Moduli Space of Sheaves on Fans

by

Koopa Hakimi

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

April, 2011

Abstract

A conjecture of H. Hopf states that if M^{2n} is a closed, Riemannian manifold of nonpositive sectional curvature, then its Euler characteristic $\chi(M^{2n})$, should satify $(-1)^n\chi(M^{2n}) \geq 0$. Ruth Charney and Michael Davis investigated the conjecture in the context of piecewise Euclidean manifolds having "nonpositive curvature" in the sense of Gromov's CAT(0) inequality. In that context the conjecture can be reduced to a local version which predicts the sign of a "local Euler characteristic" at each vertex. They stated precisely various conjectures in their paper which we are interested in one of them stated as Conjecture D (see [1]) which is equivalent to the Hopf Conjecture for piecewise Euclidean manifolds cellulated by cubes.

The goal of this thesis is to study the Charney - Davis Conjecture stated as Conjecture (D) by using sheaves on fans.

Table of Contents

Acknowledgements

I would like to express my deepest appreciation to my supervisor, Professor Kalle Karu, for introducing me to the field and allowing me to work on this project. I am grateful for all the help he provided, and his genuine desire to see me succeed. I would like to extend special thanks to Professor Zinovy Reichstein at the University of British Columbia for taking the time to read and evaluate my thesis. Last but not least, I express my sincere appreciation to all the helpful staff of the Mathematics Department.

Koopa Hakimi

The University of British Columbia
May 2011

Chapter 1

Introduction

Definition 1.1 (flag complex)**.** Let K be a simplicial complex. A set V of vertices in K spans a complete graph if any two distinct elements of V span an edge in K. A simplicial cell complex K is a *flag complex* if any set of vertices which spans a complete graph actually spans a simplex. Thus, a flag complex is a simplicial complex with no "empty simplices".

Let K be a simplicial complex and f_i the number of i-simplices in K. Put

$$\lambda(K) = 1 + \sum \left(-\frac{1}{2}\right)^{i+1} f_i.$$

Now let K^{2n-1} be a simplicial complex which is homeomorphic to S^{2n-1} and $\lambda(K)$ the quantity defined as above

Conjecture 1 (Conjecture D)**.** *If K^{2n-1} is a flag complex, then*

$$(-1)^n \lambda(K^{2n-1}) \geq 0.$$

Definition 1.2 (simplicial complete fan)**.** A **simplicial complete fan** Δ consists of cones over a simplicial complex $K \cup \{\varnothing\}$. Actually we think of an element $\sigma \in \Delta$ as a cone over the simplex, with $\varnothing$ the zero cone. Then $\dim(\sigma)$ is the dimension of the simplex plus 1. We call the fan Δ *complete* if $K \simeq S^{2n-1}$.

Remark 1. Actually the conjecture can be made more generally for simplicial cell complexes. One can correspond a fan to any **piecewise spherical cell complex** (see [1, p.120]) by taking a **convex polyhedral cone** associated to any **spherical cell** as the cones of the fan. Thus note that in order the fan to be a **complete** fan the necessary condition is simply our piecewise spherical cell complex to be homeomorphic to S^{n-1}. Note that what we defined above is an **abstract** fan, with no embedding in $\mathbb{R}^n$.

Definition 1.3 (flag fan)**.** A simplicial complete fan is *flag* if the simplicial complex K in the previous definition is a flag complex.

Definition 1.4 (Poset of a fan)**.** Let Δ be a simplicial fan we define a binary relation "$\leq$" over Δ as follows: for each $\tau, \sigma \in \Delta$ $\sigma \leq \tau$ if and only if $\sigma \subset \tau$. Clearly this is a partial order over the set Δ.

By the definition of a simplicial complete fan we can introduce the new quantity $\lambda^{'}(\Delta)$ as follows :

$$\lambda^{'}(\Delta) = \sum_{i=0}^{\dim \Delta} \left(-\frac{1}{2}\right)^{i} f_i^{'}$$

where $f_i^{'}$ is the number of i-dimensional cones in Δ which is equal to the number of $(i-1)$-dimensional simplices in K, $f_0^{'} = 1$. Assume Δ has dimension $2n$. Note that since multiplication by a positive constant does not change the sign, by multiplying with 2^{2n}, we can introduce a new quantity $\lambda^{''}(\Delta)$:

$$\lambda^{''}(\Delta) = \sum (-2)^{2n-i} f_i^{'}.$$

It is therefore convenient to state the following equivalent formulation of Conjecture 1.

Conjecture 2. *Suppose that Δ is a simplicial complete flag fan and that $\lambda^{''}$ is defined by the formula above. then:*

$$(-1)^n \cdot \lambda^{''}(\Delta) \geq 0.$$

We approach this conjecture using sheaves on the fan Δ. A sheaf $\mathcal{F}$ on a fan consists of vector spaces $\mathcal{F}_\tau$ for all $\tau \in \Delta$ and restriction maps $\mathcal{F}_\tau \to \mathcal{F}_\sigma$ for $\tau \geq \sigma$, satisfying compatibility conditions spelled out in Definition 2.1.
Now consider the following *cellular complex* of the sheaf $\mathcal{F}$:

$$C^\bullet(\mathcal{F}, \Delta) : 0 \to C^0 \to C^1 \to C^2 \to \cdots \to C^{\dim(\Delta)} \to 0$$

where $C^i = \bigoplus_{\mathrm{codim}(\tau)=i} \mathcal{F}_\tau$ and the boundary maps are the restriction maps with ±signs.
Suppose $\dim(\mathcal{F}_\tau) = 2^{\mathrm{codim}\,\tau}$ and consider the Euler characteristic of the given

cellular complex $C^\bullet(\mathcal{F}, \Delta)$

$$\begin{aligned}\chi(C^\bullet(\mathcal{F}, \Delta)) &= \dim\ C^0 - \dim\ C^1 + \dim\ C^2 - \cdots \\ &= \sum_i (-1)^i \sum_{\operatorname{codim}(\tau)=i} \dim \mathcal{F}_\tau = \lambda^{''}(\Delta).\end{aligned}$$

Now assume there exists a sheaf $\mathcal{F}$ where $C^\bullet(\mathcal{F}, \Delta)$ has only the non-zero cohomology in the middle degree n, thus we conclude that

$$\lambda^{''}(\Delta) = \chi(C^\bullet(\mathcal{F}, \Delta)) = \chi\left(H^\bullet(C^\bullet(\mathcal{F}, \Delta))\right) = (-1)^n \cdot \dim H^n(C^n(\mathcal{F}))$$

and this simply shows

$$(-1)^n \lambda^{'}(\Delta) \geq 0.$$

Thus in view of what mentioned about the sheaf cohomology above, Conjecture 1 is implied by the following

Conjecture 3. *There exist a sheaf $\mathcal{F}$ on Δ with $\dim(\mathcal{F}_\tau) = 2^{\operatorname{codim}(\tau)}$ which has non-zero cohomology in middle degree only.*

One guess would be the **general** point (sheaf) of the given dimension vector is expected to satisfy the above condition. Thus in order to make sense of the **general** sheaf, we need to introduce an irreducible moduli space of all such sheaves. Thus in proceeding chapters we construct the corresponding moduli space and prove its irreducibility only for 2-dimensional flag fans. In order to prove irreducibility, we will give two proofs, one by studying the tangent space and the embedding dimension and one by stratifying the moduli space and fibre dimensions.

Chapter 2

The Moduli Problem

2.1 Sheaves on Fans

From now on we work on the field $\mathbb{C}$.

