MATHEMATICAL FLEXIBILITY

by

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Abstract

Mathematicians, mathematics researchers and educators are now arguing that an essential aim of mathematics education should be to equip students so they can adapt to new mathematical situations and use mathematics to solve authentic problems that arise in day-to-day life. This, *mathematical flexibility* – defined here as adaptation when dealing with number, magnitude or form – is important to mathematics researchers and educators, but the classroom context may not always promote flexibility. Building across converging lines of cognitive, social-psychological, and neuro-biological research, this study investigated whether mathematical flexibility might be profitably understood as a network of functional components. This study was designed to: 1) investigate the functional components of mathematical flexibility and contrast them with functional components of mathematical competence; and 2) evaluate the effectiveness of a network approach for understanding the relationship between environmental and individual components of mathematical flexibility. Results indicated that flexibility appeared to be associated with network activity which co-activated two or more other networks, while competence appeared to be characterized by a series of network activations which occurred individually and in sequence. Further, results suggested that the case study approach used here to identify network activity could reveal meaningful dynamics in network activity, and these dynamics could be related to flexible or competent performance. Implications for researchers and practitioners are identified in the discussion. However, because this study was constrained by the ways in which flexibility was conceptualized and features of the methodology, limitations and directions for future research are also suggested.

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Activity Across Time Graph (AAT Graph)
British Columbia (BC)
Semi-Structured Interview (SSI)
Self-Regulated Learning (SRL)

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CHAPTER ONE

Introduction

This study was designed to investigate the components of *mathematical flexibility* – a construct used in this study to describe mathematical activity that is contextually sensitive and adaptive. This investigation is important because some mathematicians, mathematics researchers and educators are now arguing that an essential aim of mathematics education should be to equip students so they can adapt to new mathematical situations and use mathematics to solve authentic problems that arise in day-to-day life. This skill set has been increasingly emphasized in Western Canada, where recent curriculum documents argue that successful mathematics education occurs only if students take mathematical risks, are curious and engage in mathematical projects (Governments of Yukon, Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut and Saskatchewan, 2008). These documents suggest that the entire project of mathematics education exists to “prepare students to use mathematics confidently to solve problems” (Government of Yukon et al., 2008, p. 4).

Similarly, researchers like Allan Schoenfeld have emphasized a view of mathematics education in which learners are "capable of interpreting the vast amounts of quantitative data they encounter on a daily basis, and of making balanced judgments on the basis of those interpretations. … They are flexible thinkers with a broad repertoire of techniques and perspectives for learning to think mathematically, dealing with novel problems and situations" (Schoenfeld, 1992, pp. 4-5). The study is based on the assertion, defended below, that deeper understandings of this mathematical flexibility can support the aims of both mathematics educators and academics investigating this phenomenon.

What is Mathematical Flexibility?

Adaptive navigation of a changing mathematical environment is what I term mathematical flexibility. This phenomenon has been widely investigated across a variety of research programs. Across programs, terminology and research methods vary. These research programs have used a variety of other monikers: for example, number sense (Greeno, 1991; Dehaene 1999), central conceptual structures (Case & Okamoto, 1996) and mathematical thinking (Schoenfeld, 1992). Each construct foregrounds particular aspects of mathematical flexibility.

One of the goals in this study was to triangulate across these research programs, and so I am using "mathematical flexibility" in a literal, dictionary sense, so that this construct encompasses enough to be consonant with multiple theoretical perspectives. *Flexible* refers to "a ready capability to adapt to new, different, or changing requirements." *Mathematics* refers to "activity involving quantity, magnitude or form." Thus, *flexible mathematics* describes adaptation to new, different or changing requirements when engaging with quantity, magnitude, or form.

Researchers have already identified important aspects of mathematical flexibility. Drawing across research programs, four possible elements of mathematical flexibility can be identified. Table 1 relates these elements of mathematical flexibility with the sources of evidence which support their validity.

Distinct Modules in the Brain

A team of cognitive scientists led by Robbie Case has argued that specific, semi-stable, networks of concepts and conceptual relations seem to undergo constant evolution and periodic revolution in the minds of students doing math (Case & Okamato, 1996). These networks identified by Case et al. (1996) seem to evolve and merge to support increasing levels of mathematical flexibility, first with whole numbers and eventually with rational numbers.

Case's work (e.g., Case & Okamoto, 1996) specifies particular modules required for whole number flexibility and suggests how these mental structures develop. In this account, Case and Okamoto argue that a visual-spatial mental structure which recognizes which of two quantities is "more or less" becomes integrated with a structure for a socially-learned verbal counting routine. This composite structure functions as a mental number line for single-digit whole numbers. After exposure to specific learning experiences and endogenous cognitive development, multiple counting lines are created for the 10s, 100s, and so on. Finally, Case and Okamoto suggest, these multiple counting lines are coordinated to create an "integrated bidimensional" structure – what teachers might refer to as an understanding of place value (Case & Okamoto, 1996).

Using brain imaging research, Dehaene and his team have provided converging evidence for the existence of the structures proposed by Case: modules for 'more and less' and verbal understandings of number. These studies have identified a horizontal section of intraparietal sulcus (HIPS) which shows increased activation when perceiving number, no matter what form the number is presented in. In Dehaene's (2004) research, the HIPS has been the only area which shows increased activation when subjects have been asked to identify which of a set of stimuli are numbers and it has not activated when subjects have been asked to look for colors or letters. The HIPS has been found to respond to subliminal presentation of number. It also shows increased activation in approximation tasks more than exact or rote ones (see Dehaene, Molko, Cohen & Wilson, 2004 for a review of this literature). Thus, Dehaene argues that the HIPS appears to function to recognize magnitude.

However, studies of subjects with brain lesions suggest that this fuzzy number line is distinct from other, verbal understandings of number. Dehaene, Dehaene-Lambertz and Cohen (1998) suggest that lesions to the HIPS "can be selective for calculation, with reading, writing, spoken recognition and pronunciation of Arabic digits and number words not being affected" (p. 358). In one lesion study, the subject could recall some multiplication facts, though they were unable to discern the meaning of these facts (Dehaene et al., 2004). On the other hand, a patient with an intact HIPS but lesions to verbal areas could no longer retrieve multiplication facts, but could compare the magnitude of two numbers, do simple arithmetic and bisect an interval (Dehaene et al., 1998). Thus, the HIPS appears to have functions that are distinct from other mental structures used in interacting with number, and Dehaene (2004) has found evidence of other brain areas responsible for interacting with number in a verbal, nominal way.

These two research programs converge to suggest that there is a structure in the brain which functions to evaluate more and less and another that facilitates verbal interaction with number. It appears that these structures are distinct because brain lesion studies demonstrate their separability, and developmental evidence from Case et al. (1996) suggests that they begin as separate structures, merging only as experience and cognitive development allow. This modular structure can be considered a characteristic of cognitive systems, including systems for mathematical flexibility.

New Structures May Be Old Structures, Newly Integrated

In many theories emerging in cognitive psychology, complex cognitive activity is conceptualized as the integration of simpler cognitive structures. For example, Hauser (2006) conceptualizes human morality as the integration of structures for processing emotional experience and structures for higher level social decision making, using a third metacognitive structure in the frontal lobe to coordinate the activity of these structures. Similarly, Adams (1998) argues that cognitive structures for reading are composed of phonological, orthographic, meaning and contextual “processors” which integrate to facilitate the rich cognitive activity we call ‘reading.’

In studies of mathematical flexibility, complex mathematical cognition also tends to be conceptualized as integration of simpler cognitive units. For example, Dehaene’s neuropsychological account of number sense argues that three distinct structures for processing number – as Arabic numerals, verbal sounds, and analogical magnitudes – combine to produce our robust sense of number (Dehaene, Piazza, Pinel & Cohen, 2003).

Modeling development as the integration of already-present mental structures is also nicely illustrated by Moss and Case’s (1999) work on rational numbers. Their 1999 paper on rational number sense (RNS) traces the development of a conceptual network which integrates visual-spatial and symbolic representations of rational numbers into a flexible understanding of rational numbers.

In Moss and Case’s theory, for which they have considerable empirical support, the RNS network consists of two integrated networks: one for doubling and halving numbers using verbal mental structures and another for visual-spatial conceptions like “half-full” and “mostly-empty.” My interpretation of this theory suggests that, if a student with robust RNS is asked to figure out how much liquid is in a quarter-full, 600 ml jar, their visual spatial structure for dividing "full" quantities into halves and quarters will connect with a symbolic/numerical structure for "halving numbers,” allowing them to simultaneously quarter both the visual-spatial quantity and the number. If a student’s conception of doubling or halving is not connected with visual-spatial understandings of “full,” and “quarter full” their understandings are not integrated. This student may be able to mechanically calculate ¼ of 600, but might be unable to intuit that "a fifth-full" of 600 ml is a little less than 150 ml. Or, they may be able to fill the jar so that it is ¼ full, but have no idea of how to figure out how much liquid is in the jar. They may even be able to do both without understanding that the two operations are connected. However, Case and Moss posit, once a student has developed RNS (a new cognitive structure composed of integrated old structures), one conception is associated with the other, and students are able to estimate, solve "word problems," detect errors that are off by an order of magnitude, and otherwise interact competently and fluidly with rational numbers. Recent fMRI research has supported this conceptualization, suggesting that by adulthood fractions are perceived as a whole and linguistic and visual spatial systems are fully integrated, so that linguistic conceptions of fractions are immediately perceived in terms of their magnitude and size and vice versa (Jacob & Neider, 2009).

Thus, research suggests that new mental structures, including those for mathematical flexibility, are created through the modification and integration of existing structures; and that these new structures function in ways that are qualitatively different from the functions of the structures that compose it.

Same Behaviour, Different Thoughts

Case's work (e.g., Case et al., 1996; Case & Okamoto, 1996; Moss & Case, 1999) has suggested that cognitive structures for mathematical flexibility tend to develop uniformly and evenly in terms of the observable behaviours produced by those structures. However, this same research suggests that similar behaviors can emerge from different cognitive activity. For example, Moss and Case (1999) asked three Grade 4 students to fill in 3/4 of a pie which had been broken into 8 pieces. Three students gave different explanations as they successfully filled 3/4th of the pie:

- “There are two slices in a quarter so you need six [slices] to make three quarters [shades them]”
- “[Shades six sections] I just keep the quarters and forget about the eighths.”
- “I don't know ...Well let me see .... This is a half [student shaded four sections]... so you would need two more to make 3/4 [shades two more sections]”

(Moss & Case, 1999, pp. 137-138)

Moss and Case argue that, in each of these examples, students made use of the same underlying structures (the visual-spatial and halving/doubling structures mentioned above). However, the particular way these structures integrated and activated depended on the particular student and the context.

Case’s research suggests that different thinking paths can lead to the same outcome. Each student considered the novel problem in his or her own way, utilizing a point-of-view which allowed seeing the solution, yet each shaded the pie in the same way. It follows that mathematical flexibility requires adaptability, not just the use of a particular strategy or point-of-view to achieve an outcome.

Seizing Opportunities for Flexibility

Even if structures for mathematical flexibility exist and are well integrated, adaptive performance requires that a person see and utilize an opportunity to be flexible. For example, Greeno (1991) cites an account of a Brazilian street vendor who sold coconuts. The vendor was accustomed to selling coconuts in units of three or one coconut. One day a buyer requested ten coconuts while a researcher was observing. The researcher reported that the vendor, unused to working with multiples of ten, reasoned aloud: "Three will be 105 ... with three more, that will be 210 ... I need four more....That is (pause) 315 ... I think it's 350." (Carraher, Carraher & Schliemann, as cited in Greeno, 1991, p. 172).

I interpret this example as showing distinct cognitive structures combining in novel ways to produce adaptation to a novel context. The street vendor, unaccustomed to thinking about ten coconuts, used a "landmark" in his "quantity space" (Greeno, 1991), namely the price of three coconuts, to adaptively construct a price for ten coconuts.

Importantly, the vendor detected an opportunity to interact with this customer, using modules he had already acquired, to complete a unique and novel transaction. One can easily imagine the vendor inflexibly saying "you must buy nine, or twelve coconuts" because he could not think of a way to calculate the cost.

Thus, it follows that seeing and using this opportunity to recombine his existing knowledge was essential to the flexibility the vendor demonstrated. These "characteristics of ... arrangements in the environment that support their contributions to interactive activity" have been termed affordances (Greeno, 1994, p.341). Without detecting and using opportunities for novel mathematical activity – using affordances – mathematical flexibility may be impossible.

Mathematical Flexibility in Context

The concept of mathematical flexibility is applied within a complex and multi-faceted set of research and teaching contexts. This section surfaces some of these contexts and discusses their implications for investigating mathematical flexibility.

Mathematical Competence

In contrast to mathematical flexibility, I argue that students may also engage in *mathematical competence*. In this study, mathematical competence was defined as successfully navigating mathematical activities *without* adapting preexisting mathematical knowledge. This definition suggests that mathematical competence exists relative to what a person has already been prepared to do and thus cannot be detected without knowledge of a person's past activity in a domain. If people adapt what they bring to a situation in light of particular mathematical affordances, they exhibit flexibility. If they navigate the situation without any adaptation or recombination of existing knowledge, and without attending to unique affordances in that particular situation, they exhibit competence.

Teaching Mathematical Flexibility

There is evidence that secondary teachers struggle when attempting to help their students become more flexible. For example, an investigation of teacher perceptions towards flexibility found that half the teachers studied did not believe lower achieving students should be taught "higher-order" mathematics at all because it would be too difficult and frustrating (Zohar, Degani, & Vaaknin, 2001). In the context of the curriculum goals discussed above, this indicates that 45% of teachers in this study believed that the main goal of mathematical education – building mathematical flexibility – is unachievable for a sizeable subset of their students.

Another study (Harel, 2008) focused on two teachers who had spent four weeks at a workshop where the

> workshop curriculum strongly emphasized meaning, and the workshop leader reinforced this at every opportunity. Asking participants to compare different solutions to nonstandard problems was one way of stressing meaning. The problems were unlike textbook exercises in that a correct solution method, and indeed the underlying mathematical content were not initially obvious (p. 117).

After the workshop, the two teachers were observed in the classroom. Subsequent analysis found that, even after training, meaning was de-emphasized in instruction in favor of procedural strategies justified on the basis of teacher authority rather than meaning. This analysis also suggested that, despite explicit training and teacher buy-in, students in both of the teachers' classes believed mathematics primarily involved procedural tasks.

Flexibility and Linear Equations

The present research study focuses attention on how students are mathematically flexible in the context of linear equations. In their research, Leinhardt, Zaslavsky and Stein (1990) have also focused on flexibility in linear equations, making a case for why this is important as an area of interest both to researchers and educators. Leinhardt et al. (1990) highlight two aspects of linear equations which make it an important topic for students. First, they suggest that "functions and graphs is a topic in which two symbolic systems are used to illuminate each other" (p. 3). From the lens of mathematical flexibility, I conceptualize this domain as one where one symbol system (ordered pairs) substitutes for another (graphical) as students integrate structures that were previously distinct. Second, Leinhardt et al. (1990) suggest that linear equations are a good way of examining patterns, because they allow focus on both the specific relation between two values and the graph or function that holistically describes that relationship. Linear equations facilitate simultaneous perception of holistic and individuated perceptions of mathematical relationships.

Curriculum documents also suggest linear equations are an important domain for students. Linear functions are an essential part of the current BC Math curriculum (BC Ministry of Education, 2006) and have an even larger role in the new curriculum, which is being phased in in 2011. A third of the current standard Grade 10 math course is devoted to linear functions (BC

Ministry of Education, 2006). Curriculum documents for Grade 11 and 12 assume knowledge of linear equations and curriculum documents for earlier grades emphasize developing structures for working with graphs and coordinates.

Yet, despite the pedagogical and academic emphasis on flexibility in linear equations, insufficient research has examined flexibility in linear equations and in British Columbia (BC) it is underemphasized in instruction and evaluation. For example, 13 out of 15 problems on the most recently released provincial exam in BC have analogues in Math textbooks used in the province. Teachers, who expect these analogues on the exam, tend to prepare students for specific question types, removing the need for mathematical flexibility (personal experience). In 12 years of working with students enrolled in Math 10 courses, I cannot remember a single instance of a teacher giving a student a task that required flexibility; almost every teacher I have observed has given explicit, step-by-step instructions for each problem type students will come across in independent work. Thus, while mathematical flexibility in linear equations is an important curricular goal, researchers know little about it and teachers may struggle to help students become flexible.

Fruitful Investigations into Mathematical Flexibility

Understanding mathematical flexibility in terms of modular structures which integrate over time and experience to create new, functionally different structures may support the design and development of useful interventions. Two separate research groups studying mathematical flexibility have used this approach to design educational interventions.

As discussed above, Dehaene (2006) identified two sub-systems for encoding number: one for "non-verbal quantity representation and [the other for] developing symbolic representations of numbers, such Arabic numerals or number words" (p. 3). His proposal is that these subsystems work together to allow for flexible estimation, calculation and numerical manipulation. After discovering these subsystems, Dehaene, along with Anna Wilson, built a software program that seeks to cement these modules together in the minds of students (Wilson, Dehaene, Pinel, Revkin, Cohen & Cohen, 2006). A trial of this program found improvement in measures of non-verbal quantity representation and subtraction.

Griffin (2004a) used a functional analysis she did with her research partner Robbie Case to inform an effective intervention to help students develop a "mental number line." Case et al. (1996) argue that two structures (subsystems) – one for detecting magnitude and another that can execute the counting routine – integrate, and produce this "mental number line." Griffin casts these systems in functional terms, suggesting that a counting schema "enables them to verbally count from one to five, use the one-to-one correspondence rule, and to use the cardinality rule" and a quantity schema "can compare two groups of objects that differ in size and tell which has a lot or a little" (Griffin, 2004b, p. 328). By looking at developmental progression of students across these functional capabilities, Griffin (2004b) designed a piece of software where each section build on the previous section, because they connect these subsystems in a way that is similar to natural development. After using this program, students were superior on all measures of mathematical and scientific tests used in the study (Griffin, 2004b).

Thus, identifying the functional subsystems that underlie a particular instance of flexibility has been a fruitful way of designing interventions that promote flexibility.

The Present Investigation

Mathematical flexibility, as defined in this study, appears to be a characteristic of successful math students and a major goal of mathematics education. However, flexibility is generally difficult to teach at the high-school level. In particular, linear functions is a domain of high-school math that successful students must be able to navigate; but mathematical flexibility is unstudied in this domain, and seems to be rarely emphasized in instruction or evaluation.

Further, following on the promising research conducted by Daheane, Griffin and Case, it appears that understanding mathematical flexibility in terms of functional sub-systems can lead to curriculum design that is effective. From this context I form my two research questions:

1. *How do functional components instrumental to mathematical flexibility interact with each other to produce flexibility in high-school students engaging with linear equations?*
2. *How do these functional components differ from the operation of functional components instrumental to inflexible but procedurally competent performance in linear equations?*

Addressing these research questions may advance understanding about an important element of learning (mathematical flexibility) in an important but unstudied domain (linear equations), using an approach for understanding (functional analysis) that has potential to directly benefit designers of educational curricula and interventions.

To address these research questions, I: 1) set up a multiple case-study framework in which diverse subjects were given affordances to demonstrate both mathematical flexibility and procedural competency when working with linear equations, 2) observed activity in a variety of functional subsystems thought to be relevant to mathematical flexibility in linear equations in those contexts; and 3) identified patterns that emerged across these particular cases to inform understandings of how particular functional sub-systems might contribute to mathematical flexibility.

In the next two chapters I elaborate on this study's purpose, theoretical grounding, and method. Chapter Two examines two existing strategies used to understand mathematical flexibility and establishes a theoretical lens and method designed to build on the strengths of both approaches. In that chapter, I also propose a set of structures – networks – that may be active as students exhibit flexibility. Chapter Three specifies the methodology for the study. In the final two chapters I present and discuss the results. Chapter Four builds on student-level data to report results in cross-student patterns of network activity and flexibility. Chapter Five discusses these results in the context of the study and suggests extensions and limitations of the protocol.

CHAPTER TWO

Investigating Flexibility

Two Approaches to Investigating Flexibility

Two approaches to understanding mathematical flexibility have been used by researchers. Each has its own strengths and weaknesses, and below, I elucidate both, and suggest a method and study design designed to capture the strengths of both approaches.

Identifying Structures.

Researchers like Robbie Case and Stanlais Dehaene have investigated mathematical flexibility by identifying in-the-head structures which are instrumental for mathematical flexibility. These are conceptualized both as biological structures (e.g., a particular region of cortex) and functional structures (e.g., a structure for interacting with whole number). Dehaene emphasizes the relationship between biological structures and conscious experience / task performance (e.g., Dehaene et al., 2004); Case emphasizes development of functional structures over time (e.g., Case & Okamoto, 1996). However, both researchers switch between functional and biological perspectives; these are not mutually exclusive. Mental functions happen within the physical body, thus, they must have a biological basis (Dennett, 1991). Both researchers position particular in-the-head structures as instrumental for complex mathematical activity.

Dehaene's work is wide ranging, but tends to relate brain images to the subjective experience of subjects and the objective behaviors they display. These studies investigate a wide variety of subjects (for example, patients with brain lesions or Aboriginal tribes in South America) across a wide variety of experimental paradigms.

For example, in one study, Dehaene (2004) subliminally presented stimuli to subjects, some of which were numbers, some of which were letters. His subjects reported being unable to see these stimuli, yet Dehaene reported seeing activation in the horizontal interparietal cortex (HIPS) when number was presented and not when letters were presented. This, says Dehaene, is “evidence for a micro-coding of quantities in intraparietal cortex” (Naccache & Dehaene, 2001, p. 972).

Because individual experiments can be too narrow to warrant broad interpretation, an important part of Dehaene’s contributions (e.g,. Dehaene et al., 2003; Dehaene, 1999) are theoretical syntheses of multiple experiments (some of these experiments are done by his team; some not). For example, Dehaene et al. (2003) differentiated the HIPS from another region implicated in mathematical flexibility – the angular gyrus (AG) – by reviewing eight neuroimaging experiments (across several experimental paradigms) and 11 case studies of patients who had suffered lesions to their parietal lobe. Synthesizing these experiments and the case studies, Dehaene et al. (2003) suggested a reliable distinction between the AG and the HIPS – the AG facilitates verbal interaction with number, while the HIPS facilitates spatial interaction with number. Thus, a key element of structure identification is to synthesize across many small studies in order to hypothesize the role of particular physical structures.

Case’s research synthesized empirical evidence in order to propose a developmental progression for mental structures that support mathematics (Case et al., 1996). Once he had hypothesized a specific developmental path, he looked for the predicted developmental progression, and, from that, inferred that the hypothesized mental structures exist.

For example, after looking across a set of developmental studies, Case (1992) argued that cognitive development tended to be even across people and across domains within a single person. He thus concluded that minds contain “central conceptual structures” that facilitate similar activity across different cognitive domains. From there, he hypothesized a specific developmental sequence of task performance capabilities and suggested a relationship between age and task performance in the domains of mathematics, social understanding and drawing.

In mathematics, Case and Okamoto (1996) hypothesized that most 6 year olds can use a single mental number line, but eight year olds can coordinate two mental number lines – one for ones, and one for tens. They administered a variety of tasks, some of which required a single mental number line, others which required coordination of two number lines. When they looked for correlations (through factor analyses), they found that success on one of the coordinated number line questions was correlated with successful performance on other coordinated number line questions. They also found that nearly all 8 year olds in the study could do coordinated number line tasks, but almost no 6 year olds could. From this they concluded that there is a structure in the mind supporting a single number line, and, around age eight, that structure changes to allow coordination of two number lines (Case et al., 1996). This approach was cross-sectional – they examined several students at one time, rather than the same student over time – but succeeded in identifying development of mental structures because they had a specific hypothesis about how these structures could be detected in the relationship between age and task performance.

I position the structure identification approach to mathematical flexibility as a tool rather than a methodology. This tool allows researchers to identify which particular structures facilitate which particular types of activity. In general, both Case and Dehaene's methodologies build on theoretical insight to make specific predictions about the contributions of particular structures to particular measurable aspects of mathematical cognition and flexibility. However, their specific methods differ greatly, and Dehaene, in particular, makes wide use of a variety of methodologies, from brain scan studies, to comparisons of Aboriginal South Americans and

French citizens, to literature syntheses in order to understand the structures underlying mathematical flexibility.

From a practical point-of-view, structure identification is useful for educational designers. As described above, Dehaene's work has been used to create an intervention for dyscalculia that has been successful at improving performance on math. The author of the software (Anna Wilson) reports it is designed to link the activity between verbal and spatial systems for number – a design consideration based specifically on the identification of these two brain regions as dissociable but integrated and essential to interacting fluidly with whole number (Wilson et al., 2006).

Structure identification, because it is so concrete, can also constrain theory in important ways. For example, Dehaene's triple-code theory of number suggests three structures that integrate to facilitate conception of number. This research is extremely widely cited (A Google Scholar search indicates that Dehaene's seminal book *The Number Sense* (1997, 1999) has been cited 1337 times); the triple-code theory is synthesized from a wide variety of experimental and research paradigms; and the theory is grounded in an empirically supported theory of mental structure and the role of consciousness in mental activity (Dehaene & Naccache, 2001). Thus, theories of mathematical cognition which do not take structures identified in the triple-code model into account risk ignoring structures which may be instrumental to mathematical cognition.

Perhaps structures can be easily integrated into interventions and research because the view that minds contain structures is an extremely intuitive way for humans to think. Carl Bereiter (2002) calls this intuition "mind as container," and suggests that it has evolved as a fundamental human point of view, where "people [are] credited with a mind, and all of the relevant cognitive or emotional stuff is thought of as residing in one or another of those minds," (Bereiter, 2002, p. 13). This stance is fruitful in predicting the behavior of a particular person – which is why it evolved (Dennett, 1978) – but it may break down when trying to uncover how the system that produces intentional behavior actually works because it emphasizes what is in the mind, rather than how the mind functions (See Dennett (1991) or Bereiter (2002) for a defense of this position). For example, Dehaene's triple code theory specifies the effect of specific brain lesions on simple task performance very accurately, without consideration of the relation between the three structures Dehaene posits. However, the triple code model has not been used to explain how these structures produce sophisticated mathematical flexibility deemed important by other researchers (like estimating the size of North America (Greeno, 1991)). Dehaene concurs: "Higher-level cultural developments in arithmetic emerge through the establishment of linkages between this core analogical representation (the "number line") and other verbal and visual representations of number notations" (Dehaene, 2001, p. 2).

Tracing Relationships.

In contrast to "in-the-head" conceptualizations of mathematical cognition, flexibility can be conceptualized as a type of practice that occurs in transactions between mental and contextual structures as they support flexibility.

This approach has been utilized by social psychologists like James Greeno. Greeno's relational approach suggests that "conceptual understanding and conceptual growth are … achievements of interaction" (Greeno, 2007, p. 10) between individuals, groups of individuals and contextual factors. Interactions with other knowledgeable people and with the texts and symbols of mathematics allow people to build new knowledge. This new knowledge is manifest in an individual's new capacity to navigate mathematical environments. From this point of view, flexibility refers to the ability to more or less competently navigate mathematical situations, rather than the presence or absence of any particular mental structure.

For Greeno, this flexibility depends on affordances, not on in-the-head structures. Affordances are “characteristics of objects and arrangements in the environment that support interactive activity [with these objects]” (Greeno, 1994, p. 241). Students are flexible when they utilize features of objects in the environment to adjust to a novel mathematical situation. For example, Greeno suggests that the most useful affordance in a mathematical conceptual object – like a number – is its ability to combine with other objects (as in addition, or multiplication). The numbers 4 and 6 have compositional affordances that allow them to combine to make 10 through addition or 24 through multiplication. Utilizing this affordance means combining 4 and 6 within an internal mental model of numbers and “seeing” 10 or 24 emerge within that internal mental model, depending on the affordance utilized. Thus, flexibility, which depends on individual-contextual interaction, results as students are able to utilize particular affordances in the environment to achieve flexible activity.

To uncover the individual-contextual transactions that support flexibility, Greeno emphasizes post-hoc analysis of discourse and traces of particular mathematical transactions. From these records, Greeno infers affordances that students have utilized and the social interactions from which this capacity has arisen. Rather than interpreting these data from a particular viewpoint, he suggests an approach where new ways of understanding the transaction flow from a process where existing conceptions are tried out, improved and melded together in order to fit the data (Greeno & MMAP, 1997).

An example of this process can be found in Greeno’s 1991 treatise on “number sense.” Similar to flexibility, number sense refers to fluid and flexible activity in the domain of number.

After a deep synthesis of the existing research, Greeno suggests an extended metaphor for number sense, suggesting it requires navigation of a conceptual environment much like one might navigate a physical environment, like a city. Like a physical environment, flexibly navigating mathematical environments requires: 1) knowing what sites in the particular mathematical environment are important and how to utilize them; 2) how to move from place to place within that environment; 3) how to plan sequences of movement through that environment; and 4) how to create representations of the environment that can be used to support productive activity (Greeno, 1994).

The difference in emphasis between a relational, environmental understanding of mathematical cognition and an approach which emphasizes structures can be seen by contrasting how Greeno and Case view the development of basic whole number sense (being able to count, add and subtract). As described above, Case views whole number sense as emerging when visual-spatial structures for less and more merge with the counting routine to create a robust mental number line. This number line then allows students to reason about whole numbers and demonstrate basic number sense. From Case's point of view, this process is fully dependent on the symbol systems and cultural tools with which a child interacts and develops. Students develop number lines when cultures have counting routines and numbers, and words for more and less, and when families provide ample opportunity for interaction with these cultural structures. However, Case suggests that a variety of activities all support similar development. He points to contrasts between Japanese and American students, who, despite very different specific experiences – Japanese students spend more time calculating with number, Americans spend more time using clocks and money - seem to have developed very similar general conceptual structures (Okamoto, Case, Bleiker & Henderson, 1996). Yet, though variation across culture is small, within a culture, students with lower socio-economic status (SES) develop conceptual structures more slowly than those with higher SES. Case concludes that culture impacts development through the types of activities it promotes, rather than the specific experiences it affords. Low SES students develop more slowly, argues Case, because they have a relative paucity of types of the types of experiences that lead to the development of central conceptual structures. And Japanese and American students, despite very different specific experiences, develop similarly because they have the same types of experiences.

Like Case, Greeno suggests that counting operations and spatial properties of objects ground students' early understanding of addition and subtraction (Greeno, 1991). Greeno, too, emphasizes how whole number sense requires the internalization of spatial relationships found in the physical world. Numbers begin as objects that are arranged linearly – that is, spatially – in the physical world, and over time, Greeno argues, students internalize this spatial relationship, and create a mental model, where numbers are related by their position on the number line. The affordances that students use to operate with this model of a number line are also derived from features of the physical context. For example, Greeno cites research that suggests that flexible students use phrases like 'tacking on' or 'knocking off' to describe addition and subtraction, suggesting that a visual-spatial model is active within students' mind.

As Greeno points out, Case and Greeno's accounts of whole number sense may be consistent (Greeno, 2007). In both cases students use a mental structure with both symbolic and spatial properties to reason about numbers. However, Greeno sees this mental structure as being dependent on the specific contexts in which it developed. Because the counting routine is taught spatially (by counting a set of objects) and fairly universally, it is easily internalized into a mental model that has spatial and physical properties. Other more complex mathematical flexibility – like linear equations – may require specific social experiences to facilitate the development of internal structures required for such activity. Thus, though Greeno suggests that internal structures are essential to mathematical flexibility, he foregrounds the specific experiences from which these structures develop and the ways in which these in-the-head structures relate to each other and contextual structures.

This approach – explaining abilities in terms of the social/contextual processes in which they are grounded – allows Greeno to construct rich understandings of complex examples of flexibility. An example of these rich constructions can be found in Greeno and Van De Sand's (2007) account of a transaction between two students and a teacher. The teacher had presented a complicated word problem to the class and asked the class to name the variable that would represent one of the features of the problem. When a student gave an unexpected answer, the teacher inquired further, and, with input from a second student, the teacher and students generated a solution to the problem that was fundamentally different from what the teacher had intended to teach, and what the student had intended when answering the teacher's question. Greeno and Van De Sand's account of this transaction is based on audio transcripts and records of what the teacher wrote during the interaction. By interpreting and connecting student utterances and traces of activity, Greeno and Van De Sand demonstrate how both student and teacher's mental structures (which they refer to as schemata) are changed through interaction with each other. This example illustrates the importance of situating the mental structures active during flexibility in their contexts. Seeing how these structures develop in light of the discourse taking place between student and teacher allows the researcher to position the interaction of mental structure and context as instrumental to flexibility, providing a rich, dynamic explanation.

Greeno's approach has helped development in research and tools for practitioners. For researchers, he has emphasized individual-contextual interactions as a bridge between cognitive and situated perspectives (Greeno, 2007). This approach situates cognition within the environmental contexts which are instrumental to its development. As result, unique research environments have been developed such as the MMAP environment, where students interact with software designed to afford complex and flexible interactions with mathematical objects. Through this work, researchers have constructed richer depictions of students developing mathematical flexibility in environments that are complex relative to those typically studied by most in-the-head researchers.

For educators, Greeno's research has supported the development of tools – like the MMAP environment – that teachers can use in the classroom. Greeno points out that these programs drive students to "uncover the math" rather than "cover the math" (Greeno & MMAP, 1997, p. 101), suggesting that they are designed to cause learning through interactions with materials that afford exploration. Because these programs are designed with an individual-contextual interaction in mind, they are designed to afford complex and dynamic interactions that can affect learning and flexibility. Unlike the computer program designed by Dehaene's team (The Number Race), the MMAP environment does not explicitly attempt to change any particular structure within the mind; instead it offers opportunities to interact with mathematical objects, because the situated theory on which it is based predicts that these objects should become internalized into mental models and then become available for future reasoning and activity.

However, foregrounding relational understandings of flexibility, derived through post-hoc analysis, may underemphasize the efficiency afforded by contributions like Case and

Dehaene's. These contributions help focus research, because they specify some of the design principles at work within the individual-contextual system, and thus, place important constraints on the types of theoretical analyses which are appropriate. For example, the establishment of the HIPS as the seat of visual-spatial numerical cognition (through the cross-paradigm synthesis discussed above) tells us that a theory of numerical cognition might benefit by including a role for a fuzzy analogical number line. The social processes emphasized by Greeno do not allow unlimited modification of existing in-the-head structures – what is in-the-head is constrained by millions of years of evolutionary design and the specific ontogeny of the individual under investigation. Thus, knowledge of what structures are instrumental in certain types of cognition can focus research efforts on targets which are likely to be productive.

As well, foregrounding contextual-individual interactions may underemphasize the ways in which individuals are themselves collections of structures interacting in complex and dynamic ways to produce the patterns of behavior we recognize as individuals. That individuals are collections of functional processes is a point emphasized in the field of cognitive science (Dennett, 1991; Minsky, 1988) because it facilitates powerful explanations that come from breaking down the individual into functional parts. This point-of-view makes the actions of the individual more easily interpretable because they are seen as emanating from a chaotic set of tractable functional processes.

An Integrative Approach

Table 2 highlights the key features of the identifying structures approach to understanding mathematical flexibility, and the relating structures approach. In this section, I suggest a theoretical and empirical point-of-view that may capture the strengths of both approaches. I then briefly trace the logic of my particular study in light of this theoretical and methodological point-of-view.

This study is designed to illuminate the question: *How do functional components instrumental to mathematical flexibility interact with each other to produce flexibility in high-school students engaging with linear equations? How do these differ from the operation of functional components instrumental to inflexible but procedurally competent performance in linear equations?* In light of the above discussion, I suggest that a thorough approach to answering this question should allow for identification of in-the-head structures instrumental to mathematical flexibility and trace the relations between those in-the head-structures and contextual structures as flexibility or competence is demonstrated. This study explores the role of particular in-the-head structures, discerned through an extensive literature synthesis, while still capturing the unexpected and dynamic interplays of personal and contextual factors that constitute flexibility.

Networks as a Model for Mathematical Flexibility

Though structural and relational approaches to mathematical flexibility contain different points of emphasis, both argue that particular mental structures are required for flexibility and that those structures develop out of contextual interactions. The research question for this study asks how structures instrumental to mathematical flexibility differ from those instrumental to competence. These structures may differ in terms of what they are or they may differ in terms of how they relate. Thus, there may be a benefit to an approach to flexibility which simultaneously emphasizes the structures that exist within a mind and the way in which these develop and relate.

Such an approach might be found in a view of mathematical flexibility (and cognition in general) as emerging from a network of mental and environmental structures. This viewpoint seems useful for two reasons. First, a network model accounts for the features of mathematical flexibility already identified by researchers like Case, Dehaene and Greeno. Second, a network model allows for both contextual and mental structures to be viewed as functional sub-systems within a larger "transaction" between organism and environment. Transactions are a way of conceptualizing human activity as fundamentally inseparable from the contexts in which it occurs, and to see activity as a whole rather than as separable into distinct entities or components (Dewey, 1946). As I will discuss below, network models of mathematical flexibility allow for a view of context and individual as cohered, without losing the contributions particular individual or contextual factors might make to the nature of the transaction.

Networks as Effective Models of Minds.

A network is a mathematical model of a dynamic system, consisting of nodes joined by edges (Girvin & Newman, 2002). At the most abstract level, nodes represent structures in a system and edges represent the connections between these structures. Nodes send information to other nodes through their edges, which can be modeled as having both weight and direction. An edge with a high connection weight will allow information to flow more easily between the nodes it connects than an edge with a low connection weight. Edges that are directed send information in one direction only; networks that are undirected permit the flow of information in both directions.

Networks can be used to model a wide variety of phenomena, including the Internet, social networks, industrial relationships, metabolic networks, blood vessels, postal delivery routes, and, importantly for this study, mental processes (Newman, 2003). Networks are effective models of systems that are complex and non-linear, like the brain. The brain is composed of 100 million neurons, which have 100 billion connections between them. Research has shown that the processing that happens between these neurons occurs in parallel, and non-linearly, suggesting that networks might be an effective way to model the activity of a mind. In fact, the use of the term ‘network’ when referring to cognitive structures is widespread across the different areas of psychology.

One way of conceptualizing mental networks is to see nodes as representing a particular function performed in the mind and edges as connections that allow the activity of a structure to spread and influence other mental functions. This approach can be considered functionalist. Functionalism refers to the view that "what makes something a mind (or a belief, or a pain, or a fear) is not what it is made of, but what it can do. We appreciate this principle as uncontroversial in other areas, especially in our assessment of artifacts. What makes something a spark plug is that it can be plugged into a particular situation and deliver a spark when called upon” (Dennett, 1996, p. 68). From this perspective, networks can be said to contain a particular structure when a unique function for that structure is identified.

Research on networks (Girvin & Newman, 2002) suggests that five properties of networks may be consistent with known properties of mathematical cognition and flexibility, which may suggest that networks are a useful way of modeling mathematical flexibility. First, networks have a modular structure; so do the components of mathematical flexibility. Second, networks have hubs – structures instrumental to particular activity – and thus are vulnerable to system-wide breakdown when a hub is unable to function; similarly, research suggests that mathematical flexibility depends on particular modules (like the HIPS), and without them, even the most basic mathematical activity is impossible. Third, hubs can connect to a variety of sub-networks, meaning that similar network output can be generated by different in-network activity, as long as those networks feed into the same hub; similarly, mathematics researchers have discovered that a wide variety of cognitive activity that can support a whole number sense that looks similar across people (Okamoto et al., 1996). Fourth, cognitive networks are auto-associative, so that activation of any sub-network will tend to activate the entire larger network of which it is a part; similarly, mathematical cognition seems to require that a small piece of the context (an affordance) trigger a wide network of cognitive processes. Fifth, the hierarchical nature of cognitive networks allows specific experiences to facilitate progressively more productive representations of the external world as top-down modules learn to recognize the general in the specific; mathematical flexibility requires mental models that model mathematical objects, a reciprocal interaction between specific learning and general understandings that leads to more adaptive responses over time. Because these five features of networks are also features of mathematical cognition, I suggest that networks are an appropriate model for mathematical cognition.

Networks as Equating Mental and Contextual Structures.

Networks allow mathematical flexibility and competence to be viewed as a transaction between individual and context. From a network perspective, both context and individual are thought to combine to form a qualitatively different 'object' – a transaction – distributed in time and space, which cannot be reduced to independent contributions of context and individual. However, past research which has emphasized the role of context in internal mental activity has still posited the individual and the environment as distinct and described them as distinct entities.

A network view of mathematical flexibility situates the contribution of individual and context in a larger system (network), representing the irreducible, dynamic entity that individual and context combine to make. At any moment in the network, individual and contextual structures (nodes) are in some state, as a result of the states and connections of connected nodes at previous moments. Nodes transmit their states across edges, to other nodes, affecting how those nodes exist at the next point in time.

From this point of view, parts of the network are distinct when they have distinct functions. For example, based on Case and Greeno's research programs, it seems reasonable to suggest that visual-spatial nodes that compare relative size and verbal nodes for counting both make distinct functional contributions to the activity of the whole number network. Although this approach may appear to isolate the contribution of particular nodes, creating undue simplicity, isolated node activity can only be interpreted in light of its connection to other nodes in the present, past and future. Case's developmental account of whole number sense suggests that the visual-spatial and verbal networks merge over time, as their activation is amplified and fused by other nodes which function as monitors of mental activity. At some developmental point, each of these nodes communicates their activity to the monitoring nodes. The monitoring nodes in turn amplify and stabilize the activity of the visual-spatial and verbal nodes. Over time, these nodes fuse into a composite node that functions to interact with whole number. The extent to which they are fused at any moment depends on how they activate; this, in turn, depends on what networks were active at previous moments, and how their activity was transmitted. Thus, understanding whole number sense requires both an understanding of the structures involved, and how they interact and develop over time.

A network approach foregrounds the ways in which mental and contextual structures (nodes) operate as a whole and isolates parts of the network by function rather than location. As a result, contextual and individual factors can be seen in the same light – as functional components of a larger system. As a hypothetical example, consider how a calculator functions in a network where a dyscalculic student works on a math problem. For a student who cannot calculate internally (perhaps because of lesions to the HIPS), the calculator functions as a node for whole number sense, functioning to allow quick and fluid interaction with numerical entities. The information produced by this node is transmitted through photons (light particles), which hit the retina, and become spike trains of electricity, moving through neurons in the brain. This electrical activity in the brain excites other nodes, which use this information in the service of more complex and widely distributed activity (like solving a word problem, for example). In a non-dyscalculic student, one can imagine observing identical behavior, except that the function of interacting with whole number takes place within the HIPS instead of the calculator. Whether the calculator and the HIPS represent the same structure (functioning to interact with whole number) may be an open question; but from a functionalist, network perspective, this question will be decided by comparing the relative functions of the calculator and HIPS. If they are identical, then the networks may be said to be the same, even though the location and substrate of the function differs. If the functions are different, the networks can be said to be different.

Implications of a Network Theory for Mathematical Flexibility with Linear Equations, and the Research Questions of the Study

The ways in which features of networks are analogous to descriptions of mathematical flexibility and cognition suggests that networks can be used to fruitfully model networks of individuals-in-context exhibiting mathematical flexibility. A network model also suggests that network nodes are best understood in functional terms, and should be described in terms of what they can do, rather than what or where they are.[1].

[1] Conceptualizing nodes as 'doing' things – as though nodes are themselves intentional beings with beliefs, wants and goals – may be an important tool for conceptual progress within cognitive science, though it caries the risk of oversimplifying complex processes, (Dennett, 1991, Minsky, 1988). As the functions of these nodes are further specified, there may come a point where intentional explanations of nodes are not as powerful as biological design explanations (e.g., explaining neuronal activity in terms of potassium/sodium channels). The network perspective I have put forward does not privilege human beings as 'real' or 'genuine' intentional systems vs. 'pseudo' intentional

Thus, a network theory has the potential to account for the situated and relational nature of cognitive processes emphasized by Greeno, while preserving the contribution of specific mental structures within the individual-contextual network. A network theory allows for an individual to be both a structure within a larger system and itself a combination of the sub-networks that compose individuals. Thus, observing activity as though occurring in a network preserves the benefits of both the identifying structures and tracing relationships approaches to understanding flexibility.

In this study I used a case study framework that allowed tracing complex and dynamic contextual-individual interactions to investigate the qualities of mathematical flexibility. The approach I used was based on the above network theory of mathematical flexibility. My plan was to:

1. Examine prior research to determine what network nodes have been associated with mathematical flexibility in linear equations, and hypothesize how they contribute to mathematical flexibility in linear equations.
2. Choose a heterogeneous set of participants, so that the networks students brought to the study would be as different as possible. Identify, at the start of the study, as best as possible, how network nodes had developed within each participant, as a result of past individual-contextual transactions.
3. Ask subjects to engage with a set of tasks, some of which were designed to afford mathematical flexibility, some of which were designed to afford mathematical

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systems like thermostats, dogs or trees; instead, what makes a system intentional is how effective intentional explanations are at predicting the behavior of the system. This can create a situation that may be intuitively confusing – that nodes can act on their intentions without being conscious or aware of them. A thorough defense of this position is beyond the scope of this thesis.

competence. Observe activity in the nodes known to be relevant to mathematical flexibility.

Candidate functional components of mathematical flexibility have been identified by researchers. In the next section, I summarize from across research programs to identify a preliminary set of functional components (nodes) that are likely to be involved when individuals flexibly engage with linear equations.

Candidates for Networks Involved in Mathematical Flexibility

In this section, I propose a set of networks likely to be recruited in mathematical flexibility with linear equations. For each, I also build from literature to describe how I attempted to measure activity in those networks in this study.

Domain Specific Cognitive Networks

The existence of hubs in networks implies that every domain of human activity has networks instrumental to that activity. In cognitive psychology, a variety of these domain specific networks have been identified. For example, a network involving a region of orbitalfrontal cortex has been identified as essential to moral reasoning (Hauser, 2006). In any domain in which we are examining mathematical competence or mathematical flexibility, we should expect to find networks specific to the functions required in that domain.

Measurement Strategies.

Within linear equations, it is unclear what networks are essential to flexibility. However, curriculum documents contain descriptions of seven distinct functional capacities that constitute successful performance in the linear equations section of the course. The capacities reflected in curriculum documents are typically the outcome of an extensive curriculum creation process which, in turn, depends on the cultural history of linear equations in the minds of those creating the curriculum. Thus these seven functional capacities provide a good starting point for investigating the domain-specific functional components of linear equations flexibility:

- Solve problems involving slopes of line-segments
- Give the equation of a line, once it's uniquely specified
- Solve problems using slopes of parallel or perpendicular line segments
- Relate ordered pairs, a 'rule' and a graph
- Identify x/y-intercepts, domain/range, and slope of a graph from an equation
- Sketch linear equations given the equation (in either standard or general form)
- Make linear models (equations) of real situations.

Linear equations may also depend on networks that function to support sub-competencies within mathematics. Mathematics is a cumulative field and linear equations require competence with algebra, coordinate systems, and fractions (personal experience). This study was designed to assess both what students brought to the study in terms of these networks and notable activations of those networks during the study itself. Again, what students brought to the study was assessed in ways consistent with the BC learning outcomes.

Metacognitive Networks

Metacognition is an area of cognition that has been studied from a variety of perspectives, under a large number of names, and across disciplines as diverse as economics, psychology, philosophy, education and biology. For this study, I positioned metacognition as networks which function to modulate and adapt other active networks. Schoenfeld (1992, 1997) suggests that metacognitive processes in mathematics can be differentiated into "(a) individuals' declarative knowledge about their cognitive processes, (b) self-regulatory procedures, including monitoring and 'on-line' decision-making, (c) beliefs and affects, and their effects on performance" (Schoenfeld 1992, pp. 38-39). Individuals' declarative knowledge comprises their knowledge about their own and others' thinking and learning processes while self-regulation refers to students' approaches to managing their engagement in mathematics activity. Students' metacognition has also been associated with students' beliefs and affect about the discipline of mathematics and their work in mathematics, and how those influence their activity in mathematics (Schoenfeld, 1992).

Effective metacognitive networks may be instrumental to mathematical flexibility. Declarative knowledge of cognitive processes is important for helping students make realistic assessments of what they can accomplish and for choosing effective strategies to handle complex problems (Schoenfeld, 1987). Effective self-regulation during mathematical activity is essential for seeing and utilizing affordances (Schoenfeld, 1987). Beliefs and affects about mathematics influence the ways students engage in mathematical activity (Schoenfeld, 1987).

The lens with which Schoenfeld views metacognition highlights aspects of metacognition which can be observed directly or by asking students targeted questions – stated beliefs and affect, propositional self-knowledge, and how activity changes during problem solving – and the relationship between those actions and mathematical flexibility. This approach can be viewed as relational, emphasizing the way in which metacognitive activity relates to other cognitive strategies and how metacognitive activity is part of a larger cluster of mental activity supporting complex problem solving. However other approaches, which can be understood as identifying structures, provide complementary insights which can be used to further organize understandings of metacognition.

Narrative and Experiential Metacognition.

Structural approaches to understanding metacognition have identified two distinct networks supporting metacognition. The first network generates beliefs about the self, declarative affect and declarative knowledge through "higher order self-reference characterized by neural processes supporting awareness of a self that extends across time" (Farb, Segal, Mayberg, Bean, McKeon, Fatima & Anderson, 2007, p. 7). I will refer to this network as the narrative-metacognitive network. This network's function is to model the network itself. This self-modeling forms a "strange loop" that is fundamentally self-referential and circular (Hofstadter, 2006). The "declarative knowledge about … cognitive processes," and "beliefs and affects" identified by Schoenfeld (1987) may be outputs of this network because they seem to require awareness of a self that exists across time and the use of linguistic networks.

Farb et al. (2007) argue that the narrative-metacognitive network may "represent an overlearned mode of information processing that has become automatic through practice, consistent with established findings on training-induced automaticity" (p. 7) and that "narrative generation as a default state of self-reference is increasingly supported by neural evidence" (p. 2).

The second network involves "viscerosomatic cortical areas [which] support an immediate information processing network of identity, distinct from abstract and narrative representations of the self" (Farb et al., 2007, p. 8). It also inhibits "cognitive elaboration on any one mental event in favor of broadly attending to more temporally proximal sensory objects, canvassing thoughts, feelings and physical sensations without selecting any one sensory object. This network avoids rumination by disengaging attentional processes of self-referential elaboration," (Farb et. al, 2007, p. 2). I will refer to this as the experiential-metacognitive network.

The “self-regulatory” procedures, including monitoring and "on-line" decision-making” proposed by Schoenfeld (1992) may be conceptualized as part of this experiential network. Because this network supports "broad attentional focus centered on momentary experience, including internal thoughts, emotions and external sensory events, in addition to bodily sensations" (Farb et. al, 2007, p. 7), it may provide the information needed to make on-line adjustments to cognitive strategies.

Christoff, Gordon, and Smith (in press) have argued that a state of defocused attention may be one of the key factors facilitating creative thought. I expect that the experiential metacognitive network will be essential to flexible mathematics because a key characteristic of flexible mathematics is the creative reactivation and recombination of networks that already exist.

However, metacognition may be a function of both experiential and narrative networks acting in integrated ways. In Farb et al.’s (2007) work, these networks could only be observed separately because the subjects were trained mediators who had been specifically trained to activate the experiential network without activating the narrative network. Christoff, Gordon, and Smith (in press) suggest that executive functions, found close to areas recruited for narrative generation, are required to evaluate the quality of outputs from the networks supporting this more "defocused" state. Thus it is likely that these two networks interact, providing a robust metacognitive network capable of integrating the self across time with current experience. This study was designed to explore how these two networks operated as students exhibited flexibility with linear equations.

