$m$-Liouville theorems and regularity results for elliptic PDEs

by

Mostafa Fazly

B.Sc., Razi University, 2004
M.Sc., Sharif University of Technology, 2006

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in
The Faculty of Graduate Studies
(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA
(Vancouver)
December 2012
© Mostafa Fazly 2012
Abstract

In this thesis, we explore the behaviour of solutions of various semilinear elliptic equations and systems. Here are our main observations:

- For
  \[- \text{div}(\gamma(x) \nabla u) = \lambda(x)f(u) \quad x \in \mathbb{R}^n.\]
  we prove 0-Liouville theorems for either a general nonlinearity \( f \in C^1(\mathbb{R}) \) or specific nonlinearities. Applying generalized Hardy inequalities we show that some of these results are optimal for specific \( \gamma(x) \) and \( \lambda(x) \).

  On the other hand, for double-well potential nonlinearities we prove \( m \)-Liouville theorems for any \( 0 \leq m < n \). This can be seen as an extension of De Giorgi’s conjecture (1978) to higher dimensional solutions.

- For
  \[ \Delta u = \nabla H(u) \quad \text{in} \quad \mathbb{R}^n, \]
  where \( u : \mathbb{R}^n \to \mathbb{R}^m, H \in C^2(\mathbb{R}^m) \), we state the following 1-Liouville theorem as a conjecture which is a counterpart of the De Giorgi’s conjecture (1978) and we prove it in dimensions \( n \leq 3 \).

**Conjecture 1.** Suppose \( u = (u_i)_{i=1}^m \) is an \( H \)-monotone bounded entire solutions of the above system, then at least in dimensions \( n \leq 8 \), each component \( u_i \) must be one-dimensional.

- Following ideas given for the above gradient system, we study the extremal solutions of:
  \[- \Delta u = \lambda f'(u)g(v), \quad -\Delta v = \gamma f(u)g'(v) \quad \text{in} \ \Omega, \]
  and
  \[- \Delta u = \lambda f(u)g'(v), \quad -\Delta v = \gamma f'(u)g(v) \quad \text{in} \ \Omega, \]
with zero Dirichlet boundary conditions and where \( \lambda, \gamma \) are positive parameters. We prove various regularity results for either general nonlinearities or explicit nonlinearities.

- For Hénon-Lane-Emden system

\[
\begin{aligned}
-\Delta u &= |x|^n v^p \quad \text{in} \quad \mathbb{R}^n, \\
-\Delta v &= |x|^b u^q \quad \text{in} \quad \mathbb{R}^n,
\end{aligned}
\]

we prove the following conjecture is indeed the case in dimension \( n = 3 \) provided the solution is bounded.

**Conjecture 2.** Suppose \( (p, q) \) is under the critical hyperbola, i.e.,

\[
\frac{n + a}{p + 1} + \frac{n + b}{q + 1} > n - 2.
\]

Then nonnegative solutions must be zero.

Also, assuming stability of the solutions, we prove 0-Liouville theorems in higher dimensions for various cases of parameters.

- For the following nonlocal eigenvalue problem

\[
\begin{aligned}
(-\Delta)^{1/2} u &= \lambda g(x) f(u) \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

we show that the extremal solution associated with the extremal parameter is the unique solution and also we prove that when \( f \) is suitably supercritical and \( \Omega \) satisfies certain geometrical conditions then there is a unique solution for small positive \( \lambda \).
Preface

This thesis is a compendium of seven papers. Chapters 2 and 5 consist of two separate articles and Chapters 3, 4, and 6 consist of one article.

- Section 2.1 of Chapter 2 is a version of published joint work with Dr. Cowan in [32].
- Section 2.2 of Chapter 2 is a version of independent submitted work in [50].
- Chapter 3 is a version of published joint work with Prof. Ghoussoub in [51].
- Chapter 4 is a version of submitted joint work with Dr. Cowan in [30].
- Chapter 5 contains an independent published work in [49] and a version of submitted joint work with Prof. Ghoussoub in [52].
- Chapter 6 is a version of published joint work with Dr. Cowan in [31].

Articles:
# Table of Contents

Abstract ................................................................. ii

Preface ................................................................. iv

Table of Contents ....................................................... v

Acknowledgements ..................................................... vii

Dedication ............................................................... ix

1 Introduction ........................................................... 1

2 $m$-Liouville theorems of a non-uniformly elliptic equation .... 5
   2.1 Specific nonlinearities .......................................... 6
   2.1.1 Introduction .................................................. 6
   2.1.2 Proof of main results ....................................... 11
   2.2 General nonlinearity ............................................. 16
   2.2.1 Introduction and main results .............................. 16
   2.2.2 Proofs .......................................................... 21
   2.3 Summary and conclusions ..................................... 30

3 One dimensional symmetry results for gradient systems ....... 31
   3.1 Introduction ..................................................... 31
   3.2 A linear Liouville theorem for systems and first applications 34
   3.3 De Giorgi type results ......................................... 40
   3.4 Summary and conclusions ..................................... 47

4 Regularity results for gradient and nongradient systems ..... 48
   4.1 Introduction ..................................................... 48
   4.2 Arbitrary nonlinearities ....................................... 53
   4.3 Explicit nonlinearities ......................................... 64
   4.4 Summary and conclusions ..................................... 71
# Table of Contents

5 **The Hénon-Lane-Emden conjecture** ........................................... 72
  5.1 Introduction and main results .............................................. 72
    5.1.1 Liouville theorems for bounded non-negative solutions .......... 74
    5.1.2 Liouville theorems for stable non-negative solutions .......... 75
  5.2 Proof in the case of non-negative solutions ............................. 76
  5.3 On solutions of the second order Hénon equation with finite Morse index ................................................................. 83
  5.4 On solutions of the fourth order Hénon equation with finite Morse index ................................................................. 89
  5.5 On stable solutions of the Hénon-Lane-Emden system .............. 97
  5.6 Summary and conclusions .................................................... 101

6 **Uniqueness of solutions for a nonlocal eigenvalue problem** 102
  6.1 Introduction ................................................................. 102
    6.1.1 The local eigenvalue problem ....................................... 103
    6.1.2 The nonlocal eigenvalue problem ................................... 104
  6.2 Uniqueness of the extremal solution .................................... 108
  6.3 Uniqueness of solutions for small λ ................................... 113
  6.4 Summary and conclusions ................................................... 118

Bibliography ................................................................. 119
Acknowledgements

I would like to express my sincere gratitude and appreciation to my supervisor Professor Nassif Ghoussoub for his mentorship, encouragement, support and most of all friendship throughout the course of this degree. And for all his contributions to mathematics community; in particular by being founder and scientific director of the prestigious research institutes such as PIMS and BIRS that have had tremendous impacts on my studies.

I am enormously grateful to Professor Changfeng Gui and Professor Juncheng Wei for helpful discussions, for fruitful comments and for their wonderful talks given at UBC, PIMS and BIRS. I also thank Professor del Pino for his minicourse and his talk on Allen-Cahn equation at BIRS 2010 and 2012.

I appreciate the supervisory committee Professor Tai-Peng Tsai and Professor Stephen Gustafson for courses that they offered and for their comments. I also appreciate the organizers of the weekly PDE seminar, Professor Young-Heon Kim, for giving me the opportunity to be the audience of many interesting talks and also giving two talks.

It has been a great pleasure to do mathematics on chalkboard with friend of mine Craig Cowan (usually at midnight while we are constantly drinking coffee!) and with Professor Denis Bonheur during his visit.

I am thankful of the University Examiners Professor Michael Ward and Professor Chen Greif for their valuable comments and questions and Professor Rabab Ward for being a chair of the defense.

Thanks to all the organizers and speakers of these wonderful workshops and summer schools that I have attended during my doctoral studies:

- Workshop on recent trends in geometric and nonlinear analysis, BIRS, Banff, Aug 6 -10, 2012.

- Workshop on deterministic and stochastic front propagation, BIRS, Banff, March 21 - 26, 2010

- PDE Summer School, PIMS-UBC, July-August 2009
Acknowledgements


Last but not the least, I am grateful to friends, faculties, staffs at UBC, PIMS and BIRS that have provided me with a very friendly environment to focus on my studies. In particular, I am grateful to Professor Adem Alejandro, Danny Fan, Wynne Fong, Do-Rim Joo.
Dedication

To my family; my lovely parents, brothers, sister-in-law and new born nephew, Parham.
Chapter 1

Introduction

This thesis is based on seven research papers [30–32, 49–52] found in Chapters 2–6. Each of the chapters begins with a detailed introduction to the results it contains. In this chapter, we explain the relevance of all the chapters in this thesis and we give a general point of view about the problems that we study.

A main part of this thesis focuses on the study of the semilinear elliptic equations originating in the problems of geometry, analysis and physics, most importantly the Allen-Cahn equation and the De Giorgi’s conjecture (1978). This conjecture brings together two groups of mathematicians: one specializing in nonlinear partial differential equations and another in differential geometry, more specifically on minimal surfaces and constant mean curvature surfaces.

De Giorgi’s Conjecture (1978), [36]. Suppose that \( u \) is a bounded solution of the Allen-Cahn equation

\[
-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n,
\]

such that \( \partial_x u(x) > 0 \) for all \( x \in \mathbb{R}^n \), i.e. \( u \) is monotone. Then for at least \( n \leq 8 \) the level sets of \( u \) are all hyperplanes, i.e. \( u \) is a one-dimensional solution.

Equation (1.1) derived by Cahn-Hilliard and Allen-Cahn in the gradient theory of phase transitions in connection with the

\[
I_\epsilon(u) = \frac{\epsilon}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\epsilon} \int_{\Omega} (1 - u^2)^2
\]

where \( \epsilon > 0 \) and \( \Omega \subset \mathbb{R}^n \) is a bounded domain. The theory of \( \Gamma \)-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces. It is known that a family of local minimizers \( u_\epsilon \) with uniformly bounded energy as \( \epsilon \to 0^+ \) approaches a function of the form \( \chi_E - \chi_{E^c} \); \( \chi \) is the characteristic function and \( \partial E \) is a set with minimal perimeter, so that the limiting interface between the stable
Chapter 1. Introduction

phases $u = 1$ and $u = -1$ is expected to approach a minimal hypersurface. This connection led De Giorgi to formulate his conjecture inspired by the Bernstein’s theorem for minimal graphs. This theorem states that any minimal hypersurface in $\mathbb{R}^n$, which is also a graph of a function of $n - 1$ variables, must be a hyperplane when $n \leq 8$.

In 1997, Ghoussoub and Gui [53] proved the De Giorgi’s conjecture for $n = 2$. They used a 0-Liouville theorem for the ratio $\sigma := \frac{\partial_{x_1} u}{\partial_{x_2} u}$ developed by Berestycki, Caffarelli and Nirenberg in [9] for the study of symmetry properties of positive solutions of semilinear elliptic equations in half spaces. Unfortunately, it is not known whether or not this 0-Liouville theorem is optimal.

Ambrosio and Cabré [5] and later in a joint work with Alberti [4] extended these results up to dimension $n = 3$. The De Giorgi’s conjecture for higher dimensions is still open. However, Ghoussoub and Gui showed in [54] that the conjecture is true for $n = 4$ or $n = 5$ for a special class of solutions that satisfy an antisymmetry condition. In 2003, Savin [84] assuming the additional natural hypothesis

$$\lim_{x_n \to \pm \infty} u(x', x_n) \to \pm 1,$$  

proved that the conjecture is true in dimension $n \leq 8$. The proof is nonvariational and it uses the sliding method for a special family of radially symmetric functions. Finally in 2008, del Pino-Kowalczyk-Wei [37] gave a counterexample to De Giorgi’s conjecture in dimension $n \geq 9$ which has long been believed to exist.

Under the much stronger assumption that the limits in (1.2) are uniform in $x'$, the conjecture is known as the Gibbons’ conjecture. This conjecture was first proved for $n \leq 3$ by Ghoussoub and Gui in [53] and then for all dimensions independently with different methods by Barlow, Bass and Gui [7], Berestycki, Hamel and Monneau [10] and Farina [45]. In this thesis, we consider various equations and systems and prove $m$-Liouville theorem for stable (monotone) solutions.

Motivated by this conjecture, I have introduced two main concepts.

The first concept is the “$H$-monotone solutions” that allows us to formulate a counterpart of the De Giorgi’s conjecture for system of equations stating that the $H$-monotone and bounded solutions of the gradient systems on the whole space of dimension $n \leq 8$ must be 1-dimensional solutions. We prove the conjecture in lower dimensions, $n \leq 3$, and using an extension of a certain geometric Poincaré inequality we show that gradients of the various components of the solutions are parallel. We believe this is in the
right track of extending the De Giorgi’s conjecture to system of equations. The $H$-monotonicity assumption seems to be crucial for concluding that the solutions are 1-dimensional. Indeed, it was shown in [1] that when $H$ is a multiple-well potential on $\mathbb{R}^2$, the system has entire heteroclinic solutions $(u, v)$, meaning that for each fixed $x_2 \in \mathbb{R}$, they connect (when $x_1 \to \pm \infty$) a pair of constant global minima of $H$, while if $x_2 \to \pm \infty$, they connect a pair of distinct one dimensional stationary wave solutions $z_1(x_1)$ and $z_2(x_1)$. Note that these convergence are even uniform, which means that the corresponding Gibbon’s conjecture for systems of equations is not valid in general, without the assumption of $H$-monotonicity.

The second concept is the “$m$-Liouville theorem” for $m = 0, \cdots, n - 1$ that allows us to formulate a counterpart of the De Giorgi’s conjecture for equations but this time with higher-dimensional solutions as opposed to 1-dimensional solutions. We use the induction idea that is to use 0-Liouville theorem (0-dimensional solutions) to prove 1-Liouville theorem (1-dimensional solutions) and then to prove $(n - 1)$-Liouville theorem ($(n - 1)$-dimensional solutions). The reason that we call this “$m$-Liouville theorem” is because of the great mathematician Joseph Liouville (1809-1882) who proved a classical theorem in complex analysis stating that bounded harmonic functions on the whole space must be constant and constants are 0-dimensional objects, see Chapter 2. Moreover, in Chapter 5, we prove various 0-Liouville theorems regarding Hénon-Lane-Emden systems which gives a positive answer to the Hénon-Lane-Emden conjecture in dimension three. We also provide a new stability inequality that gives us the chance to treat systems the same way as the second order equations.

On the other hand, it is well known that there is a close relationship between the regularity of solutions on bounded domains and 0-Liouville theorem for related “limiting equations” on the whole space, via rescaling and blow up procedures. For the last few years, various people at UBC have been studying the following nonlinear eigenvalue problem

\[
(P)_\lambda : \quad Lu = \lambda g(x)f(u) \quad \text{in } \Omega,
\]

where $L$ is an elliptic operator of the form $L = \alpha \Delta^2 - \beta \Delta$ for various values of $\alpha$ and $\beta$, including $L = -\Delta$ and $L = \Delta^2$, under either Dirichlet or Navier boundary conditions. Note that in the above equation $\lambda > 0$ is a parameter, $\Omega$ is a smooth bounded domain in $\mathbb{R}^n (n \geq 2)$, $0 < g(x) \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$ and the function $f \in C^1(\mathbb{R})$ satisfies $f(0) = 1$ and one of the following conditions

(R) $f$ is smooth, increasing, convex on $\mathbb{R}$ and is superlinear at $\infty$ (i.e.
Chapter 1. Introduction

\[
\lim_{t \to \infty} \frac{f(t)}{t} = \infty.
\]

(S) \( f \) is smooth, increasing, convex on \([0, 1)\) and \( \lim_{t \to 1} f(t) = +\infty \).

For example \( f(t) = e^t \) and \( f(t) = (1 + t)^p, p > 1 \) known as the Gelfand and the Lane-Emden nonlinearities satisfy (R) and \( f(t) = (1 - t)^p, p < 0 \) known as the MEMS (Micro-Electro-Mechanical Systems) nonlinearity satisfies (S). We now list the properties one comes to expect when studying \((P)_{\lambda}\). It is well known that there exists a critical parameter \( \lambda^* \in (0, \infty) \), called the extremal parameter, such that for all \( 0 < \lambda < \lambda^* \) there exists a smooth, minimal solution \( u_{\lambda} \) of \((P)_{\lambda}\). In addition for each \( x \in \Omega \) the map \( \lambda \mapsto u_{\lambda}(x) \) is increasing in \((0, \lambda^*)\). This allows one to define the pointwise limit \( u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_{\lambda}(x) \) which can be shown to be a weak solution, in a suitably defined sense, of \((P)_{\lambda^*}\). For this reason \( u^* \) is called the extremal solution. It is also known that for \( \lambda > \lambda^* \) there are no weak solutions of \((P)_{\lambda}\). In Chapter 4 we study the regularity of extremal solutions for gradient and twisted gradient systems and we extend known regularity results in the scalar case to the system of equations.

In Chapter 6, we study a nonlocal eigenvalue problem and prove the above standard facts about the \( \lambda^* \) and the extremal solutions. Then applying Pohozaev type arguments, we prove the uniqueness of solutions.

This diagram illustrates the relevance of the chapters.
Chapter 2

$m$-Liouville theorems of a non-uniformly elliptic equation

In this chapter we study the following semilinear elliptic equation with an advection term

\[ -\Delta u + a(x) \cdot \nabla u = b(x)f(u) \quad x \in \mathbb{R}^n \]  

(2.1)

for any locally bounded vector field \( a : \mathbb{R}^n \to \mathbb{R}^n \), any function \( b \in C(\mathbb{R}^n) \) and \( f \in C^1(\mathbb{R}) \). We mainly focus on the cases that \( a \) and \( b \) are not constant and study how a lower order perturbation would change the behaviour of the solutions.

Note that if \( a(x) \) is of gradient form, that is there exists a smooth \( c(x) \) such that \( a(x) = \nabla c(x) \), then one can rewrite (2.1) as

\[ -\Delta u + \nabla c(x) \cdot \nabla u = b(x)f(u) \quad x \in \mathbb{R}^n. \]  

(2.2)

If we set \( \gamma(x) = e^{-c(x)} \) and \( \lambda(x) = e^{-c(x)}b(x) \) then we can rewrite (2.2) as the following equation in divergence form

\[ -\text{div}(\gamma(x)\nabla u) = \lambda(x)f(u) \quad x \in \mathbb{R}^n. \]  

(2.3)

We assume that \( \gamma(x) \) and \( \lambda(x) \), which we call weights, are smooth positive functions (we allow \( \lambda \) to be zero at say a point) and which satisfy various growth conditions at \( \infty \). Since \( \gamma(x) > 0 \), the operator \( \text{div}(\gamma(x)\nabla \cdot) \) is a non-uniformly elliptic operator.

**Definition 2.1.** We say that (2.3) satisfies $m$-Liouville theorem if for certain \( \lambda \) and \( \gamma \) solutions of (2.3) are $m$-dimensional for $0 \leq m < n$, i.e., they only depend on $m$ variables.

**Definition 2.2.** We call a classical solution $u$ of (2.3) to be

(i) monotone if \( \partial_{x_n} u > 0 \) for all $x \in \mathbb{R}^n$. 

5
2.1. Specific nonlinearities

(ii) asymptotically convergent if
\[ \lim_{x_n \to \pm \infty} u(x^m, x_n) \to \pm 1 \text{ for any } x^m \in \mathbb{R}^{n-1}. \] (2.4)

If this limit is uniform then we call it uniformly asymptotically convergent.

(iii) pointwise stable if there exists a function $0 < v$ that satisfies the linearized equation
\[ -\text{div}(\gamma(x)\nabla v) = \lambda(x)f'(u)v \text{ on } \mathbb{R}^n. \]

(iv) stable provided
\[ \int \lambda(x)f'(u)\psi^2 \leq \int \gamma(x)|\nabla \psi|^2 \quad \forall \psi \in C^2_c. \]

Note that monotonicity $\rightarrow$ pointwise stability $\rightarrow$ stability.

In Section 2.1 we prove 0-Liouville theorems for (2.3) with specific nonlinearities of the form $e^u, u^p$ where $p > 1$ and $-u^{-p}$ where $p > 0$. Also applying a generalized Hardy inequality, we prove optimality of some results. This section is taken from [32].

In Section 2.2 we prove various $m$-Liouville theorems for any $0 \leq m < n$ for (2.3) with a general nonlinearity, possibly of sign changing nonlinearities. This section is taken from [50].

2.1 Specific nonlinearities

2.1.1 Introduction

In this section, we are interested in the existence versus non-existence of stable sub- and super-solutions of equations of the form
\[ -\text{div}(\gamma(x)\nabla u) = \lambda(x)f(u) \quad \text{in } \mathbb{R}^n, \]

where $f(u)$ is one of the following non-linearities: $e^u, u^p$ where $p > 1$ and $-u^{-p}$ where $p > 0$. Recall that we say that a solution $u$ of $-\Delta u = f(u)$ in $\mathbb{R}^n$ is stable provided
\[ \int f'(u)\psi^2 \leq \int |\nabla \psi|^2, \quad \forall \psi \in C^2_c, \]
where $C_c^2$ is the set of $C^2$ functions defined on $\mathbb{R}^n$ with compact support. Note that the stability of $u$ is just saying that the second variation at $u$ of the energy associated with the equation is non-negative. In our setting this becomes: We say a $C^2$ sub/super-solution $u$ of (2.3) is stable provided

$$\int \lambda f'(u)\psi^2 \leq \int \gamma |\nabla \psi|^2 \quad \forall \psi \in C_c^2. \quad (2.5)$$

**Remark 2.1.** Note that if $\gamma$ has enough integrability then it is immediate that if $u$ is a stable solution of (2.3) we have $\int \lambda f'(u) = 0$ (provided $f$ is increasing). To see this let $0 \leq \psi \leq 1$ be supported in a ball of radius $2R$ centered at the origin $(B_{2R})$ with $\psi = 1$ on $B_R$ and such that $|\nabla \psi| \leq \frac{C}{R}$ where $C > 0$ is independent of $R$. Putting this $\psi$ into (2.5) one obtains

$$\int_{B_R} \lambda f'(u) \leq \frac{C}{R^2} \int_{R<|x|<2R} \gamma,$$

and so if the right hand side goes to zero as $R \to \infty$ we have the desired result.

The existence versus non-existence of stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^n$ or $-\Delta u = g(x)f(u)$ in $\mathbb{R}^n$ is now quite well understood, see [23, 34, 39, 41, 43, 44, 46, 47, 65]. We remark that some of these results are examining the case where $\Delta$ is replaced with $\Delta_p$ (the $p$-Laplacian) and also in many cases the authors are interested in finite Morse index solutions or solutions which are stable outside a compact set. Much of the interest in these Liouville type theorems stems from the fact that the non-existence of a stable solution is related to the existence of a priori estimates for stable solutions of a related equation on a bounded domain.

In [77] equations similar to $-\Delta u = |x|^\alpha u^p$ where examined on the unit ball in $\mathbb{R}^n$ with zero Dirichlet boundary conditions. There it was shown that for $\alpha > 0$ that one can obtain positive solutions for $p$ supercritical with respect to Sobolev embedding and so one can view that the term $|x|^\alpha$ is restoring some compactness. A similar feature happens for equations of the form

$$-\Delta u = |x|^\alpha f(u) \quad \text{in } \mathbb{R}^n;$$

the value of $\alpha$ can vastly alter the existence versus non-existence of a stable solution, see [23, 40, 41, 43, 44].

We now come to our main results and for this we need to define a few quantities:
2.1. Specific nonlinearities

\[ I_G := R^{-4t-2} \int_{|x| < 2R} \frac{\gamma^{2t+1}}{\lambda^{2t}} dx, \]
\[ J_G := R^{-2t-1} \int_{|x| < 2R} \frac{\| \gamma \|^2t+1}{\lambda^{2t}} dx, \]
\[ I_L := R^{-2(2t+p-1)p-1} \int_{|x| < 2R} \left( \frac{\gamma^{p+2t-1}}{\lambda^{2t}} \right)^{\frac{1}{p-1}} dx, \]
\[ J_L := R^{-p+2t-1} \int_{|x| < 2R} \left( \frac{\| \gamma \|^{p+2t-1}}{\lambda^{2t}} \right)^{\frac{1}{p-1}} dx, \]
\[ I_M := R^{-2p+2t+1p+1} \int_{|x| < 2R} \left( \frac{\gamma^{p+2t+1}}{\lambda^{2t}} \right)^{\frac{1}{p+1}} dx, \]
\[ J_M := R^{-p+2t+1p+1} \int_{|x| < 2R} \left( \frac{\| \gamma \|^{p+2t+1}}{\lambda^{2t}} \right)^{\frac{1}{p+1}} dx. \]

The three equations we examine are

\[- \text{div}(\gamma \nabla u) = \lambda e^u \quad \text{in} \quad \mathbb{R}^n \quad (G), \]
\[- \text{div}(\gamma \nabla u) = \lambda u^p \quad \text{in} \quad \mathbb{R}^n \quad (L), \]
\[- \text{div}(\gamma \nabla u) = -\lambda u^{-p} \quad \text{in} \quad \mathbb{R}^n \quad (M), \]

and where we restrict \((L)\) to the case \(p > 1\) and \((M)\) to \(p > 0\). By solution we always mean a \(C^2\) solution. We now come to our main results in terms of abstract \(\gamma\) and \(\lambda\). We remark that our approach to non-existence of stable solutions is the approach due to Farina, see [39, 46, 47].

**Theorem 2.1.**

1. There is no stable sub-solution of \((G)\) if \(I_G, J_G \to 0\) as \(R \to \infty\) for some \(0 < t < 2\).

2. There is no positive stable sub-solution (super-solution) of \((L)\) if \(I_L, J_L \to 0\) as \(R \to \infty\) for some \(p - \sqrt{p(p-1)} < t < p + \sqrt{p(p-1)} \ (0 < t < \frac{1}{2})\).

3. There is no positive stable super-solution of \((M)\) if \(I_M, J_M \to 0\) as \(R \to \infty\) for some \(0 < t < p + \sqrt{p(p+1)}\).

If we assume that \(\gamma\) has some monotonicity we can do better. We will assume that the monotonicity conditions is satisfied for big \(x\) but really all ones needs is for it to be satisfied on a suitable sequence of annuli.
2.1. Specific nonlinearities

**Theorem 2.2.** 1. There is no stable sub-solution of \((G)\) with \(\nabla \gamma(x) \cdot x \leq 0\) for big \(x\) if \(I_G \to 0\) as \(R \to \infty\) for some \(0 < t < 2\).

2. There is no positive stable sub-solution of \((L)\) provided \(I_L \to 0\) as \(R \to \infty\) for either:

   - some \(1 \leq t < p + \sqrt{p(p-1)}\) and \(\nabla \gamma(x) \cdot x \leq 0\) for big \(x\), or
   - some \(p - \sqrt{p(p-1)} < t \leq 1\) and \(\nabla \gamma(x) \cdot x \geq 0\) for big \(x\).

There is no positive super-solution of \((M)\) provided \(I_M \to 0\) as \(R \to \infty\) for some \(0 < t < \frac{1}{2}\) and \(\nabla \gamma(x) \cdot x \leq 0\) for big \(x\).

3. There is no positive stable super-solution of \((M)\) provided \(I_M \to 0\) as \(R \to \infty\) for some \(0 < t < p + \sqrt{p(p+1)}\).

**Corollary 2.1.** Suppose \(\gamma \leq C\lambda\) for big \(x\), \(\lambda \in L^\infty\), \(\nabla \gamma(x) \cdot x \leq 0\) for big \(x\).

1. There is no stable sub-solution of \((G)\) if \(n \leq 9\).

2. There is no positive stable sub-solution of \((L)\) if

   \[
   n < 2 + \frac{4}{p-1} \left(p + \sqrt{p(p-1)}\right).
   \]

3. There is no positive stable super-solution of \((M)\) if

   \[
   n < 2 + \frac{4}{p+1} \left(p + \sqrt{p(p+1)}\right).
   \]

If one takes \(\gamma = \lambda = 1\) in the above corollary, the results obtained for \((G)\) and \((L)\), and for some values of \(p\) in \((M)\), are optimal, see [44, 46, 47].

We now drop all monotonicity conditions on \(\gamma\).

**Corollary 2.2.** Suppose \(\gamma \leq C\lambda\) for big \(x\), \(\lambda \in L^\infty\), \(|\nabla \gamma| \leq C\lambda\) for big \(x\).

1. There is no stable sub-solution of \((G)\) if \(n \leq 4\).

2. There is no positive stable sub-solution of \((L)\) if

   \[
   n < 1 + \frac{2}{p-1} \left(p + \sqrt{p(p-1)}\right).
   \]
3. There is no positive super-solution of \((M)\) if

\[
    n < 1 + \frac{2}{p+1} \left( p + \sqrt{p(p+1)} \right).
\]

Some of the conditions on \(\lambda\) and \(\gamma\) in Corollary 2.2 seem somewhat artificial. If we shift over to the advection equation (and we take \(\gamma = \lambda\) for simplicity)

\[-\Delta u + \nabla \gamma \cdot \nabla u = f(u),\]

the conditions on \(\gamma\) become: \(\gamma\) is bounded from below and has a bounded gradient.

In what follows we examine the case where \(\gamma(x) = (|x|^2 + 1)^{\frac{\alpha}{2}}\) and \(\lambda(x) = g(x)(|x|^2 + 1)^{\frac{\beta}{2}}\), where \(g(x)\) is positive except at say a point, smooth and where \(\lim_{|x| \to \infty} g(x) = C \in (0, \infty)\). For this class of weights we can essentially obtain optimal results.

**Theorem 2.3.** Take \(\gamma\) and \(\lambda\) as above.

1. If \(n + \alpha - 2 < 0\) then there is no stable sub-solution for \((G)\), \((L)\) (here we require it to be positive) and in the case of \((M)\) there is no positive stable super-solution. This case is the trivial case, see Remark 2.1.

**Assumption:** For the remaining cases we assume that \(n + \alpha - 2 > 0\).

2. If \(n + \alpha - 2 < 4(\beta - \alpha + 2)\) then there is no stable sub-solution for \((G)\).

3. If \(n + \alpha - 2 < \frac{2(\beta - \alpha + 2)}{p-1} \left( p + \sqrt{p(p-1)} \right)\) then there is no positive stable sub-solution of \((L)\).

4. If \(n + \alpha - 2 < \frac{2(\beta - \alpha + 2)}{p+1} \left( p + \sqrt{p(p+1)} \right)\) then there is no positive stable super-solution of \((M)\).

5. Further more 2,3,4 are optimal in the sense if \(n + \alpha - 2 > 0\) and the remaining inequality is not satisfied (and in addition we assume we don’t have equality in the inequality) then we can find a suitable function \(g(x)\) which satisfies the above properties and a stable sub/super-solution \(u\) for the appropriate equation.

**Remark 2.2.** Many of the above results can be extended to the case of equality in either the \(n + \alpha - 2 \geq 0\) and also the other inequality which depends on the equation we are examining. We omit the details because one cannot prove the results in a unified way.
2.1. Specific nonlinearities

In showing that an explicit solution is stable we will need the weighted Hardy inequality given in [25].

Lemma 2.1. Suppose $E > 0$ is a smooth function. Then one has

$$(\tau - \frac{1}{2})^2 \int E^{2r-2} |\nabla E|^2 \phi^2 + (\frac{1}{2} - \tau) \int (-\Delta E) E^{2r-1} \phi^2 \leq \int E^{2r} |\nabla \phi|^2,$$

for all $\phi \in C_\infty^\infty(\mathbb{R}^n)$ and $\tau \in \mathbb{R}.$

By picking an appropriate function $E$ this gives,

Corollary 2.3. For all $\phi \in C_\infty^\infty$ and $t, \alpha \in \mathbb{R}.$ We have

$$\int (1 + |x|^2)^\alpha |\nabla \phi|^2 \geq (t + \alpha)^2 \int |x|^2 (1 + |x|^2)^{-1+\frac{\alpha}{2}} \phi^2$$

$$+ (t + \alpha) \int (n - 2(t + 1) \frac{|x|^2}{1 + |x|^2})(1 + |x|^2)^{-1+\frac{\alpha}{2}} \phi^2.$$

2.1.2 Proof of main results

Proof of Theorem 2.1 (1). Suppose $u$ is a stable sub-solution of (G) with $I_G, J_G \to 0$ as $R \to \infty$ and let $0 \leq \phi \leq 1$ denote a smooth compactly supported function. Put $\psi := e^{tu} \phi$ into (2.5), where $0 < t < 2,$ to arrive at

$$\int \lambda e^{2tu} |\nabla u|^2 \phi^2 \leq t^2 \int \gamma e^{2tu} |\nabla \phi|^2 \phi^2$$

$$+ \int \gamma e^{2tu} |\nabla \phi|^2 + 2t \int \gamma e^{2tu} \phi \nabla u \cdot \nabla \phi.$$

Now multiply (G) by $e^{2tu} \phi^2$ and integrate by parts to arrive at

$$2t \int \gamma e^{2tu} |\nabla u|^2 \phi^2 \leq \int \lambda e^{(2t+1)u} \phi^2 - 2 \int \gamma e^{2tu} \phi \nabla u \cdot \nabla \phi,$$

and now if one equates like terms they arrive at

$$\frac{(2 - t)}{2} \int \lambda e^{(2t+1)u} \phi^2 \leq \int \gamma e^{2tu} \left( |\nabla \phi|^2 - \frac{\Delta \phi}{2} \right) dx$$

$$- \frac{1}{2} \int e^{2tu} \phi \nabla \gamma \cdot \nabla \phi. \quad (2.6)$$

Now substitute $\phi^m$ into this inequality for $\phi$ where $m$ is a big integer to obtain

$$\frac{(2 - t)}{2} \int \lambda e^{(2t+1)u} \phi^{2m} \leq C_m \int \gamma e^{2tu} \phi^{2m-2} \left( |\nabla \phi|^2 + \phi |\Delta \phi| \right) dx$$

$$- D_m \int e^{2tu} \phi^{2m-1} \nabla \gamma \cdot \nabla \phi \quad (2.7)$$
2.1. Specific nonlinearities

where $C_m$ and $D_m$ are positive constants just depending on $m$. We now estimate the terms on the right but we mention that when ones assume the appropriate monotonicity on $\gamma$ it is the last integral on the right which one is able to drop.

\[
\int \gamma e^{2tu} \phi^{2m-2} |\nabla \phi|^2 = \int \lambda^{\frac{2t}{2t+1}} e^{2tu} \phi^{2m-2} \frac{\gamma}{\lambda^{\frac{2t}{2t+1}}} |\nabla \phi|^2 \\
\leq \left( \int \lambda e^{(2t+1)u} \phi^{2m-2} \left( \frac{(2t+1)}{2t} \right) dx \right)^{\frac{2t}{2t+1}} \\
\left( \int \frac{\gamma^{2t+1}}{\lambda^{2t}} |\nabla \phi|^{2(2t+1)} \right)^{\frac{1}{2t+1}} .
\]

Now, for fixed $0 < t < 2$ we can take $m$ big enough so $(2m - 2) \frac{(2t+1)}{2t} \geq 2m$ and since $0 \leq \phi \leq 1$ this allows us to replace the power on $\phi$ in the first term on the right with $2m$ and hence we obtain

\[
\int \gamma e^{2tu} \phi^{2m-2} |\nabla \phi|^2 \leq \left( \int \lambda e^{(2t+1)u} \phi^{2m} dx \right)^{\frac{2t}{2t+1}} \left( \int \frac{\gamma^{2t+1}}{\lambda^{2t}} |\nabla \phi|^{2(2t+1)} \right)^{\frac{1}{2t+1}} .
\]  

(2.8)

We now take the test functions $\phi$ to be such that $0 \leq \phi \leq 1$ with $\phi$ supported in the ball $B_{2R}$ with $\phi = 1$ on $B_R$ and $|\nabla \phi| \leq \frac{C}{R}$ where $C > 0$ is independent of $R$. Putting this choice of $\phi$ we obtain

\[
\int \gamma e^{2tu} \phi^{2m-2} |\nabla \phi|^2 \leq \left( \int \lambda e^{(2t+1)u} \phi^{2m} dx \right)^{\frac{2t}{2t+1}} I_1^{\frac{1}{2t+1}} .
\]  

(2.9)

One similarly shows that

\[
\int \gamma e^{2tu} \phi^{2m-2} |\nabla \phi|^2 \leq \left( \int \lambda e^{(2t+1)u} \phi^{2m} \right)^{\frac{2t}{2t+1}} I_G^{\frac{1}{2t+1}} .
\]

(2.10)
2.1. Specific nonlinearities

We now estimate this last term. A similar argument using Hölder’s inequality shows that

\[ \int e^{2tu} \phi^{2m-1} |\nabla \gamma| |\nabla \phi| \leq \left( \int \lambda \phi^{2m} e^{(2t+1)u} \, dx \right)^{\frac{2t}{2t+1}} J_{G}^{\frac{1}{2t+1}}. \]

Combining the results gives that

\[ (2 - t) \left( \int \lambda e^{(2t+1)u} \phi^{2m} \, dx \right)^{\frac{1}{2t+1}} \leq I_{G}^{\frac{1}{2t+1}} + J_{G}^{\frac{1}{2t+1}}, \tag{2.11} \]

and now we send \( R \to \infty \) and use the fact that \( I_{G}, J_{G} \to 0 \) as \( R \to \infty \) to see that

\[ \int \lambda e^{(2t+1)u} = 0, \]

which is clearly a contradiction. Hence there is no stable sub-solution of \( G \).

(2). Suppose that \( u > 0 \) is a stable sub-solution (super-solution) of \( L \). Then a similar calculation as in (1) shows that for \( p - \sqrt{p(p-1)} < t < p + \sqrt{p(p-1)}, \ (0 < t < \frac{1}{2}) \) one has

\[
(p - \frac{t^2}{2t - 1}) \int \lambda u^{2t+p-1} \phi^{2m} \leq D_{m} \int \gamma u^{2t} \phi^{2(m-1)} (|\nabla \phi|^2 + \phi |\Delta \phi|) \\
+ C_{m} \frac{(1-t)}{2(2t - 1)} \int u^{2t} \phi^{2m-1} \nabla \gamma \cdot \nabla \phi. \tag{2.12}
\]

One now applies Hölder’s argument as in (1) but the terms \( I_{L} \) and \( J_{L} \) will appear on the right hand side of the resulting equation. This shift from a sub-solution to a super-solution depending on whether \( t > \frac{1}{2} \) or \( t < \frac{1}{2} \) is a result from the sign change of \( 2t - 1 \) at \( t = \frac{1}{2} \). We leave the details for the reader.

(3). This case is also similar to (1) and (2).

\[ \square \]

**Proof of Theorem 2.2** (1). Again we suppose there is a stable sub-solution \( u \) of \( G \). Our starting point is (2.7) and we wish to be able to drop the term

\[ -D_{m} \int e^{2tu} \phi^{2m-1} \nabla \gamma \cdot \nabla \phi, \]

from (2.7). We can choose \( \phi \) as in the proof of Theorem 2.1 but also such that \( \nabla \phi(x) = -C(x)x \) where \( C(x) \geq 0 \). So if we assume that \( \nabla \gamma \cdot x \leq 0 \)
2.1. Specific nonlinearities

for big $x$ then we see that this last term is non-positive and hence we can drop the term. The the proof is as before but now we only require that $\lim_{R \to \infty} I_G = 0$.

(2). Suppose that $u > 0$ is a stable sub-solution of $(L)$ and so (2.12) holds for all $p - \sqrt{p(p-1)} < t < p + \sqrt{p(p-1)}$. Now we wish to use monotonicity to drop the term from (2.12) involving the term $\nabla \gamma \cdot \nabla \phi$. $\phi$ is chosen the same as in (1) but here one notes that the co-efficient for this term changes sign at $t = 1$ and hence by restriction $t$ to the appropriate side of 1 (along with the above condition on $t$ and $\gamma$) we can drop the last term depending on which monotonicity we have and hence to obtain a contraction we only require that $\lim_{R \to \infty} I_L = 0$. The result for the non-existence of a stable super-solution is similar be here one restricts $0 < t < \frac{1}{2}$.

