Finding Conservation Laws for Partial Differential Equations

by

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Abstract

In this thesis, we discuss systematic methods of finding conservation laws for systems of partial differential equations (PDEs). We first review the direct method of finding conservation laws. In order to use the direct method, one first seeks a set of conservation law multipliers so that a linear combination of the PDEs with the multipliers will yield a divergence expression. Once a set of conservation law multipliers is determined, one proceeds to find the fluxes of the conservation law.

As the solution to the problem of finding conservation law multipliers is well-understood, in this thesis we focus on constructing the fluxes assuming the knowledge of a set of conservation law multipliers. First, we derive a new method called the flux equation method and show that, in general, fluxes can be found by at most computing a line integral. We show that the homotopy integral formula is a special case of the line integral formula obtained from the flux equations. We also show how the line integral formula can be simplified in the presence of a point symmetry of the PDE system and of the set of conservation law multipliers. By examples, we illustrate that the flux equation method can derive fluxes which would be otherwise difficult to find. We also review existing known methods of finding fluxes and make comparison with the flux equation method.

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Chapter 1

Introduction

A conservation law of a system of partial differential equations (PDEs) is a divergence expression which vanishes on solutions of the PDE system. The origin of conservation laws stems from physical principles such as conservation of mass, momentum and energy. Furthermore, conservation laws have applications in the study of PDEs such as in showing existence and uniqueness of solutions for hyperbolic systems of conservation laws [1], and as well as in developing numerical methods such as finite element methods [2, 3]. Naturally, two questions that one might ask are:

1) How does one find conservation laws for a given PDE system?

2) And if so, can one find conservation laws systematically?

Traditionally, conservation laws were derived from rather ad-hoc approaches. Although there is a well-known systematic method of finding conservation laws for variational PDEs due to Noether [4], the applicability of Noether's method is limited by the fact that there are many interesting PDE systems which are not variational as written. To tackle the above two questions at once, a systematic approach of finding conservation laws, called the direct method, has been developed recently [5–7]. There are two main steps to the direct method:

1) Determine a set of conservation law multipliers so that a linear combination of the PDEs with the conservation law multipliers yields a divergence expression.

2) Having determined a set of conservation law multipliers, find the corresponding fluxes to obtain the conservation law.

It is known that conservation law multipliers can be found using the method of Euler operator. In this thesis, our focus is on tackling the general problem of finding fluxes once a set of conservation law multipliers has been determined. In Chapter 2, we present the background on conservation law multipliers and their equivalence classes. Then, we will highlight the intimate connection of conservation law multipliers and conservation laws of a given PDE system.

In Chapter 3, we present a new method called the flux equation method. From the flux equations, the key result is that, in general, fluxes can be found by at most computing a line integral. Various examples will illustrate the computational efficiency and simplicity of using the flux equation method to derive conservation laws. We also show how the line integral formula can be simplified in the presence of a point symmetry of the PDE system and a set of conservation law multipliers. By examples, we will use the flux equation method to derive conservation laws which are otherwise difficult to find with existing known methods.

In Chapter 4, we review known methods of finding fluxes and make comparison with the flux equation method. We will discuss general methods of constructing fluxes such as the matching method and the homotopy integral formula, as well as more specialized methods such as Noether's Theorem and the method of a non-critical scaling symmetry. In particular, we will show that the homotopy integral formula is in fact a special case of the line integral formula obtained from the flux equations.

Chapter 2

Conservation Laws and Conservation Law Multipliers

This chapter introduces the background material on conservation laws for PDEs. In general, there are many conservation laws for a given PDE system such as trivial conservation laws. Since trivial conservation laws do not give new information specifically about a PDE system, it is natural to consider conservation laws up to equivalence in a manner to be made precise later. The central objects in the study of conservation laws are the sets of conservation law multipliers. The key property of sets of conservation law multipliers is that their existence implies the existence of conservation laws. Conversely, it turns out for non-degenerate PDE systems, every conservation law up to equivalence must arise from a set of conservation law multipliers. In general, the correspondence between sets of conservation law multipliers and equivalent conservation laws can be many-to-one. However, if a PDE system admits a Cauchy-Kolvalevskaya form and the sets of conservation law multipliers satisfy some mild conditions, then there is a one-to-one correspondence between each set of conservation law multipliers and each set of equivalent conservation laws.

2.1 Conservation Laws

Before defining conservation laws for PDEs, we first introduce some notation.

Let $\mathcal{R} = \{R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0\}_{\sigma=1}^N$ be a system of N PDEs defined on a domain[1] $\mathcal{D} \subset \mathbb{R}^n$ with at most k-th order partial derivatives of $u(x) = (u^1(x), \ldots, u^m(x))$ with respect to $x = (x^1, \ldots, x^n)$.

Using standard notations, we denote $C^k(\mathcal{D})$ as the family of functions which are continuously differentiable in $\mathcal{D}$ up to the k-th order. We call functions belonging to the family $C^\infty(\mathcal{D})$ smooth functions.

[1]We will always refer to a domain as a connected open subset of $\mathbb{R}^n$.

To avoid confusion that may arise, we use $u(x) = (u^1(x), \ldots, u^m(x))$ exclusively to denote a solution of the PDE system $\mathcal{R}$ with $C^k(\mathcal{D})$ components. By equality of mixed partial derivatives, for any $V(x) = (V^1(x), \ldots, V^m(x))$ with $C^k(\mathcal{D})$ components and a fixed $r = 0, \ldots, k$, there are $\binom{n+r-1}{r}$ different r-th order partial derivatives of $V^\rho(x)$ for each component $\rho = 1, \ldots, m$. Hence for each $r = 0, \ldots, k$, we define the Euclidean space $\mathcal{U}_r \simeq \mathbb{R}^{m\binom{n+r-1}{r}}$ labelled by $U = (U^1, \ldots, U^m)$ and its *prolongation*. The concept of prolongation is best explained by an example. Let $n = 2$, $m = 2$ and $U = V(x, y)$ with $C^2(\mathcal{D})$ components. Then at each $(x, y) \in \mathcal{D}$, we have the following points:

$$\begin{aligned}
(U^1, U^2) &= (V^1, V^2)\big|_{(x,y)} \in \mathcal{U}_0 \simeq \mathbb{R}^2, \\
(U^1_x, U^1_y, U^2_x, U^2_y) &= (V^1_x, V^1_y, V^2_x, V^2_y)\big|_{(x,y)} \in \mathcal{U}_1 \simeq \mathbb{R}^4, \\
(U^1_{xx}, U^1_{xy}, U^1_{yy}, U^2_{xx}, U^2_{xy}, U^2_{yy}) &= (V^1_{xx}, V^1_{xy}, V^1_{yy}, V^2_{xx}, V^2_{xy}, V^2_{yy})\big|_{(x,y)} \in \mathcal{U}_2 \simeq \mathbb{R}^6.
\end{aligned}$$

We denote all the Euclidean spaces $\mathcal{U}_0, \ldots, \mathcal{U}_k$ by the k-th prolonged space[2] $\mathcal{U}^{(k)} = \mathcal{U}_0 \times \cdots \times \mathcal{U}_k$ and we denote the k-th prolongation of a point $U \in \mathcal{U}_0$ by the point $U^{(k)} \in \mathcal{U}^{(k)}$. For example, if $n = 2$, $m = 2$, and $U = V(x, y)$ with $C^2(\mathcal{D})$ components, then $U^{(2)}$ is the point in the second prolonged space $\mathcal{U}^{(2)}$ given by

$$U^{(2)} = (V^1, V^2, V^1_x, V^1_y, V^2_x, V^2_y, V^1_{xx}, V^1_{xy}, V^1_{yy}, V^2_{xx}, V^2_{xy}, V^2_{yy})\big|_{(x,y)} \in \mathcal{U}^{(2)} \simeq \mathbb{R}^{12}.$$

Note that since each $\mathcal{U}_r$ has $m\binom{n+r-1}{r}$ many components, $\mathcal{U}^{(k)}$ has in total $m\sum_{r=0}^{k}\binom{n+r-1}{r} = m\binom{n+k}{k}$ components; i.e. $\mathcal{U}^{(k)} \simeq \mathbb{R}^{m\binom{n+k}{k}}$.

For convenience, we often use square brackets $[x, U]$ to denote the dependences for functions defined on $\mathcal{D} \times \mathcal{U}^{(k)}$. For example, given a function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$, we use $f[x, U]$ to denote the value of f evaluated at $x \in \mathcal{D}$ and $U^{(k)} \in \mathcal{U}^{(k)}$; i.e. $f(x, U^{(k)})$ and $f[x, U]$ mean the same expression. For example, the expressions of the PDEs of $\mathcal{R}$ can be denoted as $R^\sigma[x, U]$ for each $\sigma = 1, \ldots, m$. In particular, $R^\sigma[x, U]$ vanishes for all $\sigma = 1, \ldots, m$ on any $C^k(\mathcal{D})$ solution $U = u(x)$ of the PDE system $\mathcal{R}$.

As usual, a function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ is continuous at $(x, U^{(k)}) \in \mathcal{D} \times \mathcal{U}^{(k)}$ if f is continuous at both $x \in \mathcal{D}$ and $U^{(k)} \in \mathcal{U}^{(k)} \simeq \mathbb{R}^{m\binom{n+k}{k}}$. Similarly, $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ is differentiable at $(x, U^{(k)}) \in \mathcal{D} \times \mathcal{U}^{(k)}$ if f is differentiable at both $x \in \mathcal{D}$ and $U^{(k)} \in \mathcal{U}^{(k)} \simeq \mathbb{R}^{m\binom{n+k}{k}}$. For example, the function

[2] The k-th prolonged space $\mathcal{U}^{(k)}$ defined here is a simplified version of the *k-th order jet space*. For details, see [8] for generalizations.

$f[x,U] = \sqrt{(U_x)^2 + (U_y)^2}$ is differentiable (in fact smooth) everywhere on $\mathcal{D}$ and $\mathcal{U}^{(k)} \backslash \{0\}$.

We also use the repeated index summation convention throughout unless otherwise specified.

Definition 2.1.1. *Let $\mathcal{D}' \times \mathcal{V}^{(k)}$ be an open connected subset of $\mathcal{D} \times \mathcal{U}^{(k)}$. A local conservation law of the PDE system $\mathcal{R}$ is a divergence expression which vanishes on solutions $u(x)$ of the PDE system $\mathcal{R}$ defined on $\mathcal{D}' \times \mathcal{V}^{(k)}$; more precisely, there exists smooth functions $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ such that for any $x \in \mathcal{D}'$ and any $C^k(\mathcal{D}')$ solution $u(x) \in \mathcal{V}^{(k)}$,*

$$D_i \Phi^i[x,U]\big|_{U=u(x)} = 0. \tag{2.1}$$

A global conservation law is a local conservation law which holds on $\mathcal{D} \times \mathcal{U}^{(k)}$; i.e. $\mathcal{D}' \times \mathcal{V}^{(k)}$ can be extended to all of $\mathcal{D} \times \mathcal{U}^{(k)}$.

The expressions $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ are called *fluxes* and D_i denotes the *total derivative* with respect to x^i which is a differential operator acting on smooth functions defined on $\mathcal{D} \times \mathcal{U}^{(k)}$ given by,

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|J| \le k} U^\rho_{J+\hat{i}} \frac{\partial}{\partial U^\rho_J},$$

where we have used the *multi-index* notation for the capital index J and $\hat{i}$.

Definition 2.1.2. *A multi-index $J = (j_1, \dots, j_n)$ is a vector with n components of nonnegative integers, where $|J| = \sum_{i=1}^n j_i$ denotes the length of J.*

For example, points in $\mathcal{U}^{(k)}$ can be efficiently labelled by using the multi-index notation; i.e. if $U = V(x)$ with $C^{|J|}(\mathcal{D})$ components, then we denote for each component $\rho = 1, \dots, m$,

$$U^\rho_J = \frac{\partial^{j_1}}{\partial x^{j_1}} \cdots \frac{\partial^{j_n}}{\partial x^{j_n}} V^\rho(x),$$

where we define $U^\rho_J = U^\rho$ if J is the null multi-index $0 = (0, \dots, 0)$. The multi-index notation is also useful when we need to take multiple total derivatives at once. For example, we denote

$$D_J = \underbrace{D_1 \cdots D_1}_{j_1 \text{times}} \underbrace{D_2 \cdots D_2}_{j_2 \text{times}} \cdots \underbrace{D_n \cdots D_n}_{j_n \text{times}}.$$

On occasions, we use $\hat{i}$ to denote the multi-index which has a one in the i-th component and zeros in all other components.

In general, two multi-indices I, J can be added and subtracted componentwise provided that each component remains nonnegative. For example, if $U = V(x)$ with $C^{|J|+1}(\mathcal{D})$ components, we denote

$$U^{\rho}_{J+\hat{i}} = \frac{\partial^{j_1}}{\partial x^{j_1}} \cdots \frac{\partial^{j_i+1}}{\partial x^{j_i+1}} \cdots \frac{\partial^{j_n}}{\partial x^{j_n}} V^{\rho}(x).$$

Example 2.1.3. *Let* $R[(t,x^1,\ldots,x^n), u(t,x^1,\ldots,x^n)] = u_t - \sum_{i=1}^n u_{x^i x^i} = 0$ *be the heat equation in* n *spatial dimensions with* $\mathcal{D} = [0,\infty) \times \mathbb{R}^n$*.*

Defining $\Phi^t[(t,x^1,\ldots,x^n),U] = U$ and $\Phi^i[(t,x^1,\ldots,x^n),U] = -U_{x^i}$ for all $i = 1,\ldots,n$, one can see that these fluxes yield a global conservation law

$$\begin{aligned} &D_t\Phi^t[(t,x^1,\ldots,x^n),U] + D_i\Phi^i[(t,x^1,\ldots,x^n),U]\Big|_{U=u(t,x^1,\ldots,x^n)} \\ &= \left(U_t - \sum_{i=1}^n U_{x^i x^i}\right)\Bigg|_{U=u(t,x^1,\ldots,x^n)} = 0. \end{aligned}$$

Example 2.1.4. *Let* $R[(t,x),u(t,x)] = u_t + u^c u_x + u_{xxx} = 0$ *be the generalized Korteweg-de Vries (KdV) equation in one spatial dimension with* $\mathcal{D} = [0,\infty) \times \mathbb{R}$ *and* $c > -2$*.*

Letting $\Phi^t[(t,x),U] = \frac{U^2}{2}$ and $\Phi^x[(t,x),U] = \frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U_x^2}{2}$, these fluxes yield a global conservation law

$$\begin{aligned} D_t\Phi^t[(t,x),U] + D_x\Phi^x[(t,x),U]\Big|_{U=u(t,x)} &= U(U_t + U^c U_x + U_{xxx})\big|_{U=u(t,x)} \\ &= 0. \end{aligned}$$

Example 2.1.5. *Let* $R[(t,x),u(t,x)] = u_{tt} - (c^2(u)u_x)_x = 0$ *be a nonlinear wave equation with smooth wave speed* $c(u)$ *in one spatial dimension with* $\mathcal{D} = [0,\infty) \times \mathbb{R}$*.*

Defining $\Phi^t[(t,x),U] = xtU_t - xU$ and $\Phi^x[(t,x),U] = -xtc^2(U)U_x$ $+t\int^U c^2(\mu)d\mu$, one can verify these fluxes yield a global conservation law

$$\begin{aligned} D_t\Phi^t[(t,x),U] + D_x\Phi^x[(t,x),U]\Big|_{U=u(t,x)} &= xt(U_{tt} - (c^2(U)U_x)_x)\Big|_{U=u(t,x)} \\ &= 0. \end{aligned}$$

The PDEs presented so far all possess global conservation laws.

Example 2.1.6. *Let* $R[(t,x,y),u(t,x,y)] = u_t - \sqrt{u_x^2 + u_y^2} = 0$ *be the 2D flame equation.*

Since $R[(t,x,y),U]$ is smooth everywhere except at $0 \in \mathcal{U}^{(k)}$, the 2D flame equation has only local conservation laws, as we will see in Chapter 3.

To motivate the definition of conservation laws, let's discuss how conservation laws can arise naturally in the study of PDEs. Firstly, if the PDE system $\mathcal{R}$ has an additional time variable $t \in [0,\infty)$; i.e. $\mathcal{D} = [0,\infty) \times \Omega$ for some bounded spatial domain $\Omega \subset \mathbb{R}^n$, then integrating a global conservation law over $x = (x^1, \ldots, x^n) \in \Omega$ and applying the divergence theorem yields

$$D_t \int_\Omega \Phi^t[(t,x),u(t,x)]d^n x = -\int_{\partial\Omega} \Phi^i[(t,x),u(t,x)]dS_i.$$

Hence, if each $\Phi^i[(t,x),u(t,x)]$ vanishes on the boundary $\partial\Omega$ for all $t \in [0,\infty)$, then $\int_\Omega \Phi^t[(t,x),u(t,x)]d^n x$ is a conserved quantity in time. Secondly, if we multiply a global conservation law by any compactly supported smooth function $\phi(x)$ on $\mathcal{D}$, i.e. $\phi(x)$ is smooth and vanishes on the boundary $\partial\Omega$, then integrating by parts yields

$$0 = \int_\Omega \phi(x) D_i \Phi^i[x,u(x)]d^n x = -\int_\Omega \Phi^i[x,u(x)]D_i\phi(x)d^n x. \tag{2.2}$$

Thus, this could yield a weaker formulation of the PDE system $\mathcal{R}$; since each of the $\Phi^i[x,u(x)]$ may contain only up to $(k-1)$-th order of derivatives of $u(x)$[3].

The study of existence and regularity of solutions of the PDE system $\mathcal{R}$ and their relation to conservation laws is not the subject matter of this thesis. Our main focus here is to derive fluxes for a given PDE system $\mathcal{R}$. Thus to avoid vacuous statements such as the existence of conservation laws to a PDE system with no $C^k(\mathcal{D})$ solutions, we adopt a simplifying assumption that the PDE system $\mathcal{R}$ of interest has a solution $u(x)$ of the PDE system $\mathcal{R}$ defined on some open neighbourhood of $x \in \mathcal{D}$ that is sufficiently smooth[4]. How smooth the solutions need to be will depend on the particular PDE system $\mathcal{R}$. For example, we will assume throughout that a solution $u(x)$ of

[3] In general, the set of fluxes can depend up to the maximal order of derivatives appearing in the PDE systems and in the set of *conservation law multipliers*; see [12] or later in Chapter 2 and 3.

[4] This is the case if the PDE system $\mathcal{R}$ admits a Cauchy-Kovalevskaya form. See Appendix B.

a k-th order PDE system $\mathcal{R}$ has at least k times continuously differentiable components. We leave the extension of the results on conservation laws of weak solutions for future investigations.

2.2 Equivalence Class of Conservation Laws

Given two local conservation laws of the PDE system $\mathcal{R}$ both defined on $\mathcal{D}' \times \mathcal{V}^{(k)}$, adding them together yields another local conservation law and multiplying a local conservation law by a scalar over $\mathbb{R}$ also yields a local conservation law. Thus, the set of local conservation laws for a given PDE system $\mathcal{R}$ defined on $\mathcal{D}' \times \mathcal{V}^{(k)}$ satisfies the axioms of a vector space.