Measurement Strategies.

Although Schoenfeld (1987) has created a taxonomy of metacognitive activity – beliefs and affect, self-regulatory procedures and knowledge of cognition – his empirical work tends to focus on the dynamics between activities that emerge as students engage with novel problems rather than looking for examples of metacognition as he has defined it. Typically, Schoenfeld (e.g., 1999) has used graphics to present these dynamics across time. The horizontal axis indicates time; the vertical axis contains categorical descriptions of student cognitive activity (i.e., reading, analyzing, exploring, planning, implementing, and verifying). Filled in regions indicate that a particular activity occurred at a particular time. Figure 1 gives an example of an Activity Across Time (AAT) graph from a previous study. In this example, Schoenfeld has traced the activity of a dyad as they solved a nonstandard problem.[2] The benefit of this graphic is that it highlights both what students were doing and the relationships between these actions across time. For example, it captures how students moved from exploring the conceptual space of the problem to planning to implementing this plan. For parsimony, I will refer to this type of graphic as an AAT graph.

This study tracked metacognition by building on this approach. Similar to Schoenfeld, this study captured self-regulatory activity by looking at dynamics in cognitive activity across time. However, for this study, the categories of activity tracked also included the networks that may be essential to mathematical flexibility, rather than only the cognitive activity Schoenfeld tracks. The study involved explicitly tracking active beliefs or affects, self-regulatory activity, and stated knowledge about cognitive processes. As well, the study attempted to track metacognitive activity that was narrative, experiential, or both, across time. As a result, it was

[2] The specific problem is not available, except in a book I have not been able to locate.

hoped that I could track the activity of metacognitive networks and how they interact. A more detailed description of how AAT graphs were used to detect network activation can be found in Chapter 3.

Up to now, assessing narrative or experiential metacognition has been done with brain imaging techniques. Observing this activity indirectly and without using brain imaging techniques has not to my knowledge been done; thus, a coding procedure was designed and was tested as a preliminary analytical step, to ensure coding procedures were consonant with the nature of the data. This coding procedure contrasted ways in which traces and utterances indicated reference to the self across time from ways in which they indicated on-line and experiential awareness. Because this was a new approach, this study offered a test of the feasibility of this approach in investigating these kinds of metacognitive networks.

The above approach suggests an effective way to observe metacognition as it occurs, but it does not capture the metacognitive profile that students bring to activities. Although every activity occurs as a transaction in context, individuals acquire mental networks that increase the propensity of certain network activation states. Thus, strategies for measuring what students brought to the study in terms of metacognition were also important. Schoenfeld (1989) has created an instrument which allows exploration of a set of beliefs about mathematics. The instrument contains over 70 questions, too many to include in the study without taking an unreasonable amount of time. In this study, subsets of those questions were administered before students began the problems in the study as a way to understand the beliefs about themselves and mathematics that students brought to the study. A discussion of the specific questions used in the instrument can be found below.

Capturing what students can say in general about their cognitive and self-regulatory processes has not been done in Schoenfeld's work, nor have I been able to find other instruments based on Schoenfeld's taxonomy. It is not clear that capturing self-regulatory or cognitive processes is possible in general as the transactional/contextual nature of mental activity suggests an irreducible relation between mental and contextual activity. However, in order to see if students' general self-perceptions did relate meaningfully to measures of self-regulation and cognitive processes, a set of open-ended questions were designed to elicit what students knew about their cognitive and self-regulatory processes in math, for flexibility, and for linear equations specifically. These are detailed in Chapter 3.

Core Mathematical Networks

Daheane (Dehaene et al., 2004; Dehaene, 1999) has suggested that three networks serve to perceive and interact with number in general.

The first is a core quantity system, analogical to an internal "number line." This network functions to output proximity relations between number and is used primarily in addition and subtraction, particularly operations which are not memorized (i.e., that have a sum or difference greater than 10).

The second is a network that manipulates numbers in verbal form. This "region is part of the language system, and contributes to number processing only inasmuch as some arithmetic operations, such as multiplication, make particularly strong demands on a verbal coding of numbers" (Dehaene et al., 2003, p. 494). It is used primarily in multiplication and exact calculation, particularly addition or subtraction facts which are memorized.

A third network interacts with number in Arabic numeral form. This region recognizes the meaning of Arabic numerals without having to convert them to verbal representations (Dehaene, 1999).

It is unclear how relative strengths or deficits within these networks affect competence and flexibility. Participants for this study were selected to ensure that they were competent with linear equations, thus it was unlikely that severe deficits in any of these networks existed. It is unclear how differences in these networks across participants affect flexibility; one of the aims of this study was to understand if and how these networks are instrumental to flexibility.

Measurement Strategies.

Deficits in the core analogical networks are known to affect estimation, and exact calculation with addends greater than 10 (Dehaene et al., 2004). Deficits in the verbal region are known to cause difficulty with overlearned facts. It is unclear how difficulties processing Arabic digits can be observed outside of brain imaging paradigms. For this study, a set of timed estimation, exact calculation and multiplication tasks was given to each subject as a rough measure of the activity of each network. To discriminate between Arabic and verbal forms of number, tests of multiplication and exact calculation were given with both Arabic and linguistic numbers presented.

Contextual Networks

This study had several contextual features which may have influenced the ways in which flexibility and competence were demonstrated. First, students engaged with problems designed by the researcher. These problems formed part of the context and their structure afforded particular ways of engaging flexibly or competently. Second, subjects had opportunities to interact with the researcher, including opportunities to explain and ask about problems they had just worked on. The way in which the questions were asked and the nature of the interaction with the researcher may have afforded particular ways of responding or have activated particular networks within a subject. Third, the study itself allowed students access to a calculator, paper, and a sheet with the problems on them. Ways in which students interacted with these tools represented components of the individual-contextual transaction. Finally, the specific protocol of the study formed part of the context. In particular, the intake interview may have activated certain mental networks that were then more available to subjects during the study and the follow up questions may have activated or amplified the activity of mental networks.

Measurement Strategies.

As the above analysis of Greeno's (1991) research suggests, capturing interactions between context and individuals requires rich description of complex interactions and analysis of those interactions that is relational and dynamic. Central to this approach is recording all activity that occurs in an integrated and high fidelity way. This study was designed to support rich dynamic analysis in many ways. Subjects were tape and video recorded, their utterances transcribed, and traces of all work or activity were collected; all of these were linked to a central timeline that recorded timing of key features of the study protocol (e.g., changes between problems, asking of follow up questions). The visual format adapted from Schoenfeld's (1989) work allowed dynamic descriptions of complex mathematical activity.

Thus, this study facilitated the types of rich understandings researchers like Greeno have deemed central to understanding individual-context transactions instrumental to mathematical flexibility.

Consciousness

I define consciousness in a functional way: It serves to make the activity of modular processes in the cognitive network available to other modular processes, including those which are not connected by a hub. Heuristically, consciousness can be considered momentary "fame in the brain" (Dennett, 2001) – when activity from a particular network becomes famous throughout the entire mental network. As Dehaene puts it: "This model emphasizes the role of distributed neurons with long-distance connections, particularly dense in prefrontal, cingulate, and parietal regions, which are capable of interconnecting multiple specialized processors and can broadcast signals at the brain scale in a spontaneous and sudden manner" (Dehaene & Changeux, 2004, p. 1148).

It is unclear what the role of consciousness is in mathematical flexibility or mathematical competence. On the one hand many researchers have emphasized linguistic aspects of consciousness (Dennett, 1991), and Christoff, Gordon, and Smith (2007) have suggested that this linguistic network may interfere with creative thought. So being more conscious may interfere with mathematical flexibility, but support mathematical competence by allowing access to prelearned verbal sequential memories of over-learned tasks. On the other hand, the deep auto-association required to activate a wide ranging network may depend on consciousness to achieve this auto-association, particularly between distant brain regions. Thus, though I predicted that the content of consciousness would affect mathematical flexibility and mathematical competence, I did not predict how.

Measurement Strategies.

Dennett (1991) has suggested heterophenomenology as an approach to capturing what people are conscious of from self-reports, without assuming they have privileged access to their own consciousness. A commitment to heterophenomenology entails enlisting the participant in the task of reporting their conscious experience and considering this data not as a true fact about reality, but as a usable interpretation of the participant's point of view. In terms of accessing conscious experience, this can be done either during or after the activity in question.

Because this study attempted to distinguish the activity of narrative and experiential metacognitive networks and narrative networks may, as suggested above, be a dominant thinking mode in day-to-day life, it was essential not to promote the activity of this network by asking students to reflect verbally mid-problem. Thus, in this study, participants were asked to reflect on what they were thinking as they solved problems after attempting the problems. These utterances were interpreted as indicating what subjects believed they were conscious of while working on these problems.

Summary

Table 3 shows the networks expected to be active in students demonstrating flexibility with linear equations. For each I define the function of the network, its expected effect on mathematical flexibility, the sources of evidence for the network's potential role, and research supported measurement strategies. The next chapter describes the specific methodology I used in the present study to explore how these functional components could be observed and related to mathematical flexibility and competence.

CHAPTER THREE

The purpose of this study was to uncover the dynamic relationships between different kinds of networks (see Table 3) as students exhibited flexibility and competence with linear equations. To achieve this goal, I used a multiple case study design. Five students were selected and observed approaching a set of linear equations problems designed by the researcher to afford competence or flexibility. Before beginning the problems, students were interviewed in order to identify the state of the mental networks believed to be instrumental in linear equations flexibility. Then students were engaged in solving math problems and asked follow up questions about their thinking while doing so. Throughout, activation of networks specified in Table 3 was observed. Analysis traced the interaction between mental networks and linked these interactions to the data collected during the semi-structured interview. Finally, the researcher looked across cases to identify generalities that held across the particular cases examined in the study.

Participants met with the researcher once for two hours. Participants were interviewed, and then engaged in, and reflected on linear equations problems designed to afford competence or flexibility.

Research Design

To investigate functional components underlying mathematical flexibility as they are engaged in context, this study used a case study framework in which a combination of methods (i.e., interviews, traces, observations) were employed to better understand students' engagement in mathematical problem-solving. The case study methodology was designed to capture the dynamic activity of mental and contextual structures which are potentially instrumental to flexibility, as suggested by research on mathematical flexibility. This section elaborates and justifies this approach.

Yin (1994) suggests that a case study approach can be useful when asking “Why” or “How” questions of a phenomenon which is not easily divorced from its context. Mathematical flexibility may be such a phenomenon, as Greeno’s (1991) relational approach to investigating flexibility suggests. Yin also points out that case studies are well equipped to address transactions “in which there will be many more variables of interest than data points” (Yin, 2004, p. 13). Because mathematical flexibility depends on the complex interplay of a host of known and unknown cognitive and contextual structures, it seems reasonable to expect that many instrumental variables will need to be investigated simultaneously. Thus, a case study approach may provide a productive approach for investigating mathematical flexibility.

In some cases case studies can rely on theory development before data are collected to focus data collection and analysis (Yin, 1994). This analysis can then, in turn, be used to evaluate the theory from which it was generated. This study is based on a network theory of mathematical flexibility (established in Chapter 2), and a corresponding literature synthesis; measures of activity during the case study are based on this theoretical approach. The interpretation strategy (described below) affords evaluation of these theoretical predictions.

However, it was unclear how useful the theoretical approach described above would be for understanding mathematical flexibility. Exploring the explanatory effectiveness of a network theory was also supported by a case study approach. Central to a network theory is the notion of hubs – functional structures within the network which are instrumental to mathematical flexibility. It was unclear if the functional components under investigation would be discernable or relatable to flexibility and competence. It was possible that a network theory of linear equations would add little parsimony to understanding mathematical flexibility and that analysis would best done from a different theoretical framework (Greeno’s environmental metaphor of mathematics, for example). Though the study used network theory to structure observation and analysis, it is possible that this theoretical approach may have been less helpful than other theoretical lenses, and a case study framework allowed this possibility to be evaluated.

Because the goal of this study was to uncover what is instrumental to exhibiting mathematical flexibility in linear equations, a multiple-case study with a theoretical replication logic was used. Theoretical replication refers to searching for contrasting results across cases, all of which support or reject aspects of the underlying theory (Yin, 1994). This is different from a literal replication study in that results are not expected to be identical, but are expected to support or weaken theoretical propositions. In this study several students (cases) were observed interacting with problems that may have afforded flexibility or competence. Analysis looked across these cases and interpreted observations, survey responses and traces in light of the network theory of mathematical flexibility defended below. The goal was to use this approach to identify relationships between mental and contextual structures as they supported flexibility. It was hoped that these relationships were expressed in general enough terms that they held within and across cases. Thus, multiple-case studies provided several person-context interactions from which to evaluate theory and a theoretical replication logic allowed for multiple sources of theoretical support without expecting identical results between cases.

Yin (2004) suggests five distinct phases to the design and implementation of multiple-case, theoretical replication case studies. In the first phase, a theory is developed to inform study design, data collection and analysis. The theory developed for the present study can be found in Chapter 2. In the second phase, case selection and the design of data collection protocols occur. Case selection and data collection protocols can be found below. In the third phase, multiple case studies are conducted and individual case reports are written. The procedures used in this case study can be found below. In the fourth and final phase, cross-case analysis is used to modify initial theory. The interpretation strategy that was used in the present study can be found at the end of this chapter and the results and corresponding discussion comprise Chapters Four and Five of this thesis.

Participant Selection

The study included five students in high-school enrolled in Principles of Mathematics 11 and 12 (or the equivalent) who could all perform competently with linear equations and who received a 73% or above in Principles of Mathematics 10. Contrasting mathematical competence with flexibility was central to the study and sampling from students who have succeeded in mathematics, continue to take mathematics and who can perform competently with linear equations may have made students more likely to exhibit mathematical competence in linear equations during the study. Because the study design required in-depth interaction with the researcher, only students with sufficiently strong English language skills (to read problems and engage in a semi-structured interview) were included in the study.

In addition, the researcher attempted to attract a set of students who had a diverse set of mental networks. Each participant had his or her own history of specific experiences with mathematics, and those experiences likely shaped the design of their particular mental network. Having variation across student experience may have increased the chances that their particular mental networks varied in design, which in turn increased the validity of generalizations made across cases. From a network perspective, learning differences represent differences in network structure. Thus, students with distinct learning strengths and challenges were included where possible. To that end, volunteers were enrolled in the study until reasonable diversity in the sample was achieved.

Students were recruited through a poster indicating the purpose of the study, the time involved, compensation, the researchers and an e-mail parents or guardians of interested participants could use to contact the researchers. This poster was posted in public locations, like coffee shops, or public spaces intended for postering. The poster was also distributed to contacts of the researcher who made it available to prospective participants (with permission from any associated organizations).

Prospective participant parent/guardians contacted the researcher to find out about the study. If a student wanted to participate, a tentative appointment time was set and they were e-mailed a consent and assent form to return at their appointment.

Data Collection

Understanding which networks were active while participants were working on problems and how they interacted was a major challenge and goal of the study. Networks activate without necessarily causing any behavioral sign of their activation, so they may be hard to observe; some of the specific networks under investigation are not usually observed in the context of complex, flexible mathematics; and there is minimal research holistically observing the particular group of networks specified for the study. In light of these challenges, a main goal of the study was exploring if and how the study protocol supported observation and inference of network activation. In the study, data were collected to attempt to identify both the mental networks that students brought to the study and the ways that those networks interacted with contextual features of the study to produce more or less flexible mathematics.

Semi-Structured Interview.

The semi-structured interview (SSI) asked a variety of questions intended to surface the state of the mental networks students brought to the study. The protocol for this interview can be found in Appendix A. Table 4 connects particular questions in the SSI with the functional nodes it was designed to uncover. Some of the questions designed to uncover metacognitive activity were taken from Schoenfeld (1989); others were created by the researcher to explore new ways of getting at network activation. Measures for core mathematical networks were based on the triple code model (Dehaene, 1999). Measures of cognitive networks were based on the core skills required for successful performance in linear equations as defined in Chapter Two.

Traces of Student Activity While Solving Problems.

Students were asked to work on six problems, three of which were designed to afford competence, and three of which were designed to afford flexibility (see Table 5 for the questions used). As they worked on problems, audio and video recordings were made for transcription. In addition, traces of student work, found on paper distributed to participants to use to record their work and answers, were collected. It was hoped that these data would facilitate understanding of which networks were active and when they were active during problem solving. It was unclear if these data would facilitate fruitful analysis of network activity as set out in Chapter Two. However, an adaptation of Schoenfeld's AAT graphs (described below) was included in the protocol to allow these data to inform understandings of which networks were active and when.

Student Perceptions of Their Own Activity.

Structured introspection may be a promising way of surfacing another source of data about which mental networks are active and when they are active. After each problem, students were asked a series of follow up questions (see Appendix B). I designed the follow-up questions myself, based on personal experience, because they were largely designed to probe metacognitive networks (experiential and narrative) for which self-reflection protocols have not yet been generated. In the first (Follow1), students were asked, "Can you take me through how you solved this problem? Give me as much detail as possible about what you were thinking, feeling, and what you were trying to do at each stage of the problem." This question was designed to probe the experiential and narrative metacognitive networks simultaneously, by putting students "back in the moment" and asking them to describe what they experienced.

After they responded, participants were asked a second follow-up question (Follow2): "If a friend were about to attempt this problem, what should they know?" This question was designed to isolate narrative metacognitive network activity by asking students to consider themselves as existing across time, relating to another person in verbal form (which I hypothesized focuses attention away from the defocused, in-the-moment processes active in the experiential metacognitive network).

As a final follow-up question (Follow3), students were asked, "Do you have any questions about this problem? I am happy to answer any questions you have, if you'd like to clarify anything, or see where you went wrong [if their performance was not successful]." The researcher attempted to answer their questions as literally as possible. This question was designed to surface where conflicts between mental networks occurred, and how students attempted to resolve those conflicts. By doing this, I hoped to identify relationships between network nodes as they were active while solving the problem.

Procedure

Overview

Participants met with the researcher for a single session. This session took a maximum of 2 hours. The session had two components: in the first, the networks students brought to the study were surfaced using the SSI; in the second, students engaged with linear equation problems designed to afford either competence or flexibility.

Surfacing The Networks Students Brought to the Study

As discussed above, participants were interviewed as to what networks they brought to the study. This part of the study took less than 1 hour. As described above, the SSI used as a guide in interviewing students was an exploratory protocol, drawing on Schoenfeld (1989), the work of Dehaene (2004), the core cognitive networks used in linear equations, and the hunches of the researcher.

Working With, and Reflecting On, Problems Which Afforded Flexibility or Competence

After completing the SSI, students engaged with six problems (two sets of three problems) involving linear equations (see Table 5) and answered a series of follow up questions for each problem. This part of the study took between 30-60 minutes. Students answered follow-up questions immediately after completing each problem.

Within each set, one problem was designed to afford competence, the other designed to afford flexibility. Problems designed to afford competence could be solved by straightforward application of one or more of the seven competencies specific to linear equations, as detailed in Chapter 2. Problems that afforded flexibility could not be solved with basic application of these competences, but could be solved by recombining these competencies in novel ways. Each flexibility problem had multiple paths to the correct solution, so it was unclear what cognitive path students would take, but the problems could not be solved with standard application of procedures as they are generally taught in BC schools.

The two problems within each set were matched in terms of linguistic demands, the linear equations concepts that were relevant, and problem structure. Each problem required use of particular domain-specific networks drawn from curriculum documents. Two problem elements distinguished problems designed to afford competency from those that afforded flexibility: 1) researcher experience with students taking Principles of Mathematics 10 suggested a set of problems which are typical and used across schools and on provincial exams – problems designed to afford competence were drawn from this set, while those affording flexibility were not; and 2) problems designed to afford flexibility contained an added element not usually found in typical linear equations problems. For example, one problem asked students to find the slope of the line joining (a, b) and (3a, 2b). Most students would have seen many problems asking them to calculate slope given two points; or given slope and one point, to find a possible second point. However, these problems typically use specific rational numbers, rather than variables. The inclusion of variables required students to also simplify like terms and then express the resulting slope as a relation between two variables, rather than as a rational number. Though simplifying like terms is a basic mathematical operation taught in Grade 9 and used throughout high-school mathematics, its use in this context appears to require flexibility, because it is not typically used in this context.

After each problem, students were asked a series of follow up questions designed to allow them to recollect and report on their mental experiences while working on the problems, as described above. Because it was essential to observe on-line activity with minimal amplification of the narrative metacognitive network, students were not asked to reflect on their activity until they had finished the problem. As discussed above, this might have made observing network activity during the activity more challenging.

At the end of the sessions, students were thanked for participating and asked if they would like to hear about the results of the study when the study was complete. If they asked to learn about the results, the researcher agreed to contact students to set up a meeting once the study was complete..

Data Synthesis and Interpretation

Interpretation of Network Activation

Understanding which networks were active while participants were working on problems, and how they interacted, was a major challenge of this project. It was also central to the data interpretation strategy described below. As discussed above, networks activate without necessarily causing any behavioral sign of their activation; some of the specific networks under investigation are not usually observed in the context of complex, flexible mathematics; and there is minimal research holistically observing the particular group of networks specified for the study. In light of these challenges, a main goal of the study was exploring if and how the study protocol might support observation and inference of network activation.

The study protocol was designed to provide rich descriptions of participant activity as they worked on problems. Traces of student work were collected from paper participants used to record their answers. Video and audio tapes recorded utterances, gestures and facial expressions. Follow up questions gave reports of participants' internal experience and afforded them opportunities to ask questions which might have revealed more about their internal experience. For each case, rich descriptions of student activity were generated and analyzed by the researcher. Data analysis approaches were used to interpret these rich descriptions and try out different ways of interpreting network activation.

Analysis of rich description to generate parsimonious interpretations of network activation may be limited in its efficacy. It may be that particular utterances, activity, and traces are too vague to be reliably considered the product of a particular network. Because students were not asked to reflect mid-problem, follow up questions may not have reliably indicated which networks were active at which times.

To provide a complementary, more structured lens from which to view data, the protocol included creation of two versions of Schoenfeld's AAT graphs which could be used to inform data interpretation. AAT graphs track activity across time as students work. It was hoped that analysis of AAT graphs in the study would allow dynamics of student activity, and network activity, to emerge.

The first type of AAT graph (AAT1) was based loosely on the Schoenfeld (1992) example discussed earlier. These graphs track student activity across time during problem solving. Schoenfeld tracked categories of activity germane to the dyad activity he was observing – reading, analyzing, exploring, planning, implementing, and verifying. In this study, AAT1 graphs were constructed in an attempt to replicate the concrete types of activity Schoenfeld discussed, but using categories relevant to the activity under study here. These categories were finalized during initial data analysis to ensure they matched what students actually did.

It was hoped that analysis of this type of AAT1 graph would allow temporal patterns of network activation to emerge. These graphs show dynamics of activity across time; within a network theory, changes in activity represent changes in network states. Thus, from each change, the activity of particular networks might be inferred. However, because activity was tracked using categories of activity, rather than network states, it was not clear up front whether or not these inferences would be valid.

To provide a second view of cognitive dynamics across the study, a second type of AAT graph (AAT2) was used to analyze data. Rather than track participant level activity, AAT2 graphs tracked activity within the networks under investigation. In-context observation of these networks (as they are defined above) is a new research aim, and the method I tried was exploratory. Table 6 details each of the networks under investigation and gives exploratory criteria that were applied to try to identify network activation. For example, student utterances relating to belief, affect, or assessment of cognitive capacity and utterances in which the participant is the subject were interpreted as narrative-metacognitive activity.

Analysis of AAT2 graphs was designed to focus on looking for patterns of network activation within students and then across students. It was hoped that relating these patterns to exhibited flexibility and competence –within and across students – would illuminate how network activation can support flexibility. However, AAT2 graphs specify only which networks are active – not how they are active or how they are transmitting activation to other networks. This limitation may have hindered the ability of the study to uncover how network activation related to flexibility. As well, the exploratory nature of the criteria used may have hindered their validity, because they were not explicitly derived from other research. Thus, while it was hoped that the AAT2 graphs would illuminate relationships between functional networks and flexibility; but, because activation was measured only as off/on and using exploratory criteria, whether or not the graphs would ultimately prove useful was unclear.

In sum, three types of observation were used to support detection of network activation. First, rich descriptions of student activity afforded opportunities to posit parsimonious inferences about network activity. Second, AAT1 graphs were designed to illuminate activity dynamics and seeing change in activity across time, and to facilitate understanding of which networks are active and how they supported or inhibited flexibility. Third, AAT2 graphs were designed to attempt to track network activity itself, allowing direct analysis of network activation (though threats to validity made this approach difficult).

Every utterance and recordable action, by definition, results from activation of a variety of functional nodes, including ones I have not specified. Thus, detecting node activation was necessarily fuzzy and exploratory. Reliably and validly detecting activation in functional nodes was a test of my methodology. If I found that differentiating one node from another was too difficult, this would suggest that this methodology might not suffice to accomplish what I set out to do.

Relating Flexibility, Competence, and Network Activation

My main synthetic strategy was to triangulate across transactions within a case and then to triangulate across cases to identify increasingly general conclusions about flexibility and competence in linear equations. Triangulation refers to substantiating hypotheses by synthesizing across distinct lines of evidence, preferably with distinct data collection methods as well, in order to evaluate theoretical predictions (Eistentat, 1989). In this study, each student engaged in six separate problems – in which traces of written work, and audio and videotapes of activity were collected; as well, follow-up questions were asked– giving audio and video data; and students completed the SSI, giving audio and video data that was transcribed. Table 4 relates affordances for observation during the study with the functional networks observed. This set of data was designed to be broad enough to afford triangulation that could inform theory. Within a case, I used an iterative method of triangulation, using a single piece of data, or a hunch, to inform a hypothesized relationship between functional nodes. I then examined documentary evidence (artifacts and audio/video tapes) to attempt to falsify or qualify these hypothesized relationships.

Relationships that stood up more strongly to this process were considered to be better triangulated, and a more valid picture of relationships that existed within that student's mental networks. After this process, each case had a set of hypothesized relationships between functional nodes and their contribution to flexibility. I then used the same process to look for increasingly general patterns across students. Again, I examined documentary evidence (traces, utterances, and audio/video tapes), to attempt to falsify or qualify these hypothesized relationships. Relationships that stood up more strongly to this process were considered better triangulated and a more valid picture of relationships that existed across students.

Importantly, I did not expect to find stable general relationships like "experiential metacognitive networks always cause mathematical flexibility." The network model is more dynamic than traditional causal models, where constructs are expected to have an independent and consistent effect on a dependent variable. Instead, I looked for general relationships like "experiential metacognitive networks can inhibit narrative metacognitive networks, and if they do, they tend to activate both cognitive networks for linear equations, and facilitate switching fluidly between those networks." This type of generalization is consistent with the stable but non-linear patterns found within networks.

CHAPTER FOUR

Results

The purpose of this study was to contrast network activity as students exhibited mathematical flexibility with network activity when students exhibited mathematical competence, while working in linear equations. In this report of findings, I focus attention on what I learned about the relationship between student problem-solving, network activity, and flexible or competent performance. I provide a descriptive report of each student's activity while working on each of the six problems, including characterizing elements of their activity on each problem as competent or flexible performance in mathematics. These reports include descriptions of network activity in mental and contextual networks thought important to mathematical flexibility. I then integrate across these problem-by-problem activity reports to identify student-specific patterns in network activation and relate these patterns to student flexibility and competence. Finally, I offer an integrative analysis of patterns in network activity/flexibility that were observable across cases. But because a parallel goal in this thesis was to test out a relatively new methodological approach, in this chapter I begin by drawing attention to affordances and limitations I discovered in my methodology, and explain what steps were taken to improve the quality of inference that could be made from the data.

Characterizing Students' Participation in Mathematical Problem-Solving

The methodological plan in this study was to analyze data from written, audio, and video traces of student activity in order to characterize network activation while participants were working through mathematics problems and to relate that activity to flexible or competent performance. As described earlier, a variety of data were collected to inform understanding of students' activity during problem solving. Participant work on each problem was recorded on sheets provided by the researcher. Video records of students' problem-solving performance were collected on a video camera with an SD memory card. To aid in analyzing the rich combination of data, participant answers to follow up questions were transcribed by a research assistant. In this section, I briefly describe how I ultimately treated the data in order to best derive an understanding of students' network activity.

Describing Students' Problem-Solving Activity.

In order to make sense of students' *problem-solving activity*, I started by segmenting problem-solving into time intervals that seemed to represent coherent units of activity, and assigned these with "activity codes." The particular activity codes were derived from participant activity, and designed to be as theory-neutral as possible. The codes used included *thinking*; *writing*; *talking*; *reading*; and *using calculator*. Activity codes were supplemented with "activity descriptions" – brief but specific descriptions of what that participant was doing during the coded period. The specific descriptions were created by triangulating activity captured on video with written work, and with participant perceptions of their work as reflected in responses to the follow up questions. As such, descriptions represent the best effort of the researcher to detect what participants were thinking and doing during each problem, but were subject to errors stemming from undetectable activity, or misperception of activity.

The *a priori* analysis plan called for activity reports to be summarized in two types of activity graphs that could characterize each student's engagement in each of six problems: AAT and AAT2 graphs. AAT graphs, based on Schoenfeld's (1997) research, were planned to represent activity across time. AAT2 graphs were proposed as a way of tracing network activation across time. However, once the data were examined, it became clear that detection of network activation was possible at some points, but not at all points. Creation of AAT2 graphs, then, risked giving a misleading visual impression, as a large portion of network activity was not observable.

As a result, the decision was made to use AAT graphs as the framework for characterizing problem-solving performance. AAT graphs were created with assistance from a research assistant, working from a template set of charts. Tables of coded activity were converted into graphs, where each activity code was 'on' as horizontal bars (see Figure x for an example). The x-axis represents the time elapsed in the problem, and the y-axis represents the categories of activities. The bars are annotated with a number, which corresponds to a chart of "activity details," which are located under the x-axis.

Interpreting Network Activity.

Next, I conducted a careful analysis of the extent to which networks were activated during these activity sequences, based on the scheme found in Table 5 focused on network coding. I provide descriptions of network activation observed within "activity reports." Network activity did not map perfectly onto the time intervals generated for AAT graphs, so network activity is coded as corresponding to particular sets of activity details. However, this coding is temporally imprecise – network activity is likely to be occurring in time intervals that do not correspond directly to observable behavior – and thus, the recorded activities to which network activity corresponds should be considered accurate, but loosely connected to observable activity.

Note that I also found, during data coding, that it was difficult to consistently characterize metacognitive statements or behaviour as narrative or experiential based on the data collected. Instead, I focused on describing activities that could be classed metacognitive, and drew conclusions about activation of metacognitive networks, including an analysis of their experiential and narrative nature, based on patterns observed in those more broadly coded data.

In order to create coding criteria for recognizing activity in metacognitive networks, I drew on Schoenfeld's (1992) three part taxonomy of metacognition in mathematical problem solving (as described earlier). In line with this framework, network activity was coded as metacognitive if there was evidence of: 1) expression of belief or affect; 2) declarative knowledge about cognitive processes; or 3) self-regulatory procedures like monitoring or on-line decision-making. Note that applying these coding criteria did not associate a metacognitive label with all actions that may be positioned as metacognitive in other research. Specifically, activity unaccompanied by expression of belief or affect, or declarative knowledge was not termed metacognitive network activation, unless I could detect specific activity related to "making a decision." This coding choice was made to account for the possibility that particular networks may produce activity that looks 'metacognitive' (in the broadest sense), because of the ways in which networks interact, but does not actually require active networks that state belief and affect, declarative knowledge, or make decisions. If this transaction level metacognition occurred, it could be considered a property of the transaction as a whole, emerging from the interactions of non-metacognitive networks, rather than from particular metacognitive network activity within that transaction.

The approach of using clear evidence of a "decision" as evidence for metacognitive network activity may have had some limitations. This approach was most challenging in cases of participants shifting and applying strategies. When inferring metacognitive activity in the absence of clearly articulated declarative knowledge about cognitive processes, I looked for evidence of activity related to shifting activity, or enacting a strategy. This evidence included activity that seemed to function to shift activity or strategize (e.g., pausing, thinking, subvocalizing), or evidence of cognition related to this "decision." This approach to associating metacognitive activation with decision-making risked missing important metacognitive network activity, but had the benefit of increasing the chance that activity identified represented metacognitive network activation, rather than “metacognition” emerging from non-metacognitive network interactions.

Identifying Flexibility and Competence.

Student activity was also coded as *flexible or competent*. In the theoretical framework presented earlier, I conceptualized activity as flexible when there is adaptation to new, different or changing requirements when engaging with quantity, magnitude, or form. Activity was conceptualized as competent when a student navigates the situation without any adaptation or recombination of existing knowledge and without attending to unique affordances in that particular situation. Following these definitions, if triangulation between data sources suggested that particular activity was an adaptation of competent activity, this activity was coded as flexible. If evidence of adaptation was not present, the activity was coded as competent.

One implication of this coding scheme is that activity was coded as competent whenever there was insufficient evidence to view the activity as flexible. Therefore, behavior coded as competent may have reflected either competent activity, or flexible activity which did not leave a trace of that flexibility. In order to increase the validity of competence coding, evidence of the competent nature of the activity is noted where possible. In particular, student self-reports of their work were used to construct understandings of when students were interacting with linear equations competently. In cases where the competent nature of the activity was unclear, this was noted in the activity reports.

In contrast with the research plan, flexible and competent activity were ultimately coded on the activity level, rather than the problem level. Though the initial research question intended to contrast flexible and competent problem solving, the results indicate that flexibility and competence may be more effectively conceptualized as referring to particular units of activity within a problem, rather than to the nature of the problem solving as a whole. As will be shown below, in several cases, students showed evidence of both flexible and competent activity within a single problem. Thus, the decision was made to code activity as flexible or competent, rather than characterizing problem solving performance overall as flexible or competent.

Initial analysis plans also suggested framing three categories of activity: competent, flexible, and incompetent. However, when coding, it appeared that problem-solving included a mix of competent and flexible activity, and "incompetence" was hard to define, except as a relatively unmeaningful judgment of whether activity lead to a successful or unsuccessful outcome. The decision was made to code activity as flexible or competent, and to omit the incompetent category.

Activity Reports

To characterize students' problem-solving, I ultimately pulled together activity reports for each problem solved by each participant. Activity reports represent the core results of the study. Activity reports were generated through an iterative process. First, student activity was coded to indicate what students were doing at each moment during the problem. This activity was translated by the researcher into narrative descriptions of student activity, which were then combined with transcripts of how students described their activity during follow-up questions. Together, with the AAT graph, these narratives provide a rich description of how the student acted during problem solving. From these results, I coded activity in terms of network activation, and in terms of flexibility and competence. Descriptions of network activity and flexibility follow the rich illustrations of that activity. At the end of each student's question reports, I summarize key patterns of network activation and flexibility, and their connections, that can be found for a particular student.

Clayton[3]

At the time of this study, Clayton was a student going into Grade 12 at a private school for students who have a language-based learning disability. He reported getting a B in Math 10 (73%-86%) but could not remember his mark exactly. Clayton was easy going and relaxed. He was comfortable interacting with the researcher. When asked about his strengths and challenges, Clayton did not identify strengths, but discussed his weaknesses: "I'm dyslexic but I don't know if ...I guess sometimes it's different but I'm not sure for math really. I guess math was … in elementary school [it] was sort of the problem…but then when I came here it started to really unfold a bit a bit better."   Table 7 indicates which of Clayton's activities were competent or flexible.

Problem C1

Activity Description.

The AAT graph describing Clayton's activity in this first problem is presented in Figure 2. After using the wrong formula initially, Clayton was able to solve this problem successfully. Clayton began his work on this problem by clarifying what the researcher's expectations were ("Do you want me to draw a graph?"). Once the researcher informed him that he could provide any work that gave insight to his thinking, Clayton restated this ("so basically, show you what I'm thinking"). After returning his attention to his paper, he wrote down the mid-point formula. After a brief pause, he drew an x/y axis, and sketched a line with the points given in the question.

[3] Names of participants have been changed.

While in thought, Clayton (C) realized that the midpoint formula did not afford him a solution to the problem, and he crossed it out and clarified with the researcher (R):

C: Uh, if I can't remember the formula for slope or whatever?

R: Can I see what you did there?

C: I think that's the midpoint formula

R: That is the midpoint formula, that's right. I'll tell you the formula, if you'd like. It's y2-y1 over x2-x1

C: Oh, ok, yeah yeah yeah.

Armed with the slope formula, Clayton made quick work of the rest of the problem, writing the slope formula, and plugging in the values given to get the correct answer.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

C: Well, I guess first I got mixed up with the formula. I started using midpoint formula, then I realized that that gives you two points so then I was like, ok that's not it. So I decided to draw out - sketch a graph. Then you told me what the slope formula is, and I was able to just plug in y2 and y2 and x2 and x1.

R: And what do you think would have happened if I hadn't give you the formula, what would you have done?

C: Probably would make my life harder, and try to find the area in here, which would also give you the slope, right?

Affordances to Detect Network Activation.

First, Clayton's problem solving activity suggested use of *domain specific networks* for midpoint formula, joining points to make a line, and the slope formula. That Clayton wrote the midpoint formula immediately after beginning the question suggests that a 'midpoint formula network' was activated by the question text itself. It is possible that the two coordinates in the question text triggered activation in cognitive networks that can deal with sets of coordinates (which might include a network for the midpoint formula). From Clayton's activity, I infer that the midpoint formula activated most strongly. However, later, Clayton realized that the midpoint would not help him, and after asking for help, he dampened the midpoint network activity and used other domain specific networks to solve the problem. Clayton's plotting and joining of the two points early in the question indicates that domain specific networks for algebraic/visual-spatial representation of coordinate systems were also active during the question.

Networks supporting algebraic use of the slope formula were instrumental to Clayton's successful performance. After seeking a reminder as to the slope formula, Clayton easily used this formula to solve the problem, without pausing to think or ask questions. Though he needed a reminder as to the content of this formula, it is likely that network began to activate earlier in the question, as he sought out the formula once the midpoint formula had failed. His speed and competence with the formula suggests that, once activated, for Clayton, this network supported a well-practiced algebraic procedure for finding the slope, given two points.

Core mathematical networks for exact calculation were active late in the question, when Clayton mentally calculated the result of the slope formula.

It may be the case that *metacognitive networks* supported Clayton's shift away from the midpoint formula, and his parsing of the expectations for the question. After two minutes of trying to use the midpoint formula to determine the answer, Clayton said "Uh – if I can't remember the formula for slope of whatever?" This utterance represents declarative knowledge about cognitive processes, and thus may be understood as metacognitive network activity.

Finally, Clayton made use of two key *contextual* networks: the paper, and the researcher. For Clayton, interactions with the researcher seemed to afford activation of relevant domain specific networks. Twice during the problems, Clayton sought clarification or information from the researcher which impacted the domain specific networks active immediately following. For example, when asking about the slope formula, the researcher's response activated the domain specific network for slope, affording him a cognitively simple path to the solution. Previous to his utterance, Clayton must have known the slope formula was important (since he asked about it), and therefore that network must have been activated to some extent; but his use of the contextual network (i.e. the researcher) strengthened the activation of the domain specific network in such a way that he could easily solve the problem.

Clayton also used the paper to highlight the visual-spatial aspects of the problem, and to store formula and calculation information. Early in his activity, Clayton sketched the graph depicted in the question. As soon as formula was in his head, he wrote it down. He also wrote down all calculation steps, doing none in his head. Thus, it seems possible that, for Clayton, the paper affords storing of information and problem representations, so that they do not need to be stored in his brain.

Clayton showed *competence* in the way he applied the midpoint and slope formulas, and wrote down their results and intermediate calculations. Though only the slope formula contributed to his successful solution, both formulas were applied easily and accurately. The activity seems to have been supported by relevant domain specific and contextual networks.

Clayton showed *flexibility* during this problem by obtaining the slope formula from the researcher and employing it. The mid-point formula, which Clayton initially used, returns two distinct values (one for x and one for y). Once Clayton had obtained these two values, he crossed out the midpoint formula, and asked the researcher “So, if I can’t remember the slope?” At this point in the question, Clayton could see from his work that there were two points in the answer, rather than a single value, and thus realized that his formula wouldn’t afford him an answer. He adapted to this situation by probing the researcher for the information he was missing. Ultimately, this flexibility was essential to solving the problem, as it afforded him the necessary domain specific network activation needed to competently solve the problem.

This flexibility may have been supported by network activity. In this flexible activity, Clayton used a contextual network (the two points he wrote on his paper) to activate metacognitive and contextual networks (saying to the researcher: “Uh, if I can’t remember the slope of whatever?”, which, in turn, activated domain specific networks (for slope formula), which he then used to correctly and competently solve the problem. This adaptation seemed to be triggered by activity in the contextual network of his paper, which in turn, activated the relevant metacognitive and domain specific networks.

Problem F1

Activity Description.

The AAT graph describing Clayton’s activity in this second problem is presented in Figure 3. Clayton solved this problem quickly and easily. He began by reading the question, possibly twice (given the time that he took, and that the particular school he attends has very small classes and gives explicit strategy instruction for problem solving). After he read the problem, he wrote the slope formula, inserted the relevant terms, and simplified the algebraic expressions to obtain the slope of the line in simplified form. He did pause to think twice during the problem, though only for a few seconds.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

C: Basically the same was solving the other problem using the slope formula. I was also thinking like since 3a - I'm guessing if this point was a or something, but it's 3a so it's 3 times as long. Same with 2b - it'd be twice as long.

R: How did you use that in solving the problem?

C: I didn't actually. I don't know. Maybe I was just thinking about drawing a graph and then it occurred to me that I could just use the slope formula

Affordances to Detect Network Activation.

Clayton’s work on this problem afforded detection of activation of domain specific, core mathematical, metacognitive, and contextual networks. First, Clayton’s activity shows evidence that *domain specific* networks for slope, algebraic manipulation, and visual-spatial coordinates were active. Clayton’s quick use of the slope formula suggests that it was highly active – his work in the previous question may have activated it. At the end of the problem, Clayton simplified two easy algebraic expressions. His detection of this affordance suggests that networks for algebraic simplification were active during the problem, and were recruited to complete the problem. Early in the problem, Clayton drew an x/y axis, suggesting that initially visual spatial networks for coordinate graphing were active.

This question does not require the use of *core mathematical* networks, except for very simple, highly automatic calculations (3-1, and 2-1). Clayton was able to solve this problem quickly and easily, suggesting a minimal role for *metacognitive* activity. However, Clayton’s shift away from his initial step – drawing an x/y axis – to an algebraic method may have represented a metacognitive shift in strategy (or it may have been a direct interaction between contextual and domain specific networks).

Finally, *contextual* networks were used to store information. As in the previous question, Clayton used the paper to highlight the visual-spatial aspects of the problem, and to store formula and calculation information. Throughout the problem, he wrote all steps, doing none in his head. Thus, it seems possible that, for Clayton, the paper affords storing of information and problem representations, so that they do not need to be stored in his brain.

Clayton's activity in this question appeared to be a *competent* extension of his competent activity in the previous problem. He quickly and easily solved the problem, and appeared to use the same domain specific networks, in the same way, suggesting competence supported by these slope networks. Clayton's activity in the question did not appear to have detectible examples of *flexibility*.

Problem C2

Activity Description.

The AAT graph describing Clayton's activity in this third problem is presented in Figure 4. Again Clayton seemed to perceive this problem as fairly simple, and he solved it quickly and accurately. After reading the question, he divided the page into two columns and labeled them Company A and Company B. In each column he wrote the access fee, determined the portion of the cost that relates to minutes by multiplying the relevant numbers, and added this to the system access fee.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

C: Ok, well basically there's Company A and B. Company A, you'd start off having to pay 12.50....that's for the plan right? And then 20 cents a minute. Uh, so the monthly fee is 12.50 for company A, and then if you 30 minutes that month you'd times how much it cost per minute by the 30 minutes - that equals to 6 dollars. Then you add the two together – the monthly fee and how many minutes you used so the total is 18.50 for Company A. And basically the same thing for company B but with different number, but the total would be 16.60 for company B.

Affordances to Detect Network Activation.

In this question, Clayton may have activated *domain specific* networks for knowledge of cell-phone plan pricing. These likely included networks for calculating total costs given a per minute cost, as he demonstrated competence in this area. Although it is possible that these networks are in fact more general mathematical networks for linear equations, the fact that he did not write general equations or talk about the problem as requiring an equation suggests that the networks were more specific to cell-phone pricing or at least to consumer spending.

Though Clayton used the calculator to do most arithmetic, he did once add two numbers in his head, indicating activity in *core mathematical* networks. Clayton's choice to divide the page in two represents a *metacognitive* decision to attack the two companies in the question separately. Activity in *contextual* networks was evident when, as in previous problems, Clayton used the paper to store calculation and information, including the results of intermediate calculation. In this problem, Clayton also used a calculator to multiply (rather than using core mathematical or domain specific networks).

All students solved this problem in a *competent* manner, including Clayton. He immediately calculated the costs accurately, suggesting that this competence was supported by domain specific networks for cell-phone pricing. Clayton's activity in the question did not appear to have detectible examples of *flexibility*.

Problem F2

Activity Description.

The AAT graph describing Clayton's activity in this fourth problem is presented in Figure 5. Clayton struggled with this question, and ultimately gave up before positing an answer. Clayton began by reading the question. This question is similar linguistically to the previous question and uses the same companies. Initially, he interpreted the question as asking "which of the companies in the previous question was least expensive?" He clarified that understanding with the researcher:

C: Can I just link them together? Or do want me to go through the same steps I just did?

R: Link them together? What do you mean by link them together?

C: Cause it says your friend asks you is Company A cheaper than Company B.

R: <u>When</u> is Company A cheaper?

C: Is that all I have to answer? Do I have to go through the same steps?

R: No you don't have to show anything, you just have to answer that question. Maybe that question is not worded clearly.

C: No I think I got it.

R: You think you got what it's asking?

At this point, the researcher was unsure how Clayton was understanding the question, but Clayton began to write "Company A is cheaper when dealing with the cost of its minutes," and the researcher did not continue the interaction. However, this writing suggests that Clayton still saw this problem as requiring him to summarize the results of the previous problem. Clayton then stopped, thinking that he had solved the question. However, the researcher, concerned that the problem had been worded poorly, offered some clarification:

R: I think that question is poorly worded, so I'm going to re-ask it… I want to know up to what point - in terms of how many minutes per month you're using - is company A cheaper than company B. So I'm not looking for something like, at 15 minutes company A is cheaper, I'm looking for a range. Do you get what I mean?

C: Like?

R: It might be that up to 500 minutes Company A is cheaper than Company B, but if you're going to use more than 500 minutes, Company B is cheaper.

Clayton then suggested using the T(i) formula from a different unit of Grade 10 math (sequence and series). The researcher suggested that this approach was not the intended approach, though it may have worked. The researcher and Clayton continued to discuss how to interpret the task. Ultimately, they were not able to achieve understanding, and Clayton looked back at the previous question for a minute. After a minute, he indicated to the researcher that he was done with the question.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

C: Well I guess I was just trying to figure out what - like - how many days or months it would take for Company A to be cheaper in the long run.

R: And how were you thinking about that?

C: Well I was looking back to this other question- where it gave you the numbers for how much it would cost per month. So basically in one month Company A was more expensive *but not by much...like a buck or something. And uh, so then, I guess probably two months it'd be....Company A would be cheaper to have, but I didn't know how to find that amount.*

Affordances to Detect Network Activation.

Clayton's work on this problem afforded detection of activation of domain specific, metacognitive, and contextual networks. This question clearly activated all the *domain specific* networks that had been active in the previous question. This is evidenced by the fact that Clayton was cognitively stuck on the idea that the previous question contained the insight relevant for this question. Networks for other formulas from Grade 10 math, specifically the formula to find a term in an arithmetic sequence, were also active. The researcher's comment that this formula was probably not helpful may have dampened activity in this network.

As with most other students, networks for representing linear relationships with an equation did not activate. Even during a follow-up explanation, Clayton did not seem to note the affordance for comparison that creating an equation for each company would provide. A likely hypothesis is that his way of understanding the problem – as a continuation of the last problem – blocked any other understandings of the problem. As well, it seems that the researcher's attempts to explain the problem were not helpful in promoting use of these networks.

Clayton did no calculation or number work during the question, and thus did not activate his *core mathematical networks.*

Metacognition networks supported Clayton in asking for clarification and in looking back at his previous work. Early in the problem, Clayton asked for clarification, believing that his strategy (to report the results of the previous question) was too simple. However, Clayton cut off this discussion – "No, I got it" – and returned to his strategy. Although clearly the researcher was ineffective in explaining the question, it is also likely that Clayton was having trouble shifting strategies. This suggests a lack of *metacognitive* activation where it may have been useful. Even if the question was unclear, inhibitory metacognitive networks may have helped Clayton to stop network activity related to the last question that may have been interfering with his cognition. Late in the problem, it was clear to Clayton that he did not understand how to approach the problem. At that point, he asked if he could look back at his work, and spent a minute combing through the previous question, looking for insight or clues, suggesting metacognitive activation. Clayton thought for a moment, crossed out his previous work, thought for a moment, then asked if he could look back at his work.

As in previous questions, Clayton used multiple discussions with the researcher to extend and confirm his understanding of the question. This was not fruitful in this problem, as Clayton and the researcher were not able to communicate about the task demands in a way that afforded success on the problem. Because relevant domain-specific cognitive networks were not active, Clayton had nothing to outsource to the paper. Thus unlike previous problems, there was no meaningful interaction with the paper or other *contextual* networks.

Clayton's confusion in the problem suggests that it may not be appropriate to code any of Clayton's activity as *competent.* Follow-up questions suggest Clayton did not understand the question, despite repeated conversations with the researcher. And it was difficult to code his activity as coming from competent use of networks that he brought to the study.

Clayton did exhibit *flexibility* during the problem when he suggested using the t(i) formula for identifying terms in an arithmetic sequence to solve the problem. Clayton was stuck, and he adapted by searching internally for a cognitive network related to the task at hand, and checking the utility of that network by talking to the researcher. Although the formula Clayton activated did not easily afford a successful solution, his searching for an alternative represents adaptation to the question. From a network lens, flexibility was triggered by activity in contextual networks (the researcher's comment that the proposed answer did not answer the intended question), which in turn activated domain specific networks (for the t(i) formula) and use of contextual networks (for checking the domain specific network with the researcher).

Clayton continued to exhibit flexibility by then looking back on his previous work for possible insight (though he did not find any). Adapting to the state of not knowing how to answer the question, Clayton's metacognitive networks (for shifting away from the approach he had been using) triggered activity in contextual networks (his reading work on previous questions) in an attempt to activate relevant domain specific networks. This contextual activity did not trigger domain-specific networks that could have helped solving the problem, and thus Clayton's flexibility in this problem did not support successful performance.

Problem C3

Activity Description.

The AAT graph describing Clayton's activity in this fifth problem is presented in Figure 6. Clayton solved this question easily and quickly. After reading the question, Clayton spoke to the researcher to check his understanding. After clarifying that finding the slopes of both lines would afford him a correct solution, he made quick work of the question. First he separated the points into two lines and called their slopes m_1 and m_2. He then wrote the slope formula and calculated these slopes directly.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

C: The lines aren't parallel because they have 2 different slopes.

R: Take me through what you were thinking and feeling.

C: So they have two lines, I separated them from line 1 and line 2 so that it'd be finding two different slopes, m1 and m2. So I found m2 and the slope for that line is 2, and the slope for the second line is 4, so I used the slope formula y2-y1 / x2-x1, so I found the slopes and they're not equal. Is that correct?

