The Conformal Anomaly and Dark Energy

How the Conformal Anomaly can Introduce an IR Scale into the Vacuum Energy

by

Evan Cameron Thomas

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Abstract

We suggest that the solution to the cosmological vacuum energy puzzle may come from the infrared sector of the low energy effective quantum field theory of gravity, where the impact of the conformal anomaly is of the upmost importance. We proceed by introducing two auxiliary fields, which describe the quantum state by the specifications of their macroscopic IR boundary conditions, in contrast to UV quantum effects. Our investigation aims at finding a realistic cosmological solution which interprets the observed dark energy as a well defined deficit in the zero point energy density of the Universe. The energy density arises from a 'phase transition' that alters the quantum ground state, wherein we give a precise 'renormalization' scheme and definition for a 'renormalized vacuum energy' that we identify with the dark energy.

Preface

The majority of this document is based wholly on the paper [1], which was the result of a collaboration between myself, Federico Urban, and Ariel Zhitnitsky. The relevant calculations were carried out in parallel between myself and Federico. The driving ideas came primarily from Ariel. Writing was done collaboratively by the three of us.

Table of Contents

Appendix

Chapter 1

Introduction

In cosmology, the 'Concordance Model', also called ΛCDM, has become the standard phenomenological model for the cosmology of our Universe; it gives a spacetime manifold which accurately describes the observational evidence for the evolution of the Universe. ΛCDM requires some rather unsettling assumptions. First, the majority of clustered matter is some kind of exotic 'dark' matter, which means it does not interact electromagnetically. Second, the total energy density is dominated by some sort of non-clustered and homogeneously and isotropically distributed "dark" energy [2, 3, 4]. The dark energy seems to have the same equation of state as a cosmological constant and accounts for around 75% of the total energy density, which strikingly, seems to be the required ratio to produce an approximately flat spacial geometry. WMAP gives [5],

$$\Omega_\Lambda \approx 0.728^{+0.015}_{-0.016}. \tag{1.1}$$

Although describing the dark energy as a cosmological constant put in by hand is certainly plausible, it is somewhat unsatisfactory. Ideally physicists could describe the dark energy as a natural parameter coming from a physical theory, and indeed many theories have been proposed [6]. The most naive idea would be to associate the dark energy with some vacuum fluctuations in Quantum Field Theory (QFT), the'zero-point' energy, but this immediately leads to an energy density proportional to the forth power of the cutoff scale. The cutoff scale is the scale of the highest energy modes, the scale at which the theory breaks down ($\sim M_{Pl}^4$). Unfortunately, observation differs from this naive prediction by many orders of magnitude. Most proposed theories try to employ some scheme to cancel or suppress short distance vacuum fluctuations [6]. Many theories also require some new physics, such as an additional scalar field or some modifications to General Relativity (GR). In the following, we discuss an effective field theory viewpoint for gravity which sidesteps the need to consider the high energy vacuum fluctuations and which requires no new physics, only the Standard Model of particle physics and relativistic gravity. We then examine an application of this viewpoint to describe mechanisms that could give rise to an appropriate background energy: the QFT Conformal Anomaly.

1.1 Conventions

In the notation of [7] we use the sign conventions $(-++)$, which is to say, the Minkowski metric is given by

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the Reimann curvature tensor is given by

$$\mathfrak{R}^{\mu}{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha},$$

and the Ricci curvature tensor is given by

$$\mathfrak{R}_{\mu\nu} = \mathfrak{R}^{\alpha}{}_{\mu\alpha\nu}.$$

Chapter 2

Gravity as an Effective Theory

Semi-classical quantum gravity gives the following prescription for coupling gravity to quantum fields. Take the Einstein Equation,

$$\mathfrak{G}_{\mu\nu} + \Lambda \mathfrak{g}_{\mu\nu} = \alpha_G T_{\mu\nu}, \tag{2.1}$$

where $\alpha_G = \frac{8\pi G}{c^4}$, and replace the stress-energy tensor ($T_{\mu\nu}$) with its quantum expectation value ($\langle T_{\mu\nu} \rangle$), so that,

$$\mathfrak{G}_{\mu\nu} + \Lambda \mathfrak{g}_{\mu\nu} = \alpha_G \langle T_{\mu\nu} \rangle. \tag{2.2}$$

In the above equations the Einstein Tensor ($\mathfrak{G}_{\mu\nu}$) is given by,

$$\mathfrak{G}_{\mu\nu} = \mathfrak{R}_{\mu\nu} - \frac{1}{2} \mathfrak{g}_{\mu\nu} \mathfrak{R}, \tag{2.3}$$

and Λ is the 'cosmological constant'. The statement that the dark energy should arise from a physical theory rather than an ad-hoc parameter is just the statement that in (2.2) the Λ term should really be part of the $\langle T_{\mu\nu} \rangle$ term. So that

$$\mathfrak{G}_{\mu\nu} = \alpha_G \langle T_{\mu\nu} \rangle, \tag{2.4}$$

with $\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}^{(particles)} \rangle + \langle T_{\mu\nu}^{(vacuum)} \rangle$, where by particles we mean any energetic excitations in any relevant fields (including gravity itself) and by 'vacuum' we mean any energy in the physical vacuum itself. So we make the association,

$$\langle T_{\mu\nu}^{(vac)} \rangle = -\frac{\Lambda}{\alpha_G} \mathfrak{g}_{\mu\nu}, \tag{2.5}$$

and the dark energy problem is the problem of describing an observed Λ in terms of a theoretical value $\langle T_{\mu\nu}^{(vac)} \rangle$.

Viewing gravity as a fundamental interaction, it should couple to the highest energy vacuum fluctuations, and this is the source for the naive calculation,

$$\langle T_{\mu}^{\mu} \rangle \sim M_{pl}^4 \approx (10^{28} eV)^4. \tag{2.6}$$

If the vacuum contributions of the various fields are cancelled by supersymmetry, instead

$$\langle T^{\mu}_{\mu} \rangle \sim M^4_{SUSY} \gtrsim (10^{13} eV)^4. \tag{2.7}$$

Notice that these are nowhere near the observed value [8],

$$\rho_{obs} \approx (10^{-3} eV)^4. \tag{2.8}$$

If however, gravity is viewed as a low energy effective interaction, the 'gravitons' should be viewed as quasiparticles which do not interact with all of the microscopic degrees of freedom. Rather, as quasiparticles, the gravitons are sensitive to excitations at some effective scale, possibly much lower than the relevant microscopic scales. This idea is the standard effective lagrangian approach to problems across physics, including for example, phonons in a condensed matter system. A typical scale for a phonon system is in the range of eV, despite the microscopic scales for the present fundamental interactions being MeV (electron mass) and GeV (nuclear mass). The phonons don't experience the microscopic physics and so the effective scale is unrelated to the scales for the underlying fundamental theory. In this way the problem of explaining the 'smallness' of the dark energy is replaced by the question of what relevant scale does enter the effective theory of gravity. As an effective field theory, rather than addressing the microscopic reasons for the size of the cosmological constant, we 'renormalize' and absorb any high-energy (UV) divergence into a 'bare cosmological constant', so that we are left with a dynamical theory in terms of a 'renormalized energy density'. The new scale associated with the renormalized energy density then has nothing to do with the UV cutoff scale, but instead is related to some low-energy (IR) scale which enters as a result of this subtraction procedure. Thus, (2.4) becomes

$$\mathfrak{G}_{\mu\nu} = \alpha_G \langle T_{\mu\nu} \rangle_{ren}. \tag{2.9}$$

So the process is as follows. We choose a specific 'point of normalization' at which we know (or at least have some idea) what the stress-energy tensor should be, a point at which we know the relevant values in (2.9). We then set a bare cosmological constant so that it cancels any UV divergences and leaves a finite renormalized stress-energy tensor which matches the quantity we know. This subtraction procedure is done once and forever at our 'point of normalization' and so is set at all other points. Afterward, any other low-energy quantity (including the dark energy) can be derived from the effective QFT for gravity. This process does not address the underlying 'fundamental' problem, in that it has nothing to say about why the bare quantity has the structure that it does. The theory is not based on any 'quantum gravity' and so cannot explain the microscopic situation. That said, as a low-energy effective theory, it does still have predictive power. Once the renormalization is carried out, whatever quantum fields we consider will have some dynamical structure as will the spacetime governed by (2.9).