Definition 2.1. Let Δ be a partially ordered set. By a sheaf on Δ we mean a data, $\mathcal{F}$, consisting of:

(1) a finite dimensional complex vector space $\mathcal{F}_\sigma$, for each $\sigma \in \Delta$

(2) linear maps (restrictions), $\mathrm{res}_{\tau,\sigma} : \mathcal{F}_\tau \to \mathcal{F}_\sigma$ for $\sigma \leq \tau$

In order to define a (pre)sheaf the maps above should satisfy the restriction morphism conditions. By restriction morphism we mean:

(1) for every $\sigma \in \Delta$ the map $\mathrm{res}_{\sigma,\sigma} : \mathcal{F}_\sigma \to \mathcal{F}_\sigma$ is the identity

(2) if $\zeta \leq \tau \leq \sigma$ then $(\mathrm{res}_{\tau,\zeta}) \circ (\mathrm{res}_{\sigma,\tau}) = \mathrm{res}_{\sigma,\zeta}$

We can define the dimension vector of a sheaf $\mathcal{F}$ by $dim(\mathcal{F}) = (d_\tau)$ where $d_\tau = \dim(\mathcal{F}_\tau)$.

Definition 2.2. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on Δ, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of vector spaces $\varphi_\tau : \mathcal{F}_\tau \to \mathcal{G}_\tau$ for each $\tau \in \Delta$, such that whenever $\tau \leq \sigma$, the diagram

$$\begin{tikzcd}
\mathcal{F}_\sigma \arrow[r, "\varphi_\sigma"] \arrow[d, "\mathrm{res}_{\sigma,\tau}"'] & \mathcal{G}_\sigma \arrow[d, "\mathrm{res}_{\sigma,\tau}"] \\
\mathcal{F}_\tau \arrow[r, "\varphi_\tau"] & \mathcal{G}_\tau
\end{tikzcd}$$

commutes, where $\mathrm{res}_{\sigma,\tau}$ are the restriction maps in $\mathcal{F}$ and $\mathcal{G}$.

Example 1. The constant sheaf $\underline{\mathbb{C}^n}_\Delta$ is the sheaf comprised of the vector space $\mathbb{C}^n$ at each element of Δ, with identity morphisms as the restriction morphisms.

Definition 2.3 (Families of sheaves over S)**.** A family of sheaves $\mathcal{F}$ over a scheme S is the families of vector bundles over S consists of vector bundles F_τ over S for each $\tau \in \Delta$ and homomorphisms of bundles $\mathrm{res}_{\tau,\sigma} : F_\sigma \to F_\tau$ satisfying the same conditions as in Definition 2.2. A family $\mathcal{F}$ has a well defined dimension vector $d = (d_\tau)$ where $d_\tau = rank(F_\tau)$.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ between families of sheaves over S consists of vector bundle homomorphisms $\varphi_\tau : F_\tau \to G_\tau$ for all $\tau \in \Delta$ making the diagram in Definition 2.2 commutative.

If $\mathcal{F}, \mathcal{G}$ are families of sheaves over S, then $\mathcal{F}$ is a subfamily of $\mathcal{G}$ if $F_\tau \subset G_\tau$ is a subbundle for all τ and the restriction maps of $\mathcal{F}$ are the restrictions of the restrictions maps of $\mathcal{G}$.

When $\mathcal{G}$ is a sheaf on Δ, we let $G \times S$ be the constant family of sheaves over S, such that $(G \times S)_\tau$ is the trivial vector bundle with fibre $\mathcal{G}_\tau$ and the maps $\mathrm{res}_{\tau,\sigma} : (G \times S)_\tau \to (G \times S)_\sigma$ are given by $\mathrm{res}_{\tau,\sigma} : \mathcal{G}_\tau \to \mathcal{G}_\sigma$.

Definition 2.4 (Families of subsheaves)**.** Fix one sheaf $\mathcal{G}$ on Δ. A family of subsheaves of $\mathcal{G}$ over S is a subfamilies $\mathcal{F} \subset \mathcal{G} \times S$.

Example 2. Let $\mathcal{G}$ to be the constant sheaf $\mathbb{C}^n$ then by a family of subsheaves of $\mathbb{C}^n$ we mean a collection of vector bundles $F_\sigma \subset \mathbb{C}^n$ such that $\forall \sigma \geq \tau$ then $F_\sigma \subset F_\tau \subset \mathbb{C}^n$, and the following diagram

$$\begin{tikzcd}
F_\sigma \arrow[r, "\varphi_\sigma"] \arrow[d, dotted] & \mathbb{C}^n \arrow[d, "id"] \\
F_\tau \arrow[r, "\varphi_\tau"] & \mathbb{C}^n
\end{tikzcd}$$

commutes.

2.2 Construction of a Moduli Space $\mathcal{M}_\Delta$

Fix a sheaf $\mathcal{G}$ on Δ and also fix a dimension vector $\mathbf{d} = (d_\tau)_{\tau\in\Delta}$. To define our moduli problem let $\mathbf{Sub}^{\boldsymbol{\Delta}}_{\mathcal{G},\mathbf{d}}$ be a functor from a category of Schemes to Sets, $\mathbf{Sub}^{\boldsymbol{\Delta}}_{\mathcal{G},\mathbf{d}}$: **Schemes** $\to$ **Sets** by

$$\mathbf{Sub}^{\boldsymbol{\Delta}}_{\mathcal{G},\mathbf{d}}(\mathbf{S}) = \big\{\text{families of subsheaves of } \mathcal{G} \text{ such that } \dim(\mathcal{F}) = \mathbf{d}\big\}.$$

The fundamental question to answer in studying a given moduli problem is whether the functor $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\mathbf{\Delta}}$ is representable in the category of schemes i.e. if there is a scheme $\mathcal{M}_{\Delta}$ and an isomorphism ψ (of functors from Schemes to Sets) between $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\mathbf{\Delta}}$ and the functor of points of $\mathcal{M}_{\Delta}$. This last is the functor $\mathrm{Hom}_{Sch}(S, \mathcal{M}_{\Delta})$. In this section we will show that this functor is indeed representable.

Theorem 2.1. *The functor* $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\mathbf{\Delta}}$ *is representable.*

Example 3. (Grassmannian Functor). For the single point set $\Delta = \{\tau\}$ we define the grassmannian functor $\mathbf{G}$ to be $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\{\tau\}}$.
Let W be a finite-dimensional vector space. By $G(k, W)$ we denote the Grassmannian of k-dimensional linear subspaces in W. Thus, to every such subspace $V \subset W$ there corresponds a point $[V] \in G(k, W)$. It is well known that this functor is representable and in fact $G(d, \mathcal{G}_{\tau})$ is its fine moduli space.
By $\mathcal{U}_{\tau}$ we mean a universal bundle over $G(d_{\tau}, W)$ with fibre over $[V]$ the vector space $V \subset W$. Then $\mathcal{U}_{\tau}$ is a sub-bundle of the trivial bundle $G(d_{\tau,W}) \times W$.

Example 4. (Flag Functor). Let $\Delta = \{\sigma, \tau\}$ where $\sigma \leq \tau$ and $\mathcal{G}$ a sheaf on Δ, and define our Flag functor to be $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\{\tau,\sigma\}}$.
At first let $\mathcal{G}$ be the constant sheaf $\underline{W}_{\Delta}$ where W a finite dimensional vector space. Thus for the families of subsheaf of $\mathcal{G}$ over S we have the flag $\mathcal{F}_{\tau} \subset \mathcal{F}_{\sigma} \subset W$

$$\begin{tikzcd}
\mathcal{F}_{\tau} \arrow[r, hook] \arrow[d] & W \arrow[d, "id"] \\
\mathcal{F}_{\sigma} \arrow[r, hook] & W .
\end{tikzcd}$$

Functor $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\{\tau,\sigma\}}$ is know to be representable by the flag variety $F(d_{\sigma}, d_{\tau}, W)$. It can be constructed as a closed subvariety of $G(d_{\tau}, W) \times G(d_{\sigma}, W)$ as follows. Let $\mathcal{U}_{\tau}$ and $\mathcal{U}_{\sigma}$ be the Universal bundles on $G(d_{\tau}, W)$ and $G(d_{\sigma}, W)$ respectively and let Q_{τ} as the quotient $Q_{\tau} = G(d_{\tau}, W) \times W/\mathcal{U}_{\tau}$. Similarly for $\mathcal{U}_{\sigma}$ denote Q_{σ} the corresponding quotient.
Consider the morphisms of vector bundles:

$$\begin{tikzcd}
\pi_1^*\mathcal{U}_{\tau} \arrow[rr, "\psi"] \arrow[dr] & & \pi_2^*Q_{\sigma} \arrow[dl] \\
& G(d_{\tau}, W) \times G(d_{\sigma}, W) &
\end{tikzcd}$$

where π_1, π_2 are the projections to first and second factors. Thus we can write $F(d_\tau, d_\sigma, W)$ as a zero locus of ψ.