R: Yes, that's correct. Can you tell me why you drew an x and y axis?

C: It was just going through my head that maybe it would be easier, but I just realized that I can plug them into the equation

Affordances to Detect Network Activation.

In this problem, Clayton used *domain specific*, algebraic networks for slope that had been active throughout the study. As well, Clayton's choice to confirm with the researcher that finding the slopes of both lines would lead to success suggests that networks relating parallel lines to slopes were active throughout the problem. Clayton performed several in-the-head calculations, suggesting networks for *exact calculation* were also active.

This problem was fairly easy for Clayton and thus required less metacognition. However, it is likely that metacognition networks supported Clayton in checking his understanding of the problem with the researcher. He confirmed his understanding that finding the slopes would allow him to check if the lines are parallel and thus was able to attack the rest of the problem with competence and confidence.

As in previous problems, Clayton used paper to record calculations and the slope formula. Again, this included intermediate steps. As well, Clayton used the researcher to check the usefulness of his proposed strategy. This strategy afforded him an assurance that he was on the right track, but the exact function of this affordance is unclear.

Clayton’s activity in this question was coded as competent. He made use of the slope formula and quickly calculated and compared the slopes of both lines. This activity seemed to be supported by domain specific networks for slope and parallel lines, core mathematical networks, and contextual/metacognitive network transactions.

Problem F3

Activity Description.

The AAT graph describing Clayton’s activity in this final problem is presented in Figure 7. Clayton was able to construct equations for both lines but did not equate them and thus did not get the correct answer. After reading the question, Clayton directly asked the researcher “so what am I looking for?” The researcher did not answer directly, to give Clayton a chance to try the question. Clayton labeled the points in the question text, drew an x/y axis, and drew line CD. After a moment of thought, he subvocalized “ok, I understand.” He then calculated the slope of CD and created an expression for the slope of AB (which involves variables and thus is not equal to a number). However, after about 15 seconds of thought, Clayton indicated he was done with the problem. Though he had correctly identified the slope of CD and an expression for the slope of AB, he did not equate the two, and thus could not provide a solution to the question.

In response to a follow-up question asking him to describe his thinking through the problem, Clayton replied:

> *C: Basically I drew a line – maybe to try and help me and don't think it really did - where c and d were on a graph. So, I got that, and then I found the slope of CD to see if it would do anything. And maybe if I were to find the slope of AB, I could use that, to find the values of a and b with the slope from CD.*

Affordances to Detect Network Activation.

Clayton's work on this problem afforded him detection of activation of domain specific, core mathematical, metacognitive, and contextual networks. In this problem, Clayton used networks for slope that had been active throughout the study. However, unlike in the previous question, *domain specific* networks for relating slope to parallel lines were not active - at least not enough to afford equating the slope of CD and the expression for the slope of AB.

Networks for *exact calculation* were active when the slope of AB was calculated In this problem, *metacognitive* networks seemed to support Clayton's perception that he could successfully solve the problem. He subvocalized to himself "ok, I understand." However, because this utterance was followed by a period of quiet thought, it is possible that this narrative understanding did not support actual performance on the problem. Its function from this point of view might have been to give him the confidence to continue. However, because this utterance was surrounded by quiet thought, the nature of the metacognition occurring cannot be known for sure. Metacognitive networks must have also helped Clayton decide he did not know how to approach the problem once expressions for the slopes had been made. However, there was no behavioral evidence or information in the follow up question to indicate how he decided that problem should be over; thus, we can't be sure of the exact function of this metacognition.

Clayton tried to use discussion with the researcher to directly ask for task demands. Because his request was so specific, the researcher told Clayton he couldn't answer the question. Given Clayton's frequent use of this strategy, it is likely that domain experts are a key element of *contextual* networks for Clayton, and that this lack of help constrained him from using this contextual network to support his successful answering of the problem. As in previous problems, Clayton used paper to store all information and calculations.

Clayton showed *competence* during this problem when he set up expressions for the slopes of both lines and when he graphed CD; this suggests use of domain specific networks in this competent activity. Clayton's activity in the question did not appear to have detectible examples of flexibility.

Patterns in Network Activation and Flexibility Across Problems: Clayton

Across problems, Clayton consistently activated useful *domain specific* networks for midpoint formula, slope formula, algebraic manipulation, cell-phone pricing, and relating slopes to parallel lines. Clayton's networks for slope seemed to be largely algebraic and were essential to his success across tasks. He did not make notable use of visual spatial representations of slope during the study; though he graphed some of the lines in the problem, those representations seemed to represent a problem-solving routine, rather than informing his solutions (he never referred to these drawings). Once primed, this algebraic slope network afforded Clayton success on several problems; however, the network for slope needed to be primed by the researcher. Initially, Clayton tried to use the midpoint formula, but stopped when it outputted two values. However, once this network was active, Clayton showed the ability to use the formula adaptively and to connect it to his networks for parallel lines.

Clayton showed effective use of *core mathematical* networks for exact calculation. He was able to calculate easily and made no errors. There is no evidence that Clayton used analogical mathematical networks; he showed no sign of estimation or intuition with relative sizes of number.

In one case, Clayton's *metacognitive* activity – subvocalizing – seemed to function to sustain the activation of previously active domain specific networks. In problem C3, Clayton subvocalized to himself – "OK, I understand" – and then paused to think. As suggested above, in this case, the activity may have served to keep active networks for slope and parallel lines. Subvocalization can occur undetected (e.g., in the head) and so it is unclear if this example represents a general pattern of subvocalization to keep domain specific networks active.

In other cases, Clayton's *metacognitive* activity seemed to function to indirectly activate inactive domain specific networks through use of *contextual* networks – the paper, and the researcher. As Clayton became aware that he was unsure of what to do in a problem (e.g., in problems C1, F2, and F3), he asked the researcher, directly or indirectly, for support. The support he sought was either a specific formula or a procedure, suggesting that he was hoping to activate the relevant domain specific networks. Once he felt these networks were active, he moved on to apply the procedure or formula he had discerned, sometimes with success (i.e., problem C1); sometimes not (i.e., problems F2 and F3).

The paper was also an important contextual network for Clayton. Twice Clayton noticed a problem with his work on the page and shifted strategies, suggesting that activity in the contextual network (changes to what was written on the paper) triggered metacognitive network activation. For example, his discussion with the researcher, which primed the slope formula, occurred after he wrote the two values which outputted from the mid-point formula and realized he wanted only one value. Though "two values" are intrinsic to the midpoint formula, and he likely could have predicted that two values would be produced if he were asked, Clayton did not notice this until he had written and viewed the values on the page.

For Clayton, then, metacognitive and contextual networks may be highly integrated, and metacognitive networks may function to interact with contextual networks; combined, these networks seemed to function to indirectly activate domain specific networks. When talking with the researcher, metacognitive networks seemed to trigger contextual activity (i.e., he decided to interact with the researcher). When interacting with the paper, activity in the contextual network (i.e., changes to what was written) seemed to trigger metacognitive network activity (i.e. a new strategy decision).

These contextual and metacognitive transactions sometimes afforded Clayton *flexibility*. When Clayton abandoned the midpoint formula in the first question and indirectly asked for the slope formula, he may have been responding to a disconnect between the result of the midpoint formula and his knowledge that the correct answer would have only one number. It seems as though the result of the midpoint formula (as revealed through contextual networks) triggered this flexibility.

Clayton also showed flexibility when he became stuck in question F2, and tried, unsuccessfully, to find the solution. The flexibility in this question also seemed to be explainable as an interaction between contextual and metacognitive networks, in the service of activating the correct domain specific network. Clayton, realizing that he needed a new strategy (metacognitive activation) asked the researcher (contextual network) if there was a link to a formula from another unit in math (domain specific network). His interaction with the researcher dampened that domain specific network (as the researcher suggested it would not be useful). Clayton then probed another contextual network (looking back at his work booklet), unsuccessfully searching for an affordance to utilize in the question. Thus, across these examples of flexibility, it appeared that Clayton's flexible adjustments involved transactions between contextual and metacognitive networks.

Adam

At the time of this study, Adam was a student entering Grade 12. He was a full-time badminton player and thus worked out of a distance school with support from tutors. He reported strong performance in Math 10 (89%). Adam was relaxed and confident in his work. He discussed his work easily with the researcher and seemed to enjoy the process of being in the study. Adam described himself as good at math, reporting "my strength I guess is that I'm naturally, I think, pretty good at math...so...like once as soon as I have the concept I'm gonna like get it really fast but..my weakness is, I lack the...I don't put in enough time...into it."

Note that, as was the case with other students, Adam's activity report was constructed based on observations, video records, and responses to follow-up questions. Unfortunately, written traces of his problem-solving activity were accidentally discarded. But little information was ultimately lost, because Adam was clear in his descriptions of his activity during the problems and most traces of written activity were also captured on video. Table 8 indicates which of Adam's activities were competent or flexible.

Problem C1

Activity Description.

The AAT graph describing Adam's activity in this first problem is presented in Figure 8. Adam worked through this question quickly and easily, but because of a small calculation error, he did not obtain the correct answer. Adam began by quickly labeling the points in the question text and writing the points down in the slope equation (without actually writing the equation itself). After pausing briefly to think, he realized he had mislabeled the points and corrected the labeling. He then calculated the slope, reduced the fraction, and circled an answer that was incorrect because of a small calculation error from subtracting.

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

A: First I try to remember the formula of it…cause that's the easiest way to answer it...and I remembered it's y2 minus y1 over x2 minus x1...just to help with this. So this is my first thought...pretty sure that my first thought is like, if my brain thinks it, no point in me like, changing it...cause...I dunno I've heard rumours that...you know the first thought it's the true instinct, just go with it. So uh then I uh put down here which one is my x2 and which one … I decided y2 minus y1...did the math, over x2 or...x1, did the math...although…I just realized that I did the math wrong which is pretty sad. Um...it would be...over 4, not 2...3. So yeah I just didn't check the problem over which I should have done, but technically I did here so it still counts. Yeah I just did the math…?

Affordances to Detect Network Activation.

In this question Adam showed use of *domain specific* networks for finding slope given two points. His labeling of the points in the question text showed evidence of a domain specific network for working with slope questions – he later mentioned this labeling as something a student working on this problem should know to do, suggesting that he may have domain specific networks for these types of problems.

Mathematical networks supporting *exact calculation* were activated to subtract the coordinates; ultimately, they did not afford the correct solution. However, during follow up, as he talked through the question, Adam noticed himself that the calculation was wrong, suggesting that this network is capable, at some times, of doing this calculation.

During the question, Adam showed *metacognitive* activity when he stopped to think about the question and then corrected his labeling of the question text. He did not change any of the numbers in his slope equation as a result, suggesting that this relabeling helped him think, in his head, about which subtraction equations was required. As mentioned above, this subtraction was ultimately incorrect.

In terms of *contextual networks,* Adam did not speak to the researcher during the question. He used the paper to write an equation with the points as well as to record his fractional answer and a reduced form of that answer. Later, after the question, Adam noticed his calculation error by reviewing the paper, suggesting the contextual networks supported detection of this error.

Adam's performance on this question mostly consisted of *competent* activity, consistent with his perception that he "just did the math." Adam showed competence in his annotation of the points in the question text, use of the slope formula, and calculation of the slope. Follow-up questions suggest this was easy for Adam and relatively automatic. This activity seemed to be supported by domain specific and calculation networks for using slope formula and exact calculation. Adam did show *flexibility* when he corrected the annotation he had made to the points – initially, he had mislabeled x2, y2 and x1, y1. In this case a contextual network (the writing on the paper) triggered a metacognitive shift (to correcting the points) which impacted the contextual network (what was written on the paper). It is unclear if this activity supported Adam's fluid use of the slope formula or if it had no impact on his performance.

Problem F1

Activity Description.

The AAT graph describing Adam's activity in this second problem is presented in Figure 9. Adam solved this problem quickly and correctly. As in the previous problem, he labeled the points in the question text as x_2, y_2 and x_1, y_1. He then correctly wrote the slope equation using the points from the question, subtracted them in his head, and wrote down the answer.

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

> *A: Uh well there's no variable…or number variable, so I uh...I think that's what the term is, trying to sound smart on camera...uh...so I did the exact same thing and then the last stage I numbered them to make sure I knew which one was x1, y1, y2, x2...and then I again plugged them again to where they should be using the formula...and..then I just subtracted them...and I knew 2b minus b would be 1b and 3a minus a is 2a….so.*

Affordances to Detect Network Activation.

Adam's work on this problem afforded detection of activation of domain specific, metacognitive, and contextual networks. *Domain specific* networks for slope, and for working with algebraic equations were active during this problem: Adam correctly transferred the variables into the slope equations and performed the algebraic calculation required easily. This question does not require the use of *core mathematical networks,* except for very simple, likely highly automatic calculations (3-1 and 2-1).

Adam answered this question quickly and fluidly and without notable *metacognitive* activity. This may be because he followed a procedure identical to the one he used in the previous problem, thus minimizing the need for modification of network activation. Adam's report that he "did the exact same thing" as in the previous question was not coded as a metacognitive decision, as it was challenging to detect if this was a report of what Adam thought when starting the problem or if it was a post-hoc explanation for a direct interaction between contextual networks (the question text) and domain specific networks (for slope questions). Adam did not interact with the researcher. He did access *contextual* networks when he used the paper to label the points in the question text, create a slope equation, and write the answer.

Adam's activity in this question appeared to be a *competent* extension of his competent activity in the previous problem. He quickly and easily solved the problem and appeared to use the same domain specific networks, in the same way, suggesting competence. Adam's activity in the question did not appear to have detectible examples of *flexibility*.

Problem C2

Activity Description.

The AAT graph describing Adam's activity in this third problem is presented in Figure 10. To begin this question, Adam wrote the system access fees for both companies. After a brief pause, he wrote an equation for Company A and solved it in his head, fluidly writing the answer. He began an equation for Company B, noticed a mistake, and after a search for the eraser, corrected his equation. Adam then asked if he could use a calculator and, when he was told yes, he used the calculator to find the total cost for Company B and wrote it down.

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

> *A: Um..well first I was just trying to pick out the important numbers? Um, and then I figured out what's the question...so at first I took out...uh...12.5 no matter what and that it's...20 cents times each minute. The second I took 10 no matter what and it's 22 cents per...times each minute. So then I uh...read like on that it's 30 minutes a month, so I ...um...plugged in the...I did 30 times .2, like 30 times the minutes...like the amount of minutes, which gave me ...um...6 for the first month...and then I added with the 12.5 access fee no matter what and that gave me $2.5 a month. The second one I did 30 times 22 and it gave me 10...yeah it gave me 6.6 and I added that with the 10 and that gave me a...$16.6 per month bill which was cheaper.*

Affordances to Detect Network Activation.

Domain specific networks for real-life linear equations were active during this problem. During follow-up Adam indicated that this was a very straightforward problem, suggesting that networks for situations with this structure were active and easily available. It is unclear if these networks are particular to cell-phones or general to real-life linear equations situations. Networks for exact calculation were used to find the total of Company A.

Adam showed use of *metacognitive* networks when he noticed and erased an error he had made in the equation for Company B. As well, he recognized the benefit of a calculator, and asked for access to one mid-problem, suggesting metacognitive network activity.

Adam interacted with the researcher by asking if it was possible to use a calculator. He wrote important information at the start of the problem and created written expressions for the calculations required to get the correct answer. He also wrote the correct answer. He interacted with a calculator to find the cost of Company B.

Adam's creation of expressions for both companies and calculation of the relevant costs may have represented *competent* activity; his follow-up answer suggests that this activity was done through activating a train of domain-specific networks which supported each step of the problem solving process. Adam exhibited *flexibility* when he asked the researcher if he could use a calculator. Though many students used calculators, in other cases these formed part of the problem context, because they were on the table. However, the researcher forgot to offer a calculator to Adam, and thus his behaviour represents an adaptation to his evaluation that manual calculation would take too long for his taste, a calculator could speed him up, and the researcher might have access to a calculator. From a network point of view, this represents metacognitive activity (awareness that a calculator would speed things up and that he didn't have one) triggering contextual activity (discussion with the researcher) which changed the nature of relevant contextual networks (in that the context now contained a calculator). This flexibility was not essential to the problem solution, but allowed Adam to speed up, suggesting that he was adjusting his pace through this flexibility.

Problem F2

Activity Description.

The AAT graph describing Adam's activity in this fourth problem is presented in Figure 11. Adam solved this problem successfully by finding the difference in system access fees and dividing that by the difference in per-minute costs to find the number of minutes at which Company B would be equal to Company A. Because the meaning of the question text had been confused by earlier participants, the researcher asked Adam to read the question and confirm his understanding of the problem. After discussion, it appeared that Adam had correctly interpreted the problem and he began. First, he wrote the difference between the two companies' system access fees - $2.50. Next, he wrote the difference between their per-minute fees - $0.02. He then used the calculator to divide $2.50 by $0.02, and got a result of 125. He thought that might be the answer and began to guess and test; he first calculated the per-minute cost of 125 minutes at $0.20, and then the cost of 125 minutes at $0.22. After finding that the difference between his answers was $2.50 – equal to the difference he had calculated between the access fees – he concluded that the answer was 125 and announced that to the researcher.

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

A: Um first I took out um...first I thought like...uh it's...gonna be difficult, then I uh...realized that...basically all I had to do was...what's the difference, so how much do I *need to make so that's...12.50 minus 10 which gives me my 250 number...then what's the difference in these 2…uhh .22 minus .02 so that gives me that number...and then I took a random guess and hoped that if I divide them it would give me my number which was 125 which was the right number...um...and it's the in between number where they'll be the same. I then guessed and checked….so predominantly checked after that...so I did .20 times 125 which gave me 25 for the first one, second one .22 gave me 27.5, the difference there was my 250 variable….so then I had….*

R: Now when you say you took a random guess, was it really a random guess?

A: I...figured somewhere it would be divide but I knew even if it wasn't divide if it was add I would still...I mean if it was multiplication, there was only two things it could be so….

R: How do you know that?

A: Um...I don't.

R: Or maybe I should ask you, why did you think that? I'm not trying to challenge your knowledge.

A: Um...I just realized that...and I knew from the previous question that it couldn't be like...2.5 minutes that would be enough. So I knew if you divide 250 by .02 it would be a number. If you multiply minutes, I'm pretty sure you'd get a smaller number...so to divide this is really the only option now that I think about it.

In this problem, Adam showed evidence of *domain specific* networks for linear equation situations, as his activity suggested competence interacting with the structure of the cost functions for both companies.

In this question, it seemed as though Adam's successful performance was highly supported by *core mathematical* networks, in particular those for analogical calculation. For example, Adam chose to divide the difference in system access fees by the difference in per-minute cost; he later reported this was a "random guess." Upon further questioning, Adam suggested that division was the intuitive choice, though he was uncertain it was correct. This suggests possible activation of visual-spatial networks for number, possibly activating without verbal mediation. Alternatively, it could be chance that division was the first operation that occurred to Adam and is unrelated to his core mathematical networks.

Throughout this problem, Adam made several choices that facilitated his successful solution. First, he chose to calculate the difference between the per-minute and system access fees. Second, he chose to divide those numbers. Third, he chose to guess and test the answer from division, because it seemed intuitive. It is possible that each of these choices was supported by *metacognitive* activation. It is clear that the decision to calculate the differences in cost was explicit and clear, though how useful the outcome would be was unknown to Adam. Similarly, the "random" choice to divide the differences represents a decision to play with the relevant numbers in order to produce a plausible answer that could be used for guess and test. In both cases, metacognitive networks seem to have supported Adam in choosing an operation and to provide a plausible value in order to afford activity in the following stage. Additionally, his choice to guess and test with 125, once division had provided that number, reflects metacognitive awareness that 125 is a reasonable answer and an appropriate decision is to check that number (compared to, for example, the .008 he would have got had he reversed the numerator and denominator). Thus, metacognitive activity in this question supported decisions that produced values that moved him forward in the question.

Though I chose to code this activity as metacognitive, there is some reason to believe this activity does not represent metacognitive network activation. There is no direct evidence of declarative knowledge of cognitive processes, affect or belief, or strategy shifting. Instead, I infer strategy shifting from the fact that Adam described thinking narratively about his choices at the moment he chose to divide and to find the differences between the fees and that it seemed as though such a move might require explicit decision making. However, it is also conceivable that this response from Adam is a post-question artifact of being asked to construct an explanation of this activity. It is possible that core mathematical activation directly triggered domain specific and other core mathematical networks rather than requiring mediation by metacognitive networks.

Adam's overall strategy – to generate a plausible number using the information given, and guess and test it – may reflect metacognitive activation at the beginning of the problem. Describing the beginning of the question, Adam said "first I thought like...uh it's...gonna be difficult, then I uh...realized that...basically all I had to do was…[explains entire procedure]." This phrasing suggests that the strategy of creating candidates for guess and test was chosen explicitly, and before beginning the problem, rather than coming as Adam seized an affordance later in the problem, or being automatically triggered by the question text. If this activity represents metacognitive network activation, then the choices to find the differences, divide, and guess and test may have stemmed from this metacognitive activity, rather than in-problem metacognitive activity.

Use of *contextual* networks occurred when Adam spoke with the researcher (at the researcher's request) to ensure the question made sense. He also used the paper to record intermediate calculations, and the calculator to guess and test the answer.

Though Adam's activity in this question was largely flexible, Adam demonstrated *competence*, at the end of the question, when he used domain specific networks for guessing and testing to verify the answer that emerged from his flexible activity. Adam showed *flexibility* when he obtained the difference between the access fees and the monthly costs, and when he divided these differences. Follow-up questions suggest that Adam was unsure how this would play out, but that he suspected this information might be useful for finding candidate values of minutes to guess and test. In both finding the difference and dividing, Adam was unsure of the effect of his step, but he recognized that he did not know what to do and that these actions were likely to lead to candidate values for minutes that could be plugged into the equations to verify the answer.

From a network point of view, Adam seemed to use activation in analogical core math networks (his sense of the differences; his sense of the result of dividing) to activate metacognitive activity (choosing a calculation strategy), which activated other core math networks (for finding exact values), in turn producing candidate values that could be used in domain specific networks (for guessing and testing). As discussed above, it is also possible that this flexibility occurred without support from metacognitive networks. However, it does seem clear that interactions between core mathematical networks and domain specific networks were instrumental to this flexibility.

Problem C3

Activity Description.

The AAT graph describing Adam's activity in this fifth problem is presented in Figure 12. Adam solved this problem quickly and announced the correct answer. He did not prove his answer algebraically, but his justification for his response seemed valid. Adam began this problem by drawing an x/y axis and circling the points in the question text. After a moment of thought, he plotted the first pair of points and joined them. After a second moment of thought, he plotted and joined the second pair of points. This process took over 30 seconds and as he drew the points, Adam kept tracing over the line and putting circles at evenly spaced intervals across the lines. He then announced, "they are not parallel." After some discussion with the researcher, he provided a justification for his answer: "to prove it, if I extend their lines, they cross, and parallel lines wouldn't cross."

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

A: Um...I read it. At first I was kinda like well this seems pretty intense and then I read it again and thought okay just plug the points and...I knew...I ? They knew they were not going to be parallel because they had different slopes, I could tell just by looking at it…and…

R: And that occurred to you early in the question that they had different slopes?

A: Uh not...no. Not so much that they had different slopes but it occurred to me that...um...well that they just looked different, so…that's how I solved the question.

R: What looked different?

A: Just the…

R: The lines?

A: No, just the two um…

R: Points

A: …points looked different. Like...these…they looked different. The numbers looked different.

Affordances to Detect Network Activation.

Domain specific networks for graphing coordinate systems seemed to be active during this question. It appears that both algebraic and visual-spatial networks for slope were active. Adam's reported "I could tell they were going to be parallel because they had different slopes. I could tell just by looking," suggesting a role for visual spatial networks for slope. However, he also suggested that the particular points afforded him an explanation, suggesting algebraic conceptions of slope were also active. Networks for parallel lines were also active, and connected to visual-spatial representations of slope.

Adam did not calculate or work with number in any obvious way, suggesting that *core mathematical* networks were not active as he solved the problem. However, it is possible that, as he traced over his lines, he was calculating proportions between points.

It was unclear if Adam showed *metacognitive* network activity during this question. He did not shift strategies or show verbal mediation of the task. He seemed to simply draw lines and infer from inspection that they were not parallel. However, his inspection took 30 seconds and may have represented metacognitive activation of two domain specific networks simultaneously (for algebraic and visual-spatial understandings of slope). Though there is not explicit evidence for this interpretation, it may be the most parsimonious way of understanding how and why Adam sustained his attention for such a long time on his problem representations. *Contextually,* Adam used the paper to create a visual-spatial representation of the problem.

The way Adam constructed a visual-spatial representation of the problem – by plotting and joining the points – appears to represent *competent* activity, supported by domain specific networks for visual spatial understandings of slope. Adam showed *flexibility* when he stared at the problem for 30 seconds, tracing over it with his pen. He was unsure of how to adapt his work to provide a coherent solution and used the experience of staring at the problem to generate an appropriate solution. Though his flexibility did not afford a successful solution (in that he did not prove his answer algebraically), it did afford a consistent and correct interpretation of the problem. From a network point of view, Adam's flexibility seemed to occur as metacognitive networks sustained activation of two domain-specific networks (for algebraic and visual-spatial representations of slope) which, after connecting, supported an accurate and coherent solution.

Problem F3

Activity Description.

The AAT graph describing Adam's activity in this final problem is presented in Figure 13. Adam solved this problem by successfully equating expressions for the slopes of the two lines and solving for a pair of coordinates that satisfied the equation. Adam began the problem by subvocalizing to himself "[on the first line] the slope is one over two so…." He then wrote the slope "1/2" and an expression for the slope of the second line. He then gave the researcher a solution. As the researcher began to ask the follow up questions, Adam quickly interrupted "Arrrrrrrrr. That wouldn't make them parallel. Crap!" When the researcher indicated he could go back to the problem, Adam thought for nearly 30 seconds. He asked for lined paper, but when told there was none, continued to think. He then drew an x/y axis and drew one of the lines. After more thought, and tracing of lines with his pen, he began to do calculations on the paper. What he wrote could not be specifically detected from video evidence, but follow-up questions suggest that Adam put in a value for one of the variables and found the other, using the fact that the slopes needed to be equal. Throughout the cycles of thinking and writing, Adam subvocalized, seemingly with number. Eventually, Adam gave a correct answer orally.

In response to a follow-up question asking him to describe his thinking through the problem, Adam replied:

> *A: Uh...well first I've realized that….to get these...the slopes have to be even...and so I noticed okay...5 and this has to be 2.5 and 4, has to be 8 to get the slope...and I thought I had the question solved and I realized they're the same the line and...lines can't be touching so I can't use that…so then I had to um...essentially I figured out the slope for the other line which was 2...and I had to make...using these numbers…make this slope so I chose...used the same numbers that would be here, although I didn't really have to...so I made it 2.5...and uh...add those cause they're negative 2.5 so you get 6.5...when subtracted...and something minus 5 over 6.5 gets you 2, so just do the math it's 18 minus 5 gives you over 6.5 gets you 2.*

Affordances to Detect Network Activation.

Domain specific networks for parallel lines and their relationship to slope appeared to be active throughout the problem. Networks for algebraic relations involving two variables appeared to be active; Adam was able to mentally compute what a second variable would be after choosing a value to plug in for variable number. Adam did mental calculations to find the second variable, given the first, suggesting activation in *core mathematical* networks.

Adam showed evidence of *metacognitive* activation when he realized he had the incorrect answer – as the researcher asked him to verbalize his solution, he interrupted to say he realized his solution was incorrect. He then spent nearly a minute thinking, clearly looking for a strategy. After this long period, he began the algebraic solution that ultimately worked. However, mid-thought, he asked for lined paper, which suggests he may have had a visual-spatial solution in mind, before turning to an algebraic strategy.

Adam interacted with the researcher to determine that his first solution was wrong. He also created a visual spatial representation of the lines on the paper, as well as algebraic expressions for the slopes of the two lines.

Adam's activity in this question was coded as *competent*, even though some elements of Adam's activity suggested some flexibility. During this problem, Adam had several long periods of inaction, suggesting that he may have been adapting mentally. However follow up answers and Adam's activity in the question may suggest that these adaptations were competent: they were automatically done, without any adjustment of activity. Each step appeared to flow easily from the previous step, including rumination and error checking. This competent activity seemed to be supported by domain specific networks for finding slopes and equating parallel lines.

Patterns in Network Activation and Flexibility: Adam

Adam seems to have activated *domain specific* networks that were capable of supporting a variety of activities. Networks for finding slope were active and used throughout the problems. Adam's activity suggested he had well integrated networks for visual spatial and algebraic notions of slope that were integrated with networks for parallel lines. This was evidenced by his use of a mostly visual spatial strategy in question C3 (parallel lines), but an algebraic strategy in C1 (slopes). These two networks were likely to be integrated, given Adam's follow up response to C3, which indicated an intuitive understanding of the relationship between the points and the slope. As well, questions C2 and F2 were solved correctly, suggesting domain specific networks for real-life linear situations were available to support performance.

In general, *core mathematical networks* were well developed. However, in the first question, Adam did make a small calculation error, which he later caught. It is unclear what this implies for the nature of his core mathematical networks. Adam showed a network for approximate calculation that was well integrated with domain specific and metacognitive networks. His solution to the flexible cell phone question (F2) indicates an intuitive sense of the difference between the system access fees and the per minute costs and what operation (division) might afford information that could lead to answering the question.

Adam's *metacognitive* networks seemed to support him in detecting errors, possibly using a narrative metacognitive network. Twice, through talking to the researcher, he detected errors in his work – one a calculation error, one a subtle error in question F3 (parallel lines). Throughout the problems, Adam showed evidence of subvocalization - he talked to himself frequently. His explanations for his work were clear and narrative in form, while other students gave more disjointed, stream-of-consciousness-style explanations. Thus, this may indicate that for Adam, error checking can be effective, and is supported by linguistic metacognitive networks.

However, twice Adam noticed small errors online, while working. Though subvocalization was present often in his work, it was not clear if he was subvocalizing when he found errors himself. This may indicate activity in experiential networks, which may integrate with narrative networks to provide robust metacognition.

Adam's metacognitive networks also supported him in asking the researcher for contextual tools that would support his performance. In one question, he asked for a calculator mid-problem; in another he asked for lined paper. In both cases, the request came mid-question, suggesting that this metacognitive activity emerged from an interaction with the question, rather than being generated when Adam was planning how to approach the question. This may imply that experiential metacognition may have supported Adam's interaction with contextual networks.

Finally, Adam showed experiential metacognitive activity when he made strategic choices in the flexible cell-phone question (F2). His follow-up answers to that question suggest that he was unsure how dividing the differences in system access fees by the difference in per minute costs would help, but he intuitively suspected it would. It is possible that experiential networks, working with analogical core mathematical networks, activated in response to his perception of the size of that ratio, and supported Adam in choosing a strategy that would lead to the correct answer.

In terms of *contextual* networks, Adam used the paper to record his work and to construct visual spatial and algebraic representations of the questions. He interacted with the researcher to ask for tools that would support him, like a calculator and lined paper.

Adam demonstrated three types of *flexibility*: adaptively correcting his work, constructing novel solutions, and intentional rumination. Each of these seemed to be a function of different network activation. First, Adam found errors in his annotations of the points in the first slope question (C1); as mentioned above, this occurred while checking over the problem on his paper. This flexibility may have been afforded by an interaction between metacognitive network activity (for checking over problems) and contextual networks (the paper). This flexibility seemed to require the use of contextual networks – specifically the paper. Adam was able to see his error because it was right in front of him. In other questions, the follow-up step of looking over his work seemed to trigger awareness of errors. So it seems that Adam's flexible corrections were a function of metacognitive networks interacting with contextual networks.

Second, Adam successfully used an intuitive strategy to construct a novel solution for a challenging cell-phone problem (F2). This problem has no straightforward solution, and it is not typically seen in Grade 10 classrooms. Adam, however, saw the relationships between the difference in system access fees and the difference in per-minute costs. His responses to follow-up questions suggested that this awareness may have come from activating analogical core mathematical networks. It appears that this activity in analogical core mathematical networks flowed to other networks to afford a correct and novel solution (i.e. the relative sizes of the differences in these numbers activated metacognitive networks that in turn activated networks for particular exact calculation strategies).

Finally, Adam also exhibited flexibility as he solved question C3 (parallel). Adam spent time staring at the question, tracing over the lines, and filling in the points. This activity cannot have contributed logically to the solution, but it may have afforded the network activity required to see that the lines were not parallel. He ultimately gave a non-algebraic, visual-spatial explanation for his answer. His problem-solving activity may have represented integrated activity between experiential and narrative metacognitive networks, domain specific networks for algebraic and visual-spatial networks for slope. Specifically, experiential metacognitive networks may have been activated by his tracing over the shapes on the page (a contextual network), while narrative metacognitive networks may have made sense of that experience, in turn activating domain specific networks for algebraic and visual spatial networks. His oral solution to the problem may thus have represented a transaction between five networks: two metacognitive, two domain specific, and one contextual. Ultimately, Adam demonstrated robust applications of flexibility. Across instances, this flexibility varied in function, in effectiveness, and in terms of the networks implicated.

George

At the time of this study, George was entering Grade 12 at a school for children who have learning disabilities. This school is known locally for providing highly scaffolded instruction and explicit strategy instruction. In terms of testing in mathematics, most students at this school are tested on questions quite similar to those that they have seen during instruction. In the semi-structured interview, George did not report any significant challenges in math and suggested he was a kinesthetic and auditory learner. George worked very quickly in the study and seemed eager to show that he could do the work with minimal help. Table 9 indicates which of George's activities were competent or flexible.

Problem C1

Activity Description.

The AAT graph describing George's activity in this first problem is presented in Figure 14. George solved this problem quickly, easily and correctly. He wrote the slope formula, filled in the coordinates from the question text, mentally performed the subtraction required, and simplified the resulting fraction to get the correct answer.

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

> *G: First stage was just write out the formula...which is y2 - y1 or x2 - x1. Then put in the values… 9-3 over 10-2. Then do the subtraction and then simplify the... which is 6/8, then simplify that down to 3/4.*

Affordances to Detect Network Activation.

Domain specific networks for slope formula and for working for coordinate systems appeared to be active, as evidenced by George's writing of the slope formula and ease with the coordinates. It does not appear that visual spatial networks were active, as all representations of the problem were algebraic.

Core mathematical networks for exact calculation seemed to support George when he simplified the expressions within the slope equation. George showed no overt signs of *metacognitive* activation. He didn't shift strategies or comment on belief, affect, or cognitive processes, and solved the entire problem quickly and fluidly. *Contextual* networks were active when George used the paper to record the results of calculations at each step. He did not speak with the researcher during the problem.

George's responses to follow-up questions suggested that an easily accessible train of domain specific network activations facilitated a quick and *competent* solution to the problem; these included networks for slope equations and for exact calculation. George's activity in the question did not appear to have detectible examples of flexibility.

Problem F1

Activity Description.

The AAT graph describing George's activity in this second problem is presented in Figure 15. George solved this problem correctly in a nearly identical fashion to the first problem. First he wrote the slope formula. Next he inserted the points. Then he mentally subtracted the algebraic expressions to give the correct answer.

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

G: It's very similar …with the formula first...then fill in the values...in this case variables, 2b-2 over 3a-a. You get…. b/2a, which can't be reduced...so I'd say this is b/2a

Affordances to Detect Network Activation.

As in the previous problem, *domain specific* networks for algebraic understandings of slope appeared to be active, but there was no evidence of visual-spatial networks being active. In addition, George seemed to make use of domain-specific networks supporting subtraction of algebraic expressions; this was evident when he mentally subtracted the algebraic terms.

This question does not require the use of *core mathematical* networks, except for very simple, likely highly automatic calculations (3-1, and 2-1). Again, George showed no overt signs of *metacognitive* activation. He didn't shift strategies and solved the entire problem quickly and fluidly. *Contextual* networks were implicated when George used the paper to record the results of calculations at each step. He did not speak with the researcher during the problem.

George appeared to solve this problem *competently*, as all his activity was accurate and quick. George also suggested in his responses to follow-up questions that he was able to implement the same competent routine as in the previous question. It seems as though domain-specific networks for slope were instrumental in George's competence. George's activity in the question did not appear to have detectible examples of flexibility.

Problem C2

Activity Description.

The AAT graph describing George's activity in this third problem is presented in Figure 16. George solved this problem easily and correctly. George began by reading the problem, while the researcher got a calculator. Once George had a calculator, he divided the page in two. He wrote an equation for Company A, then used the calculator to solve it, and write down the answer. He did the same for Company B and thus completed the problem.

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

G: Uh...you have to...take the time...amount per minute, multiply it by the number of minutes to get the rates for actual use...and then add in the system access fee...which for A...the total was $18.50 and for B it was $16.60.

Affordances to Detect Network Activation.

This question showed evidence of *domain specific* networks for linear equation relationships. George was easily able to see from the question text how to construct equations for the cost of the two plans. It is unclear from his activity and responses to follow-up questions if the activated network was primarily related to math questions or to cell phone plans, but his follow-up answer may suggest activation in networks for cell-phones.

George used the calculator to do calculations during the question, so it is likely that *core mathematical* networks were not active during the question. As in the previous two questions, George showed minimal *metacognitive* activation. His activity was quick, fluid, and seemed not to change during the question. *Contextual* networks were active when George used the paper to record the results of calculations at each step. He did not speak with the researcher during the problem.

George's follow-up questions suggest that this problem was solved *competently*, using domain specific networks for real-life cell phone plans. George's activity in the question did not appear to have detectible examples of *flexibility*.

Problem F2

Activity Description.

The AAT graph describing George's activity in this fourth problem is presented in Figure 17. George struggled with this problem, trying two approaches before giving an answer (through guess and test) that was close to the correct answer, but not right. George began this question by talking it through with the researcher (at the researcher's request). After showing that he understood the problem, he created a table of values to show the results of both companies for different amounts of minutes. He used the calculator to fill in the table of values, and said to the researcher, “I know it’s between the 30th and 60th minute.” However, after pausing to think for a moment, he changed strategies. George then wrote equations for Companies A and B. He divided the equation for Company A by .2 on both sides, creating a true, but unhelpful relation (y/.2=y+10). As well, he used the same variable (y) in two places in the same equation, to represent two different things. He stopped to think and then said, “I can’t remember how to write the formula for that right now.” After the researcher informed him that he could stop the question at any time, George went back to the calculator, and finished the question by announcing, incorrectly, that the two companies are equal at the 116th minute.

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

> *G:Uhm...initial I started with the table of values but the end trying to … I wasn’t properly accounting for the difference in the monthly access fee…so then I tried to write it out as an equation but...I couldn’t quite remember how to write it out properly...so I wasn’t able to work with that...so...that to an approximation with the guess and check.*
>
> *R: Mm hmm. So you did some guess and check?*
>
> G: Yeah.
>
> …
>
> *R: Oh, how were you feeling during the problem?*
>
> *G: Uh, a bit irritated that I couldn’t remember.*

Affordances to Detect Network Activation.

George’s work on this problem affording detection of activation of domain specific, core mathematical, metacognitive, and contextual networks. *Domain specific* networks for algebraic representations of linear equation situations appeared to be active in this problem. However, because George used the same variable for cost and minutes, it seems like these networks were unable to support correct solutions to the problem and may not have been well developed. As well, domain specific networks for guessing and testing with values were active, as evidenced by the table of values George created.

George did no mental calculation during the problem, and thus *core mathematical networks* were probably not active during this question.

George showed *metacognitive* awareness of his progress on the problem and its relation to the researcher's perception of his performance. This was manifest in declarative knowledge – twice he made comments to the researcher explaining his inability to complete a certain part of the problem. For example, during the table of values, right before he shifted to a new strategy, he said to the researcher, "I know it's between the 30th and 60th minute." While the answer did not actually fall within that range, it may have been that metacognitive networks were active and trying to find useful information in George's table of values.

George also decided to switch strategies once he realized that the table of values was not fruitful. He was clearly trying to move from guess and test to an equation-based solution. However, once he shifted, he was unable to solve the problem, and thus was "irritated" that he "can't remember how to write the formula for that." This suggests that his metacognitive switch was from "guess and test" to equations, but no more specific than that. He assumed the existence of a formula for "that" and when it didn't occur to him, he felt frustrated.

Finally, George showed metacognitive activity when he, after expressing that he couldn't find the right equation, continued guessing and testing. He didn't find a correct solution, but instead stopped once he reached a number close to the solution. It may be that this metacognitive shift represented an attempt to finish the problem quickly, but close to the solution, as a way of ensuring the researcher believed he understood the problem. Possibly, if he had really been shifting to the right solution, he would have continued guessing and testing until he found the correct answer.

George used the paper to create a table of values and to write out potential equations for the company costs. Additionally, *contextual* networks were utilized when George spoke to the researcher frequently during this problem. He expressed his frustration at not finding the right "equation," and indicated that he thought the answer was between the 30th and 60th minute. Although there is no direct evidence, it's possible that George expressed his inability to find the right equation as a way of getting the researcher's permission to be finished. However, when the researcher told George that he would have to decide if the problem was finished, George returned to guessing and testing until he had a more respectable (but still wrong) solution. Thus, George may have been interacting with the researcher in order to ensure the researcher saw him as smart and capable, or to ensure he understood when the question was considered complete.

The way in which George guessed and tested in this problem may have represented *competent* activity. He seemed able to plug in values to the equation easily, and calculate the cost at a variety of minute values. This seemed to be supported by domain specific networks for guess and test.

George exhibited *flexibility* when he created general equations for the two lines. Though these equations were neither true nor helpful, given that George was stuck, this action represents adaptation to the problem. As George progressed, he continued to show flexibility, when he attempted to divide the two equations for the lines. Once the equations had been made, George needed to find a way to have them give an answer, so he attempted to apply division to produce a correct answer. After this technique didn't succeed, George continued to show flexibility by his expression of frustration to the researcher. George reported feeling irritated; this expression of frustration may have served to modulate that irritation, by indirectly asking the researcher for support. It is unclear if George knew that his utterance would motivate a response from the researcher, but this seems reasonable, as the utterance was clearly directed at the researcher, and George appeared to expect a response. Finally, after George was told he could end the problem when he liked, he returned to his initial strategy – guess and test. George may have been managing the researcher's perception of his ability and thus returned to his initial strategy (though he could have stopped the question) to get a response closer to the actual answer.

When this flexibility is examined from a network perspective, it appears that George metacognitively activated domain specific networks (for creating equations), which impacted contextual networks (his paper), in turn activating domain specific networks (for working with equations). When this did not afford a solution, he used contextual networks (the researcher) to impact domain specific networks for task demands ("end the question when you like"), leading to reactivation of a domain specific network (for guessing and testing) in the service of an incorrect, but close solution.

Problem C3

Activity Description.

The AAT graph describing George's activity in this fifth problem is presented in Figure 18. George answered this question quickly and correctly. George began this question by reading the problem and subvocalizing to himself that the slopes must be equal. After a moment of thought, he wrote the slope formula and filled it in twice, once for each line. After pausing for moment, he mentally calculated the slopes, wrote them down, and said to the researcher "No, they won't be truly parallel."

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

G: Uh...no they won't be truly parallel.

R: Yeah.

G: Uh...the slope doesn't match...which means that...they will eventually cross.

Affordances to Detect Network Activation.

Domain specific networks for slope appeared to be active during this question, as evidenced by George's use of the slope formula. As well, his subvocalization early in the question may have afforded activation of networks for parallel lines and their relation to slope. During this subvocalization, he told himself that he was looking for parallel lines.

George did two simple subtraction equations in his head, suggesting that he was using *core mathematical* networks for exact calculation.

Early in the problem, George told himself that the slopes of the lines must be equal. This may have represented a *metacognitive* activation of the domain specific networks for parallel lines and their relationship to slope. Alternatively, they may have represented a verbal restatement of that domain specific activation *after* the activation had occurred. In the first case, this subvocalization would have represented metacognitive activity in the sense of indicating a decision to activate networks for slope; in the second, this subvocalaization would have represented metacognitive activity in the sense of revealing declarative knowledge of cognitive processes. Although both explanations are possible, the fact the George launched into a successful solution immediately after subvocalizing suggests that the first case is more likely, and that his activity represents metacognitive activation of domain specific networks.

Throughout the problem, George wrote down subcalculations on paper, utilizing *contextual* networks. He did not use a calculator or talk to the researcher.

George showed evidence of *competence* throughout this problem, for example in writing the expressions for both lines and checking if they were equal. Competence is suggested by the ease with which George did these activities and was likely supported by domain specific networks (for parallel lines, slope, and algebra), and metacognitive activity (subvocalization). He did not show detectable examples of *flexibility*.

Problem F3

Activity Description.

The AAT graph describing George's activity in this final problem is presented in Figure 19. George solved the problem quickly and with a correct method, but, because of a small calculation error, he got an incorrect answer. George began by reading the question while subvocalizing to himself. He then muttered, "so the lines must be parallel," and began to calculate the slope in his head. He miscalculated, getting 5 as the answer to 10-4, and wrote an incorrect slope of 5/3. He then wrote an expression for the slope of the second line. After inspecting the equation for a moment, he wrote b=10, a=1. This answer is incorrect; however, had 5/3 been the correct slope, George's answer would have been correct.

In response to a follow-up question asking him to describe his thinking through the problem, George replied:

> *G: Uh, for them to never to touch they'd have to be parallel, which means their slope must be equal. So then you...write out...you solve for the slope of … which is 5 over 3. And then you can...run that through which…pretty much...shows what you need to put in...and then...*

Affordances to Detect Network Activation.

*Domain specific n*etworks for slope appeared to be active throughout the problem, as well as networks relating slope to parallel lines. The network for slope likely included the slope formula, as that was written immediately. As well, networks for algebraic representations of slope seemed to be active, as evidenced by George's expression for the slope of the second line.

George mentally calculated both the slope of the first line and a set of points that would satisfy the expression for the second line. This suggests that networks for *core math* were active, and coactive with domain specific networks for algebra. Early in the question, George subvocalized that the lines would be parallel. As in the last question, this may have represented *metacogntive* activation of necessary domain specific networks. George recorded all intermediate calculations on paper, using that *contextual* network. He did not interact with the researcher.

George showed evidence of *competence* throughout this problem, in writing the expressions for both lines, equating them, and solving them. Competence is suggested by the ease with which George did these activities,and was likely supported by domain specific networks (for parallel lines, slope, and algebra) and metacognitive activity (subvocalization). He did not show detectable examples of *flexibility.*

Patterns in Network Activation and Flexibility: George

George's problem-solving activity suggested activation of highly accessible and useful *domain specific* networks. These networks tended to be algebraic rather than visual-spatial, as George did not draw any graphs during the study. He showed easy activation of networks for slopes, the relationship to slopes and parallel lines, and for real-life linear equations. In most cases, activation of these networks supported successful performance.

In terms of *core mathematical networks,* although he made some important errors, overall George showed capacity with exact calculation. Still, he seemed to prefer using the calculator. Though George did not mention any learning differences, the school he is at may suggest that he has a learning issue. If this is the case, use of the calculator as a strategy may make sense, if it reduces the memory demands of calculation. Calculator use may also have been an explicit strategy taught at school.

George showed low levels of *metacognitive* activation across the problems, perhaps because he solved many questions quickly and easily. Most of his metacognitive activation seemed to be narrative in nature. He seemed to use narrative metacognitive networks to subvocalize while solving problems; these subvocaliations (about what the question entails) appeared to activate and keep active the relevant domain specific networks for equating slope that he needed to solve questions involving parallel lines. In addition, he made comments twice to the researcher explaining his inability to solve a problem ("I can't remember the formula"). George's lack of acknowledgment of any challenges in math in the SSI, the statement he made, and his tone and reported irritation at his errors, may indicate that these metacognitive networks serve to sustain George's perceptions of himself as a good learner in math; showing frustration when he was unable to be successful may represent a violation to his metacognitive narrative expectation that he will not make such errors, and, when errors occur, he may feel compelled to explain them. Of course, this explanation, though it seems parsimonious and consistent with George, is speculative, and thus, may not be correct.

George used paper in all questions to record nearly all of his work. This suggests, that, for George, the paper is an important component of his *contextual* network for math. However, he did not show any evidence of using this activity to support activation of other networks - he didn't look back at his work, and he didn't correct it when it was wrong - which may indicate that this use of paper is automatic and an artifact of his competence in his mathematics with no functional role. George only spoke to the researcher to express frustration and seek implicit permission to finish the problem he was working on. This may suggest that, for George, contextual networks involving experts in the domain may afford him information about the task and his performance relative to expectations.

George was highly competent with most of the questions, and showed no *flexibility*, except in question F2 (cell-phone). In that question, George saw early that guess and test would be a lengthy process. At that point, he switched to trying to create equations for both lines; however, errors in those equations impaired successful performance. George then continued to exhibit flexibility and expressed his frustration to the researcher. When the researcher did not offer explicit support, he returned to guess and test, in order to get a more respectable answer. Again, this may suggest that George was managing his perceptions of himself as a successful math student -- giving up, though explicitly allowed by the researcher, would not have afforded him a perception of himself as having “almost got it.” A return to guess and test facilitated an answer that was close to the correct answer, and thus was close to successful performance from George's point of view. Again, this is a speculative interpretation, and though it seems a reasonable reflection of the data, it requires significant inference and thus is susceptible to error. From this point of view, George’s only discernable flexibility involved adapting to the problem in order to keep his self-perception consistent.

Because George showed flexibility in only one question, it may not be valid to generalize from this example to explain how flexibility is related to network activity for George. However, within that question, George showed several examples of flexibility and each of these seemed to use metacognitive activation to connect activity between contextual and domain specific networks. On the other hand, George showed examples of *competence* throughout the study. In each case of competence, George seemed to rely heavily on domain specific networks, making relatively little use of contextual and core mathematical networks. These domain specific networks seemed well developed and easily accessible.

Farley

At the time of this study, Farley had just completed principles of Math 10, and he estimated that his mark was a B at the end of term. He worked through the problems as quickly as he could and with a minimum of writing. His body and tone suggested that he wanted to move the study along quickly, though he was pleasant and friendly. Initially, Farley did not identify any differences in the way he learns, but, once asked how he differs from his peers, he said, "I think that I learn faster...I'm less attentive definitely so it makes me...it takes me more time but when I...when I…[am attentive], I tend to be very attentive, and I learn very quickly."   Table 10 indicates which of Farley's activities were competent or flexible.

Problem C1

Activity Description.

The AAT graph describing Farley's activity in this first problem is presented in Figure 20. Farley solved this problem correctly, in his head. After the researcher asked Farley to begin, Farley thought, looking around the room. After a few seconds he began to read the problem. After thinking for 10 seconds, he wrote the correct answer – ¾ - down on the sheet.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

F: Well at first I tried to remember the formula for a slope...like n equals........and then I calculate the difference between 2 and 10, for me 9...so then 3 and 9...so 6...then 2 and 10..2 and 10 is 8...and then simplify that by dividing by 2, 3 and 4.

Affordances to Detect Network Activation.

Across the problems, it was challenging to discern Farley's network activation, as a great deal of his work was in his head. However, based on his follow-up questions it seems that *domain specific* networks for understanding slope were active, but they did not support explicit representation of the slope formula. Farley knew that slope required him to find the ratio of the vertical and horizontal differences between the points, but he pointed out that "for optimal performance, you would use the slope formula." This suggests that his network didn't include nodes for the explicit representation of the slope formula.

Farley did mental subtraction during this question, suggesting *core mathematical* networks for exact calculation were active. Farley demonstrated no overt *metacognitive* activity. It seemed as though the strategy he used came quickly and intuitively and required little metacognitive control. Farley made minimal use of the paper – he used it only to write down his final answer. He did not interact with the researcher. There was almost no use of *contextual* networks.

