Dimensional Reduction and Spacetime Pathologies

by

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Abstract

Dimensional reduction is a well known technique in general relativity. It has been used to resolve certain singularities, to generate new solutions, and to reduce the computational complexity of numerical evolution. These advantages, however, often prove costly, as the reduced spacetime may have various pathologies, such as singularities, poor asymptotics, negative energy, and even superluminal matter flows. The first two parts of this thesis investigate when and how these pathologies arise.

After considering several simple examples, we first prove, using perturbative techniques, that under certain reasonable assumptions any asymptotically flat reduction of an asymptotically flat spacetime results in negative energy seen by timelike observers. The next part describes the topological rigidity theorem and its consequences for certain reductions to three dimensions, confirming and generalizing the results of the perturbative approach. The last part of the thesis is an investigation of the claim that closed timelike curves generically appearing in general relativity are a mathematical artifact of periodic coordinate identifications, using, in part, the dimensional reduction techniques. We show that removing these periodic identifications results in naked quasi-regular singularities and is not even guaranteed to get rid of the closed timelike curves.

Statement of Collaboration

The research covered by this thesis is a collection of projects done either alone or in collaboration with or under supervision of Kristin Schleich, Don Witt and Johan Brannlund. Specifically, chapter 3 is largely an unassisted effort, most of the chapter 4 is a presentation of the work done in collaboration with Kristin Schleich, Don Witt and Johan Brannlund. Chapter 5 describes and extends the results first obtained by Kristin Schleich and Don Witt. Chapter 6 describes my original work, with guidance and feedback provided by Kristin Schleich.

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Dedication

To my mother, who believed in me even when I didn't.

Chapter 1

Introduction

Einstein's general relativity relates curvature of the spacetime to its matter content: the Einstein equation states that the Einstein curvature tensor G_{ab} is proportional to the stress-energy tensor T_{ab} of the matter, i.e.,

$$G_{ab} \propto T_{ab}.$$

This relation can be interpreted in two ways: one can calculate the stress-energy tensor if the spacetime metric is given, or one can specify the desired stress-energy tensor and solve for the spacetime metric. The former approach is easy and determines the stress energy tensor uniquely, but the stress-energy tensor corresponding to an arbitrary metric is not guaranteed to describe anything like the "normal" matter we are accustomed to, such as stars, planets, electromagnetic radiation etc. It might, for example, correspond to negative energy, or to superluminal matter flows, often described informally as "tachyons". A more useful problem is, in a sense, an opposite one: one specifies the desired matter content and the curvature in space at a given point in time (known as the *initial surface*) and uses it to calculate what happens to the matter and the space as it evolves in time. This is known as the initial value problem or *Cauchy problem* in general relativity. Due to the non-linearity of the Einstein equation, the Cauchy problem is impossible to solve in closed form in all but trivial cases, so numerical methods must be used instead. In fact, the situation is even worse than that. The Cauchy problem has only been proven to be *well-posed* — to have a unique solution, at least for some finite time, and to have continuous dependence on initial conditions — only when the stress-energy tensor on the *initial surface* is well behaved. Specifically, it must describe subluminal matter sources with non-negative energy density. This is known as the *dominant energy condition*. The initial surface must not have *naked singularities*, those singularities not inside a blackhole. If any of these restrictions do not hold, the spacetime in question and its matter content are often described as *pathological*. While it is not out of the question that some pathological matter sources can be used to construct a well-posed Cauchy problem, the issue has to be investigated on a case-by-case basis.

The motivation for this thesis was initially provided by the need to reconcile the $2+1+1$ numerical formalism (see below) with the topological censorship theorems, discussed in section 5.2. The $2+1+1$ numerical schemes aim to exploit the *axisymmetry* of some initial data sets in order to reduce the computational complexity of constructing a four-dimensional spacetime from an initial data set (see e.g., [1]). The standard approach to the Cauchy problem in general relativity (GR) is known as the $3+1$ split [2], where 3 is the dimension of the spatial slice being evolved along a (one-dimensional) time coordinate. In the $2+1+1$ approach one first dimensionally reduces the Einstein equation from 4 to 3 dimensions, by trading the dimension corresponding to the axisymmetry for additional terms in the stress-energy tensor. The resulting three-dimensional Einstein equation is then subjected to a $2+1$ split, where now a two-dimensional spatial slice is evolved in time. The four-dimensional spacetime can then be reconstructed from the evolved three-dimensional one, if desired.

An example of a problem where the $2+1+1$ approach was hoped to produce interesting results is the head-on collision of two static or co-axially rotating blackholes. However, it follows from the topological censorship theorems originally proven in the 1990s that the Cauchy surface in a globally hyperbolic $(2+1)$-dimensional spacetime without negative energy must either be simply connected — any loop can be contracted into a point — or have topology of a circle [3]. This condition is violated by axisymmetric reduction of a two-blackhole spacetime. To avoid a contradiction, one is then led to suspect that the stress-energy tensor induced by the reduction in the reduced spacetime results in a negative energy density.

The first step was to check this suspicion on some simple axisymmetric spacetimes. This investigation led at first to a surprising (or, in the hindsight, maybe not so surprising) result, that even a spacetime as simple as the spherically symmetric vacuum blackhole in four dimensions, when dimensionally reduced with respect to one of its rotational symmetries, yields a spacetime where the stress-energy tensor has some unusual properties. In particular, it has negative energy density. It also appears superluminal to some timelike observers. Adding electric charge, angular momentum or even gravitational radiation does not cure the problem, as the pathologies of the stress energy tensor of the reduced spacetime persist, at least asymptotically.

Given that all the asymptotically flat reductions of various asymptotically flat spacetimes resulted in a pathological stress energy tensor, it is reasonable to suspect that this is a general feature of such reductions. To that end, we next investigated the conditions under which the presence of pathologies can be proven. One approach is to consider any asymptotically flat spacetime metric to be asymptotically a perturbation on the Schwarzschild metric, and estimate the contribution of such perturbations to the induced stress-energy tensor of the reduced spacetime. We have proven that the weak energy condition (WEC) is violated, subject to some reasonable assumptions, for the axisymmetric dimensional reduction from any dimension $D > 3$ (see chapter 4).

Since an axisymmetric reduction creates a boundary in the reduced spacetime where the symmetry axis used to be, care must be taken when calculating quantities such as the total energy of the reduced spacetime, as there is no closed surface ($(D-2)$-sphere) over which the corresponding Komar integral can be calculated (see section 2.4). Fortunately, for the standard dimensional reduction the boundary is regular — the curvature of the reduced spacetime remains bounded near the boundary. This allows one to "double" the spacetime and remove the boundary by gluing another copy of it along the boundary. Now the total energy of a spacetime (also known as the *ADM mass* or *ADM energy*, after Arnowitt, Deser, and Misner, who originally derived a Hamiltonian formulation of general relativity [4]) and other asymptotic quantities can be computed, and, surprisingly, the total energy vanishes identically after the reduction. This suggests that if one were to apply a suitable version of the positive energy theorem to the reduced spacetime (decribed in section 4.2), it follows that the reduction breaks the dominant energy condition, unless the reduced spacetime is flat.

The perturbative calculation of the (pathological) induced stress-energy tensor has motivated a search for an underlying reason for the violations, which resulted in the topological rigidity theorem, formulated and proven in [5]. The theorem states that any globally hyperbolic bundle reduction to an asymptotically flat (AF) or an asymptotically locally (ALADS) spacetime that satisfies the null energy condition (NEC) restricts the topology of the domain of outer communications of both the original and the reduced spacetimes to just two possible cases when the reduced spacetime is three-dimensional: that of a disk or an annulus (a disk with a hole, no matter how small, cut in it). A corollary to the theorem is that there is no $U(1)$ *reduction* of a four-dimensional AF globally hyperbolic spacetime to a three-dimensional AF or ALADS spacetime that satisfies NEC.

The domains of applicability of the perturbative argument and the topological rigidity theorem overlap, but are not identical. The perturbative calculation shows that any axisymmetric reduction of a $(D > 3)$-dimensional asymptotically flat axisymmetric spacetime to a $(D-1)$-dimensional asymptotically flat spacetime results in a pathological stress-energy tensor. The topological rigidity theorem implies that a $U(1)$ reduction of a four-dimensional

AF spacetime with a $U(1)$ Killing vector to a three-dimensional AF spacetime must include negative energy.

The above conclusion has significant implications for any $2+1+1$ numerical scheme, as the initial value formulation in this case is not guaranteed to be well posed or even to exist. The proof of existence, uniqueness and continuous dependence of evolution on initial data assumes subluminal matter flows with everywhere non-negative energy density [2]. A numerical implementation of the $2+1+1$ approach that does not address these issues explicitly is likely to face serious obstacles. The $2+1+1$ approach has been attempted by different groups [1, 6], with only a modest success, and it has been all but superceded by more general (and now more successful) fully $(3+1)$-dimensional techniques, where the amount of calculations is reduced when the axisymmetry is taken into consideration [7–9]. It is possible that some of the problems encountered by the pioneers of the $2+1+1$ schemes are at least in part due to the pathological properties of the reduced spacetimes.

While axisymmetric asymptotically flat reductions of asymptotically flat spacetimes necessarily result in negative energy and superluminal matter flows, other reductions do not have to. For example, one can avoid negative energy by conformally rescaling the reduced metric. We analyze a well-known conformal rescaling of the reduced metric by the norm of the axisymmetric Killing vector in section 3.9. In this case there is a trade-off for keeping the energy density positive everywhere: the rescaled metric has a naked timelike singularity on the axis of symmetry and is no longer asymptotically flat, making it problematic to define asymptotic quantities, such as mass and charge.

There are also reductions that do not create any (additional) pathologies. Some of them are described in section 5.6. For example, an axisymmetric reduction of a four-dimensional black string results in a well-behaved three-dimensional spacetime. This is not a counterexample to the corollary, as this spacetime is not asymptotically flat to begin with. Another example is the dimensional reduction of the *anti-de Sitter* (AdS) solution. There both the original and the reduced spacetimes are asymptotically AdS, not asymptotically flat.

While analyzing the properties of dimensional reductions of different exact solutions, I have come across the fact that a simple reduction of the Gödel spacetime [10] with respect to a timelike Killing vector with open orbits produces a flat Euclidean space. This provided a motivation to inquire how the well-known Gödel closed timelike curves (CTC) [10, 11] arise from such a simple spacetime, given that there exists a nowhere singular timelike Killing vector field with a trivial topology ($\mathbb{R}$), and whether the existence of CTCs in the total spacetime can be inferred from the properties of the reduced spacetime. While this question remains open, a related claim was investigated, namely that of Cooperstock and Tieu [12], that the CTCs of the Gödel spacetime, as well as those of the Gott spacetime [13], are nothing but an artifact of a bad coordinate identification.

To that end, I have recast their suggestion to remove the "artificial" periodic identification of the angular coordinate in the polar chart into the formal "unwrapping" procedure. This procedure is investigated in detail using the simple examples of Minkowski, Gott and Gödel spacetimes. I have shown that unwrapping necessarily produces a naked quasi-regular singularity in a previously regular spacetime, thus confirming the accepted point of view that CTCs are not a mathematical artifact, but a necessary consequence of general relativity, contrary to the claims of Cooperstock and Tieu.

While unwrapping the Gott spacetime does indeed remove all CTCs from it, the unwrapped Gödel space still had CTCs through every point. I next asked whether there is any way to remove the remaining CTCs from the Gödel spacetime, without creating an artificial boundary or matching it to a CTC-free spacetime with a different metric. As a result, I have constructed a "multiple unwrapping" procedure, where the Gödel space is first tessellated with a regular lattice of timelike lines, such that any CTC has to wrap around at least one of them, then the timelike lines are removed from the spacetime, and finally the resulting multiply connected spacetime is replaced with its universal cover, which no longer contains CTCs, but instead has a countable infinity of naked quasi-regular singularities. However, at any regular point the spacetime metric is still that of Gödel. While I have not proven that the suggested tessellation removes all CTCs from Gödel, this conjecture is supported by analyzing a different kind of a CTCs, a sector-like CTCs, which are also removed by this multiple unwrapping.

As the results presented in this thesis show, dimensional reduction, while a useful tool, is also a dangerous one, as it more often than not results in pathologies of one kind of another, even if the original spacetime is perfectly regular. On the flip side, various dimensional reductions can also cure (or mask) such pathologies as singularities, CTCs and "bad" stress-energy tensor.

The thesis is structured as follows: chapter 2 describes the standard definitions, results and techniques that are extensively used in the rest of the thesis. Another important concept referred to in this thesis many times is that of singular spacetimes. The discussion of singularities is discussed in section 6.3.1.

Chapter 3 analyzes the properties of several examples of axisymmetric reduction and the resulting pathologies, if any. Chapter 4 describes the perturbative arguments for the violations of the energy conditions. The topological reasons are presented in chapter 5. Finally, the investigation of CTC unwrapping is the subject of chapter 6.

Throughout the thesis, we assume that the Einstein equation holds in any spacetime under consideration, and we use the units where $8\pi G/c^2 = 1$ so that the Einstein equation in the presence of the cosmological constant reads $G_{ab} + \Lambda g_{ab} = T_{ab}$. Thus some of the expressions may differ from those given in other places by a factor of 8π.

Chapter 2

Background

We make repeated use of a number of concepts in general relativity, such as global hyperbolicity, energy conditions, asymptotic flatness and axisymmetry. While these are described in many standard texts on the subject, some of the definitions often differ slightly between them. This section reviews the necessary concepts and results. The formalism of dimensional reduction of a spacetime admitting a Killing vector is also reviewed.

In the following we use a concept of a *spacetime-with-boundary*. It is obtained from a “usual” non-compact spacetime by attaching a boundary to it. The metric of such a spacetime can be conformally rescaled in a way that brings the boundary at infinity, usually denoted as $\mathscr{I}$, to a finite distance, making its structure easy to see on a diagram. For AF spacetimes one makes a distinction between past and the future infinities, $\mathscr{I}_0^-$ and $\mathscr{I}_0^+$ respectively, (see section 2.3). One such diagram is shown on Fig. 2.1.

2.1 Causality and Hyperbolicity

A spacetime $\mathscr{M}$ is strongly causal if for any point $p \in \mathscr{M}$ there exists a neighborhood of any point $p \in \mathscr{M}$ through which no timelike curve passes more than once.

A spacetime with boundary $\mathscr{M}'$ is *globally hyperbolic* if it is strongly causal and the sets $J^+(p,\mathscr{M}') \cap J^-(q,\mathscr{M}')$ (i.e., the intersection of the causal future[1] of p and the causal past of q) are compact for all points $p, q \in \mathscr{M}'$. A *Cauchy surface* V' is a spacelike hypersurface such that every non-spacelike curve intersects this surface exactly once. A partial Cauchy surface is a surface that satisfies the weaker condition that each non-spacelike curve intersects the surface at most once[2]. Note V' for a manifold with boundary

[1]The timelike future (causal future) of a set S relative to U, $I^+(S,U)$ ($J^+(S,U)$), is the set of all points that can be reached from S by a future directed timelike curve (causal curve) in U. The interchange of the past with future in the previous definition yields $I^-(S,U)$ ($J^-(S,U)$).

[2]These definitions are the usual extensions of those for manifolds without boundary used in proofs of topological censorship for locally asymptotically anti-de Sitter spacetimes.

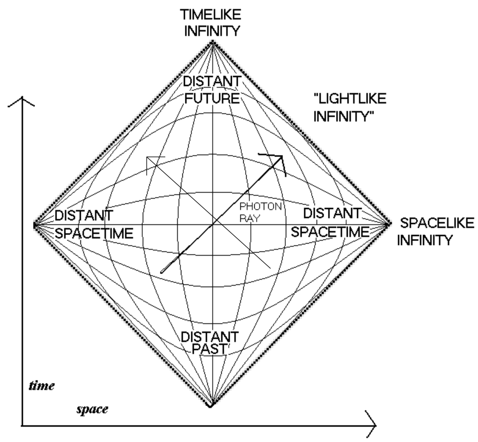

Figure 2.1: Conformal diagram of an infinite Minkowski universe. Courtesy of Wikimedia Commons.

will have a boundary on $\mathscr{I}$.

The *domain of outer communications* (DOC) is the portion of a spacetime $\mathscr{M}$ which is exterior to event horizons. Precisely $D = I^-(\mathscr{I}_0^+) \cap I^+(\mathscr{I}_0^-)$ for a connected component $\mathscr{I}_0$ for an AF spacetime and $D = I^-(\mathscr{I}_0) \cap I^+(\mathscr{I}_0)$ for an ALADS spacetime is the subset of $\mathscr{M}$ that is in causal contact with $\mathscr{I}$. Note that D is the interior of an $(n+1)$-dimensional $D' = D \cup \mathscr{I}$ and that D' is itself a globally hyperbolic spacetime with boundary.

A second-order linear partial differential equation (PDE) for a field ϕ is said to be hyperbolic if and only if it can be expressed in the form $\nabla^2\phi + A^a\nabla_a\phi + B\phi + C = 0$, where A_a is a smooth vector field [14, Eq. 10.1.19]. A second order quasi-linear[3] PDE for a field ϕ is said to be hyperbolic if in can be expressed in the form $g^{ab}(x, \phi, \nabla_c\phi)\nabla_a\nabla_b\phi = F(x, \phi, \nabla_a\phi)$ [14, Eq. 10.1.21]. Here x represents the coordinates on the manifold and ϕ may consist of multiple fields.

A concept important for numerical evolution is *strong hyperbolicity*, for it limits the propagation speed of high-frequency perturbations. See [15]. The precise definition of the term is discussed in [16], the important part for our purposes is that the ADM formulation of the initial value problem can be cast into a strongly hyperbolic form.

Causality and hyperbolicity are essential if one wants to reconstruct the full spacetime from a given initial surface. Some of the spacetimes considered

[3] Linear in second derivatives.

in this thesis, most notably the Gödel one, contain CTCs, and so are neither globally hyperbolic, nor causal.

2.2 Energy Conditions

To be physically relevant, a matter source must have a stress-energy tensor that obeys a number of restrictions. Since most of this thesis revolves around violations of energy conditions, we review these conditions on the stress-energy tensor and the implications of their violations in some detail. Further discussion can be found in [2, §4.3].

2.2.1 Weak Energy Condition

In most cases observed classical physical matter has non-negative energy density. Energy density is observer-dependent, and for a timelike test observer with a velocity u^a (normalized to $g_{ab}u^a u^b = -1$) moving through the matter source with the stress-energy tensor T_{ab} it is

$$\rho = T_{ab}u^a u^b. \tag{2.1}$$

The *weak energy condition* (WEC) can then be formulated as

$$\rho \geq 0 \tag{2.2}$$

for all timelike observers. It is usually assumed that the vector u^a is future-pointing, but since ρ is quadratic in u^a, this restriction is not necessary. If the Einstein equation (without the cosmological constant written separately, $G_{ab} = T_{ab}$) is presumed to hold, then the energy density can be inferred from the spacetime curvature, $\rho = G_{ab}u^a u^b$.

Often there exist an orthonormal frame at each point where the stress energy tensor can be written in a canonical form $T^{ab} = \mathrm{diag}\{\rho_0, P_1, \ldots, P_{n-1}\}$, where ρ_0 is the energy density and $(P_1, \ldots, P_{n-1})$ are the components of the spatial momentum "vector" in this frame. This type of the stress energy tensor is called Type I in [2]. In this case, while the magnitude of the observed energy density varies between observers, its sign remains non-negative as long as $\rho_0 \geq 0$ and $\rho_0 \geq -P_i$ for all i.

2.2.2 Null Energy Condition

The *null energy condition* (NEC) is a variation of the WEC, but applied to null, rather than timelike, vectors:

$$T_{ab}k^a k^b \geq 0, \tag{2.3}$$

where $g_{ab}k^a k^b = 0$. For a diagonal stress-energy tensor the NEC holds as long as $\rho_0 \geq -P_i$ for all i. This condition is weaker than the weak energy condition (for example, negative energy density is allowed, as long as the pressure is strong enough to compensate). The NEC is necessary for the proofs of the singularity theorems, because it ensures focusing of null geodesics. Indeed, according to the Raychaudhuri equation [2] the null geodesic expansion θ is

$$k^a \nabla_a \theta = -R_{ab}k^a k^b + \omega^{ab}\omega_{ab} - \sigma^{ab}\sigma_{ab} - \frac{1}{3}\theta^2, \tag{2.4}$$

and, given that vorticity of null geodesics ω_{ab} is zero, it becomes (using the fact that for null vectors $R_{ab}k^a k^b = T_{ab}k^a k^b$, whether or not the cosmological constant is present)

$$k^a \nabla_a \theta = -T_{ab}k^a k^b - \sigma^{ab}\sigma_{ab} - \frac{1}{3}\theta^2, \tag{2.5}$$

which is manifestly negative. So in this case the NEC is equivalent to the null convergence condition.

2.2.3 Average Null Energy Condition

Both weak and null energy conditions can be violated in isolated regions of space. The well known Casimir effect [17], an attractive force between two parallel plates in vacuum, is one such example. Other examples include certain kinds of electromagnetic radiation states [18] and in certain entangled states in quantum mechanics [19]. In all these examples the localized regions of negative energy are surrounded by the regions of positive energy. If there is enough of it to "outweigh" the negative energy, then at least some of the results that assume NEC can still go through. To that end one can formulate the *average null energy condition* (ANEC), which holds when the average energy density measured along all complete null geodesics is non-negative: Add the mathematical formulation of ANEC:

$$\int_{-\infty}^{\infty} R_{ab}k^a k^b(\eta', \eta')\,\mathrm{d}s \geq 0 \tag{2.6}$$

along any complete null geodesic $s \mapsto \eta(s)$ with tangent k^a.

This energy condition is sufficient to rule out a number of exotic phenomena in GR. In particular, the ANEC restriction on the stress-energy tensor is sufficient to prove the topological censorship theorems [20]. It is, however, possible to construct examples of spacetimes where ANEC does not hold [21].

2.2.4 Dominant Energy Condition

Another reasonable restriction on physical matter is that it does not travel faster than light or *non-tachyonic*. Mathematically this can be expressed as follows: the momentum density p^a of the matter with the stress energy tensor T_{ab} seen by a future-pointing timelike observer with velocity u^a is

$$p^a = T^a_b u^b. \tag{2.7}$$

This vector is non-spacelike when

$$p^a p_b \leq 0. \tag{2.8}$$

If this condition holds for all physical (i.e., future-pointing and non-spacelike) observers, and the WEC also holds, then the *dominant energy condition* (DEC) is said to hold. The DEC is necessary to prove the existence of the Cauchy problem in general relativity [2], as it establishes a lightcone and prevents matter from coming from infinity into the future domain of dependence. If this condition is not imposed, matter may essentially pop in and out of existence at will.

Specifically, the conservation theorem [2, §4.3] states that

Theorem. *If the energy-momentum tensor obeys the dominant energy condition and is zero on a timelike boundary of a compact region, and on the initial surface of the region, then it is zero everywhere in the region.*

The DEC is only a sufficient condition for the Cauchy development to exist. It may still be possible to have a Cauchy development in some cases where the DEC is violated, however each specific case has to be analyzed separately.

2.2.5 Timelike Convergence and the Strong Energy Condition

As discussed in section 2.2.2, the NEC implies the null convergence condition. For a similar condition to hold for the timelike vectors, the following restriction on the stress energy tensor is required:

$$R_{ab}u^a u^b = T_{ab}u^a u^b + \frac{1}{2}T - \Lambda \geq 0. \tag{2.9}$$

This timelike convergence condition is known as the *strong energy condition* SEC when $\Lambda = 0$. When this condition holds, gravity appears as an attractive, rather than a repulsive force, as the geodesic expansion is necessarily negative for a hypersurface-orthogonal geodesic congruence.

Out of all the energy conditions described here, the NEC and DEC are of special interest in this thesis. In particular, the NEC is a necessary condition for the topological rigidity theorem, and the DEC is the one most commonly violated by Killing reductions.

2.3 Asymptotic Flatness

An asymptotically flat spacetime is a representation of an isolated system in an "otherwise flat" spacetime (see [14]). There are different ways to define an asymptotically flat spacetime. The main idea is that the curvature (and the matter density) should fall off quickly enough to appear in the leading order as a "point mass" for an observer "at infinity". One should be able to take a weak field (Newtonian) limit and obtain the total mass, total charge and total angular momentum of such a system, given by a suitable generalization of the Gauss law. A covariant way to do this was first introduced by Penrose [22], where a physical spacetime $\mathscr{M}$ is conformally embedded into an unphysical spacetime $\tilde{\mathscr{M}}$ with a boundary $\partial\mathscr{M} = \tilde{\mathscr{M}} \setminus \mathscr{M}$. See [2, 14] for the detailed discussion and illuminating diagrams. The boundary describes the infinity of the original spacetime, and one can then talk about observers in the neighborhood of infinity. The properties of the spatial infinity in four dimensions were investigated in detail by Ashtekar and Hansen [23, 24].

It is worth noting the well known fact that the four-dimensional case is rather special as far as timelike geodesics in a spacetime produced by a point mass (described by a generalization of a Schwarzschild metric to an arbitrary number of dimensions) are concerned. Only in four dimensions do bound and stable geodesics exist. In three dimensions the spacetime is locally flat, and the total deficit angle is below[4] 2π, so the timelike geodesics necessarily start and end at infinity, except for the radial ones, which start or end in the singularity. In five or more dimensions all non-circular geodesics end either at infinity or in the singularity, and the circular geodesics are unstable against small perturbations. Thus, higher dimensional asymptotically flat spacetimes are not associated with gravitationally bound orbits of compact bodies in vacuum, and, in that sense, are not as "natural" as the four-dimensional asymptotically flat spacetime. In contrast, when a negative cosmological constant is introduced, such as in the Schwarzschild-AdS

[4] Otherwise the spacetime is closed [25].

spacetime, stable bound orbits exist in any dimension. See e.g., [26] for the details.

We start with a coordinate-independent definition of AF and ALADS spacetimes.

A spacetime $(\mathscr{M},)$ is *asymptotically flat* (AF) if it can be conformally included into a spacetime-with-boundary $\mathscr{M}' = \mathscr{M} \cup \mathscr{I}$, with metric g'_{ab}, such that

(a) for some conformal factor $\Omega \in C^1(\mathscr{M}')$, $g'_{ab} = \Omega^2 g_{ab}$ on $\mathscr{M}$ and Ω vanishes on $\mathscr{I}$ but has null gradient which is non-vanishing point-wise on $\mathscr{I}$, and

(b) the boundary $\partial\mathscr{M}' = \mathscr{M}' \setminus \mathscr{M} = \mathscr{I}$ is a disjoint union of past and future parts $\mathscr{I}^+ \cup \mathscr{I}^-$, each having topology $S^2 \times \mathbb{R}$ with $\mathbb{R}$'s complete null generators.

Strictly speaking, this only defines null asymptotic flatness, as the spatial asymptotic flatness requires extra conditions to be imposed [14]. Since null asymptotic flatness is all that will be required, we do not make this distinction.