Now in general consider $\mathcal{G}$ not necessarily to be a constant sheaf. A subsheaf $\mathcal{F} \subset \mathcal{G}$ is a commutative diagram:

$$\begin{tikzcd}
\mathcal{F}_\tau \arrow[r, hook] \arrow[d] & \mathcal{G}_\tau \arrow[d, "\mathrm{res}_{\tau,\sigma}"] \\
\mathcal{F}_\sigma \arrow[r, hook] & \mathcal{G}_\sigma
\end{tikzcd}$$

We claim that subsheaves $\mathcal{F} \subset \mathcal{G}$ are again parametrized by a closed subscheme of

$$G := G(d_\tau, \mathcal{G}_\tau) \times G(d_\sigma, \mathcal{G}_\sigma).$$

Let $\mathcal{U}_\tau, \mathcal{U}_\sigma, Q_\tau, Q_\sigma$ be as before and let π_1, π_2 be the projections:

$$\begin{tikzcd}
& G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G(d_\tau, \mathcal{G}_\tau) & & G(d_\tau, \mathcal{G}_\tau)
\end{tikzcd}$$

Note that the linear map $\mathrm{res}_{\tau,\sigma} : \mathcal{G}_\tau \to \mathcal{G}_\sigma$ induces a homomorphism of trivial vector bundles over G:

$$\begin{tikzcd}
G \times \mathcal{G}_\tau \arrow[r] & G \times \mathcal{G}_\sigma.
\end{tikzcd}$$

Now construct a homomorphism of vector bundles $\psi : \pi_1^* \mathcal{U}_\tau \to \pi_2^* Q_\sigma$ over G as the composition

$$\begin{tikzcd}
\pi_1^* \mathcal{U}_\tau \arrow[r] \arrow[rrr, bend left, "\psi"] & G \times \mathcal{G}_\tau \arrow[r] & G \times \mathcal{G}_\sigma \arrow[r] & \pi_2^* Q_\sigma.
\end{tikzcd}$$

and define the zero locus of the map ψ to be $\mathfrak{F}(d_\tau, d_\sigma, \mathcal{G}) \subset G(d_\sigma, \mathcal{G}_\sigma) \times G(d_\tau, \mathcal{G}_\tau)$ and call it the *general flag* scheme.

Lemma 1. *The general flag scheme* $\mathfrak{F}(d_\tau, d_\sigma, \mathcal{G})$ *is the fine moduli space for the Flag Functor* $\mathbf{Sub}_{\mathcal{G},\mathbf{d}}^{\{\tau,\sigma\}}$.

Proof. Let $\varphi \in \mathrm{Hom}(S, \mathfrak{F}(\tau, \sigma, \mathcal{G}))$ be given, we need to construct a family of sheaves $\mathcal{F}$ on S. let $\iota : \mathfrak{F} \hookrightarrow G(d_\sigma, \mathcal{G}_\sigma) \times G(d_\tau, \mathcal{G}_\tau)$ and consider the Universal bundles $\mathcal{U}_\tau$ and $\mathcal{U}_\sigma$ over the corresponding grassmannians then to give a families of sheaves over S define the vector bundles $F_\tau := \varphi^*\iota^*\pi_1^*(\mathcal{U}_\tau)$ and $F_\sigma := \varphi^*\iota^*\pi_2^*(\mathcal{U}_\sigma)$. Now by our definition of the general flag $\mathfrak{F}$ as a zero locus of ψ then we get a morphism $\iota^*\pi_1^*(\mathcal{U}_\tau) \to \iota^*\pi_2^*(\mathcal{U}_\sigma)$ and consequently a morphism $F_\tau \to F_\sigma$.

Now given a family of subsheaves $\mathcal{F}$ of a sheaf $\mathcal{G}$ over S. Since the grassmannian functor is representable it would imply we would get morphisms for each family $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$ over S

$$\varphi_\tau \in \mathrm{Hom}(S, G(d_\tau, \mathcal{G}_\tau)),\ \varphi_\sigma \in \mathrm{Hom}(S, G(d_\sigma, \mathcal{G}_\sigma)).$$

Now let our morphism be $\varphi := (\varphi_\sigma, \varphi_\tau) : S \to G(d_\sigma, \mathcal{G}_\sigma) \times G(d_\tau, \mathcal{G}_\tau)$. Since the general flag was defined as the zero locus of the map ψ then it would imply that φ factors through $\mathfrak{F}(\tau, \sigma, \mathcal{G})$. ☐

Definition 2.5 (Moduli Space)**.** Let $\Delta = \{\tau_1, ..., \tau_n\}$ be an arbitrary finite poset and consider the projections

$$\pi_{i,j} : \prod_{\tau_k} G(d_{\tau_k}, \mathcal{G}_{\tau_k}) \to G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \times G(d_{\tau_j}, \mathcal{G}_{\tau_j})$$

for each two comparable elements $\tau_i \leq \tau_j$. Now let $\mathcal{M}_\Delta$ to be the scheme theoretic intersection of $\pi_{i,j}^{-1}\mathfrak{F}(\tau, \sigma, \mathcal{G})$ i.e. $\mathcal{M}_\Delta := \bigcap_{i,j} \pi_{i,j}^{-1}\mathfrak{F}(\tau, \sigma, \mathcal{G})$.

Now we will prove the Theorem 2.1 by showing the $\mathcal{M}_\Delta$ is our moduli space.

Proof of Theorem 2.1. To show $\mathcal{M}_\Delta$ is the fine moduli space for our functor $\mathbf{Sub}^{\mathbf{\Delta}}_{\mathcal{G},\mathbf{d}}$. Let $\varphi \in \mathrm{Hom}_{Sch}(S, \mathcal{M}_\Delta)$, To define $\mathcal{F}_\tau$, consider the projections

$$\prod_\tau G(d_\tau, \mathcal{G}_\tau) \xrightarrow{\pi_\tau} G(d_\tau, \mathcal{G}_\tau).$$

Take the universal bundle $\mathcal{U}_\tau$ over $G(d_\tau, \mathcal{G}_\tau)$. Then define the vector bundles

$$F_\tau := \varphi^*\pi_\tau^*(\mathcal{U}_\tau).$$

Now since S maps into $\pi_{ij}^{-1}\mathfrak{F}$ then by the restriction map $\mathrm{res}_{\sigma,\tau} : \mathcal{G}_\tau \to \mathcal{G}_\sigma$ we get a map

$$F_\tau \longrightarrow F_\sigma.$$

To show the converse correspondence let $\mathcal{F}$ be a family of subsheaves of a fixed sheaf $\mathcal{G}$ over S, to construct a morphism $\varphi : S \to \mathcal{M}_\Delta$ since $F_{\tau_i} \subset G_{\tau_i}$ from representability of the grassmannian functor by $G(d_{\tau_i}, \mathcal{G}_{\tau_i})$ we get morphisms $\varphi_{\tau_i} : S \to G(d_{\tau_i}, \mathcal{G}_{\tau_i})$ for each τ_i, now let a map $\varphi : S \to \prod_\tau G(d_{\tau_i}, \mathcal{G}_{\tau_i})$ to be $\varphi := (\varphi_{\tau_i})_{\tau_i \in \Delta}$.

