Regularity in second and fourth order nonlinear elliptic problems

by

Craig Cowan

M.Sc., Simon Fraser University, 2006
B.Sc., Simon Fraser University, 2004

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August, 2010

© Craig Cowan 2010

Abstract

This thesis consists of six research papers.

In "Regularity of the extremal solution in a MEMS model with advection," we examine the equation given by $-\Delta u + c(x)\cdot\nabla u = \lambda f(u)$ in Ω with Dirichlet boundary conditions and where $f(u) = (1-u)^{-2}$ or $f(u) = e^u$. Our main result is that the associated extremal solution is smooth provided this is the case for the advection free case; $c(x) = 0$.

In "Estimates on pull-in distances in MEMS models and other nonlinear eigenvalue problems" we prove some results, which were observed numerically, regarding equations of the form $-\Delta u = \lambda|x|^\alpha F(u)$ in B where B is the unit ball in $\mathbb{R}^N$. In addition we obtain upper and lower estimates on the extremal solutions associated with various nonlinear eigenvalue problems.

In "The critical dimension for a fourth order elliptic problem with singular nonlinearity," we examine the equation given by $\Delta^2 u = \lambda(1-u)^{-2}$ in B with Dirichlet boundary conditions where B is the unit ball in $\mathbb{R}^N$. Our main result is that the extremal solution u^* is smooth if and only if $N \le 8$.

In "Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains" we examine the equation $\Delta^2 u = \lambda f(u)$ in Ω with Navier boundary conditions where Ω is a general bounded domain in $\mathbb{R}^N$. We obtain various results concerning the regularity of the associated extremal solution.

In "Regularity of the extremal solutions in elliptic systems" we examine the elliptic system given by $-\Delta u = \lambda e^v, \quad -\Delta v = \gamma e^u$ in Ω where λ and γ are positive constants and we obtain results concering the regularity of the extremal solutions.

In "Optimal Hardy inequalities for general elliptic operators with improvements" we examine some very general Hardy inequalities. Optimal constants are obtained and we characterize the improvements of these general Hardy inequalities. In addition we prove various weighted versions of these inequalities with improvements and many other results.

Table of Contents

Acknowledgements

I am indebted to my supervisor Nassif Ghoussoub for his guidance and constant support. He introduced me to many interesting problems and was always available for help despite his busy schedule. He took me on as a student when many others would not. For all of this I am truly grateful.

Dedication

To my brother and my parents who have shown me immense support during my academic pursuits.

Statement of Co-Authorship

Chapters 2 and 3 were jointly authored by Craig Cowan and Nassif Ghoussoub. Chapter 4 was jointly authored by Craig Cowan, Pierpaolo Esposito, Nassif Ghoussoub and Amir Moradifam. Chapter 5 was jointly authored by Craig Cowan, Pierpaolo Esposito and Nassif Ghoussoub. Chapter 6 was authored by Craig Cowan. Chapter 7 was authored Craig Cowan.

In all of the joint papers of this thesis, all authors contributed equally to the identification and design of the research problem, performing the research, data analysis, and manuscript preparation.

Chapter 1

Introduction

The main focus of research in this thesis is the study of the regularity of the extremal solution associated with various semi-linear elliptic problems and also generalized Hardy inequalities. This thesis consists of six papers which have either been published, accepted or submitted: [17, 19, 20, 18, 16, 15].

1.1 Second order nonlinear eigenvalue problems

1.1.1 Introduction to second order problems

The following equation

$$\left\{\begin{array}{rcll} v_t - \Delta v &=& \lambda(1-\varepsilon v)^m e^{\frac{v}{1+\varepsilon v}} & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega, \end{array}\right.$$

is used to model a combustion process. The model is known as the *solid fuel ignition model* see [26]. It is a model for the thermal reaction process in a combustible, nondeformable material of constant density during the ignition period. Here λ is known as the Frank-Kamenetskii parameter, v is a dimensionless temperature, and $\frac{1}{\varepsilon}$ is the activation energy.

The following model has been proposed, see [36] and [37], for the description of the steady state of a simple Electrostatic MEMS device:

$$\begin{cases} \alpha\Delta^2 u = \left(\beta\int_\Omega |\nabla u|^2 dx + \gamma\right)\Delta u + \frac{\lambda g(x)}{(1-u)^2\left(1+\chi\int_\Omega \frac{dx}{(1-u)^2}\right)} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \alpha\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \chi \geq 0$, $g \in C(\overline{\Omega}, [0,1])$ are fixed, Ω is a bounded domain in $\mathbb{R}^N$ and $\lambda \geq 0$ is a varying parameter. The function $u(x)$ denotes the height above a point $x \in \Omega \subset \mathbb{R}^N$ of a dielectric membrane clamped on $\partial\Omega$, once it deflects torwards a ground plate fixed at height $z = 1$, whenever a positive voltage – proportional to λ – is applied.

A first step in understanding the above models is to examine a simplified and generalized stationary version of the above models given by

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) \qquad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter and where Ω is a bounded domain in $\mathbb{R}^N$. One also generally restricts the nonlinearities f to one of two classes:

(R): f is smooth, increasing, convex with $f(0) = 1$ and $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$,

or

(S): f is smooth, increasing, convex with $f(0) = 1$ and $\lim_{u\nearrow 1} f(u) = \infty$.

We now recall some well known facts, see [8, 11, 14, 21, 32, 23, 4, 31]. Define

$$\lambda^* := \sup \{\lambda \geq 0 : \text{there exists a smooth solution of } (P)_\lambda \}.$$

Then $0 < \lambda^* < \infty$ and for each $0 < \lambda < \lambda^*$ there exists a smooth solution of $(P)_\lambda$ and for $\lambda > \lambda^*$ there are no solutions of $(P)_\lambda$ even in a very weak sense. For $0 < \lambda < \lambda^*$ there exists a smooth *minimal* solution, which we denote by u_λ, here minimal means that if v is a solution of $(P)_\lambda$ then $u_\lambda \leq v$ a.e. in Ω. It is also known that the minimal solution u_λ is *semi-stable* in the sense that the (possibly formal) energy associated with the problem $(P)_\lambda$ has nonnegative second variation at u_λ. In other words

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 dx \leq \int_\Omega |\nabla \psi|^2 dx, \qquad \forall \psi \in H_0^1(\Omega). \tag{1.2}$$

For each $x \in \Omega$, $\lambda \mapsto u_\lambda$ is increasing in $(0, \lambda^*)$. This allows one to define, at least pointwise, the *extremal solution*

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x).$$

The extremal solution is indeed a weak solution of $(P)_{\lambda^*}$ and in fact can be shown to be the unique weak solution. Now one can ask whether the extremal solution is a classical solution which means bounded when f satisfies (R) (or bounded away from 1 in the case where f satisfies (S)) or is indeed a honest to goodness weak solution of $(P)_{\lambda^*}$. It turns out that both situations can occur. This issue of the regularity of the extremal solution is the main focus of this thesis, at least as far as nonlinear eigenvalue problems are concerned. We now list some of the known results concerning the regularity of the extremal solution. Recall again that Ω is a bounded domain in $\mathbb{R}^N$.

1. [30] Suppose $f(u) = e^u$ and Ω is the unit ball in $\mathbb{R}^N$. For $N \geq 10$, $u^* = -2\log(|x|)$. Hence u^* is unbounded.
2. [21] Suppose $f(u) = e^u$ and $N \leq 9$. Then u^* is bounded.
3. [34] Suppose f satisfies (R) and $N \leq 3$. Then u^* is bounded. In addition L^p estimates are obtained on $f(u^*)$ in the case $N \geq 4$. .
4. [13] Suppose f satisfies (R) and Ω is the unit ball in $\mathbb{R}^N$ with $N < 10$. Then u^* is bounded. Considering the first result above shows this is optimal.
5. [12] Suppose f satisfies (R) and Ω is a bounded convex domain in $\mathbb{R}^4$. Then u^* is bounded.

In [39] various results concerning the regularity of u^* are also obtained.

One may ask why is the regularity of the extremal solution important. The ultimate goal is to understand the full bifurcation diagram associated with the problem $(P)_\lambda$ and the first step is to understand whether u^* is a classical solution of $(P)_{\lambda^*}$. If this is the case then the results of [21] allows one to start a second branch of solutions emanating from (λ^*, u^*).

1.1.2 Second order models with advection

A natural generalization of the problem $(P)_\lambda$ (in the case of the MEMS nonlinearity) is to consider the problem

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x)\cdot\nabla u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$ and where $c(x)$ is a smooth vector field defined in Ω.

Modifying the proofs used in analyzing the advection free case ($c = 0$) one can again show the existence of a positive finite critical parameter λ^* such that for $0 < \lambda < \lambda^*$ there exists a smooth minimal solution u_λ of $(S)_\lambda$, while there are no smooth solutions of $(S)_\lambda$ for $\lambda > \lambda^*$. Moreover, the minimal solutions are also *semi-stable* in the sense that the principal eigenvalue of the corresponding linearized operator

$$L_{u,\lambda,c} := -\Delta + c(x)\cdot\nabla - \frac{2\lambda}{(1-u_\lambda)^3}$$

in $H_0^1(\Omega)$ is non-negative. See [6] where these results are proved for general C^1 convex nonlinearities which are superlinear at ∞.

In Chapter 2 we consider the question of the regularity of the extremal solution associated with $(S)_\lambda$. Our main result is given by the following:

Theorem 1. *If $1 \leq N \leq 7$, then the extremal solution u^* of $(S)_{\lambda^*}$ is smooth.*

It should be noted that $N = 7$ is optimal here after one considers the case of $c(x) = 0$ on the unit ball in $\mathbb{R}^8$, see [23]. In this paper we also show that if $\frac{1}{(1-u)^2}$ is replaced with e^u then the extremal solution is bounded provided $N \leq 9$, which, again is optimal. One should note that the interesting case of $c(x)$ is given when $c(x)$ is divergence free. If $c(x)$ is given by the gradient of a scalar function then $(S)_\lambda$ is of variational nature and the standard approach (when $c(x) = 0$) is easily modified to handle this case.

1.1.3 Pull in distance in the MEMS model

One can study slight generalizations of $(P)_\lambda$ given by

$$(P)_{\lambda,g(x)} \qquad \begin{cases} -\Delta u &= \lambda g(x) f(u) \qquad \text{in } B, \\ u &= 0 \qquad \text{on } \partial B, \end{cases}$$

where $\lambda > 0$ is a parameter, B is the unit ball in $\mathbb{R}^N$ and where $g(x) \geq 0$ is a nonzero Hölder continuous function. In [29], in the case of the MEMS nonlinearity $f(u) = (1-u)^{-2}$, it was noticed numerically that the L^∞ norm of the extremal solution associated with $(P)_{\lambda,|x|^\alpha}$, on the unit ball in $\mathbb{R}^2$, was independent of α. In Chapter 3 we show, using a change of variables, that this is true for any nonlinearity f. Using the results from [13] we show that the extremal solution associated with $(P)_{\lambda,|x|^\alpha}$ is bounded provided $N < 10 + 4\alpha$, for any general nonlinearity f satisfying (R).

It was also noted numerically in [37] that maximum stable deflection of the membrane was always at least one third of the distance from the undeflected membrane to the ground plate, in our setting this translates to: the extremal solution u^* associated with $(P)_\lambda$ in the case where $f(u) = (1-u)^{-2}$ satisfies $\|u^*\|_{L^\infty} \geq \frac{1}{3}$. In Chapter 3 we show this is indeed the case. In fact this estimate is essentially independent of the operator $-\Delta$. The estimate holds for any reasonable second or fourth order linear operator which satisfies a weak maximum principle.

The initial motivation of writing Chapter 3 was to obtain upper estimates on the *pull in distance* in the case of the MEMS model, here we are defining the pull in distance as $\|u^*\|_{L^\infty}$. Before our work generally one was only interested in whether $\|u^*\|_{L^\infty} < 1$ or $= 1$. We obtain various upper estimates and we also examine some asymptotics of the minimal solution u_λ for λ close to λ^*.

1.2 Fourth order nonlinear eigenvalue problems

1.2.1 Introduction to fourth order problems

We now turn our attention to fourth order problems of the form

$$\Delta^2 u = \lambda f(u) \qquad \text{in } \Omega, \tag{1.3}$$

where, for the time being, we won't specify the boundary conditions. If one examines the second order case $(P)_\lambda$ and in particular the proofs showing: the extremal parameter is finite, the existence of minimal solutions, the monotonicity of the minimal solutions in λ, they will realize that the availability of a weak maximum principle is crucial. By a weak maximum principle we mean that if $f \geq 0$ is any reasonable function defined in Ω and $\Delta^2 v = f$ in Ω with the imposed boundary conditions then $v \geq 0$ a.e. in Ω. In other words the associated Greens function is nonnegative. Generally there are two popular boundary conditions for which one does indeed have the weak maximum principle:

- (Dirichlet) The Dirichlet boundary conditions $u = \partial_\nu u = 0$ on $\partial\Omega$ where Ω is the unit ball in $\mathbb{R}^N$. The positivity of the Greens function here is due to [7] and is called Boggio's Principle.

- (Navier) The Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ where Ω is a general domain. Here two applications of the second order maximum principle shows the desired result.

The basic properties, not including the regularity of the extremal solution, associated with the fourth order problems were established in [2, 5]. In addition some results concerning the regularity of the extremal solution were obtained but these results can be considered as subcritical and critical results. A major breakthrough in the regularity of the extremal solution in fourth order problems is due to [22] where they examined the fourth order analog of the Gelfand Problem given by

$$(D)_\lambda \qquad \begin{cases} -\Delta u = \lambda e^u & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ \partial_\nu u = 0 & \text{on } \partial B \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$. They showed that the extremal solution u^* is bounded if and only if $N \leq 12$. Their proof is very particular to the unit ball but can be extended to domains with enough symmetry, we omit the details. It is important to note that their proof does not use the usual method of energy estimates. Since this work there has been a flurry of results concerning the regularity of the extremal solution in fourth order problems with either Dirichlet or Navier boundary conditions but all are restricted to the unit ball.

1.2.2 The critical dimension in a fourth order MEMS model

In Chapter 4 we examine the fourth order MEMS model given by

$$(D)_\lambda \qquad \begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } B \\ 0 < u < 1 & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$. Our main result is given by the following theorem:

Theorem 2. *Let u^* denote the extremal solution associated with $(D)_\lambda$. Then $\sup_B u^* < 1$ if and only if $N \leq 8$.*

Our approach is heavily inspired by the approach taken in [22]. We remark that we did not need to resort to a computer assisted proof to show that the extremal solution is singular in the intermediate dimensions as was the case in [22].

1.2.3 Fourth order models on general domains

In Chapter 5 we are interested in the regularity of the extremal solution u^* associated with

$$(N)_\lambda \qquad \begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \geq 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and where f satisfies (R).

Our main result is given by the following theorem:

Theorem 3. *Suppose f,N and Ω is as above. The extremal solution u^* is bounded in all of the following cases:*

1. *$N \leq 5$. (For $N \geq 6$ we show that $f(u^*) \in L^p(\Omega)$ for all $p < \frac{N}{N-2}$.)*

2. *$f(t) = e^t$ or $f(t) = (t+1)^p$ where $p > 1$ for $N \leq 8$.*

3. *$N \geq 9$ and $f(t) = (t+1)^p$ where $1 < p < \frac{N}{N-8}$.*

4. *$\liminf_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} > 0$ and $N \leq 7$.*

5. *$\gamma := \limsup_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2}$ and $N < \frac{8}{\gamma}$.*

Remark 1.2.1. *Recall that in [22] they show that provided Ω is the unit ball in $\mathbb{R}^N$ then the extremal solution associated with*

$$\Delta^2 u = \lambda e^u \qquad \Omega,$$

with Dirichlet boundary conditions is bounded if and only if $N \leq 12$. With this in mind one fully expects that the extremal solution associated with $(N)_\lambda$, in the case where $f(t) = e^t$, is bounded on a general domain provided $N \leq 12$. One should note that we have vastly improved on the best result to date: u^* is bounded for $N \leq 4$ [5], but there is still much room for improvement.

We also examine the case of singular nonlinearities of the form $f(t) = \frac{1}{(1-t)^p}$ where $p > 1$. Now the question of the regularity of the extremal solution is whether $\sup_\Omega u^* < 1$ or $\sup_\Omega u^* = 1$. Our result with respect to these singular nonlinearities is given by:

Theorem 4. *Suppose $f(t) = \frac{1}{(1-t)^p}$ where $p > 1$ and $p \neq 3$. If $N \leq \frac{8p}{p+1}$ then $\sup_\Omega u^* < 1$.*

1.2.4 Elliptic systems

Closely related to fourth order models with Navier boundary conditions are elliptic systems. One can examine systems of nonlinear eigenvalue problems given by

$$(P)_{\lambda,\gamma} \qquad \left\{ \begin{array}{rcll} -\Delta u &=& \lambda f(v) & \Omega \\ -\Delta v &=& \gamma g(u) & \Omega \\ u &=& 0 & \partial\Omega \\ v &=& 0 & \partial\Omega \end{array} \right.$$

where λ, γ are positive parameters, f and g satisfy (R) and where Ω is a bounded domain in $\mathbb{R}^N$. We now follow the work of M. Montenegro [33], where all of the following results are taken from. We also mention that he obtains many more results and also that he studies a much more general system then $(P)_{\lambda,\gamma}$. We let $\mathcal{Q} = \{(\lambda,\gamma) : \lambda, \gamma > 0\}$ and we define

$$\mathcal{U} := \{(\lambda,\gamma) \in \mathcal{Q} : \text{ there exists a smooth solution } (u,v) \text{ of } (P)_{\lambda,\gamma}\}.$$

We set $\Upsilon := \partial\mathcal{U} \cap \mathcal{Q}$. The curve Υ is well defined and separates $\mathcal{Q}$ into two connected components $\mathcal{Q}$ and $\mathcal{V}$.

We omit the various properties of Υ but the interested reader should consult [33]. One point we mention is that if for $x, y \in \mathbb{R}^2$ we say $x \leq y$ provided $x_i \leq y_i$ for $i = 1, 2$ then it is easily seen, using the method of sub/supersolutions, that if $(0,0) < (\lambda_0, \gamma_0) \leq (\lambda, \gamma) \in \mathcal{U}$ then $(\lambda_0, \gamma_0) \in \mathcal{U}$. Now it can be shown that for each $(\lambda,\gamma) \in \mathcal{U}$ there exists a smooth minimal solution $(u_{\lambda,\gamma}, v_{\lambda,\gamma})$ of $(P)_{\lambda,\gamma}$ and if $(0,0) < (\lambda_1,\gamma_1) \leq (\lambda_2,\gamma_2) \in \mathcal{U}$ then

$$(u_{\lambda_1,\gamma_1}, v_{\lambda_1,\gamma_1}) \leq (u_{\lambda_2,\gamma_2}, v_{\lambda_2,\gamma_2}).$$

Now for $(\lambda^*,\gamma^*) \in \Upsilon$ there is some $0 < \sigma < \infty$ such that $\gamma^* = \sigma\lambda^*$ and we can define the extremal solution (u^*, v^*) at (λ^*,γ^*) by passing to the limit along the ray given by $\gamma = \sigma\lambda$ for $0 < \lambda < \lambda^*$. Moreover it can be shown that (u^*, v^*) is indeed a weak solution of $(P)_{\lambda^*,\gamma^*}$. We now come to the issue of stability.

Theorem 5. *([33]) Let $(\lambda,\gamma) \in \mathcal{U}$ and let (u,v) denote the minimal solution of $(P)_{\lambda,\gamma}$. Then (u,v) is semi-stable in the sense that there is some smooth $0 < \phi, \psi \in H_0^1(\Omega)$ and $0 \leq K$ such that*

$$-\Delta\phi = \lambda e^v \psi + K\phi, \qquad -\Delta\psi = \gamma e^u \phi + K\psi, \qquad \Omega.$$

In Chapter 6 we examine the system given by

$$(P)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u &= \lambda e^v & \Omega \\ -\Delta v &= \gamma e^u & \Omega \\ u &= 0 & \partial\Omega \\ v &= 0 & \partial\Omega \end{cases}$$

where λ, γ are positive parameters and where Ω is a smooth bounded domain in $\mathbb{R}^N$. As before we let $\mathcal{Q} = \{(\lambda,\gamma) : \lambda,\gamma > 0\}$ and we define

$$\mathcal{U} := \{(\lambda,\gamma) \in \mathcal{Q} : \text{ there exists a smooth solution } (u,v) \text{ of } (P)_{\lambda,\gamma}\}.$$

We set $\Upsilon := \partial\mathcal{U} \cap \mathcal{Q}$. Our main result is given by the following theorem.

Theorem 1.4. *Let $3 \leq N \leq 9$ and suppose that $(\lambda,\gamma) \in \Upsilon$ with*

$$\frac{N-2}{8} < \frac{\gamma}{\lambda} < \frac{8}{N-2}.$$

Then the associated extremal solution (u^*, v^*) is smooth.

1.3 Hardy inequalities

We begin by recalling the various Hardy inequalities. Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin and where $n \geq 3$. Then Hardy's inequality (see [35]) asserts that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx, \tag{1.5}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\left(\frac{n-2}{2}\right)^2$ is optimal and not attained. An analogous result asserts that for any bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and $\delta(x) := dist(x, \partial\Omega)$ (the euclidean distance from x to $\partial\Omega$), there holds (see [9])

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{\delta^2} dx, \tag{1.6}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\frac{1}{4}$ is optimal and not attained. We will refer to this inequality as Hardy's boundary inequality.

Recently Hardy inequalities involving more general distance functions than the distance to the origin or distance to the boundary have been studied (see [3]). Suppose Ω is a domain in $\mathbb{R}^n$ and M a piecewise smooth surface of co-dimension k, $k = 1, ..., n$. In case $k = n$ we adopt the convention that M is a point, say, the origin. Set $d(x) := dist(x, M)$. Suppose $k \neq 2$ and $-\Delta d^{2-k} \geq 0$ in $\Omega \backslash M$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(k-2)^2}{4} \int_\Omega \frac{u^2}{d^2} dx, \tag{1.7}$$

for all $u \in H_0^1(\Omega \backslash M)$. We comment that the above inequalities all have L^p analogs.

In the last few years improved versions of the above inequalities have been obtained, in the sense that non-negative terms are added to the right hand sides of the inequalities; see [11], [9], [3], [10],[24], [25],[38]. One common type of improvement for the above Hardy inequalities are the so called potentials; we call $0 \le V(x)$, defined in Ω, a potential for (1.5) provided

$$\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \ge \int_\Omega V(x) u^2 dx, \qquad u \in H_0^1(\Omega).$$

Most of the results in this direction are explicit examples of potentials V where, in the best results, V is an infinite series involving complicated inductively defined functions. Recently Ghoussoub and Moradifam [27] gave necessary and sufficient conditions for a nonnegative function V to be a potential for an improved Hardy inequality on a radial domain, which allowed them to unify all of the known results concerning explicit potentials, and to prove new ones, including Hardy-Rellich type inequalities [28].

In another direction people have considered Hardy inequalities for operators more general than the Laplacian. One case of this is the results obtained by Adimurthi and A. Sekar [1]: Suppose Ω is a smooth domain in $\mathbb{R}^n$ which contains the origin, $A(x) = ((a^{i,j}(x)))$ denotes a symmetric, uniformly positive definite matrix with suitably smooth coefficients and for $\xi \in \mathbb{R}^n$ we define $|\xi|_A^2 := |\xi|_{A(x)}^2 := A(x)\xi \cdot \xi$. Now suppose E is a solution of $\mathcal{L}_{A,p}(E) := -div\left(|\nabla E|_A^{p-2} A \nabla E\right) = \delta_0$ in Ω with $E = 0$ on $\partial\Omega$ where δ_0 is the Dirac mass at 0. Then for all $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx \ge 0.$$

Improvements of this inequality were also obtained and they posed the following question: Is $\left(\frac{p-1}{p}\right)^p$ optimal?

In Chapter 7 we establish Hardy inequalities of the form

$$\int_\Omega |\nabla u|_A^2 dx \ge \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx, \qquad u \in H_0^1(\Omega) \tag{1.8}$$

where E is a positive function defined in Ω, $-div(A\nabla E)$ is a nonnegative nonzero finite measure in Ω which we denote by μ and where $A(x)$ is a $n \times n$ symmetric, uniformly positive definite matrix defined in Ω with $|\xi|_A^2 := A(x)\xi \cdot \xi$ for $\xi \in \mathbb{R}^n$. We show that (1.8) is optimal if $E = 0$ on $\partial\Omega$ or $E = \infty$ on the support of μ and is not attained in either case. When $E = 0$ on $\partial\Omega$ we show

$$\int_\Omega |\nabla u|_A^2 dx \ge \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \frac{1}{2} \int_\Omega \frac{u^2}{E} d\mu, \qquad u \in H_0^1(\Omega) \tag{1.9}$$

is optimal and not attained. Optimal weighted versions of these inequalities are also established. Optimal analogous versions of (1.8) and (1.9) are established for $p \neq 2$ which, in the case that μ is a Dirac mass, answers a best constant question posed by Adimurthi and Sekar, see [1].

Since the above inequalities do not attain a standard question to ask is for what functions $0 \leq V(x)$ do we have

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \int_\Omega V(x) u^2 dx, \qquad u \in H_0^1(\Omega). \tag{1.10}$$

Necessary and sufficient conditions on V are obtained (in terms of the solvability of a linear pde) for (1.10) to hold. Analogous results involving improvements are obtained for the weighted versions.

We establish optimal inequalities which are similar to (1.8) and are valid for $u \in H^1(\Omega)$. We obtain results on improvements of this inequality which are similar to the above results on improvements. In addition weighted versions of this inequality are also obtained.

Provided E satisfies various properties we show (1.8) holds and is optimal on exterior and annular domains for u non-zero on the inner boundary.

We remark that most of the known Hardy inequalities (in the case where $p = 2$) can be obtained, via the above approach, by making suitable choices for E and $A(x)$.

Bibliography

[1] Adimurthi, Anusha Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proceedings of the Royal Society of Edinburgh, 136A, 1111-1130, 2006

[2] G. Arioli, F. Gazzola, H.C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. **36**, 1226-1258 (2005).

[3] B. Barbatis, S. Filippas, A. Tertikas, *Series expansions for L^p Hardy inequalities*, (August 20, 2007) preprint

[4] L. Ben Chaabane,*On the extremal solutions of semilinear elliptic problems*, Abstr. Appl. Anal. 2005, no. 1, 1-9.

[5] E. Berchio and F. Gazzola, *Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities*, Electronic Journal of Differential Equations, Vol. 2005(2005), No. 34, pp. 120.

[6] H. Berestycki, A. Kiselev, A. Novikov and L. Ryzhik, *The explosion problem in a flow*, preprint (2008).

[7] T. Boggio, Sulle funzioni di Green dordine m, Rend. Circ. Mat. Palermo (1905), 97-135.

[8] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa *Blow up for $u_t - \Delta u = g(u)$ revisited,* Ad. Diff. Eq. **1** (1996), 73-90.

[9] H. Brezis, M. Marcus, *Hardy's inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1-2, 217–237 (1998).

[10] H. Brezis, M. Marcus, I.Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal. 171 (2000), 177-191

[11] H. Brezis and L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 443–469.

[12] X. Cabré, *Regularity of minimizers of semilinear elliptic problems up to dimension four*, (preprint) (2009).

[13] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. 238 (2006), no. 2, 709–733.

[14] T. Cazenave, M. Escobedo and M. Assunta Pozio *Some stability properties for minimal solutions of $-\Delta u = \lambda g(u)$*. Port. Math. (N.S.) 59 (2002), no. 4, 373-391.

[15] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements,* Commun. Pure Appl. Anal. 9 (2010), no. 1, 109–140.

[16] C. Cowan, *Regularity of the extremal solutions in elliptic systems,* submitted (2010).

[17] C. Cowan, P. Esposito and N. Ghoussoub, *Regularity of Extremal Solutions in Fourth Order Nonlinear Eigenvalue Problems on General Domains*, Discrete Contin. Dyn. Syst., In press (2010).

[18] C. Cowan, P. Esposito, N. Ghoussoub, A. Mordifam, *The critical dimension for a fourth order elliptical problem with singular nonlinearity*, Arch. Ration. Mech. Anal., In press (2009) 19 pp

[19] C. Cowan and N. Ghoussoub, *Regularity of the extremal solution in a MEMS model with advection*, Methods and Applications of Analysis, Vol. 15, No 3 (2008) p. 355-362.

[20] C. Cowan and N. Ghoussoub, *Estimates on pull-in distances in MEMS models and other nonlinear eigenvalue problems*, Submitted (2009).

[21] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variation methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal., 58 (1975), pp.207-218.

[22] J. Davila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. 39 (2007), 565-592.