A spacetime $(\mathscr{M}, g_{ab})$ is *asymptotically locally anti-de Sitter* (ALADS) if it can be conformally included into a spacetime-with-boundary $\mathscr{M}' = \mathscr{M} \cup \mathscr{I}$, with metric g'_{ab}, such that $\partial\mathscr{M}' = \mathscr{I}$ is timelike (i.e., is a Lorentzian hypersurface in the induced metric) and $\mathscr{M} = \mathscr{M}' \setminus \mathscr{I}$.

The conformal factor $\Omega \in C^1(\mathscr{M}')$ satisfies

(a) $\Omega > 0$ and $g'_{ab} = \Omega^2 g_{ab}$ on $\mathscr{M}$, and

(b) $\Omega = 0$ and $\mathrm{d}\Omega \neq 0$ point-wise on $\mathscr{I}$. $\mathscr{I}$ may in general have multiple components.

Another way to define the asymptotically flat $(3+1)$-dimensional spacetime is through the initial value formulation [14]: Fix a positive-definite 3-metric h_{ab} and extrinsic curvature tensor K_{ab} on the initial surface. The spacetime can only be asymptotically flat if the metric approaches the flat metric fast enough (as $O(1/r)$) and the extrinsic curvature falls off sufficiently quickly (as $O(1/r^2)$) as $r \to \infty$. More precisely the asymptotic flatness conditions can be stated as follows [27]: an initial data set $(\Sigma^3, h_{ab}, K_{ab})$ is asymptotically flat if there exists a diffeomorphism (a differentiable map with a differentiable inverse) F from $\mathbb{R}^3$ minus a ball B_1 to the initial data surface Σ^3 minus a ball B_2 such that

$$
\begin{aligned}
F_* h_{ab} &= \left(1 + \frac{M}{2r}\right)^4 \delta_{ab} + O\left(\frac{1}{r^2}\right) \\
F_* K_{ab} &= O\left(\frac{1}{r^2}\right),
\end{aligned}
\tag{2.10}
$$

where δ_{ab} is the flat Riemannian metric and $r^2 = x_a x^a$ in asymptotically Cartesian coordinates x^a. The pullback F_* will be omitted for notational simplicity.

We note that the conformally flat metric $(1 + m/2r)^4 \delta_{ab}$ is spherically-symmetric — it is in fact the spatial part of the Schwarzschild metric (discussed in section 3.1) written in isotropic coordinates and so possesses a $U(1)$ Killing vector ξ^S. An equivalent definition exists that uses a perturbation on the flat background [27]: Let $\tilde{h}_{ab} = h_{ab} - \delta_{ab}$, where h_{ab} is again the metric of an achronal initial data surface Σ^3, δ_{ab} is the flat Euclidean 3-metric, and $\tilde{h}_{ab}$ satisfies the following fall-off conditions: $\tilde{h}_{ab} = O(1/r)$, $\nabla_c \tilde{h}_{ab} = O(1/r^2)$ and $\nabla_c \nabla_d \tilde{h}_{ab} = O(1/r^3)$. The derivatives are taken with respect to the "background" flat metric δ_{ab}.

The definition (2.10) can be generalized to an arbitrary number of spacetime dimensions $D > 3$ as

$$
h_{ab} = f(r)\delta_{ab} + O\left(\frac{1}{r^{D-2}}\right), \tag{2.11}
$$

with the conformal factor

$$
f(r) = \left(1 + \frac{16\pi M}{(D-2) S_{D-2} r^{D-3}}\right)^{4/(D-3)}, \tag{2.12}
$$

where

$$
S_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}
$$

is the surface area of an n-dimensional sphere of unit radius. Three dimensions is a special case: there the spacetime is always flat outside of matter sources, and the concept of asymptotic flatness loses meaning.

We this definition in chapter 4 to prove that an asymptotically flat dimensional reduction of an asymptotically flat spacetime results in a pathological induced stress-energy tensor.

2.4 Total Energy and Angular Momentum

Owing to being asymptotically Minkowski, the asymptotically flat spacetimes can be ascribed a finite total energy and angular momentum. These quantities are calculated as Komar integrals of the corresponding asymptotic symmetry: time translation and rotation [14]. Specifically, total energy E of an AF spacetime can be calculated as

$$E = -\frac{1}{8\pi} \lim_{S \to \mathscr{I}_0} \int_S *\nabla_a t_b, \tag{2.13}$$

where S is a $(D-2)$-dimensional sphere on the initial surface Σ, t^a is the asymptotically Killing time translation vector in $\mathscr{M}$ satisfying $\nabla^a t^a = 0$ in the neighborhood of $\mathscr{I}^+$ and $*\nabla_a t_b$ is the $(D-2)$-form dual to $\nabla_a t_b$ (it can be written using the Levi-Civita tensor as $*\nabla_a t_b = \varepsilon_{ab..cd} \nabla^c t^d$).

In a coordinate form this becomes

$$E = -\frac{1}{8\pi} \lim_{S \to \mathscr{I}_0} \int_S \frac{t^a n^b - t^b n^a}{|t^c t_c|} \nabla_a t_b \mathrm{d}A, \tag{2.14}$$

where n^a is a normal to the surface and $\mathrm{d}A$ denotes integration over the surface.

If an initial slice is given, the total energy can be expressed using the ADM mass. The ADM mass M of an asymptotically flat spacetime can be calculated from the spatial metric h_{ab} of an asymptotically flat initial data set [28] (Σ, h_{ab}, K_{ab}) as a limit at infinity of the surface term in the ADM Hamiltonian (generalized from [14, Eq. 11.2.4] for an arbitrary $D \geq 4$):

$$M_{\mathrm{ADM}} = \frac{1}{16\pi(D-3)} \lim_{r \to \infty} \oint \mathrm{d}A^a \left(\partial^b h_{ab} - \partial_a h^b_b \right), \tag{2.15}$$

where the integral is taken over a sphere of constant r and h_{ab} is expressed in asymptotically Cartesian coordinates.

It is worth noting that total energy of the spacetime as a whole is not the sum of energies of objects inside it. For example, for a spacetime consisting of a blackhole surrounded by a perfect fluid ring analyzed in [29], the "partial" Komar mass for each of the two objects is negative, even though the total energy of the spacetime is positive, as required by the *positive energy theorem* (described in section 4.2).

The total *angular momentum* of an AF spacetime is

$$J = \lim_{S \to \mathscr{I}_0} \int_S *\nabla_a \xi_b. \tag{2.16}$$

Here ξ_b is one of the asymptotic rotational Killing vectors.

2.5 Axisymmetry

The definitions of axisymmetry given in literature vary somewhat and are not always equivalent. The two issues are whether the axis of symmetry is included in the spacetime and if so, the dimensionality of the axis. We will here follow Carter [30], who makes a distinction between an axisymmetry and a "true" $U(1)$ symmetry.

Definition 1. *A non-singular spacetime* $(\mathscr{M}, g)$ *is said to have a cyclic symmetry (also known as* $U(1)$ *symmetry) if it admits an isometry with closed orbits, i.e., there is an effective action* $\xi : U(1) \times \mathscr{M} \to \mathscr{M}$ *of the one-parameter cyclic group* $U(1)$ *which preserves the metric, i.e., for every point* $\phi \in U(1)$, $\xi_* g = g$.

It is standard to use $0 \leq \phi \leq 2\pi$ as the parameter of the group. The tangent vector to the orbits of ξ, $\xi^a = (\partial/\partial\phi)^a$ is a Killing vector of the spacetime.

Definition 2. *An* n*-dimensional spacetime with cyclic symmetry is called* axisymmetric *if the set of fixed points of the isometry is a submanifold* $\mathscr{F}$ *of* $n-2$ *dimensions, properly embedded in* $\mathscr{M}$*.* $\mathscr{F}$ *is called the* axis of symmetry*.*

Definition 2 is required by some authors but not by others (see [30–34]), and those who do require the axis to be included in the spacetime do not always place the same set of restrictions on it. Except for the discussion in section 5.4, we will always deal with axisymmetry in this thesis.

An example of a three-dimensional spacetime with a cyclic symmetry that is not an axisymmetry according to our definition is a direct product of a two-dimensional flat torus T^2 with a timelike line. In this case the spacetime is not simply connected. An example of a cyclic symmetry that is not an axisymmetry on a simply connected four-dimensional spacetime with the topology $S^3 \times R$ is the Hopf fibration of S^3 (see e.g., [35]), which has no fixed points[5]. When we discuss the $U(1)$ symmetry we assume that it is not an axisymmetry, i.e., the action of the symmetry group on the manifold is free. In other words, all the fibers of the map $\pi : \mathscr{M}/U(1) \to \bar{\mathscr{M}}$ are homeomorphic to a circle.

The axis of symmetry $\mathscr{F}$ has several important properties listed below.

Property 1. *The norm of the Killing vector* ξ^a, $\lambda = g_{ab}\xi^a\xi^b$ *vanishes on the axis* $\mathscr{F}$*.*

[5] S^3, of course, has other cyclic symmetries that are axisymmetries.

Property 2. *(Elementary Flatness Condition) On the axis of symmetry*

$$\frac{\nabla_a\lambda\,\nabla^a\lambda}{4\lambda} \to 1. \tag{2.17}$$

Elementary flatness is equivalent to regularity on the axis. This is a generalization of the Minkowski case, where in the cylindrical chart the metric (outside the axis) can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-3}^2, \tag{2.18}$$

where $0 \leq \phi \leq 2\pi$. Here $\lambda = \rho^2$. For the proof of this condition, see e.g., [32].

Property 3. *$\mathscr{F}$ is an autoparallel (*totally geodesic*) submanifold of $\mathscr{M}$.*

That is, for any two vector fields X and Y on $\mathscr{F}$ $X^a\nabla_aY$ is tangent to $\mathscr{F}$. The proof can be found in many differential geometry texts. For a succinct version, see e.g., [36]. The consequence of this is

Property 4. *$\mathscr{F}$ has identically vanishing extrinsic curvature.*

This property is what ensures the existence of a non-singular extension (a doubling) of the reduced spacetime, as described in section 3.8.

Property 5. *$\mathscr{F}$ is* timelike*, i.e., there is a timelike curve through every point.*

See e.g., [32, Theorem 1] for the proof.

An interesting special case of axisymmetric spacetimes are those admitting another Killing vector ζ, tangent to the axis of symmetry. In this case the two Killing vectors commute, $[\xi, \zeta] = 0$. Carter [30] proved that such a Killing vector exists for any stationary spacetime (in which case ζ is everywhere timelike). Kerr blackhole metric is the best known example of an asymptotically flat stationary spacetime. If ζ is spacelike, we have a cylindrical symmetry [31]. The van Stockum cylinder, discussed in section 6.2, is one example of such a spacetime.

2.6 Dimensional Reduction

The reduction procedure we employ was pioneered in four spacetime dimensions by Geroch in [37], where it was used in his solution-generating technique. The Hamiltonian approach to this reduction is discussed in e.g.,

[38]. We first present it for a four-dimensional spacetime, and then generalize to any number of dimensions. The formalism here applies to any spacetime with a Killing vector, the axisymmetric case is discussed at the end of the section. A related type of reduction, the conformal one, is discussed in section 3.9. The advantage of the reduction employed here is that it preserves the fall-off of the metric, which is essential for keeping the reduced spacetime asymptotically flat.

Consider a four-dimensional spacetime $\mathscr{M}$ with metric g_{ab}. We want to identify points along the orbits of an axisymmetric spacelike Killing field ξ, which has the norm $\lambda = \xi^a \xi_a$. The metric of the reduced three-dimensional spacetime is

$$\bar{g}_{ab} = g_{ab} - \frac{\xi_a \xi_b}{\lambda}. \tag{2.19}$$

It follows from the Gauss-Codazzi relations [2] that

$$\begin{aligned}
\bar{R}_{ab} =& \frac{1}{2\lambda^2}\left(\omega_a\omega_b - \bar{g}_{ab}\omega^c\omega_c\right) \\
&+ \frac{1}{2\lambda}\bar{\nabla}_a\bar{\nabla}_b\lambda - \frac{1}{4\lambda^2}\bar{\nabla}_a\lambda\bar{\nabla}_b\lambda + \bar{g}_a^c\bar{g}_b^d R_{cd}, \\
\bar{\nabla}^2\lambda =& \frac{1}{2\lambda}\bar{\nabla}^c\lambda\bar{\nabla}_c\lambda - \frac{1}{\lambda}\omega^c\omega_c - 2R_{cd}\xi^c\xi^d, \\
\bar{\nabla}^a\omega_a =& \frac{3}{2\lambda}\omega_c\bar{\nabla}^c\lambda, \\
\bar{\nabla}_{[a}\omega_{b]} =& -\epsilon_{abcd}\xi^c R_e^d \xi^e.
\end{aligned} \tag{2.20}$$

Here $R_{ab} = T_{ab} - \frac{1}{2}T g_{ab}$ is the full spacetime Ricci tensor, $\bar{R}_{ab}$ its counterpart in the reduced spacetime, $\omega_a = \epsilon_{abcd}\xi^b\nabla^c\xi^d$ is the twist of ξ and $\bar{\nabla}$ is the covariant derivative on the reduced spacetime.

The reduction from five to four dimensions is described in [39]. The main difference from the four-dimensional case is that the twist $\omega_{ab} = \epsilon_{abcde}\xi^b\nabla^d\xi^e$ is conveniently described by a two-form instead of a one form. In higher dimensions calculating twist as a dual to $\xi_a\nabla_b\xi_c$ does not reduce the number of indices, so it makes sense to use the projection on the reduced spacetime $\omega_{ab} = g_a{}^c g_b{}^d \nabla_c \xi_d$ instead of any other form of twist.

The following describes the reduction procedure in D spacetime dimensions. The capital indices refer to the full spacetime and the lower case ones to the reduced spacetime.

Let $(\mathscr{M}, g_{AB})$ be a D-dimensional spacetime admitting a Killing vector ξ^A with $\nabla_{(A}\xi_{B)} = 0$. In the following we use ξ as a fiber of a bundle, as a vector field, and as a parameter of a curve describing orbits of ξ^A. The meaning, when not explicitly given, should be unambiguous from the context. One can consider $\mathscr{M}$ a lift of a $(D-1)$-dimensional manifold $\bar{\mathscr{M}}$ with ξ as the fiber of a fiber bundle: $\xi : \mathscr{M} \to \bar{\mathscr{M}}$. This is possible as long as all the fibers have the same topology. Since orbits of ξ are one-dimensional, there are only two choices. If the orbits are open, then the fiber bundle is a principal $\mathbb{R}$-bundle, and if they are closed, then it is a principal $U(1)$-bundle (the rotation axis, if any, has to be treated separately). The base space $\bar{\mathscr{M}}$ is then a quotient space $\bar{\mathscr{M}} = \mathscr{M}/\xi$. The map from a tensor field $T^{C..D}_{A..B}$ on $\mathscr{M}$ to a tensor field $\bar{T}^{c..d}_{a..b}$ on $\bar{\mathscr{M}}$ is an isomorphism if the tensor in the full space is horizontal and does not explicitly depend on the Lie group parameter ξ:

$$\xi \cdot T^{C..D}_{A..B} = 0 \tag{2.21}$$

$$\mathscr{L}_\xi T^{C..D}_{A..B} = 0, \tag{2.22}$$

where the top equation means vanishing of all possible dot products. In particular, the projection

$$\bar{g}_{ab} \cong \bar{g}_{AB} = g_{AB} - \frac{\xi_A \xi_B}{\lambda},$$

where $\lambda = \xi^C \xi_C$ is isomorphic to a tensor $\bar{g}_{ab}$ in the base space, and the symbol $\cong$ denotes the isomorphism between the tensors in base and total spaces. Similarly,

$$\bar{g}^b_a \cong \bar{g}^B_A = g^B_A - \frac{\xi_A \xi^B}{\lambda},$$

and

$$\bar{g}^{ab} \cong \bar{g}^{AB} = g^{AB} - \frac{\xi^A \xi^B}{\lambda}.$$

In the following we will use '=' instead of '$\cong$', as there is no reason to make a distinction between the two quantities. For an arbitrary tensor $T^{C..D}_{A..B}$ the map is given explicitly as

$$\bar{T}^{c..d}_{a..b} = \bar{g}^A_a \cdots \bar{g}^B_b \bar{g}^c_C \cdots \bar{g}^d_D T^{C..D}_{A..B}. \tag{2.23}$$

This also applies to the derivative of any ξ-independent horizontal tensor:

$$\bar{\nabla}_e \bar{T}^{c..d}_{a..b} = \bar{g}^E_e \bar{g}^A_a \cdots \bar{g}^B_b \bar{g}^c_C \cdots \bar{g}^d_D \nabla_E T^{C..D}_{A..B}. \tag{2.24}$$

The next step is to project the Einstein equation $G_{AB} = T_{AB}$ on $\bar{\mathscr{M}}$. A symmetric tensor U_{AB} can written in terms of the horizontal and vertical tensors as

$$U_{ab} = \bar{g}^A_a \bar{g}^B_b U_{AB} + \frac{2}{\lambda} \xi_{(a} \bar{g}^A_{b)} U_{AB} \xi^B + \frac{1}{\lambda^2} \xi^A \xi^B U_{AB} \xi_a \xi_b. \tag{2.25}$$

Thus, just like in the case of reduction from four to three dimensions, the Einstein equation in the full space is equivalent to three equations in the reduced spacetime, a tensor, a vector and a scalar one. The tensor equation involves the curvature of the reduced spacetime and so can be treated as the Einstein equation in the reduced spacetime with extra matter terms. The other two are in essence the equations of state for the scalar and vector induced fields. There are different ways to project the Einstein equation [1, 37]. For example, one can use the difference of two connections, one for the original and one for the reduced metric [14]:

$$\bar{\nabla}_a \bar{\nabla}_b f = \nabla_a \nabla_b f - C^c_{ab} \bar{\nabla}_c f, \tag{2.26}$$

where f is a scalar in both the full and the reduced spacetimes and

$$C^c_{ab} = \frac{1}{2} \bar{g}^{cd} \left(\nabla_a \bar{g}_{bd} + \nabla_b \bar{g}_{ad} - \nabla_d \bar{g}_{ab} \right). \tag{2.27}$$

Alternatively, one can express the Riemann tensor of the reduced spacetime directly using that for a vector field k^a the identity $\nabla_{[a} \nabla_{b]} k_c = R_{dabc} k^d$ holds in both the original and the reduced spacetime. We then get, after some algebra,

$$\bar{R}_{abcd} = g^A_{[a} g^B_{b]} g^C_{[c} g^D_{d]} \left(R_{ABCD} + \frac{2}{\lambda} \nabla_A \xi_B \nabla_C \xi_D + \frac{2}{\lambda} \nabla_A \xi_C \nabla_B \xi_D \right). \tag{2.28}$$

It is convenient to express the quantities in the reduced spacetime using the Killing vector ξ^a itself and its two lower-dimensional characteristics, the norm λ and the twist 2-form[6]:

$$\omega_{ab} = \bar{g}^A_a \bar{g}^B_b \nabla_A \xi_B = \nabla_a \xi_b + \frac{1}{\lambda} \xi_{[a} \nabla_{b]} \lambda, \tag{2.29}$$

and (2.28) becomes

$$\bar{R}_{abcd} = g^A_{[a} g^B_{b]} g^C_{[c} g^D_{d]} R_{ABCD} + \frac{2}{\lambda} \left(\omega_{ab} \omega_{cd} + \omega_{ac} \omega_{bd} \right). \tag{2.30}$$

From this expression it is clear that in the hypersurface-orthogonal case, where the twist vanishes, the Riemann tensor in the reduced spacetime is simply the image of the Riemann tensor in the full spacetime.

The connection coefficients are:

$$C^c_{ab} = -\frac{2}{\lambda} \xi_{(a} \omega_{b)}{}^c + \frac{1}{2\lambda^2} \xi_a \xi_b \bar{\nabla}^c \lambda, \tag{2.31}$$

[6] Also known as the vorticity tensor if ξ^a is an arbitrary timelike field, see e.g., [2, §4.12].

and the Ricci tensor in the reduced spacetime $\bar{R}_{ab}$ is:

$$\bar{R}_{ab} = \bar{g}_a^A \bar{g}_b^B R_{AB} + \frac{2}{\lambda}\omega_{ac}\omega_b{}^c + \frac{1}{2\lambda}\bar{\nabla}_a\bar{\nabla}_b\lambda - \frac{1}{4\lambda^2}\bar{\nabla}_a\lambda\bar{\nabla}_b\lambda. \tag{2.32}$$

The induced curvature tensors remain finite in the dimensional reduction with respect to a Killing vector described in this section as long as the norm of the Killing vector is non-zero. However, the latter is precisely the case on the axis of symmetry, so the axis requires a special consideration. In particular, the expression (2.28) must remain bounded in the neighborhood of the axis. This follows from the fact that for a regular spacetime the projector $\bar{g}_a^b$ is bounded and the twist ω_{ab} vanishes on the axis. See also section 3.8.

Chapter 3

Examples of Dimensional Reduction

In this chapter we dimensionally reduce a few well-known spacetimes to observe the arising pathologies in a very concrete way. While the examples presented here are quite elementary, we have not been able to find a discussion of energy conditions in dimensionally reduced spacetimes in any previously published work.

We begin by applying dimensional reduction to four-dimensional asymptotically flat spacetimes. We find that, although the dimensional reduction produces a three-dimensional spacetime geometry that is asymptotically flat, it now violates all energy conditions. In addition, the resulting spacetime now exhibits a boundary at the axis of symmetry, though it can be removed by doubling.

We then consider dimensional reduction of higher dimensional asymptotically flat spacetimes and of spacetimes with cosmological constant such as (A)dS spacetime. Next we discuss removal of the boundary present in certain of these reductions by doubling. Finally we discuss an alternate reduction, *the conformal reduction.* This reduction satisfies the DEC but exhibits two pathologies:

1. it induces a naked timeline singularity in axisymmetric reductions and
2. the reduced spacetime is no longer asymptotically flat.

These pathologies limit the usefulness of this dimensional reduction for the study of axisymmetric spacetimes.

3.1 Schwarzschild Blackhole

A simple example of dimensional reduction of a non-flat axisymmetric spacetime with respect to its rotational Killing vector is that of a four-dimensional Schwarzschild blackhole. Despite being probably the simplest non-trivial asymptotically flat case, this example shows all the essential properties of an axisymmetric dimensional reduction preserving asymptotic flatness (the definitions of asymptotic flatness are discussed in section 2.3). Specifically, the pathologies we encounter in this case are the same ones seen in general.

The Schwarzschild spacetime is a vacuum spacetime, so all energy conditions (section 2.2) are trivially satisfied. The Schwarzschild singularity is shrouded by an event horizon. However, the dimensional reduction under consideration leads to a rather pathological spacetime. Specifically, the stress-energy tensor induced in the reduced spacetime by the reduction is seen as having negative energy density by all timelike and null observers everywhere. Additionally, the momentum flow seen by timelike observers is tachyonic for timelike observers with a large enough angular velocity. Thus the null, the weak and both parts of the dominant energy conditions are violated. The reduced spacetime has a boundary, but the doubling procedure (section 3.8) restores the asymptotic flatness of the spacetime, so the global quantities such as the ADM mass can be calculated. The details are described in this section.

The four-dimensional Schwarzschild metric in the Schwarzschild coordinates (t, r, θ, ϕ) is

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{3.1}$$

This coordinate chart breaks down on the event horizon, $r = 2M$, and there are other charts, such as the Kruskal-Szekeres one ([2], section 5.5), which cover all of the spacetime, all the way to the singularity at $r = 0$, but, given that we are only interested in the asymptotic properties of the reduction, the Schwarzschild chart is adequate. It covers the DOC, the part of the spacetime accessible to a remote observer. A spatial slice of the horizon is topologically a two-sphere.

This manifestly axisymmetric spacetime can be reduced with respect to the Killing vector ϕ. The metric of the reduced spacetime $\bar{g}_{ab} = g_{ab} - r^{-2}\phi_a\phi_b$ is

$$\mathrm{d}\bar{s}^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\mathrm{d}\theta^2, \tag{3.2}$$

where $0 \leq \theta \leq \pi$. While this is simply the metric (3.1) with the last term "chopped off" (because the Killing vector ϕ is hypersurface-orthogonal and the coordinate system is adapted to it), if we treat it as a metric of a three-dimensional spacetime, it will no longer correspond to a vacuum solution. The induced stress-energy tensor is

$$\bar{G}_{ab} = \frac{M}{r^3}\left(-\frac{t_a t_b}{(t_c t^c)^2} + \frac{r_a r_b}{(r_c r^c)^2} - 2\frac{\theta_a\theta_b}{(\theta_c\theta^c)^2}\right), \tag{3.3}$$

where t^a, r^a, θ^a are the coordinate vectors. In an orthonormal basis locally coinciding with the coordinate one this expression can be written as

$$\bar{G}_{\hat{a}\hat{b}} = \frac{M}{r^3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} . \tag{3.4}$$

This stress-energy tensor can also be modeled as an anisotropic fluid with negative energy density $\rho = -M/r^3$, equal and opposite positive radial pressure $p_{\|} = -\rho$, and a large negative tangential pressure $p_{\perp} = 2\rho$. From the form of the reduced stress-energy tensor ($\rho < 0$) one can already see that the WEC, NEC and DEC are all violated. To see the violations in detail, we consider a general timelike observer with the velocity u^a. In an orthonormal frame it can be written as

$$u^a = \gamma \left(\hat{t}^a + \hat{r}^a v \cos\alpha + \hat{\theta}^a v \sin\alpha \right),$$

where v is the observer's instantaneous spatial velocity relative to a static observer t^a, α defines its direction of motion and $\gamma = 1/\sqrt{1-v^2}$.

The energy density seen by this observer is

$$\rho = \gamma \frac{M}{r^3} \left(-1 + v^2 \cos^2\alpha - 2v^2 \sin^2\alpha \right) = -\frac{M}{r^3} \left(1 + 3\gamma^2 v^2 \sin^2\alpha \right). \tag{3.5}$$

This expression is negative everywhere in the reduced spacetime for a positive mass Schwarzschild solution. Thus, the weak energy condition, $\rho \geq 0$, is maximally violated in this case. The null case can be obtained by taking a limit $v \to 1$ and rescaling, resulting in

$$\rho = -\frac{M}{r^3} \sin^2\alpha, \tag{3.6}$$

so the NEC is also violated.

