{B}}}(s) = \sum_{\rho} \frac{1}{s-\rho} + O\left(\frac{\log t}{1-\theta}\right)$$

where the sum is taken over zeros of $\zeta_{\mathcal{B}}(s)$ in the disk of radius $(1-\theta)/2$ centered at $1+it$. Looking at this equality for s_1 as defined above we get that

$$\sum_{\rho} \frac{1}{s_1-\rho} \ll \frac{\log t}{1-\theta}.$$

Now suppose we have an $s = \sigma + it$ such that $1 - \frac{c(1-\theta)}{2\log t} \leq \sigma \leq 1 + \frac{1-\theta}{\log t}$ then

$$\frac{\zeta'_{\mathcal{B}}}{\zeta_{\mathcal{B}}}(s) - \frac{\zeta'_{\mathcal{B}}}{\zeta_{\mathcal{B}}}(s_1) = \sum_{\rho} \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O\left(\frac{\log t}{1-\sigma}\right)$$

again, where the sum is over zeros, ρ, in the disk or radius $(1-\theta)/2$ centered at $1+it$. For all zeros of $\zeta_{\mathcal{B}}(s)$ in this disk we have $|s-\rho| \asymp |s_1-\rho|$. So upon differencing we get the bound

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{s_1-s}{(s-\rho)(s_1-\rho)} \ll \frac{1}{|s_1-\rho|^2 \log t} \ll \Re\frac{1}{s_1-\rho}.$$

Therefore we have that $\frac{\zeta'_{\mathcal{B}}}{\zeta_{\mathcal{B}}}(s) \ll \frac{\log t}{1-\theta}$ for $\sigma > 1 - \frac{c(1-\theta)}{2\log t}$ and $|t| \geq t_0$. Moving along we turn the above work into a bound for $\log \zeta_{\mathcal{B}}(s)$, which in turn will provide us with a bound on $\zeta_{\mathcal{B}}(s)$. By the analytic continuation of ζ in (4.3.1) we have that for $\sigma > \theta$

$$\frac{A}{\sigma-1} < \zeta_{\mathcal{B}}(s) < \frac{A\sigma}{1-\sigma}.$$

Therefore if $\sigma > 1 + \frac{1-\theta}{\log t}$ then $\zeta_{\mathcal{B}}(s) < A\left(1 + \frac{\log t}{1-\theta}\right)$ and furthermore

$$|\log \zeta_{\mathcal{B}}(s)| \leq \log\log t + O(1/(1-\theta)).$$

In particular the same $s_1 = 1 + \frac{1-\theta}{\log t} + it$ satisfies the above bound. As before we take a difference and compute

$$\log \zeta_{\mathcal{B}}(s) - \log \zeta_{\mathcal{B}}(s_1) = \int_{s_1}^{s} \frac{\zeta'_{\mathcal{B}}}{\zeta_{\mathcal{B}}}(w)dw.$$

Now consider an s such that $1 - \frac{c(1-\theta)}{2\log t} \leq \sigma \leq 1 + \frac{1-\theta}{\log t}$. Since we have a big-oh bound on the integrand, in the strip, of $\frac{\log t}{1-\theta}$ we get that $|\log \zeta_{\mathcal{B}}(s)| \leq \log\log t + O\left(1/(1-\theta)\right)$ in the strip as well. Complex analysis tells us that $\log|\zeta_{\mathcal{B}}(s)| = \Re \log \zeta_{\mathcal{B}}(s)$. Therefore $\zeta_{\mathcal{B}}(s) \ll A \log t$ in whenever $\sigma > 1 - \frac{c(1-\theta)}{2\log t}$ and $|t| > t_0$.

