Modular Invariance of Closed Bosonic String Theory with a PP-Wave Background

by

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Abstract

After a brief review of the necessary parts of the theory of the bosonic string, a consistent pp-wave background with constant dilaton and constant three-index antisymmetric field strength is introduced. In particular, the gravitational background is the plane wave with constant coefficients, and the antisymmetric field strength is chosen such that the worldsheet theory is both diff$\times$Weyl invariant and stable. The one-loop closed bosonic string amplitude is evaluated and shown to be modular invariant. Then the free energy of a free closed string gas is calculated, modular invariance of it is proved, and the result is shown to be equivalent to the sum of free energies for the individual particle states.

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Dedication

This thesis is dedicated to Sheri, for her strong motivational effect.

Chapter 1

Introduction

A Feynman diagram in perturbative bosonic string theory is described by some number of bosonic fields defined on a two-dimensional *worldsheet*, and coupled to massless string background fields. For the oriented closed bosonic string, these massless strings are the graviton, axion, and dilaton. Interpreting the graviton background as a spacetime metric, the bosonic fields can be seen as embedding coordinates into this spacetime, whose number is just the dimensionality of the spacetime. This gives a notion of spacetime length, and we find that this interpretation makes sense provided the graviton background is smooth in the sense that it varies slowly on the scale of the string length.

A worldsheet metric can be introduced in the construction of the theory, provided the theory is assumed to be Weyl invariant in addition to the expected diffeomorphism invariance. This is necessary to ensure there is a BRST symmetry present, which is needed to remove all of the negative norm states, and thus ensure unitarity. Imposing these symmetries on the worldsheet theory results in dynamic field equations that the backgrounds must satisfy. The *moduli* for a given worldsheet topology are the classes of worldsheet metrics that are equivalent up to a local diff$\times$Weyl transformation, and the *modular group* group of transformations connecting equivalent moduli spaces. We are interested here in the question of whether or not physical quantities can depend on the particular choice of moduli space used to describe the worldsheet metric, that is, whether or not the theory is modular invariant.

Our study builds upon work done by Longton in [3]. It centres around the theory of the oriented closed bosonic string in a particular type of pp-wave background, known as the plane wave. This background has become of interest in string theory through the discovery of the AdS/CFT correspondence. Specifically, by a particular so called "Penrose" limit, the relevant geometry for particles with large angular momentum travelling on the sphere of $\mathrm{AdS}_5 \times \mathrm{S}^5$ looks like a plane wave, where strings can be quantized exactly. The string spectrum is reproduced in a particular limit of $\mathcal{N} = 4$ Super Yang Mills theory [4].

We will verify modular invariance of the one loop amplitude, as well as the free energy of a free string gas. As a nice check, we'll reproduce the free energy by calculating the sum of free energies for the individual particle states using the closed string spectrum calculated by Longton [3] and reproduced here.

Chapter 2

Review

In this chapter we review the theory of the oriented closed bosonic string in a general spacetime background as described by Polchinski in [7].

2.1 Strings in flat spacetime

2.1.1 The Nambu-Goto action

A string moving in D-dimensional Minkowski spacetime – with metric signature $(-,+,+,\ldots,+)$ – traces out a surface that can be described by the embedding into spacetime of a two-dimensional manifold, M, known as the worldsheet. The embedding can be defined in terms of D real-valued coordinate functions, X^μ, defined on the worldsheet.

In general, the topology of the worldsheet depends on how the string interacts – interactions introduce handles. As well, open strings in the theory introduce boundaries on the worldsheet. For our purposes, it will suffice to develop the theory with only closed strings. Thus in moving from some event to another, if the string has not interacted then the worldsheet is topologically just a cylinder.

To propose an action for a free string, we require that it be invariant under Poincaré transformations in spacetime so that strings obey special relativity. We also require that it be invariant under worldsheet diffeomorphisms so that the physics does not depend the choice of worldsheet coordinates. The simplest action consistent with these symmetries is just the invariant spacetime area of the worldsheet,

$$S_{\mathrm{NG}}\,[X] = -\frac{1}{2\pi\alpha'}\int_M d^2\sigma\,(-\det h_{ab})^{1/2}\,, \tag{2.1}$$

where $1/2\pi\alpha'$ can be interpreted as the string's tension, and $h_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ is the induced metric of the worldsheet's embedding into spacetime. This is completely analogous to the action for a free particle which is proportional to the invariant spacetime length of its world line.

2.1.2 The Polyakov action

If we introduce a Lorentzian worldsheet metric, γ_{ab}, then the following action, known as the Polyakov action, is classically equivalent to the Nambu-Goto action,

$$S_{\mathrm{P}}[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\,(-\gamma)^{1/2}\,\gamma^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{2.2}$$

provided the worldsheet metric satisfies the classical equations of motion,

$$\frac{\delta S_{\mathrm{P}}}{\delta\gamma^{ab}} = 0. \tag{2.3}$$

This equivalence can be easily shown using the formula for the variation of a determinant,

$$\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma\gamma_{ab}\delta\gamma^{ab}. \tag{2.4}$$

Being quadratic in the fields, X^μ, the Polyakov action is much simpler to work with than the Nambu-Goto action.

Notice that the introduction of a worldsheet metric has introduced the following redundancy: invariance under local Weyl transformations of the worldsheet,

$$\gamma_{ab} \mapsto e^{2\omega}\gamma_{ab}, \tag{2.5}$$

where ω is any smooth, real-valued function on M. Thus, we will denote the group of classical worldsheet symmetries, diff $\times$ Weyl.

The string spectrum could be calculated at this point using the light-cone gauge. This is detailed in [7], but since we will be performing a similar calculation in detail in section 4.3.1, we will not reproduce it here. The main result we would be interested in achieving from this is the fact that there are states which transform under Lorentz transformations as massless symmetric and antisymmetric tensors, as well as massless scalar states. These are the graviton, axion, and dilaton, respectively. But we would also find that there are only massless states in the theory as long as $D = 26$. Thus, in order to preserve Lorentz invariance, we must impose this condition.

We will find it more useful, however, to work with a Riemannian worldsheet metric, g_{ab}. Up to a conventional minus sign, the Polyakov action is then

$$S_{\mathrm{P}}[X,g] = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\,g^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{2.6}$$

It turns out that either choice of worldsheet metric signature produces the same scattering amplitudes, so we will not worry here about this sleight of hand.

2.1.3 String interactions

Assuming the worldsheet has no boundary, a natural generalization of the Polyakov action is to add a term of the form, $\lambda\chi$, where

$$\chi = \frac{1}{4\pi}\int_M d^2\sigma\sqrt{g}\,R, \tag{2.7}$$

and λ is some constant. This is because χ is simply the Euler number of the worldsheet, a quantity that depends only its topology; therefore, this addition preserves the local diff $\times$ Weyl invariance of the Polyakov action. This can easily be generalized to include worldsheet boundaries by adding a boundary integral, but we will not need this. The natural way of considering interacting strings is by simply allowing for handles in the worldsheet, so this additional term in the action also serves to give each interaction a nontrivial coupling strength. The introduction of this term will seem much more natural as we further develop our theory.

2.1.4 Scattering amplitudes

For a given worldsheet topology, we have essentially developed a classical field theory of D free scalars in two-dimensions. In the usual way then, the vacuum amplitude for a given diagram (worldsheet topology) can be written,

$$Z = \int \frac{[dX\,dg]}{V_{\mathrm{diff}\times\mathrm{Weyl}}}\exp\left(-S_{\mathrm{P}}\left[X,g\right]-\lambda\chi\right). \tag{2.8}$$

Since this is a vacuum to vacuum amplitude the worldsheet can be assumed to be compact. The path integral runs over all Riemannian metrics, g_{ab}, on the worldsheet, and all embeddings, X^μ, of the worldsheet into spacetime. $V_{\mathrm{diff}\times\mathrm{Weyl}}$ is the volume of the worldsheet symmetry group, and it is included in order to compensate for the fact that the path integral counts configurations that are physically indistinguishable from one another, that is, configurations that are related by a local diff $\times$ Weyl transformation. By including this factor we have implicitly required any diffeomorphism or Weyl anomaly to vanish. It is not absurd for us to want to require this at this point since the worldsheet metric was essentially introduced as an unphysical auxiliary variable, and because we need a source for the constraint, $D = 26$; forcing a potential Weyl anomaly to vanish would be a natural source for this in our formalism. We will have more to say on how to ensure this below. We should also note here that the path integral over X^0 is a wrong-sign Gaussian, and is only defined after an appropriate analytic continuation.

String sources can be described by local disturbances on the worldsheet. Using the diff × Weyl invariance, such a disturbance can be shrunk down to a single, arbitrary point. So for a given string source with momentum k^μ and internal state j, there corresponds a local diff × Weyl-invariant *vertex operator* of the form,

$$\int d^2\sigma \sqrt{g(\sigma)}\, \mathcal{V}_j(k;\sigma). \tag{2.9}$$

The S-matrix for a given scattering process is then gotten by inserting the appropriate vertex operators into the above path integral, then summing over all compact topologies (the ways the strings can interact),

$$S_{j_1,\ldots,j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int \frac{[dX\, dg]}{V_{\text{diff}\times\text{Weyl}}} \exp\left(-S_{\text{P}}\left[X,g\right] - \lambda\chi\right) \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{g(\sigma_i)}\, \mathcal{V}_{j_i}(k_i;\sigma_i). \tag{2.10}$$

2.2 General spacetime background

We now generalize the Polyakov action to general Lorentzian spacetimes with background fields. We will be interested only in oriented closed strings. The most general action that remains invariant under rigid Weyl transformations on the worldsheet is then,

$$S_\sigma\left[X,g\right] = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g}\, \left(g^{ab} G_{\mu\nu}\left(X\right) + i\epsilon^{ab} B_{\mu\nu}\left(X\right)\right) \partial_a X^\mu \partial_b X^\nu + \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g}\, R\Phi\left(X\right). \tag{2.11}$$

Here $G_{\mu\nu}(X)$ is a symmetric spacetime tensor background, $B_{\mu\nu}(X)$ is an antisymmetric spacetime tensor background, and $\Phi(X)$ is a spacetime scalar background. The antisymmetric tensor, ϵ^{ab}, is defined by $\sqrt{g}\,\epsilon^{12} = 1$. The spacetime background fields, $G_{\mu\nu}(X), B_{\mu\nu}(X)$, and $\Phi(X)$ are just exponentiations around the flat spacetime background in the Polyakov action of the vertex operators for the massless graviton, axion, and dilaton states, respectively, so this choice of generalization turns out to be quite natural. The previous generalization of the Polyakov action is then seen to be just a particular dilaton background. So that the form (2.10) for the S-matrix holds by simply replacing the Polyakov action with the above, we pull out a constant, λ, from the definition of the dilaton background.