Another issue with this spacetime is the range of θ and the topology of the horizon $r = 2M$ after reduction. Since the Schwarzschild coordinates break down at the horizon, we use the Kruskal-Szekeres chart, which remains regular on the horizon. The reduced metric in this chart is

$$\mathrm{d}\bar{s}^2 = \frac{16M^2}{r} e^{-\frac{r}{2M}} \left(-\mathrm{d}t'^2 + \mathrm{d}x'^2 \right) + r^2 \mathrm{d}\theta^2, \tag{3.7}$$

where

$$t'^2 - x'^2 = -(r - 2M) e^{\frac{r}{2M}}. \tag{3.8}$$

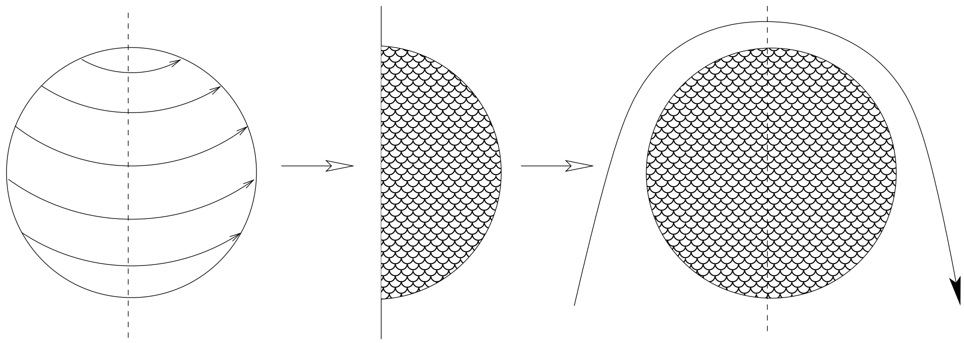

Figure 3.1: A schematic representation of axisymmetric dimensional reduction and doubling of the Schwarzschild blackhole. One spatial slice is shown. Arrows show the orbits of the Killing vector ϕ. Spherical horizon $r = 2M$ in three dimensions becomes a half-circle after reduction, connected to the boundary at $\sin\theta = 0$. After doubling, the spacetime has no boundary and there are non-contractible causal curves starting and ending on $\mathscr{I}$ in this spacetime. One such curve is shown.

The horizon $r = 2M$, $0 \le \theta \le \pi$ is manifestly regular and corresponds to a null half-cone $t' = x'$, no longer a closed surface. The set of points for which $\sin\theta = 0$ can be attached as a boundary to the reduced spacetime, thus closing the horizon. The DOC $r > 2M$ is simply connected. One can remove the boundary by identifying $\theta = 0 \cong \theta = \pi$, but this breaks the elementary flatness condition (2.17) and creates a conical singularity. A better way to remove the boundary is by "doubling" this spacetime. This corresponds to changing the range of θ to $[0, 2\pi]$ and identifying the end points. Then the only remaining singularity in this spacetime is the original one at $r = 0$. The horizon in the doubled spacetime is now $S^1 \times \mathbb{R}$, and the DOC is no longer simply connected. The consequences of this fact are discussed in chapter 5. The process of reduction and doubling is shown on Fig. 3.1. Interestingly enough, for an (unphysical) negative mass Schwarzschild solution ($M < 0$), the weak energy condition (though not DEC) is actually satisfied in the reduced spacetime, since the energy density is now positive. Of course, the latter solution has other pathologies in both four and three dimensions: it describes a naked singularity rather than a blackhole[7], and it is perturbatively unstable in four dimensions (see e.g., [40] and references therein). This example is symptomatic of how dimensional reduction can

[7]Though this can be ameliorated by matching to a "negative density star".

create, cure, and trade spacetime pathologies.

We now consider the second part of the dominant energy condition — subluminal matter flows seen by all timelike observers. We already know it is violated, since the tangential pressure exceeds the energy density in (3.4). To see which observers see superluminal matter flows we calculate the norm of observed momentum:

$$p^2 = -\frac{M^2\left(1 - 3\gamma^2 v^2 \sin^2\alpha\right)}{r^6}. \tag{3.9}$$

Thus fast-rotating observers (with the linear rotational velocity above $c/\sqrt{3}$ in the static frame) see superluminal matter flow in this spacetime, this time regardless of the sign of the blackhole mass M.

Superluminal matter flows present a challenge from an initial value point of view. In particular, as shown in [2, §4.3], DEC ensures that a subset of Cauchy surface has a future Cauchy development. See also [41, Appendix IV]. The light cone is necessary in order to bound from above any changes in the spatial metric during evolution and consequently to prove that the Cauchy problem is well posed, i.e., that the solution exists, is unique and continuously depends on the initial data. When the DEC is violated, the resulting apparent superluminal matter flows are often described as "non-local" when discussed in the context of solving a wave map (a non-linear generalization of the wave equation). A lack of bound on the propagation speed means that the standard techniques and results for hyperbolic systems do not apply and well-posedness must be examined on a case-by-case basis. For example, DeTurck [42] shows existence and uniqueness of solutions to the Einstein equation with invertible stress energy tensor (though not necessarily their continuous dependence on initial conditions).

To consider the ANEC (2.6), we first note that any geodesic without ϕ-dependence is mapped into a geodesic by the reduction (and when the Killing vector is hypersurface orthogonal any geodesic initially orthogonal to the Killing vector remains so). In fact, in a general case an image of a null geodesic commuting with the Killing vector used for the reduction is still a null geodesic in the reduced spacetime. Indeed, given any k^a such that

$$k^a k_a = 0, \qquad k^a \phi_a = 0, \qquad [k, \phi]^a = 0, \tag{3.10}$$

(that is, k^a is null and unaffected by the reduction), we have for its image,

$\bar{k}^a$, [8]

$$\bar{k}^a \bar{\nabla}_a \bar{k}^b = k^a \nabla_a k^b = 0. \tag{3.11}$$

Thus any null geodesic lying in a cross-section of the spacetime by $\phi = \mathsf{const}$ remains a null geodesic after the dimensional reduction. Any such geodesic must, of course intersect the axis of symmetry, which does not present an issue, as long as the reduced spacetime is doubled. If we integrate the everywhere negative quantity $T_{ab}k^a k^b$ along this geodesic we get a negative number. Thus ANEC is also violated in this example.

It is also interesting to consider whether an observer in the reduced Schwarzschild spacetime would see gravity as an attractive force. In the Newtonian approximation, positive energy density is associated with attraction and negative energy density is associated with repulsion. In the reduced Schwarzschild spacetime energy density is negative everywhere outside the central object. Since the stress-energy tensor induced by an axisymmetric reduction of the Schwarzschild metric is traceless, the strong energy condition is equivalent to the WEC (we always assume the Einstein equation to hold in the reduced spacetime). Since the WEC is violated, so is the SEC. On the other hand, since geodesics commuting with the Killing vector in the full spacetime are mapped into geodesics in the reduced spacetime, bound orbits are mapped into bound orbits, indicating attraction, rather than repulsion. This can be confirmed by calculating the geodesic expansion $\nabla_a \xi^a$ for a given geodesic ξ^a and its rate of change along the geodesic, which remains negative everywhere.

A convenient way to think about this is that the negative energy density manifests itself as a sort of a "vacuum polarization", screening the central object from distant observers, rather than being an independent source of repulsive gravity, in a way, say, the cosmological constant is. In other words, the central object still attracts, but its attractive force falls off faster than it would in a vacuum spacetime. This model, while valid for reductions to four or more dimensions, does not quite apply to reductions to three dimensions, because any three-dimensional vacuum spacetime is locally flat, the once parallel geodesics remain parallel, and locally there is neither attraction nor repulsion. Any interesting effects in three-dimensional gravity result from matter sources (including the cosmological constant) and/or from global identifications. Still, one can think of the $1/r$ fall-off of the "attractive

[8] The image of k^a, written $\bar{k}^a$, is defined by

$$\bar{k}^a - \frac{1}{\lambda} k^c \xi_c \xi^a = k^a.$$

force” induced by reduction from four to three dimensions as faster than the “no fall-off at all” for a vacuum spacetime.

Next we calculate the mass of the reduced Schwarzschild spacetime using (2.15). For the four-dimensional Schwarzschild metric in the Schwarzschild coordinates we get $E = M$, as expected. Note that the integrand is $-M/r^2$ and the integration measure is $r^2 \mathrm{d}\Omega^2$. This calculation can also be applied to the reduced (and doubled) Schwarzschild spacetime. The initial data set on the reduced spacetime is induced by the reduction. The integrand is unchanged, however the boundary of the resulting two-dimensional initial surface is S^1, not S^2. As a result, the integral over the one-sphere gives $-4\pi M/r$, and the limit

$$E = -\frac{1}{8\pi} \lim_{r\to\infty} \left(-\frac{4\pi M}{r} \right)$$

vanishes due to the fall-off of the integrand. Thus the total energy of the reduced Schwarzschild spacetime is zero. If the positive energy theorem (see section 4.2) can be applied to this (non-flat) spacetime, we can conclude that the dominant energy condition must be violated, which is confirmed by the calculations such as (3.5) and (3.9).

The main issues with the axisymmetric reduction of the Schwarzschild metric persist for other highly symmetric AF spacetimes, as well, as discussed below.

3.2 Charged Blackhole

In the previous section we considered the axisymmetric reduction of the Schwarzschild spacetime. This spacetime is asymptotically a good model of an isolated gravitational source of finite size, without any charge, spin or contribution from other non-gravitational interactions. The Schwarzschild spacetime is vacuum, so it is reasonable to inquire whether adding matter (with positive energy density) to the mix would prevent or alleviate the pathologies of the stress energy tensor induced by the axisymmetric reduction preserving asymptotic flatness. One of the simplest cases to analyze is the charged (Reissner-Nordstrom) blackhole, where the matter content is purely due to the static spherically symmetric electric field [14]. In the Schwarzschild coordinates the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \quad (3.12)$$

Here, Q is the electric charge (as seen from infinity, using the Gauss law). The reduced metric is simply

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2\mathrm{d}\theta^2. \tag{3.13}$$

The stress-energy tensor of the reduced spacetime in an orthonormal frame locally coinciding with the Schwarzschild coordinates is

$$G_{\hat{a}\hat{b}} = \begin{bmatrix} -\left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right) & 0 & 0 \\ 0 & \left(\frac{M}{r^3} - \frac{Q^2}{r^4}\right) & 0 \\ 0 & 0 & -\left(\frac{2M}{r^3} - \frac{3Q^2}{r^4}\right) \end{bmatrix}, \tag{3.14}$$

or, in terms of the coordinate vectors

$$\begin{aligned} \bar{G}_{ab} = &\frac{M}{r^3}\left(-\frac{t_a t_b}{(t_c t^c)^2} + \frac{r_a r_b}{(r_c r^c)^2} - 2\frac{\theta_a \theta_b}{(\theta_c \theta^c)^2}\right) \\ &+ \frac{Q}{r^4}\left(\frac{t_a t_b}{(t_c t^c)^2} - \frac{r_a r_b}{(r_c r^c)^2} + 3\frac{\theta_a \theta_b}{(\theta_c \theta^c)^2}\right). \end{aligned} \tag{3.15}$$

We see that the electric field adds a positive term to the overall energy density, however its contribution falls off as r^{-4}, faster than that of the pure reduced Schwarzschild, so adding electric charge to an isolated body does not cure the pathological stress energy tensor, at least not asymptotically. Even close to the horizon the situation is not much better. Indeed, the WEC is violated when $T_{00} < 0$, or $r > Q^2/M$. In the non-extremal case the metric has two horizons:

$$r_- = \frac{Q^2}{M + \sqrt{M^2 - Q^2}}, \qquad r_+ = \frac{Q^2}{M - \sqrt{M^2 - Q^2}} \tag{3.16}$$

and the positive energy density region lies wholly inside the outer horizon r_+. Thus the energy density in this spacetime is negative everywhere in the DOC ($r > r_+$). The second part of the dominant energy condition (subluminal matter flows) is violated for $r > 2Q^2/M$, which is barely outside the outer horizon even for near-extremal blackholes.

3.3 Spinning Blackhole

Another way an axisymmetric reduction could potentially induce a non-pathological stress-energy tensor is when the Killing vector field used for reduction has a non-zero twist (2.29). A standard example of such an axi-symmetric spacetime is the rotating (Kerr) blackhole, and its charged version, the Kerr-Newman blackhole. The Kerr metric can be written as

$$\mathrm{d}s^2 = -\frac{\Delta}{r_0^2}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 + \frac{\sin^2\theta}{r_0^2}\left(\left(\rho^2+a^2\right)\mathrm{d}\phi - a\mathrm{d}t\right)^2 + \frac{r_0^2}{\Delta}\mathrm{d}\rho^2 + r_0^2\mathrm{d}\theta^2, \tag{3.17}$$

where $\Delta = \rho^2 - 2M\rho + Q^2 + a^2$ and $r_0^2 = \rho^2 + a^2\cos^2\theta$ and M, Q, and a are respectively mass, electric charge and angular momentum.

$(\partial/\partial\phi)^a$ is a Killing vector with a non-zero norm outside the rotation axis $\sin\theta = 0$, so the metric can be dimensionally reduced with respect to it. The relevant calculations are straightforward but long, so they were generally performed with assistance of a computer algebra software.

The expressions for the energy density and the norm of the momentum density seen by a timelike observer on a worldline with the unit tangent vector (u^t, u^r, u^θ) can be expanded in the powers of $O(1/r)$, with the three leading terms being

$$\begin{aligned}\rho = &-\frac{M}{r^3}\left(1+3u^{\theta^2}\right) + \frac{Q^2}{r^4}\left(1+4u^{\theta^2}\right)\\ &-\frac{3Ma^2}{r^5}\left(\left(u^{r^2} - 7u^{\theta^2} - 2\right)\cos^2\theta + 6u^r u^\theta \sin\theta\cos\theta - u^{r^2} + u^{\theta^2}\right)\\ &+ O\left(\frac{1}{r^6}\right),\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\bar{p}^a\bar{p}_a = &-\frac{M^2}{r^6}\left(1-3u^{\theta^2}\right) + \frac{2Mq^2}{r^7}\left(-1+5u^{\theta^2}\right)\\ &+\frac{Q^4}{r^8}\left(1-8u^{\theta^2}\right)\\ &+\frac{6Ma^2}{r^8}\left(\left(u^{r^2} + 8u^{\theta^2} - 2\right)\cos^2\theta - 3u^r u^\theta\sin\theta\cos\theta - 6u^{r^2} - 2u^{\theta^2}\right)\\ &+O\left(\frac{1}{r^9}\right).\end{aligned} \tag{3.19}$$

The expansion shows that these quantities display the following property: the highest order contribution comes solely from the mass term, the next order comes solely from the charge parameter, and only the one after that depends on the angular momentum. Thus, for large enough values of r, the leading order contribution coincides with that of the reduced Schwarzschild metric, and so the energy conditions are still violated, at least asymptotically.

3.4 Axisymmetric Gravitational Radiation

All the spacetimes considered thus far in this section were either stationary or static. It is natural to ask whether non-stationary spacetimes also violate the energy conditions, at least asymptotically. An example of a non-stationary axisymmetric spacetime is axisymmetric gravitational radiation on a stationary background. While not an exact solution, it is an asymptotic approximation of radiation from axisymmetric gravitational wave emitters. In the original spacetime gravitational radiation does not contribute to the stress energy tensor, but, just like in the Schwarzschild case, it shows up as an induced stress-energy tensor in the reduced spacetime. Given that radiation falls off slower than a static field, one might conceivably expect that it becomes a dominant contribution to the stress-energy tensor asymptotically and could thus "cure" the pathology induced by the axisymmetric reduction. In this section we investigate the properties of the induced stress-energy tensor in the simplest case of gravitational radiation on a flat (Minkowski) background.

The radiation in the far zone and in the slow motion approximation from the source with the quadrupole moment Q_{ab} in the full four-dimensional spacetime can be described by the (perturbation) metric and is, up to a constant factor,

$$h_{ab} = \frac{1}{r}\frac{\mathrm{d}^2 Q_{ab}}{\mathrm{d}t^2}, \tag{3.20}$$

where r is the distance from the source (see e.g., [14, Eq. 4.4.49]).

The radiation will be axisymmetric when $\mathrm{d}^2 Q_{ab}/\mathrm{d}t^2$ is axisymmetric. Since we are working in the linear approximation, we can expand this term into Fourier modes characterized by their frequency $\omega \neq 0$. For a given frequency ω we can write the gravitational field h_{ab} in the transverse-traceless gauge as the real part of

$$h_{ab}(\omega) = f(\theta,\omega)\frac{e^{i\omega(r-t)}}{r}\left({e^\theta}_a {e^\theta}_b - {e^\phi}_a {e^\phi}_b\right), \tag{3.21}$$

where ${e^\theta}_a$ and ${e^\phi}_a$ are two of the unit vectors of the orthonormal basis locally corresponding to the spherical coordinates (t, r, θ, ϕ), ω is the oscillation frequency and $f(\theta, \omega)$ depends on the details of the source of the gravitational wave. Because of the axisymmetry, only one polarization mode is present (see e.g., [43]). We note that the expression (3.21) is only valid in a local coordinate chart, which is sufficient for our purposes. Because this metric is linearized, its Ricci tensor does not vanish identically. It is, however, traceless, as radiation ought to be. Next we perform the axisymmetric reduction with respect to the Killing vector ϕ. It is twist-free, so the reduced metric is simply a hypersurface defined by $\phi = \mathsf{const}$:

$$\bar{h}_{ab} = f(\theta)\frac{e^{i\omega(r-t)}}{r} e_{\theta a} e_{\theta b}. \tag{3.22}$$

This reduced metric in three dimensions is no longer traceless, as the radiation now corresponds to the induced "matter" term, which matches the fact that there is no "true" gravitational radiation in three dimensions. The linearized Einstein equation in the reduced spacetime reads

$$\bar{\nabla}^c \bar{\nabla}_c \bar{h}_{ab} = \bar{T}_{ab}. \tag{3.23}$$

The leading order of the energy density associated with this stress-energy tensor for a timelike vector u^a in the reduced spacetime is

$$\bar{\rho} = -\frac{f(\theta)\omega(u^\theta)^2}{r^2} \sin\left(\omega(t-r)\right) + O\left(\frac{1}{r^3}\right). \tag{3.24}$$

This induced energy density falls off as r^{-2}, which is slower than the r^{-3} fall-off from the Schwarzschild reduction, as expected. Importantly, it also vanishes for static observers ($u^\theta = 0$) and so does not affect the energy density they measure.

In this linearized approximation a general gravitational wave can be expanded in the basis of the waves of the type (3.22). Since $\omega \neq 0$, the complete expansion for the energy density has zero mean, and so is negative "half the time", thus violating the weak energy condition. Moreover, for static observers the reduced radiation does not contribute to the induced stress-energy tensor, so any negative energy present in the background reduced spacetime is also present when the contribution from the reduced gravitational radiation is accounted for. Thus gravitational radiation, while significantly affecting the asymptotics of an axisymmetrically reduced spacetime, does not cure the weak energy condition violations induced by the reduction, and might even make them worse.

3.5 Five-dimensional Schwarzschild-like Spacetime

All the examples of axisymmetric dimensional reduction so far considered created pathologies, rather than cured them. Here we mention a rather artificial case where a somewhat pathological five-dimensional spacetime is turned into a well-behaving one by dimensional reduction. We start with a spherically symmetric spacetime with the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\mathrm{d}\Omega_3^2, \tag{3.25}$$

where $\mathrm{d}\Omega_3^2$ is the metric on a unit three-sphere. This is not a Schwarzschild metric in five dimensions, since the radial fall-off is r^{-1} and not r^{-2}. Its stress-energy tensor is

$$G_{ab} = \frac{3M}{r^3}\left(\frac{t_a t_b}{(t_c t^c)^2} + \frac{r_a r_b}{(r_c r^c)^2}\right). \tag{3.26}$$

While the energy conditions manifestly hold for this spacetime, the ADM mass of a $t = \mathsf{const}$ slice diverges. Yet an axisymmetric reduction with respect to one of the Killing vectors of the three-sphere produces the standard Schwarzschild metric in four spacetime dimensions. If the chosen Killing vector corresponds to an S^1 fiber of the Hopf fibration of S^3, then the reduction is $U(1)$ (not an axisymmetric one) and the reduced spacetime is already regular, without the need for doubling. Alternatively, we can use the "standard" spherical coordinates with the metric $\mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$, then reduce with respect to the Killing vector ϕ and then double the spacetime to remove the axis.

3.6 Higher-dimensional Schwarzschild Spacetime

In section 3.1 we analyzed the effects of an axisymmetric Killing reduction on the four-dimensional Schwarzschild blackhole. The situation is similar in higher dimensions, with only the radial fall-off of the (pathological) stress-energy tensor dependent on the dimension. Specifically, for a static spherically symmetric D-dimensional vacuum blackhole defined by its horizon radius r_h, the metric in the Schwarzschild coordinates is

$$\mathrm{d}s^2 = -\left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \left(\frac{r_h}{r}\right)^{D-3}} + r^2\mathrm{d}\Omega_{D-2}^2, \tag{3.27}$$

where $d\Omega^2_{D-2}$ is the metric on a $(D-2)$-dimensional unit sphere. Like in four dimensions, this chart is singular on the horizon, however, it is always possible to construct a regular one, similar to the Kruskal-Szekeres chart discussed in section 3.1.

If we now pick an axisymmetric Killing vector in this spacetime and perform the dimensional reduction, given that the Killing vector in question is manifestly hypersurface-orthogonal, the reduced spacetime is a subspace of the original space, with the metric

$$ds^2 = -\left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right) dt^2 + \frac{dr^2}{1 - \left(\frac{r_h}{r}\right)^{D-3}} + r^2 d\Omega^2_{D-3}. \tag{3.28}$$

After doubling the reduced spacetime to get rid of the boundary created by the axis, the term $d\Omega^2_{D-3}$ describes a $(D-3)$-dimensional sphere. The Einstein tensor (i.e., the induced stress-energy tensor) of the reduced spacetime is:

$$\begin{aligned} G_{ab} = \left(\frac{r_h}{r}\right)^{D-3} \Bigg[&\frac{D-2}{2}\left(\frac{t_a t_b}{-(t_c t^c)^2} + \frac{r_a r_b}{(r_c r^c)^2}\right) \\ &- \frac{\theta_{3a}\theta_{3b}}{(\theta_{3c}\theta_3{}^c)^2} - \cdots - \frac{\theta_{D-1a}\theta_{D-1b}}{(\theta_{D-1c}\theta_{D-1}{}^c)^2}\Bigg], \end{aligned} \tag{3.29}$$

where $(\theta_3 \cdots \theta_{D-1})$ are the spherical coordinates on the $(D-3)$-dimensional unit sphere. We see that the WEC is always violated, but the tangential pressures do not exceed the energy density for $D > 4$, so in higher dimensions no observers in the reduced spacetime see superluminal matter flows. Consequently the only pathology of the reduction of a negative mass higher-dimensional Schwarzschild solution, unlike that of the one in four dimensions, is the naked singularity at $r = 0$.

3.7 Effects of the Cosmological Constant

So far all the examples considered described a dimensional reduction that preserves asymptotic flatness. However, the observed universe is known to consist mostly of the "dark energy" [44], which is currently best modeled using a positive cosmological constant. Another reason to consider non-AF spacetimes is the famous AdS/CFT duality conjecture [45], where the negative cosmological constant is an essential part of the setup. Thus it may be interesting to see how adding the cosmological constant affects the axisymmetric reduction.

3.7.1 De Sitter and Anti-de Sitter Spacetimes

For the maximally symmetric case, the Minkowski, de Sitter and anti-de Sitter spacetimes, the metric in D spacetime dimensions can be written in the static chart as

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda r^2}{(D-1)(D-2)}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{\Lambda r^2}{(D-1)(D-2)}} + r^2 \mathrm{d}\Omega^2_{D-2}, \quad (3.30)$$

where $\mathrm{d}\Omega^2_{D-2}$ is a metric on the $(D-2)$-dimensional unit sphere and $\Lambda = 0$ for the Minkowski spacetime. As before, the reduction with respect to any of the sphere's axisymmetries results in the metric of the same form, only with D reduced by one. The reduced spacetime is again (A)dS, only with a re-normalized cosmological constant, $\Lambda_{D-1} = \Lambda_D(D-3)/(D-1)$. The reduced Minkowski space ($\Lambda_D = 0$) remains flat. Thus, the reduction has no effect at all on the energy conditions.

3.7.2 Schwarzschild-(Anti-)de Sitter

Combining a Schwarzschild blackhole with the cosmological constant results in the (A)dS stress energy tensor being added to the Schwarzschild one in both the full and dimensionally reduced spacetimes. Indeed the metric is

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \left(\frac{r_s}{r}\right)^{D-3} - \frac{\Lambda r^2}{(D-1)(D-2)}\right)\mathrm{d}t^2 \\ &+ \frac{\mathrm{d}r^2}{1 - \left(\frac{r_s}{r}\right)^{D-3} - \frac{\Lambda r^2}{(D-1)(D-2)}} + r^2 \mathrm{d}\Omega^2_{D-2}, \end{aligned} \quad (3.31)$$

and the stress energy tensor is (including the contribution from the cosmological constant)

$$T_{ab} = T^S{}_{ab} - \Lambda g_{ab}, \quad (3.32)$$

where $T^S{}_{ab}$ is the stress energy tensor of the Schwarzschild blackhole alone, and Λ is the original (for the full spacetime) or the re-normalized (for the reduced spacetime) cosmological constant. The WEC (and so the DEC) is violated maximally for the anti-de Sitter case, since both terms yield negative energy, and the near-horizon energy density can become negative in the de Sitter case if the cosmological constant is not very large. However, the DEC is always violated for the de Sitter case, as the positive tangential pressure induced by the reduction adds to the Λ-term, making it always exceed the energy density. Thus adding the cosmological constant to the mix does not cure the pathologies in this example, though it may alleviate some of them.

3.7.3 Four-dimensional AdS Soliton

In sections 3.1 and 3.7 we saw that even though the axisymmetric reduction of the Minkowski spacetime does not locally change the stress-energy tensor, such a reduction of another vacuum spacetime, the Schwarzschild one, leads to a pathological stress-energy tensor. Given that the dimensional reduction of an (A)dS spacetime results in only a rescaling of the cosmological constant, it is interesting to see what happens if we axisymmetrically reduce a non-AdS spacetime with the stress energy tensor identical to that of the AdS. One of the best known examples of such a spacetime is the AdS soliton.

The AdS soliton [46] in four dimensions is described by the metric

$$\mathrm{d}s^2 = -\frac{r^2}{l^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + \frac{r^2}{l^2}\mathrm{d}\theta^2 + V(r)\mathrm{d}\phi^2, \tag{3.33}$$

where

$$V(r) = \frac{r^2}{l^2}\left(1 - \frac{r_0^3}{r^3}\right)$$

is obtained from a four-dimensional p-brane metric by a double analytic continuation. The stress-energy tensor is that of the AdS spacetime:

$$T_{ab} = -\frac{3}{l^2}g_{ab}, \tag{3.34}$$

(and so it satisfies all the energy conditions) but the spacetime is not AdS, because its curvature is not constant, as can be seen from the square of the Riemann tensor, known as the Kretschmann scalar:

$$R_{abcd}R^{abcd} = \frac{24}{l^4} + \frac{12r_0^6}{l^4 r^6}.$$

However, it is asymptotically locally AdS, with the cosmological constant $\Lambda = -3/l^2$. Just like in the Schwarzschild case, the spacetime has a singularity at $r = 0$, but its global structure is different. First, we consider the necessary identifications. Since $\partial/\partial\phi$ is a Killing vector with the norm vanishing at $r = r_0$, to avoid a conical singularity at $r = r_0$, the elementary flatness condition (2.17) must hold. This results in the coordinate ϕ being periodic, with the period $\frac{4}{3}\pi l^2/r_0$. Additionally, this restricts the range of r to $r \geq r_0$. No other coordinates have to be periodic, although imposing periodicity on θ results in the finite total energy of this spacetime [46]. The conformal infinity of the spacetime is homotopy-equivalent to a torus T^2.

