Collapsing Fibres under Kähler Ricci Flow on Hirzebruch Manifolds

by

Kael Dixon

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Abstract

In this article we study the Kähler Ricci flow on a class of $\mathbb{CP}^1$ bundles over $\mathbb{CP}^{n-1}$ known as Hirzebruch manifolds. These are defined by $M_{n,k} := \mathbb{P}(H^k \oplus \mathbb{C}_{\mathbb{P}^{n-1}})$ where H is the canonical line bundle and $n, k \in \mathbb{N}$ (we refer to §2 for a detailed description of these). We follow the work in [11], where Song and Weinkove study solutions to the Kähler Ricci flow for a Calabi symmetric Kähler metrics on Hirzebruch manifolds (see §2 for definitions). They were able to show that, depending on the initial Kähler class, the Ricci flow would reach a finite time singularity corresponding to the manifold either shrinking to a point, contracting the zero section to a point, or collapsing the fibres. In this paper, we investigate how the fibres collapse in the latter case with the further assumptions that the singularity is formed at a type I rate, and that the length of a generic vector does not decay too quickly in some sense. In this case we show that the fibres converge to round spheres after blowing up around a singular point on a fibre.

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Chapter 1

Introduction

The goal of the thisis is to prove the following theorem:

Theorem 1. *Assume* $(M_{n,k}, g(t))$ *is radially symmetric solution to the Kähler Ricci flow* (1.1) *on the Hirzebruch manifold* $M_{n,k}$ *with initial metric as in case 3 described in §2.1. Suppose the solution develops a Type I singularity at time* T *and that for some* $p \in M_{n,k}\backslash\{D_0, D_\infty\}$ *and vector* $X \in T_pM_{n,k}$ *tangent to the fibre, there exists* $c > 0$ *such that* $\|X\|^2_{g(t)} \geq c(T-t)$.

Then for any $p \in M_{n,k}$*, the singularity model at* p *is a Riemannian product* $(\mathbb{C}^{n-1} \times \mathbb{CP}^1, \delta_{\mathbb{C}^{n-1}} + h(t))$ *where* $\delta_{\mathbb{C}^{n-1}}$ *is the standard flat metric on* $\mathbb{C}^{n-1}$ *and* $(\mathbb{CP}^1, h(t))$ *is the standard shrinking sphere soliton.*

We refer to §1.1, §2 and §3 for the definitions, terms and notations in Theorem 1.

1.1 Kähler Ricci Flow

The Ricci flow is the following system of PDE's on the space of metrics of a manifold:

$$\begin{aligned} \frac{\partial g}{\partial t}(t) &= -\mathrm{Ric}_{g(t)} \\ g(0) &= g_0, \end{aligned} \tag{1.1}$$

This equation is only weakly parabolic, yet local existence and uniqueness of smooth solutions given smooth initial metrics was proved by Hamilton in [4]. We say that $T > 0$ is the *singular time* of a solution to Ricci flow if $[0, T)$ is a maximal interval on which a smooth solution to the flow exists. If $T < \infty$, we say that the solution forms a *finite time singularity.*

Kähler Ricci flow is simply the study of Ricci flow in the case when the initial metric g_0 is Kähler. In this case, solution $g(t)$ will remain Kähler as long as the solution exists (See [7]) and thus induces the corresponding flow of Kähler forms:

$$\begin{aligned} \tfrac{d}{dt}\omega(t) &= -\mathrm{Ric}^{\wedge}_{\omega(t)} \\ \omega(0) &= \omega_0 \end{aligned} \tag{1.2}$$

where ω is the Kähler form of g and $\mathrm{Ric}^{\wedge}$ is the Ricci form. Since both of these are closed real-valued (1,1)-forms, this system descends to the following flow on $H^2(M,\mathbb{R}) \cap H^{1,1}(M)$:

$$\begin{aligned} \tfrac{d}{dt}[\omega(t)] &= -[\mathrm{Ric}^{\wedge}_{\omega(t)}] \\ [\omega(0)] &= [\omega_0] \end{aligned} \tag{1.3}$$

We call the set of all cohomology classes corresponding to Kähler metrics the *Kähler cone*:

$$\{\alpha \in H^2(M,\mathbb{R}) \cap H^{1,1}(M) : \alpha > 0\},$$

where we write $\alpha > 0$ if there exists a form $\tau > 0$ such that $[\tau] = \alpha$. Since ω is Kähler, $[\mathrm{Ric}^{\wedge}_{\omega}] = 2\pi c_1(M)$, where $c_1(M)$ is the first Chern class of M which is an invariant of the complex structure, and thus is independent of ω and in particular of t. In particular the family of classes

$$\alpha(t) := [\omega_0] - 2\pi t c_1(M)$$

clearly solves the above cohomological equation. Conversely, as described in [11], if $\alpha(t)$ above lies in the Kähler cone for all $t \in [0,T)$, then (1.2) will in fact have a solution $\omega(t)$ defined for $t \in [0,T)$. It follows that if T is finite, then $\alpha(T)$ lies on the boundry of the Kähler cone.

Chapter 2

Hirzebruch Manifolds

2.1 Introduction

Let $M_{n,k} := \mathbb{P}(H^{\otimes k} \oplus \mathbb{C}_{\mathbb{P}^{n-1}})$ be an $n-$dimensional Hirzebruch manifold, where H is the canonical line bundle over $\mathbb{CP}^{n-1}$ (see §2.2 for detailed description). It is known that in this case the first Chern class $c_1(M_{n,k})$ is positive and thus the Kähler Ricci flow starting from any Kähler metric on $M_{n,k}$ must develop a singularity in finite time (see [11]). As discussed above, this corresponds to reaching the boundry of the Kähler cone. Conveniently, the Kähler cone is easy to describe in this case. In fact, it is generated by only two classes: $[D_\infty]$, the fundamental class of the divisor D_∞ corresponding to the ∞ section of the bundle, and $[\pi^{-1}\omega_{FS}]$, the pull back of the Fubini-Study metric on the base. We can write our initial Kähler class as

$$\alpha_0 = \frac{1}{k}(b_0 - a_0)[D_\infty] + a_0[\pi^{-1}\omega_{FS}],$$

for some $0 < a_0 < b_0$. Suppose now that the we have a solution to the Kähler Ricci flow on $M_{n,k}$ which becomes singular at time $T > 0$, in other words that $\alpha(t) := [\omega_0] - 2\pi t c_1(M)$ reaches the boundry of the Kähler cone at time T. Then under the assumption that the initial metric is Calabi symmetric (see §2.3 for definition) it was shown in [11] that the behaviour of the Ricci flow near time T can be classified in terms of which part of the boundry of the Kähler cone $\alpha(T)$ lies on:

1. *Case*: $a_0(n + k) = b_0(n - k)$.

   Then $\alpha(T) = 0$, so the manifold shrinks to a point. This is the situation where it is possible for the solution to be a shrinking Kähler Ricci soliton, and in fact, there exists one on Hirzebruch manifolds called the Koiso soliton. This soliton is constructed using the same symmetry assumptions that we are using. See [6] for details. A theorem of Tian and Zhu then tells us that if we normalized the Kähler Ricci flow to keep the volume constant, then the solution must converge to the soliton. See [10] for details.