To see why it is important to keep in mind that the string is assumed to be closed, notice that an open string could couple to a vector background, A_μ, through a term involving an integral over the boundary of the worldsheet. In the same way as above, we would find this to be nothing but a coherent state of photons. Furthermore, the unoriented string theory contains only string states that are invariant under worldsheet parity transformations. This transformation reverses the signs of the axion state, as well as the photon state, so we must consider only oriented strings in order to be able to couple to an axion background.

The reason we required the backgrounds to satisfy particular spacetime coordinate transformation properties is so that general spacetime coordinate invariance is satisfied, as this is the generalization to the spacetime Poincaré invariance in the Polyakov action. Also notice the presence of the generalized electromagnetic gauge symmetry,

$$\delta B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu. \tag{2.12}$$

This allows us to introduce the gauge invariant three-index antisymmetric field strength,

$$H_{\omega\mu\nu} = \partial_\omega B_{\mu\nu} + \partial_\nu B_{\omega\mu} + \partial_\mu B_{\nu\omega}. \tag{2.13}$$

This gauge invariance can be fixed for a given field strength by choosing any antisymmetric background whose exterior spacetime derivative equals the field strength, again assuming the theory only contains closed strings.

As in the flat spacetime case we assume the vanishing of any Weyl anomaly. The quantum energy momentum tensor, T^{ab}, is defined in terms of a variation with respect to the metric of the path integral with general insertions,

$$\delta\big\langle \dots \big\rangle_g = -\frac{1}{4\pi}\int_M d^2\sigma g(\sigma)^{1/2}\delta g_{ab}(\sigma)\big\langle T^{ab}(\sigma)\dots\big\rangle_g. \tag{2.14}$$

So for a Weyl variation,

$$\delta_W\big\langle \dots \big\rangle_g = -\frac{1}{2\pi}\int_M d^2\sigma g(\sigma)^{1/2}\delta\omega(\sigma)\left\langle T^a{}_a(\sigma)\dots\right\rangle_g. \tag{2.15}$$

Thus, Weyl invariance is equivalent to the operator statement that the energy momentum tensor is traceless.

It is given in [7] that

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu \partial_b Z^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R, \tag{2.16}$$

and to first order in α',

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}, \tag{2.17a}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu}, \tag{2.17b}$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\lambda \Phi \nabla^\lambda \Phi - \frac{\alpha'}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}. \tag{2.17c}$$

In these expansions of the beta functions, all terms with up to two spacetime derivatives are kept. Thus, the theory is Weyl invariant if the above beta functions vanish. This gives us the equations of motion that the background fields must satisfy, along with the number of spacetime dimensions, D.

It is immediately seen that the Minkowski spacetime metric with vanishing axion and constant dilaton satisfies these equations with $D = 26$, as stated previously in terms of the light-cone gauge formalism.

2.3 The S-matrix

2.3.1 Moduli and CKVs

The metric can not always be completely fixed by the diff $\times$ Weyl symmetry since for a given worldsheet topology, the *moduli space*,

$$\mathcal{M} = \frac{\mathcal{G}}{\text{diff} \times \text{Weyl}}, \tag{2.18}$$

is not always trivial. Here, $\mathcal{G}$ is the space of worldsheet metrics. Furthermore, a choice of gauge slice may not completely fix the gauge, i.e. the group of residual symmetries known as the *conformal Killing group (CKG)* may be nontrivial as well.

The diff $\times$ Weyl redundancy is studied at the infinitessimal level by Polchinski in [7]. Noting that there is a one-to-one correspondence between Riemann surfaces and orientable Riemannian manifolds mod Weyl, he shows that variations in the moduli correspond to *holomorphic quadratic differentials*, and the infinitessimal elements of the CKG, called *conformal Killing vectors (CKVs)*, correspond to *holomorphic vector fields*. We let their total numbers be μ and κ, respectively, and note that they depend only on the worldsheet topology. In the Riemann surface picture, the CKG is then just the group of global conformal transformations.

2.3.2 Gauge-fixing the Polyakov path integral

We are now ready to gauge-fix the Polyakov path integral for the S-matrix,

$$S_{j_1,\ldots,j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int \frac{[dX\, dg]}{V_{\text{diff}\times\text{Weyl}}} \exp\left(-S_\sigma\left[X,g\right] - \lambda\chi\right) \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{g(\sigma_i)}\, \mathcal{V}_{j_i}(k_i;\sigma_i). \tag{2.19}$$

By the discussion above, we know that a particular gauge slice, $\hat{g}(t)$, – which we refer to as the *fiducal metric* – must specify a point, t, in a μ-dimensional moduli space. Furthermore, the remaining symmetries can be fixed by fixing κ of the vertex operator coordinates, $\sigma_i^a \to \hat{\sigma}_i^a$. With $[d\zeta]$ the measure on the diff $\times$ Weyl group, we convert the integral over metrics and vertex operator positions to an integral over the gauge group, the moduli, and the unfixed positions:

$$[dg]\, d^{2n}\sigma \to [d\zeta]\, d^\mu t\, d^{2n-\kappa}\sigma. \tag{2.20}$$

To get the Jacobian for this transformation, we define the Faddeev-Popov determinant by

$$1 = \Delta_{\text{FP}}\left(g,\sigma\right) \int_{\mathcal{M}} d^\mu t \int_{\text{diff}\times\text{Weyl}} [d\zeta]\, \delta\left(g - \hat{g}(t)^\zeta\right) \prod_{(a,i)\in f} \delta\left(\sigma_i^a - \hat{\sigma}_i^{\zeta a}\right), \tag{2.21}$$

where the set of fixed coordinates, (a,i), has been labelled f. It turns out there may be a residual group of symmetries of finite order, n_R, so the arguments in the delta functions vanish at n_R points.

Following the usual Faddeev-Popov procedure [1], the scattering amplitude becomes,

$$S_{j_1,\ldots,j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int_{\mathcal{M}} d^\mu t\, \Delta_{\text{FP}}\left(\hat{g}(t),\hat{\sigma}\right) \int [dX] \int \prod_{(a,i)\notin f} d\sigma_i^a$$
$$\times \exp\left(-S_\sigma\left[X,\hat{g}(t)\right] - \lambda\chi\right) \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)}\, \mathcal{V}_{j_i}(k_i;\sigma_i). \tag{2.22}$$

Also by the Faddeev-Popov procedure, and as shown in Polchinski [7], the Faddeev-Popov determinant can be written in terms of integrals over

Grassmann variables, b_{ab} and c^a, where b is traceless and symmetric in its indices. It is

$$\Delta_{\mathrm{FP}}(\hat{g},\hat{\sigma}) = \frac{1}{n_R}\int [db\ dc]\ \exp\left(-S_g[b,c]\right)\prod_{k=1}^{\mu}\frac{1}{4\pi}\left(b,\partial_k\hat{g}\right)\prod_{(a,i)\in f} c^a(\hat{\sigma}_i), \tag{2.23}$$

where

$$S_g[b,c] = \frac{1}{4\pi}\left(b, 2\hat{P}_1 c\right) \tag{2.24}$$

is the *ghost action*, the inner product for traceless symmetric 2-tensors is defined by

$$\left(b,b'\right) = \int d^2\sigma\sqrt{g}\ b^{ab}\ b'_{ab}, \tag{2.25}$$

and the differential operator P_1 takes vectors into traceless symmetric 2-tensors by

$$\left(P_1 c\right)_{ab} = \frac{1}{2}\left(\nabla_a c_b + \nabla_b c_a - g_{ab}\nabla_d c^d\right). \tag{2.26}$$

The scattering amplitude can then be written,

$$S_{j_1,\ldots,j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}}\int_{\mathcal{M}}\frac{d^\mu t}{n_R}\int [dX\, db\, dc]\ \exp\left(-\hat{S}_\sigma - \hat{S}_g - \lambda\chi\right)$$
$$\times\prod_{(a,i)\notin f}\int d\sigma_i^a\prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_k\hat{g})\prod_{(a,i)\in f}c^a(\hat{\sigma}_i)\prod_{i=1}^{n}\sqrt{\hat{g}(\sigma_i)}\ \mathcal{V}_{j_i}(k_i;\sigma_i). \tag{2.27}$$

2.4 The torus

Using the complex coordinates,

$$z = \sigma^1 + i\sigma^2, \quad \bar{z} = \sigma^1 - i\sigma^2, \tag{2.28}$$

the worldsheet can be described as a Riemann surface. The torus can then be generally described by the lattice quotient,

$$T^2 = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}, \tag{2.29}$$

with metric, $ds^2 = dz\, d\bar{z}$, for some τ in the upper half-plane. We refer to τ as the *modular parameter*.

The lattice quotient is invariant under transformations of the modular parameter of the form,

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{2.30}$$

where $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = 1$. This is just the group, $SL(2, \mathbb{Z})$. Furthermore, these transformations are left invariant if we take all of the parameters to their negatives, so the symmetry group is really $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, which we refer to as the *modular group*, Γ. It can be shown that its generators are

$$\mathcal{T} : \tau \to \tau + 1, \quad \mathcal{S} : \tau \to -1/\tau. \tag{2.31}$$

A *fundamental domain* of Γ is any region of the upper half-plane whose points can not be reached by each other via any modular transformation, and any point outside it can be reached from unique point inside it by some modular transformation. A direct consequence of this definition is that there exits a unique modular transformation bijectively mapping any given pair of fundamental domains. It can be shown that the region, F_0, defined by

$$z \in F_0 \quad \text{if} \quad \operatorname{Im} z > 0, \quad \text{and} \quad \begin{cases} -\frac{1}{2} \leq \operatorname{Re} z \leq 0, & |z| \geq 1 \\ 0 < \operatorname{Re} < \frac{1}{2}, & |z| > 1 \end{cases}, \tag{2.32}$$

is a fundamental domain of Γ. The fundamental domains of Γ are shown graphically in figure 2.1.

The coordinate transformation,

$$\left(\sigma'^1, \sigma'^2\right) = \left(\sigma^1 - \frac{\tau_1}{\tau_2}\sigma^2, \frac{1}{\tau_2}\sigma^2\right), \tag{2.33}$$

takes the lattice's unit cell to the unit square, and the metric becomes

$$\left(\hat{g}_{ab}(\tau)\right) = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{2.34}$$

Thus, we can also describe the torus with the coordinate region, $0 \leq \sigma^1, \sigma^2 \leq 1$, appropriate periodicity conditions, and the metric,

$$ds^2 = \left|d\sigma^1 + \tau d\sigma^2\right|^2, \tag{2.35}$$

and we can identify the moduli space of the torus to be the set of all (τ_1, τ_2), with $\tau = \tau_1 + i\tau_2 \in F_0$.