Identified or not, the AdS soliton is an axisymmetric spacetime with a regular axis of symmetry $r = r_0$, where ϕ is a periodic coordinate suitable for axisymmetric dimensional reduction. The standard reduction

$$\bar{g}_{ab} = g_{ab} - \frac{1}{V(r)}\phi_a\phi_b$$

leads to the following reduced metric:

$$\mathrm{d}\bar{s}^2 = -\frac{r^2}{l^2}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r)} + \frac{r^2}{l^2}\mathrm{d}\theta^2. \tag{3.35}$$

Since this is a $U(1)$ reduction, and not an axisymmetric one, the reduced spacetime is regular and without boundary.

The stress-energy tensor of this three-dimensional spacetime in the orthonormal frame (t, r, θ) is

$$\bar{G}_{\hat{a}\hat{b}} = \begin{bmatrix} \frac{1}{l^2}\left(1 - \frac{r_0^3}{r^3}\right) & 0 & 0 \\ 0 & -\frac{1}{l^2}\left(1 - \frac{r_0^3}{2r^3}\right) & 0 \\ 0 & 0 & -\frac{1}{l^2}\left(1 - \frac{r_0^3}{2r^3}\right) \end{bmatrix}, \tag{3.36}$$

and the observed energy density for the timelike observers is

$$\bar{\rho} = \frac{r_0^3}{l^2 r^3}\gamma^2\left(1 - \frac{v^2}{2}\right), \tag{3.37}$$

while for the null observers it is

$$\bar{\rho} = \frac{r_0^3}{2l^2 r^3}. \tag{3.38}$$

Both expressions are positive everywhere, so both the WEC and NEC (and ANEC) are satisfied. The DEC, is, however, violated:

$$p^2 = \frac{1}{l^2}\gamma^2\left(-\left(1 - \frac{r_0^3}{r^3}\right)^2 + v^2\left(1 - \frac{r_0^3}{2r^3}\right)^2\right), \tag{3.39}$$

which is positive when

$$v > 1 - \frac{1}{2\frac{r^3}{r_0^3} - 1}.$$

This condition can be satisfied for some observers at any point in this spacetime, even for asymptotically static observers. In this case we have traded one pathology for another: negative energy for superluminal matter flows.

3.8 Doubling Reduced Spacetime

As we have seen, the common problem of any spacetime obtained by an axisymmetric reduction is the remnant of the axis, manifesting itself as a timelike singularity. The singularity extends “all the way to infinity”, which makes computing Komar integrals, such as the ADM mass, impossible. In this section we discuss a way to remove such a singularity by “doubling” the reduced spacetime. This procedure has already been applied in the Schwarzschild example in section 3.1. In this section we first review it for the flat four-dimensional spacetime and then generalize to arbitrary axisymmetric spacetimes.

3.8.1 Doubling Reduced Minkowski Spacetime

For the simplest possible case of the four-dimensional Minkowski spacetime with the metric in polar coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{3.40}$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, the spacetime obtained by the axisymmetric reduction with respect to the Killing vector θ^a is just

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{3.41}$$

where $r \geq 0$ and $0 \leq \theta \leq \pi$. This chart describes a half-plane for each $t = \mathsf{const}$, so the spacetime is “one-half” of the full three-dimensional Minkowski spacetime, with the boundary described by $\theta = 0$ and $\theta = \pi$, or, after transforming to the Cartesian coordinates, by $x = 0$, where $x = r\cos\theta$ and $y = r\sin\theta$.

It is clear in this example that this spacetime can be extended through the boundary by gluing another copy of it, obtained by reflecting the original across the boundary ($x \mapsto -x$). Explicitly, the doubled spacetime is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{3.42}$$

where $x, y \in \mathbb{R}$. This doubled spacetime is no longer singular, and we can calculate asymptotic quantities, such as the ADM mass (which is, of course, zero in this case).

This argument can be generalized to an axisymmetric reduction of any axisymmetric spacetime in any number of dimensions. The four-dimensional case is described in detail in [34, §2.2.3, Proposition 2.2.1]. The main points are:

- The spacetime obtained by axisymmetric reduction has a boundary where the axis of rotation used to be.
- The boundary is a totally geodesic submanifold of the total (and reduced) spacetimes.

As noted before, an axisymmetric reduction maps the symmetry axis into a timelike singularity in the reduced spacetime. Our aim is to construct an extension of the reduced spacetime that removes this singularity.

To prove the first point, we note that the symmetry axis is a boundary of the reduced spacetime. First, we recall from section 2.5 that $\mathscr{F}$ is a totally geodesic submanifold of $\mathscr{M}$ with vanishing extrinsic curvature. We can find an axisymmetric neighborhood of $\mathscr{F}$ U of $\mathscr{F}$, where we can introduce a cylindrical chart $(t, \rho, \phi, x_3, \ldots, x_D)$, $\rho > 0$, $0 \leq \phi \leq 2\pi$ mapping $U \to \mathbb{R}^n \backslash \mathscr{F}$. The $(D-1)$-dimensional manifold defined by $\phi = 0$ is a cross-section for the action of $U(1)$ on U, with the coordinates $(t, \rho, x_3, \ldots, x_D)$, $\rho > 0$. We can then add the axis $\mathscr{F}$ (corresponding to $\rho = 0$) to the reduced manifold $\mathscr{M}_r = (\mathscr{M} \setminus \mathscr{F})/U(1)$ and obtain a manifold with boundary $\bar{\mathscr{M}} = \mathscr{M}_r \cap \mathscr{F}$.

The second point follows from the properties of the axis, described in section 2.5. Indeed, by the above construction one can see that the properties of $\mathscr{F} \in \mathscr{M}$ are retained by $\bar{\mathscr{F}} \in \bar{\mathscr{M}}$. Specifically, $\bar{\mathscr{F}}$ is totally geodesic with vanishing extrinsic curvature in $\bar{\mathscr{M}}$.

Next, we consider a copies of $\bar{\mathscr{M}}$, $\bar{\mathscr{M}}'$, covered by the charts $(t', \rho', x_3', \ldots, x_n')$ near the boundary. We now join the two spacetimes along the common boundary $\mathscr{F} = \mathscr{F}'$, $\rho = \rho' = 0$, such that at the boundary all the coordinates match: $t = t'$, $x_3 = x_3'$, etc. We note that the junction conditions [47] do not contribute to the stress-energy tensor at $\mathscr{F}$. Indeed, given that $\mathscr{F}$ is a totally geodesic submanifold, its extrinsic curvature vanishes. Moreover, by construction the metric is continuous across $\mathscr{F}$. Thus the surface stress-energy tensor vanishes and the symmetry axis essentially "disappears" after doubling.

Once the spacelike boundary has been eliminated, we can construct "spheres near infinity" and calculate the ADM mass of the doubled reduced spacetime.

If the original spacetime has an event horizon crossing the symmetry axis, as it does in the Schwarzschild case, considered in section 3.1, the horizon in the reduced space gains a boundary, which is removed by the doubling procedure. If the horizon topology before reduction is $S^{D-2} \times \mathbb{R}$, then after the reduction and doubling it is $S^{D-3} \times \mathbb{R}$. The resulting change in the topological properties of the DOC can sometimes be used to constrain its physical properties, as discussed in chapter 5.

3.9 Axisymmetric Conformal Reduction

Of course, not every reduction has the same pathologies. Here we consider an axisymmetric reduction that results in a non-asymptotically flat spacetime, the conformal reduction. We show that, while the induced stress energy tensor satisfies the energy conditions, there is a naked timelike curvature singularity in the reduced spacetime. Moreover, one cannot sensibly define any asymptotic quantities, such as the total mass and charge, in the reduced spacetime. The properties of this reduction are investigated in this section.

We rescale the metric obtained by the usual Killing reduction, $\bar{g}_{ab} = g_{ab} - \xi_a \xi_b / \lambda$ by the conformal factor equal to the norm of the Killing vector:

$$\tilde{g}_{ab} = \lambda g_{ab} - \xi_a \xi_b. \tag{3.43}$$

The Einstein-Hilbert action

$$S = \int \mathrm{d}^D x \sqrt{-g} R \tag{3.44}$$

can be written as a sum of the Einstein-Hilbert action for the conformally reduced[9] metric $\tilde{g}_{ab}$ and two fields, a scalar λ and a twist tensor

$$\omega_{ab} = \frac{1}{\lambda} \tilde{g}_a^c \tilde{g}_b^d \nabla_c \xi_d,$$

giving

$$S = \int \mathrm{d}^{D-1} x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2\lambda^2} \left(\tilde{\nabla}^a \lambda \tilde{\nabla}_a \lambda + \tilde{\nabla}^a \omega^{bc} \tilde{\nabla}_a \omega_{bc} \right) \right). \tag{3.45}$$

Because the action is quadratic in the induced fields, the induced stress energy tensor obeys all energy conditions. However, this nice property comes at a price. In particular, the following issues arise:

- The reduced spacetime is no longer asymptotically flat, making its total mass and angular momentum undefined.
- The reduced spacetime has a naked timelike curvature singularity on the symmetry axis, at least for the reduction from four to three dimensions.

[9] See [48, Eq. 9] and [49, Eq. 35] for the conformal reduction from four to three dimensions, and [39, Eq. 46] for the conformal reduction from five to four dimensions.

We now look at these pathologies in more detail. In fact, they are already apparent for the conformal reduction of the Minkowski space, so we consider this simpler example instead. The metric of the conformally reduced four-dimensional Minkowski spacetime in spherical coordinates is

$$\mathrm{d}s^2 = r^2 \sin^2\theta \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2\right). \tag{3.46}$$

After rescaling $r' = \frac{1}{2}r^2, \theta' = 2\theta$ we get

$$\mathrm{d}s^2 = \sin^2\frac{\theta'}{2} \left(-r'\mathrm{d}t^2 + \mathrm{d}r'^2 + r'^2 \mathrm{d}\theta'^2\right). \tag{3.47}$$

In this form the conformally rescaled metric has a manifest r'-dependent time-dilation term, and no asymptotic time translation symmetry, since the norm of the time translation Killing vector diverges at the "spatial infinity". Moreover, the Ricci scalar for this spacetime is

$$R = \frac{1}{2r'^2 \sin^4\frac{\theta'}{2}}, \tag{3.48}$$

which diverges on the symmetry axis, where $r\sin\theta = 0$, resulting in a naked curvature singularity. The curvature term responsible for the divergence is present in any conformal reduction and it dominates any other contribution to the stress energy tensor from a regular axisymmetric spacetime. The curvature singularity in question corresponds to the original rotation axis, and so it cannot be completely shrouded by an event horizon. Indeed, if it were, the original four-dimensional spacetime would have included a black string, which is not asymptotically flat in four dimensions.

In addition, the conformal term makes this spacetime non-asymptotically flat, even though the Ricci scalar vanishes at infinity. This becomes obvious when one considers timelike and null geodesics in this spacetime. The energy conservation for a timelike radial geodesic leads to the following first order equation of motion:

$$(\dot{r}')^2 = \frac{k^2}{(r')^4} - 1, \tag{3.49}$$

where $k = \mathsf{const}$ and corresponds to the energy of the particle. The second term tends to zero at large r, constraining the allowable values of r'. Indeed, the turning points are at $r' = \sqrt{|k'|}$. Thus a free test particle of finite energy cannot escape to infinity, no matter how far from $r = 0$ it is or how much energy it has. For the null geodesics the radial equation of motion is

$$(\dot{r}')^2 = \frac{k^2}{(r')^4}. \tag{3.50}$$

Thus light can "almost" escape to infinity, but it is infinitely red-shifted in the process. These conclusions also hold for non-radial geodesics.

In a non-Minkowski spacetime the term (3.48) is added to the induced curvature, which falls off as $1/r^3$, and so asymptotically the energy density is determined by the conformal factor, and the conclusions made for the Minkowski space remain valid.

Chapter 4

Violations of Energy Conditions in Dimensionally Reduced Spacetimes

We have seen that the pathologies arising from the dimensional reduction of the Schwarzschild spacetime with respect to one of its rotational Killing vectors to an asymptotically flat spacetime manifest themselves in other reductions that preserve asymptotic flatness. Many such spacetimes can be thought of as asymptotic perturbations on the Schwarzschild one. In this chapter we clarify the meaning of asymptotic perturbation on Schwarzschild and construct a perturbative argument to show that the weak energy condition must be asymptotically violated in an axisymmetric reduction that preserves asymptotic flatness. This argument holds for the reduced spacetimes in three or more spacetime dimensions. We then present a non-perturbative argument, based on the positive energy theorem, that the dominant energy condition is also violated.

4.1 Perturbative Calculation

The definition of a perturbation in our case will be somewhat different from the standard one in [14]. Usually one can define a metric perturbation on a given manifold $\mathscr{M}$ using a smooth one-parameter map representing a family of metrics $g_{ab}(s)$, such that $g_{ab}(0) = g^0_{ab}$, where g^0_{ab} is the unperturbed metric. The parameter s is the same for all points on the manifold. A power series expansion near $s = 0$ gives

$$g_{ab} = g^0_{ab} + s\gamma_{ab} + O\left(s^2\right). \tag{4.1}$$

The tensor γ_{ab} does not have to be "small", only $s\gamma_{ab}$ does. Additionally, the derivatives of γ_{ab} are presumed to exist and be bounded on the domain of interest. Since the metric is non-degenerate everywhere, the inverse metric can be similarly perturbatively expanded:

$$g^{ab} = g^{0\,ab} - s\gamma^{ab} + O\left(s^2\right). \tag{4.2}$$

The perturbation of the inverse metric can be expressed using (4.1) and (4.2) as

$$\gamma^{ab} = g^{0\,ac} g^{0\,bd} \gamma_{cd}, \tag{4.3}$$

where the terms of order $O\left(s^2\right)$ are omitted.

For the connection difference between the perturbed and the original metric we have [14, §7.5.12]:

$$\begin{aligned} \nabla_a u_b &= \nabla^0_a u_b + s C^{0\,c}_{ab} u_c \\ &= \nabla^0_a u_b + s\frac{1}{2} g^{0\,cd}\left(\nabla^0_a \gamma_{bd} + \nabla^0_b \gamma_{ad} - \nabla^0_d \gamma_{ab}\right) u_c + O\left(s^2\right) \end{aligned} \tag{4.4}$$

and for the Ricci tensor we have [14, §7.5.15]:

$$\begin{aligned} R_{ab} = R^0_{ab} + \\ s\left(-\frac{1}{2}\nabla^0_a \nabla^0_b \gamma - \frac{1}{2}\nabla^{0\,c}\nabla^0_c \gamma_{ab} + \frac{1}{2}\nabla^{0\,c}\nabla^0_b \gamma_{ac} + \frac{1}{2}\nabla^{0\,c}\nabla^0_a \gamma_{cb}\right), \end{aligned} \tag{4.5}$$

where $\gamma = g^{ab}\gamma_{ab}$.

This approach cannot be repeated verbatim in the case under consideration, for an asymptotically flat perturbation on an asymptotically flat spacetime. We recall that any AF spacetime $(\mathscr{M}, g_{ab})$ admits a scalar field Ω vanishing on the boundary $\partial\mathscr{M}$ of the compactified manifold $\tilde{\mathscr{M}}$, $\mathscr{M} = \tilde{\mathscr{M}} \backslash \partial\mathscr{M}$, which is at least twice differentiable and has a non-vanishing gradient on the boundary, $\nabla_a \Omega$ on $\partial\mathscr{M}$. In the case of the Schwarzschild spacetime the field Ω can be identified with the reciprocal radial coordinate, $r = 1/\Omega$. Both the original and the perturbed metric tend to the flat one at large r, so there is no r-independent parametrization of the form (4.1) for which γ_{ab} is not required to be small. Here r is the radial coordinate in the Schwarzschild chart, which is our unperturbed metric. Hence, we need a new definition of what constitutes a perturbation, one that allows for an expansion parameter that is not independent of the metric.

Without loss of generality we can assume that the perturbed spacetime has the same ADM mass as the background Schwarzschild spacetime (if it does not, we can always choose a different background Schwarzschild spacetime, with the Schwarzschild mass parameter equal to that of the perturbed spacetime, and the recalculated perturbation will not contribute to the ADM mass).

By an asymptotically flat perturbation on Schwarzschild we mean a faster radial fall-off in the components of γ_{ab} compared to the components of the Schwarzschild metric g^S_{ab} in the same orthonormal basis. More precisely, given a manifold R^D minus a timelike cylinder $B^{D-1} \times R$, parametrized in the Schwarzschild-like coordinates $(t, r, \theta, \phi, \ldots)$, $r > r_0$, we can put two metrics on this manifold, g_{ab} and g^S_{ab}, such that in an orthonormal basis $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}, \ldots)$

$$g_{\alpha\beta} - g^S_{\alpha\beta} = O\left(\frac{1}{r^{D-2}}\right) \tag{4.6}$$

for all α, β.

The excised timelike cylinder is selected in such a way as to hide all the features peculiar to the perturbed spacetime, such as any unusual topology, event horizons and singularities. Since the spacetime is asymptotically flat, it is always possible to find a large enough r_0 such that these features lie inside the excised cylinder. If a spacetime is axisymmetric, this cylindrical boundary can always be chosen to coincide with the orbits of the axisymmetric Killing vector.

In the remaining part of the spacetime the metric can be expressed as

$$g_{ab} = g^S_{ab} + \frac{1}{r^{D-2}}\gamma_{ab} + O\left(\frac{1}{r^{D-1}}\right), \tag{4.7}$$

where γ_{ab} does not depend on r (all remaining r-dependence is of the order $O(1/r^{D-1})$ or higher and so is confined to the neglected terms). This definition lets us keep the tensor part of the perturbation, γ_{ab}, non-small, and confine the "small" part to the scalar $1/r^{D-2}$ and the neglectable part of the perturbation to the terms of the order $1/r^{D-1}$ and higher. Additionally, the r-independence of γ_{ab} allows for some simplifications later on.

We next consider what happens to the axisymmetric Killing vector after the perturbation. The perturbed metric is assumed to remain axisymmetric in order for the reduction to make sense. However, the Killing vector of the perturbed metric ξ^a is not necessarily the same as that of the unperturbed metric ξ^{0a}. We can safely assume however, that the original and perturbed spacetime are homotopic. Indeed, the two relevant points are:

- the topology of the spacetime (with a suitable timelike cylinder excised) is unaffected by the perturbation (otherwise it would not be a perturbation to begin with), so there is a continuous smooth map between an unperturbed and a perturbed spacetimes, in both the full and the reduced spacetimes,

- the curves and tangent vectors to them do not depend on the metric (although their norms and other inner products do).

The first point guarantees that the same atlas can cover both the unperturbed and perturbed spacetimes, and the second point implies that the tangent vectors remain unchanged when mapped from an unperturbed metric to a perturbed one (a given atlas, which is valid in both spaces, fixes the map). If desired, we can choose an atlas where the Killing vector, perturbed or unperturbed, is tangent to one of the coordinates ($\xi = \partial/\partial_\phi$). The perturbation of the metric may affects both the norm and the twist of the perturbed Killing vector. While asymptotically the perturbed norm is that of the original Killing vector, and the twist vanishes, we need to bound the two in order to calculate the effects of the perturbation on the stress-energy tensor of the reduced spacetime.

After the axisymmetric reduction

$$\bar{g}_{ab} = g_{ab} - \frac{\xi_a \xi_b}{\xi^c \xi_c}$$

the reduced perturbed metric is

$$\bar{g}_{ab} = \bar{g}^S_{ab} + \left(\frac{1}{r}\right)^{D-2} \bar{\gamma}_{ab} + O\left(\frac{1}{r^{D-1}}\right). \tag{4.8}$$

For the hypersurface-orthogonal case the reduced spacetime $\bar{\mathscr{M}}$ is simply a hyper-surface in $\mathscr{M}$ (with zero extrinsic curvature, since $\mathscr{L}_\xi g_{ab} = 0$), so the fall-off of the reduced metric is the same as the original one. Another way to see that is that in the coordinates adapted to the Killing vector ξ, with the norm λ and parametrized by $0 \leq \phi \leq 2\pi$ the metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}\bar{s}^2 + \lambda \mathrm{d}\phi^2 \tag{4.9}$$

The effects of the twist in the non-hypersurface orthogonal case can be bounded by the contribution to the angular momentum of the original spacetime. Indeed, for the limit of the Komar integral (2.16)

$$J = \lim_{S \to \mathscr{I}_0} \int_S * \nabla_a \xi_b \tag{4.10}$$

to be finite, the fall-off of the cross term $g_{x\phi}$ in the Killing vector-adapted coordinates must be $1/r^{D-3}$ or faster, resulting in the twist contribution to the reduced metric, $-\xi_a \xi_b / \lambda$, having the fall-off at least as fast as $1/r^{2D-4}$.

For example in the case of the four-dimensional Kerr metric the cross term falls off as $1/r$, and the contribution to the reduced metric falls off as $1/r^4$.

Given this, we can safely assume that in (4.8), $\bar{g}^S_{ab}$ is reduced with respect to its Killing vector ξ^S_a, while g_{ab} and γ_{ab} are reduced with respect to their own Killing vector ξ_a.

Defining the perturbation using the asymptotic fall-off conditions of the form r^{-n} makes it possible to say that the Schwarzschild metric is an asymptotically flat perturbation on Minkowski ($n = D-3$), the charged (Reissner-Nordstrom) metric is an asymptotically flat perturbation on Schwarzschild, and the charged rotating (Kerr-Newman) metric is an asymptotically flat perturbation on Reissner-Nordstrom. In any given approximation the reduced perturbation $\bar{\gamma}_{ab}$ does not depend on r, as the r-dependence has been made explicit in all relevant orders by the expansion (4.8). The only other restriction on $\bar{\gamma}_{ab}$ is that it is twice differentiable.

Comparing with (4.1), we see that r^{-n} plays the role of the expansion parameter s. However, since it is no longer a constant for the whole spacetime, the expressions for the curvature perturbations have to be re-evaluated. As a starting point, we can use the difference between the two connections, those for the perturbed and the unperturbed reduced metrics, $\bar{\nabla}^P$ and $\bar{\nabla}^S$.

$$\bar{\nabla}^P_a u_b - \bar{\nabla}^S_a u_b = C^c_{ab} u_c = \frac{1}{2} g^{cd} \left(\bar{\nabla}^S_a g_{bd} + \bar{\nabla}^S_b g_{ad} - \bar{\nabla}^S_d g_{ab} \right) u_c. \tag{4.11}$$

In the remainder of this section we only use the unperturbed connection and only the reduced metric, so we can safely omit the overbar and the superscripts P and S for brevity. The difference between the Ricci tensors of the two spacetimes at the same point [14, §7.5.9]:

$$R_{ab} = R^S_{ab} - \nabla_a C^c_{bc} + \nabla_c C^c_{ab} + C^d_{ab} C^c_{cd} - C^d_{bc} C^c_{ad}, \tag{4.12}$$

where R^S_{ab} is the stress-energy tensor of the *reduced* Schwarzschild metric, and the connection coefficients are

$$\begin{aligned} C^c_{ab} =& \frac{1}{2} \Big(- (D-2)\, r^{-(D-1)} \left(\gamma^c_b \nabla_a r + \gamma^c_a \nabla_b r - \gamma_{ab} \nabla^c r \right) \\ & + r^{-(D-2)} \left(\nabla_a \gamma^c_b + \nabla_b \gamma^c_a - \nabla^c \gamma_{ab} \right) \Big) + O\left(\frac{1}{r^{2D-4}} \right). \end{aligned} \tag{4.13}$$

It is tempting to simply discard all terms of the order $O(1/r^{D-1})$. However, this is the order of $\bar{R}^S_{ab}$ (3.28), so for now we have to keep this order and only discard the higher order terms $O(1/r^D)$. In fact, we ought to keep one more term in the expansion (4.8), but it turns out that this term only contributes in the orders we already discard.

Moreover, $C^c_{ab} = O(1/r^{D-2})$ or smaller, so the terms quadratic in C^c_{ab} are at most $O(1/r^{2D-4})$, which is smaller than $O(1/r^{D-1})$ for $D > 3$, which covers all the interesting cases. These terms can, therefore, be discarded from (4.12).

The contribution to the reduced Ricci tensor is (after discarding the terms containing second derivatives of r, which only contribute in the order $O(1/r^D)$):

$$\begin{aligned} R_{ab} =& R^S_{ab} + \frac{D-2}{2} r^{-(D-1)} \left(\nabla_a \gamma \nabla_b r + \nabla_b \gamma \nabla_a r - \nabla_c \gamma^c_b \nabla_a r - \nabla_c \gamma^c_a \nabla_b r \right. \\ & \left. + \left(-\nabla_a \gamma^c_b - \nabla_b \gamma^c_a + 2\nabla^c \gamma_{ab} \right) \nabla_c r \right) \\ & + \frac{1}{2} r^{-(D-2)} \left(\nabla_c \nabla_a \gamma^c_b + \nabla_c \nabla_b \gamma^c_a - \nabla^2 \gamma_{ab} - \nabla_a \nabla_b \gamma \right) \\ & + O\left(\frac{1}{r^{2D-4}} \right), \end{aligned} \tag{4.14}$$

where $\gamma = \gamma^a_a$ denotes the trace of the perturbation.

The term without any derivatives of r matches that in [14, §7.5.15], as expected. Next simplification comes from recalling that γ_{ab} does not depend on r, so all r-derivatives of it[10] vanish:

$$\begin{aligned} R_{ab} =& R^S_{ab} + \frac{D-2}{r^{D-1}} \left(\nabla_{(a} \gamma \nabla_{b)} r - \nabla_c \gamma^c_{(b} \nabla_{a)} r \right) \\ & + \frac{1}{2r^{D-2}} \left(2\nabla_c \nabla_{(a} \gamma^c_{b)} - \nabla^2 \gamma_{ab} - \nabla_a \nabla_b \gamma \right) + O\left(\frac{1}{r^{2D-4}} \right) \end{aligned} \tag{4.15}$$

The Ricci scalar in the reduced space is

$$\begin{aligned} R =& \frac{D-2}{2} r^{-(D-1)} \left(2\nabla_a \gamma \nabla^a r - \nabla_c \gamma^c_a \nabla^a r - \nabla_c \gamma^c_a \nabla^a r \right) \\ & + \frac{1}{2} r^{-(D-2)} \left(2\nabla_c \nabla^a \gamma^c_a - 2\nabla^2 \gamma \right) + O\left(\frac{1}{r^{2D-4}} \right), \end{aligned} \tag{4.16}$$

where we used the fact that the Ricci tensor of the reduced Schwarzschild metric is traceless. As before, using the r-independence of γ_{ab}, we discard

[10] Represented in the above expression by the term

$$\left(-\nabla_a \gamma^c_b - \nabla_b \gamma^c_a + 2\nabla^c \gamma_{ab} \right) \nabla_c r.$$

the first term and get

$$R = \frac{1}{2} r^{-(D-2)} \left(2\nabla_c \nabla^a \gamma^c_a - 2\nabla^2 \gamma \right) + O\left(\frac{1}{r^{2D-4}} \right) \tag{4.17}$$

Before calculating energy density seen by observers in the reduced perturbed spacetime, we have to address the issue of what kind of observers we are interested in. We note that it is sufficient to only consider observers that are static in the unperturbed Schwarzschild spacetime, i.e., those moving along the trajectories of the Schwarzschild coordinate t in the unperturbed reduced spacetime. Since the metric and its perturbation are smooth, if these observers see negative energy density, so do those on sufficiently close trajectories. These "Schwarzschild-static" observers are not necessarily static in the perturbed spacetime. Indeed, the notion of a static observer might not even make sense for a general perturbation. They are only static in the background Schwarzschild metric, which is sufficient for our purposes since all the calculations are done using the background metric and connection.