2. *Case*: $a_0(n+k) < b_0(n-k)$.

   Then $\alpha(T)$ lies on the part of the boundry generated by $[D_\infty]$ and under the symmetry assumption, the Ricci flow solution exhibits Gromov-Hausdorff convergence to the orbifold $M_{n,k}/D_0$, with the divisor D_0 contracted to a point.

3. *Case*: $a_0(n+k) > b_0(n-k)$.

   This is the case that we will concern ourselves with, namely the case when $\alpha(T)$ lies on the part of the boundry generated by $[\pi^{-1}\omega_{FS}]$. In this case the volume is proportional to $(T-t)$ and under the symmetry assumption the Ricci flow solution exhibits Gromov-Hausdorff convergence to the base manifold with metric $a_T\omega_{FS}$ where ω_{FS} is the Fubini-Study metric on the base and $\alpha(T) = a_T[\pi^{-1}\omega_{FS}]$.

2.2 Local Coordinates on $M_{n,k}$

We now describe the manifolds $M_{n,k}$ through local coordinates. Let $\{x_i\}_{i=1}^n$ be homogeneous coordinates for $\mathbb{CP}^{n-1}$. Consider the charts U_i given by $x_i \neq 0$. $M_{n,k}$ can then be realised as a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ via projective fibre coordinates y_i on $\pi^{-1}(U_i)$ where the transition functions are given by $y_j = \left(\frac{x_j}{x_l}\right)^k y_i$ on $\pi^{-1}(U_i \cap U_j)$. Notice that there are two natural sections (divisors) of this bundle: D_0 and D_∞, corresponding to the fibre coordinate y_j taking the values 0 and ∞ respectively.

Consider the action of $\mathbb{Z}_k$ on $\mathbb{C}^n\backslash 0$ generated by

$$1 : \mathbb{C}^n\backslash 0 \to \mathbb{C}^n\backslash 0 : x \mapsto e^{\frac{2\pi i}{k}} x.$$

We identify $(\mathbb{C}^n\backslash 0)/\mathbb{Z}_k$ with $M_{n,k}\backslash(D_0 \cup D_\infty)$ via the following k-to-one maps on each $V_i := \{x_i \neq 0\} \subset \mathbb{C}^n\backslash 0$

$$\psi_i : V_i \to \pi^{-1}(U_i)\backslash(U_0 \cup U_\infty) \cong U_i \times \mathbb{CP}^1\backslash\{0,\infty\} : x \mapsto ([x], x_i^k).$$

Note that these maps commute with the transition maps, allowing us to extend this local identification to the whole space $\mathbb{C}^n\backslash 0$. Also note that the divisors are obtained by taking the limit as x_i goes to 0 or ∞ respectively.

2.3 Radially Symmetric Metrics

In this section we describe a class of rotationally metrics on the Hirzebruch manifolds $M_{n,k}$. These were first investigated by Calabi in [1] from where we derive most of the calculations in this section. We will begin by defining a class of radially symmetric metrics on $\mathbb{C}^n \backslash 0$. Then by assuming certain asymptotic conditions for these metrics (at 0 and ∞) we will then show that these push forward to a smooth metric on $M_{n,k}$ via the coordinate descriptions of the last section.

Consider the function on $\mathbb{C}^n \backslash 0$ given by $\rho := \log \|\vec{x}\|^2$. We will consider Kähler metrics on $\mathbb{C}^n \backslash 0$ with potentials $u(\rho)$, which are functions of ρ only, and hence are radially symmetric. In order for u to be a potential for a Kähler metric, we need $u'' > 0$, where the derivatives are taken with respect to ρ.

Now consider the Kähler metric

$$\omega = i\partial\bar{\partial}u = i\partial(u'\bar{\partial}\rho) = iu''\partial\rho \wedge \bar{\partial}\rho + iu'\partial\bar{\partial}\rho.$$

Claim 1. *The radially symmetric metric ω descends to a metric on $(\mathbb{C}^n \backslash 0)/\mathbb{Z}_k$.*

Proof. It suffices to show that the function ρ is invariant under the action. In other words $1_*\rho = \rho$.

$$1_*\rho(\vec{x}) = \rho \circ 1(\vec{x}) = \rho(e^{\frac{2\pi i}{k}}\vec{x}) = \log \|e^{\frac{2\pi i}{k}}\vec{x}\|^2 = \log \|\vec{x}\|^2 = \rho(\vec{x})$$

□

In particular, we can pushforward ω to $M_{n,k} \backslash (D_0, D_\infty)$ via the maps $\{\psi_i\}_{i=1}^n$ from §2.2. Now we further impose the condition that there exist $0 < a < b < \infty$ such that

$$a := \lim_{\rho \to -\infty} u',$$

$$b := \lim_{\rho \to \infty} u'.$$

These asymptotic conditions basicallly guarantee that the Kähler form ω on $M_{n,k} \backslash (D_0 \cup D_\infty)$ will extend smoothly to all of $M_{n,k}$ ([1]). Now we will explicitely compute ω in the local trivialization $\pi^{-1}(\{U_j\})$, for some fixed $j \in \{0...n\}$. First, choose holomorphic coordinates $\{z_i := \frac{x_i}{x_j}\}_{i \neq j}$ for U_j and y for the fibre.

Claim 2. *The function ρ pushes forward to a function ρ on $M_{n,k} \backslash (D_0 \cup D_\infty)$ which can be written in coordinates as $\rho = \frac{1}{k}\log y\bar{y} + 2\pi v_{FS}$, where v_{FS} is the potential for the Fubini-Study metric on the base $\mathbb{CP}^{n-1}$*

Proof. Our identification is through the maps $\psi_i : V_i \to \pi^{-1}(U_i)\backslash(U_0 \cup U_\infty)$, so we compute:

$$\begin{aligned}(\psi_j)_*\rho = (\psi_j)_* \log\|\vec{x}\|^2 = (\psi_j)_* \log\sum_{i=0}^n x_i\bar{x}_i = (\psi_j)_* \log\left(x_j\bar{x}_j \sum_{i=0}^n \frac{x_i\bar{x}_i}{x_j\bar{x}_j}\right)\\ = (\psi_j)_*\left(\tfrac{1}{k}\log(x_j\bar{x}_j)^k + \log\sum_{i=0}^n \frac{x_i\bar{x}_i}{x_j\bar{x}_j}\right)\\ = \tfrac{1}{k}\log y\bar{y} + \log\left(1+\sum_{i\neq j} z_i\bar{z}_i\right) = \tfrac{1}{k}\log y\bar{y} + 2\pi v_{FS}.\end{aligned}$$

□

For ease of notation, we will consider ρ as a function on each trivializatoin U_j via the identification above, writing ρ instead of $(\phi_j)_*\rho$. Note that

$$\partial\bar{\partial}\log y\bar{y} = \partial\tfrac{y}{y\bar{y}}d\bar{y} = \partial\tfrac{1}{\bar{y}}d\bar{y} = 0,$$

so that $i\partial\bar{\partial}\rho = i\partial\bar{\partial}v_{VS} = \pi^{-1}\omega_{FS}$ is the pull-back of the Fubini-Study metric on the base. Since the Kähler form of the radially symmetric metric is $iu''\partial\rho\wedge\bar{\partial}\rho + iu'\partial\bar{\partial}\rho$, this shows that the second term is degenerate along the fibre. Song and Weinkove showed that case 3 of Ricci flow on $M_{n,k}$ basically corresponds to having u' converging to a constant under the flow. From the above observation, this intuitively corresponds to: as u' moves towards being a constant, making u'' go to zero, the metric becomes proportional to the Fubini-Study metric on the base and degenerate on the fibre (see Theorem 2).