(3). The proof here is similar to (1) and (2) and we omit the details.

\[\square\]

**Proof of Corollary 2.1.** We suppose that $\gamma \leq C \lambda$ for big $x$, $\lambda \in L^\infty$, $\nabla \gamma(x) \cdot x \leq 0$ for big $x$.

(1). Since $\nabla \gamma \cdot x \leq 0$ for big $x$ we can apply Theorem 2.2 to show the non-existence of a stable solution to $(G)$. Note that with the above assumptions on $\omega_i$ we have that

$$I_G \leq CR^n R^{4t+2}.$$  

For $N \leq 9$ we can take $0 < t < 2$ but close enough to 2 so the right hand side goes to zero as $R \to \infty$.

Both (2) and (3) also follow directly from applying Theorem 2.2. Note that one can say more about (2) by taking the multiple cases as listed in Theorem 2.2 but we have choice to leave this to the reader.

\[\square\]

**Proof of Corollary 2.2.** Since we have no monotonicity conditions now we will need both $I$ and $J$ to go to zero to show the non-existence of a stable solution. Again the results are obtained immediately by applying Theorem 2.1 and we prefer to omit the details.

\[\square\]

**Proof of Theorem 2.3.** (1). If $n + \alpha - 2 < 0$ then using Remark 2.1 one easily sees there is no stable sub-solution of $(G)$ and $(L)$ (positive for $(L)$) or a positive stable super-solution of $(M)$. So we now assume that $n + \alpha - 2 > 0$. Note that the monotonicity of $\gamma$ changes when $\alpha$ changes sign and hence one would think that we need to consider separate cases if we hope to utilize the monotonicity results. But a computation shows that in fact $I$ and $J$ are just multiples of each other in all three cases so it suffices to show, say, that $\lim_{R \to \infty} I = 0$.  

14
2.1. Specific nonlinearities

(2). Note that for $R > 1$ one has

\[
I_G \leq \frac{C}{R^{4t+2}} \int_{R < |x| < 2R} |x|^{\alpha(2t+1)-2t\beta}
\]

\[
\leq \frac{C}{R^{4t+2}} R^{N+\alpha(2t+1)-2t\beta},
\]

and so to show the non-existence we want to find some $0 < t < 2$ such that

\[
4t + 2 > n + \alpha(2t+1)-2t\beta,
\]

which is equivalent to $2t(\beta-\alpha+2) > (n + \alpha - 2)$. Now recall that we are assuming that $0 < n + \alpha - 2 < 4(\beta - \alpha + 2)$ and hence we have the desired result by taking $t < 2$ but sufficiently close. The proof of the non-existence results for (3) and (4) are similar and we omit the details.

(5). We now assume that $n + \alpha - 2 > 0$. In showing the existence of stable sub/super-solutions we need to consider $\beta - \alpha + 2 < 0$ and $\beta - \alpha + 2 > 0$ separately.

- $(\beta - \alpha + 2 < 0)$ Here we take $u(x) = 0$ in the case of $(G)$ and $u = 1$ in the case of $(L)$ and $(M)$. In addition we take $g(x) = \varepsilon$. It is clear that in all cases $u$ is the appropriate sub or super-solution. The only thing one needs to check is the stability. In all cases this reduces to trying to show that we have

\[
\sigma \int (1 + |x|^2)^{\frac{n}{2} - 1} \phi^2 \leq \int (1 + |x|^2)^{\frac{n}{2}} |\nabla \phi|^2,
\]

for all $\phi \in C_c^\infty$ where $\sigma$ is some small positive constant; its either $\varepsilon$ or $p\varepsilon$ depending on which equation were are examining. To show this we use the result from Corollary 2.3 and we drop a few positive terms to arrive at

\[
\int (1 + |x|^2)^{\frac{n}{2}} |\nabla \phi|^2 \geq (t + \frac{\alpha}{2}) \int \left( n - 2(t+1) \frac{|x|^2}{1 + |x|^2} \right) (1 + |x|^2)^{-1 + \frac{n}{2}}
\]

which holds for all $\phi \in C_c^\infty$ and $t, \alpha \in \mathbb{R}$. Now, since $n + \alpha - 2 > 0$, we can choose $t$ such that $-\frac{\alpha}{2} < t < \frac{n-2}{2}$. So, the integrand function in the right hand side is positive and since for small enough $\sigma$ we have

\[
\sigma \leq (t + \frac{\alpha}{2})(n - 2(t+1) \frac{|x|^2}{1 + |x|^2}) \text{ for all } x \in \mathbb{R}^n
\]

we get stability.
• \((\beta - \alpha + 2 > 0)\) In the case of \((G)\) we take \(u(x) = - \frac{\beta - \alpha + 2}{2} \ln(1 + |x|^2)\) and \(g(x) := (\beta - \alpha + 2)(n + (\alpha - 2)\frac{|x|^2}{1 + |x|^2})\). By a computation one sees that \(u\) is a sub-solution of \((G)\) and hence we need now to only show the stability, which amounts to showing that
\[
\int \frac{g(x)\psi^2}{(1 + |x|^2)^{\frac{n}{2} + 1}} \leq \int \frac{|
abla \psi|^2}{(1 + |x|^2)^{\frac{n}{2}}},
\]
for all \(\psi \in C_c^\infty\). To show this we use Corollary 2.3. So we need to choose an appropriate \(t\) in \(-\frac{\alpha}{2} \leq t \leq \frac{n - 2}{2}\) such that for all \(x \in \mathbb{R}^n\) we have
\[
(\beta - \alpha + 2) \left( n + (\alpha - 2) \frac{|x|^2}{1 + |x|^2} \right) \leq (t + \frac{\alpha}{2})^2 \frac{|x|^2}{1 + |x|^2} + (t + \frac{\alpha}{2}) \left( n - 2(t + 1) \frac{|x|^2}{1 + |x|^2} \right).
\]
With a simple calculation one sees we need just to have
\[
(\beta - \alpha + 2) \leq (t + \frac{\alpha}{2}) \\
(\beta - \alpha + 2) (n + (\alpha - 2) \leq (t + \frac{\alpha}{2}) \left( n - t - 2 + \frac{\alpha}{2} \right).
\]
If one takes \(t = \frac{n - 2}{2}\) in the case where \(n \neq 2\) and \(t\) close to zero in the case for \(n = 2\) one easily sees the above inequalities both hold, after considering all the constraints on \(\alpha, \beta\) and \(n\).

We now consider the case of \((L)\). Here one takes \(g(x) := \frac{\beta - \alpha + 2}{p-1} (n + \frac{\alpha - 2}{p-1}) \frac{|x|^2}{1 + |x|^2} \) and \(u(x) = (1 + |x|^2)^{\frac{\beta - \alpha + 2}{2(p-1)}}\). Using essentially the same approach as in \((G)\) one shows that \(u\) is a stable sub-solution of \((L)\) with this choice of \(g\).

For the case of \((M)\) we take \(u(x) = (1 + |x|^2)^{\frac{\beta - \alpha + 2}{2(p+1)}}\) and \(g(x) := \frac{\beta - \alpha + 2}{p+1} (n + (\alpha - 2 + \frac{\beta - \alpha + 2}{p+1}) \frac{|x|^2}{1 + |x|^2} \).

\[
= 2.2 \quad \text{General nonlinearity}
\]

\[
2.2.1 \quad \text{Introduction and main results}
\]

In this section, we are consider the following equation
\[
- \text{div}(\gamma(x) \nabla u) = \lambda(x)f(u) \quad \text{in} \ \mathbb{R}^n,
\]
(2.13)
2.2. General nonlinearity

where \( f(u) \) is a general nonlinearity.

We assume that \( \gamma, \lambda \) are positive functions such that \( \frac{\nabla \gamma}{\gamma}, \frac{\lambda}{\gamma} \in L^\infty_{\text{loc}}(\mathbb{R}^n) \).

Let us first fix the following notation.

**Notation 1.** \( x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^s = \mathbb{R}^n \) for \( n = d + s \) and also \( x = (x'', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \).

Throughout this section we assume that \( \gamma(x) = \gamma(x') \) and \( \lambda(x) = \lambda(x') \) and

\[ \mathcal{G} := \{ g : \mathbb{R}^+ \to \mathbb{R}^+, \text{ is nondecreasing and} \int_1^\infty \frac{1}{rg(r)} \, dr = \infty \}. \]

Note that \( \mathcal{G} \) is not empty, e.g. \( g(r) = \log(1 + r) \) is in \( \mathcal{G} \). This class of nonlinearities has been defined by Karp [61, 62] and has also been used by Moschini [75].

Needless to say that \( m \)-Liouville theorems, especially for \( m = 0, 1 \), have been of great interest for authors. Let us mention some this results that are related to our main results.

As shown by Gilbarg and Serrin in [60] (see P. 324) a 0-Liouville theorem holds for bounded solutions of the linear equation

\[ -\Delta u + a(x) \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^n \]  \hspace{1cm} (2.14)

where \( n \geq 2 \) and \( a(x) = O(|x|^{-1}) \). If we replace the equality with the inequality \( \geq \) in (2.14), then it is easy to see that there exist nonconstant bounded solutions satisfying \( a(x) = O(|x|^{-1}) \). This implies a natural question that under what assumptions on \( a, b \) and solutions one can prove a 0-Liouville theorem for the nonlinear case, (2.3), with a general nonlinearity \( f \geq 0 \). We prove such a theorem for bounded pointwise stable solutions as Theorem 2.5. To prove this result we follow ideas given by Dupaigne-Farina in [39], where they considered (2.3) with \( a = 0 \) and \( b = 1 \).

In contrast to 0-Liouville theorems, usually 1-Liouville theorems are more challenging to prove. In fact, one of the most well known conjectures stated in elliptic PDE is the following 1-Liouville theorem.

**De Giorgi’s Conjecture (1978),** [36]. Suppose that \( u : \mathbb{R}^n \to [-1, 1] \) is a classical monotone solution of (2.1) for \( a = 0, b = 1 \) and \( f(u) = u - u^3 \).

Then for at least \( n \leq 8 \) equation (2.1) satisfies 1-Liouville theorem.

With the additional hypothesis that solutions are uniformly asymptotically convergent, the conjecture is known as the Gibbons’ conjecture. We
2.2. General nonlinearity

will give more details on the known results to this conjecture in the next chapter.

Berestycki, Hamel and Monneau, Theorem 2 in \[10\], have shown that a 1-Liouville theorem holds for uniformly asymptotically convergent solutions of \( (2.3) \) under the assumption that \( a \) is a constant vector, \( b(x) = b(x_n) \) is bounded and \( f \) is Lipschitz continuous on \([-1,1]\) satisfying \( f(\pm 1) = 0 \) and furthermore there exists \( \delta > 0 \) such that \( f \) is non-increasing on \([-1,-1+\delta]\) and on \([1-\delta,1]\).

However, a counterexample given by Bonnet-Hamel in \[15\] shows that this result no longer holds if we drop the "uniformly" assumption. In other words, they constructed a 2-dimensional monotone and asymptotically convergent solution such that for \( \alpha \in (0, \pi/2] \)

\[
\begin{align*}
  u(t \cos \theta, t \sin \theta) & \rightarrow -1 \text{ as } t \rightarrow \infty \text{ for } -\frac{\pi}{2} - \alpha < \theta < -\frac{\pi}{2} + \alpha \\
  u(t \cos \theta, t \sin \theta) & \rightarrow 1 \text{ as } t \rightarrow \infty \text{ for } -\frac{\pi}{2} + \alpha < \theta < \frac{3\pi}{2} - \alpha
\end{align*}
\]

for

\[
- \Delta u + C \partial_{x_2} u = f(u) \quad \text{in } \mathbb{R}^2 \tag{2.15}
\]

where \( C \) is a constant and for some particular \( f \) that satisfies the same conditions. The level sets of such a solution are parallel lines and cannot be 1-dimensional. Therefore, De Giorgi's conjecture does not hold for \( (2.15) \).

Note that this is a sharp result. Since when \( C = 0 \), it follows from a result of Ghoussoub and Gui \[53\] that \( (2.15) \) satisfies a 1-Liouville theorem.

Moreover, Berestycki, Hamel and Monneau, Theorem 3 in \[10\], have proved that the 1-Liouville theorem no longer holds if \( a \) is a non constant vector, even for uniformly asymptotically convergent solutions. More precisely, they proved that the following equation in two dimensions

\[
- \Delta u - a(x_1) \partial_{x_1} u = f(u) \quad \text{in } \mathbb{R}^2 \tag{2.16}
\]

admits both a solution depending on only \( x_2 \) and infinitely many nonplanar solutions, that is, solutions whose level sets are not parallel. The construction of nonplanar solutions is very technical and relies on the subsolution supresolution method. The assumption on \( a \) is

\[
\lim_{R \to \infty} \int_{-R}^{R} e^{-\int_{0}^{t} a(t') \, dt'} \, dx_1 < \infty. \tag{2.17}
\]

This assumption is used in defining a subsolution-supersolution in \((3.21)\) and \((3.9)\) in \[10\]. Note that \( a(x_1) \equiv C \) does not satisfy this condition. However,
2.2. General nonlinearity

either \( a(x_1) = tx_1 + s \) for \( t > 0 \) and any \( s \in \mathbb{R} \) or \( a(x_1) = t \tanh x_1 + s \) for \( t > |s| \) can be chosen to fulfill the assumption (2.17).

Note also that taking \( \lambda(x_1, x_2) = \gamma(x_1, x_2) = e^{\int_0^{x_1} a(s) ds} \) we can write (2.16) in the divergence form (2.3), and (2.17) will be

\[
\lim_{R \to \infty} \int_{-R}^{R} \frac{1}{\gamma(x_1)} \, dx_1 < \infty. \tag{2.18}
\]

As a conclusion, the Gibbons’ conjecture (and therefore De Giorgi’s conjecture) cannot be extended to (2.16).

In this section, we prove \( m \)-Liouville theorems for solutions of (2.3) for any \( 0 \leq m < n \) under various assumptions on \( \gamma \) and \( \lambda \). In particular, for equation (2.16) on \( \mathbb{R}^n \) up to dimension \( n \leq 4 \) we prove that a 2-Liouville theorem holds for monotone and asymptotically convergent solutions provided

\[
\lim_{R \to \infty} \int_{-R}^{R} \gamma(x_1) \, dx_1 < \infty. \tag{2.19}
\]

This could be seen as extension of De Giorgi’s conjecture to higher dimensional solutions. Here are the main results.

**Theorem 2.4.** Let \( u \) be a monotone and asymptotically convergent solution of (2.3). Assume that \( f \in C^1([-1, 1]) \) and \( F(t) \leq \min\{F(-1), F(1)\} \) for all \( t \in (-1, 1) \), where \( F' = f \). Moreover, suppose that either

1. \[
\int_{B_R} \gamma(x') \, dx' \leq C g(R), \tag{2.20}
\]

and \( n \leq d + 3 \) or

2. \[
\int_{B_R} \gamma(x') \, dx' \leq C R g(R), \tag{2.21}
\]

and \( n = d + 2 \)

for any \( g \in \mathcal{G} \). Then, (2.3) satisfies \((d+1)\)-Liouville theorem.

**Remark 2.1.** Note that the assumptions of Theorem 2.4 satisfy if the nonlinearity is the double-well potential \( f(t) = t - t^3 \) and therefore \( F(t) = -\frac{1}{4}(1-t^2)^2 \). For \( \lambda = \gamma = 1 \) and \( d = 0 \) this result is given by Ambrosio and Cabré [5] and Ghoussoub-Gui [53]. Also note that for \( d = 0 \), the righthand side of the above integral estimates is just a constant.
2.2. General nonlinearity

Now if we put some extra assumptions on the sign of nonlinearity $f$, we can prove a 0-Liouville theorem for equation (2.3) with a general nonlinearity.

**Theorem 2.5.** Let $u$ be a bounded pointwise stable solution for (2.3) and let either $0 \leq f(t)$ or $tf(t) \leq 0$ for all $t$ on the range of $u$. If either

1. \[ \int_{B_R} \gamma(x')dx' \leq Cg(R), \] \hspace{1cm} (2.22)

and $n \leq d + 4$ or

2. \[ \int_{B_R} \gamma(x')dx' \leq CRg(R), \] \hspace{1cm} (2.23)

and $n \leq d + 3$

for any $g \in \mathcal{G}$. Then, (2.3) satisfies 0-Liouville theorem.

Note that the above double-well potential nonlinearity $f(t) = t - t^3$ does not satisfy neither $0 \leq f(t)$ or $tf(t) \leq 0$. Before we start the proofs, we will give some examples that shows the application of Theorem 2.4.

**Some applications:**

In this part, we give examples to show the applications of the main results. We pick specific $\lambda$ and $\gamma$ such that assumptions of Theorem 2.4 and 2.5 are fulfilled. For these examples we assume that $F(t) \leq \min\{F(-1), F(1)\}$ for all $t \in (-1, 1)$, e.g. the double well potential $f(t) = t - t^3$ and $F(t) = -\frac{1}{4}(1 - t^2)^2$ satisfies this assumption.

**2-Liouville theorem**

**Corollary 2.4.** Assume that $d = 1$. Let $a \in L^\infty(\mathbb{R})$ and

\[ \int_{-R}^{R} e^{-\int_{0}^{t} a(t)dt}dx_1 \leq CR^{1-\epsilon}g(R) \] \hspace{1cm} (2.24)

Then monotone and asymptotically convergent solutions of

\[-\Delta u + a(x_1)u_{x_1} = f(u) \quad \text{in} \quad \mathbb{R}^n\]

satisfy 2-Liouville theorem for $n \leq 3 + \epsilon$. 20
2.2. General nonlinearity

Example 1. Take \( a(t) = \frac{2t}{1+t^2} \) or \( a(t) = \alpha \tanh t + \beta \) for \( \alpha > |\beta| \). Then, (2.24) holds for \( \epsilon = 1 \). Therefore,

\[-\Delta u + a(x_1)u_{x_1} = f(u) \quad \text{in } \mathbb{R}^n\]

satisfies 2-Liouville theorem for \( n \leq 4 \).

3-Liouville theorem

Corollary 2.5. Assume that \( d = 2 \). Let \( a_1, a_2 \in L^\infty(\mathbb{R}) \) and

\[
\int_{x_1^2 + x_2^2 \leq R^2} e^{-\int_0^{x_1} a_1(t)dt} e^{-\int_0^{x_2} a_2(t)dt} dx_1 dx_2 \leq CR^{2-\epsilon} g(R) \tag{2.25}
\]

Then monotone and asymptotically convergent solutions of

\[-\Delta u + a_1(x_1)u_{x_1} + a_2(x_2)u_{x_2} = f(u)\]

satisfy 3-Liouville theorem for \( n \leq 3 + \epsilon \).

Example 2. Take \( a_1(t) = a_2(t) = \frac{2t}{1+t^2} \). Then, (2.25) holds for \( \epsilon = 2 \). Therefore,

\[-\Delta u + \frac{2x_1}{1 + x_1^2} u_{x_1} + \frac{2x_2}{1 + x_2^2} u_{x_2} = f(u)\]

satisfies 3-Liouville theorem for \( n \leq 5 \).

2.2.2 Proofs

We start this section with the following 0-Liouville theorem that is given by Berestycki-Caffarelli-Nirenberg [9] and Ghoussoub-Gui [53] for bounded \( h\sigma \) and then improved by Ambrosio-Cabré [5] and Moschini [75].

Proposition 2.1. Let \( 0 < h \in L^\infty_{\text{loc}}(\mathbb{R}^n) \) and \( \sigma \in H^1_{\text{loc}}(\mathbb{R}^n) \). If

\[
\sigma \text{ div}(h(x)\nabla \sigma) \geq 0 \quad \text{in } \mathbb{R}^n, \tag{2.26}
\]

and

\[
\int_{B_{2R} \setminus B_R} h(x)\sigma^2 \leq CR^2 g(R), \quad \text{for } R > 1 \tag{2.27}
\]

for any \( g \in \mathcal{G} \) then \( \sigma \) is constant.
Remark 2.2. In two dimensions Proposition 2.1 is sharp in the sense that the following example

\[ h \equiv 1 \quad \text{and for } R_0 > e^{3/4} \text{ set } \sigma := \begin{cases} \log R_0 + \frac{r^2}{R_0^2} - \frac{r^4}{4R_0^4} - \frac{3}{4} & \text{for } r < R_0, \\ \log r & \text{for } r \geq R_0 \end{cases} \]

given in [75] (Remark 5.4) shows that this result does not hold if \( g(R) = \log^2(R) \). Note that \( \log^2(1 + r) \) is not in \( G \), however \( \log(1 + r) \) is in \( G \).

In Remark 5.5 [75], it’s asked to prove or disprove Proposition 2.1 for \( g(R) = R^{n-3} \) and \( 4 \leq n \leq 8 \). Note that for \( n \geq 9 \), Ghoussoub and Gui gave a counterexample for this choice of \( g \) as Proposition 2.6 in [54]. This counterexample is very well-constructed and even holds with the equality in (2.26), i.e. \( \sigma \text{ div}(h(x)\nabla \sigma) = 0 \).

Here we give an elementary example that shows for the supersolution case (inequality \( \geq \) holds in (2.26)) Proposition 2.1 for \( g(R) = R^{n-3} \) and \( 4 \leq n \leq 8 \) does not hold.

Let \( n \geq 2, \delta_n > 2 \). Set \( h(x) = (1 + |x|^2)^{-\frac{n+\delta_n-4}{2}} \) and \( \sigma(x) = (1 + |x|^2)^{\frac{\delta_n-2}{2}} \), then these functions are smooth and \( h \in L^\infty(\mathbb{R}^n) \). By a simple calculation one can see that (2.26) holds and moreover

\[ \int_{B_R} h(x)\sigma^2 \leq R^{\delta_n} \]

Also note that for \( \delta_n < n \) we have \( h\sigma^2 \in L^\infty(\mathbb{R}^n) \). Now, take \( \delta_n = n - 1 \) then \( R^{\delta_n} = R^2 g(R) \) for \( g(R) = R^{n-3} \). But \( \sigma \) is not a constant.

This means that to prove the De Giorgi’s conjecture in dimensions \( 4 \leq n \leq 8 \), using a counterpart of Proposition 2.1 for \( g(R) = R^{n-3} \), one needs to only assume the equality holds in (2.26).

Here we apply the given 0-Liouville theorem, i.e. Proposition 2.1, to prove a \((d + 1)\)-Liouville theorem. Note that using geometric inequalities, similar results are proved by Savin and Valdinoci [85] for \( \gamma \) and \( g \) to be constant.

Proposition 2.2. Let \( u \) be a monotone solution of (2.3). If there exists \( C(n,d) > 0 \) such that

\[ \int_{B_{2R}\setminus B_R} \gamma(x)|\nabla_{x'} u|^2 dx \leq CR^2 g(R), \]  

(2.28)

for any \( g \in G \). Then, (2.3) satisfies \((d + 1)\)-Liouville theorem.
Proof: Define \( \phi_i(x) := \frac{\partial u}{\partial x_i}(x), i = d + 1, \ldots, n \) for \( x \in \mathbb{R}^n \). So, \( \phi_i \) satisfies the following linearized equation

\[
- \text{div}(\gamma(x) \nabla \phi_i) = \lambda(x) f'(u) \phi_i.
\]

It’s straightforward to see that

\[
\text{div}(\gamma(x) \phi_i^2 \nabla \sigma_i) = 0, \quad \text{for} \quad i = d + 1, \ldots, n
\]

where \( \sigma_i := \frac{\phi_i}{\phi_n} \).

Note that \( \phi_i^2 \sigma_i^2 = \vert \partial_i u \vert^2 \) and from \((2.28)\), for all \( i = d + 1, \ldots, n \) we have

\[
\int_{B_{2R} \setminus B_R} \gamma(x) \phi_i^2 \sigma_i^2 \, dx = \int_{B_{2R} \setminus B_R} \gamma(x) \vert \partial_i u \vert^2 \, dx \\
\leq \int_{B_{2R} \setminus B_R} \gamma(x) \vert \nabla x' u \vert^2 \, dx \\
\leq CR^2 g(R).
\]

Applying Proposition 2.1 with \( h(x) = \gamma(x) \phi_n^2 \), we get that \( \sigma_i \) are all constant. Therefore, there exit \( C_i \) such that \( \sigma_i(x) = C_i \) on \( \mathbb{R}^n \). Note that \( C_n = 1 \).

From the definition of \( \sigma_i \) we get \( \frac{\partial u}{\partial x_i}(x) = C_i \frac{\partial u}{\partial x_n}(x) \) for all \( i = d + 1, \ldots, n - 1 \). Therefore, \( \nabla x' u(x) = \frac{\partial u}{\partial x_n}(x)(C_{d+1}, C_{d+2}, \ldots, C_{n-1}, 1) \). Since \( u \) is monotone, i.e. \( \frac{\partial u}{\partial x_n} > 0 \), we conclude that \( \nabla x' u(x) \) does not change sign for all \( x \in \mathbb{R}^n \). Also, note that \( u \) is constant along the following directions:

\[
(0, 0, \ldots, 0, 1, 0, \ldots, 0, -C_{d+1}), \ldots, \quad (0, 0, \ldots, 0, 1, 0, \ldots, 0, -C_{d+2}), \ldots,
\]

\[
(0, 0, \ldots, 0, 1, 0, \ldots, 0, -C_{d+1}), \ldots,
\]

Therefore, \( u \) is a function of \( (x', C \cdot x'') \) where \( C = (C_{d+1}, \ldots, C_{n-1}, 1) \).

Applying the above proposition we can prove the second theorem.

Proof of Theorem 2.5: Let \( u \) be a bounded pointwise stable solution of \((2.3)\). Since \( u \) is a pointwise stable solution, there exists \( v > 0 \) such that

\[
- \text{div}(\gamma(x) \nabla v) = \lambda(x) f'(u) v.
\]
2.2. General nonlinearity

It’s straightforward to see that

\[ \text{div}(\gamma(x)v^2\nabla \sigma_i) = 0, \quad \text{for} \quad i = d + 1, \ldots, n \]

where \( \sigma_i := \frac{\partial u}{\partial x_i}v \). Therefore, \((\sigma_i v)^2 \leq |\nabla u|^2\) and

\[ \int_{B_R} \gamma(x)v^2\sigma_i^2 \leq \int_{B_R} \gamma(x)|\nabla u|^2. \quad (2.29) \]

To apply Proposition 2.1 we need to find an upper bound for the right hand side of the above inequality. First, assume that \( f(u) \geq 0 \). Multiply both sides of (2.3) with \((u - ||u||_{\infty})\phi^2\) where \( 0 \leq \phi \leq 1 \) is a test function. Since \( \lambda(x)f(u)(u - ||u||_{\infty}) \leq 0 \) we have

\[ -\text{div}(\gamma(x')\nabla u)(u - ||u||_{\infty})\phi^2 \leq 0 \quad \text{in} \quad \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^s. \quad (2.30) \]

For the case that \( uf(u) \leq 0 \), multiply both sides of (2.3) with \( u\phi^2 \) and use the assumption \( \lambda(x)uf(u) \leq 0 \) to get a similar equation as (2.30)

\[ -u\text{div}(\gamma(x')\nabla u)\phi^2 \leq 0 \quad \text{in} \quad \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^s. \quad (2.31) \]

Now, integrate both sides of (2.30) and (2.31) and use the fact that \( u \) is bounded to get

\[ \int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2\phi^2 \leq C \int_{\mathbb{R}^n} \gamma(x')|\nabla u||\nabla \phi| \]

\[ \leq C \left( \int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2\phi^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \gamma(x')|\nabla \phi|^2 \right)^{\frac{1}{2}} \quad (2.32) \]

Now if we set \( \phi = 1 \) in \( B_R \) and \( \phi = 0 \) in \( \mathbb{R}^n \setminus B_{2R} \) where \( |\nabla \phi| \leq \frac{C}{R} \) we have

\[ \int_{B_R} \gamma(x)|\nabla u|^2 \leq CR^{-2} \int_{B_{2R}\setminus B_R} \gamma(x')dx \]

\[ \leq CR^{s-2} \int_{B_{2R}} \gamma(x')dx' \]

Let (2.22) hold in dimensions \( n \leq d + 4 \), then \( n - d - 2 = s - 2 \leq 2 \) and

\[ R^{s-2} \int_{B_{2R}} \gamma(x')dx' \leq CR^2g(R). \]

The same inequality holds if we assume that (2.23) and \( n \leq d + 3 \) hold. Therefore

\[ \int_{B_R} \gamma(x)|\nabla u|^2 \leq CR^2g(R). \]
2.2. General nonlinearity

From this and (2.29) one can see (2.27) holds. Now, applying Proposition 2.1 we obtain that all \( \sigma_i \) are constant and then by the same discussion as Proposition 2.2 we have \( u(x', x'') = w(x', a \cdot x'') \) such that \( |a| = 1 \). Note that \( w \) satisfies

\[
(w - ||w||_\infty) \nabla (w - ||w||_\infty) \geq 0 \quad \text{in} \quad \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R},
\]

where \( f(w) \geq 0 \) and similarly

\[
w \nabla (w - ||w||_\infty) \geq 0 \quad \text{in} \quad \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}, \tag{2.34}
\]

where \( w f(w) \geq 0 \). Since \( w \) is bounded and \( \int_{B_R} \gamma(x')dx' \leq CRg(R) \), if either (2.22) or (2.23) holds,

\[
\int_{B_R} \gamma(x)(w - ||w||_\infty)^2 dx \leq CR \int_{B_R} \gamma(x)dx' \leq R^2 g(R)
\]

Hence applying Proposition 2.1 again for (2.33) and (2.34) we obtain that \( w \) is constant.

Note that by the following elementary lemma we can have an \( L^2 \) upper bound for \( |\nabla u| \) in terms of \( \lambda \) and \( \gamma \). The following lemma holds for subsolutions of (2.3). By subsolution we mean "\( \leq \)" holds in (2.3).

**Lemma 2.2.** Let \( u \) be a bounded subsolution of (2.3) with any \( f \in C^1(\mathbb{R}) \). Then

\[
\int_{B_R} \gamma(x)|\nabla u|^2 dx \leq CR^d \int_{B_R} \{\lambda(x') + R^{-2} \gamma(x')\} dx', \tag{2.35}
\]

where the positive constant \( C \) is independent of \( R \).

**Proof:** Multiply both sides of (2.3) with \((||u||_\infty + u)\phi^2\) for a test function \( 0 \leq \phi \leq 1 \). Then, integrating by parts we get

\[
\int_{\mathbb{R}^n} \gamma(x) \nabla u \cdot \nabla (\phi^2 (||u||_\infty + u)) \leq \int_{\mathbb{R}^n} \lambda(x) f(u)(||u||_\infty + u)\phi^2
\]

We keep the gradient of \( u \) in right hand side and we get

\[
\int_{\mathbb{R}^n} \gamma(x)|\nabla u|^2 \phi^2 \leq \int_{\mathbb{R}^n} \lambda(x) f(u)(||u||_\infty + u)\phi^2 + 4||u||_\infty \int_{\mathbb{R}^n} \gamma(x)|\nabla u||\nabla \phi|\phi
\]
2.2. General nonlinearity

Now take \( C > 2 \max |f(u)||u|\infty \) and \( 0 < \epsilon < (4||u|\infty)^{-1} \). Applying the Young’s inequality, i.e. \( ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \), for the last term in right hand side of the above estimate we get

\[
(1 - 4||u|\infty\epsilon) \int_{\mathbb{R}^n} \gamma(x)|\nabla u|^2 \phi^2 \leq C \int_{\mathbb{R}^n} \lambda(x)\phi^2 + \frac{||u|\infty}{\epsilon} \int_{\mathbb{R}^n} \gamma(x)|\nabla \phi|^2.
\]

Set \( \phi \) to be a standard smooth test function such that \( \phi = 1 \) in \( B_R \) and \( \phi = 0 \) in \( \mathbb{R}^n \setminus B_{2R} \) with \( ||\nabla \phi||_{L^\infty(B_{2R})} < CR^{-1} \). This proves (2.35).

Now, assuming the monotonicity of solutions we get stronger upper bounds on the energy of the solutions. First, note that if \( u \in C^{2,\alpha} \) on the domain \( \Omega = B_1(y) \) is a bounded solution of

\[-\Delta u - \nabla \frac{\gamma}{\gamma} \cdot \nabla u = \frac{\lambda}{\gamma} f(u)\]

where \( \frac{\nabla \gamma}{\gamma}, \frac{\lambda}{\gamma} \in L^\infty(\mathbb{R}^n) \) and \( f \in C^1(\mathbb{R}) \). Then,

\[u \in W^{2,p}_{loc}(\mathbb{R}^n), \text{ for any } 1 < p < \infty.\]

Using the Sobolev embedding \( W^{2,p}(B_1(y)) \subset C^1\left(B_1(y)\right) \) for \( p > n \) and any \( y \in \mathbb{R}^n \), we have \( u \in C^1(\mathbb{R}^n) \) and

\[|\nabla u| \in L^\infty(\mathbb{R}^n).\]

**Lemma 2.3.** Let \( u \) be a bounded monotone solution of (2.3) for any \( f \in C^1(\mathbb{R}) \) and

\[
\lim_{x_n \to \infty} u(x'', x_n) = 1 \quad \forall x = (x'', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R},
\]

then

\[
E_R(u) \leq C \int_{\partial B_R} \gamma(x) dS(x), \quad (2.37)
\]

where the positive constant \( C \) is independent of \( R \) and \( E_R(u) \) is the energy functional defined by

\[
E_R(u) := \frac{1}{2} \int_{B_R} \gamma(x)|\nabla u|^2 dx - \int_{B_R} \lambda(x)(F(u) - F(1)) dx.
\]
2.2. General nonlinearity

**Proof:** Define \( u^t(x) = u(x'', x_n + t) \) for \( t \in \mathbb{R} \). Therefore \( u^t \) is a bounded solution of (2.3) with \( |\nabla u^t| \in L^\infty(\mathbb{R}^n) \). We prove (2.37) in two steps. Note that the shifted function \( u^t \) satisfies (2.3), i.e.,

\[-\text{div}(\gamma(x)\nabla u^t) = \lambda(x)f(u^t) \quad \text{in} \quad \mathbb{R}^n\]  

(2.38)

Moreover, the following monotonicity and decay conditions hold.

\[
\begin{aligned}
\partial_t u^t(x) &> 0 \quad \text{in} \quad \mathbb{R}^n, \\
\lim_{t \to \infty} u^t(x) & = 1 \quad \text{in} \quad \mathbb{R}^n.
\end{aligned}
\]  

(2.39)

**Step 1:** We claim that the following decay holds

\[
\lim_{t \to \infty} E_R(u^t) = 0.
\]

To prove the claim note that from (2.39), one can see

\[
\lim_{t \to \infty} \int_{B_R} \lambda(x)(F(u^t) - F(1)) dx \to 0.
\]

Therefore, we need to prove the following

\[
\lim_{t \to \infty} \int_{B_R} \gamma(x)|\nabla u^t|^2 dx \to 0.
\]

To do so, multiply both sides of (2.38) with \( u^t - 1 \), respectively and do integration by parts to end up with

\[
\int_{B_R} \gamma(x)|\nabla u^t|^2 - \int_{\partial B_R} \gamma(x)\partial_\nu u^t(u^t - 1) = \int_{B_R} \lambda(x)f(u^t)(u^t - 1),
\]

taking the limit of both sides as \( t \to \infty \) finishes the proof of Step 1.

**Step 2:** Now we claim that the following upper bound for energy holds

\[
E_R(u) \leq E_R(u^t) + M \int_{\partial B_R} (u^t - u) dS(x) \quad \text{for all} \quad t \in \mathbb{R}^+.
\]  

(2.40)

Differentiating the energy functional that is tested by \( u^t \) gives us the following

\[
\partial_t E_R(u^t) = \int_{B_R} \gamma(x)\nabla u^t \cdot \nabla(\partial_t u^t) - \int_{B_R} \lambda(x)f(u^t)\partial_t u^t.
\]  

(2.41)

Now, multiply (2.38) with \( \partial_t u^t \) and integrate to get

\[
\int_{B_R} \gamma(x)\nabla u^t \cdot \nabla(\partial_t u^t) - \int_{\partial B_R} \gamma(x)\partial_\nu u^t\partial_t u^t = \int_{B_R} \lambda(x)f(u^t)\partial_t u^t. \]  

(2.42)
2.2. General nonlinearity

Combine (2.42) and (2.41) to get
\[
\partial_t E_R(u^t) = \int_{\partial B_R} \gamma(x) \partial_\nu u^t \partial_t u^t.
\] (2.43)

Note that \(-\|\nabla u\|_\infty \leq \partial_\nu u^t \leq \|\nabla u\|_\infty\) and \(\partial_t u^t > 0\). Therefore,
\[
\partial_t E_R(u^t) \geq -\|\nabla u\|_\infty \int_{\partial B_R} \gamma(x) \partial_t u^t dS(x).
\] (2.44)

On the other hand,
\[
E_R(u) = E_R(u^t) - \int_0^t \partial_s E_R(u^s) ds,
\]
\[
\leq E_R(u^t) + \|\nabla u\|_\infty \int_0^t \int_{\partial B_R} \gamma(x) \partial_\nu u^s dS(x) ds
\]
\[
= E_R(u^t) + \|\nabla u\|_\infty \int_{\partial B_R} \gamma(x)(u^t - u) dS(x).
\] (2.45)

To finish the proof of the theorem just note that \(u(x) < u^t(x)\) for all \(x \in \mathbb{R}^n\) and \(t \in \mathbb{R}^+\). Moreover, by Step 1 we have \(\lim_{t \to \infty} E_R(u^t) = 0\). Therefore, (2.45) states
\[
E_R(u) \leq C \int_{\partial B_R} \gamma(x) dS(x).
\]

Now, we will use the following elementary inequality which compares the surface integral with the volume integral.

**Lemma 2.4.** Let \(s \geq 2\), \(d \geq 1\) and \(x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^s = \mathbb{R}^n\), and \(\gamma \in C^\infty(\mathbb{R}^d)\) be positive. Then
\[
\int_{\partial B_R} \gamma(x') dS(x) \leq CR^{s-1} \int_{B_R} \gamma(x') dx'
\]
where \(C\) is independent of \(R\).

**Proof:** For a general surface \(x_n = \phi(x_1, x_2, \cdots, x_{n-1})\), the surface area element is \(dA = (1 + |D\phi|^2)^{1/2} dx_1 \cdots dx_{n-1}\). For the sphere \(\phi(x_1, x_2, \cdots, x_{n-1}) = (R^2 - |x_1|^2 - |x_2|^2 - \cdots |x_{n-1}|^2)^{1/2}\) and therefore
\[
dA = (1 + |D\phi|^2)^{1/2} dx_1 \cdots dx_{n-1} = \frac{R}{\phi} dx_1 \cdots dx_{n-1}.
\]
2.2. General nonlinearity

Integrating out the \(x''\)-variable, we have

\[
\int_{\partial B_R} \gamma(x') dS(x) = \int_{B_R} \gamma(x') w(R, x') dx'
\]

for some weight function \(w(x') \geq 0\).

We now prove that

\[
w(R, x') = CR(R^2 - |x'|^2)^{\frac{s-2}{2}},
\]

which will obviously prove the lemma since \(w \leq CR^{s-1}\) whenever \(s \geq 2\).