Definition 2.2.1. *The vector space over $\mathbb{R}$ of local conservation laws of the PDE system $\mathcal{R}$ defined on $\mathcal{D}' \times \mathcal{V}^{(k)}$ with component addition and scalar multiplication is denoted by*

$$\widetilde{CL}(\mathcal{R};\mathcal{D}'\times\mathcal{V}^{(k)}) = \left\{ (\Phi^1,\ldots,\Phi^n) \;\middle|\; \begin{array}{l} \text{Each } \Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R} \text{ smooth,} \\ D_i\Phi^i[x,U]\big|_{U=u(x)} = 0 \end{array} \right\}.$$

We simply use $\widetilde{CL}(\mathcal{R})$ to denote $\widetilde{CL}(\mathcal{R};\mathcal{D}\times\mathcal{U}^{(k)})$.

In practice, some (local or global) conservation laws are not as useful as others. For example, this is the situation if a conservation law arises through differential identities, such as

$$D_x(e) + D_y(\pi) = 0,$$

$$D_x(U_y) + D_y(-U_x) = U_{yx} - U_{xy} = 0.$$

Moreover, if for some smooth functions $C^i_{\sigma,J}[x,U]$, a conservation law has fluxes of the form

$$\Phi^i[x,U] = \sum_{\sigma,|J|\leq k} C^i_{\sigma,J}[x,U] D_J R^\sigma[x,U],$$

then $\Phi^i[x,u(x)] = 0$ identically on any solution $u(x)$ of the PDE system $\mathcal{R}$. For example,

$$D_x(e^{-U^2}R^1[(x,y),U]) + D_y(\cos(xy)D_xR^2[(x,y),U])\Big|_{U=u(x)} = 0.$$

These two types of conservation laws are called trivial conservation laws. Since trivial conservation laws do not provide new information specifically about the PDE system $\mathcal{R}$, we are only interested in finding non-trivial conservation laws. Hence, we consider two conservation laws as equivalent if the difference between their flux components yields a trivial conservation law. This leads to the following definition of an equivalence class of local conservation laws.

Definition 2.2.2. *$CL(\mathcal{R};\mathcal{D}'\times\mathcal{V}^{(k)}) = \{[\Phi] : \Phi \in \widetilde{CL}(\mathcal{R};\mathcal{D}'\times\mathcal{V}^{(k)})\}$ is the set of equivalence classes of local conservation laws of the PDE system $\mathcal{R}$, where $\Phi, \Psi \in \widetilde{CL}(\mathcal{R})$ are equivalent if and only if $\Phi - \Psi$ is a trivial conservation law. Again, we simply use $CL(\mathcal{R})$ to denote $CL(\mathcal{R};\mathcal{D}\times\mathcal{U}^{(k)})$.*

To keep the notations simple, we will often just use Φ in short to denote the equivalence class of $[\Phi]$, while keeping in mind that conservation laws are distinguished up to their equivalence.

2.3 Euler Operator

Definition 2.3.1. *The Euler operator with respect to component U^ρ for $\rho = 1,\dots,m$, denoted by $\mathcal{E}_\rho$, is a differential operator acting on smooth functions defined on $\mathcal{D}\times\mathcal{U}^{(k)}$ given by*

$$\mathcal{E}_\rho = \sum_{|J|\le k} (-1)^{|J|} D_J \frac{\partial}{\partial U^\rho_J}. \tag{2.3}$$

If $m = 1$, we omit the subscript ρ and simply write $\mathcal{E}$ as the Euler operator.

The fundamental property of the Euler operator is captured in the following theorem.

Theorem 2.3.2. *Let $\mathcal{D}'$ be a bounded[5] simply-connected open subset of $\mathcal{D}$ and let $\mathcal{V}^{(k)}$ be a connected open subset of $\mathcal{U}^{(k)}$. For any smooth function $f : \mathcal{D}'\times\mathcal{V}^{(k)} \to \mathbb{R}$, there exists smooth functions $\{\Phi^i : \mathcal{D}'\times\mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ such that $f[x,U] = D_i\Phi^i[x,U]$ everywhere on $\mathcal{D}'\times\mathcal{V}^{(k)}$ if and only if $\mathcal{E}_\rho(f[x,U]) = 0$ everywhere on $\mathcal{D}'\times\mathcal{V}^{(k)}$ for all $\rho = 1,\dots,m$.*

[5] If $\mathcal{D}'$ is unbounded, an additional assumption on f is needed to warrant interchanging the order of differentiation and integration as we will see in the course of the proof. For example, the order of differentiation and integration can be interchanged provided there is an integrable function $g(x)$ on $\mathcal{D}'$ such that $\left|\dfrac{\partial f[x,U]}{\partial U^\rho_I}\right| \le g(x)$ for every multi-index I with $|I| \le k$ and at every point in $\mathcal{V}^{(k)}$.

Proof. Suppose $f[x,U] = D_i\Phi^i[x,U]$ identically on $\mathcal{D}' \times \mathcal{V}^{(k)}$. Choose any $U^{(k)} \in \mathcal{U}^{(k)}$. Then for any $s \in \mathbb{R}$ and any smooth function $\xi(x) = (\xi^1(x), \ldots, \xi^m(x))$ compactly supported on $\mathcal{D}'$, the divergence theorem yields:

$$\begin{aligned} \frac{d}{ds}\int_{\mathcal{D}'} f[x,U+s\xi(x)]d^n x &= \frac{d}{ds}\int_{\partial\mathcal{D}'} \Phi^i[x,U+s\xi(x)]dS_i \\ &= \frac{d}{ds}\int_{\partial\mathcal{D}'} \Phi^i[x,U]dS_i = 0. \end{aligned}$$

Since $\overline{\mathcal{D}'}$ is compact, we can interchange the derivative and the integral sign of the above expression and evaluate the integral at $s = 0$:

$$\begin{aligned} 0 &= \left(\frac{d}{ds}\int_{\mathcal{D}'} f[x,U+s\xi(x)]d^n x\right)\Bigg|_{s=0} = \int_{\mathcal{D}'} \frac{d}{ds}\Big(f[x,U+s\xi(x)]\Big)\Bigg|_{s=0} d^n x \\ &= \int_{\mathcal{D}'} \sum_{|J|\leq k} \frac{\partial f}{\partial U^\rho_J}[x,U] D_J\left(\xi^\rho(x)\right) dx \\ &= \int_{\mathcal{D}'} \sum_{|J|\leq k} (-1)^{|J|}\left(D_J \frac{\partial f}{\partial U^\rho_J}\right)[x,U]\xi^\rho(x)dx = \int_{\mathcal{D}'} \mathcal{E}_\rho(f[x,U])\xi^\rho(x)dx, \end{aligned}$$

where the second last equality follows from repeatedly integrating by parts for each multi-index J and the fact that $\xi(x)$ vanishes on the boundary of $\mathcal{D}'$. Since $\xi^\rho(x)$ can be chosen to be any smooth functions compactly supported on $\mathcal{D}'$ and $\mathcal{E}_\rho(f[x,U])$ is continuous on $\mathcal{D}'$, $\mathcal{E}_\rho(f[x,U]) = 0$ identically for all $x \in \mathcal{D}'$ for all $\rho = 1, \ldots, m$. Since this is true for an arbitrary choice of $U^{(k)} \in \mathcal{V}^{(k)}$, the forward implication is proved.

We delay the proof of the converse of this theorem until Chapter 3 when we have access to a fundamental divergence identity relating the Euler operator. □

The connectedness assumption on $\mathcal{V}^{(k)}$ in Theorem 2.3.2 is crucial. Consider the expression $f[x,U] = \frac{U_x}{U}$ which is smooth on two disconnected subsets $\mathcal{V}^{(k)}_+ = \{U^{(k)} \in \mathcal{U}^{(k)} \big| \, U > 0\}$ and $\mathcal{V}^{(k)}_- = \{U^{(k)} \in \mathcal{U}^{(k)} \big| \, U < 0\}$.

On $\mathcal{V}^{(k)}_+$, $\ln(U)$ is well-defined and $D_x(\ln(U)) = \frac{U_x}{U} = f[x,U]$. While on $\mathcal{V}^{(k)}_-$, $\ln(-U)$ is well-defined and $D_x(\ln(-U)) = \frac{-U_x}{-U} = f[x,U]$. Thus even though $\mathcal{E}(f[x,U]) = 0$ everywhere on $\mathcal{V}^{(k)} = \mathcal{V}^{(k)}_+ \cup \mathcal{V}^{(k)}_-$, there cannot be a single smooth function $\Phi^x[x,U]$ which equals $\ln(U)$ on $\mathcal{V}^{(k)}_+$ and simultaneously equals $\ln(-U)$ on $\mathcal{V}^{(k)}_-$!

Example 2.3.3. *Let* $R[(t,x^1,\dots,x^n),u(t,x^1,\dots,x^n)] = u_t - \sum_{i=1}^n u_{x^ix^i} = 0$ *be the heat equation in* n *spatial dimensions with* $\mathcal{D} = [0,\infty) \times \mathbb{R}^n$.

The expression of the heat equation itself is a divergence expression defined on the entire $\mathcal{D} \times \mathcal{U}^{(k)}$ since

$$\begin{aligned} R[(t,x^1,\dots,x^n),U] &= U_t - \sum_{i=1^n} U_{x^ix^i} \\ &= D_t\,(U) + D_{x^i}\,(-U_{x^i})\,. \end{aligned}$$

Thus, according to Theorem 2.3.2, the expression $R[(t,x^1,\dots,x^n),U]$ should vanish identically on $\mathcal{D} \times \mathcal{U}^{(k)}$ upon applying the Euler operator $\mathcal{E}$. Indeed, this is the case since

$$\begin{aligned} -D_t\frac{\partial}{\partial U_t}\Big(\,R[(t,x^1,\dots,x^n),U]\Big) &= -D_t(1) = 0, \\ D_{x^ix^i}\frac{\partial}{\partial U_{x^ix^i}}\Big(\,R[(t,x^1,\dots,x^n),U]\Big) &= D_{x^ix^i}(-1) = 0. \end{aligned}$$

Hence, summing the above expressions yields $\mathcal{E}(R[(t,x^1,\dots,x^n),U]) = 0$ identically on $\mathcal{D} \times \mathcal{U}^{(k)}$.

Example 2.3.4. *Let* $R[(t,x),u(t,x)] = u_t + u^c u_x + u_{xxx} = 0$ *be the generalized KdV equation in one spatial dimension with* $\mathcal{D} = [0,\infty) \times \mathbb{R}$ *and* $c > -2$.

The product of U with the expression of the generalized KdV equation itself is a divergence expression defined on $\mathcal{D} \times \mathcal{U}^{(k)}$ since

$$\begin{aligned} U \cdot R[(t,x),U] &= UU_t + U^{c+1}U_x + UU_{xxx} \\ &= D_t\left(\frac{U^2}{2}\right) + D_x\left(\frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U_x^2}{2}\right). \end{aligned}$$

Hence, according to Theorem 2.3.2, the expression $U \cdot R[(t,x),U]$ should vanish identically on $\mathcal{D} \times \mathcal{U}^{(k)}$ upon applying the Euler operator $\mathcal{E}$. Indeed, this is the case since

$$\begin{aligned} \frac{\partial}{\partial U}\Big(\,U \cdot R[(t,x),U]\Big) &= U_t + (c+1)U^cU_x + U_{xxx}, \\ -D_t\frac{\partial}{\partial U_t}\Big(\,U \cdot R[(t,x),U]\Big) &= -D_t(U) = -U_t, \\ -D_x\frac{\partial}{\partial U_x}\Big(\,U \cdot R[(t,x),U]\Big) &= -D_x(U^{c+1}) = -(c+1)U^cU_x, \\ -D_{xxx}\frac{\partial}{\partial U_{xxx}}\Big(\,U \cdot R[(t,x),U]\Big) &= -D_{xxx}(U) = -U_{xxx}. \end{aligned}$$

Thus, summing the above expressions yields $\mathcal{E}\left(U \cdot R[(t,x),U]\right) = 0$ identically on $\mathcal{D} \times \mathcal{U}^{(k)}$.

2.4 Conservation Law Multipliers

Definition 2.4.1. *Let $\mathcal{D}' \times \mathcal{V}^{(k)}$ be a subdomain (i.e. a connected open subset) of $\mathcal{D} \times \mathcal{U}^{(k)}$. A set of smooth functions $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is called a set of local conservation law multipliers of the PDE system $\mathcal{R}$ if there exists smooth functions $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ such that everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:*

$$\Lambda_\sigma[x,U]R^\sigma[x,U] = D_i\Phi^i[x,U]. \tag{2.4}$$

A set of global conservation law multipliers of the PDE system $\mathcal{R}$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$ which is defined on all of $\mathcal{D} \times \mathcal{U}^{(k)}$.

Thus equivalently, by Theorem (2.3.2), $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$ if and only if

$$\mathcal{E}_\rho(\Lambda_\sigma[x,U]R^\sigma[x,U]) = 0, \text{ everywhere on } \mathcal{D}' \times \mathcal{V}^{(k)} \text{ for all } \rho = 1,\dots,m. \tag{2.5}$$

Hence to find a set of conservation law multipliers, one can proceed to solve the system of PDEs (2.5) for the unknowns $\Lambda_\sigma[x,U]$.[6]

From the definition of a set of local conservation law multipliers, we see immediately that the existence of a set of local conservation law multipliers implies the existence of a local conservation law.

Theorem 2.4.2. *Suppose $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$, then there exists a corresponding local conservation law for the PDE system $\mathcal{R}$.*

Proof.

$$D_i\Phi^i[x,U]\Big|_{U=u(x)} = \Lambda_\sigma[x,U]R^\sigma[x,U]\Big|_{U=u(x)} = \Lambda_\sigma[x,u(x)]R^\sigma[x,u(x)] = 0.$$

□

Example 2.4.3. *Let $R[(t,x^1,\dots,x^n),u(t,x^1,\dots,x^n)] = u_t - \sum_{i=1}^n u_{x^ix^i} = 0$ be the heat equation with $\mathcal{D} = [0,\infty) \times \mathbb{R}^n$.*

[6] This procedure using Euler operators to find sets of conservation law multipliers is discussed thoroughly in [7].

One can see easily that one choice of a global conservation law multiplier is $\Lambda[(t,x),U] = 1$ since

$$1 \cdot R[(t,x),U] = D_t(U) - D_{x^i}(U_{x^i}).$$

In general, 1 is always a conservation law multiplier if $R[(t,x),U]$ is already a divergence expression.

Example 2.4.4. *Let $R[(t,x),u(t,x)] = u_t + u^c u_x + u_{xxx} = 0$ be the generalized KdV equation in 1D with $\mathcal{D} = [0,\infty) \times \mathbb{R}$ and $c > -2$.*

One can see that $\Lambda[x,U] = U$ is a global conservation law multiplier since

$$U(U_t + U^c U_x + U_{xxx}) = D_t\left(\frac{U^2}{2}\right) + D_x\left(\frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U_x^2}{2}\right).$$

Example 2.4.5. *Let $R[(t,x),u(t,x)] = u_{tt} - (c^2(u)u_x)_x = 0$ be the nonlinear wave equation with smooth wave speed $c(u)$ with $\mathcal{D} = [0,\infty) \times \mathbb{R}$.*

One can check that $\Lambda[x,U] = xt$ is a global conservation law multiplier since

$$xt(U_{tt} - (c^2(U)U_x)_x) = D_t(xtU_t - xU) + D_x\left(-xtc^2(U)U_x + t\int^U c^2(\mu)d\mu\right).$$

There is a partial converse to Theorem 2.4.2, i.e. every global conservation law of the PDE system $\mathcal{R}$ up to equivalence of trivial conservation laws arises from a set of global conservation law multipliers. In particular, this is the case if we assume the PDE system $\mathcal{R}$ is *non-degenerate*[7].

Theorem 2.4.6. *Suppose the PDE system $\mathcal{R}$ is non-degenerate. If there exist smooth functions $\{\Phi^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{i=1}^n$ where $D_i\Phi^i[x,U]\big|_{U=u(x)} = 0$ on any smooth solution $u(x)$ of the PDE system $\mathcal{R}$, then there exists a set of global conservation law multipliers $\{\Lambda_\sigma : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ and a trivial conservation law with fluxes $\{\Psi^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{i=1}^n$ such that everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:*

$$D_i(\Phi^i[x,U] - \Psi^i[x,U]) = \Lambda_\sigma[x,U]R^\sigma[x,U].$$

[7]See Appendix A for the definition of non-degenerate PDEs.

Proof. From Theorem A.0.5 from Appendix A, $D_i\Phi^i[x,U]\big|_{U=u(x)} = 0$ on any smooth solution of a non-degenerate system $\mathcal{R}$ if and only if there exist smooth functions $A_{\sigma,J}[x,U]$ such that

$$D_i\Phi^i[x,U] = \sum_{\sigma,|J|\leq k} A_{\sigma,J}[x,U]D_J R^\sigma[x,U] \tag{2.6}$$

for any $(x,U^{(k)}) \in \mathcal{D}\times\mathcal{U}^{(k)}$. By repeatedly integrating by parts on each multi-index J of equation (2.6), we see that

$$D_i\Phi^i[x,U] = D_i\Psi^i[x,U] + \sum_{\sigma,|J|\leq k} (-1)^{|J|}D_J(A_{\sigma,J}[x,U])R^\sigma[x,U],$$

where $\Psi^i[x,U] = \sum_{\sigma,|J|\leq k} C^i_{\sigma,J}[x,U]D_J R^\sigma[x,U]$ for some smooth $C^i_{\sigma,J}[x,U]$, i.e. $\{\Psi^i[x,U]\}_{i=1}^n$ are trivial fluxes.
Letting $\Lambda_\sigma[x,U] = \sum_{|J|\leq k} (-1)^{|J|}D_J(A_{\sigma,J}[x,U])$, then everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$:

$$D_i(\Phi^i[x,U] - \Psi^i[x,U]) = \Lambda_\sigma[x,U]R^\sigma[x,U].$$

□

Hence, given a non-degenerate PDE system $\mathcal{R}$, equivalence classes of global conservation laws are intimately connected with global conservation law multipliers. In general, given a PDE system $\mathcal{R}$, there can be many sets of conservation law multipliers that give rise to the same equivalence class of conservation laws. However, one can show that if the PDE system $\mathcal{R}$ admits a Cauchy-Kovalevskaya form[8] with a certain condition on the form of global conservation law multipliers, then each equivalence class of global conservation laws of the PDE system $\mathcal{R}$ can be identified with one set of global conservation law multipliers of the PDE system $\mathcal{R}$. To state this more precisely, we define the vector space over $\mathbb{R}$ of global conservation law multipliers for the PDE system $\mathcal{R}$ as:

Definition 2.4.7. *The vector space over $\mathbb{R}$ of global conservation law multipliers of the PDE system $\mathcal{R}$ with component-wise addition and scalar multiplication is denoted by*

[8] See Appendix B for the definition of PDEs with a Cauchy-Kovalevskaya form.

$$M(\mathcal{R}) = \left\{ (\Lambda_1, \ldots, \Lambda_m) \;\middle|\; \begin{array}{l} \Lambda_\sigma : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R} \textit{ smooth for } \sigma = 1, \ldots, m \\ \Phi^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R} \textit{ smooth for } i = 1, \ldots, n \\ \Lambda_\sigma[x, U] R^\sigma[x, U] = D_i \Phi^i[x, U] \textit{ on } \mathcal{D} \times \mathcal{U}^{(k)} \end{array} \right\}.$$

Theorem 2.4.8. *If the PDE system $\mathcal{R}$ has a Cauchy-Kovalevskaya form with respect to the variable x^i and each set of global conservation law multipliers in $M(\mathcal{R})$ does not contain $U^\rho_{\hat{i}}, U^\rho_{2\hat{i}}, \ldots, U^\rho_{k\hat{i}}$, then there is an one-to-one linear correspondence between $CL(\mathcal{R})$ and $M(\mathcal{R})$.*

Proof. See [6]. □

From now on, we will focus on PDE systems that admits a Cauchy-Kovalevskaya form. In particular, $N = m$, i.e. the number of PDEs of $\mathcal{R}$ must match the number of dependent variables present in the PDE system $\mathcal{R}$.