Farley's activity in this question appeared entirely *competent*, as Farley easily subtracted and divided the relevant values to obtain slope. This activity appeared to be supported by domain specific networks for slope (which didn't include the slope formula) and core mathematical networks for calculation. Farley's activity in the question did not appear to have detectible examples of flexibility.

Problem F1

Activity Description.

The AAT graph describing Farley's activity in this second problem is presented in Figure 21. Again Farley solved this problem in his head. After briefly reading the question, he wrote 2a on the paper as a numerator on a fraction. He then crossed that out and wrote the correct answer. The entire question was answered in under 25 seconds.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

F: The same way I solved the last one, difference between 3a and a is 2a, and the difference between b and 2b is b.

R: Okay. And here's a follow up, were you using the slope formula?

F: Ah. No.

Affordances to Detect Network Activation.

Domain specific networks for slope were active throughout the question, but again, they didn't support use of the slope formula. In a follow-up question the researcher asked Farley if he had used the slope formula, and he said no. Later, he indicated that the slope formula "wouldn't work because you can't subtract 3a and a." In fact, that is exactly what Farley had done to answer the question. *Core mathematical* networks for very simple exact calculation appeared to be active in this question.

Again, *metacognitive* activity was challenging to discern. However, Farley showed a disconnect between his narrative understandings of the problem and the strategy he used. As discussed above, he subtracted and followed the exact procedures of the slope formula; however, he explicitly denied that he used that formula and suggested that it is unuseable in such a situation. This suggests that certain networks within Farley's mind functioned to interact with slope relations; but these networks did not always activate narrative metacognitive awareness, thus leading to a disconnect between what the explanations generated by narrative networks and what domain specific networks are doing. Farley made minimal use of the paper, and did not talk to the researcher. There was almost no use of *contextual* networks.

Farley's activity in this question appeared *competent,* with Farley subtracting the relevant values and dividing, quickly, easily and accurately. This competence was supported by Farley's domain specific network activity (in slope and algebraic networks). Farley's activity in the question did not appear to have detectible examples of flexibility.

Problem C2

Activity Description.

The AAT graph describing Farley's activity in this third problem is presented in Figure 22. In this problem, Farley cycled through periods of writing and thinking until he wrote an answer that was correct for company A, but not for company B. He wrote on the paper 4 times during the problem and appeared to spend about 30 seconds writing in total. However, by the end of the question he had written only "A: $18.50 / B: $19.10." It was challenging to discern what was written at each point during the question, because so much time was spent writing, but very little ink was on the paper.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

> *F: Uh, just try...to...uh trying to do question … which is...uh...just adding…multiplying 20 from the first company and 22 for the second company by 30, and then adding it to either $12.50 or $10.00.*

Affordances to Detect Network Activation.

Farley's work on this problem afforded detection of activation of domain specific, core mathematical, metacognitive, and contextual networks. Based on Farley's responses to follow-up questions, it seems as though *domain specific* networks for cell-phone plans were active, as well as those for linear relations. Farley indicated that this question was exceptionally easy, suggesting that more "formal" mathematical networks were not required (e.g., networks for creating linear equations to represent real life situations).

Core mathematical networks for calculation appeared to be active throughout the question, as Farley did all calculations in his head. However, his calculation for Company B was not correct. Again, Farley showed little overt evidence of *metacognitive* activity. His belief that the question was very easy may suggest that the problem required little metacognitive monitoring or adjustment. Farley did not store intermediate calculations on the paper, only final results. He did not interact with the researcher. Again, he made no meaningful use of *contextual* networks.

Farley's activity in this question seemed entirely *competent*, though it was mostly in his head. He generated answers quickly and reported simply applying the requisite addition and multiplication (though, his final answer was not correct). Farley's activity in the question did not appear to have detectible examples of *flexibility*.

Problem F2

Activity Description.

The AAT graph describing Farley's activity in this fourth problem is presented in Figure 23. Farley struggled to understand what this problem was asking, and ultimately was unable to give a correct answer, instead giving true, but very specific statements about the relative costs of the companies (his final answer was "Company A costs more after 4 hours"). After briefly reading the problem, Farley began by discussing what type of answer was appropriate with the researcher. At that point Farley appeared to have a good idea of what the problem was asking; after a brief period he wrote and said, "Company A is cheaper when you talk more." After the researcher clarified that he was looking for a numerical answer, Farley began to write (it was unclear what he was writing). He then used his calculator to check the relative costs of both companies with four hours of talking. This took over a minute. The researcher and Farley then had a brief dialogue:

F: Uh...if you talk 4 hours a month, company A would be cheaper than company B.

R: Okay. Is that the answer?

F: Uh, **is** *that the answer?*

R: Uh, I can tell you in two [follow up] questions.

F: I just checked it on the calculato.r

Farley's answer was true, but not what the researcher was looking for as a correct answer.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

F: I just plugged in a value...for how many minutes you talk and how many hours you talk...and there's 60 minutes in an hour so 4 hours gives you 240 minutes….240 x 20 + $12.50 would give you the cost of company a, 240 x 22 + $10.00 would give you the cost of company b, and in these circumstances company a would be cheaper than company b.

Affordances to Detect Network Activation.

Domain specific networks for cell phone costs appeared to be active, as evidenced by Farley's facility in finding the cost of cell phone use for each company. It appears as though networks for comparing and relating two linear equations were not active. It is unclear whether networks for algebraic representation of linear equations situations were active, as Farley did not write a general equation at any time. Most of the calculation in this question occurred on the calculator, instead of using *core mathematical* networks.

Farley showed little evidence of *metacognitive* activation. Though he did shift strategy (from giving a general answer, to using specific costs for a specific number of minutes), he did so in response to a specific request from the researcher for a numerical answer, and there appeared to be no other activation of metacognitive networks. However, Farley may have used metacognitive activation to manage the effort he put forth in this question. In this question, the researcher initiated a conversation with Farley to ensure the question was correctly understood, and it seemed as though Farley understood. Though it is possible that Farley's ultimate answer was a literal response to the researcher's request for a "numerical answer," it is also possible that Farley's activity represented a deliberate and metacognitive choice to use a low effort strategy. This interpretation is consistent with the researcher's observations of Farley's approaches to problem-solving and with Farley's stated preference for doing as little work as possible, and thus, this activity was coded as metacognitive.

This transaction was the only one in the study which seemed to be characterized only by flexibility, with no detectible examples of *competence.* Although Farley's calculation of the cost of both companies at 4 hours of talking time may have reflected application of a competent routine, in this context, his activity was interpreted as part of a larger flexible adaptation to minimize task demands.

Farley seemed to exhibit *flexibility* through this question by strategically answering the question in a way that minimized calculation and writing. Conversations with the researcher seemed to make clear that a correct answer would require a statement of the conditions under which one company would be cheaper than the other. However, Farley's strategies were low calculation, low writing strategies. This may suggest an adaptation to the question; rather than engaging in the effortful mental processes required to give an answer, Farley chose a strategy which afforded him minimal work.. Though I used this reasoning to code the activity as flexible, it is also possible that he simply did not understand the question; if this was the fact of the matter, than this activity would not be flexible, nor should his network activity be understood as metacognitive. This type of flexibility was challenging to relate to network activation. Farley may have used metacognitive activation (to evaluate task demands) to activate domain specific networks (for activity that would be acceptable, but that didn't require guess and test or equations).

Problem C3

Activity Description.

The AAT graph describing Farley's activity in this fifth problem is presented in Figure 24. Farley solved this problem correctly and with ease. He first mentally calculated the slope of the first line; he then calculated the slope of the second. After a brief pause to think, he wrote "no, they have different slopes," ending the question.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

> *F: Well I know that uhm...in order for the lines to be parallel they have to have the same slope...so I calculated the slope of both lines and they had different slopes so they aren't parallel.*

Affordances to Detect Network Activation.

Domain specific networks for finding slope, given points, appeared to be active, as were domain specific networks for parallel lines. Farley did several in-the-head calculations to find the slopes of the two lines suggesting activation of *core mathematical* networks. Again, there was minimal evidence of *metacognitive* activation. Farley seemed to know intuitively and quickly what to do, and he dispatched the problem with ease. Farley did not use the paper for intermediate calculation, but did use it to write the two slopes and to indicate that the lines were not parallel. Thus he did not use *contextual* networks to solve the problem.

Farley appeared to draw on networks for slope throughout the study. He seemed to apply them easily and quickly in this question, suggesting *competent* activity supported by domain specific networks for slopes, relating parallel lines, and core mathematical networks. Farley's activity in the question did not appear to have detectible examples of flexibility.

Problem F3

Activity Description.

The AAT graph describing Farley's activity in this final problem is presented in Figure 25. Farley solved this problem correctly, after a long period of rumination. Farley began the question by asking the researcher if he was looking for "actual values" since there were many possible solutions. After answering yes, the researcher asked for Farley to expand on his idea that there were many possible values, but after realizing this discussion could be distracting, asked Farley to return to the problem. After joking with his mom (who was in the room) that she would struggle with this problem, Farley began to think for a long period of time, almost motionless. He later reported that he was cycling between thoughts about his evening work shift and the question. After almost a minute, he wrote a 4. A few more seconds later he announced, “Ok, it’s really hard for me to envision this … it’s going to take a while.” After another minute of silent, almost motionless thought, he crossed out the four, and wrote a correct solution.

In response to a follow-up question asking him to describe his thinking through the problem, Farley replied:

F: Well first I had no idea what the hell I was gonna do, but I figured if I could make the second line parallel...not parallel, horizontal or vertical...for some reason that would stop it from crossing paths with the other one...I realized … that if they were parallel they wouldn’t touch. So I just gave the second one the same slope as the first one.

R: And when you were sitting there thinking, what was going through your mind?

F: Lots of things.

R: What kinds of things. Tell me as much as you can remember.

F: Well I was...because of the way my brain works while I was trying to solve the problem, I was also paying attention to the fact that my mother had put on her jacket...which means that she must be cold, which means that it’s probably not very nice out today, which means that I’m probably not going to end up selling very many [Dairy Queen] blizzards today when I got to work, which means it’s going to be a slow night at work.

R: So you’re thinking about that? Okay, were you going back and forth between that and the math thought, or was that sort of a self-contained thought?

F: No, there’s a back and forth.

R: So what math stuff was going through your head while you were going back and forth through your daily plan.

F: Uh...well...what math stuff that was going on through my head was I don't understand, I don't understand...I have no idea how to do this, and yeah...and then I realized that I could make them parallel and they would never cross.

Affordances to Detect Network Activation.

Domain specific networks for parallel lines, and their relationship to slope appeared to be active in this question. Farley indicated that he understood that he could "make the lines parallel" (i.e., give them the same slope). Domain specific networks for predicting the conditions for his evening job were also active during this problem, as evidenced in his transcript, above. Algebraic networks also appeared to be active, as Farley solved for one variable, given a value for another, in his head. During this problem Farley did several in the head calculations as part of getting the slope of one line and finding values that work in the second line, suggesting use of *core mathematical networks*.

Farley realized early in the problem that there was an infinity of solutions and sought clarification from the researcher. This indicated *metacognitive* activity in the service of correct task interpretation. Farley recognized that he didn't understand what a good answer looked like, and asked for clarification.

In his follow up questions, Farley also told the researcher that, as he was thinking about the answer (for almost two minutes) he was switching between thinking about the problem, and thinking about his evening. It is possible that this represented a lack of metacognitive control. Farley may simply have had trouble controlling what he was thinking about. However, it's also possible that Farley's activity represented a specific metacognitive strategy, namely to distract himself long enough to take a fresh look at the problem. Farley indicated that he was saying to himself, "I don't understand. I don't understand." It is possible that he shifted to thinking about work as way of giving him a break from those thoughts, allowing his mathematical networks to solve the problem, unencumbered by narrative networks which, at that point, were activating in an unhelpful way. During that break, those networks were able to see a path to a solution, evidenced by Farley's statement, "I realized I could make them parallel," and offered their activation to other networks by putting that realization into consciousness.

Contextual networks were highly implicated in this question relative to other questions. Farley interacted with the researcher to ensure that he had a correct interpretation of the task. He also used the paper to store an intermediate number (4), which he ultimately did not use in the problem. This represented the only time Farley used the paper to store intermediate calculations during the study.

Because much of Farley's work was in his head, it was difficult to discern exactly the nature of his activity; however, it seems as though Farley's calculation of a pair of values that satisfied the question represented *competent* activity. There is little direct evidence, but this was done correctly, and it seems reasonable that Farley would have been comfortable with such a procedure, given his propensity to calculate in his head. This activity was likely supported by domain specific networks for solving algebraic equations.

Farley exhibited *flexibility* when he clarified his understanding of the task demands with the researcher. For other students, clarifying task demands represents competent behaviour, but in this context, this activity may have represented an adaptation, as Farley had not interacted with the researcher during any questions at that point. However, in this question, Farley saw that the most efficient use of his effort was to get clarification up front. From a network point of view, metacognitive activity (the decision to ask the researcher) may have activated a contextual network (the researcher) to aid in activating the right type of domain specific networks.

As well, Farley may have exhibited flexibility when ruminating, cycling between thoughts about his evening and thoughts about the question. This repeated distraction may have afforded him a fresh look at the problem each time his thoughts cycled back to the question. The suggestion that this cycle may have been flexible is supported by the observation that he did ultimately obtain a correct answer for the question, suggesting his activity functioned to afford him the answer. From a network point of view, his sense that he was not getting the answer (metacognitive activity) co-activated domain specific networks (for slopes of parallel lines, and the demands of his evening) to produce new domain specific activity (for setting the expressions equal mentally and solving) that allowed him to correctly solve the problem.

Patterns in Network Activation and Flexibility: Farley

Like other students, Farley appeared to make extensive use of *domain specific* networks for slope. There is evidence, though, that for Farley, this network did not include an explicit representation of the slope formula. Farley did most slope calculations mentally and he never wrote the formula down. At one point, when the researcher asked him what a friend should know about the problem, he suggested they should use the slope formula. The researcher then asked if he had just used the slope formula, and Farley said, “no.” This suggests that, although Farley’s networks for slope supported the exact calculations associated with the slope formula, his networks did not include explicit, algebraic representation of that formula. In addition, Farley drew on domain specific networks for parallel lines, their relationship to slope, and manipulation of algebraic expressions. These networks appeared to be active while solving problems and to support successful performance.

It also appeared that networks for cell-phone plans were active in questions C2 and F2 (cell phone). The way in which Farley answered those questions suggested a primarily real-life network, rather than a linear relationship network. As a result, Farley was not able to solve question F2. It also appeared that domain specific networks for relating external conditions (the weather) to predictions of Farley's evening job were active, given his report of the impact of weather on his sales in question F3 (parallel lines).

Farley did a great deal of mental math, rarely using the paper. Thus, it appears that Farley's *core mathematical* networks were well established and easy to recruit. He did multi-step algebra in his head.

Farley showed very little *metacognitive* activity through the study. Only two questions showed metacognitive network activity. In question C1 (slopes), Farley used the slope formula, but was not aware of it, suggesting a disconnect between narrative and experiential metacognitive networks. Farley also once asked for clarification of a specific question (in F3). In this case, he was ensuring that the researcher was looking for a single pair of values. It is possible, given Farley's desire to move quickly through the material, and his lack of metacognitive activity elsewhere, that he was trying to ensure that his interpretation was right, because other interpretations (i.e., a general algebraic equation describing the relationship) may have taken longer. From this point of view, metacognition may have served to help Farley reach his goal of completing the study quickly.

Farley's cycling between thinking about his evening work and his task may have represented a general metacognitive strategy of distraction to afford insight. The researcher has heard several anecdotal reports from students with attention problems indicating that distracting oneself from a problem, and then immediately returning to it, can support insight and flexibility. Given Farley's reported attention issues, this may be partly what happened, when he successfully solved the problem during cycles of thinking about work and the problem.

As suggested above, Farley’s metacognitive activity may have been characterized by a disconnect between experiential metacognitive/domain-specific networks and narrative metacognitive networks. In questions C1 and F1, Farley did not realize narratively that he was using the slope formula, even though he was. In question F3, Farley was telling himself “I don’t understand, I don’t understand” while, at the same time, domain-specific and experiential metacognitive networks were correctly solving the problem. Thus, narrative networks did not seem to activate in an integrated way with domain-specific and experiential networks for Farley.

Farley did not seem to see the paper as part of a useful *contextual* network for doing problems. He only recorded answers on them, and that may be because that's what the study entailed, rather than him seeing an affordance in writing down answers. Similarly, Farley rarely interacted with the researcher, except briefly in the last question, suggesting that he did not see a need to use that contextual network in solving the problems.

It is possible that Farley's activity in question F2 (cell-phone) represented a *flexible* adaptation to a challenging problem by using a metacognitive strategy of minimizing calculation and writing. In Farley's semi-structured interview he indicated feelings of extreme negative emotions to describe working on problems he already knows how to do (he used words like despise, avoid and unmotivating) to describe his experience with mathematics. Despite extensive conversations with the researcher, Farley did not seem to apply the task interpretation discussed, possibly because he realized how lengthy a process it might be to answer that question. At the same time, Farley’s activity may not have represented a desire to minimize work, and instead may have simply represented a misinterpretation of the problem.

Farley also exhibited flexibility when he afforded himself insight into question F3, by thinking about his evening work shift. This may seem counterintuitive, but as discussed above, it is a strategy that kids with attention issues have anecdotally reported using; and in this case, it was effective. It is difficult to pinpoint exactly what type of network activation affords this flexibility. In follow-up, Farley seemed to believe that this “happened to him” rather than him choosing to distract himself. At the same time, experiential metacognitive activity (which, for Farley, may not be well linked with narrative metacognition) may have been dynamically changing domain specific network activation in response to internal vicero-somatic feedback about the effectiveness of that domain specific activation (i.e., experimental networks may monitor the predicted reward from amplifying to a particular network, and thus, cycle back and forth between two networks, shifting when the predicted reward from attending to a particular network goes below a particular point).

For Farley, flexibility may have involved combinations of a variety of experiential and domain specific networks, but included minimal activation of narrative metacognitive networks. On the other hand, Farley’s competence seemed to have been largely supported by domain specific network activation. Metacognitive, core mathematical and contextual networks were largely ignored by Farley during the study, particularly when he exhibited competence.

Harriet

At the time of this study, Harriet was entering Grade 12 at a local school that provides "Jewish Education." She reported an 83 in Math 11. Harriet's semi-structured interview gave a strong impression of how she sees herself as a learner. Harriet showed an explicit preference for flexible mathematics over competent mathematics when she discussed her strengths and weaknesses:

H: I’ve always been pretty good at math. I’m not...that great at like…you know, linear systems, which is apparently what we’re doing. …. But um...I don’t know. I really like *trig...Uh, I like...things that...I can like...figure out, but now like...if it's like...something that requires like thinking, as opposed to just plugging in a formula, but like not where it's like...I don't know, just it's something about the lines and graphs that just…I don't really understand it.*

R: Okay. And you don't like the formulaic stuff as much.

H: I don't mind formulas...it's just like...it's more fun for me...when it's something I can like figure out. Like if I see a formula and I can figure out a way to do it...not using the formula.

R: Yeah. You prefer that.

H: Yeah. But I like having the formula because if I can like look at something that's like already been done, I can figure out how to do it without someone telling me. ... But like...so I like to like look at the formula and think "well, what if I do this instead...would it still come out the same?" and stuff like that.

R: ... So then what would you say your weaknesses are in math?

H: Um...well I'd say my weaknesses are regarding like school math…. I think I'm really really good at it so I never practice and sometimes that gets me really behind.

R: Yeah. And so what do you do when that happens?

H: Nothing.

R: Has that been a problem forever, and across different subjects, or…?

H: Yeah. I...I, I don't...I have a really poor work ethic.

R: Poor work ethic. Can you tell me more about that?

H: Um...I like...I rarely do any homework at all. Um...I've always had this problem where...this sounds really conceited but like where I think that like I'm so smart that like I *don't have to do any homework. Um...kind of like...it's like "Oh whatever, if I don't do this I'll still be ok" not like "I'm so smart I can just make it up" it's just...it doesn't really click in my head that if I don't do it, it could really affect me later. Even though like I know it does, I just like go like uh...go on Facebook and just do it eventually. .... Like I don't think I've done any math homework all year.*

R: Is there anything different or distinct about the way you learn in general? That you know of ....

H: Well I have ADD. So there's that, I don't really know how that affects my learning math, but um...I...I have like a really short attention span. I don't know, I like math though. I..um...I just...I find it hard to concentrate on some things...um...what was the question again? Sorry.

Table 11 indicates which of Harriet's activities were competent or flexible.

Problem C1

Activity Description.

The AAT graph describing Harriet's activity in this first problem is presented in Figure 26. Harriet solved this question correctly by counting the rise and run and dividing the two. Harriet began this question by drawing an x/y axis. She then created ten marks along each axis, representing units. She graphed both points, then went back to ensure she had the correct number of unit marks. Finally she counted the rise and the run, tracing the distance with her pen. She then wrote the rise over run and simplified the fraction to get the correct answer.

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

H: "Ok, well first I was like, ok, we'll I'll draw a graph. Then I drew the graph and realized I didn't make enough space, so I made it bigger. And then I drew the points on the graph, and I connected the line, poorly. And then I did rise over run... so I just counted up and over between the two points....and I think that's what I supposed to do [trails off].

Affordances to Detect Network Activation.

Domain specific networks for slope appeared to be active during this problem, as evidenced by her use of "rise over run." These networks seemed to support counting the rise and run, rather than subtracting the x and y values to obtain the rise and run. It is possible that the phrase "rise over run" for Harriet, supports a counting approach to slope, since neither the slope formula or even the word slope were mentioned in her responses to the follow up questions. Very basic *core mathematical* networks for counting were used during this problem, but given her approach, it seems as though networks for calculation were not active.

Harriet used *metacognitive* networks to monitor the appropriateness of her graph and to ensure the unit marks on the axes allowed her to make the graph she needed. This was evidenced by the fact that she returned to this task after graphing the line. It is unclear if this activity was essential to solve the problem or graph the line more accurately, or if it was an intentional distraction to allow her mind time to process the next step.

In terms of *contextual* networks, Harriet used the paper to construct a visual-spatial representation of the line, which she used to correctly solve the problem. She also recorded an intermediate calculation for slope on the paper, as well as a reduced form of the final answer.

Harriet showed *competence* in the way she solved this problem. She easily and quickly drew the graph and counted the slope. This activity appeared to be supported by domain specific networks for slope (which, though they afforded calculation of slope, did not explicitly include the slope formula). This competence also seemed to be supported by contextual networks, which afforded her drawing of the question. Harriet's activity in the question did not appear to have detectible examples of *flexibility*.

Problem F1

Activity Description.

The AAT graph describing Harriet's activity in this second problem is presented in Figure 27. Harriet was not able to solve this problem, because early in the problem she chose to use specific values for a and b, rather than solve the problem in general. Though her final answer was correct, given her choices for a and b, it was not correct in general. Harriet began this problem by drawing an x/y axis and creating unit marks along the axes. She then selected possible values for a and b (a=1, b=2). Based on these values, she graphed the point (a,b) as (1,2) and (3a,2b) as (3,4). She then joined the points, and traced the line with her pen, estimating rise and run. She drew a 1x1 grid over the 1st quadrant of the graph, and counted her rise and run. With a shrug, she wrote m=1 on the top of the page.

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

> *H: Well I don't think I know how to do this, but then I drew a graph, and then I was like I have no idea if like a and b are supposed to be something. Well I mean like, obviously they're supposed to be something, but I mean if I'm supposed to be able to figure out what it was. So I was like k let's assume that 1 would be 1 and b would be 2...and then I plotted 1,2 and then that would make this one 3 and 4 because you know . 1 times 1 and two times two .... um... and so I plotted the second point and then I did rise over run.*

Affordances to Detect Network Activation.

Domain specific networks for rise and run appeared to be active during this problem. As in the previous problem, it seems as though these networks supported counting for Harriet, but not necessarily finding the differences between the points using subtraction or using the explicit slope formula. Domain specific networks supporting “plugging in values for variables” were also evident. Harriet approached this problem specifically, assigning values to a and b, and working out the implications of that assignment.

Very basic *core mathematical* networks for counting were used during this problem, but again networks for calculation did not appear to be active. Harriet showed *metacognitive* activation when she tried to count the rise over run, realized that this counting was unreliable, and drew a 1x1 grid to help her count more accurately. It may be that discovering counting was difficult triggered a network for a new strategy (drawing a grid).

Contextual networks were implicated when the paper was used to create a visual-spatial representation of the problem. Ultimately, this representation was too specific (assigning specific values to a and b), and didn’t support a more general answer. Harriet did not interact with the researcher during the problem.

Harriet seemed to exhibit *competence* early in this problem when she drew an x/y axis, and late in the problem, when she counted slope. In both cases she seemed to be using domain specific networks to support this competence, one for drawing axes, another for counting slope. Harriet exhibited *flexibility* in her choice of specific values for a and b. Her follow up answer suggests that she was sure that a and b should "be something" but was not sure if she should give them. However, after thought, she realized that choosing values would afford her a graph, which she could then use to count slope, and so she did it. As well, Harriet’s choice to use a grid to make counting rise and run easier also represents flexibility. When her initial counting strategy seemed too imprecise, she realized a grid would clarify the rise and run.

This flexibility may be understood as the result of network activation. Harriet metacognitively activated a domain specific network (plugging in values) to create a situation in a contextual network (her paper). She then created a new element in the contextual network (drawing a grid over the lines) to facilitate use of a domain specific network (for counting slope), all in the service of an accurate, but non-general (and thus incorrect) answer.

Problem C2

Activity Description.

The AAT graph describing Harriet's activity in this third problem is presented in Figure 28. Harriet solved this question easily and correctly. Harriet began by reading the first sentence aloud and then silently reading the rest of the problem in her head. She quickly wrote expressions for the cost of both companies at 30 minutes. She calculated the cost of Company A on her calculator, wrote it down, mentally calculated the cost of Company B, and wrote that down as well.

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

> *H: Well first I was like 'I don't know how to do this" because I was looking at it, and at first I thought it was one of the ones where you like let a equal and do the like.....the two equations and find the variables...but then I was like there really are no variables since they give you all the information, and so t...I was like well.... it's gonna be 20 cents per one minute and there's 30 of them plus whatever the access fee is, and then I was like 'Oh that's easy, why didn't I see that before.'*

Affordances to Detect Network Activation.

It appears that *domain specific* networks for situations with a linear structure were active. Unlike other participants, Harriet's answers to follow up questions indicated that she understood this problem as an academic math problem, rather than relying on less academic understanding of cell phone prices ("I was looking at it, and it first I thought it was [a systems of equations problem] … but then I [realized] … they give you all the information"). Harriet was able to infer the linear structure of this problem and make the requisite calculations, suggesting that networks supporting these situations were active. Harriet did some calculating in her head – in particular finding the cost of Company B – which suggests that *core mathematics* networks for exact calculation were active.

Harriet's follow-up response cited above suggests that she shifted strategies early in the problem. Her perception of the task changed from a linear system to plugging values into two separate equations. It is possible that this realization/strategy switch was function of *metacognitive* networks monitoring the usefulness of particular task interpretations and adjusting network activity accordingly. Harriet used the paper to record equations for the cost of both companies at 30 minutes of talking and to record the answers, indicating use of *contextual* networks.

Harriet's activity during this problem seemed to be *competent*. She easily shifted strategies into a plugging-in approach that afforded her a quick and easy solution. This competence seemed to be supported by domain specific networks (including those for real-life cell phones and linear equations) and core mathematical networks. Harriet did not show evidence of *flexibility* during this problem.

Problem F2

Activity Description.

The AAT graph describing Harriet's activity in this fourth problem is presented in Figure 29. Harriet spent 10 minutes coming up with the correct solution to this problem. To begin, she read the question, asked if she could look back at her previous work, and recorded the results of the last question on the page. She then started using the calculator – she later reported that she was trying to divide the difference between the system access fees ($2.50) by the per minute costs. When she couldn't make sense of her answer, she started to guess and test. Harriet would choose a number of minutes, and calculate the cost of both companies for that number of minutes. After nearly 7 minutes of guessing and testing with different values for number of minutes, she got the correct answer and wrote it down.

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

> *H: Ok, well, I didn't know how to do it. Ahhh...I'm sure there's an easy way to do it that I overlooked. Ummm so, after a while of not understanding like what to do, I was like, K, well I'm just going to..... do trial and error until I figure it out - well actually, first I was like, k, it was like the difference between these two is like 2.5 and then I was like k, so, I did like 2.5/something, and I don't remember why I did that. I did 2.5/.2 because it's 20 cents a minute, and I was like well, that's how much is .... like....how much more this one has to be than that one? or something? I don't really know what I was doing, I just kind of ...I just kind of try things and I don't remember why I was thinking that that would work. But it came up with 12.5 which was the same number as that, so I assumed it meant something, I just didn't know what. ....uh...and then I gave up on that because it didn't*

really seem to be working, and I just, went down along the number line, until eventually I got to 125, and they both came out to be 37.5.

Affordances to Detect Network Activation.

Early in the question, it seemed as though *domain specific* networks that could relate the difference in system access fees and the per-minute cost were active. However, these networks were either not active enough to support effective performance, or not specific enough, as they did not lead to a successful solution. Later in the question, domain specific networks for calculating the result of a linear equation, given an input variable (minutes) seemed to be active, as Harriet searched for the solution. It also appears as though a network provided candidate minute values. As Harriet progressed through guess and testing, her candidate values got closer and closer to the answer. This suggests some organized mental activity in terms of selecting candidate values, which may have been associated with a domain specific network, a metacognitive network, or a combination of both.

Harriet used the calculator to do all calculations during the questions, so it is unlikely *exact calculation* networks were involved. The selection of candidate values for m may have required core mathematical networks for detection of magnitude, as her selections became increasingly accurate as the question progressed.

Harriet shifted strategies two minutes into the problem. Initially, she tried to come to a logical conclusion by working with the difference in system access fees. However, after realizing the numbers coming from this process didn't make sense, she decided to guess and test. It may be that *metacognitive* networks supported this decision, by helping Harriet detect that she was not getting closer to the answer. It's also possible that metacognitive networks helped Harriet stay focused during her guess and test period. She guessed and tested for seven minutes – a long time relative to other students. Though there is no direct evidence for this, it seems possible that metacognitive networks might have kept the guessing and testing networks active for a long enough period to obtain the correct response.

Contextual networks were also implicated in how Harriet solved the problem. She asked the researcher about the parameters for the study ("Can I look back?"). She also used the paper to record 17 different costs for different numbers of minute values. She never referred to these, so it is unclear why she wrote them down.

Harriet's guess and test approach may have represented *competent* activity. The phrasing of her follow up answer, "I'm just going to do …. trial and error," suggests that she considered this an inelegant but obvious route to the answer, and her fluidity and ease with the strategy suggest competence as well. This competence seemed to be supported by domain specific networks for guess and test, linear equations, cell-phone plans and contextual networks (the paper and the calculator).

Harriet showed *flexibility* when she divided the system access fee by the per minute costs as an initial strategy. Harriet suggested that she was "just kinda [trying] things" – which suggests that she may have been searching for a strategy to solve the problem that didn't require guess and test. Once this approach failed to be fruitful, she moved to guess and test, which may also have represented a flexible adaptation to the problem, by adjusting to use a strategy that was slow but sure (guess and test).

Though Harriet solved the problem, her initial flexibility was not essential to her solution. Harriet first went down the wrong path (flexibly) by using activation in analogical core math networks (about the relative size of the costs of the companies) to activate metacognitive activity (choosing to divide numbers) that she thought might be useful (but were not). Metacognitive activation (detection that this strategy did not work) then activated domain specific networks (for guessing and testing) that were used to solve the problem.

Problem C3

Activity Description.

The AAT graph describing Harriet's activity in this fifth problem is presented in Figure 30. Harriet did not solve this problem correctly because she made an incorrect inference from the points given in the question. Harriet began this problem by verbalizing her strategy to the researcher ("so draw it?"). Though the researcher did not confirm that this was the correct strategy, Harriet graphed all four points and joined the two lines. After looking at her graph, Harriet announced "I can tell you how they're parallel without drawing the lines."

The researcher and Harriet engaged in a discussion about Harriet's solution. She gave a lengthy explanation relating the points in one line to the points in the other. She indicated that adding '1' to the points from one line would prove that they were parallel (this is not true). Even with follow-up questions and triangulation between written records and video evidence, I have been unable to discern what strategy she was using here. However, it seems as though assumptions about using addition to relate points afforded an opportunity to see lines as parallel, even though they were not.

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

> *H: Ok, well, I can tell you how I know they're parallel without drawing this graph. Ok, well this one is 2 and 4 and you just add one to either side which means they're going to be slightly more over but like exactly in the same place. And it's the same as this one, but*

you minus 2. I mean 1. ....so like the coordinates are like equal, but like one less or one more on either side.

…

H: I drew the graph, just to be sure it was parallel. And then as I was looking at the coordinates I noticed the relationships between them, and then I was like I guess I didn't need the graph but I guess it kind of proves I did it.

Affordances to Detect Network Activation.

Domain specific networks for parallel lines appeared to be active, but it was unclear what type of activity they supported. Until late in the follow-up questions, it did not appear that Harriet understood that parallel lines must have the same slope. So, although networks supporting both algebraic and visual spatial understandings of parallel lines were active (as evidenced by her visual spatial and algebraic solutions), it is unclear what actions these networks afforded Harriet and it seems that these networks did not relate slope to parallel lines.

Based on Harriet's oral proof for her answer, it seems as though *core mathematical* relationships were central to her conclusion that the lines were parallel. However, because I have been unable to discern what Harriet was thinking, it is unclear what these networks contributed to the activity.

Harriet began the problem by drawing both lines, but then abruptly launched into an oral, algebraic justification of her conclusion. This switch occurred rapidly, right after she completed her second line graph. This may indicate *metacognitive* activity. Harriet detected an affordance in the points, which allowed her to make a proof that made sense to her. As a result, she shifted away from her graph and launched into an algebraic explanation.

Contextual networks were active when Harriet used the paper to construct a visual-spatial representation of the problem. She also discussed her proof orally with the researcher rather than providing formal written mathematics.

Harriet began this problem *competently*, by drawing an x/y axis and graphing both lines, a process which seemed to be supported by domain specific and contextual networks (the paper). Harriet's construction of an algebraic explanation for why the lines are parallel may have indicated *flexibility*. Though the explanation was incorrect, it seemed to represent an attempt by Harriet to deal with the fact that her graphs did not afford her a proof. She saw a potential affordance in the points given and used that to construct an explanation she thought might prove what she could see in her diagram. From a network point of view, Harriet appeared to metacognitively co-activate two domain specific networks (visual-spatial and algebraic networks for slope) to generate a domain specific and incorrect algebraic explanation for how the question works.

Problem F3

Activity Description.

The AAT graph describing Harriet's activity in this final problem is presented in Figure 31. Harriet solved this problem correctly. She began by drawing an x and y axis. After mentioning to the researcher "I talk to myself while I'm doing this," she graphed and joined the points in the given lines. After a moment of thought, she wrote down an equation for the slope of the second two points. She corrected this twice and finally got a correct expression.

Harriet then informed the researcher that there were many possible values that would make the lines parallel and gave the example of (11,1), but suggested that this was not an acceptable answer, as she thought the question asked for values that would <u>not</u> make the lines parallel. The researcher, inferring that she had misinterpreted the problem (which asked for points that would make the lines parallel) confirmed this understanding ("so it couldn't be 11 and 1?"). As she responded Harriet realized her misinterpretation, and corrected her answer ("it would be 11 and 1").

In response to a follow-up question asking her to describe her thinking through the problem, Harriet replied:

H: Can I just....say that....I mean like there could be like 100 of things that it could be but basically anything ...it can't be anything that would make this fraction 2. so like. uh. ummmm.......11 and ....uh 7, no, not 7, it's the other way around..... 11 and 1. What? ...Yes

R: Yeah that's right. So it couldn't be 11 and 1?

H: Uhhh, no ...well that would make them parallel if it was 11 and 1.

R: And what's the question asking you?

H: If the lines never touch, so yeah if they're parallel, so it would be 11 and 1.

R: So tell me what you did to solve the problem.

H: Well, I drew this line, and then I found the slope of it, even though I didn't really need to draw it. and then I got the slope of that because of what you told me in the first one...the y1 ...the y minus the y and the x minus the x ... and then I wrote that and I was thinking about ....uh.....when we do factoring, we were like - oh - cause it has to - don't know - equal 0 or something.

R: So that popped into your mind?

H: You know how it's like x does not equal 5 or something like that?

R: Oh, well then they can't have the same slope because minutes ago because you just said two minutes ago that if they have the same slope then they're parallel. Well I guess I *read the question wrong, because I thought that meant that if they like -- anything that they're not parallel. But I understood the question - I just kind of read it backwards.*

Affordances to Detect Network Activation.

Domain specific networks for parallel lines and their relation to slope appeared to be active in this problem. In the previous problem, this network did not seem to be active. However, during the follow up to the previous question the researcher explained the relationship, and Harriet was able to activate that network during this problem. It also seems as though *algebraic* networks were active during this problem as Harriet solved for one variable, given a value for the second variable. As well, she created an algebraic expression of the second slope. Harriet solved for the correct solution in her head, suggesting that *core mathematical* networks for exact calculation activated with networks for algebra to afford her a correct solution.

Metacognitive networks appeared to be active early in the problem, when Harriet noticed that she was talking aloud and mentioned to the researcher that she talks to herself. I coded this activation as metacogntive because it seemed as though a network activated in response to the verbal activity and afforded Harriet an opportunity to mention that.

Later in the problem, metacognitive network activation may have supported a shift from using a visual-spatial representation of the problem to working on an algebraic solution. Though this activity was coded as metacognitive, it is unclear if the visual representation directly activated domain specific networks for algebraic representations of slopes, or if metacognitive networks functioned to shift from a visual representation which was not working to an algebraic slope relation –which had been mentioned in the follow ups to the previous question – that might afford a correct solution. Because the move seemed deliberate, and Harriet described cycling through domain specific networks looking for a useful one, this activity was coded as metacognitive.

Harriet used the paper to construct a visual-spatial representation of the problem and to construct expressions for both slopes, showing use of *contextual* networks. Harriet also used interaction with the researcher as a way of giving her answer. She didn't write it, but instead provided a detailed justification to the researcher of her solution. This interaction allowed her to find and correct her mistake in the task interpretation.

Harriet showed evidence of *competent* activity at the start of the problem when she drew a graph of the line. Additionally, after flexibly using the slope to construct expressions for the slopes of both lines, she appeared to competently equate these expressions and mentally solve for a pair of values that satisfied this equation. This competence may have been supported by network activity in domain specific networks parallel lines, slope and algebra, a metacognitive activation shifting towards an algebraic solution, and networks for exact calculation.

Harriet showed *flexibility* when she used the slope formula to construct expressions for the slopes of both lines. Unlike other students (Farley excepted), Harriet had not used the slope formula in any of the previous questions. However, her response to follow-up questions on the previous question had included discussion of the slope formula, and during this question, Harriet used it fruitfully to obtain the correct answer. This represents adaptation to the problem and to interacting with slope in general. The previous follow-up questioning had afforded her use of a new tool -- the slope formula – which she put to work right away.

From a network viewpoint, it is possible to infer that, in the previous question (C2), Harriet had made a metacognitive request (to hear the correct answer) that activated a domain specific network (for the slope equation) which Harriet metacognitively activated in F3 as part of flexible activity which afforded a correct solution.

Patterns in Network Activation and Flexibility: Harriet

Across problems, it appeared that Harriet made effective use of *domain specific* networks for working with slope. Initially, these networks did not seem to include explicit representation of the slope formula. However, in the follow up to C3 (parallel lines), the researcher and Harriet discussed the slope formula, and, in the final question, F3 (parallel lines), Harriet used the formula explicitly to solve the problem correctly. This may suggest that Harriet's domain specific networks for slope changed over the course of the study, accommodating explicit representation of the slope formula and connections to parallel lines that could be used productively.

Harriet also showed evidence of understanding general linear equation relationships in her work on the cell-phone questions (C2 and F2). She wrote linear equations for both, and reported thinking about the problem in a formal mathematical, rather than 'cell-phone' modality. In addition, Harriet used visual-spatial domain specific networks for slope during questions that were solved by other students without such a diagram. This may suggest that Harriet saw visual-spatial relationships as primary within this domain. As well, domain specific networks for calculating with linear equations and for plugging in values were evident across problems.

Harriet used a counting approach to find rise and run, rather than subtraction. This observation does not necessarily indicate that exact calculation networks for subtraction were not functioning for Harriet, but it does suggest that her *core mathematical* networks included those for counting.

Harriet also showed evidence of network activation for approximate calculation when she selected candidate values for the number of minutes in question F2 (cellphone) and in her incorrect oral answer for question C3 (parallel lines). In F2, each guess from guessing and testing was closer to the answer and thus suggested that approximate calculation networks afforded candidate values to domain specific networks. In addition, Harriet initially tried to divide the system access fees and the per minute costs, possibly suggesting an approximate sense that these numbers might afford successful performance. In C3 (parallel lines), Harriet relied on an intuitive, non-specific relationship between the coordinates to justify her incorrect answer. These examples imply that approximate calculation networks may have formed an important part of Harriet's networks for mathematics.

Harriet reported that she "just tries things" and that was evident throughout the study. Harriet made a complete shift in strategy in three of the questions, suggesting a *metacognitive* network for strategy shifting (e.g., her shift from algebraic approaches to guess and test in F2). It is unclear this was supported by narrative or experiential activity, but it is possible that experiential networks detected incongruities in the problem and interacted with narrative networks to affect a strategy shift. This shifting seems to have had two metacognitive components, one to detect a lack of success with the current strategy and another to suggest candidate strategies. Harriet also showed metacognitive awareness of her experience when she reported that she “talks to herself” mid-problem. It seems as though she noticed her behavior and decided to explain it to the researcher.

Harriet used the paper to construct visual spatial representations of the situations in problems C1 and F1 (slopes) and C3 and F3 (parallel lines). She then used these representations actively in her problem solving, looking at them, tracing over them, and reasoning about them. Thus, for Harriet, the paper may have provided an important *contextual* network where understandings could be constructed and analyzed. Harriet did not generally interact with the researcher, except to provide oral justification for her answers. This suggests that Harriet did not see asking for help as an important contextual affordance.

In general, Harriet's performance was highly *flexible*, though not always successful. She demonstrated flexibility in 4 of the 6 questions. One thread in Harriet's flexibility was a tendency to try strategies without any clear idea of where they would lead. For example, in question F1 (slopes), Harriet chose values for a and b. She later reported that she was unsure of what this strategy would afford, but she was confident it could be used fruitfully. Similarly, in question F2 (cell-phone), Harriet divided the system access fee by the per minute cost because she was "just trying things." In both cases, the trigger for flexibility seems to be an awareness that a strategy was needed and a hunch that a particular strategy might change the way she was understanding the problem, thus affording a solution. This type of activation may have been experientially metacognitive. Her responses to follow-up questions suggested that Harriet could not describe linguistically the expected effect of these strategies, yet she seemed to have a “feeling” that they would work. This pattern of activity might be associated with an experiential network proposing candidate solutions, with some input from narrative networks.

Harriet also seemed to use contextual affordances to trigger flexible activity. For example, in the follow up to question C3 (parallel lines), the researcher and Harriet talked about the explicit slope formula. In the following question, she used this formula to construct expressions for slope, though she had not done that previously. This suggests that Harriet, hearing the slope formula, detected an opportunity for its use and recruited it. In question C3, Harriet used the drawing of the lines and the specific points in the question to construct an oral explanation for an incorrect solution. Again, Harriet seemed to be looking in the contextual networks for activation she could internalize, and to extrapolate from that to find strategies for problems.

Patterns in Network Activity Across Students

In this section, I report findings of a cross case analysis of patterns in network activity apparent across students. First, across students, it appears that *domain specific* networks were instrumental to successful performance. All students showed evidence of networks for slope, for connecting slope to parallel lines, and for modeling real world equations. The results presented here focus on patterns in slope and real-world equations networks, as those networks appeared to show meaningful patterns across students.

It appears as though networks for slope were required for successful performance by these students in some problems focused on linear equations (specifically, questions C1, C2, F1 and F3). That said, the networks appeared to afford success in different ways, varying in the extent to which they supported explicit use of the slope formula and included visual-spatial and algebraic networks.

For example, problem-solving activity on these problems suggested that three students (George, Clayton and Adam) had networks for slope that facilitated interaction with the explicit slope formula, while two students (Farley and Harriet) appeared to have networks for slope that did not include the explicit slope formula. More specifically, George's, Clayton's and Adam's, networks for slope seemed to function to facilitate algebraic calculation using an explicit representation of the slope formula. For example, in each question, George wrote the slope formula explicitly, and then in a step-by-step fashion proceeded to plug in values and simplify accordingly. The instrumental nature of this algebraic form for these students could be seen in Clayton's activity - he was unable to solve the problem until he asked the researcher for the formula. Once the network for slope was active, he was able to use it in a variety of questions and it supported successful performance.

Farley completed most calculations in his head, denying that he had used the slope formula. However, his activity descriptions indicated that his operations were identical to what would be done with that formula (i.e., he subtracted the relevant values and divided them). It is unclear, then, exactly how Farley was perceiving and working with the activity of his slope network, but his responses suggest that the explicit formula may not have contributed to his competent use of slope on many problems. In Harriet's case, networks for slope also seemed implicit, as well as visual-spatial. Harriet drew graphs in each question. She found slope by counting rise and run rather than subtracting. The activation pattern may have arisen because Harriet could not remember the slope formula and did not want to ask. Later, once the formula was brought up in follow-up questions, she did use it explicitly. However, the fluidity with which she drew visual-spatial representations and ease with which found slope using these representations suggest that this type of representation was more easily accessible than algebraic notions of slope.

Students' networks for slope also seemed to differ in the extent to which visual-spatial representations were included. Again, Harriet seemed to depend more on a visual-spatial than on an algebraic representation (even if she used an algebraic representation easily when reminded of the formula). For George and Clayton, algebraic networks seemed to represent their sole network for slope, while no visual-spatial representation of slope appeared (e.g., using a drawing of line to work with slope). Adam's networks seemed to fluidly integrate network structures for visual spatial and algebraic understandings of slope. His performance on question C3 (parallel lines) suggested that he understood the problem simultaneously in algebraic and visual-spatial modes, using the joint activation to construct an answer for the question. Thus, across students, results suggest that slope networks supported successful performance, but that the form of those networks varied across students in the extent to which they included visual-spatial representations of slope and the explicit slope formula.

Across students, in addition to networks for slope, all students seemed to activate networks supporting real-life applications of linear equations. These networks varied in the extent to which they were integrated with other domain specific networks for formal mathematics. For example, slope formulas did not seem to contribute to successful performance on the two problems involving cell phone plans. Instead, all five students activated domain specific networks for real-life linear equations in the first cell-phone problem and used these networks to solve the first question correctly (except Farley, who made a calculation error). It is reasonable to assume that most students in the study had experience with cell phone plans and how they are priced. Follow-up responses and activity reports indicate that this question was considered exceptionally easy by students, perhaps because they were so used to working mathematically in this context.

However, for most students, this network activation in C2 may have interfered with performance on the second cell-phone problem (F2). All students except Harriet seemed to interpret question C2 in a very literal and contextual way. If 'real-life' contextual networks for cell-phones were active in C2, they may have stayed active in the following question, impairing activation of formal mathematical domain specific networks. The two students who successfully solved question C2 ultimately verified their solutions with guess and test approaches, perhaps because networks for formal algebraic networks were less active than those for real-life cell phone pricing. That said, George did attempt to construct an algebraic relationship. So it may not have been that students were unaware that more formal representations of the problem were possible, but the activity of relevant networks may not have been sufficient at that time to support their use.

Only one student, Harriet, seemed to activate domain specific networks for linear equations when working on the cell-phone questions. In her case, there may have been a link between her activation of networks for mathematical equations in C2 and her competence with the guess and test strategy in F2. Framing C2 in algebraic terms may have afforded Harriet a more formal mathematical network state in F2, allowing her to succeed where three other students had not.

Second, *core mathematical* networks for exact calculation were utilized by all students, but only some seemed to make use of analogical networks for number approximation. Across participants, exact calculation was used throughout the problems. There were two errors – one by Adam and one by George. Adam caught his error during follow-up questioning. George did not. But in both cases, the error did not impact the activity of other networks and the questions were solved in a logically sound manner. In general, exact calculation networks seemed to function as expected and supported performance with minimal effect on other networks.

Only Adam and Harriet showed any activation of analogical core mathematical networks, and this was in questions F2 (both students, cell-phone) and C3 (Adam, parallel lines). Both started F2 by trying to relate the system access fee to the per-minute costs. Harriet divided one by the other, while Adam divided the differences between both companies. However, in both cases, it seems as though the relative sizes of those numbers triggered a strategy for relating those numbers. In Adam's case, he acknowledged this explicitly in his response to follow-up questions, indicating that his intuition suggested these differences might produce useful values for guess and test. This method did not work for Harriet, as the numbers she divided did not produce good candidate values. Yet, it seemed that later in the question, this analogical network was still active, as her candidates for guess and testing got increasingly accurate. It is of note that only Adam and Harriet were able to solve this problem, and thus it's possible that successful performance on this problem was supported by analogical networks for core mathematics.

Third, participants appeared to vary in terms of activity in their *metacognitive* networks. For George and Farley, metacognitive activation was seen less frequently, seemed to sustain activity in domain specific networks, and may have been related to their goals of completing the study quickly. For Adam, Clayton and Harriet, metacognition activation was seen more frequently, seemed to support increasing activation in domain specific and contextual networks, was less-determinate, and may have been related to their goals of engaging deeply with the problems in the study.

George and Farley showed metacognitive activation in three and two questions, respectively, compared to Adam, Harriet and Clayton, who showed metacognitive activation in five, six and six questions, respectively. Within problems, Adam, Harriet and Clayton appeared to activate metacognitive networks more frequently, and for longer periods of time.

George's metacognitive activity seemed to function to manage his self-perceptions and to keep himself on-track. For example, at the start of both parallel lines questions, George reminded himself that the lines would have equal slopes, perhaps sustaining activity of networks relating slopes and parallel lines. In question F3, Farley's metacognitive networks appeared to activate two disparate networks simultaneously, to afford insight. In both cases, metacognitive activation appeared to sustain activation in active domain specific networks.

On the other hand, Adam, Harriet and Clayton's metacognition seemed to facilitate strategic shifts by increasing activation in domain specific networks which were not sufficiently active to support performance. For example, strategic shifts occurred for all three students as a result of metacognitive interaction with the work they had done on the paper. For example, as Adam began to trace the paper and ruminate in question C3 he seemed to have no strategy for showing how the lines were parallel. However, this metacognitive network activation appeared to facilitate interaction between the paper and his in-the-head networks, in turn integrating activation between visual spatial and algebraic networks for slope. This integrated network then appeared to facilitate an oral algebraic and visual spatial justification for his answer, suggesting that this metacognitive activation afforded a strategy shift that led to the correct answer. For Harriet, Clayton and Adam, network activity also appeared to support spreading activation from contextual to domain specific networks, to recruit needed domain specific networks. For example, both Harriet and Clayton used discussions with the researcher to activate networks for slope which they later used in both competent and flexible activity.