2.4 A Closer Look at the Fibres

Claim 3. *Restricting ω to the fibres produces pairwise isometric Riemannian manifolds.*

Proof. It suffices to prove the claim on every local trivialization, since we can then compose isometries to map any fibre onto any other one. Choose a local trivialization $\pi^{-1}(U_j)$ for some $j \in \{1,\dots,n\}$. Using the holomorphic coordinates $\{z_i\}_{i\neq j}$ on U_j, it suffices to show that the fibre at z is isometric to the fibre at 0 with respect to the the induced metric for each $z \in U_j$.

Using the previous claim and the fact that the v_{FS} term does not contribute to the fibre direction, we get that the induced metric on the fibre at z is

$$\begin{aligned}
\omega|_{\pi^{-1}(z)} &= iu''|_{\pi^{-1}(z)}\partial\rho|_{\pi^{-1}(z)} \wedge \bar{\partial}\rho|_{\pi^{-1}(z)} + iu'|_{\pi^{-1}(z)}\partial\bar{\partial}\rho|_{\pi^{-1}(z)} \\
&= iu''|_{\pi^{-1}(z)}\partial(\tfrac{1}{k}\log y\bar{y}) \wedge \bar{\partial}(\tfrac{1}{k}\log y\bar{y}) + iu'|_{\pi^{-1}(z)}\partial\bar{\partial}(\tfrac{1}{k}\log y\bar{y}) \\
&= \frac{iu''|_{\pi^{-1}(z)}}{k^2 y\bar{y}} dy \wedge d\bar{y}.
\end{aligned}$$

Note that the only dependence on z is from u'' being a function of ρ, and hence of both y and z. We will define our isometry by

$$f_z : \pi^{-1}(0) \to \pi^{-1}(z) : y \mapsto \left(1 + \|z\|^2\right)^{\frac{k}{2}} y.$$

Then

$$\begin{aligned}
(f_z)_*\rho|_{\pi^{-1}(0)} &= (f_z)_*\frac{1}{k}\log y\bar{y} = \tfrac{1}{k}\log y\bar{y}\left(1 + \|z\|^2\right)^k \\
&= \tfrac{1}{k}\log y\bar{y} + \log\left(1 + \|z\|^2\right) \\
&= \rho|_{\pi^{-1}(z)}.
\end{aligned}$$

Hence $(f_z)_*u''|_{\pi^{-1}(0)} = u''|_{\pi^{-1}(z)}$. This gives

$$\begin{aligned}
(f_z)_*\omega|_{\pi^{-1}(0)} &= (f_z)_*\frac{iu''|_{\pi^{-1}(0)}}{k^2 y\bar{y}} dy \wedge d\bar{y} \\
&= \frac{\left(1 + \|z\|^2\right)^k}{(1 + \|z\|^2)^k}\frac{iu''|_{\pi^{-1}(z)}}{k^2 y\bar{y}} dy \wedge d\bar{y} \\
&= \frac{iu''|_{\pi^{-1}(z)}}{k^2 y\bar{y}} dy \wedge d\bar{y} \\
&= \omega|_{\pi^{-1}(z)},
\end{aligned}$$

so that f_z is an isometry. □

The above claim allows us to concentrate on the fibre $\pi^{-1}(0)$ without losing any generality. By definition, $\pi^{-1}(0) \cong \mathbb{CP}^1 \cong \mathbb{C} \cup \infty$. Our coordinate y will act as a coordinate for $\mathbb{C}$ here. On $\pi^{-1}(0)$, we have $\rho = \frac{1}{k}\log y\bar{y}$.

Chapter 3

Singularity Analysis for Ricci flow

In this section we set up the necessary background to study singularities for the Ricci flow in our setting: case 3 from §2.1. In [11], Song and Weinkove show that under the condition $a_0(n+k) > b_0(n-k)$ (case 3), the Ricci flow with an initial metric satisfying the Calabi symmetry condition will form a singularity at time $T := \frac{b_0 - a_0}{2k}$. They showed that this corresponds to studying a corresponding family $u(\rho, t)$ of evolving potentials as above such that $\partial\bar{\partial}u(\rho, t)$ is the corresponding family of evolving metrics. In particular, the the assymptotics "$a(t)$ and $b(t)$" of $u(\rho, t)$ from the previous section will also vary in time such that

Theorem 2. *[Song-Weinkove [11]] In case 3 for the Ricci flow of a Calabi symmetric metric on $M_{n,k}$, if T is the singular time then $(b(t) - a(t)) = 2k(T - t)$. In particular, the Ricci flow solution exhibits Gromov-Hausdorff convergence to the base manifold with metric $a_T\pi^{-1}\omega_{FS}$ where ω_{FS} is the Fubini-Study metric on the base.*

We now discuss singularities for the Ricci flow in more detail. We will define and discuss type I singularities of the flow. In particular, we will show that the assumption on the existence of a singular point p on $M_{n,k}$ in Theorem 1.1 is natural under the assumption of type I singularity. As for the additional assumption we make on the rate of metric decay at p, we observe that this is natural in the sense as shown in §4.2 Lemma 4, the volume of the fiber with respect to $g(t)$ is precisely $a(T - t)$ for some constant a.

To reduce clutter in the following sections, I will often write $f \lesssim g$ for functions f and g to mean that there exists a positive constant C such that $f \leq Cg$ on the common domain of f and g. This constant will often only be dependent on the initial metric, but occasionally it may depend on the point of evaluation (but not the time), in which case it will be mentioned that it is a pointwise estimate.

For Ricci flow to form a singularity at time T, one must necessarily have

$$\sup_M \|\mathrm{Rm}_{g(t)}\|_{g(t)} \gtrsim \frac{1}{T-t}.$$

This motivates the following definitions:

Definition 1. *A sequence* $(p_i, t_i)_{i\in\mathbb{N}} \subset M \times [0,T)$ *is an* essential blow-up sequence *if* $t_i \nearrow T$ *and*

$$\|Rm_{g(t_i)}\|_{g(t_i)}(p_i) \gtrsim \frac{1}{T-t_i}$$

Definition 2. *A point* $p \in M$ *is a* singular point *if there exists an essential blow up sequence* $(p_i, t_i)_{i\in\mathbb{N}}$ *with* $p = \lim_{i\to\infty} p_i$.[1]

Finite time singularities are classified according to the speed at which

$$\sup_M \|\mathrm{Rm}_{g(t)}\|_{g(t)}$$

diverges to ∞. We say that a solution for the Ricci flow equation is a *Type I solution* if

$$\sup_M \|\mathrm{Rm}_{g(t)}\|_{g(t)} \lesssim \frac{1}{T-t}.$$

We say that a finite time singular solution is a *Type IIa solution* if it is not of type I. It is conjectured that Type IIa singularities will only form under highly symmetric initial conditions.