Chapter 3

Flux Equations and Line Integral Formula

In this chapter, we present the main result of this thesis; i.e. the flux equations and a line integral formula for the equivalent fluxes. In particular, given a set of conservation law multipliers of a PDE system $\mathcal{R}$, the corresponding equivalent fluxes must satisfy the flux equations. Moreover, the equivalent fluxes can be found by using a line integral formula derived from the flux equation. We also show how the line integral formula can be simplified in the presence of a point symmetry of the PDE system and a set of conservation law multipliers. This chapter will begin by showing a fundamental divergence identity. From the divergence identity, we derive the flux equations and consequently the line integral formula. Examples will be presented to illustrate the applicability and computational efficiency of the flux equation method.

3.1 The Flux Equations

We first derive an elementary divergence identity which essentially comes from integration by parts. Before presenting the result, we first need to extend the utility of the multi-index notation introduced in Definition 2.1.2.

Definition 3.1.1. *A finite sequence of multi-indices* $\{J_l\}_{l=0}^{r}$ *is an* incrementally increasing sequence *if* $J_l = J_{l-1} + \hat{i_l}$ *for all* $l = 1, \ldots, r$ *and for some sequence* $\{i_l\}_{l=1}^{r} \subset \mathbb{N}$.

For example, if $J_0 = (0, 2, 1)$, $J_1 = (1, 2, 1)$, $J_2 = (1, 2, 2)$ and $J_3 = (1, 3, 2)$, then the sequence $\{J_l\}_{l=0}^{3}$ is incrementally increasing, where $i_1 = 1$, $i_2 = 3$ and $i_3 = 2$. Note that if $\{J_l\}_{l=0}^{r}$ is an incrementally increasing sequence, $|J_l| = |J_0| + l$.

To save writing, we will from now on implicitly assume the dependence on $[x, U]$ when the argument $[x, U]$ is omitted for functions defined on $\mathcal{D}' \times \mathcal{V}^{(k)}$.

Theorem 3.1.2. *Let $f : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ and $g : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ be smooth functions. Then for any multi-index P with $r = |P| \leq k$ and any incrementally increasing sequence $\{J_l\}_{l=0}^{r}$ with $J_0 = 0$ and $J_r = P$, everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$, one has*

$$(D_P f)g = f(-1)^{|P|}(D_P g) + \sum_{l=1}^{r} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i_l}} f)(D_{J_{l-1}} g) \right].$$

Proof. For any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$, we prove this by induction on r.
For $r = 0$, $J_0 = 0 = P$ and hence employing the empty sum notation:

$$\begin{aligned}(D_P f)g &= fg = f(-1)^{|P|}(D_P g) \\ &= f(-1)^{|P|}(D_P g) + \sum_{l=1}^{r} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i_l}} f)(D_{J_{l-1}} g) \right].\end{aligned}$$

For $r > 0$, notice that $\{J_l\}_{l=0}^{r-1}$ is also an incrementally increasing sequence with $J_0 = 0$ and $J_{r-1} = P - \hat{i_r}$. Hence by the induction hypothesis, since $|J_{r-1}| = r - 1$:

$$\begin{aligned}(D_P f)g &= (D_{J_{r-1}+\hat{i_r}} f)g = (D_{J_{r-1}}(D_{i_r} f))g \\ &= (D_{i_r} f)(-1)^{r-1}(D_{J_{r-1}} g) \\ &\quad + \sum_{l=1}^{r-1} (-1)^{l-1} D_{i_l} \left[(D_{J_{r-1}-J_{l-1}-\hat{i_l}}(D_{i_r} f))(D_{J_{l-1}} g) \right] \\ &= f(-1)^{r}(D_{i_r}(D_{J_{r-1}} g)) + (-1)^{r-1} D_{i_r} \left[f(D_{J_{r-1}} g) \right] \\ &\quad + \sum_{l=1}^{r-1} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i_l}} f)(D_{J_{l-1}} g) \right] \\ &= f(-1)^{|P|}(D_P g) + (-1)^{r-1} D_{i_r} \left[(D_{P-J_{r-1}-\hat{i_r}} f)(D_{J_{r-1}} g) \right] \\ &\quad + \sum_{l=1}^{r-1} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i_l}} f)(-D_{J_{l-1}}) g \right] \\ &= f(-1)^{|P|}(D_P g) + \sum_{l=1}^{r} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i_l}} f)(D_{J_{l-1}} g) \right].\end{aligned}$$

□

There is one particular incrementally increasing sequence which will be convenient to use.

Definition 3.1.3. *A finite sequence $\{J_l\}_{l=0}^r$ is an ordered incrementally increasing sequence if it is an incrementally increasing sequence and the sequence $\{i_l\}_{l=1}^r \subset \mathbb{N}$ satisfies $i_l \le i_{l+1}$ for $l = 1, \ldots, r-1$.*

For example, if $J_0 = (0,0,0)$, $J_1 = (1,0,0)$, $J_2 = (1,1,0)$ and $J_3 = (1,2,0)$, then the sequence $\{J_l\}_{l=0}^3$ is ordered incrementally increasing.

Given any multi-index $P = (p_1, p_2, \ldots, p_n)$, let $\{J_l\}_{l=0}^r$ be the unique ordered incrementally increasing sequence such that $J_0 = 0$ and $J_r = P$. It's straightforward to verify that

$$\begin{aligned}
\text{for } 0 \le l \le |P_{(1)}| : &\qquad J_l = l\hat{1}; \\
\text{for } |P_{(1)}| \le l \le |P_{(2)}| : &\qquad J_l = P_{(1)} + (l - |P_{(1)}|)\hat{2}; \\
\text{(by induction)} &\qquad \vdots\ ; \\
\text{for } |P_{(i-1)}| \le l \le |P_{(i)}| : &\qquad J_l = P_{(i-1)} + (l - |P_{(i-1)}|)\hat{i},
\end{aligned}$$

where $P_{(i)}$ denotes the first i indices of P; i.e. $P_{(i)} = (p_1, p_2, \ldots, p_i, 0, \ldots 0)$ and for convenience we define $P_{(0)} = 0$.

Given two multi-indices $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$, we denote $J \le I$ if $j_l \le i_l$ for all $l = 1, \ldots, n$. Thus, by choosing the ordered incrementally increasing sequence for Theorem 3.1.2, we obtain the following.

Corollary 3.1.4. *Let $f : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ and $g : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ be smooth functions, then for any multi-index P with $|P| \le k$, everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:*

$$(D_P f)g = f(-1)^{|P|}(D_P g) + \sum_{i=1}^{n} D_i \left[\sum_{P_{(i-1)} \le J \le P_{(i)} - \hat{i}} (-1)^{|J|} (D_{P-J-\hat{i}} f)(D_J g) \right].$$

Proof. Pick any point in $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$. Choosing $\{J_l\}_{l=0}^r$ be the ordered incrementally increasing sequence in Theorem 3.1.2 yields:

$$(D_P f)g = f(-1)^{|P|}(D_P g) + \sum_{l=1}^{r} (-1)^{l-1} D_{i_l} \left[(D_{P-J_{l-1}-\hat{i}_l} f)(D_{J_{l-1}} g) \right]$$

For any l such that $|P_{(i-1)}| < l \le |P_{(i)}|$, $J_l = P_{(i-1)} + (l - |P_{(i-1)}|)\hat{i}$ and $i_l = i$. In other words, as $l-1$ ranges from $|P_{(i-1)}|$ to $|P_{(i)}| - 1$, $P_{(i-1)} \le J_{l-1} \le P_{(i-1)} + (p_i - 1)\hat{i} = P_{(i)} - \hat{i}$. Thus the set of multi-indices $\{J_{l-1} : |P_{(i-1)}| <$

$l \leq |P_{(i)}|\}$ is the same as the set of multi-indices $\{J : P_{(i-1)} \leq J \leq P_{(i)} - \hat{i}\}$. Hence, by summing over $i = 1, \ldots, n$ first, we can rearrange the sum as:

$$(D_P f)g = f(-1)^{|P|}(D_P g) + \sum_{i=1}^{n} D_i \left[\sum_{P_{(i-1)} \leq J \leq P_{(i)} - \hat{i}} (-1)^{|J|}(D_{P-J-\hat{i}} f)(D_J g) \right].$$

□

Now we are in the position to prove a fundamental divergence identity that relates to the Euler operator $\mathcal{E}_\rho$.

Theorem 3.1.5. *Let $f : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ be a smooth function and also let $\gamma : [a, b] \to \mathcal{V}^{(k)}$ be a differentiable curve*[9]*. Then for any $x \in \mathcal{D}$ and $s \in [a, b]$,*

$$\frac{d}{ds} f[x, \gamma(s)] = \eta^\rho[x, \gamma(s)]\mathcal{E}_\rho(f[x, \gamma(s)]) + D_i \Psi^i(\eta, f)[x, \gamma(s)]$$

where $\eta^\rho[x, \gamma(s)] = \frac{d\gamma^\rho(s)}{ds}$ and

$$\Psi^i(\eta, f)[x, \gamma(s)]$$
$$= \sum_{\substack{|I| \leq k-1 \\ I_{(i-1)}=0}} \sum_{\substack{|J| \leq k-|I|-1 \\ J_{(i)}=J}} (-1)^{|J|} \left((D_I \eta^\rho) \left(D_J \frac{\partial f}{\partial U^\rho_{I+J+\hat{i}}} \right) \right) [x, \gamma(s)].$$

Proof. Since by definition of $\mathcal{V}^{(k)}$, $\gamma^\rho_P(s) = D_P(\gamma^\rho(s))$, applying the chain rule on any $x \in \mathcal{D}'$ yields

$$\begin{aligned} \frac{d}{ds} f[x, \gamma(s)] &= \sum_{|P| \leq k} \frac{\partial f}{\partial U^\rho_P}[x, \gamma(s)] \frac{dD_P(\gamma^\rho(s))}{ds} \\ &= \sum_{|P| \leq k} \frac{\partial f}{\partial U^\rho_P}[x, \gamma(s)] D_P \eta^\rho[x, \gamma(s)]. \qquad (3.1) \end{aligned}$$

[9]A differentiable curve $\gamma : [a, b] \to \mathcal{V}^{(k)} \subset \mathcal{U}^{(k)}$ is a differentiable curve which preserves the structure of $\mathcal{U}^{(k)}$; i.e. if $U^\rho = \gamma^\rho(s)$ then each components of $\gamma(s)$ in the prolonged space $\mathcal{U}^{(k)}$ must satisfy $U^\rho_I = D_I(U^\rho) = D_I(\gamma^\rho(s)) = \gamma^\rho_I(s)$ for any multi-index I. For example, for any $x \in \mathcal{D}$ and any smooth function $V(x) = (V^1(x), \ldots, V^m(x))$, we can define the linear curve $\gamma : [0, 1] \to \mathcal{U}_0$ defined by $\gamma^\rho(s) = sV^\rho(x)$. By preserving the differentiable structure of $\mathcal{U}^{(k)}$, $\gamma(s)$ prolongs to a differentiable curve $\gamma : [0, 1] \to \mathcal{U}^{(k)}$ with components $\gamma^\rho_I(s) = D_I(sV^\rho(x)) = sV^\rho_I(x)$.

Using Corollary 3.1.4, for each multi-index P and at each point in $\mathcal{D}' \times \mathcal{V}^{(k)}$:

$$(D_P\eta^\rho)\frac{\partial f}{\partial U_P^\rho} = \eta^\rho(-1)^{|P|}\left(D_P\frac{\partial f}{\partial U_P^\rho}\right) + \sum_{i=1}^{n} D_i\left[\sum_{P_{(i-1)}\leq J\leq P_{(i)}-\hat{i}}(-1)^{|J|}(D_{P-J-\hat{i}}\eta^\rho)\left(D_J\frac{\partial f}{\partial U_P^\rho}\right)\right].$$

Combining equation (3.1) with the definition of the Euler operator $\mathcal{E}_\rho$ yields

$$\begin{aligned}
\frac{d}{ds}f[x,\gamma(s)] &= \eta^\rho[x,\gamma(s)]\sum_{|P|\leq k}(-1)^{|P|}\left(D_P\frac{\partial f}{\partial U_P^\rho}\right)[x,\gamma(s)] \\
&\quad+\sum_{i=1}^{n}D_i\left[\sum_{\substack{|P|\leq k\\ P_{(i-1)}\leq J\leq P_{(i)}-\hat{i}}}(-1)^{|J|}\left((D_{P-J-\hat{i}}\eta^\rho)\left(D_J\frac{\partial f}{\partial U_P^\rho}\right)\right)[x,\gamma(s)]\right] \\
&= \eta^\rho[x,\gamma(s)]\mathcal{E}_\rho(f[x,\gamma(s)]) \\
&\quad+\sum_{i=1}^{n}D_i\left[\sum_{\substack{|P|\leq k\\ P_{(i-1)}\leq J\leq P_{(i)}-\hat{i}}}(-1)^{|J|}\left((D_{P-J-\hat{i}}\eta^\rho)\left(D_J\frac{\partial f}{\partial U_P^\rho}\right)\right)[x,\gamma(s)]\right].
\end{aligned}$$

To simplify this further, notice that $J \leq P_{(i)} - \hat{i}$ implies J has zero entries after the i-th element; i.e. $J_{(i)} = J$. Moreover, for $P_{(i-1)} \leq J$, the multi-index $I = P - J - \hat{i}$ has zero entries in the first $(i-1)$-th elements; i.e. $I_{(i-1)} = 0$. Hence, we can substitute for $P = I + J + \hat{i}$ with conditions $J_{(i)} = J$ and $I_{(i-1)} = 0$. In particular, since $|I| + |J| + 1 = |I + J + \hat{i}| = |P| \leq k$, we can rewrite the sum as

$$\begin{aligned}
\frac{d}{ds}f[x,\gamma(s)] &= \eta^\rho[x,\gamma(s)]\mathcal{E}_\rho(f[x,\gamma(s)]) \\
&\quad+D_i\left[\sum_{\substack{|I|\leq k-1\\ I_{(i-1)}=0}}\ \sum_{\substack{|J|\leq k-|I|-1\\ J_{(i)}=J}}(-1)^{|J|}\left((D_I\eta^\rho)\left(D_J\frac{\partial f}{\partial U_{I+J+\hat{i}}^\rho}\right)\right)[x,\gamma(s)]\right].
\end{aligned}$$

□

Using this divergence identity, we can now complete the proof of the Euler operator property; i.e. the converse of Theorem 2.3.2 from Chapter 2.

Proof of the converse of Theorem 2.3.2. Suppose the converse is true, i.e. $\mathcal{E}_\rho(f[x,U]) = 0$ everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$ for all $\rho = 1, \dots, m$. Then by Theorem 3.1.5, for any differentiable curve $\gamma : [a,b] \to \mathcal{V}^{(k)}$:

$$\frac{d}{ds} f[x, \gamma(s)] = D_i \Psi^i(\eta, f)[x, \gamma(s)]. \tag{3.2}$$

Fix any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$ and pick a smooth function $c(x) = (c^1(x), \dots, c^m(x))$ such that the k-th prolongation at $c(x)$ is in $\mathcal{V}^{(k)}$. Since $\mathcal{V}^{(k)}$ is connected and hence $\mathcal{V}_0$ is path-connected, we can find a differentiable curve $\gamma : [a,b] \to \mathcal{V}_0$ such that $\gamma(a) = c(x)$ and $\gamma(b) = U$. Prolonging γ to the curve $\gamma : [a,b] \to \mathcal{V}^{(k)}$ and integrating equation (3.2) for such a curve γ yields

$$\begin{aligned} & f[x,\gamma(b)] - f[x,\gamma(a)] = \int_a^b D_i \Psi^i(\eta, f)[x, \gamma(s)] ds \\ \Rightarrow \qquad & f[x,U] - f[x,c(x)] = D_i \int_a^b \Psi^i(\eta, f)[x, \gamma(s)] ds. \end{aligned}$$

Note that the resulting value of the integral is independent of the choice of $\gamma(s)$ and hence it is well-defined. Since $\mathcal{D}$ is simply-connected, we can find smooth $\Theta^i(x)$ for all $1 \leq i \leq n$ such that $D_i \Theta^i(x) = f[x, c(x)]$ everywhere in $\mathcal{D}$. Hence, everywhere on $\mathcal{D}$:

$$f[x,U] = D_i \left[\Theta^i(x) + \int_0^1 \Psi^i(\eta, f)[x, \gamma(s)] ds \right].$$

Since this is true for any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$, the converse is proved. □

Using Theorem 3.1.5, we also obtain the following corollary by restricting $\gamma(s)$ to *flows* under *evolutionary vector fields*[10].

Definition 3.1.6. *A differential operator X acting on smooth functions defined on $\mathcal{D} \times \mathcal{U}^{(k)}$,*

$$X = \xi^i(x,U) \frac{\partial}{\partial x^i} + \eta^\rho(x,U) \frac{\partial}{\partial U^\rho} + \sum_{0<|I|\leq k} \eta_I^\rho[x,U] \frac{\partial}{\partial U_I^\rho},$$

[10] See Appendix C for more details on flows and evolutionary vector fields.

is called a vector field if the smooth functions $\{\xi^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{i=1}^n$, $\{\eta^\rho : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ *and* $\{\eta_I^\rho : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{\rho=1,I}^m$ *satisfy the relation everywhere on* $\mathcal{D} \times \mathcal{U}^{(k)}$ *given by*

$$\eta_I^\rho[x,U] = D_I\left(\eta^\rho(x,U) - \xi^i(x,U)U_i^\rho\right) + \xi^i(x,U)U_{I+\hat{i}}^\rho.$$

Definition 3.1.7. *A vector field* $\hat{X}$ *is an called an evolutionary vector field if* $\xi^i(x,U) = 0$ *for all* $i = 1, \dots, n$, *i.e.,*

$$\hat{X} = \sum_{|I|\leq k} (D_I \eta^\rho[x,U]) \frac{\partial}{\partial U_I^\rho}.$$

Definition 3.1.8. *The flow under an evolutionary vector field* $\hat{X}$ *starting at* $U \in \mathcal{U}_0^{(k)}$ *is the unique differentiable curve* $\gamma : (a-\epsilon, a+\epsilon) \to \mathcal{U}_0$ *such that* $\gamma(a) = U$ *and for all* $\rho = 1, \dots, m$,

$$\frac{d}{ds}\gamma^\rho(s) = \eta^\rho[x, \gamma(s)], \tag{3.3}$$

for all $s \in (a-\epsilon, a+\epsilon)$, *i.e., the flow* $\gamma(s)$ *of* $\hat{X}$ *is the unique solution to the ODE system* (3.3) *satisfying the initial condition* $\gamma(a) = U$. *The prolonged flow* $\gamma : (a-\epsilon, a+\epsilon) \to \mathcal{U}^{(k)}$ *is the prolonged curve obtained from preserving the differentiable structure of* $\mathcal{U}^{(k)}$, *i.e.,* $\gamma_I^\rho(s) = D_I\gamma^\rho(s)$.