[23] P. Esposito, N. Ghoussoub, Y. J. Guo: *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Research Monograph, In press (2009) 260 p.

[24] S. Filippas, A. Tertika, *Optimizing improved Hardy inequalities*, J.Funct. Anal. 192 (2002), n0.1, 186-233

[25] J.Fleckinger, E.M. Harrell II, F. Thelin, *Boundary behaviour and estimates for solutions of equations containing the p-Laplacian*, Electron. J. Differential Equations 38 (1999), 1-19

[26] G. R. Gavalas, "Nonlinear Differential Equations of Chemically Reacting Systems," Springer-Verlag, New York Inc., New York, 1968.

[27] N. Ghoussoub, A. Moradifam, *On the best possible remaining term in the Hardy inequality*, Proc. Natl. Acad. Sci. USA 105 (2008), no. 37, 13746–13751.

[28] N. Ghoussoub and A. Moradifam, *Bessel Pairs and Optimal Hardy and Hardy-Rellich Inequalities*, (preprint) 2008

[29] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a mems device with varying dielectric properties*, SIAM J. Appl. Math 66 (2005), 309-338.

[30] D. Joseph and T. Lundgren, *Quasilinear Dirichlet problems driven by positive source, Arch. Rational Mech. Anal.* **49** (1973), 241269.

[31] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, J. Diff. Eq. **16** (1974), 103-125.

[32] F. Mignot and J-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations 5 (1980), 791-836.

[33] M. Montenegro, *Minimal solutions for a class of elliptic systems*, Bull. London Math. Soc.**37** (2005) 405-416.

[34] G., Nedev, *Regularity of the extremal solution of semilinear elliptic equations,* C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), no. 11, 997–1002.

[35] A. Kufner and B. Opic, *Hardy Type Inequalities*, Pitman Research Notes in Mathematics, Vol. 219, Longman, New York, 1990

[36] J.A. Pelesko, *Mathematical modeling of electrostatic mems with tailored dielectric properties*, SIAM J. Appl. Math. 62 (2002), 888-908.

[37] J.A. Pelesko and A.A. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

[38] L. Vazquez and E. Zuazua, *The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential*, J. Funct. Anal. 173, 103-153 (2000).

[39] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems,* Commun. Contemp. Math. **4** (2002), no. 3, p. 547-558.

Chapter 2

Regularity of the extremal solution in a MEMS model with advection[1]

2.1 Introduction

The following equation has often been used to model a simple *Micro-Electro-Mechanical System* (MEMS) device:

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \frac{\lambda}{(1-u)^2} \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is proportional to the applied voltage and $0 < u(x) < 1$ denotes the deflection of the membrane. This model has been extensively studied, see [10], [11] in regards to the model and [7], [6], [8] for mathematical aspects of $(P)_\lambda$. It is well known (see above references) that there exists some positive finite critical parameter λ^* such that for all $0 < \lambda < \lambda^*$, the equation $(P)_\lambda$ has a smooth minimal stable (see below) solution u_λ, while for $\lambda > \lambda^*$ there are no weak solutions of $(P)_\lambda$ (see [7] for a precise definition of weak solution). Standard elliptic regularity theory yields that a solution u of $(P)_\lambda$ is smooth if and only if $\sup_\Omega u < 1$. One can also show that $\lambda \mapsto u_\lambda(x)$ is increasing and hence one can define the extremal solution

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

which can be shown to be a weak solution of $(P)_{\lambda^*}$.

Recall that a smooth solution u of $(P)_\lambda$ is said to be *minimal* if any other solution v of $(P)_\lambda$ satisfies $u \le v$ a.e. in Ω. Such solutions are then

[1] A version of this chapter has been accepted for publication: C. Cowan and N. Ghoussoub, *Regularity of the extremal solution in a MEMS model with advection*, Methods and Applications of Analysis, Vol. 15, No 3 (2008)

semi-stable meaning that the principal eigenvalue of the linearized operator

$$L_{u,\lambda} := -\Delta - \frac{2\lambda}{(1-u)^3}$$

in $H_0^1(\Omega)$ is nonnegative. This property can be expressed variationally by the inequality

$$2\lambda \int_\Omega \frac{\psi^2}{(1-u)^3} \leq \int_\Omega |\nabla \psi|^2, \qquad \forall \psi \in H_0^1(\Omega). \tag{2.1}$$

which can be viewed as the nonnegativeness of the second variation of the energy functional associated with $(P)_\lambda$ at u.

Now a question of interest is whether u^* is a smooth solution of $(P)_{\lambda^*}$. It is shown in [7] that this is indeed the case provided $N \leq 7$. This result is optimal in the sense that u^* is singular in dimension $N \geq 8$ with Ω taken to be the unit ball.

Our main interest here will be in the regularity of the extremal solution associated with

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x)\cdot\nabla u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $c \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ and where again Ω is a smooth bounded domain in $\mathbb{R}^N$. Modifying the proofs used in analyzing $(P)_\lambda$ one can again show the existence of a positive finite critical parameter λ^* such that for $0 < \lambda < \lambda^*$ there exists a smooth minimal solution u_λ of $(S)_\lambda$, while there are no smooth solutions of $(S)_\lambda$ for $\lambda > \lambda^*$. Moreover, the minimal solutions are also *semi-stable* in the sense that the principal eigenvalue of the corresponding linearized operator

$$L_{u,\lambda,c} := -\Delta + c(x)\cdot\nabla - \frac{2\lambda}{(1-u_\lambda)^3}$$

in $H_0^1(\Omega)$ is non-negative. See [3] where these results are proved for general C^1 convex nonlinearities which are superlinear at ∞. Our main result concerns the regularity of the extremal solution of $(S)_\lambda$.

Theorem 2.2. *If $1 \leq N \leq 7$, then the extremal solution u^* of $(S)_{\lambda^*}$ is smooth.*

Remark 2.1.1. A crucial (in fact the main) ingredient in proving the regularity of u^* in $(P)_\lambda$, is the energy inequality (2.1) which is used in conjunction with the equation $(P)_\lambda$, to obtain uniform (in λ) L^p-estimates on $(1-u_\lambda)^{-2}$ whenever u_λ is the minimal solution (See [7]). However, the semi-stability of u_λ in the case of $(S)_\lambda$, does not translate into an *energy inequality* which allows the use of arbitrary test functions. Overcoming this will be the major hurdle in proving Theorem 6.2.

We point out, however, that if $c(x) = \nabla\gamma$ for some smooth function γ on $\bar{\Omega}$, then the semi-stability condition on the minimal solution u_λ of $(S)_\lambda$ translates into

$$2\lambda \int_\Omega \frac{e^{-\gamma}\psi^2}{(1-u_\lambda)^3} \leq \int_\Omega e^{-\gamma}|\nabla\psi|^2, \qquad \forall \psi \in H_0^1(\Omega). \tag{2.3}$$

Then, with slight modifications, one can use the standard approach for $(P)_\lambda$ to obtain the analogous result for $(S)_\lambda$ stated in Theorem 6.2.

The novel case is therefore when c is a divergence free vector field. Actually, we shall use the following version of the Hodge decomposition, in order to deal with general vector fields c.

Lemma 2.4. *Any vector field $c \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$ can be decomposed as $c(x) = -\nabla\gamma + a(x)$ where γ is a smooth scalar function and $a(x)$ is a smooth bounded vector field such that* $\mathrm{div}(e^\gamma a) = 0$.

Proof. By the Krein-Rutman theory, the linear eigenvalue problem

$$\left\{ \begin{array}{lcll} \Delta\alpha + \mathrm{div}(\alpha c) & = & \mu\alpha & \Omega, \\ (\nabla\alpha + \alpha c)\cdot n & = & 0 & \partial\Omega, \end{array} \right. \tag{2.5}$$

where n is the unit outer normal on $\partial\Omega$, has a positive solution α in Ω when μ is the principal eigenvalue. Integrating the equation over Ω, one sees that $\mu = 0$. The positivity of α on the boundary follows from the boundary condition and the maximum principle. In other words, we have that $\Delta\alpha + \mathrm{div}(\alpha c) = 0$ on Ω, and $\alpha > 0$ on $\bar{\Omega}$.

Now define $\gamma := \log(\alpha)$ and $a := c + \nabla\gamma$. An easy computation shows that $\mathrm{div}(e^\gamma a) = 0$. □

Throughout the rest of this note c, a, γ will be defined as above.

2.2 A general Hardy inequality and non-selfadjoint eigenvalue problems

Consider the linear eigenvalue problem

$$\begin{cases} -\Delta\phi + c\cdot\nabla\phi - \rho\phi &= K\phi \qquad \Omega, \\ \phi &= 0 \qquad \partial\Omega, \end{cases} \tag{2.6}$$

where c is a smooth bounded vector field on Ω, $\rho \in C^\infty(\overline{\Omega})$ and K is a scalar. We assume that (ϕ, K) is the principal eigenpair for (2.6) and that $\phi > 0$ in Ω, and $K \geq 0$. Note that elliptic regularity theory shows that ϕ is then smooth.

We shall now use a general Hardy inequality to make up for the lack of a variational characterization for the pair (ϕ, K). The following result is taken from [4], which we duplicate here for the convenience of the reader. For a complete discussion on general Hardy inequalities including best constants, attainability and improvements of, see [4]. We should point out that this approach to Hardy inequalities is not new, but it is generally restricted to specific functions E which yield known versions of Hardy inequalities; see [1] and reference within.

Lemma 2.7. *Let $A(x)$ denote a uniformly positive definite $N\times N$ matrix with smooth coefficients defined on Ω. Suppose E is a smooth positive function on Ω and fix a constant β with $1 \leq \beta \leq 2$. Then, for all $\psi \in H_0^1(\Omega)$ we have*

$$\int_\Omega |\nabla\psi|_A^2 \geq \frac{\beta(2-\beta)}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}\psi^2 + \frac{\beta}{2}\int_\Omega \frac{-\mathrm{div}(A\nabla E)}{E}\psi^2, \tag{2.8}$$

where $\int_\Omega |\nabla\psi|_A^2 = \int_\Omega A(x)\nabla\psi\cdot\nabla\psi$.

Proof. For simplicity we prove the case where $A(x)$ is given by the identity matrix. For the general case, we refer to [4]. Let E_0 denote a smooth positive function defined in Ω and let $\psi \in C_c^\infty(\Omega)$. Set $v := \frac{\psi}{\sqrt{E_0}}$. Then

$$|\nabla\psi|^2 = E_0|\nabla v|^2 + \frac{|\nabla E_0|^2}{4E_0^2}\psi^2 + v\nabla v\cdot\nabla E_0. \tag{2.9}$$

Integrating the last term by parts gives

$$\int_\Omega v\nabla v\cdot\nabla E_0 = \frac{1}{2}\int_\Omega \frac{-\Delta E_0}{E_0}\psi^2$$

and so integrating (6.8) gives

$$\int_\Omega |\nabla \psi|^2 \geq \frac{1}{4}\int_\Omega \frac{|\nabla E_0|^2}{E_0^2}\psi^2 + \frac{1}{2}\int_\Omega \frac{-\Delta E_0}{E_0}\psi^2, \tag{2.10}$$

where we dropped a nonnegative term. So we have the desired result for $\beta = 1$. When $\beta \neq 1$ one puts $E_0 := E^\beta$ into (2.10) and collects like terms to obtain the desired result. $\square$

We now use the above lemma to obtain an energy inequality valid for the principal eigenpair of (2.6).

Theorem 2.11. *Suppose that the principal eigenpair (ϕ, K) of (2.6) are such that $\phi > 0$ and $K \geq 0$. Then, for $1 \leq \beta \leq 2$ we have for all $\psi \in H_0^1(\Omega)$,*

$$\int_\Omega e^\gamma |\nabla \psi|^2 \geq \frac{\beta(2-\beta)}{4}\int_\Omega \frac{e^\gamma |\nabla \phi|^2}{\phi^2}\psi^2 + \frac{\beta}{2}\int_\Omega e^\gamma \rho(x)\psi^2 - \frac{\beta}{2}\int_\Omega \frac{e^\gamma a\cdot\nabla\phi}{\phi}\psi^2. \tag{2.12}$$

Proof. Note that (2.6) can be rewritten as

$$-\mathrm{div}(e^\gamma \nabla\phi) + e^\gamma a\cdot\nabla\phi = e^\gamma\left(\rho(x) + K\right)\phi \qquad \text{in } \Omega,$$

where as mentioned above we are using the decomposition $c = -\nabla\gamma + a$. We now set $E := \phi$ and $A(x) = e^\gamma I$ (where I is the identity matrix) and use (2.8) along with the above equation to obtain the desired result. Note that we have dropped the nonnegative term involving K.

$\square$

2.3 Proof of theorem 2.2

For $0 < \lambda < \lambda^*$, we denote by u_λ the smooth minimal semi-stable solution of $(S)_\lambda$. Let (ϕ, K) denote the principal eigenpair associated with the linearization of $(S)_\lambda$ at u_λ. Then $0 < \phi$ in Ω, $0 \leq K$ and (ϕ, K) satisfy

$$\left\{\begin{array}{rcll} -\Delta\phi + c\cdot\nabla\phi &=& (\frac{2\lambda}{(1-u_\lambda)^3} + K)\phi & \Omega, \\ \phi &=& 0 & \partial\Omega. \end{array}\right. \tag{2.13}$$

Again, elliptic regularity theory shows that ϕ is smooth. Consider $c = -\nabla\gamma + a$ to be the decomposition of c described in Lemma 1. We now obtain the main estimate.

Theorem 2.14. *For $0 < \lambda < \lambda^*$, $1 < \beta < 2$ and $0 < t < \beta + \sqrt{\beta^2 + \beta}$, we have the following estimate:*

$$\lambda\left(\beta - \frac{t^2}{2t+1}\right)\int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{2t+3}} \leq 2\beta\lambda\int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{t+3}} + \frac{\beta\|a\|_{L^\infty}^2}{4(2-\beta)}\int_\Omega \frac{e^\gamma}{(1-u_\lambda)^{2t}}.$$

Proof. Fix $0 < \beta < 2$, let $0 < t$ and u denote the minimal solution associated with $(S)_\lambda$. We shall use Theorem 2.11 with $\rho(x) = \frac{2\lambda}{(1-u_\lambda)^3}$. Put $\psi := \frac{1}{(1-u)^t} - 1$ into (2.12) to obtain

$$t^2\int_\Omega \frac{e^\gamma|\nabla u|^2}{(1-u)^{2t+2}} \geq \beta\lambda\int_\Omega \frac{e^\gamma}{(1-u)^3}\left(\frac{1}{(1-u)^t} - 1\right)^2 + \frac{\beta}{2}\int_\Omega e^\gamma\left(\frac{(2-\beta)}{2}\frac{|\nabla\phi|^2}{\phi^2} - \frac{a\cdot\nabla\phi}{\phi}\right)\psi^2.$$

Now note that $(S)_\lambda$ can be rewritten as

$$-\mathrm{div}(e^\gamma\nabla u) + e^\gamma a\cdot\nabla u = \frac{\lambda e^\gamma}{(1-u)^2} \qquad \text{in } \Omega,$$

and test this on $\bar{\phi} := \frac{1}{(1-u)^{2t+1}} - 1$ to obtain

$$(2t+1)\int_\Omega \frac{e^\gamma|\nabla u|^2}{(1-u)^{2t+2}} + H = \lambda\int_\Omega \frac{e^\gamma}{(1-u)^2}\left(\frac{1}{(1-u)^{2t+1}} - 1\right),$$

where

$$H := \int_\Omega e^\gamma a\cdot\nabla u\left(\frac{1}{(1-u)^{2t+1}} - 1\right).$$

One easily sees that $H = 0$ after considering the fact H can be rewritten in the form $\int_\Omega(e^\gamma a)\cdot\nabla G(u)$ for an appropriately chosen function G with $G(0) = 0$. Combining the above two inequalities and dropping some positive terms gives

$$\lambda\left(\beta - \frac{t^2}{2t+1}\right)\int_\Omega \frac{e^\gamma}{(1-u)^{2t+3}} \leq 2\beta\lambda\int_\Omega \frac{e^\gamma}{(1-u)^{3+t}} + \frac{\beta}{2}\int_\Omega e^\gamma\Lambda(x)\left(\frac{1}{(1-u)^t} - 1\right)^2$$

where
$$\Lambda(x) := \frac{a\cdot\nabla\phi}{\phi} - \frac{(2-\beta)}{2}\frac{|\nabla\phi|^2}{\phi^2}.$$
Simple calculus shows that
$$\sup_{\Omega}\Lambda(x) \le \frac{\|a\|^2_{L^\infty}}{2(2-\beta)},$$
which, after substituting into the above inequality, completes the proof of the main estimate. $\square$

Note now that the restriction $t < \beta + \sqrt{\beta^2+\beta}$ is needed to ensure that the coefficient $\beta - \frac{t^2}{2t+1}$ is positive. It follows then that $\frac{1}{(1-u_\lambda)^2}$ is uniformly bounded (in λ) in $L^p(\Omega)$ for all $p < p_0 := \frac{7}{2}+\sqrt{6} \approx 5.94...$ and after passing to limits we have the same result for the extremal solution u^*.

To conclude the proof of Theorem 6.2, it suffices to note the following result.

Lemma 2.15. *Suppose $3 \le N \le 7$ and the extremal solution u^* satisfies $\frac{1}{(1-u^*)^2} \in L^{\frac{3N}{4}}(\Omega)$, then u^* is smooth.*

Proof. First note that by elliptic regularity one has $u^* \in W^{2,\frac{3N}{4}}(\Omega)$ and after applying the Sobolev embedding theorem one has $u^* \in C^{0,\frac{2}{3}}(\overline{\Omega})$. Now suppose $\|u\|_{L^\infty} = 1$ so that there is some $x_0 \in \Omega$ such that $u(x_0) = 1$. Then
$$\frac{1}{1-u(x)} \ge \frac{C}{|x-x_0|^{\frac{2}{3}}}$$
and hence
$$\infty > \int_\Omega \frac{1}{((1-u^*)^2)^{\frac{3N}{4}}} \ge C\int_\Omega \frac{1}{|x|^N} = \infty,$$
which is a contradiction. It follows that $\frac{1}{(1-u^*)^2} \in L^\infty(\Omega)$, and u^* is therefore smooth. $\square$

Using this lemma and the above L^p-bound on $\frac{1}{(1-u^*)^2}$, one sees that u^* is smooth for $3 \le N \le 7$. To show the result in dimensions $N = 1, 2$, one needs a slight variation of the above argument. We omit the details, and the interested reader can consult [7] for the proof when $c(x) = 0$.

Remark 2.3.1. As mentioned in the abstract, this method applies to most non-selfadjoint eigenvalue problems of the form

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x)\cdot\nabla u &= \lambda f(u) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $f(u)$ is an appropriate convex nonlinearity such as $f(u) = e^u$, and $f(u) = (1+u)^p$. It shows in particular that the presence of an advection does not change the critical dimension of the problem, hence addressing an issue raised recently by Berestycki et al [3]. One can also extend the general regularity results of Nedev [9] (for general convex f in dimensions 2 and 3) and those of Cabre and Capella [2] (for general radially symmetric f on a ball, and up to dimension 9). All these questions are the subject of a forthcoming paper [5].

Bibliography

[1] Adimurthi and Anusha Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proceedings of the Royal Society of Edinburgh, 136A, 1111-1130, 2006

[2] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. **238** (2006), no. 2, 709–733.

[3] H. Berestycki, A. Kiselev, A. Novikov and L. Ryzhik, *The explosion problem in a flow*, preprint (2008).

[4] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, (preprint).

[5] C. Cowan, N. Ghoussoub, *Regularity and stability of solutions in non-linear eigenvalue problems with advection*, (In preparation) (2008).

[6] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731–1768.

[7] N. Ghoussoub and Y. Guo, *On the partial differential equations of electro MEMS devices: stationary case*, SIAM J. Math. Anal. 38 (2007), 1423-1449.

[8] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a mems device with varying dielectric properties*, SIAM J. Appl. Math 66 (2005), 309-338.

[9] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations.* C. R. Math. Acad. Sci. Paris **330** (2000), 997–1002.

[10] J.A. Pelesko, *Mathematical modeling of electrostatic mems with tailored dielectric properties*, SIAM J. Appl. Math. 62 (2002), 888-908.

[11] J.A. Pelesko and A.A. Berstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

Chapter 3

Estimates on pull-in distances in MEMS models and other nonlinear eigenvalue problems[2]

3.1 Introduction

We examine problems of the form

$$\begin{cases} -\Delta u = \lambda f(x) F(u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,f}$}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \lambda$, f is a nonnegative nonzero bounded Hölder continuous function, usually dubbed as the *permittivity profile*, and where F is a suitably defined nonlinearity. We have in mind the examples $F(u) = e^u$, $F(u) = (u+1)^p$ where $p > 1$ and $F(u) = (1-u)^{-p}$ where $p > 0$. We now formalize the classes of of nonlinearities we examine. Suppose F is a smooth, increasing, convex nonlinearity on its domain $0 \in D_F \subset \mathbb{R}$, such that $F(0) = 1$. If $D_F = [0, \infty)$ and F is superlinear at ∞, then F is said to be a *regular nonlinearity*. When $D_F := [0, 1)$ and $\lim_{u \nearrow 1} F(u) = +\infty$ then we say that F is a *singular nonlinearity*.

We say that a solution u of $(P_{\lambda,f})$ is *classical* provided $\|u\|_{L^\infty} < \infty$ (resp., $\|u\|_{L^\infty} < 1$) if F is a regular (resp., singular) nonlinearity. Note that by elliptic regularity theory, this is equivalent to saying that a classical solution is in $C^{2,\alpha}$ for some $\alpha > 0$.

It is by now well-known (see [6], [25], [7], [19], [26], [3]) that – regardless whether F is a regular or singular nonlinearity – there exists an extremal

[2] A version of this chapter has been submitted for publication: C. Cowan and N. Ghoussoub *Estimates on pull-in distances in MEMS models and other nonlinear eigenvalue problems*

parameter $\lambda^* \in (0, +\infty)$ depending on Ω, f and N, and which can be defined as

$$\lambda^*(\Omega, f) := \sup\{\lambda > 0 : (P_{\lambda,f}) \text{ has a classical solution}\},$$

such that $(P_{\lambda,f})$ has a *minimal* classical solution u_λ for every $\lambda \in (0, \lambda^*)$, and no solution for $\lambda > \lambda^*$. By a *"minimal solution"* u, we mean one such that any other solution v of $(P_{\lambda,f})$ satisfies $v \geq u$ a.e. in Ω. One can then also show that $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$. This allows us to define the *extremal solution* by

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

which can be seen as some kind of a "weak solution" for $(P_{\lambda^*,f})$, see [25], [3].

We shall also need the notion of *stability*. Given a solution u of $(P_{\lambda,f})$, we say that u is *stable* (resp., *semi-stable*) provided $\mu_1(\lambda, u) > 0$, (resp., $\mu_1(\lambda, u) \geq 0$) where

$$\mu_1(\lambda, u) := \inf\left\{ \int_\Omega (|\nabla\psi|^2 - \lambda f(x) F'(u)\psi^2) dx : \ \psi \in C_c^\infty(\Omega), \int_\Omega \psi^2 = 1 \right\}.$$

Under our assumptions on the nonlinearity F, and whether it is regular or singular, one can show that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is stable. If in addition, u^* is a classical solution of $(P_{\lambda^*,f})$, then necessarily $\mu_1(\lambda^*, u^*) = 0$, since otherwise one could use the Implicit Function Theorem, in a suitable function space, to obtain solutions to $(P_{\lambda,f})$ for $\lambda > \lambda^*$, which would be a contradiction.

The question of the regularity of the extremal solution has attracted a lot of attention in the last decade. For general regular nonlinearities, with $f(x) = 1$, the extremal solution is classical provided one of the following holds:

- Ω is contained in $\mathbb{R}^N$ with $N \leq 3$ [27]. This has recently been improved to $N \leq 4$, see [8].
- Ω is a ball in $\mathbb{R}^N$ with $N \leq 9$, see [9].
- Ω is contained in $\mathbb{R}^N$ where $N \leq 9$ under various weak assumptions on $F(u)$, see [31].

The second result is optimal after one considers $F(u) = e^u$ on the unit ball in $\mathbb{R}^{10}$. It is an open question as to whether for $4 \leq N \leq 9$, there is a regular nonlinearity F and a domain $\Omega \subset \mathbb{R}^N$ on which the corresponding extremal solution is unbounded. In the case of the MEMS model, where $F(u) = (1-u)^{-2}$, it is known that the extremal solution is classical provided $N \leq 7$ and that this result is optimal (see [19]). On the other hand, for any dimension $N > 2$, there exists a singular nonlinearity, namely $F(u) = (1-u)^{-p}$ for some $p := p(N) > 0$, such that the corresponding extremal solution is not classical (see Chapter 3 of [16]).

In this paper, we are mostly interested in the quantitative aspects of the regularity of the extremal solution u^*, which were initially motivated by the equation

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$M_{\lambda,f}$}$$

In dimension $N = 2$ this equation models a simple *Micro-Electromechanical-Systems* MEMS device, which roughly consists of a dielectric elastic membrane that is attached to the boundary of Ω, by which upper surface has a thin conducting film. At a distance of 1 above the undeflected membrane sits a grounded plate, i.e., a plate held at zero voltage. When a voltage $V > 0$ is applied to the thin film of the membrane, it deflects towards the ground plate. After various physical limits of the parameters involved, a dimensional argument and a simplification, ones arrives at $(M_{\lambda,f})$ for the steady state of the membrane. Here λ is proportional to the applied voltage V and the *permittivity profile* $f(x)$ allows for varying dielectric properties of the membrane.

As seen above (see [26] or [19]), one expects the extremal solution u^* in small dimension N to be bounded away from 1, hence to be a classical solution. Since the parameter λ^* corresponds to the critical voltage beyond which there is a snap-through, and since u^* is the optimal deflection of the membrane, it is therefore important for the design of MEMS devices to know how the critical voltage λ^* and the *pull-in-distance* – defined as $\|u^*\|_{L^\infty}$ – depend on the geometry of the membrane and on the permittivity profile. Several analytical and numerical estimates on λ^* have been derived by Pelesko [28], Guo-Pan-Ward [22], Guo-Ghoussoub [19] and others in the case of the MEMS non-linearity $F(u) = \frac{1}{(1-u)^2}$. On the other hand, only numerical estimates have been obtained for the pull-in distance in the case of power-law (resp., exponential) permittivity profiles $f(x) = |x|^\alpha$ (resp., $f(x) = e^{\alpha x}$). In this paper, we shall see that one can give rigorous proofs and estimates for phenomena, which so far have only been observed numerically by various authors. We shall also include corresponding results for general – not necessarily MEMS-type – nonlinearities.

Here is a brief description of the paper. In section 2, we give upper estimates on the pull-in voltage $\lambda^*(\Omega, f)$ in fairly general situations, which will in turn yield lower bounds on $\|u^*\|_{L^\infty}$. What is remarkable here is that the estimates – which are valid for general nonlinearities – turn out to only depend on the permittivity profiles and not on the domain, nor on the dimension. Actually, they also apply to any reasonable differential operator, see Remark 3.2.1.

In section 3, we give upper estimates on $\|u^*\|_{L^\infty}$ which are computationally friendly. Just as in the proof of the regularity of u^* in low dimensions, we use the energy estimates on the minimal solutions coupled with L^p to L^∞ Sobolev-type constants related to corresponding linear equations. While the result is satisfactory for exponential nonlinearity, it is not so for the MEMS model, which led us to reconsider this nonlinearity in the case of the ball where more precise L^p to Hölder estimates can be used. We stress here that we are not interested in optimal upper estimates but rather estimates for which, if given a specific domain Ω and a nonlinearity F, one can easily obtain some numerical parameters by plotting a function of a single variable – possibly – using a Computer Algebra System.

Section 4 was motivated by an intriguing phenomenon observed numerically by Guo-Pan-Ward [22], namely that on a two dimensional disc, the pull-in distance does not depend on the power of the permittivity profile $f(x) = |x|^\alpha$. We prove that this is indeed the case by a simple scaling argument which relates the problem $(P_{\lambda,|x|^\alpha})$ on the unit ball of $\mathbb{R}^N$ to $(P_{\lambda,1})$ (which for simplicity we denote by (P_λ)) on a ball in a fractional dimension $N(\alpha)$. (Note that when f is radial and Ω is the unit ball in $\mathbb{R}^N$, all stable solutions of $(P_{\lambda,f})$ are then radial and hence we can examine the problem in fractional dimensions). One can then easily transfer many results established for (P_λ) to $(P_{\lambda,|x|^\alpha})$. This observation, combined for example with the results of Cabre and Capella [9], leads to new regularity results for the extremal solution associated with $(P_{\lambda,|x|^\alpha})$.