In the Minkowski vacuum, with metric $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, the Einstein tensor, $\mathfrak{g}_{\mu\nu}$ vanishes identically, and so the renormalization must be defined such that the renormalized energy density also vanishes. Thus, the renormalized vacuum energy, what we will associate with the dark energy, will only be non-zero in non-trivial spacetimes. We therefore expect it will carry some dependance on the deviation from flat space. In a deSitter space for example the dark energy should depend on some (positive) power of the Hubble parameter (H). In the conformally flat Friedmann-Robertson-Walker (FRW) deSitter spacetime with metric

$$ds^2 = dt^2 - \Omega^2(t)d\vec{r}^2, \tag{2.10}$$

the Hubble parameter is defined as

$$H = \frac{1}{\Omega}\frac{d\Omega}{dt} \approx 10^{-10} year^{-1}. \tag{2.11}$$

The situation described above is in no way new or revolutionary (see fore example [9, 10, 11]) and the idea is similar to situations across Quantum Field Theories. As an example, consider the electron mass and charge in Quantum Electrodynamics (QED). Although we generally consider QED to be a fundamental theory of nature, the renormalization logic we apply is the same. We do not know why the bare mass or bare charge have the structure that they do, but we can measure both and so we know what structure they do have. Perhaps some yet undiscovered underlying theory can explain the microscopic picture better, but if we simply renormalize the mass and charge to the measured values at that 'point of normalization', we get a theory for QED which has predictive power. It describes various physical quantities (such as collision cross-sections) extremely accurately.

Chapter 3

The Conformal Anomaly

3.1 Anomalies in QFT

In a QFT, an anomaly is a classical symmetry which is broken by quantum effects, that is the quantization procedure itself destroys the symmetry. One way to look at it is, in a classical theory, the Lagrangian Action,

$$\mathfrak{S} = \int dtL = \int d^4x \mathfrak{L}[\partial_\mu \varphi, \varphi], \tag{3.1}$$

describes the dynamics for a field φ by the Euler-Lagrange equation of motion,

$$\partial_\mu \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathfrak{L}}{\partial \varphi} = 0. \tag{3.2}$$

Thus, if the Lagrangian density, $\mathfrak{L}$, is invariant under a transformation, the equations of motion will be as well. In this way, the symmetries of the theory are completely encoded in the Lagrangian density. Once we go to a quantum picture however, there are no deterministic equations of motion; the deterministic picture has been destroyed by 'quantum fluctuations'. The equations of motion are replaced instead by a statistical picture considering correlations. We write down as a fundamental object, the generating functional,

$$Z[J] = \int \mathfrak{D}\phi \, e^{i \int d^4x \{\mathfrak{L}[\phi(x)] + J(x)\phi(x)\}}, \tag{3.3}$$

and derive any correlation functions by successive differentiation with respect to J. In this form, we see that a symmetry of the Lagrangian density no longer implies a symmetry of the theory, because although the statistical (phase) weight will be invariant, the functional integral measure may not be. Then we must include a Jacobian determinant which can in turn adjust any classical conservation law by adding an 'anomalous contribution'.

The Chiral (Adler-Bell-Jackiw) Anomaly is an instance of a QFT anomaly that has been extensively studied and has real observable consequences; it gives the leading contribution to the decay amplitude $\pi^0 \longrightarrow \gamma + \gamma$. More on this topic can be found in [12] and references therein. Derivations from the functional integral measure can be found in [13, 14].

3.2 The Conformal Anomaly

In the framework of our effective QFT picture for gravity, we are interested in contributions to the stress-energy tensor and so there is a particular anomaly of interest: the conformal (also called trace or scale) anomaly. The conformal anomaly is the breaking of the conformal symmetry by quantum effects. By conformal symmetry, we mean symmetry under the transformation

$$\mathfrak{g}_{\mu\nu}(x) \longrightarrow \mho^2(x)\mathfrak{g}_{\mu\nu}(x), \tag{3.4}$$

where $\mho$ is a real valued function which we will often write as

$$\mho(x) = e^{\sigma(x)}. \tag{3.5}$$

Then the conformal degree of freedom is (equivalently) $\sigma(x)$ or $\mho(x)$. Looking a little more closely at our action, now with the possibility of more general spacetime manifolds explicitly included in the measure,

$$\mathfrak{S} \equiv \int d^4x\, \sqrt{-\mathfrak{g}}\, \mathfrak{L}, \tag{3.6}$$

the stress-energy tensor $T_{\mu\nu}$ is given by the variation of the action with respect to the metric,

$$T_{\mu\nu} = \frac{2}{\sqrt{-\mathfrak{g}}}\frac{\delta\mathfrak{S}}{\delta\mathfrak{g}^{\mu\nu}}. \tag{3.7}$$

Thus, the trace of the stress-energy tensor describes the variation of the action with respect to conformal changes in the metric (3.4),

$$T^{\mu}_{\mu} = -\frac{\Omega}{\sqrt{-\mathfrak{g}}}\frac{\delta\mathfrak{S}}{\delta\Omega} = -\frac{1}{\sqrt{-\mathfrak{g}}}\frac{\delta\mathfrak{S}}{\delta\sigma}. \tag{3.8}$$

This means that a theory which is invariant under (3.4) will give a traceless stress-energy. In particular, we consider a gauge theory (in our case it will be QCD) with the Lagrangian density

$$\mathfrak{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m + g\gamma^\mu A^a_\mu T^a)\psi, \tag{3.9}$$

where the field strength tensor is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{3.10}$$

with f^{abc} the structure constants and T^a the generators for the gauge group. In the chiral limit ($m = 0$), the Lagrangian density (3.10) is invariant under the transformation (3.4) and so classically the stress-energy tensor must be traceless. Once the theory is quantized however, there can be anomalous terms in this trace, and indeed in flat space ($\mathfrak{g}_{\mu\nu} = \eta_{\mu\nu}$) the trace is given by [15]

$$\langle T^\mu_\mu \rangle = \frac{\beta(g)}{2g} \langle F^a_{\mu\nu} F^{a\mu\nu} \rangle + (1 + \gamma_m) m \langle \overline{\psi}\psi \rangle \sim \Lambda^4_{QCD}, \tag{3.11}$$

where β describes the scale dependance of the coupling g, and γ_m the scale dependance of the fermion mass m. Even in the chiral limit, wherein the term involving $\langle \overline{\psi}\psi \rangle$ vanishes, the term involving $\langle FF \rangle$ remains, and so there is an anomalous breaking of scale invariance. Already, in Minkowski space, we can see the need to introduce our 'renormalization procedure' because the Einstein tensor identically vanishes and so the physical stress energy tensor must also. Notice however, that if we consider a 'free' chiral theory ($m = 0$ and $g = 0$), then this anomalous contribution does vanish.