Corollary 3.1.9. *Let* $\hat{X}_\eta$ *be an evolutionary vector field and* $f : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ *be a smooth function. Then everywhere on* $\mathcal{D}' \times \mathcal{V}^{(k)}$,

$$\hat{X}_\eta(f[x,U]) = \eta^\rho[x,U]\mathcal{E}_\rho(f[x,U]) + D_i\Psi^i(\eta, f)[x,U],$$

where $\Psi^i(\eta, f)[x,U]$ *is as given in Theorem 3.1.5.*

Proof. Choose any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$ and let $\gamma : (-\epsilon, \epsilon) \to \mathcal{V}_0$ be the corresponding flow of the evolutionary vector field of $\hat{X}_\eta$ with $\gamma(0) = U$. Prolong γ to the curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{V}^{(k)}$. Then by the property of evolutionary vector fields and Theorem 3.1.5,

$$\begin{aligned}
\hat{X}_\eta(f[x,U]) &= \frac{d}{ds} f[x, \gamma(s)]\bigg|_{s=0} \\
&= \Big(\eta^\rho[x,\gamma(s)]\mathcal{E}_\rho(f[x,\gamma(s)]) + D_i\Psi^i(\eta, f)[x,\gamma(s)]\Big)\Big|_{s=0} \\
&= \eta^\rho[x,U]\mathcal{E}_\rho(f[x,U]) + D_i\Psi^i(\eta, f)[x,U],
\end{aligned}$$

where the last step follows from the definition of flows. Since this is true for any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$, the corollary is proved. □

In particular, applying Corollary 3.1.9 to the special case when $f[x,U] = \Lambda_\sigma[x,U]R^\sigma[x,U]$, where $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$, yields:

Corollary 3.1.10. *Let $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ be a set of local conservation law multipliers of the PDE system $\mathcal{R}$ and let $\hat{X}_\eta$ be an evolutionary vector field. Then everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$,*

$$\hat{X}_\eta(\Lambda_\sigma[x,U]R^\sigma[x,U]) = D_i\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U],$$

where $\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U]$ is as given in Theorem 3.1.5.

Proof. Set $f[x,U] = \Lambda_\sigma[x,U]R^\sigma[x,U]$ in Corollary 3.1.9 and apply equation (2.5). □

Now we are in the position to present the main result of this thesis; i.e. the *flux equations.*

Theorem 3.1.11. *Let $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ be a set of local conservation law multipliers of the PDE system $\mathcal{R}$. Then for any $i = 1, \ldots, n$, $\rho = 1, \ldots, m$ and multi-index I, the equivalent fluxes $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ must satisfy everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:*

$$\frac{\partial \Phi^i}{\partial U_I^\rho}[x,U] = \begin{cases} \displaystyle\sum_{\substack{|J| \le k-|I|-1 \\ J_{(i)}=J}} (-1)^{|J|} \left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\rho_{I+J+\hat{i}}} \right)[x,U], & \text{if } I_{(i-1)} = 0, \\ 0, & \text{if } I_{(i-1)} \neq 0. \end{cases} \tag{3.4}$$

Proof. Choose any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$. By Corollary 3.1.10 and the commutativity property of total derivatives with evolutionary vector fields [11],

$$\begin{aligned} D_i\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U] &= \hat{X}_\eta(\Lambda_\sigma[x,U]R^\sigma[x,U]) \\ &= \hat{X}_\eta(D_i\Phi^i[x,U]) \\ &= D_i\hat{X}_\eta(\Phi^i[x,U]), \end{aligned}$$

for any smooth functions $\{\eta^\mu : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\mu=1}^m$. Thus, up to equivalences, for each $i = 1, \ldots, n$ and everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:

[11] See Appendix C.

$$\hat{X}_\eta(\Phi^i[x,U]) = \Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U]$$
$$\Rightarrow \sum_{|P|\le k} (D_P\eta^\mu)\frac{\partial \Phi^i}{\partial U_P^\mu} = \sum_{\substack{|P|\le k-1\\ P_{(i-1)}=0}} (D_P\eta^\mu) \sum_{\substack{|J|\le k-|I|-1\\ J_{(i)}=J}} (-1)^{|J|}\left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\mu_{I+J+\hat{i}}}\right). \tag{3.5}$$

Fix any $i = 1,\ldots,n$, $\rho = 1,\ldots,m$ and multi-index $I = (i_1,\ldots,i_n)$. We now prove the flux equations by induction on $r = |I|$. For $r = 0$, i.e. $I = 0$, choose $\eta^\mu[x,U] = \delta^\mu_\rho$, where $\delta^\rho_\rho = 1$ and $\delta^\mu_\rho = 0$ if $\mu \neq \rho$. Thus, $D_P\eta^\mu[x,U] = 0$ if $0 < |P|$ or $\mu \neq \rho$ and equation (3.5) simplifies to

$$\frac{\partial \Phi^i}{\partial U^\rho} = \sum_{\substack{|J|\le k-1\\ J_{(i)}=J}} (-1)^{|J|}\left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\rho_{I+J+\hat{i}}}\right),$$

which agrees with the flux equation for the case when $I = 0$, as $I_{(i-1)} = 0$ is automatically satisfied for all $i = 1,\ldots,n$.

By the induction hypothesis for $r = |I| > 0$, the flux equations are satisfied for all multi-indices P such that $|P| < r$. Hence, for any smooth functions $\{\eta^\mu : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\mu=1}^m$, equation (3.5) reduces to

$$\sum_{r\le|P|\le k} (D_P\eta^\mu)\frac{\partial \Phi^i}{\partial U_I^\mu} = \sum_{\substack{r\le|P|\le k-1\\ P_{(i-1)}=0}} (D_P\eta^\mu) \sum_{\substack{|J|\le k-|I|-1\\ J_{(i)}=J}} (-1)^{|J|}\left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\mu_{I+J+\hat{i}}}\right). \tag{3.6}$$

Now choose $\eta^\mu[x,U] = \dfrac{\delta^\mu_\rho x^I}{I!}$, where $x^I = (x^1)^{i_1}\cdots(x^n)^{i_n}$ and $I! = i_1!\cdots i_n!$. Thus, $D_P\eta^\mu[x,U] = 0$ if $r < |P|$ or $P \neq I$ or $\mu \neq \rho$, i.e. the only summand left in equation (3.6) is the term involving $P = I$ and $\mu = \rho$. If $I_{(i-1)} = 0$, the right hand side of equation (3.6) further simplifies to

$$\frac{\partial \Phi^i}{\partial U_P^\rho} = \sum_{\substack{|J|\le k-r-1\\ J_{(i)}=J}} (-1)^{|J|}\left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\rho_{I+J+\hat{i}}}\right).$$

Otherwise, the right hand side of equation (3.6) is an empty sum, i.e., if $I_{(i-1)} \neq 0$, then $\dfrac{\partial \Phi^i}{\partial U^\rho_I} = 0.$ $\square$

For convenience, we write out explicitly the flux equations that will be useful for the subsequent examples.

3.1.1 Example: The Flux Equations for a Third-order Scalar PDE

Consider the case $n = 3$, $m = 1$, $k = 3$ with $(x^1, x^2, x^3) = (t, x, y)$. This is the case for any third order scalar PDE, $R[(t, x, y), u(t, x, y)] = 0$, with three variables t, x and y and with a set of conservation law multipliers involving derivatives up to at most third order.

Given a multiplier $\Lambda[(t, x, y), U]$ of the scalar PDE, we first write out the flux equations (Theorem 3.4) for $\Phi^t[(t, x, y), U]$. Since t is identified with x^1, then for any multi-index $I = (i_1, i_2, i_3)$, the condition $0 = I_{(1-1)} = 0$ is always satisfied. Also for the multi-index $J = (j_1, j_2, j_3)$, the condition $J_{(1)} = J$ implies $J = (j_1, 0, 0)$. Hence the flux equations for $\Phi^t[(t, x, y), U]$ should only sum over all $J = (j_1, 0, 0)$ with $|J| \leq 2 - |I|$. This leads to the following set of equations:

$$
\begin{aligned}
\frac{\partial \Phi^t}{\partial U} &= \frac{\partial}{\partial U_t}(\Lambda R) - D_t \frac{\partial}{\partial U_{tt}}(\Lambda R) + D_t D_t \frac{\partial}{\partial U_{ttt}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_t} &= \frac{\partial}{\partial U_{tt}}(\Lambda R) - D_t \frac{\partial}{\partial U_{ttt}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_x} &= \frac{\partial}{\partial U_{tx}}(\Lambda R) - D_t \frac{\partial}{\partial U_{ttx}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_y} &= \frac{\partial}{\partial U_{ty}}(\Lambda R) - D_t \frac{\partial}{\partial U_{tty}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{tt}} &= \frac{\partial}{\partial U_{ttt}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{tx}} &= \frac{\partial}{\partial U_{ttx}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{ty}} &= \frac{\partial}{\partial U_{tty}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{xx}} &= \frac{\partial}{\partial U_{txx}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{xy}} &= \frac{\partial}{\partial U_{txy}}(\Lambda R), \\
\frac{\partial \Phi^t}{\partial U_{yy}} &= \frac{\partial}{\partial U_{tyy}}(\Lambda R).
\end{aligned}
\tag{3.7}
$$

Next, we write out the flux equations (Theorem 3.4) for $\Phi^x[(t,x,y),U]$. Since x is identified with x^2, then for any multi-index $I=(i_1,i_2,i_3)$, the condition $0=I_{(2-1)}=I_{(1)}$ is satisfied when $I=(0,i_2,i_3)$. Also for the multi-index $J=(j_1,j_2,j_3)$, the condition $J_{(2)}=J$ implies $J=(j_1,j_2,0)$. Hence the flux equations for $\Phi^x[(t,x,y),U]$ should only sum over all $J=(j_1,j_2,0)$ with $|J|\leq 2-|I|$. This leads to the following set of equations:

$$
\begin{aligned}
\frac{\partial\Phi^x}{\partial U} &= \frac{\partial}{\partial U_x}(\Lambda R) - D_t\frac{\partial}{\partial U_{tx}}(\Lambda R) - D_x\frac{\partial}{\partial U_{xx}}(\Lambda R)\\
&\quad + D_tD_t\frac{\partial}{\partial U_{ttx}}(\Lambda R) + D_tD_x\frac{\partial}{\partial U_{txx}}(\Lambda R) + D_xD_x\frac{\partial}{\partial U_{xxx}}(\Lambda R),\\
\frac{\partial\Phi^x}{\partial U_t} &= 0,\\
\frac{\partial\Phi^x}{\partial U_x} &= \frac{\partial}{\partial U_{xx}}(\Lambda R) - D_t\frac{\partial}{\partial U_{txx}}(\Lambda R) - D_x\frac{\partial}{\partial U_{xxx}}(\Lambda R),\\
\frac{\partial\Phi^x}{\partial U_y} &= \frac{\partial}{\partial U_{xy}}(\Lambda R) - D_t\frac{\partial}{\partial U_{txy}}(\Lambda R) - D_x\frac{\partial}{\partial U_{xxy}}(\Lambda R),\\
\frac{\partial\Phi^x}{\partial U_{tt}} &= 0,\\
\frac{\partial\Phi^x}{\partial U_{tx}} &= 0,\\
\frac{\partial\Phi^x}{\partial U_{ty}} &= 0,\\
\frac{\partial\Phi^x}{\partial U_{xx}} &= \frac{\partial}{\partial U_{xxx}}(\Lambda R),\\
\frac{\partial\Phi^x}{\partial U_{xy}} &= \frac{\partial}{\partial U_{xxy}}(\Lambda R),\\
\frac{\partial\Phi^x}{\partial U_{yy}} &= \frac{\partial}{\partial U_{xyy}}(\Lambda R).
\end{aligned}
\tag{3.8}
$$

Finally, we write out the flux equations (Theorem 3.4) for $\Phi^y[(t,x,y),U]$. Since y is identified with x^3, then for any multi-index $I=(i_1,i_2,i_3)$, the condition $0=I_{(3-1)}=I_{(2)}$ is satisfied when $I=(0,0,i_3)$. Also for the multi-index $J=(j_1,j_2,j_3)$, the condition $J_{(3)}=J$ is always satisfied. Hence the flux equations for $\Phi^y[(t,x,y),U]$ should sum over all J with $|J|\leq 2-|I|$. This leads to the following set of equations:

$$
\begin{aligned}
\frac{\partial \Phi^y}{\partial U} &= \frac{\partial}{\partial U_y}(\Lambda R) - D_t \frac{\partial}{\partial U_{ty}}(\Lambda R) - D_x \frac{\partial}{\partial U_{xy}}(\Lambda R) - D_y \frac{\partial}{\partial U_{yy}}(\Lambda R) \\
&\quad + D_t D_t \frac{\partial}{\partial U_{tty}}(\Lambda R) + D_t D_x \frac{\partial}{\partial U_{txy}}(\Lambda R) + D_t D_y \frac{\partial}{\partial U_{tyy}}(\Lambda R) \\
&\quad + D_x D_x \frac{\partial}{\partial U_{xxy}}(\Lambda R) + D_x D_y \frac{\partial}{\partial U_{xyy}}(\Lambda R) + D_y D_y \frac{\partial}{\partial U_{yyy}}(\Lambda R)\,, \\
\frac{\partial \Phi^y}{\partial U_t} &= 0, \\
\frac{\partial \Phi^y}{\partial U_x} &= 0, \\
\frac{\partial \Phi^y}{\partial U_y} &= \frac{\partial}{\partial U_{yy}}(\Lambda R) - D_t \frac{\partial}{\partial U_{tyy}}(\Lambda R) - D_x \frac{\partial}{\partial U_{xyy}}(\Lambda R) - D_y \frac{\partial}{\partial U_{yyy}}(\Lambda R)\,, \\
\frac{\partial \Phi^y}{\partial U_{tt}} &= 0, \\
\frac{\partial \Phi^y}{\partial U_{tx}} &= 0, \\
\frac{\partial \Phi^y}{\partial U_{ty}} &= 0, \\
\frac{\partial \Phi^y}{\partial U_{xx}} &= 0, \\
\frac{\partial \Phi^y}{\partial U_{xy}} &= 0, \\
\frac{\partial \Phi^y}{\partial U_{yy}} &= \frac{\partial}{\partial U_{yyy}}(\Lambda R)\,.
\end{aligned} \tag{3.9}
$$

At first glance, the flux equations do not appear to be symmetric with respect to each of the variables t, x and y. Indeed, one can derive a symmetric set of flux equations but at the expense of introducing weighting constants. However, as we will see shortly in applications, whether or not the flux equations are symmetric is immaterial because any solution of the flux equations will lead to the equivalent fluxes that correspond to a given set of conservation law multipliers.

3.1.2 Example: The Flux Equations for a Second-order System of Two PDEs

Now consider the case $n = 2$, $m = 2$, $k = 2$ with $(x^1, x^2) = (t, x)$. This is the case for any second order system of two PDEs $\{R^\sigma[(t, x), (u^1(t, x),$

$u^2(t,x))] = 0\}_{\sigma=1}^2$, with two variables t and x and with a set of conservation law multipliers involving second order derivatives.

Given a set of multipliers $\{\Lambda^\sigma[(t,x),(U^1,U^2)]\}_{\sigma=1}^2$, we first write out the flux equations (Theorem 3.4) for $\Phi^t[(t,x),(U^1,U^2)]$. Since t is identified with x^1, then for any multi-index $I=(i_1,i_2)$, the condition $0 = I_{(1-1)} = 0$ is always satisfied. Also for the multi-index $J=(j_1,j_2)$, the condition $J_{(1)} = J$ is satisfied when $J=(j_1,0)$. Hence the flux equations for $\Phi^t[(t,x),(U^1,U^2)]$ should sum over all J with $\leq |J| \leq 1-|I|$. This leads to the following set of equations:

$$\begin{aligned}
\frac{\partial\Phi^t}{\partial U^1} &= \frac{\partial}{\partial U^1_t}(\Lambda R) - D_t\frac{\partial}{\partial U^1_{tt}}(\Lambda R)\,,\\
\frac{\partial\Phi^t}{\partial U^2} &= \frac{\partial}{\partial U^2_t}(\Lambda R) - D_t\frac{\partial}{\partial U^2_{tt}}(\Lambda R)\,,\\
\frac{\partial\Phi^t}{\partial U^1_t} &= \frac{\partial}{\partial U^1_{tt}}(\Lambda R)\,,\\
\frac{\partial\Phi^t}{\partial U^2_t} &= \frac{\partial}{\partial U^2_{tt}}(\Lambda R)\,,\\
\frac{\partial\Phi^t}{\partial U^1_x} &= \frac{\partial}{\partial U^1_{tx}}(\Lambda R)\,,\\
\frac{\partial\Phi^t}{\partial U^2_x} &= \frac{\partial}{\partial U^2_{tx}}(\Lambda R)\,.
\end{aligned}\tag{3.10}$$

Similarly, we write out the flux equations (Theorem 3.4) for $\Phi^x[(t,x),U]$. Since x is identified with x^2, then for any multi-index $I=(i_1,i_2)$, the condition $0 = I_{(2-1)} = I_{(1)}$ is satisfied when $I=(0,i_2)$. Also for the multi-index $J=(j_1,j_2)$, the condition $J_{(2)} = J$ is always satisfied. Hence the flux equations for $\Phi^x[(t,x),U]$ should sum over all J with $|J| \leq 1-|I|$. This leads to the following set of equations:

$$
\begin{aligned}
\frac{\partial \Phi^x}{\partial U^1} &= \frac{\partial}{\partial U^1_x}(\Lambda R) - D_t \frac{\partial}{\partial U^1_{tx}}(\Lambda R) - D_x \frac{\partial}{\partial U^1_{xx}}(\Lambda R),\\
\frac{\partial \Phi^x}{\partial U^2} &= \frac{\partial}{\partial U^2_x}(\Lambda R) - D_t \frac{\partial}{\partial U^2_{tx}}(\Lambda R) - D_x \frac{\partial}{\partial U^2_{xx}}(\Lambda R),\\
\frac{\partial \Phi^x}{\partial U^1_t} &= 0,\\
\frac{\partial \Phi^x}{\partial U^2_t} &= 0,\\
\frac{\partial \Phi^x}{\partial U^1_x} &= \frac{\partial}{\partial U^1_{xx}}(\Lambda R),\\
\frac{\partial \Phi^x}{\partial U^2_x} &= \frac{\partial}{\partial U^2_{xx}}(\Lambda R).
\end{aligned}
\tag{3.11}
$$

3.2 The Line Integral Formula

Given a set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$, a solution to the flux equations yields a corresponding equivalent set of fluxes $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$. In particular, the solution can be given by the following line integral formula.