Now in order to complete the proof we need to show the image of S lies in $\mathcal{M}_\Delta$. Consider the following composition of morphisms where $\pi_{i,j}$ is our projection in Definition 2.5.

$$S \xrightarrow{\varphi} \prod_{\tau_i} G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \xrightarrow{\pi_{ij}} G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \times G(d_{\tau_j}, \mathcal{G}_{\tau_j})$$

thus all we have to show is the image would lie in the $\mathfrak{F}(d_{\tau_i}, d_{\tau_j}, \mathcal{G}) \subset G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \times G(d_{\tau_j}, \mathcal{G}_{\tau_j})$, and this is because

$$\begin{tikzcd}
F_{\tau_i} \arrow[r, hook] \arrow[d] & \mathcal{G}_{\tau_i} \times G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \arrow[d, "\mathrm{res}_{\tau_j,\tau_i}"] \\
F_{\tau_j} \arrow[r, hook] & \mathcal{G}_{\tau_j} \times G(d_{\tau_j}, \mathcal{G}_{\tau_j})
\end{tikzcd}$$

commutes and gives a family on $\mathfrak{F}(d_{\tau_i}, d_{\tau_j}, \mathcal{G})$. $\square$

Chapter 3

Tangent Space

3.1 The Tangent Space of the Functor

In this section let k be any algebraically closed field.
One approach to study irreducibility of the moduli space would be by studying its tangent space. So in these section I would define the tangent space of $\mathcal{M}_\Delta$ and in the proceeding section I will give some applications to irreducibility of the moduli space of 2-dimensional fans.

Definition 3.1. Given a functor

$$F : \{\text{rings over k}\} \to \{sets\}$$

and an element $x \in F(k)$, we define the tangent space of F at x to be

$$T_x(F) = \left\{ \begin{array}{l} \text{inverse image of } x \in F(k) \text{ under} \\ F_{k[\epsilon]\to k} : F(k[\epsilon]) \to F(k) \end{array} \right\} \subset F(k[\epsilon])$$

This has the structure of a vector space over k.

Example 5. Let R be any k-algebra. An algebraic variety (or, more generally, a scheme) X determines a functor

$$\underline{X} : \{\text{rings}\} \to \{sets\}$$

where $\underline{X}(R)$ is the set of R-valued points of the variety X.
Then after taking an affine open cover of X, at each k-valued point $x \in \underline{X}(k)$ the tangent space $T_x(\underline{X}) \subset \underline{X}(k[\epsilon])$ in the sense of Definition 3.1.1 coincides with the Zariski tangent space.

3.2 The Tangent Space of $\mathcal{M}_\Delta$

Proposition 3.1. *The tangent space to the Grassmannian* $G(r,n)$ *at a point* $\Lambda \in G(r,n)$ *corresponding to an r-dimensional subspace* $\Lambda \subset k^n$ *is canonically isomorphic to* $\mathrm{Hom}_k(\Lambda, k^n/\Lambda)$.

Proof. Consider the grassmannian functor defined in Example 3. To the point $[U] \in G(r,n)$ there corresponds a morphism

$$U \hookrightarrow k^n,$$

and a tangent vector at $[U]$ is then morphism of $k[\epsilon]$-modules

$$\widetilde{U} \longrightarrow k[\epsilon]^n$$

whose reduction module (ϵ) coincides with $U \to k^n$. Let $u_1, ..., u_r \in k^n$ be a basis of U, and let

$$u_1 + \epsilon v_1, ..., u_r + \epsilon v_r \in k[\epsilon]^n$$

be a free basis of $\widetilde{U}$ as a $k[\epsilon]$-module. Since $\epsilon^2 = 0$, it follows that $\epsilon u_1, ..., \epsilon u_r \in \widetilde{U}$, and this is a basis of $U\epsilon$. This shows that the given tangent vector determines a well-defined linear map

$$U \to V = k^n/U, \qquad u_i \mapsto v_i \mod U$$

and this correspondence defines an isomorphism $T_{[U]}G \xrightarrow{\sim} \mathrm{Hom}(U, k^n/U)$. $\square$

Remark 2. (Tangent space to the flag variety) For each k, l we consider the flag manifold $F(k,l,n)$, i.e., the incidence correspondence

$$\Omega = \{(\Lambda, \Gamma) : \Lambda \subset \Gamma \subset k^n\} \subset G(k,n) \times G(l,n).$$

It is not hard to see that Ω is smooth. We want to identify the tangent space to Ω as a subspace of the tangent space to the product $G(k,n) \times G(l,n)$. To do this, let $(\Lambda, \Gamma) \in \Omega$ be any point of Ω. The tangent vector, viewed as an element of the tangent space $T_{\Lambda,\Gamma}(G(k,n) \times G(l,n)) = T_\Lambda(G) \times T_\Gamma(G)$, is (η, φ) where $\eta \in T_\Lambda(G)$ and $\varphi \in T_\Gamma(G)$ respectively. Now it is well known [2] the tangent space is the space of pairs

$$T_{\Lambda,\Gamma}(\Omega) = \left\{ (\eta, \varphi) : \begin{array}{c} \eta \in \mathrm{Hom}(\Lambda, k^{n+1}/\Lambda) \\ \varphi \in Hom(\Gamma, k^{n+1}/\Gamma) \end{array} \text{ and } \varphi|_\Lambda \equiv \eta \text{ mod } \ \Gamma \right\}$$

Proposition 3.2. *The tangent space to the general flag scheme at a point* $(U_\tau, U_\sigma) \in G \times G$ *is given by morphisms* $\varphi_\tau \in \mathrm{Hom}(U_\tau, k^n/U_\tau)$ *and* $\varphi_\sigma \in \mathrm{Hom}(U_\sigma, k^m/U_\sigma)$ *such that*

$$\begin{tikzcd} U_\tau \arrow[r, "\varphi_\tau"] \arrow[d] & \mathcal{G}_\tau/U_\tau \arrow[d] \\ U_\sigma \arrow[r, "\varphi_\sigma"] & \mathcal{G}_\sigma/U_\sigma. \end{tikzcd} \tag{3.1}$$

Proof. By the same argument in Proposition 3.2 consider the following morphisms

$$\begin{tikzcd} U_\tau \arrow[r, hook] \arrow[d] & \mathcal{G}_\tau \arrow[d, "\psi"] & = k^n \\ U_\sigma \arrow[r, hook] & \mathcal{G}_\sigma & = k^m \end{tikzcd}$$

The tangent vector at U_τ, U_σ is then is given by morphisms of $k[\epsilon]$-modules

$$\begin{tikzcd} \widetilde{U_\tau} \arrow[r, hook] \arrow[d] & \mathcal{G}_\tau[\epsilon] \arrow[d, "\widetilde{\psi}"] & = k[\epsilon]^n \\ \widetilde{U_\sigma} \arrow[r, hook] & \mathcal{G}_\sigma[\epsilon] & = k[\epsilon]^m \end{tikzcd} \tag{3.2}$$

Suppose the diagram (3.1) is given. Let $u_1, ..., u_l, u_{l+1}, ..., u_r \in k^n$ be a basis of U_τ and $u_1^{'}, ..., u_l^{'}, u_{l+1}^{'}, ..., u_s^{'} \in k^m$ be a basis of U_σ where $Span\{u_{l+1}, ..., u_r\} = \ker \varphi$ and $\psi(u_i) = u_i^{'}$, and let

$$u_1 + \epsilon v_1, ..., u_r + \epsilon v_r \in k[\epsilon]^n$$

$$u_1^{'} + \epsilon v_1^{'}, ..., u_s^{'} + \epsilon v_s^{'} \in k[\epsilon]^m,$$