Rewrite \(\phi = (\rho^2 - y^2)^{1/2}\), where \(\rho^2 = R^2 - |x'|^2\) and \(x'' = (y, x_n)\). The weight is then

\[
w(x') = \int_{|y| < \rho} R dy = \frac{R}{\rho}. \text{ surface area of } \partial B^s_p = \frac{R}{\rho} C_s \rho^{s-1} = C_s R \rho^{s-2},
\]

and we are done.

We are ready to see the proof of the first theorem.

**Proof of Theorem [2.4]**: Without loss of generality assume that \(F(-1) \geq F(1)\). Then from the assumptions we get \(F(u) - F(1) \leq 0\). From Lemma [2.3] we get

\[
\int_{B_R} \gamma(x') |\nabla u|^2 dx \leq C \int_{\partial B_R} \gamma(x') dS(x). \quad (2.46)
\]

Applying Lemma [2.4] we get

\[
\int_{B_R} \gamma(x') |\nabla u|^2 dx \leq C R^{s-1} \int_{B_R} \gamma(x') dx'. \quad (2.47)
\]

Note that from Proposition [2.2] we know if (2.28) satisfies then solutions are \(d + 1\)-dimensional. Therefore, we need

\[
\int_{B_R} \gamma(x') dx' \leq C R^{3-s} g(R), \quad (2.48)
\]

for any \(g \in G\) and \(3 \geq s = n - d \geq 2\). This finishes the proof. Note that if \(F(-1) < F(1)\), replace \(u(x'', x_n)\) with \(-u(x'', -x_n)\).

\(\square\)
2.3 Summary and conclusions

In this chapter, for the following nonuniformly elliptic equation

\[- \text{div}(\gamma(x)\nabla u) = \lambda(x)f(u) \quad x \in \mathbb{R}^n.\]

we proved 0-Liouville theorems for either a general nonlinearity \( f \in C^1(\mathbb{R}) \) or specific nonlinearities \( f(u) = e^u, u^p \) where \( p > 1 \) and \( -u^{-p} \) where \( p > 0 \) known as the Gelfand, Lane-Emden and negative exponent nonlinearities, respectively. For a specific class of weights \( \gamma(x) = (|x|+1)^\alpha \) and \( \lambda(x) = (|x|+1)^\beta g(x) \), where \( g(x) \) is a positive function with a finite limit at infinity, we showed that these results are optimal using various generalized Hardy inequalities.

On the other hand, for double-well potential nonlinearities we proved \( m \)-Liouville theorems for any \( 0 \leq m < n \). This can be seen as an extension of De Giorgi’s conjecture (1978) to higher dimensional solutions.
Chapter 3

One dimensional symmetry results for gradient systems

3.1 Introduction

As it was mentioned in Chapter 2 in 1978, Ennio De Giorgi proposed the following conjecture.

De Giorgi’s Conjecture (1978), [36]. Suppose that $u$ is an entire solution of the Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{on } \mathbb{R}^N \quad (3.1)$$

satisfying $|u(x)| \leq 1$, $\frac{\partial u}{\partial x_N}(x) > 0$ for $x = (x', x_N) \in \mathbb{R}^N$. Then, at least in dimensions $N \leq 8$ the level sets of $u$ must be hyperplanes, i.e. there exists $g \in C^2(\mathbb{R})$ such that $u(x) = g(ax' - x_N)$, for some fixed $a \in \mathbb{R}^{N-1}$.

The first positive result on the De Giorgi conjecture was established in 1997 by Ghoussoub and Gui [53] for dimension $N = 2$. Their proof used the following linear Liouville type theorem for elliptic equations in divergence form, which (only) holds in dimensions 1 and 2 ([7, 53]). If $\phi > 0$, then any solution $\sigma$ of

$$\text{div}(\phi^2 \nabla \sigma) = 0, \quad (3.2)$$

such that $\phi \sigma$ is bounded, is necessarily constant. This result is then applied to the ratio $\sigma := \frac{\partial u}{\partial x_2}$ to conclude in dimension 2. Ambrosio and Cabré [5] extended the result to dimension $N = 3$ by noting that for the linear Liouville theorem to hold, it suffices that

$$\int_{B_R} \phi^2 \sigma^2 \leq CR^2, \quad (3.3)$$

and by proving that any solution $u$ satisfying $\partial_N u > 0$ satisfies the energy estimate

$$\int_{B_R} \phi^2 \sigma^2 \leq CR^{N-1}. \quad (3.4)$$

31
3.1. Introduction

The conjecture remains open in dimensions $4 \leq N \leq 8$. However, Ghoussoub and Gui also showed in [54] that it is true for $N = 4$ or $N = 5$ for solutions that satisfy certain antisymmetry conditions, and Savin [84] established its validity for $4 \leq N \leq 8$ under the following additional natural hypothesis on the solution,

$$
\lim_{x_N \to \pm \infty} u(x', x_N) \to \pm 1. 
$$

Unlike the above proofs in dimensions $N \leq 5$, the proof of Savin is non-variational and does not use a Liouville type theorem. Our proofs below for analogous results corresponding to systems are more in the spirit of Ghoussoub-Gui and Ambrosio-Cabré, which mostly rely on notions of stability and on an interesting linear Liouville theorem that is suitable for non-linear elliptic systems of the following type:

$$
\Delta u = \nabla H(u) \text{ in } \mathbb{R}^N, 
$$

where $u : \mathbb{R}^N \to \mathbb{R}^m$, $H \in C^2(\mathbb{R}^m)$ and $\nabla H(u) = (H_{u_i}(u_1, u_2, \ldots, u_m))_i$. The notation $H_{u_i}$ is for the partial derivative $\partial H/\partial u_i$.

**Definition 3.1.** We shall say that the system (3.6) (or the non-linearity $H$) is orientable, if there exist nonzero functions $\theta_k \in C^1(\mathbb{R}^N)$, $k = 1, \ldots, m$, which do not change sign, such that for all $i, j$ with $1 \leq i < j \leq m$, we have

$$
H_{u_i u_j} \theta_i(x) \theta_j(x) \leq 0 \text{ for all } x \in \mathbb{R}^N. 
$$

Note that the above condition on the system means that none of the mixed derivative $H_{u_i u_j}$ changes sign. It is clear that a system consisting of two equations (i.e., $m = 2$) is always orientable as long as $H_{u_1 u_2}$ does not change sign. On the other hand, if $m = 3$, then the system (3.6) cannot be orientable if, for example, all three mixed derivatives $H_{u_i u_j}$ with $i < j$ are positive. This concept of ”orientable system” seems to be the right framework for dealing with systems of three or more equations. We shall see for example, that for such systems, the notions of variational stability and pointwise stability coincide.

We shall consider solutions of (3.6) whose components $(u_1, u_2, \ldots, u_m)$ are strictly monotone in the last variable $x_N$. However, and in contrast to the case of a single equation, the various components need not be all increasing (or decreasing). This leads us to the following definition of monotonicity.

**Definition 3.2.** Say that a solution $u = (u_k)_{k=1}^m$ of (3.6) is $H$-monotone if the following hold:
3.1. Introduction

1. For every $i \in \{1, \ldots, m\}$, $u_i$ is strictly monotone in the $x_N$-variable (i.e., $\partial_N u_i \neq 0$).

2. For $i < j$, we have

$$H_{u_i u_j} \partial_N u_i(x) \partial_N u_j(x) \leq 0 \text{ for all } x \in \mathbb{R}^N. \quad (3.8)$$

We shall then write $I \cup J = \{1, \ldots, m\}$, where

$$\partial_N u_i > 0 > \partial_N u_j \text{ for } i \in I \text{ and } j \in J. \quad (3.9)$$

It is clear that the mere existence of an $H$-monotone solution for (3.6) implies that the system is orientable, as it suffices to use $\eta_i = \partial_N u_i$. We now recall two notions of stability that will be considered in the sequel.

**Definition 3.3.** A solution $u$ of the system (3.6) on a domain $\Omega$ is said to be

(i) **stable**, if the second variation of the corresponding energy functional is nonnegative, i.e., if

$$\sum_i \int_\Omega |\nabla \zeta_i|^2 + \sum_{i,j} \int_\Omega H_{u_i u_j} \zeta_i \zeta_j \geq 0,$$

for every $\zeta_k \in C^1_c(\Omega), k = 1, \ldots, m$.

(ii) **pointwise stable**, if there exist $(\phi_i)_{i=1}^m$ in $C^1(\Omega)$ that do not change sign and $\lambda \geq 0$ such that

$$\Delta \phi_i = \sum_j H_{u_i u_j} \phi_j - \lambda \phi_i \text{ in } \Omega \text{ for all } i = 1, \ldots, m, \quad (3.11)$$

and $H_{u_i u_j} \phi_j \phi_i \leq 0$ for $1 \leq i < j \leq m$.

Note that a system that possesses a pointwise stable solution is necessarily orientable. On the other hand, we shall prove in Section 3.2 that for solutions of orientable systems, the two notions of stability are equivalent. The main focus of this paper is to provide some answers to the following conjecture, which extends the one by De Giorgi on Allen-Cahn equations to more general systems.

**Conjecture 1.** Suppose $u = (u_i)_{i=1}^m$ is an $H$-monotone bounded entire solutions of the system (3.6), then at least in dimensions $N \leq 8$, the level sets of each component $u_i$ must be a hyperplane.
3.2 A linear Liouville theorem for systems and first applications

The following Liouville theorem plays a key role in this paper. Note that for the case $m = 1$, this type of Liouville theorem was noted by Berestycki, Caffarelli and Nirenberg in [9] and used by Ghoussoub-Gui [53] and later by Ambrosio and Cabré [5] to prove the De Giorgi conjecture in dimensions two and three. Also, Ghoussoub and Gui in [54] used a slightly stronger version to show that the De Giorgi’s conjecture is true in dimensions four and five for a special class of solutions that satisfy an antisymmetry condition.

**Proposition 3.1.** Assume that $\phi_i \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ are such that $\phi_i^2 > 0$ a.e., and $\sigma_i \in H^1_{\text{loc}}(\mathbb{R}^N)$ satisfy

$$
\sum_{i=1}^{m} \int_{B_{2R}\setminus B_R} \phi_i^2 \sigma_i^2 \leq CR^2. 
$$

(3.12)

If $(\sigma_i)_{i=1}^m$ are solutions of

$$
\text{div}(\phi_i^2 \nabla \sigma_i) + \sum_{j=1}^{m} h_{i,j}(x)f(\sigma_i - \sigma_j) + k_i(x)g(\sigma_i) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{3.13}
$$

for $i = 1, \cdots, m$ where $0 \geq h_{i,j}, k_i \in L^1_{\text{loc}}(\mathbb{R}^N)$, $h_{i,j} = h_{j,i}$ and $f, g \in L^1_{\text{loc}}(\mathbb{R})$ are odd functions such that $f(t), g(t) \geq 0$ for $t \in \mathbb{R}^+$. Then, for all $i = 1, ..., m$, the functions $\sigma_i$ are constant.

**Proof:** Multiply both sides of (3.13) by $\sigma_i \zeta_R^2$ where $\zeta_R \in C^1_c(\mathbb{R}^N)$ with $0 \leq \zeta_R \leq 1$ being the following test function;

$$
\zeta_R(x) = \begin{cases} 
1, & \text{if } |x| < R, \\
0, & \text{if } |x| > 2R,
\end{cases}
$$

where $||\nabla \zeta_R||_\infty \leq R^{-1}$. By integrating by parts, we get

$$
\int_{B_{2R}} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 + 2 \int_{B_{2R}} \phi_i^2 \nabla \sigma_i \cdot \nabla \zeta_R \zeta_R \sigma_i 
- \int_{B_{2R}} \sum_{j=1}^{m} h_{i,j}(x)f(\sigma_i - \sigma_j)\sigma_i \zeta_R^2 - \int_{B_{2R}} k_i(x)g(\sigma_i)\sigma_i \zeta_R^2 = 0.
$$

Summing the above identity over $i$, we get

$$
\sum_{i=1}^{m} \int_{B_{2R}} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 = -2 \sum_{i=1}^{m} \int_{B_{2R}\setminus B_R} \phi_i^2 \nabla \sigma_i \cdot \nabla \zeta_R \zeta_R \sigma_i + \int_{B_{2R}} I(x)\zeta_R^2 
+ \int_{B_{2R}} J(x)\zeta_R^2.
$$

34
3.2. A linear Liouville theorem for systems and first applications

where

\[ I(x) := \sum_{i,j} h_{ij}(x)\sigma_i f(\sigma_i - \sigma_j) \quad \text{and} \quad J(x) := \sum_i k_i(x)g(\sigma_i)\sigma_i. \]

Note that

\[ I(x) = \sum_{i,j} h_{ij}(x)\sigma_i f(\sigma_i - \sigma_j) = \sum_{i<j} h_{i,j}\sigma_i f(\sigma_i - \sigma_j) + \sum_{i>j} h_{i,j}\sigma_i f(\sigma_j - \sigma_i) \quad \text{since} \quad h_{ij} = h_{ji}. \]

Since \( h_{i,j}(x) \leq 0 \) and \( (\sigma_i - \sigma_j)f(\sigma_i - \sigma_j) \geq 0 \) for all \( i,j \), we have \( I(x) \leq 0 \). Similarly, \( J(x) \leq 0 \). Therefore, for \( 0 < \epsilon < 1 \), we get Young’s inequality that

\[
\sum_{i=1}^{m} \int_{B_{2R}} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 \leq 2 \sum_{i=1}^{m} \int_{B_{2R}\setminus B_R} \phi_i^2 |\nabla \sigma_i||\nabla \zeta_R|\zeta_R \sigma_i \\
\leq \epsilon \sum_{i=1}^{m} \int_{B_{2R}\setminus B_R} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 \\
+ C\epsilon \sum_{i=1}^{m} \int_{B_{2R}\setminus B_R} (\sigma_i\phi_i)^2 |\nabla \zeta_R|^2. \quad (3.14)
\]

By assumption (3.12) we see that

\[
\sum_{i=1}^{m} \int_{\mathbb{R}^N} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 < \infty \quad \text{and} \quad \sum_{i=1}^{m} \int_{\mathbb{R}^N} (\sigma_i\phi_i)^2 |\nabla \zeta_R|^2 < \infty.
\]

Estimate (3.14) then yields

\[
\sum_{i=1}^{m} \int_{\mathbb{R}^N} \phi_i^2 |\nabla \sigma_i|^2 \zeta_R^2 = 0,
\]

which means that \( \sigma_i \) for all \( i = 1, ..., m \) must be constant. \( \square \)

Our first application is the following extension of a recent result by Berestycki, Lin, Wei and Zhao [11] who considered a system of \( m = 2 \) equations and the nonlinearity \( H(t,s) = \frac{1}{2}t^2s^2 \), which also appear as a limiting elliptic system arising in phase separation for multiple states Bose-Einstein condensates.
Theorem 3.1. Suppose the nonlinearity \( H \) satisfies the condition:

\[
u_i H u_i \geq 0 \quad \text{for all } 1 \leq i \leq m. \tag{3.15}\]

Then, any pointwise stable solution \( u \) of the system \( (3.6) \), which satisfies

\[
\sum_i \int_{B_2 \setminus B_R} u_i^2 \leq CR^4, \tag{3.16}
\]

is necessarily one-dimensional.

Proof: Note that we do not assume here that \( u \) is bounded solution. Multiply both sides of \( (3.6) \) with \( u \zeta^2 \) to get

\[
u_i \Delta u_i \zeta^2 = u_i H u_i^2 \zeta^2 \geq 0 \quad \text{in } \mathbb{R}^N. \tag{3.17}\]

An integration by parts yields,

\[
\int_{B_R} |\nabla u_i|^2 \zeta^2 \leq 2 \int_{B_R} |\nabla u_i||\nabla \zeta| u_i \zeta. \tag{3.18}\]

Now, use the same test function as in the proof of Proposition 3.1 to obtain

\[
\sum_i \int_{B_R} |\nabla u_i|^2 \leq CR^{-2} \sum_i \int_{B_2 \setminus B_R} u_i^2 \leq CR^2. \tag{3.19}\]

Since \( u \) is a pointwise stable solution of \( (3.6) \), there exist eigenfunctions \((\phi_i)_i\) such that

\[
\Delta \phi_i = \sum_j H_{u_i u_j} \phi_j - \lambda \phi_i \quad \text{in } \mathbb{R}^N, \tag{3.20}\]

where \( \phi_i \) does not change sign, \( H_{u_i u_j} \phi_i(x) \phi_j(x) \leq 0 \) and \( \lambda \geq 0 \). For any fixed \( \eta = (\eta',0) \in \mathbb{R}^{N-1} \times \{0\} \), define \( \psi_i := \nabla u_i \cdot \eta \) and observe that \( \psi_i \) satisfies the following equation

\[
\Delta \psi_i = \sum_j H_{u_i u_j} \psi_j \quad \text{in } \mathbb{R}^N. \tag{3.21}\]

It is straightforward to see that \( \sigma_i := \frac{\psi_i}{\phi_i} \) is a solution of system \( (3.13) \) with \( h_{i,j}(x) = H_{u_i u_j} \phi_i(x) \phi_j(x), k_i(x) = -\lambda \phi_i^2 \) and \( f, g \) equal the identity. Apply now Proposition 3.1 to deduce that \( \sigma_i \) is constant for every \( i = 1, \ldots, m \), which clearly yields our claim. \( \square \)
3.2. A linear Liouville theorem for systems and first applications

Remark 3.1. Note that in the case where \( m = 2 \) and \( H(t,s) = \frac{1}{2} t^2 s^2 \), the above theorem yields that any positive solution \((u,v)\) of the corresponding system (3.6), which satisfies the growth assumption \( u(x), v(x) = O(|x|^k) \) and such that \( \partial_N u > 0, \partial_N v < 0 \), is necessarily one-dimensional provided \( N \leq 4 - 2k \). Note that Noris et al. [78] have recently shown that a solution such that \( u(x), v(x) \leq C(1 + |x|^\alpha) \) is necessarily constant for \( \alpha \in (0,1) \).

We can also deduce the following Liouville theorem for bounded solutions of (3.6) with general non-positive nonlinearities. The approach to this Liouville theorem seems to be new, even for single equations. It is worth comparing to the general results of Nedev [76] and Cabré [16] regarding the regularity of stable solutions of semilinear equations with general nonlinearities up to dimension four.

Theorem 3.2. Suppose \( H \) is a nonlinearity verifying
\[
H_{u_i} \leq 0 \quad \text{for all } i = 1, \ldots, m. \tag{3.22}
\]
If the dimension \( N \leq 4 \), then any bounded pointwise stable solution of the system (3.6) is necessarily constant.

Proof: Multiply both sides of system (3.6) with \((u_i - \|u_i\|_\infty)\zeta^2\). Since \( H_{u_i}(u_i - \|u_i\|_\infty) \geq 0 \) we have
\[
\Delta u_i(u_i - \|u_i\|_\infty)\zeta^2 \geq 0 \quad \text{in } \mathbb{R}^N. \tag{3.23}
\]
After an integration by parts, we end up with
\[
\int_{B_R} |\nabla u_i|^2 \zeta^2 \leq 2 \int_{B_R} |\nabla u_i| |\nabla \zeta| (\|u_i\|_\infty - u_i) \zeta \quad \text{for all } 1 \leq i \leq m. \tag{3.24}
\]
Using Young’s inequality and adding we get
\[
\sum_i \int_{B_R} |\nabla u_i|^2 \leq R^{N-2}. \tag{3.25}
\]
As in the preceding theorem, one can apply Proposition 3.1 to quotients of partial derivatives to obtain that each \( u_i \) is one dimensional solutions as long as \( N \leq 4 \). Note now that \( u_i \) is a bounded solution for (3.23) in dimension one, and the corresponding decay estimate (3.25) now implies that \( u_i \) must be constant for all \( 1 \leq i \leq m \).

We now show that stability and pointwise stability are equivalent for solutions of orientable elliptic systems.
3.2. A linear Liouville theorem for systems and first applications

Lemma 3.1. A $C^2$-function is a pointwise stable solution of the system (3.6) if and only if it is a stable solution and the system is orientable.

Proof: Assume first that $u$ is a pointwise stable solution for (3.6). It is clear that the system is then obviously orientable. In order to show that $u$ is a stable solution, we consider test functions $\zeta_i \in C^1_c(\mathbb{R}^N)$ and multiply both sides of (3.11) with $\frac{\zeta_i^2}{\phi_i}$ to obtain

$$-\int_{\mathbb{R}^N} |\nabla \phi_i|^2 \frac{\zeta_i^2}{\phi_i^2} + 2 \int_{\mathbb{R}^N} \nabla \phi_i \cdot \nabla \zeta_i \frac{\zeta_i}{\phi_i} + \sum_j \int_{\mathbb{R}^N} H_{u_iu_j} \frac{\phi_j}{\phi_i} \zeta_i^2 - \lambda \zeta_i^2 = 0.$$ 

By applying Young’s inequality for the first two terms and taking sums, we get

$$\sum_i \int_{\mathbb{R}^N} |\nabla \zeta_i|^2 + \sum_{i,j} \int_{\mathbb{R}^N} H_{u_iu_j} \frac{\phi_j}{\phi_i} \zeta_i^2 \geq \lambda \sum_i \int_{\mathbb{R}^N} \zeta_i^2 \geq 0.$$ 

Note now that

$$\sum_{i,j} H_{u_iu_j} \frac{\phi_j}{\phi_i} \zeta_i^2 = \sum_i H_{u_iu_i} \zeta_i^2 + \sum_{i \neq j} H_{u_iu_j} \frac{\phi_j}{\phi_i} \zeta_i^2$$

$$= \sum_i H_{u_iu_i} \zeta_i^2 + \sum_{i < j} H_{u_iu_j} \frac{\phi_j}{\phi_i} \zeta_i^2 + \sum_{i > j} H_{u_iu_j} \frac{\phi_i}{\phi_j} \zeta_i^2$$

$$= \sum_i H_{u_iu_i} \zeta_i^2 + \sum_{i < j} H_{u_iu_j} (\phi_i \phi_j)^{-1} (\phi_j^2 \zeta_i^2 + \phi_i^2 \zeta_j^2)$$

$$\leq \sum_i H_{u_iu_i} \zeta_i^2 + 2 \sum_{i < j} H_{u_iu_j} \zeta_i \zeta_j \text{ since } H_{u_iu_j} (\phi_i \phi_j)^{-1} \leq 0$$

$$= \sum_{i,j} H_{u_iu_j} \zeta_i \zeta_j,$$

which finishes the proof.

For the reverse implication, we assume the system orientable and consider a stable solution $u$. We shall follow ideas of Ghoussoub-Gui in [53] (see also Berestycki-Caffarelli-Nirenberg in [9]) to show that $u$ is pointwise stable.

Define for each $R > 0$,

$$\lambda_1(R) := \min_{(\zeta_i)_{i=1}^n \in H_0^1(B_R(0)) \setminus \{0\}} I_R \left( (\zeta_i)_{i=1}^n \right),$$

(3.26)
3.2. A linear Liouville theorem for systems and first applications

where

\[ I_R((\zeta_i)_{i=1}^m) := \left\{ \sum_i \int_{B_R(0)} |\nabla \zeta_i|^2 + \sum_{i,j} \int_{B_R(0)} H_{u_i,u_j} \zeta_i \zeta_j, \sum_i \int_{B_R(0)} \zeta_i^2 = 1 \right\}. \]

Since \( u \) is a stable solution, we have that \( \lambda_1(R) \geq 0 \) and there exist eigenfunctions \( \zeta_i^R \) such that

\[
\begin{align*}
\Delta \zeta_i^R &= \sum_j H_{u_i,u_j} \zeta_j^R - \lambda_1(R) \zeta_i^R, & \text{if } |x| < R, \\
\zeta_i^R &= 0, & \text{if } |x| = R. 
\end{align*}
\]

(3.27)

Since the system is orientable, there exists \((\theta_k)_{k=1}^m\) such that \( H_{u_i,u_j} \theta_i \theta_j \leq 0 \).

We can then use the signs of the \( \theta_k \)'s to assign signs for the eigenfunctions \((\zeta_k^R)_{k}\) so that they satisfy

\[
\sum_{j \neq i} \text{sgn}(H_{u_i,u_j})H_{u_i,u_j} \zeta_j^R - \lambda_1(R) \zeta_i^R \leq 0.
\]

(3.28)

For that it suffices to replace \( \zeta_i^R \) with \( \text{sgn}(\theta_i)|\zeta_i^R| \) if need be. We can also normalize them so that

\[
\sum_k |\zeta_k^R(0)| = 1.
\]

(3.29)

Note that \( \lambda_1(R) \downarrow \lambda_1 \geq 0 \) as \( R \to \infty \). Define \( \chi_i^R := \text{sgn}(\zeta_i^R)\zeta_i^R \) and multiply system (3.27) with \( \text{sgn}(\zeta_i^R) \) to get that

\[
\begin{align*}
\Delta \chi_i^R &= H_{u_i,u_i} \chi_i^R - \sum_{j \neq i} \text{sgn}(H_{u_i,u_j})H_{u_i,u_j} \chi_j^R - \lambda_1(R) \chi_i^R, & \text{if } |x| < R, \\
\chi_i^R &= 0, & \text{if } |x| = R. 
\end{align*}
\]

(3.30)

Note that to get (3.30) we have used (3.28), i.e.,

\[
\text{sgn}(\zeta_i^R) = -\text{sgn}(H_{u_i,u_j})\text{sgn}(\zeta_j^R).
\]

Since now \( \chi_i^R \) is a nonnegative solution for (3.30), Harnack’s inequality yields that for any compact subset \( K \), \( \max_K |\chi_i^R| \leq C(K) \min_K |\chi_i^R| \) for all \( i = 1, \ldots, m \) with the latter constant being independent of \( \chi_i^R \). Standard elliptic estimates also yield that the family \((\chi_i^R)_R\) have also uniformly bounded derivatives on compact sets. It follows that for a subsequence \((R_k)_k\) going to infinity, \((\chi_i^R)_k\) converges in \( C^2_{\text{loc}}(\mathbb{R}^N) \) to some \( \chi_i \in C^2(\mathbb{R}^N) \) and that \( \chi_i \geq 0 \). From (3.30) we see that \( \chi_i \) satisfies

\[
\Delta \chi_i = H_{u_i,u_i} \chi_i - \sum_{j \neq i} \text{sgn}(H_{u_i,u_j})H_{u_i,u_j} \chi_j - \lambda_1 \chi_i \leq (H_{u_i,u_i} - \lambda_1) \chi_i
\]

(3.31)

\[ \leq (H_{u_i,u_i} - \lambda_1) \chi_i \]
3.3. De Giorgi type results

Since \( \chi_i \geq 0 \) and \( H_{u_i u_j} \) is bounded, the strong maximum principle yields that either \( \chi_i = 0 \) or \( \chi_i > 0 \) in \( \mathbb{R}^N \). If now \( \chi_i = 0 \), then from (3.31) we have \( \sum_{i \neq j} \text{sgn}(H_{u_i u_j}) H_{u_i u_j} \chi_j = 0 \) which means \( \chi_j = 0 \) if \( j \neq i \), which contradicts (3.29). It follows that \( \chi_i > 0 \) for all \( i = 1, \ldots, m \). Set now \( \phi_i := \text{sgn}(\theta_i) \chi_i \) for \( i = 1, \ldots, m \) and observe that \( (\phi_i) \) satisfy (3.11) and that \( H_{u_i u_j} \phi_j \phi_i \leq 0 \) for \( i < j \), which means that \( u \) is a pointwise stable solution. \( \square \)

3.3 De Giorgi type results

We first establish a geometric Poincaré inequality for stable solutions of system (3.6), which will enable us to get not only De Giorgi type results but also certain rigidity properties on the gradient of the solutions.

**Theorem 3.3.** Assume that \( m, N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) is an open set. Then, for any \( \eta = (\eta_1, \ldots, \eta_m) \in C^1_c(\Omega) \), the following inequality holds for any classical stable solution \( u \in C^2(\Omega) \) of (3.6)

\[
\sum_i \int_\Omega |\nabla u_i|^2 |\nabla \eta_i|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} (|\nabla u_i|^2 A_i^2 + |\nabla_T|\nabla u_i|^2) \eta_i^2 \\
+ \sum_{i \neq j} \int_\Omega (\nabla u_i \cdot \nabla u_j \eta_i^2 - |\nabla u_i||\nabla u_j| \eta_i \eta_j) H_{u_i u_j}, \tag{3.32}
\]

where \( \nabla_T \) stands for the tangential gradient along a given level set of \( u_i \) and \( A_i^2 \) for the sum of the squares of the principal curvatures of such a level set.

**Proof:** Let \( \eta = (\eta_1, \ldots, \eta_m) \) and \( \eta_i \in C^1_c(\Omega) \). Test the stability inequality (3.10) with \( \zeta_i = |\nabla u_i| \eta_i \) to get

\[
0 \leq \sum_i \int_\Omega |\nabla (|\nabla u_i| \eta_i)|^2 + \sum_{i,j} \int_\Omega H_{u_i u_j} |\nabla u_i||\nabla u_j| \eta_i \eta_j \\
= \sum_i \int_\Omega |\nabla |\nabla u_i||^2 \eta_i^2 + \sum_i \int_\Omega |\nabla \eta_i|^2 |\nabla u_i|^2 + \frac{1}{2} \sum_i \int_{\mathbb{R}^N} \nabla |\nabla u_i|^2 \cdot \nabla \eta_i^2 \\
+ \sum_i \int_\Omega H_{u_i u_i} |\nabla u_i|^2 \eta_i^2 + \sum_{i \neq j} \int_\Omega H_{u_i u_j} |\nabla u_i||\nabla u_j| \eta_i \eta_j. \tag{3.33}
\]

Differentiate the \( i^{th} \) equation of (3.6) with respect to \( x_k \) for each \( i = 1, 2, \ldots, m \) and multiply with \( \partial_k u \) to get

\[
\partial_k u_i \Delta \partial_k u_i = \sum_j H_{u_i u_j} \partial_k u_j \partial_k u_i = H_{u_i u_j} |\partial_k u_i|^2 + \sum_{j \neq i} H_{u_i u_j} \partial_k u_i \partial_k u_j.
\]
Multiply both sides with $\eta_i^2$ and integrate by parts to obtain
\[
\int_{\Omega} H_{u_{i,j}} |\partial_k u_i|^2 \eta_i^2 = \int_{\Omega} \partial_k u_i \Delta \partial_k u_i \eta_i^2 - \sum_{j \neq i} \int_{\mathbb{R}^N} H_{u_{i,j}} \partial_k u_i \partial_k u_j \eta_i^2
\]
\[
= - \int_{\Omega} |\nabla \partial_k u_i|^2 \eta_i^2 - \frac{1}{2} \int_{\Omega} |\nabla u_i|^2 \cdot \nabla \eta_i^2
\]
\[
- \sum_{j \neq i} \int_{\Omega} H_{u_{i,j}} \partial_k u_i \partial_k u_j \eta_i^2.
\]

By summing over the index $k$, we obtain
\[
\int_{\Omega} H_{u_{i,j}} |\nabla u_i|^2 \eta_i^2 = - \sum_k \int_{\Omega} |\nabla \partial_k u_i|^2 \eta_i^2 - \frac{1}{2} \int_{\Omega} |\nabla u_i|^2 \cdot \nabla \eta_i^2
\]
\[
- \sum_k \sum_{j \neq i} \int_{\Omega} H_{u_{i,j}} \partial_k u_i \partial_k u_j \eta_i^2.
\]

Combine (3.33) and (5.29) to get
\[
\sum_i \int_{\Omega} |\nabla \eta_i|^2 |\nabla u_i|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} \left( \sum_k |\nabla \partial_k u_i|^2 - |\nabla \nabla u_i|^2 \right) \eta_i^2
\]
\[
\quad \quad \quad + \sum_{i \neq j} \int_{\Omega} (|\nabla u_i| \cdot |\nabla u_j| \eta_i^2 - |\nabla u_i||\nabla u_j| \eta_i \eta_j) H_{u_{i,j}}.
\]

According to formula (2.1) given in [93], the following geometric identity between the tangential gradients and curvatures holds. For any $w \in C^2(\Omega)$
\[
\sum_{k=1}^{N} |\nabla \partial_k w|^2 - |\nabla |\nabla w||^2
\]
\[
= \left\{ \begin{array}{ll}
|\nabla w|^2 (\sum_{l=1}^{N-1} \kappa_l^2) + |\nabla_T|\nabla w|^2 & \text{for } x \in \{|\nabla w| > 0 \cap \Omega\} \\
0 & \text{for } x \in \{|\nabla w| = 0 \cap \Omega\},
\end{array} \right.
\]

where $\kappa_l$ are the principal curvatures of the level set of $w$ at $x$ and $\nabla_T$ denotes the orthogonal projection of the gradient along this level set. In light of this formula, we finally get (3.32).

**Remark 3.1.** Note that for the case of $m = 1$ the use of (3.35) and of $\zeta = |\nabla u|\eta$ in the stability (or semi-stability) condition (3.10) was first exploited by Sternberg and Zumbrun [93] to study semilinear phase transitions.
problems. Later on, Farina, Sciunzi, and Valdinoci [48] used it to reprove the De Giorgi’s conjecture in dimension two, and Cabré used it (see Proposition 2.2 in [16]) to prove the boundedness of extremal solutions of semilinear elliptic equations with Dirichlet boundary conditions on a convex domain up to dimension four.

Here is an application of the above geometric Poincaré inequality for stable solutions of (3.6).

**Theorem 3.4.** Any bounded stable solution \( u \) of an orientable system (3.6) in \( \mathbb{R}^2 \) is one-dimensional. Moreover, if \( H_{u_i u_j} \) is not identically zero, then for \( i \neq j \),

\[
\nabla u_i = C_{i,j} \nabla u_j \text{ for all } x \in \mathbb{R}^2,
\]

(3.36)

where \( C_{i,j} \) are constants whose sign is opposite to the one of \( H_{u_i u_j} \).

**Proof:** Fix the following standard test function

\[
\chi(x) := \begin{cases} 
\frac{1}{2}, & \text{if } |x| \leq \sqrt{R}, \\
\frac{\log \frac{R}{|x|}}{\log R}, & \text{if } \sqrt{R} < |x| < R, \\
0, & \text{if } |x| \geq R.
\end{cases}
\]

Since the system (3.6) is orientable, there exist nonzero functions \( \theta_k \in C^1(\mathbb{R}^N), k = 1, \cdots, m \), which do not change sign such that

\[
H_{u_i u_j} \theta_i \theta_j \leq 0, \quad \text{for all } i, j \in \{1, \cdots, m\} \text{ and } i < j.
\]

(3.37)

Consider \( r_k := \text{sgn}(\theta_k) \chi \) for \( 1 \leq k \leq m \), where again \( \text{sgn}(x) \) is the Sign function. The geometric Poincaré inequality (3.32) yields

\[
\int_{B_R \setminus B_{\sqrt{R}}} \sum_i |\nabla u_i|^2 |\nabla \chi|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} \left( |\nabla u_i|^2 \kappa_i^2 + |\nabla \chi| |\nabla u_i|^2 \right) \chi^2
\]

\[
+ \sum_{i \neq j} \int_{\mathbb{R}^N} (\nabla u_i \cdot \nabla u_j - \text{sgn}(\theta_i) \text{sgn}(\theta_j) |\nabla u_i||\nabla u_j|) H_{u_i u_j} \chi^2
\]

\[
= I_1 + I_2.
\]

(3.38)

Note that \( I_1 \) is clearly nonnegative. Moreover, (3.37) yields that

\[
H_{u_i u_j} \text{sgn}(\theta_i) \text{sgn}(\theta_j) \leq 0
\]
for all $i < j$, and therefore, $I_2$ can be written as

$$I_2 = \sum_{i \neq j} \int_{\mathbb{R}^N} (\text{sgn}(H_{u_i u_j}) \nabla u_i \cdot \nabla u_j + \| \nabla u_i \| \text{sgn}(H_{u_i u_j}) \chi^2),$$

which is also nonnegative.

On the other hand, since

$$\int_{B_R \backslash B_{\sqrt{R}}} \sum_i \| \nabla u_i \|^2 \| \nabla \chi \|^2 \leq C \begin{cases} \frac{1}{\log R}, & \text{if } N = 2, \\ \frac{1}{R^{N-2} + R(N-2)/2}, & \text{if } N \neq 2, \end{cases}$$

one can see that in dimension two the left hand side of (3.38) goes to zero as $R \to \infty$. Since $I_1 = 0$, one concludes that all $u_i$ for $i = 1, \ldots, m$ are one-dimensional and from the fact that $I_2 = 0$, provided $H_{u_i u_j}$ is not identically zero, we obtain that for all $x \in \mathbb{R}^2$,

$$-\text{sgn}(H_{u_i u_j}) \nabla u_i \cdot \nabla u_j = \| \nabla u_i \| \| \nabla u_j \|,$$

which completes the proof of the theorem.

Now, we are ready to state and prove the main result of this chapter.

**Theorem 3.5.** Conjecture (1) holds for $N \leq 3$.

**Proof:** Let again $\phi_i := \partial_N u_i$ and $\psi_i := \nabla u_i \cdot \eta$ for any fixed $\eta = (\eta', 0) \in \mathbb{R}^{N-1} \times \{0\}$ in such a way that $\sigma_i := \frac{\psi_i}{\phi_i}$ is a solution of system (3.13) for $h_{i,j}(x) = H_{u_i u_j} \phi_i(x) \phi_j(x)$ and $f$ to be the identity. Since $\| \nabla u_i \| \in L^\infty(\mathbb{R}^N)$, we have $\| \phi_i \sigma_i \|_{L^\infty(\mathbb{R}^N)} < \infty$.

In dimension $N = 2$, assumption (3.12) holds and Proposition 3.1 then yields that $\sigma_i$ is constant, which finishes the proof as argued before.

In dimension $N = 3$, we shall follow ideas used by Ambrosio-Cabré [5] and Alberti-Ambrosio-Cabré [4] in the case of a single equation. We first note that $u$ being $H$-monotone means that $u$ is a stable solution of (3.6). Moreover, the function $v(x_1, x_2) := \lim_{x_3 \to \infty} u(x_1, x_2, x_3)$ is also a bounded stable solution for (3.6) in $\mathbb{R}^2$. Indeed, it suffices to test (3.10) on $\zeta_k(x) = \eta_k(x') \chi_R(x_N)$ where $\eta_k \in C_c^1(\mathbb{R}^{N-1})$ and $\chi_R \in C_c^1(\mathbb{R})$ is defined as

$$\chi_R(t) := \begin{cases} 1, & \text{if } R + 1 < t < 2R + 1, \\ 0, & \text{if } t < R \text{ or } t > 2R + 2, \end{cases}$$

for $R > 1$, $0 \leq \chi_R \leq 1$ and $0 \leq \chi_R' \leq 2$. Note also that since $u$ is an $H$-monotone solution, the system (3.6) is then orientable. It follows from Theorem 3.4 that $v$ is one dimensional and consequently the energy of $v$ in
3.3. De Giorgi type results

A two-dimensional ball of radius $R$ is bounded by a multiple of $R$, which yields that
\[
\limsup_{t \to \infty} E(u^t) \leq CR^2, \tag{3.39}
\]
where here $u^t(x') := u(x', x_n + t)$ for $t \in \mathbb{R}$ and $E_R(u) = \int_{B_R} \frac{1}{2} |\nabla u|^2 + H(u) - c_u d\mathbf{x}$ for $c_u := \inf H(u)$.