From now on we assume that $(M_{n,k}, g(t))$ is a solution to Kähler Ricci flow as in Theorem 1.1.

Lemma 1. *Let* $p \in M$. *If there exists* $c > 0$ *and* $t_0 \in [0,T)$ *such that*

$$\|Rm_{g(t)}\|_{g(t)}(p) \leq \tfrac{c}{T-t}, \forall t \in (t_0, T),$$

then $\|X_p\|_{g(t)} \gtrsim (T-t)^c$ *for all* $X \in T_pM$.

Proof. Let $\alpha > 0$ be a constant to be chosen later. Then for $t \in (t_0, T)$, we have

$$\frac{d}{dt}\log\|\alpha X\|_g^2 = -\frac{\mathrm{Ric}_g(\alpha X, \alpha X)}{\|\alpha X\|_g^2} \geq -\|\mathrm{Ric}_g\|_g \geq -\|\mathrm{Rm}_g\|_g \geq -\frac{c}{T-t}.$$

[1] This is not the most general definition of singular point, but for type I solutions they are equivalent. See [3] for details.

Integrating from t_0 to t gives

$$\log \|\alpha X\|^2_{g(s)}|^{s=t}_{s=t_0} \geq c \log(T-s)|^{s=t}_{s=t_0}.$$

Choosing α so that $\log \|\alpha X\|^2_{g(t_0)} = c\log(T-t_0)$ gives

$$\log \|\alpha X\|^2_{g(t)} \geq c \log(T-t).$$

Finally, exponentiating gives

$$(T-t)^c \leq \|\alpha X\|^2_{g(t)} \lesssim \|X\|^2_{g(t)}.$$

□

Proposition 1. *For any $p \in M_{n,k} \backslash \{D_0 \cup D_\infty\}$, there exists a sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \nearrow T$ such that $(p, t_i)_{i\in\mathbb{N}}$ is an essential blow-up sequence, and thus p is a singular point.*

Proof. Assume that there does not exist such an essential blow-up sequence. Then for any $c > 0$, the set

$$\left\{ t \in [0,T) : \|\mathrm{Rm}_{g(t)}\|_{g(t)}(p) \geq \frac{c}{T-t} \right\}$$

is bounded. Thus there exists a $t_0 \in [0,T)$ such that

$$\|\mathrm{Rm}_{g(t)}\|_{g(t)}(p) < \frac{c}{T-t}, \forall t \in (t_0, T).$$

Without loss of generality, by the symmetry of the metric, we can consider p to be in $\pi^{-1}(U_n)$. Then, as before, we have local holomorphic coordinates $\{y, z_i\}_{i=1}^{n-1}$. Moreover, the symmetry allows us to assume that the z coordinate of p is 0. It is clear by looking at the metric in these coordinates that

$$\|\tfrac{\partial}{\partial y}\|^2_{g(t)}(p) = \frac{u''(t)}{k^2 y\bar{y}}(p).$$

From Lemma 4.3 in [11], we $u''(t) \lesssim (T-t)$, so that $\|\frac{\partial}{\partial y}\|^2_{g(t)} \lesssim (T-t)$. Combining this with the preceeding lemma gives

$$(T-t)^c < \|\tfrac{\partial}{\partial y}\|^2_{g(t)} \lesssim T-t.$$

In other words, $(T-t)^{c-1}$ is bounded, which is impossible for $c<1$. This contradicts c being chosen arbitrarily. □

Note that it is not surprising that the singularity is always found in our case by using an essential blow-up sequence with each term using the same point. In fact, in [3], Enders, Müller and Topping show that every singularity for a type I Ricci flow solution on a complete manifold occurs in this way.

For the rest of the discussion, we will fix a fibre and the singular point p such that without loss of generality, in local coordinates we have $y(p) = 0$ and $z(p) = 0$. In particular, $\rho(p) = 0$.

3.1 Parabolic Dilations

Let $(p_i, t_i)_{i\in\mathbb{N}}$ be an essential blow-up sequence along the Ricci flow. We will now describe the parabolic scalings of the flow along such a sequence. We will use this in the next section to define the singularity model for the flow along a blow up sequence.

For each i, we define

$$K_i = \|\mathrm{Rm}_g\|_g(p_i, t_i)$$

and a time dependent metric

$$g_i : \big[-K_i t_i, K_i(T - t_i)\big) \to \mathrm{Sym}^2 TM^* : t \mapsto K_i g\left(t_i + \frac{t}{K_i}\right).$$

Note that each g_i is a solution to the Ricci flow equation, since

$$\frac{d}{dt} g_i(t) = \frac{d}{dt} K_i g(t_i + tK_i^{-1}) = -\mathrm{Ric}_{g(t_i + tK_i^{-1})} = -\mathrm{Ric}_{K_i^{-1} g_i(t)} = -\mathrm{Ric}_{g_i(t)}.$$

Definition 3. *The sequence of Ricci flow solutions* $(g_i)_{i\in\mathbb{N}}$ *are the* parabolic dilations *associated to the essential blow-up sequence* $(p_i, t_i)_{i\in\mathbb{N}}$.

We will analyze the parabolic dilations through their behaviour at $t = 0$. Note that $g_i(0) = K_i g(t_i)$. Thus

$$\|\mathrm{Rm}_{g_i(0)}\|_{g_i(0)} = \|\mathrm{Rm}_{K_i g(t_i)}\|_{K_i g(t_i)} = K_i^{-1} \|\mathrm{Rm}_{g(t_i)}\|_{g(t_i)} = K_i^{-1} K_i = 1.$$

Let $p \in M_{n,k} \backslash \{D_0 \cup D_\infty\}$. From proposition 1, there exists an essential blow-up sequence $(p, t_i)_{i\in\mathbb{N}}$.

Proposition 2. *The distance between fibres with respect to* $g_i(0)$ *diverges to infinity as* $i \to \infty$.

Proof. Recall that $M_{n,k}, g(t)$ converges in the Gromov-Hausdorff sense to the base $\mathbb{CP}^{n-1}$ with the metric $a_T g_{FS}$ under the Ricci flow as t approaches T. Thus, the distance between points in two different fibres converges to the distance between the corresponding points on the base with respect to $a_T g_{FS}$. With respect to the parabolic dilations $g_i(0) = K_i g(t_i)$, this allows us to approximate the distance between fibres by the distance between the corresponding base points with respect to $K_i a_T g_{FS}$, which diverges since K_i does. $\square$

Chapter 4

The Singularity Model

Let $(M_{n,k}, g(t), p)$ be the solution to Ricci flow as in Theorem 1 where p is the singular point. We now consider the corresponding parabolic dilations $(M_{n,k}, g_i, p)_{i\in\mathbb{N}}$ of our flow as a sequence of pointed solutions to the Ricci flow. We will see that this sequence converges in some sense to some $(M_\infty, g_\infty, p_\infty)$, which is the *singularity model* corresponding to $(M_{n,k}, g_i, p)_{i\in\mathbb{N}}$. In this section, we will review some basic definitions and results about singularity models, and describe the one corresponding to $(M_{n,k}, g_i, p)_{i\in\mathbb{N}}$ as described above.