For these observers the calculation of the energy density can be simplified considerably. Indeed, the terms containing $\nabla_a r$ do not contribute to the energy density and we get

$$\begin{aligned} G_{ab} u^a u^b =& R^S_{ab} u^a u^b \\ &+ \frac{1}{2} r^{-(D-2)} \Big(\left(\nabla_c \nabla_a \gamma^c_b + \nabla_c \nabla_b \gamma^c_a - \nabla^2 \gamma_{ab} - \nabla_a \nabla_b \gamma \right) u^a u^b \\ &+ \left(\nabla_c \nabla^a \gamma^c_a - \nabla^2 \gamma \right) \Big) + O\left(\frac{1}{r^{2D-4}} \right). \end{aligned} \tag{4.18}$$

The first term is $R^S_{ab} u^a u^b = -(D-3) r_S^{D-3} / r^{D-1}$, where r_S is the Schwarzschild radius in D dimensions, which is determined by the blackhole's ADM mass. To estimate the remaining terms it is convenient to introduce a projection operator on the $D-2$ spacelike hypersurface orthogonal to u^a (i.e., the one given by setting $t = \mathsf{const}$ in the reduced Schwarzschild chart):

$$p^{ab} = \bar{g}^{ab} + u^a u^b. \tag{4.19}$$

For Schwarzschild-static observers this operator is time-independent. Explicitly, in the Schwarzschild spherical coordinates it is

$$p_a^b = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \tag{4.20}$$

The observed energy density can now be expressed as

$$\begin{aligned} G_{ab}u^a u^b = & -\frac{(D-3)\, r_S^{D-3}}{r^{D-1}} \\ & + \frac{1}{2r^{D-2}}\left(-p^{ab}p^{cd} + p^{ac}p^{bd}\right)\nabla_c\nabla_d\gamma_{ab} + O\left(\frac{1}{r^{2D-4}}\right). \end{aligned} \tag{4.21}$$

This shows that any time-dependence in the perturbation does not contribute to the energy density seen by static observers. We have already noted that in section 3.4 in the special case of linearized axisymmetric radiation. Any other dependence is guaranteed to drop at least two orders of r when differentiated twice, bringing the fall-off to the slower of $O(1/r^{2D-4})$ and $O(1/r^D)$. This implies that the contribution of the perturbation to the energy density seen by static observers falls off faster than that of the energy density seen by these observers in an unperturbed reduced Schwarzschild spacetime, and ensures that an asymptotically flat axisymmetric reduction of an asymptotically flat spacetime with positive energy density (and hence positive ADM mass) will violate the weak energy condition.

This conclusion has been checked in a special case, a conformal perturbation, where the expression (4.21) can be computed by hand. For example, in the conformal case,

$$g_{ab}^P = g_{ab} + f(r,\ldots)g_{ab}, \tag{4.22}$$

where $f\,(r,\ldots) = O(1/r)$ we get

$$G_{ab}u^a u^b = -\frac{(D-3)r_S^{D-3}}{r^{D-1}} - \frac{D-2}{2r^{D-2}}p^{ab}\nabla_a\nabla_b f + O\left(\frac{1}{r^{2D-4}}\right). \tag{4.23}$$

The second term is of the order $O(1/r^D)$, so it does not asymptotically contribute to the observed energy density.

A similar argument can be advanced to show that the NEC is also violated, however in this case we cannot use static observers, as null geodesics are never static. The relevant calculations are beyond the scope of this thesis. From analyzing the axisymmetric reduction of the Schwarzschild metric in higher dimensions (section 3.6), we know that in general the reduction does not induce superluminal matter flows, so the second part of the DEC may well survive the reduction.

4.2 Violation of the Dominant Energy Condition in Axisymmetric Reduction

Here we offer an argument that shows that the dominant energy condition is violated by any axisymmetric reduction that preserves asymptotic flatness. This argument is based on the positive energy theorems (PET), see [50–53]. The combined statement of the PET is (see e.g., [54] for review) as follows:

Theorem 1 (Positive Energy Theorem)**.** *Let $(\mathscr{M}, g)$ be a a D-dimensional spacetime admitting an asymptotically flat spacelike hypersurface $(\bar{\mathscr{M}}, \bar{g})$ satisfying the dominant energy condition. Then the total ADM mass m satisfies $m \geq 0$, with equality if and only if $(\mathscr{M}, g)$ is flat (Minkowski).*

We next perform the dimensional reduction with respect to the axisymmetric Killing vector. In general a spacetime obtained by an axisymmetric reduction is not geodesically complete, due to a timelike quasi-regular singularity [55] on the axis. We remove the singularity by doubling the reduced spacetime (see section 3.8). This enables us to calculate the surface integrals at infinity and also to apply the PET. The doubled reduced spacetime is again asymptotically flat by assumption, and we can calculate its ADM mass. We show that it necessarily vanishes. The PET applied to the doubled reduced spacetime implies that the dominant energy condition is violated or that the spacetime is flat. The DEC can be violated one of two ways: the WEC is violated, i.e., there is negative energy density somewhere in the spacetime, or the spacetime contains regions of superluminal matter flows. The examples considered in chapter 3 show that either or both can happen. Given that the doubled spacetime consists of two identical copies of the reduced spacetime, the same applies to the non-doubled reduced spacetime, as well.

This result can be stated as follows:

Theorem 2. *Let $(\mathscr{M}, g)$ be an asymptotically flat D-dimensional axisymmetric spacetime ($D \geq 4$) obeying the geometric conditions (but not necessarily the DEC) required for the proof of one of the positive energy theorems. Let $(\bar{\mathscr{M}}, \bar{g})$ be the $(D-1)$-dimensional spacetime obtained by an* *axisymmetric reduction with respect to the Killing vector ξ generating the axisymmetry in $\mathscr{M}$. Then the stress-energy tensor of the reduced spacetime violates the dominant energy condition.*

Note that we do not actually require that the DEC be valid in the full spacetime, only that any other conditions required for the proof of the PET are satisfied. For example, the negative mass Schwarzschild solution violates the WEC and consequently the DEC. The axisymmetrically reduced negative mass Schwarzschild actually does obey the WEC, but not the DEC, as some timelike observers can see spacelike matter flows (see section 3.1).

We prove the following lemma first:

Lemma 1. *The ADM mass of an asymptotically flat spacetime obtained by an axisymmetric reduction of a regular asymptotically flat spacetime in four or more dimensions vanishes.*

Proof. We note that in the expression for the ADM mass (2.15) the integration measure is proportional to r^{D-2}, as it describes a $(D-2)$-dimensional sphere of radius r. Given that the limit (2.15) is finite, the integrand must have a radial fall off $1/r^{D-2}$ or faster. After the reduction and doubling the new integration measure is proportional to r^{D-3}. As shown in section 4.1, the reduction does not affect the fall-off on the metric and its derivatives, so the integral in (2.15) for the reduced spacetime falls off at least as $1/r$ and the corresponding limit is zero. □

Theorem 2 immediately follows once we assume that the conditions of the PET are satisfied.

We illustrate this result by a direct calculation of the ADM mass of the reduced four-dimensional Reissner-Nordstrom metric. The spatial part of the metric is:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2. \tag{4.24}$$

The reduction with respect to the azimuthal Killing vector ϕ gives

$$\mathrm{d}\bar{s}^2 = \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2\mathrm{d}\theta^2 = \left(1 + \frac{2M}{r}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + O\left(\frac{1}{r^2}\right). \tag{4.25}$$

Since θ is a Killing vector of the reduced metric, the metric is regular everywhere except $r = 0$, including the boundary resulting from the reduction, $\theta = 0$ and $\theta = \pi$, and so we can extend this spacetime by extending the range of the coordinate θ to $0 \leq \theta \leq 2\pi$ from a half-plane to a full plane. The ADM mass of this spacetime is

$$\bar{M}_{ADM} = -\lim_{r\to\infty} 2\pi r \frac{\mathrm{d}}{\mathrm{d}r}\left(1 + \frac{2M}{r}\right) = 0. \tag{4.26}$$

This theorem does not specify which part of the DEC is violated, the positive energy or the non-spacelike matter flows, or both. From the examples in section 3 we see that both parts are violated at least asymptotically for the reduction to three dimensions, but in higher dimensions it is the WEC that is likely to break, as there no induced superluminal matter flows in the higher-dimensional Schwarzschild reduction (see 3.6).

Chapter 5

Topological Restrictions on the Properties of Dimensionally Reduced Spacetimes

5.1 Introduction

In chapter 3 we gave several examples of axisymmetric reduction preserving asymptotic flatness. These examples show what goes wrong with the energy conditions in the reduced spacetime. In chapter 4 we gave an argument for why this happens, based on the fall-off conditions. Specifically, The fall-off conditions on the metric tensor and the Killing vector used for the reduction force the violation of the weak and null energy conditions in the reduced spacetime, at least asymptotically. In section 4.2 we invoked the positive energy theorem to show that the DEC is also asymptotically violated in such reductions. Given the pervasiveness of these pathologies, it is natural to ask whether there is a more general reason for the reductions to be pathological. In this chapter we provide two such arguments, based on the principle of topological censorship (PTC).

First, we discuss the nicer case of the $U(1)$ reduction, rather than the axisymmetric reduction, and only from four to three spacetime dimensions. We show that there is no $U(1)$ reduction that preserves global hyperbolicity, asymptotic flatness and positive energy at the same time. This is a consequence of the topological rigidity theorem [5], a proof of which is given for the $U(1)$ reduction and outlined for a reduction with respect to a general Lie group. This theorem is an application of the PTC, which restricts the topology of the interior of the spacetime, given the topology of its (conformal) boundary.

Next we use the PTC to show that there is no axisymmetric reduction of a spacetime with two or more spherical horizons resulting in a three-dimensional AF or ALADS globally hyperbolic spacetime with everywhere positive energy. We then discuss the implications of this to the $2+1+1$ formalism in numerical relativity.

These topological arguments still do no imply that any axisymmetric reduction is necessarily pathological. Indeed, the Minkowski spacetime can be reduced from four to three dimensions without any ill effects, other than the presence of a timelike boundary where the axis of rotation used to be. However, the arguments in the preceding chapters show that this is the only such case, as any spacetime not devoid of matter is either already pathological or results in a pathological spacetime after the reduction.

This chapter is organized as follows: in section 5.2 we review the principle of topological censorship, which is central to the arguments that follow. In section 5.3 we review the PTC corollary restricting the topology of any Cauchy surface in a "nice" three-dimensional spacetime. In section 5.4 we prove the topological rigidity of a $U(1)$ reduction from four to three dimensions. In section 5.5 we use the topological rigidity and the PTC to show that there is no $U(1)$ reduction from a four-dimensional AF globally hyperbolic spacetime to a three-dimensional AF or ALADS spacetime satisfying NEC. In section 5.6 we review the examples of spacetimes that are and are not a subject to these restrictions. In section 5.7 we return to the axisymmetric reduction and show that there can be at most one blackhole in any "nice" reduced spacetime. In section 5.8 we consider the implications of the arguments presented in this chapter for axisymmetric numerical schemes.

5.2 Topological Censorship

The (weak) cosmic censorship conjecture, advanced by Roger Penrose in [56] states that any singularities forming as a result of a Cauchy development of regular initial data are hidden by an event horizon. A related topological censorship conjecture, asserting that not only singularities, but also any non-trivial topology cannot be observed by an asymptotic non-spacelike observer, and so must be hidden behind an event horizon, as well, was first advanced in [57]. It was proven in [20] for globally hyperbolic asymptotically flat spacetimes obeying the average null energy condition. Multiple generalizations and related results for the topology of event horizons followed: [3, 58–64]. The result essential to the topological rigidity was first proven in [3, Proposition 3.1], and it can be stated as follows:

Let $\mathscr{D}$ be the DOC of a boundary at infinity $\mathscr{I}$ of a conformally compactified spacetime $\mathscr{M}$. If the PTC holds for this spacetime, i.e., that if every causal curve in $\mathscr{D}$ with endpoints on $\mathscr{I}$ can be deformed to $\mathscr{I}$, then the map of the fundamental group of $\mathscr{I}$ onto the fundamental group of $\mathscr{D}$, induced by inclusion, is surjective.

5.3 PTC Corollary for Three-dimensional Spacetimes

In a general case the topological restriction on the DOC by the topology of its conformal boundary is rather mild, but for the case of three dimensions the topology of the DOC is restricted quite severely. This has been formulated in[60, Theorem 6]. We will use an equivalent statement given in [5]:

Corollary (Topological Censorship)**.** *Let $\mathscr{D}^{2+1}$ be the DOC of a globally hyperbolic AF or ALADS spacetime-with-boundary satisfying NEC, and let V be a (two-dimensional) Cauchy surface in $\mathscr{D}^{2+1}$. Then V is either a two-ball B^2 or an annulus, $I \times S^1$.*

We review the necessary definitions first. Let $\mathscr{M}$ be an AF or an ALADS spacetime, as described in section 2.3 with $\mathscr{I}$ as its boundary. Let $\mathscr{M}' = \mathscr{M} \cap \mathscr{I}$ be the conformally compactified spacetime with boundary. In four spacetime dimensions each connected component of $\mathscr{I}$ is homeomorphic to $S^2 \times \mathbb{R}$. If the spacetime $\mathscr{M}'$ is globally hyperbolic, it admits a Cauchy surface V'. V' is a two-dimensional manifold with boundary, and its boundary Σ_∞ is a subspace of $\mathscr{I}$. A manifold without boundary $V = V' \setminus \Sigma_\infty$ is a Cauchy surface in the original spacetime $\mathscr{M}$.

The DOC is the portion of the spacetime exterior to the event horizons, the intersection of the causal past and the causal future of $\mathscr{I}$. A spacetime with boundary $D' = D \cap \mathscr{I}$ is the DOC of $\mathscr{M}'$.

Proof. First we summarize the proof strategy.

1. The conformal boundary Σ_∞ of V is S^1.

2. The compactified Cauchy surface is a subspace of the D'.

3. The inclusion map $i : \Sigma_\infty \to V'$ implies that the homomorphism of fundamental groups $i_* : \pi_1(\Sigma_\infty) \to \pi_1(V')$ is surjective.

4. $\pi_1(V')$ is a subgroup of $\pi_1(\Sigma_\infty)$

5. Any subgroup of integers is isomorphic to a cyclic group $\mathbb{Z}_m$.

6. the only choices for m for two-dimensional manifolds are 0 and 1, corresponding to the annulus and the disk respectively.

Now, we prove each step.

Step 1 V is a two dimensional manifold, so its conformal compactification boundary Σ_∞ is a one-dimensional manifold. The only closed connected one-dimensional manifold is S^1, and the only connected non-compact manifold is R, so these are the only boundary choices. For the AF and ALADS spacetimes the boundary is compact, which leaves S^1 as the only choice.

Step 2 $\Sigma_\infty \subset \mathscr{I} \in D'$ and $D \subset D'$, so $V' \subset D'$.

Step 3 First, we recall the definition of a map induced by inclusion (see [65, §52]). The inclusion map $i : \Sigma_\infty \to V'$ induces a map between fundamental groups of the two spaces. Specifically, for a given basepoint $x_0 \in \Sigma_\infty$ and the map i, the map $i_* : \pi_1(\Sigma_\infty) \to \pi_1(V')$ is defined by the equation $i_*([f]) = [i \circ f]$, where $f : I \to \Sigma_\infty$ is a curve in V and $[f]$ is the set of equivalence classes of curves under homotopy. For each path connected piece the fundamental groups with different base points are isomorphic, so we can safely omit mention of the base point.

Similarly, the inclusion map $V' \to D'$ also induces a homomorphism of fundamental groups between these spaces. The PTC states that every causal curve in D' with endpoints on $\mathscr{I}$ can be deformed to a curve on $\mathscr{I}$, and it is shown in [3, Proposition 3.1] that the map $\pi_1(\mathscr{I}) \to \pi_1(D')$ is surjective. Moreover, the map of their respective Cauchy surfaces, i_*, is also surjective (see [60, Theorem 4]).

Step 4 Surjectivity of homomorphisms implies that $\pi_1(V')$ is a subgroup of $\pi_1(\Sigma_\infty)$. Since Σ_∞ is S^1, its fundamental group is the group of integers $\mathbb{Z}$.

Step 5 $\mathbb{Z}$ is the only infinite cyclic group, up to an isomorphism (see e.g., [66, Chapter 6, Example 2]), and every subgroup of a cyclic group is cyclic[11]. A finite cyclic group of order m is isomorphic to the group of integers under addition modulo m, $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.

Step 6 Any compact real two-dimensional manifold without boundary is a connected sum of spheres S^2, tori T^2 and real projective spaces RP^2

[11] This is sometimes referred to as the fundamental theorem of cyclic groups.

[65, Chapter 12]. A manifold with boundary can be obtained from the one without boundary by removing a number of disks B^2. For a two-dimensional manifold with boundary, one also gets the contribution from the removed disks. It follows from the Van Kampen theorem that the fundamental group of a two-dimensional manifold is the free group on the generators of the fundamental group of each of the spaces used in its construction. This group is non-Abelian except when the compact manifold is S^2, T^2 or RP^2. Furthermore, an orientable two-dimensional manifold with a non-empty boundary with an Abelian fundamental group is either a sphere with one disk removed, or a sphere with two disks removed (since a torus with a disk removed has a non-Abelian fundamental group and so does a sphere with three or more disks removed). In the first case the fundamental group is trivial, $m = 1$, and in the second case it is $\mathbb{Z}$, corresponding to $m = 0$.

In the spacetimes under consideration (AF or ALADS) the $\mathscr{I}$ is orientable, and by [3, Corollary 3.3] the DOC D' is orientable, too. This means that the Cauchy surface V' of D' is also orientable. Since V' is an orientable two-dimensional manifold with boundary, it satisfies the above conditions. The Cauchy surface V of D is V' with the boundary removed, and so it is either a disk B^2 or an annulus, $I \times S^1$.

□

5.4 Topological Rigidity of $U(1)$ Reduction to Three Dimensions

The topological censorship corollary for three-dimensional AF or ALADS spacetimes restricts the topology of the Cauchy surface to being either a disk or an annulus. This topological restriction makes it possible to constrain the topology of a total spacetime dimensionally reduced to a "nice" three-dimensional spacetime, subject to some reasonable conditions. Specifically, the total space topologically is necessarily a product of the reduced space and the group of spacelike symmetries generated by the Killing vectors employed in the reduction. In this section we prove this for a spacelike $U(1)$ reduction. As discussed in the section 2, this applies to any cylindrical reduction without fixed points (there is no rotation axis in the spacetime). For example, a toroidal wormhole spacetime admits such a reduction. The metric of one such spacetime metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (r^2 + a^2)(\mathrm{d}\theta^2 + \mathrm{d}\phi^2). \tag{5.1}$$

Since $-\infty < r < \infty$, the spatial part of this metric is topologically $\mathbb{R} \times T^2$, and its spatial Killing vectors with closed orbits, such as ∂_θ, have no fixed points and so can serve as principal $U(1)$ fibers. Thus a reduction with respect to one of those vectors is a $U(1)$ reduction. The resulting three-dimensional spacetime is regular, with the topology $\mathbb{R} \times S^1$ and the metric

$$\mathrm{d}\bar{s}^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (r^2 + a^2)\mathrm{d}\theta^2. \tag{5.2}$$

We note that neither the original nor the reduced spacetime obey the NEC or are asymptotically flat, and the topology of the full spacetime is a product topology, so this example, while simple, is not very illuminating for the purposes of this discussion. An example of a non-trivial bundle is the $U(1)$ reduction of a four-dimensional spacetime with a spherical spatial part, $S^3 \times \mathbb{R}$, such as the closed FLRW model (see e.g., [14]):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}s_3^2, \tag{5.3}$$

where $\mathrm{d}s_3^2$ is a metric on S^3. Since the Hopf fiber is an isometry of S^3, we can use it as a Killing vector field for dimensional reduction. Applying the Hopf fibration $U(1) : S^3 \to S^2$, we can reduce the spacetime to be topologically $\mathbb{R} \times S^2$, with the metric

$$\mathrm{d}\bar{s}^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}s_2^2, \tag{5.4}$$

where $\mathrm{d}s_2^2$ is a metric on S^2. An explicit construction for electrovacuum spacetimes is given in [67]. Both the total and the reduced spacetimes are globally hyperbolic and satisfy the energy conditions, but are not AF (or ALADS).

Theorem 3 (Topological Rigidity of $U(1)$ Reduction)**.** *Let D^{2+1+1} be the DOC of a globally hyperbolic spacetime with boundary with a spacelike $U(1)$ symmetry group. If D^{2+1+1} admits a globally hyperbolic bundle reduction $D^{2+1} = D^{2+1+1}/U(1)$ that is both AF or ALADS and satisfies NEC, then either $D^{2+1+1} = \mathbb{R}^3 \times U(1)$ or $D^{2+1+1} = \mathbb{R}^2 \times S^1 \times U(1)$.*

Proof. The topological censorship corollary applied to $D^{2+1+1}/U(1)$ directly implies that $D^{2+1+1}/U(1) = \mathbb{R}^3$ or $D^{2+1+1}/U(1) = \mathbb{R}^2 \times S^1$. If the base space is contractible, any bundle over it is trivial (a product bundle), since the local trivialization can be extended to be global. This proves the theorem for the case of the contractible base space $D^{2+1} = \mathbb{R}^3$. For the non-contractible case $D^{2+1} = \mathbb{R}^2 \times S^1$ we need to consider the $U(1)$ bundles over S^1. There are two possible bundles of S^1, the topological space of $U(1)$, over S^1: the trivial bundle, resulting in the T^2 total space, and the twisted bundle, known as the Klein bottle K. The latter is non-orientable, making the potential Cauchy surface $K \times \mathbb{R}$ non-orientable, as well, which contradicts [3, Corollary 3.3]. Alternatively, one can show that there is no non-trivial principal $U(1)$ bundle over S^1, without invoking the orientability argument. Indeed, the classifying space $BU(1)$ is CP^∞, which is the model space of Eilenberg-MacLane $K(Z, 2)$ and so is simply connected (see [35, Chapter 4]). □

Theorem 3 can be extended to cover an arbitrary Lie group as a principal bundle. This general theorem and its proof were first given in [5] and also generalized for a special case of a non-bundle reduction, specifically, for Seifert fiber manifolds.

5.5 No AF $U(1)$ reduction from Four to Three Dimensions

The topological rigidity theorem requires the reduced three-dimensional spacetime to obey the PTC, but it imposes no such restriction on the total (four-dimensional) spacetime. If one further requires the four-dimensional spacetime to be asymptotically flat, the already short list of possible reductions disappears completely. This is a straightforward corollary from the PTC in combination with the topological rigidity theorem 3:

Corollary (No $U(1)$ reduction)**.** *There is no $U(1)$ bundle reduction of a four-dimensional asymptotically flat spacetime to a three-dimensional asymptotically flat or ALADS spacetime that satisfies the NEC.*

Proof. If the PTC holds in the full four-dimensional spacetime, its DOC D^{2+1+1} is simply connected if $\mathscr{I}$ is. Consequently, the DOC of a four-dimensional asymptotically flat globally hyperbolic spacetime obeying NEC is simply connected. Theorem 3 implies that its topology is a product topology, where one of the terms is S^1. But a product space where one of the terms is non-simply connected is again non-simply connected[12] (see e.g., [65, Theorem 60.1]). Thus no such reduction exists. □

The proof relies on the four-dimensional spacetime having a simply connected DOC, hence the restriction to the AF spacetimes. Other spacetimes obeying the PTC may still be reduced to a PTC-obeying three dimensional spacetime. Such a spacetime may, however, still turn out to be pathological for other reasons, as we will see in the following section.

[12]Specifically, $D^{2+1+1} = \mathbb{R}^2 \times S^1 \times \mathbb{R}$ or $D^{2+1+1} = S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}$.

5.6 Examples of Reduction

A simple example of a $U(1)$ reduction from four to three dimensions where both the total and the base space obey the PTC is the $U(1)$ reduction of the four-dimensional black string[13] solution. We can construct the latter from a three-dimensional BTZ blackhole [68] by taking a direct product with another spatial dimension. For an explicit construction, see e.g., [69]. The resulting metric is

$$\mathrm{d}s^2 = -\left(\alpha^2 r^2 - m + J^2/r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\alpha^2 r^2 - m} + r^2\mathrm{d}\phi^2 - 2J\mathrm{d}t\mathrm{d}\phi + \mathrm{d}z^2, \quad (5.5)$$

where $\alpha^2 = -\Lambda$, m is the blackhole mass (in three dimensions) and J is its angular momentum. The important point is that the (quasi-regular) singularity at $r = 0$ (hidden from external observers by event horizons when $m > 2\alpha J$), is not a part of the spacetime, so the orbits of the Killing vector ∂_ϕ are circles everywhere. This is different from, say, the Schwarzschild case, where the axis of rotation corresponding to the fixed points of the Killing vector ∂_ϕ is a part of the spacetime. Thus a Killing reduction with respect to ϕ is a $U(1)$ reduction.

We now discuss the properties of the original and the reduced spacetimes. The stress energy tensor is that of an AdS space, $G_{ab} = -\alpha^2 g_{ab}$, so the WEC and NEC hold. The spacetime is also globally hyperbolic and ALADS. The metric of the reduced spacetime after the $U(1)$ reduction with respect to ϕ is

$$\mathrm{d}\bar{s}^2 = -\left(\alpha^2 r^2 - m + J^2/r^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\alpha^2 r^2 - m + J^2/r^2} + \mathrm{d}z^2. \quad (5.6)$$

The stress-energy tensor of the reduced spacetime is

$$G_{ab} = \left(\alpha^2 + \frac{3J^2}{r^4}\right) z_a z_b,$$

where z^a is the unit tangent vector along z (note that the Weyl tensor vanishes and so the spacetime is conformally flat). Thus NEC (and WEC) is preserved: for a null (or timelike) unit vector k^a the observed energy density is

$$k^a k^b T_{ab} = \left(\alpha^2 + \frac{3J^2}{r^4}\right)(k^z)^2 \geq 0,$$

[13]Here a *black string* is a cylindrical AdS blackhole.

where k^z is the z-component of k^a in an orthonormal frame. Note, however that the dominant energy condition does not hold, as the pressure in the z direction exceeds the (vanishing) energy density.

While the reduced spacetime is neither AF nor ALADS, the PTC still applies and the equivalent of theorem 3 holds: the DOC of the four-dimensional spacetime is topologically $\mathbb{R}^3 \times U(1)$. This is an example of a $U(1)$ reduction from a four-dimensional globally hyperbolic ALADS spacetime satisfying NEC to a three-dimensional globally hyperbolic simply connected spacetime satisfying NEC. It is not a counterexample to the corollary in section 5.5, since the four-dimensional spacetime is not AF.