To finish the proof, we shall show that
\[
\int_{B_R} |\nabla u|^2 \leq CR^2. \tag{3.40}
\]
Note that shifted function $u^t$ is also a bounded solution of (3.6) with $|\nabla u^t| \in L^\infty(\mathbb{R}^N)$, i.e.,
\[
\Delta u^t = \nabla H(u^t) \text{ in } \mathbb{R}^N, \tag{3.41}
\]
and also
\[
\partial_t u^t_i > 0 > \partial_t u^t_j \quad \text{for all } i \in I \text{ and } j \in J \text{ and in } \mathbb{R}^N. \tag{3.42}
\]
Since $u^t_i$ converges to $v_i$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ for all $i = 1, \cdots, m$, we have
\[
\lim_{t \to \infty} E(u^t) = E(v).
\]

Now, we claim that the following upper bound for the energy holds.
\[
E_R(u) \leq E_R(u^t) + M \int_{\partial B_R} \left( \sum_{i \in I} (u^t_i - u_i) + \sum_{j \in J} (u_j - u^t_j) \right) dS \quad \text{for all } t \in \mathbb{R}^+, \tag{3.43}
\]
where $M = \max_i ||\nabla u_i||_{L^\infty(\mathbb{R}^N)}$. Indeed, by differentiating the energy functional along the path $u^t$, one gets
\[
\partial_t E_R(u^t) = \int_{B_R} \nabla u^t \cdot \nabla (\partial_t u^t) + \int_{B_R} \nabla H(u^t) \partial_t u^t, \tag{3.44}
\]
where $\nabla H(u^t) \partial_t u^t = \sum_i H_{ui}(u^t) \partial_t u^t_i$. Now, multiply (3.41) with $\partial_t u^t$, to obtain
\[
- \int_{B_R} \nabla u^t \cdot \nabla (\partial_t u^t) + \int_{\partial B_R} \partial_v u^t \partial_t u^t = \int_{B_R} \nabla H(u^t) \partial_t u^t. \tag{3.45}
\]
From (3.45) and (3.44) we obtain
\[
\partial_t E_R(u^t) = \int_{\partial B_R} \partial_v u^t \partial_t u^t = \sum_i \int_{\partial B_R} \partial_v u^t_i \partial_t u^t_i. \tag{3.46}
\]
3.3. De Giorgi type results

Note that $-M \leq \partial_\nu u_t \leq M$ and $\partial_t u_i^t > 0 > \partial_t u_j^t$ for $i \in I$ and $j \in J$. Therefore,

$$\partial_t E_R(u^t) \geq M \int_{\partial B_R} \left( \sum_j \partial_t u_j^t - \sum_i \partial_t u_i^t \right) dS. \quad (3.47)$$

On the other hand,

$$E_R(u) = E_R(u^t) - \int_0^t \partial_t E_R(u^s) ds,$$

$$\leq E_R(u^t) + M \int_0^t \int_{\partial B_R} \left( \sum_i \partial_s u_i^s - \sum_j \partial_s u_j^s \right) dS ds$$

$$= E_R(u^t) + M \int_{\partial B_R} \left( \sum_i (u_i^t - u_i) + \sum_j (u_j - u_j^t) \right) dS. \quad (3.48)$$

To finish the proof of the theorem just note that $u_i < u_i^t$ and $u_j^t < u_j$ for all $i \in I$, $j \in J$ and $t \in \mathbb{R}^+$. Moreover, from (3.39) we have $\lim_{t \to \infty} E_R(u^t) \leq CR^2$. Therefore, (3.48) yields

$$E_R(u) \leq C |\partial B_R| \leq CR^2,$$

and we are done. \qed

The above proof suggests that –just as in the case of a single equation– any $H$-monotone solution $u$ of (3.6) must satisfy the following estimate

$$\int_{B_R} |\nabla u|^2 \leq CR^{N-1} \text{ for any } R > 1, \quad (3.49)$$

for some constant $C > 0$. This can be done in the following particular case.

**Theorem 3.6.** If $u$ is a bounded $H$-monotone solution of (3.6) such that for $i = 1, \ldots, m$,

$$\lim_{x_N \to \infty} u_i(x', x_N) = a_i, \quad \forall x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$$

where $a_i$ are constants, then

$$E_R(u) = \int_{B_R} \frac{1}{2} |\nabla u|^2 + H(u) - H(a) dx \leq CR^{N-1}, \quad (3.50)$$

where $a = \{a_i\}_{i=1}^m$ and $C$ is a positive constant independent of $R$. 45
3.3. De Giorgi type results

**Proof:** We first note the following decay on the energy of the shifted function $u^t$ as defined above,

$$\lim_{t \to \infty} E_R(u^t) = 0. \quad (3.51)$$

Indeed, since $u^t$ is convergent to a pointwise, one can see that

$$\lim_{t \to \infty} \int_{B_R} (H(u^t) - H(a)) dx \to 0.$$

Therefore, we need to prove that

$$\lim_{t \to \infty} \int_{B_R} |\nabla u^t_i|^2 dx \to 0.$$

To do so, multiply both sides of (3.41) with $u^t_i - a_i$ and integrate by parts to get

$$- \int_{B_R} |\nabla u^t_i|^2 + \int_{\partial B_R} \partial_{\nu} u^t_i (u^t_i - a_i) = \int_{B_R} \nabla H(u^t)(u^t_i - a_i),$$

which yields (3.51).

To get the energy bound in (3.50), one can follow the proof of the previous theorem to end up with

$$E_R(u) \leq E_R(u^t) + C|\partial B_R| \quad \text{for all} \quad t \in \mathbb{R}^+.$$

To conclude, it suffices to send $t \to \infty$ and to use the fact that $\lim_{t \to \infty} E_R(u^t) = 0$ to finally obtain that

$$E_R(u) \leq C|\partial B_R| \leq CR^{N-1}.$$

**Remark 3.2.** Using Pohozaev type arguments one can see that

$$\Gamma_R = \frac{E_R(u)}{R^{N-1}} \quad \text{is increasing} \quad (3.52)$$

provided the following pointwise estimate holds:

$$|\nabla u|^2 \leq 2H(u). \quad (3.53)$$

Note that this is an extension of the pointwise estimate that Modica [68] proved in the case of a single equation. It is still not known for systems, though Caffarelli-Lin in [24] and later, Alikakos in [3] have shown, in the case where $H \geq 0$, the following weaker monotonicity formula, namely that

$$\Lambda_R = \frac{E_R(u)}{R^{N-2}} \quad \text{is increasing in} \quad R. \quad (3.54)$$
Remark 3.3. The $H$-monotonicity assumption seems to be crucial for concluding that the solutions are one-dimensional. Indeed, it was shown in [1] that when $H$ is a multiple-well potential on $\mathbb{R}^2$, the system has entire heteroclinic solutions $(u, v)$, meaning that for each fixed $x_2 \in \mathbb{R}$, they connect (when $x_1 \to \pm \infty$) a pair of constant global minima of $W$, while if $x_2 \to \pm \infty$, they connect a pair of distinct one dimensional stationary wave solutions $z_1(x_1)$ and $z_2(x_1)$. Note that these convergence are even uniform, which means that the corresponding Gibbons conjecture for systems of equations is not valid in general, without the assumption of $H$-monotonicity.

3.4 Summary and conclusions

In this chapter, we considered the following gradient system

$$\Delta u = \nabla H(u) \quad \text{in} \quad \mathbb{R}^n,$$

where $u : \mathbb{R}^n \to \mathbb{R}^m$, $H \in C^2(\mathbb{R}^m)$. and proved, under various conditions on the nonlinearity $H$ that, at least in low dimensions $n \leq 3$, a solution $u = (u_i)_{i=1}^m$ is necessarily one-dimensional whenever each one of its components $u_i$ is monotone in one direction. We stated the following 1-Liouville theorem as a conjecture which is a counterpart of the De Giorgi’s conjecture (1978) for the above system and we proved it in dimensions $n \leq 3$.

**Conjecture.** Suppose $u = (u_i)_{i=1}^m$ is an $H$-monotone bounded entire solutions of the above system, then at least in dimensions $n \leq 8$, each component $u_i$ must be one-dimensional.

Just like in the proofs of the classical De Giorgi’s conjecture in dimension 2 (Ghoussoub-Gui) and in dimension 3 (Ambrosio-Cabrè), the key step is a 0-Liouville theorem for linear systems. We also gave an extension of a geometric Poincaré inequality to systems and used it to establish De Giorgi type results for stable solutions as well as additional rigidity properties stating that the gradients of the various components of the solutions must be parallel. We introduced and exploited the concept of an orientable system, which seems to be key for dealing with systems of three or more equations. For such systems, the notion of a stable solution in a variational sense coincide with the pointwise (or spectral) concept of stability.
Chapter 4

Regularity results for gradient and nongradient systems

4.1 Introduction

In this chapter we examine the following systems:

\[(G)_{\lambda,\gamma}\]
\[
\begin{align*}
-\Delta u &= \lambda f'(u)g(v) & \Omega \\
-\Delta v &= \gamma f(u)g'(v) & \Omega, \\
u &= v = 0 & \partial\Omega
\end{align*}
\]

and

\[(H)_{\lambda,\gamma}\]
\[
\begin{align*}
-\Delta u &= \lambda f(u)g'(v) & \Omega \\
-\Delta v &= \gamma f'(u)g(v) & \Omega, \\
u &= v = 0 & \partial\Omega
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) and \(\lambda, \gamma > 0\) are positive parameters. The nonlinearities \(f\) and \(g\) will satisfy various properties but will always at least satisfy

\[(R)\] \(f\) is smooth, increasing and convex with \(f(0) = 1\) and \(f\) superlinear at \(\infty\).

To deal with these systems we develop stability inequalities by following ideas given in Chapter 3.

We begin by recalling the scalar analog of the above systems.

Given a nonlinearity \(f\) which satisfies (R), the following equation

\[(Q)_\lambda\]
\[
\begin{align*}
-\Delta u &= \lambda f(u) & \Omega \\
u &= 0 & \partial\Omega
\end{align*}
\]

48
4.1. Introduction

is now quite well understood whenever $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [12, 13, 16, 17, 43, 55, 71, 76]. We now list the properties one comes to expect when studying $(Q)_\lambda$. It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$, called the extremal parameter, such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution $u_\lambda$ of $(Q)_\lambda$. Here minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. For this reason $u^*$ is called the extremal solution. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also one can show the minimal solution $u_\lambda$ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 \leq \int_\Omega |\nabla \psi|^2, \quad \forall \psi \in H^1_0(\Omega).$$

A question that has attracted a lot of attention is the regularity of the extremal solution. It is known that the extremal solution can be a classical solution or it can be a singular weak solution. We now list some results in this direction:

- ([33]) $u^*$ is bounded if $f(u) = e^u$ and $N \leq 9$.
- ([76]) $u^*$ is bounded if $f$ satisfies (R) and $N \leq 3$.
- ([16]) $u^*$ is bounded if $f$ satisfies (R) (can drop the convexity assumption) and $\Omega$ a convex domain in $\mathbb{R}^4$.
- ([17]) $u^*$ is bounded if $\Omega$ is a radial domain in $\mathbb{R}^N$ with $N < 10$ and $f$ satisfies (R) (can drop the convexity assumption).

It is precisely these type of results which we are interested in extending to the case of systems. Before we can discuss the regularity of the extremal solutions associated with $(G)_{\lambda, \gamma}$ and $(H)_{\lambda, \gamma}$ we need to introduce some notation.

Under various conditions on $f$ and $g$ the above systems fit into the general framework of developed in [74], who examined a generalization of

$$\begin{cases}
-\Delta u &= \lambda F(u, v) & \Omega \\
-\Delta v &= \gamma G(u, v) & \Omega, \\
u &= v = 0 & \partial \Omega.
\end{cases}$$

49
4.1. Introduction

The following results are all taken from [74]. Let \( Q = \{(\lambda, \gamma) : \lambda, \gamma > 0\} \) and we define

\[
U := \{(\lambda, \gamma) \in Q : \text{there exists a smooth solution } (u, v) \text{ of } (P)_{\lambda, \gamma}\}.
\]

Firstly we assume that \( F(0, 0), G(0, 0) > 0 \). A simple argument shows that if \( F \) is superlinear at \( u = \infty \), uniformly in \( v \), then the set of \( \lambda \) in \( U \) is bounded. Similarly we assume that \( G \) is superlinear at \( v = \infty \), uniformly in \( u \) and hence we get \( U \) is bounded. We also assume that \( F, G \) are increasing in each variable. This allows the use of a sub/supersolution approach and one easily sees that if \( (\lambda, \gamma) \in U \) then so is \( (0, \lambda \times 0) \]. One also sees that \( U \) is nonempty.

We now define \( \Upsilon := \partial U \cap Q \), which plays the role of the extremal parameter \( \lambda^* \). Various properties of \( \Upsilon \) are known, see [74]. Given \( (\lambda^*, \gamma^*) \in \Upsilon \) set \( \sigma := \frac{\lambda^*}{\gamma^*} \in (0, \infty) \) and define

\[
\Gamma_\sigma := \{(\lambda, \lambda \sigma) : \frac{\lambda^*}{2} < \lambda < \lambda^*\}.
\]

We let \( (u_\lambda, v_\lambda) \) denote the minimal solution \( (P)_{\lambda, \sigma \lambda} \) for \( \frac{\lambda^*}{2} < \lambda < \lambda^* \). One easily sees that for each \( x \in \Omega \) that \( u_\lambda(x), v_\lambda(x) \) are increasing in \( \lambda \) and hence we define

\[
u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x), \quad v^*(x) := \lim_{\lambda \nearrow \lambda^*} v_\lambda(x),
\]

and we call \( (u^*, v^*) \) the extremal solution associated with \( (\lambda^*, \gamma^*) \in \Upsilon \). Under some very minor growth assumptions on \( F \) and \( G \) one can show that \( (u^*, v^*) \) is a weak solution of \( (P)_{\lambda^*, \gamma^*} \).

We now come to the issue of stability.

**Theorem 4.1.** [74] Let \( (\lambda, \gamma) \in U \) and let \( (u, v) \) denote the minimal solution of \( (P)_{\lambda, \gamma} \). Then \( (u, v) \) is semi-stable in the sense that there is some smooth \( 0 < \zeta, \chi \in H^1_0(\Omega) \) and \( 0 \leq \eta \) such that

\[
-\Delta \zeta = \lambda F_u(u, v) \zeta + \lambda F_v(u, v) \chi + \eta \zeta, \quad \Omega.
\]
\[
-\Delta \chi = \gamma G_u(u, v) \zeta + \gamma G_v(u, v) \chi + \eta \chi, \quad \Omega.
\]

In this chapter, we prove that the extremal solution of \( (G)_{\lambda^*, \gamma^*} \) with general nonlinearities, either on a general domain and lower dimensions or on a radial domain and higher dimensions are regular. Moreover, for explicit nonlinearities we prove regularity on a general domain in higher dimensions.
4.1. Introduction

The following stability inequalities play a key role in this paper and we shall refer to them many times through proofs. The idea of getting such inequalities comes from Chapter 3 where the De Giorgi type results and Liouville theorems have been proved for a much more general gradient system.

Lemma 4.1. 1. Let \((u,v)\) denote a semi-stable solution of \((G)_{\lambda,\gamma}\) in the sense of (4.1). Then

\[
\int f''(u)g(v)\phi^2 + \int f(u)g''(v)\psi^2 + 2 \int f'(u)g'(v)\phi\psi \leq \frac{1}{\lambda} \int |\nabla \phi|^2 + \frac{1}{\gamma} \int |\nabla \psi|^2. \tag{4.2}
\]

2. Let \((u,v)\) denote a semi-stable solution of \((H)_{\lambda,\gamma}\) in the sense of (4.1). Then

\[
\int f'(u)g'(v)(\phi^2 + \psi^2) + 2 \int \sqrt{ff''gg''}\phi\psi \leq \frac{1}{\lambda} \int |\nabla \phi|^2 + \frac{1}{\gamma} \int |\nabla \psi|^2. \tag{4.3}
\]

Proof. (1) By Theorem 4.1 there is some \(0 < \zeta, \chi\) such that

\[
-\Delta \zeta \geq \lambda f''(u)g(v)\zeta + \lambda f'(u)g'(v)\chi \quad \text{in } \Omega
\]

\[
-\Delta \chi \geq \gamma f'(u)g'(v)\zeta + \gamma f(u)g''(v)\chi \quad \text{in } \Omega.
\]

Consider test functions \(\phi, \psi \in H_0^1(\Omega)\) and multiply both sides of the above inequalities with \(\frac{\phi^2}{\zeta}\) and \(\frac{\psi^2}{\chi}\) to obtain

\[
-\int |\nabla \phi|^2 \frac{\phi^2}{\zeta} + 2 \int \nabla \phi \cdot \nabla \zeta \frac{\zeta}{\phi} \geq \int \lambda f''(u)g'(v)\phi^2 \frac{\chi}{\zeta} + \int \lambda f(u)g''(v)\phi^2,
\]

\[
-\int |\nabla \psi|^2 \frac{\chi^2}{\psi^2} + 2 \int \nabla \psi \cdot \nabla \chi \frac{\chi}{\psi} \geq \int \gamma f'(u)g'(v)\psi^2 + \int \gamma f''(u)g'(v)\psi^2 \frac{\zeta}{\chi}.
\]

Apply Young’s inequality for the left hand side of each inequality and add them to get

\[
\lambda \int f''(u)g(v)\phi^2 + \gamma \int f(u)g''(v)\psi^2 + \int f'(u)g'(v) \left( \lambda \phi^2 \frac{\chi}{\zeta} + \gamma \psi^2 \frac{\zeta}{\chi} \right) \leq \int |\nabla \phi|^2 + \int |\nabla \psi|^2,
\]
Simple calculations show that the third term is an upper bound for
\[ 2\sqrt{\lambda\gamma} \int f'(u)g'(v)\phi\psi. \]

Then, replacing \( \phi \) with \( \frac{\phi}{\sqrt{\lambda}} \) and \( \psi \) with \( \frac{\psi}{\sqrt{\gamma}} \) gives the desired result.

(2) Proof is quite similar to (1). By Theorem 4.1 there is some \( 0 < \zeta, \chi \) such that
\[ -\frac{\Delta \zeta}{\zeta} \geq \lambda f'(u)g'(v) + \lambda f(u)g''(v)\frac{\chi}{\zeta} \text{ in } \Omega, \]
\[ -\frac{\Delta \chi}{\chi} \geq \gamma f''(u)g(v)\frac{\zeta}{\chi} + \gamma f'(u)g'(v) \text{ in } \Omega, \]
and we now multiply the first equation by \( \phi^2 \) and the second by \( \psi^2 \) and add the equations and integrate over \( \Omega \). In addition we use the fact that
\[ \int_{\Omega} -\frac{\Delta E}{E} \phi^2 \leq \int |\nabla \phi|^2, \]
for any \( E > 0 \) and \( \phi \in H^1_0(\Omega) \). Doing this one obtains
\[ \int_{\Omega} f'(u)g'(v)(\lambda \phi^2 + \gamma \psi^2) + \int_{\Omega} \lambda f(u)g''(v)\phi^2\frac{\chi}{\zeta} + \gamma f''(u)g(v)\psi^2\frac{\zeta}{\chi} \]
\[ \leq \int_{\Omega} |\nabla \phi|^2 + |\nabla \psi|^2. \]
Again some simple algebra shows that
\[ 2\sqrt{\lambda\gamma} \int_{\Omega} \sqrt{f(u)f''(u)g(v)g''(v)}\phi\psi, \]
is a lower bound for the second integral. Using this lower bound and replacing \( \phi \) with \( \frac{\phi}{\sqrt{\lambda}} \) and \( \psi \) with \( \frac{\psi}{\sqrt{\gamma}} \) finishes the proof.

\[ \square \]

In Section 4.2, we explore the regularity of extremal solutions for systems \((G)_{\lambda, \gamma}\) and \((H)_{\lambda, \gamma}\) with arbitrary nonlinearities and, in then Section 4.3 we consider explicit nonlinearities. We finish the current section by noting that in \cite{[26]} the system
\[ (E)_{\lambda, \gamma} \quad \begin{cases} -\Delta u = \lambda e^v & \Omega \\ -\Delta v = \gamma e^u & \Omega, \\ u = v = 0 & \partial \Omega, \end{cases} \]
was examined. It was shown that if $\Omega$ is a bounded domain in $\mathbb{R}^N$ where $N \leq 9$, then the extremal solution $(u^*, v^*)$ associated with $(\lambda^*, \gamma^*) \in \Upsilon$ is bounded if

$$\frac{N - 2}{8} \leq \frac{\gamma^*}{\lambda^*} \leq \frac{8}{N - 2}.$$ 

Note that as one gets closer to the diagonal parameter range $\gamma = \lambda$ that better regularity results are obtained. At the diagonal the system can be shown to reduce to the scalar equation $-\Delta u = \lambda e^u$. This phenomena will also be present in Section 3 where we consider explicit nonlinearities.

### 4.2 Arbitrary nonlinearities

We begin by examining $(G)_{\lambda, \gamma}$ in the case of arbitrary nonlinearities and we show the extremal solutions are bounded in low dimensions and our methods of proof are close to [76].

**Theorem 4.2.** Suppose that $\Omega$ is a bounded smooth convex domain in $\mathbb{R}^N$ where $N \leq 3$ and suppose $f$ and $g$ both satisfy condition (R). We also assume that $a := f'(0) > 0$ and $b := g'(0) > 0$. In addition we assume that $f', g'$ are convex and there is some $\xi > 0$ such that

$$\liminf_{u \to \infty} \frac{f''(u)}{u^\xi} > 0, \quad \liminf_{v \to \infty} \frac{g''(v)}{v^\xi} > 0. \tag{4.4}$$

Let $(\lambda^*, \gamma^*) \in \Upsilon$. Then the associated extremal solution of $(G)_{\lambda^*, \gamma^*}$ is bounded.

For radial domains we obtain similar results but in higher dimensions and our methods of proof follow very closely to [17] and [96].

**Theorem 4.3.** Let $\Omega = B_1$, $N \geq 3$, and $f$ and $g$ both satisfy condition (R) and in addition we assume that there is some $\xi > 0$ such that

$$\liminf_{u \to \infty} \frac{f'(u)}{u^{1+\xi}} > 0, \quad \liminf_{v \to \infty} \frac{g'(v)}{v^{1+\xi}} > 0.$$ 

Let $(\lambda^*, \gamma^*) \in \Upsilon$ and let $(u^*, v^*)$ denote the extremal solution associated with $(G)_{\lambda^*, \gamma^*}$. Then

1. if $N < 10$, then $u^*, v^* \in L^\infty(B_1)$,
2. if $N = 10$, then $u^*(r), v^*(r) \leq C_{\lambda^*, \gamma^*}(1 + |\log r|)$ for $r \in (0, 1]$,
3. if $N > 10$, then $u^*(r), v^*(r) \leq C_{\lambda^*, \gamma^*, N} r^{-\frac{N}{2} + \sqrt{N-1} + 2}$ for $r \in (0, 1]$. 

53
4.2. Arbitrary nonlinearities

We are unable to prove the analogous version for the system $(H)_{\lambda,\gamma}$ and hence we restrict our attention to the special case.

**Theorem 4.4.** Suppose $\Omega$ a bounded smooth convex domain in $\mathbb{R}^3$ and $1 < q < \infty$. Assume $f$ satisfies (R) and we also assume that $f'' \geq C > 0$. Let $(\lambda^*, \gamma^*) \in \Upsilon$. Then the associated extremal solution of $(H)_{\lambda^*,\gamma^*}$ for $g(v) = (1 + v)^q$ is bounded.

The following lemma is used to prove Theorem 4.2 where a convex domain is assumed but we prove the lemma for general domains.

**Lemma 4.2.** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $f$ and $g$ satisfy the conditions from Theorem 4.2 but one can weaken condition (4.4) to $f'', g'' \to \infty$ at $\infty$. Let $(\lambda^*, \gamma^*) \in \Upsilon$ and let $(u^*, v^*)$ denote the extremal solution associated with $(G)_{\lambda^*,\gamma^*}$. Then there is some $C < \infty$ such that

(i) $\int f'(u^*)g'(v^*)(f'(u^*) - a)(g'(v^*) - b) \leq C$

(ii) $\int (f'(u^*) - a)f''(u^*)g(v^*) \leq C$

(iii) $\int (g'(v^*) - b)g''(v^*)f(u^*) \leq C$

**Remark 4.1.** Let $f_i, g_i$ denote smooth increasing nonlinearities with $f_i(0), g_i(0) > 0$ and we also assume there is some $\xi > 0$ such

$$\liminf_{u \to \infty} \frac{f_i(u)}{u^{1+\xi}} > 0, \quad \liminf_{v \to \infty} \frac{g_i(v)}{v^{1+\xi}} > 0. \quad (4.5)$$

Let $(u_m, v_m)$ denote a sequence of smooth solutions of $(P)_{\lambda_m, \sigma \lambda_m}$, where $0 < \sigma < \infty$ is fixed and $\lambda_m$ is restricted to a compact subset of $(0, \infty)$ and $F(u, v) = f_1(u)g_1(u)$, $G(u, v) = f_2(u)g_2(v)$. Then we have the estimate

$$\int f_1(u_m)g_1(v_m)\delta + \int f_2(u_m)g_2(v_m)\delta \leq C,$$

where $\delta(x) := \text{dist}(x, \partial \Omega)$. Applying regularity theory shows that $u_m, v_m$ are bounded in $L^1(\Omega)$.

On occasion we will restrict our attention to smooth convex domains where many of the proofs are much more compact. For this we will use a result proven using the Moving Plane Method. So we assume that $\Omega$ is a smooth convex domain. Then there is some $\varepsilon_0 > 0$ small such that any smooth solution $(u, v)$ of $(P)_{\lambda, \gamma}$ satisfies some monotonicity properties in in the small strip $\{x \in \Omega : \delta(x) \leq \varepsilon_0\}$ near the boundary. (Essentially one can assume $u, v$ are strictly decreasing as one approaches the boundary). This
4.2. Arbitrary nonlinearities

coupled with the $L^1$ bounds on $u_m$ and $v_m$ shows that there is some constant $C_0$ such that $u_m, v_m \leq C_0$ on some small strip, say $\{x \in \Omega : \delta(x) \leq \epsilon_1\}$ near the boundary. Assuming the same restrictions on $\lambda_m, \gamma_m$ one can use the maximum principle to show that there is some $C_1 > 0$ such that $u_m, v_m \geq C_1$ in the compliment of this small strip.

**Proof.** All integrals are over $\Omega$ unless otherwise stated. Our approach will be to obtain uniform estimates for any minimal solution $(u, v)$ of $(G)_{\lambda, \gamma}$ on the ray $\Gamma_\sigma$ and then one sends $\lambda \nearrow \lambda^*$ to obtain the same estimate for $(u^*, v^*)$. Let $(u, v)$ denote a smooth minimal solution of $(G)_{\lambda, \gamma}$ on the ray $\Gamma_\sigma$ and put $\phi := f'(u) - a$ and $\psi := g'(v) - b$ into (4.2) to obtain

$$
\int f''(u)g(v)(f'(u) - a)^2 + \int f(u)g''(v)(g'(v) - b)^2 + 2\int f'(u)g'(v)(f'(u) - a)(g'(v) - b),
$$

is less than or equal

$$
\frac{1}{\lambda} \int \nabla(f'(u) - a)f''(u) \cdot \nabla u + \frac{1}{\gamma} \int \nabla(g'(v) - b)g''(v) \cdot \nabla v. \quad (4.6)
$$

Integrating (4.6) by parts shows that this is equal to

$$
-\frac{1}{\lambda} \int (f'(u) - a)f'''(u) |\nabla u|^2 + \frac{1}{\lambda} \int (f'(u) - a)f''(u)(-\Delta u),
$$

plus a similar term in involving $v$. We use the equation $(G)_{\lambda, \gamma}$ to replace $-\Delta u$ and $-\Delta v$ in the last line and simplify to arrive at

$$
\frac{1}{\lambda} \int (f'(u) - a)f'''(u) |\nabla u|^2 + \frac{1}{\gamma} \int (g'(v) - b)g'''(v) |\nabla v|^2 + 2\int f'(u)g'(v)(f'(u) - a)(g'(v) - b),
$$

is less than or equal to

$$
a \int f''(u)(f'(u) - a)g(v) + b \int g''(v)(g'(v) - b)f(u).
$$

We now define $h_1(u) := \int_0^u (f'(t) - a) f''(t) dt$ and $h_2(v) := \int_0^v (g'(t) - b) g''(t) dt$. Subbing this into the previous inequality and integrating by
4.2. Arbitrary nonlinearities

parts and using \((G)_{\lambda,\gamma}\) again we arrive at

\[
\int h_1(u)f'(u)g(v) + \int h_2(v)f(u)g'(v) + 2 \int f'(u)g'(v)(f'(u) - a)(g'(v) - b) \\
\leq a \int f''(u)(f'(u) - a)g(v) + b \int g''(v)(g'(v) - b)f(u)
\]  \hspace{1cm} (4.7)

Now suppose \(u > \alpha > 0\). Then we have

\[
h_1(u) \geq \int_{\alpha}^{u} (f'(t) - a)\frac{f''(t)dt}{f''(u)} \geq (f'(\alpha) - a)(f''(u) - f''(\alpha)),
\]

and so using the condition on \(f''(u)\) we see that

\[
\liminf_{u \to \infty} \frac{h_1(u)}{f''(u)} \geq f'(\alpha) - a,
\]

for any \(\alpha > 0\). But since \(f\) is convex and superlinear at infinity we see that \(\lim_{u \to \infty} \frac{h_1(u)}{f''(u)} = \infty\). Similarly \(\lim_{v \to \infty} \frac{h_2(v)}{g''(v)} = \infty\).

We now estimate the integral \(\int f''(u)g(v)(f'(u) - a)\). There is some \(T > 1\) large such that for all \(u \geq T\) we have \(h_1(u) \geq 100(a + 1)f''(u)\) for all \(u \geq T\). Then we have

\[
\int f''(u)g(v)(f'(u) - a) = \int_{u \geq T} + \int_{u < T} \\
\leq \frac{1}{100(a + 1)} \int h_1(u)g(v)(f'(u) - a) \\
+ \int_{u < T} \int f''(u)g(v)(f'(u) - a).
\]

We now estimate this last integral. Let \(T\) be as above and fixed and we let \(k \geq 1\) denote a natural number.

\[
\int_{u < T} f''(u)g(v)(f'(u) - a) = \int_{u < T, v < kT} + \int_{u < T, v \geq kT} \\
= C(k, T) + \int_{u < T, v \geq kT} f''(u)g(v)(f'(u) - a)
\]

and we now estimate this last integral. One easily sees that this last integral is bounded above by

\[
\sup_{u < T} \frac{f''(u)}{f'(u)} \sup_{v > kT} \frac{g(v)}{(g'(v) - b)g'(v)} \int (f'(u) - a)(g'(v) - b)f'(u)g'(v).
\]
Combining this all together we see that for all sufficiently large $T$ and all $1 \leq k$ there is some constant $C(k, T)$ such that

\[
\int f''(u)g(v)(f'(u) - a) \leq \frac{1}{100(a + 1)} \int h_1(u)f'(u)g(v) + C(k, T) + \sup_{u < T} \frac{f''(u)}{f'(u)} \sup_{v > kT} g(v)(f'(v) - b)g'(v)
\]

\[
\int \int (f''(u) - a)(g'(v) - b)f'(u)g'(v) + sup_{u > kT} f''(u)\cdot sup_{v > kT} g'(v)\cdot (g''(v) - b)g'(v).
\]

Using the same argument one can show for all sufficiently large $T$ and for all $1 \leq k$ there is some $C(k, T)$ such that

\[
\int g''(v)f(u)(g'(v) - b) \leq \frac{1}{100(b + 1)} \int h_2(v)g'(v)f(u) + C(k, T) + \sup_{v < T} \frac{g''(v)}{g'(v)} \sup_{u > kT} f(u)(f'(u) - a)f'(u)
\]

\[
\int (f'(u) - a)(g'(v) - b)f'(u)g'(v).
\]

Since $f''$, $g'' \to \infty$ we see that

\[
\lim_{k \to \infty} sup_{u > kT} \frac{f(u)}{f'(u) - a}f'(u) = 0,
\]

and similarly for the other term. Hence by taking $k$ sufficiently large we can substitute everything back into (4.7) and see that all the integrals in (4.7) are bounded independent of $\lambda$.

\[\square\]

**Proof of Theorem 4.2.**

We assume that $N = 3$ and $\Omega$ is convex domain in $\mathbb{R}^3$. The case of $N = 1, 2$ is easier and we omit their proofs. We suppose that $(\lambda^*, \gamma^*) \in \Upsilon$ and $(u^*, v^*)$ is the associated extremal solution of $(G)_{\lambda^*, \gamma^*}$. Set $\sigma = \frac{\gamma^*}{\lambda^*}$. Using Remark 4.1 along with Lemma 4.2 we see that $f'(u^*)g'(v^*) \in L^2(\Omega)$. Note that this and the convexity of $g$ show that

\[
\int_\Omega \frac{f'(u^*)^2g(v^*)^2}{(v^* + 1)^2} \leq C.
\]

From Lemma 4.2 and Remark 4.1 we have $-\Delta u^*, -\Delta v^* \in L^1$ and hence we have $u^*, v^* \in L^3$, i.e. $L^p$ for any $p < 3$.  

57
4.2. Arbitrary nonlinearities

We now use the domain decomposition method as in [76]. Set
\[
\Omega_1 := \left\{ x : f'(u^*)^2 g(v^*) \geq f'(u^*)^2 - \alpha g(v^*)^2 \right\},
\]
\[
\Omega_2 := \Omega \setminus \Omega_1 = \left\{ x : f'(u^*) g(v^*) \leq (v^* + 1)^2 \right\},
\]
where 0 < \alpha is to be picked later. First note that
\[
\int_{\Omega_1} (f'(u^*) g(v^*))^{2-\alpha} \leq \int_{\Omega} (f'(u^*) g(v^*))^{2} \leq C.
\]
Similarly we have
\[
\int_{\Omega_2} (f'(u^*) g(v^*))^{p} \leq \int_{\Omega} (v^* + 1)^{2p/\alpha}.
\]
Taking \alpha = \frac{4}{5} and using the \(L^3\)-bound on \(v\) shows that \(f'(u^*) g(v^*) \in L^{6/5}(\Omega)\). By a symmetry argument we also have \(f(u^*) g'(v^*) \in L^{6/5}(\Omega)\).

By elliptic regularity we have \(u^*, v^* \in W^{2, \frac{6}{5}}\) and this is contained in \(L^{6/5}(\Omega)\) after considering the Sobolev imbedding theorem. Using these estimates and again using the domain decomposition \(\Omega_1\) and \(\Omega_2\) but taking \alpha = \frac{1}{2} gives that \(f'(u^*) g(v^*) \in L^{\frac{3}{2}}(\Omega)\) and by symmetry we have the same for \((u^*) g'(v^*)\). Elliptic regularity now shows that \(u^*, v^* \in W^{2, \frac{3}{2}}\) and this is contained in \(L^{p}\) for any \(p < \infty\). One last iteration with \alpha = \frac{4}{5} shows that \(f'(u^*) g(v^*) \in L^{\frac{4}{5}}(\Omega)\) and after considering elliptic regularity and the Sobolev imbedding we have \(u^*\) is bounded. By symmetry we see \(v^*\) is also bounded.

\[\square\]

**Proof of Theorem 4.4.** Let \((\lambda^*, \gamma^*) \in \mathcal{Y}\) and let \((u^*, v^*)\) denote the extremal solution associated with \((H)_{\lambda^*, \gamma^*}\). We let \((u, v)\) denote a minimal solution on the ray \(\Gamma_{\sigma}\) where \(\sigma = \frac{\gamma^*}{\alpha^*}\). Without loss of generality we can assume \(\lambda = \sigma = 1\) to simplify the calculations. Note the assumption on \(f''\) shows there is some \(\xi > 0\) such that \(\frac{f(u)}{u^2 + 1} \to \infty\) as \(u \to \infty\). Using this along with the fact that \(q > 1\) and Remark 4.1 shows that \(\Delta u^*, \Delta v^* \in L^1(\Omega)\).

Set \(\alpha := f(u) - 1\) and \(\beta := (v + 1)^t - 1\) where \(1 < t < t_+(q) := q + \sqrt{q(q - 1)}\) into the stability inequality to arrive at
\[
(q - \frac{t^2}{2t - 1}) \int f'(u)(v + 1)^{2t + q - 1} + \int (f(u) - 1) f''(u) |\nabla u|^2
\]
\[
+ 2\sqrt{q(q - 1)} \int \sqrt{f(u)} f''(u)(v + 1)^{q - 1}(f(u) - 1)((v + 1)^t - 1)
\]
\[58\]
4.2. Arbitrary nonlinearities

is less than or equal to

\[ 2q \int f(u)f'(u)(v + 1)^{q-1} + 2q \int f'(u)(v + 1)^{q+t-1}. \]

We label these integrals as \( I_i \) for \( 1 \leq i \leq 5 \) from left to right. The condition on \( t \) ensures the coefficient in front \( I_1 \) is positive. We can rewrite

\[ I_2 = \int qh_1(u)f(u)(v+1)^{q-1}, \quad \text{where} \quad h_1(u) = \int_0^u (f(\tau)-1)f''(\tau) d\tau. \]

One easily sees that \( \frac{h_1(u)}{f'(u)} \to \infty \) as \( u \to \infty \). Let \( T \) be sufficiently large such that \( h_1(u) \geq 10f'(u) \) for all \( u \geq T \). Then one easily sees that

\[ \int f(u)f'(u)(v+1)^{q-1} \leq \frac{1}{10} \int h_1(u)f(u)(v+1)^{q-1} + f'(T) \int f(u)(v+1)^{q-1}. \]

We also have

\[ \int f'(u)(v + 1)^{q+t-1} \leq T^{q+t-1} \int f'(u) + \frac{1}{T^t} \int f'(u)(v + 1)^{q+2t-1}, \]

and so after combining the estimates we have

\[ \left( q - \frac{t^2}{2t-1} - \frac{2q}{T^t} \right) \int f'(u)(v + 1)^{q+2t-1} + \frac{4q}{5} \int h_1(u)f(u)(v + 1)^{q-1} + \right. \]

\[ + 2\sqrt{q(q-1)} \int \sqrt{f(u)f''(u)}(v + 1)^{q-1}(f(u) - 1)((v + 1)^t - 1), \]

is less than or equal

\[ 2qf'(T) \int f(u)(v + 1)^{q-1} + 2qT^{q+t-1} \int f'(u). \]

Passing to the limit shows this inequality holds with \( (u^*, v^*) \) in place of \( (u, v) \).

But these last two integrals are finite and hence we have an estimate provided the first coefficient is positive, which is indeed the case provided we take \( T \) bigger if necessary. Hence each of the following integrals is finite

(i) \( \int f'(u^*)(v^* + 1)^{q+2t-1}, \)

(ii) \( \int f'(u^*)f(u^*)(v^* + 1)^{q-1}, \)

(iii) \( \int f(u^*)^2(v^* + 1)^{q+t-1}, \)
4.2. Arbitrary nonlinearities

for all $1 < t < t_+(q)$.

Now note that $-\Delta f(u) = -f''(u)|\nabla u|^2 + f'(u)f(u)(v+1)^{q-1}$ and since $f$ is convex and since $f''(u)f(u)(v+1)^{q-1}$ is uniformly bounded in $L^1(\Omega)$ along the ray $\Gamma_\sigma$ shows that $f(u)$ is uniformly bounded in $L^{\frac{N}{N-2}} = L^{3-}$; and hence $f(u_\ast) \in L^{3-}$.