Be the basis for $\widetilde{U}_\tau$ and $\widetilde{U}_\sigma$ respectively then by Proposition 3.2 the tangent space $T_{U_\tau}\mathbb{G}$ is isomorphic to $\mathrm{Hom}(U_\tau, k^n/U_\tau)$ and also similarly $T_{U_\sigma}\mathbb{G}$ is isomorphic to $\mathrm{Hom}(U_\sigma, k^n/U_\sigma)$ via the linear maps

$$\begin{aligned} U_\tau \to k^n/U_\tau, \quad & u_i \mapsto v_i \mod U_\tau \\ U_\sigma \to k^m/U_\sigma, \quad & u_i' \mapsto v_i' \mod U_\sigma. \end{aligned} \tag{3.3}$$

Since for each i we can write

$$\widetilde{\psi}(u_i + \epsilon v_i) = \sum (u_j' + \epsilon v_j')(a_j + \epsilon b_j) = \sum u_j' a_j + \epsilon \sum (a_j v_j' + b_j u_j')$$

we conclude that

$$\widetilde{\psi}(u_i) = \sum a_j u_j', \qquad \widetilde{\psi}(v_i) = \sum (a_j v_j' + b_j u_j').$$

Now for the case where $i \geq l+1$

$$\begin{tikzcd} u_i \arrow[r, mapsto] \arrow[d, mapsto] & v_i \arrow[d, mapsto] \\ 0 \arrow[r, mapsto] & 0 = \varphi(v_i) \end{tikzcd}$$

from $u_i \mapsto 0$ we conclude that $a_j = 0$ for all j thus

$$\varphi(v_i) = \sum_j b_j u_j' \equiv 0 \mod U_\sigma. \tag{3.4}$$

for the case where $i \leq l$

$$\begin{tikzcd} u_i \arrow[r, mapsto] \arrow[d, mapsto] & v_i \arrow[d, mapsto] \\ u_i' \arrow[r, mapsto] & v_i' = \varphi(v_i) \end{tikzcd}$$

since $u_i \mapsto u_i'$ implies $a_i = 1$ and $a_j = 0$ for $j \neq i$ thus we have

$$\varphi(v_i) = v_i' + \sum b_j u_j' = v_i' \mod U_\sigma. \tag{3.5}$$

From (3.4) and (3.5) we conclude that mod U_σ

$$\varphi(v_i) = [0] \quad \text{for } i \geq l+1 \text{ and } \quad \varphi(v_i) = [v_i'] \quad \text{for } i \leq l.$$

Thus we get the following commutative diagram of morphisms

$$\begin{tikzcd}
U_\tau \arrow[r] \arrow[d] & \mathcal{G}_\tau/U_\tau \arrow[d] \\
U_\sigma \arrow[r] & \mathcal{G}_\sigma/U_\sigma
\end{tikzcd}$$

Actually the other correspondence direction is straightforward. Suppose the diagram (3.2) is given then by morphisms (3.3) we would get the diagram (3.1) □

Theorem 3.1. *The tangent space of* $\mathbf{Sub}^{\mathbf{\Delta}}_{\mathcal{G},\mathbf{d}}$ *at a point* $\mathcal{F}$ *is* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}/\mathcal{F})$.

Proof. By the construction of our fine moduli space $\mathcal{M}_\Delta$ as the scheme theoretic intersection of inverse image of each general flag $\mathfrak{F}$ for each $\tau_i \leq \tau_j$ by projections

$$\pi_{ij} : \prod_{\tau\in\Delta} G(d_\tau, \mathcal{G}_\tau) \to G(d_{\tau_i}, \mathcal{G}_{\tau_i}) \times G(d_{\tau_j}, \mathcal{G}_{\tau_j})$$

we conclude that

$$T_\mathcal{F}(\mathcal{M}_\Delta) \subset \prod T_\mathcal{F}(G(d_\tau, \mathcal{G}_\tau) = \prod \operatorname{Hom}(\mathcal{F}_\tau, \mathcal{G}_\tau/\mathcal{F}_\tau)$$

is the intersection of $(\varphi_\tau)_{\tau\in\Delta}$, satisfying

$$\begin{tikzcd}
\mathcal{F}_\tau \arrow[r, "\varphi_\tau"] \arrow[d] & \mathcal{G}_\tau/\mathcal{F}_\tau \arrow[d] \\
\mathcal{F}_\sigma \arrow[r, "\varphi_\sigma"] & \mathcal{G}_\sigma/\mathcal{F}_\sigma
\end{tikzcd}$$

□

Chapter 4

Irreducibility of the Moduli Space

We will give 2 different proofs of the irreducibility in the 2-dimensional case for certain $\mathcal{G}$ and dimension vector $\mathbf{d} = (d_\tau)$.

Definition 4.1 (poset of a 2-dimensional complete fan)**.** The poset of a 2-dimensional complete fan consist of a set $\Delta = \{0, \tau_1, ..., \tau_k, \sigma_1, ..., \sigma_k\}$ and the relations $\sigma_i, \sigma_{i+1} \geq \tau_i$ for $1 \leq i \leq k-1$ and for $i = k$, $\tau_k \leq \sigma_k, \sigma_1$. Also the dimension vector is given by fixing $\dim \tau_i = 2$ and $\dim \sigma_i = 1$. Note that as mentioned in the introduction we are interested in such sheaves on Δ where in this case

$$\mathcal{G} = \underline{\mathbb{C}}^4_\Delta,$$
$$\dim(\mathcal{F}_{\sigma_i}) = 1, \ \dim(\mathcal{F}_{\tau_i}) = 2 \text{ and } \dim(\mathcal{F}_0) = 4$$

We will fix this $\mathcal{G}$ and this dimension vector $\mathbf{d} = (d_\tau)$ for the rest of the section.

Proposition 4.1. *The dimension of every component of $\mathcal{M}_\Delta$ is $\geq 3k$.*

Proof. Since $\mathcal{M}_\Delta \subset (G(1,4))^k \times (g(2,4))^k$ we can consider the projections $\pi_{\tau,\sigma}$: $\mathcal{M}_\Delta \to \mathbb{P}^3 \times \mathbb{G}(1,3)$ for each $\tau \leq \sigma$. now take the flag variety $F(1,2;\mathbb{C}^4) \subset G(1,3) \times G(2,4)$, since it is nonsingular and has codimension 2 it is defined locally by 2 equations so we get locally $4k$ equations in total. And since the dimension of the ambient space is $3k + 4k$ we conclude that every component of $\mathcal{M}_\Delta$ has dimension

$$\geq 3k + 4k - 4k = 3k$$

□

Theorem 4.1. *The moduli space $\mathcal{M}_\Delta$ is singular at a point corresponding to $\mathcal{F} \subset \mathcal{G}$ if and only if:*

(i) $\mathcal{F}_{\tau_i} = \mathcal{F}_{\tau_j}$ for all $\tau_i, \tau_j \in \Delta$.

(ii) $Span\{\mathcal{F}_{\sigma_i}\} \neq \mathbb{C}^4$.

Proof. First consider in contrary the first condition is not satisfied and we will prove in this case any point is smooth by showing the dimension of the tangent space is exactly $3k$.

Partition the maximum cones $\{\tau_1, ..., \tau_n\}$ into m sets as follows

$$\{\tau_1, ..., \tau_n\} = \Delta_1 \cup \Delta_2 \cup ... \cup \Delta_m$$

where if τ_i, τ_{i+1} are neighbours, then they lie in the same Δ_j if and only if $\mathcal{F}_{\tau_i} = \mathcal{F}_{\tau_j}$

In what follows for simplicity we shall write $\mathcal{Q}$ for the quotient $\mathbb{C}^4/\mathcal{F}$.