In section 5, we study the asymptotics in λ, and we obtain upper and lower pointwise bounds on the minimal solutions u_λ, in the case where u^* is singular. The upper estimates are valid on arbitrary domains and we restrict ourselves to radial domains for the lower estimates since more explicit bounds can then be found. For that we exploit the fact that both u^* and $\frac{d}{d\lambda}u_\lambda|_{\lambda=\lambda^*}$ are explicitly known in the case where Ω is a ball and u^* is singular.

We now list our main notation. For a nonlinearity F, we denote by a_F the upper bound of the domain D_F, which means that $a_F := \infty$ if F is regular, and $a_F := 1$ if F is singular, in such a way that $D_F := [0, a_F)$.

We shall also associate to F the numbers

$$B_F := \sup_{\tau\in(0,a_F)} \tfrac{\tau}{F(\tau)} \quad \text{and} \quad C_F := \textstyle\int_0^{a_F} \tfrac{d\tau}{F(\tau)}. \tag{3.1}$$

The ball of radius R centred at x_0 in $\mathbb{R}^N$ will be denoted by $B_R(x_0)$. If $x_0 = 0$ then we omit x_0 and if $R = 1$ then we just write B. Given a set Ω in $\mathbb{R}^N$ we let $|\Omega|$ denote its N-dimensional Lebesgue measure, while ω_N denotes the volume of the unit ball B in $\mathbb{R}^N$. The conjugate index of p will be denoted by p' in such a way that $\frac{1}{p} + \frac{1}{p'} = 1$. For a radial function u we write $u(r) = u(|x|)$. The first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ will be denoted by $\lambda_1(\Omega)$ and the corresponding positive eigenfunction will be ϕ_Ω, assuming the normalization $\int_\Omega \phi_\Omega = 1$.

Before we proceed further, we would like to express our gratitude to the referees of this paper for their very pertinent comments and suggestions.

3.2 Lower estimates for the $L^\infty-$norm of the extremal solution

This section is devoted to the proof of the following result.

Theorem 3.2. *Suppose F is either a regular or singular nonlinearity and that u^* is the extremal solution of $(P_{\lambda,f})$, which we assume to be classical. Then,*

$$\|u^*\|_{L^\infty} \geq (F')^{-1}\left(\max\left\{\frac{1}{B_F}\frac{\inf_\Omega f}{\sup_\Omega f}, \frac{1}{C_F}\frac{\int_\Omega f\phi_\Omega dx}{\sup_\Omega f}\right\}\right), \tag{3.3}$$

where we define $(F')^{-1}(z) = 0$ for $z < F'(0)$.

Before proceeding with the proof, we give some applications.

Corollary 3.4. *Suppose f is a non-negative bounded Hölder continuous permittivity profile and that the extremal solution u^* of $(P_{\lambda,f})$ on a bounded domain Ω is regular.*

1. *If $F(u) = \frac{1}{(1-u)^p}$, $p > 0$, then*

$$\|u^*\|_{L^\infty} \geq 1 - \min\left\{\frac{p}{p+1}\left(\frac{\sup_\Omega f}{\inf_\Omega f}\right)^{\frac{1}{p+1}}, \left(\frac{p}{p+1}\frac{\sup_\Omega f}{\int_\Omega f\phi_\Omega\, dx}\right)^{\frac{1}{p+1}}\right\}. \tag{3.5}$$

In particular, when the permittivity $f \equiv 1$*, then for any dimension* $1 \le N \le 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}}$*, and any bounded domain* $\Omega \subset \mathbb{R}^N$*, we have*

$$\|u^*\|_{L^\infty} \ge \frac{1}{p+1}. \tag{3.6}$$

2. *If* $F(u) = (u+1)^p$*,* $p > 1$*, then*

$$\|u^*\|_{L^\infty} \ge \max\left\{\frac{p}{p-1}\Big(\frac{\inf_\Omega f}{\sup_\Omega f}\Big)^{\frac{1}{p-1}}, \Big(\frac{p-1}{p}\frac{\int_\Omega f\phi_\Omega dx}{\sup_\Omega f}\Big)^{\frac{1}{p-1}}\right\} - 1 \tag{3.7}$$

In particular, when $f \equiv 1$*, then for any dimension* $1 \le N \le 10$*, and any bounded domain* $\Omega \subset \mathbb{R}^N$*, we have* $\|u^*\|_{L^\infty} \ge \frac{1}{p-1}$*.*

3. *If* $F(u) = e^u$*, then*

$$+\infty > \|u^*\|_{L^\infty} \ge \max\left\{1 + \log\Big(\frac{\inf_\Omega f}{\sup_\Omega f}\Big), \log\Big(\frac{\int_\Omega f\phi_\Omega\, dx}{\sup_\Omega f}\Big)\right\}. \tag{3.8}$$

In particular, when the permittivity $f \equiv 1$*, then for any dimension* $1 \le N \le 9$*, and any bounded domain* $\Omega \subset \mathbb{R}^N$*, we have* $+\infty > \|u^*\|_{L^\infty} \ge 1$*.*

Note that the dimension restrictions above (which are actually sharp, see [26] and [23]) are exactly what is needed to ensure that the corresponding extremal solution is regular. The proof of Theorem 3.2 follows immediately from the combination of the following two propositions. The first provides upper estimates on λ^*, in terms of F, Ω and f.

Proposition 3.2.1. *Suppose* F *is either a regular or singular nonlinearity. Then*

$$\lambda^*(\Omega, f) \le \lambda_1(\Omega)\min\left\{\frac{B_F}{\inf_\Omega f}, \frac{C_F}{\int_\Omega f\phi_\Omega dx}\right\}, \tag{3.9}$$

where B_F *and* C_F *are given in* (3.1)*.*

Proof. Supposing u is a classical solution of $(P_{\lambda,f})$, we multiply both sides of the equation by ϕ_Ω and integrate to obtain

$$\int_\Omega (\lambda F(u) f - \lambda_1(\Omega) u)\, \phi_\Omega dx = 0.$$

Since $\phi_\Omega > 0$ we must have

$$\lambda \le \lambda_1(\Omega)\sup_\Omega \frac{u}{fF(u)} \le \frac{\lambda_1(\Omega)}{\inf_\Omega f}\sup_{z\in D_F}\frac{z}{F(z)} = \frac{\lambda_1(\Omega)B_F}{\inf_\Omega f}.$$

For the second bound, multiply $(P_{\lambda,f})$ by $\frac{\phi_\Omega}{F(u)}$ and integrate to obtain

$$\begin{aligned}
\int_\Omega \lambda f \phi_\Omega dx &= \int_\Omega (-\Delta u)\frac{\phi_\Omega}{F(u)} dx \\
&= \int_\Omega \frac{\nabla u \cdot \nabla \phi_\Omega}{F(u)} dx - \int_\Omega \frac{\phi_\Omega F'(u)|\nabla u|^2}{F(u)^2} dx \\
&\le \int_\Omega \frac{\nabla u \cdot \nabla \phi_\Omega}{F(u)} dx \\
&= \int_\Omega \nabla \phi_\Omega \cdot \nabla \left(\int_0^{u(x)} \frac{1}{F(\tau)} d\tau \right) dx, \\
&= \lambda_1(\Omega) \int_\Omega \phi_\Omega \left(\int_0^{u(x)} \frac{1}{F(\tau)} d\tau \right) dx \\
&\le \lambda_1(\Omega) C_F,
\end{aligned}$$

after recalling the normalization of ϕ_Ω. □

Proposition 3.2.2. *Suppose u^* is the extremal solution of $(P_{\lambda,f})$ which we assume to be classical. Then*

$$\lambda_1(\Omega) \le \lambda^* \|f F'(u^*)\|_{L^\infty}. \tag{3.10}$$

Proof. Since u^* is a classical solution we have $\mu_1(\lambda^*, u^*) = 0$ and hence there is some $0 < \psi \in H_0^1(\Omega)$ such that $-\Delta \psi = \lambda^* f(x) F'(u^*)\psi$ in Ω. Multiplying this by ψ gives

$$\int_\Omega \lambda^* f(x) F'(u^*)\psi^2 dx = \int_\Omega |\nabla \psi|^2 dx \ge \int_\Omega \lambda_1(\Omega)\psi^2 dx,$$

which shows the desired result.

□

Remark 3.2.1. *Note that the lower bounds (when $f = 1$) are independent of the domain. It is also fairly easy to adapt the proof below to show that they are not particularly exclusive to the Laplacian $-\Delta$. Indeed, the same lower bounds can be obtained if we replace it by any operator of the form $L(u) := -div(A(x)\nabla u)$ where $A(x)$ is a symmetric uniformly positive definite $N \times N$ matrix defined in Ω.*

Moreover, the same arguments show that the extremal solution associated with

$$\Delta^2 u = \lambda F(u) \qquad \Omega,$$

also satisfies the same lower bound, where for general domains Ω we restrict our attention to the Navier boundary conditions: $u = \Delta u = 0$ on $\partial\Omega$, while in the case of Ω being a ball we can use the Dirichlet boundary conditions: $u = \partial_\nu u = 0$ on ∂B.

For recent advances on fourth order nonlinear eigenvalue problems, we refer to [10], [13], [15], [12], [4] and [2].

3.3 Upper estimates for the $L^\infty-$norm of the extremal solution

In this section we look for upper estimates on the extremal solution u^* associated with (P_λ), where F is one of the three linearities considered in Corollary 3.4, and where we take $f(x) = 1$ for simplicity. The methods consist of combining the energy estimates – which are critical in showing that the extremal solution is regular in low dimension – with various L^∞ and Hölder estimates for linear equations. We point out that our main interest here is to show how one can obtain explicit upper estimates on u^* using rigorous analysis. We are not interested in (or claim to have) sharp estimates.

The following simple observation can be useful when looking for upper estimates.

Observation 3.3.1. *Suppose u^* is the extremal solution associated with (P_λ) in Ω with extremal parameter λ^*. Then the extremal solution associated with (P_λ) in the domain $\Omega_\rho := \rho\Omega$ (where $\rho > 0$) is given by $v^*_\rho(x) := u^*(\frac{x}{\rho})$ with extremal parameter $\lambda^*(\Omega_\rho) = \frac{\lambda^*(\Omega)}{\rho^2}$.*

3.3.1 Upper estimates on general domains

We begin with the case of exponential nonlinearities.

Theorem 3.11. *Suppose $F(u) = e^u$, Ω is a bounded domain in $\mathbb{R}^N$ and u^* is the extremal solution associated with (P_λ).*

1. *If $3 \le N \le 9$, then*

$$\|u^*\|_{L^\infty} \le \frac{\lambda_1(\Omega)\beta_N}{e(N-2)}\Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{2}{N}}, \tag{3.12}$$

   where

$$\beta_N := \inf\Big\{N^{\frac{-1}{2t+1}}\Big(\frac{2t}{4t+2-N}\Big)^{\frac{2t}{2t+1}}\Big(\frac{4}{2-t}\Big)^{\frac{1}{t}};\quad \frac{N-2}{4} < t < 2\Big\}.$$

2. If $\Omega \subset B_{\frac{1}{2}} \subset \mathbb{R}^2$, then

$$\|u^*\|_{L^\infty} \leq \frac{\lambda_1(\Omega)}{e} \inf \{M; \quad 0 < t < 2\},$$

where

$$M := \Big(\frac{4}{2-t}\Big)^{\frac{1}{t}} \big(\frac{|\Omega|}{2\pi}\big)^{\frac{1}{2t+1}} \Lambda\Big(\frac{2t+1}{2t}, \big(\frac{|\Omega|}{\pi}\big)^{\frac{1}{2}}\Big)^{\frac{2t}{2t+1}}$$

and where we define for $p > 1$ and $0 < R < 1$,

$$\Lambda(p, R) := \int_0^R (-\log(r))^p r dr.$$

We also consider the case of a MEMS nonlinearity.

Theorem 3.13. *Suppose $F(u) = (1-u)^{-2}$, Ω is a bounded domain in $\mathbb{R}^N$ and u^* is the extremal solution associated with (M_λ) in Ω. If $3 \leq N \leq 7$, then*

$$\|u^*\|_{L^\infty} \leq 1 - e^{-\frac{\lambda_1(\Omega)\gamma_N}{2(N-2)}\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{2}{N}}}, \tag{3.14}$$

where

$$\gamma_N := \inf \left\{ \frac{16N^{\frac{-3}{2t+3}}}{27} \left(\frac{2t}{4t+6-3N}\right)^{\frac{2t}{2t+3}} \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}}; \right.$$
$$\left. \frac{3(N-2)}{4} < t < 2 + \sqrt{6} \right\}.$$

Remark 3.3.1. *Using a similar approach one can show that if u^* is the extremal solution associated with (P_λ), in the case where $F(u) = (u+1)^p$, $p > 1$, and $N = 3$ or $N = 4$ then*

$$\|u^*\|_{L^\infty} \leq \frac{(p-1)^{p-1}\lambda_1(\Omega)\beta_{N,p}}{p^p(N-2)} \Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{2}{N}},$$

where

$$\beta_{N,p} = \inf \left\{ \frac{(2tp - p - t^2)^{\frac{-p}{t}} (2t-1)^{\frac{2t-1}{2t+p-1} + \frac{p}{t}} (2p)^{\frac{p}{t}}}{N^{\frac{p}{2t+p-1}} (4t + 2p - 2 - Np)^{\frac{2t-1}{2t+p-1}}} : \right.$$
$$\left. \max\{t_p^-, t_{N,p}\} < t < t_p^+ \right\},$$

and where

$$t_p^- := p - \sqrt{p^2 - p}, \quad t_p^+ := p + \sqrt{p^2 - p}, \quad t_{N,p} := \frac{pN}{4} - \frac{p}{2} + \frac{1}{2}.$$

We have omitted $N = 2$ just for simplicity. To obtain estimates for $N \leq 10$ one has to perform a bootstrap argument or restrict the range of values for p.

For proving the above theorems we shall need the following easy lemmas.

Lemma 3.15. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$.*

1. *If $N \geq 3$ and $\tau > \frac{N}{2}$, then for all $x \in \Omega$,*

$$\left(\int_\Omega \frac{1}{|y-x|^{(N-2)\tau'}} dy\right)^{\frac{1}{\tau'}} \leq \frac{\omega_N^{1-\frac{2}{N}} N^{1-\frac{1}{\tau}} (\tau-1)^{\frac{\tau-1}{\tau}} |\Omega|^{\frac{2}{N}-\frac{1}{\tau}}}{(2\tau - N)^{\frac{\tau-1}{\tau}}}.$$

2. *If $N = 2$, $\tau > 1$ and $\Omega \subset B_{\frac{1}{2}} \subset \mathbb{R}^2$, then for all $x \in \Omega$,*

$$\left(\int_\Omega (-\log(|y-x|))^{\tau'} dy\right)^{\frac{1}{\tau'}} \leq (2\pi)^{\frac{\tau-1}{\tau}} \Lambda\left(\frac{\tau}{\tau-1}, \frac{|\Omega|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\right)^{\frac{\tau-1}{\tau}}.$$

We shall need L^∞ bounds on the solutions of Poisson linear equations. We settle here for the following elementary ones, though the constants involved could possibly be improved by using the approach of A. Brandt in [5].

Lemma 3.16. *Suppose $-\Delta u = g(x) \geq 0$ in Ω with $u = 0$ on $\partial\Omega$ where Ω a bounded domain in $\mathbb{R}^N$ and g is smooth.*

1. *If $N \geq 3$, then for all $\tau > \frac{N}{2}$,*

$$\|u\|_{L^\infty} \leq \frac{\|g\|_{L^\tau} (\tau-1)^{\frac{\tau-1}{\tau}} |\Omega|^{\frac{2}{N}-\frac{1}{\tau}}}{N^{\frac{1}{\tau}} (N-2) \omega_N^{\frac{2}{N}} (2\tau - N)^{\frac{\tau-1}{\tau}}}.$$

2. *If $N = 2$ and $\Omega \subset B_{\frac{1}{2}}$, then for all $\tau > 1$,*

$$\|u\|_{L^\infty} \leq \frac{\|g\|_{L^\tau} \Lambda\left(\frac{\tau}{\tau-1}, \frac{|\Omega|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\right)^{\frac{\tau-1}{\tau}}}{(2\pi)^{\frac{1}{\tau}}}.$$

Proof. In both cases, we let $v(x)$ denote the Newtonian potential of g, i.e.,

$$v(x) := \frac{1}{N(N-2)\omega_N}\int_\Omega \frac{g(y)}{|y-x|^{N-2}}dy,$$

for $N \geq 3$ and

$$v(x) := \frac{1}{2\pi}\int_\Omega (-\log(|y-x|))g(y)dy,$$

for $N = 2$. Since $0 \leq u(x) \leq v(x)$ in Ω, it suffices to show the desired L^∞ estimate on v. To do this, one uses (for $N \geq 3$) Hölder's inequality to write

$$v(x) \quad \leq \quad \frac{1}{N(N-2)\omega_N}\|g\|_{L^\tau}\left(\int_\Omega \frac{1}{|y-x|^{(N-2)\tau'}}dy\right)^{\frac{1}{\tau'}}.$$

and then use the integral estimate in the previous lemma. □

For the convenience of the reader we now derive some standard energy estimates for stable solutions of (P_λ).

Lemma 3.17. *([14], [26]) Suppose u is a classical semi-stable solution of (P_λ).*

1. *If $F(u) = e^u$, then for all $0 < t < 2$, we have*

$$\|e^u\|_{L^{2t+1}} \leq \Big(\frac{4}{2-t}\Big)^{\frac{1}{t}}|\Omega|^{\frac{1}{2t+1}}.$$

2. *If $F(u) = (1-u)^{-2}$, then for all $0 < t < 2+\sqrt{6}$, we have*

$$\|(1-u)^{-2}\|_{L^{t+\frac{3}{2}}} \leq \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}}|\Omega|^{\frac{2}{2t+3}}.$$

Proof. 1) Using the test function $\psi := e^{tu} - 1$, where $0 < t < 2$, in the stability conditions gives

$$\frac{\lambda}{t^2}\int_\Omega e^u(e^{tu}-1)^2 \leq \int_\Omega e^{2tu}|\nabla u|^2.$$

Now testing (P_λ) on $\phi = e^{2tu} - 1$ and rearranging, gives

$$\int_\Omega e^{2tu}|\nabla u|^2 = \frac{\lambda}{2t}\int_\Omega e^u(e^{2tu}-1).$$

Comparing the last two inequalities and dropping some positive terms gives

$$\left(\frac{1}{t}-\frac{1}{2}\right)\int_\Omega e^{(2t+1)u} \le \frac{2}{t}\int_\Omega e^{(t+1)u},$$

and after an application of Hölder's inequality on the right one obtains

$$\|e^u\|_{L^{2t+1}} \le \frac{4^{\frac{1}{t}}}{(2-t)^{\frac{1}{t}}}|\Omega|^{\frac{1}{2t+1}}. \tag{3.18}$$

2) Take $\psi := (1-u)^{-2}-1$, $\phi := (1-u)^{-2t-1}-1$ and proceed as in 1) by putting ψ into the stability condition and testing (M_λ) on ϕ. We obtain

$$\left(\frac{2}{t^2}-\frac{1}{2t+1}\right)\int_\Omega \frac{1}{(1-u)^{2t+3}} \le \frac{4}{t^2}\int_\Omega \frac{1}{(1-u)^{t+3}},$$

after dropping a couple of positive terms. Hölder's inequality then yields

$$\left(\frac{2}{t^2}-\frac{1}{2t+1}\right)\left\|\frac{1}{1-u}\right\|^t_{L^{2t+3}} \le \frac{4}{t^2}|\Omega|^{\frac{t}{2t+3}}. \tag{3.19}$$

□

We now combine the energy estimates with the linear estimates to obtain upper estimates on u^*.

Proof of Theorem 3.11: Use Lemma 3.16 with $g(x) := \lambda^* e^{u^*}$ and $\tau = 2t+1$ along with the estimate $\lambda^* \le \frac{\lambda_1(\Omega)}{e}$ to arrive at an estimate of the form

$$\|u^*\|_{L^\infty} \le C(t,N,|\Omega|)\frac{\lambda_1(\Omega)}{e}\|e^{u^*}\|_{L^{2t+1}},$$

where $C(t,N,|\Omega|)$ is provided by Lemma 3.16. Now replace the L^p−norm on the right using the energy estimates from Lemma 3.17 to arrive at the desired result. The restrictions on t are a result of the restrictions on τ in the linear estimates along with the restrictions on t from the energy estimates. □

Proof of Theorem 3.13: Let Ω denote a bounded domain in $\mathbb{R}^N$ where $3 \le N \le 7$ and let u^* denote the extremal solution associated with (M_λ) in Ω. Since the reasoning works for any log-convex nonlinearity F (i.e., $u \mapsto \log(F(u))$ is convex), we define $v := \log(F(u^*))$, and so

$$-\Delta v = -\frac{d^2}{du^2}\log(F(u))\Big|_{u=u^*}|\nabla u^*|^2 + \lambda^* F'(u^*) \qquad \text{in } \Omega,$$

with $v = 0$ on $\partial\Omega$. Since F is log convex, the first term on the right is negative. We now define w by

$$\begin{aligned} -\Delta w &= \lambda^* F'(u^*) && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and so $0 \le v(x) \le w(x)$ a.e. in Ω by the maximum principle. Using the linear estimates from Lemma 3.16 with $g(x) := \lambda^* F'(u^*)$ one has

$$\| \log \frac{1}{(1-u)^2} \|_{L^\infty} = \|v\|_{L^\infty} \le \|w\|_{L^\infty} \le \tilde{C}_\tau \lambda^* \|F'(u^*)\|_{L^\tau} = \tilde{C}_\tau \lambda^* \Big\| \frac{1}{1-u^*} \Big\|^3_{L^{3\tau}}.$$

Taking now $\tau = \frac{2t}{3} + 1 > \frac{N}{2}$, we can then replace the $L^\tau-$norm on the right by using the energy estimates from Lemma 3.17, which will give the desired conclusion. □

3.3.2 Upper estimates on radial domains

While the upper estimate on general domains obtained in the last subsection is quite satisfactory for the exponential nonlinearity, it is not so for the case of the MEMS nonlinearity. Indeed, using Maple one sees that if $\Omega := (0,1)^3$ the unit cube in $\mathbb{R}^3$ (and so $\lambda_1(\Omega) = 3\pi^2$), Formula (3.14) would then give that

$$\|u^*\|_{L^\infty} \le .993..., \tag{3.20}$$

which is clearly not a very good upper estimate. This is mainly due to the fact that we drop a potentially large term in the proof of Theorem 3.2, when we replaced v by w in order to apply the linear estimate of Lemma 3.2. Note that this was not needed for the exponential nonlinearity in the proof of Theorem 3.1.

In this section we examine radial domains, where better results are available on u^*, at least in the case of $F(u) = (1-u)^{-2}$. One can also examine the exponential nonlinearity using this approach but we won't do this since the last section seems to give satisfactory results. For simplicity, we shall also restrict our attention to the case of $f \equiv 1$. The main difference is that we use here Hölder estimates on linear equations versus the L^∞ estimates of the last subsection.

For the remainder of this section we assume that Ω is the unit ball B in $\mathbb{R}^N$ and $F(u) = (1-u)^{-2}$. We define the following parameter:

$$\gamma(\tau, N) := \begin{cases} \frac{\tau}{2\tau-1} & N=1 \\ \frac{\tau}{4(\tau-1)} & N=2 \\ \frac{(\tau-1)^{\frac{\tau-1}{\tau}}}{(N-2)N^{\frac{1}{\tau}}(2\tau-N)^{\frac{\tau-1}{\tau}}} & N \geq 3. \end{cases}$$

Lemma 3.21. *Let u denote a smooth radially decreasing solution of $-\Delta u = g(r) \geq 0$ in the unit ball B of $\mathbb{R}^N$. If $\max\{1, \frac{N}{2}\} < \tau < \infty$, then one has the estimate:*

$$u(0) \geq u(R) \geq u(0) - \frac{\gamma(\tau,N)\|g\|_{L^\tau}}{\omega_N^{\frac{1}{\tau}}} R^{2-\frac{N}{\tau}} \quad \textit{for all } R \in (0,1). \tag{3.22}$$

Proof. When $N=1$, we integrate the equation between 0 and r, and apply Hölder's inequality to obtain $-u'(r) \leq \frac{\|g\|_\tau r^{\frac{1}{\tau'}}}{2}$. Now integrate both terms between 0 and R, and use again Hölder's inequality to obtain the desired result.

When $N \geq 2$, we multiply the equation by r and integrate over $(0,R)$ to arrive at

$$R(-u'(R)) + (N-2)(u(0)-u(R)) = \int_0^R rg(r)dr.$$

If now $N=2$, then one has

$$R(-u'(R)) = \int_0^R rg(r)dr \leq \frac{\|g\|_{L^\tau} R^{\frac{2}{\tau'}}}{2\pi^{1-\frac{1}{\tau'}}}.$$

Dividing by R and integrating the result over $(0,R)$ gives the claim.

Now take $N \geq 3$. Since $-u'(R) \geq 0$ we can drop a term to arrive at

$$\begin{aligned}(N-2)(u(0)-u(R)) \leq \int_0^R rg(r)dr &= \frac{1}{N\omega_N}\int_{B_R}\frac{g(x)}{|x|^{N-2}}dx \\ &\leq \frac{\|g\|_{L^\tau}}{N\omega_N}\left(\int_{B_R}\frac{1}{|x|^{(N-2)\tau'}}dx\right)^{\frac{1}{\tau'}},\end{aligned}$$

and then use Lemma 3.15 to evaluate the integral on the right and finish the proof. □

We now come to the result which will yield our upper estimates on u^*.

Theorem 3.23. *Suppose u is a smooth semi-stable solution of (P_λ) on the unit ball B in $\mathbb{R}^N$, where $1 \le N \le 11$. Then, for* $\max\left\{0, \frac{N-3}{2}\right\} < t < 2+\sqrt{6}$, *we have*

$$\int_0^1 \frac{R^{N-1}\, dR}{\left(1 - \|u\|_{L^\infty} + \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}} R^{\frac{4t+6-2N}{2t+3}}\right)^{2t+3}} \le M, \quad (3.24)$$

where

$$M := \frac{1}{N}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}}.$$

Remark 3.3.2. Note that the above theorem only shows that $\|u\|_{L^\infty}$ is bounded away from 1 if $4t+6-2N \ge 2N-1$ which, once coupled with the other condition on t cannot be satisfied in the higher dimensions. This is to be expected since the extremal solution u^* satisfies $u^*(0) = 1$ for $N \ge 8$.

Proof. Suppose u is a smooth semi-stable (so radial) solution of (P_λ). Then, the above linear estimate applied with $g(r) := \lambda(1-u)^{-2}$, gives that for all $R \in (0,1)$,

$$1 - u(R) \le 1 - u(0) + \frac{\lambda\gamma(\tau, N)\|(1-u)^{-2}\|_{L^\tau} R^{2-\frac{N}{2}}}{\omega_N^{\frac{1}{\tau}}}.$$

Now use the upper bound $\lambda^* \le \frac{4\lambda_1(\Omega)}{27}$ from Proposition 3.2.1, take $\tau = t+\frac{3}{2}$, and replace the L^τ−norm on the right via the energy estimate from Lemma 3.17, to obtain

$$1 - u(R) \le 1 - u(0) + \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}} R^{\frac{4t+6-2N}{2t+3}}.$$

This yields the inequality

$$N\omega_N \int_0^1 \frac{R^{N-1}\, dR}{\left(1 - \|u\|_{L^\infty} + \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}} R^{\frac{4t+6-2N}{2t+3}}\right)^{2t+3}} \le M_0$$

where

$$M_0 := N\omega_N \int_0^1 \frac{R^{N-1}\, dR}{(1-u(R))^{2t+3}}.$$

But the right hand side is actually equal to $\|(1-u)^{-2}\|_{L^{t+\frac{3}{2}}}^{t+\frac{3}{2}}$, hence we can again use the energy estimate from Lemma 3.17 to majorize it and complete the proof.

□

Remark 3.3.3. *Using Maple to approximate the integral in (3.24) while optimizing over t, we get the following estimates on the extremal solution u^* of (P_λ) on the unit ball in $\mathbb{R}^N$.*

1. *If $N = 1$, then $\|u^*\|_{L^\infty} \leq .49...$*
2. *If $N = 2$, then $\|u^*\|_{L^\infty} \leq .55...$*

We note here that even though these estimates are not better than those obtained numerically (for example in [22] where $\|u\|_{L^\infty} = .4444...$), the idea here is that we only use elementary analysis that is also applicable to other domains and nonlinearities.