Another example of a $U(1)$ reduction that is not a counterexample to the corollary is the four-dimensional AdS solution reviewed in section 3.7.3. The total space is ALADS, not AF, and its DOC is not simply connected, so a $U(1)$ reduction preserving NEC is allowed. As mentioned in section 3.7.3, the solution is still rather pathological because some observers see superluminal matter flows.

5.7 No Blackhole Horizons in Reduction to Three Dimensions

In four dimensions the PTC does not restrict the number of spherical blackholes allowed in an asymptotically flat spacetime, since a spatial cut of each horizon is a two-sphere and so is simply connected. However, in three dimensions a spatial cut of the blackhole horizon is S^1, the only closed connected one-manifold. Similarly, the spatial boundary at infinity Σ_∞ is also S^1. So far there is no contradiction to the PTC. But if we consider a three-dimensional spacetime with more than one horizon, the topology of the DOC becomes more complicated than that of Σ_∞, and no surjective homomorphism from $\pi_1(\Sigma_\infty)$ to $\pi_1(\mathrm{DOC})$ exist. Concretely, for a two-blackhole spacetime the fundamental group of the DOC is the free group on two generators, which is non-Abelian, and so not a subgroup of $\pi_1(\Sigma_\infty) = \mathbb{Z}$. Since any globally hyperbolic AF or ALADS spacetime satisfying ANEC is a subject to the PTC, any globally hyperbolic three-dimensional AF or ALADS spacetime with two or more horizons must violate ANEC.

It would be premature, however, to conclude that an AF dimensional reduction from four to three dimensions of a spacetime with multiple horizons contradicts the PTC. To see the problem we first return to the axisymmetric reduction of a Schwarzschild spacetime and its topology. The reduced spacetime is described in section 3.1. Due to the timelike boundary remaining from the rotation axis (whether included into the spacetime or not), the DOC of the reduced spacetime is simply connected and so its fundamental group is trivial. Thus, the PTC is trivially satisfied (though the reduced spacetime itself does, of course, maximally violate all the energy conditions).

To make the fundamental group non-trivial without changing the local properties of the spacetime we can extend the reduced spacetime through its timelike boundary by gluing a second copy to it, as described in section 3.8. This procedure removes the boundary, and the DOC of the resulting spacetime is no longer simply connected. The fundamental group of both the boundary at infinity and the DOC is still $\mathbb{Z}$, so the PTC holds and we cannot use it to infer that the reduced spacetime is pathological. However, we can apply the results of [70, 71] to the doubled spacetime to show that the DEC is violated. In particular, the result states that any $2+1$ initial data set, which obeys the dominant energy condition with non-negative cosmological constant and which satisfies a mild asymptotic condition (a "null mean convexity condition"), must have trivial topology. Moreover, any data set obeying these conditions cannot contain horizons (or, more precisely, marginally outer trapped surfaces). Since the doubled spacetime is asymptotically flat, it satisfies the required conditions and so must be either horizon-free or violate the DEC. However, less restrictive energy conditions, such as ANEC, might still hold.

We will now consider an axisymmetric AF spacetime with two or more blackholes. After the reduction with respect to the rotational Killing vector we get a spacetime with two horizons, but still simply connected. However, after extending the reduced spacetime through the timelike boundary we get a spacetime with the DOC whose fundamental group is not only non-trivial, but also non-Abelian, as this doubled reduced spacetime is homeomorphic to the one discussed in the beginning of this section. Thus ANEC is violated in this spacetime, and since it consists of two copies of the reduced spacetime, ANEC is also violated in the reduced spacetime. Thus any axisymmetric reduction of a four-dimensional two-blackhole spacetime that preserves asymptotic flatness necessarily induces a pathological stress energy tensor in the reduced spacetime.

5.8 Considerations for Axisymmetric Numerical Schemes

Until the recent breakthroughs in the full 3D numerical evolution codes [72] the axisymmetric numerical schemes were often considered for suitable problems, such as axisymmetric blackhole collisions, collapsing Brill waves [6], axisymmetric scalar field collapse [1]. So far the success has been intermittent, apparently due to numerical stability issues. While the topic of numerical relativity is outside of the scope of this thesis, we would like to remark that it is possible that some of these difficulties are related to the pathological stress-energy tensor of the reduced $(2+1)$-dimensional spacetimes. Specifically, as shown in chapters 3 and 4, the induced stress-energy tensor in reduced asymptotically flat spacetimes generically has negative energy density, and appears to have superluminal matter flows for the observers with large lateral velocity.

As discussed in section 3.9, it is possible to trade a pathological stress-energy tensor for a naked timelike curvature singularity on the axis, by rescaling the reduced metric by the norm of the Killing vector. Then the numerical evolution challenge is to control the divergent curvature terms near the rotation axis, rather than to deal with the misbehaving stress energy tensor in the bulk. However, if both the total and the reduced spacetime are AF, the argument of the section 5.7 can be applied to show that a numerically evolved reduced spacetime must violate ANEC. We discuss the point in more detail in this section.

We assume that a strongly hyperbolic [16] well-posed $2+1+1$ numerical scheme is used to evolve a two-dimensional spatial slice, using some form of evolution equations derived from the Einstein equation reduced from four to three dimensions. The resulting three-dimensional spacetime is then necessarily isomorphic to what one would obtain by numerically evolving the full three-dimensional axisymmetric spatial slice and then reducing with respect to the spacelike Killing vector corresponding to the axisymmetry, though the type of the reduction may depend on the particulars of the $2+1+1$ numerical scheme used. We further assume, for the purposes of this analysis, that the two-dimensional spacelike slice being evolved numerically is asymptotically flat, and so is the full four-dimensional spacetime, once reconstructed. Given these assumptions, we can apply the PTC to both the reduced and the total spacetimes. But first we have to get rid of the timelike boundary, using the spacetime doubling, discussed in section 3.8. The Cauchy surface of this doubled spacetime no longer has a timelike boundary and we can apply the argument of [3, Section IV] to establish the conditions required for the PTC to hold in the reduced doubled spacetime. Since the PTC does not hold in an AF spacetime with two or more blackholes, the reduced spacetime must violate ANEC. The retraction by the timelike field argument in the remark to [3, Theorem 4.1] establishes the Lemma 4.2 and the remainder of the proof of the topological rigidity theorem. Then if an axisymmetric reduction preserves asymptotic flatness (or it is a reduction from an AF spacetime to an ALADS one), it must violate the NEC. The paper [73] claims to have constructed "a strongly hyperbolic first-order evolution system" for an axisymmetric reduction. If such a hyperbolic scheme is constructed, or is claimed to be constructed, then the topological rigidity corollary should imply that the reduced spacetime is pathological.

Chapter 6

Unwrapping Closed Timelike Curves

6.1 Introduction

Closed timelike curves (CTCs) are closed curves in a spacetime that a timelike test observer can trace [2]. Although spacetimes with CTCs cannot be constructed by evolution, they are still solutions of the Einstein equation, just not of its initial value formulation. CTCs are considered problematic because their presence appears to lead to causality violations. Spacetimes with CTCs are usually dealt with in one of two ways: either a spacetime with CTCs is declared not physically relevant or it is modified globally in such a way that the CTCs are absent. In many cases the CTCs are manifest isometries of the spacetime and follow coordinate curves of a periodically identified coordinate. If this periodic identification is removed, the global structure of the spacetime changes. This global modification is informally known as unwrapping. One is then tempted to declare the unwrapped spacetime to be "more natural" than the original one. This is the content of the claim of Cooperstock and Tieu in [12], who declare that "the imposition of periodicity in a timelike coordinate is the actual source of CTCs, rather than the physics of general relativity". We investigate this claim in detail.

Two natural questions that arise are: Do spacetimes extended by unwrapping contain any CTCs not explicitly removed by unwrapping? Do any other pathologies arise as a result of unwrapping CTCs? To answer these questions we consider the two spacetimes where CTCs are concluded to be artificial by Cooperstock and Tieu, the Gödel universe [10] and the Gott spacetime [13]. These spacetimes serve as nice toy models, as they are highly symmetric and essentially $(2+1)$-dimensional (each is a direct products of a $(2+1)$-dimensional spacetime with a spacelike real line).

Implicit in the claim of [12] seems to be the requirement that the metric of the spacetime without the periodic identifications is locally the same as of the original spacetime with identifications. An important issue is whether, if the original spacetime is regular everywhere, the spacetime where the CTCs are unwrapped should be as well. Otherwise one ends up trading CTCs for singularities. We find that in axisymmetric spacetimes, where the axis of symmetry is a regular $(n-2)$-dimensional subspace, such as the Gödel spacetime, there is an obstacle to having this regularity everywhere, which manifests itself as a special kind of singularity in the unwrapped spacetime. This singularity is of the type known in literature as a *quasi-regular singularity* [55], where the spacetime curvature is bounded along each incomplete curve. This result also holds for the Gott spacetime.

This "unwrapping" singularity is similar to the one describing infinitely thin straight cosmic strings in $3+1$ dimensions, where strings are $(1+1)$-dimensional timelike singularities. If a $(3+1)$-dimensional spacetime is a direct product of a $(2+1)$-dimensional spacetime and a real line, the third spatial dimension can be projected out, transforming the string into a point particle. The main difference between the unwrapping singularity and a point particle singularity is topological: there are no closed curves winding around the unwrapping singularity. Observationally, the effect of such a singularity is also significantly different from a regular cosmic string: instead of the usual gravitational lensing we observe a total lack of correlation between the images from the opposite sides of an unwrapping singularity, as they come from completely different parts of the universe.

We show that, while unwrapping the Gott spacetime results in a (singular) spacetime with no CTCs, unwrapping the Gödel space does not remove all of the CTCs.

In the case of the Gödel spacetime, where the CTCs still persist after unwrapping, we investigate a possibility of a "multiple unwrapping", where multiple families of CTCs are unwrapped all at once. In this procedure multiple "strings" are removed, and the resulting multiply connected spacetime is then unwrapped by constructing its universal cover in order to get rid of the CTCs winding around each removed string. This removes all of the circular CTCs winding around each such string and we conjecture that no other CTCs remain, either.

Our example of extending a locally Gödel chart in a way that apparently does not give rise to CTCs results in a spacetime with pervasive quasi-regular naked singularities instead. Similarly, a CTC-free extension of the Gott spacetime, even though it is locally Minkowski almost everywhere, also results in a naked quasi-regular singularity. These examples support the view that any attempt to get rid of CTCs by alternate extensions, even if successful, is likely to result in other pathologies.

This chapter is organized as follows. In section 6.2 we give a brief review of some of the spacetimes admitting CTCs and describe how the and Gott solutions fit into the picture. In section 6.3 we define what we mean by unwrapping and investigate the nature of the resulting singularities. In section 6.4 we unwrap the Gott spacetime and show that there are no CTCs in the unwrapped (singular) spacetime. In section 6.5 we unwrap the Gödel space and show that the unwrapped space is singular, and moreover, that CTCs are still present. In section 6.6 we "improve" our unwrapping procedure in order to remove the remaining circular CTCs and discuss the properties of the resulting spacetime. The research presented in this chapter has been published in [74].

6.2 Spacetimes with CTCs

Spacetimes with CTCs can arise in a variety of ways. In some cases, such as the van Stockum cylinder, the Gödel universe and the Kerr blackhole, CTCs are produced by the "frame dragging" effect of the rotating matter. In other cases, such as the spinning string, they are due to coordinate identifications. In yet other cases the CTCs arise due to the non-trivial topology of the spacetime itself (wormholes). We give a brief overview of some of these spacetimes in this section, with the emphasis on the Gödel spacetime.

The first spacetime where the CTCs are manifest, the van Stockum cylinder, was constructed by Lanczos in 1924, then rediscovered by van Stockum [75] and analyzed by Tipler [76]. This spacetime is stationary and axisymmetric (it admits two commuting Killing vectors, one timelike and one spacelike with closed orbits and a regular $(1+1)$-dimensional axis), its metric is of the Weyl-Papapetrou type [33]:

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}t\mathrm{d}\phi + C(r)\mathrm{d}\phi^2 + H(r)(\mathrm{d}r^2 + \mathrm{d}z^2). \tag{6.1}$$

Here ϕ is 2π-periodic, and so the CTCs appear whenever $C(r) < 0$. Conventionally, the solution consists of a spinning dust cylinder matched to an external vacuum solution. For certain values of the cylinder size and angular momentum the CTCs occur in the external vacuum only. See [76] for details.

One of the most famous solutions of the Einstein equation which admit CTCs was obtained in 1949 by Gödel [10]. Its geodesics were computed in [11] and its properties are discussed in [2]. Following [2, §5.7], we write the metric of the Gödel universe as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 - \frac{1}{2}e^{2\sqrt{2}\omega x}\mathrm{d}y^2 - 2e^{\sqrt{2}\omega x}\mathrm{d}t\mathrm{d}y + \mathrm{d}z^2. \tag{6.2}$$

Here $\omega = \mathsf{const}$ and (t, x, y, z) take all real values. In the following we will call this coordinate system Cartesian. The manifold of the Gödel metric is $\mathbb{R}^4$ and the spacetime is homogeneous. The matter source in this space can be written as

$$T_{ab} = \rho u_a u_b + \frac{1}{2}\rho g_{ab}, \tag{6.3}$$

where $\rho = 2\omega^2$ is the energy density in the units where $8\pi G = 1$ and $c = 1$. Here u_a is the timelike unit vector field tangent to the coordinate curves of t in (6.2). If the second term in (6.3) is associated with a negative cosmological constant $\Lambda = -\rho/2$, then the matter content of the Gödel universe is rotating dust (pressureless perfect fluid) with density ρ, and ω is the magnitude of its vorticity flow. This spacetime is a direct product of a three-dimensional spacetime with a real line $\mathbb{R}$, parametrized by the coordinate z, which does not add any interesting features and can be safely ignored. In the following we set $\omega = 1/\sqrt{2}$, so that $\rho = 1$. This is equivalent to rescaling the coordinates, up to a constant overall factor in the metric.

The Gödel space is highly symmetric, admitting 4 out of a possible 6 Killing vectors in the (t, x, y) subspace. These are

$$\partial_t, \quad \partial_y, \quad \partial_x - y\partial_y, \quad -2e^{-x}\partial_t + y\partial_x + (e^{-2x} - \tfrac{1}{2}y^2)\partial_y.$$

The first one of these commutes with the rest, the last three form an $SO(2,1)$ Lie algebra.

The rotating matter in the Gödel space leads to CTCs, some of which are manifest after a coordinate transformation to a cylindrical-like chart (τ, r, ϕ), where $r > 0$, τ can take any real values and ϕ is a 2π-periodic angular coordinate:

$$\begin{aligned} e^x &= \cosh 2r + \cos\phi \sinh 2r \\ ye^x &= \sqrt{2}\sin\phi \sinh 2r \\ \tan\frac{1}{2}\left(\phi + \frac{t}{\sqrt{2}} - \sqrt{2}\tau\right) &= e^{-2r}\tan\frac{1}{2}\phi. \end{aligned} \tag{6.4}$$

The new metric, after omitting the irrelevant z-coordinate, is

$$\mathrm{d}s^2 = -d\tau^2 + \mathrm{d}r^2 + \sinh^2 r\left(1 - \sinh^2 r\right)\mathrm{d}\phi^2 - 2\sqrt{2}\sinh^2 r d\tau \mathrm{d}\phi. \tag{6.5}$$

The validity of imposing periodicity on ϕ follows from the fact that the last of the equations (6.4) is 2π-periodic in ϕ, and from the regularity condition on the axis. Specifically, a spacetime admitting an axial ($U(1)$) Killing vector ξ^a, parametrized by a 2π-periodic coordinate ϕ is regular on the rotation axis (a set of fixed points of ξ^a) if and only if the elementary flatness condition (2.17) holds:

$$\frac{\nabla_a(\xi^c\xi_c)\nabla^a(\xi^c\xi_c)}{4\xi^c\xi_c} \to 1, \tag{6.6}$$

where the limit corresponds to the rotation axis [33]. This is further discussed in section 6.3. This condition holds on the axis $r = 0$ of (6.5). The tangent to the coordinate curve of ϕ is future pointing when it is timelike, resulting in CTCs. This occurs for $\sinh r > 1$.

Given that the Gödel spacetime has CTCs passing through every point, it has generally been discarded as pathological. Other CTC-admitting solutions have been harder to dismiss, however.

There are several known examples of non-simply connected spacetimes with matter sources satisfying the Null, Weak and Dominant energy conditions where CTCs are present. In such cases Carter, in his investigation of the spinning blackhole metric [77], makes a distinction between the "trivial" CTCs (those that are not homotopic to zero, i.e., non-contractible) and the "non-trivial" (contractible) ones, such as those present in the Gödel spacetime. The trivial CTCs can be removed by going from a given non-simply connected spacetime to its universal cover, without changing the metric locally. The non-trivial ones obviously persist even in that case.

A standard example of a spacetime admitting trivial CTCs is Minkowski spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{6.7}$$

with the periodically identified timelike coordinate ($t \sim t + T$). This spacetime is homeomorphic to a cylinder $\mathbb{S} \times \mathbb{R}^3$, all CTCs are trivial and can be removed by going to its universal covering space, the $\mathbb{R}^4$ manifold with Minkowski metric.

Another example, relevant to some of the unwrapping constructs below, is the spacetime of an infinite spinning cosmic string. A single straight non-spinning cosmic string along z-direction is described by the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2\mathrm{d}\phi^2 + \mathrm{d}z^2, \tag{6.8}$$

$0 \leq \phi \leq 2\pi$, $m \neq 0$. This was first analyzed by Marder [78], who described the conical singularity at $r = 0$, without assigning any physical meaning to it. For the spinning string the metric is

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \frac{a}{2\pi}\mathrm{d}\phi\right)^2 + \mathrm{d}r^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2\mathrm{d}\phi^2 + \mathrm{d}z^2, \tag{6.9}$$

$0 \leq \phi \leq 2\pi$, where a is the angular momentum per unit length, and m is the mass per unit length (string tension), respectively. This spacetime can be obtained from that of a non-rotating string (6.8) by replacing the identification $(t, r, \phi, z) \sim (t, r, \phi + 2\pi, z)$ with $(t, r, \phi, z) \sim (t + a, r, \phi + 2\pi, z)$ and substituting $t \mapsto t + a\phi/2\pi$ (see e.g., [25]). This is an example of "topological frame dragging", where an observer on the Killing horizon (corresponding to the zeros of the norm of ∂_ϕ) appears to be rotating relative to an observer at infinity ([79, 80]). The manifold of (6.9) is regular and flat everywhere except at $r = 0$, where there is a conical singularity due to the mass term (the deficit angle is equal to m). The coordinate curves of ϕ are closed and become timelike sufficiently close to the string, i.e., $r < a/(2\pi - m)$. The conical singularity can be smeared out by a suitable matter distribution [81], in which case the CTCs become contractible.

Instead of a single rotating string one can produce CTCs with a pair of non-rotating strings moving with respect to each other with a non-zero impact parameter, as discovered by Gott [13]. The coordinate chart in the vicinity of each string is just (6.8), and the two charts can be smoothly connected, such that there is a boost along the junction, as discussed in detail in section 6.4.1. This causal structure of this spacetime is, in a sense, an opposite of the spinning string one, as the CTCs in it have minimum size and extend all the way to infinity [82]. Thus the Gott spacetime violates the "physical acceptability conditions" proposed in [83]. The CTCs of this spacetime are discussed in detail in section 6.4.2.

In all of the above examples the CTCs exist for all times, and so such spacetimes are usually considered unphysical, since they do not admit a Cauchy surface from which such a spacetime could evolve.

Another well-known example of a vacuum spacetime with CTCs is the Kerr blackhole. In this metric there exist both the CTCs that wrap around the ring singularity and those that do not [77]. Both kinds are hidden from an external observer by the event horizon of the blackhole.

A different class of spacetimes admitting CTCs are those with wormholes and "exotic" matter sources (those violating the energy conditions), but without singularities. Exotic matter is required to keep the wormholes traversable. The first traversable wormhole metric was given in [84]. An example of such a metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(r^2 + R\left(r\right)^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{6.10}$$

where $R(r)$ has compact support, $r \in \mathbb{R}$ and $R(0) \neq 0$. Positive and negative values of r correspond to two different flat asymptotic regions. These can be patched together far enough from the wormhole throats, where $R(r) = 0$. To do that we can identify a 3-plane (e.g., $x = r \sin\theta \cos\phi = \mathsf{const}$) in one of these asymptotic regions with a corresponding 3-plane in the other. Since the spacetime is flat there, both the intrinsic and extrinsic curvatures of the planes vanish, and so a spacetime with such an identification satisfies the Einstein equation without the need for any additional matter sources. Once a wormhole exists in a given spatial slice $t = \mathsf{const}$, it can be manipulated into producing CTCs by a variety of means, all based on changing the relative rates of time flow between the throats, as seen by an asymptotic observer, until there is a CTC threading through them. This can be achieved using either the special relativistic time dilation effect, as described in the original paper, or the gravitational one [85]. Since all such CTCs result from the underlying spacetime not being simply connected, going to the simply connected universal cover gets rid of the CTCs.

Another evidence of the ubiquity of CTCs is the solution constructed by Ori [86], where a regular Cauchy horizon bounded by a closed null geodesic may be constructed as a limit of an evolution of a regular spatial slice and the matter sources satisfying energy conditions.

There is a number of conjectures and theorems that deal with the CTCs and their appearance. Tipler [87] has shown that CTCs cannot evolve from non-singular initial data in a regular asymptotically flat spacetime. Hawking [88] has advanced the chronology protection conjecture, which states that, if the NEC holds, then the Cauchy horizon (the null boundary of the domain of validity of the Cauchy problem, see e.g., [2]) cannot be compactly generated. Moreover, even if the NEC is violated, quantum effects are likely to prevent the Cauchy horizon from appearing.

6.3 Unwrapping

Since one of our goals is to show that unwrapping contractible CTCs creates singularities, we first review the definition of singularities in general relativity. Next we discuss the quasi-regular singularities resulting from changing the angular coordinate identifications in Minkowski spacetime, as described in [55]. Finally, we consider one particular kind of quasi-regular singularities, the unwrapping singularity, one that results from unwrapping closed timelike curves in axisymmetric spacetimes with a regular symmetry axis.

6.3.1 Singular Spacetimes

Intuitively, one tends to think of a singularity in a spacetime as a place in it where the curvature diverges or something else pathological happens. This approach, while satisfactory in other field theories, where the spacetime is provided *a priori*, does not work in general relativity, because the spacetime is not given in advance. Instead it is determined by solving the Einstein equation, which relates the matter content to the spacetime curvature. This means that the manifold and the metric must be smooth enough to keep the Einstein tensor finite everywhere. Hence, any singular point is not a part of the spacetime manifold and cannot be described as a "place" [2], so a different definition of singularity is required.

A singularity can be detected by the existence of curves of "finite length" which are inextendible in at least one direction. An inextendible curve γ is defined as a continuous map $\gamma : [0,1) \to \mathscr{M}$ for which there is no end point, i.e., there is no continuous map $\gamma' : [0,1] \to \mathscr{M}$, $\gamma \subset \gamma'$. However, while "length along a curve" is well-defined for manifolds with Riemannian metric, a Lorentzian metric does not give rise to a distance function for an arbitrary curve. For a geodesic curve one can use its affine length [2], but there is no unique or natural prescription for a distance between two points on a general curve.

The definition of singularities through the existence of incomplete inextendible geodesics is successfully used in the proofs of the singularity theorems [2]. These theorems are generally associated with the most familiar type of singularities, the curvature singularities, such as that of the Schwarzschild metric for $r \to 0$. This is, however, not the only type of singularities possible. A commonly accepted classification of singularities is given in [55]. In the case of curvature singularities some of the curvature invariants, such as R, $R_{ab}R^{ab}$ or $R_{abcd}R^{abcd}$, grow unbounded along an incomplete curve. A different case is the parallel-propagated curvature singularities, where some of the components of the Riemann tensor cannot be bounded along some incomplete curves, even though all of the curvature invariants remain finite or even vanish, such as in the case of singularities formed by gravitational waves.

Yet another case, and the one most relevant to the subject of this chapter, is that of quasi-regular singularities. There the curvature tensors remain smooth and bounded everywhere along an incomplete curve, yet the curve is still inextendible. To avoid the case where singularities are created artificially by removing a regular point from a given spacetime, only inextendible spacetimes are considered. A spacetime is inextendible if it is not isometric to a proper subset of another regular spacetime of the same dimension.

A classic example of a quasi-regular singularity is the two-dimensional cone. The differentiable structure and the metric on that space can be taken to be induced from its embedding into a three-dimensional flat space. The space is smooth (even flat) everywhere outside the apex of the cone. However, an attempt to include the apex into the space and continue the geodesics through it leads to problems. One can see that there is a curvature singularity at the apex by considering a cone with a spherical cap (as seen from its three-dimensional embedding) and taking a limit where the cap radius goes to zero (see e.g., [89]). The induced differentiable structure breaks down at the apex of the cone. In fact, this space is an example of a conifold, which is a generalization of a manifold, and it allows quasi-regular singularities of the conical type [90]. Cosmic string, described by the metric (6.8), is another example of a conical singularity. Mars and Senovilla [32] have shown that an axisymmetric spacetime is regular on the axis if and only if the regularity condition (6.6) holds. It is obviously violated in the case of a cosmic string with non-zero mass (6.8).

Here we briefly review some of the definitions of a singularity, each tailored to a particular class of problems. For the simple examples of quasi-regular singularities considered in this paper, all the definitions agree. A detailed review of the topic can be found in [91].

One of the first definitions was Geroch's g-boundary [92], defined by existence of incomplete geodesics, whether timelike, spacelike or null. The timelike geodesic incompleteness is the most severe case, as there would exist inertial observers whose existence comes to an end within a finite proper time. This definition does not address the case of singularities that can only be reached by non-geodesic curves.

Another definition of singularity, originally due to Penrose [22], uses the concept of a conformal boundary. There a spacetime is embedded in another "unphysical" Lorentzian manifold conformally, rather than properly, in effect bringing the "infinity" to the finite values of the coordinates. This allows one to attach a conformal boundary to the spacetime. See e.g., [2]. If such a boundary is reached by a geodesic curve with a finite affine parameter value in the original spacetime, the conformal boundary is singular. This idea works well in highly symmetric cases, where the unphysical spacetime is easy to construct.

A famous attempt to assign a causal boundary to a spacetime without resorting to an "external" concept, such as the unphysical conformal spacetime, is that of Geroch, Kronheimer and Penrose [93]. They attach a "causal boundary" to any spacetime subject to certain causality restrictions.

Schmidt [94] generalized the geodesic affine parameter to non-geodesic curves, in order to characterize singularities that cannot be reached by a freely falling observer. The basic idea is to assign a positive-definite distance function to points on an arbitrary curve using a Riemannian metric on a frame bundle parallel-propagated along the curve. See e.g., [2]. This in turn enabled the use of Cauchy completion to assign an end point to an incomplete curve. These endpoints define a so called b-boundary. Once the b-boundary points are found, one can talk about a neighborhood of a singularity and the behavior of tensors, such as the Riemann tensor, along a curve approaching the singularity.