Now order cones τ_i, σ_j so that τ_i, τ_k lie in different partitions Δ_i, $\tau_1 \cap \tau_k = \sigma_k$. We choose morphisms $\varphi_{\tau_i}, \varphi_{\tau_j}$ in the following order

$$\tau_1, \sigma_1, \tau_2, \sigma_2, ..., \tau_k, \sigma_k.$$

Choose φ_{τ_1} arbitrary. By induction, suppose $\varphi_{\tau_1}\varphi_{\sigma_1}, ..., \varphi_{\tau_i}$ are chosen. Let $\tau = \tau_i, \sigma = \sigma_i$ and $\tau^{'} = \tau_{i+1}$. We explain how to choose $\varphi_\sigma, \varphi_{\tau^{'}}$ in two cases:

Let $\tau \in \Delta_1$ and $\tau^{'} \in \Delta_2$ where this two have a common face namely σ. How from the following conditions on morphisms for our corresponding choices of σ, τ and $\tau^{'}$:

$$\begin{tikzcd}
\mathbb{C} \simeq \mathcal{F}_\tau \arrow[r, "\varphi_\tau"] \arrow[d, hook] & \mathcal{Q}_\tau \simeq \mathbb{C}^3 \arrow[d] \\
\mathbb{C}^2 \simeq \mathcal{F}_\sigma \arrow[r, "\varphi_\sigma"] & \mathcal{Q}_\sigma \simeq \mathbb{C}^2 \\
\mathbb{C} \simeq \mathcal{F}_{\tau^{'}} \arrow[r, "\varphi_{\tau^{'}}"] \arrow[u, hook] & \mathcal{Q}_{\tau^{'}} \simeq \mathbb{C}^3 \arrow[u]
\end{tikzcd}$$

Case 1. ($\tau, \tau^{'}$ lie in different partition. i.e. $\mathcal{F}_\tau \neq \mathcal{F}_{\tau^{'}}$)

If φ_τ is given, there are 2 dimension choices for φ_σ, and there is 1 dimension of choice for $\varphi_{\tau^{'}}$ since

$$\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau^{'}}.$$

Case 2. ($\tau, \tau^{'}$ lie in the same partition. i.e. $\mathcal{F}_\tau = \mathcal{F}_{\tau^{'}}$)

Again if φ_τ is given, there are 2 dimension choices for φ_σ, and there is 1 dimension of choice for $\varphi_{\tau^{'}}$.

Thus in both cases we have:

$$\dim[\mathrm{Hom}(\mathcal{F}_\sigma, \mathbb{C}^4/\mathcal{F}_\sigma)] = 2, \dim[\mathrm{Hom}(\mathcal{F}_{\tau^{'}}, \mathbb{C}^4/\mathcal{F}_{\tau^{'}})] = 1$$

Now note that the last step in order to complete the fan is when $\varphi_{\tau_1}, \varphi_{\tau_k}$ are given and we want to choose φ_{σ_k}, but as in case 1, φ_{τ_1} and φ_{τ_k} together determine φ_{σ_k} uniquely since like in case 1:

$$\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}.$$

and as a result we have:

$$\dim[\mathrm{Hom}(\mathcal{F}, \mathbb{C}^4/\mathcal{F})] = 3 + (k-1) \times 1 + (k-1) \times 2 = 3k.$$

Now in order to complete the proof let $\mathcal{F}$ be any arbitrary sheaf on Δ and assume $\mathcal{F}_{\tau_i} = \mathcal{F}_{\tau_j}$ for all i, j which would decompose our sheaf $\mathcal{F}$ into $\mathcal{F} = \underline{\mathbb{C}}_\Delta \oplus \mathcal{F}^{'}$. Thus we have

$$\mathbb{C}^4/\mathcal{F} = \mathbb{C}^4/\mathbb{C} \oplus \mathcal{F}^{'} = \mathbb{C}^3/\mathcal{F}^{'}$$

and consequently the tangent space at $\mathcal{F}$ would be

$$\begin{aligned} T_{\mathcal{F}}(\mathcal{M}_\Delta) = \mathrm{Hom}(\mathcal{F}, \mathbb{C}^4/\mathcal{F}) &= \mathrm{Hom}(\mathbb{C} \oplus \mathcal{F}^{'}, \mathbb{C}^3/\mathcal{F}^{'}) \\ &= \mathrm{Hom}(\mathbb{C}, \mathbb{C}^3/\mathcal{F}^{'}) \oplus \mathrm{Hom}(\mathcal{F}^{'}, \mathbb{C}^3/\mathcal{F}^{'}) \end{aligned}$$

which would imply

$$\dim(T_{\mathcal{F}}(\mathcal{M}_\Delta)) = \dim[\mathrm{Hom}(\mathbb{C}, \mathbb{C}^3/\mathcal{F}^{'})] + \dim[\mathrm{Hom}(\mathcal{F}^{'}, \mathbb{C}^3/\mathcal{F}^{'})].$$

Note that $\mathcal{F}^{'}_{\tau_i} = 0$ for all $\tau_i \in \Delta$.
Consider $\varphi \in \mathrm{Hom}(\mathcal{F}^{'}, \mathbb{C}^3/\mathcal{F}^{'})$, thus for each σ the morphism φ_σ should make the following diagram commutative

$$\begin{tikzcd}
\mathbb{C} \simeq \mathcal{F}^{'}_{\sigma} \arrow[r, "\varphi_\sigma"] \arrow[d, hook] & \mathcal{Q}_\sigma \simeq \mathbb{C}^2 \arrow[d] \\
\mathbb{C}^3 \simeq \mathcal{F}_0 \arrow[r, "\varphi_\sigma"] & 0
\end{tikzcd}$$

which would imply that the space of all such morphisms (φ_σ) would be of dimension 2 and since there are k such σ's we conclude that

$$\dim[\mathrm{Hom}(\mathcal{F}^{'}, \mathcal{Q})] = 2k$$

To compute the dimension of $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^3/\mathcal{F}^{'})$ let $\varphi \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^3/\mathcal{F}^{'})$ consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\varphi_\tau"] \arrow[d, hook] & \mathcal{Q}_\tau \simeq \mathbb{C}^3 \arrow[d] \\
\mathbb{C} \arrow[r, "\varphi_\sigma"] & \mathcal{Q}_\sigma \simeq \mathbb{C}^2 \\
\mathbb{C} \arrow[r, "\varphi_{\tau'}"] \arrow[u, hook] & \mathcal{Q}_{\tau'} \simeq \mathbb{C}^3 \arrow[u]
\end{tikzcd}$$

where $\tau, \tau^{'} \geq \sigma$, let $\varphi_\tau(1) = q_\tau \in Q_\tau$ and $\varphi_{\tau'}(1) = q_{\tau'} \in Q_{\tau'}$, subject to conditions:

$$\mathrm{res}_{\tau,\sigma}(q_\tau) = \mathrm{res}_{\tau',\sigma}(q_{\tau'}) \quad \text{if} \quad \tau, \tau^{'} \geq \sigma.$$

Now we can write this condition as:

$$(q_\tau, q_{\tau'}) \in \ker(\mathcal{Q}_\tau \oplus \mathcal{Q}_{\tau'} \xrightarrow{(\mathrm{res}_{\tau,\sigma},\ -\mathrm{res}_{\tau',\sigma})} \oplus \mathcal{Q}_\sigma)$$