Variation in metacognitive activity may have also been related to the goals of students. George and Farley completed the study very quickly, and, within each question, seemed to be working as fast as possible. The researcher had the impression that George seemed eager to prove that he could do these problems very quickly and correctly, and that Farley was eager to do these problems quickly, and with a minimum of writing. In follow-up questions, neither took the opportunity (offered by the researcher) to obtain clarification about problems they had struggled with. On the other hand, Adam, Harriet and Clayton spent longer doing the study, and all three asked frequently about their answers, and were eager to understand the correct solutions; they also used network activation afforded by interactions with the researcher in future questions

(Adam excepted). In addition, Adam, Harriet, and Clayton seemed to enjoy doing the problems, and were more willing to try alternative approaches.

Adam, Harriet and Clayton also appeared to show less automatic activation of cognitive networks, and more flexible domain specific networks, relative to Farley and George. In each question, Farley and George seemed to pick definite strategies quickly, and tried to implement them sequentially. For example, once Farley selected an approach for F2, he seemed unable to adjust it, even though repeated interactions with the researcher may have suggested to him that this approach was not leading to the correct answer. When George worked on F2, he did metacognitively switch strategies – from guess and test, to creating equations, back to guess and test. However, in each case his shift was quick, and he gave up at the first sign of trouble. He seemed to expect that, if a strategy worked, it would work instantly, and as soon as he ran into trouble, he shifted again, back to his original, more linear strategy of guess and test.

In contrast, within the more metacognitively active students, strategies tended to be less determinate, and more exploratory – both Harriet and Adam reported trying things to see what would happen. These metacognitive activations could be considered exploratory in the sense that they did not seem to support a predetermined activation of domain specific networks. Instead, they appear to support the activation and recombination of domain specific and contextual networks in novel ways. For example, on F2, both Harriet and Adam used networks that detected analogical magnitude to activate domain specific networks for manipulating the differences between the costs of the two companies. They both reported being unsure of what this would produce, but they both expressed hope that it would produce good candidate values for guess and test. In Adam's case, it did, and he used this to solve the problem. In Harriet's case it did not, and she abandoned this approach in service of a more successful one. In both cases, metacognitive strategies supported activity that had no clear outcome, but that could affect contextual-individual transaction (i.e. the task conditions) in a way that would support further network activation.

The differences in metacognitive activity observed across students may be related to the relative activation of narrative and experiential networks. Previous research has suggested distinct roles for narrative and experiential metacognitive networks: narrative networks construct metacognitive explanations of behavior across time, while experiential networks function to monitor experience without verbal mediation. George's metacognitive activity seemed highly narrative, and neither he nor Farley seemed to have an experiential connection to the work. In addition, Farley may have shown a disassociation between narrative and experiential networks when he was unaware narratively that he was using the slope formula. For other students, metacognitive activity seemed to be characterized by sensitivity to the features of the problem, as they were unfolding before the participants. For example, Adam and Harriet corrected errors as they looked over their own work; they also tried strategies when they couldn't predict the outcome, and then used the results of that strategy to afford further action. It may be the case that narrative and experiential networks were more tightly integrated for these students, and that allowed narrative networks to elaborate and amplify experiential metacognitive activity; this integration may imply a qualitatively different sort of metacognitive network than students whose narrative and experiential networks are more separate. The validity of this conclusion is unclear, as it was challenging to differentiate narrative and experiential activity.

In sum, these results appear to indicate a difference in metacogntive network activity activation across students. Some students showed more metacognitive activation, and slower, less rigid activation of domain specific cognitive networks; these students' metacognitive activation seemed to function to afford strategy shifts, activation of cognitive networks, and error detection. Other students showed considerably lower levels of metacognitive activation; and rigid strategy choice and rigid activation of domain specific networks. These students' metacognitive activation seemed to function to afford sustaining cognitive networks, and seeking to maintain perceptions of self, and reinforce the goals of these students – to finish the study quickly and competently.

Fourth, *contextual* networks seemed to function to: 1) be a record of intermediate steps; 2) activate metacognitive networks that supported strategy shifts and error detection; 3) change domain specific or contextual network activity; and 4) facilitate task interpretation. These functions could be seen in student interactions with two main contextual networks: the paper, and the researcher.

In some cases, students recorded intermediate calculations on the paper. For example, George and Clayton both wrote down every intermediate calculation that they did during the problems. The function of recording calculations was unclear, but, for some students, it did not appear to support strategy shifts or error detection. Neither used their paper to afford error detection, and they did not seem to review what they had written. For George especially, there was no evidence that use of the paper led to any successful performance – competent or flexible. It's possible that use of the paper to write intermediate calculations is supported by a network which is specific to the domain 'doing math problems in an academic context.' This domain specific network may transfer the output of intermediate steps to paper, when paper is provided. However, this network may be unconnected to other cognitive and metacognitive networks, suggesting that, in some cases, recording intermediate calculations may not have an instrumental role in the way students solve problems.

In addition to recording calculations, for some students the paper appeared to support students in detecting errors, and promoting shifts in strategy, mediated by metacognitive network activity. Like George and Clayton, Harriet and Adam also wrote down a great deal of their work, and constructed visual spatial representations of several problems. However, comparison of their work with George and Clayton indicates a less linear, less narrative progression to their work. While George and Clayton's work followed step by step, and would make marking easy for most teachers, Harriet and Adam's work was disjointed, unclear and strewn across the page. This may suggest that, for Harriet and Adam, the paper serves as an explicit representation of internal mental activity, and a place to hold representations in a 'virtual working memory,' thus reducing the mental energy required to keep these representations active. Both Harriet and Adam appeared to follow examinations of these representations with metacognitive activity, and then strategy shifts or detection errors. Thus, for some students, the paper functioned to facilitate error detection and shifts in strategy, and these shifts were often mediated by metacognitive network activity.

Another contextual network students made use of was the researcher. Discussions with the researcher were used by some students to activate needed domain specific and contextual networks, and to aid in task interpretation. Adam requested both a calculator, and lined paper from the researcher, suggesting that he saw the researcher as able to activate other relevant contextual networks, that would in turn, afford successful performance. Clayton spoke frequently to the researcher to obtain additional information about relevant formulas, and his task interpretations, suggesting that, for Clayton, activity in contextual networks (i.e. things the researcher said) activated domain specific networks. Similarly, Harriet used discussion during the follow-up questions to activate domain specific networks she later used. Contextual networks may, for some students, function to affect the activity of other networks.

In sum, the results indicate that network activity could be detected, and that patterns could be drawn across students. Successful performance on some questions appeared to require well-developed networks for slope. All students showed evidence of such a network, but networks varied in the extent to which they supported algebraic, visual spatial, and formula based interaction with slope. All students appeared to activate 'real-life' networks for cell-phone pricing in C2. This appeared to interfere with performance in question F2, by blocking activation of networks for linear equations in all students except Harriet, who, in C2, appeared to have more formally mathematical networks active. In core mathematical networks, analogical detection of magnitude appeared to support Harriet and Adam's successful performance on F2; but, networks for exact calculation did not seem to impact performance for most students in most questions. Metacognitive activation appeared to function differently for different students. For some (George and Farley), metacognitive activation appeared to be less frequent, more determinate, and appeared to maintain network activation. For others (Adam, Clayton and Harriet), metacognitive activation was more frequent, less determinate, and appeared to activate networks that had not yet been used. Contextual networks appeared to support a variety of activities. The paper provided by the researcher served to record intermediate steps and representations. For some students (George and Clayton), this activity appeared unconnected to other network activity; for others (Adam and Harriet), it appeared to afford metacognitive activation, in turn, changing activation in other contextual or domain specific networks. Interactions with the researcher appeared to afford more robust task interpretation and affect activation in other domain specific or contextual networks.

Patterns in Flexibility and Competence

For this study, flexibility was defined as activity demonstrating adjustment to new or changing conditions when engaging with quantity, number or form. Competence was conceptualized as mathematical activity that does not contain adaptation or recombination of existing knowledge or attunement to the affordances of that situation.

The results of this study suggest that 1) particular questions afforded flexibility more than others, but that flexibility could be detected in all problems; 2) across students competence seemed to be inversely related to flexibility, and flexibility was exhibited by different students for different reasons, and 3) flexibility appeared to occur as metacognitive networks co-activated or integrated activity of two or more non-metacognitive networks, while competence appeared to occur as a sequence of network activations.

Patterns in Flexibility Across Questions.

The study protocol contained six problems for students to work on: three designed to afford competence (C1, C2, C3) and three designed to afford flexibility (F1, F2, F3). These problems were matched linguistically, and in terms of the domain specific linear equations networks thought to be required. In this section, I describe patterns of flexible activity observed within each question, and draw across these results to suggest patterns in flexibility, and their relationship to question conditions. Briefly, the results appear to indicate that, while question F3 afforded flexibility to most students, all questions afforded flexibility. In all cases, flexibility seemed to be generated by task conditions in which the state of the transaction did not appear (to the participant) to support successful performance. Students seemed to be adapting to different elements of the problems and flexibility did not tend to extend across an entire problem. Though notions of successful performance seemed to vary across students, in each case, once a student became aware that they needed help to achieve their goal, a variety of flexible behaviors manifested.

Question *C1 (Slope)* was, for most students, an opportunity to demonstrate competence working with slope. Most students did not demonstrate adjustment to any elements of the task, and further, follow up questions indicated that straightforward set of activities were employed competently to obtain the answer. However, without a way of working with slope, competence on this question was challenging, and the one student who had forgotten this formula exhibited flexibility in indirectly asking the researcher for the relevant formula. Thus, most students had the domain specific networks required for the question, and did not tend to exhibit flexibility on this question.

Question *F1 (Slope)* was, also, for most students, an opportunity to show competence. Nearly all students reported using the same competent routine as in question C1. However, Harriet demonstrated how this question can promote flexibility, if the relevant domain specific networks are not sufficiently active. Harriet's networks for slope did not include those for the slope formula – her only way of working with slope was to count the rise and the run. As a result, she adapted, showing flexibility, and chose specific values for a and b – though she was not sure that would work – and calculated slope, given those values. Though this didn't afford her a correct solution, it is a good fit for the notion of flexibility put forward in this study. Harriet, unsure of what to do, adapted to the question by trying a strategy she had seen work in other contexts – plugging in values.

Question *C2* (*Cell Phone)* was also considered by all students to be easy, and all students except Farley got the question correct. All students appeared to competently use domain specific knowledge for cell-phones competently in sequence, and the only errors were a result of miscalculation. Conversely, most students appeared to find question *F2 (Cell Phone)* very challenging, and all students demonstrated some flexibility during the problem. F2 was also the most difficult question in that all students reported finding it challenging (except Farley), and only Adam and Harriet solved it correctly. For nearly all students, even interpreting this question was challenging; the researcher, after seeing this with Clayton, made sure to ensure students understood what was being asked of them, but still, Farley and Clayton did not seem to understand what the question asked. This difficulty with the question appeared to afford a variety of examples of flexibility

Clayton demonstrated flexibility by looking back at his previous work to see if he could detect an affordance that could be used in the question. Though he did not, during that time he came up with a new strategy – using the T(i) formula from another unit. This strategy is related, but unlikely to help him solve the problem, and the researcher, in response to his asking, suggested it was not the intended solution, and Clayton abandoned it. Both Harriet and Adam detected that the relative ratios of the system access fees, and the per-minute costs might be related to the correct solution, and both tried to work with these values, flexibly, unsure of where it might lead. Though both solved the question correctly, only Adam was able to use this flexible move to afford correct performance; Harriet switched to a guess and test method.

George demonstrated flexibility in the question several times. He began by trying guess and test, which I coded as competent, as it seemed fluid and automatic. However, he could see that it wasn't working, and tried to construct equations that related minutes and cost for both companies. However, a misconception – that minutes and cost could be represented by the same variable – meant that he was unable to reason or work with these equations. He then expressed his frustration to the researcher, which I interpreted as seeking permission to stop the problem. When the researcher suggested he could stop when he liked, he returned to guess and test to get a more reasonable strategy. Given this motivational component to George's flexibility, it may be that George's flexibility activity in F2 supports movement through a variety of domain specific network activations, in an attempt to succeed at 'appearing to be reasonably competent on this problem' rather than 'finding the point at which Company A is cheaper than Company B.' This interpretation explains why some of his flexibility seemed to modulate emotional and task related activity, rather than the intellectual content of the problem.

Finally, Farley showed flexibility in choosing a strategy that minimized the work, in written form, and in terms of the mental work required to solve the problem. He used the calculator to guess and test, but seemed in general to want to solve the problem as quickly as possible, and with a minimum of effort. His ultimate answer, though technically a "numerical answer," as requested by the researcher, may not reflect the absolute function of his networks in general, but rather the activity of those networks given his motivation for the task.

Thus, question F2 afforded many opportunities to observe flexibility; this flexibility appeared to represent adaption to different elements of the problem. Adam, George and Clayton, appeared to adapt in an attempt to get the correct answers. George and Farley appeared to adapt emotional and task related activity, modulating effort and emotional challenge. For all students, this flexibility appeared when students realized they did not know what to do.

Question *C3* (*parallel lines)* was solved competently by George, Clayton and Farley, but appeared to afford flexible activity to both Harriet and Adam. George, Clayton and Farley all solved the problem easily with no notable adjustment to their activity. However, for Adam, flexibility was manifest as he started at the problem, tracing over the lines, and filling the dots. This silent thought was followed by an explanation of the problem that indicated he had related his visual spatial and algebraic understandings of the problem. This explanation seemed correct, and I coded Adam has having solved the problem correctly though he did not ‘prove’ his answer. Harriet got the wrong answer, after staring at the problem for a while. She, as well, had constructed visual spatial representations, and was looking at the relationships between these representations and the points given in the question. Her explanation was difficult to parse, and not correct, but it did indicate a flexible combination of those two networks. For both Harriet and Adam, flexibility seemed to be related to their sense that they could solve the problem, but that they did not know what steps were next.

Question *F3 (parallel lines)* afforded flexibility to Harriet and Farley. Harriet, in C3, had discussed the slope formula with the researcher. She had not yet used it explicitly in the study, but she made use of it immediately. Farley’s activity was hard to code. He reported cycling between the question and his evening plans. I coded this behavior as flexible because it seemed to be an adaption to his state of not knowing what to do, though it may not have been conscious. Farley did get the correct answer, and it may be that becoming “distracted” (as he suggested he does) is actually an adaption to not knowing what to do. Simply allowing himself to ruminate may have facilitated the network activation required to solve the problem. Farley later reported that he began this strategy because he had no idea what to do, suggesting that this flexibility was generated by task conditions where he was unsure what to do.

In sum, of the three problems designed to afford flexibility, only F2 appeared to reliably afford flexibility across students. Within this problem, what was being adapted appeared to vary across students. George and Farley appeared to be adjusting elements of the motivational context of the problem, while Harriet, Clayton and Adam appeared to adjust elements related to the problem solution. However, within F2, all flexibility appeared to flow from task conditions where students were unsure of the next step, or how to get a solution.

This result appears to hold across problems. Though most questions did not consistently afford flexibility, when task conditions suggested to a student that they did not know what to do, flexibility tended to appear. As a result, questions designed to afford competence sometimes afforded flexibility, because particular students did not have the network resources to deal with particular problems (for example, Harriet's lack of an explicit slope formula in C2). Appreciating these two findings together may imply that F2 tended to afford flexibility because it created task conditions where a student didn't know what to do. As a result, students attempted to promote activity in networks they thought might help (some contextual, some domain specific); to the extent that these networks did help (e.g. Adam, Harriet), this flexibility promoted successful performance.

Patterns of Flexibility and Competence Across Students.

The results suggest that students varied in how often they exhibited flexibility, and the purpose of their flexibility. Additionally, levels of flexibility appeared to be inversely related to levels of competence. Students with high levels of flexibility in the study (Adam and Harriet) showed flexibility that focused on obtaining a correct solution; students with lower levels of flexibility (George and Farley) showed high levels of competence. The more flexible students appeared to adapt their activity to support success on the problem, while less flexible students appeared to adapt their activity to modulate their internal emotional state.

George and Farley showed high levels of competence, and low levels of flexibility, employing domain specific networks quickly, and usually correctly. For George and Farley, their competence seemed to imply that flexibility was rarely required – and when it was, the adjustments appeared to affect emotional network activity, rather than the networks observed in this study. For example, George, in F2, reported being "irritated, " and Farley appeared frustrated both times he exhibited flexibility (minimizing work in F2, and dealing with the fact that "I don't get it, I don't get it" in F3). These transactions represent flexibility as defined – as an adjustment of activity when dealing with number or form – but the question of whether these constitute mathematical flexibility in the way that researchers and teachers understand it will be taken up in the discussion. Additionally, the lack of emotional networks in the protocol and theoretical approach used makes it difficult to fully interpret this flexibility, another point which will be taken up in the discussion.

Clayton appeared to show flexibility more often than Farley or George, and when he did (questions C1 and F2), it was centered around the use of contextual networks (the researcher, his previous work), to activate domain specific networks that would help him solve the problem (for example, his activation of the slope formula through discussion with the researcher in C1). Clayton also appeared to exhibit competence frequently. Thus Clayton appeared to be a student with a variety of competent strategies, who flexibly activated contextual networks, to afford him activation in domain specific networks that would, in turn, afford more competent activity.

Adam and Harriet both demonstrated flexibility frequently, and the adaptations shown appeared to be centered around affording success on the problem. For example, both played with the numbers given in F3 to try and generate a candidate answer they could guess and test. In question C3, Adam ruminated, merging domain specific networks for algebra and visual spatial networks, and obtaining a correct solution. Even when this flexibility did not afford success – for example Harriet's failed attempt to use values for a and b in question F1 – it was clear that the adaptive activity functioned to afford adjustment of the answer, rather than the emotional state of the student. Adam and Harriet did also demonstrate competence, but less frequently than other students.

Thus, the results appear to indicate that some students responded to the problems in the study with high levels of competence, and low levels of flexibility. When they did exhibit flexibility, it tended to be related to emotional network activity. Other students appeared to exhibit flexibility more often, and showed less competent activity. These students tended to be flexible in order to obtain the correct answer.

Patterns in Flexibility, Competence and Network Activation.

The results seem to indicate that across all examples of flexibility, metacognitive networks appear to merge the activity of at least two other networks: either a contextual and a domain specific network, or a combination of in-the-head networks. The network activity may be contrasted with competent activity, which appeared to be associated with a string of networks activating in sequence.

Metacognitive networks appeared to co-activate other networks in all examples of flexibility. In some cases flexibility occurred as metacognitive networks co-activated two in-the-head networks. For example, metacognitive activation appeared to merge the activity of visual spatial and algebraic networks for slope, as Adam flexibly constructed an oral explanation for F3. Similarly, Adam co-activated networks for exact calculation, guess and test, and analogical magnitude in his correct and flexible solution for F2. In other cases, metacognitive networks seemed to function to activate contextual networks that would in turn activate a domain specific network. For example, in C1, Clayton used metacognitive network activity to activate a contextual network (discussion with the researcher), which, in turn, activated his domain specific network for working with slope algebraically. Similarly, Harriet's success on F2 may be a result of contextual activity with the researcher during the previous problem's follow up questions. This contextual activity appears to have triggered metacognitive activity, which in turn activated domain specific networks for slope that included the slope formula.

In addition, flexibility generally involved the co-activation, or sequential activation of two or more networks, that were not yet fully integrated. For example, Adam's activity in C3 indicated rumination and co-activation of networks for algebraic and visual spatial conceptions of slope. His accurate oral answer drew on both networks; but the time spent ruminating may suggest that this co-activation was instrumental in affording this explanation. Similarly, Farley's co-activation of domain specific networks for work and slope of parallel lines in F3, appeared to afford him activation of a new domain specific network that integrated networks for solving algebra problems, and relating parallel lines and slopes.

These two patterns in flexibility- metacognitive networks bridging the activity of two other networks, and the emerging integration of not-yet-integrated networks – can be contrasted with patterns of activation in competent activity. Unlike flexible behavior, competent behavior seems to be associated with a string of metacognitively unconnected domain specific activity. For example, in C1, George used a contextual network (the paper) to record the result of a connected series of domain specific activities (writing slope formula, plugging points, calculating, simplifying) that afforded him the correct answer. In C2, almost all students activated a series of domain specific networks (for finding the variable component of the cost, the fixed component of the cost, and calculating them) that were applied rapidly and accurately. In F1, Adam used a series of domain specific activations to get the correct answer (annotating points, writing slope formula, inserting points, simplifying algebraic expression).

Though metacognition does not appear to link these network activations, metacognitive network activity may be part of the series of competent network activity. There were examples of metacognitive activity appearing during competent activity. For example, Harriet appeared to use metacognitive network activity to look back at her previous work in question F2. Unlike Clayton, who also looked back at his work, Harriet's metacognitive activity appeared to be automatic, and there was no evidence that her looking back represented an adjustment. Similarly, George appeared to use subvocalization in C3 and F3 (metacognitive network activation) to keep active the domain specific networks supporting his competence.

Thus, the results appear to indicate that, within this study, flexibility occurred when metacognitive network activity co-activated or integrated activity of two or more networks, while competence was characterized asa more rigid sequence of domain specific, contextual, and occasionally metacognitive, network activities. Flexibility did not appear to be accounted for by the presence of metacognitive activity. Instead, metacognitive activity needed to function to co-activate other networks, in order to afford flexibility.

CHAPTER FIVE

Discussion

This purpose of this study was to investigate the functional components of mathematical flexibility and contrast them to functional components of mathematical competence. Promoting flexible performance by students is considered an important aim of mathematics education, and understanding flexibility is a goal of education, psychology and neuroscience research. However, educators report challenges in affording their students mathematical flexibility, and current research paradigms have struggled to integrate the diverse strands of research that contain knowledge about flexibility. Detecting how particular functional components are instrumental to flexibility may serve to inform both the practice of instructors and the models researchers use to understand flexibility.

In order to investigate these questions, a three stage process occurred. In the first stage, I integrated across research programs in neuroscience, psychology, and education / social psychology to construct a network approach to conceptualizing flexibility that might afford investigation that integrates across these disciplines. In the second stage, I constructed a research protocol that might allow me to both answer the research question and evaluate the effectiveness of the protocol to answer questions about flexibility. Finally, after implementing this protocol, I engaged in an iterative process of results generation, where I drew on descriptions and traces of student activity to infer the activity of particular cognitive and contextual networks and relate that network activity to patterns in student flexibility.

In this chapter, I first discuss what those results may imply for methodological approaches for studying flexibility. Next, I examine what could be learned about mathematical flexibility from the results of this study. Third, I discuss limitations to the study, Finally, I suggest directions for future research that could complement or refine the results of the study.

Methodological Affordances and Challenges

The protocol for this study was designed to afford observation of network activation, the networks students bring to the study, and mathematical flexibility. In this section, I suggest ways in which this protocol facilitated these observations, and limitations to the methodology that impaired this observation.

Network Activation.

The study was designed to examine if relevant network activation implicated in problem-solving could be inferred by triangulating between observations of activities, and artifacts generated through activity. In general, it seems that the protocol and study as implemented did facilitate observation of activity in the networks hypothesized to be important to mathematical flexibility in linear equations. However, challenges emerged in detecting activity in networks for consciousness, in discriminating between experiential and narrative metacognitive activity, and in accounting for all activity that appeared essential to problem solving.

In some cases, I was able to infer network dynamics in a way that informed understanding of students' problem solving. For example, in question C3, Adam ruminated and stared at the problem for a long time, tracing over the line he had drawn and over the points on the line. From a network perspective, it appeared that Adam was co-activating visual-spatial and algebraic networks for slope, possibly triggered by metacognitive network activity. In his oral answer to the question, Adam appeared to draw on both visual-spatial and algebraic notions of slope in constructing his explanation. The dynamics of this activity may be usefully accounted for by a network theory, which would suggest that as activation spread from metacognitive nodes to domain specific nodes, integrated networks emerged that could support new activity. The period of rumination may also be explained by a network approach. According to the network theory I propose, spreading activation occurs across time and co-activation of this network activity must hit some critical activation level before it influences activity.

In addition, the results suggested that meaningful patterns in network activity could be drawn across students and problems, and that those patterns could be related to flexible mathematics. For example, metacognitive networks appeared to function in two distinct ways: to sustain domain specific network activation and to co-activate two or more networks. Results suggested that these patterns in network activation varied across students: two students' metacognitive activation appeared to function to sustain domain specific network activity, while three other students' metacognitive activity appeared to support co-activation of two or more networks. The students who tended to co-activate also appeared to be more frequently flexible, and flexible in the service of obtaining a correct answer. This may suggest that this pattern of metacognitive network activation is associated with successful problem solving, which in turn, may suggest that this protocol is able to discern patterns in network activity and relate them to flexible mathematics.

Although the methodological approach used here had promise, in that meaningful patterns in network activity could be detected and related to flexible mathematics, there were limitations to what could be observed. Networks of consciousness and narrative/experiential metacognition were challenging to observe. First, activity in networks for consciousness was not observed in this study. For this study, consciousness can be conceptualized as network activity that functions to transmit information from one network to another. It was hoped that answers to follow-up questions would indicate what students were conscious of as they solved problems. However, even though follow-up questions were designed to ask students to probe their experience of the question, it was difficult to interpret these as accurate descriptions of conscious activity. Instead, student responses seemed to be narratives that made sense of student activity retrospectively. There is some evidence (Dennett, 1991) that narrative descriptions of previous activity may not reflect actual conscious experience. Practically, I found this to be the case, as follow-up answers gave little insight into the ways in which networks were communicating. In fact, at key points where networks seemed to be co-activating and communicating – for example, Farley's co-activation of work and parallel lines networks in F3 – introspection seemed to produce little useful information about conscious activity. Although Farley knew he was thinking about these two ideas, he had no way of describing how this communication functioned to facilitate his correct solution. Thus, the study protocol did not afford detection of activity in networks for consciousness.

Narrative/experiential networks were also difficult to observe. Previous research suggests distinct roles for narrative and experiential metacognitive networks (Farb et al., 2007). Narrative networks construct metacognitive explanations of behavior across time, while experiential networks function to monitor experience without verbal mediation. This protocol had two features designed to afford discriminations between those networks. First, coding criteria differentiated between these networks, associating narrative network activation with students' using terminology that related the self across time (use of the pronoun "I" for example), and experiential networks with using terminology related to experience. Second, follow-up questions were designed to discriminate between these networks by first asking students to recount their experience (experiential) and then asking them what they would tell a friend about the problem (narrative). In practice, neither of these protocol features appeared to afford reliable discrimination between different forms of metacognitive network activity. It was difficult to apply coding criteria reliably, given that utterances from students included both descriptions of experience and notions of the self across time. The first follow up question elicited integrated responses that drew on both narrative and experiential factors, sometimes in highly combined ways. When students indicated what they would tell a friend, they generally restated their answer to the first question in more concise terms. Further, during observations of problem-solving performance, I found that much metacognitive activity was not accompanied by utterances at all. It was inferable from activity, but not from how the participants talked about their activity (rendering the coding criteria irrelevant).

The study did contain examples of students using narrative to subvocalize and keep themselves on track, or showing metacognitive activity without any trace of narrative activity. Further, when students subvocalized, using narrative networks, this tended to help them maintain current network activation. Experiential networks were anecdotally associated with insight and flexibility. However, these conclusions should be considered as very preliminary and robust patterns were not drawn for the above reasons. Thus, the experiential/narrative distinction may be important, but reliable detection of this activity may require more nuanced methodological tools.

Challenges in differentiating between narrative and experiential forms of metacognition might have been a function of the grain size of the study, which focused on coherent activity as the smallest unit of behavior. Research on metacognitive networks (Farb et al., 2007) suggests that the activity of these two types of networks is highly integrated in most people, and there is some evidence that mindfulness training is required to disassociate their activity. From this point of view, it makes sense that verbal and behavioral traces of activity are not fine grained enough to discern the activity of these two networks. Instead, measurement tools which can detect meaningful activity across very short time spans (e.g., 200 milliseconds), may be required to fully understand the interplay between these networks.

The above analysis suggests that the protocol used in this study was not able to reveal dynamics in some networks thought to be important to flexibility (i.e., consciousness; metacognitive). In addition, there was evidence that other networks instrumental to flexibility were not factored into the initial model, or systematically observed. For example, activation in networks for emotion was observed during the study. Both Farley and George appeared frustrated during the only times they exhibited flexibility and this frustration appeared to influence that flexibility. They appeared to adjust their activity to reduce their negative emotion, rather than to get the correct answer. Some other theories of adaptive behaviour (like self-regulation) have suggested that regulation of emotion is essential to effective academic work and to adaptability (Butler, Cartier, Schnellert, Gagnon, Higginson & Giammarino, 2005).The lack of affordance to systematically identify emotion from a network point of view may have limited the study's power. Emotional networks may play a powerful role in flexibility.

Similarly, some research has suggested that networks which maintain predictions of future activity affect how other networks activate (Hawkins, 2004), but these networks were not accounted for in the protocol. In the results above, students tended to exhibit flexibility when they didn't know what to do. However, detection of when they didn't know what to do was not accounted for in my description of network activation. This detection may be accounted for by networks that function to predict what should happen. Once other networks contradict that prediction, meta-networks may detect this conflict, and spur network activity to adjust. A possible instance of this occurred when Clayton realized that the midpoint formula, which he used in C1, would not afford him a correct answer. He noticed that the midpoint formula yielded two points, when he needed a single point. He then flexibly obtained the formula he needed. Though this explanation accounts for this activity from an activity point of view, it does not explain all of the network activity that afforded this flexibility, and in particular, how networks for predicting outcomes might have been implicated.

In sum, the protocol used in this study seemed to afford detection of important forms of network activity that could be related meaningfully to flexibility, suggesting the usefulness of this general approach. That said, the methodological tools used here were not effective in detecting networks for consciousness and experiential as opposed to narrative metacognition and failed to target some forms of network activity (i.e., in emotional and predictive networks) that might have been central to flexibility. Future research could extend the approaches used here to more sensitively tap into these networks and assess their role in problem-solving activity.

Networks Students Bring.

The network theory put forward in this study suggests that students bring network structures to activity (in the sense that a particular student's networks can function to afford particular activity), and that these structures are a result of previous activity. A semi-structured interview was included in the protocol, in order to facilitate triangulation of activity with the networks students appeared to bring to the study. Though the semi-structured interview provided a useful lens into the general approaches and self-perceptions students brought to the study, in general, it did not appear to contribute greatly to the results, or serve a particularly useful source of data for triangulating network activity.

Specifically, the SSI was designed to capture network structures brought to the study (metacognitive, domain specific, core mathematical), as well as students' perceptions of their strengths and weaknesses in mathematics. Data about students' narrative understandings of their strengths and weaknesses in math did prove useful in interpreting some of their activities, but the SSI did not appear to afford investigation of network structure.

The way that the SSI facilitated interpretation of student activity can be seen in Farley's and Harriet's responses. Farley exhibited strong negative emotion during the SSI when discussing "boring" math work. This was later useful in interpreting what flexibility appeared to afford for Farley (i.e., management of that negative emotion). Harriet, in the question about her strengths and weaknesses, reported an explicit preference for solving problems flexibly rather than competently. This finding helped me interpret Harriet's frequent exhibitions of flexibility as consistent with her preferred ways of working.

However, other questions designed to probe network structure did not give data that could be used with reliability to understand the nature of these networks. Further, tests of domain specific and core mathematical networks did not prove useful in uncovering the networks students brought to the study, perhaps because the questions were too easy. All students answered all domain specific networks accurately. Core mathematical tests, again, were too easy, and did not afford more meaningful triangulation of network activity that occurred during the study. As a result, most of the SSI was not used in inferring network activity for the study. However, an anecdotal experience with Harriet suggests that had the SSI been written with a focus on flexibility, investigating stated perceptions of flexibility, and perhaps students' propensity to find novel solutions to problems, more robust triangulation around student patterns of flexibility could have been made. Future research may employ redesigned SSIs which focus on student perceptions of flexibility, or from new ways of uncovering the networks students bring to the study.

More generally, a major limitation in this study was with challenges in determining what students bring in terms of networks. To the extent that future research can solve this challenge, more accurate and reliable triangulation of results can be expected. Thus, it may be important for future research to focus on ways of detecting what students bring in terms of networks.

Flexibility.

Flexibility was defined broadly in the coding criteria as an adaption of previously learned activity to the problem. Applying these criteria did appear to afford identification of flexibility, as defined, and in some cases, was in line with research and teacher perceptions of mathematical flexibility. Multiple sources of data – traces, student follow up questions, and activity reports – facilitated rich descriptions of student activity, and a large number of flexible activities were observed, across questions and students. Some of these examples appear to reflect flexibility as it is understood by researchers. Schoenfeld, for example, frames flexibility as about developing "thinkers with a broad repertoire of techniques and perspectives for learning to think mathematically, dealing with novel problems and situations" (Schoenfeld, 1992, pp. 4-5). Clayton's activity in F2 may be an example of this type of flexibility: he used a variety of techniques (guess and test, discussion with the researcher, looking back at his work) to parse a problem he did not understand. Similarly, Adam's activity in F2 showed flexible activity centered around playing with the numbers in the problem to afford a useful guess and test value. In both cases, participants appear to use a broad variety of techniques to handle a new, challenging situation.

However, the broadness of the flexibility definition may have also resulted in labeling activity as flexible in ways that were not consistent with what researchers and educators mean when they talk about flexibility. Researchers (e.g., Greeno 1991, Schoenfeld 1998) have spoken of flexibility in a way that foregrounds adaptive proficiency with the mathematical situation. Educators have focused on the need for mathematically capable citizens (Schoenfeld, 1992, pp. 4-5). Not all flexible activity in this study appeared to be related to proficiency with the situation, or to indicate a mathematical capable citizen. For example, in C2, Adam asked for a calculator. This activity was coded as flexible, as there was no calculator on the table, and this activity represented an adjustment of activity to Adam's desire to go faster. It is not clear if Adam's activity represented flexibility in the sense of contributing to his proficiency. In one sense, it did not – his activity did not really help him deal with the problem, which he could have solved without a calculator. In another sense, it did – adjusting the time a problem is going to take by using contextual affordances may be consistent with facility in dealing with novel problems or situations. So, although the broad definition of flexibility used in the study may have included some activity that is outside the scope of flexibility as understood by educators and researchers, other aspects of flexibility that were detected may have been consistent with flexibility as it is generally understood by educators and researchers.

This protocol may identify flexibility more effectively if future research contains a rigorous theoretical investigation and refinement of the constructs in this study. The constructs in this study are fuzzy, and detecting flexibility was challenging. However, the phenomenon of flexibility has been studied across a wide variety of paradigms. A deep theoretical investigation could serve to clarify the relevant controversies, and consolidate language around points of agreement; the more successful this is, the more that researchers will be able to 'speak the same language' across disciplines, and the more likely that studies of flexibility will identify flexibility in ways that make sense to researchers and teachers.

The grain size of the study may have also limited affordances to observe flexibility in student activity. Activity was coded as flexible when adaption to the problem occurred. However, the fundamental unit detected was coherent behavioral activity. Other research (e.g. Dehaene 2004, Dennett 1991) suggests that important mental events happen on the scale of milliseconds, not seconds, and their detection may require sensitive physiological measures that were missing from this protocol. As a result, it is possible that important adjustments occurred that could not be inferred from traces of activity, and therefore, the flexible activity identified should be considered a meaningful subset of the flexible activity which actually occurred (i.e., the "behaviorally observable" subset of activity). Future research may benefit from approaches which observe flexibility on other time scales.

What Was Learned About Mathematical Flexibility and Networks That Support It

This section highlights what was learned about mathematical flexibility and the networks that support it. First, I address how the results of this study support and extend existing understandings of mathematical flexibility. Second, I discuss the way that contextual/in-the-head interactions were observed in this study and how this may inform understanding of individual-environment transactions. Third, I contrast the functional components of mathematical flexibility with the components of mathematical competence, to provide a provisional answer to the research question that motivated this study.

The results of this study appear consistent with previous research on mathematical flexibility, which has suggested four key features of mathematical flexibility: 1) distinct modules in the brain contribute separately to flexibility (Case et al., 1996; Case & Okamoto, 1996; Dehaene 2004); 2) structures supporting flexibility may be combinations of older structures which integrate over time (Moss & Case, 1999); 3) different patterns of activity can lead to similar behavior (Moss and Case, 1999); and 4) a student must detect an affordance in order to exhibit flexibility – it is not enough simply to 'have' the needed structures (Greeno, 1991). The results of this study support these features, and can afford a deeper understanding of each of these features.

First, results indicate that particular networks were required, in particular contexts, to support flexibility. For example, domain specific modules for slope appeared central to flexibility in many of the problems posed here. For example, Adam, who appeared to have robust, integrated visual-spatial and algebraic networks for slope, was able to use this integrated network to solve C2. On the other hand, Clayton's networks for slope appeared more rigid and were less automatic (as they had to be primed by the researcher). This appeared to impair Clayton's ability to use these networks flexibly in F3. He failed to equate his expressions for slope, which would have afforded him a solution. This failure of flexibility may have been a result of slope networks that were less adaptable and robust than Adam's, whose slope networks appeared to be supportive of his flexibility.

Networks for modeling real life situations with formal linear equations may also have been essential to successful flexibility in question F2 (cell phone), while the strong activation of domain specific networks for real-life cell phone pricing in question C2 (cell phone) appeared to interfere with student performance on F2, by blocking activation of these networks for working with linear equations. For most students, activity in F2 did not include activation of networks for mathematical linear equations, and as a result, students were not able to flexibly activate those domain specific networks in this new context (it appeared no students had seen problems like this one previously). However, as discussed above, in C2, Harriet did appear to activate more formal mathematical networks, which appeared to contribute to her successful and flexible performance on F2. Thus, it may be the case that particular domain specific networks contributed to successful applications of flexibility in this question.

However, though domain specific networks for slope appeared essential to examples of flexibility that supported successful performance, in all cases of flexibility, metacognitive networks appeared to sequentially or co-activate two or more networks in novel ways. This mutual activation appeared to afford the adjustments that constitute flexibility, suggesting that this type of metacognitive activity – simultaneous co-activation of other networks – may be central to flexibility in all cases.

Thus, the results from this study support the finding that particular modules may be required to show flexibility in particular contexts. Activity in slope networks and linear equations networks may have been instrumental to some mathematical flexibility. Activity in metacogntive networks which served to co-activate or sequentially activate other networks appeared instrumental to all mathematical flexibility. This appears consistent with the findings of other researchers that particular networks may be central to flexibility in particular domains (e.g. Dehaene, 2004).

However, these results also indicate that domain specific networks that seem very similar may differ in subtle ways that affect flexibility. For example, domain specific networks for slope varied greatly across participants. For some, these networks supported algebraic activity with slope only (e.g., George). For others, these networks seemed to support visual-spatial activity (e.g., Harriet). In others, these networks seemed to integrate visual-spatial and algebraic activity (e.g., Adam). Each of these students exhibited different patterns of flexibility which appeared to relate to differences in those networks. For example, a lack of algebraic networks for Harriet seemed to force her to rely on visual-spatial representations in F1. As a result, she got a solution that worked for the particular visual-spatial representation she chose, but was not as general as a result that could have been obtained with an algebraic representation. This may suggest that, within a type of module, many variations of network structure can exist and these variations may have implications for performance. As a result, caution should be exercised in generalizing about how types of modules will affect performance; instead, differentiating within a type of module should be emphasized as a way of predicting differences in flexibility.

Second, the results are consistent with research by Case (1999) and others which indicates that complex modules may be created as modules connect and integrate over time. This can be seen by contrasting the activity of Harriet and Adam. Both Harriet and Adam showed flexibility throughout the study, and often used similar approaches to solve the problems (i.e., F1, C3). However, Adam was more able to solve the problems than Harriet. It is possible that these differences in performance could be explained by differences in the integration of their algebraic and visual-spatial networks. For example, in C3, both Harriet and Adam stared at their visual-spatial representations of the problem and then gave oral solutions. However, Adam's was accurate, while Harriet's was not. Harriet, who appeared to have a weaker integration with algebraic networks for slope, seemed to construct an algebraic explanation that was not coherent and drew false conclusions. On the other hand, Adam's behavior as he looked over the graph suggests that he was merging visual-spatial and algebraic interpretations – he circled and re-circled points and drew his hand up and down the line. His explanation was coherent, accurate and drew a true conclusion. Thus, results indicate that structures may need to integrate in particular ways to support flexible behavior.

Third, previous research on mathematical flexibility has indicated that similar activity can be generated by different internal mental activity (Moss and Case, 1999). The results of this study seem to support this finding. For example, all students successfully solved question C1. However, some students used a rigid sequence of algebraic domain specific network activities to solve the problem (George, Adam and Farley), another student invoked contextual networks (Clayton), and another used visual-spatial, domain-specific networks (Harriet). Similarly, both Harriet and Adam adapted flexibly to question F2 and got the same, correct answer. Both started the question by using analogical mathematical networks. However, Adam was able to follow that strategy to completion, while Harriet had to move to guess and test to find a solution, as that network did not support generation of useful candidate values. In this case, initial network activity and final answers were very similar, but intermediate mental activity was quite different.

This example also illustrates the converse to the above finding: that similar mental activity can lead to very different outcomes. Both Adam and Harriet began the problem with the same intuition and used very similar strategies. However, a small difference – that Adam worked with the differences in values and Harriet used the values themselves – ended up affording very different paths to the correct solution. This indicates that small differences in the structure of mental networks can lead to large differences in observable behaviour, a finding that has implications for flexibility, but that I have not seen mentioned in research.

Finally, previous research on mathematical flexibility indicates that, even when relevant networks are active, an affordance must be seized for flexibility to occur (Greeno, 1994).

George's performance in F3 is a clear example of that finding. Although he successfully constructed equations for both slopes, and activity in previous questions indicated strong networks for relating parallel lines and slopes and solving algebra problems, he was not able to solve the problem. This suggests that, for George, it was not enough to have the relevant networks. He also needed to see an affordance to combine those networks in a way that would support a correct answer.

In addition to supporting research findings related to the four key features of mathematical flexibility, the results of this study may be useful in resolving a tension in the research between approaches to understanding mathematical problem-solving that foreground contextual factors and approaches that foreground in-the-head networks. Results from this study suggested significant ways in which in-the-head and contextual networks interact. In several examples of flexibility observed in the study, metacognitive networks appeared to facilitate communication between contextual and in-the-head networks. For example, in C1, Clayton requested the slope formula from the researcher. Once he had it, the domain specific network for it was active (including how to use it) and he was successful in solving the problem. In question F2, George expressed frustration at not having the solution (i.e., interacting with a contextual network). The response of the researcher – that he could finish when he wanted – triggered metacognitive activity which functioned to select a strategy that would afford him a better, but not right, answer. This in turn, triggered domain specific networks for guess and test to reactivate. These findings suggest that these coordinations between in-the-head and contextual networks may be mediated by particular metacognitive networks that function to co-activate networks.

Further, these results suggest that flexibility can be usefully conceptualized as interactions between contextual networks and in-the-head networks, rather than having contextual networks interact with 'individuals' or in-the-head networks interacting with "the environment." For example, in Clayton's case, metacognition triggered contextual networks which triggered domain specific activity; in George's, contextual networks triggered metacognitive networks which triggered domain specific networks. Theoretically considering both individuals and environments to be composed of interconnecting networks affords a nuanced view of individual-environment transactions, where activity does not need to be mediated by the whole individual or whole environment. This affordance is a major feature of this model, as it privileges neither the cognitive nor situated perspective, but instead affords interactions between these points of view.

In addition to extending findings on flexibility, and affording nuanced views of contextual-individual interactions, the results of this study may offer a provisional answer to the original research question: *what functional components underlie flexibility, and how do those contrast with those that underlie competence?* The results suggest that metacognitive activity may serve to connect contextual networks with domain specific networks, or connect two or more domain specific networks, and that this combination of network activity is central to flexibility. At the same time, this type of metacognitive activity is not sufficient to affect flexibility; results suggest that domain specific and contextual networks must also be able to activate. In contrast, competence in this study seemed to be characterized by a sequence of domain specific and contextual network activity occurring without metacognitive co-activation. When metacognitive activity was associated with competence, it appeared to function to sustain activation of already-active networks.

In addition, the results seem to suggest an essential condition that must occur in the individual-contextual transaction before flexibility will occur. The student must recognize a need to adapt. Besides Harriet, all students seemed to default to competent approaches, invoking flexibility only when the transaction demanded it. This may have occurred when the sequence of domain specific network activations in the competent approach triggered a contextual or metacognitive network state that suggested a need for an adaptation. For example, George began F3 by using a competent strategy of guess and test. This strategy broke down, as George realized that finding the correct answer this way would take a long time. He detected this by seeing in his chart that he was far from a solution. This contextual activity triggered a metacognitive strategy shift, which, in turn, triggered domain specific networks for algebraic relationships. Thus, failed competence can trigger flexibility.

In addition, it appears that competence is not a prerequisite for flexibility, but may, in fact interfere with flexibility. Students who showed fluid and automatic competence (like George and Farley) also showed far fewer examples of flexibility, and that flexibility was less focused on correctly solving the problem. This suggests that students who depend on competence for success may struggle to be flexible. On the other hand, students like Adam and Harriet showed many examples of flexibility, but they seemed less competent in the sense that they were slower to consider problems, stated a preference for flexibility (Harriet), and successfully worked themselves out of tricky situations through adaptation. Thus, as the constructs of flexibility and competence develop, it will be important to consider the ways in which exhibiting one of these constructs may interfere with the other.

That said, it is possible that interference between competence and flexibility observed in this study was an artifact of the way they were defined. Flexibility requires adaptation;

competence requires lack of adaptation. Thus, that one appears to preclude the other may, at first, appear tautological. However, it is theoretically possible that students who apply procedures quickly, easily and automatically also would be first to find opportunities to be flexible. In this case their work could have been characterized by frequent rapid shifts between flexible and competent behaviour. On the other hand, students who showed low levels of flexibility could have, in theory, also shown incompetence, stopping questions immediately, or early during a question. Because the results appear to indicate a reciprocal relationship between competence and flexibility, and there are other relationships between competence and flexibility that are theoretically conceivable, this finding may be understood as a genuine finding, rather than an artifact of coding criteria.

Thus, these results suggest that flexibility and competence were significantly different in this study. Flexibility was characterized by metacognitive co-activation of two or more networks, while competence was characterized by more rigid sequences of network activity. Further, competence appeared to be a default mode, with flexibility having appeared only in response to challenge during the problem. These findings may have implications for theory, research and practice, and these implications are discussed in the following section.

Implications for Theory, Research, and Practice

The results of this study may have uses for theories of mathematical flexibility, by demonstrating the effectiveness of a functional approach that avoids foregrounding of situated or contextual factors in transactions. In the 1990s, a debate emerged in the research about the benefits of situated vs. cognitive perspectives in understanding human activity (see Greeno 2007 for a review of both sides of this debate). Both sides of this debate agree that both situated and personal factors impact activity; however they disagree in terms of which factors should be foregrounded, or seen as 'causing' the activity of the other factor. The results of this study suggest an integrative approach that emphasizes the function of different factors affecting activity, rather than their location in 3D space (i.e. inside a human body, or in a 'context'). For example, in this study, both George and Clayton solved question C1 using an identical behavioral approach – they activated the domain specific network for working with algebraic slopes, plugged in the values, and simplified. However, George activated the network for slope internally, after reading the problem and then metacognitively amplifying the activity of that domain specific network. Clayton, activated the network for slope by asking the researcher for it. From a network theory, these two activities are very similar. From this lens, both situated and cognitive findings can 'speak the same language' by referring to the function of particular networks, and attempting to differentiate networks by function rather than location.

This theoretical move has implications for research designs that afford investigation of individual-contextual transactions, including those investigating mathematical flexibility. Investigations that use a functional network theory to inform study design can frame individual and contextual networks in terms of what they do, rather than what and where they are. As a result, researchers can look at the situated/cognitive debate as an empirical one; there may be a difference in how situated and cognitive factors function, and this difference may be best explained by their location (inside a human body, or an environment); or, they may turn out to be genuinely interchangeable, with contextual and internal nodes functioning identically for different people. There may be particular network activity that must occur in a body, and particular network activity that must occur in a context; or both bodies and environments may be capable of supporting the same sorts of network activities. From a network perspective, a commitment is not required in the research design; this may represent a significant theoretical contribution.

This study also contributes to academic conceptions of mathematical flexibility by highlighting the ways in which flexibility emerges from interactions between in-the-head and environmental networks, supported by metacognitive co-activation of these networks. Further, the results may suggest that overlearned domain specific activity may interfere with flexibility. These findings have been highlighted in a variety of research on cognition but have been underemphasized in research on mathematical flexibility.

In addition, the results may help researchers generalize other research findings about flexibility, by showing them in a new population. The findings about mathematical flexibility upon which this study is based are largely based on studies of elementary students. These results support and extend that knowledge (as described above), but they do so in a population of 15-17 year old students. This may suggest that findings about flexibility in elementary students may hold as students get older. Future research may benefit from observing the relationship between network activity and flexibility in other contexts, including other grades, or other areas of math. For example, understanding flexibility in more heavily visual spatial areas of math (like geometry) could either validate the above findings in other contexts or provide useful nuance for educators and researchers.

The present study may also have implications for teachers, though its findings need to be replicated and extended for these implications to warrant action. However, results may suggest that typical instruction in mathematics may support competence at the cost of flexibility. Often students are shown a set of domain specific activities, and then, those networks are reinforced with a series of questions that incrementally use those networks in more sophisticated ways.

Because each question builds on the previous in clear ways, and instruction often includes explicit strategies for these questions, students are often given opportunities to exhibit competence, but not flexibility. The results of this study suggest that students must be genuinely confused as to what to do before they will exhibit flexibility. If instruction does not afford these opportunities, students will not practice being flexible. Since there was also some evidence that students differed in how often they exhibited flexibility, this lack of practice may be self-reinforcing, causing reliance on competent approaches, in turn further reducing practice with flexibility. Thus, there may be value for teachers in providing opportunities for students to be confused or unsure of what to do, and to allow those to resolve. At the same time, these results should not be taken to suggest that domain specific instruction has no value; without the relevant domain specific networks to draw on, there are no mental resources to use flexibly, and this was also evident in the results . There were students who used the metacognitive co-activation that appears to be a hallmark of flexibility, but because they could not employ the correct domain specific network, this flexibility did not appear to contribute to performance. But there may be value in supplementing direct instruction in domain specific activity with ambiguous, multi-network problems.

In addition, this study may have implications for teachers who believe that most students may not be capable of showing flexibility, or thinking flexibly about math. This belief may come from a conception of flexibility as requiring competence as a precondition. However, these results indicate that flexibility is afforded when the individual-contextual transaction requires adaptation, and that students both successful and not, will show flexibility. As a result, these students' lack of flexibility may not be a function of inherent 'inflexibility', but rather, a lack of relevant contextual and domain specific networks to draw on when exhibiting flexibility.

Ironically, struggling students may receive more practice in flexibility, since they are generally less able to access relevant domain specific networks, and may need training in how to be more competent in order to be more successfully flexible.

Conversely, it may be challenging to teach flexibility to highly competent learners. George may have been an example of such a learner. Exhibiting flexibility was very frustrating for him, and he was eager to end the state of confusion that motivated his flexibility. Teaching George to tolerate that experience, after years of developing competence, may be challenging. Thus, competent students may need both emotional support in tolerating the conditions that give rise to flexibility, and opportunities to do math where competent routines will not support successful performance.

Limitations

The way in flexibility was conceptualized and observed in this study may have limited the results. Flexibility was assumed to be operating on the level of the question, with particular questions in the protocol designed to afford flexibility or competence. In practice, flexibility was often fleeting, manifesting for a moment before the student would return to more rigid sequences of domain specific activity. This suggests that flexibility should be a construct that operates on the grain size of a coherent activity, rather than a question or task. Future research may benefit from protocols which take this into account, and include questions designed to afford both flexibility and competence.