Scott and Szekeres [95] suggested that the boundary definition need not be restricted to using only the objects intrinsic to the Lorentzian manifold. Moreover, it should be definable for any differentiable manifold with an affine connection, thus accommodating theories other than just the Einstein's general relativity, such as the gauge theories, Einstein-Cartan and others. The abstract boundary approach is based on the idea of an "envelopment", a way to embed a manifold in a "larger" manifold of the same dimension. The boundary points are then regular points of the topological boundary of the embedding. The abstract boundary points and sets are formed by equivalence classes of envelopments that cover each other. The set of all abstract boundary points is called the abstract boundary, or a-boundary. The singularities are classified as either removable, if they can be covered by a non-singular boundary set or essential if they cannot.

For our purposes the timelike geodesic incompleteness provides an adequate definition of singularities. Indeed, any singularity resulting from a change in coordinate identifications appears in place of a formerly regular point. Since there are timelike geodesics passing through any regular point, these geodesics become incomplete after the change in identifications.

In the case of singularities constructed by changing periodic coordinate identifications in an axisymmetric spacetime, the singular boundary coincides with the set of the fixed points of the periodic coordinate before the identification is changed. For example, for the two-dimensional cone with the metric $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2$, where $0 \leq \phi \leq \Phi < 2\pi$, $r = 0$ is its singular boundary, due to the deficit angle. A conical singularity in the four-dimensional spacetime with the metric (6.8) is an example of a two-dimensional singular boundary which is a flat $\mathbb{R}^2$ manifold. The $r = 0$ singularity of the Schwarzschild spacetime can be described as a singular b-boundary with a rather peculiar structure [96].

It is worth noting that the singular boundary is different from the boundary of an n-dimensional "manifold with boundary". The latter is itself an

$(n-1)$-dimensional manifold without boundary. In contrast, the singular boundary can be of any dimension, it may or may not be a manifold itself, and may or may not have a boundary (or a singular boundary). See [55] for examples and counter-examples.

6.3.2 Quasi-regular Singularities in a Flat Spacetime

Here we describe the construction of a conical as well as unwrapped singularities in a flat spacetime. Since changing the coordinate identifications does not change the metric at any of the regular points, the unwrapping singularities are always quasi-regular (they are not associated with a curvature divergence). Provided the original spacetime is asymptotically flat and has no event horizon, neither does the unwrapped one, so the resulting quasi-regular singularity is *naked*, i.e., there are future directed null curves originating arbitrarily close to the singularity that reach the future null infinity. We first describe the unwrapping singularity for the four-dimensional Minkowski spacetime and then generalize the definition to apply to the spacetimes of interest.

A conical singularity in a four-dimensional Minkowski spacetime can be obtained as follows [55]:

1. Remove the timelike two-plane $x = y = 0$ in the Cartesian chart. This is precisely where the cylindrical chart is not defined ($r^2 = x^2+y^2 = 0$). This space is no longer simply connected and has the topology of $\mathbb{S}^1 \times \mathbb{R}^3$.

2. Unwrap the resulting space to obtain its universal covering $(\bar{\mathscr{M}}, \bar{g})$, with the same flat metric in the cylindrical chart, but with the range of the new angular coordinate $\bar{\phi}$ extended to $\bar{\phi} \in \mathbb{R}$, instead of $0 \leq \phi \leq 2\pi$. A local Cartesian chart (with the non-negative x-axis removed) is obtained by the standard transformation ($x = r\cos\phi, y = r\sin\phi$). A two-dimensional slice of this spacetime in the $x - y$ plane ($t = z = 0$) is shown in Fig. 6.1 and the three-dimensional embedding of this slice into $\mathbb{R}^3$ is shown in Fig. 6.2. A well-known version of this space is the Riemann surface forming the domain of the complex log function.

3. Identify the points under translation through an angle $\Phi \neq 2\pi$, taking care to preserve the rotational isometry $r = \mathsf{const}$, i.e., $(t, r, \phi, z) \sim (t, r, \phi + \Phi, z)$.

For $\Phi \neq 2\pi$ we get the metric of the cosmic string (6.8) with the string tension m equal to the deficit angle $m = 2\pi - \Phi$. If $\Phi > 2\pi$, the string

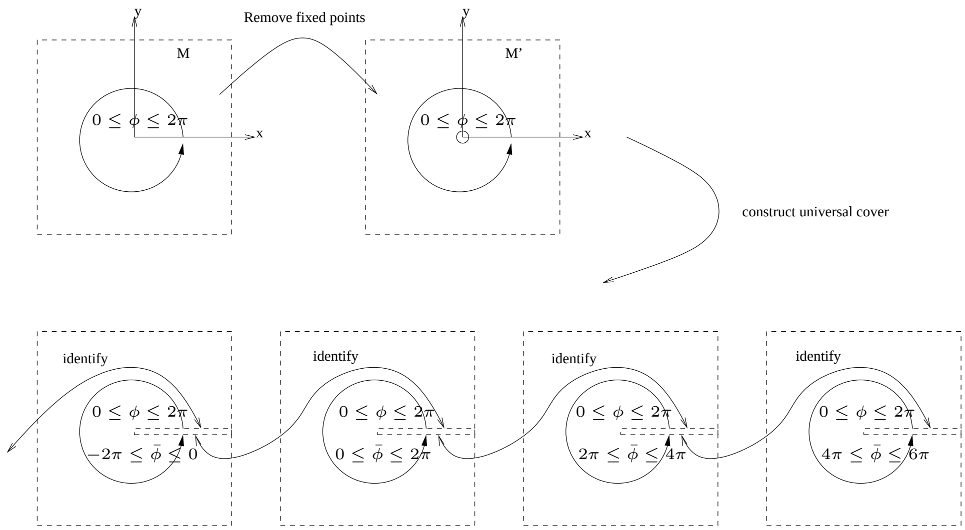

Figure 6.1: Unwrapping Minkowski space: a) pick a cylindrical chart, b) remove the fixed points of ϕ at $r = 0$, c) go to the universal cover by extending the range of ϕ to all reals. A $t = z = 0$ slice is shown as a collection of Cartesian charts with $y = 0, x \geq 0$ removed and the surfaces $\phi = 2\pi$ of one chart identified with the surface $\phi = 0$ of the next chart. $\bar{\phi}$ is the global angular coordinate, $\bar{\phi} \in \mathbb{R}$.

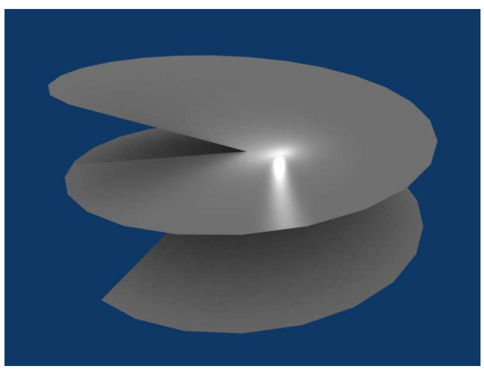

Figure 6.2: A visualization of a two-dimensional spatial slice of the unwrapped Minkowski space.

has negative tension. The presence of a conical singularity is reflected in the focusing (or defocusing) of geodesics passing on the opposite sides of the singularity. This can also be seen from the violation of the regularity condition (6.6). This condition is obviously satisfied for the ordinary Minkowski spacetime, where $\xi^c\xi_c = r^2$ in the cylindrical chart, but not after the identifications where the deficit angle $m \neq 0$, as can be seen by a coordinate transformation where the deficit angle is traded for a constant factor in $\xi^c\xi_c$, such as in (6.8), breaking (6.6).

If the last step is omitted, we get the unwrapped space $(\bar{\mathscr{M}}, \bar{g})$, and the singularity is not conical in the standard sense, as there is no closed curve wrapping around it. We will call this type of quasi-regular singularity the unwrapping singularity.

The subspace $x = y = r = 0$ removed in the step 1 forms the "boundary" of each of the three spaces corresponding to the three steps above. (We put the word boundary in quotes to indicate that it is neither a boundary of a manifold, nor yet a singular boundary, but rather an artificial "hole" in the spacetime.) After step 1 this boundary is regular, since it can be included back in to form the original inextendible Minkowski spacetime.

After step 2 the boundary is no longer regular. Indeed, if it were regular, we would be able to include it back into the spacetime and show that any neighborhood of a point on the boundary point is homeomorphic to an open ball in $\mathbb{R}^4$. This would in turn imply that there are closed curves around any such point. Furthermore, any such closed curve is homotopic to a closed curve $r = \mathsf{const}$ lying inside the open ball. However, step 2 explicitly removes all such curves, contradicting the regularity assumption. The singularity is quasi-regular, because the Riemann tensor vanishes for all points $r > 0$.

After the step 3 the boundary is still singular, as any arbitrarily small curve around it has the same fixed deficit angle, breaking the regularity condition (6.6).

The argument that the singular boundary of the unwrapped spacetime is quasi-regular applies to a class of regular spacetimes that is larger than just the Minkowski space. For the steps 1 and 2 to be applicable for a certain spacetime, it is sufficient to have the spacetime covered by a single Cartesian chart. In particular, it is valid for the Gott and Gödel spacetimes and is generalized in section 6.3.3 to locally axisymmetric spacetimes.

Moreover, for any future-directed null curve connecting the origin point O in this Cartesian chart with a certain point P in the original spacetime there is a future-directed null curve connecting the boundary in the unwrapped spacetime with one of the infinitely many copies of the point P resulting from unwrapping. Thus, if the unwrapping point is not hidden by an event horizon in the original asymptotically flat spacetime, the quasi-regular singularity in the unwrapped spacetime is necessarily naked.

For completeness, we mention another type of quasi-regular singularity, the Misner singularity, constructed by removing a spacelike two-plane $t = z = 0$ from Minkowski space, then periodically identifying points under a given boost $t^2 - z^2 = \mathsf{const}$, to obtain the four-dimensional Misner space[14]. The topology of the resulting space is $\mathbb{S} \times \mathbb{R}^3$, it contains CTCs through every point, and the surface $t = z = 0$ is a quasi-regular singularity. See [55] for detailed examples. If we omit the identification step, we obtain an "unwrapped" Misner space.

One can construct rather complicated quasi-regular singularities by cutting and gluing together spacetime pieces with different properties. For example, Krasnikov [98] describes string-like singularities that are loops or spirals embedded in otherwise flat spacetime.

6.3.3 Singularities Created by Unwrapping

We now generalize the unwrapping procedure to a general axisymmetric spacetime and review the properties of the resulting singularity.

As discussed in section 2.5, we define an axisymmetric spacetime $(\mathscr{M}, g)$ as the one which is invariant under the action $\Phi : SO(2) \times \mathscr{M} \to \mathscr{M}$ of the one parameter rotation group $SO(2) \equiv U(1)$, such that the set of the fixed points of Φ is an $(n-2)$-dimensional embedded surface $\mathscr{F}$. If the latter condition is not required to hold, the spacetime is cyclic. A cyclic, but not axisymmetric, spacetime does not necessarily becomes singular after unwrapping - for example one can unwrap a two-dimensional torus in an obvious way (by going to its universal cover) and get a two-dimensional plane.

Note that the axis of symmetry $\mathscr{F}$ is a set of regular points in $\mathscr{M}$. By this definition, the cosmic string spacetime is, while cyclic, not axisymmetric, unless the infinitely thin string is thickened and smoothed into a one of finite diameter.

To unwrap an axisymmetric spacetime we follow the same steps as in section 6.3.2. See also [36, 98]:

- Start with an axisymmetric spacetime $(\mathscr{M}, g)$
- Remove the fixed point set $\mathscr{F}$ (the symmetry axis) to obtain $(\mathscr{M}' = \mathscr{M} \setminus \mathscr{F}, g)$

[14] A four-dimensional Misner space is a direct product of the two-dimensional Misner space introduced in [97] with the $x - y$ plane

- Go to the universal covering space $\bar{\mathscr{M}}$ of $\mathscr{M}'$, $\mathbb{Z} : \bar{\mathscr{M}} \to \mathscr{M}'$.

The unwrapped spacetime $(\bar{\mathscr{M}}, g)$ is singular by construction (we excised a set of regular points from its base space), and inextendible, provided $(\mathscr{M}, g)$ is inextendible. We cannot use the regularity criterion (6.6) to show inextendibility, as it only applies to axisymmetric spacetimes, and $\bar{\mathscr{M}}$ is not axisymmetric, because the rotation Φ is lifted to the translation $\bar{\Phi}$. Instead we mirror the argument from [55] reproduced in section 6.3.2 for the unwrapped Minkowski spacetime. Suppose there is an extension $\hat{\mathscr{M}}$ of $\bar{\mathscr{M}}$ that includes a point of $\mathscr{F}$ as a regular point. Any compact neighborhood $\hat{\mathscr{O}}$ of a point $p \in \mathscr{F} \subset \hat{\mathscr{M}}$ includes a neighborhood $\bar{\mathscr{O}}$ of the corresponding point on the singular boundary of the unwrapped space $\bar{\mathscr{M}}$. However, $\bar{\mathscr{O}}$ includes a lift $\hat{\bar{\Phi}}$ of some orbits of the rotation Φ, which are non-compact (and infinitely long, as measured by the generalized affine parameter) in the unwrapped spacetime $\bar{\mathscr{M}}$. Thus $\hat{\mathscr{O}}$ is also non-compact, leading to a contradiction. Any attempt to compactify $\hat{\mathscr{O}}$ while keeping $\hat{\bar{\Phi}}$ an isometry would make the orbits of $\hat{\bar{\Phi}}$ closed, thus violating the regularity condition.

The boundary of the unwrapped spacetime is again a quasi-regular singularity[15], as there is no curvature divergence anywhere along a lift of the curves from $\mathscr{M}$ to $\bar{\mathscr{M}}$. The singularity is naked, provided the rotation axis is not hidden by the event horizon in $\mathscr{M}$.

The steps described in this section can be generalized to remove the assumption of axisymmetry. Indeed, since we are interested in unwrapping contractible CTCs, we only need to make them non-contractible by removing a relevant subspace $\mathscr{F}$ (typically $(D-2)$-dimensional, such as the axis of symmetry) from the original manifold $\mathscr{M}$. We can then go to the universal cover of the resulting non-simply connected manifold $\bar{\mathscr{M}}$. This lift "unwraps" the CTCs in question, at the expense of creating quasi-regular singularities. In this thesis we restrict our attention to just two examples of spacetimes containing CTCs, and their unwrapping, so we do not use this more general form.

6.4 Unwrapped Gott Spacetime

The Gott spacetime, discussed in detail in this section, admits CTCs. Cooperstock and Tieu suggested that such a matching is artificial and that identification "before the Lorentz boost is applied" is "more natural" [12]. Since

[15] A singularity resulting from unwrapping of a blackhole, such as BTZ one, [68] may or may not remain shrouded by its (unwrapped) event horizon.

their claim is based on a different spacetime[16] and relies on a closed curve crossing these ribbon singularities, it is hard to evaluate. Instead, in keeping with the procedure described in section 6.3.3, we unwrap the Gott spacetime, such that the Gott CTCs correspond to open curves in the unwrapped spacetime. To do that without changing the metric locally, we remove a timelike line from the Gott spacetime and construct the universal cover of the resulting multiply connected (and now singular) spacetime. We show that no other CTCs are present in the new spacetime. We also construct some alternative extensions of the Gott spacetime with a timelike line removed and discuss their properties.

6.4.1 Construction

We first review the way the (2 + 1)-dimensional Gott spacetime is constructed. Following Gott [13], we start with the (2 + 1)-dimensional version of the straight cosmic string spacetime (6.8), where the string is represented by a point particle. The metric of a single point-particle in 2 + 1 dimensions

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(1 - \frac{m}{2\pi}\right)^2 r^2 \mathrm{d}\phi'^2, \tag{6.11}$$

where $0 \le \phi' \le 2\pi$ and the particle mass m is the deficit angle. The deficit angle can be made explicit by the substitution $\phi' = \phi/(1 - m/2\pi)$, where now $0 \le \phi \le 2\pi - m$:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2. \tag{6.12}$$

The coordinate identification $\phi \sim \phi + 2\pi - m$ corresponds to removing a timelike wedge centered at the particle and identifying the opposite faces of the wedge. The angle ϕ_0 the wedge makes with the horizontal axis corresponds to the remaining coordinate gauge freedom and can be chosen in a way that simplifies a particular calculation. Two such strings can be joined together, and the relative orientation of the wedges is a matter of convenience. Gott has chosen the wedge angle in a way that identifies the surfaces

$$\phi_0 = \frac{\pi}{2} - \frac{m}{2} \sim \phi_1 = \frac{\pi}{2} + \frac{m}{2},$$

which simplifies his proof of the existence of CTCs. Cutler [99] used the identification $\phi_0 = \pi/2 - m \sim \phi_1 = \pi/2$ for one of the strings to show

[16] They appear to remove two timelike ribbons from Minkowski space and boost the resulting singularities relative to each other.

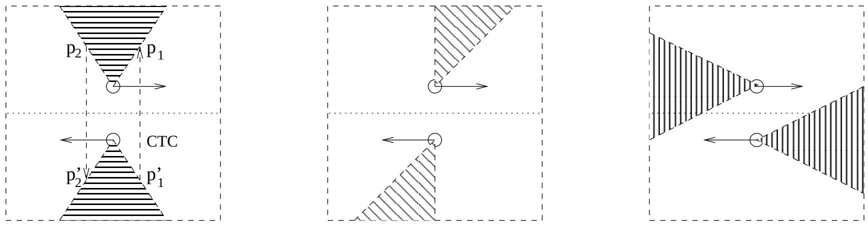

Figure 6.3: Conical wedge identification choices in the Gott spacetime. Wedge fill lines indicate the identified points. The strings are shown at the moment of the closest approach. Left: Gott identification, where CTCs are manifest (one CTC is shown). Center: Cutler identification, used to prove the existence of points not lying on any CTCs. Right: Carroll identification, used to visualize the existence of CTC-free spacelike hypersurfaces, as the opposite sides of the wedge are identified at equal times.

the existence of a timelike cylinder enclosing both strings which no CTCs enter. Carroll et al. [100] identified $\phi_0 = -m/2 \sim \phi_1 = m/2$ to visualize the existence of spacelike hypersurfaces through which no CTCs pass. These identification choices are illustrated on Fig. 6.3. As our goal is to investigate the CTCs in the Gott spacetime, we use the Gott's choice of ϕ_0.

We write each face of the wedge (denoted by the indices 1 and 2) in the Cartesian coordinate system, expressed in terms of two parameters t and x:

$$\begin{aligned} t_1 &= t, \quad x_1 = x, \quad\;\; y_1 = x \cot\frac{m}{2}, \\ t_2 &= t, \quad x_2 = -x, \quad y_2 = x \cot\frac{m}{2}. \end{aligned} \tag{6.13}$$

Points corresponding to the same values of (t, x) on both faces are identified.

It is possible to have multiple strings in the same spacetime, as long as the total deficit angle does not exceed 2π, otherwise the topology of the spacetime becomes $\mathbb{S}^2 \times \mathbb{R}$ instead of $\mathbb{R}^3$, where the total deficit angle is equal to 4π [25].

Next step is to boost the wedge in the positive x direction with the velocity $v < c = 1$. The coordinates of the faces are then Lorentz-transformed into the laboratory frame as $t_L = \gamma(t + vx)$, $x_L = \gamma(x + vt)$, $y_L = y$:

$$\begin{aligned} t_{1L} &= \gamma(t + vx), \quad x_{1L} = \gamma(x + vt), \quad\;\; y_{1L} = x \cot\frac{m}{2}, \\ t_{2L} &= \gamma(t - vx), \quad x_{2L} = \gamma(-x + vt), \quad y_{2L} = x \cot\frac{m}{2}. \end{aligned} \tag{6.14}$$

One can see that a pair of identified points $p_1 = (t_{1L}, x_{1L}, y_{1L})$ and $p_2 =$

(t_{2L}, x_{2L}, y_{2L}) have different values of the time coordinate t_L in the laboratory (center of momentum) frame. Specifically, the time difference between the two is

$$\Delta t_L = t_{2L} - t_{1L} = -2\gamma x = v(x_{2L} - x_{1L}), \tag{6.15}$$

and is always negative. To describe two boosted strings of masses m moving along the x-axis in opposite directions with the velocities v and $-v$ and the impact parameter $2b$, we shift the boosted wedge (6.14) by b in the positive y direction and introduce a second wedge, $v \mapsto -v'$ and $y \mapsto -y$:

$$\begin{aligned}
t_{1L} &= \gamma(t + vx), & x_{1L} &= \gamma(x + vt), & y_{1L} &= b + x \cot \frac{m}{2}, \\
t_{2L} &= \gamma(t - vx), & x_{2L} &= \gamma(-x + vt), & y_{2L} &= b + x \cot \frac{m}{2}, \\
t'_{1L} &= \gamma(t' - vx'), & x'_{1L} &= \gamma(x' - vt'), & y'_{1L} &= -b - x' \cot \frac{m}{2}, \\
t'_{2L} &= \gamma(t' + vx'), & x'_{2L} &= \gamma(-x' - vt'), & y'_{2L} &= -b - x' \cot \frac{m}{2},
\end{aligned} \tag{6.16}$$

where primed variables describe the second wedge. We have now constructed the Gott spacetime in a single Cartesian chart (t_L, x_L, y_L), corresponding to the laboratory frame, subject to the two wedge identifications $(t_{1L}, x_{1L}, y_{1L}) \sim (t_{2L}, x_{2L}, y_{2L})$ and $(t'_{1L}, x'_{1L}, y'_{1L}) \sim (t'_{2L}, x'_{2L}, y'_{2L})$. If we choose t' and x' such that $\gamma(t+vx) = \gamma(t'-vx')$ and $\gamma(x+vt) = \gamma(x'-vt')$, then the closest approach of the strings corresponds to $t_L = 0$.

6.4.2 CTCs of the Gott Spacetime

Following Gott, we now consider the curves composed of two pieces of geodesics with $x_L = \mathsf{const}$, as shown on the Fig. 6.3 left. The first piece connects the two wedges at the points $p'_1 = (t'_{1L} = -T, x'_{1L} = a, y'_{1L} = -VT)$ and $p_1 = (t_{1L} = T, x_{1L} = a, y_{1L} = VT)$. Here V is the velocity of the observer traveling the geodesics. The travel time along the geodesic T is

$$T = \frac{b + a\gamma \cot \frac{m}{2}}{V + v\gamma \cot \frac{m}{2}}.$$

The second piece connects the two wedges at the points $p_2 = (t_{2L} = -T, x_{2L} = -a, y_{2L} = VT)$ and $p'_2 = (t'_{2L} = T, x'_{2L} = -a, y'_{2L} = -VT)$. For the two geodesics to form a closed curve, the initial point of one must be the final point of another: $p_1 \sim p_2$ and $p'_1 \sim p'_2$. In this case the coordinate time $2T$ taken to travel from one wedge to another is balanced exactly by the backward time jump across the wedge between $x_{1L} = a$ and $x_{2L} = -a$. Using (6.15), we get

$$2T + \Delta t_L = 2T + v(x_{2L} - x_{1L}) = 0, \tag{6.17}$$

or $T = av$, resulting in the relation $V = b/(av) + \frac{1}{\gamma v} \cot m/2$. The curve is a CTC for $V < 1$, which is possible for large enough a whenever $\gamma v > \cot m/2$. This also sets the lower limit on a for a given boost:

$$a_{\min} = \frac{\gamma b}{\gamma v - \cot \frac{m}{2}}$$

for this choice of geodesics.

These $x_L = \mathsf{const}$ CTCs are not the only ones possible. Cutler [99] has determined the (null) boundary of the region containing CTCs, and it turns out that, while there are closed null curves passing closer to the origin than $a_{\min}$, no CTCs pass through the origin and all CTCs go counter-clockwise around the origin.

Moreover, even when CTCs are present, there always exist a neighborhood of each string free of CTCs. As a consequence, "smoothing out" the strings in a small enough neighborhood does not affect the CTCs, as noted by Cutler.

6.4.3 Unwrapping Gott Spacetime

The existence of the minimum value for a CTC's distance from the $x = y = 0$ line implies that all CTCs wrap around it in the direction opposite to the relative motion of the strings (they also wrap around both strings at some finite distance from each). The Gott spacetime is flat away from the strings, so we can apply the same unwrapping procedure as in section 6.3: we remove the subspace $x = y = 0$ and construct the universal covering space of the resulting non-simply connected manifold (we assume that the original Gott spacetime is simply connected, as the strings can be smoothed out). Since in the Gott's choice of identifications neither wedge crosses the x-axis, the resulting space can be described using countably many copies of the Cartesian (t, x, y) charts, with the charts n and $n + 1$ joined along the non-negative x-axis of each one, as shown on Fig. 6.4a.

This unwrapped Gott spacetime is simply connected, has a quasi-regular singularity at $x = y = 0$ and admits no CTCs, no matter how fast the strings are moving relative to each other. It also contains a countable infinity of pairs of boosted strings. The presence of the singularity removes the

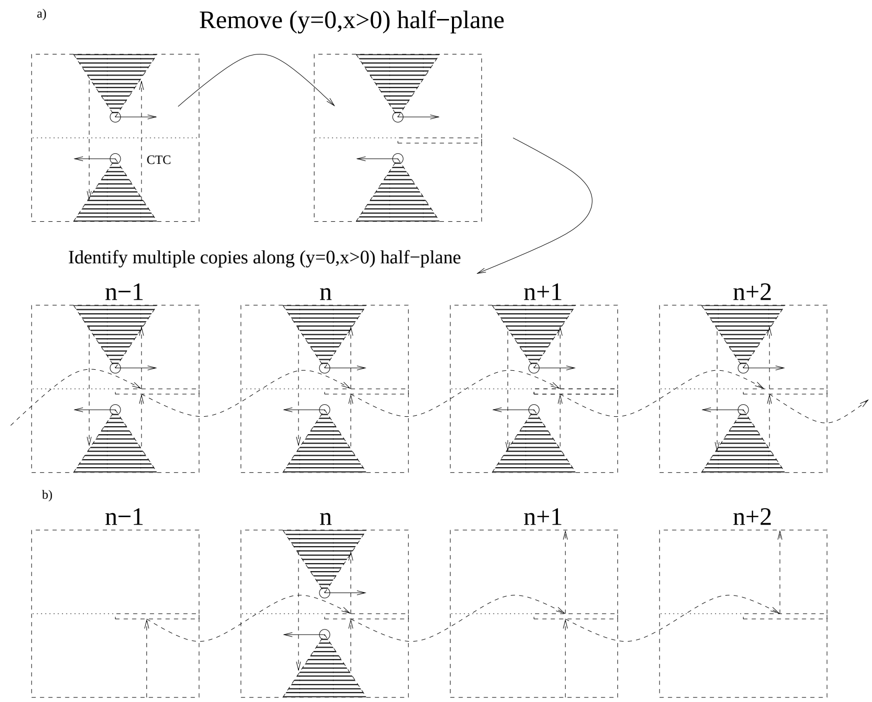

Figure 6.4: Unwrapping Gott spacetime. a) The $x = y = 0$ subspace is removed and the universal cover is constructed by patching multiple Cartesian charts together. The former CTCs (indicated by the vertical arrows) are now open curves passing from chart to chart. The identifications between charts are indicated by the wavy arrows. b) An alternative way to unwrap: a single Gott chart is matched to a collection of Minkowski charts. The former CTCs wrap around just the two strings in the single Gott chart.

restriction on the total mass (deficit angle) of all strings, which can now be arbitrarily large, though each string's mass still cannot exceed π.