A bound for $\zeta_{\mathcal{B}}(s)/s$ on the region where $1-\frac{c(1-\theta)}{2\log t}<\sigma<2$ and $|t|<2$ is seen by examining the Laurent series expansion about $s=1$. Since $\zeta_{\mathcal{B}}(s)$ has a simple pole at $s=1$ and is elsewhere analytic we have that on this region

$$\frac{\zeta_{\mathcal{B}}(s)^z}{s}=\frac{A^z}{(s-1)^z}+O\left(\frac{1}{|s-1|^{\Re z-1}}\right).$$

□

4.4 The Integrations of $\zeta_{\mathcal{B}}$

Now we are ready to start evaluating some integrals of interest. By our version of Perron's formula and the subsequent bounds on the error term – Theorem 4.2 – we can use the information gained about $\zeta_{\mathcal{B}}$ through Lemma 4.3 to evaluate the partial summation of $d_z(n)$, defined for any $z\in\mathbb{C}$. This arithmetic function is defined by $\zeta_{\mathcal{B}}(s)^z=\sum_{n\in\mathcal{B}}d_z(n)n^{-s}$. Call $D_z(x)=\sum_{n\le x}d_z(n)$. When $k\in\mathbb{N}$ the function $d_k(n)$ has the simple interpretation that it is the number of all ordered k-tuples $(m_1,...,m_k)\in\mathcal{B}^k$ such that $m_1\cdots m_k=n$.

We start by evaluating $D_z(x)$ this when $z=\ell$ is an integer. We will extend this to the general case by using the bound, $|d_z(n)|\le d_{|z|}(n)\le d_R(n)$ for any integer $R\ge|z|$, along with Perron's formula (4.2.2) and then evaluating a contour integral using Lemma 4.3. First we need some simple calculations.

Lemma 4.4. *Let $\mathcal{P}$ be a set of Beurling primes such that the associated integer counting function is as in (4.1.6). Furthermore define $\epsilon(x):=N_{\mathcal{B}}(x)-Ax$, that is the error term of the integer summation formula. Then note that $\epsilon(x)\ll x^\theta$. Then we can evaluate $\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{n^v}$ for $v\in(0,1]$ as follows:*

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{n^v}=\begin{cases}A\log x+C_{\mathcal{B}}+O\left(x^{\theta-1}\right) & v=1\\ A\cdot\theta\cdot\frac{x^{1-\theta}}{1-\theta}+O(\theta\cdot\log x). & v=\theta\\ A\cdot v\cdot\frac{x^{1-v}}{1-v}+\int_{1^-}^{\infty}\frac{\epsilon(u)}{u^{1+v}}\,du+O_{\theta,v}(x^{\theta-v}) & \text{otherwise}\end{cases}$$

where $C_{\mathcal{B}}=\int_{1^-}^{\infty}\frac{\epsilon(u)}{u^2}\,du$ is the analog for the Euler constant, usually referred to as C_0 in the natural prime case. Note that the integral in the case when $v\neq1,\theta$ is a constant if $v>\theta$ and would fit into the error term if $v<\theta$. Furthermore the sum

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{\log^a n}{n}=A\frac{\log^{a+1}}{a+1}x+O(x^{\theta-1}\log^a x)$$

Proof. First consider the case when $v=\theta$ and evaluate by converting the sum to a Riemann–Stieltjes integral and applying integration by parts to see that

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{n^\theta} = \int_{1^-}^x \frac{1}{u^\theta} dN_{\mathcal{B}}(u)$$
$$= \int_{1^-}^x [Au + O(u^\theta)]\frac{\theta}{u^{\theta+1}}du$$
$$= A\cdot\theta\cdot\frac{x^{1-\theta}}{1-\theta} + O_\theta(\log x).$$

The case where $v \in (0,1)\setminus\theta$ is calculated very similarly. We see that in this case

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{n^v} = \int_{1^-}^x \frac{1}{u^v} dN_{\mathcal{B}}(u)$$
$$= \int_{1^-}^x [Au + O(u^\theta)]\frac{v}{u^{v+1}}du$$
$$= A\cdot v\cdot\frac{x^{1-v}}{1-v} + \int_{1^-}^\infty \frac{\epsilon(u)}{u^{1+v}} + O_{\theta,v}(x^{\theta-v}).$$