We now obtain some explicit upper bounds on u^* in dimensions $N = 1, 2$. For that, we define

$$C(t,N) := \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}}.$$

Corollary 3.25. *Suppose u^* is the extremal solution of (P_λ) on the unit ball in $\mathbb{R}^N$.*

1. *If $N = 1$, then*

$$\|u^*\|_{L^\infty} \leq 1 - \sup\left\{M_1 : 0 < t < 2+\sqrt{6}\right\},$$

where

$$M_1 := \left(2C(t,1)(t+1)\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}} + \frac{1}{C(t,1)^{2+2t}}\right)^{\frac{-1}{2t+2}}.$$

2. *If $N = 2$, then*

$$\|u^*\|_{L^\infty} \leq 1 - \sup\left\{M_2 : \frac{1}{2} \leq t < 2+\sqrt{6}\right\},$$

where

$$M_2 := \left(C(t,2)^2(t+1)\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}} + \frac{2t+2}{C(t,2)^{2t+1}}\right)^{\frac{-1}{2t+1}}.$$

Proof. 1) For $0 < t < 2+\sqrt{6}$ one has $\frac{4t+6-2N}{2t+3} \geq 1$ and so we can replace the power on R in (3.24) by 1, so as to be able to explicitly calculate the integral in (3.24). One then drops a few positive terms to arrive at the desired result.

2) For $\frac{1}{2} \leq t < 2+\sqrt{6}$ one has $\frac{4t+6-2N}{2t+3} \geq 1$, so again we replace the power on R in (3.24) by 1 and carry on as in the first part. □

3.4 Effect of power-law profiles on pull-in distances

Our goal in this section is to study the effect of power-like permittivity profiles $f(x) = |x|^\alpha$ on the problem $(P_{\lambda,\alpha})$ (our notation for $(P_{\lambda,|x|^\alpha})$) on the unit ball $B = B_1(0)$. Numerical results – in particular those obtained by Guo, Pan and Ward in [22] for MEMS nonlinearities– give lots of information, but the most intriguing one is their observation that on a 2-dimensional disc, the pull-in distance does not depend on α, at least in the case where $F(u) = (1-u)^{-2}$, and that the solution develops a boundary-layer structure near the boundary of the domain as α is increased. In other words, the L^∞−norm of the extremal solution of $(M_{\lambda,\alpha})$ is independent of $\alpha \geq 0$. In this section, we shall give a simple proof of this observation and other interesting phenomena, which actually holds true for more general nonlinearities.

We first observe that since $r \mapsto r^\alpha$ is increasing, the moving plane method of Gidas, Ni and Nirenberg [21] does not guarantee the radial symmetry of all solutions to $(S_{\lambda,f})$.

However if ones assumes the solution is minimal or semi-stable then the solution is indeed radial, see for instance [19]. Similar results are proven in [1]. We include the proof from [19] for the readers convenience.

Proposition 3.4.1. *Let Ω be a radially symmetric domain and assume f is a radial profile on Ω. Then, the minimal solutions of $(P_{\lambda,f})$ on Ω are necessarily radially symmetric and consequently*

$$\lambda^*(\Omega, f) = \lambda_r^*(\Omega, f) = \sup\left\{\lambda;\ (P_{\lambda,f}) \text{ has a radial solution}\right\}.$$

Moreover, if Ω is a ball, then any radial solution of $(P_{\lambda,f})$ attains its maximum at 0.

Proof. It is clear that $\lambda_r^*(\Omega, f) \leq \lambda^*(\Omega, f)$, and the reverse will be proved if we establish that every minimal solution of $(P_{\lambda,f})$ with $0 < \lambda < \lambda^*(\Omega, f)$ is radially symmetric. The recursive linear scheme that is used to construct the minimal solutions, gives a radial function at each step, and the resulting limiting function is therefore radially symmetric.

For a solution $u(r)$ on the ball of radius R, we have $u_r(0) = 0$ and

$$-u_{rr} - \frac{N-1}{r} u_r = \lambda f(r) F(u) \quad \text{in} \quad (0, R)\,.$$

Hence, $-\frac{d(r^{N-1}u_r)}{dr} = \lambda f(r) r^{N-1} F(u) \geq 0$, and therefore $u_r < 0$ in $(0, R)$ since $u_r(0) = 0$. This shows that $u(r)$ attains its maximum at 0, and that – just as in the case where $f \equiv 1$ – we have $\|u^*\|_\infty = u^*(0)$. □

It follows from this proposition that for radially symmetric domains Ω and profiles f, the extremal solution u^* is necessarily radially symmetric and that the pull-in distance is nothing but $u^*(0)$. We shall denote by $\lambda^*_\alpha(N)$ (resp., u^*_α) the pull-in voltage (resp., the extremal solution) of $(P_{\lambda,f})$ when $f(x) = |x|^\alpha$, and Ω is the unit ball in $\mathbb{R}^N$. Given a radial function u we define, the possibly, fractional n dimensional Laplacian of u by $\Delta_n u = u'' + \frac{n-1}{r} u'$.

We now make the following crucial observation.

Proposition 3.4.2. *For any $\alpha > -2$, the change of variable $u(r) = w(r^{1+\frac{\alpha}{2}})$ gives a correspondence between the radially symmetric solutions of the equation*

$$\begin{cases} -\Delta_N u = \lambda(1+\frac{\alpha}{2})^2 |x|^\alpha F(u) & \text{in } B, \\ \quad u = 0 & \text{on } \partial B, \end{cases} \tag{3.26}$$

in dimension N and those of the equation

$$\begin{cases} -\Delta_{\frac{2(N+\alpha)}{2+\alpha}} w = \lambda F(w) & \text{in } B, \\ \quad w = 0 & \text{on } \partial B, \end{cases} \tag{3.27}$$

in – the potentially fractional – dimension $N(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$. Moreover, we have

$$\lambda^*_\alpha(N) = (1+\tfrac{\alpha}{2})^2 \lambda^*_0(N(\alpha)) \quad \text{and} \quad u^*_\alpha(r) = w^*(r^{1+\frac{\alpha}{2}}), \tag{3.28}$$

where u^*_α is the extremal solution for (3.26) and w^* is the extremal solution of (3.27).

Proof: A straightforward calculation gives that

$$\Delta_N u(r) + (1+\frac{\alpha}{2})^2 \lambda r^\alpha F(u(r)) = (1+\frac{\alpha}{2})^2 r^\alpha \Big(\Delta_{N(\alpha)} w(r^{1+\frac{\alpha}{2}}) + \lambda F(w(r^{1+\frac{\alpha}{2}}\Big),$$

where $N(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$. To finish the proof it suffices to use the fact that the extremal solution is unique in either problem, including the case of a fractional dimension. The change of variables will therefore take the extremal solution to the extremal solution. We leave the details to the interested reader.

□

The above transformation allows us to deduce many results for the case of a power-law profile, from corresponding ones associated to constant profiles. The fact that it preserves the L^∞-norm has consequences on the pull-in distance and on the role of the profile in the critical dimension. It does also give proofs for various intriguing phenomena displayed by the numerical results below, especially in the case of a two dimensional disc, where the transformation does not alter the dimension since then $N(\alpha) = 2$.

The following corollary summarizes these consequences.

Corollary 3.29. *With the above notations, the following hold:*

1. *For any dimension $N \geq 1$, we have for $\alpha >> 1$,*

$$\lambda_\alpha^*(N) \sim (1 + \frac{\alpha}{2})^2 \lambda_0^*(2). \tag{3.30}$$

2. *If $N = 2$, then*

$$\lambda_\alpha^*(2) = (1 + \tfrac{\alpha}{2})^2 \lambda_0^*(2) \quad \textit{and} \quad \|u_\alpha^*\|_{L^\infty} = \|u_0^*\|_{L^\infty} \textit{ for all } \alpha > -2. \tag{3.31}$$

Proof. 1) From the above proposition, we have $\lambda_\alpha^*(N) = (1+\frac{\alpha}{2})^2 \lambda_0^*(\frac{2N+2\alpha}{\alpha+2})$, and $\lambda_0^*(\frac{2N+2\alpha}{\alpha+2}) \sim \lambda_0^*(2)$ whenever α is large.
2) follows from the fact that for $N = 2$, we then have $N_\alpha = 2$ for each α which means that

$$\lambda_\alpha^*(2) = \frac{(\alpha+2)^2}{4} \lambda_0^*(2), \tag{3.32}$$

and the pull-in distance in dimension 2 on the ball is $\|u_\alpha^*\|_{L^\infty} = \|w^*\|_{L^\infty}$, where $u_\alpha^*(r) = w^*(r^{1+\frac{\alpha}{2}})$. The pull-in distance is therefore independent of α. □

Corollary 3.33. *The following estimates hold in a MEMS model with a power-law permittivity profile, i.e., if $F(u) = (1-u)^{-2}$ and $f(x) = |x|^\alpha$.*

1. *For any dimension $N \geq 1$, we have for $\alpha >> 1$,*

$$\lambda_\alpha^*(N) \sim 0.789(1 + \frac{\alpha}{2})^2. \tag{3.34}$$

2. *If* $N = 2$, *then*

$$\lambda^*_\alpha(2) = 0.789(1 + \tfrac{\alpha}{2})^2 \quad \textit{and} \quad \|u^*_\alpha\|_{L^\infty} = 0.445 \textit{ for all } \alpha > -2. \tag{3.35}$$

3. *If* $1 \leq N \leq 7$ *or if* $N \geq 8$ *and* $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, *then the extremal solution* u^*_α *of* $(M_{\lambda,\alpha})$ *on the ball is classical and the pull-in distance* $\|u^*_\alpha\|_{L^\infty} < 1$.

4. *If the dimension* $N \geq 8$, *and* $0 \leq \alpha \leq \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, *then the extremal solution is exactly* $u^*_\alpha(x) = 1 - |x|^{\frac{2+\alpha}{3}}$, *which means that*

$$\lambda^*_\alpha(N) = \tfrac{(2+\alpha)(3N+\alpha-4)}{9} \quad \textit{and} \quad \|u^*_\alpha\|_{L^\infty} = 1. \tag{3.36}$$

Proof. 1) and 2) follow from the above proposition and the fact that $\lambda^*_0(2) = 0.789$ and $\|u^*_0\|_{L^\infty} = 0.445$.

3) The extremal solution u^*_α of $(M_{\lambda,\alpha})$ is regular if and only if $\|u^*_\alpha\|_{L^\infty} = \|w^*\|_{L^\infty} < 1$, where w^* is the extremal solution for (M_λ) in dimension $N(\alpha)$. According to [19], this happens if $\frac{N(\alpha)}{2} < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$ which means that $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$.

4) Note first that $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ is a $H^1_0(B)-$weak solution of $(M_{\lambda,|x|^\alpha})$ for any $\alpha > 4 - 3N$. The voltage is then $\lambda_\alpha(N) = \frac{(2+\alpha)(3N+\alpha-4)}{9}$. Since now $\|u^*\|_{L^\infty} = 1$, then by the above proposition, it remains only to show that for all $\phi \in H^1_0(B)$,

$$\int_B |\nabla\phi|^2 \geq \int_B \frac{2\lambda|x|^\alpha}{(1-u^*)^3}\phi^2. \tag{3.37}$$

But Hardy's inequality gives for $N \geq 3$ that $\int_B |\nabla\phi|^2 \geq \frac{(N-2)^2}{4}\int_B \frac{\phi^2}{|x|^2}$ for any $\phi \in H^1_0(B)$, which means that (3.37) holds whenever $2\lambda_\alpha(N) \leq \frac{(N-2)^2}{4}$ or, equivalently, if $N \geq 8$ and $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. □

The above scaling has also the following direct consequences.

Corollary 3.38. *Suppose* F *is a regular nonlinearity and* $N < 10 + 4\alpha$, *then the extremal functional* u^*_α *of* $(P_{\lambda,\alpha})$ *on the ball is classical.*

Proof. Cabre and Capella [9] showed that the extremal solution on the ball is always bounded for $N \leq 9$. They were only interested in integer dimensions, but an inspection of their proof indicates that the same result holds for any fractional dimensions $N < 10$. Combining this with our observation in Proposition 3.4.2 completes the proof. To see that this is optimal one recalls that when $F(v) = e^v$ the extremal solution is unbounded in $N = 10$. Using this fact and the change of variables above yields the optimality of this result. □

Remark 3.4.1. One can also use this change of variables to study permittivity profiles with negative powers (i.e., $f(x) = |x|^\alpha$ for $0 > \alpha > -2$. For example suppose $F(u) = e^u$, then using the above change of variables, one can show that for a fixed N ($3 \leq N \leq 9$), the extremal solution associated with

$$-\Delta u = |x|^\alpha e^u \quad \text{ on } B,$$

is singular for $\alpha \in (-2, \frac{10-N}{4}]$, while it is a classical solution for $\alpha \in (\frac{10-N}{4}, 0)$.

3.5 Asymptotic behavior of stable solutions near the pull-in voltage

We now establish pointwise upper and lower estimates on the minimal solutions u_λ in terms of λ, λ^*, the extremal solution u^* and $\frac{d}{d\lambda}u_\lambda\big|_{\lambda=\lambda^*}$. For simplicity we restrict our attention to $F(u) = e^u$ and $F(u) = (1-u)^{-2}$. In addition we allow fractional dimensions for results on the unit ball since then, one can apply the results of the previous section to deal with power-law profiles $(P_{\lambda,\alpha})$. We first recall that on a ball, one can explicitly write the extremal solutions whenever they are singular. See for example [7, 9, 19].

- If $F(u) = e^u$, then $u^*(x) = \log(\frac{1}{|x|^2})$ is an extremal solution on the unit ball in $\mathbb{R}^N$ at $\lambda^* = 2N - 4$, provided $N \geq 10$.
- If $F(u) = \frac{1}{(1-u)^2}$, then $u^*(x) = u^*(x) = 1 - |x|^{\frac{2}{3}}$ is an extremal solution on the unit ball in $\mathbb{R}^N$ at $\lambda^* = \frac{6N-8}{9}$, provided $N \geq \frac{14+\sqrt{6}}{3}$.

Theorem 3.39. *Let u^* denote the extremal solution of (P_λ) on a smooth bounded domain Ω in $\mathbb{R}^N$.*

1. *If $F(u) = (1-u)^{-2}$, then for $0 < \lambda < \lambda^*$, we have*

$$u_\lambda(x) \leq \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{3}} u^*(x) \qquad \textit{for a.e. } x \in \Omega. \tag{3.40}$$

Moreover, if Ω *is the unit ball in* $\mathbb{R}^N$ *with* $N \geq \frac{14+4\sqrt{6}}{3} = 7.93...$, *then for* $0 < \lambda < \lambda^* = \frac{6N-8}{9}$ *we have*

$$1 - |x|^{\frac{2}{3}} - \frac{3(\lambda^* - \lambda)}{(6N-8)} \left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{9N^2-84N+100}}{6}} - 1 \right) \leq u_\lambda(x)$$

$$u_\lambda(x) \leq \left(\frac{\lambda}{\lambda^*} \right)^{\frac{1}{3}} (1 - |x|^{\frac{2}{3}}$$

for a.e. $x \in \Omega$.

2. *If* $F(u) = e^u$, *then for* $0 < \lambda < \lambda^*$,

$$u_\lambda(x) \leq \log \left(\frac{\lambda^*}{\lambda^* - \lambda + \lambda e^{-u^*}} \right) \qquad \textit{for a.e. } x \in \Omega. \tag{3.41}$$

Moreover, if Ω *is the unit ball in* $\mathbb{R}^N$ *with* $N \geq 10$, *then for* $0 < \lambda < \lambda^* = 2N - 4$ *we have*

$$\log(\frac{1}{|x|^2}) - \frac{(\lambda^* - \lambda)}{(2N-4)} \left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{N^2-12N+20}}{2}} - 1 \right) \leq u_\lambda(x)$$

$$u_\lambda(x) \leq \log \left(\frac{\lambda^*}{\lambda^* - \lambda + \lambda |x|^2} \right),$$

for a.e. $x \in \Omega$.

Proof. The upper estimates follow easily from the minimality of u_λ and the fact that $x \mapsto \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{3}} u^*(x)$ (resp., $x \mapsto \log\left(\frac{\lambda^*}{\lambda^*-\lambda+\lambda e^{-u^*}}\right)$ is a supersolution of (P_λ) in the case that $F(u) = (1-u)^{-2}$ (resp., $F(u) = e^u$).

For the lower bound, we shall proceed as follows: First, recall that $\lambda \mapsto u_\lambda$ is differentiable (this is a result of the Implicit Function Theorem) and increasing on $(0, \lambda^*)$, and so if one defines $v_\lambda := \frac{d}{d\lambda} u_\lambda$, then v_λ is positive and solves the linear equation

$$\begin{cases} -\Delta v = F(u_\lambda) + \lambda F'(u_\lambda) v & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

where, F is given by either e^u or $(1-u)^{-2}$. We shall need the following notion.

Definition 3.42. *An extremal solution* u^* *associated with* (P_λ) *is said to be* super-stable *provided there exists* $\varepsilon > 0$ *such that*

$$(\lambda^* + \varepsilon) \int_\Omega F'(u^*) \psi^2 \leq \int_\Omega |\nabla \psi|^2 \qquad \textit{for all } \psi \in H_0^1(\Omega).$$

Note that if u^* is a super-stable extremal solution then $\mu_1(\lambda^*, u^*) > 0$. We shall see at the end of this section that the converse is however not true. We shall need the following result, a part of which is contained in [11], where many other interesting properties of the mapping $\lambda \mapsto u_\lambda(x)$ are exhibited.

Lemma 3.43. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$. Then,*

1. *For $0 < \lambda < \lambda^*$, v_λ is the unique H_0^1-weak solution of (Q_λ).*
2. *$\lambda \mapsto v_\lambda$ is increasing on $(0, \lambda^*)$, and therefore $v^*(x) := \lim_{\lambda \to \lambda^*} v_\lambda(x)$ is defined for a.e. $x \in \Omega$.*
3. *$\lambda \mapsto u_\lambda$ is convex on $(0, \lambda^*)$, and therefore for $0 < \lambda < \lambda^*$ we have for a.e. $x \in \Omega$,*
$$u_\lambda(x) \geq u^*(x) + (\lambda - \lambda^*)v^*(x). \tag{3.44}$$
4. *If u^* is super-stable, then v^* is the unique H_0^1-weak solution of $(Q)_{\lambda^*}$.*

Proof. (1) One can use the fact that $\mu_1(\lambda, u_\lambda) \geq 0$, and a standard minimization argument to show the existence of an H_0^1-solution to (Q_λ). Using the fact that $\mu_1(\lambda, u_\lambda) > 0$ one can see that the solution is unique.

(2) Let $0 < \lambda < \lambda^*$ and $\varepsilon > 0$ small. Note first that

$$\begin{aligned} -\Delta(v_{\lambda+\varepsilon} - v_\lambda) &= F(u_{\lambda+\varepsilon}) - F(u_\lambda) + \varepsilon F'(u_{\lambda+\varepsilon})v_{\lambda+\varepsilon} \\ &\quad + \lambda F'(u_{\lambda+\varepsilon})v_{\lambda+\varepsilon} - \lambda F'(u_\lambda)v_\lambda \\ &= g(x) + \lambda F'(u_\lambda)(v_{\lambda+\varepsilon} - v_\lambda), \end{aligned}$$

where

$$g(x) := F(u_{\lambda+\varepsilon}) - F(u_\lambda) + \varepsilon F'(u_{\lambda+\varepsilon})v_{\lambda+\varepsilon} + \lambda\big(F'(u_{\lambda+\varepsilon})v_{\lambda+\varepsilon} - F'(u_\lambda)v_{\lambda+\varepsilon}\big)$$

is in $H^1(\Omega)$ and is positive. Now set $w := v_{\lambda+\varepsilon} - v_\lambda$ in such a way that w solves

$$\begin{aligned} -\Delta w &= g(x) + \lambda F'(u_\lambda)w && \text{on } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Testing this equation on w^- gives

$$-\int_\Omega g w^- \geq \mu_1(\lambda, u_\lambda) \int_\Omega (w^-)^2,$$

and hence $w^- = 0$ a.e. in Ω. By the maximum principle one then get that $w > 0$ in Ω and hence that $\lambda \to v_\lambda$ is increasing. We can therefore define the limit $v^*(x) := \lim_{\lambda\to\lambda^*} v_\lambda(x)$, which exists a.e. x in Ω, though it might be infinite on a large set.

(3) The convexity of $\lambda \mapsto u_\lambda$ follows from the fact that $\lambda \mapsto v_\lambda$ is increasing. We can therefore write $u_\lambda \geq u_t + (\lambda - t)v_t$ for $0 < \lambda, t < \lambda^*$ and a.e. $x \in \Omega$. The claim now follows by letting t go to λ^*.

(4) Since u^* is super-stable one has

$$(\lambda + \varepsilon)\int_\Omega F'(u_\lambda)\psi^2 \leq \int_\Omega |\nabla\psi|^2 \qquad \forall \psi \in H_0^1.$$

Using this and testing (Q_λ) on v_λ gives

$$\varepsilon\int_\Omega F'(u_\lambda)v_\lambda^2 \leq \int_\Omega F(u_\lambda)v_\lambda.$$

Since F is either $F(u) = e^u$ or $F(u) = (1-u)^{-2}$, the left hand side is necessarily bounded. From this and again by testing (Q_λ) on v_λ one sees that v_λ is bounded in H_0^1. Passing to limits, one sees that v^* is a H_0^1-weak solution of (Q_{λ^*}). The uniqueness follows from the fact that $\mu_1(\lambda^*, u^*) > 0$. □

We now complete the proof of Theorem 3.39. For that we assume that Ω is the unit ball in $\mathbb{R}^N$. It is then easy to show using Hardy's inequality that the explicit extremal solutions for (P_λ) given above, are super-stable provided $N > 10$ (resp., $N > \frac{14+4\sqrt{6}}{3} = 7.93...$) when $F(u) = e^u$ (resp., $F(u) = (1-u)^{-2}$). An easy calculation also shows that

$$v^*(x) = \frac{1}{2N-4}\left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{N^2-12N+20}}{2}} - 1\right),$$

(when $F(u) = e^u$) resp.,

$$v^*(x) = \frac{3}{6N-8}\left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{9N^2-84N+100}}{6}} - 1\right),$$

(when $F(u) = (1-u)^{-2}$) are H_0^1-weak solutions of $(Q)_{\lambda^*}$ in the respective cases, assuming the dimension restrictions above. Using this and the earlier convexity result gives the desired lower bounds for $N > 10$ ($N > \frac{14+4\sqrt{6}}{3}$) in the exponential and MEMS cases respectively. To obtain the result for the critical dimensions one passes to the limit in N. We omit the details. □

Bibliography

[1] N.D. Alikakos, and P.W. Bates, *On the singular limit in a phase field model of phase transitions.* Analyse Nonlinéaire, Ann. Inst. Henri Poincaré **5** (1988), 141-178.

[2] G. Arioli, F. Gazzola, H.C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. **36**, 1226-1258 (2005).

[3] L. Ben Chaabane,*On the extremal solutions of semilinear elliptic problems*, Abstr. Appl. Anal. 2005, no. 1, 1-9.

[4] E. Berchio and F. Gazzola, *Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities*, Electronic Journal of Differential Equations, Vol. 2005(2005), No. 34, pp. 120.

[5] A. Brandt, *Interior estimates for second-order elliptic differential (or finite-difference) equations via the maximum principle*, Israel J. Math. 7 1969 95-121.

[6] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa *Blow up for* $u_t - \Delta u = g(u)$ *revisited,* Ad. Diff. Eq. **1** (1996), 73-90.

[7] H. Brezis and L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 443–469.

[8] X. Cabré, *Regularity of minimizers of semilinear elliptic problems up to dimension four*, (preprint) (2009).

[9] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. 238 (2006), no. 2, 709–733.

[10] D. Cassani, J. do O and N. Ghoussoub, *On a fourth order elliptic problem with a singular nonlinearity*, Advances Nonlinear Studies, **9**, (2009), 177-197

[11] T. Cazenave, M. Escobedo and M. Assunta Pozio *Some stability properties for minimal solutions of* $-\Delta u = \lambda g(u)$. Port. Math. (N.S.) 59 (2002), no. 4, 373-391.

[12] C. Cowan, P. Esposito and N. Ghoussoub, *Regularity of Extremal Solutions in Fourth Order Nonlinear Eigenvalue Problems on General Domains*, preprint (2010).

[13] C. Cowan, P. Esposito, N. Ghoussoub, A. Mordifam, *The critical dimension for a fourth order elliptical problem with singular nonlinearity*, Arch. Ration. Mech. Anal., In press (2009) 19 pp

[14] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variation methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal., 58 (1975), pp.207-218.

[15] J. Davila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. 39 (2007), 565-592.

[16] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731–1768.

[17] P. Esposito, N. Ghoussoub, Y. J. Guo: *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Research Monograph, In press (2009) 260 p.

[18] F. Gazzola and H.-Ch. Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, NoDEA 8 (2001), 35-44.

[19] N. Ghoussoub and Y. Guo, *On the partial differential equations of electro MEMS devices: stationary case*, SIAM J. Math. Anal. 38 (2007), 1423-1449.

[20] N. Ghoussoub and A. Moradifam, *Bessel Pairs and Optimal Hardy and Hardy-Rellich Inequalities*, (preprint) 2008

[21] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[22] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a mems device with varying dielectric properties*, SIAM J. Appl. Math 66 (2005), 309-338.

[23] D. Joseph and T. Lundgren, *Quasilinear Dirichlet problems driven by positive source, Arch. Rational Mech. Anal.* **49** (1973), 241269.

[24] F.H. Lin and Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic acutation*, Proc. R. Soc. London, Ser. A 463 (2007), 1323-1337.

[25] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, J. Diff. Eq. **16** (1974), 103-125.

[26] F. Mignot and J-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations 5 (1980), 791-836.

[27] G., Nedev, *Regularity of the extremal solution of semilinear elliptic equations,* C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), no. 11, 997–1002.

[28] J.A. Pelesko, *Mathematical modeling of electrostatic mems with tailored dielectric properties*, SIAM J. Appl. Math. 62 (2002), 888-908.

[29] J.A. Pelesko and A.A. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

[30] L. Vazquez and E. Zuazua, *The Hardy Inequality and the Asymptotic Behaviour of the Heat Equation with an Inverse-Square Potential*, J. Funct. Anal. 173, 103-153 (2000).

[31] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems,* Commun. Contemp. Math. **4** (2002), no. 3, p. 547-558.

Chapter 4

The critical dimension for a fourth order elliptic problem with singular nonlinearity[3]

4.1 Introduction

The following model has been proposed for the description of the steady-state of a simple Electrostatic MEMS device:

$$\begin{cases} \alpha\Delta^2 u = \left(\beta\int_\Omega |\nabla u|^2 dx + \gamma\right)\Delta u + \frac{\lambda f(x)}{(1-u)^2\left(1+\chi\int_\Omega \frac{dx}{(1-u)^2}\right)} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \alpha\partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\alpha, \beta, \gamma, \chi \geq 0$, $f \in C(\overline{\Omega}, [0,1])$ are fixed, Ω is a bounded domain in $\mathbb{R}^N$ and $\lambda \geq 0$ is a varying parameter (see for example Bernstein and Pelesko [20]). The function $u(x)$ denotes the height above a point $x \in \Omega \subset \mathbb{R}^N$ of a dielectric membrane clamped on $\partial\Omega$, once it deflects torwards a ground plate fixed at height $z = 1$, whenever a positive voltage – proportional to λ – is applied.

In studying this problem, one typically makes various simplifying assumptions on the parameters $\alpha, \beta, \gamma, \chi$, and the first approximation of (4.1) that has been studied extensively so far is the equation

$$(S)_{\lambda,f} \qquad \begin{cases} -\Delta u = \lambda\frac{f(x)}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have set $\alpha = \beta = \chi = 0$ and $\gamma = 1$ (see for example [7, 9, 10] and the monograph [8]). This simple model, which lends itself to the

[3] A version of this chapter has been accepted for publication: C. Cowan, P. Esposito, N. Ghoussoub and A. Mordifam: *The critical dimension for a fourth order elliptical problem with singular nonlinearity*, Arch. Ration. Mech. Anal., In press (2009).

vast literature on second order semilinear eigenvalue problems, is already a rich source of interesting mathematical problems. The case when the "permittivity profile" f is constant ($f = 1$) on a general domain was studied in [17], following the pioneering work of Joseph and Lundgren [14] who had considered the radially symmetric case. The case for a non constant permittivity profile f was advocated by Pelesko [19], taken up by [12], and studied in depth in [7, 9, 10]. The starting point of the analysis is the existence of a pull-in voltage $\lambda^*(\Omega, f)$, defined as

$$\lambda^*(\Omega, f) := \sup\Big\{\lambda > 0 : \text{there exists a classical solution of } (S)_{\lambda,f}\Big\}.$$

It is then shown that for every $0 < \lambda < \lambda^*$, there exists a smooth minimal (smallest) solution of $(S)_{\lambda,f}$, while for $\lambda > \lambda^*$ there is no solution even in a weak sense. Moreover, the branch $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in \Omega$, and therefore the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution – which, by elliptic regularity theory, is equivalent to whether $\sup_\Omega u^* < 1$ – is an important question for many reasons, not the least of which being the fact that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*). This issue turned out to depend closely on the dimension and on the permittivity profile f. Indeed, it was shown in [10] that u^* is regular in dimensions $1 \le N \le 7$, while it is not necessarily the case for $N \ge 8$. In other words, the dimension $N = 7$ is critical for equation $(S)_\lambda$ (when $f = 1$, we simplify the notation $(S)_{\lambda,1}$ into $(S)_\lambda$). On the other hand, it is shown in [9] that the regularity of u^* can be restored in any dimension, provided we allow for a power law profile $|x|^\eta$ with η large enough.