Putting these together:

$$(q_\tau)_\tau \in \ker(\oplus \mathcal{Q}_{\tau_i} \xrightarrow{\psi} \oplus \mathcal{Q}_{\sigma_i})$$

where the map ψ is $\pm\,\mathrm{res}_{\tau,\sigma}$. The signs are chosen as in the cellular complex, depending on the chosen orientation of τ_i, σ_i. Thus we conclude that

$$\dim[\mathrm{Hom}(\mathbb{C}, \mathcal{Q})] = \dim(\ker(\oplus \mathcal{Q}_{\tau_i} \xrightarrow{\psi} \oplus \mathcal{Q}_{\sigma_j})).$$

Now since $\dim(\oplus \mathcal{Q}_{\tau_i}) = 3k$ and $\dim(\oplus \mathcal{Q}_{\sigma_j}) = 2k$ we conclude that

$$\dim[\mathrm{Hom}(\mathbb{C}, \mathcal{Q})] \geq k.$$

Note that since $T_{\mathcal{F}}(\mathcal{M}_\Delta)$ is the Zariski tangent space for any point $\mathcal{F} \in \mathcal{M}_\Delta$, $\dim(T_{\mathcal{F}(\mathcal{M}_\Delta)}) \geq \dim(\mathcal{M}_\Delta)$, with equality if and only if $\mathcal{F}$ is nonsingular. Now from above we can derive that a point is singular if and only if dimension of $\mathrm{Hom}(\mathbb{C}, \mathcal{Q})$ is strictly greater that k and consequently this is the case if and only if ψ is not surjective.

Now in order to complete the proof we will show non-surjectivity of ψ is equivalent to condition (II). From the following exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \oplus\mathcal{F}^{'}_{\tau_i} \arrow[r] \arrow[d] & \oplus\mathbb{C}^3_{\tau_i} \arrow[r] \arrow[d, "\alpha"] & \oplus Q_{\tau_i} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \oplus\mathcal{F}^{'}_{\sigma_j} \arrow[r] & \oplus\mathbb{C}^3_{\sigma_j} \arrow[r] & \oplus Q_{\sigma_j} \arrow[r] & 0
\end{tikzcd}$$

we get a long exact cohomology sequence:

$$0 \to \ker\alpha \longrightarrow \ker\varphi \longrightarrow \oplus\mathcal{F}^{'}_{\sigma_j} \longrightarrow \text{coker } \alpha = \mathbb{C}^3 \longrightarrow \text{coker } \psi \to 0.$$

Thus coker $\psi \neq 0$ iff $(\oplus\mathcal{F}^{'}_{\sigma_j} \to \mathbb{C}^3)$ is not surjective iff $(\oplus\mathcal{F}_{\sigma_j} \to \mathbb{C}^4)$ is not surjective. This gives, in particular, that

$$\dim(\text{coker } \psi) = \text{codim}(\text{Span}\{\mathcal{F}^{'}_{\sigma_j}\}) \subset \mathbb{C}^3 = \text{codim}(\text{Span}\{\mathcal{F}_{\sigma_j}\}) \subset \mathbb{C}^4. \qquad \square$$

Remark 3. Before proving irreducibility by studying the tangent space, now that we know which points are singular lets say more about the dimension of singular points. Consider the singular point $\mathcal{F}$. We know from condition (I), $\mathcal{F}_{\tau_i} = \mathcal{F}_{\tau_j}$. Consider 2 types of singular points:

1 $\dim \text{Span}\{\mathcal{F}_{\sigma_i}\} = 3$

2 $\dim \text{Span}\{\mathcal{F}_{\sigma_i}\} = 2$ which is equivalent to $\mathcal{F}_{\sigma_i} = \mathcal{F}_{\sigma_j}$ for all i, j.

The second type of singular points are the most singular points with tangent space dimension $3k + 2$ coming from the fact that

$$\dim T_{\mathcal{F}}(\mathcal{M}_\Delta) = 2k + \dim(\ker\psi) = 3k + \dim(\text{coker}\psi).$$

Lemma 2. *The dimension of Sing*$(\mathcal{M}_\Delta)$ *is* $5 + k$.

Proof. Since we know a point $\mathcal{F}$ is singular if and only if the two conditions above is satisfied we can simply compute the singular locus as follow:
Choose a 3-dimensional space $V \subset \mathbb{C}^4$. The space of all such V is of dimension 3 (The dimension of $G(3, \mathbb{C}^4)$). Then choose a 1-dimensional space $\mathcal{F}_\tau \subset V = \mathbb{C}^3$. The corresponding space is of dimension 2 (The dimension of $\mathbb{P}^2$) and at last for each σ we have to choose a 2-dimensional space $\mathcal{F}_\sigma$ such that $\mathcal{F}_\tau \subset \mathcal{F}_\sigma \subset V$ or

equivalently

$$\mathcal{F}_\sigma/\mathcal{F}_\tau = \mathbb{C} \subset V/\mathcal{F}_\tau \simeq \mathbb{C}^2$$

which has dimension k, i.e. The dimension of $(\mathbb{P}^1)^k$. $\square$

Theorem 4.2 (Affine Dimension Theorem)**.** *Let Y, Z be varieties of dimensions r, s in $\mathbb{A}^n$. Then every irreducible component W of $Y \cap Z$ has dimension $\geq r+s-n$.*

Proof. (see [3], p. 48.) $\square$

Lemma 3. *$\mathcal{M}_\Delta$ is connected.*

Proof. Let $\mathcal{U}_4$ be the set of all 4×4 upper triangular matrices with diagonal entries 1:

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{U}_4$ acts on $\mathbb{C}^4$, hence acts on $\mathcal{M}_\Delta$. Since $\mathcal{U}_4$ is solvable, every projective variety on which $\mathcal{U}_4$ acts has a fixed point. Thus $\mathcal{M}_\Delta$ has a unique fixed point $\mathcal{F}$ where $\mathcal{F}_{\tau_i} = Span\{e_1\}$, $\mathcal{F}_{\sigma_j} = Span\{e_1, e_2\}$. This fixed point must lie in every component of $\mathcal{M}_\Delta$. $\square$

Theorem 4.3. *For 2-dimensional complete fans the moduli space is irreducible for $k \geq 4$.*

Proof. Suppose $\mathcal{M}_\Delta$ has two irreducible components then each component is of dimension $\geq 3k$. Now let $\mathcal{F}$ be the fixed point of $\mathcal{U}_4$ as in Lemma 3. then

$$T_{\mathcal{F}}(\mathcal{M}_\Delta) = 3k + 2.$$

Now since locally (analytically) we can embed $\mathcal{M}_\Delta$ to its tangent space thus by the "Affine Dimension Theorem" we have

$$5 + k = \dim(Sing(\mathcal{M}_\Delta)) \geq \dim Y \cap Z \geq 3k + 3k - (3k + 2)$$

which would imply $k \leq 3$. $\square$

Chapter 5

Stratification of $\mathcal{M}_\Delta$

Another way to prove Theorem 4.3 would be by means of stratification of $\mathcal{M}_\Delta$ and calculating the fibre dimensions of each stratas.
Consider the map

$$\pi : \mathcal{M}_\Delta \to \prod_\tau G(d_\tau, \mathbb{C}^4) = (\mathbb{P}^3)^k.$$

We can stratify $\mathcal{M}_\Delta$ according to the combinatorial type of the image of $\mathcal{F} \in \mathcal{M}_\Delta$.