Furthermore, in more general spaces, there are anomalous contributions even from free fields [16]. In particular, in our deSitter space defined in (2.10) we obtain,

$$\langle T^\nu_\mu \rangle = \frac{q(s)}{960\pi^2\alpha^4} \delta^\nu_\mu \sim H^4. \tag{3.12}$$

In the above, $q(s)$ is a factor which depends on the spin of the field we are considering, and $\alpha = 2H^{-1}$.

So, in a flat space non-interacting theory ($H = 0$ and $g = 0$) there is no anomaly, but this is not the case in general. In a flat space interacting theory ($H = 0$ and $g \neq 0$) the conformal anomaly is given by (3.11) and in a curved space non-interacting theory ($H \neq 0$ and $g = 0$) the conformal anomaly is given by (3.12). The true picture of course needs to be a curved space fully interacting theory, which is a technically difficult problem. There have been some arguments [9, 10, 11] presented that the proper description should give

$$\langle T^\mu_\mu \rangle \sim H \Lambda^3_{QCD}. \tag{3.13}$$

It is encouraging that this order of magnitude coincides with the observed value (2.8), and astonishingly, a similar prediction was made as early as 1967 in [17]. Arguments presented in [11] suggest that this sort of scaling behavior can only arise in a strongly coupled QFT, and that in a weakly coupled theory, such as the electro-weak sector of the Standard Model, the above effect will be exponentially suppressed ($\sim e^{-\frac{2\pi}{\alpha}}$). In QED for example the effects arising from the dynamical breaking of scale invariance will be suppressed by a factor of 10^{-370}.

It should also be noted that the above behavior contradicts some covariance arguments, which demand that only even powers of H appear in expressions for the vacuum energy [18], but we claim this is not a problem; these dimensional arguments are not generally applicable in the case of a nontrivial global topology. For example, if we introduce a nonzero temperature, implemented by considering a cylindrical topology (one dimension becomes compact with size $L \sim \frac{1}{T}$), we should expect powers of T in the expansion for the vacuum energy. If the size scales as $L \sim HL_0$ [19], it would lead to the desired scaling.

As a quick aside, notice that QCD playing a key role in the Dark Energy problem would also explain, in part, the so called 'Cosmic Coincidence Problem', which is the similarity among the scales of dark energy, dark matter, and baryonic matter ($\Omega_{DE} \approx 4\Omega_{DM}$ and $\Omega_{DM} \approx \Omega_{Baryons}$). See [20] and references therein for a discussion of how the dark matter could also be related to QCD and the QCD scale (Λ_{QCD}).

In the next section we explore the dynamics of the conformal factor in order to understand under what conditions the anomaly could give rise to a finite remnant after our 'renormalization' subtraction, and particularly under what conditions that finite residual takes the form (3.13). To that end, we now consider the anomalous contribution as generated by some specific nonlocal induced interactions as argued in [21, 22, 23, 24]. For more on these ideas and applications thereof see [25, 26, 27, 28, 29].

3.3 Dynamics of the Conformal Factor

In four spacetime dimensions the trace anomaly takes the form

$$\langle T^{\mu}_{\mu} \rangle = bF + b'\left(E - \frac{2}{3}\Box\mathfrak{R}\right) + b''\Box\mathfrak{R}, \tag{3.14}$$

where

$$\begin{array}{lllll} E & \equiv & *\mathfrak{R}_{\mu\nu\rho\sigma} * \mathfrak{R}^{\mu\nu\rho\sigma} & = & \mathfrak{R}_{\mu\nu\rho\sigma}\mathfrak{R}^{\mu\nu\rho\sigma} - 4\mathfrak{R}_{\mu\nu}\mathfrak{R}^{\mu\nu} + \mathfrak{R}^2, \\ F & \equiv & \mathfrak{C}_{\mu\nu\rho\sigma}\mathfrak{C}^{\mu\nu\rho\sigma} & = & \mathfrak{R}_{\mu\nu\rho\sigma}\mathfrak{R}^{\mu\nu\rho\sigma} - 2\mathfrak{R}_{\mu\nu}\mathfrak{R}^{\mu\nu} + \frac{1}{3}\mathfrak{R}^2. \end{array} \tag{3.15}$$

In the above $*\mathfrak{R}_{\mu\nu\rho\sigma}$ is the dual of the the Reimann curvature tensor given by

$$*\mathfrak{R}_{\mu\nu\rho\sigma} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\mathfrak{R}^{\lambda\kappa}{}_{\rho\sigma},$$

and $\mathfrak{C}_{\mu\nu\rho\sigma}$ is the Weyl curvature tensor given by

$$\mathfrak{C}_{\mu\nu\rho\sigma} \equiv \mathfrak{R}_{\mu\nu\rho\sigma} - \mathfrak{g}_{\mu[\rho}\mathfrak{R}_{\sigma]\nu} + \mathfrak{g}_{\nu[\rho}\mathfrak{R}_{\sigma]\mu} + \frac{1}{3}\mathfrak{R}\,\mathfrak{g}_{\mu[\rho}\mathfrak{g}_{\sigma]\nu}.$$

The coefficients b and b' in 3.14 are dimensionless parameters which depend only on the number of free massless degrees of freedom in the theory [16]. The statement that only free and effectively massless fields contribute to the anomaly, is just a statement that the relevant loop diagrams are saturated at large length scales (IR), which is exactly what we might expect for a presumably IR effect, like the dark energy. Those coefficients are, in terms of the numbers of massless scalars (N_S), spinors (N_F), and vectors (N_V),

$$\begin{aligned} b &= -\tfrac{1}{120(4\pi)^2}(N_S + 6N_F + 12N_V), \\ b' &= \tfrac{1}{360(4\pi)^2}(N_S + 11N_F + 62N_V). \end{aligned} \tag{3.16}$$

The meaning of the third coefficient b'' is somewhat unclear since it is possible to redefine this coefficient by adding a local counterterm proportional to $\mathfrak{R}^2$ to the gravitational action. Thus, it seems that this coefficient is regularization dependent and could be removed by a suitable gauge choice. As such, we do not consider it a part of the 'true' anomaly. The terms with coefficients b and b' however cannot be derived from a local action containing only the metric $\mathfrak{g}_{\mu\nu}$ and it's derivatives unless that action is singular [16, 30]. There are in fact three ways to describe the action: as a singular local quantity, as a finite nonlocal quantity, or as a nicely behaved finite local action involving two additional auxiliary fields.

To see how these actions come about, we proceed along the lines of [21] and consider the conformally related metrics

$$\mathfrak{g}_{\mu\nu} = e^{2\sigma(x)}\bar{\mathfrak{g}}_{\mu\nu}. \tag{3.17}$$

We can then consider the first two terms from (3.14) and find,

$$\sqrt{-\mathfrak{g}}F = \sqrt{-\bar{\mathfrak{g}}}\bar{F}, \tag{3.18}$$

$$\sqrt{-\mathfrak{g}}\left(E - \frac{2}{3}\Box R\right) = \sqrt{-\bar{\mathfrak{g}}}\left(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\right) + 4\sqrt{-\bar{\mathfrak{g}}}\bar{\triangle}_4\sigma, \tag{3.19}$$

where $\triangle_4$ is a conformally covariant fourth order differential operator given by

$$\triangle_4 \equiv \Box^2 + 2\mathfrak{R}^{\mu\nu}D_\mu D_\nu - \frac{2}{3}\mathfrak{R}\Box + \frac{1}{3}(D^\mu\mathfrak{R})D_\mu, \tag{3.20}$$

with $\Box = D^\mu D_\mu$ and $D_\mu = \partial_\mu + \Gamma_\mu$ the usual covariant derivative. Then we can write down an action which will, upon differentiation with respect to the conformal factor σ, give the conformal anomaly (3.14). This is called the Wess-Zumino action and is given by

$$\Gamma[\bar{\mathfrak{g}};\sigma] = b\int d^4x\sqrt{-\bar{\mathfrak{g}}}\bar{F}\sigma + b'\int d^4x\sqrt{-\bar{\mathfrak{g}}}\left\{\left(\bar{E} - \frac{2}{3}\Box\bar{\mathfrak{R}}\right)\sigma + 2\sigma\bar{\triangle}_4\sigma\right\}. \tag{3.21}$$