We now use (i) and (iii) to show that $u_\ast$ is bounded. Pick $p > \frac{3}{2}$ but close and $\alpha > 0$ and $1 < \tau < \infty$ such that

$$(p-\alpha)\tau < 3, \quad \tau^\prime \alpha = \frac{3}{2}, \quad (q-1)p\tau' < q-1 + t_+(q).$$

Then we have

$$\int (f(u_\ast)(v_\ast + 1)^{(q-1)p}) \leq \left( \int f(u_\ast)^{(p-\alpha)\tau} \right)^{\frac{1}{2}} \left( \int f(u_\ast)^{\frac{3}{2}(v_\ast + 1)^{\tau'p(q-1)}} \right)^{\frac{1}{2^\prime}},$$

but the right hand side is finite and hence by elliptic regularity we have $u_\ast$ is bounded. We now show that $v_\ast$ is bounded. First note that we have $\int (v_\ast + 1)^{2t+q-1} < \infty$ for any $1 < t < t_+(q)$ and hence one has $\int (v_\ast + 1)^{q+1} < \infty$. Since $u_\ast$ is bounded this shows that $v_\ast \in H^1_0$. Now to complete the proof of $v_\ast$ being bounded it is sufficient to show (since $u_\ast$ is bounded) that $(v_\ast + 1) \in L^{(q-1)p}$ for some $p > \frac{3}{2}$ but this easily follows after considering the above estimate.

**Proof of Theorem 4.3.** Step 1. Let $(u,v)$ denote a smooth minimal solution of $(G)_{\lambda,\gamma}$ on the ray $\Gamma_\sigma$ where $\sigma := \frac{\lambda}{N^2}$. Then taking a derivative of $(G)_{\lambda,\gamma}$ with respect to $r$ gives

$$\begin{cases}
-\Delta u_r + \frac{N-1}{r^2}u_r &= \lambda f''(u)g(v)u_r + \lambda f'(u)g'(v)v_r \quad \text{for } 0 < r < 1, \\
-\Delta v_r + \frac{N-1}{r^2}v_r &= \gamma f'(u)g'(v)u_r + \gamma f''(u)g''(v)v_r \quad \text{for } 0 < r < 1. \quad (4.8)
\end{cases}$$

Multiply the first and the second equations of (4.8) with $u_r\phi^2$ and $v_r\phi^2$ where $\phi \in C^{0,1}(B_1) \cap H^1_0(B_1)$ gives

$$\begin{align*}
\int |\nabla u_r|^2 \phi^2 + \frac{1}{2} \nabla u_r \cdot \phi^2 + \frac{N-1}{r^2}u_r^2 \phi^2 &= \int \lambda f''(u)g(v)u_r^2 \phi^2 + \lambda f'(u)g'(v)v_r u_r \phi^2 \\
\int |\nabla v_r|^2 \phi^2 + \frac{1}{2} \nabla v_r \cdot \phi^2 + \frac{N-1}{r^2}v_r^2 \phi^2 &= \int \gamma f'(u)g'(v)u_r v_r \phi^2 + \gamma f''(u)g''(v)v_r^2 \phi^2 \quad (4.9)
\end{align*}$$
On the other hand, testing (4.2) on $\phi \rightarrow u_r\phi$ and $\psi \rightarrow v_r\phi$ where $\phi$ is as above, we get

$$\int f''(u)g'(v)u_r^2\phi^2 + f(u)g''(v)v_r^2\phi^2 + 2 \int f'(u)g'(v)u_r v_r\phi^2 \leq \frac{1}{\lambda} \int |\nabla (u_r\phi)|^2 + \frac{1}{\gamma} \int |\nabla (v_r\phi)|^2.$$ 

Expanding the right hand side we get

$$\frac{1}{\lambda} \int \left( |\nabla u_r|^2 \phi^2 + u_r^2 |\nabla \phi|^2 + \frac{1}{2} |\nabla \phi^2 \cdot \nabla u_r^2 \right)$$

$$+ \frac{1}{\gamma} \int \left( |\nabla v_r|^2 \phi^2 + v_r^2 |\nabla \phi|^2 + \frac{1}{2} |\nabla \phi^2 \cdot \nabla v_r^2 \right)$$

Applying (4.9) the above will be

$$\frac{1}{\lambda} \int u_r^2 |\nabla \phi|^2 + \frac{1}{\gamma} \int v_r^2 |\nabla \phi|^2 - \frac{N - 1}{\lambda} \int u_r^2 \phi^2 - \frac{N - 1}{\gamma} \int v_r^2 \phi^2$$

$$+ 2 \int f'(u)g'(v)u_r v_r\phi^2 + \int f(u)g''(v)v_r^2\phi^2 + \int f''(u)g(v)u_r^2\phi^2$$

Therefore one obtains, after substituting $r\phi$ for $\phi$, (4.10)

$$(N - 1) \int \left( \frac{u_r^2}{\lambda} + \frac{v_r^2}{\gamma} \right) \phi^2 \leq \int \left( \frac{u_r^2}{\lambda} + \frac{v_r^2}{\gamma} \right) |\nabla (r\phi)|^2,$$

for all $\phi \in C^{0,1}(B_1) \cap H^1_0(B_1)$. Note that there is no $f$ and $g$ in this estimate.

**Step 2.** We show that (4.10) implies that $u^*$, $v^* \in H^1_0(B_1)$. Firstly we argue there is some $C_R > 0$ such that for any $0 < R < 1$ we have $\sup_{r > R} (u(r) + v(r)) \leq C_R$ and $C_R$ is independent of $\lambda$ (and hence the estimate also holds for $u^*, v^*$). To see this we first note that by Remark 4.1 we have $\|u\|_{L^1}, \|v\|_{L^1} \leq C$ (uniformly in $\lambda$) and since $u, v$ are radially decreasing we have the desired result otherwise we couldn’t have the $L^1$ bound. We now let $0 \leq \phi \leq 1$ be a smooth function supported in $B_1$ with $\phi = 1$ on $B_{1/2}$. Putting this into (4.10) and rearranging gives

$$(N - 2) \int_{B_{1/2}} \frac{u_r^2}{\lambda} + \frac{v_r^2}{\gamma} \leq C \int_{B_1 \setminus B_{1/2}} \frac{u_r^2}{\lambda} + \frac{v_r^2}{\gamma}.$$
4.2. Arbitrary nonlinearities

Now let $0 \leq \psi \leq 1$ denote smooth function with $\psi = 0$ in $B_{\frac{1}{4}}$ and $\psi = 1$ for $|x| > \frac{1}{2}$. Multiply $-\Delta u = \lambda f'(u)g(v)$ by $u\psi^2$ and integrate by parts and use Young’s inequality to arrive at

$$\int |\nabla u|^2 \psi^2 \leq 2\lambda \int f'(u)g(v)u\psi^2 + 4\int u^2 |\nabla \psi|^2,$$

and hence we have

$$\int_{B_{1}\setminus B_{\frac{1}{2}}} |\nabla u|^2 \leq 2\lambda \int_{B_{1}\setminus B_{\frac{1}{4}}} f'(u)g(v)u + C \int_{B_{1}\setminus B_{\frac{1}{4}}} u^2 \quad (4.12)$$

and we now use the pointwise bound to see that

$$\int_{B_{1}\setminus B_{\frac{1}{2}}} |\nabla u|^2 \leq C,$$

where $C$ is independant of $\lambda$. Similarly we obtain the same estimate of $v$ and combining this with $(4.11)$ we see that $u, v$ are bounded in $H^{1}_{0}(B_{1})$ independant of $\lambda$ and hence $u^*, v^* \in H^{1}_{0}(B_{1})$.

**Step 3.** Let $(u, v)$ denote a minimal solution of $(G)_{\lambda, \gamma}$ on the $\Gamma_{\sigma}$ where $\sigma := \frac{2}{\sqrt{N}}$.

For $0 < r < \frac{1}{2}$ define $\phi$ to be the following test function

$$\phi(t) = \begin{cases} 
  r^{-\sqrt{N-1}} & \text{if } 0 \leq t \leq r, \\
  t^{-\sqrt{N-1}} & \text{if } r < t \leq 1/2, \\
  2^{\sqrt{N-1}+2}(1-t) & \text{if } 1/2 < t \leq 1.
\end{cases}$$

Putting $\phi$ into $(4.10)$ gives

$$\int_{0}^{r} \left( \frac{u_{r}^{2}(t)}{\lambda} + \frac{v_{r}^{2}(t)}{\gamma} \right) t^{N-1} dt \leq C_{N}r^{2\sqrt{N-1}+2} \left( \frac{1}{\lambda} ||\nabla u||_{L^{2}(B_{1}\setminus B_{1/2})}^{2} + \frac{1}{\gamma} ||\nabla v||_{L^{2}(B_{1}\setminus B_{1/2})}^{2} \right),$$

for all $0 < r < \frac{1}{2}$ and one easily extends this to all $0 < r < 1$ by taking $C_{N}$
4.2. Arbitrary nonlinearities

bigger if necessary. From this, by simple calculations we get

\[
\frac{1}{\sqrt[\gamma]{N-1}}|u(r) - u(\frac{r}{2})| + \frac{1}{\sqrt{\gamma}}|v(r) - v(\frac{r}{2})| \\
\leq \int_{r/2}^{r} \left( \frac{1}{\sqrt[\gamma]{N-1}}|u_r(t)| + \frac{1}{\sqrt{\gamma}}|v_r(t)| \right) t^{\frac{N-1}{2}} t^{\frac{1-N}{2}} dt \\
\leq \sqrt{2} \left( \int_{r/2}^{r} \left( \frac{u_r(t)}{\lambda} + \frac{v_r(t)}{\gamma} \right) t^{N-1} dt \right)^{1/2} \left( \int_{r/2}^{r} t^{1-N} dt \right)^{1/2} \\
\leq C_N \sqrt{N-1} + 2 - \frac{N}{2} \left( \frac{1}{\sqrt[\gamma]{N-1}} \|u_1\|_{L^2(B_1 \setminus B_{1/2})} + \frac{1}{\sqrt{\gamma}} \|v_1\|_{L^2(B_1 \setminus B_{1/2})} \right). \\
\]

Let \(0 < r \leq 1\). Then, there exist \(m \in N\) and \(1/2 < r_1 \leq 1\) such that \(r = \frac{r_1}{2m+1}\). Since \(u, v\) are radial, we have \(u(r_1) \leq \|u\|_{L^\infty(B_1 \setminus B_{1/2})} \leq C_N\|u\|_{H^1(B_1 \setminus B_{1/2})}\) and \(v(r_1) \leq \|v\|_{L^\infty(B_1 \setminus B_{1/2})} \leq C_N\|v\|_{H^1(B_1 \setminus B_{1/2})}^2\).

\[
\frac{1}{\sqrt[\gamma]{N-1}}|u(r)| + \frac{1}{\sqrt{\gamma}}|v(r)| \\
\leq \frac{1}{\sqrt[\gamma]{N-1}}|u(r_1) - u(r)| + \frac{1}{\sqrt{\gamma}}|v(r_1) - v(r)| + \frac{1}{\sqrt[\gamma]{N-1}}|u(r_1)| + \frac{1}{\sqrt{\gamma}}|v(r_1)| \\
\leq \frac{1}{\sqrt[\gamma]{N-1}} \sum_{i=1}^{m-1} \left| \frac{r_1}{2i-1} - \frac{r_1}{2i} \right| + \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{m-1} \left| \frac{r_1}{2i-1} - \frac{r_1}{2i} \right| \\
+ \frac{C_N}{\sqrt[\gamma]{N-1}} |u|_{H^1(B_1 \setminus B_{1/2})}^2 + \frac{C_N}{\sqrt{\gamma}} |v|_{H^1(B_1 \setminus B_{1/2})}^2 \\
\leq C_N \sum_{i=1}^{m-1} \left( \frac{r_1}{2i-1} \right)^{-N/2 + \sqrt{N-1} + 2} + \frac{C_N}{\sqrt[\gamma]{N-1}} |u|_{H^1(B_1 \setminus B_{1/2})}^2 + \frac{C_N}{\sqrt{\gamma}} |v|_{H^1(B_1 \setminus B_{1/2})}^2 \\
\leq C_N \left( \sum_{i=1}^{m-1} \left( \frac{r_1}{2i} \right)^{-N/2 + \sqrt{N-1} + 2} + 1 \right) \\
\left( \frac{1}{\sqrt[\gamma]{N-1}} |u|_{H^1(B_1 \setminus B_{1/2})}^2 + \frac{1}{\sqrt{\gamma}} |v|_{H^1(B_1 \setminus B_{1/2})} ^2 \right).
\]

Note that the sign of \(\sqrt{N-1} + 2 - \frac{N}{2}\) is crucial in getting estimates. Since \(\sqrt{N-1} + 2 - \frac{N}{2} = 0\) if and only if \(N = 10\), this dimension is the
4.3 Explicit nonlinearities

critical dimension. From the above, for any $0 < r \leq 1$ we get if $2 \leq N < 10$, 
$$ \frac{1}{\sqrt{\lambda}} |u(r)| + \frac{1}{\sqrt{\gamma}} |v(r)| \leq C_N \left( \frac{1}{\sqrt{\lambda}} ||u||_{H^1(B_1 \setminus B_{1/2})} + \frac{1}{\sqrt{\gamma}} ||v||_{H^1(B_1 \setminus B_{1/2})} \right), $$
if $N = 10$,
$$ \frac{1}{\sqrt{\lambda}} |u(r)| + \frac{1}{\sqrt{\gamma}} |v(r)| $$
$$ \leq C_N (1 + |\log r|) \left( \frac{1}{\sqrt{\lambda}} ||u||_{H^1(B_1 \setminus B_{1/2})} + \frac{1}{\sqrt{\gamma}} ||v||_{H^1(B_1 \setminus B_{1/2})} \right), $$
if $N > 10$,
$$ \frac{1}{\sqrt{\lambda}} |u(r)| + \frac{1}{\sqrt{\gamma}} |v(r)| $$
$$ \leq C_N r^{-\frac{N}{2} + \sqrt{N-1} + 2} \left( \frac{1}{\sqrt{\lambda}} ||u||_{H^1(B_1 \setminus B_{1/2})} + \frac{1}{\sqrt{\gamma}} ||v||_{H^1(B_1 \setminus B_{1/2})} \right). $$

Passing to limits we obtain the desired estimates for $u^*, v^*$.

\[ \square \]

4.3 Explicit nonlinearities

We now examine the case of polynomial nonlinearities and for this we recall the definition $t_+(p) = p + \sqrt{p(p-1)}$ and note that $t_+$ is increasing on $[1, \infty)$.

We begin with the gradient system.

Theorem 4.5. Let $f(u) = (u + 1)^p, g(v) = (v + 1)^q$ and suppose $p, q > 2$.

1. Let $(\lambda^*, \gamma^*) \in \Upsilon$. Then the associated extremal solution of $(G)_{\lambda^*, \gamma^*}$ is bounded provided

$$ \frac{N}{2} < 1 + \frac{2}{p + q - 2} \max\{t_+(p - 1), t_+(q - 1)\}. \quad (4.13) $$

2. Let $(\lambda^*, \gamma^*) \in \Upsilon$ and define

$$ I_{p,q,\lambda,\gamma}(t) := p + q - 1 - \frac{t^2}{2t - 1} + \frac{2pq}{p + q} \left( \frac{\gamma q}{\lambda p} \right)^{t+q-1} - 1. $$

Let $t_0 := \max\{t_+(p - 1), t_+(q - 1)\}$ and suppose that $I_{p,q,\lambda^*,\gamma^*}(t_0) > 0$ and $I_{q,p,\lambda^*,\gamma^*}(t_0) > 0$. 

64
4.3. Explicit nonlinearities

The map \( t \mapsto \min \{ I_{p,q,\lambda^*,\gamma^*}(t), I_{q,p,\lambda^*,\gamma^*}(t) \} \) is decreasing and has a root in \((t_0, \infty)\), which we denote by \( T \). Suppose

\[
\frac{N}{2} < 1 + \frac{2}{p+q-2} T.
\] (4.14)

Then the associated extremal solution of \((G)_{\lambda^*,\gamma^*}\) is bounded.

Remark 4.2. Note that the condition on \( t_0 \) from the second part of Theorem 4.5 is really a condition on how close the parameters \((\lambda^*, \gamma^*)\) are to the “diagonal” given by \( \lambda^* p = \gamma^* q \). On the diagonal one trivially sees the condition is satisfied. Some algebra shows that the condition is satisfied provided \((\lambda^*, \gamma^*)\) lie within the cone

\[
\left( 1 \frac{p+q}{2pq} \left[ p+q-1-\max\{p-1,q-1\} \right] \right) \frac{1}{q+q-1} < \frac{\gamma^* q}{\lambda^* p} < \\
\left( 1 \frac{p+q}{2pq} \left[ p+q-1-\max\{p-1,q-1\} \right] \right) \frac{1}{q+q-1}.
\]

Theorem 4.6. Let \( f(u) = (u+1)^p \), \( g(v) := (v+1)^q \) with \( p, q > 1 \).

Suppose

\[
N < \min \left\{ 4 + \frac{2}{p-1} \sqrt{\frac{p}{p-1}}, 4 + \frac{2}{q-1} + 2 \sqrt{\frac{q}{q-1}} \right\},
\]

and \((\lambda^*, \gamma^*) \in \Upsilon\). Then the associated extremal solution of \((H)_{\lambda^*,\gamma^*}\) is bounded.

Remark 4.3. Note that \( p \mapsto 4 + \frac{2}{p-1} + 2 \sqrt{\frac{p}{p-1}} \) is decreasing and goes to 6 as \( p \to \infty \). Hence for \( N \leq 6 \) we see all extremal solutions are bounded for any \( p, q \). As in the case of \((G)_{\lambda,\gamma}\) one can obtain better results provided they restrict the range of the parameters \((\lambda, \gamma)\) to a certain cone with axis given by \( \frac{\gamma}{\lambda} = \frac{q}{p} \), we omit the details.

We begin with some pointwise comparison results.

Lemma 4.3. Let \( f(u) = (u+1)^p \) and \( g(v) = (v+1)^q \) where \( p, q > 1 \).

1. Suppose that \((u, v)\) is a smooth solution of \((G)_{\lambda,\gamma}\) where \( \lambda p \geq \gamma q \). Then

\[
v \leq u \leq \frac{\lambda p}{\gamma q} v.
\]
2. Suppose \((u,v)\) is the smooth minimal solution of \((H)_{\lambda,\gamma}\) where \(q\lambda \geq \gamma p\). Then \(p\gamma u \geq q\lambda v\).

Proof: (1) Subtracting two equations of \((G)_{\lambda,\gamma}\) we get

\[-\Delta(u - v) = (1 + u)^{p-1}(1 + v)^{q-1}(\lambda p(1 + v) - \gamma q(1 + u)) \geq \gamma q(1 + u)^{p-1}(1 + v)^{q-1}(v - u)\]

multiply both sides of the above with \((u - v)_-\) to get \(\int |\nabla(u - v)_-|^2 \leq 0\) and therefore \(v \leq u\). Now, multiply the second equation of \((G)_{\lambda,\gamma}\) with \(\frac{\lambda p}{\gamma q}\) and again subtract two equations to get

\[-\Delta(u - \frac{\lambda p}{\gamma q}v) = \lambda p(1 + u)^{p-1}(1 + v)^{q-1}(v - u) \leq 0\]

From maximum principle we get \(u \leq \frac{\lambda p}{\gamma q}v\).

(2) Set \(K(x) := (u + 1)^{p-1}(v + 1)^{q-1}\). First note that

\[L(u - v) := -\Delta(u - v) - \gamma pK(x)(u - v) = K(x)((\lambda q - \gamma p)u + \lambda q - \gamma p),\]

and note that the right hand side is nonnegative. If we can show that \(L\) satisfies the maximum principle then we’d have \(u - v \geq 0\).

We now assume that \((u,v)\) is the smooth minimal solution of \((H)_{\lambda,\gamma}\) and additional we assume that \((\lambda,\gamma) \in \mathcal{U}\setminus \mathcal{Y}\). By Theorem A there is some \(\eta \geq 0\) and \(\psi > 0\) such that

\[-\Delta \psi - q\gamma K(x)\psi \geq \eta \psi. \quad (4.15)\]

Since \((\lambda,\gamma) \notin \mathcal{Y}\) one can infact show that \(\eta > 0\). Hence the linear operator on the left satisfies the maximum principle. Since \(q > 1\) we see that \(L\) must also satisfy the maximum principle and hence \(u \geq v\). In the case where \((\lambda,\gamma) \in \mathcal{U}\cap \mathcal{Y}\) we pass to the limit along the fixed parameter ray through \((0,0)\) and \((\lambda,\gamma)\) and use the above result. Hence we have shown that \(u \geq v\).

Now set \(t := \frac{\lambda q}{\gamma p}\) and then note that

\[-\Delta(u - tv) = K(x)\lambda q((u + 1) - (v + 1)) \geq 0\]

and hence \(u \geq tv\).

\[\Box\]

Before we prove Theorem 4.5 we need a general energy estimate.
4.3. Explicit nonlinearities

**Lemma 4.4.** Let \( f(u) = (u + 1)^p \) and \( g(v) = (v + 1)^q \). Suppose that \((u,v)\) is a semi-stable solution of \((G)_{\lambda,\gamma} \) with \( s, t \in \mathbb{R} \setminus \{\frac{1}{2}\} \). Then

\[
\begin{align*}
p \left( p - 1 - \frac{t^2}{2t - 1} \right) &\int (1 + u)^{2t+p-2}(1 + v)^q \\
+ q \left( q - 1 - \frac{s^2}{2s - 1} \right) &\int (1 + v)^{2s+q-2}(1 + u)^p \\
+ 2pq \int (1 + u)^{t+p-1}(1 + v)^{s+q-1} &+ p(p - 1) \int (1 + u)^{p-2}(1 + v)^q \\
+ q(q - 1) &\int (1 + u)^p(1 + v)^{q-2} + p \frac{t^2}{2t - 1} \int (1 + u)^{p-1}(1 + v)^q \\
+ q \frac{s^2}{2s - 1} &\int (1 + u)^p(1 + v)^{q-1} \\
\leq 2p(p - 1) &\int (1 + u)^{t+p-2}(1 + v)^q + 2pq \int (1 + u)^{t+p-1}(1 + v)^{q-1} \\\n+ 2q(q - 1) &\int (1 + u)^p(1 + v)^{s+q-2} + 2pq \int (1 + u)^{p-1}(1 + v)^{s+q-1}
\end{align*}
\]

*Proof.* This is an application of Lemma 4.1. Take \( \phi := (1 + u)^t - 1 \) and \( \psi := (1 + v)^s - 1 \) in (4.2), then we have

\[
\begin{align*}
p(p - 1) &\int (1 + u)^{p-2}(1 + v)^q ((1 + u)^t - 1)^2 \\
+ q(q - 1) &\int (1 + v)^{q-2}(1 + u)^p ((1 + v)^s - 1)^2 \\
+ 2pq &\int (1 + u)^{p-1}(1 + v)^{q-1} ((1 + u)^t - 1) ((1 + v)^s - 1) \\
\leq t^2 &\int |\nabla u|^2(1 + u)^{2t-2} + \frac{s^2}{\gamma} \int |\nabla v|^2(1 + v)^{2s-2}
\end{align*}
\]

(4.16)

Multiply the first and the second equation of \((G)_{\lambda,\gamma} \) with \((1 + u)^{2t-1} - 1\) and \((1 + v)^{2s-1} - 1\), respectively, to get

\[
(2t-1) \int |\nabla u|^2(1+u)^{2t-2} = \lambda p \int (1+u)^{2t+p-2}(1+v)^q - \lambda p \int (1+u)^{p-1}(1+v)^q
\]

and

\[
(2s-1) \int |\nabla v|^2(1+v)^{2s-2} = \gamma q \int (1+v)^{2s+q-2}(1+u)^p - \gamma q \int (1+v)^{q-1}(1+u)^p
\]

Using these identities and (4.16) finishes the proof.

\[\square\]
4.3. Explicit nonlinearities

Proof of Theorem 4.5. (1) Let $(\lambda^*, \gamma^*) \in \Upsilon$ and let $(u, v)$ denote a smooth minimal solution on the ray $\Gamma_\sigma$ where $\sigma := \gamma^*/\lambda^*$. Let $1 < t < t_+(p - 1)$ and $1 < s < t_+(q - 1)$ in Lemma 4.4 to arrive at an inequality of the form

$$\int (u + 1)^{2t+p-2}(v + 1)^q + \int (u + 1)^p(v + 1)^{2s+q-2} \leq C_{t,s}.$$ 

First note that

$$\int |\nabla u|^2 \leq p\lambda^* \int (u + 1)^p(v + 1)^q \leq C_{t,s},$$

provided $p < 2t_+(p - 1) + p - 2$ but this holds since $p > 1$ and by passing to the limit we see that $u^* \in H^1_0(\Omega)$. We similarly show that $v^* \in H^1_0(\Omega)$.

Without loss of generality assume that $p \geq q$ and hence $t_+(q - 1) \leq t_+(p - 1)$ and so we have

$$\int (u + 1)^{2t+p+q-2} \leq C_t,$$

for all $1 < t < t_+(p - 1)$. We now re-write the equation as

$$-\frac{\Delta u^*}{\lambda^*} = c(x)u^* + p(v^* + 1)^q,$$

where

$$0 \leq c(x) = p\frac{(u^* + 1)^{p-1} - 1}{u^*}(v^* + 1)^q \leq C(u^* + 1)^{p+q-2}.$$ 

We now apply regularity theory to see that $u^*$ is bounded provided $c(x), (v^* + 1)^q \in L^T$ for some $T > N/2$. But this holds provided

$$(p + q - 2) \frac{N}{2} < 2t_+(p - 1) + p + q - 2,$$

which is the desired result. To see $v^*$ is bounded we now use the pointwise comparison between $u$ and $v$ and pass to the limit along the ray $\Gamma_\sigma$.

(2) In (1) we only used the first two integrals from Lemma 4.4 to obtain estimates. In this part we also use the third integral. Let $(\lambda^*, \gamma^*) \in \mathcal{U}$ and let $(u, v)$ denote the minimal solution on the ray $\Gamma_\sigma$, where $\sigma := \gamma^*/\lambda^*$. The exact proof depends on the sign of $\lambda^*p - \gamma^*q$ and we suppose that $\lambda^*p \geq \gamma^*q$. Let $t_0 < t < T$ and so $I_{p,q,\lambda^*,\gamma^*}(t), I_{q,p,\lambda^*,\gamma^*}(t) > 0$ and $p - 1 - \frac{t^2}{2t - 1}, q - 1 - \frac{t^2}{2t - 1} <
0. We now set \( s = t \) and examine the estimate from Lemma 4.4. Note the coefficients in front of the first two integrals are negative and the coefficient in front of the third integral is positive. The other integrals on the left are lower order terms which we drop. Now note that \( u \geq v \) and so we can replace, since the coefficients are negative, the \( u \)'s in the first two integrals from the estimate in Lemma 4.4 with \( v \)'s. In the third integral we use the fact that \( \frac{\gamma q}{\lambda p}(u + 1) \leq v + 1 \). Writing this all out and then again using the fact that we can compare \( u \) and \( v \), one can see that the following is a lower bound for the left-hand side of the integral estimate given by Lemma 4.4:

\[
\int \left(1 + \frac{u}{1}ight)^{p - 1 - \frac{t^2}{2t - 1}} + q \left(1 - \frac{t^2}{2t - 1}\right) + 2pq \left(\frac{\gamma q}{\lambda p}\right)^{t+q-1} \]

\[
\int (1 + u)^{2t+p+q-2} + (p+q) \left(1 - \frac{t^2}{2t - 1} + 2pq \left(\frac{\gamma q}{\lambda p}\right)^{t+q-1} - 1\right) \]

\[
\int (1 + u)^{2t+p+q-2}.
\]

Combining everything gives an estimate of the form

\[
I_{p,q,\lambda^*,\gamma^*}(t) \int (1 + u)^{2t+p+q-2} \leq C_{p,q,\lambda^*,\gamma^*} \int (1 + u)^{t+p+q-2}.
\]

Since \( I_{p,q,\lambda^*,\gamma^*}(t) > 0 \) we have an estimate. We now proceed exactly as in the first part. We rewrite the equation in the alternate form and we then require that

\[
(p + q - 2) \frac{N}{2} < 2t + p + q - 2,
\]

for some \( t_0 < t \) where \( I_{p,q,\lambda^*,\gamma^*}(t) > 0 \).

\[\square\]

Lemma 4.5. Let \((\lambda^*, \gamma^*) \in \Upsilon\).

and let \((u, v)\) denote a minimal solution of \((H)_{\lambda, \gamma}\) on the ray \( \Gamma_{\sigma} \) where \( \sigma = \frac{\gamma^*}{\lambda^*} \). Then for \( 1 < t < t_+(p) \) and \( 1 < \tau < t_+(q) \) we have

1. \[
\int (u + 1)^{2t+p-1}(v + 1)^{q-1} \leq C,
\]

2. \[
\int (u + 1)^{p-1}(v + 1)^{2\tau+q-1} \leq C,
\]
3. \[
\int (u + 1)^{p+t-1} (v + 1)^{q+\tau-1} \leq C,
\]
where \( C \) is uniform on the ray \( \Gamma_\sigma \).

Proof. Set \( \phi := (u + 1)^t - 1 \) and \( \psi := (v + 1)^\tau - 1 \) and put these into the stability inequality given by (4.3) to arrive at an inequality of the form
\[
q(p - \frac{t^2}{2t - 1}) \int (u + 1)^{2t+p-1} (v + 1)^q - 1
+ p(q - \frac{\tau^2}{2\tau - 1}) \int (u + 1)^{p-1} (v + 1)^{q+2\tau-1}
+ 2\sqrt{p(p-1)}q(q-1) \int (u + 1)^{p+t-1} (v + 1)^{q+\tau-1}
\leq C(p, q) \int (u + 1)^{p+t-1} (v + 1)^q - 1
+ C(p, q) \int (u + 1)^{p-1} (v + 1)^{q+\tau-1}
\]
and note that for the given choices of \( t, \tau \) the coefficients on the left are positive. One now easily sees that the terms on the right are lower order terms and hence we obtain the desired estimates after some standard calculations.

\[
\square
\]

Proof of Theorem 4.6. Without loss of generality we suppose that \( \lambda^* q \geq \gamma^* p \). Let \((u, v)\) denote a minimal solution on the ray \( \Gamma_\sigma \) where \( \sigma := \frac{\gamma^*}{\lambda^*} \).

Note that we have \( u \geq \frac{\lambda^* q}{\gamma^*} v > v \). We first show that \( u^* \in H_0^1 \). First note that
\[
\int |\nabla u|^2 = \lambda q \int (u + 1)^p u(v + 1)^q - 1,
\]
along the ray \( \Gamma_\sigma \) and the right hand side is uniformly bounded provided
\[
p + 1 < p - 1 + 2t^+_1(p),
\]
which is the case, for any \( p > 1 \) and dimension \( N \), after considering the estimates from Lemma 4.5. We now rewrite the equation for \( u^* \) as
\[
-\Delta u^* = \lambda q \left( \frac{(u^* + 1)^p - 1}{u^*} \right) (v^* + 1)^q - 1 u^* + \lambda q (v^* + 1)^q - 1,
\]
and to show \( u^* \) is bounded it is sufficient to show that \((u^*+1)^{p-1}(v^*+1)^{q-1} \in L^r\) for some \( r > \frac{N}{2} \). Using Lemma 4.5 one sees this is the case provided
\[
\frac{N}{2}(p-1) < p-1 + t_+(p), \quad \frac{N}{2}(q-1) < q-1 + t_+(q).
\]
So we have shown that \( u^* \) is bounded and we now use the fact that \( u^* \geq v^* \) to see the same for \( v^* \).

\[
\square
\]

4.4 Summary and conclusions

In this chapter, following ideas given for the gradient systems in the last chapter, we examined the two elliptic systems on a bounded domain:

\[(G)_{\lambda, \gamma} - \Delta u = \lambda f'(u)g(v), \quad -\Delta v = \gamma f(u)g'(v) \quad \text{in} \; \Omega,\]

and

\[(H)_{\lambda, \gamma} - \Delta u = \lambda f(u)g'(v), \quad -\Delta v = \gamma f'(u)g(v) \quad \text{in} \; \Omega,\]

with zero Dirichlet boundary conditions and where \( \lambda, \gamma \) are positive parameters. We showed that for general nonlinearities \( f \) and \( g \) that the extremal solutions associated with \((G)_{\lambda, \gamma}\) are bounded provided \( \Omega \) is a convex domain in \( \mathbb{R}^n \) where \( n \leq 3 \). In the case of a radial domain we showed the extremal solutions are bounded provided \( n < 10 \).

The extremal solutions associated with \((H)_{\lambda, \gamma}\) are bounded in the case where \( f \) is arbitrary, \( g(v) = (v + 1)^q \) where \( 1 < q < \infty \) and where \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \), \( n \leq 3 \).

Results were also obtained in higher dimensions for \((G)_{\lambda, \gamma}\) and \((H)_{\lambda, \gamma}\) for the case of explicit nonlinearities of the form \( f(u) = (u + 1)^p \) and \( g(v) = (v + 1)^q \).
Chapter 5

The Hénon-Lane-Emden conjecture

5.1 Introduction and main results

We consider the following weighted system

\[
\begin{align*}
-\Delta u &= |x|^av^p \quad \text{in } \mathbb{R}^N, \\
-\Delta v &= |x|^bu^q \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

(5.1)

where \(pq > 1\) and \(p, q, a, b \geq 0\) and \(\Omega\) is a subset of \(\mathbb{R}^N\), \(N \geq 1\).

We start by noting that in the case of the Lane-Emden scalar equation (i.e., when \(p = q\) and \(a = b = 0\)) on a bounded star-shaped domain \(\Omega \subset \mathbb{R}^N\), the Pohozaev inequality shows that there is no positive solution satisfying the Dirichlet boundary condition, whenever \(p \geq \frac{N+2}{N-2}\), the critical Sobolev exponent. On the other hand, a celebrated theorem by Gidas-Spruck [58] states that there is no positive solution for the Lane-Emden equation on the whole space whenever \(p < \frac{N+2}{N-2}\) for \(N \geq 3\). This non-existence result is also optimal as shown by Gidas, Ni and Nirenberg in [57] under the assumption that \(u = O(|x|^{2-N})\), and by Caffarelli, Gidas and Spruck in [19] without the growth assumption. See also Chen and Li [24] for an easier proof based on the moving planes method. Also, Lin [63] using moving plane methods proved similar optimal non-existence results for \(p < \frac{N+4}{N-4}, N > 4\) in the case of the fourth order Lane-Emden equation (i.e., when \(p > 1 = q\) and \(a = b = 0\)).

In the case of the system (5.1), one can again use the Pohozaev identity whenever \(\Omega\) is a bounded star-shaped domain in \(\mathbb{R}^N\), to establish the following non-existence result.

**Theorem A.** [49, 82] *Let \(N \geq 3\) and let \(\Omega \subset \mathbb{R}^N\) be a star-shaped bounded domain. If*

\[
\frac{N+a}{p+1} + \frac{N+b}{q+1} \leq N - 2,
\]

(5.2)
then there is no positive solution for (5.1) on $\Omega$ that satisfy the Dirichlet boundary conditions.

By noting that the curve $\frac{N+a}{p+1} + \frac{N+b}{q+1} = N - 2$ is the critical Sobolev hyperbola, the above theorem states that the Liouville-type result for positive solutions on bounded star-shaped domain holds when $(p, q)$ is above the critical hyperbola. It is therefore expected that – just like the case of the scalar Lane-Emden equation ($p = q$ and $a = b = 0$) – the non-existence of solutions on the whole space $\mathbb{R}^N$ should occur exactly when $(p, q)$ is in the complementary domain, that is when it is under the critical hyperbola.

This is the statement of the following Hénon-Lane-Emden conjecture.

**Conjecture 2.** Suppose $(p, q)$ is under the critical hyperbola, i.e.,

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2.$$  \hspace{1cm} (5.3)

Then there is no positive solution for system (5.1).

Proving such a non-existence result seems to be challenging even for the Lane-Emden conjecture (i.e., when $a = b = 0$) for systems. The case of radial solutions was solved by Mitidieri [68] in any dimension, and both Mitidieri [68] and Serrin-Zou [90] constructed positive radial solutions on and above the critical hyperbola, i.e. $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$, which means that the non-existence theorem is optimal for radial solutions.

For non-radial solutions of the Lane-Emden system, there are the results of Souto [92], Mitidieri [68] and Serrin-Zou [93] who proved the non-existence of solutions in dimensions $N = 1, 2$, while in dimension $N = 3$, Serrin-Zou [93] gave a proof for the non-existence of polynomially bounded solutions, an assumption that was removed later by Poláčik, Quittner and Souplet [81]. More recently, Souplet [91] settled completely the conjecture in dimension $N = 4$, while providing in dimensions $N \geq 5$, a more restrictive new region for the exponents $(p, q)$ that insures non-existence.

**Theorem B. (Souplet [91])** Assume $a = b = 0$.

(i) Let $N = 4$ and $p, q > 0$. If $(p, q)$ satisfies

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$  \hspace{1cm} (5.4)

then system (5.1) has no positive solutions.
5.1. Introduction and main results

(ii) Let \( N \geq 5 \), and \( p, q > 0 \) with \( pq > 1 \). If \((p, q)\) satisfies (5.4), along with

\[
2 \max \left\{ \frac{p + 1}{pq - 1}, \frac{q + 1}{pq - 1} \right\} > N - 3, \tag{5.5}
\]

then every non-negative solution of system (5.1) is necessarily trivial.

The Lane-Emden conjecture in dimensions \( N \geq 5 \) is still open. The Hénon-Lane-Emden conjecture is even less understood. Even for the scalar case \( a = b \) and \( p = q \) (i.e., the Hénon equation), Gidas and Spruck in [58] solved the conjecture only for radial solutions, also showing that in this case, the non-existence result is optimal. For non-radial solutions, they proved some partial results such as the non-existence of positive solutions for \( a \geq 2 \) and \( p \leq \frac{N+2}{N-2} \) (the Sobolev critical exponent for \( a = 0 \)).

For systems, Mitidieri [68] gave a partial solution to the conjecture for radial solutions by showing that there is no positive radial solution for (5.1) for all \( N \geq 3 \) provided \( p, q > 1 \) satisfy

\[
\frac{N + \min\{a, b\}}{p + 1} + \frac{N + \min\{a, b\}}{q + 1} > N - 2, \tag{5.6}
\]

Recently, Bidaut-Veron-Giacomini [14] used a Pohozaev type argument and a suitable change of variables to give a complete solution in the radial case.

**Theorem C. (Bidaut-Veron-Giacomini [14])** For \( N \geq 3 \), system (5.1) admits a positive radial solution \((u, v)\) such that \( u, v \in C^2(0, \infty) \cap C([0, \infty)) \) if and only if \((p, q)\) is above or on the critical hyperbola, i.e., when (5.2) holds.

5.1.1 Liouville theorems for bounded non-negative solutions

With the lack of progress on the full conjecture, the attention turned to showing that bounded non-negative solutions are necessarily trivial. Recently, Phan and Souplet [80] showed among other things that the Hénon-Lane-Emden conjecture for the scalar case holds for bounded positive solutions in dimension \( N = 3 \).

**Theorem D. (Phan-Souplet [80])** Let \( N = 3 \), \( a = b > 0 \) and \( p = q > 1 \). Assume \((p, q)\) satisfies (5.3), then there is no positive bounded solution for the Hénon equation, i.e.,

\[
-\Delta u = |x|^a u^p \quad \text{in} \quad \mathbb{R}^N. \tag{5.7}
\]
In this note, we shall first extend the above result of Phan-Souplet \[80\] to the full Hénon-Lane-Emden system by showing the following\(^1\).