Definition 4. *A sequence $(M, g_i, p_i)_{i\in\mathbb{N}}$ of pointed Riemannian manifolds* converges in the Cheeger-Gromov sense *to $(M_\infty, g_\infty, p_\infty)$ if for each $i \in \mathbb{N}$ there exists neighbourhoods $\mathcal{U}_i$ and $\mathcal{V}_i$ of p_∞ and p_i respectively and a diffeomorphism $\phi_i : \mathcal{U}_i \cong \mathcal{V}_i$ satisfying the following three conditions:*

- $\phi_i(p_\infty) = p_i$.
- $(\phi_i^* g_i)_{i\in\mathbb{N}}$ *converges uniformly to g_∞ in every $\mathcal{C}^m$ norm on any compact subset of $M_\infty \times (\alpha, \omega)$.*
- *Every compact subset of M_∞ must be contained in each of $\{\mathcal{U}_i\}_{i\geq N}$ for some $N \in \mathbb{N}$.*

Definition 5. *Let $\kappa > 0$. A Riemannian manifold (M, g) is κ-*noncollapsed at all scales *if for any $x \in M$ and $r > 0$ satsifying* $\sup_{B_r(x)} \|Rm\| \leq \frac{1}{r^2}$*, we have $Vol(B_r(x)) \geq \kappa r^n$.*

Theorem 3. *[Hamilton-Perelman] Let $(M, g(t))$ be a smooth solution to the Ricci flow on a closed manifold M encountering a singularty at a finite time T. Let $(p_i, t_i)_{i\in\mathbb{N}}$ be an essential blow-up sequence such that*

$$\sup_{(x,t)\in M\times[0,t_i)} \|Rm_{g(t)}\|_{g(t)}(x) \lesssim K_i.$$

Then there exists a subsequence of the sequence of parabolic dilations (M, g_i, p_i) converging in the sense of Cheeger-Gromov to a complete ancient solution to the Ricci flow $(M_\infty, g_\infty, p_\infty)$. Futhermore, there exists a $\kappa > 0$ such that g_∞ is κ-noncollapsed at all scales.

This result follows from Hamilton's completeness theorem [5] and Perelman's no local collapsing [8].

Corollary 1. *For our singular point $p \in M_{n,k}$, the corresponding sequence of parabolic dilations $(M_{n,k}, g_i, p)$ converges in the sense of Cheeger-Gromov to a complete ancient solution to the Ricci flow $(M_\infty, g_\infty, p_\infty)$. Futhermore, there exists a $\kappa > 0$ such that g_∞ is κ-noncollapsed at all scales.*

Proof. Since p is a singular point, $K_i \gtrsim \frac{1}{T-t_i}$. Since we are assuming that our solution is a type I solution, we have

$$\sup_{M_{n,k}} \|\mathrm{Rm}_{g(t)}\|_{g(t)} \lesssim \tfrac{1}{T-t_i}.$$

Putting this together, we have

$$\sup_{(x,t)\in M_{n,k}\times[0,t_i)} \|\mathrm{Rm}_{g(t)}\|_{g(t)}(x) \lesssim \frac{1}{T-t_i} \lesssim K_i,$$

so that we can apply the above theorem to get the desired result. □

Lemma 2.

$$e^{-|\rho|} \lesssim \frac{u''(\rho,t)}{u''(0,t)} \lesssim e^{|\rho|}.$$

Proof. We will again use Lemma 4.3 from [11], which this time provides us with the estimate $|u'''| \lesssim u''$. This gives

$$\left|\log\frac{u''(\rho)}{u''(0)}\right| = \left|\int_0^\rho (\log u''(s))' ds\right| = \left|\int_0^\rho \frac{u'''(s)}{u''(s)} ds\right| \lesssim |\rho|\,.$$

Exponentiating then gives the desired result. □

Proposition 3. *For each $\rho \in \mathbb{R}$, we have the pointwise estimate*

$$(T-t) \lesssim u''(\rho,t) \lesssim (T-t).$$

Proof. By our assumption, there exists an $X \in TM_{n,k}$ with $\rho_0 = \rho\circ\pi(X) \in \mathbb{R}$ and $\|X\|^2_{g(t)} \gtrsim (T-t)$. By symmetry, we can assume that $\pi(X)$ is in the same fibre as p, where we have local coordinates $\{y, z\}$ with $z = 0$ on the fibre. Then we can write the metric as

$$g(t) = u''(\rho,t)\tfrac{dy}{ky}\tfrac{d\bar{y}}{k\bar{y}}$$

for $\rho = \log(y\bar{y}) \in \mathbb{R}$. Thus $\|X\|^2_{g(t)} \propto u''(\rho_0, t)$. It follows from our assumption that $u''(\rho_0,t) \gtrsim (T-t)$. Combining this with the preceeding lemma gives the pointwise estimate $u''(\rho,t) \gtrsim (T-t)$ for $\rho \in \mathbb{R}$. This combines with the uniform estimate $u''(\rho,t) \lesssim (T-t)$ from Lemma 4.3 of [11] to give the desired result. □

4.1 A Candidate for the Limit Manifold

In this section, we will consider $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ as a candidate for the limit manifold M_∞. We will begin by defining a sequence of holomorphic maps $\psi_i : \mathbb{C}^{n-1} \times \mathbb{CP}^1 \to M_{n,k}$ with respect to which we will study the pullback metrics $((\psi_i)^* g_i(0))$. The main goal of this section will be to show that the limit of the pullback metrics factors as product metric. Later we will show this sequence converges to a limit in some sense away from $(\cdot, 0), (\cdot, \infty)$. We will later use this fact to show that in fact $\mathbb{C}^{n-1} \times \mathbb{CP}^1 = M_\infty$.

Let $\{w_j, y\}_{j=1}^{n-1}$ be holomorhic coordinates for $\mathbb{C}^{n-1} \times \mathbb{CP}^1$, where we abuse notation by allowing y to take the value ∞, thus parametrizing $\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$. Recall that

$$U_n = \{[x_1, \ldots, x_n] : x_n \neq 0\} \ni \pi^{-1}[0, \ldots, 0, 1]$$

Recall that $\pi^{-1}(U_n)$ has holomorphic fibred coordinates $\{z_j, y\}_{j=1}^{n-1}$, again allowing y to take the value ∞ in order to parametrize each fibre $\mathbb{CP}^1$. We will identify $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ and $\pi^{-1}(U_n)$ for each i via the biholomorhisms:

$$\psi_i : \mathbb{C}^{n-1} \times \mathbb{CP}^1 \to \pi^{-1}(U_n) : (w, y) \mapsto (\sqrt{K_i} w, y),$$

In particular, recall that we chose our singular point $p = (0, 1)$ in the fiber coordinates on $\pi^{-1}(U_n)$, and thus we have $\psi_i^{-1}(p) = (0, 1)$ in (w, y) coordinates on $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ for all i.