Notice that this effective action is a function of the conformal factor σ and the metric $\bar{\mathfrak{g}}_{\mu\nu}$, because it is impossible to find a corresponding action as a function of only the original metric $\mathfrak{g}_{\mu\nu}$. This is illustrative of the fact that in order to describe the conformal anomaly we are forced to consider additional degrees of freedom, in this case the conformal factor. It is also possible to formally remove the conformal factor by solving (3.19) and (3.18), and realizing that

$$\Gamma[\bar{\mathfrak{g}};\sigma] = S_A[\mathfrak{g} = e^{2\sigma}\bar{\mathfrak{g}}] - S_A[\bar{\mathfrak{g}}], \tag{3.22}$$

where the nonlocal anomalous action S_A is

$$S_A[\mathfrak{g}] = \frac{1}{8}\int d^4x\sqrt{-\mathfrak{g}}\left[E - \frac{2}{3}\Box\mathfrak{R}\right]_x \triangle_4^{-1}(x,x')\left[2bF + b'\left(E - \frac{2}{3}\Box\mathfrak{R}\right)\right]_{x'}. \tag{3.23}$$

This action is clearly nonlocal; the next step is to render this action local by introducing two auxiliary scalar fields which through their equations of motion encode the information in (3.21) or (3.23).

3.4 The Infrared Auxiliary Fields

We consider (3.23) to be a fundamental part of the low-energy effective theory of gravity, and as such it will contribute to the total stress-energy tensor (the 'anomalous' part). Therefore we endeavor to express the nonlocal action (3.23) as a local action containing some extra degrees of freedom, two auxiliary scalar fields obeying the Euler-Lagrange equations of motion

$$\triangle_4\phi = \frac{1}{2}\left(E - \frac{2}{3}\Box\mathfrak{R}\right), \tag{3.24}$$

$$\triangle_4\psi = \frac{1}{2}F. \tag{3.25}$$

In terms of these new fields ϕ and ψ, the anomalous action (3.23) becomes

$$\begin{aligned} S_A[\mathfrak{g};\phi,\psi] &= \frac{b}{2}\int d^4x\sqrt{-\mathfrak{g}}\left\{-\psi\triangle_4\phi - \phi\triangle_4\psi + F\phi + \left(E - \frac{2}{3}\Box\mathfrak{R}\right)\psi\right\} \\ &+ \frac{b'}{2}\int d^4x\sqrt{-\mathfrak{g}}\left\{-\phi\triangle_4\phi + \left(E - \frac{2}{3}\Box\mathfrak{R}\right)\phi\right\}. \end{aligned} \tag{3.26}$$

The action (3.26) is the starting point for the practical calculations we will perform. Notice that the action is now entirely local and that all dependance on the conformal factor have dropped out. We view this as the true effective action for the anomaly. Therefore, the full action for the low-energy effective gravity is

$$S_{total} = S_{EH} + S^{(2)}_{Weyl} + \sum_{n\geq 3}^{\infty} S^{(n)}_{local} + S_A, \tag{3.27}$$

where the first term is the usual Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\alpha_G}\int d^4x\sqrt{-\mathfrak{g}}\mathfrak{R}, \tag{3.28}$$

and the second is the conformally invariant Weyl action

$$S^{(2)}_{Weyl} = \alpha_W \int d^4x\sqrt{-\mathfrak{g}}\mathfrak{C}^{\mu\nu\rho\sigma}\mathfrak{C}_{\mu\nu\rho\sigma}. \tag{3.29}$$

The sum appearing in place of a third term represents all possible higher order curvature invariants, each term suppressed by $(2n-4)$ powers of the Planck mass M_{pl} at low-energy. Finally, the last term is the local 'anomalous' action (3.26). As usual, we can derive a corresponding anomalous energy momentum tensor by differentiating with respect to the metric,

$$T^A_{\mu\nu} \equiv -\frac{2}{\sqrt{-\mathfrak{g}}}\frac{\delta S_A}{\delta \mathfrak{g}^{\mu\nu}} = bF_{\mu\nu} + b' E_{\mu\nu}, \tag{3.30}$$

where the explicit functional forms for the two anomalous tensors $E_{\mu\nu}$ and $F_{\mu\nu}$ are given in [21]; the full general forms are quite messy and not terribly relevant here. As it should be, the trace of this stress-energy tensor is is proportional to the equations of motion for the auxiliary ϕ and ψ, which, once employed, return exactly the (on-shell) conformal anomaly

$$T^{A\mu}_{\mu} = 2b\triangle_4\psi + 2b'\triangle_4\phi = bF + b'\left(E - \frac{2}{3}\Box\mathfrak{R}\right). \tag{3.31}$$

This equation (3.31) is identical to the equation (3.14), but it was obtained from from the auxiliary field formalism rather than directly renormalizing the gravitational action or the resulting divergent stress-energy tensor as in [16]. Although in the trace of the stress-energy, the auxiliary fields appear only as their equations of motion, and therefore do not appear in the final expression for the trace, this is not the case when we consider the individual components of the stress-energy tensor (3.30) separately. The auxiliary fields appear in combinations which cannot be removed by simple application of the equations of motion. In general, the stress-energy is dependent on not only the form of the Euler-Lagrange equations (3.24) and (3.25), but also on the particular solutions for ϕ and ψ to those equations. In particular, the choice of boundary conditions for the auxiliary fields, in solving their differential equations of motion, will pick out a particular state from all possibilities, and will explicitly enter the general expression for the stress-energy tensor (3.30), along with the coefficients b and b'.

Looking at the full action (3.27), at low energy, the relevant terms are the Einstein-Hilbert action, the conformally invariant Weyl action, and the anomalous action. In general, the Weyl contribution and the anomalous contribution to the stress-energy would be of the same order of magnitude, as they both contain four derivatives of the metric. Moreover, the anomalous action is not uniquely defined by (3.22) since we can add to it any conformally invariant term without changing the the equations of motion. However, if the spacetime we consider is conformally flat, then the Weyl invariant vanishes identically, leaving the anomalous contribution as the only (and unique) contribution to the stress-energy. In this case, the anomalous stress-energy (3.30) can be shown to provide the information about the tracefree part of the complete stress-energy $T_{\mu\nu}$. These ideas are explored more closely in [21].

3.5 Horizon Divergences

Any spacetime with a static Killing horizon is conformally related to flat spacetime near that horizon, and so the stress-energy tensor is well approximated by the sole contribution (3.30) coming from the anomalous action (3.26). Hence, we will now consider spacetimes with horizons, and in particular the deSitter spacetime in static coordinates (and some closely related coordinate systems), which is in addition conformally flat. In general, in dealing with spacetimes with horizons, in which we have a coordinate singularity, we expect the solutions of the Euler-Lagrange equations for any field in this background to diverge at that coordinate singularity. We therefore expect that the stress-energy, $T_{\mu\nu}$, being derived from singular fields, will inherit this singularity at the horizon. Indeed, this is the case for a number of examples in QFT in curved space, for instance, the Boulware state in static deSitter space, or in Schwarzschild space. As argued in [21], this does not imply that the quantum state is to be rejected as nonphysical.