Theorem 3.2.1. *Let $\mathcal{D}'$ be a simply-connected subdomain of $\mathcal{D}$ and $\mathcal{V}^{(k)}$ be a connected open subset of $\mathcal{U}^{(k)}$. Pick any $(x, U^{(k)}) \in \mathcal{D}' \times \mathcal{V}^{(k)}$ and any differentiable curve $\gamma : [a,b] \to \mathcal{V}^{(k)}$ such that $\gamma(a) = c(x)$ and $\gamma(b) = U$, where $c(x) = (c^1(x), \dots, c^m(x))$ is any smooth function such that the k-th prolongation of $c(x)$ is in $\mathcal{V}^{(k)}$. If $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$, then $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ are the corresponding equivalent fluxes if and only if for all $i = 1, \dots, n$,*

$$
\Phi^i[x,U] = \int_a^b \Psi^i(\eta, \Lambda_\sigma R^\sigma)[x, \gamma(s)] ds, \tag{3.12}
$$

where $\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U]$ is as given in Theorem 3.1.5.

Proof. Suppose $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ are the equivalent fluxes for the set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$. By the flux equations and the fundamental theorem on line integrals:

$$\begin{aligned}
\Phi^i[x,U] &= \Phi^i[x,c(x)] + \int_a^b \sum_{|I|\le k-1} \frac{\partial \Phi^i}{\partial U_I^\rho}[x,\gamma(s)]\frac{d\gamma_I^\rho(s)}{ds}ds \\
&= \Phi^i[x,c(x)] \\
&\quad + \int_a^b \sum_{\substack{|I|\le k-1 \\ I_{(i-1)}=0}} \sum_{\substack{|J|\le k-|I|-1 \\ J_{(i)}=J}} (-1)^{|J|}\left(D_J \frac{\partial(\Lambda_\sigma R^\sigma)}{\partial U^\rho_{I+J+\hat{i}}}\right)[x,\gamma(s)]\frac{d\gamma_I^\rho(s)}{ds}ds \\
&= \Phi^i[x,c(x)] + \int_a^b \Psi^i(\eta,\Lambda_\sigma R^\sigma)[x,\gamma(s)]ds,
\end{aligned}$$

where the last step follows from $\frac{d\gamma(s)}{ds} = \eta[x,\gamma(s)]$. Since $\{\Theta^i(x) = \Phi^i[x,c(x)]\}_{i=1}^n$ is a set of trivial fluxes, the forward implication is proved. Conversely, suppose $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ satisfies equation (3.12). By Corollary 3.1.5 and since $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers, for all $s \in [a,b]$ one has

$$\frac{d}{ds}(\Lambda_\sigma R^\sigma)[x,\gamma(s)] \quad = \quad D_i\Psi^i(\eta,\Lambda_\sigma R^\sigma)[x,\gamma(s)].$$

Hence, integrating the above equation for s yields:

$$\begin{aligned}
(\Lambda_\sigma R^\sigma)[x,U] - (\Lambda_\sigma R^\sigma)[x,c(x)] &= D_i \int_a^b \Psi^i(\eta,\Lambda_\sigma R^\sigma)[x,\gamma(s)]ds \\
&= D_i\Phi^i[x,U].
\end{aligned}$$

Since $\mathcal{D}'$ is simply-connected, we can find trivial fluxes $\{\Theta^i(x)\}_{i=1}^n$ such that $D_i\Theta^i(x) = (\Lambda_\sigma R^\sigma)[x,c(x)]$. In other words, $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ is the corresponding set of equivalent fluxes for the set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$. $\square$

In order to use the line integral formula, we must choose a differentiable curve $\gamma(s)$ such that its range is defined on all of $\mathcal{V}^{(k)}$ and as well as a smooth function $c(x)$ such that its k-th prolongation is defined in $\mathcal{V}^{(k)}$ for all $x \in \mathcal{D}'$. Moreover, in practice, we would like the resulting integrals to be "simple" enough to be computed explicitly in order to determine the explicit form of the fluxes. To the best knowledge of the author, unfortunately there isn't a systematic way to choose $\gamma(s)$ and $c(x)$ that guarantees simple integrations for the line integral formula.

Nonetheless, typically in applications, $\mathcal{D}' \times \mathcal{V}^{(k)}$ is the entire $\mathcal{D} \times \mathcal{U}^{(k)}$ (i.e. $0 \in \mathcal{V}^{(k)}$). Often in this case, a linear curve $\gamma : [a,b] \to \mathcal{U}_0$ defined by $\gamma(s) = sU$ will lead to simple integrations for line integral formula. If $0 \notin \mathcal{V}^{(k)}$, then we must in general choose an appropriate curve $\gamma(s)$ so that its range avoids the singularity at 0. As we will discuss in Chapter 4, a special case of this is the *homotopy integral formula* in [5] where we would choose $\gamma(s) = sU + (1-s)c(x)$ for some non-vanishing smooth function $c(x)$. However, there is still no guarantee that this choice of curve will lead to simple integrations.

3.2.1 Example: Generalized KdV Equation

Consider the generalized KdV equation with $c > -2$:

$$R[(t,x), u(t,x)] = u_t + u^c u_x + u_{xxx} = 0. \tag{3.13}$$

Since in this case, $n = 2 < 3$, $m = 1$ and $k = 3$, we can use the flux equations (3.7) and (3.8). Using the Euler operator method to find conservation law multipliers, it can be shown that $\Lambda[(t,x), U] = U$ is a global conservation law multiplier for the generalized KdV equation (3.13). Hence, for this choice of $\Lambda[(t,x), U] = U$, the flux equations for $\Phi^t[(t,x), U]$ and $\Phi^x[(t,x), U]$ are given by

$$\begin{aligned}
\frac{\partial \Phi^t}{\partial U} &= U, \\
\frac{\partial \Phi^x}{\partial U} &= U^{c+1} + U_{xx}, \\
\frac{\partial \Phi^x}{\partial U_x} &= -U_x, \\
\frac{\partial \Phi^x}{\partial U_{xx}} &= U,
\end{aligned}$$

where all other partial derivatives of $\Phi^t[(t,x), U]$ and $\Phi^x[(t,x), U]$ are zero. Since both $R[(t,x), U]$ and $\Lambda[(t,x), U]$ are smooth everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$, we can choose $\gamma : [0,1] \to \mathcal{U}_0$ to be the linear curve defined by $\gamma(s) = sU$. By prolonging $\gamma(s)$, the line integral formula (Theorem 3.2.1) yields

$$\begin{aligned}\Phi^t[(t,x),U] &= \int_0^1 \frac{\partial \Phi^t}{\partial U}[(t,x),\gamma(s)]\frac{d\gamma(s)}{ds}ds \\ &= \int_0^1 (sU)U ds \\ &= \frac{U^2}{2},\end{aligned}$$

$$\begin{aligned}\Phi^x[(t,x),U] &= \int_0^1 \left(\frac{\partial \Phi^x}{\partial U}[(t,x),\gamma(s)]\frac{d\gamma(s)}{ds} + \frac{\partial \Phi^x}{\partial U_x}[(t,x),\gamma(s)]\frac{dD_x\gamma(s)}{ds}\right. \\ &\quad \left. + \frac{\partial \Phi^x}{\partial U_{xx}}[(t,x),\gamma(s)]\frac{dD_xD_x\gamma(s)}{ds}\right) ds \\ &= \int_0^1 \Big(((sU)^{c+1} + sU_{xx})U + (-sU_x)U_x + (sU)U_{xx}\Big) ds \\ &= U^{c+2}\int_0^1 s^{c+1}ds + UU_{xx}\int_0^1 2sds - U_x^2\int_0^1 sds \\ &= \frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U_x^2}{2}.\end{aligned}$$

Indeed, one can readily verify that these fluxes correspond to the global conservation law multiplier $\Lambda[(t,x),U] = U$ of equation (3.13):

$$U(U_t + U^cU_x + U_{xxx}) = D_x\left(\frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U_x^2}{2}\right) + D_t\left(\frac{U^2}{2}\right).$$

3.2.2 Example: 1D Nonlinear Wave Equation

Consider the 1D nonlinear wave equation with smooth wave speed:

$$R[(t,x),u(t,x)] = u_{tt} - (c^2(u)u_x)_x = u_{tt} - 2c(u)c'(u)u_x^2 - c^2(u)u_{xx} = 0. \tag{3.14}$$

Since in this case, $n = 2 < 3$, $m = 1$, $k = 2 < 3$, we can again use the flux equations (3.7) and (3.8). Using the Euler operator method to find conservation law multipliers, it can be shown that $\Lambda[(t,x),U] = xt$ is a global conservation law multiplier for the nonlinear wave equation (3.14). Hence, for this choice of $\Lambda[(t,x),U] = xt$, the flux equations for $\Phi^t[(t,x),U]$ and $\Phi^x[(t,x),U]$ are given by

$$
\begin{aligned}
\frac{\partial \Phi^t}{\partial U} &= -x, \\
\frac{\partial \Phi^t}{\partial U_t} &= xt, \\
\frac{\partial \Phi^x}{\partial U} &= -2xtc(U)c'(U)U_x + tc^2(U), \\
\frac{\partial \Phi^x}{\partial U_x} &= -xtc(U)^2,
\end{aligned}
$$

where again all other partial derivatives of $\Phi^t[(t,x),U]$ and $\Phi^x[(t,x),U]$ are zero. As both $R[(t,x),U]$ and $\Lambda[(t,x),U]$ are smooth everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$, we could again choose the linear curve for $\gamma(s)$. Instead, we illustrate in this example that choosing $\gamma : [0,1] \to \mathcal{U}_0$ to be the polynomial curve defined by $\gamma(s) = s^p U$ for any $p > 0$ leads to the same equivalent fluxes for all $p > 0$. Indeed, this is true in general since the the line integral formula holds for any differentiable curve $\gamma(s)$. Hence, the line integral formula yields

$$
\begin{aligned}
\Phi^t[(t,x),U] &= \int_0^1 \left(\frac{\partial \Phi^t}{\partial U}[(t,x),\gamma(s)]\frac{d\gamma(s)}{ds} + \frac{\partial \Phi^t}{\partial U_t}[(t,x),\gamma(s)]\frac{dD_t\gamma(s)}{ds} \right) ds \\
&= \int_0^1 \Big((-x)ps^{p-1}U + (xt)ps^{p-1}U_t \Big) ds \\
&= xtU_t - xU,
\end{aligned}
$$

$$
\begin{aligned}
\Phi^x[(t,x),U] &= \int_0^1 \left(\frac{\partial \Phi^x}{\partial U}[(t,x),\gamma(s)]\frac{d\gamma(s)}{ds} + \frac{\partial \Phi^x}{\partial U_x}[(t,x),\gamma(s)]\frac{dD_x\gamma(s)}{ds} \right) ds \\
&= \int_0^1 \Big((-2xtc(s^pU)c'(s^pU)s^pU_x + tc^2(s^pU))ps^{p-1}U \\
&\quad +(-xtc^2(s^pU))ps^{p-1}U_x \Big) ds \\
&= -xtU_x \int_0^1 \Big(2ps^{2p-1}Uc(s^pU)c'(s^pU) + ps^{p-1}c^2(s^pU) \Big) ds \\
&\quad + t\int_0^1 c^2(s^pU)ps^{p-1}U ds \\
&= -xtU_x \int_0^1 \frac{d}{ds}\left(s^pc^2(s^pU)\right) ds + t\int_0^U c^2(\mu)d\mu \\
&= -xtU_xc^2(U) + t\int_0^U c^2(\mu)d\mu.
\end{aligned}
$$

Indeed, one can readily verify that these are the equivalent fluxes for the global conservation law multiplier $\Lambda[(t,x),U] = xt$ of equation (3.14):

$$\begin{aligned} xt\left(U_{tt} - 2c(U)U_x^2 - c^2(U)U_{xx}\right) &= D_x\left(-xtU_xc^2(U) + t\int_0^U c^2(\mu)d\mu\right) \\ &\quad + D_t\left(xtU_t - xU\right). \end{aligned}$$

3.2.3 Example: Nonlinear Telegraph System

Consider the nonlinear telegraph system:

$$\begin{aligned} R^1[(t,x),(u^1(t,x),u^2(t,x))] &= u_t^2 - ((u^1)^2+1)u_x^1 - u^1 = 0, \\ R^2[(t,x),(u^1(t,x),u^2(t,x))] &= u_t^1 - u_x^2 = 0. \end{aligned} \tag{3.15}$$

Since in this case, $n = 2$, $m = 2$, $k = 1 < 2$, we can use the flux equations (3.10) and (3.11). Using the Euler operator method to find conservation law multipliers, it can be shown that $\Lambda_1[(t,x),(U^1,U^2)] = t$ and $\Lambda_2[(t,x),(U^1,U^2)] = x - \frac{t^2}{2}$ together form a set of global conservation law multipliers for the nonlinear telegraph system (3.15). Hence, for this set of conservation law multipliers, the flux equations for $\Phi^t[(t,x),(U^1,U^2)]$ and $\Phi^x[(t,x),(U^1,U^2)]$ are given by

$$\begin{aligned} \frac{\partial\Phi^t}{\partial U^1} &= x - \frac{t^2}{2}, \\ \frac{\partial\Phi^t}{\partial U^2} &= t, \\ \frac{\partial\Phi^x}{\partial U^1} &= -t((U^1)^2+1), \\ \frac{\partial\Phi^x}{\partial U^2} &= \frac{t^2}{2} - x. \end{aligned}$$

Since $R^\sigma[(t,x),(U^1,U^2)]$ and $\Lambda_\sigma[(t,x),(U^1,U^2)]$ are smooth everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$ for $\sigma = 1,2$, we can choose $\gamma : [0,1] \to \mathcal{U}_0$ to be the linear curve defined by $\gamma(s) = (\gamma^1(s),\gamma^2(s)) = s(U^1,U^2)$. By prolonging $\gamma(s)$, the line integral formula yields

$$
\begin{aligned}
&\Phi^t[(t,x),(U^1,U^2)]\\
&\quad= \int_0^1 \left(\frac{\partial \Phi^t}{\partial U^1}[(t,x),\gamma(s)]\frac{d\gamma^1(s)}{ds} + \frac{\partial \Phi^t}{\partial U^2}[(t,x),\gamma(s)]\frac{d\gamma^2(s)}{ds} \right) ds\\
&\quad= \int_0^1 \left(\left(x - \frac{t^2}{2} \right) U^1 + tU^2 \right) ds\\
&\quad= \left(x - \frac{t^2}{2} \right) U^1 + tU^2,\\
&\Phi^x[(t,x),(U^1,U^2)]\\
&\quad= \int_0^1 \left(\frac{\partial \Phi^x}{\partial U^1}[(t,x),\gamma(s)]\frac{d\gamma^1(s)}{ds} + \frac{\partial \Phi^x}{\partial U^2}[(t,x),\gamma(s)]\frac{d\gamma^2(s)}{ds} \right) ds\\
&\quad= \int_0^1 \left(-t((sU^1)^2+1)U^1 + \left(\frac{t^2}{2} - x \right) U^2 \right) ds\\
&\quad= -t(U^1)^3 \int_0^1 s^2 ds + \left(\left(\frac{t^2}{2} - x \right) U^2 - tU^1 \right) \int_0^1 ds\\
&\quad= -\left(\frac{(U^1)^3}{3} + U^1 \right) t + \left(\frac{t^2}{2} - x \right) U^2.
\end{aligned}
$$

One can easily verify that these form the equivalent fluxes for the set of global conservation law multipliers $\Lambda_1[(t,x),(U^1,U^2)] = t, \Lambda_2[(t,x),(U^1,U^2)] = x - \frac{t^2}{2}$ of equation (3.15):

$$
\begin{aligned}
&t(U_t^2 - ((U^1)^2+1)U_x^1 - U^1) + \left(x - \frac{t^2}{2} \right) (U_t^1 - U_x^2)\\
&\quad= D_x \left(-\left(\frac{(U^1)^3}{3} + U^1 \right) t + \left(\frac{t^2}{2} - x \right) U^2 \right) + D_t \left(\left(x - \frac{t^2}{2} \right) U^1 + tU^2 \right).
\end{aligned}
$$

In [9], integral formulas for fluxes were derived by other means. Since the line integral formula is a general theorem, it is not a coincidence that the integral formulas from [9] correspond precisely with the line integral formula in this case.

Rather than picking $\gamma(s)$ randomly and seeing if it will lead to simple integrations, we should take advantage of the freedom in choosing $\gamma(s)$ in the line integral formula. In particular, we can simplify the line integral formula by making use of the *symmetries*[12] of both the PDE system $\mathcal{R}$ and the sets of conservation law multipliers.

[12] See Appendix C for details on symmetries.

Theorem 3.2.2. *Suppose the hypotheses of Theorem 3.2.1 are satisfied and further suppose $\hat{X}_\eta$ is the generator of a point symmetry of the PDE equations of $\mathcal{R}$. Let $\gamma : [a,b] \to \mathcal{V}^{(k)}$ be the corresponding flow of $\hat{X}_\eta$ such that $\gamma(a) = c(x)$ and $\gamma(b) = U$. The equivalent fluxes from Theorem 3.2.1 simplify to*

$$\Phi^i[x,U] = \int_a^b \sum_{\substack{|I|\le k-1 \\ |J|\le k-|I|-1 \\ I_{(i-1)}=0 \\ J_{(i)}=J}} (-1)^{|J|} \left((D_I \eta^\rho) \left(D_J \left(\Lambda_\sigma \frac{\partial R^\sigma}{\partial U^\rho_{I+J+\hat{i}}} \right) \right) \right) [x, \gamma(s)] ds.$$

Proof. We can rewrite each of the fluxes $\Phi^i[x,U]$ from Theorem 3.2.1 as

$$\Phi^i[x,U] = \int_a^b \sum_{\substack{|I|\le k-1 \\ |J|\le k-|I|-1 \\ I_{(i-1)}=0 \\ J_{(i)}=J}} (-1)^{|J|} \left((D_I \eta^\rho) \left(D_J \left(\Lambda_\sigma \frac{\partial R^\sigma}{\partial U^\rho_{I+J+\hat{i}}} \right) \right) \right.$$
$$\left. + (D_I \eta^\rho) \left(D_J \left(\frac{\partial \Lambda_\sigma}{\partial U^\rho_{I+J+\hat{i}}} R^\sigma \right) \right) \right) [x, \gamma(s)] ds. \tag{3.16}$$

Since $\hat{X}_\eta$ is the generator of a point symmetry of a (non-degenerate) PDE system $\mathcal{R}$, by Theorem C.0.14 there exists a smooth matrix $\{A^\sigma_\mu[x,U;s]\}^m_{\sigma,\mu=1}$ which satisfies $R^\sigma[x,\gamma(s)] = A^\sigma_\mu[x,U;s]R^\mu[x,U]$ under the flow $\gamma(s)$ of $\hat{X}_\eta$ for all $s \in [a,b]$. Hence, substituting $R^\sigma[x,\gamma(s)] = A^\sigma_\mu[x,U;s]R^\mu[x,U]$ in equation (3.16) shows that

$$\int_a^b \sum_{\substack{|I|\le k-1 \\ |J|\le k-|I|-1 \\ I_{(i-1)}=0 \\ J_{(i)}=J}} (-1)^{|J|} \left((D_I \eta^\rho) \left(D_J \left(\frac{\partial \Lambda_\sigma}{\partial U^\rho_{I+J+\hat{i}}} R^\sigma \right) \right) \right) [x, \gamma(s)] ds$$

is a trivial flux. □

In other words, there are potentially fewer integrations required to find the equivalent fluxes if we choose the curve $\gamma(s)$ to be the flow under a symmetry generator $\hat{X}_\eta$ of the PDE system $\mathcal{R}$. Furthermore, we can obtain further simplifications for the equivalent fluxes if $\hat{X}_\eta$ is *a generator of a point symmetry for the set of local conservation law multipliers* $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}^m_{\sigma=1}$.