5.1 $k = 3$

By Proposition 4.1 the dimension of $\mathcal{M}_\Delta$ is 9. Let

$$(\mathbb{P}^3)^3 = \mathcal{U} \sqcup \mathcal{S}_{1,2} \sqcup \mathcal{S}_{1,3} \sqcup \mathcal{S}_{2,3} \sqcup \mathcal{S}_{1,2,3}$$

be the stratification of $\mathbb{P}^3$, where $\mathcal{S}_{i,j}$ is the set consisting of 3 choices of points namely $p_i := \mathbb{P}(\mathcal{F}_{\tau_i})$ in $\mathbb{P}$ such that the two points $p_i = p_j$ and the third point is different from the other two and $\mathcal{S}_{1,2,3}$ is a set consisting of three such points $(p_1, p_2, p_3) \in (\mathbb{P}^3)^3$ where all three are equal and $\mathcal{U}$ would be a set where p_i's are all distinct.
Now consider the restriction of π on each inverse image of these stratas. Since we have

$$\mathcal{M}_\Delta = \pi^{-1}(\mathcal{U}) \sqcup \pi^{-1}(\mathcal{S}_{1,2}) \sqcup \pi^{-1}(\mathcal{S}_{1,3}) \sqcup \pi^{-1}(\mathcal{S}_{2,3}) \sqcup \pi^{-1}(\mathcal{S}_{1,2,3})$$

where each piece is irreducible.
Since $\mathcal{S}_{1,2,3}$ is an irreducible closed set in $(\mathbb{P}^3)^3$ (it is the diagonal set in the product) the inverse image of it under π is a closed set in $\mathcal{M}_\Delta$. The dimension of the fiber is 6 since by fixing one point in $\mathbb{P}^3$ we are looking for all the possible 3 lines passing through this one point which the space is of dimension $(\mathbb{P}^2)^3$ by fibre dimension

theorem we conclude that

$$\dim(\pi^{-1}(\mathcal{S}_{1,2,3})) = 6 + 3 = 9$$

Consider the strata $\mathcal{S}_{1,2}$. We need to choose 2 points p_1, p_3 in $\mathbb{P}^3$ thus $\dim \mathcal{S}_{1,2} = 6$. Now in order to compute the fibre dimension note that by fixing two points p_1, p_3 these two points uniquely determine a line passing through them thus we only need to choose one line so the fibre would be $\mathbb{P}^2$. Thus we conclude that

$$\dim(\pi^{-1}(\mathcal{S}_{1,2})) = 2 + 6 = 8.$$

Similarly for strats $\mathcal{S}_{1,3}$ and $\mathcal{S}_{2,3}$ we have:

$$\dim(\pi^{-1}(\mathcal{S}_{1,3})) = 2 + 6 = 8, \quad \dim(\pi^{-1}(\mathcal{S}_{2,3})) = 2 + 6 = 8.$$

Thus $\mathcal{M}_\Delta$ is reducible since it has 2 components. Actually the other component is $\overline{\pi^{-1}(\mathcal{U})}$ since each fiber on $\mathcal{U}$ is a single point (there exists only one line passing through 2 points) thus $\dim(\overline{\pi^{-1}(\mathcal{U})}) = 0 + \dim(\mathcal{U}) = 0 + 9 = 9$.

5.2 $k \geq 4$

In order to define the stratas consider the partitions of the cones $\{\tau_1, ..., \tau_k\}$ introduced in The proof of Theorem 4.0.2. in to m pieces. Thus we can associate a cyclic partition of the set $\{1, ..., k\}$, namely $\{i_1, ..., i_m\}$ to each partition of $\{\tau_1, ..., \tau_k\}$, as follows :

$$\begin{aligned}\{1, \cdots, k\} &= \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_m \\ &= \{i_1, i_1 + 1, \cdots, i_2 - 1\} \sqcup \{i_2, i_2 + 1, \cdots, i_3 - 1\} \sqcup \cdots\end{aligned}$$

where $m = |\{i_1, \cdots, i_m\}|$ is equal the number of partitions of cones $\{\tau_1, ..., \tau_k\} = \Delta_1 \cup ... \cup \Delta_m$.

Now by the same argument for the case $k = 3$, consider the following stratification of $(\mathbb{P}^3)^3$

$$(\mathbb{P}^3)^3 = \mathcal{U} \sqcup \mathcal{S}_{\{1\}} \sqcup \mathcal{S}_{\{i_1, i_2\}} \sqcup ... \sqcup \mathcal{S}_{\{i_1, ..., i_m\}} \sqcup ... \sqcup \mathcal{S}_{\{1,2,...,k\}}.$$

Then if $1 < m < k$

$$\begin{aligned}\dim \pi^{-1}(\mathcal{S}_{\{i_1,\ldots,i_m\}}) &= \dim(\text{fibre}) + \dim(\mathcal{S}_\Sigma) \\ &= \dim(\mathbb{P}^2)^{\left(\sum_i |\Sigma_i|\right)-m} + \dim(\mathbb{P}^3)^m \\ &= 2(k-m) + 3m = 2k + m \\ &< 3k\end{aligned}$$

If $m = 1$, then

$$\begin{aligned}\dim \pi^{-1}(\mathcal{S}_{\{1\}}) &= \dim(\text{fibre}) + \dim(\mathcal{S}_\Sigma) \\ &= \dim(\mathbb{P}^2)^k + \dim(\mathbb{P}^3) \\ &< 3k\end{aligned}$$

Now since in each case the dimension of each inverse image of components are strictly less than $3k$ we conclude that $\mathcal{M}_\Delta$ is irreducible and its only component is $\pi^{-1}(\mathcal{S}_{\{1,2,\ldots,k\}})$.

Chapter 6

Conclusion

Following the construction of our moduli space and its irreducibility in 2 dimensional case we prove Conjecture 3 for 2 dimension case by showing the general sheaf $\mathcal{F}$ on $\Delta = \{0, \tau, \sigma\}$ with dimension vectors $\dim(\mathcal{F}_\tau) = 1$, $\dim(\mathcal{F}_\sigma) = 2$ and $\dim(\mathcal{F}_0) = 4$ has non-zero cohomology in middle degree only.

Theorem 6.1. *Let $\mathcal{F}$ be a general sheaf in $\mathcal{M}_\Delta$ for $k \geq 4$. Then the cellular complex $C^\bullet(\mathcal{F}, \Delta)$ has nonzero cohomology in degree 1 only.*

Proof. By Theorem 4.1. we need that

(1). $Span\{\mathcal{F}_\sigma\} = \mathbb{C}^4$.

(2). $\bigoplus_\tau \mathcal{F}_\tau \xrightarrow{\psi} \bigoplus_\sigma \mathcal{F}_\sigma$ is injective.

If $(f_\tau) \in \ker \psi$, then

$$\mathrm{res}_{\tau,\sigma}(f_\tau) = \mathrm{res}_{\tau',\sigma}(f_{\tau'}) \quad \text{for all } \tau, \tau^{'} \geq \sigma.$$

Thus if $\mathcal{F}_\tau \neq \mathcal{F}_{\tau'}$, then $f_\tau = f_{\tau'} = 0$ □

Remark 4. Note that $C^\bullet(\mathcal{F}, \Delta)$ has nonzero cohomology in middle dimension only if and only if both conditions of Theorem 4.1 are false

Example 6. For the case where $k = 3$ the component $\overline{\pi^{-1}(\mathcal{U})}$ satisfies condition (ii) in the Theorem 4.1., hence $C^\bullet(\mathcal{F}, \Delta)$ has cohomology in degree 2. The component $\pi^{-1}(\mathcal{S}_{1,2,3})$ satisfies the condition (I), and $C^\bullet(\mathcal{F}, \Delta)$ has cohomology in degree 0.

Bibliography

[1] R. Charney and M. Davis, *The Euler characteristic of a non-positively curved, piecewise Euclidean manifold*, Pacific J. Math. 171 (1995), pp. 117-137,

[2] J. Harris, *Algebraic Geometry: A First Course*. GTM No. 133. Springer-Verlag, first edition, 1992.

[3] R. Hartshorne, *Algebraic Geometry*. GTM, vol. 52, Springer Science + Business Media, LLC, 2006.

[4] S. Mukai, *An Introduction to Invariants and Moduli*, Cambridge Studies in Advanced Mathematics, 81. Cambridge Univ. Press, Cambridge, 2003.