**Theorem 5.1.** Suppose \( N = 3 \) and \((p, q)\) satisfy (5.3). Then, there is no positive bounded solution for (5.1).

We also give a few partial results for the Hénon equation whether of second order or fourth order in all dimensions \( N \geq 3 \) or \( N \geq 5 \).

We note that Miditieri and Pohozaev \[70\] have shown that the above result holds in higher dimension provided the following stronger condition holds:

\[
\max\{\alpha, \beta\} \geq N - 2,
\]

where \( \alpha := \frac{(b+2)p+(a+2)}{pq-1} \) and \( \beta := \frac{(a+2)q+(b+2)}{pq-1} \). For that they used a rescaled test-function method (as in Lemma 5.1 below) to prove the result for \( p, q \geq 1 \). More recently, Armstrong and Sirakov \[6\] proved –among other things– similar results for \( p, q > 0 \), by developing new maximum principle type arguments. We are thankful to P. Souplet for informing us of these latest developments by Armstrong and Sirakov.

### 5.1.2 Liouville theorems for stable non-negative solutions

We shall also consider in the scalar case the question of existence of solutions with finite Morse index solutions (as opposed to bounded solutions). For scalar equations, we get the following counterpart to the Phan-Souplet result in higher dimensions \( (N \geq 3) \).

**Theorem 5.2.** Let \( a \geq 0, \ p > 1 \) and \( N \geq 3 \). Then, for any Sobolev sub-critical exponent, i.e.,

\[
1 < p < \frac{N + 2 + 2a}{N - 2},
\]

equation (5.7) has no positive solution with finite Morse index.

We also have the following result for the fourth order equation,

\[
\Delta^2 u = |x|^au^p \quad \text{in} \quad \mathbb{R}^N.
\]  

\(^1\)Upon receiving our preprint, P. Souplet informed us that Q.H. Phan has also proved the same result in dimension \( N = 3 \), as well as other interesting results in higher dimensions. Our proofs are quite similar since both are essentially refinements of those of P. Souplet in his groundbreaking work on the Lane-Emden conjecture for systems.
5.2 Proof in the case of non-negative solutions

**Theorem 5.3.** Let $a \geq 0$, $p > 1$ and $N \geq 5$. Then, for any Sobolev sub-critical exponent, i.e.,

$$1 < p < \frac{N + 4 + 2a}{N - 4},$$

equation (5.8) has no positive solution with finite Morse index.

For systems, we have the following result.

**Theorem 5.4.** Suppose that $0 \leq a - b \leq (N - 2)(p - q)$. Then, system (5.1) has no positive stable solution whenever the dimension satisfy

$$N < 2 + 2 \left(\frac{p(b + 2) + a + 2}{pq - 1}\right) \left(\sqrt{\frac{pq(q + 1)}{p + 1}} + \frac{pq(q + 1)}{p + 1} - \sqrt{\frac{pq(q + 1)}{p + 1}}\right).$$

(5.9)

The case when $a = b = 0$ (i.e., the Lane-Emden system) was already established by by Cowan in [29]. Note that this result contains the result of Fazly in [49], who had considered the case $q = 1 < p$, $a = b$ and shown the result under the condition,

$$N < 8 + 3a + \frac{8 + 4a}{p - 1},$$

(5.10)

which is already larger than the domain under the critical hyperbola, i.e. $N < 4 + a + \frac{8 + 4a}{p - 1}$. Also, this contains the result of Wei-Ye in [97] who had considered the case $q = 1 < p$, $a = b = 0$. There are also various results for the cases where $-2 < a, b < 0$ and $pq \leq 1$. For that we refer to [14, 49, 56, 58, 59, 70, 80].

5.2 Proof in the case of non-negative solutions

In this section, we shall prove here Theorem 5.1. The main tools will be Pohozaev-type identities as well as various integral estimates.

The proof is heavily inspired by ideas of Souplet [91] and Serrin-Zou [93]. We use Pohozaev-type identities, various integral estimates, as well as some elliptic estimates on the sphere. Throughout this section, all norms refer to functions defined on the unit sphere, i.e. $\|u\| := \|u\|_{L^p(S^{N-1})}$. We start with the following estimate on the non-linear terms. Note that for $a = b = 0$, this was proved by Serrin and Zou [93] via ODE techniques,
and by Miditieri and Pohozaev [70] who used the following rescaled test functions approach for \(a, b > -2\). For the sake of convenience of readers, we recall the proof. Interested readers can find more details for both scalar and system cases in [83].

**Lemma 5.1.** For any positive entire solution \((u, v)\) of (5.1) and \(R > 1\), there holds

\[
\int_{B_R} |x|^a v^p \leq C R^{N-2-\frac{(b+2)p+(a+2)}{pq-1}},
\]

(5.11)

\[
\int_{B_R} |x|^b u^q \leq C R^{N-2-\frac{(a+2)p+(b+2)}{pq-1}},
\]

(5.12)

where the positive constant \(C\) does not depend on \(R\).

**Proof:** Fix the following function \(\zeta_R \in C^2_c(\mathbb{R}^N)\) with \(0 \leq \zeta_R \leq 1\);

\[
\zeta_R(x) = \begin{cases} 
  1, & \text{if } |x| < R; \\
  0, & \text{if } |x| > 2R;
\end{cases}
\]

where \(||\nabla \zeta_R||_\infty \leq \frac{C}{R}\) and \(||\Delta \zeta_R||_\infty \leq \frac{C}{R^2}\). For fixed \(m \geq 2\), we have

\[
|\Delta \zeta_R^m(x)| \leq C \begin{cases} 
  0, & \text{if } |x| < R \text{ or } |x| > 2R; \\
  R^{-2} \zeta_R^{m-2}, & \text{if } R < |x| < 2R;
\end{cases}
\]

For \(m \geq 2\), test the first equation of (5.1) by \(\zeta_R^m\) and integrate to get

\[
\int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m = -\int_{\mathbb{R}^N} \Delta u \zeta_R^m \\
= -\int_{\mathbb{R}^N} u \Delta \zeta_R^m \leq C R^{-2} \int_{B_{2R} \setminus B_R} u \zeta_R^{m-2}.
\]

Applying Hölder’s inequality we get

\[
\int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m \leq C R^{-2} \left( \int_{B_{2R} \setminus B_R} |x|^{-\frac{b}{q'}} \right)^{\frac{1}{q'}} \left( \int_{B_{2R} \setminus B_R} |x|^b u^q \zeta_R^{(m-2)q} \right)^{\frac{1}{q}} \\
\leq C R^{(N-\frac{b}{q'})\frac{1}{q'}-2} \left( \int_{B_{2R} \setminus B_R} |x|^b u^q \zeta_R^{(m-2)q} \right)^{\frac{1}{q}}.
\]

By a similar calculation for \(k \geq 2\), we obtain

\[
\int_{\mathbb{R}^N} |x|^b u^q \zeta_R^k \leq C R^{(N-\frac{a}{p'})\frac{1}{p'}-2} \left( \int_{B_{2R} \setminus B_R} |x|^a u^p \zeta_R^{(k-2)p} \right)^{\frac{1}{p}},
\]

77
5.2. Proof in the case of non-negative solutions

where $\frac{1}{p} + \frac{1}{p'} = 1$. Since $pq > 1$, for large enough $k$ we have $2 + \frac{k}{q} < (k - 2)p$. So, we can choose $m$ such that $2 + \frac{k}{q} \leq m \leq (k - 2)p$ which means that $m \leq (k - 2)p$ and $k \leq (m - 2)q$. By collecting the above inequalities we get for $pq > 1$,

$$
\left( \int_{\mathbb{R}^N} |x|^a v^p \zeta_R^m \right)^{pq} \leq C R^{(N - \frac{a}{pq} - 2)\frac{1}{q} - \frac{a}{pq}} \left( \int_{B_R} |x|^b u^q \zeta_R^m \right)^p
$$

and

$$
\left( \int_{\mathbb{R}^N} |x|^b u^q \zeta_R^k \right)^{pq} \leq C R^{(N - \frac{b}{pq} - 2)\frac{1}{p} - \frac{b}{pq}} \left( \int_{B_R} |x|^a v^p \zeta_R^m \right)^q
$$

By using Hölder’s inequality, we can now get the following $L^1$-estimates.

**Corollary 5.1.** With the same assumptions as Lemma 5.1, we have

$$
\int_{B_R} v^s \leq C R^{N - \frac{(a+2)q + (b+2)}{pq-1} s},
$$

$$
\int_{B_R} u^t \leq C R^{N - \frac{(b+2)p + (a+2)}{pq-1} t},
$$

for any $0 < t < q$ and $0 < s < p$ where the positive constant $C$ does not depend on $R$.

We now recall the following fundamental elliptic estimates.

**Lemma 5.2.** (Sobolev inequalities on the sphere $S^{N-1}$) Let $N \geq 2$, integer $j \geq 1$ and $1 < k < m \leq \infty$. For $z \in W^{j,k}(S^{N-1})$, we have

$$
\|z\|_{L^m(S^{N-1})} \leq C (\|D^j_{\theta} z\|_{L^k(S^{N-1})} + \|z\|_{L^1(S^{N-1})}),
$$

where

$$
\left\{ \begin{array}{ll}
\frac{1}{k} - \frac{1}{m} = \frac{j}{N-1}, & \text{if } k < (N - 1)/j, \\
 m = \infty, & \text{if } k > (N - 1)/j,
\end{array} \right.
$$

and $C = C(j, k, N) > 0$.  

78
5.2. Proof in the case of non-negative solutions

**Lemma 5.3.** (Elliptic $L^p$-estimate on $B_R$). Let $1 < k < \infty$ and $R > 0$. For $z \in W^{2,k}(B_{2R})$, we have
\[
\int_{B_R} |D^2_x z|^k \leq C \left( \int_{B_{2R}} |\Delta z|^k + R^{-2k} \int_{B_{2R}} |z|^k \right),
\]
where $C = C(k, N) > 0$.

**Lemma 5.4.** (An interpolation inequality on $B_R$). Let $R > 0$. For $z \in W^{2,1}(B_{2R})$, we have
\[
\int_{B_R} |D_x z| \leq C \left( R \int_{B_{2R}} |\Delta z|^1 + R^{-1} \int_{B_{2R}} |z| \right),
\]
where $C = C(N) > 0$.

By applying Lemma 5.1, Corollary 5.1 and Lemma 5.4, we obtain the following estimates on the derivatives of $u$ and $v$.

**Lemma 5.5.** We have
\[
\int_{B_R} |D_x v| \leq C R^{N-1-\frac{(a+2)q+(b+2)}{pq-1}},
\]
\[
\int_{B_R} |D_x u| \leq C R^{N-1-\frac{(b+2)p+(a+2)}{pq-1}},
\]
where the positive constant $C$ does not depend on $R$.

**Lemma 5.6.** ($L^1$-regularity estimate on $B_R$) Let $N > 2$ and $1 \leq k \leq \frac{N}{N-2}$. For any $z \in L^1(B_{2R})$ we have
\[
||z||_{L^k(B_R)} \leq C \left( R^{2+N(\frac{1}{k}-1)} ||\Delta z||_{L^1(B_{2R})} + R^{N(\frac{1}{k}-1)} ||z||_{L^1(B_{2R})} \right),
\]
where $C = C(k, N) > 0$.

For $a = b = 0$, the following Pohozaev identity has been obtained by Mitidieri [69], Serrin and Zou [93]. It has also been used by Souplet in [91].

**Lemma 5.7.** (Pohozaev identity). Suppose $\lambda, \gamma \in \mathbb{R}$ satisfy $\lambda + \gamma = N - 2$. If $(u, v)$ is a positive solution of (5.1), then it necessarily satisfy
\[
\left( \frac{N + a}{p + 1} - \lambda \right) \int_{B_R} |x|^a u^{p+1} + \left( \frac{N + b}{q + 1} - \gamma \right) \int_{B_R} |x|^b v^{q+1} = R^{N+a} \int_{S^{N-1}} u^{p+1} + R^{N+b} \int_{S^{N-1}} v^{q+1} + R^N \int_{S^{N-1}} (u_r v_r - R^{-2} u_\theta v_\theta) \\
+ R^{N-1} \int_{S^{N-1}} (\lambda u_r v + \gamma v_r u).
\]
5.2. Proof in the case of non-negative solutions

Now, we are in the position to prove Theorem 5.1.

Proof of Theorem 5.1: Since \((p, q)\) satisfy (5.3), then we can choose \(\lambda\) and \(\gamma\) such that \(N + \frac{a}{p + 1} > \lambda\) and \(N + \frac{b}{q + 1} > \gamma\). Now, for all \(R > 0\) define

\[
F(R) := \left(\frac{N + a}{p + 1} - \lambda\right) \int_{B_R} |x|^a v^{p+1} + \left(\frac{N + b}{q + 1} - \gamma\right) \int_{B_R} |x|^b u^{q+1}.
\]

From Lemma 5.7, we have

\[
F(R) \leq C \left(G_1(R) + G_2(R)\right),
\]

where

\[
G_1(R) := R^{N+a} \int_{S^{N-1}} v^{p+1} + R^{N+b} \int_{S^{N-1}} u^{q+1},
\]

and

\[
G_2(R) := R^N \int_{S^{N-1}} \left(|D_x u(R)| + R^{-1} u(R)\right) \left(|D_x v(R)| + R^{-1} v(R)\right).
\]

Step 1. Upper bounds for \(G_1\) and \(G_2\). Set \(m = \infty\) in Lemma 5.2 to get for either \(t = p + 1\) or \(t = q + 1\)

\[
||u||_t \leq ||u||_{\infty} \leq C(||D^2_x u||_{1+\epsilon} + ||u||_1) \leq C(R^2 ||D^2_x u||_{1+\epsilon} + ||u||_1),
\]

where \(\epsilon > 0\) is small enough and will be chosen later. So,

\[
G_1(R) \leq R^{N+\alpha+2(p+1)} \left(||D^2_x v||_{1+\epsilon} + R^{-2} ||v||_1\right)^{1+p} + R^{N+b+2(q+1)} \left(||D^2_x u||_{1+\epsilon} + R^{-2} ||u||_1\right)^{1+q}.
\]

We now look for the same type bounds for \(G_2\). Apply Schwarz’s inequality to get

\[
G_2(R) \leq R^N \left(\int_{S^{N-1}} (|D_x u(R)| + R^{-1} u(R))^2\right)^{1/2} \left(\int_{S^{N-1}} (|D_x v(R)| + R^{-1} v(R))^2\right)^{1/2} \leq R^N \left(||D_x u||_2 + R^{-1} ||u||_1\right) \left(||D_x v||_2 + R^{-1} ||v||_1\right).
\]

Then, using Lemma 5.2 we obtain the following upper bounds.

\[
||D_x u||_2 \leq C \left(||D^2_x u||_{1+\epsilon} + ||D_x u||_1\right) \leq C \left(R^2 ||D^2_x u||_{1+\epsilon} + ||D_x u||_1\right),
||D_x v||_2 \leq C \left(||D^2_x v||_{1+\epsilon} + ||D_x v||_1\right) \leq C \left(R^2 ||D^2_x v||_{1+\epsilon} + ||D_x v||_1\right).
\]

80
5.2. Proof in the case of non-negative solutions

It follows that

\[ G_2(R) \leq R^{N+2} \left( ||D_x^2 u||_{1+\epsilon} + R^{-1}||D_x u||_1 + R^{-2}||u||_1 \right) \left( ||D_x^2 v||_{1+\epsilon} + R^{-1}||D_x v||_1 + R^{-2}||v||_1 \right). \]  \tag{5.17}

**Step 2.** The following \( L^1 \)-estimates hold in the annulus domain \( B_R \setminus B_{R/2} \):

\[ \int_{R/2}^{R} ||v(r)||_{1} r^{N-1} dr \leq C R^{N-(a+2)p+(b+2)q/pq-1}, \]  \tag{5.18}

\[ \int_{R/2}^{R} ||u(r)||_{1} r^{N-1} dr \leq C R^{N-(b+2)p+(a+2)q/pq-1}, \]  \tag{5.19}

\[ \int_{R/2}^{R} ||D_x v||_{1} r^{N-1} dr \leq C R^{N-1-(a+2)p+(b+2)q/pq-1}, \]  \tag{5.20}

\[ \int_{R/2}^{R} ||D_x u||_{1} r^{N-1} dr \leq C R^{N-1-(b+2)p+(a+2)q/pq-1}, \]  \tag{5.21}

\[ \int_{R/2}^{R} ||D_x^2 v||_{1+\epsilon} r^{N-1} dr \leq C R^{N-2-(a+2)p+(b+2)q/pq-1} + b\epsilon, \]  \tag{5.22}

\[ \int_{R/2}^{R} ||D_x^2 u||_{1+\epsilon} r^{N-1} dr \leq C R^{N-2-(b+2)p+(a+2)q/pq-1} + a\epsilon. \]  \tag{5.23}

To prove \(5.18\)-\(5.21\), we just apply Corollary 5.1 and Lemma 5.5. Here is for example the proof for \(5.23\). Apply Lemma 5.3, Corollary 5.1 and Lemma 5.1 to get

\[ \int_{R/2}^{R} ||D_x^2 u||_{1+\epsilon} r^{N-1} dr = \int_{R/2}^{R} ||D_x^2 u||_{1+\epsilon} dx \]

\[ \leq C \int_{B_{2R}} |\Delta u|^{1+\epsilon} dx + C R^{-2(1+\epsilon)} \int_{B_{2R}} u^{1+\epsilon} dx \]

\[ \leq C R^{N-2-(b+2)p+(a+2)q/pq-1} + a\epsilon + C R^{N-2-(b+2)p+(a+2)q/pq-1} - 2(1+\epsilon) \]

\[ \leq C R^{N-2-(b+2)p+(a+2)q/pq-1} + a\epsilon. \]

The proof of \(5.22\) is similar.
5.2. Proof in the case of non-negative solutions

**Step 3** For large enough $M$, define following sets;

\[
\begin{align*}
\Gamma_1(R) & := \{ r \in (R, 2R); \|v(r)\|_1 > MR^{-\frac{(a+2)q+(b+2)}{pq-1}} \}, \\
\Gamma_2(R) & := \{ r \in (R, 2R); \|u(r)\|_1 > MR^{-\frac{(b+2)p+(a+2)}{pq-1}} \}, \\
\Gamma_3(R) & := \{ r \in (R, 2R); \|D_x v\|_1 > MR^{-1-\frac{(a+2)q+(b+2)}{pq-1}} \}, \\
\Gamma_4(R) & := \{ r \in (R, 2R); \|D_x u\|_1 > MR^{-1-\frac{(b+2)p+(a+2)}{pq-1}} \}, \\
\Gamma_5(R) & := \{ r \in (R, 2R); \|D^2_x v\|_{1+\epsilon} > MR^{-2-\frac{(a+2)q+(b+2)}{pq-1}} \}, \\
\Gamma_6(R) & := \{ r \in (R, 2R); \|D^2_x u\|_{1+\epsilon} > MR^{-2-\frac{(b+2)p+(a+2)}{pq-1}} + \epsilon \}.
\end{align*}
\]

Using (5.23), we get

\[
C \geq R^{-N+2+\frac{(b+2)p+(a+2)}{pq-1}} - \epsilon \int_R^{2R} \|D^2_x u\|_{1+\epsilon}^{N-1} dr
\]

\[
\geq R^{-N+2+\frac{(b+2)p+(a+2)}{pq-1}} - \epsilon \|\Gamma_6(R)\| R^{N-1} MR^{-2-\frac{(b+2)p+(a+2)}{pq-1}} + \epsilon
\]

\[
= M\|\Gamma_6(R)\| R^{-1}.
\]

Therefore, choosing large enough $M$, we get $|\Gamma_6(R)| \leq R/7$. Similarly, using (5.18)-(5.22), one can see $|\Gamma_i(R)| \leq R/7$ for $1 \leq i \leq 5$. Hence, for each $R \geq 1$, we can find

\[
\hat{R} \in (R, 2R) \setminus \bigcup_{i=1}^{6} \Gamma_i(R) \neq \phi. \tag{5.24}
\]

We now have the following upper bounds on (5.16) and (5.17) for the radius $\hat{R}$ given by (5.24);

\[
\begin{align*}
G_1(\hat{R}) & \leq C \hat{R}^{N+a+2(p+1)} \left( \hat{R}^{-\frac{(a+2)q+(b+2)}{pq-1}} - 2 + be \right) \frac{1}{1+\epsilon} + \hat{R}^{-2-\frac{(a+2)q+(b+2)}{pq-1}} \right)^{p+1} \\
& + C \hat{R}^{N+b+2(q+1)} \left( \hat{R}^{-\frac{(b+2)p+(a+2)}{pq-1}} - 2 + \alpha \right) \frac{1}{1+\epsilon} + \hat{R}^{-2-\frac{(b+2)p+(a+2)}{pq-1}} \right)^{q+1} \\
& \leq C \left( \hat{R}^{-a_1(\epsilon)} + \hat{R}^{-a'_1(\epsilon)} \right),
\end{align*}
\]

where

\[
\begin{align*}
a_1(\epsilon) &= (p+1) \left[ \left( \frac{2 + (a+2)q+(b+2)}{pq-1} - be \right) - \frac{1}{1+\epsilon} - 2 - \frac{N+a}{p+1} \right], \\
a'_1(\epsilon) &= (q+1) \left[ \left( \frac{2 + (b+2)p+(a+2)}{pq-1} - \alpha \right) - \frac{1}{1+\epsilon} - 2 - \frac{N+b}{q+1} \right].
\end{align*}
\]
5.3. On solutions of the second order Hénon equation with finite Morse index

Also,

\[ G_2(\hat{R}) \leq C \hat{R}^{N+2} \left( \hat{R}^{-\frac{(b+2)p+(a+2)}{pq-1}} \frac{1}{1+\epsilon} + \hat{R}^{-2-\frac{(a+2)p+(a+2)}{pq-1}} \right) \]

\[ \left( \hat{R}^{-\frac{(a+2)q+(b+2)}{pq-1}} \frac{1}{1+\epsilon} + \hat{R}^{-2-\frac{(a+2)q+(b+2)}{pq-1}} \right) \],

\[ \leq C \hat{R}^{-a_2(\epsilon)}, \]

where

\[ a_2(\epsilon) = -N - 2 + \frac{1}{1+\epsilon} \left( 4 - (a+b)\epsilon + \frac{(b+2)(p+1) + (a+2)(q+1)}{pq-1} \right). \]

Hence, from (5.15) we get

\[ F(R) \leq C \left( G_1(\hat{R}) + G_2(\hat{R}) \right) \leq C R^{-\eta_\epsilon}, \]

where \( \eta_\epsilon := \min\{a_1(\epsilon), a_1'(\epsilon), a_2(\epsilon)\} \) and the positive constant \( C \) does not depend on \( R \). By a straightforward calculation, we have

\[ a_2(0) = -N - 2 + \frac{(b+2)(p+1) + (a+2)(q+1)}{pq-1} > 0 \]

if and only if

\[ \frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2. \]

Also,

\[ a_1(0) > 0, \quad \text{iff} \quad \frac{(a+2)q+(b+2)}{pq-1} > \frac{N+a}{p+1}, \quad (5.25) \]

\[ a_1'(0) > 0, \quad \text{iff} \quad \frac{(b+2)p+(a+2)}{pq-1} > \frac{N+b}{q+1}. \quad (5.26) \]

Now, if \( p \) and \( q \) satisfy (5.3), then (5.25) and (5.26) hold, and we can therefore choose \( \eta_\epsilon > 0 \) for small enough \( \epsilon > 0 \). We now conclude by sending \( R \to \infty \) and get the contradiction.

\[ \square \]

5.3 On solutions of the second order Hénon equation with finite Morse index

We shall prove here Theorem 5.2. For that we recall that a critical point \( u \in C^2(\Omega) \) of the energy functional

\[ I(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |x|^a u^{p+1}. \]
5.3. On solutions of the second order Hénon equation with finite Morse index

is said to be

- a stable solution of (5.7) if for any \( \phi \in C^1_c(\Omega) \), we have
  \[
  I_{uu}(\phi) := \int_\Omega |\nabla \phi|^2 - p \int_\Omega |x|^a u^{p-1} \phi^2 \geq 0.
  \]

- a stable solution outside a compact set \( \Sigma \subset \Omega \) if \( I_{uu}(\phi) \geq 0 \) for all \( \phi \in C^1_c(\Omega \setminus \Sigma) \), also \( u \) has a Morse index equal to \( m \geq 1 \) if \( m \) is the maximal dimension of a subspace \( X_m \) of \( C^1_c(\Omega) \) such that \( I_{uu}(\phi) < 0 \) for all \( \phi \in X_m \setminus \{0\} \).

- a solution with Morse index \( m \) if there exist \( \phi_1, \ldots, \phi_m \) such that \( X_m = \text{Span}\{\phi_1, \ldots, \phi_m\} \subset C^1_c(\Omega) \) and \( I_{uu}(\phi) < 0 \) for all \( \phi \in X_m \setminus \{0\} \).

Note that if \( u \) is of Morse index \( m \), then for all \( \phi \in C^1_c(\Omega \setminus \Sigma) \) we have
\[
I_{uu}(\phi) \geq 0, \quad \text{where } \Sigma = \bigcup_{i=1}^{m_i} \text{supp}(\phi_i),
\]
and therefore \( u \) is stable outside the compact set \( \Sigma \subset \Omega \).

We shall need the following lemma.

Lemma 5.8. Let \( \Omega \subset \mathbb{R}^N \) and let \( u \in C^2(\Omega) \) be a positive stable solution of (5.7). Set \( f(x) = |x|^a, a > 0 \), then, for any \( 1 \leq t < -1 + 2p + 2\sqrt{p(p-1)} \) we have
\[
\int_\Omega (|\nabla u|^2 u^{t-1} + f(x) u^{t+p}) \phi^{2m} \leq C \int_\Omega f(x)^{\frac{t+1}{p-1}} |\nabla \phi|^{\frac{2t+1}{p-1}}, \quad (5.27)
\]
for all \( \phi \in C^1_c(\Omega) \) with \( 0 \leq \phi \leq 1 \) and for large enough \( m \). The constant \( C \) does not depend on \( \Omega \) and \( u \).

Proof: The following proof also holds true for weak solutions. The ideas are adapted from [43–45]. Note first that for any stable solution of (5.7) and \( \eta \in C^1_c(\Omega) \), we have the following:
\[
p \int_\Omega |x|^a u^{p-1} \eta^2 \leq \int_\Omega |\nabla \eta|^2, \quad (5.28)
\]
\[
p \int_\Omega |x|^a u^p \eta = \int_\Omega \nabla u \cdot \nabla \eta. \quad (5.29)
\]

Test (5.29) on \( \eta = u^t \phi^2 \) for \( \phi \in C^1_c(\Omega) \) for an appropriate \( t \in \mathbb{R} \) that will be chosen later, to get
\[
\int_\Omega |x|^a u^{t+p} \phi^2 = \int_\Omega \nabla u \cdot \nabla (u^t \phi^2)
\]
\[
= t \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + 2 \int_\Omega u^t \nabla u \cdot \nabla \phi \phi.
\]

84
5.3. On solutions of the second order Hénon equation with finite Morse index

Apply Young’s inequality\(^2\) to \((\nabla u | u^{t+1} \phi \nabla \phi)\) to obtain
\[
(t - \epsilon) \int_\Omega |\nabla u|^2 u^{t-1} \partial \phi^2 \leq C_\epsilon \int_\Omega u^{t+1} |\nabla \phi|^2 + \int_\Omega |x|^a u^{t+p} \phi^2. \tag{5.30}
\]

Now, test (5.28) on \(u^{t+1} \phi\) to get
\[
p \int_\Omega |x|^a u^{t+p} \phi^2 \leq \frac{(t+1)^2}{4} \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + \int_\Omega u^{t+1} |\nabla \phi|^2 + (t+1) \int_\Omega u^{t} \nabla u \cdot \nabla \phi \phi \leq \frac{(t+1)^2}{4} + 2\epsilon \int_\Omega |\nabla u|^2 u^{t-1} \phi^2 + (C'_{\epsilon,t} + C''_{\epsilon,t}) \int_\Omega u^{t+1} |\nabla \phi|^2,
\]
where again we have used Young’s inequality in the last estimate. Combine now this inequality with (5.30) to see
\[
\left( p - \frac{(t+1)^2}{4} + 2\epsilon \right) \int_\Omega |x|^a u^{t+p} \phi^2 \leq \left( \frac{(t+1)^2}{4} + 2\epsilon \right) C_\epsilon + C'_{\epsilon,t} + C''_{\epsilon,t} \int_\Omega u^{t+1} |\nabla \phi|^2. \tag{5.31}
\]

For an appropriate choice of \(t\), given in the assumption, we see that the coefficient in L.H.S. is positive for \(\epsilon\) small enough. Therefore, replacing \(\phi\) with \(\phi^m\) for large enough \(m\) and applying Hölder’s inequality with exponents \(\frac{t+p}{t+1}\) and \(\frac{t+p}{p-1}\) we obtain
\[
\int_\Omega |x|^a u^{t+p} \phi^{2m} \leq D_{\epsilon,t,m} \int_\Omega |x|^{-\frac{t+1}{p-1}} |\nabla \phi|^2 \frac{t+p}{p-1}. \tag{5.32}
\]

Note that both exponents are greater than 1 for \(t\) given in (i) and (ii).

On the other hand, combining (5.30) and (5.31) gives us
\[
\int_\Omega |\nabla u|^2 u^{t-1} \phi^2 \leq D'_{\epsilon,t} \int_\Omega u^{t+1} |\nabla \phi|^2.
\]

Similarly, replace \(\phi\) by \(\phi^m\) and apply Hölder’s inequality with exponents \(\frac{t+p}{t+1}\) and \(\frac{t+p}{p-1}\) to get
\[
\int_\Omega |\nabla u|^2 u^{t-1} \phi^{2m} \leq D''_{\epsilon,t,m} \int_\Omega |x|^{-\frac{t+1}{p-1}} |\nabla \phi|^2 \frac{t+p}{p-1}.
\]

\(^2\)For any \(a, b, \epsilon > 0\), \(ab \leq \epsilon a^2 + C(\epsilon)b^2\), for some \(C(\epsilon)\).
5.3. On solutions of the second order Hénon equation with finite Morse index

This inequality and (5.32) finish the proof of (5.27).

Now, we are in the position to prove the theorem.

**Proof of Theorem 5.2:** We proceed in the following steps.

**Step 1:** We have the following standard Pohozaev type identity on any \( \Omega \subset \mathbb{R}^N \).

\[
\frac{N + a}{p + 1} \int_{\Omega} |x|^a u^{p+1} - \frac{N - 2}{2} \int_{\Omega} |\nabla u|^2 = \frac{1}{p + 1} \int_{\partial \Omega} |x|^a u^{p+1} x \cdot \nu \\
+ \int_{\partial \Omega} x \cdot \nabla u \nu \cdot \nabla u \\
- \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 x \cdot \nu. \tag{5.33}
\]

To get (5.33), just multiply both sides of (5.7) by \( x \cdot \nabla u \), do integration by parts and collect terms.

**Step 2:** The following estimates hold:

\[
|\nabla u| \in L^2(\mathbb{R}^N),
\]

\[
|x|^a u^{p+1} \in L^1(\mathbb{R}^N).
\]

First recall that \( u \) is stable outside a compact set \( \Sigma \subset \Omega \). To prove our claim, we use (5.27) with the following test function \( \xi_R \in C^1_c(\mathbb{R}^N \setminus \Sigma) \) for \( R > R_0 + 3 \) and \( \Sigma \subset B_{R_0} \):

\[
\xi_R(x) := \begin{cases} 
0, & \text{if } |x| < R_0 + 1; \\
1, & \text{if } R_0 + 2 < |x| < R; \\
0, & \text{if } |x| > 2R;
\end{cases}
\]

which satisfies \( 0 \leq \xi_R \leq 1, ||\nabla \xi_R||_{L^\infty(B_{2R} \setminus B_R)} < \frac{C}{R} \) and \( ||\nabla \xi_R||_{L^\infty(B_{R_0+2} \setminus B_{R_0+1})} < C_{R_0} \). Therefore,

\[
\int_{R_0+2 < |x| < R} (|\nabla u|^2 u^{t-1} + |x|^a u^{t+p}) \leq C_{R_0} + \hat{C} R^{N - \frac{2(p+2)}{p+1} - \frac{p+1}{p+1}a},
\]

for all \( 1 \leq t < -1 + 2p + 2\sqrt{p(p-1)}. \)

Now, set \( t = 1 \) and send \( R \to \infty \). Since \( N < \frac{2(p+a+1)}{p-1} \), we see \( \int_{\mathbb{R}^N} |\nabla u|^2 < \infty \) and \( \int_{\mathbb{R}^N} |x|^a u^{p+1} < \infty \).
5.3. On solutions of the second order Hénon equation with finite Morse index

Step 3: The following equality holds

\[ \int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} |x|^{a(p+1)}. \]  \hspace{1cm} (5.34)

Multiply (5.7) with \( u\zeta_R \) for \( \zeta_R \in C_0^1(\mathbb{R}^N) \) which satisfies \( 0 \leq \zeta_R \leq 1 \), \( ||\nabla \zeta_R||_{\infty} < \frac{C}{R} \) and \( \zeta_R(x) := \begin{cases} 1, & \text{if } |x| < R; \\ 0, & \text{if } |x| > 2R. \end{cases} \)

Then, integrate over \( B_{2R} \) to get

\[ \int_{B_{2R}} |x|^{a(p+1)} \zeta_R - \int_{B_{2R}} |\nabla u|^2 \zeta_R = \int_{B_{2R}} \nabla \zeta_R \cdot \nabla u. \] \hspace{1cm} (5.35)

By Hölder’s inequality, we have the following upper bound for R.H.S. of (5.35),

\[ |\int_{B_{2R}} \nabla \zeta_R \cdot \nabla u| \leq R^{-1} \int_{B_{2R}} \nabla u \left( |x|^\frac{a}{p+1} u \right) |x|^{-\frac{a}{p+1}} \leq R^{-1} \left( \int_{B_{2R}} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^{a(p+1)} \right)^{\frac{1}{p+1}} \left( \int_{B_{2R}} |x|^{-\frac{2a}{p+1}} \right)^{\frac{p-1}{2(p+1)}} = R^{\frac{N(p-1)-2(p+a+1)}{2(p+1)}} \left( \int_{B_{2R}} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^{a(p+1)} \right)^{\frac{1}{p+1}}. \]

Therefore, from Step 2, there exists a positive constant \( C \) independent of \( R \) such that

\[ |\int_{B_{2R}} \nabla \zeta_R \cdot \nabla u| \leq CR^{\frac{N(p-1)-2(p+a+1)}{2(p+1)}}. \]

Since \( N < \frac{2(p+a+1)}{p-1} \), we have \( \lim_{R \to \infty} |\int_{B_{2R}} \nabla \zeta_R \cdot \nabla u| = 0 \). Hence (5.35) implies (5.34).

Step 4: we have

\[ \left( \frac{N+a}{p+1} - \frac{N-2}{2} \right) \int_{\mathbb{R}^N} |x|^{a(p+1)} = 0. \]
5.3. On solutions of the second order Hénon equation with finite Morse index

Apply Lemma 5.8 for \( t = 1 \) with the following test function \( \phi_R \in C_c^1(\mathbb{R}^N \setminus \Sigma) \) for \( R > 2R_0 \);
\[
\phi_R(x) := \begin{cases} 
0, & \text{if } |x| < R/2; \\
1, & \text{if } R < |x| < 2R; \\
0, & \text{if } |x| > 3R;
\end{cases}
\]
which satisfies \( 0 \leq \phi_R \leq 1 \), \( ||\nabla \phi_R||_{L^\infty(B_{3R} \setminus B_{R/2})} < \frac{C}{R} \) to get
\[
\int_{B_{2R} \setminus B_R} (|\nabla u|^2 + |x|^a u^{p+1}) \leq CR^{N-\frac{2(p+a+1)}{p-1}}. \tag{5.36}
\]

Now, define the following sets for large enough \( M \);
\[
\theta_1(R) := \{ r \in (R, 2R); ||D_x u(r)||_2^2 > MR^{-\frac{2(p+a+1)}{p-1}} \}, \\
\theta_2(R) := \{ r \in (R, 2R); ||u(r)||_{p+1} > MR^{-\frac{2(p+a+1)}{p-1}} - a \}.
\]
From (5.36), we have
\[
C \geq R^{-N+\frac{2(p+a+1)}{p-1}+a} \int_R^{2R} ||u(r)||_{p+1}^{p+1} r^{N-1} dr \\
\geq R^{-N+\frac{2(p+a+1)}{p-1}+a} \theta_2(R) |R|^{N-1} MR^{-\frac{2(p+a+1)}{p-1}} - a = M|\theta_2(R)|R^{-1}.
\]
Similarly, one can show \( |\theta_1(R)| \leq R/M \). By choosing \( M \) large enough we conclude \( |\theta_i(R)| \leq R/3 \) for \( i = 1, 2 \). Therefore, for each \( R \geq 1 \), we can find \( \tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{i=2} \Lambda_i(R) \neq \phi \).

Now, apply Pohozaev identity, (5.33), with \( \Omega = B_{\tilde{R}} \) to see that R.H.S. converges to zero if \( R \to \infty \) for subcritical \( p \), i.e. \( N < \frac{2(p+a+1)}{p-1} \). Hence,
\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 = \frac{N+a}{p+1} \int_{\mathbb{R}^N} |x|^a u^{p+1}.
\]
From this and (5.34), we finish the proof of Step 4.

Remark: For the Sobolev critical case \( p = \frac{N+2+2a}{N-2} \), using the change of variable \( w := u(r^{1+\frac{a}{2}}) \) and applying well-known classifying-type results mentioned in the introduction for the Lane-Emden equation, one can see all radial solutions of (5.7) are of the following form
\[
u_\epsilon(r) := k(\epsilon)(\epsilon + r^{2+a})^{\frac{2-N}{2+a}}, \tag{5.37}
\]
where $k(\epsilon) = (\epsilon(N + a)(N - 2))^{\frac{N-2}{p+1}}$. Then, from the classical Hardy’s inequality it is straightforward to see $u_\epsilon$ is stable outside a compact set $\overline{B_{R_0}}$, for an appropriate $R_0$. Note that for $-2 < a \leq 0$, by Schwarz symmetrization (or rearrangement), it is shown in [56] that all radial solutions of (5.7) with $p = \frac{N+2+2a}{N-2}$ and $N > 2$ are of the form (5.37).

5.4 On solutions of the fourth order Hénon equation with finite Morse index

We shall prove here Theorem 5.3. For that we recall that a critical point $u$ of the energy functional

$$I(u) := \int_{\Omega} \frac{1}{2} |\Delta u|^2 - \frac{1}{p+1} \int_{\Omega} |x|^a u^{p+1},$$

is said to be a stable solution of (5.8), if for any $\phi \in C^4_c(\Omega)$, we have

$$I_{uu}(\phi) := \int_{\Omega} |\Delta \phi|^2 - p \int_{\Omega} |x|^a u^{p-1} \phi^2 \geq 0.$$

Similarly to the second order case, one can define the notion of stability outside a compact set, which contains the notion of solutions with finite Morse index. We first prove the following estimate.

**Lemma 5.9.** Let $\Omega \subset \mathbb{R}^N$ and let $u \in C^4(\Omega)$ be a positive stable solution of (5.8). Then, for large enough $m$, we have for all $\phi \in C^4_c(\Omega)$ with $0 \leq \phi \leq 1$,

$$\int_{\Omega} (|\Delta u|^2 + |x|^a u^{p+1}) \phi^{2m} \leq C \int_{\Omega} |x|^{-\frac{2}{p-1}a} |T(\phi)| \frac{p+1}{p-1},$$

(5.38)

where $T(\phi) := |\Delta \phi|^2 + |\nabla \phi|^4 + |\Delta |\nabla \phi|^2| + |\nabla \phi \cdot \nabla \Delta \phi|$. The constant $C$ does not depend on $\Omega$ and $u$.