We calculate the sum of reciprocals or Beurling integers much in the same way as we would for the natural numbers. Now see that

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{1}{n} = \int_{1^-}^x \frac{1}{u} dN_{\mathcal{B}}(u)$$
$$= \int_{1^-}^x [Au + \epsilon(u)]\frac{1}{u^2}\, du$$
$$= A\log u + \int_{1^-}^\infty \frac{\epsilon(u)}{u^2}du + O(x^{\theta-1}).$$

Next calculate

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}}\frac{\log^a(n)}{n} = \int_1^x \frac{\log^a(u)}{u} dN_{\mathcal{B}}(u)$$
$$=A\int_1^x \frac{\log^a(u)}{u}du + \int_1^x \frac{\log^a(u)}{u}d\epsilon(u)$$
$$=A\frac{\log^{a+1}}{a+1}x + \int_1^x \frac{\log^a(u)}{u}d\epsilon(u)$$

Where $\epsilon(u) := N_{\mathcal{B}}(u) - Au$. Then repeated integration by parts gives us that

$$\int_1^x \frac{\log^a(u)}{u} d\epsilon(u) \ll x^{\theta-1}\log^a x + \int_1^x \frac{\log^a(u)}{u^{2-\theta}}du \ll x^{\theta-1}\log^a x.$$

$\square$

Lemma 4.5. *Let ℓ be a natural number. Given a set of Beurling primes, $\mathcal{P}$, such that the associated set of Beurling integers, $\mathcal{B}$, is as in (4.1.6), then*

$$D_\ell(x) = A^\ell x \cdot P_\ell(\log x) + O\left(x^{1-\frac{1-\theta}{2^{\ell-1}}}\right)$$

where P_ℓ is a polynomial of degree $\ell - 1$, which is independent of the choice of $\mathcal{P}$.

Proof. We proceed by induction. First note that $D_1(x) = N_{\mathcal{B}}(x) = Ax + O(x^\theta)$. Now we have the following induction hypothesis: for all $r < \ell$

$$D_r(x) = A^r \cdot xP_r(\log x) + O(x^{1-\frac{1-\theta}{2^{r-1}}})$$

where P_r is a polynomial of degree $r - 1$. Now we use the hyperbola method to write

$$\begin{aligned}
D_\ell(x) = \sum_{\substack{km\le x\\ k,m\in\mathcal{B}}} D_{\ell-1}(m) &= \sum_{\substack{k\le y\\ k\in\mathcal{B}}} d_{\ell-1}\left(\frac{x}{k}\right) + \sum_{\substack{m\le x/y\\ m\in\mathcal{B}}} d_{\ell-1}(m) D_1\left(\frac{x}{m}\right) - D_1(y)D_{\ell-1}\left(\frac{x}{y}\right) \\
&= A^\ell x \sum_{\substack{k\le y\\ k\in\mathcal{B}}} \frac{1}{k}\cdot P_{\ell-1}(\log x/k) + Ax \sum_{\substack{m\le x/y\\ m\in\mathcal{B}}} \frac{1}{m}\cdot d_{\ell-1}(m) \\
&- \left(Ay + O\left(y^\theta\right)\right)\cdot\left(A^{\ell-1}\frac{x}{y}\cdot P_{\ell-1}(\log x/y) + O\left((x/y)^{1-\frac{1-\theta}{2^{\ell-2}}}\right)\right) \qquad (4.4.1) \\
&+ \ O\left(\sum_{\substack{k\le y\\ k\in\mathcal{B}}} \left(\frac{x}{k}\right)^{1-\frac{1-\theta}{2^{\ell-2}}} + \sum_{\substack{m\le x/y\\ m\in\mathcal{B}}} d_{\ell-1}(m)\left(\frac{x}{m}\right)^{1-\frac{1-\theta}{2^{\ell-2}}}\right).
\end{aligned}$$