The case where $\beta = \gamma = \chi = 0$ (and $\alpha = 1$) in the above model, that is when we are dealing with the following fourth order analog of $(S)_\lambda$

$$(P)_\lambda \qquad \begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

was also considered by [4, 15] but with limited success. One of the reasons is the lack of a "maximum principle" which plays such a crucial role in developing the theory for the Laplacian. Indeed, it is a well known fact that such a principle does not normally hold for general domains Ω (at least for the clamped boundary conditions $u = \partial_\nu u = 0$ on $\partial\Omega$) unless one restricts attention to the unit ball $\Omega = B$ in $\mathbb{R}^N$, where one can exploit a positivity preserving property of Δ^2 due to T. Boggio [3]. This is precisely what was done in the references mentioned above, where a theory of the minimal branch associated with $(P)_\lambda$ is developed along the same lines as for $(S)_\lambda$. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties. This means that the methods used to analyze the regularity of the extremal solution for $(S)_\lambda$ could not carry to the corresponding problem for $(P)_\lambda$.

This is the question we address in this paper as we eventually show the following result.

Theorem 4.2. *The unique extremal solution u^* for $(P)_{\lambda^*}$ in B is regular in dimension $1 \leq N \leq 8$, while it is singular (i.e, $\sup_B u^* = 1$) for $N \geq 9$.*

In other words, the critical dimension for $(P)_\lambda$ in B is $N = 8$, as opposed to being equal to 7 in $(S)_\lambda$. We add that our methods are heavily inspired by the recent paper of Davila et al. [5] where it is shown that $N = 12$ is the critical dimension for the fourth order nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B, \end{cases}$$

while the critical dimension for its second order counterpart (i.e., the Gelfand problem) is $N = 9$. There is, however, one major difference between our approach and the one used by Dávila et al. [5]. It is related to the most delicate dimensions – just above the critical one – where they use a computer assisted proof to establish the singularity of the extremal solution, while our method is more analytical and relies on improved and non standard Hardy-Rellich inequalities recently established by Ghoussoub-Moradifam [11].

We remark that very recently, see [6], Dávila et al. examined the question of the multiplicity of solutions to the problem

$$\begin{cases} \Delta^2 u = \lambda(1 + sign(p)u)^p & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B. \end{cases}$$

In our case, where $p = -2$, they have shown that for $3 \leq N \leq 8$ there exists a critical parameter $0 < \lambda_S < \lambda^*$ where the problem has an infinite number of smooth solutions and also a singular solution.

Throughout this paper, we will always consider problem $(P)_\lambda$ on the unit ball B. We start by recalling some of the results from [4] concerning $(P)_\lambda$, that will be needed in the sequel. We define

$$\lambda^* := \sup \Big\{ \lambda > 0 : \text{ there exists a classical solution of } (P)_\lambda \Big\},$$

and note that we are not restricting our attention to radial solutions. We will deal also with weak solutions defined as follows.

Definition 4.3. *We say that u is a weak solution of $(P)_\lambda$ if $0 \le u \le 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and*

$$\int_B u\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B).$$

We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_\lambda$ if the equality is replaced with the inequality $\ge$ (resp. $\le$) for all $\phi \in C^4(\bar{B}) \cap H_0^2(B)$ with $\phi \ge 0$.

We also recall the notions of regularity and stability.

Definition 4.4. *Say that a weak solution u of $(P)_\lambda$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $=$) and stable (resp. semi-stable) if*

$$\mu_1(u) = \inf \left\{ \int_B (\Delta\phi)^2 - 2\lambda \int_B \frac{\phi^2}{(1-u)^3} : \phi \in H_0^2(B), \|\phi\|_{L^2} = 1 \right\}$$

is positive (resp. non-negative).

The following extension of Boggio's principle will be frequently used in the sequel (see [2, Lemma 16] and [5, Lemma 2.4]).

Lemma 4.5 (Boggio's Principle)**.** *Let $u \in L^1(B)$. Then $u \ge 0$ a.e. in B, provided one of the following conditions hold:*

1. *$u \in C^4(\overline{B})$, $\Delta^2 u \ge 0$ on B, and $u = \frac{\partial u}{\partial n} = 0$ on ∂B.*
2. *$\int_B u\Delta^2\phi\, dx \ge 0$ for all $0 \le \phi \in C^4(\overline{B}) \cap H_0^2(B)$.*
3. *$u \in H^2(B)$, $u = 0$ and $\frac{\partial u}{\partial n} \le 0$ on ∂B, and $\int_B \Delta u \Delta\phi \ge 0$ for all $0 \le \phi \in H_0^2(B)$.*

Moreover, either $u \equiv 0$ or $u > 0$ a.e. in B.

The following theorem summarizes the main results in [4] that will be needed in the sequel.

Theorem 4.6. *The following assertions hold:*

1. *For each $0 < \lambda < \lambda^*$, there exists a classical minimal solution u_λ of $(P)_\lambda$. Moreover u_λ is radial and radially decreasing.*

2. *For $\lambda > \lambda^*$, there are no weak solutions of $(P)_\lambda$.*

3. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*

4. *The pull-in voltage λ^* satisfies the following bounds:*

$$\max\left\{\frac{32(10N - N^2 - 12)}{27}, \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})\right\} \leq \lambda^* \leq \frac{4\nu_1}{27}$$

   where ν_1 denotes the first eigenvalue of Δ^2 in $H_0^2(B)$.

5. *For each $0 < \lambda < \lambda^*$, u_λ is a stable solution (i.e., $\mu_1(u_\lambda) > 0$).*

Using the stability of u_λ, it can be shown that u_λ is uniformly bounded in $H_0^2(B)$ and that $\frac{1}{1-u_\lambda}$ is uniformly bounded in $L^3(B)$. Since now $\lambda \mapsto u_\lambda(x)$ is increasing, the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is well defined (in the pointwise sense), $u^* \in H_0^2(B)$, $\frac{1}{1-u^*} \in L^3(B)$ and u^* is a weak solution of $(P)_{\lambda^*}$. Moreover u^* is the unique weak solution of $(P)_{\lambda^*}$.

4.2 The effect of boundary conditions on the pull-in voltage

As in [5], we are led to examine problem $(P)_\lambda$ with non-homogeneous boundary conditions such as

$$(P)_{\lambda,\alpha,\beta} \qquad \begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } B \\ \alpha < u < 1 & \text{in } B \\ u = \alpha\,,\ \partial_\nu u = \beta & \text{on } \partial B, \end{cases}$$

where α, β are given.

Notice first that some restrictions on α and β are necessary. Indeed, letting $\Phi(x) := (\alpha - \frac{\beta}{2}) + \frac{\beta}{2}|x|^2$ denote the unique solution of

$$\begin{cases} \Delta^2 \Phi = 0 & \text{in } B \\ \Phi = \alpha\,,\ \partial_\nu \Phi = \beta & \text{on } \partial B, \end{cases} \tag{4.7}$$

we infer immediately from Lemma 4.5 that the function $u - \Phi$ is positive in B, which yields to

$$\sup_B \Phi < \sup_B u \leq 1.$$

To insure that Φ is a classical sub-solution of $(P)_{\lambda,\alpha,\beta}$, we impose $\alpha \neq 1$ and $\beta \leq 0$, and condition $\sup\limits_B \Phi < 1$ rewrites as $\alpha - \frac{\beta}{2} < 1$. We will then say that the pair (α,β) is *admissible* if $\beta \leq 0$, and $\alpha - \frac{\beta}{2} < 1$.

This section will be devoted to obtaining results for $(P)_{\lambda,\alpha,\beta}$ when (α,β) is an admissible pair, which are analogous to those for $(P)_\lambda$. To cut down on notation, we shall sometimes drop α and β from our expressions whenever such an emphasis is not needed. For example in this section u_λ and u^* will denote the minimal and extremal solution of $(P)_{\lambda,\alpha,\beta}$.

The notion of weak solution for $(P)_{\lambda,\alpha,\beta}$ is defined as follows.

Definition 4.8. *We say that u is a weak solution of $(P)_{\lambda,\alpha,\beta}$ if $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B (u-\Phi)\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (4.7). We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_{\lambda,\alpha,\beta}$ if the equality is replaced with the inequality $\geq$ (resp. $\leq$) for $\phi \geq 0$.

We now define as before

$$\lambda^*(\alpha,\beta) := \sup\{\lambda > 0 : (P)_{\lambda,\alpha,\beta} \text{ has a classical solution}\}$$

and

$$\lambda_*(\alpha,\beta) := \sup\{\lambda > 0 : (P)_{\lambda,\alpha,\beta} \text{ has a weak solution}\}.$$

Observe that by the Implicit Function Theorem, one can always solve $(P)_{\lambda,\alpha,\beta}$ for small λ's. Therefore, λ^* (and also λ_*) is well defined.

Let now U be a weak super-solution of $(P)_{\lambda,\alpha,\beta}$. Recall the following standard existence result.

Theorem 4.9 ([2])**.** *For every $0 \leq f \in L^1(B)$, there exists a unique $0 \leq u \in L^1(B)$ which satisfies*

$$\textstyle\int_B u\Delta^2\phi = \int_B f\phi, \quad \text{for all } \phi \in C^4(\bar{B}) \cap H_0^2(B).$$

We can now introduce the following "weak iterative scheme": Start with $u_0 = U$ and (inductively) let u_n, $n \geq 1$, be the solution of

$$\int_B (u_n - \Phi)\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u_{n-1})^2} \qquad \forall\, \phi \in C^4(\bar{B}) \cap H_0^2(B)$$

given by Theorem 4.9. Since 0 is a sub-solution of $(P)_{\lambda,\alpha,\beta}$, one can easily show inductively by using Lemma 4.5 that $\alpha \le u_{n+1} \le u_n \le U$ for every $n \ge 0$. Since

$$(1-u_n)^{-2} \le (1-U)^{-2} \in L^1(B),$$

we get by Lebesgue Theorem, that the function $u = \lim\limits_{n\to+\infty} u_n$ is a weak solution of $(P)_{\lambda,\alpha,\beta}$ such that $\alpha \le u \le U$. In other words, the following result holds.

Proposition 4.2.1. *Assume the existence of a weak super-solution U of $(P)_{\lambda,\alpha,\beta}$. Then there exists a weak solution u of $(P)_{\lambda,\alpha,\beta}$ so that $\alpha \le u \le U$ a.e. in B.*

In particular, we can find a weak solution of $(P)_{\lambda,\alpha,\beta}$ for every $\lambda \in (0,\lambda_*)$. Now we show that this is still true for regular weak solutions.

Proposition 4.2.2. *Let (α,β) be an admissible pair and let u be a weak solution of $(P)_{\lambda,\alpha,\beta}$. Then for every $0<\mu<\lambda$, there is a regular solution for $(P)_{\mu,\alpha,\beta}$.*

Proof. Let $\varepsilon \in (0,1)$ be given and let $\bar u = (1-\varepsilon)u + \varepsilon\Phi$, where Φ is given in (4.7). We have that

$$\sup_B \bar u \le (1-\varepsilon) + \varepsilon \sup_B \Phi < 1\,, \quad \inf_B \bar u \ge (1-\varepsilon)\alpha + \varepsilon \inf_B \Phi = \alpha,$$

and for every $0 \le \phi \in C^4(\bar B)\cap H_0^2(B)$ there holds:

$$\begin{aligned}\int_B (\bar u - \Phi)\Delta^2\phi &= (1-\varepsilon)\int_B (u-\Phi)\Delta^2\phi = (1-\varepsilon)\lambda \int_B \frac{\phi}{(1-u)^2}\\ &= (1-\varepsilon)^3\lambda \int_B \frac{\phi}{(1-\bar u+\varepsilon(\Phi-1))^2}\\ &\ge (1-\varepsilon)^3\lambda \int_B \frac{\phi}{(1-\bar u)^2}.\end{aligned}$$

Note that $0 \le (1-\varepsilon)(1-u) = 1-\bar u + \varepsilon(\Phi-1) < 1-\bar u$. So $\bar u$ is a weak super-solution of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ satisfying $\sup\limits_B \bar u < 1$. From Proposition 4.2.1 we get the existence of a weak solution w of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ so that $\alpha \le w \le \bar u$. In particular, $\sup\limits_B w < 1$ and w is a regular weak solution. Since $\varepsilon \in (0,1)$ is arbitrarily chosen, the proof is complete. □

Proposition 4.2.2 implies in particular the existence of a regular weak solution U_λ for every $\lambda \in (0, \lambda_*)$. Introduce now a "classical" iterative scheme: $u_0 = 0$ and (inductively) $u_n = v_n + \Phi$, $n \geq 1$, where $v_n \in H_0^2(B)$ is the (radial) solution of

$$\Delta^2 v_n = \Delta^2(u_n - \Phi) = \frac{\lambda}{(1-u_{n-1})^2} \qquad \text{in } B. \tag{4.10}$$

Since $v_n \in H_0^2(B)$, u_n is also a weak solution of (4.10), and by Lemma 4.5 we know that $\alpha \leq u_n \leq u_{n+1} \leq U_\lambda$ for every $n \geq 0$. Since $\sup_B u_n \leq \sup_B U_\lambda < 1$ for $n \geq 0$, we get that $(1-u_{n-1})^{-2} \in L^2(B)$ and the existence of v_n is guaranteed. Since v_n is easily seen to be uniformly bounded in $H_0^2(B)$, we have that $u_\lambda := \lim_{n\to+\infty} u_n$ does hold pointwise and weakly in $H^2(B)$. By Lebesgue Theorem, we have that u_λ is a radial weak solution of $(P)_{\lambda,\alpha,\beta}$ so that $\sup_B u_\lambda \leq \sup_B U_\lambda < 1$. By elliptic regularity theory [1], we have $u_\lambda \in C^\infty(\bar{B})$ and $u_\lambda - \Phi = \partial_\nu(u_\lambda - \Phi) = 0$ on ∂B. So we can integrate by parts to get

$$\int_B \Delta^2 u_\lambda \phi = \int_B \Delta^2(u_\lambda - \Phi)\phi = \int_B (u_\lambda - \Phi)\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u_\lambda)^2},$$

for every $\phi \in C^4(\bar{B}) \cap H_0^2(B)$. Hence, u_λ is a radial classical solution of $(P)_{\lambda,\alpha,\beta}$ showing that $\lambda^* = \lambda_*$. Moreover, since Φ and $v_\lambda := u_\lambda - \Phi$ are radially decreasing in view of [21], we get that u_λ is radially decreasing too. Since the argument above shows that $u_\lambda < U$ for any other classical solution U of $(P)_{\mu,\alpha,\beta}$ with $\mu \geq \lambda$, we have that u_λ is exactly the minimal solution and u_λ is strictly increasing as $\lambda \uparrow \lambda^*$. In particular, we can define u^* in the usual way: $u^*(x) = \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$.

Finally, we show the finiteness of the pull-in voltage.

Lemma 4.11. *If (α, β) is an admissible pair, then $\lambda^*(\alpha,\beta) < +\infty$.*

Proof. Let u be a classical solution of $(P)_{\lambda,\alpha,\beta}$ and let (ψ, ν_1) denote the first eigenpair of Δ^2 in $H_0^2(B)$ with $\psi > 0$. Now, let C be such that

$$\int_{\partial B} (\beta\Delta\psi - \alpha\partial_\nu\Delta\psi) = C\int_B \psi.$$

Multiplying $(P)_{\lambda,\alpha,\beta}$ by ψ and then integrating by parts one arrives at

$$\int_B \left(\frac{\lambda}{(1-u)^2} - \nu_1 u - C\right)\psi = 0.$$

Since $\psi > 0$ there must exist a point $\bar{x} \in B$ where $\frac{\lambda}{(1-u(\bar{x}))^2} - \nu_1 u(\bar{x}) - C \leq 0$. Since $\alpha < u(\bar{x}) < 1$, one can conclude that $\lambda \leq \sup_{\alpha<u<1}(\nu_1 u + C)(1-u)^2$, which shows that $\lambda^* < +\infty$. □

The following summarizes what we have shown so far.

Theorem 4.12. *If (α, β) is an admissible pair, then $\lambda^* := \lambda^*(\alpha, \beta) \in (0, +\infty)$ and the following hold:*

1. *For each $0 < \lambda < \lambda^*$ there exists a classical, minimal solution u_λ of $(P)_{\lambda,\alpha,\beta}$. Moreover u_λ is radial and radially decreasing.*
2. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*
3. *For $\lambda > \lambda^*$ there are no weak solutions of $(P)_{\lambda,\alpha,\beta}$.*

4.2.1 Stability of the minimal branch of solutions

This section is devoted to the proof of the following stability result for minimal solutions. We shall need yet another notion of $H^2(B)-$weak solutions, which is an intermediate class between classical and weak solutions.

Definition 4.13. *We say that u is a $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2(B)$, $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B \Delta u \Delta \phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (4.7). We say that u is a $H^2(B)-$weak super-solution (resp. $H^2(B)-$weak sub-solution) of $(P)_{\lambda,\alpha,\beta}$ if for $\phi \geq 0$ the equality is replaced with $\geq$ (resp. $\leq$) and $u \geq \alpha$ (resp. $\leq$), $\partial_\nu u \leq \beta$ (resp. $\geq$) on ∂B.

Theorem 4.14. *Suppose (α, β) is an admissible pair.*

1. *The minimal solution u_λ is then stable and is the unique semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$.*
2. *The function $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$ is a well-defined semi-stable $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*
3. *When u^* is a classical solution, then $\mu_1(u^*) = 0$ and u^* is the unique $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*

4. *If v is a singular semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$, then $v = u^*$ and $\lambda = \lambda^*$*

The crucial tool is a comparison result which is valid exactly in this class of solutions.

Lemma 4.15. *Let (α, β) be an admissible pair and u be a semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$. Assume U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ so that $U - \Phi \in H_0^2(B)$. Then*

1. *$u \leq U$ a.e. in B;*

2. *If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.*

Proof. (1) Define $w := u - U$. Then by the Moreau decomposition [18] for the biharmonic operator, there exist $w_1, w_2 \in H_0^2(B)$, with $w = w_1 + w_2$, $w_1 \geq 0$ a.e., $\Delta^2 w_2 \leq 0$ in the $H^2(B)-$weak sense and $\int_B \Delta w_1 \Delta w_2 = 0$. By Lemma 4.5, we have that $w_2 \leq 0$ a.e. in B.
Given now $0 \leq \phi \in C_c^\infty(B)$, we have that

$$\int_B \Delta w \Delta \phi \leq \lambda \int_B (f(u) - f(U))\phi,$$

where $f(u) := (1-u)^{-2}$. Since u is semi-stable, one has

$$\lambda \int_B f'(u) w_1^2 \leq \int_B (\Delta w_1)^2 = \int_B \Delta w \Delta w_1 \leq \lambda \int_B (f(u) - f(U)) w_1.$$

Since $w_1 \geq w$ one also has

$$\int_B f'(u) w w_1 \leq \int_B (f(u) - f(U)) w_1,$$

which once re-arranged gives

$$\int_B \tilde{f} w_1 \geq 0,$$

where $\tilde{f}(u) = f(u) - f(U) - f'(u)(u - U)$. The strict convexity of f gives $\tilde{f} \leq 0$ and $\tilde{f} < 0$ whenever $u \neq U$. Since $w_1 \geq 0$ a.e. in B one sees that $w \leq 0$ a.e. in B. The inequality $u \leq U$ a.e. in B is then established.

(2) Since u is a classical solution, it is easy to see that the infimum in $\mu_1(u)$ is attained at some ϕ. The function ϕ is then the first eigenfunction of $\Delta^2 - \frac{2\lambda}{(1-u)^3}$ in $H_0^2(B)$. Now we show that ϕ is of fixed sign. Using the

Moreau decomposition, one has $\phi = \phi_1 + \phi_2$ where $\phi_i \in H_0^2(B)$ for $i = 1, 2$, $\phi_1 \geq 0$, $\int_B \Delta\phi_1 \Delta\phi_2 = 0$ and $\Delta^2 \phi_2 \leq 0$ in the $H_0^2(B)-$weak sense. If ϕ changes sign, then $\phi_1 \not\equiv 0$ and $\phi_2 < 0$ in B (recall that either $\phi_2 < 0$ or $\phi_2 = 0$ a.e. in B). We can write now:

$$0 = \mu_1(u) \leq \frac{\int_B (\Delta(\phi_1 - \phi_2))^2 - \lambda f'(u)(\phi_1 - \phi_2)^2}{\int_B (\phi_1 - \phi_2)^2} < \frac{\int_B (\Delta\phi)^2 - \lambda f'(u)\phi^2}{\int_B \phi^2}$$

but the right hand is equal to $\mu_1(u)$ and so in view of $\phi_1\phi_2 < -\phi_1\phi_2$ in a set of positive measure, leading to a contradiction.
So we can assume $\phi \geq 0$, and by the Boggio's principle we have $\phi > 0$ in B. For $0 \leq t \leq 1$ define

$$g(t) = \int_B \Delta\left[tU + (1-t)u\right] \Delta\phi - \lambda \int_B f(tU + (1-t)u)\phi,$$

where ϕ is the above first eigenfunction. Since f is convex one sees that

$$g(t) \geq \lambda \int_B \left[t f(U) + (1-t) f(u) - f(tU + (1-t)u)\right] \phi \geq 0$$

for every $t \geq 0$. Since $g(0) = 0$ and

$$g'(0) = \int_B \Delta(U - u)\Delta\phi - \lambda f'(u)(U - u)\phi = 0,$$

we get that

$$g''(0) = -\lambda \int_B f''(u)(U - u)^2 \phi \geq 0.$$

Since $f''(u)\phi > 0$ in B, we finally get that $U = u$ a.e. in B. $\square$

Based again on Lemma 4.5(3), we can show a more general version of Lemma 4.15.

Lemma 4.16. *Let (α, β) be an admissible pair and $\beta' \leq 0$. Let u be a semi-stable $H^2(B)-$weak sub-solution of $(P)_{\lambda,\alpha,\beta}$ with $u = \alpha$, $\partial_\nu u = \beta' \geq \beta$ on ∂B. Assume that U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ with $U = \alpha$, $\partial_\nu U = \beta$ on ∂B. Then $U \geq u$ a.e. in B.*

Proof. Let $\tilde{u} \in H_0^2(B)$ denote a weak solution to $\Delta^2 \tilde{u} = \Delta^2(u - U)$ in B. Since $\tilde{u} - u + U = 0$ and $\partial_\nu(\tilde{u} - u + U) \leq 0$ on ∂B, by Lemma 4.5 one has that $\tilde{u} \geq u - U$ a.e. in B. Again by the Moreau decomposition [18], we may write $\tilde{u}$ as $\tilde{u} = w + v$, where $w, v \in H_0^2(B)$, $w \geq 0$ a.e. in B, $\Delta^2 v \leq 0$ in a

$H^2(B)-$weak sense and $\int_B \Delta w \Delta v = 0$. Then for $0 \leq \phi \in C^4(\bar{B}) \cap H_0^2(B)$ one has

$$\int_B \Delta \tilde{u} \Delta \phi = \int_B \Delta(u-U)\Delta\phi \leq \lambda \int_B (f(u)-f(U))\phi.$$

In particular, we have that

$$\int_B \Delta \tilde{u} \Delta w \leq \lambda \int_B (f(u)-f(U))w.$$

Since by semi-stability of u

$$\lambda \int_B f'(u)w^2 \leq \int_B (\Delta w)^2 = \int_B \Delta \tilde{u} \Delta w,$$

we get that

$$\int_B f'(u)w^2 \leq \int_B (f(u)-f(U))w.$$

By Lemma 4.5 we have $v \leq 0$ and then $w \geq \tilde{u} \geq u - U$ a.e. in B. So we see that

$$0 \leq \int_B \big(f(u) - f(U) - f'(u)(u-U)\big)\, w.$$

The strict convexity of f implies as in Lemma 4.15 that $U \geq u$ a.e. in B. $\square$

We shall need the following a-priori estimates along the minimal branch u_λ.

Lemma 4.17. *Let (α, β) be an admissible pair. Then one has*

$$2\int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3} \leq \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^2},$$

where Φ is given in (4.7). In particular, there is a constant $C > 0$ so that for every $\lambda \in (0, \lambda^*)$, we have

$$\int_B (\Delta u_\lambda)^2 + \int_B \frac{1}{(1-u_\lambda)^3} \leq C. \tag{4.18}$$

Proof. Testing $(P)_{\lambda,\alpha,\beta}$ on $u_\lambda - \Phi \in C^4(\bar{B}) \cap H_0^2(B)$, we see that

$$\lambda \int_B \frac{u_\lambda - \Phi}{(1-u_\lambda)^2} = \int_B \Delta u_\lambda \Delta(u_\lambda - \Phi) = \int_B (\Delta(u_\lambda - \Phi))^2 \geq 2\lambda \int_B \frac{(u_\lambda - \Phi)^2}{(1-u_\lambda)^3}$$

in view of $\Delta^2\Phi = 0$. In particular, for $\delta > 0$ small we have that

$$\begin{aligned}\int_{\{|u_\lambda-\Phi|\geq\delta\}} \frac{1}{(1-u_\lambda)^3} &\leq \frac{1}{\delta^2}\int_{\{|u_\lambda-\Phi|\geq\delta\}} \frac{(u_\lambda-\Phi)^2}{(1-u_\lambda)^3} \leq \frac{1}{\delta^2}\int_B \frac{1}{(1-u_\lambda)^2}\\ &\leq \delta\int_{\{|u_\lambda-\Phi|\geq\delta\}} \frac{1}{(1-u_\lambda)^3} + C_\delta,\end{aligned}$$

by means of Young's inequality. Since for δ small,

$$\int_{\{|u_\lambda-\Phi|\leq\delta\}} \frac{1}{(1-u_\lambda)^3} \leq C',$$

for some $C' > 0$, we can deduce that for every $\lambda \in (0,\lambda^*)$,

$$\int_B \frac{1}{(1-u_\lambda)^3} \leq C,$$

for some $C > 0$. By Young's and Hölder's inequalities, we now have

$$\begin{aligned}\int_B (\Delta u_\lambda)^2 &= \int_B \Delta u_\lambda \Delta\Phi + \lambda\int_B \frac{u_\lambda-\Phi}{(1-u_\lambda)^2}\\ &\leq \delta\int_B (\Delta u_\lambda)^2 + C_\delta\\ &\quad +C\left(\int_B \frac{1}{(1-u_\lambda)^3}\right)^{\frac{2}{3}}\end{aligned}$$

and estimate (4.18) is therefore established. □

We are now ready to establish Theorem 5.2.

Proof of Theorem 5.2: (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0,\lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in B$, it is easily seen that $\lambda \mapsto \mu_1(u_\lambda)$ is an increasing, continuous function on $(0,\lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* :\ \mu_1(u_\lambda) > 0\}.$$

We claim that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have that $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**},\lambda^*)$ the function u_μ would be a classical super-solution of $(P)_{\lambda_{**},\alpha,\beta}$. A contradiction arises since Lemma 4.15 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 4.15 guarantees uniqueness in the class of semi-stable $H^2(B)-$weak solutions. (2) By estimate (4.18) it follows that $u_\lambda \to u^*$ in a

pointwise sense and weakly in $H^2(B)$, and $\frac{1}{1-u^*} \in L^3(B)$. In particular, u^* is a $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$ which is also semi-stable as limiting function of the semi-stable solutions $\{u_\lambda\}$.
(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 4.15, u^* is then the unique $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$. An alternative approach – which we do not pursue here– based on the very definition of the extremal solution u^* is available in [4] when $\alpha = \beta = 0$ (see also [16]) to show that u^* is the unique weak solution of $(P)_{\lambda^*}$, regardless of whether u^* is regular or not.
(4) Assume now that v is a singular semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$. If $\lambda < \lambda^*$, then by the uniqueness of the semi-stable solution, we have $v = u_\lambda$. So v is not singular and a contradiction arises.