Definition 3.2.3. *An evolutionary vector field $\hat{X}_\eta$ is called the generator of a point symmetry of the set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ if for any solution $v(x)$ of the system of equations $\{\Lambda_\sigma[x, v(x)] = 0\}_{\sigma=1}^m$, the flow $\gamma(s)$ under $\hat{X}$ with $\gamma(a) = v(x)$ satisfies $\Lambda_\sigma[x, \gamma(s)] = 0$ for all $x \in \mathcal{D}'$ and all $s \in (a - \epsilon, a + \epsilon)$ for some $\epsilon > 0$.*

Note that if the system of equations $\{\Lambda_\sigma[x, v(x)] = 0\}_{\sigma=1}^m$ is non-degenerate and $\hat{X}$ is the generator of a point symmetry of the set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$, then by Theorem C.0.14, there exists a smooth matrix $\{B_\sigma^\nu[x, U; s]\}_{\sigma,\nu=1}^m$ such that $\Lambda_\sigma[x, \gamma(s)] = \Lambda_\nu[x, U] B_\sigma^\nu[x, U; s]$ for all $s \in (a - \epsilon, a + \epsilon)$. From now on, we will always assume that the system of equations $\{\Lambda_\sigma[x, v(x)] = 0\}_{\sigma=1}^m$ is non-degenerate.

Theorem 3.2.4. *Suppose the hypotheses of Theorem 3.2.1 are satisfied and suppose $\hat{X}_\eta$ is the generator of a point symmetry of the PDEs of $\mathcal{R}$ and of the set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$. Let $\gamma : [a, b] \to \mathcal{V}^{(k)}$ be the corresponding flow of $\hat{X}_\eta$ such that $\gamma(a) = c(x)$ and $\gamma(b) = U$. The equivalent fluxes simplify to*

$$\Phi^i[x, U] = \int_a^b \sum_{\substack{|I| \leq k-1 \\ |J| \leq k-|I|-1 \\ I_{(i-1)}=0 \\ J_{(i)}=J}} (-1)^{|J|} (D_I \eta^\rho[x, \gamma(s)]) \left[D_J \left(\Lambda_\nu[x, U] \right.\right.$$
$$\left.\left. B_\sigma^\nu[x, U; s] \frac{\partial R^\sigma}{\partial U^\rho_{I+J+\hat{i}}}[x, \gamma(s)] \right) \right] ds$$

where $\{B_\sigma^\nu[x, U; s]\}_{\sigma,\nu=1}^m$ is a smooth matrix that satisfies $\Lambda_\sigma[x, \gamma(s)] = \Lambda_\nu[x, U] B_\sigma^\nu[x, U; s]$.

Proof. Substituting $\Lambda_\sigma[x, \gamma(s)] = \Lambda_\nu[x, U] B_\sigma^\nu[x, U; s]$ for the integral formula in Theorem 3.2.2 yields the desired result. □

The main advantage of Theorem 3.2.4 is that the dependence on the parameter s in the set of local conservation law multipliers $\{\Lambda_\nu[x, \gamma(s)]\}_{\nu=1}^m$ is factored out from the line integral formula which can lead to simpler integrations. As we will see in the next example, this theorem can lead to explicit algebraic formulas for the equivalent fluxes. For convenience, we state Theorem 3.2.4 for the scalar case, i.e. $m = 1$.

Corollary 3.2.5. *Suppose the hypotheses of Theorem 3.2.1 are satisfied and suppose $\hat{X}_\eta$ is the generator of a point symmetry of the PDEs of $\mathcal{R}$*

and the set of local conservation law multipliers $\Lambda : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$*. Let* $\gamma : [a,b] \to \mathcal{V}^{(k)}$ *be the corresponding flow of* $\hat{X}_\eta$ *such that* $\gamma(a) = c(x)$ *and* $\gamma(b) = U$*. The equivalent fluxes further simplify to*

$$\Phi^i[x,U] = \int_a^b \sum_{\substack{|I|\le k-1\\ |J|\le k-|I|-1\\ I_{(i-1)}=0\\ J_{(i)}=J}} (-1)^{|J|}(D_I\eta[x,\gamma(s)]) \left[D_J \left(\Lambda[x,U] \right.\right.$$
$$\left.\left. B[x,U;s]\frac{\partial R}{\partial U^\rho_{I+J+\hat{i}}}[x,\gamma(s)] \right) \right] ds$$

where $B[x,U;s]$ *is a smooth function satisfying* $\Lambda[x,\gamma(s)] = \Lambda[x,U]B[x,U;s]$.

3.2.4 Example: 2D Flame Equation

Consider the 2D flame equation:

$$R[(t,x,y),u(t,x,y)] = u_t - \sqrt{u_x^2 + u_y^2} = 0. \tag{3.17}$$

Note that $R[(t,x,y),U]$ is only smooth on $\mathcal{D}$ and $\mathcal{V}^{(k)} = \mathcal{U}^{(k)}\backslash\{0\}$. Thus the conservation laws for the 2D flame equation must be local. Nonetheless, we will show that the line integral formula leads to new local conservation laws for the 2D flame equation.

Since $n = 3$, $m = 1$, $k = 1 < 3$ for the 2D flame equation, we can use the flux equations (3.7), (3.8) and (3.9). Using the Euler operator method to find conservation law multipliers, it can be shown that $\Lambda[(t,x,y),U] = f(U_x,U_y)(U_{xx}U_{yy} - U_{xy}^2)$ is a local conservation law multiplier of the flame equation (3.17), where $f(\cdot,\cdot)$ is any smooth function of its arguments. To save writing, denote $H[(t,x,y),U] = U_{xx}U_{yy} - U_{xy}^2$. Hence, for this conservation law multiplier, the flux equations for $\Phi^t[(t,x,y),U]$, $\Phi^x[(t,x,y),U]$ and $\Phi^y[(t,x,y),U]$ are given by

$$
\begin{aligned}
\frac{\partial \Phi^t}{\partial U} &= fH, \\
\frac{\partial \Phi^x}{\partial U} &= \frac{\partial f}{\partial U_x} HR - fH \frac{U_x}{\sqrt{U_x^2 + U_y^2}} - D_x \left(fU_{yy}R \right), \\
\frac{\partial \Phi^x}{\partial U_x} &= fU_{yy}R, \\
\frac{\partial \Phi^x}{\partial U_y} &= -2fU_{xy}R, \\
\frac{\partial \Phi^y}{\partial U} &= \frac{\partial f}{\partial U_y} HR - fH \frac{U_y}{\sqrt{U_x^2 + U_y^2}} + 2D_x \left(fU_{xy}R \right) - D_y \left(fU_{yy}R \right), \\
\frac{\partial \Phi^y}{\partial U_y} &= fU_{xx}R,
\end{aligned}
$$

where all other partial derivatives of $\Phi^t[(t,x,y),U]$, $\Phi^x[(t,x,y),U]$ and $\Phi^t[(t,x,y),U]$ are zero.

Since $R[(t,x,y),U]$ is smooth everywhere except at $0 \in \mathcal{U}^{(k)}$, we can still choose $\gamma : [\epsilon, 1] \to \mathcal{U}_0$ to be a linear curve $\gamma(s) = sU$ provided that $\epsilon > 0$. It will turn out in the end that the fluxes are still well-defined in the limit $\epsilon \to 0$. Thus, by prolonging $\gamma(s)$, the line integral formula yields

$$
\begin{aligned}
\Phi^t[(t,x,y),U] - \Phi^t[(t,x,y),\epsilon U] &= \int_\epsilon^1 \frac{\partial \Phi^t}{\partial U}[(t,x,y),\gamma(s)] \frac{d\gamma(s)}{ds} ds \\
&= \int_\epsilon^1 f(sU_x, sU_y) H[(t,x,y), sU] U ds \\
&= UH[(t,x,y),U] \int_\epsilon^1 s^2 f(sU_x, sU_y) ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Phi^x[(t,x,y),U] - \Phi^x[(t,x,y),\epsilon U] \\
&\qquad = \int_\epsilon^1 \left(\frac{\partial \Phi^x}{\partial U}[(t,x,y),\gamma(s)] \frac{d\gamma(s)}{ds} \right. \\
&\qquad\quad + \frac{\partial \Phi^x}{\partial U_x}[(t,x,y),\gamma(s)] \frac{dD_x\gamma(s)}{ds} \\
&\qquad\quad \left. + \frac{\partial \Phi^x}{\partial U_y}[(t,x,y),\gamma(s)] \frac{dD_y\gamma(s)}{ds} \right) ds \\
&\qquad = \int_\epsilon^1 \left[\left(\frac{\partial f}{\partial U_x}(sU_x, sU_y) H[(t,x,y),sU] R[(t,x,y),sU] \right. \right. \\
&\qquad\quad - f(sU_x,sU_y) H[(t,x,y),sU] \frac{sU_x}{\sqrt{(sU_x)^2+(sU_y)^2}} \\
&\qquad\quad \left. -D_x \Big(f(sU_x,sU_y) sU_{yy} R[(t,x,y),sU] \Big) \right) U \\
&\qquad\quad + \Big(f(sU_x,sU_y) sU_{yy} R[(t,x,y),sU] \Big) U_x \\
&\qquad\quad \left. -2 \Big(f(sU_x,sU_y) sU_{xy} R[(t,x,y),sU] \Big) U_y \right] ds \\
&\qquad = -\frac{UU_x H[(t,x,y),U]}{\sqrt{U_x^2+U_y^2}} \int_\epsilon^1 s^2 f(sU_x,sU_y) ds \\
&\qquad\quad + R[(t,x,y),U] \left(UH[(t,x,y),U] \int_\epsilon^1 s^3 \frac{\partial f}{\partial U_x}(sU_x,sU_y) ds \right. \\
&\qquad\quad \left. + (U_{yy}U_x - 2U_{xy}U_y) \int_\epsilon^1 s^2 f(sU_x,sU_y) ds \right) \\
&\qquad\quad - UD_x \left(U_{yy} R[(t,x,y),U] \int_\epsilon^1 s^2 f(sU_x,sU_y) ds \right),
\end{aligned}
$$

$$
\begin{aligned}
&\Phi^y[(t,x,y),U] - \Phi^y[(t,x,y),\epsilon U] \\
&\qquad = \int_\epsilon^1 \left(\frac{\partial \Phi^y}{\partial U}[(t,x,y),\gamma(s)]\frac{d\gamma(s)}{ds} \right. \\
&\qquad\quad \left. + \frac{\partial \Phi^y}{\partial U_y}[(t,x,y),\gamma(s)]\frac{dD_y\gamma(s)}{ds} \right) ds \\
&\qquad = \int_\epsilon^1 \left[\left(\frac{\partial f}{\partial U_y}(sU_x,sU_y)H[(t,x,y),sU]R[(t,x,y),sU] \right. \right. \\
&\qquad\quad - f(sU_x,sU_y)H[(t,x,y),sU]\frac{sU_y}{\sqrt{(sU_x)^2+(sU_y)^2}} \\
&\qquad\quad + 2D_x\Big(f(sU_x,sU_y)sU_{xy}R[(t,x,y),sU]\Big) \\
&\qquad\quad \left. -D_y\Big(f(sU_x,sU_y)sU_{yy}R[(t,x,y),sU]\Big)\right) U \\
&\qquad\quad \left. + \Big(f(sU_x,sU_y)sU_{xx}R[(t,x,y),sU]\Big) U_y \right] ds \\
&\qquad = -\frac{UU_yH[(t,x,y),U]}{\sqrt{U_x^2+U_y^2}}\int_\epsilon^1 s^2 f(sU_x,sU_y)ds \\
&\qquad\quad + R[(t,x,y),U]\left(UH[(t,x,y),U]\int_\epsilon^1 s^3\frac{\partial f}{\partial U_y}(sU_x,sU_y)ds \right. \\
&\qquad\quad \left. + (U_{xx}U_y)\int_\epsilon^1 s^2 f(sU_x,sU_y)ds \right) \\
&\qquad\quad + 2UD_x\left(U_{xy}R[(t,x,y),U]\int_\epsilon^1 s^2 f(sU_x,sU_y)ds \right) \\
&\qquad\quad - UD_y\left(U_{yy}R[(t,x,y),U]\int_\epsilon^1 s^2 f(sU_x,sU_y)ds \right).
\end{aligned}
$$

Since both $\Phi^x[(t,x,y),U]$ and $\Phi^y[(t,x,y),U]$ contain terms proportional to $R[(t,x,y),U]$ or total derivatives of $R[(t,x,y),U]$ (i.e. those terms form trivial fluxes), the equivalent fluxes for the multiplier $\Lambda[(t,x,y),U]$ are given by

$$\Phi^t[(t,x,y),U] - \Phi^t[(t,x,y),\epsilon U] = UH[(t,x,y),U] \int_\epsilon^1 s^2 f(sU_x, sU_y)ds,$$

$$\Phi^x[(t,x,y),U] - \Phi^x[(t,x,y),\epsilon U] = -\frac{UU_x H[(t,x,y),U]}{\sqrt{U_x^2+U_y^2}} \int_\epsilon^1 s^2 f(sU_x, sU_y)ds,$$

$$\Phi^y[(t,x,y),U] - \Phi^y[(t,x,y),\epsilon U] = -\frac{UU_y H[(t,x,y),U]}{\sqrt{U_x^2+U_y^2}} \int_\epsilon^1 s^2 f(sU_x, sU_y)ds.$$

Since $\{\Phi^t[(t,x,y),0], \Phi^x[(t,x,y),0], \Phi^y[(t,x,y),0]\}$ are all trivial fluxes, substituting the expression for $H[(t,x,y),U]$ and taking the limit as $\epsilon \to 0$ yields the equivalent fluxes for the multiplier $\Lambda[(t,x,y),U]$ given by

$$\Phi^t[(t,x,y),U] = U(U_{xx}U_{yy} - U_{xy}^2) \int_0^1 s^2 f(sU_x, sU_y)ds, \tag{3.18}$$

$$\Phi^x[(t,x,y),U] = -\frac{UU_x(U_{xx}U_{yy} - U_{xy}^2)}{\sqrt{U_x^2+U_y^2}} \int_0^1 s^2 f(sU_x, sU_y)ds, \tag{3.19}$$

$$\Phi^y[(t,x,y),U] = -\frac{UU_y(U_{xx}U_{yy} - U_{xy}^2)}{\sqrt{U_x^2+U_y^2}} \int_0^1 s^2 f(sU_x, sU_y)ds. \tag{3.20}$$

Indeed, through a long and involved calculation, one can verify that for any smooth function $f(U_x, U_y)$, the fluxes $\{\Phi^t[(t,x,y),U], \Phi^x[(t,x,y),U], \Phi^y[(t,x,y),U]\}$ given by equations (3.18), (3.19) and (3.20) satisfy

$$\begin{aligned} f(U_x,U_y)&(U_{xx}U_{yy} - U_{xy}^2)(U_t - \sqrt{U_x^2+U_y^2}) \\ &= D_t\left(\Phi^t[(t,x,y),U] + \Theta^t[(t,x,y),U]\right) \\ &+ D_x\left(\Phi^x[(t,x,y),U] + \Theta^x[(t,x,y),U]\right) \\ &+ D_y\left(\Phi^y[(t,x,y),U] + \Theta^y[(t,x,y),U]\right), \end{aligned}$$

where $\{\Theta^t[(t,x,y),U], \Theta^x[(t,x,y),U], \Theta^y[(t,x,y),U]\}$ is a set of trivial fluxes.

Without knowing the explicit expression of f, equations (3.18), (3.19) and (3.20) are the general expressions for the equivalent fluxes that correspond to the conservation law multiplier $\Lambda[(t,x,y),U] = f(U_x,U_y)(U_{xx}U_{yy} - U_{xy}^2)$ of the flame equation (3.17). However, if f is a homogeneous function in its arguments (i.e. $f(U_x,U_y)$ has the property that for some constant p, $f(sU_x, sU_y) = s^p f(U_x,U_y)$ for all $s \in \mathbb{R}$), then we would obtain algebraic expressions for the equivalent fluxes. In particular, if $p > -3$, then equations (3.18), (3.19) and (3.20) simplify to

$$\begin{aligned}
\Phi^t[(t,x,y),U] &= U(U_{xx}U_{yy}-U_{xy}^2)\int_0^1 s^{p+2}f(U_x,U_y)ds \\
&= \frac{U(U_{xx}U_{yy}-U_{xy}^2)f(U_x,U_y)}{p+3}, \qquad (3.21)\\
\Phi^x[(t,x,y),U] &= -\frac{UU_x(U_{xx}U_{yy}-U_{xy}^2)}{\sqrt{U_x^2+U_y^2}}\int_0^1 s^{p+2}f(U_x,U_y)ds \\
&= -\frac{UU_x(U_{xx}U_{yy}-U_{xy}^2)f(U_x,U_y)}{(p+3)\sqrt{U_x^2+U_y^2}}, \qquad (3.22)\\
\Phi^y[(t,x,y),U] &= -\frac{UU_y}{\sqrt{U_x^2+U_y^2}}\int_0^1 s^{p+2}f(U_x,U_y)ds \\
&= -\frac{UU_y(U_{xx}U_{yy}-U_{xy}^2)f(U_x,U_y)}{(p+3)\sqrt{U_x^2+U_y^2}}. \qquad (3.23)
\end{aligned}$$

Note that we can arrive at the same result much quicker by using the line integral formula (Corollary 3.2.5) which makes use of a point symmetry of the PDE system and of the conservation law multiplier. Indeed, the evolutionary vector field $\hat{X} = \sum_{|I|\le k} U_I \frac{\partial}{\partial U_I}$ is the generator of a scaling symmetry of $R[(t,x,y),U]$ and of $\Lambda[(t,x,y),U]$, provided that $f(\cdot,\cdot)$ is a homogeneous function. In particular, the flow $\gamma : [\epsilon, 1] \to \mathcal{V}^{(k)}$ under $\hat{X}$ starting at $\gamma(1) = U$ is given by $\gamma(s) = sU$. Since $\Lambda[(t,x,y),\gamma(s)] = s^{p+2}\Lambda[(t,x,y),U]$, $B[(t,x,y),U;s] = s^{p+2}$. Thus using the simplified integral formula from Corollary 3.2.5, we get the same result as in (3.21), (3.22) and (3.23) by taking $\epsilon \to 0$.

In [7] and [10], the algebraic fluxes given by equations (3.21), (3.22) and (3.23) were found using a specialized method that employs a *non-critical scaling symmetry* ([11]). As discussed in Chapter 4, the non-critical scaling symmetry method is in fact a special case of the line integral formula obtained from Corollary 3.2.4. Moreover, the simplified line integral formula from Corollary 3.2.4 also works for other point symmetries as well. In other words, if both of the PDE system $\mathcal{R}$ and the set of conservation law multipliers of the PDE system $\mathcal{R}$ admit a common point symmetry, then Corollary 3.2.4 can be used to simplify the integrations required in the line integral formula.