The global conformal transformations on the torus are just the rigid translations on the torus, so there are two CKVs for the torus.

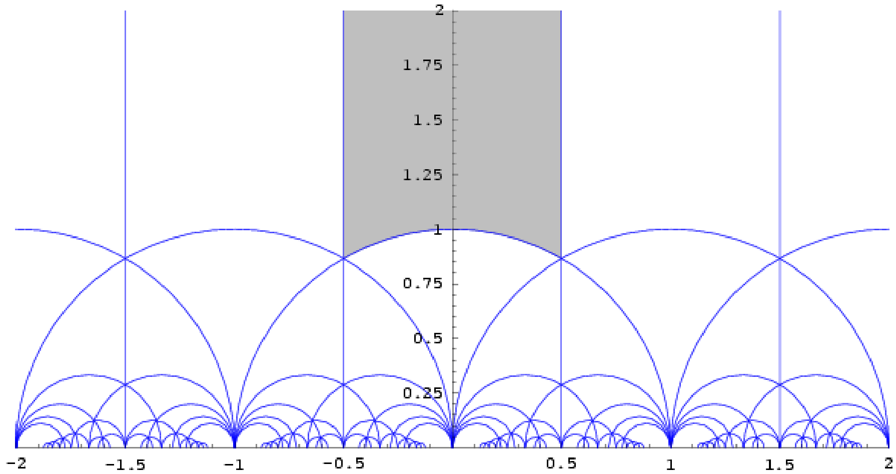

Figure 2.1: The fundamental domains of Γ, with F_0 shown explicitly. [2]

2.4.1 The one loop vacuum amplitude

The two CKVs require one vertex operator to be fixed in the contribution from tori in the scattering amplitude, (2.27). But the vertex operators are invariant under rigid translations – the CKG for the torus – so we can just take the fixed vertex operator's average over all points since the volume of the CKG is just the volume of the torus, and is finite. In terms of the complex coordinates with unit metric described above, the result is

$$S_{T^2}(1,\ldots,n) = \int_{F_0} \frac{d\tau\, d\bar{\tau}}{4\tau_2} \int [dX\, db\, d\tilde{b}\, dc\, d\tilde{c}] \exp\left(-\hat{S}_\sigma - \hat{S}_g\right) \\ \times b(0)\tilde{b}(0)\tilde{c}(0)c(0) \prod_{i=1}^{n} \int dz_i d\bar{z}_i\, \mathcal{V}_i(z_i,\bar{z}_i), \quad (2.36)$$

where the dilaton term has vanished since the Euler number, χ, vanishes for the torus.

The one loop vacuum amplitude is the above, but without any vertex operators, and it is shown in [7] to be

$$Z_{T^2} = \int_{F_0} \frac{d\tau\, d\bar{\tau}}{4\tau_2} |\eta(\tau)|^4 \int [dX]_{\hat{g}} \exp\left(-\hat{S}_\sigma\right), \quad (2.37)$$

where $\eta(\tau)$ is the *Dedekind eta function* described in appendix B.

Chapter 3

PP-Wave Background

We now describe the pp-wave spacetime background we will be working in. Light-cone spacetime coordinates are defined in terms of Cartesian ones by

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^1 \right). \tag{3.1}$$

In terms of these, the Minkowski spacetime metric is

$$ds^2 = -2dX^+ dX^- + dX^i dX^i. \tag{3.2}$$

The remaining Cartesian coordinates here, $i = 2, \ldots, D-1$, are referred to as the transverse coordinates.

In light-cone spacetime coordinates, the pp-wave spacetime is the same as the Minkowski spacetime, but with one additional term:

$$ds^2 = -\mu_{ij} X^i X^j dX^+ dX^+ - 2dX^+ dX^- + dX^i dX^i, \tag{3.3}$$

where μ_{ij} is some $D-2$ dimensional, symmetric, real matrix.

$$\frac{\partial}{\partial X^0} \tag{3.4}$$

still serves as a globally timelike Killing vector in this spacetime, so we can still interpret X^0 as the time coordinate. This is actually a special case of a pp-wave spacetime, known as a plane wave spacetime.

We consider as a background for the theory we will study, the above pp-wave spacetime metric along with a constant three-index antisymmetric field strength and constant dilaton, $\Phi(X) = \lambda$, which we pull out of our definition of the action by convention.

It is shown in [3] that for this background, the vanishing of the beta functions (2.17) yields the conditions, $D = 26$, all components of $H_{\mu\nu\rho}$ vanish except for H_{+ij} and the others related by antisymmetry, and

$$\sum_{i,j=2}^{25} {H_{+ij}}^2 = 4\,\mathrm{tr}(\mu). \tag{3.5}$$

We can fix the generalized electromagnetic gauge symmetry (2.12) by choosing the only nonvanishing components of $B_{\mu\nu}$ to be

$$B_{+i}(X) = \frac{1}{2} H_{+ij} X^j, \tag{3.6}$$

and those related by antisymmetry.

The action is then

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \Big(-\mu_{ij} X^i X^j g^{ab} \partial_a X^+ \partial_b X^+ - 2 g^{ab} \partial_a X^+ \partial_b X^- \\ + g^{ab} \partial_a X^i \partial_b X^i + i \epsilon^{ab} H_{+ij} X^j \partial_a X^+ \partial_b X^i \Big). \tag{3.7}$$

Completing the square, we find

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{ab} \Bigg[-2 \partial_a X^+ \partial_b X^- \\ + \left(\partial_a X^i - \frac{i}{2} \epsilon_a{}^{\bar{a}} \partial_{\bar{a}} X^+ H_{+ij} X^j \right) \left(\partial_b X^i - \frac{i}{2} \epsilon_b{}^{\bar{b}} \partial_{\bar{b}} X^+ H_{+i\bar{j}} X^{\bar{j}} \right) \\ + \left(\frac{1}{4} H_{+ij} H_{+i\bar{j}} - \mu_{j\bar{j}} \right) X^j X^{\bar{j}} \partial_a X^+ \partial_b X^+ \Bigg]. \tag{3.8}$$

To obtain this we have used the identity,

$$\epsilon^{a\bar{a}} \epsilon^{b\bar{b}} = g^{ab} g^{\bar{a}\bar{b}} - g^{a\bar{b}} g^{\bar{a}b}, \tag{3.9}$$

from which we find

$$g_{ab} \epsilon^{a\bar{a}} \epsilon^{b\bar{b}} = g^{\bar{a}\bar{b}}. \tag{3.10}$$

From condition (3.5), the trace of

$$\frac{1}{4} H_{+ij} H_{+i\bar{j}} - \mu_{j\bar{j}} \tag{3.11}$$

must vanish. This means that either the matrix itself vanishes or it has at least one negative eigenvalue. In the latter case, we would find worldsheet instabilities in the theory. This could be interesting; however, we will not consider this possibility here. Thus, we take

$$H_{+ij} H_{+i\bar{j}} = 4 \mu_{j\bar{j}}. \tag{3.12}$$

Furthermore, because $[H_{+ij}]$ is skew-symmetric, we can assume that it is skew-diagonal,

$$[H_{+ij}] = \begin{bmatrix} 0 & 2\mu_1 & 0 & 0 & \cdots & 0 & 0 \\ -2\mu_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2\mu_2 & \cdots & 0 & 0 \\ 0 & 0 & -2\mu_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2\mu_{12} \\ 0 & 0 & 0 & 0 & \cdots & -2\mu_{12} & 0 \end{bmatrix}, \tag{3.13}$$

provided we make the substitution in the action, $H_{+ij} \to O_{i\bar{i}} O_{j\bar{j}} H_{+\bar{i}\bar{j}}$, for some orthogonal matrix, O. But then the original form of the action can be preserved by making the transverse coordinate transformation, $X^i \to O_{i\bar{i}} X^{\bar{i}}$. Note that the path integral measure is left invariant by such a rotation of the transverse coordinates, so there is no loss of generality in just assuming at this point that $[H_{+ij}]$ is skew-diagonal. We will assume for simplicity that $\mu_j \neq 0$ for any j, although our following results can be easily generalized if this is not the case.

If we organize the coordinates into complex pairs,

$$Z^j = \frac{1}{\sqrt{2}} \left(X^{2j} + i X^{2j+1} \right), \quad \bar{Z}^j = \frac{1}{\sqrt{2}} \left(X^{2j} - i X^{2j+1} \right), \quad j = 1, \ldots, 12, \tag{3.14}$$

then we find for our action,

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{ab} \Bigg[-\partial_a X^+ \partial_b X^- + \left(\partial_a - \mu_j \epsilon_a{}^{\bar{a}} \partial_{\bar{a}} X^+ \right) \bar{Z}^j \left(\partial_b + \mu_j \epsilon_b{}^{\bar{b}} \partial_{\bar{b}} X^+ \right) Z^j \Bigg]. \tag{3.15}$$

Note that the path integral measure is invariant under both the previous rotation of our spacetime coordinates, and the change to complex coordinates.

Chapter 4

Modular Invariance

We now calculate the one loop vacuum amplitude, as well as the thermal partition function for a free string gas, and show the results are modular invariant. As a check, we reproduce the free energy of the free string gas by summing the free energies of the individual particle states.

4.1 The one loop vacuum amplitude

In section 2.4, the one loop vacuum amplitude (2.37) was calculated for a general spacetime background in terms of the fiducal metric, (2.35). So we start by evaluating the matter path integral for the pp-wave background,

$$Z_m\left[g\right] = \int \left[dX\right]_g \, \exp\left(-S\right), \tag{4.1}$$

with action, S, given by (3.15).