By Theorem 4.12(3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$ and u^* is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*,\alpha,\beta}$, we can apply Lemma 4.15 to get $v \le u^*$ a.e. in B. Since u^* is a semi-stable solution too, we can reverse the roles of v and u^* in Lemma 4.15 to see that $v \ge u^*$ a.e. in B. So equality $v = u^*$ holds and the proof is complete.

4.3 Regularity of the extremal solution for $1 \le N \le 8$

We now turn to the issue of the regularity of the extremal solution in problem $(P)_\lambda$. Unless stated otherwise, u_λ and u^* refer to the minimal and extremal solutions of $(P)_\lambda$. We shall show that the extremal solution u^* is regular provided $1 \le N \le 8$. We first begin by showing that it is indeed the case in small dimensions:

Theorem 4.19. *u^* is regular in dimensions $1 \le N \le 4$.*

Proof. As already observed, estimate (4.18) implies that $f(u^*) = (1 - u^*)^{-2} \in L^{\frac{3}{2}}(B)$. Since u^* is radial and radially decreasing, we need to show that $u^*(0) < 1$ to get the regularity of u^*. The integrability of $f(u^*)$ along with elliptic regularity theory shows that $u^* \in W^{4,\frac{3}{2}}(B)$. By the Sobolev imbedding theorem we get that u^* is a Lipschitz function in B.
Now suppose $u^*(0) = 1$ and $1 \le N \le 3$. Since

$$\frac{1}{1-u} \ge \frac{C}{|x|} \qquad \text{in } B,$$

for some $C > 0$, one sees that

$$+\infty = C^3 \int_B \frac{1}{|x|^3} \leq \int_B \frac{1}{(1-u^*)^3} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \leq N \leq 3$.
For $N = 4$ we need to be more careful and observe that $u^* \in C^{1,\frac{1}{3}}(\bar{B})$ by the Sobolev imbedding theorem. If $u^*(0) = 1$, then $\nabla u^*(0) = 0$ and

$$\frac{1}{1-u^*} \geq \frac{C}{|x|^{\frac{4}{3}}} \qquad \text{in } B,$$

for some $C > 0$. We obtain a contradiction exactly as above. □

We now tackle the regularity of u^* for $5 \leq N \leq 8$. We start with the following crucial result.

Theorem 4.20. *Assume $N \geq 5$ and let (u^*, λ^*) be the extremal pair of $(P)_\lambda$. When u^* is singular, then*

$$1 - u^*(x) \leq C_0 |x|^{\frac{4}{3}} \qquad \textit{in } B,$$

where $C_0 := \left(\frac{\lambda^*}{\bar{\lambda}}\right)^{\frac{1}{3}}$ and $\bar{\lambda} = \bar{\lambda}_N := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})$.

Proof. First note that Theorem 4.6(4) gives the lower bound:

$$\lambda^* \geq \bar{\lambda}_N = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{4.21}$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{3}}$ with $C_\delta := \left(\frac{\lambda^*}{\bar{\lambda}} + \delta\right)^{\frac{1}{3}} > 1$. Since $N \geq 5$, we have that $u_\delta \in H^2_{loc}(\mathbb{R}^N)$, $\frac{1}{1-u_\delta} \in L^3_{loc}(\mathbb{R}^N)$ and u_δ is a H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta\bar{\lambda}}{(1-u_\delta)^2} \qquad \text{in } \mathbb{R}^N.$$

We claim that $u_\delta \leq u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set $\Gamma := \{r \in (0,1) : u_\delta(r) > u^*(r)\}$ is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and one infers that

$$\alpha := u^*(r_1) = u_\delta(r_1)\,, \quad \beta := (u^*)'(r_1) \geq u_\delta'(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-\frac{4}{3}}(u_\delta(r_1 r) - 1) + 1$, we easily see that the function u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$, where

$$\alpha' := r_1^{-\frac{4}{3}}(\alpha - 1) + 1\,, \quad \beta' := r_1^{-\frac{1}{3}}\beta.$$

Similarly, define $u^*_{r_1}(r) = r_1^{-\frac{4}{3}}(u^*(r_1 r) - 1) + 1$. The dilation map

$$w \to w_{r_1}(r) = r_1^{-\frac{4}{3}}(w(r_1 r) - 1) + 1 \tag{4.22}$$

is a correspondence between solutions of $(P)_\lambda$ on B and of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ which preserves the H^2-integrability. In particular, $(u^*_{r_1},\lambda^*)$ is the extremal pair of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ (defined in the obvious way). Moreover, $u^*_{r_1}$ is a singular semi-stable $H^2(B)-$ weak solution of $(P)_{\lambda^*,\alpha',\beta'}$.

Since u^* is radially decreasing, we have that $\beta' \le 0$. Define the function $w(x) := (\alpha' - \frac{\beta'}{2}) + \frac{\beta'}{2}|x|^2 + \gamma(x)$, where γ is a solution of $\Delta^2\gamma = \lambda^*$ in B with $\gamma = \partial_\nu\gamma = 0$ on ∂B. Then w is a classical solution of

$$\begin{cases} \Delta^2 w = \lambda^* & \text{in } B \\ w = \alpha'\,, \quad \partial_\nu w = \beta' & \text{on } \partial B. \end{cases}$$

Since $\frac{\lambda^*}{(1-u^*_{r_1})^2} \ge \lambda^*$, by Lemma 4.5 we have $u^*_{r_1} \ge w$ a.e. in B. Since $w(0) = \alpha' - \frac{\beta'}{2} + \gamma(0)$ and $\gamma(0) > 0$, the bound $u^*_{r_1} \le 1$ a.e. in B yields to $\alpha' - \frac{\beta'}{2} < 1$. Namely, (α',β') is an admissible pair and by Theorem 5.2(4) we get that $(u^*_{r_1},\lambda^*)$ coincides with the extremal pair of $(P)_{\lambda,\alpha',\beta'}$ in B.

Since (α',β') is an admissible pair and u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$, we get from Proposition 4.2.1, the existence of a weak solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$. Since $\lambda^*+\delta\bar\lambda_N > \lambda^*$, we contradict the fact that λ^* is the extremal parameter of $(P)_{\lambda,\alpha',\beta'}$. □

Thanks to this lower estimate on u^*, we get the following result.

Theorem 4.23. *If $5 \le N \le 8$, then the extremal solution u^* of $(P)_\lambda$ is regular.*

Proof. Assume that u^* is singular. For $\varepsilon > 0$ set $\psi(x) := |x|^{\frac{4-N}{2}+\varepsilon}$ and note that

$$(\Delta\psi)^2 = (H_N + O(\varepsilon))|x|^{-N+2\varepsilon},$$

where

$$H_N := \frac{N^2(N-4)^2}{16}.$$

Given $\eta \in C_0^\infty(B)$, and since $N \geq 5$, we can use the test function $\eta\psi \in H_0^2(B)$ into the stability inequality to obtain

$$2\lambda \int_B \frac{\psi^2}{(1-u^*)^3} \leq \int_B (\Delta\psi)^2 + O(1),$$

where $O(1)$ is a bounded function as $\varepsilon \searrow 0$. By Theorem 4.20 we find that

$$2\bar{\lambda}_N \int_B \frac{\psi^2}{|x|^4} \leq \int_B (\Delta\psi)^2 + O(1),$$

and then

$$2\bar{\lambda}_N \int_B |x|^{-N+2\varepsilon} \leq (H_N + O(\varepsilon)) \int_B |x|^{-N+2\varepsilon} + O(1).$$

Computing the integrals one arrives at

$$2\bar{\lambda}_N \leq H_N + O(\varepsilon).$$

As $\varepsilon \to 0$ finally we obtain $2\bar{\lambda}_N \leq H_N$. Graphing this relation one sees that $N \geq 9$. □

We can now slightly improve the lower bound (4.21).

Corollary 4.24. *In any dimension $N \geq 1$, we have*

$$\lambda^* > \bar{\lambda}_N = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{4.25}$$

Proof. The function $\bar{u} := 1 - |x|^{\frac{4}{3}}$ is a $H^2(B)-$ weak solution of $(P)_{\bar{\lambda}_N, 0, -\frac{4}{3}}$. If by contradiction $\lambda^* = \bar{\lambda}_N$, then $\bar{u}$ is a $H^2(B)-$weak super-solution of $(P)_\lambda$ for every $\lambda \in (0, \lambda^*)$. By Lemma 4.15 we get that $u_\lambda \leq \bar{u}$ for all $\lambda < \lambda^*$, and then $u^* \leq \bar{u}$ a.e. in B.

If $1 \leq N \leq 8$, u^* is then regular by Theorems 4.19 and 4.23. By Theorem 5.2(3) there holds $\mu_1(u^*) = 0$. Lemma 4.15 then yields that $u^* = \bar{u}$, which is a contradiction since then u^* will not satisfy the boundary conditions.

If now $N \geq 9$ and $\bar{\lambda} = \lambda^*$, then $C_0 = 1$ in Theorem 4.20, and we then have $u^* \geq \bar{u}$. It means again that $u^* = \bar{u}$, a contradiction that completes the proof. □

4.4 The extremal solution is singular for $N \geq 9$

We prove in this section that the extremal solution is singular for $N \geq 9$. For that we have to distinguish between three different ranges for the dimension. For each range, we will need a suitable Hardy-Rellich type inequality that will be established in the last section, by using the recent results of Ghoussoub-Moradifam [11]. As in the previous section (u^*, λ^*) denotes the extremal pair of $(P)_\lambda$.

• **Case** $N \geq 17$**:** To establish the singularity of u^* for these dimensions we shall need the following well known improved Hardy-Rellich inequality, which is valid for $N \geq 5$. There exists $C > 0$, such that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\, dx \geq \frac{N^2(N-4)^2}{16} \int_B \frac{\phi^2}{|x|^4}\, dx + C \int_B \phi^2\, dx. \tag{4.26}$$

• **Case** $10 \leq N \leq 16$**:** For this case, we shall need the following inequality valid for all $\phi \in H_0^2(B)$

$$\begin{aligned}\int_B (\Delta\phi)^2\, dx \quad \geq \quad & \frac{(N-2)^2(N-4)^2}{16} I \\ & + \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}\, dx.\end{aligned} \tag{4.27}$$

where

$$I := \int_B \frac{\phi^2}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})}.$$

• **Case** $N = 9$**:** This case is the trickiest and will require the following inequality for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2\, dx \geq \int_B Q(|x|) \left(P(|x|) + \frac{N-1}{|x|^2} \right) \phi^2\, dx, \tag{4.28}$$

where

$$P(r) = \frac{p''(r) + \frac{(N-1)}{r} p'(r)}{p} \quad \text{and} \quad Q(r) = \frac{q''(r) + \frac{(N-3)}{r} q'(r)}{q},$$

with p and q being two appropriately chosen polynomials, namely

$$p(r) := r^{-\frac{N}{2}+1} + r - 1.9,$$

$$q(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48.$$

We shall first show the following upper bound on u^*.

Lemma 4.29. *If* $N \geq 9$*, then* $u^* \leq 1 - |x|^{\frac{4}{3}} =: \bar{u}$ *in* B.

Proof. Recall from Corollary 4.24 that $\bar{\lambda} := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}) < \lambda^*$. We now claim that $u_\lambda \leq \bar{u}$ for all $\lambda \in (\bar{\lambda}, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \leq R \leq 1 : u_\lambda < \bar{u} \text{ in the interval} \, (R, 1)\} > 0.$$

From the boundary conditions, one has that $u_\lambda(r) < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1$, $\alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u'_\lambda(R_1) \leq \bar{u}'(R_1)$. Introduce, as in the proof of Theorem 4.20, the functions u_{λ,R_1} and $\bar{u}_{R_1}$. We have that u_{λ,R_1} is a classical super-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(\alpha - 1) + 1\,, \quad \beta' := R_1^{-\frac{1}{3}}\beta.$$

Note that $\bar{u}_{R_1}$ is a $H^2(B)-$weak sub-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$ which is also semi-stable in view of the Hardy-Rellich inequality (4.26) and the fact that

$$2\bar{\lambda}_N \leq H_N := \frac{N^2(N-4)^2}{16}.$$

By Lemma 4.16, we deduce that $u_{\lambda,R_1} \geq \bar{u}_{R_1}$ in B. Note that, arguing as in the proof of Theorem 4.20, (α', β') is an admissible pair. We have therefore shown that $u_\lambda \geq \bar{u}$ in B_{R_1} and a contradiction arises in view of the fact that $\lim\limits_{x \to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \leq \bar{u}$ in B for every $\lambda \in (\bar{\lambda}_N, \lambda^*)$, and in particular $u^* \leq \bar{u}$ in B. □

The following lemma is the key for the proof of the singularity of u^* in higher dimensions.

Lemma 4.30. *Let* $N \geq 9$*. Suppose there exist* $\lambda' > 0$*,* $\beta > 0$ *and a singular radial function* $w \in H^2(B)$ *with* $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ *such that*

$$\begin{cases} \Delta^2 w \leq \frac{\lambda'}{(1-w)^2} & \text{for } 0 < r < 1, \\ w(1) = 0, \; w'(1) = 0, \end{cases} \tag{4.31}$$

and

$$2\beta \int_B \frac{\phi^2}{(1-w)^3} \leq \int_B (\Delta\phi)^2 \quad \text{for all} \quad \phi \in H_0^2(B). \tag{4.32}$$

1. *If* $\beta \geq \lambda'$*, then* $\lambda^* \leq \lambda'$.

2. *If either $\beta > \lambda'$ or if $\beta = \lambda' = \frac{H_N}{2}$, then the extremal solution u^* is necessarily singular.*

Proof. 1) First, note that (4.32) and $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ yield to $\frac{1}{(1-w)^2} \in L^1(B)$. By a density argument, (4.31) implies that w is a $H^2(B)-$weak sub-solution of $(P)_{\lambda'}$ whenever $N \geq 4$. If now $\lambda' < \lambda^*$, then by Lemma 4.16, w would necessarily be below the minimal solution $u_{\lambda'}$, which is a contradiction since w is singular while $u_{\lambda'}$ is regular.
2) Suppose first that $\beta = \lambda' = \frac{H_N}{2}$ and that $N \geq 9$. Since by part 1) we have $\lambda^* \leq \frac{H_N}{2}$, we get from Lemma 4.29 and the improved Hardy-Rellich inequality (4.26) that there exists $C > 0$ so that for all $\phi \in H^2_0(B)$

$$\int_B (\Delta\phi)^2 - 2\lambda^* \int_B \frac{\phi^2}{(1-u^*)^3} \geq \int_B (\Delta\phi)^2 - H_N \int_B \frac{\phi^2}{|x|^4} \geq C \int_B \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.
Suppose now that $\beta > \lambda'$, and let $\frac{\lambda'}{\beta} < \gamma < 1$ in such a way that

$$\alpha := (\frac{\gamma\lambda^*}{\lambda'})^{1/3} < 1. \tag{4.33}$$

Setting $\bar{w} := 1 - \alpha(1-w)$, we claim that

$$u^* \leq \bar{w} \ \text{ in } \ B. \tag{4.34}$$

Note that by the choice of α we have $\alpha^3\lambda' < \lambda^*$, and therefore to prove (4.34) it suffices to show that for $\alpha^3\lambda' \leq \lambda < \lambda^*$, we have $u_\lambda \leq \bar{w}$ in B. Indeed, fix such λ and note that

$$\Delta^2\bar{w} = \alpha\Delta^2 w \leq \frac{\alpha\lambda'}{(1-w)^2} = \frac{\alpha^3\lambda'}{(1-\bar{w})^2} \leq \frac{\lambda}{(1-\bar{w})^2}.$$

Assume that $u_\lambda \leq \bar{w}$ does not hold in B, and consider

$$R_1 := \sup\{0 \leq R \leq 1 \mid u_\lambda(R) > \bar{w}(R)\} > 0.$$

Since $\bar{w}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have $R_1 < 1$, $u_\lambda(R_1) = \bar{w}(R_1)$ and $(u_\lambda)'(R_1) \leq (\bar{w})'(R_1)$. Introduce, as in the proof of Theorem 4.20, the functions u_{λ,R_1} and $\bar{w}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(P)_{\lambda,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(u_\lambda(R_1) - 1) + 1\,, \quad \beta' := R_1^{-\frac{1}{3}}(u_\lambda)'(R_1).$$

Since $\lambda < \lambda^*$ and then

$$\frac{2\lambda}{(1-\bar{w})^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} = \frac{2\lambda'}{\gamma(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

by (4.32) $\bar{w}_{R_1}$ is a stable $H^2(B)-$weak sub-solution of $(P)_{\lambda,\alpha',\beta'}$. By Lemma 4.16, we deduce that $u_\lambda \geq \bar{w}$ in B_{R_1} which is impossible, since $\bar{w}$ is singular while u_λ is regular. Note that, arguing as in the proof of Theorem 4.20, (α', β') is an admissible pair. This establishes claim (4.34) which, combined with the above inequality, yields

$$\frac{2\lambda^*}{(1-u^*)^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

and therefore

$$\inf_{\phi \in H_0^2(B)} \frac{\int_B (\Delta\phi)^2 - \frac{2\lambda^*\phi^2}{(1-u^*)^3}}{\int_B \phi^2} > 0.$$

It follows that again $\mu_1(u^*) > 0$ and u^* must be singular, since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*. □

Consider for any $m > 0$ the following function:

$$w_m := 1 - \frac{3m}{3m-4} r^{4/3} + \frac{4}{3m-4} r^m, \tag{4.35}$$

which satisfies the right boundary conditions: $w_m(1) = w_m'(1) = 0$. We can now prove that the extremal solution is singular for $N \geq 9$.

Theorem 4.36. *Let $N \geq 9$. The following upper bounds on λ^* hold:*

1. *If $N \geq 31$, then Lemma 4.30 holds with $w := w_2$, $\lambda' = 27\bar{\lambda}_N$ and $\beta = \frac{H_N}{2}$, and therefore $\lambda^* \leq 27\bar{\lambda}_N$.*

2. *If $17 \leq N \leq 30$, then Lemma 4.30 holds with $w := w_3$, $\lambda' = \beta = \frac{H_N}{2}$, and therefore $\lambda^* \leq \frac{H_N}{2}$.*

3. *If $10 \leq N \leq 16$, then Lemma 4.30 holds with $w := w_3$, $\lambda'_N < \beta_N$ given in Table 4.1, and therefore $\lambda^* \leq \lambda'_N$.*

4. *If $N = 9$, then Lemma 4.30 holds with $w := w_{2.8}$, $\lambda'_9 := 366 < \beta_9 := 368.5$, and therefore $\lambda^* \leq 366$.*

The extremal solution is therefore singular for dimension $N \geq 9$.

Table 4.1: Summary

N	w	λ'_N	β_N
9	$w_{2.8}$	366	366.5
10	w_3	450	487
11	w_3	560	739
12	w_3	680	1071
13	w_3	802	1495
14	w_3	940	2026
15	w_3	1100	2678
16	w_3	1260	3469
$17 \leq N \leq 30$	w_3	$H_N/2$	$H_N/2$
$N \geq 31$	w_2	$27\bar{\lambda}_N$	$H_N/2$

Proof. 1) Assume first that $N \geq 31$, then $27\bar{\lambda} \leq \frac{H_N}{2}$. We shall show that w_2 is a singular $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$ so that (4.32) holds with $\beta = \frac{H_N}{2}$. Indeed, write

$$w_2 := 1 - |x|^{\frac{4}{3}} - 2(|x|^{\frac{4}{3}} - |x|^2) = \bar{u} - \phi_0,$$

where $\phi_0 := 2(|x|^{\frac{4}{3}} - |x|^2)$, and note that $w_2 \in H_0^2(B)$, $\frac{1}{1-w_2} \in L^3(B)$, $0 \leq w_2 \leq 1$ in B, and

$$\Delta^2 w_2 = \frac{3\bar{\lambda}}{r^{\frac{8}{3}}} \leq \frac{27\bar{\lambda}}{(1-w_2)^2} \qquad \text{in } B \setminus \{0\}.$$

So w_2 is $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$. Moreover, by $\phi_0 \geq 0$ and (4.26) we get that

$$H_N \int_B \frac{\phi^2}{(1-w_2)^3} = H_N \int_B \frac{\phi^2}{(|x|^{\frac{4}{3}} + \phi_0)^3} \leq H_N \int_B \frac{\phi^2}{|x|^4} \leq \int_B (\Delta\phi)^2$$

for all $\phi \in H_0^2(B)$. It follows from Lemma 4.30 that u^* is singular and that $\lambda^* \leq 27\bar{\lambda} \leq \frac{H_N}{2}$.
2) Assume $17 \leq N \leq 30$ and consider the function

$$w_3 := 1 - \frac{9}{5} r^{\frac{4}{3}} + \frac{4}{5} r^3.$$

We show that w_3 is a semi-stable singular $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. Indeed, we clearly have that $0 \leq w_3 \leq 1$ in B, $w_3 \in H_0^2(B)$ and

$\frac{1}{1-w_3} \in L^3(B)$. To show the stability condition, we consider $\phi \in H_0^2(B)$ and write

$$\begin{aligned} H_N \int_B \frac{\phi^2}{(1-w_3)^3} &= 125 H_N \int_B \frac{\phi^2}{(9r^{\frac{4}{3}} - 4r^3)^3} \\ &\leq 125 H_N \sup_{0<r<1} \frac{1}{(9-4r^{\frac{5}{3}})^3} \int_B \frac{\phi^2}{r^4} \\ &= H_N \int_B \frac{\phi^2}{r^4} \leq \int_B (\Delta\phi)^2 \end{aligned}$$

by virtue of (4.26). An easy computation shows that

$$\begin{aligned} \frac{H_N}{2(1-w_3)^2} - \Delta^2 w_3 &= \frac{25 H_N}{2(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{9\bar{\lambda}}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r} \\ &= \frac{25N^2(N-4)^2}{32(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{8(N-\frac{2}{3})(N-\frac{8}{3})}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r}. \end{aligned}$$

By using Maple, one can verify that this final quantity is nonnegative on $(0,1)$ whenever $17 \leq N \leq 30$, and hence w_3 is a $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. It follows from Lemma 4.30 that u^* is singular and that $\lambda^* \leq \frac{H_N}{2}$.

3) Assume $10 \leq N \leq 16$. We shall prove that again $w := w_3$ satisfies the assumptions of Lemma 4.30. Indeed, using Maple, we show that for each dimension $10 \leq N \leq 16$, inequality (4.31) holds with λ'_N given by Table 4.1. Then, by using Maple again, we show that for each dimension $10 \leq N \leq 16$, the following inequality holds

$$\begin{aligned} &\frac{(N-2)^2(N-4)^2}{16} \frac{1}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\ &\quad + \frac{(N-1)(N-4)^2}{4} \frac{1}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})} \\ &\qquad \geq \frac{2\beta_N}{(1-w_3)^3}. \end{aligned}$$

where β_N is again given by Table 4.1. The above inequality and the Hardy-Rellich inequality (4.27) guarantee that the stability condition (4.32) holds with $\beta := \beta_N$. Since $\beta_N > \lambda'_N$, we deduce from Lemma 4.30 that the extremal solution is singular for $10 \leq N \leq 16$.

4) Suppose now $N = 9$ and consider $w := w_{2.8}$. Using Maple one can see that

$$\Delta^2 w \leq \frac{366}{(1-w)^2} \quad \text{in} \quad B$$

and
$$\frac{723}{(1-w)^3} \leq Q(r)\left(P(r) + \frac{N-1}{r^2}\right) \quad \text{for all} \quad r \in (0,1),$$
where P and Q are given in (4.28). Since $723 > 2 \times 366$, by Lemma 4.30 the extremal solution u^* is singular in dimension $N = 9$. □

4.5 Improved Hardy-Rellich inequalities

We now prove the improved Hardy-Rellich inequalities used in section 4. They rely on the results of Ghoussoub-Moradifam in [11] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions.

Definition 4.37. *Say that a couple V, W of positive functions in $C^1(0,R)$ is a Bessel pair on the interval $(0,R)$, if the ordinary differential equation*
$$y''(r) + (\frac{N-1}{r} + \frac{V_r(r)}{V(r)})y'(r) + \frac{W(r)}{V(r)}y(r) = 0,$$
has a positive solution on the interval $(0,R)$.

Let B_R denotes a ball centered at zero with radius R in $\mathbb{R}^N$ $(N \geq 1)$. The space of radial functions in $C_0^\infty(B_R)$ will be denoted by $C_{0,r}^\infty(B_R)$. The needed inequalities will follow from the following result.

Theorem 4.38. (Ghoussoub-Moradifam [11]) *Let V and W be positive C^1-functions on the interval $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr = +\infty$ and $\int_0^R r^{N-1}V(r)dr < +\infty$. The following statements are then equivalent:*

1. *(V,W) is a Bessel pair on $(0,R)$.*
2. *$\int_{B_R} V(|x|)|\nabla\phi|^2 dx \geq \int_{B_R} W(|x|)\phi^2 dx$ for all $\phi \in C_0^\infty(B_R)$.*
3. *If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < N-2$, then the above are equivalent to*
$$\begin{aligned}\int_{B_R} V(|x|)(\Delta\phi)^2 dx \quad &\geq \quad \int_{B_R} W(|x|)|\nabla\phi|^2 dx \\ &\quad +(N-1)\int_{B_R} (\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nabla\phi|^2 dx\end{aligned}$$
for all $\phi \in C_{0,r}^\infty(B_R)$.

4. *If in addition,* $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ *on* $(0, R)$*, then the above are equivalent to*

$$\int_{B_R} V(|x|)(\Delta\phi)^2 dx \geq \int_{B_R} W(|x|)|\nabla\phi|^2 dx + (N-1)\int_{B_R} \left(\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla\phi|^2 dx$$

for all $\phi \in C_0^\infty(B_R)$.

We shall use the above theorem to deduce the following corollary.

Corollary 4.39. *Let* $N \geq 5$ *and* B *be the unit ball in* $\mathbb{R}^N$*. Then the following improved Hardy-Rellich inequality holds for all* $\phi \in C_0^\infty(B)$*:*

$$\int_B (\Delta\phi)^2 \, dx \geq \frac{(N-2)^2(N-4)^2}{16} \int_B \frac{\phi^2 \, dx}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2 \, dx}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}. \tag{4.40}$$

Proof. Let $0 < \alpha < 1$ and define $y(r) := r^{-\frac{N}{2}+1} - \alpha$. Since

$$-\frac{y'' + \frac{(N-1)}{r} y'}{y} = \frac{(N-2)^2}{4} \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}},$$

the couple $\left(1, \frac{(N-2)^2}{4} \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}\right)$ is a Bessel pair on $(0, 1)$. By part (4) of Theorem 4.38, the following inequality then holds:

$$\int_B (\Delta\phi)^2 dx \geq \frac{(N-2)^2}{4} \int_B \frac{|\nabla\phi|^2 \, dx}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} + (N-1)\int_B \frac{|\nabla\phi|^2 \, dx}{|x|^2}, \tag{4.41}$$

for all $\phi \in C_0^\infty(B)$. Set $V(r) := \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}$ and note that

$$\frac{V_r}{V} = -\frac{2}{r} + \frac{\alpha(N-2)}{2} \frac{r^{\frac{N}{2}-2}}{1 - \alpha r^{\frac{N}{2}-1}} \geq -\frac{2}{r}.$$

The function $y(r) = r^{-\frac{N}{2}+2} - 1$ is decreasing and is then a positive super-solution on $(0, 1)$ for the ODE

$$y'' + \left(\frac{N-1}{r} + \frac{V_r}{V}\right)y'(r) + \frac{W_1(r)}{V(r)} y = 0,$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2 - r^{\frac{N}{2}})(r^2 - \alpha r^{\frac{N}{2}+1})}.$$

Hence, by part 2) of Theorem 4.38, we can deduce that

$$\int_B \frac{|\nabla\phi|^2\, dx}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2\, dx}{(|x|^2 - \alpha|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Similarly, for $V(r) = \frac{1}{r^2}$ we have that

$$\int_B \frac{|\nabla\phi|^2\, dx}{|x|^2} \geq (\frac{N-4}{2})^2 \int_B \frac{\phi^2\, dx}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Combining the above two inequalities with (4.41) and letting $\alpha \to 1$ we get inequality (4.40). □

Corollary 4.42. *Let $N = 9$ and B be the unit ball in $\mathbb{R}^N$. Define $p(r) := r^{-\frac{N}{2}+1} + r - 1.9$ and $q(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48$. Then the following improved Hardy-Rellich inequality holds for all $\phi \in C_0^\infty(B)$:*

$$\int_B (\Delta\phi)^2\, dx \geq \int_B Q(|x|)\left(P(|x|) + \frac{N-1}{|x|^2}\right)\phi^2\, dx, \tag{4.43}$$

where

$$P(r) := -\frac{p''(r) + \frac{N-1}{r}p'(r)}{p(r)} \quad \textit{and} \quad Q(r) := -\frac{q''(r) + \frac{N-3}{r}q'(r)}{q(r)}.$$

Proof. By definition $(1, P(r))$ is a Bessel pair on $(0,1)$. One can easily see that $P(r) \geq \frac{2}{r^2}$. Hence, by Theorem 4.38(4) the following inequality holds:

$$\int_B (\Delta\phi)^2 dx \geq \int_B P(|x|)|\nabla\phi|^2\, dx + (N-1)\int_B \frac{|\nabla\phi|^2\, dx}{|x|^2}, \tag{4.44}$$

for all $\phi \in C_0^\infty(B)$. Using Maple it is easy to see that

$$\frac{P_r}{P} \geq -\frac{2}{r} \text{ in } (0,1),$$

and therefore $q(r)$ is a positive super-solution for the ODE

$$y'' + (\frac{N-1}{r} + \frac{P_r(r)}{P(r)})y'(r) + \frac{P(r)Q(r)}{P(r)}y = 0,$$

on $(0,1)$. Hence, by Theorem 4.38(2) we have for all $\phi \in C_0^\infty(B)$

$$\int_B P(|x|)|\nabla\phi|^2\,dx \geq \int_B P(|x|)Q(|x|)\phi^2\,dx,$$

and similarly

$$\int_B \frac{|\nabla\phi|^2\,dx}{|x|^2} \geq \int_B \frac{Q(|x|)}{|x|^2}\phi^2\,dx,$$

since $q(r)$ is a positive solution for the ODE

$$y'' + \frac{N-3}{r}y'(r) + Q(r)y = 0.$$

Combining the above two inequalities with (4.44) we get (4.43). □

Bibliography

[1] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math. **12** (1959), 623-727.