The singular behavior signals the breakdown of the semiclassical approximation used throughout the computation leading to the renormalized stress-energy tensor because of the enormous backreaction on the background geometry. Thus, is we are to trust our low energy effective theory in the region of space near the horizon, we must employ some process from removing the divergence. The original proposal [21] involves constructing a static system in which the metric flips to another allowed solution to the Einstein Equations before crossing the horizon and flips back after crossing (having masked the singularity with this other nonsingular metric).

We propose a rather different process, which is, 'renormalize' away the divergences through a subtraction scheme of the sort alluded to in Chapter 2, and this idea is the focus of the next chapter.

Chapter 4

Renormalizing Away the Divergences

In figuring out how to remove the divergences just discussed, we will make use of the fact that during the evolution of the Universe the relevant quantum fields in calculating the anomalous stress-energy change their properties, such as their (effective) masses, thereby changing their contributions to the anomalous action. In the most simplistic view, without considering for the moment the changes in the solutions to (3.24) and (3.25) for the auxiliary fields, the number of effectively massless fields will change and so the coefficients b and b' will change at the very least. In general of course, we expect that the auxiliary fields ϕ and ψ will adjust as well so that the total change in the anomalous stress-energy could be rather complicated.

The general picture however is straightforward, the stress-energy associated with the anomaly should go through some 'phase transitions' as the Universe evolves. It then becomes natural to ask wether the divergent part of the anomalous stress-energy changed, or only the finite part. If the divergent part of the stress-energy has the same form on both sides of these 'phase transitions', then 'renormalizing' the stress-energy tensor in the early Universe (before the 'phase transitions') would yield a finite energy density afterward. Our proposal is that, in this case, the finite difference between the divergent quantities derived before and after this 'phase transition' will give a finite, non-clustered, homogeneously and isotropically distributed (dark) energy density, which should be identified with the Dark Energy. This finite term originates precisely at the moment that the QFT spectrum contributing to the anomaly drastically changes, and so we expect that the scale at which this happens should then be the IR scale of the problem and so the scale which enters our effective QFT for gravity. To that end, we would like to see how an internal QFT scale, such as Λ_{QCD} can manifest in the final expression for the Dark Energy.

Before proceeding to a discussion of the sort of situations which could be relevant to the ideas expressed above, we will address the comment we made earlier that only free effectively massless fields contribute to the anomaly.

4.1 Free and Massless Fields

In order to make this idea more precise, we rephrase as follows. If a particle has a massless pole (and so is a candidate to be a part of the anomaly), and this pole remains massless once interactions are included, then this particle will be a part of the anomaly; if however, the interactions shift the pole, then the corresponding particle field no longer contributes to the degrees of freedom participating in the anomaly calculations.

To understand how this comes about, we need to understand how the anomaly (3.14) was originally calculated. The anomaly can be understood as a result of regularization of the quantum field theory, in which a regulator field φ_R with mass M_R is introduced for each quantum field (this is the so call 'Pauli-Villars' regulator). The statement that the conformal symmetry is explicitly broken on the quantum level is then equivalent to the statement that there is a finite contribution to the anomalous stress-energy trace after taking the limit $M_R \to \infty$. If a field φ has a non-vanishing mass m_φ, the trace of the stress-energy tensor receives a canonical contribution $\langle T^\mu_\mu \rangle \sim m^2_\varphi \langle \varphi^2 \rangle$ (see [31] for this derivation for a massive scalar) along with the corresponding anomalous contribution resulting from the regulator φ_R. If m_φ is infinite, the corresponding canonical contribution exactly cancels the anomalous one. If however, m_φ is large but finite, the resulting cancelation is not exact, but gives rise instead to a suppression of order $\sim \frac{\mathfrak{R}}{m^2_\varphi}$. Apparently, the distinction between massive and effectively massless degrees of freedom is defined by the ratio $\mathfrak{R}/m^2_\varphi$. In the case of FWR deSitter spacetime, this ratio of interest is $\sim H^2/m^2_\varphi$ so that in order to be a part of the anomaly they must be free on the hubble scale.

Precisely this same pattern is also realized for the chiral anomaly where a quark Q has non-vanishing mass m_Q (see the appendix of [32] for a detailed discussion of these calculations). Indeed, when the quark is strictly massless, the corresponding axial current has the standard anomalous form originating from the regulator field, $\langle \partial_\mu J^\mu_5 \rangle = \langle \frac{\alpha_s}{4\pi} F\tilde{F} \rangle$. If the quark is massive however, this expression for the axial current receives also a canonical term, so that $\langle \partial_\mu J^\mu_5 \rangle = \langle 2m_Q \bar{Q} i\gamma_5 Q \rangle + \langle \frac{\alpha_s}{4\pi} F\tilde{F} \rangle$. Assuming m_Q is sufficiently large ($m^2_Q \gg F \sim \Lambda^2_{QCD}$), we can expand the canonical term and the leading order term in this expansion will cancel the anomalous term so that $\langle \partial_\mu J^\mu_5 \rangle \sim m_Q^{-2} \langle g^3 F^3 \rangle + \mathcal{O}(1/m^4_Q)$. So the anomalous terms coming from this field are suppressed by a factor $\sim g\Lambda^2_{QCD}/m^2_Q$, just as above.

Another way to think about the requirement that fields be free on the Hubble scale is that in calculating the anomaly diagrammatically, there are some loop diagrams that we calculate using free propagators of the various fields going around the loop. If however the mean free path of the particles in question is shorter than the length scale on which the integrals are saturated, than it does not make sense to use the free propagator on those length scales. In the case of the conformal anomaly, the integrals in question are saturated in the far IR (which is exactly why we expect it to play a crucial role in the IR physics we are describing). The story from this picture is the same, we only consider particles with a mean free path much longer than the Hubble length.

In the next two sections we discuss situations in which the sort of drastic adjustments to the anomaly we are interested in could occur and and the relative importance we expect for these effects. We discuss, respectively, phase transitions and the decoupling of species from the early Universe thermal bath.

4.2 Phase Transitions

The first situation we consider in which the arguments presented in this chapter are likely to play a role is phase transitions that occurred during the evolution of the Universe. This is because, as stated, only effectively massless fields contribute to the anomaly, so consider a phase transition during the course of which some fields change mass, having different values on either side of the boundary of the transition. This sort of behavior is know to be the case for for a number of examples in the early Universe, for instance the Electro-Weak (EW) transition giving masses to, in its simplest version, all the Standard Model (SM) particles; or the QCD phase transition, which exhibits a peculiar time dependance for the axion mass (as well as several other hadronic states); or finally, also considering axions, the breaking of the Peccei-Quinn (PQ) symmetry which gives birth to an effectively massless axion.

4.2.1 The Electro-Weak Transition ($\sim 100\ GeV$)

At first, the EW transition may seem an ideal playground for the ideas presented here, but upon closer inspection, we are considering situations in which the numbers or types of particles contributing to the anomaly are changing. In the EW theory however, all of the particles which gain a mass due to interaction with the Higgs boson, posses electroweak and/or color charges, which in turn implies they cannot be treated as free in the early Universe and so do not contribute to the degrees of freedom count in

(3.16). We should comment that the number of relevant fields is not the only possible consideration since the vacuum structure undergoing a profound reorganization will also have an effect through the boundary conditions for the auxiliary fields, ϕ and ψ. In this first analysis however, we will look more closely at situations in which the number of relevant fields is in fact changing. Thus, we will not concern ourselves with the EW transition here.