This path integral is ill-defined as written, so we analytically continue the spacetime coordinate X^0 by performing a Wick-rotation: $X^0 = iX_E$. Note that the path integral measure is left invariant under this transformation. The light-cone spacetime coordinates (3.1) then satisfy $X^+ = -\bar{X}^-$, which motivates the definition of the new coordinates $Z^0 = X^+$ and $\bar{Z}^0 = -X^-$. The action is then

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{ab} \Bigg[\partial_a \bar{Z}^0 \partial_b Z^0 + \left(\partial_a - \mu_j \epsilon_a{}^{\bar{a}} \partial_{\bar{a}} Z^0\right) \bar{Z}^j \left(\partial_b + \mu_j \epsilon_b{}^{\bar{b}} \partial_{\bar{b}} Z^0\right) Z^j \Bigg]. \tag{4.2}$$

Using the fact that the torus has no boundary, the matter path integral can be written,

$$Z_m[g] = \int [dZ^0 d\bar{Z}^0]_g F[Z^0, g] \exp\left(\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \bar{Z}^0 \nabla^2 Z^0\right), \tag{4.3}$$

where

$$F[Z^0, g] = \prod_{j=1}^{12} \int [dZ^j d\bar{Z}^j]_g \exp\left(-\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \bar{Z}^j \mathcal{O}_j(Z^0, g) Z^j\right), \quad (4.4)$$

and

$$\mathcal{O}_j(Z^0, g) = -\frac{1}{\sqrt{g}} \left(\partial_a + \mu_j \epsilon_a{}^{\bar{a}} \partial_{\bar{a}} Z^0\right) \sqrt{g} g^{ab} \left(\partial_b + \mu_j \epsilon_b{}^{\bar{b}} \partial_{\bar{b}} Z^0\right). \quad (4.5)$$

Since $F[c, g] = F[0, g]$ for any constant, c, the functional integral appearing in equation (4.3) is of the same type appearing in equation (A.13), so

$$Z_m[g] = \left(\frac{\beta L^1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\right) \left[\det{}' \left(-\nabla^2/2\pi\right)\right]^{-1} F[0, g], \quad (4.6)$$

where β and L_1 are the lengths of the compactified X_E and X^1 directions, respectively, and the prime omits the contribution to the determinant from the zero mode.

Now, $\mathcal{O}_j(0, g) = -\nabla^2$, so we can use equation (A.13) again to find that

$$\begin{aligned} F[0, g] &= \prod_{j=1}^{12} \left[\left(\frac{L^{2j} L^{2j+1}}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\right) \left[\det{}' \left(-\nabla^2/2\pi\right)\right]^{-1}\right] \\ &= \left(\prod_{i=2}^{25} L^i\right) \left(\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\right)^{12} \left[\det{}' \left(-\nabla^2/2\pi\right)\right]^{-12}, \end{aligned} \quad (4.7)$$

where for $i = 2, \ldots, 25$, L^i is the length of the compactified X^i direction.

Letting V be the volume of the compactified spatial coordinates,

$$V = \prod_{i=1}^{25} L^i, \quad (4.8)$$

we can write,

$$Z_m[g] = \beta V \left(\frac{2\pi\alpha'}{\int d^2\sigma \sqrt{g}} \det{}' \left(-\nabla^2/2\pi\right)\right)^{-13} \quad (4.9)$$

This is the same result obtained in [6] for the matter path integral in the Minkowski spacetime background. It is shown there that the one loop vacuum amplitude is indeed Weyl invariant, provided $D = 26$ and an appropriate worldsheet cosmological constant counterterm is added to the action.

So this serves as a nice check to see that we have constructed our pp-wave background correctly.

Evaluated at the fiducal metric (2.35), the matter path integral is

$$\begin{aligned} Z_m[\hat{g}(\tau)] &= \beta V \left[\frac{2\pi\alpha'}{\tau_2} \det{}' \left(-\frac{1}{2\pi{\tau_2}^2} \left|\partial_2 - \tau\partial_1\right|^2 \right) \right]^{-13} \\ &= \beta V \left[\frac{2\pi\alpha'}{\tau_2} \prod_{n,m\in\mathbb{Z}}{}' \left(\frac{2\pi}{{\tau_2}^2} \left|n + m\tau\right|^2 \right) \right]^{-13} \\ &= \beta V \left(4\pi^2\alpha'\tau_2 \left|\eta(\tau)\right|^4 \right)^{-13}, \end{aligned} \tag{4.10}$$

where the prime on the product in the second equation omits the term with n and m both zero, and in the third equation we have used the regularization (C.21) from the appendix.

Combining this with equation (2.37), the one loop vacuum amplitude is thus,

$$Z_{T^2} = \beta V \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \left(4\pi^2\alpha'\tau_2\right)^{-13} \left|\eta(\tau)\right|^{-48}, \tag{4.11}$$

where the formal limits, $\beta \to \infty$, and $V \to \infty$ are understood to be taken.

To show this is modular invariant, it suffices to prove modular invariance under the modular group generators, $\mathcal{T} : \tau \to \tau + 1$, and $\mathcal{S} : \tau \to -1/\tau$. Noting that under $\mathcal{S}$, $d\tau d\bar{\tau} \mapsto d\tau d\bar{\tau}/|\tau|^4$, and $\tau_2 \mapsto \tau_2/|\tau|^2$, we see that

$$\frac{d\tau d\bar{\tau}}{{\tau_2}^2} \tag{4.12}$$

is modular invariant, and by equations (B.3), $\tau_2 \left|\eta(\tau)\right|^4$ is as well. Modular invariance of the one loop vacuum amplitude is then easy to see.

4.2 The thermal partition function for a free string gas

If we instead keep β finite, then the sum over all connected diagrams is interpreted as the thermal partition function for a gas of free strings with inverse temperature β. The leading β-dependence in this comes from tori with nontrivial windings around the X_E direction.

The spacetime coordinates Z^0 and $\bar{Z}^0$ should now include labels identifying winding numbers, $r_1, r_2 \in \mathbb{Z}$, not both zero. We split these up into separate periodic and linear terms:

$$
\begin{aligned}
Z_r^0(\sigma) &= Z^0(\sigma) + \frac{i\beta}{\sqrt{2}} r_a \sigma^a, \\
\bar{Z}_r^0(\sigma) &= \bar{Z}^0(\sigma) - \frac{i\beta}{\sqrt{2}} r_a \sigma^a.
\end{aligned}
\tag{4.13}
$$

Performing this change of variables in the matter path integral, we find

$$
\begin{aligned}
Z_m[g] = \sum_{r_1,r_2} & \exp\left(-\frac{\beta^2}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}r_a r_b\right) \\
& \int [dZ^0 d\bar{Z}^0]_g F_r[Z^0,g] \exp\left(\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\bar{Z}^0\nabla^2 Z^0\right),
\end{aligned}
\tag{4.14}
$$

where

$$
F_r[Z^0,g] = \prod_{j=1}^{12}\int [dZ^j d\bar{Z}^j]_g \exp\left(-\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\bar{Z}^j\mathcal{O}_{j,r}(Z^0,g)Z^j\right),
\tag{4.15}
$$

and

$$
\begin{aligned}
\mathcal{O}_{j,r}(Z^0,g) = -\frac{1}{\sqrt{g}} & \left(\partial_a + \mu_j \epsilon_a{}^{\bar{a}}\partial_{\bar{a}}Z^0 + \frac{i\beta}{\sqrt{2}}\mu_j\epsilon_a{}^{\bar{a}}r_{\bar{a}}\right) \\
& \times \sqrt{g}g^{ab}\left(\partial_b + \mu_j\epsilon_b{}^{\bar{b}}\partial_{\bar{b}}Z^0 + \frac{i\beta}{\sqrt{2}}\mu_j\epsilon_b{}^{\bar{b}}r_{\bar{b}}\right).
\end{aligned}
\tag{4.16}
$$

The functional integral,

$$
\int [dZ^0 d\bar{Z}^0]_g F_r[Z^0,g]\exp\left(\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\bar{Z}^0\nabla^2 Z^0\right),
\tag{4.17}
$$

is evaluated exactly as in the previous section, and we obtain for it,

$$
\left(\frac{\beta L^1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\right)\left[\det{}'\left(-\nabla^2/2\pi\right)\right]^{-1} F_r[0,g].
\tag{4.18}
$$

To evaluate $F_r[0,g]$, we assume that $\mathcal{O}_{j,r}(0,g)$ has no zero modes, and use the result (A.8) from the appendix. This assumption will prove to be correct later on when we evaluate the resulting functional determinant. We find

$$
F_r[0,g] = \prod_{j=1}^{12}\left[\det\left(\mathcal{O}_{j,r}(0,g)/2\pi\right)\right]^{-1}.
\tag{4.19}
$$

Thus,

$$Z_m[g] = \left(\frac{\beta L^1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\right)\left(\det'\left(-\nabla^2/2\pi\right)\right)^{-1}$$
$$\times \sum_{r_1,r_2}{}' \exp\left(-\frac{\beta^2}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}r_a r_b\right)$$
$$\times \prod_{j=1}^{12}\left[\det\left(-\frac{1}{2\pi\sqrt{g}}\left(\partial_a + \frac{i\beta}{\sqrt{2}}\mu_j {\epsilon_a}^{\bar{a}} r_{\bar{a}}\right)\sqrt{g}g^{ab}\left(\partial_b + \frac{i\beta}{\sqrt{2}}\mu_j {\epsilon_b}^{\bar{b}} r_{\bar{b}}\right)\right)\right]^{-1}, \tag{4.20}$$

where the prime on the sum omits the $r_1 = r_2 = 0$ term, and we have explicitly written out $\mathcal{O}_{j,r}(0, g)$.

Evaluated at the fiducal metric (2.35), the determinant in the square brackets in the above becomes,

$$\prod_{n_1,n_2\in\mathbb{Z}}\left[\frac{2\pi}{{\tau_2}^2}\left|\left(n_2 + \frac{\beta}{2\sqrt{2}\pi}\mu_j {\epsilon_2}^a r_a\right) - \tau\left(n_1 + \frac{\beta}{2\sqrt{2}\pi}\mu_j {\epsilon_1}^a r_a\right)\right|^2\right]$$
$$= \prod_{n,m\in\mathbb{Z}}\left[\frac{2\pi}{{\tau_2}^2}\left|n + m\tau - i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r_1 - \tau r_2\right)\right|^2\right]$$
$$= \left|\eta(\tau)^{-1}\vartheta_{11}\left(-i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r_1 - \tau r_2\right);\tau\right)\right|^2 \exp\left(-\frac{\beta^2{\mu_j}^2\left(r_1 - \tau_1 r_2\right)^2}{4\pi\tau_2}\right), \tag{4.21}$$

where in the second equation we used

$$\left(-{\epsilon_2}^a + \tau{\epsilon_1}^a\right)r_a = \left(-g_{2b} + \tau g_{1b}\right)\epsilon^{ba}r_a$$
$$= i\left(\delta_b^1 + \tau\delta_b^2\right)\tau_2\epsilon^{ba}r_a = i\left(r_1 - \tau r_2\right), \tag{4.22}$$

and in the last equation we have used the regularization (C.15) from the appendix. It is clear now that our assumption that $\mathcal{O}_{j,r}(0, g)$ has no zero modes was correct.