Flexibility was also assumed to be built on competence, and this may have impacted the sample selection in a way that may have limited observation of different types of flexibility. Some of the most insightful opportunities to observe flexibility (e.g. Clayton, C1; Harriet, F1) occurred when students appeared not to exhibit competence. However, because I assumed students must be competent to be flexible, participants were required to have grade of 73% or higher in Math 10. Had the inverse relationship between flexibility and competence been apparent, the selection pool might have been opened up to allow observation of flexibility in less competent students, which might have afforded different observations about how flexibility works.

The choice of which networks to observe also limited how the protocol accounted for flexible behaviour. For example, the lack of a role for emotional networks in the model impacted the ability of the protocol to account for some flexible behaviour. A construct of 'negative emotion' was brought in during analysis to help make sense of George and Farley's flexibility, but it is likely that including emotional networks as part of literature review and theoretical approach could have afforded more robust understanding of emotional networks. For example, it islikely that measuring anxiety from galvanic skin response, facial expressions, and emotion self-reports could afford rich triangulation of emotional network activity, which could be related to flexibility.

The conceptualization of flexibility may have been limited by an overlap with the conceptualization of metacognitive network activity. Flexibility, as defined above, refers to adaptation when dealing with number and form; and, networks for metacognitive activity may include those for self-regulatory procedures, including on-line decision making and strategy shifting. If metacognitive activity includes on-line decision making, and flexibility requires on-line adaptation, it may be that these two constructs measure aspects of the same underlying phenomena. This possibility seems supported by the fact that these constructs were used as labels for specific student activity, and metacognitive and flexible labels were frequently applied to the same activities.

Though flexibility and metacognitive network activity may correlate, they may foreground distinct aspects of the student-environment transaction. Though metacognitive activation occurred in all flexible transactions, so did domain specific network activation, suggesting that metacognition, by itself, may not 'be' or 'explain' flexibility. Additionally, metacognitive activation was not sufficient to predict flexibility – there were several cases of metacognitive activation supporting competent activity. This suggests that metacognitive activity and flexibility may be usefully differentiated: metacognition refers to activity which functions to state belief, affect, declarative knowledge of cognitive strategy, or facilitate on-line decision making; flexibility refers to the set of activities (usually accompanied by metacognitive network activity) that can be coherently understood as representing the 'adaptation' to the mathematical situation. That said, the overlap between these conceptualizations may limit the validity of the results, and future research may benefit from more carefully differentiating these constructs.

In addition to issues with conceptualization, the procedure used had intrinsic limitations that may have affected the interpretation of these results. For example, the study was preliminary and very small. The sample used in the problem consisted of high SES students, three of whom attended expensive private schools. The problems were consistent with British Columbia curricula. Thus, it may be the case that the above conclusions do not apply across SES or cross-culturally, or outside the small group of students who were observed. The conclusions above, therefore, are tentative, and require triangulation with other studies in the same paradigm, and other paradigms. Conclusions should be interpreted cautiously, and seen primarily as contributions to the methodological puzzle of how to study flexibility. Future research may benefit from extending this protocol to other students, and in other contexts.

In addition, the grain size in the study was ‘coherent units of activity.’ Flexibility and competence which occurred on scales smaller then a unit of activity were not observed; but, this does not mean they do not exist. As a result, the conclusions about flexibility found above should be limited to transactions that are understood on the level of the activity. Studies that investigate micro-activity, on the millisecond scale, or activity over the life of an organism may find different patterns in flexibility.

Finally, the specific problems that students worked on may have afforded only a subset of possible flexible behaviors. It is conceivable that other types of questions afford flexibility differently, and may not have the same properties (for example, metacognition may not feature in all forms of flexibility). Thus, conclusions should be limited to types of problems that students faced in this study, and future research may benefit from inclusion of a wider variety of problems within protocols designed to study mathematical flexibility.

Directions for Research

In addition to the suggestions already made in relation to future directions for research, intersections with brain research, intervention studies, and theories of human adaption like self-regulated learning may afford richer understandings of flexibility and its relationship to network activation.

The power of brain research to afford understandings of network activity has not yet been fully realized, and may be a useful direction for research on mathematical flexibility. For example, EEG studies have the potential to provide a fine grained record of mental activity that can be triangulated with behavioral data to increase both reliability and validity of conclusions drawn from that data. EEG allows temporal precision of brain activity to the 100 millisecond;

and approximate locational precision. Had EEG been used in this study, the question of narrative vs. experiential metacognition might have been addressed in a more satisfying way. Farb et al (2007) have established the location of these networks, and their activity could have been triangulated with exhibited flexibility and domain specific activity.

In addition to brain research, this study would be complemented by intervention studies that attempt to increase the extent and effectiveness of student flexibility. These studies could be performed by action researchers –teachers – who work collaboratively in groups to design and test interventions that increase flexibility. The results of these studies could then be triangulated with behavioral and brain studies to provide a robust multi-diciplinary view of mathematical flexibility with implications for researchers and practitioners.

In addition, an important goal for future research may be to triangulate findings from a network theory of mathematical flexibility with findings from other research and theoretical approaches that try to make sense of human adaptive behaviour. Outside of mathematical flexibility, other research has provided complex and useful accounts of adaptive behaviour. For example, research on self-regulated learning (SRL) (e.g., Butler et al., 2005) can inform understanding of how layers of context can interact with personal factors, and cognitive and metacognitive strategies to produce recursive cycles of 'strategic' behaviour. Findings from these approaches may inform design of studies of mathematical flexibility

Additionally, studies in other diciplines like SRL may benefit from insight gained from studies of flexibility. For example, the results of this study may inform the relationship between what, in self-regulated terms, might be called 'cognitive strategies.' Some theories of self-regulation suggest that student actions are supported by 'cognitive strategies' for doing the work.

The results of this study suggest ways that an activation of a particular cognitive strategy (e.g. using a real-life model for cell-phone prices), can interfere with activation of another cognitive strategy (e.g. using a formal linear equation to represent cell-phone prices). This finding may inform inferences that can be made when studying adaptive behaviour from a self-regulated perspective.

As a network theory of flexibility is integrated with other research paradigms, it is possible that cross-disciplinary insights will occur. Brain research can afford observation of network activity on different temporal scales. Intervention research may surface the classroom implications of flexibility in a way that informs a network theory. And other theories of adaptive behaviour can inform, and be informed by, studies of flexibility that foreground the role of network activation.

Closing and Conclusion

The results of this study suggest that network approaches may have a key role to play in understanding mathematical flexibility, as the contributions they afford seem to be applicable across a wide variety of research programs, and can serve to integrate the findings of programs that do not easily speak the same language. The relationship between functional components of individual-contextual transactions and flexibility has implications for teachers, who are increasingly asked to help students become flexible, and for researchers, who see flexibility as important to understanding cognition in mathematics. This study employed a network approach to understanding flexibility, and results indicate that this type of approach may have been an effective way to understand functional components of flexibility. These results are based on a protocol that drew across research programs in a variety of disciplines; the findings, too, appear to inform a wide variety of research programs. Thus, it appears that a network approach may facilitate deep understandings of mathematical flexibility that can be useful across a wide variety of disciplines, and for a wide variety of purposes.

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Appendices

Appendix A – Semi-Structured Interview

Note: This protocol was used in "Exploring Flexible Mathematics in Linear Equations" to uncover how mental networks supporting mathematical flexibility are structured within that particular participant.

Protocol notes: Students answered questions orally, in person. The format was semi-structured. While I will cover the topics identified here, the exact wording may have changed, and follow-up questions might have been asked, depending on responses from the student. Students were informed that they may choose not to answer any question, at any time.

Protocol

1. I thanked the students for volunteering to participate.
2. I provided some context about the study and purpose for the session, such as "In this research study I'm interested in learning more about how students think when they are solving math problems. So, in our conversation here, I'm just going to ask you a little bit about your experience with mathematics and ask you to solve a couple of math problems".

Demographics.

3. Name:
4. School:
5. Grade:
6. First term math mark
7. Current math mark

8. Can you tell me a little bit about your history with mathematics? What are your strengths in mathematics? What are some of the challenges you've had? Is there anything distinct or different about the way you learn? How do you know?

Metacognition - Experience of Math.

9. Can you describe as vividly as possible what it's like for you to do math homework? Tell me about where you work, how you're working and how you ¬feel when working.
10. Can you describe as vividly as possible what it's like for you to be in math class? Tell me what the class is like, how you feel, and what you're thinking about during class.
11. Can you describe as vividly as possible what it's like for you to take a math test? How do you usually feel during the test?
    a. [Optional probes "Can you tell me more about that? Can you give me a specific example of that? Is that always the case, or are there cases when that hasn't applied"]

Metacognition - Understandings of Self In Math.

12. Tell me about your strengths and weaknesses in mathematics.
13. Tell me the history of math in your life. Tell me about your best years and years that were more of a challenge. What has been easy, what has been hard? Start from when you were young, and build to today.
14. What do you tell yourself when you're thinking about doing your math homework?
15. What changes should be made to how math is taught?
16. What do your friends think about math? Do you agree?
17. What strategies do you use when you run into a problem with your homework?

Metacognition – Beliefs and Emotions.

Rate from ‘not true’ to ‘very true’ (4 point scale)

18. The reason I try to learn mathematics is…
19. To help me think more clearly in general.
20. It’s required for my program.
21. I want to do well in the course.
22. It’s interesting.
23. I’ll get in trouble if I don’t.
24. I feel stupid if I can’t understand something.
25. I don’t want to look dumb.
26. To make the teacher think I’m a good student.
27. When you get the wrong answer to a math problem
28. It is absolutely wrong – there’s no room for argument.
29. You only find out when it’s different from the book’s answer or when the teacher tells you.
30. You have to start all over in order to do it right.

a. General Perceptions.

31. Some people are good at math and some just aren’t.
32. In mathematics something is either right or it’s wrong.
33. Good mathematics teachers show students lots of different ways to look at the same question.
34. Good math teachers show you the exact way to answer the math questions you’ll be tested on.

35. Everything important about mathematics is already known by mathematicians.

36. In mathematics you can be creative and discover things by yourself.

37. Math problems can be done correctly in only one way.

Required Mathematical Skills.

Fractions.

38. Represent 3/8ths as a fraction

39. Add 3 ½ and (-5/7)

40. Divide ½ by (-2/3)

Algebra.

41. 4(x-2)=3x-3

42. 7/x=4

43. 6(x-3)=3x+3(x-6)

44. 5(x+3)-3(2x-1)=2x+10

Coordinates.

45. Graph the line joining (1/2, 2 1/2) and (-.2, 5)

46. Where on the coordinate plane do you find points with negative x's and negative y's

Core Mathematical Networks.

Verbal.

47. Answer 20 multiplication facts as quickly as you can (presented in verbal form)

Magnitude.

48. Estimate the answers to 20 addition, subtraction, multiplication and division questions with non-typical answers (i.e. not standard multiplication facts or addition subtraction with answers less than 10).

Arabic.

49. Answer twenty, two-step calculation questions as fast as possible (presented in verbal form).

50. Answer twenty, two-step calculation questions as fast as possible (presented in Arabic form).

Appendix B – Follow Up Questions

Note: This protocol was used in "Exploring Flexible Mathematics in Linear Equations." These questions allowed participants to indicate what they were thinking and doing as they solved linear equations problems.

Protocol notes: These questions were asked orally, in sequence, after participants had completed each 'Linear Equation Problem." The format was semi-structured. While I will cover the topics identified here, the exact wording may have changed, and follow-up questions may have been asked, depending on responses from the student.

Protocol

1. Can you take me through how you solved this problem? Give me as much detail about what you were thinking, feeling, and what you were trying to do at each stage of the problem
2. If a friend were about to attempt this problem, what should they know?
3. Do you have any questions about this problem? I am happy to answer any questions you have, if you'd like to clarify anything, or see where you went wrong [if their performance was not successful]."

Appendix C – Ethics Approval Certificate

The University of British Columbia
Office of Research Services
Behavioural Research Ethics Board
Suite 102, 6190 Agronomy Road,
Vancouver, B.C. V6T 1Z3

CERTIFICATE OF APPROVAL - FULL BOARD

PRINCIPAL INVESTIGATOR:	INSTITUTION / DEPARTMENT:	UBC BREB NUMBER:
Deborah L. Butler	UBC/Education/Educational & Counselling Psychology, and Special Education	H10-00808

INSTITUTION(S) WHERE RESEARCH WILL BE CARRIED OUT:

Institution	Site
UBC	Vancouver (excludes UBC Hospital)

Other locations where the research will be conducted:
The research will be conducted at UBC. But recruitment will be supported by asking contacts to make informational posters available. Permission will be requested from any associated organization (see description of recruitment strategies).

CO-INVESTIGATOR(S):
N/A

SPONSORING AGENCIES:
N/A

PROJECT TITLE:
Exploring Flexible Mathematics in Linear Equations

REB MEETING DATE:	CERTIFICATE EXPIRY DATE:
April 22, 2010	**April 22, 2011**

DOCUMENTS INCLUDED IN THIS APPROVAL:

DATE APPROVED:
May 25, 2010

Document Name	Version	Date
Protocol:		
Tables - Butler and Giammarino	N/A	May 7, 2010
Proposal - Butler and Giammarino	N/A	May 5, 2010
Consent Forms:		
Parent Consent Form	N/A	May 4, 2010
Assent Forms:		
Student Assent	N/A	May 4, 2010
Advertisements:		
Paper Recruiting	N/A	May 4, 2010
Internet Recruiting	N/A	May 4, 2010
Questionnaire, Questionnaire Cover Letter, Tests:		
Linear Equations Problems	N/A	April 6, 2010
Follow Up Questions	N/A	April 6, 2010
Semi-Structured Interview	N/A	May 4, 2010

The application for ethical review and the document(s) listed above have been reviewed and the procedures were found to be acceptable on ethical grounds for research involving human subjects.

Approval is issued on behalf of the Behavioural Research Ethics Board
and signed electronically by one of the following:

Dr. M. Judith Lynam, Chair
Dr. Ken Craig, Chair
Dr. Jim Rupert, Associate Chair
Dr. Laurie Ford, Associate Chair
Dr. Anita Ho, Associate Chair

Appendix D – Tables and Figures

Table A1

Four Possible Aspects of Mathematical Flexibility

Aspect of Flexibility	Sources of evidence
There are distinct modules in the brain which support mathematical flexibility. Each module is responsible for a particular functional component of mathematical flexibility.	Robbie Case's developmental model of central conceptual structures (Case et al., 1996); Stanlais Dehaene's (2004) neuropsychological 'number sense' research.
New functional components are built as these modules integrate.	Moss and Case's (1999) developmental model of 'rational number sense'; Case et al.'a (1996) research on 'whole number sense'; Dehaene's (2004) triple code model of 'number sense.'
There is wide variation in the way that these modules connect and integrate.	Examples of student reasoning from Moss and Case's (1999) rational number sense work.
In any particular context, flexibility requires that a person must recognize and utilize an opportunity use these functional components.	Greeno's (1991) situated account of number sense; Gibson's (cited in Greeno 1991) theory of affordances.

Table A2

Features of two distinct but related approaches to understanding mathematical flexibility

	Identifying Structures	Tracing Relationships
Exemplars	Case et al. (1996); Dehaene (2004)	Greeno (1991, 2007)
Function	To identify in-the-head structures instrumental to mathematical cognition and flexibility.	To relate individual-context transactions with changes how students act in mathematical situations.
Tools	Syntheses of studies to produce hypotheses about structures relevant to particular activity; cross-paradigm programs of research to evaluate these hypotheses	Analysis of discourse and traces to posit relationships between individual and contextual structures
Key Benefits	Specification of functional structures instrumental to mathematical flexibility; hypothesis generation and testing.	Captures dynamic and rich interplay of personal and contextual factors in mathematical flexibility; sensitive to unhypothesized features of flexibility.
Potential Limitations	May underemphasize the role of the environment in affecting cognition.	May oversimplify the 'individual,' and underemphasize the underlying machinery of which people are composed.

Table A3 – Part 1
Types of Networks Potentially Instrumental to Mathematical Flexibility

	Function	Evidence That Suggests the Network Exists	Measurement Strategies to Understand What Students Bring to the Proposed Study	Measurement Strategies to Detect Network Activation During the Study
Domain-Specific Networks for Linear Equations	To do the cognitive functions associated with linear equations.	Theoretical claim that suggest 'hubs' exist in networks to support domain-specific activity; output of curriculum design process specifies distinct functional capabilities required for linear equations.	Ask students to demonstrate functional components of linear equations suggested by curriculum documents.	Observe traces of work for evidence of particular cognitive functions.
Domain-Specific Networks for other required Mathematical Competencies	To perform domain-specific mathematical activity required in linear equations, but not specific to the linear equations domain.	Personal experience; competence with algebra, coordinate systems, and fractions is required for competence with linear equation.	Ask students to demonstrate functional components of algebra, coordinate systems and fractions suggested by curriculum documents.	Observe traces of work for evidence of particular cognitive functions.
Narrative-Metacognitive Networks / Networks for belief, affect, and declarative knowledge of cognitive capacity	To generate narrative knowledge about the self, declarative affect, and declarative knowledge.	fMRI evidence from Farb et al.; Schoenfeld's taxonomy of metacognition includes declarative self-knowledge.	Ask students questions from Schoenfeld's beliefs about mathematics survey; ask open-ended questions.	Use Schoenfeld's graphic method of tracing dynamics of activity; examine traces and audio/video recordings for reference to the self.

Table A3 – Part 2
Networks potentially instrumental to mathematical flexibility

	Function	Evidence That Suggests the Network Exists	Measurement Strategies to Understand What Students Bring to the Proposed Study	Measurement Strategies to Detect Network Activation During the Study
Experiential-Metacognitive Networks / Networks for monitoring of on-line activity	Modeling temporally proximal sensory objects, canvassing thoughts, feelings and physical sensations without selecting any one sensory object	fMRI evidence from Farb et al.; Schoenfeld's taxonomy of metacognition includes on-line decision making; Christoff e. al.'s emphasis on defocused attention as fundamental to creativity	Open-ended questions	Use Schoenfeld's graphic method of tracing dynamics of activity; examine traces of work and audio/video recordings for reference to experience.
Core Mathematical Networks	To detect magnitude, and interact with number in Arabic and verbal forms.	Dehaene's triple-code theory of Number Sense	Ask students to perform a set of estimation, calculation, and multiplication tasks	Examine traces, utterances, and videos for examples of estimation, calculation or multiplication.
Contextual Networks	To activate, inhibit, and change in-the-head networks through the sensory organs .	Theory of affordances; situated cognition research; Greeno's environmental metaphor for number sense.	N/A	Trace interactions between specific aspects of context (specific problems, interactions with researcher) and activity of in-the-head structures.
Consciousness Networks	To make the outputs of other networks available across the cortex.	Dennett and Dehaene's respective theories of consciousness.	Open-ended questions about math and metacognition	Questions which ask students to reflect on their thinking and activity; opportunities to ask the researcher

Table A4
Affordances for Observation During the Proposed Study

Network	Affordances for observation	Specific Item Numbers
Domain-Specific Networks for Linear Equations	Mini quiz in SSI; traces and utterances during activity; utterances and behavior during follow up questions	SSI(5) C(1-3); F (1-3) Follow (1-3)
Domain-Specific Networks for other required Mathematical Competencies	Mini quiz in SSI; traces and utterances during activity; utterances	SSI (6)
Narrative-Metacognitive Networks / Networks for belief, affect, and declarative knowledge of cognitive capacity	Questions in SSI; utterances and strategies while working on tasks; utterances and behavior during follow up questions	SSI(2,4) C (1-3); F(1-3) Follow (1- 3)
Experiential-Metacognitive Networks / Networks for monitoring of on-line activity	Questions in SSI; utterances and strategies while working on tasks; utterances and behavior during follow up questions	SSI(3) C (1-3); F(1-3) Follow (1- 3)
Core Mathematical Networks	Questions in SSI; fluidity in calculation and numerical intuition while working on tasks.	SSI(7) C(1-3); F(1-3)
Contextual Networks	Traces and utterances during activity; utterances and behavior during follow up questions; analysis of activity dynamics	C (1-3); F(1-3) Follow (3)
Networks for Consciousness	Utterances throughout the study indicate what a student is aware of at that time.	All items

Table A5

Questions Designed to Afford Flexibility or Competence

Affords Competence	Affords Flexibility
C1 Find the slope of the line joining (2,3) and (10,9)	F1 Find the slope of the line joining (a,b) and (3a,2b)
C2 A friend is trying to figure out what cellphone plan to purchase. Company A offers a monthly system access fee of $12.50, and charges $.20/minute; Company B offers a monthly fee of $10, and charges $.22/minute. How much is 30 minutes a month if you use Company A? Company B?	F2 A friend is trying to figure out what cellphone plan to purchase. Company A offers a monthly system access fee of $12.50, and charges $.20/minute; Company B offers a monthly fee of $10, and charges $.22/minute. Your friend asks you "When is Company A cheaper than Company B?"
C3 A line passes through (2,4) and (5,10). Another line passes through (3,5) and (4,9). Are the lines parallel? Can you prove it?	F3 A line passes through (2,4) and (5,10). Another line passes through (a,5) and (4,b). Give possible values for 'a' and 'b' if the lines never touch.

Table A6
Exploratory Criteria for Detecting Network Activation

Network	Detecting Activation
Domain-Specific Networks for Linear Equations	Traces of linear equations 'operations,' ; utterances in follow up questions referring to particular linear equations processes used;
Domain-Specific Networks for other required Mathematical Competencies	Traces of algebra, fractions or coordinate use.; utterances in follow up questions referring to particular processes used;
Narrative-Metacognitive Networks / Networks for belief, affect, and declarative knowledge of cognitive capacity	Utterances relating to belief, affect or cognitive capacity; utterances with the participant as subject;
Experiential-Metacognitive Networks / Networks for monitoring of on-line activity	Changes in activity or strategy during problem solving, utterances or videos representing changes in emotional state, notable changes in behavioral affect
Core Mathematical Networks	Participants calculate or interact with number; participant descriptions of calculation during problem solving.
Contextual Networks	Interaction with contextual features (paper, calculator); interactions with researcher
Networks for Consciousness	Student descriptions of the content of their conscious experience during follow up questions.

Table A7

Competent and Flexible Behaviours Exhibited by Clayton For Each Question

	C1 – Slope Competent	F1 – Slope Flexible	C2 – Cell Phone Competent
Competent Behaviours	Writing mid-point formula (2) Creating x/y axis (3) Writing slope formula (6) Clarifying what the question is asking for (1)	Use of slope formula (1-5) Simplification of algebraic expression (6) Algebraic expressions (1,2)	Parsing question text (2) Creation of expressions for companies a and b (5-11) Calculation of company costs (12,13)
Flexible Behaviours	Crossing out midpoint formula and indirectly asking for the slope formula (2,3) Clayton realized that the midpoint formula returned two values, but that he needed a single value. He then realized that the researcher would know best what to do, and asked "So, if I can't remember the slope?"		

Table A7 – Part 2

Competent and Flexible Behaviours Exhibited by Clayton For Each Question

	F2 – Cell Phone Flexible	C3 – Parallel Lines Competent	F3 – Parallel Lines Competent
Competent Behaviours		Use of slope formula for each line (3,4) Calculation of slope for each line (5,6)	Annotating points and drawing axis (3,4) Writing slope formula (6) Creating expressions for slope (7,8)
Flexible Behaviours	Linking question to T(i) formula. (4) Unsure of how to answer the question, Clayton searched in his head for other related tools that might be of use. Looking back at previous question (6) Unsure of how to answer the question, Clayton looked through his previous work for something that might trigger knowledge of an affordance		

Table A8

Competent and Flexible Behaviours Exhibited by Adam For Each Question

	C1 – Slope Competent	F1 – Slope Flexible	C2 – Cell Phone Competent
Competent Behaviours	Annotating slope points (1) Putting points into equation (2) Calculating slope (4)	Annotating slope points (1) Use of slope formula (2-3) Simplification of algebraic expression (4)	Creation of expressions for companies a and b (2,4,) Calculation of company costs (3,6,7)
Flexible Behaviours	Correcting annotation (3) Once Adam looked at the points he had put into the slope equation, he noticed that they were incorrect, and chose to adapt them. This 'looking over' step suggests flexibility; contextual features triggered and adaptation so that the variables would be more flexible.		Asking if he can use a calculator (5) After looking at his equation for Company B, he realized that the question would be easier if he could use a calculator. The researcher had not explicitly offered one, but Adam seized this opportunity, and asked the researcher for a calculator.

Table A8 – Part 2

Competent and Flexible Behaviours Exhibited by Adam For Each Question

	F2 – Cell Phone Flexible	C3 – Parallel Lines Competent	F3 – Parallel Lines Competent
Competent Behaviours	Verifying that cost is equal for both companies at 125 minutes (6)	Creation of x/y axis and plotting points (1-4)	Creation of initial solution (1-3) Realization that solution is wrong (4) Plugging in one value to find another (5-13)
Flexible Behaviours	Finding difference in access fees and monthly costs (2,3) This interaction may be flexible, as follow-up questions indicate that Adam was not entirely sure how this problem would play out given this strategy. Dividing differences in access fees and monthly costs (4) Follow-up questions indicate that Adam felt this was likely to lead to a useable number, but unsure of the meaning of that, and unsure that it would work.	Starting at problem, tracing with pen, filling in dots on line (5) Unsure exactly what was happening, but possibly, Adam was mentally activating his notions of slope and the visual spatial representation of that line. After tracing for over 20 seconds, he was able to construct an explanation for why they are not parallel.	

Table A9

Competent and Flexible Behaviours Exhibited by George For Each Question

	C1 – Slope Competent	F1 – Slope Flexible	C2 – Cell Phone Competent
Competent Behaviours	Use of slope formula (1-2) Simplification of fractions (3)	Use of slope formula (2-3) Simplification of algebraic expression (4)	Creation of expressions for companies a and b (2,5) Calculation of company costs (3,6)
Flexible Behaviours			

Table A9 – Part 2

Competent and Flexible Behaviours Exhibited by George For Each Question

	F2 – Cell Phone Flexible	C3 – Parallel Lines Competent	F3 – Parallel Lines Competent
Competent Behaviours	Using 'Trial and error" to solve the question (2,3)	Use of slope formula for each line (1-4) Calculation of slope for each line 5	Use of slope formula for each line (1-4) Creation of slope expressions for both lines (5-6) Plugging in a value for 'a' to find a value for 'b' (7)
Flexible Behaviours	Creation of general equation for lines (5,6) Equations neither right, nor helpful, but given that George was stuck, it was all he could think to do. Attempt to divide equations for lines (7) Once equations made, George needed to find a way to have them give an answer, so he attempted to apply division Expression of frustration to researcher (8) George reported feeling irritated; this expression may serve to modulated (flexibility) that irritation, but indirectly asking the researcher for support. Return to guess and test (9) George may have been managing the researcher's perception of his ability, and thus returned to his initial strategy (though he could have stopped the question) to get a response closer to the actual answer.		

Table A10

Competent and Flexible Behaviours Exhibited by Farley For Each Question

	C1 – Slope Competent	F1 – Slope Flexible	C2 – Cell Phone Competent
Competent Behaviours	Mentally calculating slope (1)	Mentally calculating slope and simplifying algebraic expressions (1,2)	Mentally calculating cost of both companies (one incorrectly) (1-3)
Flexible Behaviours			

Table A10 – Part 2

Competent and Flexible Behaviours Exhibited by Farley For Each Question

	F2 – Cell Phone Flexible	C3 – Parallel Lines Competent	F3 – Parallel Lines Competent
Competent Behaviours		Mentally calculating, then writing slope (1-3)	Mentally calculating values (9)
Flexible Behaviours	Answering question in way that minimizes calculation and writing. (1-4) This interaction may be flexible, as follow-up questions indicate that Adam was not entirely sure how this problem would play out given this strategy.		Clarifying that correct answers include actual values (1-2) Farley recognized that, to correctly answer the question, he could clarify tasks demands, and thus, reduce his work. Varying thinking between nightly plans, and the question (7-9) Though 'unintentional' this strategy may have facilitated his correct answer by allowing him 'find' the correct answer in his mind.

Table A11

Competent and Flexible Behaviours Exhibited by Harriet For Each Question

	C1 – Slope Competent	F1 – Slope Flexible	C2 – Cell Phone Competent
Competent Behaviours	Creation of x/y axis with unit marks (1-2) Graphing and joining of points (3-5) Counting and dividing rise and run (6-8)	Making x/y axis and making unit marks to start question (1-2) Counting slope of equation (8-9)	Creation of expressions for companies a and b Calculation of company costs
Flexible Behaviours		Choosing values for a and b and graphing (3-6) Follow up answer suggests that she was sure that a and b should "be something" but not sure if she should give them. However, after thought, she realized that choosing values would afford her a graph, which she could then use to count slope, and so she did it. Drawing grid to make counting rise and run easier (7) Adaptation for fact that she did not have slope formula ; once she realized she couldn't count precisely she needed to add a counting aid (the grid) to facilitate calculation of slope.	

Table A11 – Part 2

Competent and Flexible Behaviours Exhibited by Harriet For Each Question

	F2 – Cell Phone Flexible	C3 – Parallel Lines Competent	F3 – Parallel Lines Competent
Competent Behaviours	Using 'Trial and error" to solve the question (7-10) Asks to look back at the last question and copies result (1-2)	Drawing axis and graphing points. (2-7)	Creation of axis (1,3) Drawing of graph (4-6) Plugging in pairs of number (9)
Flexible Behaviours	Dividing system access by per minute costs (2-3) Harriet suggests that she was 'just kinda [trying] things"	Constructing algebraic explanation for why the lines are parallel (8-9) Harriet explained that, as she looked at the points, she realized that they were parallel, and that it was a function of their relationship. Ultimately, the relationship Harriet saw was too hard to discern. However, it makes sense to interpret this explanation as flexible, a s Harriet was clearly trying to use knowledge of the arithmetic relationship between the points to explain this situation.	Using the slope formula to construct expressions for slope (7) Though there had been many affordances to use the slope formula, she had not used it independently. However, our discussion in the follow up to the previous question included discussion of the slope formula. Harriet then utilized this formula to successfully get the answer.

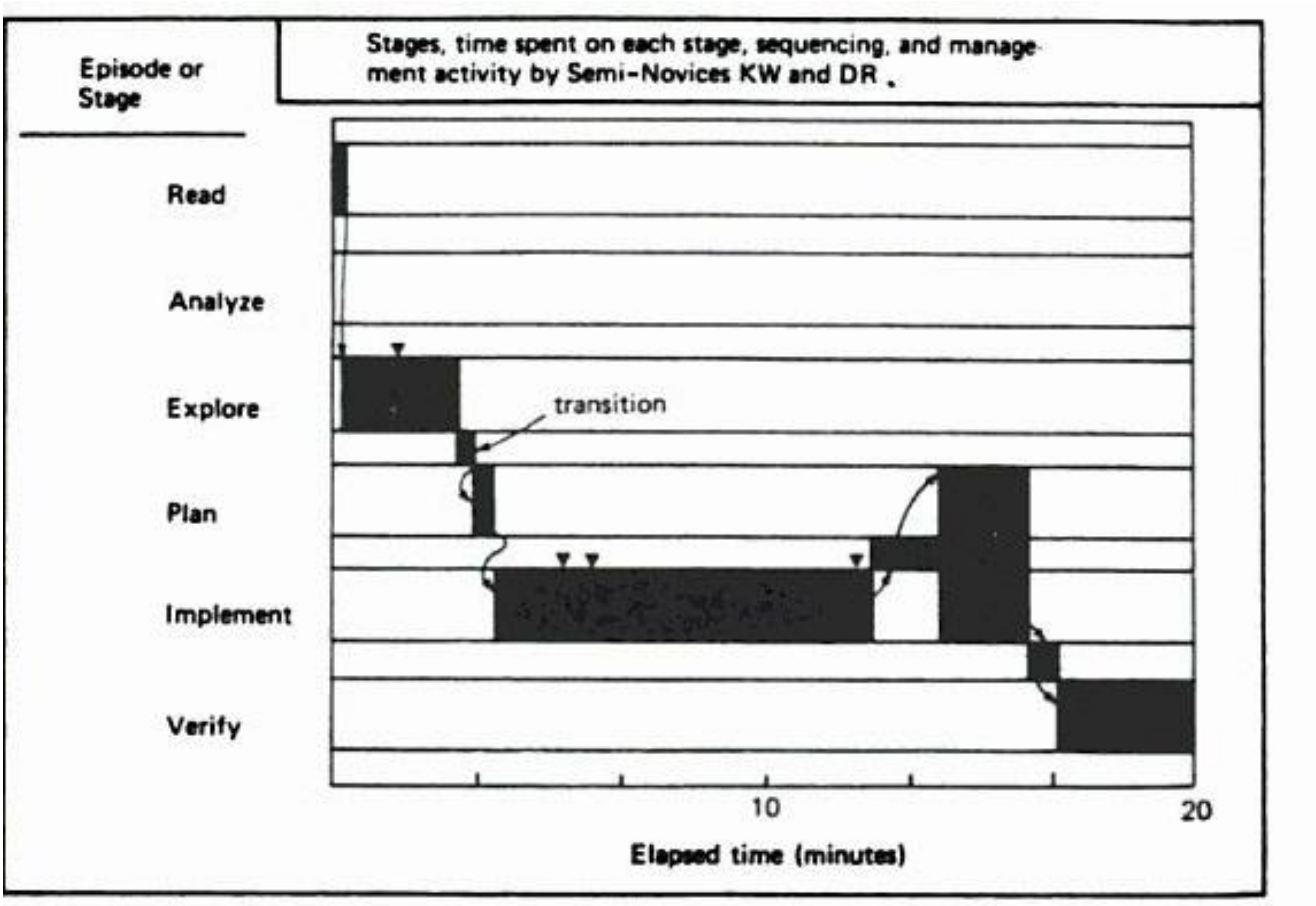
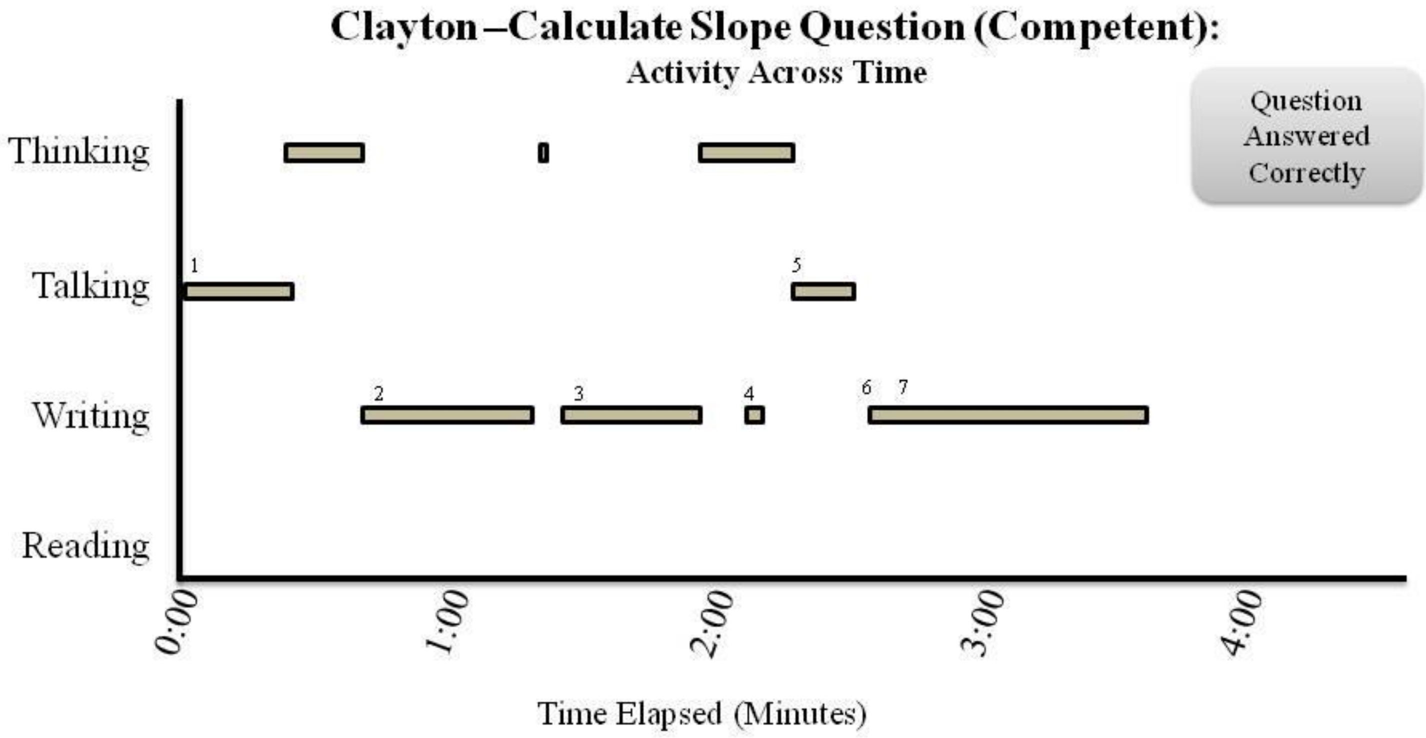


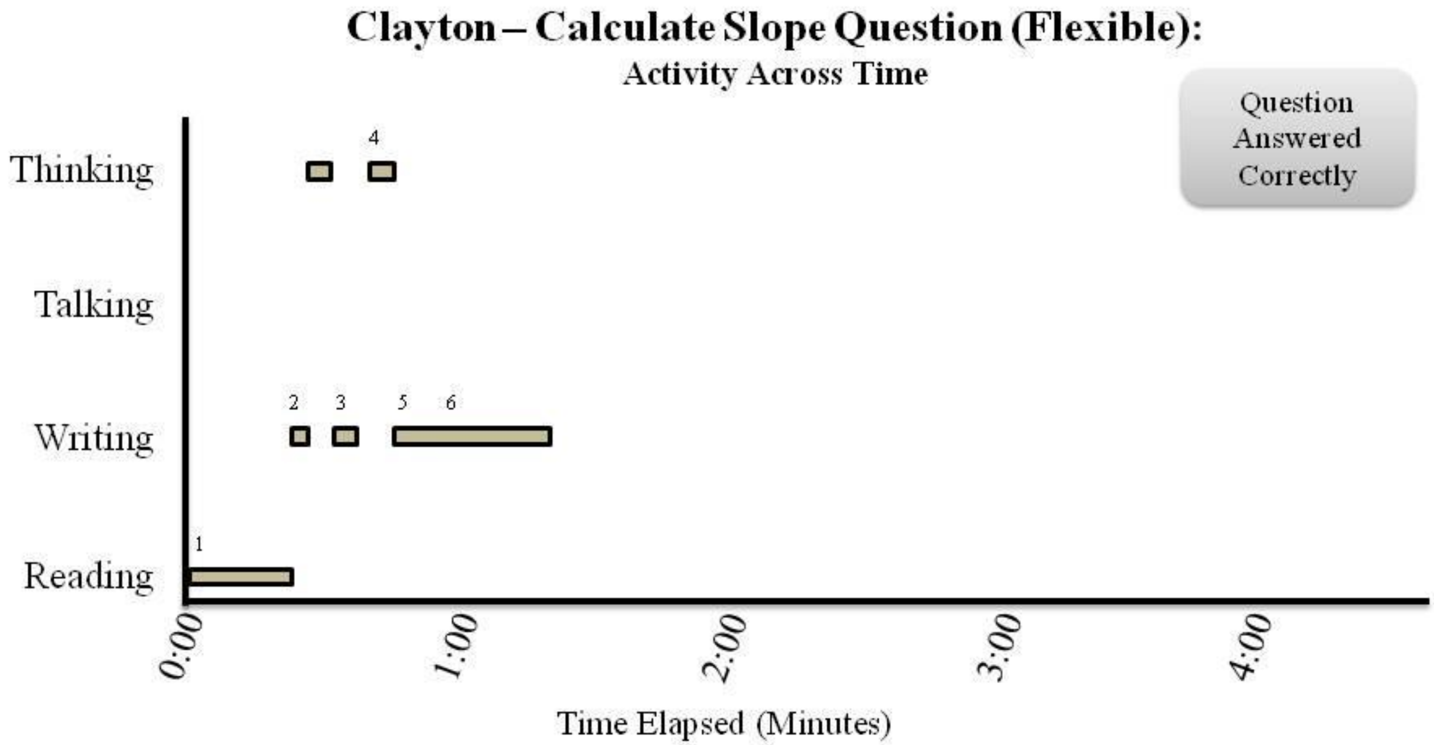
Figure A1. Schoenfeld's (1992) strategy for tracing problem-solving activity.



Activity Details

1. Clarifying what the question is asking for.
2. Writing the midpoint formula.
3. Writing x/y axis, plotting and joining points.
4. Crossing out the midpoint formula.
5. Indirectly asking for slope formula.
6. Write slope formula.
7. Plugging in values to get slope.

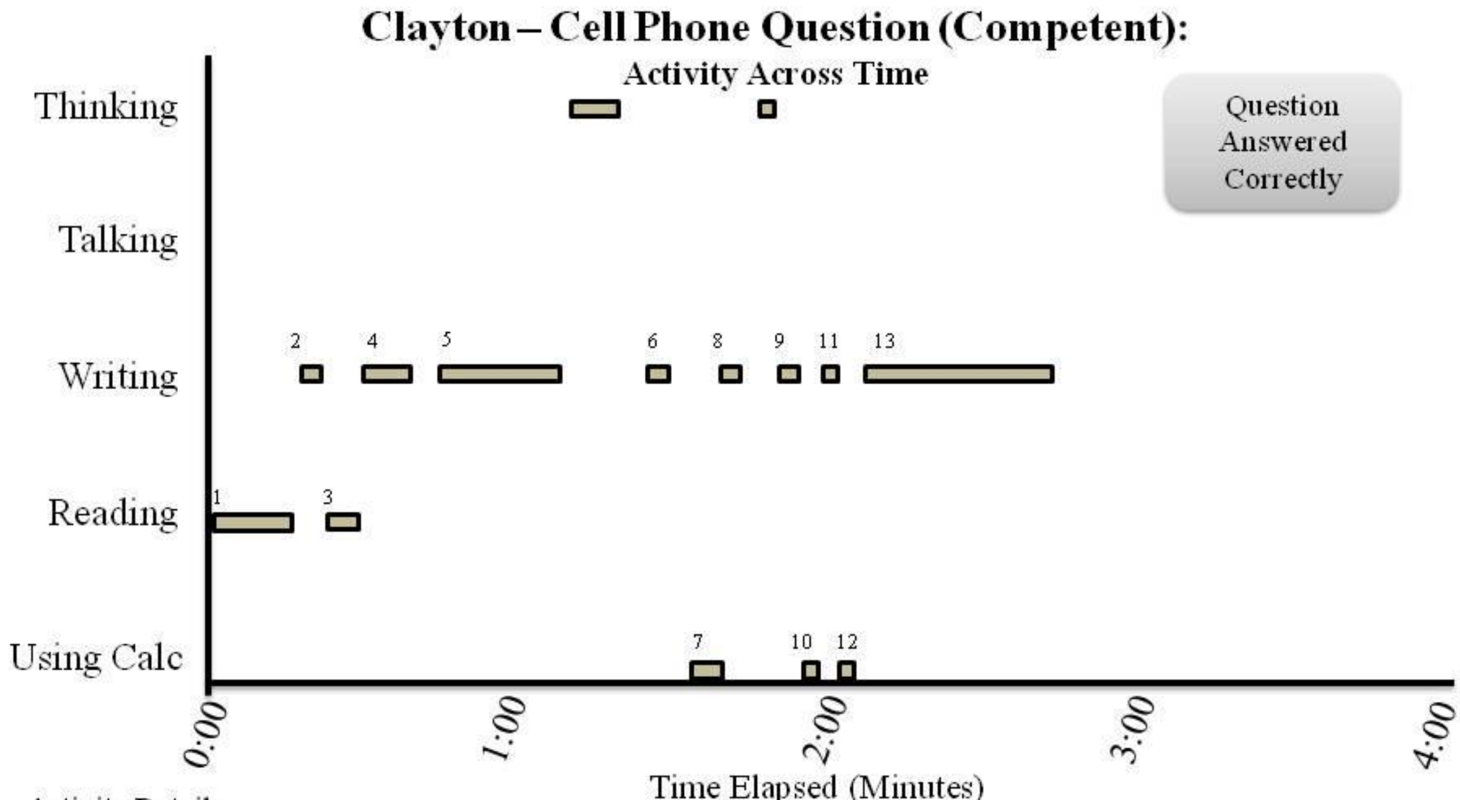
Figure A2. Clayton’s AAT for the Calculate Slope (Competent) Question



Activity Details

1. Reading the question.
2. Drawing x/y axis.
3. Writing slope formula.
4. Pen poised to write.
5. Inserting variables into equation.
6. Simplifying equation.

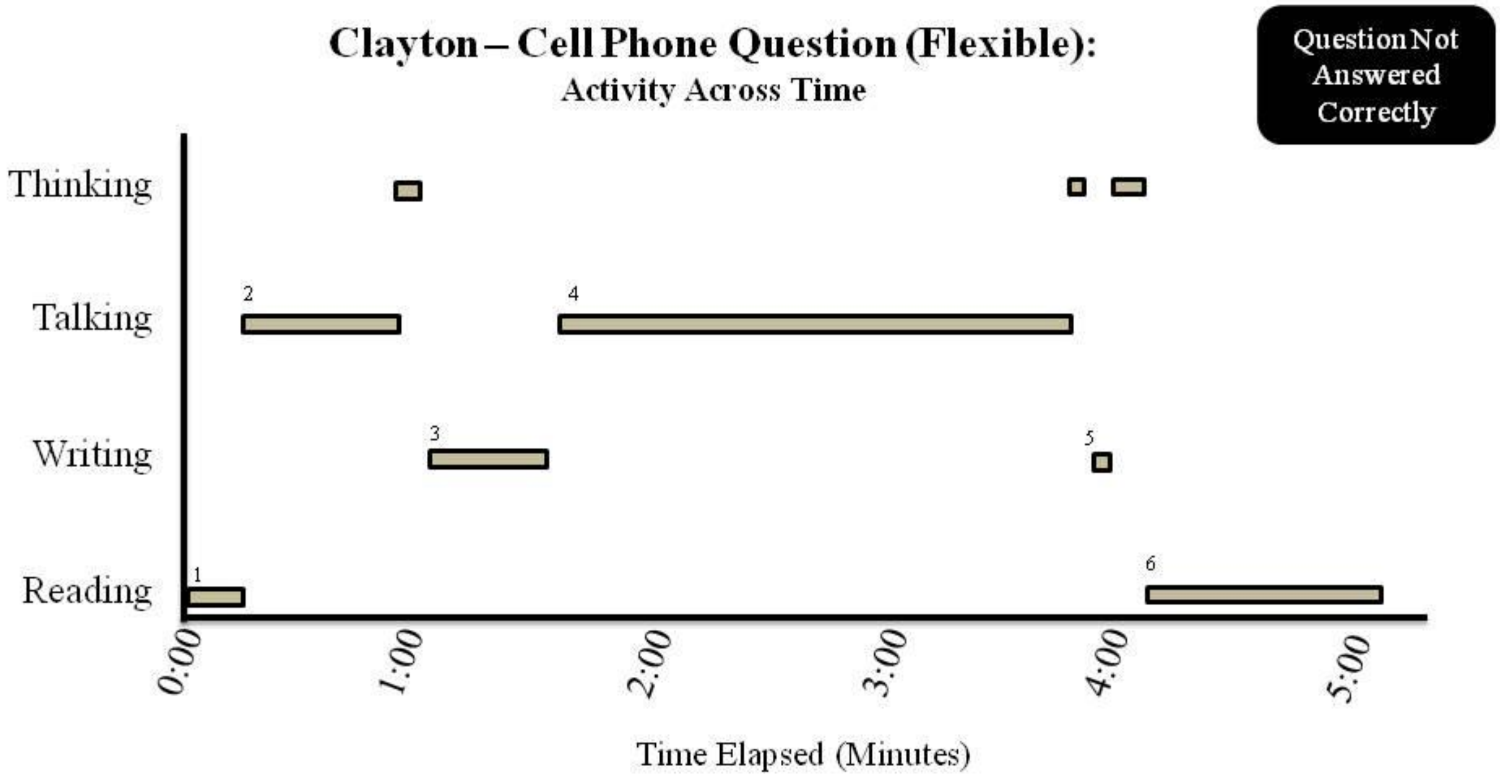
Figure A3. Clayton's AAT for the Calculate Slope (Flexible) Question.



Activity Details

1. Reading the question.
2. Parsing question text.
3. Reading the question.
4. Creating two columns (A and B).
5. Writing monthly fees.
6. Equation for minutes cost (A).
7. Finding total minutes (A).
8. Writing total minutes (A).
9. Equation for minutes cost (B).
10. Finding total minutes (B).
11. Writing total minutes (B).
12. Adding minutes and access fee .
13. Totaling Cost for Companies (A and B)

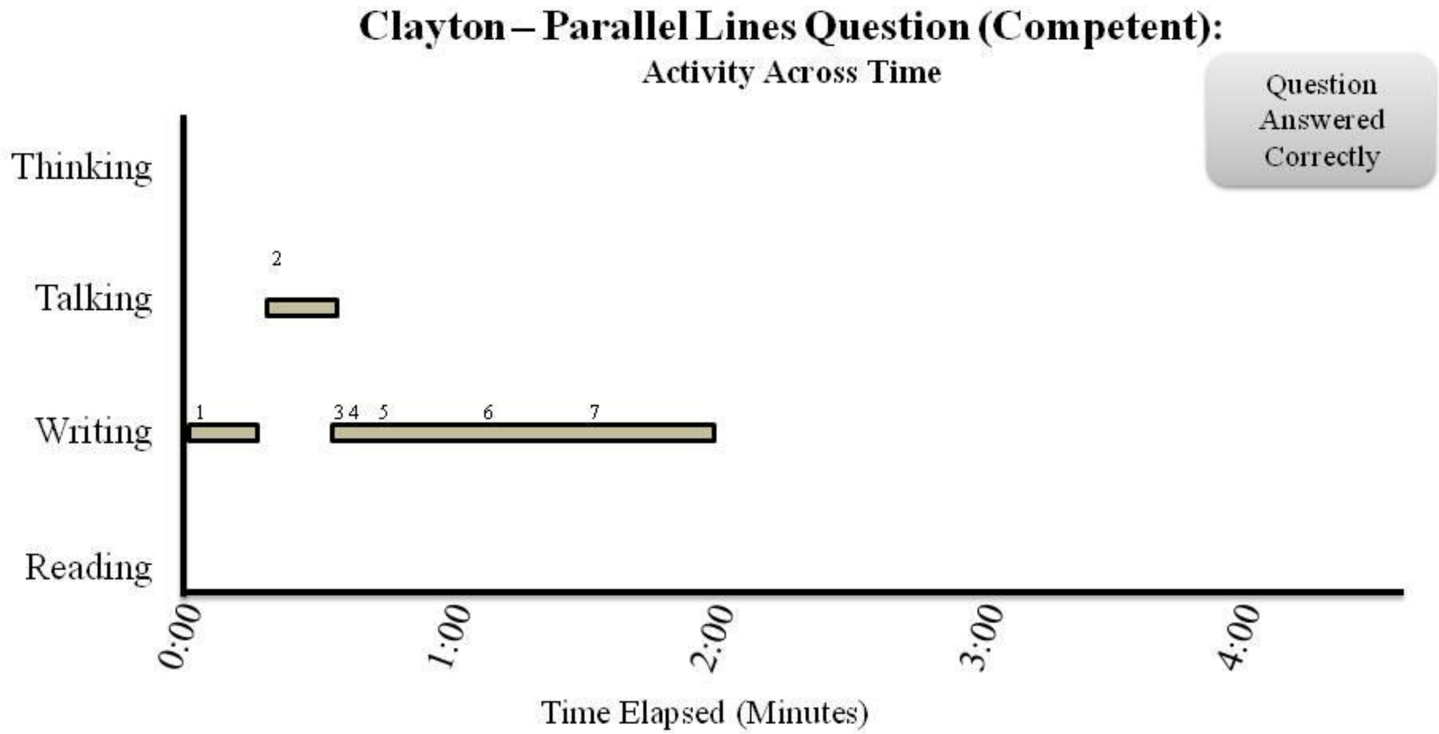
Figure A4. Clayton's AAT for the Cell Phone (Competent) Question.