4.2.2 The Peccei-Quinn Symmetry ($\sim 10^{12}\ GeV$)

The axion field was introduced by Peccei and Quinn in order to solve the 'strong' $\mathcal{CP}$ problem, which remains one of the most outstanding puzzles of the Standard Model. (See the original papers [33, 34, 35, 36, 37, 38, 39].) Thirty years after the axion was invented, it is still considered as a viable solution to the $\mathcal{CP}$ problem, and one of the plausible candidates for dark matter. At present, there are several experimental groups searching for the axion, see [40, 41] and references therein.

The primary characteristic of the mechanism proposed by Peccei and Quinn that is of interest here is the introduction of a new scalar degree of freedom, the axion, which appears once the PQ symmetry is spontaneously broken. At high temperatures the axion feels no potential and so is effectively massless, but as the system cools, the axion develops an effective mass through interaction with other particles, see below.

4.2.3 The QCD Phase Transitions ($\sim 170\ MeV$)

We should note that it is believed that the confinement-deconfinement transition is really a cross-over for the realistic quark mass rather than a true phase transition. Also, it is still unclear wether the confinement-deconfinement transition and the chiral phase transition occur at exactly the same temperature or slightly different temperatures. However, for our purposes, these subtle distinctions are not terribly relevant. The crucial point here is that the fields are in fact changing properties around this temperature. In particular, the axion field, as mentioned above, changes properties around the quark-gluon to hadron transition. Indeed, the cosmological history of this field is extremely nontrivial around the QCD epoch, the main feature of which is the time dependance of the axion mass.

At the QCD transition, the axion field acquires a small mass (which increases as temperature decreases) due to the appearance of a periodic potential induced by instantons. Once the temperature drops below the critical value and the chiral condensate forms, the axion mass m_a assumes its final form, $m_a^2 f_{PQ}^2 \sim m_q \langle \bar{q}q \rangle$. Moreover, the axion is a scalar field with no electric or color charge, and tiny cross sections when interacting with Standard Model particles, thereby making it an ideal candidate for exploring our arguments.

Hence, the 'effectively massless' axion becomes 'effectively massive' at a time, t_0 defined by $\mathfrak{R}/m_a^2(t_0) \sim 1$. Thus, we have a scenario in which there is an initial 'phase' (unconfined) with a free massless axion on some background quantum state and a final 'phase' (confined) with a massive axion in a different quantum vacuum configuration. Both properties should be tracked in the expression for the anomalous stress-energy (3.30), via, respectively, the change in the coefficients (3.16) and in the boundary conditions for the auxiliary fields ϕ and ψ.

4.3 Decoupling of Species

The second situation wherein we expect the anomalous stress-energy tensor could experience a transition from one state to another is the decoupling of some species from the thermal bath in which, at early times, most fields are immersed. For example, consider the cosmological history of photons. Again we discount charged particles, since even when decoupled from the thermal plasma, they have a mean free path shorter than the hubble length, and so cannot be treated as free on these length scales.

A photon in the early Universe, that is, before decoupling around the last scattering epoch, is in close thermal contact with other SM particles that have not yet fallen out of kinetic equilibrium. This translates to the fact that photons will have a small (momentum dependent) mean free path, and so is not involved in the anomaly. Once the photons fall out of thermal equilibrium however, they become free and massless and so are included in the anomaly. Again we have a situation in which a field takes part in the anomaly on one side of a transition (kinetic decoupling) but not on the other.

This same idea of a decoupling transition could be applied to the axion field (depending on the PQ scale f_{PQ}) or to neutrinos which underwent a similar kinetic decoupling in the MeV range.

Chapter 5

Calculations and Results

Having laid the physical basis for our arguments, we now turn to a detailed analysis of the stress-energy tensor $T_{\mu\nu}$ in the sort of scenario outlined in the previous chapter. We will try to keep the discussion quite general and not restrict ourselves to one specific setup among those mentioned above. Another reason not to choose a specific setup is that, although expected on general grounds, there is not currently a method for relating the semiclassical vacuum expectation values for such configurations with the stress-energy tensors coming from the auxiliary fields with different boundary conditions.

We will be working exclusively in deSitter spacetime, wherein the Ricci tensor and scalar are given by

$$\begin{aligned} \mathfrak{R}_{\mu\nu} &= \tfrac{1}{4}\mathfrak{g}_{\mu\nu}\mathfrak{R}, \\ \mathfrak{R} &= -12H^2, \end{aligned} \tag{5.1}$$

where H is the Hubble parameter (2.11). The fourth order curvature invariants (3.15) are given by

$$\begin{aligned} E &= 24H^4, \\ F &= 0. \end{aligned} \tag{5.2}$$

The coordinate systems considered are some of those typically employed in deSitter (dS) spacetime, namely the spatially flat Friedman-Robertson-Walker (FWR) system, the conformally flat (CF) system, the Tortoise system, the static coordinate system, and finally a coordinate system in which the deSitter spacetime is conformal to the four dimensional Rindler spacetime, which we call the Rindler coordinate system. Notice that in all these systems we will tamper with the 'radial' and 'temporal' coordinates, preserving the $\mathcal{O}(3)$ symmetry of spatial rotations.

5.1 Anomalous $T_{\mu\nu}$ in Static deSitter Coordinates

We choose, as a reference metric, the Static deSitter coordinates (Appendix.4), and having solved, in some special cases, the equations of motion for the auxiliary fields (Appendix.8), it is possible to get an explicit expression for the anomalous stress-energy, $T^A_{\mu\nu}$, in terms of the coordinates and boundary conditions. The boundary conditions imposed on the auxiliary fields appear as the integration constants, the c's and d's in (Appendix.8). First, notice that the solutions for the auxiliary fields are singular at both $r = 0$ and $r = 1/H$. We remove the singularity at the coordinate origin can by taking $c_\infty = d_\infty = 0$. In deSitter spacetime, only the diagonal components of the anomalous stress-energy tensor survive. Also, due to spherical symmetry, the $\theta\theta$ component and the $\varphi\varphi$ component are the same, so we need only the tt, rr, and $\theta\theta$ components, which are given in the appendix of [21]. Even after removing the $r = 0$ singularity from the auxiliary fields, there may, in general, still be terms that diverge as $1/r$ and $1/r^2$ in the stress-energy tensor. We remove these by setting $c_1 = d_1 = 0$.

The process we employ then is as follows. We expand the stress-energy in inverse powers of $(1 - Hr)$ in order to study the singular behavior near the horizon. In general, we expect singular behaviors $1/(1 - Hr)^2$, $1/(1 - Hr)$, and $ln(1 - Hr)$ for each of the three independent components of the stress-energy, but in fact some of the coefficients are the same or linearly dependent, so that we can describe the divergences in $T^A_{\mu\nu}$ by means of only three coefficients,

$$\begin{aligned}\alpha &= b' + 2b'c_2 + b'c_2^2 + 2bd_2 + 2bc_2d_2,\\ \beta &= 3b' + 6b'c_2 + b'c_2^2 + 6bd_2 + 2bc_2d_2,\\ \gamma &= b' + 2b'c_2 - b'c_2^2 + 2bd_2 - 2bc_2d_2.\end{aligned} \tag{5.3}$$

If we want to implement our idea of renormalization, we need the divergence on either side of our 'phase transition' the have the same form, so that there is only a finite part left after our subtraction procedure. This requirement is

$$\begin{cases}\alpha_i = \alpha_f\\ \beta_i = \beta_f\\ \gamma_i = \gamma_f\end{cases} \tag{5.4}$$

which is

$$\begin{cases} c_{2f} = \frac{1}{2b'_f}(b'_i - b'_f + 2b'_ic_{2i} + 2b_id_{2i} - 2b_fd_{2f})\\ \\ d_{2f} = \frac{1}{2b'_f}\sqrt{(b'_i - b'_f + 2b'_ic_{2i} + 2b_id_{2i})^2 - 4b'_fc_{2i}(b'_ic_{2i} + 2b_id_{2i})}\end{cases} \tag{5.5}$$

where the subscripts i and f denote the initial and final state respectively. The conditions (5.5) ensure that the horizon divergences cancel dynamically, so that a 'renormalization', in which we require the anomalous stress-energy to vanish before the transition will leave a finite stress-energy afterward. This 'renormalization prescription' demonstrates how the UV divergences are removed, leaving some IR related physics as the only relevant scale.