Thus,

$$Z_m\left[\hat{g}(\tau)\right] = \beta L^1 \left(4\pi^2\alpha'\tau_2\left|\eta(\tau)\right|^4\right)^{-1} {\sum_{r,s\in\mathbb{Z}}}' \exp\left(-\frac{\beta^2\left|r+s\tau\right|^2}{4\pi\alpha'\tau_2}\right)$$
$$\times\prod_{j=1}^{12}\left[\left|\eta\left(\tau\right)^{-1}\vartheta_{11}\left(i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r+s\tau\right);\tau\right)\right|^{-2}\exp\left(\frac{\beta^2{\mu_j}^2\left(r+s\tau_1\right)^2}{4\pi\tau_2}\right)\right]. \tag{4.23}$$

Combining this with equation (2.37) then gives us the thermal partition function for a gas of free strings to leading order in β. To get its free energy, divide this by $-\beta L^1$:

$$F(\beta) = -\frac{1}{4\pi^2\alpha'}{\sum_{r,s\in\mathbb{Z}}}'\int_{F_0}\frac{d\tau d\bar{\tau}}{4{\tau_2}^2}\exp\left(-\frac{\beta^2\left|r+s\tau\right|^2}{4\pi\alpha'\tau_2}\right)$$
$$\times\prod_{j=1}^{12}\left[\left|\eta\left(\tau\right)^{-1}\vartheta_{11}\left(i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r+s\tau\right);\tau\right)\right|^{-2}\exp\left(\frac{\beta^2{\mu_j}^2\left(r+s\tau_1\right)^2}{4\pi\tau_2}\right)\right]. \tag{4.24}$$

To see that this is modular invariant, we need only prove it for the generators, $\mathcal{T}$, and $\mathcal{S}$. If we accompany $\mathcal{T}$ with a change of summation variables, $(r', s') = (r+s, s)$, then invariance of the free energy is clear from the identities, (B.3). For $\mathcal{S}$, we need to accompany it with the change of summation variables, $(r', s') = (-s, r)$. Letting

$$\nu = i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r+s\tau\right), \tag{4.25}$$

this has transformed as $\nu \mapsto \nu/\tau$. So again using (B.3), we see that the free energy is invariant since

$$\begin{aligned}-2\,\mathrm{Re}\left(\frac{i\pi\nu^2}{\tau}\right) &+ \frac{2\pi\big(\mathrm{Im}\left(\nu/\tau\right)\big)^2}{\mathrm{Im}\left(-1/\tau\right)}\\ &= -\frac{\pi}{2\tau_2\tau\bar{\tau}}\left[\left(\tau-\bar{\tau}\right)\left(\bar{\tau}\nu^2-\tau\bar{\nu}^2\right)+\left(\bar{\tau}\nu-\tau\bar{\nu}\right)^2\right]\\ &= -\frac{\pi}{2\tau_2}\left(\nu-\bar{\nu}\right)^2 = \frac{2\pi{\nu_2}^2}{\tau_2}.\end{aligned} \tag{4.26}$$

4.3 Consistency check

In this section we find the closed string spectrum in our plane wave background using the light-cone gauge, and we use this to check that the sum of the free energies of the individual particle states reproduces the free energy of the free string gas previously calculated.

4.3.1 The closed string spectrum

The spectrum was produced by Longton in [3], but we will reproduce it here using a slightly different method. The calculation we perform closely parallels that done by Polchinski in [7] for the Minkowski spacetime background.

We start by rewriting the action (3.15) in terms of a Lorentzian worldsheet metric, γ_{ab}:

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma \, (-\gamma)^{1/2} \, \gamma^{ab}\Bigg[-\partial_a X^+ \partial_b X^- + \left(\partial_a - i\mu_j \epsilon_a{}^{\bar{a}} \partial_{\bar{a}} X^+\right) \bar{Z}^j \left(\partial_b + i\mu_j \epsilon_b{}^{\bar{b}} \partial_{\bar{b}} X^+\right) Z^j\Bigg]. \quad (4.27)$$

Then we consider a free string travelling from $X^0 = -\infty$ to $X^0 = \infty$. Because the string doesn't undergo any interactions, it has cylindrical worldsheet topology, so the dilaton term vanishes because the cylinder has vanishing Euler number. We fix the gauge symmetries in the following way: Choose one of the worldsheet coordinates to be $-\infty \leq \tau \leq \infty$, with $X^+ = \tau$. Then the fixed-τ worldsheet cross sections must be topologically circular, so we take the other coordinate to be $0 \leq \sigma \leq l$ with the embedding coordinates and worldsheet metric periodic in σ with period l. To determine the $\sigma = 0$ line up to an overall σ-translation, set it to be orthogonal to lines of constant τ, that is, set $\gamma_{\tau\sigma}(\tau, 0) = 0$. Now note that with $f = \gamma_{\sigma\sigma}\left(-\det \gamma_{ab}\right)^{-1/2}$, $f\, d\sigma$ is invariant under τ-independent reparametrizations of σ, and thus defines an invariant line element on fixed-τ worldsheet cross sections. On each of these cross sections, choose σ at each point to be proportional to the invariant length between that point and the point where $\sigma = 0$, where the proportionality constant is chosen such that $\sigma = l$ for the path going exactly once around the cross section. Note that the direction we go around the cross section is fixed by the orientation on the worldsheet since l is assumed to be positive. We have $\partial_\sigma f = 0$, but since f is Weyl invariant, we can preserve this while setting $\det \gamma_{ab} = -1$, so that we also have $\partial_\sigma \gamma_{\sigma\sigma} = 0$. We have thus fully fixed the gauge up to a choice of orientation on the worldsheet, and up to an overall σ-translation, both of which we will deal with later on. [1] Summarizing these results,

$$X^+ = \tau, \tag{4.28a}$$
$$\gamma_{\tau\sigma}(\tau, 0) = 0, \tag{4.28b}$$
$$\det \gamma_{ab} = -1, \tag{4.28c}$$
$$\partial_\sigma \gamma_{\sigma\sigma} = 0. \tag{4.28d}$$

Now, (4.28c) implies $\gamma^{\tau\tau} = -\gamma_{\sigma\sigma}$ and $\gamma^{\tau\sigma} = \gamma_{\tau\sigma}$, so if we let

$$x^-(\tau) = \frac{1}{l}\int_0^l d\sigma X^-(\tau, \sigma), \tag{4.29}$$
$$Y^-(\tau, \sigma) = X^-(\tau, \sigma) - x^-(\tau) \tag{4.30}$$

so that the mean value of Y^- for any given τ vanishes, then, using our gauge-fixing conditions, we can write the Lagrangian,

$$L = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- - \frac{1}{2\pi\alpha'}\int_0^l d\sigma\Bigg[-\gamma_{\tau\sigma}\partial_\sigma Y^- + \gamma^{ab}\left(\partial_a + i\mu_j\gamma_{a\sigma}\right)\bar{Z}^j\left(\partial_b + i\mu_j\gamma_{b\sigma}\right)Z^j\Bigg], \tag{4.31}$$

where we have taken τ to be the time coordinate. We see that Y^- simply acts as a Lagrange multiplier, constraining $\partial_\sigma\gamma_{\tau\sigma}$ to vanish. But then by equation (4.28b), this must vanish everywhere. So we find, $\gamma^{\sigma\sigma} = {\gamma_{\sigma\sigma}}^{-1}$, and the Lagrangian becomes,

$$L = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{2\pi\alpha'}\int_0^l d\sigma\Bigg[\gamma_{\sigma\sigma}\partial_\tau\bar{Z}^j\partial_\tau Z^j - {\gamma_{\sigma\sigma}}^{-1}\left(\partial_\sigma + i\mu_j\gamma_{\sigma\sigma}\right)\bar{Z}^j\left(\partial_\sigma - i\mu_j\gamma_{\sigma\sigma}\right)Z^j\Bigg]. \tag{4.32}$$

The conjugate momenta are then,

$$p_- = -p^+ = \frac{\partial L}{\partial\left(\partial_\tau x^-\right)} = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}, \tag{4.33}$$
$$\Pi_j = \Pi^j = \frac{\delta L}{\delta\left(\partial_\tau Z^j\right)}, \tag{4.34}$$

[1] There is a potential complication with this gauge for paths that double back through an X^+ hyperplane, but we will not dwell here on this fine point.

so the Hamiltonian is

$$H = p_- \partial_\tau x^- + \int_0^l d\sigma \left(\Pi_j \partial_\tau Z^j + \bar{\Pi}_j \partial_\tau \bar{Z}^j \right) - L \tag{4.35}$$

$$= \frac{l}{2\pi\alpha' p^+} \int_0^l d\sigma \Bigg[2\pi\alpha' \bar{\Pi}^j \Pi^j \tag{4.36}$$

$$+ \frac{1}{2\pi\alpha'} \left(\partial_\sigma + \frac{2\pi i \alpha' \mu_j p^+}{l} \right) \bar{Z}^j \left(\partial_\sigma - \frac{2\pi i \alpha' \mu_j p^+}{l} \right) Z^j \Bigg]. \tag{4.37}$$

Which gives us the following equations of motion,

$$\partial_\tau p^+ = -\frac{\partial H}{\partial x^-} = 0, \tag{4.38}$$

$$\partial_\tau Z^j = \frac{\delta H}{\delta \Pi^j} = 2\pi\alpha' c \, \bar{\Pi}^j, \tag{4.39}$$

$$\partial_\tau \Pi^j = -\frac{\delta H}{\delta Z^j} = \frac{c}{2\pi\alpha'} \left(\partial_\sigma + i \frac{\mu_j}{c} \right)^2 \bar{Z}^j, \tag{4.40}$$

where we have let $c = l/2\pi\alpha' p^+$. The last two can be combined into

$$\left[\left(\partial_\sigma - i \frac{\mu_j}{c} \right)^2 - \frac{1}{c^2} {\partial_\tau}^2 \right] Z^j = 0. \tag{4.41}$$

The mode expansions gotten from these equations of motion are then,

$$Z^j(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \left(\frac{2\pi\alpha'}{l\omega_n^j} \right)^{1/2} e^{-2\pi i n \sigma / l} \left(\alpha_n^j e^{i\omega_n^j c\tau} + \tilde{\alpha}_n^j e^{-i\omega_n^j c\tau} \right), \tag{4.42}$$

$$\Pi^j(\tau, \sigma) = \sum_{n=-\infty}^{\infty} i \left(\frac{\omega_n^j}{2\pi\alpha' l} \right)^{1/2} e^{2\pi i n \sigma / l} \left(-\bar{\alpha}_n^j e^{-i\omega_n^j c\tau} + \bar{\tilde{\alpha}}_n^j e^{i\omega_n^j c\tau} \right), \tag{4.43}$$

where

$$\omega_n^j = \left| \frac{2\pi n}{l} + \frac{\mu_j}{c} \right|. \tag{4.44}$$

We will see the reason for this choice of normalization shortly.