As in the case for $\ell = 2$ we can use partial summation (and the induction hypothesis) to evaluate all the sums that appear, expand and cancel terms in the error term and see that when we choose $y = \sqrt{x}$ we get an essentially minimal error term to get the desired expression for $D_\ell(x)$. To follow this plan we use the previos lemma, but in addition we need to evaluate one more sum. Namely note that

$$\sum_{\substack{n\le x/y\\ n\in\mathcal{B}}} \frac{d_{\ell-1}(m)}{m} = \int_{u=1}^{x/y} \frac{1}{u} d[D_{\ell-1}(u)]$$
$$= \frac{D_{\ell-1(x/y)}}{x/y} + \int_1^{x/y} \frac{D_{\ell-1}(u)}{u^2} du.$$

Use the induction hypothesis and partial summation to see that the above is

$$\sum_{\substack{n\le x/y\\ n\in\mathcal{B}}} \frac{d_{\ell-1}(m)}{m} = \frac{D_{\ell-1}(x/y)}{x/y} + A\int_1^{x/y} \frac{P_{\ell-1}(u)}{u} du + O\left(\int_1^{x/y} u^{-1-\frac{1-\theta}{2^{\ell-2}}} du\right).$$

Repeated integration by parts then shows us that if $y = \sqrt{x}$ then

$$\sum_{\substack{n\le x/y\\ n\in\mathcal{B}}} \frac{d_{\ell-1}(m)}{m} = P_\ell(\log x) + O\left((x/y)^{\frac{1-\theta}{2^{\ell-2}}}\right)$$

Where P_ℓ is a polynomial of degree $\ell-1$. Therefore we see that the main term of (4.4.1) is, what we expect, namely $A^\ell \cdot xP_\ell(\log x)$ for P_ℓ some $\ell-1$ degree polynomial. To take care of the error term expand, apply Lemma 4.4 and the induction hypothesis to show that the error term of (4.4.1) is

$$\ll xy^{\frac{\theta-1}{2^{\ell-2}}} + x^{1-\frac{1-\theta}{2^{\ell-2}}} y^{\frac{1-\theta}{2^{\ell-1}}} \ll x^{1+\frac{1-\theta}{2^{\ell-1}}}$$

because we choose $y = \sqrt{x}$.

□

Corollary 4.6. *For d_z defined above we have the following bounds on these two sums for any $R > |z|$:*

$$\sum_{\frac{x}{2}<n<2x} |d_z(n)| \cdot \min\left(1, \frac{x}{T|x-n|}\right) \ll A^R x \log(x)^{-R-2} \tag{4.4.2}$$

and

$$\frac{x^a}{T} \sum_{n\in\mathcal{B}} |d_z(n)| n^{-a} \ll A^R x \log(x)^{-R-2} \tag{4.4.3}$$

where T is chosen to be $T = \exp(\sqrt{\log x})$.

Proof. Consider the first sum and split it up by summing over the subset, $\mathcal{S} \subset (x/2, x) \cap \mathcal{B}$ which are close to x and those far away from x. Specifically set $\mathcal{S} := \{n \in \mathcal{B} | |n-x| \le x/(\log x)^{2R-1}\}$. Fist note that

$$\sum_{n\in\mathcal{S}} |d_z(n)| \min\left(1, \frac{x}{T|x-n|}\right) \ll \sum_{n\in\mathcal{S}} |d_z(n)|.$$

Now we use the fact that $|d_z(n)| \leq d_R(n)$ whenever $R > |z|$ is an integer and recall (4.5), our formula for $D_R(x)$, set $y = (\log x)^{-2R+1}$ to get that

$$\begin{aligned}\sum_{n\in\mathcal{S}} |d_R(n)| &= D_R(x+xy) - D_R(x-xy) \\ &= A^R\left[x\cdot\left(P_R(\log x + \log(1+y)) - P_R(\log x + \log(1-y))\right)\right] \\ &+ A^R \cdot xy\left(P_R(\log x + \log(1+y)) + P_R(\log x + \log(1-y))\right).\end{aligned}$$