We thus have a sequence $((\psi_i)^* g_i(t))_{i \in \mathbb{N}}$ of solutions to the Ricci flow on $\mathbb{C}^{n-1} \times \mathbb{CP}^1$. The goal in this section will be to calculate the components of $(\psi_i)^* g_i(0)$ with respect to the (w, y) coordinates. Recall that the Kaher form corresponding to $g_i(0)$ is simply

$$\omega_i(0) = K_i \omega(t_i) = i K_i u''(t_i) \partial\rho \wedge \bar{\partial}\rho + i K_i u'(t_i) \partial\bar{\partial}\rho.$$

We will expand the two terms above separately. For the $u'(t_i)\partial\bar{\partial}\rho$ term above we have:

$$\begin{aligned}
(\psi_i)^* K_i u'(t_i) \partial\bar{\partial}\rho &= K_i (\psi_i)^* u'(t_i) \partial\bar{\partial} \left(\tfrac{1}{k} \log y\bar{y} + \log(1 + \|z\|^2) \right) \\
&= K_i (\psi_i)^* u'(t_i) \left(\frac{dz \wedge d\bar{z}}{1 + \|z\|^2} + \frac{(\bar{z} \cdot dz) \wedge (z \cdot d\bar{z})}{(1 + \|z\|^2)^2} \right) \\
&= K_i u'(t_i) \left(\frac{K_i^{-1} dw \wedge d\bar{w}}{1 + K_i^{-1}\|w\|^2} + \frac{K_i^{-2} (\bar{w} \cdot dw) \wedge (w \cdot d\bar{w})}{(1 + K_i^{-1}\|w\|^2)^2} \right),
\end{aligned}$$

where we simplify notation by using

$$dz \wedge d\bar{z} := \sum_{i=1}^{n-1} dz^i \wedge d\bar{z}^i$$
$$z \cdot dz := \sum_{i=1}^{n-1} z_i dz^i.$$

Thus since $\lim_{i\to\infty} K_i = \infty$ and $\lim_{i\to\infty} u'(t_i) = \lim_{t\to T} u'(t) = a_T$, we have

$$\lim_{i\to\infty} (\psi_i)^* K_i u'(t_i)\partial\bar{\partial}\rho = a_T dw \wedge d\bar{w}. \tag{4.1}$$

uniformly on compact subsets of $(0,1) \in \mathbb{C}^{n-1} \times \mathbb{CP}^1$ in the C^0 sense.

For the $u''\partial\rho \wedge \bar{\partial}\rho$ term, first we compute:

$$\begin{aligned}(\psi_i)^*\partial\rho &= (\psi_i)^*\left(\tfrac{dy}{ky} + \partial \log(1 + \|z\|^2)\right)\\ &= (\psi_i)^*\left(\tfrac{dy}{ky} + \tfrac{\bar{z}\cdot dz}{1+\|z\|^2}\right)\\ &= \tfrac{dy}{ky} + \tfrac{K_i^{-1}\bar{w}\cdot dw}{1+K_i^{-1}\|w\|^2},\end{aligned}$$

thus giving

$$\begin{aligned}(\psi_i)^* K_i u''(t_i)\partial\rho \wedge \bar{\partial}\rho &= K_i u''(t_i)\left(\tfrac{dy}{ky} + \tfrac{K_i^{-1}\bar{w}\cdot dw}{1+K_i^{-1}\|w\|^2}\right) \wedge \left(\tfrac{d\bar{y}}{k\bar{y}} + \tfrac{K_i^{-1} w\cdot d\bar{w}}{1+K_i\|w\|^2}\right)\\ &= \frac{K_i u''(t_i)}{k^2 y\bar{y}} dy \wedge d\bar{y} + K_i u''(t_i) o(K_i^{-1}),\end{aligned}$$

where the asymptotic above is uniform on any neighborhood of $(0,1)$ uniformly away from $(\cdot, 0), (\cdot, \infty)$. From proposition 3, we have a pointwise estimate

$$(T - t) \lesssim u''(\rho) \lesssim (T - t).$$

Since our solution is of type I, we also have

$$\tfrac{1}{T-t_i} \lesssim K_i \lesssim \tfrac{1}{T-t_i}.$$

Combining these tells us that $u''K_i$ is pointwise bounded away from $\{0, \infty\}$. This gives the convergence

$$\lim_{i\to\infty} (\psi_i)^* K_i u''(t_i)\partial\rho \wedge \bar{\partial}\rho = h \tag{4.2}$$

where h is some Kähler metric on $\mathbb{C}\backslash 0 = \mathbb{CP}^1 \setminus \{0, \infty\}$ and the convergence is uniformly on compact subsets of $\mathbb{C}\backslash 0$ in the $\mathcal{C}^0$ sense.

Combining (4.2) and (4.1) this gives

$$\lim_{i\to\infty} (\psi_i)^*\omega_i(0) = \tilde{g} := a_T\delta_{\mathbb{C}^{n-1}} + h \tag{4.3}$$

where $\delta_{\mathbb{C}^{n-1}}$ is the standard Euclidean metric on $\mathbb{C}^{n-1}$ and the convergence is uniformly on compact subsets of $(0, 1)$ in the $\mathcal{C}^0$ sense. In particular, we have convergence to a product metric on $\mathbb{C}^{n-1} \times (\mathbb{CP}^1 \setminus \{0, \infty\})$.

4.2 An Isometry Between M_∞ and $\mathbb{C}^{n-1} \times \mathbb{CP}^1$

In this section we show that $\mathbb{C}^{n-1} \times \mathbb{CP}^1 = M_\infty$. Consider the sequence of maps

$$\Psi_i = \phi_i^{-1} \circ \psi_i$$

We begin by describing the domain of these maps. We will consider $\mathbb{C}^{n-1} \times \mathbb{C}$ as a subset of $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ using the coordinates (w, y) defined in the previous section. Now it is not hard to show that from the estimates in the previous section, for any compact subset $K \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ we may have

(i) $\psi_i(K) \subset B_{g_i(0)}(r, p) \subset \pi^{-1}(U_n)$ for some $r > 0$ some $N > 0$ and all $i > N$.

(ii) Ψ_i is defined on K and $\Psi_i(K) \subset B_{g_\infty}(2r, p_\infty) \subset M_\infty$ for some $r > 0$ some $N > 0$ and all $i > N$.