To quantize the theory, take the variables to be operators, where complex conjugation is now taken to be Hermitian conjugation. Then impose the equal time canonical quantization relations,

$$\left[x^-, p^+ \right] = - \left[x^-, p_- \right] = -i, \tag{4.45}$$

$$\left[Z^j(\tau, \sigma), \Pi^k(\tau, \sigma') \right] = \left[Z^j(\tau, \sigma), \Pi_k(\tau, \sigma') \right] = i \delta^j_k \delta(\sigma - \sigma'), \tag{4.46}$$

with all other commutators between independent variables vanishing.

We then find that the nonvanishing commutation relations for the Fourier coefficients are

$$\left[\alpha_n^j, \alpha_m^{k\,\dagger}\right] = \left[\tilde{\alpha}_n^j, \tilde{\alpha}_m^{k\,\dagger}\right] = \delta^{ij}\delta_{n,m}, \tag{4.47}$$

and we see that the normalization for the mode expansions was chosen precisely so that these expressions would simplify this way. Now since the Hamiltonian is bounded from below, the Fourier coefficients are, as we should expect, just creation and annihilation operators. Thus, we can define the number operators, $N_n^j = \alpha_n^{j\,\dagger}\alpha_n^j$, and $\tilde{N}_n^j = \tilde{\alpha}_n^{j\,\dagger}\tilde{\alpha}_n^j$, which have nonnegative integer eigenvalues. The Hamiltonian then becomes,

$$H = \frac{1}{\alpha' p^+} \sum_{j=1}^{12} \sum_{n=-\infty}^{\infty} \left|n + \alpha' \mu_j p^+\right| \left(N_n^j + \tilde{N}_n^j + 1\right), \tag{4.48}$$

and we see that the Hilbert space is just the usual Fock space construction.

The Hamiltonian is just the generator for τ-translations, so since we have set $X^+ = \tau$, we can relate H with the generator of X^+-translations. We find $H = -p_+$. The relative sign arises due to the fact that H-translations are active transformations, and X^+-translations are passive. We have thus found the spectrum, that is, the spacetime Hamiltonian,

$$H_{\text{sp}} = -p_0 = -\frac{1}{\sqrt{2}}\left(p_+ + p_-\right) = \frac{1}{\sqrt{2}}\left(H + p^+\right). \tag{4.49}$$

We must remember, however, that we have left the symmetry of overall σ-translations unfixed. This can be fixed now by imposing the constraint on the Fock space that its elements are annihilated by the generator of these translations,

$$P = -\int_0^l d\sigma \left(\Pi^j \partial_\sigma Z^j + \text{h.c.}\right) \tag{4.50}$$

$$= \frac{2\pi}{l} \sum_{j=1}^{12} \sum_{n=-\infty}^{\infty} n \left(N_n^j - \tilde{N}_n^j\right). \tag{4.51}$$

This is known as the *level-matching condition.*

4.3.2 The sum of free energies of the individual particle states

With the spectrum we have just found, we can calculate the sum of free energies of the individual particle states, and verify that this reproduces the free energy of a free string gas.

With the appropriate measure on the space of light-cone spacetime momenta, p^+, and while remembering to impose the level-matching condition – which we do by inserting an integral representation for a delta function into the trace – the sum of free energies is

$$F(\beta) = \frac{1}{2\beta} \int_0^\infty \frac{dp^+}{\sqrt{2}\pi} \operatorname{Tr} \int_{-1/2}^{1/2} d\tau_1 e^{i\tau_1 lP} \ln\left(1 - e^{-\beta H_{\rm sp}}\right). \tag{4.52}$$

The extra factor of 1/2 comes from the fact that we had two physically equivalent choices for the orientation on the worldsheet.

Using the series expansion for the logarithm,

$$\ln\left(1 - e^{-\beta H_{\rm sp}}\right) = -\sum_{r=1}^{\infty} \frac{1}{r} e^{-\beta H_{\rm sp} r}, \tag{4.53}$$

and equation (4.49), this can be written,

$$F(\beta) = -\sum_{r=1}^{\infty} \int_0^\infty dp^+ \int_{-1/2}^{1/2} d\tau_1 \frac{1}{2\sqrt{2}\pi\beta r} e^{-\beta r p^+/\sqrt{2}} \times \operatorname{Tr}\left[\exp\left(i\tau_1 lP - \frac{\beta r}{\sqrt{2}} H\right)\right]. \tag{4.54}$$

With equations (4.48) and (4.50), the argument of the trace in this expression can be written,

$$\prod_{j=1}^{12} \prod_{n=-\infty}^{\infty} \left[\exp\left(2\pi i a_{n,j}\left(N_n^j + 1/2\right)\right) \times \exp\left(-2\pi i a_{n,j}{}^*\left(\tilde{N}_n^j + 1/2\right)\right)\right], \tag{4.55}$$

where

$$a_{n,j} = n\tau_1 + i\frac{\beta r}{2\sqrt{2}\pi\alpha' p^+}\left|n + \alpha'\mu_j p^+\right|, \tag{4.56}$$

and we have used the mutual commutativity of p^+ and the number operators to break up the exponential. When evaluating the trace we can again use this commutativity to bring the trace into each of the products. We have

$$\begin{aligned} \operatorname{Tr}\exp\left(2\pi i a_{n,j}\left(N_n^j + 1/2\right)\right) &= e^{\pi i a_{n,j}} \sum_{k=0}^{\infty} e^{2\pi i a_{n,j} k} \\ &= \frac{e^{\pi i a_{n,j}}}{\left(1 - e^{2\pi i a_{n,j}}\right)} = -\frac{1}{2i\sin \pi a_{n,j}}, \end{aligned} \tag{4.57}$$

and similarly for the other exponential. Thus,

$$F(\beta) = -\sum_{r=1}^{\infty} \int_0^{\infty} dp^+ \int_{-1/2}^{1/2} d\tau_1 \frac{1}{2\sqrt{2}\pi\beta r} e^{-\beta r p^+/\sqrt{2}}$$
$$\times \prod_{j=1}^{12} \prod_{n=-\infty}^{\infty} \left| 2\sin\pi\left(n\tau_1 + i\frac{\beta r}{2\sqrt{2}\pi\alpha' p^+}\left|n + \alpha'\mu_j p^+\right|\right)\right|^{-2}. \quad (4.58)$$

We then make the change of variables, $\tau_2 = \frac{\beta r}{2\sqrt{2}\pi\alpha' p^+}$, and we let $\tau = \tau_1 + i\tau_2$. Then $2d\tau_1 d\tau_2 = d\tau d\bar{\tau}$, and

$$F(\beta) = -\frac{1}{4\pi^2\alpha'} \sum_{r=1}^{\infty} \int_E \frac{d\tau d\bar{\tau}}{4\tau_2{}^2} e^{-\beta^2 r^2/4\pi\alpha'\tau_2}$$
$$\times \prod_{j=1}^{12} \prod_{n=-\infty}^{\infty} \left| 2\sin\pi\left(n\tau_1 + i\left|n\tau_2 + \frac{\beta\mu_j r}{2\sqrt{2}\pi}\right|\right)\right|^{-2}, \quad (4.59)$$

where E is the strip, $[-\frac{1}{2}, \frac{1}{2}] \times [0, \infty)$.

This expression is similar to the free energy of the free string gas, the main differences being that there is a single sum over the positive integers instead of a double sum over all integers, not both zero, and the integration is done over the strip, E, instead of a fundamental domain of the modular group. That this expression is even modular invariant is far from obvious. To get it into the same form we first assume we can pull the negative exponent out of the infinite product. Writing

$$C_j = \frac{\beta\mu_j r}{2\sqrt{2}\pi\tau_2}, \quad (4.60)$$

and $\{C_j\} = C_j - \lfloor C_j \rfloor$, the infinite product can then be manipulated as follows:

$$\begin{aligned}
&\prod_n \Big|2\sin\pi\big(n\tau_1 + i\tau_2\,|n+C_j|\big)\Big|^2 \\
&\quad = \prod_n \Big|2\sin\pi\big(n\tau_1 - \tau_1\lfloor C_j\rfloor + i\tau_2\,|n+\{C_j\}|\big)\Big|^2 \\
&\quad = \prod_{n\geq 0} \Big|2\sin\pi\big(n\tau_1 - \tau_1\lfloor C_j\rfloor + i\tau_2\,(n+\{C_j\})\big)\Big|^2 \\
&\qquad\quad \times \prod_{n<0} \Big|2\sin\pi\big(n\tau_1 - \tau_1\lfloor C_j\rfloor - i\tau_2\,(n+\{C_j\})\big)\Big|^2 \\
&\quad = \prod_{n\geq 0} \Big|2\sin\pi\big(n\tau - \tau\lfloor C_j\rfloor + i\tau_2 C_j\big)\Big|^2 \\
&\qquad\quad \times \prod_{n<0} \Big|2\sin\pi\big(n\bar\tau - \bar\tau\lfloor C_j\rfloor - i\tau_2 C_j\big)\Big|^2 \\
&\quad = \prod_n \big|2\sin\pi\left(n\tau - \tau\lfloor C_j\rfloor + i\tau_2 C_j\right)\big|^2 \\
&\quad = \prod_n \big|2\sin\pi\left(n\tau + i\tau_2 C_j\right)\big|^2 \\
&\quad = \prod_n \left|2\sin\pi\left(n\tau + i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\right)\right|^2 .
\end{aligned} \tag{4.61}$$

With the regularization (C.15) from the appendix, this product can then be written,

$$\prod_{n,m}\left(\frac{1}{\tau_2}\left|n + m\tau + i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\right|^2\right). \tag{4.62}$$

Thus, the expression (4.59) for the sum of free energies becomes

$$\begin{aligned}
F(\beta) = -\frac{1}{4\pi^2\alpha'}\sum_{r=1}^{\infty}\int_E \frac{d\tau d\bar\tau}{4\tau_2{}^2} e^{-\beta^2 r^2/4\pi\alpha'\tau_2} \\
\times \prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{1}{\tau_2}\left|n + m\tau + i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\right|^2\right)\right]^{-1},
\end{aligned} \tag{4.63}$$

It can be shown [5] that for any given pair of relative primes, (c, d), there exists a unique modular transformation, $M(c, d)$, that maps the fundamental domain, F_0, into the strip, E, such that

$$M(c,d)^{-1}(\tau) = \frac{a\tau + b}{c\tau + d} \tag{4.64}$$

for some a, b. It can also be shown that with $E(c,d) = M(c,d)\,[F_0]$, we have

$$\bigcup_{[c,d]=1} E(c,d) = E, \tag{4.65}$$

where $[c,d]$ is the greatest common divisor of c and d.