To calculate a bound on the above we need to examine cancellation of the $\log$ part in the above expression. So using a Taylor approximation or two we get that

$$(\log x + \log(1+y))^p - (\log x + \log(1-y))^p = (\log x + O(y))^p - (\log x + O(y))^p$$

$$= (\log x)^p\left[(1 + O(y/\log x))^p - (1 + O(y/\log x))^p\right]$$

$$= (\log x)^p\left[O(p\cdot y/\log x)\right] \ll p\cdot(\log x)^{p-2R-2}$$

and if $p < R$ then the above is $\ll R\cdot(\log x)^{R-3}$. Therefore we must have

$$\sum_{n\in\mathcal{S}} d_R(n) \ll A^R x(\log x)^{-R-2}.$$

When we evaluate the sum of the remaining terms of (4.4.2) we are summing over $n \in \mathcal{B}$ that are far away from x so we use the other possible value of the minimum.

$$\sum_{n\notin\mathcal{S}} |d_z(n)| \min\left(1, \frac{x}{T|x-n|}\right) \ll \sum_{n\notin\mathcal{S}} |d_z(n)|\cdot\frac{x}{T|x-n|} \ll T^{-1}\cdot(\log x)^{2R+1}\sum_{n\notin\mathcal{S}} |d_z(n)|$$

Now using our calculated bound when z is an integer in lemma 4.5 and the fact that $|d_z(n)| \leq d_R(n)$ so the above is

$$\ll A^R x\cdot(\log x)^{-R-2}.$$

Next we turn our attention to the sum in (4.4.3). We have previously seen, in Theorem 4.3, that $\zeta_{\mathcal{B}}(a) \ll A \log x$ so this sum can be seen to be

$$\frac{x^a}{T} \sum_{n \in \mathcal{B}} |d_z(n)| n^{-a} \ll \frac{x^a}{T} (\zeta_{\mathcal{B}}(a))^R \ll \frac{x}{\exp(\sqrt{\log x})} (A \log x)^R \ll A^R x (\log)^{-R-2}.$$

□

Theorem 4.7. *Let R be any positive real number. If $x \geq p_1$, then uniformly for $|z| \leq R$*

$$D_z(x) = \frac{A^z \cdot x(\log x)^{z-1}}{\Gamma(z)} + O\left(x(\log x)^{\Re z - 2}\right).$$

Proof. If $a = 1 + 1/\log x$ then by Lemma 4.2.2

$$D_z(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \zeta_{\mathcal{B}}(s)^z \cdot \frac{x^s}{s} \ll \sum_{\frac{x}{2} < n < 2x} |d_z(n)| \cdot \min\left(1, \frac{x}{T|x-n|}\right) + \frac{x^a}{T} \sum_{n \in \mathcal{B}} |d_z(n)| n^{-a}. \quad (4.4.4)$$

We must evaluate the integral to get the main term. From Lemma 4.3 we have a bound on $\zeta_{\mathcal{B}}(s)$ which, modulo a constant, is the same as in the natural prime case. Therefore the method of integration will not be any different than in the case for the natural primes. For a reference take [12] Theorem 7.17.

Now to deal with the error term in (4.4.4) we simply apply Corollary 4.6. □

4.5 The Deduction of Theorem 1.3

The following lemmas give us a way to turn our calculated values for D_z, in Theorem 4.4.4, into bounds on $\sigma_k(x)$, the number of elements of $\mathcal{B}$ less than x with exactly k prime factors.