Chapter 4

Known Methods to Find Conservation Laws

In this chapter, we review some general and specialized methods of finding conservation laws for a PDE system $\mathcal{R}$, given a known set of conservation law multipliers. Each method has its own advantages and disadvantages in its applicability and computational efficiency in finding conservation laws. We compare these methods with the flux equation method and highlight their similarities and differences in finding conservation laws.

4.1 Matching Method

We first outline the *matching method* for finding conservation laws for the PDE system $\mathcal{R}$ and illustrate this method by an example.

Given that $\{\Lambda_\sigma[x,U]\}_{\sigma=1}^m$ and $\{R^\sigma[x,U]\}_{\sigma=1}^m$ depend at most on the k-th order derivatives of U, then according to [12] the set of fluxes $\{\Phi^i[x,U]\}_{i=1}^n$ will also depend at most on the k-th order derivatives of U. Thus, from the definition of a set of local conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ of a PDE system $\mathcal{R}$, the corresponding fluxes $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ must satisfy everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:

$$\begin{aligned}\Lambda_\sigma[x,U]R^\sigma[x,U] &= D_i\Phi^i[x,U] \\ &= \frac{\partial \Phi^i}{\partial x^i}[x,U] + \sum_{|J|\leq k} U^\rho_{J+\hat{i}}\frac{\partial \Phi^i}{\partial U^\rho_J}[x,U]. \qquad (4.1)\end{aligned}$$

Thus, matching the k-th order derivatives in equation (4.1) implies everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:

$$\Lambda_\sigma[x,U]R^\sigma[x,U] = \frac{\partial \Phi^i}{\partial x^i}[x,U] + \sum_{|J|\le k-1} U^\rho_{J+\hat{i}} \frac{\partial \Phi^i}{\partial U^\rho_J}[x,U], \quad (4.2)$$

$$\sum_{|J|=k} U^\rho_{J+\hat{i}} \frac{\partial \Phi^i}{\partial U^\rho_J}[x,U] = 0. \quad (4.3)$$

Moreover, if the terms involving the k-th order derivatives of U are linear in the expression $\Lambda_\sigma[x,U]R^\sigma[x,U]$, it can be shown that the equivalent fluxes $\{\Phi^i[x,U]\}_{i=1}^n$ will depend at most on the $(k-1)$-th order derivatives of U ([12]). In particular, equation (4.3) would be automatically satisfied. Hence assuming the set of fluxes $\{\Phi^i[x,U]\}_{i=1}^n$ depends at most on the $(k-1)$-th order derivatives of U, then matching the k-th order derivatives on both sides of equation (4.2) yields a set of determining equations for $\{\Phi^i[x,U]\}_{i=1}^n$. Solving this set of determining equations yields $\{\Phi^i[x,U]\}_{i=1}^n$ up to additive functions that depend at most on the $(k-2)$-th order derivatives of U. By matching successively lower order of derivatives of U in equation (4.2), the matching method will yield $\Phi^i[x,U]$ up to an additive function depending only on x. This procedure is best illustrated by an example.

Example 4.1.1. *Generalized KdV equation*

Consider the generalized KdV equation with $c > -2$:

$$R[(t,x),u(t,x)] = u_t + u^c u_x + u_{xxx} = 0.$$

Recall from earlier examples, $\Lambda[(t,x),U] = U$ is a (global) conservation law multiplier for the generalized KdV equation, i.e., everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$:

$$U(U_t + U^c U_x + U_{xxx}) = D_t\Phi^t[(t,x),U] + D_x\Phi^x[(t,x),U]. \quad (4.4)$$

We now use the matching method to find the corresponding fluxes. First, we must deduce the dependence of the highest order of U^ρ_I appearing in $\Phi^x[(t,x),U]$ and $\Phi^t[(t,x),U]$. For the multiplier $\Lambda = U$, we find that $\Phi^t(t,x,U,U_t,U_x)$ and $\Phi^x(t,x,U,U_t,U_x,U_{xx})$ would suffice, i.e., everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$:

$$\begin{aligned} U(U_t + U^c U_x + U_{xxx}) &= [\Phi^t_t + \Phi^t_U U_t + \Phi^t_{U_t} U_{tt} + \Phi^t_{U_x} U_{xt}] \\ &\quad + [\Phi^x_x + \Phi^x_U U_x + \Phi^x_{U_x} U_{xx} + \Phi^x_{U_t} U_{tx} + \Phi^x_{U_{xx}} U_{xxx}]. \end{aligned} \quad (4.5)$$

Note that this step is unnecessary with the flux equation method because the flux equations automatically give all the dependences of $U^{(k)} \in \mathcal{V}^{(k)}$. As equation (4.5) holds everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$, the functional dependence on each of U_I^ρ must match on both sides of (4.5). Hence, we proceed sequentially by equating the coefficients of successively lower orders of U_I^ρ and solving each of the corresponding PDEs. In this case, equating the coefficients of the highest order U_{xxx} in equation (4.5) yields everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$\Phi^x_{U_{xx}} = U \Rightarrow \Phi^x = UU_{xx} + A(t, x, U, U_t, U_x) \tag{4.6}$$

for some unknown function $A(t, x, U, U_t, U_x)$. From equation (4.6), we deduce that $\Phi^x_U = U_{xx} + A_U$ and $\Phi^x_{U_x} = A_{U_x}$. Hence, equating the coefficients of U_{xx} in equation (4.5) shows that everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$A_{U_x} = -U_x \Rightarrow A(t, x, U, U_t, U_x) = -\frac{U_x^2}{2} + B(t, x, U, U_t) \tag{4.7}$$

for some unknown function $B(t, x, U, U_t)$. Next equating the coefficients of U_{tt} in equation (4.5), we find that everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$\Phi^t_{U_t} = 0 \Rightarrow \Phi^t = C(t, x, U, U_x) \tag{4.8}$$

for some unknown function $C(t, x, U, U_x)$. Finally, matching the coefficients of U_{xt} in equation (4.5) yields everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$\Phi^x_{U_t} + \Phi^t_{U_x} = 0.$$

Note from equations (4.7) and (4.8), $\Phi^x_{U_t} = B_{U_t}$, $\Phi^t_{U_x} = C_{U_x}$. Since B does not depend on U_x and C does not depend on U_t, we can conclude that everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$B_{U_t} = -C_{U_x} = E(t, x, U) \quad \Rightarrow \quad B(t, x, U, U_t) = U_t E(t, x, U) + F(t, x, U), \tag{4.9}$$

$$C(t, x, U, U_x) = -U_x E(t, x, U) + G(t, x, U), \tag{4.10}$$

where $E(t, x, U)$, $F(t, x, U)$ and $G(t, x, U)$ are some unknown functions to be determined. It's worth noting here that the efficiency of matching coefficients depends on the sequence in which they are matched. For example, since U_{tt} and U_{xt} have the same order, one could just as well equate the U_{xt} coefficients first then follow by the U_{tt} coefficients. However, we would not obtain the simplification as above until much later. Also, the possibility this kind of inefficiency does not occur with the line integral method because the procedure is not iterative. Continuing on, matching the coefficients of U_x and U_t in equation (4.5) yields respectively,

$$F_U - E_t = U^{c+1} \Rightarrow F(t,x,U) = \frac{U^{c+2}}{c+2} + \int^U E_t(t,x,\mu)d\mu + H(t,x), \quad (4.11)$$

$$G_U + E_x = U \Rightarrow G(t,x,U) = \frac{U^2}{2} - \int^U E_x(t,x,\mu)d\mu + I(t,x), \quad (4.12)$$

for some unknown functions $H(t,x)$ and $I(t,x)$. Since $H(t,x)$ and $I(t,x)$) are independent of variables in $\mathcal{U}^{(k)}$, they are trivial fluxes. Note also for any smooth function $E(t,x,U)$, everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:

$$\begin{aligned} &D_x\left(U_t E(t,x,U) + \int^U E_t(t,x,\mu)d\mu\right) \\ &+ D_t\left(-U_t E(t,x,U) + \int^U E_x(t,x,\mu)d\mu\right) = 0, \end{aligned} \quad (4.13)$$

i.e., equation (4.13) is a differential identity and hence a trivial conservation law. Thus combining equations (4.6) through (4.13), the matching method yields the equivalent fluxes

$$\begin{aligned} \Phi^x[(t,x),U] &= UU_{xx} - \frac{U_x^2}{2} + \frac{U^{c+2}}{c+2}, \\ \Phi^t[(t,x),U] &= \frac{U^2}{2}. \end{aligned}$$

Indeed from direct computation, one can verify that everywhere on $\mathcal{D}\times\mathcal{U}^{(k)}$:

$$U(U_t + U^c U_x + U_{xxx}) = D_x\left(UU_{xx} - \frac{U_x^2}{2} + \frac{U^{c+2}}{c+2}\right) + D_t\left(\frac{U^2}{2}\right).$$

4.2 Homotopy Integral Formula

Given a set of conservation law multipliers of the PDE system $\mathcal{R}$, the *homotopy integral formula* ([6]) is an integral formula for the fluxes of the corresponding conservation law. Its main advantage is that it provides an explicit formula for the fluxes, as opposed to successively solving and integrating a system of PDEs when using the matching method. It has been pointed out in [7, 10] that one drawback of the homotopy integral formula is that the convergence of the integral formula depends on making choices of functions in order to avoid singularities in $\mathcal{U}^{(k)}$ that the sets of conservation law multipliers or the PDEs themselves may have. As we will see next, the homotopy integral formula is in fact a special case of the line integral formula (Theorem 3.2.1) obtained from the flux equations and hence we expect the line integral formula to have a better chance at remedying this convergence issue.

Theorem 4.2.1. *Let $\mathcal{D}'$ be a simply-connected subdomain of $\mathcal{D}$ and let $\mathcal{V}^{(k)}$ be a convex subset of $\mathcal{U}^{(k)}$. Pick any $(x, U^k) \in \mathcal{D}' \times \mathcal{V}^{(k)}$ and any smooth function $c(x) = (c^1(x), \ldots, c^m(x))$ such that the k-th prolongation of $c(x)$ is in $\mathcal{V}^{(k)}$. If $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the PDE system $\mathcal{R}$, then the equivalent fluxes are given by*

$$\Phi^i[x, U] = \int_0^1 \Psi^i(\eta, \Lambda_\sigma R^\sigma)[x, sU + (1-s)c(x)]ds, \tag{4.14}$$

where $\Psi^i(\eta, \Lambda_\sigma R^\sigma) : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ is as defined in Theorem 3.1.5.

Proof. Let $\gamma(s) : [0,1] \to \mathcal{V}^{(k)}$ be the linear curve by prolonging $\gamma^\rho(s) = sU^\rho + (1-s)c^\rho(x)$. Hence, $\eta^\rho[x, \gamma(s)] = \frac{d\gamma^\rho(s)}{ds} = U - c^\rho(x)$. Since $\mathcal{V}^{(k)}$ is a convex subset (and hence connected), $\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x, \gamma(s)]$ is defined for all $s \in [0,1]$. Hence, applying the line integral formula (Theorem 3.2.1) obtained from the flux equations proves the homotopy integral formula. $\square$

Since the homotopy formula is a special case of the general line integral formula when we restrict to the linear curve $\gamma(s) = sU + (1-s)c(x)$, we refer the reader for examples in Chapter 3.

4.3 Noether's Theorem

In her celebrated paper [4], Noether presented a method to find local conservation laws for PDE systems which admit a *variational principle*. If a PDE system admits an *action functional*, then the *extremals* of the action functional yields precisely the PDE system given by the Euler-Lagrange equations. Noether showed that if a PDE system admits a variational principle and there exists a point symmetry of the action functional, then one can obtain the fluxes of a local conservation law explicitly without integration. Here, we outline a generalization of Noether's theorem which includes higher-order symmetries due to Boyer [13]. Before we present this result, we need to introduce a few definitions.

Definition 4.3.1. *Given a smooth function $\mathcal{L} : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$, the action functional of $J : \mathcal{U}^{(k)} \to \mathbb{R}$ is the integral expression given by*

$$J[U] = \int_{\mathcal{D}} \mathcal{L}[x, U] d^n x.$$

The smooth function $\mathcal{L}[x, U]$ is called the Lagrangian.

Definition 4.3.2. *Let $J : \mathcal{U}^{(k)} \to \mathbb{R}$ be an action functional and let $\mathcal{A}(\mathcal{D})$ be a family of functions defined on $\mathcal{D}$. A function $V(x) = (V^1(x), \ldots, V^m(x))$ belonging to $\mathcal{A}(\mathcal{D})$ is an extremal of the action functional over $\mathcal{A}(\mathcal{D})$ if for any smooth function $\xi(x) = (\xi^1(x), \ldots, \xi^m(x))$ compactly supported on $\mathcal{D}$*

$$\frac{d}{ds} J[V + s\xi]|_{s=0} = 0.$$

For our purpose, we will take $\mathcal{A}(\mathcal{D})$ to be the family of smooth functions satisfying some given boundary conditions of the PDE system $\mathcal{R}$.

Theorem 4.3.3. *If $U = V(x)$ is a smooth extremal function of the action functional $J : \mathcal{U}^{(k)} \to \mathbb{R}$ with the associated Lagrangian function $\mathcal{L} : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$, then it must satisfy the Euler-Lagrange equations:*

$$\mathcal{E}_\rho(\mathcal{L}[x, U])|_{U=V(x)} = 0 \text{ for all } \rho = 1, \ldots, m.$$

Proof. We omit the details since we have basically already shown this during the course of the proof of the Euler operator property in Theorem 2.3.2. □

Definition 4.3.4. *A PDE system $\mathcal{R}$ admits a variational principle if the PDEs of the system are precisely those given by the Euler-Lagrange equations; i.e. there exists a Lagrangian function $\mathcal{L} : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ such that the PDEs of $\mathcal{R}$ are given by*

$$R^\sigma[x, u(x)] = \mathcal{E}_\sigma(\mathcal{L}[x, U])|_{U=u(x)} = 0 \text{ for all } \sigma = 1, \ldots, m.$$

Definition 4.3.5. *The flow under the generator $\hat{X}_\eta$ is called a variational symmetry of the action functional $J : \mathcal{U}^{(k)} \to \mathbb{R}$ if $\hat{X}_\eta$ leaves the Lagrangian function $\mathcal{L} : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ to within a divergence expression, i.e., there exist some smooth functions $\{A^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{i=1}^n$ such that everywhere on $\mathcal{D} \times \mathcal{U}^{(k)}$:*

$$\hat{X}_\eta(\mathcal{L}[x, U]) = D_i A^i[x, U].$$

Now we are in the position to prove (Boyer's generalization of) Noether's theorem ([7]).

Theorem 4.3.6. *Suppose the PDE system $\mathcal{R}$ has a variational principle with the Lagrangian function $\mathcal{L} : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$. Further suppose that $\hat{X}_\eta = \sum_{|I|\leq k}(D_I\eta^\rho[x,U])\frac{\partial}{\partial U_I^\rho}$ is the generator of a variational symmetry for the action functional $J : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$. For each $i = 1,\ldots,n$, let $\Psi^i(\eta,\mathcal{L}) : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}$ be as defined in Theorem 3.1.5. Then*

- *The smooth functions $\{\eta^\rho : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ form a set of local conservation law multipliers of the PDE system $\mathcal{R}$.*
- *Everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$, the corresponding fluxes are given by*

$$\Phi^i[x,U] = A^i[x,U] - \Psi^i(\eta,\mathcal{L})[x,U]. \tag{4.15}$$

Proof. By Theorem 3.1.9 of Chapter 3, everywhere on $\mathcal{D}' \times \mathcal{V}^{(k)}$:

$$\hat{X}_\eta(\mathcal{L}[x,U]) = \eta^\rho[x,U]\mathcal{E}_\rho(\mathcal{L}[x,U]) + D_i\Psi^i(\eta,\mathcal{L})[x,U]. \tag{4.16}$$

Since the PDE system $\mathcal{R}$ admits a variational principle and has a variational symmetry generated by $\hat{X}_\eta$, equation (4.16) simplifies to

$$\begin{aligned} D_iA^i[x,U] &= \eta^\rho[x,U]R^\rho[x,U] + D_i\Psi^i(\eta,\mathcal{L})[x,U] \\ \Rightarrow \eta^\rho[x,U]R^\rho[x,U] &= D_i\left(A^i[x,U] - \Psi^i(\eta,\mathcal{L})[x,U]\right). \end{aligned} \tag{4.17}$$

Since equation (4.17) holds everywhere on $\mathcal{D}'\times\mathcal{V}^{(k)}$, $\{\eta^\rho : \mathcal{D}'\times\mathcal{V}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ forms a set of local conservation law multipliers for the PDE system $\mathcal{R}$ and the smooth functions $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ as defined by equation (4.15) are the corresponding equivalent fluxes. □

Hence, in order to use Noether's theorem to find a conservation law for a given PDE system $\mathcal{R}$, we must first determine if the PDE system $\mathcal{R}$ admits a variational principle. Due to a criteria by Volterra [14], we can determine precisely when a given PDE system $\mathcal{R}$ admits a variational principle. To state this criteria, we need to introduce a few more definitions.

Definition 4.3.7. *The linearization operator L_ρ^σ of the PDE system $\mathcal{R}$ with respect to components $\sigma = 1,\ldots,m$ is a differential operator with its action on any smooth functions $\{F^\rho : \mathcal{D}\times\mathcal{U}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ defined by*

$$L_\rho^\sigma(F^\rho[x,U]) = \sum_{|I|\leq k}\frac{\partial R^\sigma}{\partial U_I^\rho}[x,U]D_IF^\rho[x,U].$$

Definition 4.3.8. *The adjoint operator* $L^{*\sigma}_{\rho}$ *of the PDE system* $\mathcal{R}$ *with respect to components* $\rho = 1, \ldots, m$ *is a differential operator with its action on any smooth functions* $\{F^\rho : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$ *defined by*

$$L^{*\sigma}_{\rho}(F^\rho[x,U]) = \sum_{|J|\leq k}(-D_J)\left(\frac{\partial R^\sigma}{\partial U^\rho_J}[x,U]F^\rho[x,U]\right).$$

Definition 4.3.9. *A PDE system* $\mathcal{R}$ *is called self-adjoint or variational if the linearization operator* L^σ_ρ *and the adjoint operator* $L^{*\sigma}_{\rho}$ *are equal as differential operators, i.e., for any smooth functions* $\{F^\rho : \mathcal{D}\times\mathcal{U}^{(k)} \to \mathbb{R}\}_{\rho=1}^m$*, everywhere on* $\mathcal{D} \times \mathcal{U}^{(k)}$*:*

$$L^\sigma_\rho(F^\rho[x,U]) = L^{*\sigma}_{\rho}(F^\rho[x,U]).$$

Theorem 4.3.10. *Suppose* $\mathcal{D}$ *is a star-shaped domain. A PDE system* $\mathcal{R}$ *(as written) defined on the entire* $\mathcal{D} \times \mathcal{U}^{(k)}$ *admits a variational principle if and only if the PDE system* $\mathcal{R}$ *is self-adjoint. If so, the Lagrangian function is given by*

$$\mathcal{L}[x,U] = \int_0^1 U^\rho R^\rho[x,sU]ds.$$

Proof. See [8] or [14]. □

Hence, if a given PDE system $\mathcal{R}$ is not self-adjoint as written, then one cannot use Noether's theorem to find conservation laws. Moreover, in the case if the PDE system $\mathcal{R}$ is self-adjoint, there may still be difficulties in computing the Lagrangian function explicitly as given by the integral formula in Theorem 4.3.10. The direct method of finding sets of conservation law multipliers and then their corresponding equivalent fluxes does not depend on whether the PDE system $\mathcal{R}$ admits a variational principle. In particular, the flux equation method, the matching method and the homotopy integral formula can be used regardless of whether the PDE system $\mathcal{R}$ is self-adjoint or not.