This permits the following manipulations of the expression (4.63):

$$\begin{aligned}
F(\beta) = &-\frac{1}{4\pi^2\alpha'}\sum_{r=1}^{\infty}\sum_{[c,d]=1}\int_{E(c,d)}\frac{d\tau d\bar{\tau}}{4\tau_2{}^2}e^{-\beta^2 r^2/4\pi\alpha'\tau_2} \\
&\times\prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{1}{\tau_2}\left|m+n\tau+i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\right|^2\right)\right]^{-1} \\
= &-\frac{1}{4\pi^2\alpha'}\sum_{r=1}^{\infty}\sum_{[c,d]=1}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2{}^2}e^{-\beta^2 r^2|c\tau+d|^2/4\pi\alpha'\tau_2} \\
&\times\prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{|c\tau+d|^2}{\tau_2}\left|m+n\left(\frac{a\tau+b}{c\tau+d}\right)+i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\right|^2\right)\right]^{-1} \\
= &-\frac{1}{4\pi^2\alpha'}\sum_{r=1}^{\infty}\sum_{[c,d]=1}\int_{F_0}\frac{d\tau d\bar{\tau}}{4\tau_2{}^2}e^{-\beta^2 r^2|c\tau+d|^2/4\pi\alpha'\tau_2} \\
&\times\prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{1}{\tau_2}\left|(dm+bn)+(cm+an)\,\tau+i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\,(c\tau+d)\right|^2\right)\right]^{-1}.
\end{aligned} \tag{4.66}$$

We can make the change of variables, $\begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}$ in the product because $ad - bc = 1$ implies this transformation is invertible. Then,

$$
\begin{aligned}
F(\beta) = &-\frac{1}{4\pi^2\alpha'}\sum_{r=1}^{\infty}\sum_{[c,d]=1}\int_{F_0}\frac{d\tau d\bar{\tau}}{4{\tau_2}^2}e^{-\beta^2 r^2|c\tau+d|^2/4\pi\alpha'\tau_2} \\
&\times\prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{1}{\tau_2}\left|m+n\tau+i\frac{\beta\mu_j r}{2\sqrt{2}\pi}\left(c\tau+d\right)\right|^2\right)\right]^{-1} \\
= &-\frac{1}{4\pi^2\alpha'}{\sum_{r,s\in\mathbb{Z}}}'\int_{F_0}\frac{d\tau d\bar{\tau}}{4{\tau_2}^2}e^{-\beta^2|r+s\tau|^2/4\pi\alpha'\tau_2} \\
&\times\prod_{j=1}^{12}\left[\prod_{n,m}\left(\frac{1}{\tau_2}\left|m+n\tau+i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r+s\tau\right)\right|^2\right)\right]^{-1},
\end{aligned}
\tag{4.67}
$$

where the prime on the sum omits the term $r = s = 0$.

Recalling again the regularization (C.15) from the appendix, the free energy becomes

$$
\begin{aligned}
F(\beta) = &-\frac{1}{4\pi^2\alpha'}{\sum_{r,s\in\mathbb{Z}}}'\int_{F_0}\frac{d\tau d\bar{\tau}}{4{\tau_2}^2}\exp\left(-\frac{\beta^2\left|r+s\tau\right|^2}{4\pi\alpha'\tau_2}\right) \\
&\times\prod_{j=1}^{12}\left[\left|\eta\left(\tau\right)^{-1}\vartheta_{11}\left(i\frac{\beta\mu_j}{2\sqrt{2}\pi}\left(r+s\tau\right);\tau\right)\right|^{-2}\exp\left(\frac{\beta^2{\mu_j}^2\left(r+s\tau_1\right)^2}{4\pi\tau_2}\right)\right],
\end{aligned}
\tag{4.68}
$$

nicely reproducing the free energy of a free string gas.

Chapter 5

Conclusions

Using the work of Longton [3], we have further developed the theory of the oriented closed bosonic string in a pp-wave background. We were able to prove modular invariance of both the one loop amplitude, and the free energy of a free string gas. As a nice check, the latter result was reproduced by summing the free energies for the individual particle states using the spectrum calculated in [3].

Some interesting further work that may be done at this point on this theory is the construction of vertex operators. As well, one might also want to generalize the background to allow for worldsheet instabilities.

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Appendix A

Evaluation of the Path Integrals

In this appendix, we will first evaluate the path integral,

$$\int [dZd\bar{Z}]_g \exp\left(-\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\bar{Z}\mathcal{O}Z\right), \tag{A.1}$$

where Z is a complex scalar field, and $\mathcal{O}$ is any positive definite differential operator on the space of field configurations. To define the path integral measure, first define the invariant inner product on the space of field configurations as

$$(Z_1, Z_2) = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{g}\bar{Z}_1 Z_2. \tag{A.2}$$

Then the line element is

$$\|\delta Z\|^2 = (\delta Z, \delta Z). \tag{A.3}$$

Now $\mathcal{O}$ is positive definite, so it has a complete orthonormal basis of positive eigenvectors Z_n with corresponding eigenvalues λ_n. We can expand any field configuration Z in terms of this basis:

$$Z = \sum_n c_n Z_n, \tag{A.4}$$

so the line element is

$$\|\delta Z\|^2 = \sum_n |\delta c_n|^2. \tag{A.5}$$

Thus, we define the invariant path integral measure:

$$[dZd\bar{Z}]_g = \prod_n dc_n d\bar{c}_n. \tag{A.6}$$

The path integral can now be evaluated:

$$\prod_n \left(\int dc_n d\bar{c}_n \right) \exp\left(-\sum_n \lambda_n c_n \bar{c}_n \right)$$
$$= \prod_n \left(\frac{1}{\lambda_n} \int dc_n d\bar{c}_n e^{-c_n \bar{c}_n} \right) = \prod_n \left(\frac{2\pi}{\lambda_n} \right) = \left[\det\left(\mathcal{O}/2\pi \right) \right]^{-1} . \quad \text{(A.7)}$$

Summarizing, for positive definite $\mathcal{O}$

$$\int [dZ d\bar{Z}]_g \exp\left(-\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \bar{Z} \mathcal{O} Z \right) = \left[\det\left(\mathcal{O}/2\pi \right) \right]^{-1} . \quad \text{(A.8)}$$

The path integral becomes more complicated, however, when $\mathcal{O} = -\nabla^2$ acts on the torus, and therefore has a zero mode. The only harmonic functions on the torus are constants, so labelling the zero mode $n = 0$ we see that

$$Z_0 = \left(\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \right)^{-1/2} , \quad \text{(A.9)}$$

since Z_0 is assumed to be normalized to unity. We further assume that $F[c_0 Z_0] = F[0]$. Then following the same steps as above, the path integral becomes

$$\int dc_0 d\bar{c}_0 \prod_{n \neq 0} \left(\frac{1}{\lambda_n} \int dc_n d\bar{c}_n e^{-c_n \bar{c}_n} \right) F\left[c_0 Z_0 + \sum_{m \neq 0} c_m Z_m \right]$$
$$= \prod_{n \neq 0} \left(\frac{2\pi}{\lambda_n} \right) \int dc_0 d\bar{c}_0 F[c_0 Z_0]$$
$$= \left[\det{}' \left(-\nabla^2/2\pi \right) \right]^{-1} F[0] \int dc_0 d\bar{c}_0, \quad \text{(A.10)}$$

where $\det'$ is a determinant taken over the nonzero modes. In the first equality we have assumed that the integral over each nonzero mode, n, acts as a delta function on

$$F\left[c_0 Z_0 + \sum_{m \neq 0} c_m Z_m \right] ,$$

constraining c_n in its argument to vanish. This is true provided it is an entire function of c_n for each $n \neq 0$, since then it can be written as a power series about $c_n = 0$ which converges everywhere, and the constant term is the only one that survives the integration.

We regulate the divergent integral, $\int dc_0 d\bar{c}_0$, by toroidally compactifying the field Z, giving the real and imaginary parts circumference L_1 and L_2 respectively:

$$\begin{aligned} \operatorname{Re} Z &\approx \operatorname{Re} Z + L_1, \\ \operatorname{Im} Z &\approx \operatorname{Im} Z + L_2. \end{aligned} \tag{A.11}$$

We can take $0 \leq \operatorname{Re}(c_0 Z_0) \leq L_1$, which implies
$0 \leq \operatorname{Re} c_0 \leq L_1 \left(\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\right)^{1/2}$, and similarly for $\operatorname{Im} c_0$. Thus,

$$\int dc_0 d\bar{c}_0 = \frac{L_1 L_2}{2\pi\alpha'} \int d^2\sigma \sqrt{g}. \tag{A.12}$$

Summarizing, if $F[c] = F[0]$ for constant c, then

$$\begin{aligned} &\int [dZ d\bar{Z}]_g \exp\left(\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \bar{Z} \nabla^2 Z\right) F[Z] \\ &\qquad = \left(\frac{L_1 L_2}{2\pi\alpha'} \int d^2\sigma \sqrt{g}\right) \left[\det{}' \left(-\nabla^2/2\pi\right)\right]^{-1} F[0]. \end{aligned} \tag{A.13}$$

The correct result is obtained formally in the limits $L_1, L_2 \to \infty$.

Appendix B

The Dedekind Eta and Jacobi Theta Functions

We present here some identities that we will use involving the Dedekind eta function, $\eta(\tau)$, and the half-period Jacobi theta function, $\vartheta_{11}(\nu;\tau)$, each defined for all τ in the upper half-plane, and the theta function is defined for all complex ν. In terms of,

$$q = \exp(2\pi i\tau), \quad \text{and} \quad z = \exp(2\pi i\nu), \tag{B.1}$$

these have the infinite product representations,

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m), \tag{B.2a}$$

$$\vartheta_{11}(\nu;\tau) = -2q^{1/8} \sin \pi\nu \prod_{m=1}^{\infty} (1 - q^m)(1 - zq^m)\left(1 - z^{-1}q^m\right). \tag{B.2b}$$

If we define the transformations of ν under the modular group generators, $\mathcal{T} : \tau \to \tau + 1$, and $\mathcal{S} : \tau \to -1/\tau$ to be $\mathcal{T} : \nu \to \nu$, and $\mathcal{S} : \nu \to \nu/\tau$, respectively then the modular transformations for $\eta(\tau)$ and $\vartheta_{11}(\nu;\tau)$ are

$$\eta(\tau + 1) = e^{i\pi/12}\eta(\tau), \tag{B.3a}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \tag{B.3b}$$

$$\vartheta_{11}(\nu;\tau + 1) = e^{\pi i/4}\vartheta_{11}(\nu;\tau), \tag{B.3c}$$

$$\vartheta_{11}(\nu/\tau;-1/\tau) = -i\sqrt{-i\tau}e^{\pi i\nu^2/\tau}\vartheta_{11}(\nu;\tau). \tag{B.3d}$$

Appendix C

Regularization of the Infinite Products

In this appendix we describe a regularization procedure for evaluating the infinite product,

$$\prod_{n,m\in\mathbb{Z}} \left(\frac{c}{\tau_2}\left|n+m\tau+\nu\right|^2\right). \tag{C.1}$$

If we consider τ to be the modular parameter on a torus, and assume that ν transforms under the modular group generators, $\mathcal{T} : \tau \to \tau + 1$, and $\mathcal{S} : \tau \to -1/\tau$ by $\mathcal{T} : \nu \to \nu$, and $\mathcal{S} : \nu \to \nu/\tau$, respectively, then the infinite product is manifestly modular invariant. Thus, it is an important check on our regularization procedure that modular invariance is preserved.