**Proof:** For any stable solution of (5.8) and $\eta \in C^4_c(\Omega)$, we have the followings:

$$p \int_{\Omega} |x|^a u^{p-1} \eta^2 \leq \int_{\Omega} |\Delta \eta|^2,$$

(5.39)

$$\int_{\Omega} |x|^a u^p \eta = \int_{\Omega} \Delta u \Delta \eta,$$

(5.40)

Test (5.40) on $\eta = u \phi^2$ for $\phi \in C^4_c(\Omega)$ to get

$$\int_{\Omega} |x|^a u^{p+1} \phi^2 = \int_{\Omega} \Delta u \Delta (u \phi^2)$$

(5.41)
5.4. On solutions of the fourth order Hénon equation with finite Morse index

Also, test (5.39) on \( u\phi \) and use (5.41) to get

\[
(p - 1) \int_{\Omega} |x|^a u^{p+1} \phi^2 \leq \int_{\Omega} |\Delta (u\phi)|^2 - \int_{\Omega} |x|^a u^{p+1} \phi^2 = \int_{\Omega} |\Delta (u\phi)|^2 - \int_{\Omega} \Delta u \Delta (u\phi^2).
\]

By a straightforward calculation, one can see that the following identity holds:

\[
|\Delta (u\phi)|^2 - \Delta u \Delta (u\phi^2) = 4 |\nabla u \cdot \nabla \phi|^2 + u^2 |\Delta \phi|^2 - 2 u \Delta u |\nabla \phi|^2 + 2 \nabla u^2 \cdot \nabla \phi \Delta \phi. (5.42)
\]

Therefore, we have

\[
(p - 1) \int_{\Omega} |x|^a u^{p+1} \phi^2 \leq 4 \int_{\Omega} |\nabla u|^2 |\nabla \phi|^2 + \int_{\Omega} u^2 |\Delta \phi|^2 - 2 \int_{\Omega} u \Delta u |\nabla \phi|^2 + 2 \int_{\Omega} \nabla u^2 \cdot \nabla \phi \Delta \phi.
\]

A simple integration by parts yields

\[
\int_{\Omega} |\nabla u|^2 |\nabla \phi|^2 = \int_{\Omega} u(-\Delta u)|\nabla \phi|^2 + \frac{1}{2} \int_{\Omega} u^2 \Delta |\nabla \phi|^2, \quad (5.43)
\]

which then simplifies the previous inequality to become

\[
(p - 1) \int_{\Omega} |x|^a u^{p+1} \phi^2 \leq 6 \int_{\Omega} u(-\Delta u)|\nabla \phi|^2 + \int_{\Omega} u^2(-|\Delta \phi|^2 + 2\Delta |\nabla \phi|^2 - 2 \nabla \phi \cdot \nabla \Delta \phi).
\]

Therefore,

\[
\int_{\Omega} |x|^a u^{p+1} \phi^2 \leq C \int_{\Omega} u|\Delta u||\nabla \phi|^2 + \int_{\Omega} u^2 L(\phi), \quad (5.44)
\]

where \( L(\phi) := |\Delta \phi|^2 + 2|\Delta |\nabla \phi|^2| + 2|\nabla \phi \cdot \nabla \Delta \phi| \).

On the other hand, from (5.42) and (5.43), one can see

\[
\int_{\Omega} |\Delta (u\phi)|^2 = \int_{\Omega} \Delta u \Delta (u\phi^2) + 4 \int_{\Omega} |\nabla u \cdot \nabla \phi|^2 + \int_{\Omega} u^2 |\Delta \phi|^2 - 2 u \Delta u |\nabla \phi|^2 - 2 \int_{\Omega} u^2 \text{div}(\nabla \phi \Delta \phi)
\]

\[
= \int_{\Omega} |x|^a u^{p+1} \phi^2 + 6 \int_{\Omega} u(-\Delta u)|\nabla \phi|^2 + \int_{\Omega} u^2(-|\Delta \phi|^2 + 2\Delta |\nabla \phi|^2 - 2 \nabla \phi \cdot \nabla \Delta \phi).
\]
5.4. On solutions of the fourth order Hénon equation with finite Morse index

By combining (5.44), the identity \( \Delta (u\phi) = \phi \Delta u + 2V_u \cdot \nabla \phi + u \Delta \phi \) and Young’s inequality, we get the following estimate

\[
\int_{\Omega} |\Delta u|^2 \phi^2 \leq C \int_{\Omega} u |\Delta u||\nabla \phi|^2 + C \int_{\Omega} u^2 L(\phi).
\]

Therefore,

\[
\int_{\Omega} (|x|^a u^{p+1} + |\Delta u|^2) \phi^2 \leq C \int_{\Omega} u |\Delta u||\nabla \phi|^2 + C \int_{\Omega} u^2 L(\phi).
\]

Now, replacing \( \phi \) with \( \phi^m \) for large enough \( m > 0 \) and applying Young’s inequality we end up with

\[
\int_{\Omega} (|x|^a u^{p+1} + |\Delta u|^2) \phi^{2m} \leq C \int_{\Omega} u |\Delta u||\nabla \phi|^2 \phi^{2(m-1)} + C \int_{\Omega} u^2 L(\phi^m)
\]

\[
\leq \epsilon \int_{\Omega} |\Delta u|^2 \phi^{2m} + C \int_{\Omega} u^2 |\nabla \phi|^4 \phi^{2(m-2)} + C \int_{\Omega} u^2 L(\phi^m).
\]

Then, for large enough \( m \)

\[
\int_{\Omega} (|x|^a u^{p+1} + |\Delta u|^2) \phi^{2m} \leq C \int_{\Omega} u^2 \phi^{2(m-2)} T(\phi), \tag{5.45}
\]

where \( T(\phi) := |\Delta \phi|^2 + |\nabla \phi|^4 + |\Delta |\nabla \phi|^2| + |\nabla \phi \cdot \nabla \Delta \phi| \). Now, apply Hölder’s inequality to get

\[
\int_{\Omega} u^2 \phi^{2(m-2)} T(\phi) = \int_{\Omega} |x|^a u^{p+1} \phi^{2(m-2)} |x|^{-\frac{2a}{p+1}} T(\phi)
\]

\[
\leq \left( \int_{\Omega} |x|^a u^{p+1} \phi^{2(m-2)} \right)^{\frac{p+1}{p+2}} \left( \int_{\Omega} |x|^{-\frac{2a}{p+1}} T^{\frac{p+1}{p+2}} \phi \right)^{\frac{p-1}{p+1}}
\]

Choosing \( m \) large enough, say \( 2(m - 2) \frac{p+1}{2} \geq 2m \), from (5.45) we finally get the desired inequality

\[
\int_{\Omega} (|x|^a u^{p+1} + |\Delta u|^2) \phi^{2m} \leq C \int_{\Omega} |x|^{-\frac{2a}{p+1}} T^{\frac{p+1}{p+2}} \phi.
\]

\[ \square \]

**Proof of Theorem 5.3:** We proceed in the following steps.
5.4. On solutions of the fourth order Hénon equation with finite Morse index

**Step 1:** We have the following standard Pohozaev type identity on any \( \Omega \subset \mathbb{R}^N \).

\[
\frac{N + a}{p + 1} \int_{\Omega} |x|^a u^{p+1} - \frac{N - 4}{2} \int_{\Omega} |\Delta u|^2 \\
= \frac{1}{p + 1} \int_{\partial \Omega} |x|^a u^{p+1} x \cdot \nu - \frac{1}{2} \int_{\partial \Omega} |\Delta u|^2 x \cdot \nu \\
- \int_{\partial \Omega} \nabla \Delta u \cdot \nu x \cdot \nabla u + \int_{\partial \Omega} \Delta u \nabla (x \cdot \nabla u) \cdot \nu. \tag{5.46}
\]

To get (5.46), just multiply both sides of (5.8) by \( x \cdot \nabla u \), do integration by parts and collect terms.

**Step 2:** We have

\[
|\Delta u| \in L^2(\mathbb{R}^N), \\
|x|^a u^{p+1} \in L^1(\mathbb{R}^N).
\]

Since \( u \) is stable outside a compact set \( \Sigma \subset \Omega \), using (5.38) with the following test function \( \xi_R \in C^1_c(\mathbb{R}^N \setminus \Sigma) \) for \( R > R_0 + 3 \) and \( \Sigma \subset B_{R_0} \);

\[
\xi_R(x) := \begin{cases} 
0, & \text{if } |x| < R_0 + 1; \\
1, & \text{if } R_0 + 2 < |x| < R; \\
0, & \text{if } |x| > 2R;
\end{cases}
\]

which satisfies \( 0 \leq \xi_R \leq 1 \), \( ||D^i \xi_R||_{L^\infty(B_{2R} \setminus B_R)} < \frac{C}{R} \) and

\[
||D^i \xi_R||_{L^\infty(B_{R_0+2} \setminus B_{R_0+1})} \leq C_{R_0}
\]

for \( i = 1, \cdots, 4 \), we get

\[
\int_{R_0 + 2 < |x| < R} (|\Delta u|^2 + |x|^a u^{p+1}) \leq C_{R_0} + \hat{C} R^{N - \frac{4(p+1)}{p-1} - \frac{2}{p-1} a}.
\]

For subcritical exponents, \( N < \frac{2(2p+a+2)}{p-1} \), we see \( \int_{\mathbb{R}^N} |\Delta u|^2 < \infty \) and \( \int_{\mathbb{R}^N} |x|^a u^{p+1} < \infty \).

**Step 3:** The following equality holds

\[
\int_{\mathbb{R}^N} |x|^a u^{p+1} = \int_{\mathbb{R}^N} |\Delta u|^2. \tag{5.47}
\]
5.4. On solutions of the fourth order Hénon equation with finite Morse index

Multiply \((5.8)\) with \(u\zeta_R\) for \(\zeta_R \in C^2_c(B_{2R})\) which satisfies \(0 \leq \zeta_R \leq 1\), 
\(|D^i\zeta_R|_\infty < \frac{C}{R^i}\) for \(i = 1, \ldots, 4\) and 
\[
\zeta_R(x) := \begin{cases} 
1, & \text{if } |x| < R; \\
0, & \text{if } |x| > 2R.
\end{cases}
\]

Then, integrate over \(B_{2R}\) to get
\[
\int_{B_{2R}} |x|^a u^{p+1} \zeta_R - \int_{B_{2R}} |\Delta u|^2 \zeta_R = \int_{B_{2R}} u \Delta u \Delta \zeta_R + 2 \int_{B_{2R}} \Delta u \nabla u \cdot \nabla \zeta_R =: I_1(R) + I_2(R). \tag{5.48}
\]

By Hölder’s inequality, we have the following upper bound for \(I_1(R)\),
\[
|I_1(R)| \leq R^{-2} \int_{B_{2R}} |\Delta u| (|x|^a u^{p+1}) |x|^{-\frac{a}{p+1}}
\leq R^{-2} \left( \int_{B_{2R}} |\Delta u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^a u^{p+1} \right)^{\frac{1}{2+1}} \left( \int_{B_{2R}} |x|^{-\frac{2a}{p+1}} \right)^{\frac{p-1}{2(p+1)}}
= R^{N-\frac{a}{p+1} - 2} \left( \int_{B_{2R}} |\Delta u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^a u^{p+1} \right)^{\frac{1}{p+1}}.
\]

Therefore, from Step 2, there exists a positive constant \(C\) independent of \(R\) such that
\[
|I_1(R)| \leq C R^{\frac{N(\gamma - 1) - 2(p+1)}{2(p+1)}}.
\]

Since \(N < \frac{2(p+a+1)}{p-1}\), we have \(\lim_{R \to \infty} |I_1(R)| = 0\). Now, we consider the second term in R.H.S. of \((5.48)\). Apply Young’s inequality for a given \(\epsilon > 0\) (we choose it later) to get
\[
|I_2(R)| \leq \epsilon \int_{\mathbb{R}^N} |\Delta u|^2 + C_{\epsilon} \int_{B_{2R}} |
abla u|^2 |\nabla \zeta_R|^2,
\]
Using Green’s theorem we get
\[
\int_{B_{2R}} |\nabla u|^2 |\nabla \zeta_R|^2 = \int_{B_{2R}} u (-\Delta u) |\nabla \zeta_R|^2 + \frac{1}{2} \int_{B_{2R}} u^2 |\Delta \zeta_R|^2 =: I_3(R) + I_4(R).
\]

By the same discussion as given for \(I_1(R)\) one can see \(\lim_{R \to \infty} |I_3(R)| = 93\).
5.4. On solutions of the fourth order Hénon equation with finite Morse index

For the term \( I_4(R) \), we apply Hölder’s inequality again
\[
|I_4| \leq R^{-4} \int_{B_{2R}} |x|^\frac{a}{p+1} u^2 |x|^{-\frac{a}{p+1}}
\]
\[
\leq R^{-4} \left( \int_{B_{2R}} |x|^a u^{p+1} \right)^\frac{2}{p+1} \left( \int_{B_{2R}} |x|^{-\frac{2a}{p+1}} \right)^\frac{p+1}{p+1}
\]
\[
= R^{\frac{N(p-1)}{p+1} - \frac{2a}{p+1} - \frac{4}{p+1}} \left( \int_{B_{2R}} |x|^a u^{p+1} \right)^\frac{2}{p+1}.
\]

By Step 2 and sending \( R \) to infinity we get, \( \lim_{R \to \infty} |I_4(R)| = 0 \). Since \( \lim_{R \to \infty} |I_2(R)| \leq \epsilon \int_{\mathbb{R}^N} |\Delta u|^2 \) for any \( \epsilon > 0 \), we have \( \lim_{R \to \infty} |I_2(R)| = 0 \). Therefore, (5.47) follows.

**Step 4**: The following equality holds
\[
\left( \frac{N + a}{p+1} - \frac{N-4}{2} \right) \int_{\mathbb{R}^N} |x|^a u^{p+1} = 0.
\]

Apply Lemma 5.9 with the following test function \( \phi_R \in C_c^1(\mathbb{R}^N \setminus \Sigma) \) for \( R > 2R_0 \);
\[
\phi_R(x) := \begin{cases} 
0, & \text{if } |x| < R/2; \\
1, & \text{if } R < |x| < 2R; \\
0, & \text{if } |x| > 3R;
\end{cases}
\]
where \( 0 \leq \phi_R \leq 1 \), \( ||D^i \phi_R||_{L^\infty(B_{3R} \setminus B_{R/2})} < C \). Then, we get
\[
\int_{B_{2R} \setminus B_R} |\Delta u|^2 + |x|^a u^{p+1} \leq CR^{\frac{N-2}{p+1}}.
\]
(5.49)

On the other hand, we are interested in similar upper bounds for the following terms
\[
J_1(R) := \int_{B_{2R} \setminus B_R} |\Delta u||\nabla u| \quad \text{and} \quad J_2(R) := \int_{B_{2R} \setminus B_R} |\Delta u||D_x^2 u|.
\]

For the first term, \( J_1(R) \), using Schwarz’s inequality we have
\[
\int_{B_{2R} \setminus B_R} |\Delta u||\nabla u| < \left( \int_{B_{2R} \setminus B_R} |\Delta u|^2 \right)^{1/2} \left( \int_{B_{2R} \setminus B_R} |\nabla u|^2 \right)^{1/2}.
\]
5.4. On solutions of the fourth order Hénon equation with finite Morse index

From standard elliptic interpolation estimates, L^2-norm version of Lemma 5.4, we have

\[
\int_{B_{2R}\setminus B_{R}} |\nabla u|^2 \leq CR^2 \int_{B_{4R}\setminus B_{2R/2}} |\Delta u|^2 + CR^{-2} \int_{B_{4R}\setminus B_{2R/2}} u^2
\]

\[
\leq CR^{N-\frac{2(2p+2+a)}{p-1}+2} + R^{N(p-1)} \frac{2a}{p+1} -2 \left( \int_{\mathbb{R}^N} |x|^a u^{p+1} \right)^{\frac{2}{p+1}}
\]

\[
= CR^{p+1} \left( N - \frac{2(2p+2+a)}{p-1} + 2 \right) + 2 \left( R^{2} \left( N - \frac{2(2p+2+a)}{p-1} \right) + \left( \int_{\mathbb{R}^N} |x|^a u^{p+1} \right)^{\frac{2}{p+1}} \right)
\]

Since \( \int_{\mathbb{R}^N} |x|^a u^{p+1} < \infty \) and \( N < \frac{2(2p+2+a)}{p-1} \), for \( R > 1 \) we have

\[
\int_{B_{3R}\setminus B_{R}} |\nabla u|^2 \leq CR^{p+1} \left( N - \frac{2(2p+2+a)}{p-1} + 2 \right) + 2
\]

Therefore,

\[
\int_{B_{2R}\setminus B_{R}} |\Delta u||\nabla u| < CR^{p+1} \left( N - \frac{2(2p+2+a)}{p-1} \right)^{\frac{1}{p+1}}.
\] (5.50)

Similarly for the second term, \( J_2(R) \), using Lemma 5.3, i.e.,

\[
\int_{B_{2R}\setminus B_{R}} |D_x^2 u|^2 \leq C \left( \int_{B_{4R}\setminus B_{2R/2}} |\Delta u|^2 + R^{-4} \int_{B_{4R}\setminus B_{2R/2}} u^2 \right),
\]

and similar type discussions one can see

\[
\int_{B_{3R}\setminus B_{R}} |\Delta u||D_x^2 u| < CR^{p+1} \left( N - \frac{2(2p+2+a)}{p-1} \right).
\] (5.51)

Now, define the following sets for large enough \( M; \)

\[
\Lambda_1(R) := \{ r \in (R, 2R); ||\Delta_x u(r)||^2 > MR^{-\frac{2(2p+2+a)}{p-1}} \},
\]

\[
\Lambda_2(R) := \{ r \in (R, 2R); ||u(r)||^{p+1} > MR^{-\frac{2(2p+2+a)}{p-1}} \},
\]

\[
\Lambda_3(R) := \{ r \in (R, 2R); ||\Delta_x u(r)||^{p+1} > MR^{-\frac{2(2p+2+a)}{p-1}} \},
\]

\[
\Lambda_4(R) := \{ r \in (R, 2R); ||\Delta_x u(r)||^{p+1} > MR^{-\frac{2(2p+2+a)}{p-1}} \}.
\]
In the following, we shall find a bound for the measure of the above sets. From (5.49), we have
\begin{align*}
C & \geq R^{-N + \frac{2(2p + 2 + a)}{p-1} + a} \int_{R}^{2R} |u(r)|^{p+1} r^{N-1} dr \\
& \geq R^{-N + \frac{2(2p + 2 + a)}{p-1} + a} |\Lambda_2(R)| R^{N-1} M R^{-\frac{2(2p + 2 + a)}{p-1}} = M |\Lambda_2(R)| R^{-1}.
\end{align*}
Also, from (5.50)
\begin{align*}
C & \geq R^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right)^{-1} \int_{R}^{2R} ||\Delta_x u(r)|| |\nabla_x u(r)|| r^{N-1} dr \\
& \geq R^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right)^{-1} |\Lambda_3(R)| R^{N-1} M R^{-\frac{N}{p} + \frac{2(2p + 2 + a)}{p-1}} + 1 \\
& = M |\Lambda_3(R)| R^{-1}.
\end{align*}
Similarly, from (5.51) and (5.49) we get $|\Lambda_1(R)|, |\Lambda_4(R)| \leq R/M$. By choosing $M$ large enough we conclude $|\Lambda_i(R)| \leq R/5$ for $i = 1, \cdots, 4$. Therefore, for each $R \geq 1$, we can find
\begin{equation}
\tilde{R} \in (R, 2R) \setminus \bigcup_{i=1}^{i=4} \Lambda_i(R) \neq \phi. \tag{5.52}
\end{equation}
Then, from the definition of $\tilde{R}$ and $\Lambda_i$ for $i = 1, \cdots, 4$, we have
\begin{align*}
\int_{|x| = \tilde{R}} |\Delta_x u(\tilde{R})||D_x^2 u(\tilde{R})| & \leq C \tilde{R}^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right)^{-1} \tag{5.53} \\
\int_{|x| = \tilde{R}} |\Delta_x u(\tilde{R})||\nabla_x u(\tilde{R})| & \leq C \tilde{R}^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right) \tag{5.54} \\
\int_{|x| = \tilde{R}} |\Delta_x u(\tilde{R})|^2 & \leq C \tilde{R}^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right)^{-1} \tag{5.55} \\
\int_{|x| = \tilde{R}} u^{p+1}(\tilde{R}) & \leq C \tilde{R}^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right)^{-a-1} \tag{5.56}
\end{align*}
Using (5.46) with $\Omega = B_{2\tilde{R}} \setminus B_{\tilde{R}}$, one can see
\begin{equation}
\left| \int_{|x| = \tilde{R}} \nabla \Delta u \cdot \nu x \cdot \nabla u \right| \leq C \tilde{R}^{p+1} \left( -N + \frac{2(2p + 2 + a)}{p-1} \right). \tag{5.57}
\end{equation}
Now, applying the Pohozaev identity, (5.46), with $\Omega = B_{\tilde{R}}$ and using (5.53)-(5.57), R.H.S. of (5.46), converges to zero if $R \to \infty$ for subcritical $p$,
5.5. On stable solutions of the Hénon-Lane-Emden system

i.e. \( N < \frac{2(2p+2+a)}{p-1} \). Hence,

\[
\frac{N - 4}{2} \int_{\mathbb{R}^N} |\Delta u|^2 = \frac{N + a}{p + 1} \int_{\mathbb{R}^N} |x|^a u^{p+1}.
\]

From this and (5.47), we finish the proof of Step 4.

\[
\square
\]

5.5 On stable solutions of the Hénon-Lane-Emden system

We shall prove here Theorem 5.4. For that we recall that a classical solution \((u,v)\) of (5.1) is said to be pointwise stable if there exists positive smooth \(\zeta, \eta\) such that

\[
\left\{ \begin{array}{l}
-\Delta \zeta = p|x|^{a_p-1}v \eta \quad \text{in } \Omega, \\
-\Delta \eta = q|x|^{b_q-1}\zeta \quad \text{in } \Omega.
\end{array} \right.
\]

(5.58)

In what follows we give the stability inequality for system (5.1). This inequality is the novelty here and is key tool in proving Theorem 4. The idea of getting such an inequality comes from [51].

Lemma 5.10. Assume that \((u,v)\) is a pointwise stable solution of (5.1), then for any test function \(\phi \in C^1_c(\Omega)\), we have

\[
\sqrt{pq} \int_{\Omega} |x|^{\frac{a+b+1}{2}} v^\frac{p-1}{2} u^\frac{q-1}{2} \phi^2 \leq \int_{\Omega} |\nabla \phi|^2.
\]

(5.59)

Proof: Let \((u,v)\) be a pointwise stable solution of (5.1) in such a way that there exists positive smooth \(\zeta, \eta\) such that (5.58). Multiply the first equation by \(\phi^2 \zeta^{-1}\) and the second equation by \(\phi^2 \eta^{-1}\), integrate by parts and use Young’s inequality to get

\[
p \int_{B_R} |x|^{a_p-1}v \zeta \phi^2 = - \int_{B_R} \frac{\Delta \zeta}{\zeta} \phi^2 \leq \int_{B_R} |\nabla \phi|^2
\]

\[
q \int_{B_R} |x|^{b_q-1}\zeta \eta \phi^2 = - \int_{B_R} \frac{\Delta \eta}{\eta} \phi^2 \leq \int_{B_R} |\nabla \phi|^2.
\]

Adding these two equations and doing simple calculations we get

\[
2 \int_{B_R} |\nabla \phi|^2 \geq \int_{B_R} \left( p|x|^{a_p-1}v \frac{\eta}{\zeta} + q|x|^{b_q-1}\frac{\zeta}{\eta} \right) \phi^2
\]

\[
\geq 2\sqrt{pq} \int_{B_R} |x|^{\frac{a+b}{2}} v^\frac{p-1}{2} u^\frac{q-1}{2} \phi^2.
\]
5.5. On stable solutions of the Hénon-Lane-Emden system

The following pointwise estimate is taken from [79]. As was said before, the first version of this paper was done independently of [79] and without using the following lemma of Phan. This last section—which uses Lemma 5.11—was added after his paper was posted.

Lemma 5.11. [Phan, [79]] Assume that \((u,v)\) is a classical solution for (5.1), then for

\[0 \leq a - b \leq (N - 2)(p - q)\]  \hspace{1cm} (5.60)

we have

\[|x|^{a}v^{p+1} \leq \frac{p + 1}{q + 1}|x|^{b}u^{q+1}.\]  \hspace{1cm} (5.61)

Combining the above lemmas we conclude the following integral estimate which is a counterpart of Lemma 5.8 for the second order case and Lemma 5.9 for the fourth order case.

Lemma 5.12. For \(\Omega \subset \mathbb{R}^N\), assume that (5.60) holds and that \((u,v)\) is a pointwise stable solution of (5.1). Set \(\theta := \frac{pq}{p+q+1}\). Then, for any \(t\) such that

\[\sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < t < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}} ,\]

we have for all \(\phi \in C_c^2(\Omega)\) such that \(0 \leq \phi \leq 1\),

\[\int_{\Omega} |x|^{a}v^{p}u^{2t-1}\phi^{2} \leq C \int_{\Omega} u^{2t}(||\nabla \phi|^2 + |\Delta \phi|).\]  \hspace{1cm} (5.62)

The constant \(C\) does not depend on \(\Omega\) and \((u,v)\).

Proof: Note first that for \(p \geq q\), we have \(\theta \geq q^2 > 1\) and also

\[\frac{1}{2} < \sqrt{\theta} - \sqrt{\theta - \sqrt{\theta}} < 1 < \sqrt{\theta} + \sqrt{\theta - \sqrt{\theta}} .\]

Let \((u,v)\) is a pointwise stable solution of (5.1). Then, Lemma 5.10 applies and by replacing \(\phi\) with \(u^{t}\phi\) in (5.59), where \(\phi\) is a test function, we obtain

\[\sqrt{pq} \int |x|^{a+b}v^{p-1}u^{q-1}u^{2t}\phi^{2} \leq \int |\nabla (u^{t}\phi)|^{2}.\]  \hspace{1cm} (5.63)

Rewriting the left hand side as \(\sqrt{pq} \int |x|^{a+b}v^{p-1}u^{q-1}u^{2t-1}\phi^{2}\) and using Lemma 5.11 i.e. \(\sqrt{\frac{q+1}{p+1}}|x|^{a+b}v^{\frac{p-1}{2}}u^{\frac{q-1}{2}}u^{2t-1}\phi^{2} \leq u^{\frac{q+1}{2}}\), we get

\[\sqrt{\frac{pq(q+1)}{p+1}} \int |x|^{a+b}u^{2t-1}\phi^{2} \leq t^{2} \int |\nabla u|^2u^{2t-2}\phi^{2} + \int u^{2t}\phi|\Delta \phi|.\]  \hspace{1cm} (5.64)

98
5.5. On stable solutions of the Hénon-Lane-Emden system

To find an upper bound for the first term in the above inequality with the gradient term, we multiply both sides of the first equation in (5.1) to get

\[
\int |x|^a v^p u^{2t-1} \phi^2 = \int \nabla u \cdot \nabla (u^{2t-1} \phi^2) = (2t - 1) \int |\nabla u|^2 u^{2t-2} \phi^2 - \frac{1}{2t} \int u^{2t} \Delta (\phi^2).
\]

Since \( t > \frac{1}{2} \), we get

\[
t^2 \int |\nabla u|^2 u^{2t-2} \phi^2 \leq \frac{t^2}{2t - 1} \int |x|^a v^p u^{2t-1} \phi^2 + C_t \int u^{2t} (\phi |\Delta \phi| + |\nabla \phi|^2).
\]

Combining this and (5.65) we have

\[
\left( \sqrt{\frac{pq(q+1)}{p+1}} - \frac{t^2}{2t - 1} \right) \int |x|^a v^p u^{2t-1} \phi^2 \leq C_t \int u^{2t} (\phi |\Delta \phi| + |\nabla \phi|^2).
\]

(5.65)

**Proof of Theorem 5.4:** Define \( z := u^\tau \) for \( 1 < 2\sqrt{\theta} - 2\sqrt{\theta - \sqrt{\theta}} < \tau < 2\sqrt{\theta} + 2\sqrt{\theta - \sqrt{\theta}} \) and \( \theta := \frac{pq(q+1)}{p+1} \). Then,

\[ |\Delta z| \leq C \left( |\nabla u|^2 u^{\tau-2} + |x|^a u^{\tau-1} v^p \right). \]

By integrating over balls we get

\[
\int_{B_R} |\Delta z| \leq C \int_{B_R} |\nabla u|^2 u^{\tau-2} + C \int_{B_R} |x|^a u^{\tau-1} v^p. \tag{5.66}
\]

We are now after an upper bound for the right hand side of the above inequality. To control the second term, apply Lemma 5.8 for \( t := \frac{\tau}{2} \) and standard test function \( \zeta_R \) used in the proof of Theorem 5.3 to get

\[
\int_{B_R} |x|^a v^p u^{\tau-1} \leq CR^{-2} \int_{B_R} u^\tau.
\]

To bound the first term, we use the first equation of the system. Multiply both sides of (5.1) with \( u^{\tau-1} \zeta_R^2 \) and integrate by parts to get

\[
\int_{B_R} |\nabla u|^2 u^{\tau-2} \leq \int_{B_R} |\nabla u|^2 u^{\tau-2} \zeta_R^2
\]

\[
= \frac{1}{\tau - 1} \int_{B_R} |x|^a v^p u^{\tau-1} \zeta_R^2 + \frac{1}{\tau(\tau - 1)} \int_{B_R} u^\tau \Delta (\zeta_R^2)
\]

\[
\leq C \int_{B_R} |x|^a v^p u^{\tau-1} + CR^{-2} \int_{B_R} u^\tau.
\]

99
Therefore, the following upper bound holds for (5.66),

\[ \int_{B_R} |\Delta z| \leq CR^{-2} \int_{B_R} u^\tau, \]

which means \( R^2 ||\Delta z||_{L^1(B_R)} \leq C||z||_{L^1(B_R)} \). Now, applying Lemma 5.6 for \( z = u^\tau \) we get

\[ ||z||_{L^k(B_R)} \leq CR^{N(\frac{1}{k} - 1)} ||z||_{L^1(B_{2R})}, \]

where \( C = C(k, N) > 0 \) and any \( 1 \leq k < \frac{N}{N-2} \).

Now take \( 1 \leq k_i < \frac{N}{N-2} \) for \( 1 \leq i \leq n \) and \( 2\sqrt{\theta} - 2\sqrt{\theta - \sqrt{\theta}} < 2t := \tau k_{n-1}! < 2\sqrt{\theta} + 2\sqrt{\theta - \sqrt{\theta}} \). The notation "!" stands for \( k_{n-1}! := \prod_{i=0}^{n-1} k_i \) and set \( k_0 = 1 \). By induction we have

\[ ||z||_{L^{k_n}(B_R)} \leq CR^{k_n} ||z||_{L^1(B_{2R})}, \]

where \( k_n = N \sum_{i=1}^{n} \frac{1}{k_i!} = N \left( \frac{1}{k_n!} - 1 \right) \) and \( C = C(k_i, N) > 0 \). So,

\[ \left( \int_{B_R} u^{\tau k_n!} \right)^{\frac{1}{k_n!}} \leq CR^{N(\frac{1}{k_n} - 1)} \int_{B_{2R}} u^\tau. \]

Let \( 0 < \tau < q \) and from Corollary 5.1 we get

\[ \int_{B_{2R}} u^\tau \leq CR^{-\frac{p(b+2)+a+2}{pq-1}}. \]

Therefore

\[ \left( \int_{B_R} u^{\tau k_n!} \right)^{\frac{1}{k_n!}} \leq CR^{\tau \frac{N}{\tau k_n!} - \frac{p(b+2)+a+2}{pq-1}}. \]

(5.67)

So, in the following dimensions

\[ N < \frac{p(b+2)+a+2}{pq-1} \tau k_n! \]

the right hand side of (5.67) converges to zero as \( R \) tends to infinity. Note that since \( \tau k_{n-1}! < 2\sqrt{\theta} + 2\sqrt{\theta - \sqrt{\theta}} < \tau k_n! < (2\sqrt{\theta} + 2\sqrt{\theta - \sqrt{\theta}})^{\frac{N}{N-2}} \). So,

\[ N < \tau k_n! \frac{p(b+2)+a+2}{pq-1} < 2 + \frac{p(b+2)+a+2}{pq-1} \left( 2\sqrt{\theta} + 2\sqrt{\theta - \sqrt{\theta}} \right) \]

Recall that \( \theta := \frac{pq(q+1)}{p+1} \), which completes the proof.
5.6 Summary and conclusions

In this chapter, we proved 0-Liouville theorems for the following Hénon-Lane-Emden system

\[
\begin{align*}
-\Delta u &= |x|^a v^p \quad \text{in } \mathbb{R}^n, \\
-\Delta v &= |x|^b u^q \quad \text{in } \mathbb{R}^n,
\end{align*}
\]

when \(pq > 1, p, q, a, b \geq 0\). The main conjecture states that

**Conjecture.** Suppose \((p, q)\) is under the critical hyperbola, i.e.,

\[
\frac{n + a}{p + 1} + \frac{n + b}{q + 1} > n - 2.
\]

Then nonnegative solutions must be zero.

We showed that this is indeed the case in dimension \(n = 3\) provided the solution is also assumed to be bounded, extending a result established recently by Phan-Souplet in the scalar case.

Assuming stability of the solutions, we could then prove Liouville-type theorems in higher dimensions. For the scalar cases, albeit of second order \((a = b\) and \(p = q\)) or of fourth order \((a \geq 0 = b\) and \(p > 1 = q\)), we showed that for all dimensions \(n \geq 3\) in the first case (resp., \(n \geq 5\) in the second case), there is no positive solution with a finite Morse index, whenever \(p\) is below the corresponding critical exponent, i.e. 1 < \(p < \frac{n+2-a}{n-2}\) (resp., 1 < \(p < \frac{n+4+2a}{n-4}\)). Finally, we showed that non-negative stable solutions of the full Hénon-Lane-Emden system are trivial provided

\[
n < 2 + 2 \left( \frac{p(b + 2) + a + 2}{pq - 1} \right) \left( \sqrt{z} + \sqrt{z - \sqrt{z}} \right),
\]

where \(z := \frac{pq(q+1)}{p+1}\). 

101
Chapter 6

Uniqueness of solutions for a nonlocal eigenvalue problem

6.1 Introduction

We are interested in the following nonlocal eigenvalue problem

\[(P)_{\lambda} \left\{ \begin{array}{ll}
(\Delta)^{\frac{1}{2}} u = \lambda g(x)f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.\]

where \((\Delta)^{\frac{1}{2}}\) is the square root of the Laplacian operator, \(\lambda > 0\) is a parameter, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) where \(N \geq 2\), and where \(0 < g(x) \in C^{1,\alpha}(\Omega)\) for some \(0 < \alpha\). The nonlinearity \(f\) satisfies one of the following two conditions:

(R) \(f\) is smooth, increasing and convex on \(\mathbb{R}\) with \(f(0) = 1\) and \(f\) is superlinear at \(\infty\) (i.e. \(\lim_{t \to \infty} \frac{f(t)}{t} = \infty\)), or

(S) \(f\) is smooth, increasing, convex on \([0, 1)\) with \(f(0) = 1\) and \(\lim_{t \nearrow 1} f(t) = +\infty\).

In this chapter we prove there is a unique solution of \((P)_{\lambda}\) for two parameter ranges: for small \(\lambda\) and for \(\lambda = \lambda^*\) where \(\lambda^*\) is the so called extremal parameter associated with \((P)_{\lambda}\). First, let us to recall various known facts concerning the second order analog of \((P)_{\lambda}\).

Some notations: Let \(F(t) := \int_0^t f(\tau)d\tau\) and \(C_f := \int_0^{a_f} f(t)^{-1}dt\) where \(a_f = \infty\) (resp. \(a_f = 1\)) when \(f\) satisfies (R) (resp. \(f\) satisfies (S)). We say a positive function \(f\) defined on an interval \(I\) is logarithmically convex (or log convex) provided \(u \mapsto \log(f(u))\) is convex on \(I\). Also, \(\Omega\) will always denote a smooth bounded domain in \(\mathbb{R}^N\) where \(N \geq 2\).
6.1. Introduction

6.1.1 The local eigenvalue problem

For a nonlinearity $f$ which satisfies (R) or (S), the following second order analog of $(P)_\lambda$ with the Dirichlet boundary conditions

$$(Q)_\lambda \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

is by now quite well understood whenever $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [12, 13, 16, 17, 43, 44, 55, 66, 71, 76]. We now list the properties one comes to expect when studying $(Q)_\lambda$.

It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$ such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution $u_\lambda$ of $(Q)_\lambda$. Here the minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also, one can show that the minimal solution $u_\lambda$ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_{\Omega} \lambda f'(u_\lambda) \psi^2 \leq \int_{\Omega} |\nabla \psi|^2, \quad \forall \psi \in H_0^1(\Omega).$$

We now come to the results known for $(Q)_\lambda$ which we are interested in extending to $(P)_\lambda$. In [66] it was shown that the extremal solution $u^*$ is the unique weak solution of $(Q)_{\lambda^*}$. Some of the techniques involve using concave cut offs which do not seem to carry over to the nonlocal setting. Here we use some techniques developed in [8] which were used in studying a fourth order analog of $(Q)_\lambda$. In [35] the uniqueness of the extremal solution for $\Delta^2 u = \lambda e^u$ on radial domains with Dirichlet boundary conditions was shown and this was extended to log convex (see below) nonlinearities in [64]. Some of the methods used in [64] were inspired by the techniques of [8] and so will ours in the case where $f$ satisfies (R). In [22] it was shown that the extremal solution associated with $\Delta^2 u = \lambda (1 - u)^{-2}$ on radial domains is unique and our methods for nonlinearities satisfying (S) use some of their techniques.

In [67] and [87] a generalization of $(Q)_\lambda$ was examined. They showed that if $f$ is suitably supercritical at infinity and if $\Omega$ is a star-shaped domain, then for small $\lambda > 0$ the minimal solution is the unique solution of $(Q)_\lambda$. In [42] this was done for a particular nonlinearity $f$ which satisfies (S). One can weaken the star-shaped assumption and still have uniqueness, see [86], but we do not pursue this approach here. In section 3 we extend these results...
6.1. Introduction

to \((P)_\lambda\). For more results the on uniqueness of solutions for various elliptic
problems involving parameters, see [38].

For questions on the regularity of the extremal solution in fourth order
problems we direct the interested reader to [28]. We also mention the recent
preprint [27] which examines the same issues as this paper but for equations
of the form \(\Delta^2 u = \lambda f(u)\) in \(\Omega\) with either the Dirichlet boundary conditions
\(u = |\nabla u| = 0\) on \(\partial \Omega\) or the Navier boundary conditions \(u = \Delta u = 0\) on
\(\partial \Omega\). Elliptic systems of the form \(-\Delta^2 u = \lambda f(v), -\Delta v = \gamma g(u)\) in \(\Omega\) with
\(u = v = 0\) on \(\partial \Omega\) are also examined.

6.1.2 The nonlocal eigenvalue problem

We first give the needed background regarding \((-\Delta)^{1/2}\) to examine \((P)_\lambda\), for a
more detailed background see [18]. In [21] they examined the problem \((P)_\lambda\)
with \((-\Delta)^s\) replacing \((-\Delta)^{1/2}\) and with \(g(x) = 1\). They did not investigate
the questions we are interested in but they did develop much of the needed
theory to examine \((P)_\lambda\) and so we will use many of their results.