Lemma 4.8. *For a function $b_z(n)$ define $F(s,z) = \sum_{m \in \mathcal{B}} b_z(m) m^{-s}$ and suppose that the sum*

$$\sum_{m \in \mathcal{B}} |b_z(m)| (\log m)^{2R+1} / m$$

is uniformly bounded for $|z| \leq R$. For $\sigma \geq 1$ let $c_z(n) = b_z \star d_z(n)$. Let $C_z(x) := \sum_{n \leq x} c_z(n)$ be the summation function of c_z. Then

$$C_z(x) = A^z \frac{F(1,z)}{\Gamma(z)} x (\log x)^{z-1} + O\left(x(\log x)^{\Re z - 2}\right).$$

Proof. First we simply re-write the defining summation of C_z to see that

$$\begin{aligned} C_z(x) = \sum_{n\le x} c_z(n) &= \sum_{n\le x}\sum_{m|n} b_z(m) d_z(n/m) \\ &= \sum_{m\le x} b_z(m) \sum_{n\le x/m} d_z(n) \\ &= \sum_{m\le x/p_1} b_z(m) D_z(x/m) + \sum_{x/p_1<n\le x} b_z(m) \end{aligned}$$

where p_1 is the smallest prime in $\mathcal{P}$. Note that if $a < p_1$ then $D_z(a) = 1$. Then using our formula for $D_z(x)$ in Theorem 4.7 we can calculate that

$$C_z(x) = A^z \frac{z}{\Gamma(z)} \sum_{m\le x/p_1} \frac{b_z(m)}{m} (\log(x/m))^{z-1} + O\left(\sum_{m\le x} \frac{|b_z(m)|}{m} (\log(2x/m))^{\Re z-2} \right).$$

By splitting the sum at $\sqrt{x}$ and using our uniform bound on $\sum_{m\in\mathcal{B}} |b_z(m)|(\log m)^{2R+1}/m$ we see that the error term is in the above expression is

$$\begin{aligned} &= \; x(\log x)^{\Re z-2} \sum_{m\le\sqrt{x}} \frac{|b_z(m)|}{m} + x(\log x)^{\Re z-2} \sum_{\sqrt{x}<m\le x} \frac{|b_z(m)|}{m} (\log x)^{2\Re z-2} \\ &\ll x(\log x)^{\Re z-2}. \end{aligned}$$

To handle the main term see that when $m \le \sqrt{x}$ the binomial theorem gives us that

$$(\log(x/m))^{z-1} = (\log x - \log m)^{z-1} = (\log x)^{z-1} + O\left(\log m \cdot (\log x)^{\Re z-2}\right).$$

Therefore we can write

$$\begin{aligned} &\sum_{m\le x/p_1} \frac{b_z(m)}{m} \log(x/m)^{z-1} \\ =&(\log x)^{z-1} \sum_{m\le x/p_1} \frac{b_z(m)}{m} + O\left((\log x)^{\Re z-2} \sum_{m\le\sqrt{x}} \frac{b_z(m)}{m} \log m + (\log x)^{Rz-1} \sum_{m>\sqrt{x}} \frac{b_z(m)}{m} \right) \end{aligned}$$

Furthermore, since we assumed that the sum $\sum_{m\in\mathcal{B}} |b_z(n)|(\log m)^{2R+1}/m$ is uniformly bounded on the disk $|z| < R$ we get that the above is

$$=(\log x)^{z-1}F(1,z) + O\left((\log x)^{\Re z-2}\sum_{m\in\mathcal{B}}\frac{b_z(m)}{m}(\log m)^{2R+1}\right)$$

giving the result. $\square$

Now we want to use this arithmetic lemma in the case where the arithmetic function b_z is chosen to satisfy the following

$$\sum_{n\in\mathcal{B}} b_z(n)n^{-s} = F(s,z) = \prod_{p\in\mathcal{P}}\left(1-\frac{z}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{z}.$$

If we choose $R < p_1$ then for $|z| < R$ we have that

$$\sum_{m\in\mathcal{B}} b_z(m)/m = \prod_{p\in\mathcal{P}}\left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{z}$$

is uniformly bounded in this range. Therefore, Lemma 4.8 tells us information about a function c_z defined by the property

$$\sum_{n\in\mathcal{B}} c_z(n)n^{-s} = \zeta_{\mathcal{B}}(n)^z F(s,z) = \prod_{p\in\mathcal{P}}\left(1-\frac{z}{p^s}\right)^{-1} = \sum_{n\in\mathcal{B}} z^{\Omega(n)}n^{-s}.$$