[2] G. Arioli, F. Gazzola, H.-C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. **36** (2005), no. 4, 1226-1258.

[3] T. Boggio, *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo (1905), 97-135.

[4] D. Cassani, J. do O and N. Ghoussoub, *On a fourth order elliptic problem with a singular nonlinearity*, Advances Nonlinear Studies, **9**, (2009), 177-197

[5] J. Davila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. **39** (2007), 565-592.

[6] J. Dávila, I. Flores and I. Guerra, *Multiplicity of solutions for a fourth order equation with power-type nonlinearity*, (preprint).

[7] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Commun. Contemp. Math. **10** (2008), no. 1, 17-45

[8] P. Esposito, N. Ghoussoub, Y. J. Guo: *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Courant Lecture Notes (2009) In press, 293 p.

[9] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), no. 12, 1731-1768.

[10] N. Ghoussoub, Y. Guo, *On the partial differential equations of electro MEMS devices: stationary case*, SIAM J. Math. Anal. **38** (2007), 1423-1449.

[11] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities*, Submitted (2008).

[12] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a mems device with varying dielectric properties*, SIAM J. Appl. Math **66** (2005), 309-338.

[13] Z. Guo and J. Wei, *On a fourth order nonlinear elliptic equation with negative exponent*, SIAM J. Math. Anal. **40** (2008/09), no. 5, 2034–2054.

[14] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Ration. Mech. Anal. **49** (1973), 241-268.

[15] F.H. Lin and Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic acutation*, Proc. R. Soc. London, Ser. A **463** (2007), 1323-1337.

[16] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston J. Math. **23** (1997), no. 1, 161-168.

[17] F. Mignot and J.-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations **5** (1980), 791-836.

[18] J.-J. Moreau, *Decomposition orthogonale d'un espace hilbertien selon deux cones mutuellement polaires,* C.R. Acad. Sci. Paris **255** (1962), 238-240.

[19] J.A. Pelesko, *Mathematical modeling of electrostatic mems with tailored dielectric properties*, SIAM J. Appl. Math. **62** (2002), 888-908.

[20] J.A. Pelesko and A.A. Bernstein, *Modeling MEMS and NEMS*, Chapman Hall and CRC Press, 2002.

[21] R. Soranzo, *A priori estimates and existence of positive solutions of a superlinear polyharmonic equation*, Dynam. Systems Appl. **3** (1994), no. 4, 465–487.

Chapter 5

Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains[4]

5.1 Introduction

We examine the problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (N_\lambda)$$

where $\lambda \geq 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and where f satisfies one of the following two conditions:

(R): f is smooth, increasing, convex on $\mathbb{R}$ with $f(0) = 1$ and f is superlinear at ∞ (i.e. $\lim_{t \to \infty} \frac{f(t)}{t} = \infty$);

(S): f is smooth, increasing, convex on $[0, 1)$ with $f(0) = 1$ and $\lim_{t \nearrow 1} f(t) = +\infty$.

Our main interest is in the regularity of the extremal solution u^* associated with (N_λ). Before we discuss some known results concerning the problem (N_λ) we recall various facts concerning second order versions of the above problem.

[4] A version of this chapter has been accepted for publication: C. Cowan, P. Esposito and N. Ghoussoub, *Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains*, Discrete Contin. Dyn. Syst. (2010).

5.1.1 The second order case

For a nonlinearity f of type (R) or (S), the following second order analog of (N_λ) with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{Q_λ}$$

is by now quite well understood whenever Ω is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [4, 5, 6, 12, 15, 19, 20, 23]. We now list the properties one comes to expect when studying (Q_λ).

- There exists a finite positive critical parameter λ^* such that for all $0 < \lambda < \lambda^*$ there exists a **minimal solution** u_λ of (Q_λ). By minimal solution, we mean here that if v is another solution of (Q_λ) then $v \geq u_\lambda$ a.e. in Ω.
- For each $0 < \lambda < \lambda^*$ the minimal solution u_λ is **semi-stable** in the sense that
$$\int_\Omega \lambda f'(u_\lambda)\psi^2 dx \leq \int_\Omega |\nabla \psi|^2 dx, \qquad \forall \psi \in H_0^1(\Omega),$$
and is unique among all the weak semi-stable solutions.
- The map $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$. This allows one to define $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$, the so-called **extremal solution**, which can be shown to be a weak solution of (Q_{λ^*}). In addition one can show that u^* is the unique weak solution of (Q_{λ^*}). See [19].
- There are no solutions of (Q_λ) (even in a very weak sense) for $\lambda > \lambda^*$.

A question which has attracted a lot of attention is whether the extremal function u^* is a classical solution of (Q_{λ^*}). This is of interest since one can then apply the results from [10] to start a second branch of solutions emanating from (λ^*, u^*). Note that in the case where f satisfies (R) (resp. (S)) it is sufficient –in view of standard elliptic regularity theory– to show that u^* is bounded (resp. $\sup_\Omega u^* < 1$).

This turned out to depend on the dimension, and so given a nonlinearity f, we say that N is the associated **critical dimension** provided the extremal solution u^* associated with (Q_{λ^*}) is a classical solution for any bounded smooth domain $\Omega \subset \mathbb{R}^M$ for any $M \leq N-1$, and if there exists a domain $\Omega \subset \mathbb{R}^N$ such that the associated extremal solution u^* is not a classical solution. We now list some of the known results with regard to this question.

- For $f(t) = e^t$, the critical dimension is $N = 10$. For $N \geq 10$, one can show that on the unit ball the extremal solution is explicitly given by $u^*(x) = -2\log(|x|)$.
- For $\Omega = B$ the unit ball in $\mathbb{R}^N$, u^* is bounded for any f satisfying (R) provided $N \leq 9$, which –in view of the above– is optimal (see [6]).
- On general domains, and if f satisfies (R), then u^* is bounded for $N \leq 3$ [23]. Recently this has been improved to $N \leq 4$ provided the domain is convex [5].
- For $f(t) = (1-t)^{-2}$ the critical dimension is $N = 8$ and $u^* = 1 - |x|^{\frac{2}{3}}$ is the extremal solution on the unit ball for $N \geq 8$. [15].

In the previous list, we have not considered the nonlinearity $f(t) = (t+1)^p$, $p > 1$, for which the critical dimension has been also computed but takes a complicated form. The general approach to showing N is the critical dimension for a particular nonlinearity f is to use the semi-stability of the minimal solutions u_λ to obtain various estimates which translate to uniform L^∞ bounds and then passing to the limit. These estimates generally depend on the ambient space dimension. On the other hand, in order to show the optimality of the regularity result one generally finds an explicit singular extremal solution u^* on a radial domain. Here the crucial tool is the fact that if there exists a semi-stable singular solution in $H_0^1(\Omega)$, then it has to be the extremal solution. In practice one considers an explicit singular solution on the unit ball and applies Hardy-type inequalities to show its semi-stability in the right dimension. We also remark that one cannot remove the $H_0^1(\Omega)$ condition as counterexamples can be found.

5.1.2 The fourth order case

There are two obvious fourth order extensions of (Q_λ) namely the problem (N_λ) mentioned above, and its Dirichlet counterpart

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

where ∂_ν denote the normal derivative on $\partial\Omega$. The problem (Q_λ) is heavily dependent on the maximum principle and hence this poses a major hurdle in the study of (D_λ) since for general domains there is no maximum principle for Δ^2 with Dirichlet boundary conditions. But if we restrict our attention to the unit ball then one does have a weak maximum principle [3]. The problem (D_λ) was studied in [1] and various results were obtained, but results concerning the boundedness of the extremal solution (for supercritical nonlinearities) were missing.

The first (truly supercritical) results concerning the boundedness of the extremal solution in a fourth order problem are due to [11] where they examined the problem (D_λ) on the unit ball in $\mathbb{R}^N$ with $f(t) = e^t$. They showed that the extremal solution u^* is bounded if and only if $N \leq 12$. Their approach is heavily dependent on the fact that Ω is the unit ball. Even in this situation there are two main hurdles. The first is that the standard energy estimate approach, which was so successful in the second order case, does not appear to work in the fourth order case. The second is the fact that it is quite hard to construct explicit solutions of (D_λ) on the unit ball that satisfy both boundary conditions, which is needed to show that the extremal solution is unbounded for $N \geq 13$. So what one does is to find an explicit singular, semi-stable solution which satisfies the first boundary condition, and then to perturb it enough to satisfy the second boundary condition but not too much so as to lose the semi-stability. Davila et al. [11] succeeded in doing so for $N \geq 32$, but they were forced to use a computer assisted proof to show that the extremal solution is unbounded for the intermediate dimensions $13 \leq N \leq 31$. Using various improved Hardy-Rellich inequalities from [16] the need for the computer assisted proof was removed in [21]. The case where $f(t) = (1-t)^{-2}$ was settled at the same time in [9], where we used methods developed in 3DDGM to show that the extremal solution associated with (D_λ) is a classical solution if and only if $N \leq 8$.

The problem (N_λ) with Navier boundary conditions does not suffer from the lack of a maximum principle and the existence of the minimal branch has been shown in general [2, 7]. If the domain is the unit ball, then again one can use the methods of [11] and [9] to obtain optimal results in the case of $f(t) = (1-t)^{-2}$ (see for instance [13] and [22]). However, the case of a general domain is only understood in dimensions $N \leq 4$ (See [17] and [13]). This paper is a first attempt at giving energy estimates on general domains, which –as mentioned above – while they do improve known results, they still fall short of the conjectured critical dimensions that were established when the domain is a ball.

We now fix notation and some definitions associated with problem (N_λ).

Definition 5.1. *Given a smooth solution u of (N_λ), we say that u is a semi-stable solution of (N_λ) if*

$$\int_\Omega \lambda f'(u)\psi^2 dx \leq \int_\Omega (\Delta\psi)^2 dx, \qquad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{5.2}$$

Definition 5.3. *We say a smooth solution u of (N_λ) is minimal provided $u \leq v$ a.e. in Ω for any solution v of (N_λ).*

We define the **extremal parameter** λ^* as

$$\lambda^* := \sup \{0 < \lambda : \text{there exists a smooth solution of } (N_\lambda)\} .$$

It is known, see [2, 7, 17], that:

1. $0 < \lambda^* < \infty$.
2. For each $0 < \lambda < \lambda^*$ there exists a smooth minimal solution u_λ of (N_λ). Moreover the minimal solution u_λ is semi-stable and is unique among the semi-stable solutions.
3. For each $x \in \Omega$, $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$, and it therefore makes sense to define $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$, which we call the **extremal solution**.
4. There are no solutions for $\lambda > \lambda^*$.

It is standard to show that u^* is a "weak solution" of (N_{λ^*}) in a suitable sense that we shall not define here since it will not be needed in the sequel. One can then proceed to show that u^* it is the unique weak solution in a fairly broad class of solutions. Regularity results on u^* translate into regularity properties for any weak semi-stable solution. Indeed, by points (2)-(4) above we see that a weak semi-stable solution is either the classical solution u_λ or the extremal solution u^*. Our preference for not stating the results in this generality is to avoid the technical details of defining precisely what we mean by a suitable weak solution.

5.2 Sufficient L^q-estimates for regularity

In this section, we address our attention to nonlinearities f of type (R). As mentioned above, since the extremal function u^* is the pointwise monotone limit of the classical solutions u_λ as $\lambda \nearrow \lambda^*$, it suffices to consider a sequence $(u_n)_n$ of classical solutions of (N_{λ_n}), $(\lambda_n)_n$ uniformly bounded, and try to show that

$$\sup_n \|u_n\|_\infty < +\infty. \tag{5.4}$$

By standard elliptic regularity theory (5.4) follows by a uniform bound of $f(u_n)$ in $L^q(\Omega)$, for some $q > \frac{N}{4}$. The following result provides a weakening of such a statement.

Theorem 5.5. *Suppose that for some $q \geq 1$ and $0 < \beta < \alpha$ we have*

$$\sup_n \int_\Omega f^q(u_n) < +\infty \tag{5.6}$$

and

$$\sup_n \int_\Omega \frac{f^\alpha(u_n)}{u_n^\beta + 1} < +\infty. \tag{5.7}$$

Then:

1. *If $1 \leq q \leq \frac{N}{4}$ and $\alpha \leq \frac{N}{4}$, then $\sup_n \|f(u_n)\|_s < +\infty$ for every $s < \max\{\frac{(\alpha-\beta)N}{N-4\beta}, q\}$.*
2. *If either $q > \frac{N}{4}$ or $\alpha > \frac{N}{4}$, then $\sup_n \|u_n\|_\infty < +\infty$.*

Proof. We shall first show that under assumption (5.7), the following holds:

$$\text{If } \sup_n \|f(u_n)\|_{q_0} < +\infty \text{ for } 1 \leq q_0 \leq \tfrac{N}{4}$$

$$\text{then} \quad \sup_n \|f(u_n)\|_s < +\infty \text{ for every } s < q_1, \tag{5.8}$$

where $q_1 := \frac{\alpha N q_0}{N q_0 + \beta(N - 4q_0)}$.
Indeed, for $t > 0$ set

$$\Omega_{1,t}^n := \{x \in \Omega : f(u_n(x)) \leq (u_n^\beta(x) + 1)^{\frac{t}{\beta}}\} \quad \text{and} \quad \Omega_{2,t}^n = \Omega \backslash \Omega_{1,t}^n.$$

Since $1 \leq q_0 \leq \frac{N}{4}$ and $\sup_n \|f(u_n)\|_{q_0} < +\infty$, we have via the Sobolev embedding Theorem that

$$\sup_n \|u_n\|_s < +\infty \text{ for every } s < \tfrac{Nq_0}{N-4q_0},$$

and hence on $\Omega_{1,t}^n$ we have

$$\sup_n \int_{\Omega_{1,t}^n} f^s(u_n) < +\infty \text{ for all } s < \tfrac{Nq_0}{t(N-4q_0)}. \tag{5.9}$$

On $\Omega_{2,t}^n$, we have $f^{\alpha - \frac{\beta}{t}}(u_n) \leq \frac{f^\alpha(u_n)}{u_n^\beta + 1}$, therefore

$$\sup_n \int_{\Omega_{2,t}^n} f^{\alpha - \frac{\beta}{t}}(u_n) < +\infty. \tag{5.10}$$

If $q_0 < \frac{N}{4}$, then take $t = \frac{Nq_0 + \beta(N-4q_0)}{\alpha(N-4q_0)}$ in such a way that

$$\alpha - \frac{\beta}{t} = \frac{Nq_0}{t(N-4q_0)},$$

to obtain that

$$\sup_n \|f(u_n)\|_s < +\infty \text{ for all } s < \tfrac{\alpha N q_0}{Nq_0 + \beta(N-4q_0)}.$$

If $q_0 = \frac{N}{4}$, then we can let $t \to +\infty$ in (5.10) and combine with (7.17) to obtain that

$$\sup_n \|f(u_n)\|_s < +\infty \text{ for all } s < \alpha,$$

and then (5.8) is proved. Note that for $1 < q_0 \le \frac{N}{4}$ (5.8) is equivalent to:

$$\text{if } \sup_n \|f(u_n)\|_s < +\infty \text{ for every } 1 \le s < q_0, \tag{5.11}$$

$$\text{then} \quad \sup_n \|f(u_n)\|_s < +\infty \text{ for every } s < q_1,$$

where q_1 is as before.

By elliptic regularity theory, assumption (5.6) implies for $q > \frac{N}{4}$ that $\sup_n \|u_n\|_\infty < +\infty$. When $1 \le q \le \frac{N}{4}$, by (5.8) we can say that

$$\sup_n \|f(u_n)\|_s < +\infty \qquad \text{for every } s < q_1 := \frac{\alpha N q_0}{Nq_0 + \beta(N-4q_0)}.$$

If $q_1 > \frac{N}{4}$ we are done. Otherwise, thanks to (5.11) we can use an iteration argument to show that

$$\sup_n \|f(u_n)\|_s < +\infty \qquad \text{for every } s < q_{n+1} := \frac{\alpha N q_n}{Nq_n + \beta(N-4q_n)}$$

for every $n \ge 1$, as long as $q_n \le \frac{N}{4}$. Since $\frac{(\alpha-\beta)N}{N-4\beta} > \frac{N}{4}$ when $\alpha > \frac{N}{4}$ and $1 \le q \le \frac{N}{4}$, an easy induction shows that the sequence q_n is

- increasing to $\frac{(\alpha-\beta)N}{N-4\beta}$ when $\alpha \le \frac{N}{4}$ and $1 \le q < \frac{(\alpha-\beta)N}{N-4\beta}$;
- decreasing to $\frac{(\alpha-\beta)N}{N-4\beta}$ when $\alpha \le \frac{N}{4}$ and $q > \frac{(\alpha-\beta)N}{N-4\beta}$;
- increasing and passes the value $\frac{N}{4}$ after a finite number of steps when $\alpha > \frac{N}{4}$.

Claims (1) and (2) are then established. □

We can now deduce the following.

Corollary 5.12. *Suppose $(u_n)_n$ is a sequence of solutions of (N_{λ_n}) such that*

$$\sup_n \int_\Omega f^q(u_n) < +\infty \tag{5.13}$$

for $q \geq 1$. Then $\sup_n \|u_n\|_\infty < +\infty$, in either one of the following two cases:

1. *$f(t) = e^t$ and $q \geq \frac{N}{4}$;*
2. *$f(t) = (t+1)^p$ and $q > \frac{N}{4}\left(1 - \frac{1}{p}\right)$.*

Proof. (1) For $q > \frac{N}{4}$ it follows by standard regularity theory. The case $q = \frac{N}{4}$ and $f(t) = e^t$ can be treated in the following way. Since e^{u_n} is uniformly bounded in $L^{\frac{N}{4}}(\Omega)$, by elliptic regularity theory and the Sobolev embedding Theorem u_n is uniformly bounded in $W_0^{1,N}(\Omega)$. The Moser-Trudinger inequality states that, for suitable $\alpha > 0$ and $C_i > 0$, there holds

$$\int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} dx \leq C_0 + C_1 e^{C_2 \|\nabla u\|^N_{L^N}}, \qquad \forall u \in W_0^{1,N}(\Omega).$$

Now fix $\tau > \frac{N}{4}$ and pick $\tilde{C}$ big enough such that

$$e^{\tau z} \leq \tilde{C} e^{\alpha z^{\frac{N}{N-1}}},$$

for all $z \geq 0$. Then we have

$$\frac{1}{\tilde{C}} \int_\Omega e^{\tau u_n} dx \leq C_0 + C_1 e^{C_2 \|\nabla u_n\|^N_{L^N}} \leq \bar{C},$$

and so we have e^{u_n} uniformly bounded in $L^\tau(\Omega)$ for some $\tau > \frac{N}{4}$. By elliptic estimates, the validity of (5.4) follows also in this case.

(2) The case where $f(t) = (t+1)^p$ and $\frac{N}{4}\left(1 - \frac{1}{p}\right) < q \leq \frac{N}{4}$ follows from Theorem 5.5 with the choice $\alpha = q + \frac{N}{4p}$, $\beta = \frac{N}{4}$, since $\alpha > \frac{N}{4}$ and $\alpha > \beta$. Note that

$$\sup_n \int_\Omega \frac{f^\alpha(u_n)}{u_n^\beta + 1} \leq C \sup_n \int_\Omega f^q(u_n) < +\infty,$$

for some $C > 0$. □

We now show that the standard assumption $\sup_n \|f(u_n)\|_q < +\infty$, $q > \frac{N}{4}$, which guarantees the uniform boundedness of u_n can be weakened in a different way, through a uniform integrability condition on $f'(u_n)$. Indeed, we have the following result.

Theorem 5.14. *Suppose that for some $q > \frac{N}{4}$ we have*

$$\sup_n \int_\Omega f^s(u_n) < +\infty \qquad \text{for every } 1 \leq s < \frac{N}{N-2} \tag{5.15}$$

and

$$\sup_n \int_\Omega (f')^q(u_n) < +\infty. \tag{5.16}$$

Then,

$$\sup_n \|u_n\|_\infty < +\infty. \tag{5.17}$$

Proof. Observe that $\tilde{v}_n = -\Delta u_n$ satisfies

$$\begin{cases} \Delta^2 \tilde{v}_n \leq \lambda_n f'(u_n)\tilde{v}_n & \text{in } \Omega \\ \tilde{v}_n = 0, \ -\Delta\tilde{v}_n = \lambda_n & \text{on } \partial\Omega. \end{cases}$$

Introducing the function w_n as the solution of

$$\begin{cases} -\Delta w_n = \lambda_n & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega, \end{cases}$$

we are led to study uniform boundedness for $v_n = \tilde{v}_n - w_n$, a solution of

$$\begin{cases} \Delta^2 v_n \leq \lambda_n f'(u_n) v_n + \lambda_n f'(u_n) w_n & \text{in } \Omega \\ v_n = \Delta v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.18}$$

Since λ_n is bounded, by elliptic regularity theory we deduce that

$$\sup_n \|w_n\|_\infty < +\infty, \tag{5.19}$$

and then the uniform boundedness can be equivalently established on $\tilde{v}_n$ or v_n. First we show that under assumption (5.16), the following hold:
If $\sup_n \|v_n\|_s < +\infty \ \forall\, 1 \leq s < q_0$ and $q_0 \leq \frac{Nq}{4q-N}$,

$$\text{then} \sup_n \|v_n\|_s < +\infty \forall s < q_1. \tag{5.20}$$

If $\sup_n \|v_n\|_s < +\infty \ \forall\, 1 \leq s < q_0$ and $q_0 > \frac{Nq}{4q-N}$,

$$\text{then } \sup_n \|v_n\|_\infty < +\infty \tag{5.21}$$

where $q_1 := \frac{Nqq_0}{Nq_0+q(N-4q_0)}$. Indeed, by (5.16) and (5.19) we get that

$$\lambda_n f'(u_n)v_n + \lambda_n f'(u_n)w_n \text{ uniformly bounded in } L^s(\Omega),\ \forall\, 1 \le s < \frac{qq_0}{q+q_0}.$$

Thanks to (5.18), by elliptic regularity theory and the maximum principle the previous estimate translates into: if $\frac{qq_0}{q+q_0} \le \frac{N}{4}$

$$\sup_n \|v_n\|_s < +\infty \qquad \text{for every } 1 \le s < q_1,$$

and if $\frac{qq_0}{q+q_0} > \frac{N}{4}$

$$\sup_n \|v_n\|_\infty < +\infty.$$

Therefore, (5.20)-(5.21) are established.

Thanks to (5.19), by elliptic regularity theory assumption (5.15) reads on v_n as $\sup_n \|v_n\|_\infty < +\infty$ if $N = 2, 3$ and

$$\sup_n \int_\Omega (v_n)^s < +\infty \qquad \text{for every } 1 \le s < \frac{N}{N-4} \tag{5.22}$$

if $N \ge 4$. For $N \ge 4$, set $q_0 = \frac{N}{N-4}$ and inductively $q_{i+1} = \frac{Nqq_i}{Nq_i+q(N-4q_i)}$ as long as $q_i \le \frac{Nq}{4q-N}$ so to get

$$\sup_n \|v_n\|_s < +\infty \qquad \text{for every } 1 \le s < q_{n+1},$$

in view of (5.11). Since $q > \frac{N}{4}$, the sequence q_i is increasing and passes $\frac{Nq}{4q-N}$ after a finite number of steps. As soon as q_i becomes larger than $\frac{Nq}{4q-N}$, we can use (5.19) and (5.21) to get an uniform $L^\infty-$bound on $-\Delta u_n = v_n + w_n$, and in turn the validity of (5.17) follows by elliptic estimates. □

5.3 A general regularity result for low dimensions

To the best of our knowledge the only available energy estimates for smooth, semi-stable solutions u of (N_λ) so far, are given by

$$\int_\Omega f'(u)u^2 dx \le \int_\Omega f(u)u dx. \tag{5.23}$$

To see this, take $\psi = u$ in (5.2) and integrate by parts (N_λ) against u, and then equate. In view of Corollary 5.12, this yields the following

1. If $f(t) = e^t$, then $e^{u^*}(u^*)^2 \in L^1(\Omega)$ and u^* is then regular for $N \leq 4$.

2. If $f(t) = (t+1)^p$, then $(u^*+1)^p \in L^{\frac{p+1}{p}}(\Omega)$, therefore u^* is a regular solution for $2 \leq N < \frac{4(p+1)}{p-1}$ (equivalently $N \leq 4$ or $1 \leq p < \frac{N+4}{N-4}$ and $N > 4$)

3. If $f(t) = (1-t)^{-2}$, then $(1-u^*)^{-2} \in L^{\frac{3}{2}}(\Omega)$ and u^* is regular for $N \leq 4$. See Chapter 12 of [13].

We shall substantially improve on these results in the next sections. For now, we start by considering the case of a general superlinear f and establish a fourth order analogue of the results of Nedev [23] for $N \leq 3$ and Cabre [5] for $N = 4$, regarding the regularity of the extremal solution of second order eigenvalue problems with a nonlinearity of type (R).

Theorem 5.24. *Let f be a nonlinearity of type (R). Then the extremal solution u^* of (N_λ) is regular for $N \leq 5$, while $f(u^*) \in L^q(\Omega)$ for all $q < \frac{N}{N-2}$ if $N \geq 6$.*

We shall split the proof in several lemmas that may have their own interest, in particular the simple new energy estimate given in Lemma 5.27 below, coupled with a pointwise estimates on $-\Delta u$ given in Lemma 5.25, and which was motivated by the proof of Souplet of the Lane-Emden conjecture in four space dimensions [24]. We start by the latter (next two lemmas) which does not require the stability of the solutions.

Lemma 5.25. *Suppose u is a solution of (N_λ) and g is a smooth function defined on the range of u with $f(t) \geq g(t)g'(t)$ and $g(t), g'(t), g''(t) \geq 0$ on the range of u with $g(0) = 0$. Then*

$$-\Delta u \geq \sqrt{\lambda} g(u) \qquad \text{in } \Omega. \tag{5.26}$$

Proof. Define $v := -\Delta u - \sqrt{\lambda} g(u)$ and so $v = 0$ on $\partial\Omega$ and a computation shows that

$$-\Delta v + \sqrt{\lambda} g'(u) v = \lambda[f(u) - g(u)g'(u)] + \sqrt{\lambda} g''(u)|\nabla u|^2 \qquad \text{in } \Omega.$$

The assumptions on g allow one to apply the maximum principle and obtain that $v \geq 0$ in Ω. □

Now we use the stability condition on the solution.