Theorem 4. *There exists a smooth subsequence limit map $\Psi_\infty = \lim_{i\to\infty} \Psi_i$ from $\mathbb{C}^{n-1} \times (\mathbb{C}\backslash 0)$ to M_∞. Moreover, Ψ_∞ is holomorphic.*

Proof. Let K be any compact set $K \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$. Let r be the radius of K with respect to $g_\infty(0)$. Then by (ii) above, the sequence Ψ_i is a uniformly bounded sequence on K with respect to g_∞ on M_∞, and is therefore likewise uniformly bounded with respect to $(\phi^{-1})^*g_i(0)$ on M_∞ by the locally uniform convergence $(\phi^{-1})^*g_i(0) \to g_\infty$. Moreover, each Ψ_i is holomorphic into (M_∞, J_i) where $J_i = \phi_i^* J$ is the pullback of the complex structure J on $M_{n,k}$. By derivative estimates for holomorphic functions, it follows that any fixed order derivative of the Ψ_i's will be uniformly bounded with respect to $(\phi^{-1})^*g_i(0)$ on M_∞ on any compact subdisc contained strictly within K. From this fact and the smooth local convergences $(\phi^{-1})^*g_i(0) \to g_\infty$ and $J_i \to J_\infty$, we can conclude that any fixed order derivative of the Ψ_i's will be uniformly bounded with respect to g_∞ on M_∞ on any compact subdisc contained strictly within K. From this and a diagonal subsequence argument we see that some subsequence of Ψ_i converges smoothly uniformly on compact subsets of $\mathbb{C}^{n-1}\times(\mathbb{C}\backslash 0)$ to a smooth limit Ψ_∞. That Ψ_∞ is holomorphic then follows from the convergence $J_i \to J_\infty$.

□

We now want to show that Ψ_∞ in fact extends to give a bijective holomorphic map from $\mathbb{C}^{n-1}\times\mathbb{CP}^1$ to M_∞. Note that such a map must then be a biholomorphism. We will establish this through several lemma's.

Recall from (4.3) that we had the locally uniform convergence

$$\lim_{i\to\infty}(\psi_i)^*\omega_i(0) = \tilde{g} := a_T\delta_{\mathbb{C}^{n-1}} + h.$$

in the $\mathcal{C}^0$ sense on compact subsets of $\mathbb{C}^{n-1}\times(\mathbb{C}\backslash 0)$, and that the later is a product metric on $\mathbb{C}^{n-1}\times(\mathbb{C}\backslash 0)$. Note that we have

$$g_\infty = (\Psi_\infty)_*\tilde{g},$$

since $(\Psi_\infty)_*(\lim_{i\to\infty}(\psi_i)^*\omega_i) = \lim_{i\to\infty}(\Psi_i)_*(\psi_i)^*\omega_i = \lim_{i\to\infty}(\phi_i^{-1})^*\omega_i = \omega_\infty$.

Lemma 3. *Ψ_∞ is injective.*

Proof. Let $q_1 \neq q_2 \in \mathbb{C}^{n-1}\times(\mathbb{C}\backslash 0)$. Let $r = d_{\tilde{g}}(q_1,q_2)$. Since $\tilde{g} = \lim_{i\to\infty}\psi_i^* g_i(0)$, we can choose an N large enough such that $d_{\psi_i^* g_i(0)}(q_1,q_2) > \frac{r}{2}$ for all $i > N$. Since $(\Psi_i)^*\phi_i^* g_i = (\psi_i)^* g_i$,

$$\begin{aligned} d_{g_\infty(0)}(\Phi_\infty(q_1),\Psi_\infty(q_2)) &= \lim_{i\to\infty} d_{\phi_i^* g_i(0)}(\Phi_\infty(q_1),\Psi_\infty(q_2)) \\ &= \lim_{i\to\infty} d_{\psi_i^* g_i(0)}(q_1,q_2) > \tfrac{r}{2}. \end{aligned}$$

Since $r>0, \Psi_\infty(q_1)\neq\Psi_\infty(q_2)$ as required. □

Showing surjectivity is a little trickier. The proof will come after a series of lemmas:

Lemma 4. *The fibre $(\mathbb{C}\backslash 0, h)$ has finite volume.*

Proof. We will establish this by showing that the fibres of the parabolic dilations $(\pi^{-1}(p), g_i|_{\pi^{-1}(p)})$ have volume uniformly bounded independent of i. As before, we have local holomorphic coordinate y for the fibre (assuming by symmetry that $z=0$). Through the relation $y = \exp(\frac{k\rho}{2} + i\arg y)$, we can reparametrize $\pi^{-1}(p)$ in terms of ρ and $\theta := \arg y$. The metric is given by

$$g = \tfrac{u''}{k^2 y\bar{y}} dy d\bar{y} = \tfrac{u''}{k^2}\left(\tfrac{k}{2}d\rho + i d\theta\right)\left(\tfrac{k}{2}d\rho - i d\theta\right) = u''\left(\tfrac{1}{4}d\rho^2 + \tfrac{1}{k^2}d\theta^2\right).$$

The volume of the fibre with respect to the dilated metric $g_i(0) = K_i g(t_i)$ is then

$$\begin{aligned}\int_{\mathbb{CP}^1} \frac{K_i u''(t_i)}{2k} d\rho d\theta &= \frac{K_i}{2k}\int_{\mathbb{R}} u''(t_i) d\rho \int_{\mathbb{S}^1} d\theta \\ = \frac{K_i}{2k} u'(t_i)|_{-\infty}^{\infty} 2\pi &= \frac{\pi K_i}{k}(b-a)(t_i) = 2\pi K_i (T - t_i),\end{aligned}$$

where we used that $b - a = 2k(T - t)$. From our assumption that the singularity is type I, $K_i \lesssim \frac{1}{T - t_i}$. It follows that the volume of $(\pi^{-1}(p), g_i(0)|\pi^{-1}(p))$ is bounded independent of i. Thus, the limit fibre $(\mathbb{C}\backslash 0, h)$ has finite volume. □

Lemma 5. *There exists an $r > 0$ such that the $\inf_{y \in \mathbb{C}\backslash 0} Vol(B_r^h(y)) > 0$.*

Proof. Recall from the proof of corollary 1 that there exists $C > 0$ such that

$$\sup_{M_{n,k}} \|\mathrm{Rm}_{g(t_i)}\|_{g(t_i)} \leq C K_i.$$

It follows that

$$\sup_{M_{n,k}} \|\mathrm{Rm}_{g_i(0)}\|_{g_i(0)} = \sup_{M_{n,k}} \|\mathrm{Rm}_{K_i g(t_i)}\|_{K_i g(t_i)} = \frac{1}{K_i}\sup_{M_{n,k}} \|\mathrm{Rm}_{g(t_i)}\|_{g(t_i)} \leq C.$$

Let $r = \frac{1}{\sqrt{C}}$. From theorem 3, there exists a $\kappa > 0$ such that the singularity model is κ-noncollapsed at all scales. Since

$$\sup_{M_{n,k}} \|\mathrm{Rm}_{g_i(0)}\|_{g_i(0)} \leq C = \tfrac{1}{r^2},$$

we have that $\mathrm{Vol}(B_r(x)) \geq \kappa r^{2n}$ for all $x \in M_{n,k}$. Recall that Φ_∞ is an isometric embedding of the singularity model into the product of Riemannian manifolds $(\mathbb{C}^{n-1}, a_T\delta) \times (\mathbb{CP}^1, h)$. Define modified balls for each $x = \Phi_\infty^{-1}(v, y)$ by

$$B'(x) = \Phi_\infty^{-1}(B_r^{a_T\delta}(v) \times B_r^h(y)).$$

Clearly $B'(x) \supset B_r(x)$ for each $x \in M_{n,k}$. Let c be the volume of the unit ball in $\mathbb{C}^{n-1}$ with respect to $a_T\delta$. Then we have

$$\mathrm{Vol}(B'(x)) = \mathrm{Vol}(B_r^{a_T\delta}(v))\mathrm{Vol}(B_r^h(y)) = c r^{2n-2}\mathrm{Vol}(B_r^h(y)).$$