The information that 4.8 tells us is that the partial summation function of c_z has the formula

$$C_z(x) = \sum_{k\in\mathbb{N}}\sigma_k(x)z^k = A^z\frac{F(1,z)}{\Gamma(z)}x(\log x)^{z-1} + O\left(x(\log x)^{\Re z-2}\right). \tag{4.5.1}$$

If $k > \log_{p_1}(x)$ then there are no Beurling integers, b, with exactly k prime factors such that $b < x$. So $C_z(x)$ is a polynomial in z and therefore Cauchy's theorem asserts that

$$\sigma_k(x) = \frac{1}{2\pi i}\int_{|z|=r}\frac{C_z(x)}{z^{k+1}}dz \tag{4.5.2}$$

for $r < p_1$. Now we combine all of the results of this section in order to prove the Selberg–Sathe formula for Beurling primes.

Recall the statement of Theorem 1.3

Let $\mathcal{P}$ be a set of Beurling primes with corresponding set of integers $\mathcal{B}$ such that the integer counting *function has the asymptotic $N_{\mathcal{B}}(x) = Ax + O(x^{\theta})$ where $A > 0$ and $\theta < 1$. Suppose that $R < p_1$ and that*

$$F(s,z) := \prod_{p \in \mathcal{P}} \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{z}.$$

Set $G(z) = A^z \cdot F(1,z)/\Gamma(z+1)$. Then

$$\sigma_k(x) \;=\; G\left(\frac{k-1}{\log\log x}\right) \frac{x(\log\log x)^{k-1}}{(k-1)!\log x} \left(1 \;+\; O_R\left(\frac{k}{(\log\log x)^2}\right)\right)$$

uniformly for $1 \le k \le R\log\log x$.

It is perhaps a bit surprising that doubling the number of integers in a set of Beurling integers does not seem to affect the number of primes by much. The next corollary shows us that all the integers are hiding plain sight. The generalization of the Hardy–Ramanujan theorem tells us that almost all, that is 100% of, integers less than x have about $\log\log x$ prime factors. Alternatively, and perhaps more naturally, consider the system $\mathcal{P} = \mathbb{P} \cup \{a\}$. For any $A > 1$ we can chose to add a prime a that is small enough to make $N_{\mathcal{B}}(x) \sim Ax$, yet adding a single prime will not change the asymptotic formula for $\pi_{\mathcal{B}}$.

Evaluate $G(1) = A \cdot F(1,1)/\Gamma(2) = A\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right) = A$ and see that this is enough to give us the following corollary.

Corollary 4.9. *If $k \sim \log\log x$ then*

$$\sigma_k(x) \sim A\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}.$$

Before we dive into the proof let's look at a simple example. Let $\mathcal{P}$ be the set of all natural primes excluding a finite set of primes T. Then in this case it is easy to see that our integer counting function counts all natural numbers which don't contain any $t \in T$ in their prime factorizations. Hence, by (1.2.1), we get the formula $N_{\mathcal{B}}(x) = \prod_{t\in T}\left(1-\frac{1}{t}\right) \cdot x + O(1)$. Then σ_k is the counting function of natural numbers less than x with exactly k prime factors, none of which are in T. Set $z = (k-1)/\log\log x$, and set

$$G(z) = \prod_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)^{z} \cdot \prod_{p\in\mathcal{P}}\left(1-\frac{z}{p}\right)^{-1} \cdot \frac{1}{\Gamma(1+z)}$$

Theorem 1.3 tells us that for $R < p_1$ we get that the counting function of Beurling integers comprised of exactly k primes, in this case, has the formula

$$\sigma_k(x) \;=\; G(z)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\left(1 \;+\; O_R\left(\frac{k}{(\log\log x)^2}\right)\right)$$

uniformly for $k \leq R \log\log x$. Also Corollary 4.9 tells us that, when $k \sim \log\log x$ the number of integers less than x with $\log\log k$ prime factors, none of which are in T is approximately

$$\prod_{p \in T} \left(1 - \frac{1}{p}\right) \frac{x(\log\log x)^{k-1}}{(k-1)! \log x}.$$