We are now in a position to calculate the finite contribution to the energy density obtained after the divergent parts have been removed via (5.5). The result is (Δ = final state − initial state)

$$\Delta T^{At}_{t} = \Delta T^{Ar}_{r} = \Delta T^{A\theta}_{\theta} = 6H^4 \Delta b'. \tag{5.6}$$

It is encouraging that this process arrived at a finite result, but discouraging that it has reduced to the result (3.12), $\sim H^4$ rather than $\sim H$ as we would like. Notice that this result does not depend on any of the coefficients except b', so that no IR physics from the shift in quantum state, encoded by the coefficients (c_0, c_1, c_2, d_0, d_1, d_2), has penetrated this final result. As a result, the anomalous contribution has reduced to a local second order curvature term ($\sim H^4$).

5.2 Possible Modifications

We now consider the reasons that our result (5.6) is not sensitive to the IR boundary parameters and possible modifications to this theory which could resolve the problem.

5.2.1 More General Solutions for the Auxiliary Fields

The solutions we provided for the auxiliary fields ϕ and ψ, (8) contain four integration constants each, but once we demand regularity of the solutions and of the stress-energy tensor, two coefficients vanish. Moreover, the coefficients c_0 and d_0 did not enter any calculation, since the fields always appear differentiated. Consequently, it may be necessary to generalize the solutions, and look for both 'space' and 'time' dependent solutions for the auxiliary fields.

5.2.2 A More Realistic Metric

The metrics we worked with here, despite some of them not showing explicitly the full symmetries of deSitter spacetime, still posses a high degree of symmetry. Furthermore, the assumption of an idealized deSitter spacetime throughout our analysis is not really justified given what we know about the history of the Universe. Indeed, most of the physics we discussed in Chapter

4 took place not in a deSitter background but a radiation dominated FRW one. It would be more realistic to include this 'transition' of the metric as well, though we do not expect any major deviation from the results obtained here.

5.2.3 Coupling the Auxiliary Fields to Other Fields

In our view, the most promising modification would be to consider the auxiliary fields as actual physical degrees of freedom of the theory and allow for the possibility that they couple to other Standard Model fields as well. Notice that despite their fourth order equations of motion, the spectra of the auxiliary fields do not contain any negative norm states [42]. In fact general arguments presented in [11] suggest that nonlocal (large distance) interaction may play a crucial role in understanding the scaling $H\Lambda^3_{QCD}$ and the auxiliary fields provide exactly this sort of large distance effect if not integrated out. It is possible we are not seeing the IR physics enter the final expression for the stress-energy as a result of decoupling the auxiliary fields from matter fields. In fact, similar computations performed in [43, 44, 45] suggest that this kind of interaction may be crucial to getting the desired scaling (3.13).

Chapter 6

Conclusions

Here we briefly summarize the main points of the analysis performed in this paper.

6.1 Gravity as a Low Energy EFT

We have adopted and reviewed the idea that gravity is to be thought of as a low energy effective quantum field theory wherein the UV cuttoff scale should appear neither in the low energy effective Lagrangian nor in the physical definitions for the low energy observables, such as the dark energy. Furthermore, the conformal anomaly can be described by an effective action by the introduction of two auxiliary scalar fields, whose dynamics are relevant in certain specific setups, such as spacetimes with horizons. In particular, the physical stress-energy tensor diverges at the horizon and so a method must be specified to deal with these divergences. We have outlined a sort of dark energy 'renormalization' and demonstrated how it can be applied in a particular simplified setup.

6.2 The Dark Energy Problem as an IR Effect

In this framework, the UV cutoff and its associated vacuum energy are are doomed to vanish once we specify a definition and proscription for our 'renormalization', possibly leaving behind a residual energy density whose scale would be given by the relevant IR physics that enter through the boundary conditions imposed on the auxiliary fields. We have proposed that the physical vacuum energy is to be identified with this finite residual and provided a particular 'renormalization' proscription, in which that finite part is the result of a dynamical cancelation of divergences on either side of a 'phase transition' during which time some fields participating in the anomaly change their physical properties and therefore their contribution to the anomalous stress-energy tensor.

6.3 The Scale of the Vacuum Energy

Following the general path proposed, we have employed a specific, simplified solution for the auxiliary fields in static deSitter spacetime, and have shown how it is possible to cancel the divergences by imposing certain conditions on the way the boundary conditions for the auxiliary fields change. We have also found that there is indeed a finite non-vanishing relic vacuum energy. Unfortunately, this vacuum energy is proportional to H^4 and contains no information about the IR physics, so is inviable for cosmological applications. However, we expect that once some of the modifications proposed in Chapter 5 are employed the required mixing between curvature and the IR scale of the effective QFT may emerge.

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Coordinate Systems

In this appendix we list the coordinate systems, together with the relative coordinate transformations and the resulting equations of motion with some solutions for the auxiliary fields.

Line Elements

The FWR-dS coordinate system is given in terms of the coordinates $(\tau, \rho, \theta, \varphi)$ with line element

$$ds^2 = d\tau^2 - e^{2H\tau}\left(d\rho^2 + \rho^2 d\Omega^2\right), \quad \text{(FWR-dS)} \tag{1}$$

with $d\Omega^2 = d\theta^2 + sin^2(\theta)d\varphi^2$. This can be transformed into the conformally flat system by making the substitution $\tau = -ln(-H\eta)/H$. The conformally flat system is then given in terms of the coordinates $(\eta, \rho, \theta, \varphi)$ with line element

$$ds^2 = \tfrac{1}{H^2\eta^2}\left(d\eta^2 - d\rho^2 - \rho^2 d\Omega^2\right). \quad \text{(CF-dS)} \tag{2}$$

The Tortoise system is then given by the transformations

$$\begin{aligned} \rho &= \tfrac{\omega}{H} sinh(\xi) \\ \eta &= \tfrac{\omega}{H} cosh(\xi) \end{aligned} \quad \text{and} \quad \begin{aligned} \omega &= e^{-Ht} \\ \xi &= Hr^* \end{aligned}$$

so that the Tortoise system is given in terms of the coordinates $(t, r^*, \theta, \varphi)$ with line element

$$ds^2 = sech^2(Hr^*)\left(dt^2 - dr^{*2} - \tfrac{sinh^2(Hr^*)}{H^2} d\Omega^2\right). \quad \text{(Tortoise-dS)} \tag{3}$$

From this, the Static-dS system is obtained through the transformation

$$r^* = \frac{1}{2H} ln\left[\frac{1 + Hr}{1 - Hr}\right],$$

being given in terms of the coordinates (t, r, θ, φ) with corresponding line element

$$ds^2 = (1 - H^2r^2)dt^2 - \tfrac{1}{1-H^2r^2}dr^2 - r^2 d\Omega^2. \quad \text{(Static-dS)} \tag{4}$$