<u>Activity Details</u>

1. Reading the question.
2. Asking about question demands. Clarifying relationship between this question and previous.
3. Writes "Company A is cheaper when dealing with the cost of its minutes."
4. Clarification of question with researcher.
5. Crossing out what was written.
6. Looking back at work on previous question.

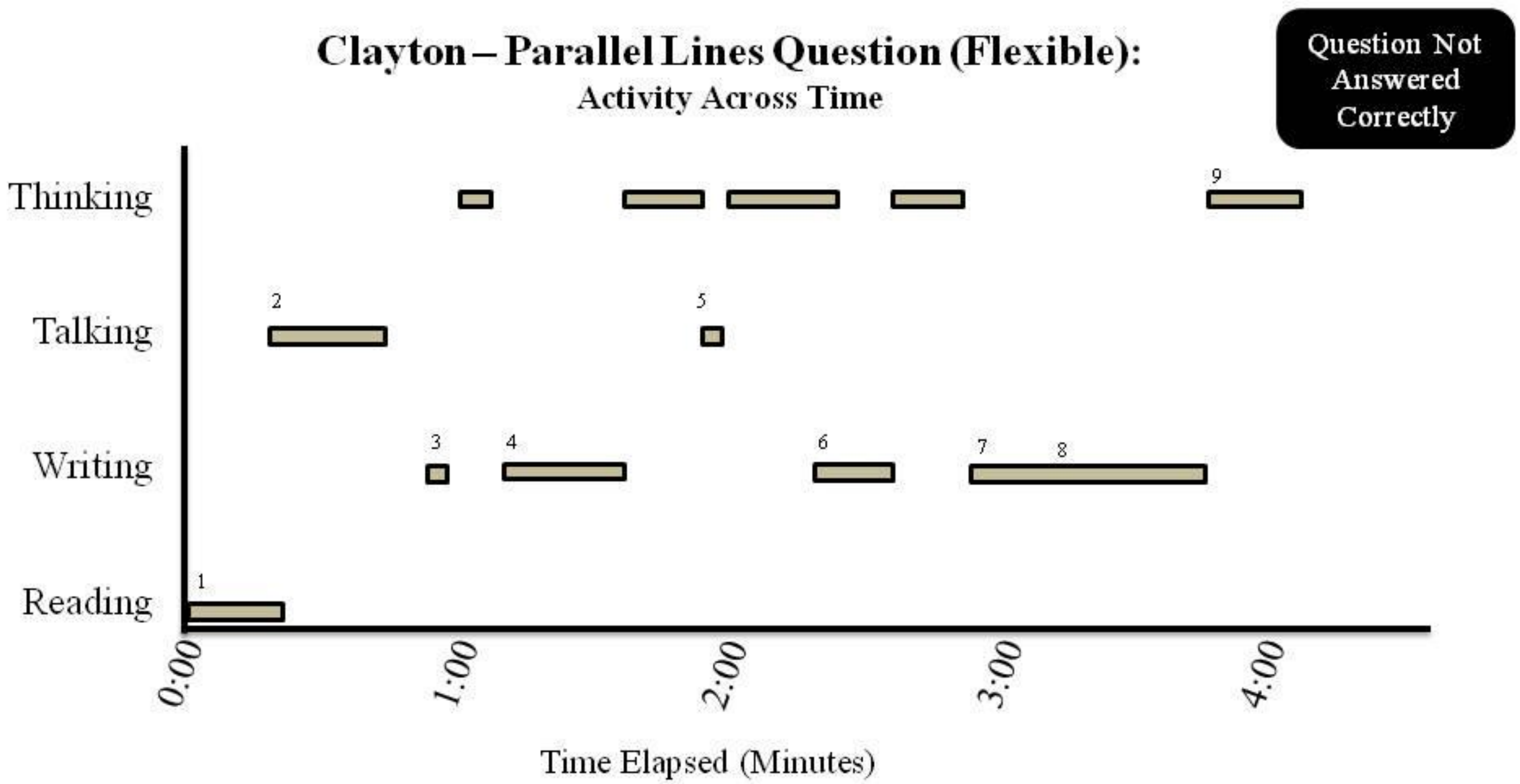
Figure A5. Clayton's AAT for the Cell Phone (Flexible) Question.



Activity Details

1. Reading the question.
2. Clarifying that "I'm basically looking for slope in this."
3. Annotating question text to label points for slope(1) and slope(2).
4. Writing slope formula.
5. Calculating slope(1).
6. Calculating slope(2).
7. Writing out answer in a full sentence.

Figure A6. Clayton's AAT for the Parallel Lines (Competent) Question.



Activity Details

1. Reading the question.
2. Asks researcher "so what am I looking for?"
3. Annotating lines AB and CD with a label.
4. Drawing axis, and line CD.
5. Subvocalizing "ok, I understand."
6. Writing slope formula.
7. Calculating slope of CD.
8. Creating expression for slope of AB.
9. Subvocalizing, while thinking.

Figure A7. Clayton's AAT for the Parallel Lines (Flexible) Question.

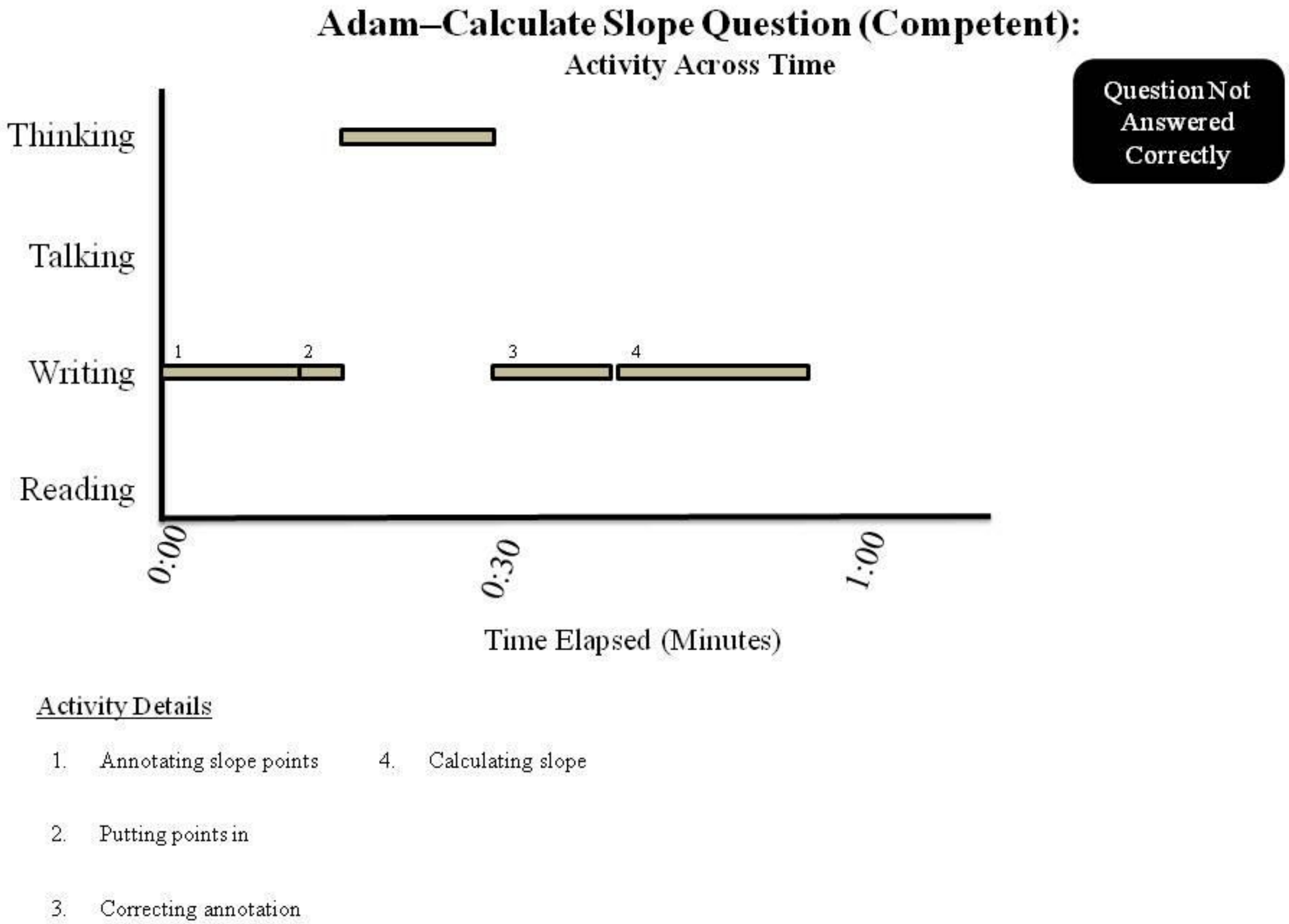


Figure A8. Adam's AAT for the Calculate Slope (Competent) Question.

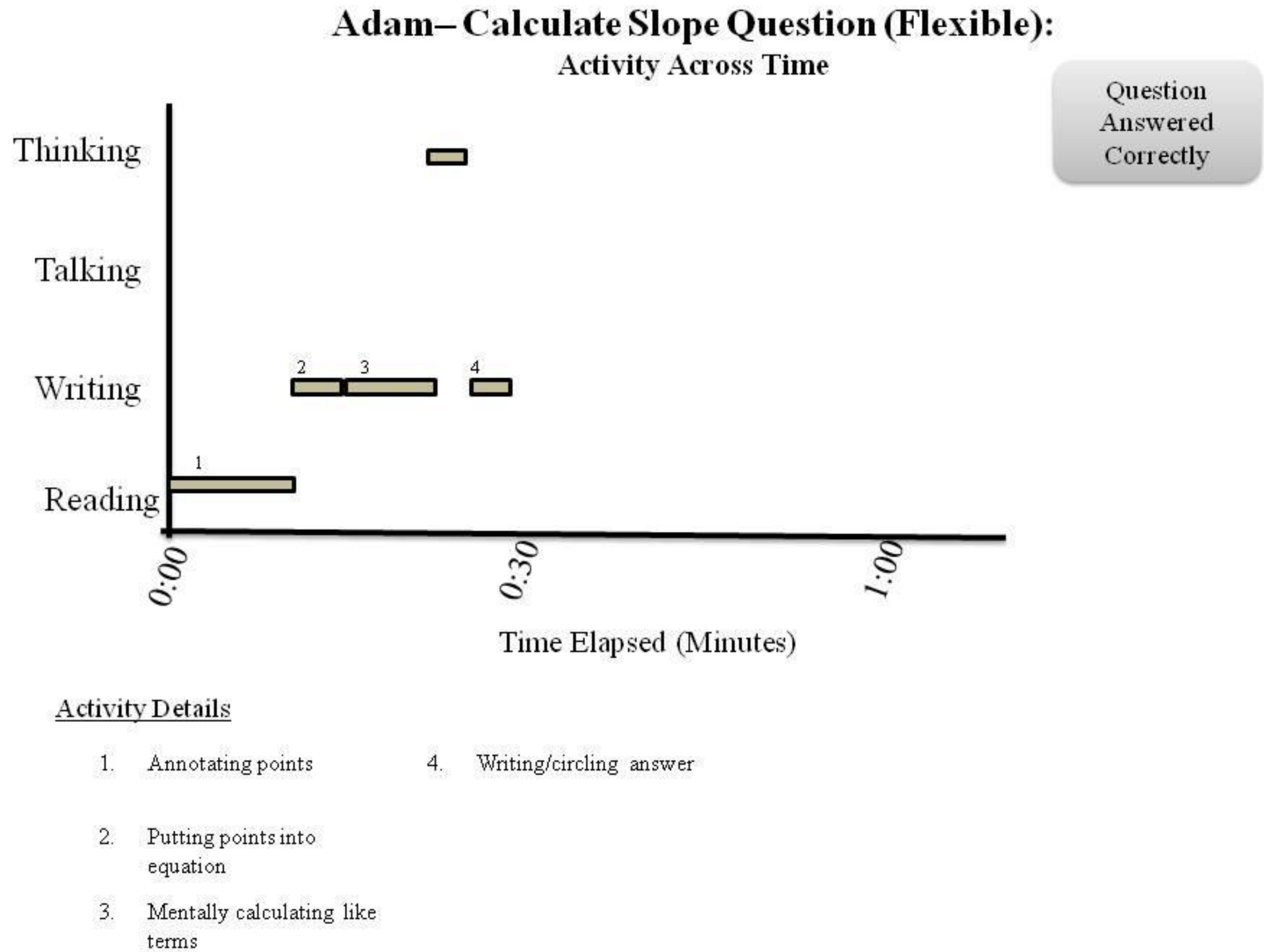
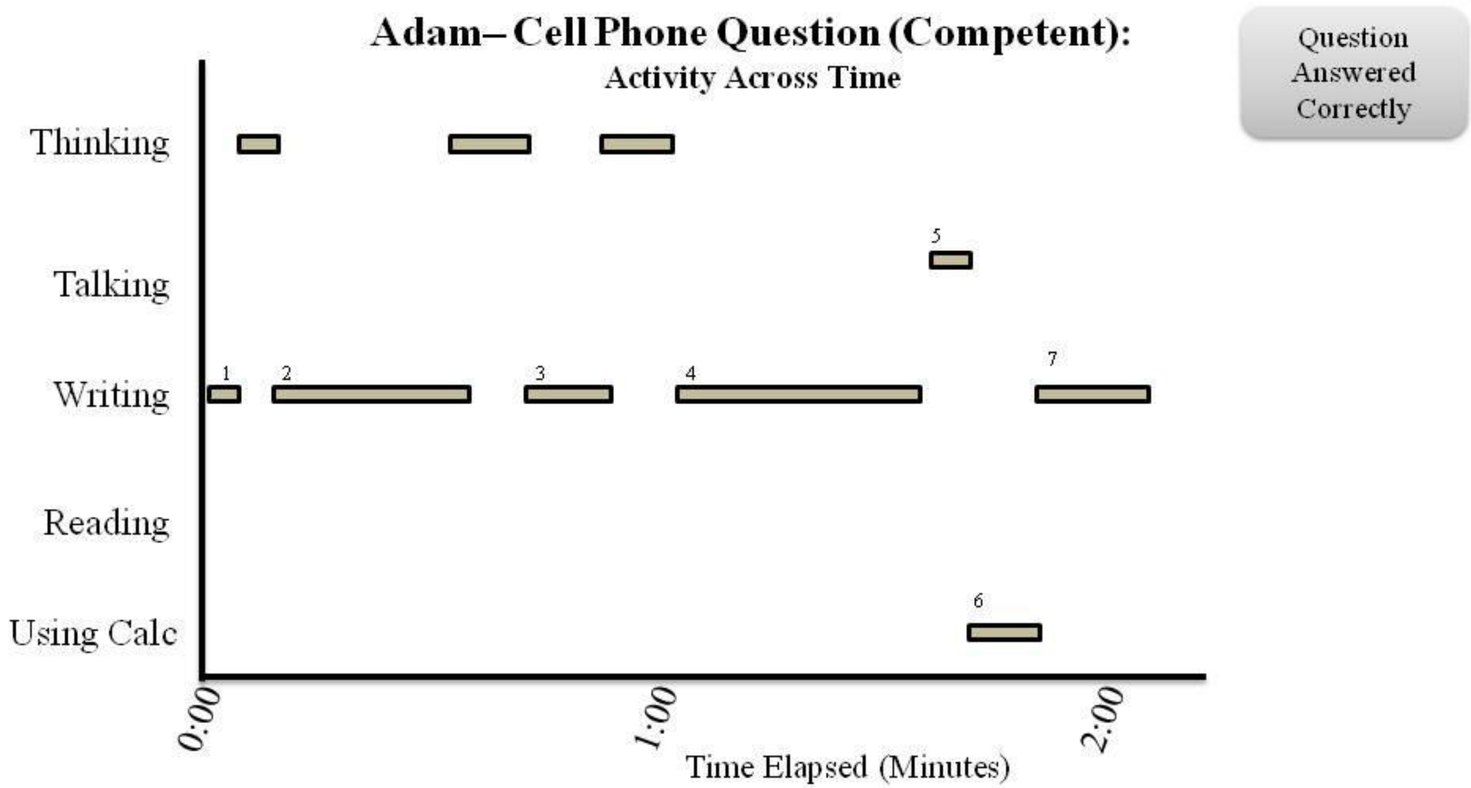


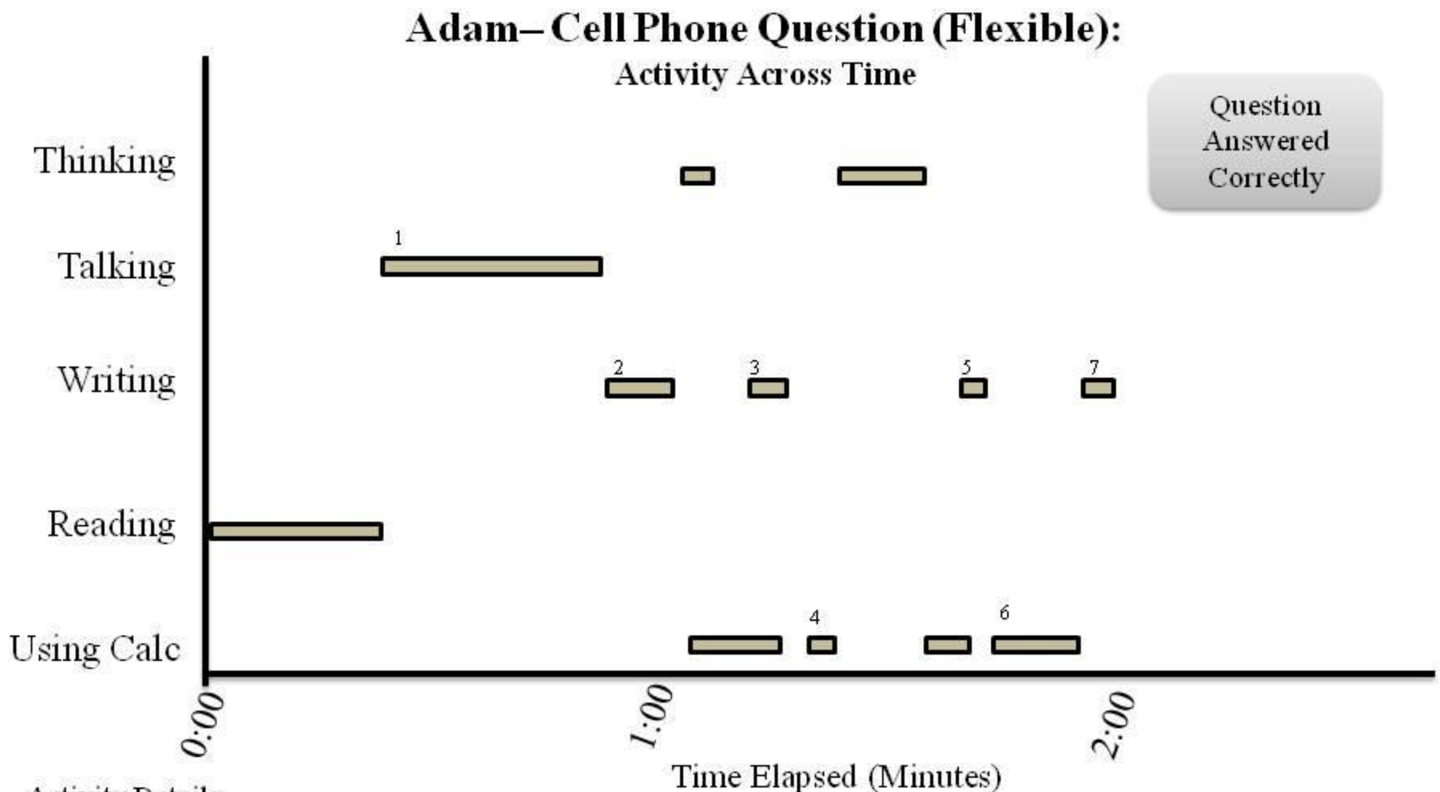
Figure A9. Adam's AAT for the Calculate Slope (Flexible) Question.



Activity Details

1. Writes system access fees for both.
2. Writes equation for Company A
3. Writes cost of Company A
4. Writes and corrects equation for Company B
5. Asks if he can use a calculator
6. Calculates cost of Company B
7. Writes cost of Company B

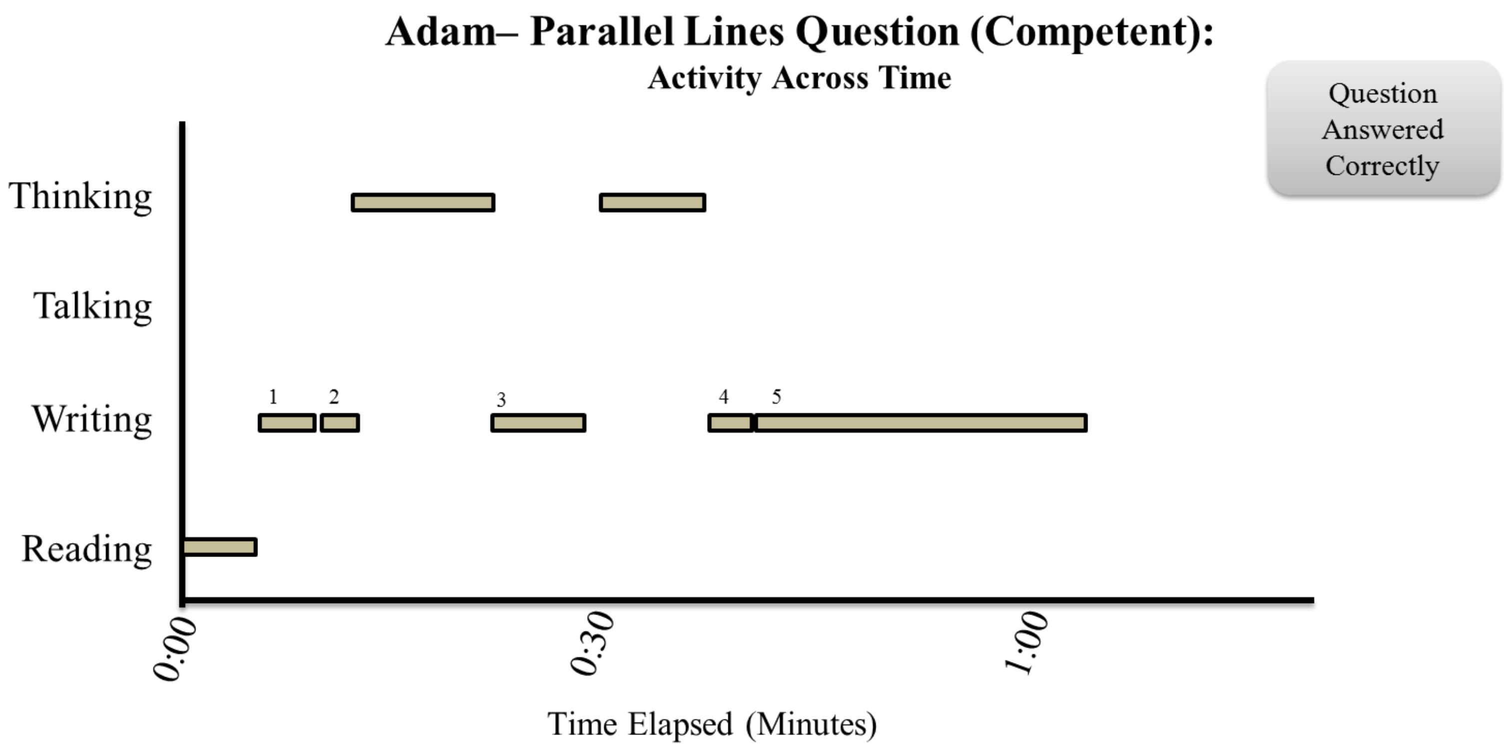
Figure A10. Adam's AAT for the Cell Phone (Competent) Question.



Activity Details

1. Interpreting question with researcher.
2. Writes difference between system access fees.
3. Writes difference between monthly costs.
4. Divides difference in access fees by difference in per minute fees.
5. Writes result of above calculation.
6. Calculates cost at 125 minutes for both companies
7. Writes down result of calculation

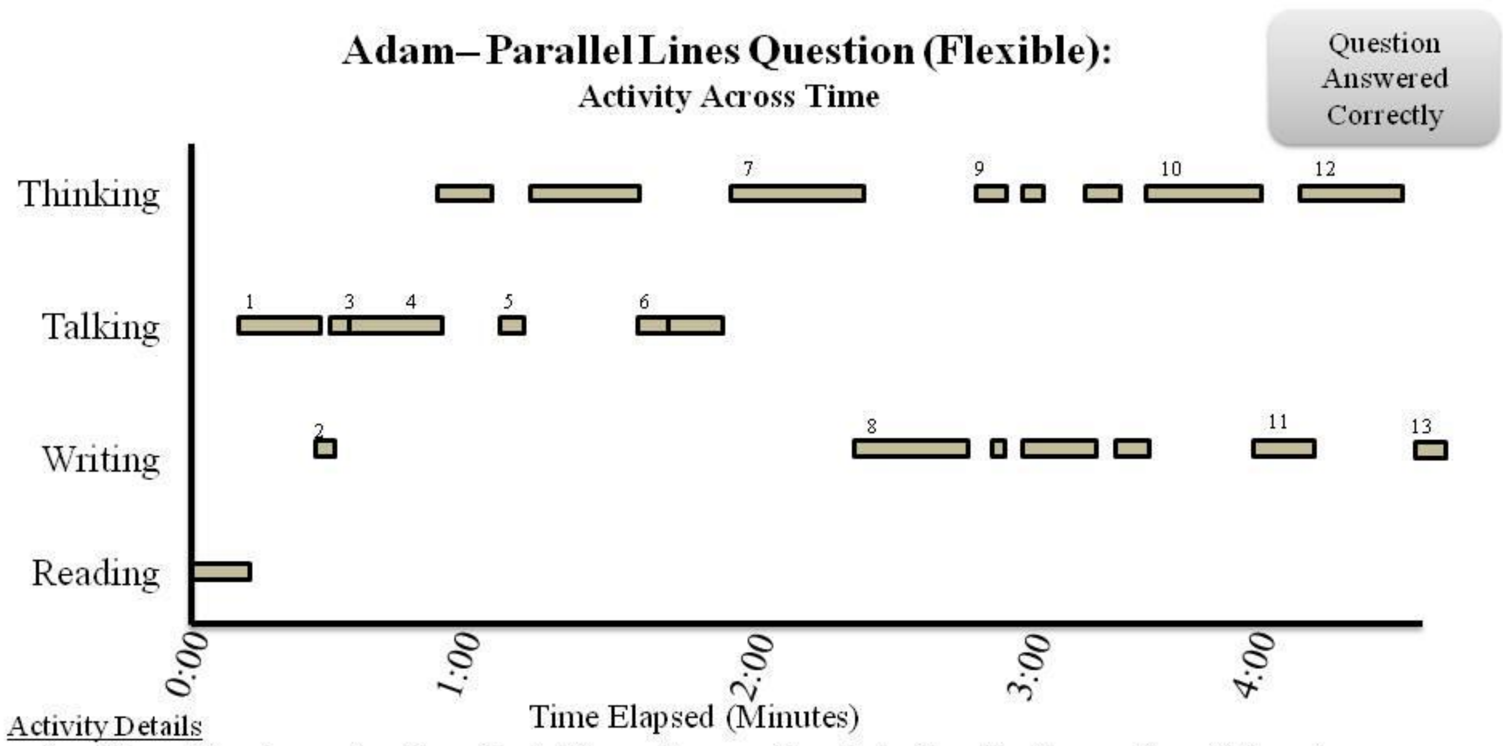
Figure A11. Adam's AAT for the Cell Phone (Flexible) Question.



Activity Details

1. Draw x and y axis
2. Identifies points
3. Joins points / erases
4. Draws / joins points
5. Tracing lines with pen / filling in dots on line

Figure A12. Adam's AAT for the Parallel Lines (Competent) Question



Activity Details

1. Sub voc: 'Here the slope is one over two…'
2. Writes coordinates
3. Expresses his solution
4. Realizes solution is wrong
5. Researcher clarifies question
6. Sub voc: 'Interesting…do you have lined paper?'
7. Pen on paper, not writing, but moving
8. Unclear what he is writing
9. Tracing lines / touching pend to page
10. Pen touching second point
11. 'Sub vocalizing' while writing
12. Sub-vocalizing with number
13. Writes and says answer

Figure A13. Adam's AAT for the Parallel Lines (Flexible) Question.

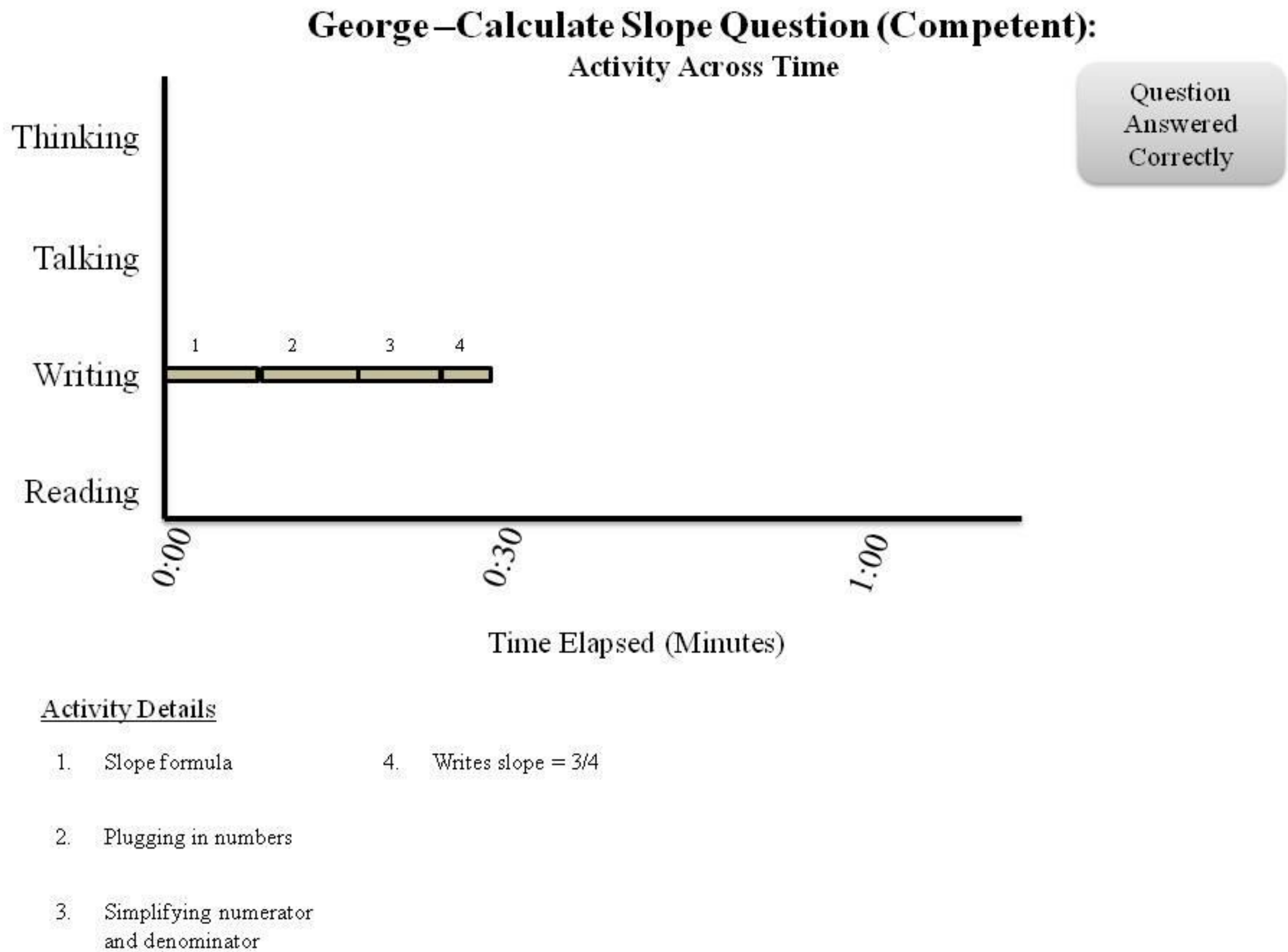


Figure A14. George's AAT for the Calculate Slope (Competent) Question

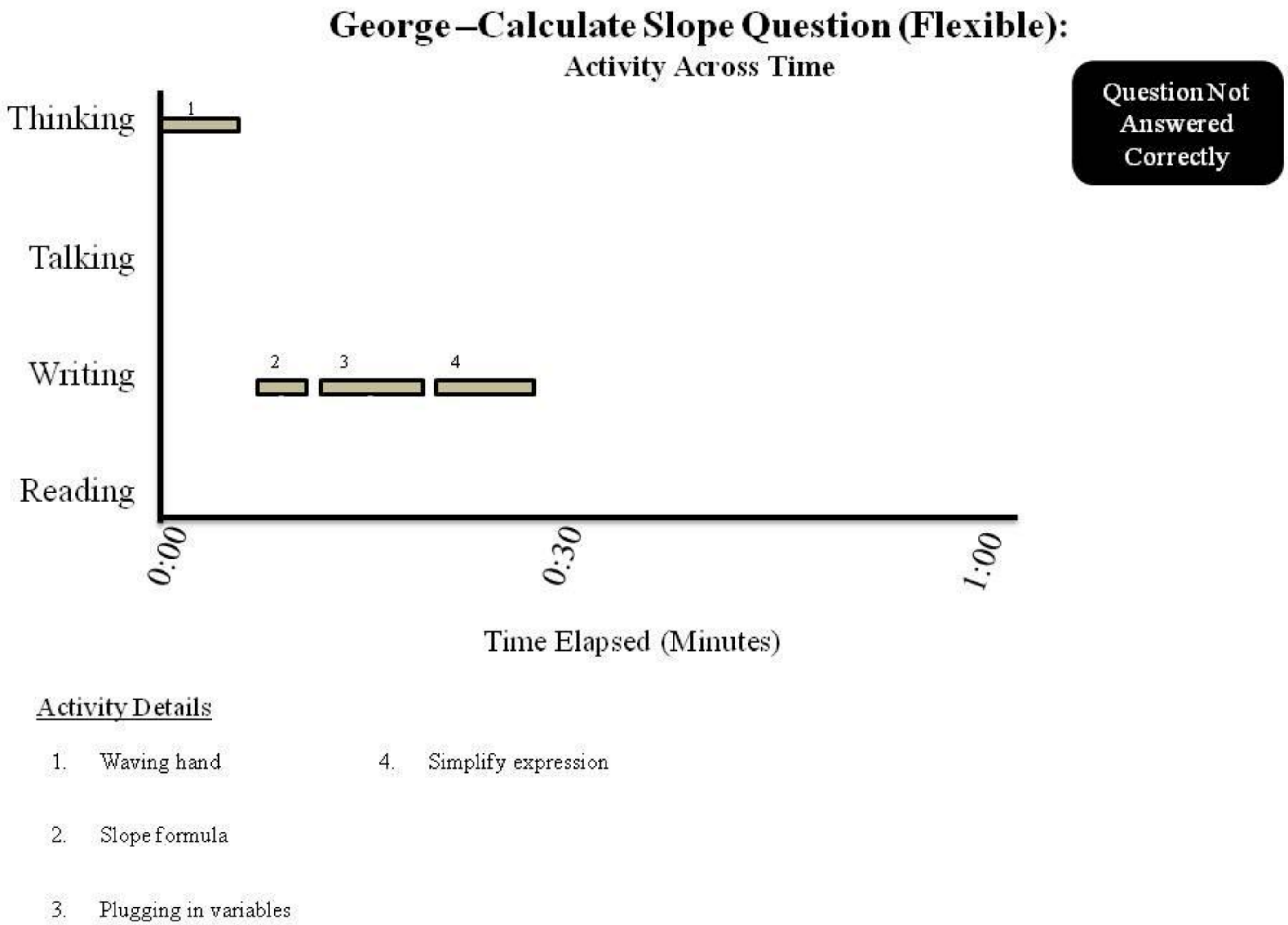
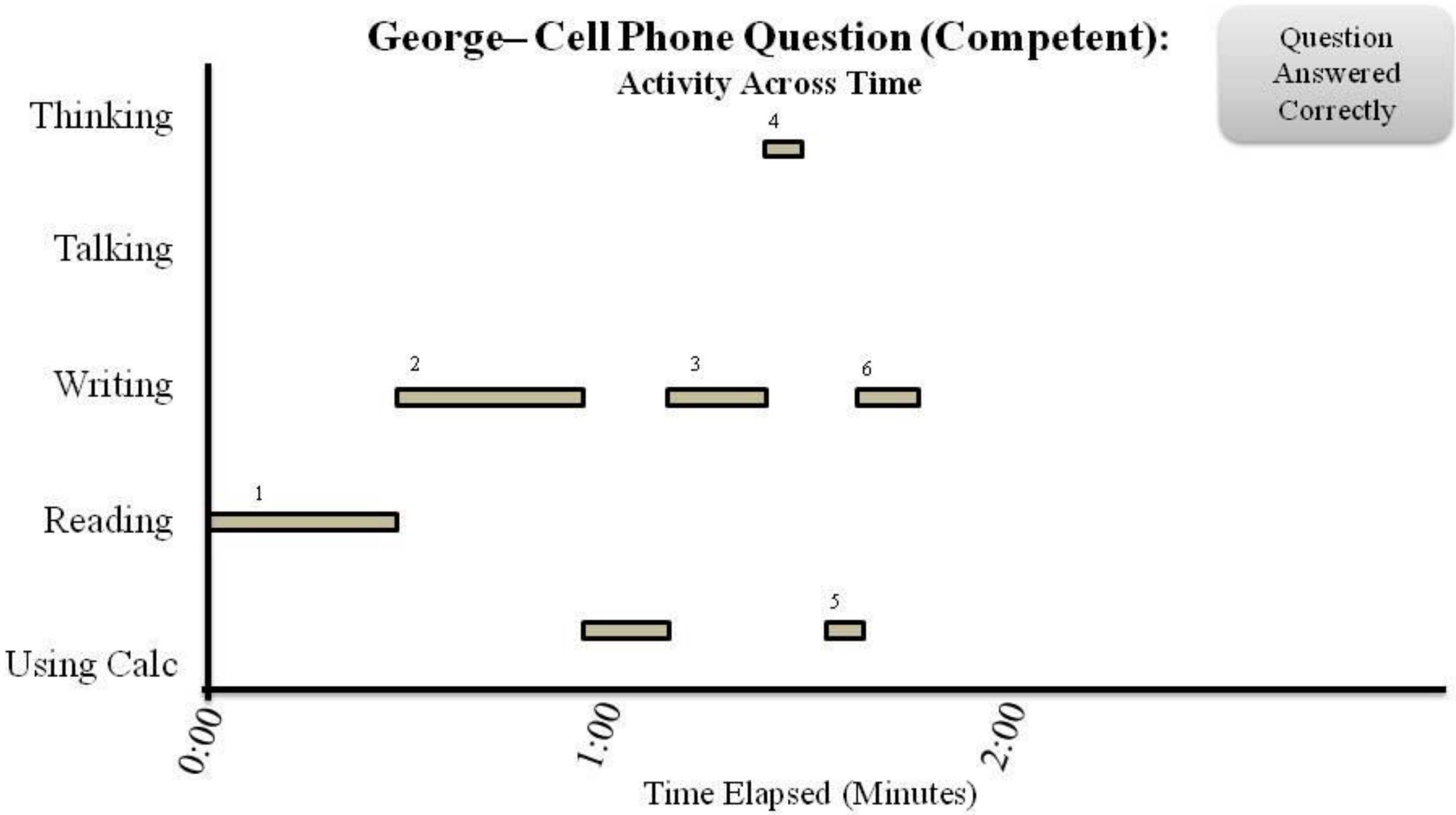


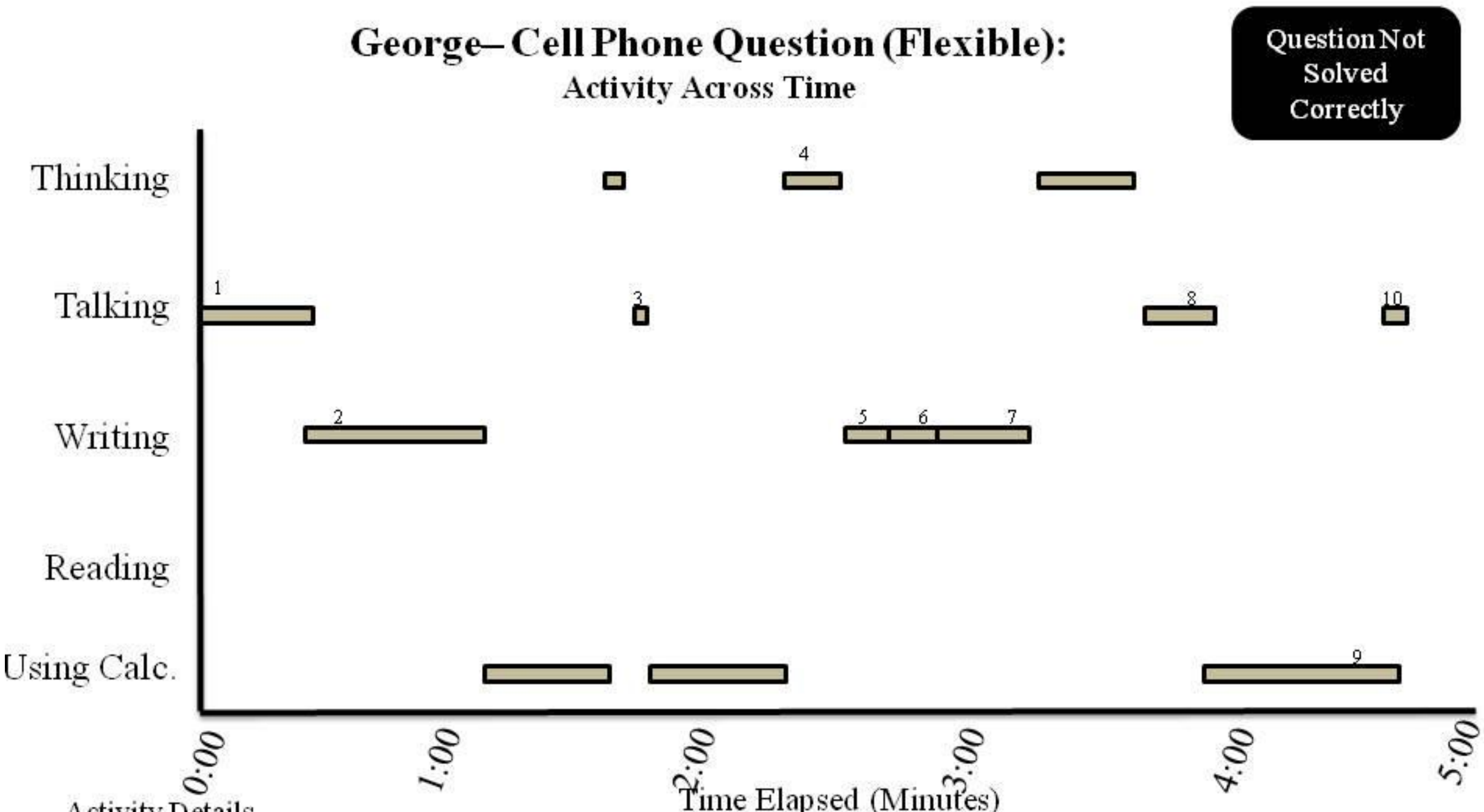
Figure A15. George's AAT for the Calculate Slope (Flexible) Question.



<u>Activity Details</u>

1. Waiting for researcher to bring calculator / reading problem.
2. Calculations for Company A
3. Total for Company A
4. Hand waving
5. Calculations for Company B
6. Total for Company B

Figure A16. George's AAT for the Cell Phone (Competent) Question.



Activity Details

1. Discussion of question
2. Creating table of values
3. "I know it's between the 30th and 60th"
4. Pausing about to write / sub-vocalizing
5. Equation for company A
6. Equation for company B
7. Dividing equation by 2 (incorrectly)
8. "I can't remember how to write the formula for that right now"
9. Guess and checking values
10. "It's going to be at the 116th minute"

Figure A17. George's AAT for the Cell Phone (Flexible) Question.

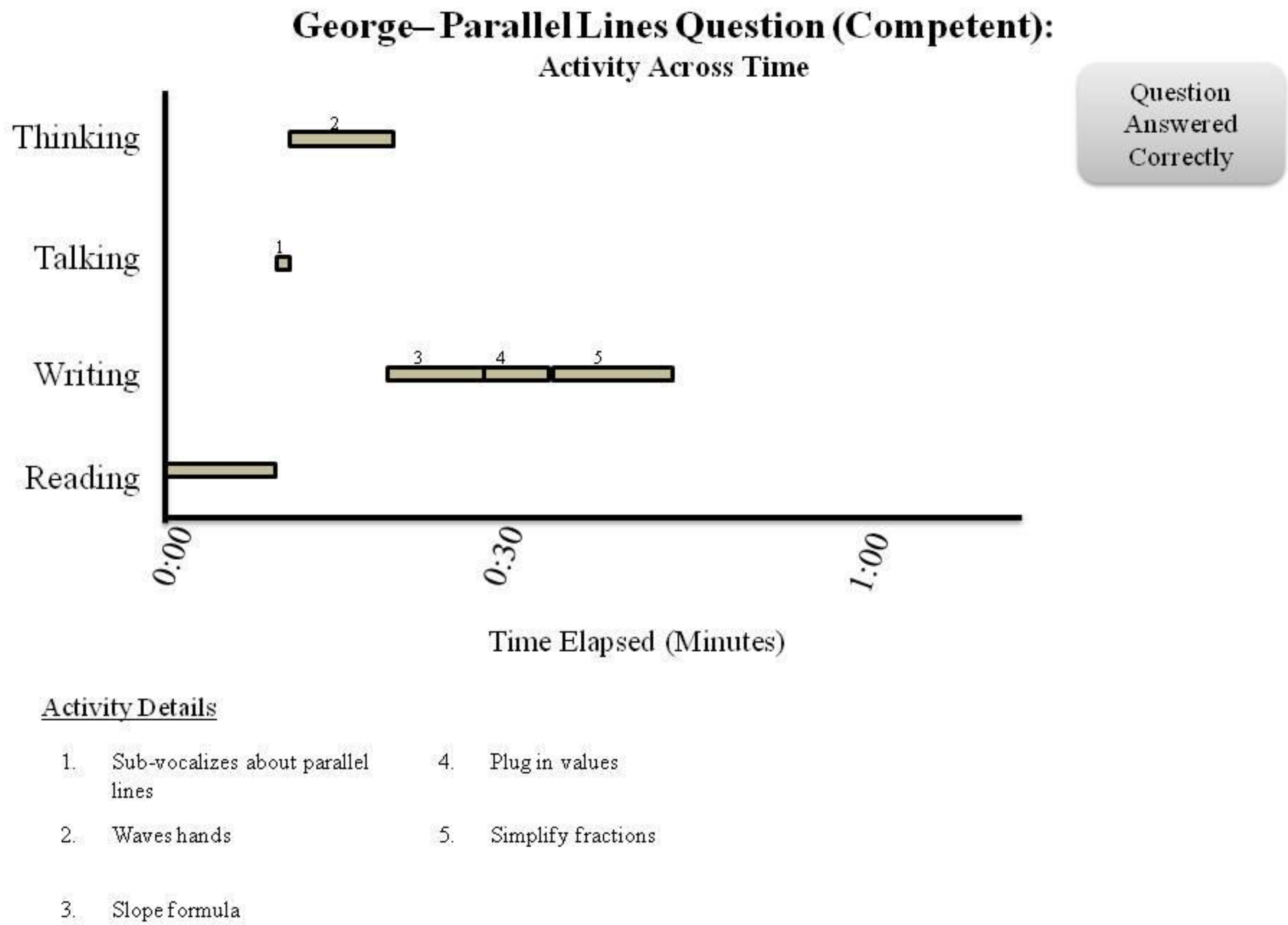


Figure A18. George's AAT for the Parallel Lines (Competent) Question.