But we also have

$$\mathrm{Vol}(B'(x)) \geq \mathrm{Vol}(B_r(x)) \geq \kappa r^{2n}.$$

Combining these gives $\mathrm{Vol}(B^h_r(y)) \geq \frac{\kappa}{c} r^2$ for all $y \in \mathbb{CP}^1$. □

Lemma 6. *The diameter of* $(\mathbb{C}\backslash 0, h)$ *is bounded.*

Proof. Let r be chosen from the preceeding lemma. Assume that $(\mathbb{C}\backslash 0, h)$ has infinite diameter. $F = \Psi_\infty(\{0\} \times (\mathbb{C}\backslash 0))$. Thus F must also have infinite diameter with respect to $g_\infty|_F = (\Psi_\infty \circ \pi_{\mathbb{C}\backslash 0})_* h$, where $\pi_{\mathbb{C}\backslash 0}$ is the projection onto the $\mathbb{C}\backslash 0$ part of $\mathbb{C}^{n-1} \times (\mathbb{C}\backslash 0) \subset \mathbb{C}^{n-1} \times \mathbb{CP}^1$. Then there exists a sequence $(y_i)_{i\in\mathbb{N}}$ such that the balls $\{B^h_r(y_i)\}_{i\in\mathbb{N}} \subset F$ are pairwise disjoint. Applying the previous lemma, we have

$$\mathrm{Vol}(F) \geq \sum_{i\in\mathbb{N}} \mathrm{Vol}(B^h_r(y_i)) \geq \inf_{y\in\mathbb{C}\backslash 0} \mathrm{Vol}(B^h_r(y)) \sum_{i\in\mathbb{N}} 1 = \infty.$$

Since $(\mathbb{C}\backslash 0, h)$ is isometrically mapped onto F, this implies that the volume of $(\mathbb{C}\backslash 0, h)$ must be infinite. This contradicts Lemma 4. Thus our assumption must be false as required. □

Lemma 7. Ψ_∞ *extends to a surjection from* $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ *to* M_∞*.*

Proof. That Ψ_∞ extends to a holomorphic map on $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ follows from the fact that Ψ_∞ is a bounded holomorphic map by the previous lemma, and derivative estimates for holomorphic functions. We now show surjectivity. Fix any $q \in M_\infty$ and consider distance r of this point to p_∞. Then for sufficiently large i we have $q \in \bar{B}_{2r}(p_\infty, g_\infty) \subset B_{3r}(p_\infty, (\phi^{-1})^* g_i(0))$. In particular, this would imply $q \in \Psi_i(K)$ for some compact set K and all i sufficiently large. From this it is not hard to see that we must have $q \in \Psi_\infty(K)$. □

The lemmas above imply the following

Theorem 5. Ψ_∞ *is a bijective holomorphic map from* $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ *to* M_∞*. Thus* Ψ_∞ *is a biholomorphism. Moreover,* h *extends to a metric on* $\mathbb{CP}^1$*, and* Ψ_∞ *is isometric with respect to the metrics* $\tilde{g} = a_T\delta + h$ *on* $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ *and* g_∞ *on* M_∞*.*

Chapter 5

Conclusion

5.1 Proof of the Main Theorem

Proof. We have shown that for any $p \in M_{n,k} \backslash \{D_0, D_\infty\}$, under the assumptions of the theorem, the Ricci flow forms a singularity at p at the singularity time. Moreover, the corresponding singularity model at time 0 is biholomorhically isometric to the product $(\mathbb{C}^{n-1}, a_T\delta) \times (\mathbb{CP}^1, h)$, where δ is the Euclidean metric on $\mathbb{C}^{n-1}$, and h is some Kähler metric on $\mathbb{CP}^1$. Since the solution is assumed to be type I, [3] tells us that singularity model must be a shrinking soliton. It follows that $(\mathbb{C}P^1, h)$ must be a shrinking soliton. By a theorem of Tian and Zhu (see [9]), Kähler Ricci solitons on compact manifolds are unique modulo automophisms, so $(\mathbb{CP}^1, h(t))$ must in fact be the standard shrinking sphere soliton. This proves the main theorem. □

5.2 Remarks

In this section, we discuss the validity of the assumptions that we made. The assumption that the lengths of of vectors tangent to the fibre bundle don't decay too fast is used to use the coordinates that we have for the fibre as coordinates for the fibre in the limit manifold. Without making this assumption, the metric in these coordinates could become degenerate even after parabolic dilations. But the limit manifold exists, so there certainly are some coordinates where the limit metric is non-degenerate. We could try to adjust the coordinates for each i, as we did in the base direction, but it is not as easy in the case. The metric in the fibre direction is invariant under linear dilations, so the adjustments would have to be non-linear, which makes things tricky. It is difficult to imagine adjustments that would be holomorphic, injective, and non-linear, so we assume that they could only be linear, so that the initial coordinates suffice. Moreover, our assumption is consistent with the actual precise decay of volume of the fiber under the flow being precisely $a(T - t)$.

The other assumption that we made was that the singularity forms at a type I rate. As mentioned before, it is conjectured that the alternative, type IIa singularities, only form with very particular initial metrics. There are other things that suggest a type I rate. The volume of the manifold and even each fibre decays at a rate of $(T-t)$, which is typical of a type I singularity (e.g. shrinking Ricci solitons). Moreover, the metric becomes degenerate only in the fibre direction. Because of the symmetry, the Ricci flow on the whole manifold can induce a modified flow $\hat{g}(t)$ on the fibre, which is of the form

$$\tfrac{d\hat{g}}{dt} = -\mathrm{Ric}_{\hat{g}} + F(u', u'', u''').$$

Moreover, the Ric term dominates the F term. From [2], we know that the Ricci flow on the $\mathbb{S}^2$ will always converge to the shrinking round sphere, and hence will form a type I singularity. Perhaps this modified flow on $\mathbb{S}^2$ is close enough to the Ricci flow to also deduce that the singularity must be type I.

It is tempting to generalize this work. In particular, one could hope that if the Ricci flow on a compact fibration coverges in the Gromov-Hausdorff sense to a metric on the base, then for a type I singularity at some p in a fibre F, the corresponding singularity model should factor as a Riemannian product of a shrinking soliton on the fibre and flat space corresponding to the dilated base. The Gromov-Hausdorff convergence would guarantee that the metric only becomes degenerate in the fibre direction. This implies that the parabolic dilations will spread disjoint fibres infinitely far apart, so that the singularity in the base direction becomes flat. One can then use a local trivialization to find coordinates for each t_i factoring apart the base and fibre direction so that they are orthogonal along F with respect to the parabolic dilation $g_i(0)$. This orthogonality should allow us to deduce that near F, the cross terms of the metric decay faster than the terms in the F direction, so that the limit metric on the singularity model have the fibre and the base orthogonal. This allows us to use the same argument with Perelman's no-local-collapsing to conclude that the fibre of the singularity model must be compact, and hence contains all of the original fibre. Of course, these are just rough ideas, and the details still need more work.

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