A timelike or null observer starting on the nth chart and traveling around both strings counter-clockwise ends up on the $(n+1)$th chart after crossing the positive x-axis. This ensures that no such curve is closed, and so no CTCs are present in the unwrapped Gott spacetime. Instead, what used to be CTCs are now open curves winding around two strings per turn.

It is worth noting that the unwrapped Gott spacetime described above is not the only way to unwrap the CTCs. Since the surface where each two Cartesian charts are joined is locally flat, we do not have to have the two moving strings present on more than one chart. For example, all but one chart can be Minkowski, as shown on Fig. 6.4.

We have shown that one can indeed change the identifications in the Gott spacetime such that no CTCs are present. The trade-off for the CTCs removal is introduction of a naked quasi-regular singularity. This singularity is timelike and so it is present in any possible initial data set, making any initial value formulation problematic.

We next turn to another $(2+1)$-dimensional spacetime with CTCs, the Gödel universe, and demonstrate that a straightforward CTCs unwrapping does not work there.

6.5 Unwrapping Gödel Spacetime

In this section we construct a spacetime which is locally Gödel at every point, but with a different global structure, such that a given set of CTCs in the original spacetime is not longer closed in the unwrapped spacetime. This is not the only way one can get rid of the Gödel CTCs. One obvious way to remove CTCs from the Gödel space is to restrict the radial coordinate in the chart (6.5), thus creating a boundary where the orbits of ϕ are still spacelike. Examples of this are given in [101, 102], where a preferred holographic screen is constructed at the radial distance $\sinh r = 1/\sqrt{2}$, where the expansion of the congruence of null geodesics emanating from a point at $r = 0$ is zero. A generalization of this approach to higher-dimensional Taub-NUT-AdS spacetimes is done in [103]. Another approach, considered in [104], is to match the Gödel interior to an exterior spacetime without CTCs. There the metric is explicitly changed locally in the regions where the CTCs used to exist.

6.5.1 Unwrapping Procedure and Circular CTCs

We now consider a particular example of unwrapping the Gödel space with respect to a given family of CTCs. The metric is given in the (τ, r, ϕ) coordinate chart by the expression (6.5). This chart is singular at $r = 0$, but this is just a coordinate singularity, as the transformation (6.4) shows. Since the spacetime is homogeneous, this chart can be constructed using any point in the spacetime as its origin. The CTCs manifest in this chart are the coordinate curves of ϕ when $\sinh r > 1$. They are not geodesics, but they are isometries of the spacetime, since the metric in this chart does not depend on ϕ. These CTCs have been extensively studied (see e.g., [2]). Cooperstock and Tieu, in [12], "question the continuation of identifying the ϕ values of 0 and 2π when ϕ becomes a timelike coordinate". Since it is impossible to abruptly change the coordinate identification of ϕ only at $\sinh r \geq 1$ without breaking the regularity of the spacetime at $\sinh r = 1$, we will instead remove this identification everywhere.

The Gödel spacetime is axisymmetric, and so the unwrapping procedure is rather straightforward. We start with the chart (6.2) of the Gödel spacetime $\mathscr{M}$. In this chart the CTCs that are coordinate curves of ϕ cross the positive x-axis in the counter-clockwise direction, like the CTCs of the Gott spacetime. We remove the $x = y = 0$ subspace (corresponding to a single fiber of the Killing vector ∂_t, which also coincides with ∂_τ at $r = 0$) and construct a universal covering space of the resulting non-simply connected manifold $\mathscr{M}'$. The resulting spacetime can be described by either a single global chart (6.5) with $\phi \in \mathbb{R}$ or by a collection of countably infinitely many Cartesian charts with the charts n and $n + 1$ joined along the positive x-axis of each one. The description using a single cylindrical chart is possible because the subspace $r = 0$, where (6.5) is not defined, has been removed, and so this chart is valid everywhere in the unwrapped spacetime.

As expected, the removed subspace makes the unwrapped Gödel space singular, and inextensible. Any curve that passed through the subspace $r = 0$ in $\mathscr{M}$ is incomplete in $\mathscr{M}'$ and hence in the unwrapped space, as well.

All circular CTCs cross the positive x-axis in $\mathscr{M}'$ at least once, so their unwrapped-space analogs in the nth Cartesian chart end up on the $n + 1$ chart after crossing the axis and so cannot be closed. This corresponds to the orbits of ϕ in the chart (6.5) being open in the unwrapped space.

6.5.2 Remaining Circular CTCs

While there are no $r = \mathsf{const} > \sinh^{-1} 1$ CTCs in the unwrapped Gödel spacetime, this is not the only kind of circular CTC present in the Gödel metric. Since the Gödel spacetime is homogeneous, we can construct a cylindrical chart (6.5) around any point and obtain CTCs winding around that point. If this new, shifted origin of the cylindrical chart is “far enough” from the old, unshifted one, then the CTCs around it will lie wholly on a single sheet of the space unwrapped around the unshifted origin, and so will remain CTCs even in the unwrapped space.

To demonstrate this, it is convenient to use the quotient space $\tilde{\mathscr{M}}$ of the $(2+1)$-dimensional Gödel space $\mathscr{M}$ with the fiber defined by the orbits of the timelike Killing vector $y : \mathscr{M} \to \tilde{\mathscr{M}}$. The metric h_{ab} of the quotient space is calculated as

$$h_{ab} = g_{ab} - \frac{y_a y_b}{y^c y_c}$$

(see e.g., [37]). Since y^a is timelike everywhere, with a non-vanishing norm, h_{ab} is nowhere singular, the reduced space is Riemannian, and its line element in the original Cartesian coordinates is just a flat two-dimensional space

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}t^2. \tag{6.18}$$

An arbitrary circular CTC in the full space, parametrized by $0 \le \phi \le 2\pi$, is completely defined by its center (t_0, x_0, y_0) and radius R. It can be written using the equivalent of (6.4) as

$$\begin{aligned} e^{x-x_0} &= \cosh 2R + \cos\phi \sinh 2R \\ (y-y_0)e^x &= \sqrt{2}\sin\phi\sinh 2R \\ \tan\frac{1}{2}\left(\phi + \frac{t-t_0}{\sqrt{2}}\right) &= e^{-2R}\tan\frac{1}{2}\phi. \end{aligned} \tag{6.19}$$

An image of this CTC in the flat quotient space $\tilde{\mathscr{M}}$ is obtained by omitting the coordinate y from (6.19):

$$\begin{aligned} e^{x-x_0} &= \cosh 2R + \cos\phi \sinh 2R \\ \tan\frac{1}{2}\left(\phi + \frac{t-t_0}{\sqrt{2}}\right) &= e^{-2R}\tan\frac{\phi}{2}, \end{aligned} \tag{6.20}$$

which can be rewritten in an explicit form as

$$\begin{aligned} x &= x_0 + \ln\cosh 2R + \cos\phi\sinh 2R \\ t &= t_0 + 2\sqrt{2}\tan^{-1}\frac{(e^{-2R}-1)\tan\frac{\phi}{2}}{1 + e^{-2R}\tan^2\frac{\phi}{2}}. \end{aligned} \tag{6.21}$$

All CTCs with the same values of x_0 and t_0, but with a different y_0 are mapped into the same closed curve. The image of the singular boundary of the unwrapped space is the point $x = t = 0$ of the quotient space.

We note the following properties of a curve described by (6.21):

1. It is inscribed in a rectangle centered at the point (x_0, t_0) and with sides $2R$ and $2\tan^{-1}\sinh R$, and

2. It circumscribes an ellipse inscribed into this rectangle, as described by equation (6.22).

$$\left(\frac{x - x_0}{2R}\right)^2 + \left(\frac{t - t_0}{2\sqrt{2}\tan^{-1}\sinh R}\right)^2 = 1. \tag{6.22}$$

The property 1 follows from the ranges of x and t in (6.21) for a given R, while the property 2 can be shown by substituting (6.21) into the left-hand side of (6.22), finding the four local minima of the resulting function of ϕ and showing that they give exactly the equality (6.22). At small R the curve (6.21) tends closer to (6.22), while at large R it asymptotically approaches the rectangle, as shown on Fig. 6.5.

We can now show that there exist CTCs that are not unwrapped by the singular boundary resulting from unwrapping around the origin in the chart (6.5). Since the image of any CTC winding around the boundary has to wrap around the image of the boundary in the reduced space, constructing the image of a CTC that does not wrap the image of the singularity is enough to show the existence of CTCs that are not unwrapped. Since the image of any CTC of radius R with, say, $|x_0| > 2R$ or $|t_0| > 2\sqrt{2}\tan^{-1}\sinh R$ does not wrap around $x = t = 0$, all such curves remain closed after unwrapping.

Thus the naïve coordinate identification change of [12] fails to unwrap at least some of the circular CTCs.

6.6 Multiple Snwrapping of the Gödel Spacetime

By unwrapping the Gödel spacetime we have introduced a quasi-regular singularity into it, yet we did not accomplish the goal of removing all CTCs from it. If one remains intent on also removing circular CTC, one may consider a multiple unwrapping instead, as described at the end of the section 6.3.3.

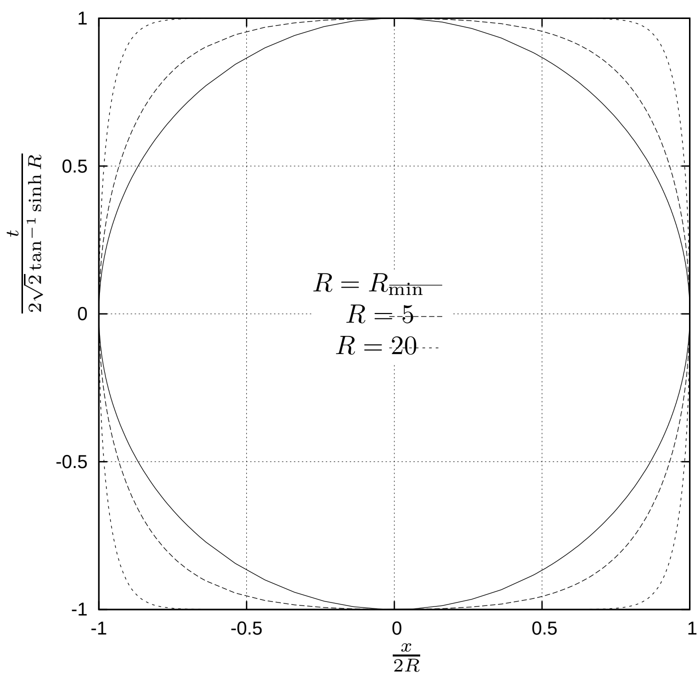

Figure 6.5: Normalized concentric CTCs of radius R illustrated in the coordinates $\left(\frac{x}{2R}, \frac{t}{2\sqrt{2}\tan^{-1}\sinh R}\right)$. The CTCs lie inside a square with the side equal to two, but outside of a circle inscribed into it.

6.6.1 Double Unwrapping

We first consider what happens when we unwrap the CTCs in just two charts. We show that even in this simple case the unwrapped space contains a countably infinite number of quasi-regular singularities.

This "double unwrapping" procedure would look as follows. We consider two different families of circular CTCs, one centered at $x = x_1, y = y_1$ and $x = x_2, t = t_2$ respectively. The subspaces $(x = x_1, t = t_1)$ and $(x = x_2, t = t_2)$ are the fixed points of the corresponding $U(1)$ isometries represented by the CTCs. As before, we first remove these fixed points from the spacetime $\mathscr{M}$, only this time we have to remove both sets, resulting in the singular space $\mathscr{M}''$. Next we construct the universal covering space $\bar{\mathscr{M}}$ of $\mathscr{M}''$ and lift the metric tensor to $\bar{\mathscr{M}}$. Since the original $(2+1)$-Gödel manifold is $\mathbb{R}^3$, we do not need to worry about accidentally removing any topological features unrelated to unwrapping. In this case $\mathscr{M}''$ is homotopy-equivalent to the wedge sum of two circles $\mathbb{S}^1 \vee \mathbb{S}^1$, known as the "figure 8" (see e.g., [35, Chapter 1]). This can be shown by explicitly constructing a deformation retraction, an operation that preserves the fundamental group of a manifold. To do that, we first note that, since the orbits of y are open lines, they can be retracted into points by the continuous map $f_s : (t, x, y) \mapsto (t, x, (1-s)y)$. f_0 is the identity map, and f_1 maps $\mathscr{M}''$ into a two-dimensional plane with two points, (x_1, t_1) and (x_2, t_2), removed. Following [35], we next retract the plane first onto two circles (one around each removed point) connected by a line segment, then contracting the connecting segment into a point. The resulting space is the "figure 8". As a result of the retraction, the singularities now "fill the inside of the circles". The fundamental group of $\mathscr{M}''$ is the fundamental group of the "figure 8", which is just the free product of two copies of $\mathbb{Z}$, $\pi_1 = \mathbb{Z} * \mathbb{Z}$. Each element of the group corresponds to winding around one of the two singularities in $\mathscr{M}''$.

The universal cover of the "figure 8" is well known, it is a tree with countably infinitely many edges and each node connecting four edges (see e.g., [35] for construction). The process of constructing the universal cover of the twice punctured plane is shown schematically on Fig. 6.6. Each edge of this graph corresponds to a CTC winding around one of the removed subspaces of fixed points. In the unwrapped space this CTC becomes open and corresponds to a given path along the graph. Traversing one edge corresponds to "going around" one of the singularities, so there is a one-to-one correspondence between the singularities in the twice-unwrapped Gödel and the edges in the graph.

We can now conclude that unwrapping around two axes at once in a

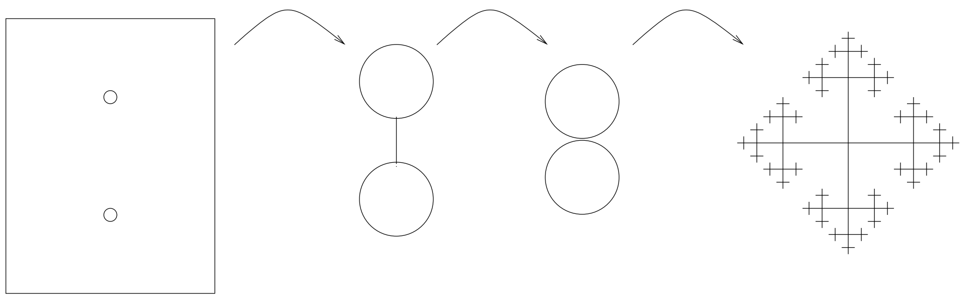

Figure 6.6: Constructing the homotopy equivalence of the Gödel space after double unwrapping. We start with the $(t, x, y = 0)$ subspace with two points removed (left), then retract the space onto first two circles connected by a line segment, then the "figure 8", and finally construct the universal cover of the "figure 8" (only the first four levels of nodes are shown). Each edge corresponds to a circle wrapping around one of the two singularities in the original space, so the unwrapped space contains infinitely many singularities.

simply connected spacetime results in a spacetime with a countable infinity of quasi-regular singularities of the type discussed in section 6.3.2.

6.6.2 Multiple Unwrapping

As discussed in section 6.5.2, there are infinitely many families of concentric CTCs parametrized by (t_0, x_0, y_0) for which at least some CTCs persist after unwrapping around $(x = 0, t = 0)$. According to the property 2, each CTC lies outside an ellipse (6.22). All such ellipses are larger than the image of a closed null curve corresponding to $\sinh R = 1$. If we modify the spacetime in a way that transforms any such ellipse into an open curve, the resulting spacetime will not have any circular CTCs. This can be accomplished by first tessellating the quotient space (x, t) with a dense enough triangular lattice, such that there is a vertex inside any ellipse with $\sinh R = 1$, then lifting it into the full three-dimensional Gödel spacetime using y as the fiber and finally by going to the universal covering space, in a procedure analogous to the one described in section 6.6.1.

The tessellation is shown schematically on Fig. 6.7. The fundamental group of the tessellated space is the free group on $\mathbb{Z}$ generators, one for each removed point. A small patch of the unwrapped quotient space is illustrated on the Fig. 6.8. Each helix corresponds to a family of unwrapped concentric circular CTCs. The price to pay for removing all circular CTCs is the introduction of a naked singular boundary consisting of a countable

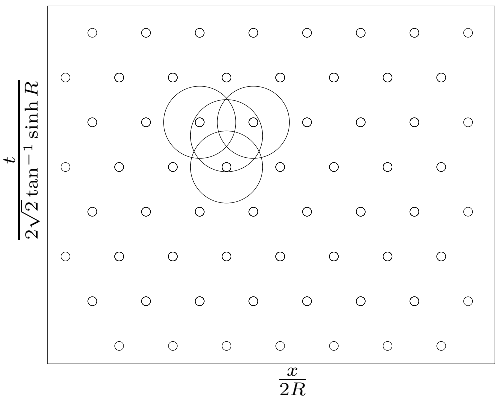

Figure 6.7: A tessellation of the two-dimensional quotient space of the Gödel spacetime. Each vertex corresponds to a timelike line in the full spacetime and each circle is an image of a closed null curve $R = R_{\text{min}}$. The tessellation is dense enough to make any circular CTCs wrap around at least one such line, as shown. Once the lines are removed and the resulting non-simply connected spacetime is lifted into the full spacetime and then into its universal cover, no circular CTCs are present in this "multiply unwrapped" spacetime.

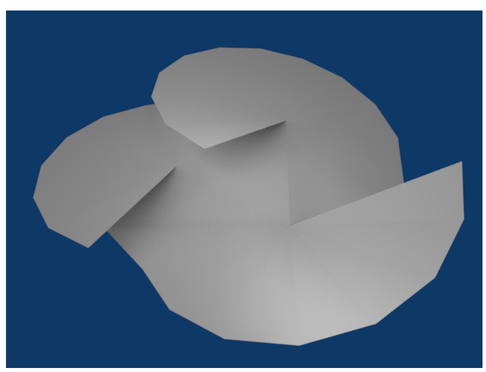

Figure 6.8: A visualization of the two-dimensional quotient space of the Gödel spacetime unwrapped at three points at once. Only a small patch of this space is shown, as unwrapping around two or more points results in a countable infinity of singular boundaries.

infinity of disjoint pieces.

6.6.3 Sector-like CTCs in the Gödel Spacetime

One can ask whether any other types of CTCs are present in the multiply unwrapped Gödel spacetime. For example, is it possible to weave one's way in between the vertexes and come back to the starting point along a CTC? This seems unlikely and we conjecture that no such CTCs exist. To support this conjecture we describe a different kind of CTCs, we call sector-like CTCs, and show that they, too, are transformed into open curves by the multiple unwrapping procedure of the section 6.6.2.

The idea of constructing a CTC surviving the tessellation of the section 6.6.2 is to exploit the property of the Gödel spacetime where an arc with a larger radius but a smaller angular distance can get us just as far back in time in the chart (6.5). The hope is then to go far along a timelike curve in a radial direction, then along an arc, then back to the starting point, thus covering a sector instead of a full circle in this chart. If the resulting sector is thin enough, then we can try to fit it in the tessellated space in such a way that no vertexes are inside the sector.

To check if this can be done, we calculate the angular and linear distance along the arc required to overcome the time lost traveling forward and back along the two radial directions. Since a closed null sector would be "thinner" than the corresponding timelike sector, we analyze the null sector first.

The (negative) time change in the coordinate τ along a null arc of radius R and angle $\Delta\phi$ in the chart (6.5) can be calculated as

$$\Delta\tau = (\sinh^2 R(\sqrt{2} - \coth R))\Delta\phi, \tag{6.23}$$

and it has to compensate for the positive time change of $2R$ along the two radii of the sector, resulting in the total angular change of

$$\Delta\phi = \frac{2R}{\sinh^2 R(\sqrt{2} - \coth R)}. \tag{6.24}$$

To see if the sector is thin enough to fit between the vertexes of the lattice for large enough R, we project it into the flat quotient space (6.18). The three legs of the path written in the (τ, r, ϕ) coordinates and parametrized

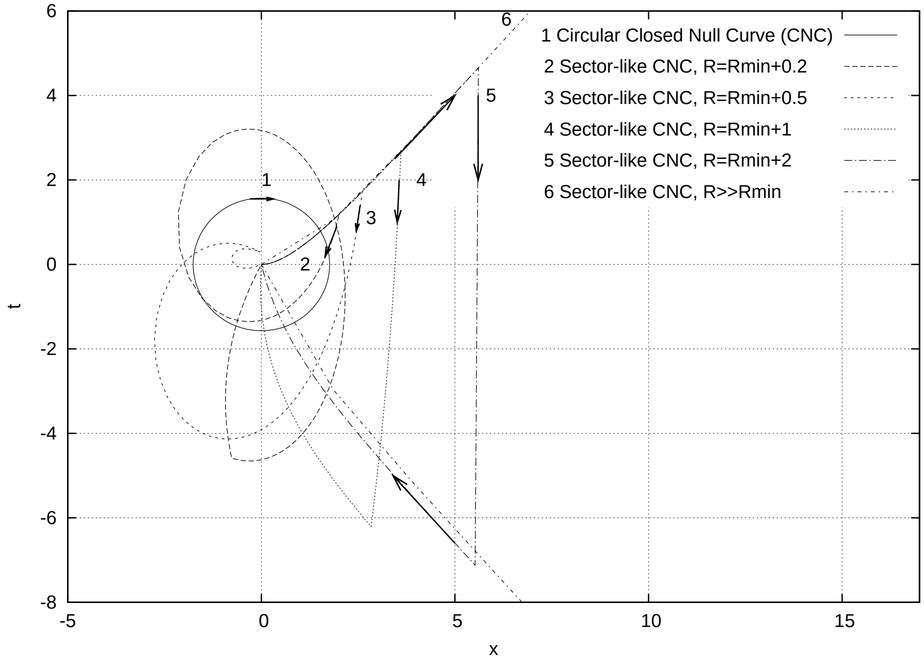

Figure 6.9: Images of sector-like closed null curves in the (x, t) coordinates for a range of values R. Arrows indicate the future null directions. For large R the curve tends to a triangle with a fixed minimum angle. The angle increases as R goes down, and for R close to the minimum value of $\sinh R_{\min} = 1$ the null curve has to wrap around the origin several times to compensate for the time lost along the radial paths.

by λ are

$$\begin{cases} (\lambda, \lambda, 0) & \text{if } 0 \le \lambda \le R, \\ \left(2R - \lambda, R, \dfrac{\lambda - R}{\sinh^2 R(\sqrt{2} - \coth R)}\right) & \text{if } R \le \lambda \le 3R, \\ \left(\lambda - 4R, 4R - \lambda, \dfrac{2R}{\sinh^2 R(\sqrt{2} - \coth R)}\right) & \text{if } 3R \le \lambda \le 4R. \end{cases} \tag{6.25}$$

The same curve in the (x, t) chart of (6.18) can be described using the explicit coordinate transformation (6.21). The resulting curves are plotted for several values of R on Fig. 6.9. For large R the three turning points of the path in the (x, t) chart asymptotically approach $(0, 0)$, $(2R, 2\sqrt{2}R)$, $(2R, -2\sqrt{2}R)$ respectively. Thus, no matter how large R is, the tessellation dense enough to unwrap circular CTCs also unwraps the sector-like closed null (and therefore timelike) curves. In this sense, the original CTCs described by Gödel appear to have the smallest "footprint" in the flat quotient space.

Whether or not there are other CTCs that persist in the multiply unwrapped spacetime, it is quite clear that removing CTCs from the Gödel spacetime solely by changing the coordinate identification results in a rather contrived space, given its countable infinity of naked quasi-regular singularities.

6.7 Conclusion

In this chapter we have defined and investigated "unwrapping" CTCs in two $(2+1)$-dimensional toy models, the Gödel spacetime and the Gott spacetime, as a concrete implementation of the claim by Cooperstock and Tieu [12] that the periodic identification of a timelike coordinate is "purely artificial". The procedure requires removing a timelike line from the spacetime and constructing a universal cover of the resulting non-simply connected spacetime. We have demonstrated that such an unwrapping creates a naked quasi-regular singularity, corresponding to the removed timelike line in the original space. The same argument was extended to any locally axisymmetric spacetime where CTC wrap around the axis, as is the case in the Gott spacetime.

While the unwrapped Gott spacetime is devoid of CTCs, the unwrapped Gödel spacetime still contains them. We have defined a "multiple unwrapping" of the Gödel spacetime in order to remove the remaining circular CTCs. As a result, this multiply unwrapped spacetime contains a countably infinite number of singularities. We conjecture that this multiple unwrapping removes all other CTCs as well, and support it by giving an explicit example of a sector-like CTC, which is also removed by the multiple unwrapping.

Our investigation into the ways of removing the CTCs by means of changing coordinate identifications resulting in unwrapping suggests that CTCs appearing in the solutions of the Einstein equation are not a mathematical artifact of arbitrary coordinate identifications, but rather are an unavoidable, if an undesirable, consequence of general relativity. Different ways to extend the same local coordinate patch of a pathological spacetime may lead to different pathologies, such as CTCs or naked quasi-regular singularities, but are unlikely to result in a physically acceptable regular spacetime.

Chapter 7

Conclusion

This thesis describes the research in the area of axisymmetric dimensional reduction in general relativity, with an emphasis on the pathologies created by the reductions preserving asymptotic flatness. Starting with some simple and highly symmetric four-dimensional examples and proceeding to rather complicated ones, we have shown that treating an axisymmetric spacetime as the one with one lower dimension without this symmetry, while an attractive idea, results in a number of issues which are likely to hamper an investigator unless the pathologies in question are explicitly dealt with. Specifically, the stress-energy tensor of a reduction preserving asymptotic flatness is shown to induce negative energy density into the reduced spacetime. In the reduction from four to three dimensions the pressures exceed energy density leading to superluminal matter flows being seen by some observers. We have shown this by treating a general asymptotically flat spacetime as a perturbation on the Schwarzschild one.

For the special, but still interesting case of a $U(1)$ reduction from four to three dimensions, we applied the topological censorship as well as the topological rigidity theorem (proved in this thesis for the reduction in question) to show that no such $U(1)$ reduction from an AF spacetime to either an AF or ALADS one can satisfy the null energy condition.

A next step in this research would be an investigation of the potential adverse effects the pathological stress-energy tensor may have on the currently popular numerical evolution schemes that take advantage of spacetime symmetries, and the ways to address them. Another potential topic of further research would be an application of the topological rigidity to an axisymmetric reduction, not just a $U(1)$ reduction.

Another application of the dimensional reduction techniques was the investigation of the nature of the closed timelike curves in the Gott and Gödel spacetimes. We have analyzed a claim that such curves are mathematical artifacts and found that any attempt to unwrap them results in the type of non-curvature singularity, dubbed here an unwrapping singularity, that cannot be removed by extending the spacetime through it. Furthermore, in the case of the Gödel spacetime even repeated unwrapping does not remove all of the closed timelike curves. An interesting question not addressed in this thesis is whether the properties of the Gödel spacetime can be inferred from the properties of the timelike Killing vector that lifts a flat Euclidean space into the Gödel spacetime.

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