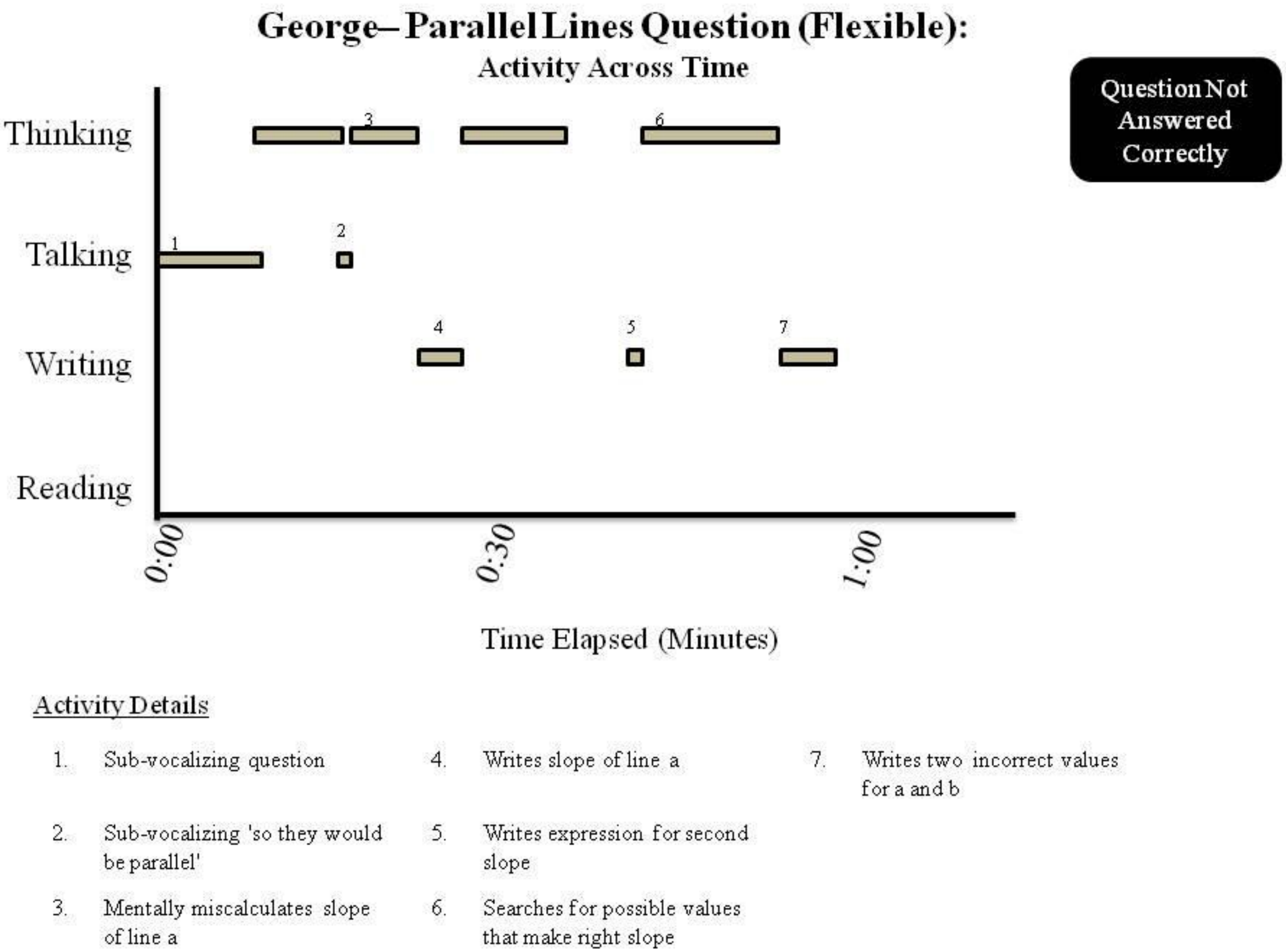
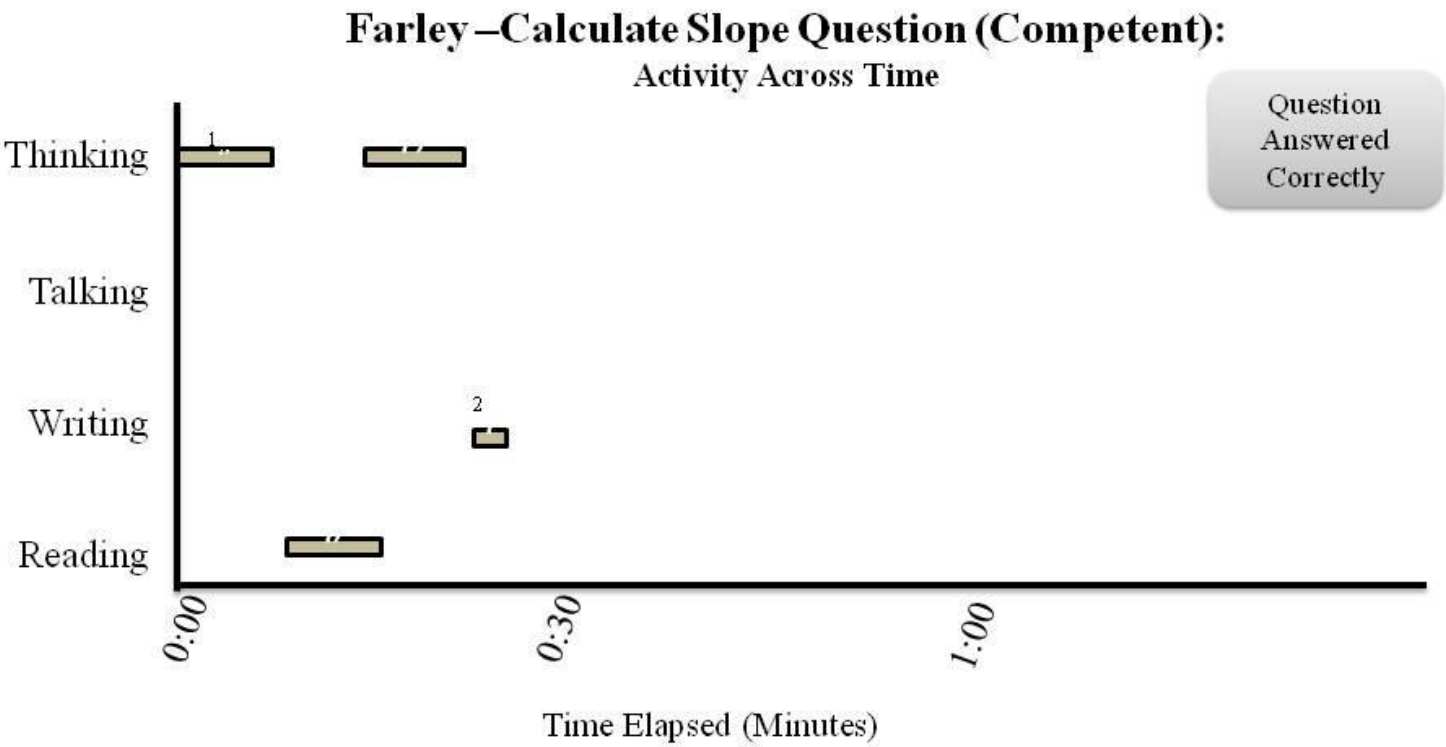


Figure A19. George's AAT for the Parallel Lines (Flexible) Question.

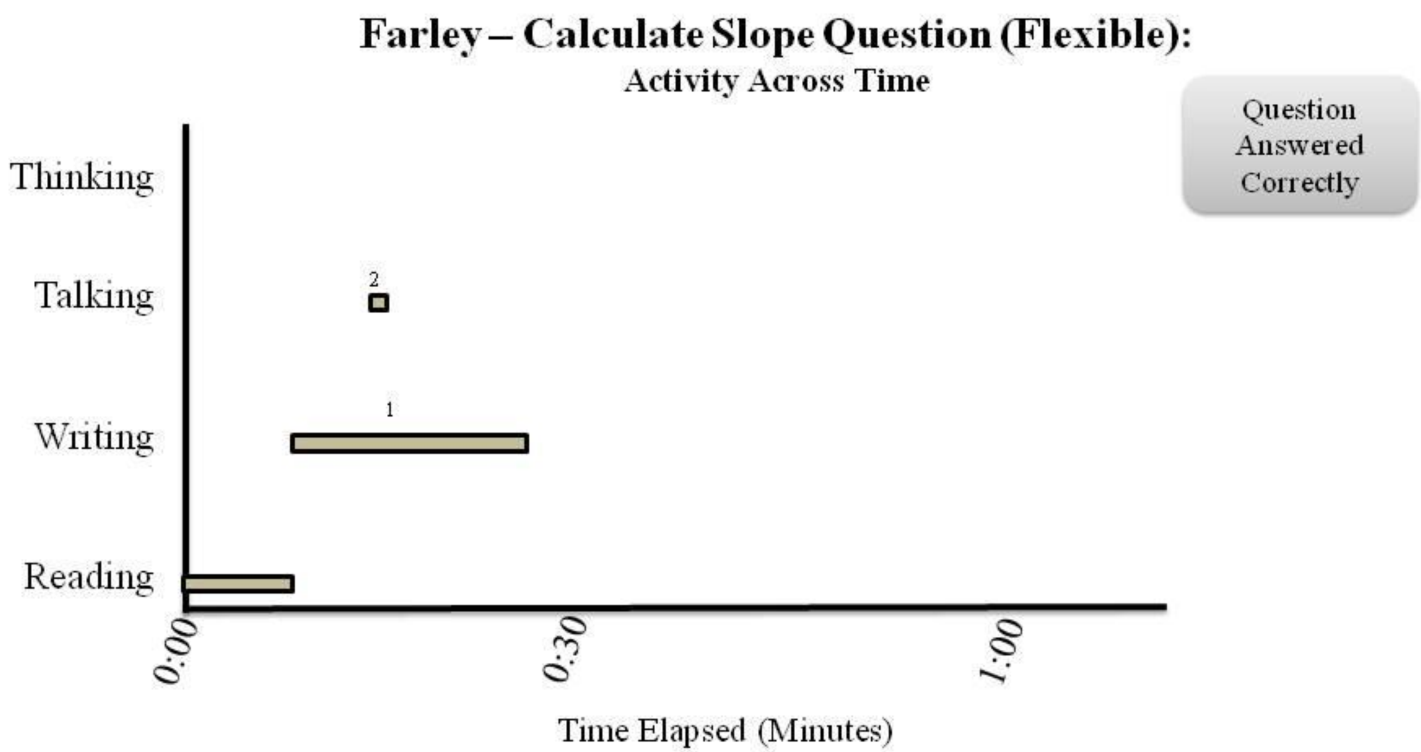


Activity Details

1. Sighs

2. Writes correct answer

Figure A20. Farley's AAT for the Calculate Slope (Competent) Question

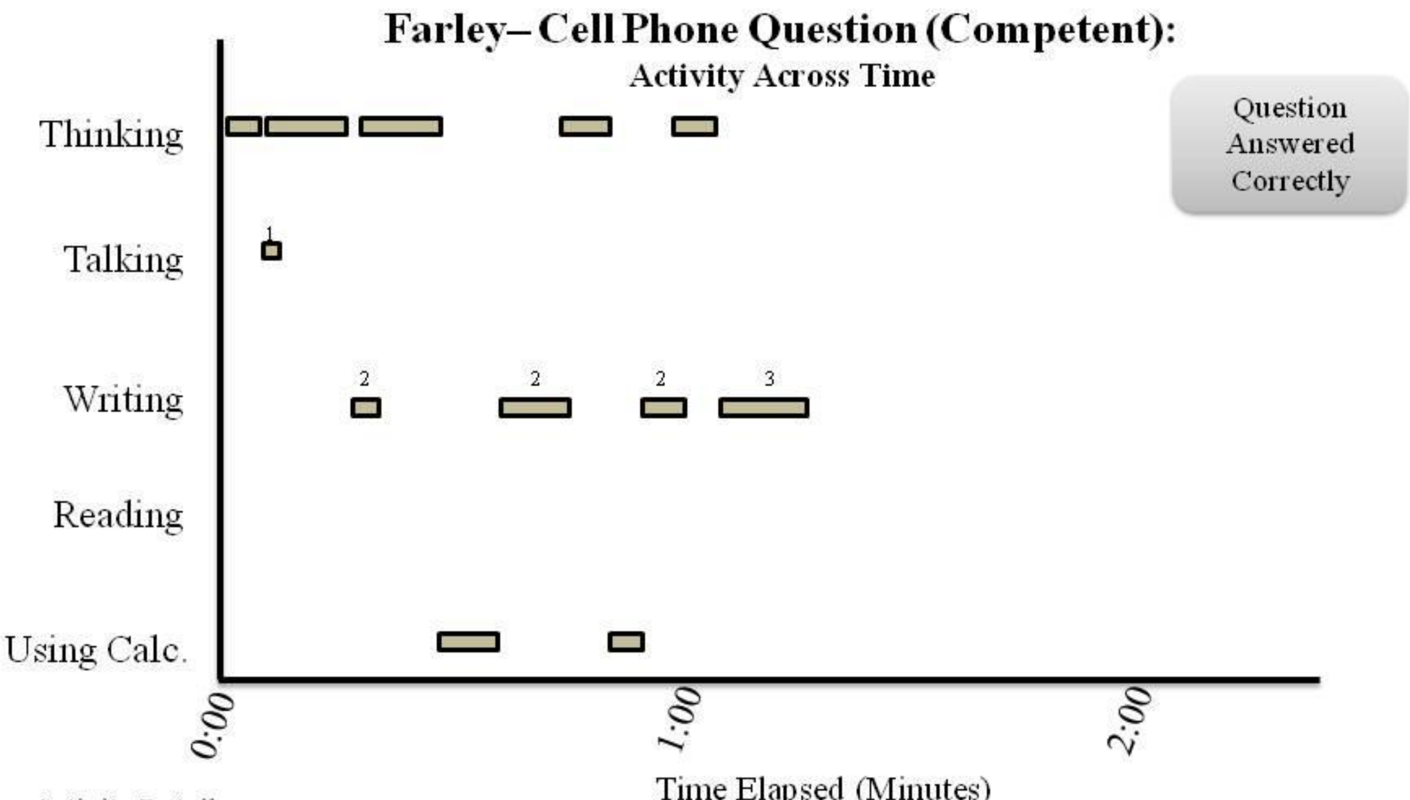


Activity Details

1. Writes 2a/ … crosses it out … writes correct answer

2. "Snorts … I've got a cold"

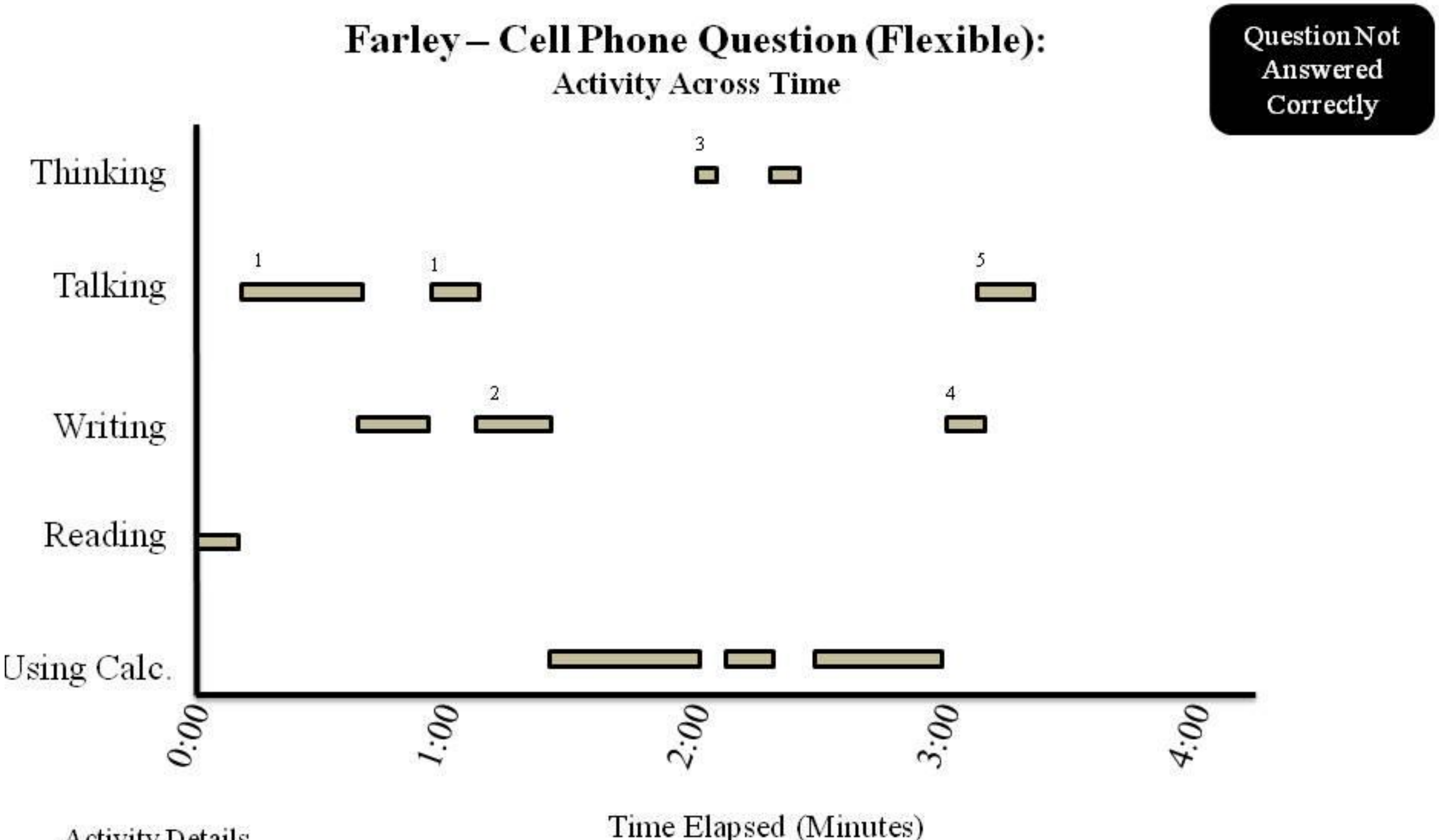
Figure A21. Farley's AAT for the Calculate Slope (Flexible) Question.



Activity Details

1. "Argh…system access fees!"
2. *Not possible to discern what was written
3. Writes cost of company b

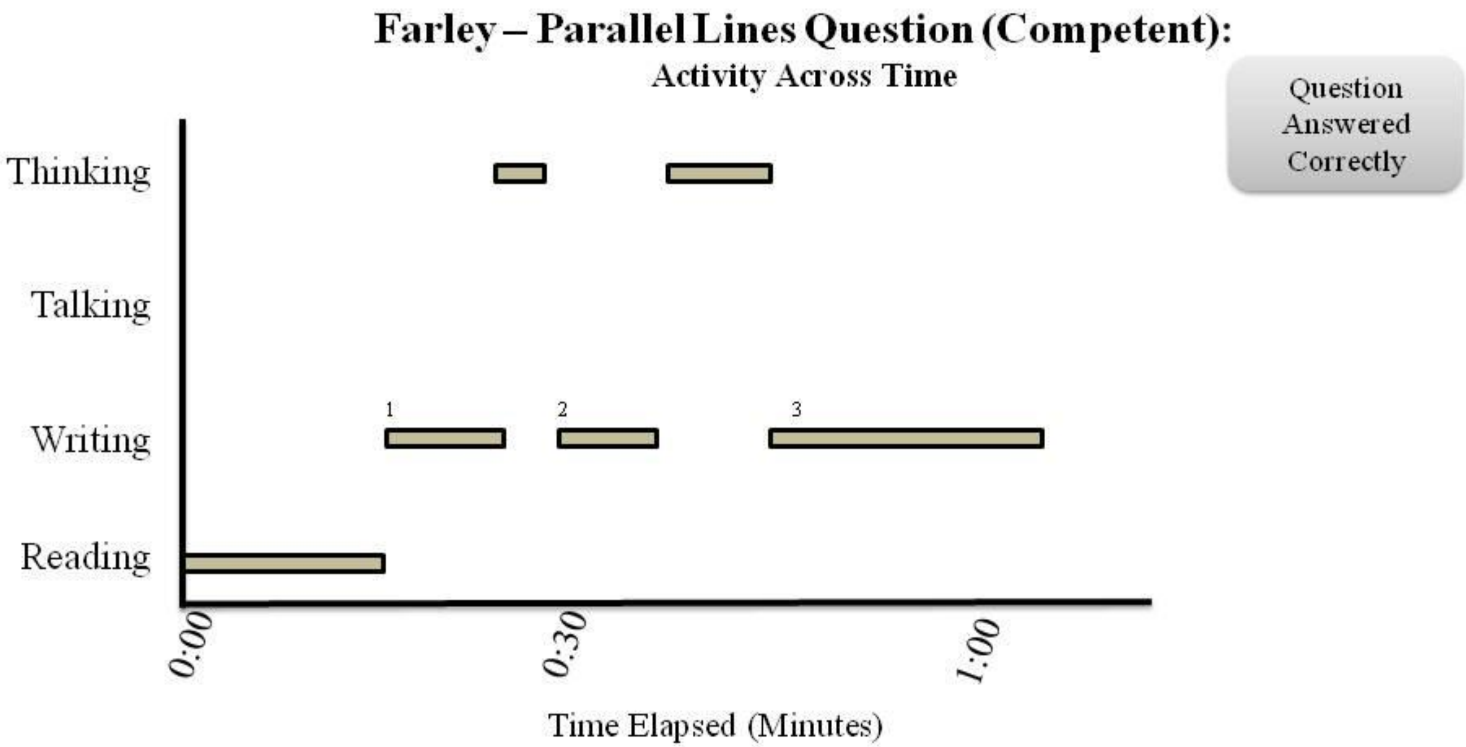
Figure A22. Farley's AAT for the Cell Phone (Competent) Question.



Activity Details

1. Clarifying acceptable answer
2. "Company A is cheaper when you talk more"
3. Scratches head
4. "Company A is cheaper if you were to talk for 4 hours a month"
5. Researcher and student discuss if problem is complete

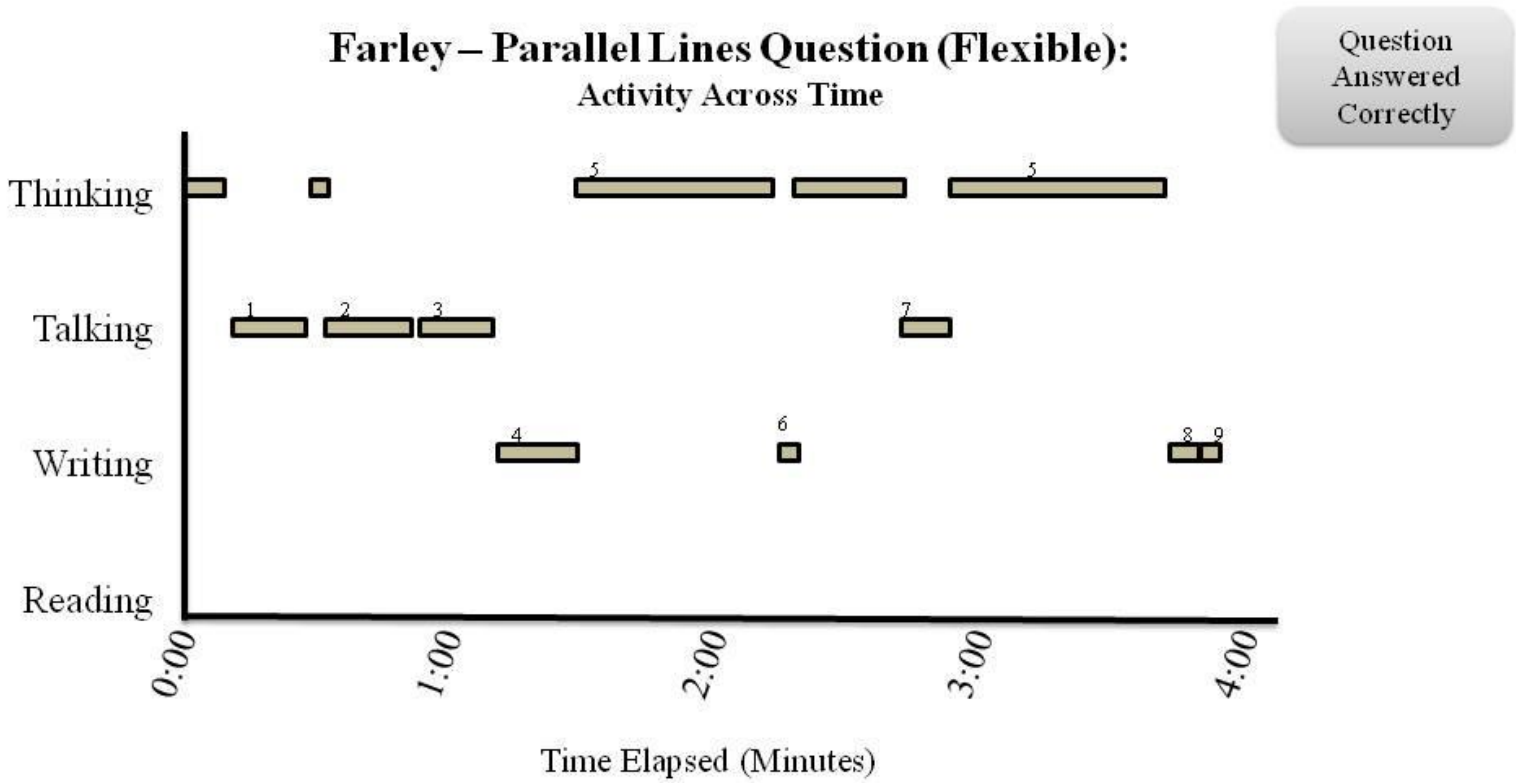
Figure A23. Farley's AAT for the Cell Phone (Flexible) Question.



Activity Details

1. Writes slope of a
2. Writes slope of b
3. "No, they have different slopes"

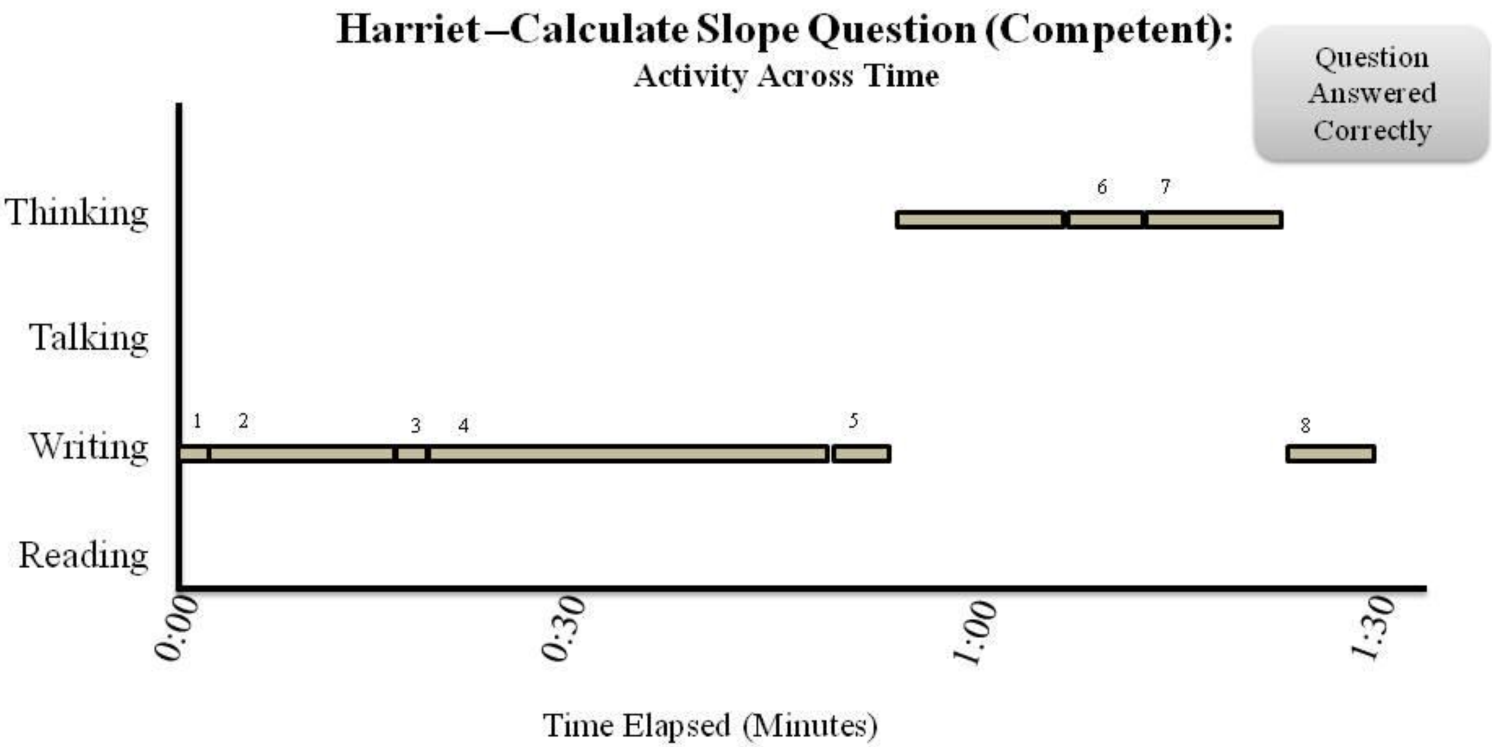
Figure A24. Farley's AAT for the Parallel Lines (Competent) Question.



Activity Details

1. Clarifying that the answer contains actual values
2. Researcher clarifies his understanding of student
3. Jokes that mom could not do this problem
4. Writes 2,1
5. Almost motionless
6. Writes 4
7. "Ok, it's really hard for me to envision this ... it's going to take a while"
8. Crosses out 4
9. Writes 7,3

Figure A25. Farley's AAT for the Parallel Lines (Flexible) Question.



Activity Details

1. Creating graph
2. Tracing over lines, creating unit points, ensuring equal numbers of unit marks
3. Graphs points
4. Ensuring correct # of unit marks
5. Draws line
6. Traces rise with pen
7. Traces run with pen
8. Writes and reduces slope

Figure A26. Harriet's AAT for the Calculate Slope (Competent) Question

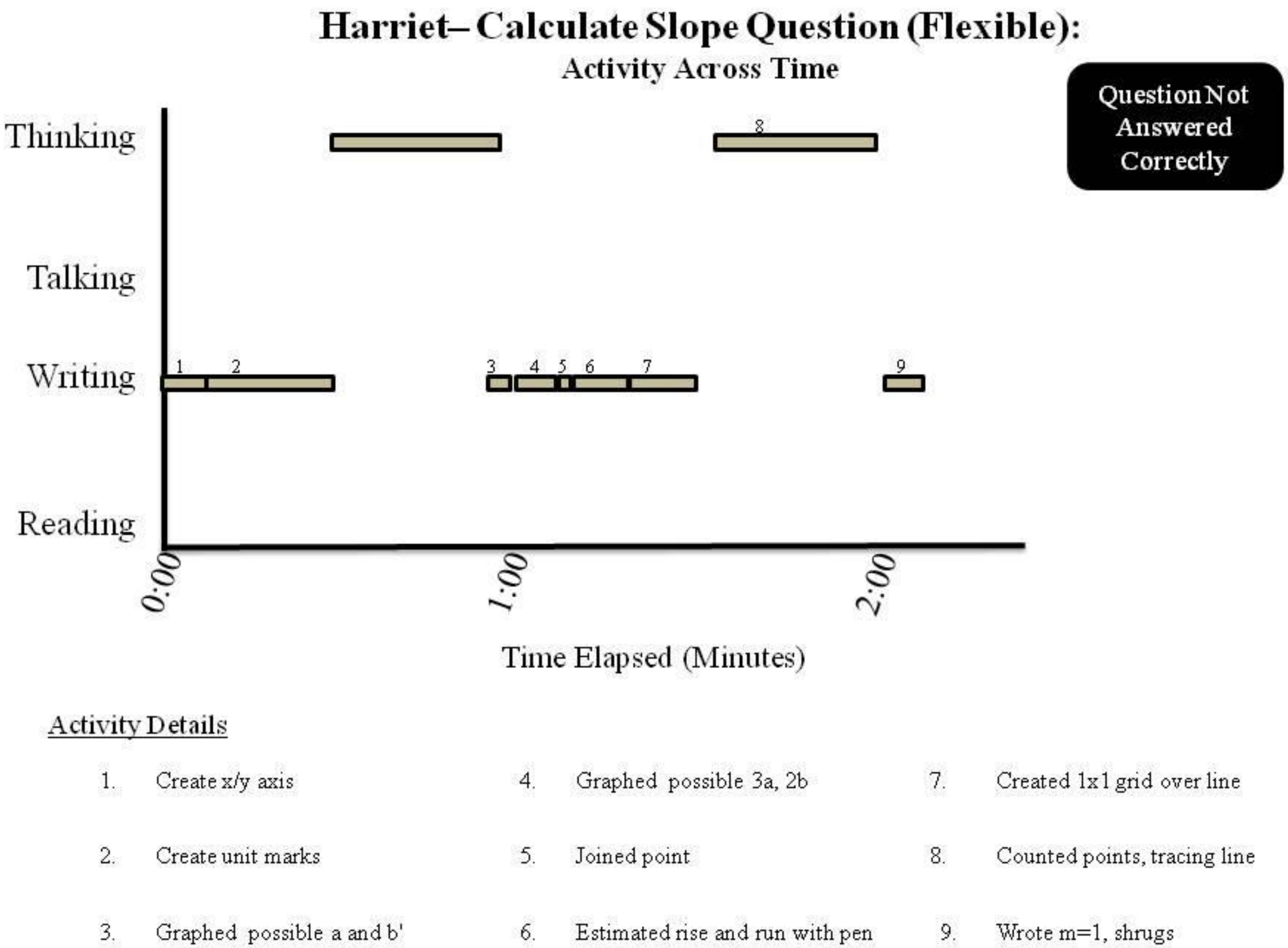
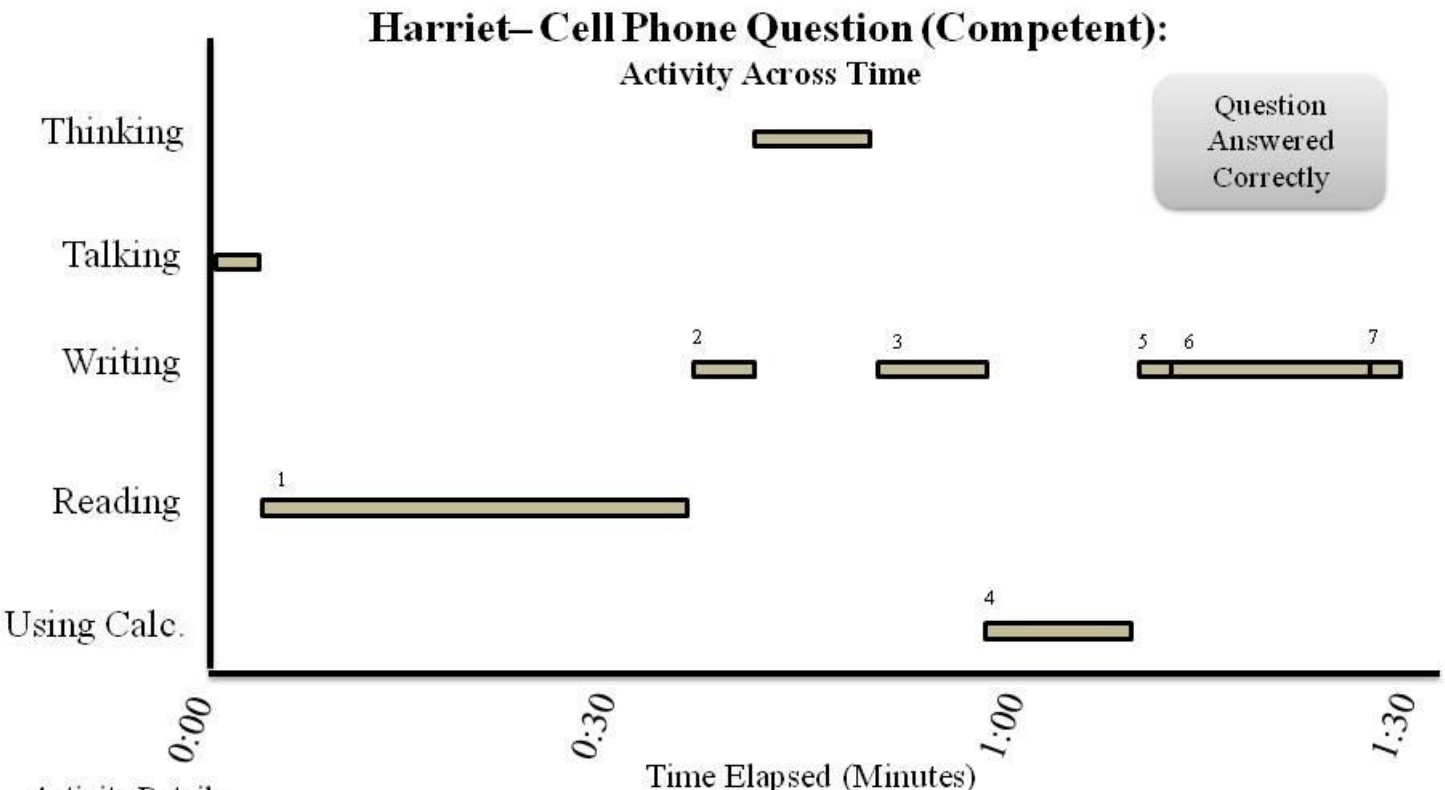


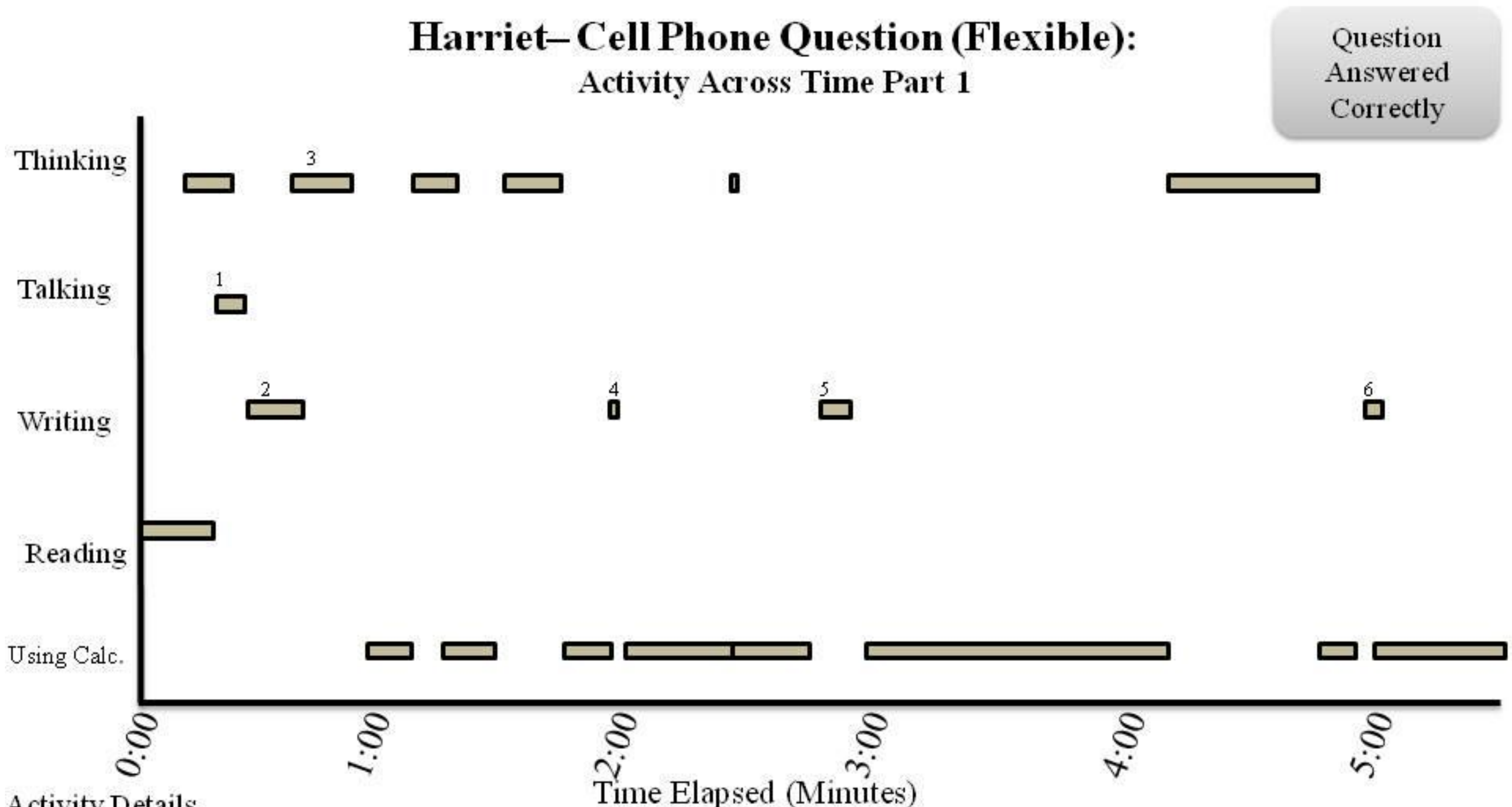
Figure A27. Harriet's AAT for the Calculate Slope (Flexible) Question.



Activity Details

1. Reads first sentence aloud
2. Writes expression for Company A
3. Writes expression for Company B
4. Calculates cost of Company A
5. Writes cost of Company A
6. Calculates cost of Company B
7. Writes cost of Company B

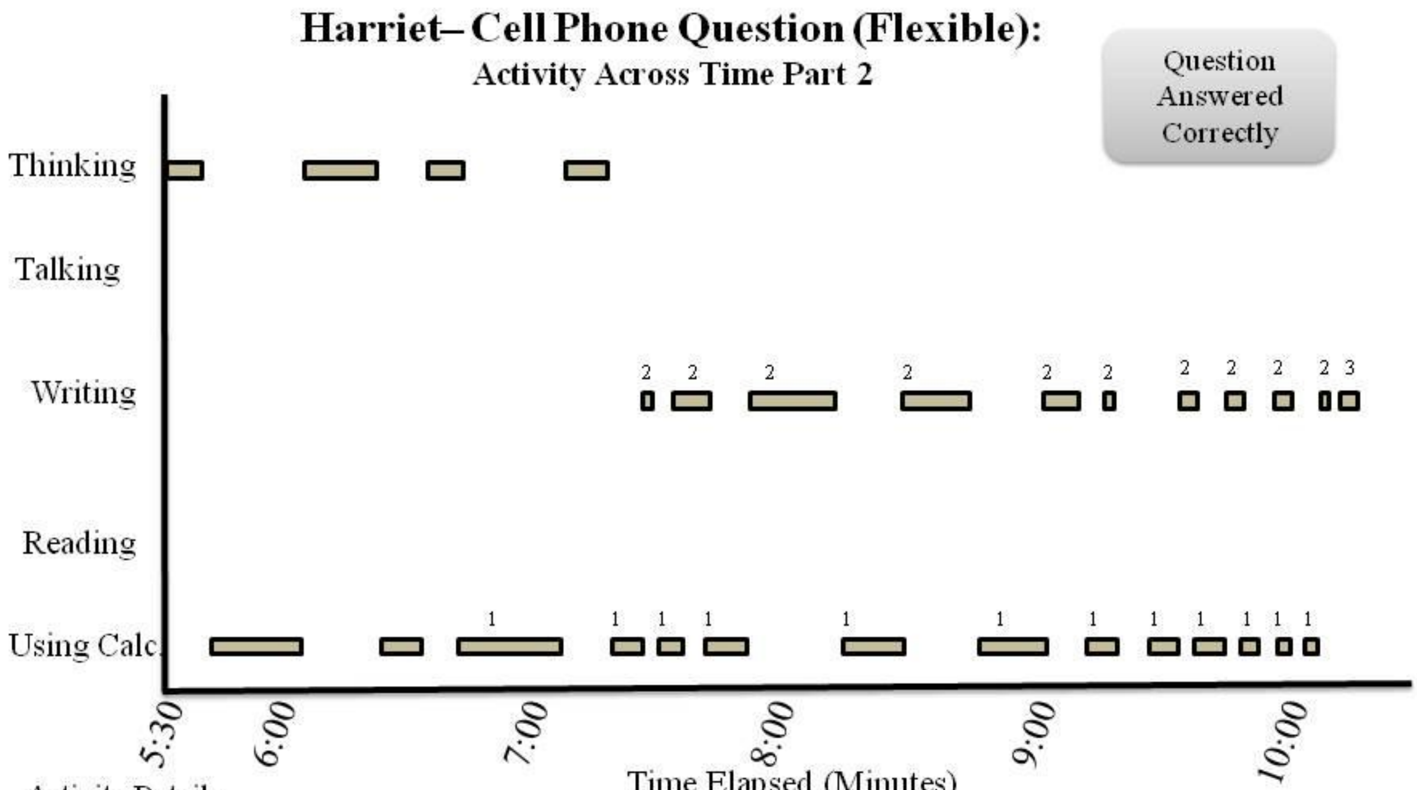
Figure A28. Harriet’s AAT for the Cell Phone (Competent) Question.



Activity Details

1. Asks to look back at last question
2. Copies result from last sheet
3. Looking up
4. Writes single number lightly
5. Writes ingle number
6. Calculates cost from both companies, given some number of minutes

Figure29.1. Harriet's AAT for the Cell Phone (Flexible) Question (0-5 min).



Activity Details

1. Calculating both companies for some 'm'
2. Writing cost from both companies, given some number of minutes
3. Writing answer

Figure A29.2. Harriet's AAT for the Cell Phone (Flexible) Question (5-10 min).

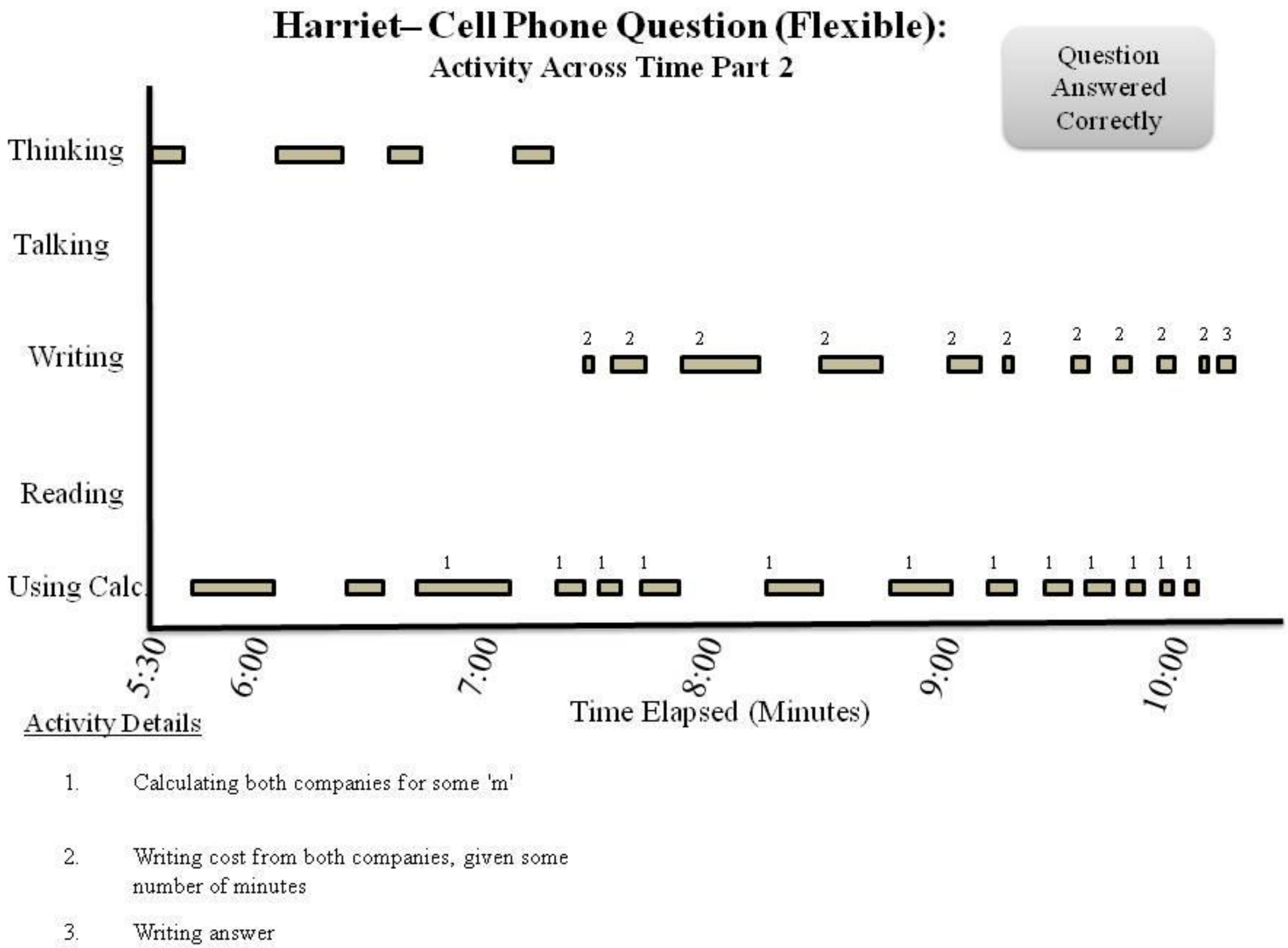
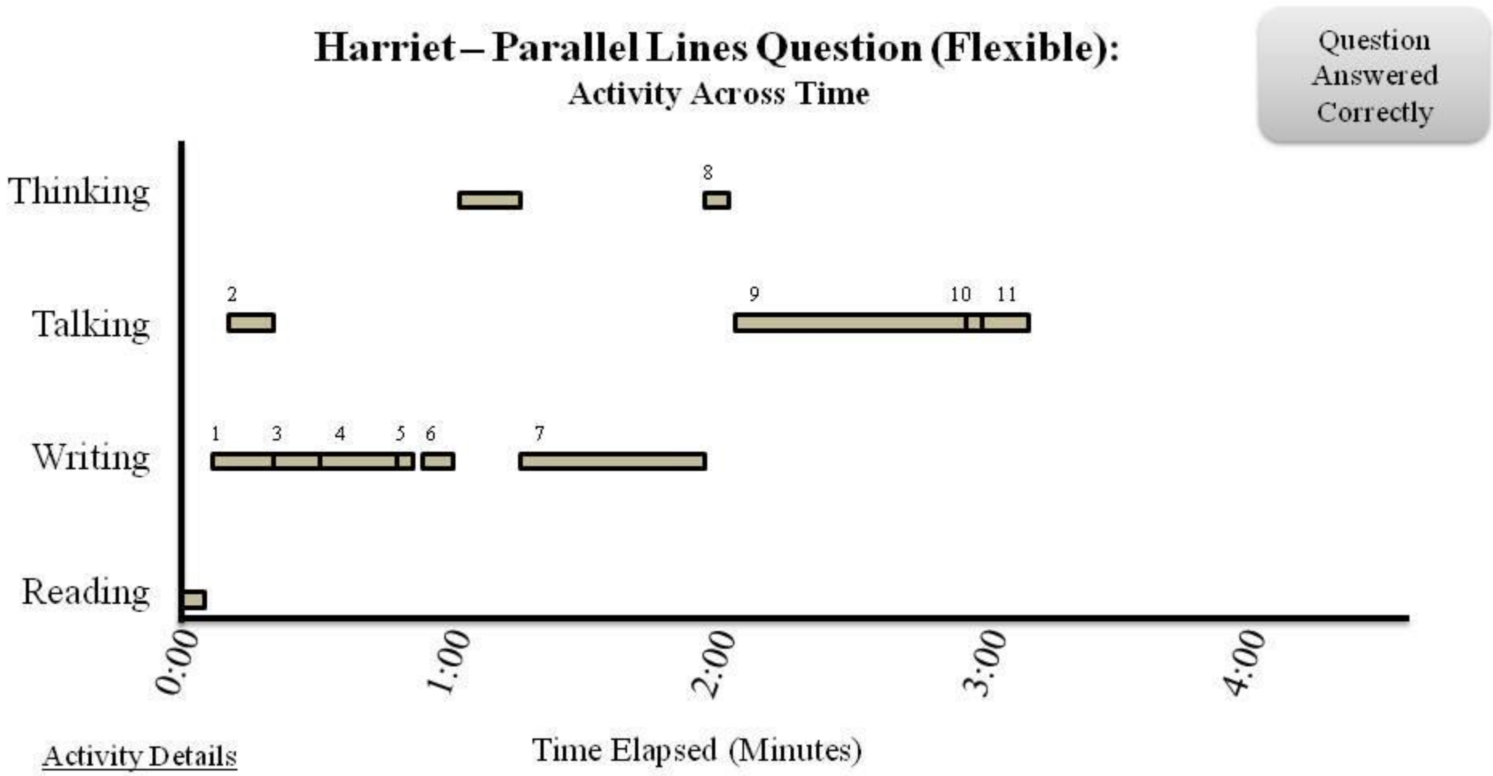


Figure A30. Harriet's AAT for the Parallel Lines (Competent) Question.



Activity Details

1. Drawing x/y axis
2. "I talk to myself while I'm doing this"
3. Creates unit marks
4. Graphs first point
5. Graphs second point
6. Joins points
7. Writes / crosses out /writes /crosses out / writes expression for both slopes
8. Picks up calculator
9. "there's 100 things it could be … so 11 and 1"
10. Researcher clarifies answer
11. Harriet restates correct answer

Figure A31. Harriet's AAT for the Parallel Lines (Flexible) Question.