Secondly, even if the given PDE system $\mathcal{R}$ admits a variational principle, we still need to find a variational symmetry for the action functional. These two obstacles highlight the key disadvantages of Noether's theorem when compared to the direct method in finding conservation laws.

Example 4.3.11. *Generalized KdV equation*

Consider the generalized KdV equation with $c > -1$:

$$R[(t,x),u(t,x)] = u_t + u^c u_x + u_{xxx} = 0.$$

After computing the linearization and adjoint operators, it follows that the generalized KdV equation as written above is not variational. However, through the change of variable $U = V_x$, the generalized KdV equation is transformed into a variational PDE [7]:

$$R'[(t,x), v(t,x)] = R[(t,x), v_x(t,x)] = v_{xt} + (v_x)^c v_{xx} + v_{xxxx} = 0,$$

where the Lagrangian function is given by,

$$\mathcal{L}[(t,x), V] = \frac{(V_{xx})^2}{2} - \frac{(V_x)^{c+3}}{(c+1)(c+2)} - \frac{V_x V_t}{2}.$$

Moreover, the evolutionary vector field $\hat{X}_\eta$ is the generator of a variational symmetry of the transformed equation where $\eta[(x,t), V] = V_x$. Indeed,

$$\begin{aligned} \hat{X}_\eta(\mathcal{L}[(t,x), V]) &= V_{xx}V_{xxx} - \frac{(c+3)(V_x)^{c+2}V_{xx}}{(c+1)(c+2)} - \frac{V_{xx}V_t}{2} - \frac{V_x V_{xt}}{2} \\ &= D_x\left(\frac{(V_{xx})^2}{2} - \frac{(V_x)^{c+3}}{(c+1)(c+2)} - \frac{V_x V_t}{2}\right). \end{aligned}$$

Hence, applying Noether's Theorem (Theorem 4.3.6), we can conclude that $\Lambda[(t,x), V] = V_x$ is a conservation law multiplier and the corresponding equivalent fluxes are given by

$$\begin{aligned} \Phi^t[(t,x), U] = \Phi^t[(t,x), V] &= \frac{(V_x)^2}{2} = \frac{U^2}{2}, \\ \Phi^x[(t,x), U] = \Phi^x[(t,x), V] &= \frac{(V_x)^{c+2}}{(c+2)} + V_x V_{xxx} - \frac{(V_x)^2}{2} \\ &= \frac{U^{c+2}}{c+2} + UU_{xx} - \frac{U^2}{2}. \end{aligned}$$

4.4 Non-critical Scaling Symmetry

This method ([7, 10, 11]) also provides an explicit algebraic formula for the fluxes without integration. The main drawback is its limited applicability since in this formula the PDEs of $\mathcal{R}$ and a set of (local) conservation law multipliers $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ of the PDE system $\mathcal{R}$ both must possess a scaling symmetry which is *non-critical*.

Definition 4.4.1. *Given a set of local conservation law multipliers* $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ *of the PDE system* $\mathcal{R}$*, the generator of a scaling symmetry* $\hat{X}_\eta$ *of the PDE system* $\mathcal{R}$ *and of the set of conservation law multipliers* $\{\Lambda_\sigma : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$ *is called non-critical, if there is a constant* $c \neq 0$ *and some trivial fluxes* $\{\Theta^i : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{i=1}^n$ *such that everywhere on* $\mathcal{D}' \times \mathcal{V}^{(k)}$*:*

$$\hat{X}_\eta(\Lambda_\sigma[x,U]R^\sigma[x,U]) = c\Lambda_\sigma[x,U]R^\sigma[x,U] + D_i\Theta^i[x,U].$$

Theorem 4.4.2. *Suppose* $\hat{X}_\eta$ *is the generator of a non-critical scaling symmetry of both the PDE system* $\mathcal{R}$ *and a set of conservation law multipliers of the PDE system* $\mathcal{R}$*,* $\{\Lambda_\sigma : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}\}_{\sigma=1}^m$*. Then the corresponding equivalent fluxes* $\{\Phi^i : \mathcal{D}' \times \mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^n$ *are given by*

$$\Phi^i[x,U] = \frac{1}{c}\Psi^i(\eta, \Lambda_\sigma R^\sigma)[x,U],$$

where $\Psi^i(\eta, \Lambda_\sigma R^\sigma)$ *is as given in Theorem 3.1.5.*

Proof. By definition of a non-critical scaling symmetry and Theorem 3.1.10:

$$\begin{aligned} D_i\Big(\Psi^i(\eta,\Lambda_\sigma R^\sigma)[x,U] - \Theta^i[x,U]\Big) &= \hat{X}_\eta(\Lambda_\sigma[x,U]R^\sigma[x,U]) - D_i\Theta^i[x,U] \\ &= c\Lambda_\sigma[x,U]R^\sigma[x,U] \\ &= cD_i\Phi^i[x,U]. \end{aligned}$$

Dividing by c on both sides yields the desired result. □

Example 4.4.3. *2D Flame Equation*

Consider again the flame equation defined on $\mathcal{D} \times \mathcal{V}^{(k)}$, where $\mathcal{V}^{(k)} = \mathcal{U}^{(k)}\backslash\{0\}$,

$$R[(t,x,y), u(t,x,y)] = u_t - \sqrt{u_x^2 + u_y^2} = 0.$$

Using the Euler operator method, it can be shown that this scalar PDE has the set of local conservation law multipliers given by linear combinations of

$$\begin{aligned} \Lambda^1[(t,x,y),U] &= \frac{U_xU_{yy} - U_yU_{xy}}{U_y^3}, \\ \Lambda^2[(t,x,y),U] &= \frac{U_yU_{xx} - U_xU_{xy}}{U_x^3}, \\ \Lambda^3[(t,x,y),U] &= f(U_x,U_y)(U_{xx}U_{yy} - U_{xy}^2), \end{aligned}$$

where $f(\cdot,\cdot)$ is any arbitrary smooth function of its arguments.

If $f(U_x, U_y)$ is a homogeneous function, then it's easy to check that each of the conservation law multipliers $\{\Lambda^j[x,U]\}_{j=1}^3$ and the flame equation is invariant under the scaling symmetry,

$$\hat{X}_\eta = \sum_{|I|\le 2} (D_I \eta[(t,x,y),U]) \frac{\partial}{\partial U_I},$$

where $\eta[(t,x,y),U] = tU_t + xU_x + yU_y$. Furthermore, the generator of a scaling symmetry satisfies

$$\hat{X}_\eta((\Lambda^j R)[(t,x,y),U]) = c^j(\Lambda^j R)[(t,x,y),U] + D_i\Theta^{ij}[(t,x,y),U]$$

for some $c^j \neq 0$ and trivial fluxes $\{\Theta^{ij} : \mathcal{D}\times\mathcal{V}^{(k)} \to \mathbb{R}\}_{i=1}^3$ for each $j = 1,2,3$, i.e. $\hat{X}_\eta$ is the generator of a non-critical scaling symmetry of the flame equation and of the three conservation law multipliers. Hence, we can apply Theorem 4.4.2 and the expression for $\Psi^{ij}[(t,x,y),U]$ given by Theorem 3.1.5 to obtain the equivalent algebraic fluxes for each of the sets of conservation law multipliers $\{\Lambda^j[x,U]\}_{j=1}^3$:

$$\Psi^{ij}[(t,x,y),U] = \frac{1}{c^j}\left(\eta^\rho \sum_{\substack{|J|\le 1\\ J^{(i)}=J}} (-1)^{|J|}\left(D_J\left(\Lambda^j \frac{\partial R}{\partial U^\rho_{J+\hat{i}}}\right)\right)\right. \tag{4.18}$$

$$\left. + \sum_{\substack{|I|=1\\ I^{(i-1)}=0}} (D_I\eta^\rho)\Lambda^j \frac{\partial R}{\partial U^\rho_{I+\hat{i}}}\right)[(t,x,y),U]. \tag{4.19}$$

Note that for the conservation law multiplier $\Lambda^3[(t,x,y),U]$, the equivalent algebraic fluxes in equation (4.19) were also obtained in Chapter 3 via the line integral formula from Theorem 3.2.1 or the simplified line integral formula from Theorem 3.2.5.

Chapter 5

Conclusion

In this thesis, we presented the flux equation method for finding conservation laws for PDEs arising from a given set of conservation law multipliers. By examples, we showed how the flux equation method generalizes some of the known methods of finding fluxes. In particular, we showed that the homotopy integral formula is in fact a special case of the line integral formula obtained from the flux equations. We also showed how the line integral formula can be simplified when there is a point symmetry of the PDE system and of the set of conservation law multipliers. In the case when the point symmetry is a non-critical scaling symmetry, the line integral formula leads to the same algebraic fluxes obtain by using the method of non-critical scaling symmetry.

In light of the flux equation method, there are many new directions for research. First, one can investigate whether the flux equation method can produce new fluxes on PDE systems where existing methods of finding conservation laws have difficulties. Secondly, using a point symmetry of the PDE system and of the set of conservation law multipliers, we have seen how the line integral formula can be simplified for finding equivalent fluxes. In some cases, this leads to algebraic expressions for the fluxes. It will be interesting to see if this result can be extended for more general classes of symmetries such as contact or higher-order symmetries. Thirdly, the flux equations provide new possibilities for computing fluxes through the use of symbolic software. Since the flux equations give the explicit dependence of the fluxes automatically from the PDEs and a set of conservation law multipliers, it will be interesting to compare the efficiency of the flux equation method with current methods using symbolic software [7, 10].

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Appendix A

Non-degenerate PDEs

Let $\mathcal{R}$ be a PDE system $\{R^\sigma[x, u(x)] = 0\}_{\sigma=1}^N$ defined on domain $\mathcal{D}$. Let $u_0(x)$ be a smooth function defined on $\mathcal{D}$ which satisfies the PDE system $\mathcal{R}$ at $x = x_0$. Then, the PDE system $\mathcal{R}$ is *locally solvable* at $x_0 \in \mathcal{D}$ and at the k-th prolongation of $u_0(x_0)$ if there exists a smooth solution $u(x)$ of the PDE system $\mathcal{R}$ defined on some neighbourhood of x_0 such that $u(x_0) = u_0(x_0)$. Furthermore, the PDE system $\mathcal{R}$ is called locally solvable if it is locally solvable at every $x_0 \in \mathcal{D}$ and every smooth function u_0 which satisfies the PDE system $\mathcal{R}$ at $x = x_0$.

The PDE system $\mathcal{R}$ has maximal rank if the $N \times (n + m\binom{n+k}{k})$ Jacobian matrix

$$J[x, U] = \left(\frac{\partial R^\sigma}{\partial x^i}, \frac{\partial R^\sigma}{\partial U_I^\rho} \right) [x, U] \tag{A.1}$$

has maximal rank on any smooth solution $U = u(x)$ of the PDE system $\mathcal{R}$.

Definition A.0.4. *A PDE system $\mathcal{R}$ is called* non-degenerate *if it is both locally solvable and has maximal rank.*

Theorem A.0.5. *Suppose the PDE system $\mathcal{R}$ is non-degenerate. Then a smooth function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ vanishes on any smooth solution of the PDE system $\mathcal{R}$ if and only if there exist smooth functions $A_{\sigma,J}[x, U]$ such that*

$$f[x, U] = \sum_{\sigma, |J| \leq k} A_{\sigma,J}[x, U] D_J R^\sigma[x, U].$$

Proof. See [8]. $\square$

Appendix B

Cauchy-Kovalevskaya Form

Definition B.0.6. *Suppose a given PDE system $\mathcal{R}$ has N equations, n independent variables and m dependent variables. Then the PDE system $\mathcal{R}$ is said to be in Cauchy-Kovalevskaya form if it has the following two properties:*

1. *N=m.*

2. *There exists a choice of variable z such that the highest derivative of u^σ with respect to z in the PDE system $\mathcal{R}$ can be isolated into an analytic system, i.e., for some positive integers $\{K^\sigma\}_{\sigma=1}^{n}$, each equation $R^\sigma[x, u(x)]$ of the PDE system $\mathcal{R}$ can be written in the form*

$$0 = R^\sigma[x, u(x)] = \frac{\partial^{K^\sigma} u^\sigma}{\partial z^{K^\sigma}} - S^\sigma[x, u(x)],$$

   where $S^\sigma[x, U]$ is analytic in its arguments and all other partial derivatives $\frac{\partial^k U^\sigma}{\partial z^k}$ appearing in $S^\sigma[x, U]$ have $k < K^\sigma$.

Definition B.0.7. *A PDE system $\mathcal{R}$ admits a Cauchy-Kovalevskaya form if there exists an analytic function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathcal{D} \times \mathcal{U}^{(k)}$ with analytic inverse such that $\{R^\sigma[f[x, u(x)]] = 0\}_{\sigma=1}^{N}$ is in Cauchy-Kovalevskaya form.*

Example B.0.8. *Suppose $\mathcal{R}$ is a scalar PDE of the form*

$$0 = R[x, u] = \frac{\partial^K u}{\partial z^K} - S[x, u],$$

where $S[.,.]$ is analytic with respect to its arguments and all partial derivatives $\frac{\partial^k U}{\partial z^k}$ in $S[x, U]$ have $k < K$. Then $\mathcal{R}$ is in Cauchy-Kovalevskaya form.

Theorem B.0.9. *(Cauchy-Kovalevskaya) Suppose a PDE system $\mathcal{R}$ is in Cauchy-Kovalevskaya form for the variable z. For any $x \in \mathcal{D}$, let $x = (z, \tilde{x})$. Then, for any $(z_0, \tilde{x}_0) \in \mathcal{D}$ and any analytic functions $\{f_k(\tilde{x})\}_{k=1}^{K-1}$ defined near the point $\tilde{x}_0$, the PDE system $\mathcal{R}$ has an unique analytic solution $u(z, \tilde{x})$ in some neighbourhood of $(z_0, \tilde{x}_0) \in \mathcal{D}$ that satisfies the data $\frac{\partial^k u}{\partial z^k}(z_0, \tilde{x}) = f_k(\tilde{x})$ for all $k < K$.*

Proof. See [8]. $\square$

Appendix C

Vector Fields, Flows and Symmetries

Theorem C.0.10. *Suppose $\hat{X}$ is an evolutionary vector field*[13] *and $\gamma : (a-\epsilon, a+\epsilon) \to \mathcal{U}_0$ is the flow*[14] *of $\hat{X}$ starting at $U \in \mathcal{U}_0$. Then for any smooth function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ and for all $s \in (a-\epsilon, a+\epsilon)$:*

$$\frac{d}{ds} f[x, \gamma(s)] = \hat{X}(f[x, \gamma(s)]).$$

Proof. This follows from the chain rule and the definition of $\gamma(s)$. □

Theorem C.0.11. *Suppose $\hat{X}$ is an evolutionary vector field. Then for any smooth function $f : \mathcal{D} \times \mathcal{U}^{(k)} \to \mathbb{R}$ and $i = 1, \ldots, n$:*

$$\hat{X}(D_i(f[x, U])) = D_i(\hat{X}(f[x, U])).$$

Proof. This follows from direct computation. □

Definition C.0.12. *Given a PDE system $\mathcal{R}$ defined on $\mathcal{D}$, an evolutionary vector field $\hat{X}$ is called the generator of a point symmetry of the PDE system $\mathcal{R}$ if for any solution $u(x)$ of the PDE system, the flow $\gamma(s)$ under $\hat{X}$ with $\gamma(a) = u(x)$ satisfies*

$$R^\sigma[x, \gamma(s)] = 0$$

for all $\sigma = 1, \ldots, m$, $x \in \mathcal{D}$ and all $s \in (a-\epsilon, a+\epsilon)$ for some $\epsilon > 0$. (I.e. the transformation generated by the flow of $\hat{X}$ maps a solution $u(x)$ of the PDE system $\mathcal{R}$ to another solution of the PDE system $\mathcal{R}$.)

Theorem C.0.13. *Given a non-degenerate PDE system $\mathcal{R}$ defined on $\mathcal{D}$, a vector field X is the generator of a point symmetry of the PDE system $\mathcal{R}$ if and only if for all $\sigma = 1, \ldots, m$:*

$$X(R^\sigma[x, U])|_{U=u(x)} = 0.$$

[13]See Definition 3.1.7 from Chapter 3.
[14]See Definition 3.1.8 from Chapter 3.

Proof. See [8]. □

Theorem C.0.14. *Suppose the PDE system $\mathcal{R}$ is non-degenerate and $\hat{X}$ is the generator of a point symmetry of the PDE system $\mathcal{R}$. Let $\gamma : (a-\epsilon, a+\epsilon) \to \mathcal{U}_0$ be the corresponding flow under $\hat{X}$ with $\gamma(a) = U$. Then there exists a smooth matrix $\{A^\sigma_\mu[x, U; s]\}^m_{\sigma,\mu=1}$ such that for all $\sigma = 1, \ldots, m$, $x \in \mathcal{D}$ and $s \in (a-\epsilon, a+\epsilon)$:*

$$R^\sigma[x, \gamma(s)] = A^\sigma_\mu[x, U; s]R^\mu[x, U].$$

Proof. Since $\hat{X}$ is the generator of a point symmetry of a non-degenerate PDE system $\mathcal{R}$ and $\gamma(s)$ implicitly depends on U, then on any solution $u(x)$ of the PDE system $\mathcal{R}$,

$$R^\sigma[x, \gamma(s)]|_{U=u(x)} = 0.$$

Hence, by Theorem A.0.5, there exists smooth functions $\{A^\sigma_\mu[x, U; s]\}^m_{\sigma,\mu=1}$ and $\{A^\sigma_{\mu,J}[x, U; s]\}_{\sigma,\mu,J}$ such that:

$$R^\sigma[x, \gamma(s)] = A^\sigma_\mu[x, U; s]R^\mu[x, U] + \sum_{\substack{0<|J|\leq k\\ \mu=1,\ldots,m}} A^\sigma_{\mu,J}[x, U; s]D_J R^\mu[x, U]. \tag{C.1}$$

By definition of the generator of a point symmetry $\hat{X}$, the image of the flow $\gamma(s)$ under $\hat{X}$ lies in $\mathcal{U}_0$. Hence, the image of the prolongation of $\gamma(s)$ at most lies in the k-th prolongation jet space $\mathcal{U}^{(k)}$. In other words, $R^\sigma[x, \gamma(s)]$ contains at most k-th order derivatives of U for all $\sigma = 1, \ldots, m$. Moreover, since each term $D_J R^\mu[x, U]$ in equation (C.1) contains derivatives strictly higher than the k-th order, the sum in (C.1) must vanish for all $s \in (a-\epsilon, a+\epsilon)$ and $x \in \mathcal{D}$. □