Finally, the Rindler-dS coordinates are given by the transformation

$$t = \tfrac{T}{H} \quad \text{and} \quad r = \tfrac{R}{H}\sqrt{1+R^2},$$

so that it is given in terms of the coordinates (T, R, θ, φ) with line element

$$ds^2 = \tfrac{1}{(1+R^2)H^2}\left(dT^2 - \tfrac{1}{1+R^2}dR^2 - R^2 d\Omega^2\right). \quad \text{(Rindler-dS)} \tag{5}$$

Equations of Motion

The equations of motion for the auxiliary fields are given by (3.24) and (3.25). The fourth order differential operator $\triangle_4$ is given in our five coordinate systems by,

$$\triangle_4 = \begin{cases} \partial_\tau^4 + 6H\partial_\tau^3 + 11H^2\partial_\tau^2 + 6H^3\partial_\tau + e^{-4H\tau}\partial_\rho^4 \\ \quad -2e^{-2H\tau}\partial_\rho^2(\partial_\tau^2 + H\partial_\tau) & \text{(FWR-dS)} \\ \\ \eta^4 H^4[\partial_\eta^4 - 2\partial_\eta^2\partial_\rho^2 + \partial_\rho^4] & \text{(CF-dS)} \\ \\ cosh^4(Hr^*)[\partial_t^4 - 4H^2\partial_t^2 + \partial_{r^*}^4 + 4H^2\partial_{r^*}^2 \\ \quad -2\partial_t^2\partial_{r^*}^2 + 4H\ cosh(Hr^*)(\partial_{r^*}^3 - \partial_{r^*}\partial_t^2) & \text{(Tortoise-dS)} \\ \\ \partial_t^4 - 2\partial_t^2\partial_r^2 - \frac{4}{r}\partial_t^2\partial_r + (1-H^2r^2)^2\partial_r^4 \\ \quad +\frac{4}{r}(1-H^2r^2)(1-3H^2r^2)\partial_r^3 & \text{(Static-dS)} \\ \quad -4H^2(7-9H^2r^2)\partial_r^2 - \frac{8}{r}H^2(1-3H^2r^2)\partial_r \\ \\ H^4(1+R^2)^2[\partial_T^4 - 4\partial_T^2 - 2(1+R^2)\partial_T^2\partial_R^2 \\ \quad -\frac{2}{R}(2+3R^2)\partial_T^2\partial_R \\ \quad +(1+R^2)^2\partial_R^4\frac{2}{R}(2+7R^2+5R^4)\partial_R^3 & \text{(Rindler-dS)} \\ \quad +(20+23R^2)\partial_R^2 + \frac{1}{R}(4+9R^2)\partial_R] \end{cases} \tag{6}$$

where we have restricted the coordinate dependance of both fields to the 'radial' and 'temporal' coordinates. These equations are a bit of a mess, and we were in fact only able to obtain a general solution for the conformally flat spacetime, but in this case all the dependence of the anomalous stress-energy tensor on the auxiliary fields enters as their equations of motion, so is independent of the particular solution [21]. We can however solve these equations if we assume a static solution (drop the 'temporal' dependence). Focusing on the last three coordinate systems (Tortoise-dS), (Static-dS), and (Rindler-dS), wherein the presence of the horizon is explicit, we can write the Euler-Lagrange equations for the auxiliary fields as follows (again having dropped the 'time' dependence)

$$\begin{bmatrix} cosh^4(Hr^*)[\partial^4_{r^*}+ \\ 4H\ cosh(hr^*)\partial^3_{r^*} + 4H^2\partial^2_{r^*}] \end{bmatrix} \begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 12H^4 \\ 0 \end{Bmatrix} \quad \text{(Tortoise-dS)}$$

$$\begin{bmatrix} (1-H^2r^2)^2\partial^4_r \\ +\frac{4}{r}(1-H^2r^2)(1-3H^2r^2)\partial^3_r \\ -4H^2(7-9H^2r^2)\partial^2_r \\ -\frac{8}{r}H^2(1-3H^2r^2)\partial_r \end{bmatrix} \begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 12H^4 \\ 0 \end{Bmatrix} \quad \text{(Static-dS)}$$

$$\begin{bmatrix} H^4(1+R^2)^2[(1+R^2)^2\partial^4_R \\ +\frac{2}{R}(2+7R^2+5R^4)\partial^3_R \\ +(20+23R^2)\partial^2_R \\ +\frac{1}{R}(4+9R^2)\partial_R \end{bmatrix} \begin{Bmatrix} \phi \\ \psi \end{Bmatrix} = \begin{Bmatrix} 12H^4 \\ 0 \end{Bmatrix} \quad \text{(Rindler-dS)}$$

Solutions

The solutions to the above Euler-Lagrange equations are given, in the Tortoise-dS system,

$$\begin{aligned} \phi(r^*) &= c_\infty ctgh(Hr^*) + c_0 - 2c_1Hr^* - 2c_2ctgh(Hr^*)Hr^* + 2ln[sech(Hr^*)], \\ \psi(r^*) &= d_\infty ctgh(Hr^*) + d_0 - 2d_1Hr^* - 2d_2ctgh(Hr^*)Hr^*, \end{aligned} \tag{7}$$

in the Static-dS system,

$$\begin{aligned} \phi(r) &= \tfrac{c_\infty}{Hr} + c_0 + c_1ln\left[\tfrac{1-Hr}{1+Hr}\right] + \tfrac{c_2}{Hr}ln\left[\tfrac{1-Hr}{1+Hr}\right] + 2ln(1-H^2r^2), \\ \psi(r) &= \tfrac{d_\infty}{Hr} + d_0 + d_1ln\left[\tfrac{1-Hr}{1+Hr}\right] + \tfrac{d_2}{Hr}ln\left[\tfrac{1-Hr}{1+Hr}\right], \end{aligned} \tag{8}$$

and in the Rindler-dS system,

$$\begin{aligned} \phi(R) &= \tfrac{c_\infty}{R}\sqrt{1+R^2} + c_0 + c_1ln\left[\tfrac{\sqrt{1+R^2}-R}{\sqrt{1+R^2}+R}\right] + \tfrac{c_2}{R}ln\left[\tfrac{\sqrt{1+R^2}-R}{\sqrt{1+R^2}+R}\right] - ln(1+R^2), \\ \psi(R) &= \tfrac{d_\infty}{R}\sqrt{1+R^2} + d_0 + d_1ln\left[\tfrac{\sqrt{1+R^2}-R}{\sqrt{1+R^2}+R}\right] + \tfrac{d_2}{R}ln\left[\tfrac{\sqrt{1+R^2}-R}{\sqrt{1+R^2}+R}\right]. \end{aligned} \tag{9}$$

Finally, we comment that the inhomogeneous solution, the last term in the expressions for ϕ, corresponds to the Bunch-Davies vacuum, which is the solution that brings every coordinate system back to the conformally flat system, see [21]. As long as this solution is employed, with the above integration constants (c's and d's) taken to be zero, all the dependance on the auxiliary fields will disappear identically, and so this description of the anomalous stress-energy in terms of the auxiliary fields cannot have any effects that are not entirely describable by the standard formalism. This is exactly what has happened in the case of the analysis [30] (see also [46]), where their results correspond only to the Bunch-Davies vacuum and no effects of the boundary conditions can possibly show up in the resulting stress-energy (the boundary conditions are explicitly discounted when taking all the integration constants to be zero).