In order to regularize this product we will make use of zeta function regularization, where divergent products are defined by

$$\prod_n f_n = \exp\left(-\frac{d}{ds}\sum_n {f_n}^{-s}\right)\Bigg|_{s=0}. \tag{C.2}$$

Letting $a = (c/\tau_2)^{1/2}$ and $b = m\tau + \nu$, we can formally manipulate the infinite product as follows:

$$\begin{aligned}\prod_{n=-\infty}^{\infty}\left|a\left(n+b\right)\right|^2 &= |ab|^2\prod_{n=1}^{\infty}\left|a\left(n+b\right)\right|^2\prod_{n=1}^{\infty}\left|a\left(n-b\right)\right|^2 \\ &= \left|ab\prod_{n=1}^{\infty}\left[a\left(n+b\right)\right]\prod_{n=1}^{\infty}\left[a\left(n-b\right)\right]\right|^2.\end{aligned} \tag{C.3}$$

But note that the last step does not necessarily hold true if we interpret these products as zeta function regularized. This could lead to problems, but remembering that the choice of regularization procedure is arbitrary anyway, we proceed ahead, and use the preservation of modular invariance as a final check on the validity of our regularization procedure.

The Hurwitz zeta function is formally defined for complex arguments s and b, with $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(b) > 0$, by

$$\zeta\left(s, b\right) = \sum_{n=0}^{\infty} \left(n + b\right)^{-s}, \tag{C.4}$$

and it satisfies the following properties:

$$\zeta\left(0, b\right) = \frac{1}{2} - b \tag{C.5a}$$

$$\zeta\left(-1, b\right) = -\frac{1}{2}b^2 + \frac{1}{2}b - \frac{1}{12} \tag{C.5b}$$

$$\frac{\partial}{\partial s}\zeta\left(s, b\right)\bigg|_{s=0} = \ln\Gamma(b) - \frac{1}{2}\ln\left(2\pi\right). \tag{C.5c}$$

Using these, we perform the following zeta function regularization:

$$\begin{aligned}
\prod_{n=1}^{\infty}\left[a\left(n+b\right)\right] &= \exp\left(-\frac{d}{ds}\sum_{n=1}^{\infty} a^{-s}\left(n+b\right)^{-s}\right)\Bigg|_{s=0} \\
&= \exp\left(-\frac{d}{ds}\left(a^{-s}\zeta\left(s,b\right) - \left(ab\right)^{-s}\right)\right)\Bigg|_{s=0} \\
&= \exp\left(\ln\left(a\right)\zeta\left(0,b\right) - \frac{\partial}{\partial s}\zeta\left(s,b\right)\bigg|_{s=0} - \ln\left(ab\right)\right) \\
&= \frac{a^{-\frac{1}{2}-b}\sqrt{2\pi}}{b\Gamma(b)}.
\end{aligned} \tag{C.6}$$

We can then write

$$ab\prod_{n=1}^{\infty}\left[a\left(n+b\right)\right]\prod_{n=1}^{\infty}\left[a\left(n-b\right)\right] = -\frac{2\pi}{b\Gamma(b)\Gamma(-b)} = -2\sin\pi b, \tag{C.7}$$

where in the last step we used the identities,

$$\Gamma(1-b)\Gamma(b) = \frac{\pi}{\sin\pi b}, \tag{C.8a}$$

$$\Gamma(b+1) = b\Gamma(b). \tag{C.8b}$$

Plugging equation (C.7) into (C.3), and using our definitions for a and b, we find

$$\prod_{n,m\in\mathbb{Z}}\left(\frac{c}{\tau_2}\left|n + m\tau + \nu\right|^2\right) = \prod_{m\in\mathbb{Z}}\left|2\sin\pi\left(m\tau+\nu\right)\right|^2. \tag{C.9}$$

Now to regularize the product over the m, we first write the right hand side of the above equations as

$$|2\sin\pi\nu|^2 \prod_{m=1}^{\infty} \left|2\sin\pi\left(m\tau+\nu\right)\right|^2 \prod_{m=1}^{\infty} \left|2\sin\pi\left(m\tau-\nu\right)\right|^2, \tag{C.10}$$

and

$$\left|2\sin\pi\left(m\tau\pm\nu\right)\right|^2 = e^{2\pi\tau_2(m\pm\nu_2/\tau_2)} \left|1-e^{i\pi(m\tau\pm\nu)}\right|^2. \tag{C.11}$$

Then we make the following formal manipulations,

$$\begin{aligned}\prod_{m=1}^{\infty} &\left(e^{2\pi\tau_2(m\pm\nu_2/\tau_2)} \left|1-e^{i\pi(m\tau\pm\nu)}\right|^2\right) \\ &= \prod_{m=1}^{\infty} e^{2\pi\tau_2(m\pm\nu_2/\tau_2)} \prod_{m=1}^{\infty} \left|1-e^{i\pi(m\tau\pm\nu)}\right|^2,\end{aligned} \tag{C.12}$$

and

$$\begin{aligned}\prod_{m=1}^{\infty} e^{2\pi\tau_2(m\pm\nu_2/\tau_2)} &= \exp\left(2\pi\tau_2 \sum_{m=1}^{\infty}\left(m\pm\frac{\nu_2}{\tau_2}\right)\right) \\ &= \exp\left(2\pi\tau_2\left(\zeta\left(-1,\pm\frac{\nu_2}{\tau_2}\right)\mp\frac{\nu_2}{\tau_2}\right)\right) \\ &= \exp\left(-\pi\frac{{\nu_2}^2}{\tau_2}\mp\pi\nu_2-\frac{\pi\tau_2}{6}\right),\end{aligned} \tag{C.13}$$

where in the last step we have used the property, (C.5b).

Putting all this together, we find

$$\begin{aligned}\prod_{m=-\infty}^{\infty} &\left|2\sin\pi\left(m\tau+\nu\right)\right|^2 = e^{-2\pi{\nu_2}^2/\tau_2} \\ &\times e^{-\pi\tau_2/3} \left|2\sin\pi\nu\right|^2 \prod_{m=1}^{\infty} \left|1-e^{i\pi(m\tau+\nu)}\right|^2 \prod_{m=1}^{\infty} \left|1-e^{i\pi(m\tau-\nu)}\right|^2.\end{aligned} \tag{C.14}$$

By using equations (B.2), we can easily write the quantity in the second line in terms of the Dedekind eta function and a Jacobi theta function. It is simply $\left|\eta\left(\tau\right)^{-1}\vartheta_{11}\left(\nu;\tau\right)\right|^2$.

Summarizing our results,

$$\prod_{n,m\in\mathbb{Z}} \left(\frac{c}{\tau_2}\left|n+m\tau+\nu\right|^2\right) = \prod_{m\in\mathbb{Z}} \left|2\sin\pi\left(m\tau+\nu\right)\right|^2 \tag{C.15a}$$

$$= \left|\eta\left(\tau\right)^{-1}\vartheta_{11}\left(\nu;\tau\right)\right|^2 e^{-2\pi{\nu_2}^2/\tau_2}. \tag{C.15b}$$

Notice that these results are independent of c.

We now check to see that modular invariance has been preserved. Referring to equations (B.3), we immediately see that it is invariant under $\mathcal{T}$, and that it is also invariant under $\mathcal{S}$ if we can show that

$$\mathrm{Re}\left(\frac{i\nu^2}{\tau}\right) - \frac{\left(\mathrm{Im}\left(\nu/\tau\right)\right)^2}{\mathrm{Im}\left(-1/\tau\right)} + \frac{{\nu_2}^2}{\tau_2} \tag{C.16}$$

vanishes. It is a simple algebraic exercise to show that this is indeed true. Thus, the regularization has preserved modular invariance.

We will also be interested in the infinite product,

$$\prod_{n,m\in\mathbb{Z}}{}' \left(\frac{c}{\tau_2}\left|n+m\tau\right|^2\right), \tag{C.17}$$

where the prime denotes the removal of the zero mode, $n = m = 0$. This can easily be obtained by dividing equation (C.15) by the $n = m = 0$ term and then taking the limit, $\nu \to 0$:

$$\prod_{n,m\in\mathbb{Z}}{}' \left(\frac{c}{\tau_2}\left|n+m\tau\right|^2\right) = \lim_{\nu\to 0}\frac{\tau_2}{c}\left|\nu^{-1}\eta\left(\tau\right)^{-1}\vartheta_{11}\left(\nu;\tau\right)\right|^2 e^{-2\pi{\nu_2}^2/\tau_2}. \tag{C.18}$$

Using equations (B.2), the term in the bars can be written,

$$-2\pi\left(\frac{\sin\pi\nu}{\pi\nu}\right)q^{1/12}\prod_{m=1}^{\infty}\left[\left(1-zq^m\right)\left(1-z^{-1}q^m\right)\right], \tag{C.19}$$

where q and z are given in equations, (B.1). In the limit, $\nu \to 0$, this becomes

$$-2\pi\left(q^{1/24}\prod_{m=1}^{\infty}\left(1-q^m\right)\right)^2 = -2\pi\eta\left(\tau\right)^2. \tag{C.20}$$

So we have

$$\prod_{n,m\in\mathbb{Z}}{}' \left(\frac{c}{\tau_2}\left|n+m\tau\right|^2\right) = \frac{4\pi^2\tau_2}{c}\left|\eta\left(\tau\right)\right|^4, \tag{C.21}$$

and by equations (B.3), it is clear that this is invariant under the generators of the modular group, and is thus modular invariant. So our regularization procedure has indeed preserved the manifest modular invariance of the infinite product.