There are various ways to make sense of \((-\Delta)^{1/2} u\). Suppose that \(u(x)\)
is a smooth function defined in \(\Omega\) which is zero on \(\partial \Omega\) and suppose that
\(u(x) = \sum_k a_k \phi_k(x)\) where \((\phi_k, \lambda_k)\) are the eigenpairs of \(-\Delta\) in \(H^1_0(\Omega)\) which
are \(L^2\) normalized. Then one defines
\[
(-\Delta)^{1/2} u(x) = \sum_k a_k \sqrt{\lambda_k} \phi_k(x).
\]

Another way is to suppose we are given \(u(x)\) which is zero on \(\partial \Omega\) and we let
\(u_e = u_e(x, y)\) denote a solution of
\[
\begin{aligned}
\Delta u_e &= 0 &\text{in } \mathcal{C} := \Omega \times (0, \infty) \\
u_e &= 0 &\text{on } \partial_L \mathcal{C} := \partial \Omega \times (0, \infty) \\
u_e &= u(x) &\text{in } \Omega \times \{0\}.
\end{aligned}
\]

Then we define
\[
(-\Delta)^{1/2} u(x) = \left. \partial_y u_e(x, y) \right|_{y=0},
\]
where \(\nu\) is the outward pointing normal on the bottom of the cylinder, \(\mathcal{C}\). We
call \(u_e\) the harmonic extension of \(u\). We define \(H^1_{0, L}(\mathcal{C})\) to be the completion
of \(C_c^\infty(\Omega \times [0, \infty))\) under the norm \(\|u\|^2 := \int_{\mathcal{C}} |\nabla u|^2\). When working on
the cylinder generally we will write integrals of the form \(\int_{\Omega \times \{y=0\}} \gamma(u_e)\) as
\(\int_{\Omega} \gamma(u)\).

Some of our results require one to examine quite weak notions of solutions
to \((P)_\lambda\) and so we begin with our definition of a weak solution.
Definition 6.1. Given $h(x) \in L^1(\Omega)$ we say that $u \in L^1(\Omega)$ is a weak solution of

$$
\begin{cases}
(-\Delta)^{\frac{1}{2}} u = h(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

provided that

$$
\int_{\Omega} w \psi = \int_{\Omega} h(x)(-\Delta)^{-\frac{1}{2}} \psi \quad \forall \psi \in C_c^\infty(\Omega).
$$

Here $(-\Delta)^{-\frac{1}{2}} \psi$ is given by the function $\phi$ where

$$
\begin{cases}
(-\Delta)^{\frac{1}{2}} \phi = \psi & \text{in } \Omega \\
\phi = 0 & \text{on } \partial \Omega.
\end{cases}
$$

The following is a weakened special case of a lemma taken from [21].

Lemma 6.1. Suppose that $h \in L^1(\Omega)$. Then there exists a unique weak solution $u$ of (6.1). Moreover if $0 \leq h$ a.e. then $u \geq 0$ in $\Omega$.

Definition 6.2. Let $f$ be a nonlinearity satisfying (R).

- We say that $u(x) \in L^1(\Omega)$ is a weak solution of $(P)_\lambda$ provided $g(x)f(u) \in L^1(\Omega)$, and

$$
\int_{\Omega} w \psi = \lambda \int_{\Omega} g(x)f(u)(-\Delta)^{-\frac{1}{2}} \psi \quad \forall \psi \in C_c^\infty(\Omega).
$$

- We say $u$ is a regular energy solution of $(P)_\lambda$ provided that $u$ is bounded, the harmonic extension $u_e$ of $u$, is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies

$$
\int_{\mathcal{C}} \nabla u_e \cdot \nabla \phi = \lambda \int_{\Omega} g(x)f(u)\phi, \quad (6.1)
$$

for all $\phi \in H^1_{0,L}(\mathcal{C})$.

- We say $\bar{u}$ is a regular energy supersolution of $(P)_\lambda$ provided that $0 \leq \bar{u}$ is bounded, the harmonic extension of $\bar{u}$ is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies

$$
\int_{\mathcal{C}} \nabla \bar{u}_e \cdot \nabla \phi \geq \lambda \int_{\Omega} g(x)f(\bar{u})\phi, \quad (6.2)
$$

for all $0 \leq \phi \in H^1_{0,L}(\mathcal{C})$. 

105
6.1. Introduction

In the case where \( f \) satisfies (S) a few minor changes are needed in the definition of solutions. For a weak solution \( u \) one requires that \( u \leq 1 \) a.e. in \( \Omega \). For \( u \) to be a regular energy solution one requires that \( \sup_\Omega u < 1 \).

We will need the following monotone iteration result, see [21]. Suppose that \( u \) and \( \overline{u} \) are regular energy sub and supersolutions of \( (P)_\lambda \). Then there exists a regular energy solution \( u \) of \( (P)_\lambda \) and \( u \leq \underline{u} \leq \overline{u} \) in \( \Omega \). By a regular energy subsolution we are using the natural analog of regular energy supersolution.

We define the extremal parameter

\[
\lambda^* := \sup \{ 0 \leq \lambda : (P)_\lambda \text{ has a regular energy solution} \},
\]

and we now show some basic properties.

**Lemma 6.2.**

1. Then \( 0 < \lambda^* \).
2. Then \( \lambda^* < \infty \).
3. For \( 0 < \lambda < \lambda^* \) there exists a regular energy solution \( u_\lambda \) of \( (P)_\lambda \) which is minimal and semi-stable.
4. For each \( x \in \Omega \) the map \( \lambda \mapsto u_\lambda(x) \) is increasing on \( (0, \lambda^*) \) and hence the pointwise limit \( u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x) \) is well defined. Then \( u^* \) is a weak solution of \( (P)_{\lambda^*} \) and satisfies \( \int_\Omega g(x)f'(u^*)f(u^*)dx < \infty \).

In this chapter we do not need the notion of a semi-stable solution other than for the proof of (4). For the definition of a semi-stable solution one can either use a nonlocal notion, see [21] or instead work on the cylinder which is what we choose to do. We say that a regular energy solution \( u \) of \( (P)_\lambda \) is semi-stable provided that

\[
\int_C |\nabla \phi|^2 \geq \lambda \int_\Omega g(x)f(u)\phi^2 \quad \forall \phi \in H^1_{0,L}(C). \tag{6.3}
\]

We now prove the lemma.

**Proof:**

1. Let \( \overline{u} \) denote a solution of \( -\Delta^{\frac{1}{2}} \overline{u} + t g(x) \) with \( \overline{u} = 0 \) on \( \partial \Omega \) where \( t > 0 \) is small enough such that \( \sup_\Omega \overline{u} < 1 \). One sees that \( \overline{u} \) is a regular energy supersolution of \( (P)_\lambda \) provided \( t \geq \lambda \sup_\Omega f(\overline{u}) \) which clearly holds for small positive \( \lambda \). Zero is clearly a regular energy subsolution and so we can apply the monotone iteration procedure to obtain a regular energy solution and hence \( \lambda^* > 0 \).

2. Suppose that either \( f \) satisfies \( (R) \) and \( C_f < \infty \) or \( f \) satisfies \( (S) \) and so trivially \( C_f < \infty \).
6.1. Introduction

Let \( u \) denote a regular energy solution of \((P)_{\lambda}\) and let \( u_e \) denote the harmonic extension. Let \( \phi \) denote the first eigenfunction of \(-\Delta\) in \( H_0^1(\Omega) \) and let \( \phi_e \) be its harmonic extension; so \( \phi_e(x, y) = \phi(x)e^{-\sqrt{\lambda_1}y} \). Multiply \( 0 = -\Delta u_e \) by \( \frac{\phi_e}{f(u_e)} \) and integrate this over the cylinder \( C \) to obtain

\[
\int_{\Omega} \lambda g(x) \phi = \int_{C} \frac{\nabla u_e \cdot \nabla \phi_e}{f(u_e)} - \int_{C} \frac{|\nabla u_e|^2 \phi_e f'(u_e)}{f(u_e)^2},
\]

and note that the second integral on the right is nonpositive and hence we can rewrite this as

\[
\int_{\Omega} \lambda g(x) \phi \leq \int_{C} \nabla \phi_e \cdot \nabla h(u_e),
\]

where \( h(t) = \int_{0}^{t} \frac{1}{f(\tau)} d\tau \). Integrating the right hand side by parts we have that it is equal to \( \int_{\Omega} (-\Delta)^{\frac{1}{2}} \phi h(u) \) which is equal to \( \sqrt{\lambda_1} \int_{\Omega} \phi h(u) \). So \( h(u) \leq C_f \) and hence we have

\[
\lambda \int_{\Omega} g(x) \phi \leq \sqrt{\lambda_1} C_f \int_{\Omega} \phi.
\]

This shows that \( \lambda^* < \infty \). The case where \( f \) satisfies (R) and where \( C_f = \infty \) needs a separate proof, see the proof of (4). Note that there are examples of \( f \) which satisfy (R) and for which \( C_f = \infty \), for example \( f(t) := (t + 1) \log(t + 1) + 1 \).

(3) The proof in the case where \( g(x) = 1 \) also works here, see [21].

(4) Again the proof used in the case where \( g(x) = 1 \) works to show the monotonicity of \( u_{\lambda} \), see [21], and hence \( u^* \) is well defined. One should note that our notion of a weak solution is more restrictive than what is typically used, i.e., we require \( g(x) f(u) \in L^1(\Omega) \) where typically one would only require that \( \delta(x) g(x) f(u) \in L^1(\Omega) \) where \( \delta(x) \) is the distance from \( x \) to \( \partial \Omega \). Hence here our proof will differ from [21].

Claim: There exists some \( C < \infty \) such that

\[
\int_{\Omega} g(x) f'(u_{\lambda}) f(u_{\lambda}) \leq C,
\]

(6.4)

for all \( 0 < \lambda < \lambda^* \) (at this point we are allowing for the possibility of \( \lambda^* = \infty \)). We first show that the claim implies that \( \lambda^* < \infty \). Note that if \( (-\Delta)^{\frac{1}{2}} \phi = g(x) \) with \( \phi = 0 \) on \( \partial \Omega \) then an application of the maximum principle along with the fact that \( f(u_{\lambda}) \geq 1 \) gives \( u_{\lambda} \geq \lambda \phi \) in \( \Omega \). This along with (6.4) rules out the possibility of \( \lambda^* = \infty \). Using a proof similar to the one in [21] one sees that \( u^* \) is a weak solution to \((P)_{\lambda^*}\) except for
6.2. Uniqueness of the extremal solution

the extra integrability condition \( g(x)f(u^*) \in L^1(\Omega) \) that we require. But sending \( \lambda \nearrow \lambda^* \) in (6.4) gives us the desired regularity and we are done.

We now prove the claim. Let \( u = u\lambda \) denote the minimal solution of \((P)\lambda\) and let \( u_e \) denote its harmonic extension. Take \( \psi := f(u_e) - 1 \) in (6.3) (\( \psi \) can be shown to be an admissible test function) and write the right hand side as

\[
\int_C \nabla f(u_e) - 1f'(u_e) \cdot \nabla u_e,
\]

and integrate this by parts. Using \((P)\lambda\) and after some cancellation one arrives at

\[
\int_C (f(u_e) - 1)f''(u_e) |\nabla u_e|^2 \leq \lambda \int_\Omega g(x)f'(u) f(u). \tag{6.5}
\]

Define \( H(t) := \int_0^t f''(\tau)(f(\tau) - 1)d\tau \) and so the left hand side of (6.5) can be written as \( \int_C \nabla H(u_e) \cdot \nabla u_e \) and integrating this by parts gives

\[
\lambda \int_\Omega g(x)f(u) H(u).
\]

Combining this with (6.5) gives

\[
\int_\Omega g(x)f(u) H(u) \leq \int_\Omega g(x)f(u) f'(u). \tag{6.6}
\]

To complete the proof we show that \( H(u) \) dominates \( f'(u) \) for big \( u \) (resp. \( u \) near 1) when \( f \) satisfies (R) (resp. (S)). If \( 0 < T < t \) then one easily sees that

\[
H(t) \geq (f(T) - 1)(f'(t) - f'(T)).
\]

Using this along with (6.6) and dividing the domain of \( \Omega \) into regions \( \{u \geq T\} \) and \( \{u < T\} \) one obtains the claim.

\( \square \)

6.2 Uniqueness of the extremal solution

**Theorem 6.1.** Suppose that either \( f \) satisfies (R) and is log convex or satisfies (S) and is strictly convex. Then the followings hold.

1. There are no weak solutions for \((P)\lambda\) for any \( \lambda > \lambda^* \).
2. The extremal solution \( u^* \) is the unique weak solution of \((P)\lambda^*\).
The following are some properties that the nonlinearity \( f \) satisfies.

**Proposition 6.1.** (1) Let \( f \) be a log convex nonlinearity which satisfies (R).

(i) For all \( 0 < \lambda < 1 \) and \( \delta > 0 \) there exists \( k > 0 \) such that
\[
f(\lambda^{-1}t) + k \geq (1 + \delta)f(t) \quad \text{for all } 0 \leq t < \infty.
\]

(ii) Given \( \varepsilon > 0 \) there exists \( 0 < \mu < 1 \) such that
\[
\mu^2 (f(\mu^{-1}t) + \varepsilon) \geq f(t) + \frac{\varepsilon}{2} \quad \text{for all } 0 \leq t < \infty.
\]

(iii) Then \( f \) is strictly convex.

(2) Let \( f \) be a nonlinearity which satisfies (S).

(i) Given \( \varepsilon > 0 \) there exists \( 0 < \mu < 1 \) such that
\[
\mu (f(\mu^{-1}t) + \varepsilon) \geq f(t) + \frac{\varepsilon}{2} \quad \text{for all } 0 \leq t \leq \mu.
\]

(ii) Then \( \lim_{t \to 1} \frac{f(t)}{F(t)} = \infty \) where \( F(t) := \int_0^t f(\tau)d\tau \).

**Proof.** See [8], [64] for the proof of (1)-(i) and (1)-(ii). Part (1)-(iii) is trivial. (2)-(i) Set \( h(t) := \mu \{ f(\mu^{-1}t) + \varepsilon \} - f(t) - \frac{\varepsilon}{2} \) and note that \( h'(t) \geq 0 \) for all \( 0 \leq t \leq \mu \) and that \( h(0) > 0 \) for \( \mu \) sufficiently close to 1 which gives us the desired result.

(2)-(ii) Let \( 0 < t < 1 \) and we use a Riemann sum with right hand endpoints to approximate \( F(t) \). So for any positive integer \( n \) we have
\[
F(t) \leq \frac{t}{n} \sum_{k=1}^{n} f\left(\frac{kt}{n}\right) \leq \frac{t(n-1)}{n} f\left(\frac{(n-1)t}{n}\right) + \frac{t}{n} f(t),
\]
and so
\[
\limsup_{t \to 1} \frac{F(t)}{f(t)} \leq \frac{1}{n},
\]
but since \( n \) is arbitrary we have the desired result.

The following is an essential step in proving Theorem 6.1. We give the proof of this lemma later.
6.2. Uniqueness of the extremal solution

Lemma 6.3. Suppose that $f$ is log convex and satisfies (R) or $f$ satisfies (S). Suppose $\varepsilon > 0$ and that $0 \leq \tau$ is a weak solution of

$$\begin{cases}
(\Delta)^{1/2} \tau = l(x) & \text{in } \Omega \\
\tau = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $g(x)(f(\tau) + \varepsilon) \leq l(x) \in L^1(\Omega)$. Then there exists a regular energy solution of

$$\begin{cases}
(\Delta)^{1/2} u = g(x)(f(u) + \varepsilon) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Proof of Theorem 6.1: Without loss of generality assume that $\lambda^* = 1$ and let $u^*$ denote the extremal solution of $(P)_{\lambda^*}$. Suppose that $v$ is also a weak solution of $(P)_{\lambda^*}$ and $v$ is not equal to $u^*$. Set $\Omega_0 := \{ x \in \Omega : u^*(x) \neq v(x), u^*(x), v(x) \in \mathbb{R} \}$ (resp. $\Omega_0 = \{ x \in \Omega : u^*(x) \neq v(x), u^*(x), v(x) < 1 \}$) when $f$ satisfies (R) (resp. (S)) and note that $|\Omega_0| > 0$. Define

$$h(x) := \begin{cases}
\frac{f(u^*(x)) + f(v(x))}{2} - f\left(\frac{u^*(x) + v(x)}{2}\right) & x \in \Omega_0 \\
0 & \text{otherwise}.
\end{cases}$$

Note that by the strict convexity of $f$, which we obtain either by hypothesis or by Proposition 6.1, we have $0 \leq h \in \Omega$ and $h > 0$ in $\Omega_0$. Also note that $h \in L^1(\Omega)$. Define $z := u^* + v$. Since $u^*$ and $v$ are weak solutions of $(P)_{\lambda^*}$, $z$ is a weak solution of

$$(-\Delta)^{1/2} z = g(x)f(z) + g(x)h(x) \quad \text{in } \Omega,$$

with $z = 0$ on $\partial \Omega$. From now on we omit the boundary values since they will always be zero unless otherwise mentioned. Let $\chi$ and $\phi$ denote weak solutions of $(-\Delta)^{1/2}\chi = g(x)h(x)$ and $(-\Delta)^{1/2}\phi = g(x)$ in $\Omega$. By taking $\varepsilon > 0$ small enough one has that $\chi \geq \varepsilon \phi$ in $\Omega$. Set $\tau := z + \varepsilon \phi - \chi$ and note that $\tau$ is a weak solution of

$$(-\Delta)^{1/2} \tau = g(x)(f(\tau) + \varepsilon) \geq 0 \quad \text{in } \Omega,$$

and by Lemma 6.1 we have that $0 \leq \tau$. Moreover, from the fact that $\tau \leq z$ in $\Omega$ we have

$$g(x)(f(\tau) + \varepsilon) \leq (-\Delta)^{1/2} \tau \in L^1(\Omega).$$

Applying Lemma 6.3 there exists a regular energy solution $u$ of

$$(-\Delta)^{1/2} u = g(x)(f(u) + \frac{\varepsilon}{2}) \quad \text{in } \Omega.$$
6.2. Uniqueness of the extremal solution

Set \( w := u + \alpha u - \frac{\varepsilon}{2} \phi \) where \( \alpha > 0 \) is chosen small enough such that \( \alpha u \leq \frac{\varepsilon}{2} \phi \) in \( \Omega \). A straightforward computation shows that \( w \) is a regular energy solution of

\[
(-\Delta)^{\frac{1}{2}} w = (1 + \alpha)g(x)f(u) + \frac{\varepsilon}{2} \alpha g(x)
\]

in \( \Omega \), and that \( w \leq u \) in \( \Omega \). By Lemma 6.1 we also have \( 0 \leq w \) in \( \Omega \). From this we see that \( w \) is a regular energy supersolution of

\[
(-\Delta)^{\frac{1}{2}} w \geq (1 + \alpha)g(x)f(w)
\]

in \( \Omega \), with zero boundary conditions. We now apply the monotone iteration argument to obtain a regular energy solution \( \tilde{u} \) of \( (-\Delta)^{\frac{1}{2}} \tilde{u} = (1 + \alpha)g(x)f(\tilde{u}) \) in \( \Omega \) which contradicts the fact that \( \lambda^* = 1 \). So, we have shown that \( |\Omega_0| = 0 \) and so \( u^* = v \) a.e. in \( \Omega \).

**Proof of Lemma 6.3:** Let \( \varepsilon > 0 \) and suppose that \( 0 \leq \tau \in L^1(\Omega) \) is a weak solution of \((-\Delta)^{\frac{1}{2}} \tau = l(x) \) in \( \Omega \) where \( 0 \leq g(x)(f(\tau) + \varepsilon) \leq l(x) \) in \( \Omega \). As in the proof of Theorem 6.1, we omit the boundary values since they will always be Dirichlet boundary conditions and we also assume that \( \lambda^* = 1 \). First, assume that \( f \) is a log convex nonlinearity which satisfies (R).

Let \( u_0 := \tau \) and let \( u_1, u_2, u_3 \) be weak solutions of

\[
(-\Delta)^{\frac{1}{2}} u_1 = \mu g(x)(f(u_0) + \varepsilon)
\]

in \( \Omega \),

\[
(-\Delta)^{\frac{1}{2}} u_2 = \mu g(x)(f(u_1) + \varepsilon)
\]

in \( \Omega \),

\[
(-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2) + \varepsilon)
\]

in \( \Omega \),

where \( 0 < \mu < 1 \) is the constant given in Proposition 6.1 such that

\[
\mu^2 \left( f\left( \frac{t}{\mu} \right) + \varepsilon \right) \geq f(t) + \frac{\varepsilon}{2}
\]

for all \( t \geq 0 \). One easily sees that \( u_2 \leq u_1 \leq \mu u_0 \). Now note that

\[
(-\Delta)^{\frac{1}{2}} u_1 = \mu g(x)(f(u_0) + \varepsilon)
\]

\[
\geq \mu g(x) f\left( \frac{u_1}{\mu} \right) + \varepsilon
\]

\[
\geq f\left( \frac{u_1}{\mu} \right) + \varepsilon
\]

By Proposition 6.1 with \( \delta := 2N - 1 > 0 \) and \( 0 < \lambda = \mu < 1 \) there exists some \( k > 0 \) such that

\[
f\left( \frac{u_1}{\mu} \right) \geq 2N f(u_1) - k
\]
6.2. Uniqueness of the extremal solution

hence one can rewrite (6.7) as

\[ (-\Delta)^{\frac{1}{2}} u_1 \geq \mu g(x) (2N f(u_1) - k + \varepsilon). \]

We let \( \phi \) be as in the proof of Theorem 6.1 and examine \( u_1 + t\phi \) where \( t > 0 \) is to be picked later. Note that

\[ (-\Delta)^{\frac{1}{2}} (u_1 + t\phi) = (-\Delta)^{\frac{1}{2}} u_1 + tg(x) \geq 2N\mu g(x) (f(u_1) + \varepsilon) + mg(x), \]

where \( m := t - \mu k + \varepsilon \mu(1 - 2N) \) and we now pick \( t > 0 \) big enough such that \( m = 0 \). Therefore, from the definition of \( u_2 \) we have that

\[ (-\Delta)^{\frac{1}{2}} (u_1 + t\phi) \geq 2N (-\Delta)^{\frac{1}{2}} u_2 \quad \text{in } \Omega. \]

So, from the maximum principle we get

\[ u_2 \leq \frac{1}{2N} (u_1 + t\phi) \quad \text{in } \Omega. \]

Since \( f \) is log convex, there is some smooth, convex increasing function \( \beta \) with \( \beta(0) = 0 \) and \( f(t) = e^{\beta(t)} \). By the convexity of \( \beta \) and since \( \beta(0) = 0 \), we have

\[ \beta(u_2) \leq \frac{1}{2N} \beta(u_1 + t\phi) \leq \frac{1}{2N} \beta(\mu u_0 + t\phi), \]

but

\[ \beta(\mu u_0 + t\phi) = \beta(\mu u_0 + (1 - \mu) \frac{t\phi}{1 - \mu}) \leq \mu \beta(u_0) + (1 - \mu) \beta(\frac{t\phi}{1 - \mu}). \]

From this we can conclude

\[ f(u_2)^{2N} \leq e^{\mu \beta(u_0)} e^{(1 - \mu) \beta(\frac{t\phi}{1 - \mu})} \leq f(u_0) f(\frac{t\phi}{1 - \mu})^{1 - \mu}. \]

So, we see that \( g(x) f(u_2)^{2N} \leq C g(x) f(u_0) \in L^1(\Omega) \) for some large constant \( C \).

Since \( g(x) \) is bounded, we conclude that \( g(x) f(u_2) \in L^{2N}(\Omega) \). But \( u_3 \) satisfies \( (-\Delta)^{\frac{1}{2}} u_3 = \mu g(x) (f(u_2) + \varepsilon) \) in \( \Omega \) and so by elliptic regularity we have that \( u_3 \) is bounded (since the right hand side is an element of \( L^p(\Omega) \) for some \( p > N \)) and now we use the fact that \( 0 \leq u_3 \leq u_2 \) and the monotone iteration argument to obtain a regular energy solution \( w \) to \( (-\Delta)^{\frac{1}{2}} w = \mu g(x) (f(w) + \varepsilon) \) in \( \Omega \).
6.3. Uniqueness of solutions for small $\lambda$

Now, set $\xi := \mu w$ and note that $\xi$ is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} \xi = \mu^2 g(x) \left( f\left( \frac{\xi}{\mu} \right) + \varepsilon \right) \quad \text{in } \Omega,$$

and from Proposition 6.1, we have

$$(-\Delta)^{\frac{1}{2}} \xi \geq g(x) \left( f(\xi) + \frac{\varepsilon}{2} \right) \quad \text{in } \Omega,$$

and so by an iteration argument, we have the desired result.

Now, assume that $f$ satisfies (S). In this case, the proof is much simpler. Define $w := \mu \tau$ where $0 < \mu < 1$ is from Proposition 6.1. Then note that

$$(-\Delta)^{\frac{1}{2}} w = \mu l(x) \geq \mu g(x) \left( f\left( \frac{w}{\mu} \right) + \varepsilon \right) \geq g(x) \left( f(w) + \frac{\varepsilon}{2} \right).$$

Hence, $w$ is a regular energy supersolution of

$$(-\Delta)^{\frac{1}{2}} w \geq g(x) \left( f(w) + \frac{\varepsilon}{2} \right),$$

and we have the desired result after an application of the monotone iteration argument.

\[\square\]

6.3 Uniqueness of solutions for small $\lambda$

In this section we prove uniqueness theorems for equation $(P)_\lambda$ for small enough $\lambda$. Throughout this section we assume that $g = 0$ on $\partial\Omega$. We need the following regularity result.

**Proposition 6.2.** [18] Let $\alpha \in (0, 1)$, $\Omega$ be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^N$ and suppose that $u$ is a weak solution of $(-\Delta)^{\frac{1}{2}} u = h(x)$ in $\Omega$ with $u = 0$ on $\partial\Omega$.

1. Suppose that $h \in L^\infty(\Omega)$. Then $u_e \in C^{0,\alpha}(\overline{\Omega})$ hence $u \in C^{0,\alpha}(\overline{\Omega})$.

2. Suppose that $h \in C^{k,\alpha}(\overline{\Omega})$ where $k = 0$ or $k = 1$ and $h = 0$ on $\partial\Omega$. Then $u_e \in C^{k+1,\alpha}(\overline{\Omega})$ hence $u \in C^{k+1,\alpha}(\overline{\Omega})$.
6.3. Uniqueness of solutions for small $\lambda$

Using this one easily obtains the following:

**Corollary 6.1.** For each $0 < \lambda < \lambda^*$ the minimal solution of $(P)_{\lambda}$, $u_{\lambda}$, belongs to $C^{2,\alpha}(\overline{\Omega})$. In addition $u_{\lambda} \to 0$ in $C^1(\overline{\Omega})$ as $\lambda \to 0$.

We now come to our main theorem of this section.

**Theorem 6.2.** Suppose that $\Omega$ is a star-shaped domain with respect to the origin and set $\gamma := \sup_{\Omega} \frac{x \cdot \nabla g(x)}{g(x)}$.

1. Suppose that $f$ satisfies (R) and that
\[
\limsup_{t \to \infty} \frac{F(t)}{f(t)t} < \frac{N - 1}{2(N + \gamma)}. \tag{6.8}
\]
Then for sufficiently small $\lambda$, $u_{\lambda}$ is the unique regular energy solution of $(P)_{\lambda}$.

2. Suppose that $f$ satisfies (S). Then for sufficiently small $\lambda$, $u_{\lambda}$ is the unique regular energy solution $(P)_{\lambda}$.

**Proof:** Let $f$ satisfy (R) and (6.8) or let $f$ satisfy (S) and suppose that $u$ is a second regular energy solution of $(P)_{\lambda}$ which is different from the minimal solution $u_{\lambda}$. Set $v := u - u_{\lambda}$ and note that $v \geq 0$ by the minimality of $u_{\lambda}$ and $v \neq 0$ since $u$ is different from the minimal solution.

A computation shows that $v$ satisfies the equation
\[
(-\Delta)^{\frac{1}{2}} v = \lambda g(x) \{ f(u_{\lambda} + v) - f(u_{\lambda}) \}. \tag{6.9}
\]

Applying Proposition 6.2 to $u$ and $u_{\lambda}$ separately shows that $v \in C^{2,\alpha}(\overline{\Omega})$.

A computation shows the following identity holds
\[
\text{div} \{(z, \nabla v_e) \nabla v_e - z \frac{\left| \nabla v_e \right|^2}{2} \} + \frac{N - 1}{2} \left| \nabla v_e \right|^2 = (z, \nabla v_e) \Delta v_e,
\]
where $z = (x, y)$. Integrating this identity over $\Omega \times (0, R)$ we end up with
\[
\frac{1}{2} \int_{\partial \Omega \times (0, R)} \left| \nabla v_e \right|^2 x \cdot \nu + \int_{\Omega} x \cdot \nabla v_e \partial_x v_e + \frac{N - 1}{2} \int_{\Omega \times (0, R)} \left| \nabla v_e \right|^2 + \varepsilon(R) = 0, \tag{6.10}
\]
where
\[
\varepsilon(R) := \int_{\Omega \times \{ y = R \}} (x \cdot \nabla v_e + R \partial_y v_e) \partial_y v_e - \frac{R}{2} \left| \nabla v_e \right|^2.
\]

114
6.3. Uniqueness of solutions for small $\lambda$

One can show that $\varepsilon(R) \to 0$ as $R \to \infty$, for details on this and the above calculations see [94]. Sending $R \to \infty$ and since $\Omega$ is star-shaped with respect to the origin, we have

$$\frac{N-1}{2} \int_{C} |\nabla v_e|^2 \leq - \int_{\Omega} x \cdot \nabla x v \, \partial_{\nu} v_e,$$

and after using (6.9) one obtains

$$\frac{N-1}{2} \int_{C} |\nabla v_e|^2 \leq -\lambda \int_{\Omega} x \cdot \nabla x v \, g(x) \{f(u_{\lambda} + v) - f(u_{\lambda})\}. \quad (6.11)$$

We now compute the right hand side of (6.11). Set $h(x, \tau) := f(u_{\lambda}(x) + \tau) - f(u_{\lambda}(x))$ and let $H(x, t) = \int_{0}^{t} h(x, \tau) d\tau$. For this portion of the proof we are working on $\Omega$ and hence all gradients are with respect to the $x$ variable. To clarify our notation note that the chain rule can be written as

$$\nabla H(x, v) = \nabla_x H(x, v) + h(x, v) \nabla v,$$

where we recall $v = v(x)$. Some computations now show that

$$H(x, t) = F(u_{\lambda} + t) - F(u_{\lambda}) - f(u_{\lambda})t,$$

and

$$\nabla_x H(x, t) = \{f(u_{\lambda} + t) - f(u_{\lambda}) - f'(u_{\lambda})t\} \nabla u_{\lambda},$$

and so the right hand side of (6.11) can be written as

$$-\lambda \int_{\Omega} g(x) \{f(u_{\lambda} + v) - f(u_{\lambda})\} x \cdot \nabla v = -\lambda \int_{\Omega} g(x) h(x, v) x \cdot \nabla v = -\lambda \int_{\Omega} g(x) x \cdot \{\nabla H(x, v) - \nabla_x H(x, v)\}$$

$$= \lambda \int_{\Omega} g(x) x \cdot \nabla_x H(x, v) + \lambda N \int_{\Omega} H(x, v) g(x) + \lambda \int_{\Omega} H(x, v) x \cdot \nabla g(x).$$

Therefore, (6.11) can be written as

$$\frac{N-1}{2} \int_{C} |\nabla v_e|^2 \leq \lambda \int_{\Omega} x \cdot \nabla x g(x) \{f(u_{\lambda} + v) - f(u_{\lambda}) - f'(u_{\lambda})v\}$$

$$+ N\lambda \int_{\Omega} g(x) \{F(u_{\lambda} + v) - F(u_{\lambda}) - f(u_{\lambda})v\}$$

$$+ \lambda \int_{\Omega} x \cdot \nabla g(x) \{F(u_{\lambda} + v) - F(u_{\lambda}) - f(u_{\lambda})v\}. \quad (6.12)$$
6.3. Uniqueness of solutions for small $\lambda$

We now assume we are in case (1). Let $\alpha$ be such that
\[
\limsup_{\tau \to \infty} \frac{F(\tau)}{\tau f(\tau)} < \alpha < \frac{N - 1}{2(N + \gamma)},
\]
so there exists some $\tau_0 > 0$ such that $F(\tau) < \alpha \tau f(\tau)$ for all $\tau \geq \tau_0$. Let
$0 < \theta < 1$ be such that $\frac{\theta(N-1)}{2} - \alpha(N + \gamma) > 0$ and we now decompose the
left hand side of (6.12) into the convex combination
\[
\frac{\theta(N-1)}{2} \int_C |\nabla v_e|^2 + \frac{(N-1)(1-\theta)}{2} \int_C |\nabla v_e|^2. \quad (6.13)
\]
Using the following trace theorem: there exists some $\tilde{C} > 0$ such that
\[
\int_C |\nabla w|^2 \geq \tilde{C} \int_\Omega w^2, \quad \forall w \in H^1_0(C), \quad (6.14)
\]
one sees that (6.13) is bounded below by
\[
\frac{\theta(N-1)}{2} \int_C |\nabla v_e|^2 + C \int_\Omega v^2.
\]
By taking $C > 0$ smaller if necessary one can bound this from below by
\[
\frac{\theta(N-1)}{2} \int_C |\nabla v_e|^2 + C \int_\Omega g(x)v^2,
\]
and after using (6.9), this last quantity is equal to
\[
\frac{\lambda \theta(N-1)}{2} \int_\Omega g(x) \{f(u_\lambda + v) - f(u_\lambda)\} v + C \int_\Omega g(x)v^2. \quad (6.15)
\]
Substituting (6.15) into (6.11) and rearranging one arrives at an inequality
of the form
\[
\int_\Omega g(x)T_\lambda(x, v) \leq 0,
\]
where
\[
T_\lambda(x, \tau) = \frac{\theta(N-1)}{2} \{f(u_\lambda + \tau) - f(u_\lambda)\} \tau + \frac{C}{\lambda} \tau^2 - N \{F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\} - x \cdot \nabla g \{F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda)\} - x \cdot \nabla u_\lambda \{f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda)\}.
\]
6.3. Uniqueness of solutions for small $\lambda$

To obtain a contradiction we show that for sufficiently small $\lambda > 0$ that $T_\lambda(x, \tau) > 0$ on $(x, \tau) \in \Omega \times (0, \infty)$ and hence we must have that $v = 0$. Define

$$ S_\lambda(x, \tau) = \frac{\theta(N - 1)}{2} \{ f(u_\lambda + \tau) - f(u_\lambda) \} \tau + \frac{C}{\lambda} \tau^2 $$

$$ - (N + \gamma) \{ F(u_\lambda + \tau) - F(u_\lambda) \} - \varepsilon_\lambda \{ f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda) \}, $$

where $\varepsilon_\lambda := \| \nabla u_\lambda \cdot x \|_{L^\infty}$. Note that since $f$ is increasing and convex that $T_\lambda(x, \tau) \geq S_\lambda(x, \tau)$ for all $\tau \geq 0$. We now show the desired positivity for $S_\lambda$ and to do this we examine large and small $\tau$ separately.

**Large $\tau$:** Take $\tau \geq \tau_0$ and $0 < \lambda \leq \lambda^*$. Since $f$ is convex and increasing

$$ S_\lambda(x, \tau) \geq \frac{\theta(N - 1)}{2} f(u_\lambda + \tau) \tau - (N + \gamma) F(u_\lambda + \tau) $$

$$ - \varepsilon_\lambda f(u_\lambda + \tau) + \frac{C}{\lambda} \tau^2 $$

$$ - \frac{\theta(N - 1)}{2} f(u_\lambda) \tau, \quad (6.16) $$

but $F(u_\lambda + \tau) < \alpha(u_\lambda + \tau) f(u_\lambda + \tau)$ for all $\tau \geq \tau_0$ and so the right hand side of (6.16) is bounded below by

$$ f(u_\lambda + \tau) \left[ \tau \left\{ \frac{\theta(N - 1)}{2} - (N + \gamma) \alpha \right\} - \varepsilon_\lambda - (N + \gamma) \alpha u_\lambda \right] $$

$$ - \frac{\theta(N - 1)}{2} f(u_\lambda) \tau + \frac{C}{\lambda} \tau^2. $$

Using the fact that $f$ is superlinear at $\infty$ there exists some $\tau_1 \geq \tau_0$ such that $S_\lambda(x, \tau) > 0$ for all $\tau \geq \tau_1$ and $0 < \lambda \leq \lambda^*$.

**Small $\tau$:** Let $0 < \lambda_0 < \frac{\lambda^*}{2}$ be such that $\| u_\lambda \|_{L^\infty} \leq 1$. Using the convexity and monotonicity of $f$ and Taylor’s Theorem there exists some $C_1 > 0$ such that

$$ F(u_\lambda + \tau) - F(u_\lambda) - f(u_\lambda) \leq C_1 \tau^2, \quad f(u_\lambda + \tau) - f(u_\lambda) - f'(u_\lambda) \tau \leq C_1 \tau^2, $$

for all $0 \leq \tau \leq \tau_0$, $0 < \lambda \leq \lambda_0$ and $x \in \Omega$. Noting that the first term of $S_\lambda(x, \tau)$ is positive for $\tau > 0$ one sees that for all $0 < \tau \leq \tau_0$, $x \in \Omega$ and $0 < \lambda < \lambda_0$ one has the lower bound

$$ S_\lambda(x, \tau) \geq \frac{C}{\lambda} \tau^2 - (N + \gamma + \varepsilon_\lambda) C_1 \tau^2, $$

117
and hence by taking $\lambda$ smaller if necessary we have the desired result.

(2) We now assume that $f$ satisfies (S). One uses a similar approach to arrive at an inequality of the form

$$\int_{\Omega} T_\lambda(x, v) \leq 0,$$

where as before $v = u - u_\lambda \geq 0$ and where we assume that $v \neq 0$. To arrive at a contradiction we show that for sufficiently small $\lambda$ that $T_\lambda(x, \tau) > 0$ for all $x \in \Omega$ and for all $0 < \tau < 1 - u_\lambda(x)$. Again the idea is to break the interval into 2 regions. For $\tau$ such that $\tau + u_\lambda(x)$ close to 1 we use Proposition 6.1, 2 (ii) to see the desired positivity. For the remainder of the interval we again use Taylor’s Theorem.

6.4 Summary and conclusions

In the last chapter, we considered a nonlocal eigenvalue problem of the form

$$\begin{cases}
(-\Delta)^{\frac{3}{2}} u = \lambda g(x) f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\lambda > 0$ is a parameter and $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$. Here $g$ is a positive function and $f$ is an increasing, convex function with $f(0) = 1$ and either $f$ blows up at 1 or $f$ is superlinear at infinity. We showed that the extremal solution $u^*$ associated with the extremal parameter $\lambda^*$ is the unique solution. We also showed that when $f$ is suitably supercritical and $\Omega$ satisfies certain geometrical conditions then there is a unique solution for small positive $\lambda$. 

\[\square\]
Bibliography


[34] E. N. Dancer; *Stable and finite Morse index solutions on $\mathbb{R}^n$ or on bounded domains with small diffusion*. Trans. Amer. Math. Soc. (2005) 357, 1225-1243.


Bibliography