Proof of Theorem 1.3. When $k = 1$ we have the classical result proved by Beurling in [1] so we may assume that $k > 1$. By Lemma 4.8, (4.5.1), and (4.5.2) we have that

$$\sigma_k(x) = \frac{1}{2\pi i} \cdot \frac{x}{\log x} \cdot \int_{|z|=r} G(z)(\log x)^z z^{-k} dz + O\left(\frac{x}{\log x} \left| \int_{|z|=r} z^{-k-1} (\log x)^z dz \right| \right). \tag{4.5.3}$$

Now we choose $r = (k-1)/\log\log x$ and see that the error term in the above is

$$\begin{aligned} &\ll \frac{x}{\log x} \cdot 2\pi r \cdot r^{-k-1} = x(\log x)^{r-1} \cdot r^{-k} \\ &= \frac{x}{(\log x)^2} e^{k-1} \frac{(\log\log x)^k}{(k-1)^k}. \end{aligned}$$

By Sterling's formula [20] we get the above to be

$$\begin{aligned} &\ll \frac{x(\log\log x)^k}{(k-1)!(\log x)} \\ &\ll \frac{x(\log\log x)^{k-3}}{(k-1)! \log x}. \end{aligned}$$

To tackle the integral in the main term just requires some technical maneuvering. See that integration by parts yields

$$E := \frac{r}{2\pi i} \int_{|z|=r} (\log x)^z z^{-k} dz = \frac{1}{2\pi i} \int_{|z|=r} (\log x)^z z^{1-k} dz.$$

So by adding and subtracting the term $\frac{G(r)}{2\pi i} \int_{|z|=r} (\log x)^z z^{-k} dz$ and the term $G'(r) \cdot E$ in two different ways we see that the integral in (4.5.3) of can be seen to be rewritten

$$\frac{1}{2\pi i}\int_{|z|=r} G(z)(\log x)^z z^{-k} dz$$

$$= \frac{G(r)}{2\pi i}\int_{|z|=r}(\log x)^z z^{-k} dz + \frac{1}{2\pi i}\int_{|z|=r}\left(G(z)-G(r)\right)(\log x)^z z^{-k} dz$$

$$= \frac{G(r)}{2\pi i}\int_{|z|=r}(\log x)^z z^{-k} dz + \frac{1}{2\pi i}\int_{|z|=r}\big(G(z)-G(r)-G'(r)(z-r)\big)(\log x)^z z^{-k} dz.$$

Cauchy's theorem can be used to show that the first integral is $(\log\log x)^{k-1}/(k-1)!$ which gives us the main term for our formula for $\sigma_k(x)$. The second integral will now be shown to bounded by our error term in (4.5.3). Consider that we can rewrite the first term in the product of the integrand as

$$G(z)-G(r)-G'(z-r) = \int_r^z (z-w)G''(w)dw \ll |z-r|^2$$

since $G''(z)$ is bounded on the region of integration. Following [12][pg. 233] yields we write $z = re^{2\pi i\theta}$ so that the second integral can be seen to be

$$\ll \int_{-1/2}^{1/2}(\sin\pi\theta)^2 e^{(k-1)\cos 2\pi\theta} d\theta.$$

Note the bounds $|\sin\pi\theta| \le |\pi\theta|$ and $\cos 2\pi\theta \le 1-8\theta^2$ for $\theta\in[-1/2,1/2]$. Therefore, using integration by parts, we can bound the above as

$$\begin{aligned}\ll r^{3-k}e^{k-1}\int_0^\infty \theta^2 e^{-8(k-1)\theta^2} d\theta &\ll r^{3-k}e^{k-1}(k-1)^{-3/2}\\ &\ll k(\log\log x)^{k-3}/(k-1)!\end{aligned}$$

since we chose $r = (k-1)/\log\log x$. This completes the proof of Theorem 1.3.

□

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