Lemma 5.27. *Suppose u is a semi-stable solution of (N_λ). Then*

$$\int_\Omega f''(u)(-\Delta u)|\nabla u|^2 dx \leq \lambda \int_\Omega f(u)dx. \tag{5.28}$$

Proof. Set $\psi = \Delta u$ in (5.2) to arrive at

$$I := \int_\Omega f'(u)(\Delta u)^2 dx \leq \int_\Omega \Delta^2 u f(u)dx =: J.$$

Now an integration by parts shows that

$$\begin{aligned} I &= \int_\Omega f''(u)(-\Delta u)|\nabla u|^2 dx - \int_\Omega f'(u)\nabla u \cdot \nabla \Delta u dx \\ J &= \lambda \int_\Omega f(u)dx - \int_\Omega f'(u)\nabla u \cdot \nabla \Delta u dx, \end{aligned}$$

in view of $f(0) = 1$. Since $I \leq J$ one obtains the result. □

Lemma 5.29. *Suppose u is a semi-stable solution of (N_λ) and that g satisfies the conditions of Lemma 5.25. If $H(u) := \int_0^u f''(\tau)g(\tau)d\tau$, then*

$$\int_\Omega g(u)H(u)dx \leq \int_\Omega f(u)dx. \tag{5.30}$$

Proof. We rewrite the result from Lemma 5.27 as

$$\begin{aligned} \lambda \int_\Omega g(u)H(u)dx &\leq \sqrt{\lambda}\int_\Omega (-\Delta u)H(u)dx = \sqrt{\lambda}\int_\Omega \nabla H(u)\cdot \nabla u dx \\ &= \sqrt{\lambda}\int_\Omega H'(u)|\nabla u|^2 dx \leq \lambda \int_\Omega f(u)dx, \end{aligned}$$

where the two inequalities use the pointwise bound from Lemma 5.25. □

Lemma 5.31. *Suppose $u \geq 0$ is a semi-stable solution of (N_λ). Then*

$$\textstyle\int_\Omega \frac{f(u)^{\frac{3}{2}}}{\sqrt{u}+1}dx \leq C \quad \textit{and} \qquad \int_\Omega f(u)dx \leq C, \tag{5.32}$$

for some constant $C > 0$ independent of λ and u.

Proof. Define for $u \geq 0$, the function

$$g(u) := \sqrt{2}\left(\int_0^u (f(t)-1)dt\right)^{\frac{1}{2}}.$$

Clearly $g(0) = 0$ and $g \geq 0$. Now square g and take a derivative to see that $2g(u)g'(u) = 2(f(u)-1)$ and so we satisfy the requirement that $f(u) \geq g(u)g'(u)$. Also from this we see that $g'(u) \geq 0$.
We now show that $g''(u) \geq 0$. Note that $g''(u)$ has the same sign as

$$\gamma(u) := f'(u) \int_0^u (f(t)-1)dt - \frac{1}{2}(f(u)-1)^2.$$

Now $\gamma(0) = 0$ and

$$\gamma'(u) = f'(u)(f(u)-1) + f''(u) \int_0^u (f(t)-1)dt - (f(u)-1)f'(u),$$

and so $\gamma'(u) \geq 0$ and hence $\gamma(u) \geq 0$.
By Lemma 5.29 we have

$$\int_\Omega g(u)H(u)dx \leq \int_\Omega f(u)dx, \tag{5.33}$$

where

$$H(u) := \int_0^u f''(\tau)g(\tau)d\tau.$$

Without loss of generality, we can assume that $\int_0^1 (f(t)-1)dt > 0$. For $u > 1$ we have

$$H(u) \geq \sqrt{2} \int_1^u f''(\tau) \left(\int_0^\tau (f(t)-1)dt \right)^{\frac{1}{2}} d\tau \geq \sqrt{2} C_0 (f'(u) - f'(1))$$

where $C_0 := \left(\int_0^1 (f(t)-1)dt \right)^{\frac{1}{2}}$. Since by convexity $f'(u) \geq \frac{f(u)-1}{u} \to \infty$ as $u \to \infty$, we can find $M > 0$ large so that

$$H(u) \geq C_0 f'(u) \qquad \forall\, u \geq M.$$

Since

$$\int_0^u (f(t)-1) \;\; \geq \;\; \int_1^u (f(t)-1) \geq (1 - \frac{1}{f(1)}) \int_1^u f(t)dt \tag{5.34}$$

(5.35)

for $u \geq 1$, from (5.33) and the above estimate we see that

$$\begin{aligned}
&\int_\Omega f'(u) \left(\int_0^{u(x)} f(t)dt \right)^{\frac{1}{2}} dx \\
&\leq \int_{\{u \geq M\}} f'(u) \left(\int_0^{u(x)} f(t)dt \right)^{\frac{1}{2}} dx + f'(M) \left(\int_0^M f(t)dt \right)^{\frac{1}{2}} |\Omega| \\
&\leq C_0^{-1}(1 - \frac{1}{f(1)})^{-1} \int_\Omega H(u) \left(\int_0^{u(x)} (f(t)-1)dt \right)^{\frac{1}{2}} dx \\
&+ f'(M) \left(\int_0^M f(t)dt \right)^{\frac{1}{2}} |\Omega| \\
&\leq C_1 \int_\Omega f(u)dx
\end{aligned}$$

for some $C_1 > 0$ (independent of λ and u), in view of $|\Omega| \leq \int_\Omega f(u)$. Defining

$$h(u) := u(f')^2(u) \int_0^u f(t)dt - \frac{1}{6}(f^{\frac{3}{2}}(u) - 1)^2,$$

we have that $h \geq 0$. Note $h(0) = 0$ and

$$h'(u) = 2uf'(u)f''(u) \int_0^u f(t)dt + u(f')^2(u)f(u) + I,$$

where

$$I = (f')^2(u) \int_0^u f(t)dt - \frac{1}{2}f^2(u)f'(u) + \frac{1}{2}f^{\frac{1}{2}}(u)f'(u).$$

Since

$$f'(u)^2 \int_0^u f(t)dt \geq f'(u) \int_0^u f'(t)f(t)dt = f'(u)\frac{f(u)^2}{2} - \frac{f'(u)}{2},$$

we have that $I \geq 0$, and then $h'(u) \geq 0$. Hence, $h(u) \geq 0$ leads to the fundamental estimate:

$$f'(u) \left(\int_0^u f(t)dt \right)^{\frac{1}{2}} \geq \frac{f^{\frac{3}{2}}(u) - 1}{\sqrt{6}(\sqrt{u} + 1)} \qquad \forall\, u \geq 0. \tag{5.36}$$

So by (5.36) we get that

$$\int_\Omega \frac{f(u)^{\frac{3}{2}}}{\sqrt{u}+1} dx \leq \int_\Omega \frac{f(u)^{\frac{3}{2}} - 1}{\sqrt{u}+1} dx + |\Omega| \leq (\sqrt{6}C_1 + 1) \int_\Omega f(u)dx.$$

Since f is superlinear at ∞ this implies the validity of (5.32). For later purposes, note that from the above estimates there holds

$$
\begin{aligned}
g(u)H(u) &\geq \sqrt{2}C_0(1-\frac{1}{f(1)})^{\frac{1}{2}}\left(\int_0^u f(t)dt\right)^{\frac{1}{2}} f'(u)\\
&\geq \sqrt{2}C_0(1-\frac{1}{f(1)})^{\frac{1}{2}}\frac{f^{\frac{3}{2}}(u)-1}{\sqrt{6}(\sqrt{u}+1)}
\end{aligned}
$$

for $u \geq M$, and then by superlinearity of f at ∞

$$\frac{g(u)H(u)}{f(u)} \to +\infty \qquad \text{as } u \to +\infty.$$

Hence, for general nonlinearities f of type (R) we can re-state Lemma 5.29 as

$$\int_\Omega g(u)H(u) \leq C \tag{5.37}$$

for every semi-stable solution u of (N_λ), where $g(u)$ is exactly as before and C is independent of λ and u. □

Proof of Theorem 5.24: Recalling that u^* is the limit of the classical solutions u_λ as $\lambda \nearrow \lambda^*$, it follows immediately from estimate (5.32) and Theorem 5.5. Indeed, in this case we can take $q = 1$, $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$ to conclude that $u^* \in L^\infty(\Omega)$ when $N \leq 5$, and $f(u^*) \in L^q(\Omega)$ for every $q < \frac{N}{N-2}$ when $N \geq 6$.

5.4 Regularity in higher dimension (I)

In this section, we will consider nonlinearities f of type (R) which satisfy the following growth condition:

$$\liminf_{t\to+\infty} \frac{f(t)f''(t)}{(f')^2(t)} > 0. \tag{5.38}$$

The aim is to gain dimensions $N = 6, 7$ in Theorem 5.24 by showing that L^2-bounds on $f(u)$ are still in order, as we state in the following theorem.

Theorem 5.39. *Let f be a nonlinearity of type (R) so that (5.38) holds. Let u be a semi-stable solution of (N_λ). Then*

$$\int_\Omega f^2(u)dx \leq C,$$

where $C > 0$ is independent of λ and u.

By standard elliptic regularity theory, Theorem 5.39 immediately yields the following improvement on Theorem 5.24.

Theorem 5.40. *Let f be a nonlinearity of type (R) so that (5.38) holds. Then the extremal solution u^* of (N_λ) is regular for $N \leq 7$.*

Proof (of Theorem 5.39) Since

$$\Big(f'(t)\int_0^t f\Big)' = f''(t)\int_0^t f + f'(t)f(t) \geq f'(t)f(t) = \Big(\frac{1}{2}f^2(t)\Big)',$$

one can integrate on $[0,u]$ to get

$$f'(u)\int_0^u f \geq \frac{1}{2}f^2(u) - \frac{1}{2}.$$

Since $f(u) \to +\infty$ as $u \to +\infty$, we can find $M \geq 1$ large so that

$$f'(u)\int_0^u f \geq \frac{1}{4}f^2(u) \qquad \forall\, u \geq M. \tag{5.41}$$

Setting

$$\delta := \liminf_{t\to+\infty} \frac{f(t)f''(t)}{(f')^2(t)} > 0,$$

we can –modulo taking a larger M – also assume that

$$f(u)f''(u) \geq \frac{\delta}{2}(f')^2(u) \qquad \forall\, u \geq M. \tag{5.42}$$

By (5.41)-(5.42) we get

$$\begin{aligned}\Big[\Big(\int_1^u f''(t)(\int_0^t f)^{\frac{1}{2}}\Big)(\int_0^u f)^{\frac{1}{2}}\Big]' &\geq f''(u)\int_0^u f \\ &\geq \frac{\delta}{2}\frac{(f')^2(u)}{f(u)}\int_0^u f \\ &\geq \frac{\delta}{8}f(u)f'(u) \\ &= \frac{\delta}{16}(f^2(u))'\end{aligned}$$

for all $u \geq M$, which, integrated once more in $[M,u]$, $u \geq M$, yields to

$$\Big(\int_1^u f''(t)(\int_0^t f)^{\frac{1}{2}}\Big)(\int_0^u f)^{\frac{1}{2}} \geq \frac{\delta}{16}f^2(u) - \frac{\delta}{16}f^2(M).$$

Then we can find $N \geq M$ large so that for $u \geq N$ we have

$$\left(\int_1^u f''(t)(\int_0^t f)^{\frac{1}{2}}\right)(\int_0^u f)^{\frac{1}{2}} \geq \frac{\delta}{32} f^2(u).$$

Setting as always $g(u) = \sqrt{2}\left(\int_0^u (f(t)-1)dt\right)^{\frac{1}{2}}$, by (5.34) we can now deduce

$$\begin{aligned} g(u)H(u) &\geq 2\left(\int_1^u f''(t)(\int_0^t (f-1)ds)^{\frac{1}{2}}\right)(\int_0^u (f-1)dt)^{\frac{1}{2}} \\ &\geq \frac{\delta}{16}(1-\frac{1}{f(1)})f^2(u) \end{aligned}$$

for $u \geq N$. By Lemma 5.29 as re-stated in (5.37) we finally get that

$$\begin{aligned} \int_\Omega f^2(u)dx &\leq \int_{\{u \geq N\}} f^2(u)dx + f^2(N)|\Omega| \\ &\leq \frac{16}{\delta}(1-\frac{1}{f(1)})^{-1}\int_\Omega g(u)H(u) \\ &\quad +f^2(N)|\Omega| \leq C \end{aligned}$$

for some $C > 0$ independent of λ and u.

Theorem 5.39 combined with Corollary 5.12 gives also immediately the following results.

Corollary 5.43. *When $f(t) = (t+1)^p$, $p > 1$, the extremal solution u^* of (N_λ) is regular if either $N \leq 8$ or if $N \geq 9$ and $p < \frac{N}{N-8}$. When $f(t) = e^t$, this is true for $N \leq 8$.*

5.5 Regularity in higher dimension (II)

We are still considering nonlinearities f of type (R). For $N \geq 6$ we want to improve upon Theorem 5.24 under the following growth condition on f

$$\gamma := \limsup_{t \to +\infty} \frac{f(t)f''(t)}{(f')^2(t)} < +\infty. \tag{5.44}$$

Typical examples of such nonlinearities are again $f(t) = e^t$ (with $\gamma = 1$) and $f(t) = (t+1)^p$ (with $\gamma = 1 - \frac{1}{p}$). The aim is to get the regularity of the extremal solution also in dimensions higher than 5 for values of γ not too large:

Theorem 5.45. *Let $N \geq 6$ and f be a nonlinearity of type (R) satisfying (5.44). The extremal solution u^* of (N_λ) is regular for $N < \frac{8}{\gamma}$.*

The validity of Theorem 5.45 follows easily Theorem 5.24, Theorem 5.14 and the following crucial estimate for stable solutions. To apply Theorem 5.14, we need to require exactly $\gamma < \frac{8}{N}$ when $N \geq 6$, and (5.44) guarantees the validity of (5.47) with $0 < \gamma + \epsilon < 2$ and $M > 0$ large enough.

Theorem 5.46. *Let f be a nonlinearity of type (R) so that*

$$f(u)f''(u) \leq \gamma (f')^2(u) \qquad \forall\, u \geq M, \tag{5.47}$$

for some $0 < \gamma < 2$ and $M > 0$. Let u be a semi-stable solution of (N_λ). Then

$$\int_\Omega (f')^{\frac{2}{\gamma}}(u)dx \leq C, \tag{5.48}$$

where $C > 0$ is a constant independent of u and λ.

Proof. Re-write (5.47) as

$$\frac{d}{dt}\log(f'(t)) \leq \frac{d}{dt}\log(f^\gamma(t)) \qquad \forall\, t \geq M$$

and integrate over $[M, u]$ to deduce that

$$f'(u) \leq \frac{f'(M)}{f^\gamma(M)} f^\gamma(u) \qquad \forall\, u \geq M. \tag{5.49}$$

Since $f(u) \geq f(0) = 1$, we can write that

$$f'(u) \leq C_0 f^\gamma(u) \qquad \forall\, u \geq 0, \tag{5.50}$$

where C_0 is a suitable large constant. Setting

$$\Gamma(u) := \int_0^u f(t)dt - \frac{1}{(2-\gamma)C_0}\left(f^{2-\gamma}(u) - 1\right),$$

one notes that $\Gamma(0) = 0$ and $\Gamma'(u) = f(u) - \frac{f^{1-\gamma}(u)f'(u)}{C_0} \geq 0$. Hence, the following estimate holds for every $u \geq 0$:

$$\sqrt{\int_0^u f(t)dt} \geq \frac{1}{\sqrt{(2-\gamma)C_0}}\left(f^{2-\gamma}(u) - 1\right)^{\frac{1}{2}}. \tag{5.51}$$

As in the previous section, set $g(u) = \sqrt{2}\left(\int_0^u (f(t)-1)dt\right)^{\frac{1}{2}}$ in such a way that it satisfies the assumptions of Lemma 5.29. By (5.34), (5.51) and the superlinearity of f at ∞, we can find $N \geq 1$ large so that for all $u \geq N$,

$$\begin{aligned} g(u) &\geq \sqrt{2}(1-\frac{1}{f(1)})^{\frac{1}{2}}\left(\int_0^u f(t)dt\right)^{\frac{1}{2}} \\ &\geq \sqrt{2}\left(\frac{f(1)-1}{(2-\gamma)C_0 f(1)}\right)^{\frac{1}{2}}(f^{2-\gamma}(u)-1)^{\frac{1}{2}} \\ &\geq \left(\frac{f(1)-1}{(2-\gamma)C_0 f(1)}\right)^{\frac{1}{2}} f^{1-\frac{\gamma}{2}}(u). \end{aligned}$$

Setting $C_1 := \left(\frac{f(1)-1}{(2-\gamma)C_0 f(1)}\right)^{\frac{1}{2}}$, we use (5.50) to find $N' \geq N$ sufficiently large so that for $u \geq N'$

$$\begin{aligned} H(u) &:= \int_0^u f''(t)g(t)dt \geq C_1 \int_N^u f''(t)f^{1-\frac{\gamma}{2}}(t)dt \\ &\geq C_1 C_0^{\frac{1}{2}-\frac{1}{\gamma}} \int_N^u f''(t)(f')^{\frac{1}{\gamma}-\frac{1}{2}}(t)dt \\ &= C_1 C_0^{\frac{1}{2}-\frac{1}{\gamma}} \frac{2\gamma}{\gamma+2}\left((f')^{\frac{1}{\gamma}+\frac{1}{2}}(u)-(f')^{\frac{1}{\gamma}+\frac{1}{2}}(N)\right) \\ &\geq C_1 C_0^{\frac{1}{2}-\frac{1}{\gamma}} \frac{\gamma}{\gamma+2}(f')^{\frac{1}{\gamma}+\frac{1}{2}}(u), \end{aligned}$$

where we have used the convexity of f and the fact that $f'(u) \to +\infty$ as $u \to +\infty$. In conclusion, setting

$$C_2 := C_1 C_0^{\frac{1}{2}-\frac{1}{\gamma}} \frac{\gamma}{\gamma+2}\left(\frac{f(1)-1}{(2-\gamma)C_0 f(1)}\right)^{\frac{1}{2}}$$

we have that

$$\int_\Omega (f')^{\frac{1}{\gamma}+\frac{1}{2}}(u)f^{1-\frac{\gamma}{2}}(u) \leq C_2^{-1}\int_\Omega H(u)g(u) + (f')^{\frac{1}{\gamma}+\frac{1}{2}}(N')f^{1-\frac{\gamma}{2}}(N')|\Omega|.$$

To complete the proof of Theorem 5.46, it suffices to couple this lower bound with (5.37) to obtain

$$\int_\Omega (f')^{\frac{1}{\gamma}+\frac{1}{2}}(u)f^{1-\frac{\gamma}{2}}(u) \leq C,$$

and then by (5.49) to get

$$\int_\Omega (f')^{\frac{2}{\gamma}}(u) \leq C'$$

for any stable solution u, where C, C' are independent of λ and u. □

Remark 5.5.1. a) As a by-product of the above theorem, one obtains again the improvements for the exponential and power nonlinearities established in Corollary 5.43. Indeed, when $f(t) = (t+1)^p$, it turns out that $\gamma = 1 - \frac{1}{p}$, and then u^* is a regular solution whenever $N \leq 5$ and $6 \leq N < 8 + \frac{8}{p-1}$. We can collect the two cases as $N \leq 8$ or $N \geq 9$ and $p < \frac{N}{N-8}$. When $f(t) = e^t$, we have that $\gamma = 1$, and then u^* is a regular solution for $N < 8$. The missing dimension $N = 8$ follows directly from Theorem 5.14 in view of the identity,

$$\int_\Omega (f')^{\frac{2}{\gamma}}(u)dx = \int_\Omega (f')^2(u)dx = \int_\Omega f^2(u)dx = \int_\Omega f^{\frac{N}{4}}(u)dx.$$

b) Recall from [11] that when $f(t) = e^t$ on the unit ball in $\mathbb{R}^N$, the extremal solution associated with (D_λ) is then smooth provided $N \leq 12$. This suggests that our regularity results are not optimal, which likely is a product of the deficient energy estimate obtained in Lemma 5.27 above.
c) Integrating once more (5.49) one sees that

- $f(u) \leq f(M)e^{\frac{f'(M)}{f(M)}(u-M)}$ for $u \geq M$ when $\gamma = 1$
- $f(u) \leq \left[f^{1-\gamma}(M) + (1-\gamma)\frac{f'(M)}{f^\gamma(M)}(u-M)\right]^{\frac{1}{1-\gamma}}$ for $u \geq M$ when $0 < \gamma < 1$.

This explains why (5.44) is sometimes referred to as a growth condition for f.
d) For exponential and power nonlinearities, the uniform bound (5.48) can be re-formulated as an L^2-bound on $f(u)$. Indeed, one can define $g(u)$ as above, and use Lemma 5.29 to deduce directly such a bound, which in turn shows that the loss in optimality is not really coming from Theorem 5.46.

5.6 Singular nonlinearities

Nonlinearities of the form $(1-u)^{-p}$, $p > 0$, have recently attracted much attention, due to their connection with the so-called MEMS (Micro Electro-Mechanical Systems) technology. Neglecting torsion effects, one is led to

study the second-order nonlinear eigenvalue problem with $p = 2$. In this case, the picture is well understood [12, 15] and we refer the interested reader to the recent monograph [13]. The fourth-order case has been firstly addressed in [18] both in the form (D_λ) and (N_λ). As already mentioned in the introduction, for problem (D_λ) the existence of the minimal branch on the unit ball has been proved in [7] along with its compactness for $N \leq 8$ [9]. Since a maximum principle holds for (N_λ), the existence of the minimal branch for (N_λ) on a general domain follows by the same argument as in [7] for (D_λ).
We will consider now the question of regularity of $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ in terms of the dimension N. We will not consider general nonlinearities of type (S) in the sequel, but we shall restrict our attention to the interesting case $(1-u)^{-p}$. The general case has been addressed in [8] for the second-order case, and growth conditions as (5.38) and (5.44) are no longer sufficient for the analysis. We now establish the following result.

Theorem 5.52. *Suppose $p > 1$ and $p \neq 3$. Then the extremal solution u^* is regular (i.e. $\sup_\Omega u^* < 1$) provided $N \leq \frac{8p}{p+1}$.*

This will follow immediately from the following two theorems.

Theorem 5.53. *Let u_n denote a sequence of solutions of (N_{λ_n}) such that there is some $\alpha > 1$ and $\alpha \geq \frac{(p+1)N}{4p}$ such that $\sup_n \|f(u_n)\|_\alpha < \infty$. Then $\sup_n \|u_n\|_\infty < 1$.*

Proof. We suppose that N is big enough so that $\frac{(p+1)N}{4p} > 1$, the lower dimensional cases being similar we omit their details. If $f(u_n)$ is bounded in $L^{\frac{(p+1)N}{4p}}$, then by elliptic regularity we have u_n bounded in $W^{4,\frac{(p+1)N}{4p}}$. By the Sobolev imbedding theorem we have u_n bounded in the space

$$C^{4-\left[\frac{4p}{p+1}\right]-1,\left[\frac{4p}{p+1}\right]+1-\frac{4p}{p+1}}(\overline{\Omega}).$$

This naturally breaks into the two cases:

- $1 < p < 3$ and then u_n is bounded in $C^{1,\frac{3-p}{p+1}}$
- $p > 3$ and u_n is then bounded in $C^{0,\frac{4}{p+1}}$.

We now let $x_n \in \Omega$ be such that $u_n(x_n) = \max_\Omega u_n$. We claim that there exists some $C > 0$, independent of n, such that

$$|u_n(x) - u_n(x_n)| \leq C|x - x_n|^{\frac{4}{p+1}}, \qquad x \in \Omega.$$

For the second case this is immediate, while for the first we use the fact that $\nabla u_n(x_n) = 0$ and the fact that there is some $0 \le t_n \le 1$ such that

$$\begin{aligned} u_n(x) - u_n(x_n) &= \nabla u_n(x_n + t_n(x - x_n)) \cdot (x - x_n) \\ &= (\nabla u_n(x_n + t_n(x - x_n)) - \nabla u_n(x_n)) \cdot (x - x_n) \end{aligned}$$

along with the fact that ∇u_n is bounded in $C^{0,\frac{3-p}{p+1}}$ to show the claim.

To complete the proof, we work towards a contradiction, and assume, after passing to a subsequence, that $u_n(x_n) = 1 - \varepsilon_n \to 1$. By passing to another subsequence, we can assume that u_n converges in $C(\overline{\Omega})$ which along with the boundary conditions guarantees that $x_n \to x_0 \in \Omega$. Then one has

$$\begin{aligned} 1 - u_n(x) &= 1 - u_n(x_n) + u_n(x_n) - u_n(x) \\ &= \varepsilon_n + u_n(x_n) - u_n(x) \\ &\le \varepsilon_n + C|x - x_n|^{\frac{4}{p+1}}, \end{aligned}$$

and so there is some $C_p > 0$ such that

$$(1 - u_n(x))^{\frac{(p+1)N}{4}} \le C_p \left(\varepsilon_n^{\frac{(p+1)N}{4}} + |x - x_n|^N \right).$$

From this one sees that

$$f(u_n(x))^{\frac{(p+1)N}{4p}} \ge \frac{C_p^{-1}}{\varepsilon_n^{\frac{(p+1)N}{4}} + |x - x_n|^N} := h_n(x).$$

But since $x_n \to x_0 \in \Omega$ and $\varepsilon_n \to 0$, ones sees that $\int_\Omega h_n(x)dx \to \infty$ which contradicts the integrability condition on $f(u_n)$. Hence we must have $\sup_n \|u_n\|_\infty < 1$.

□

We now obtain the familiar L^2 bound on $f(u)$ for semi-stable solutions. We prefer to prove this results using an explicit calculation, even if this result also follows from Theorem 4.1.

Theorem 5.54. *Suppose $p > 1$ and $u \ge 0$ is a semi-stable solution of (N_λ). Then*

$$\|f(u)\|_2 \le C,$$

where C is independent of u and λ.

Proof. Define

$$g(u) := \sqrt{\frac{2}{p-1}}\left(\frac{1}{(1-u)^{\frac{p-1}{2}}} - 1\right).$$

Note that this choice of g is different from the one used above, as it is easier to manage. It does verify the conditions of Lemma 3.2 and therefore one has $-\Delta u \geq g(u)$ a.e. in Ω, and by Lemma 3.4 we have

$$\int_\Omega g(u)H(u)dx \leq \int_\Omega f(u)dx, \tag{5.55}$$

where $H(u) := \int_0^u f''(\tau)g(\tau)d\tau$. A computation shows that

$$H(u) = C_p\left(\frac{1}{(1-u)^{\frac{3p+1}{2}}} - 1\right) + \tilde{C}_p\left(1 - \frac{1}{(1-u)^{p+1}}\right)$$

where $C_p, \tilde{C}_p > 0$. Now writing out (5.55) one obtains an estimate of the form

$$\int_\Omega \frac{1}{(1-u)^{2p}}dx \leq C(p)\int_\Omega \frac{1}{(1-u)^{\frac{3p+1}{2}}}dx + C(u)\int_\Omega \frac{1}{(1-u)^p}dx.$$

Since $p > 1$, we have that $\frac{3p+1}{2} < 2p$, from which one easily obtains the desired result. □

Acknowledgements: This work began while the second author was visiting the Pacific Institute for the Mathematical Sciences, for which he thanks Dr. Ghoussoub for the kind invitation and hospitality.

Bibliography

[1] G. Arioli, F. Gazzola, H.-C. Grunau and E. Mitidieri, *A semilinear fourth order elliptic problem with exponential nonlinearity*, SIAM J. Math. Anal. 36 (2005), 1226–1258.

[2] E. Berchio and F. Gazzola, *Some remarks on biharmonic elliptic problems with positive, increasing and convex nonlinearities*, Electronic J. Differential Equations 2005(2005), No. 34, 20 pp.

[3] T. Boggio, *Sulle funzioni di Green dordine m*, Rend. Circ. Mat. Palermo (1905), 97–135.

[4] H. Brezis and L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid 10 (1997), 443–469.

[5] X. Cabré, *Regularity of minimizers of semilinear elliptic problems up to dimension four*, preprint, arXiv:0909.4696v2.

[6] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. 238 (2006), 709–733.

[7] D. Cassani, J. do O and N. Ghoussoub, *On a fourth order elliptic problem with a singular nonlinearity*, Adv. Nonlinear Stud. **9** (2009), 177-197.

[8] D. Castorina, P. Esposito and B. Sciunzi, *Degenerate elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations **34** (2009), 279–306.

[9] C. Cowan, P. Esposito, N. Ghoussoub and A. Moradifam *The critical dimension for a fourth order elliptic problem with singular nonlinearity*, Arch. Ration. Mech. Anal., in press (2009) 19 pp.

[10] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variation methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal. 58 (1975), 207–218.

[11] J. Davila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. 39 (2007), 565–592.

[12] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), 1731–1768.

[13] P. Esposito, N. Ghoussoub and Y. Guo, *Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS*, Research Monograph, Courant Lecture Notes, in press (2009) 332 pp.

[14] L. Evans, *Partial differential equations*, Graduate Studies in Math. **19** AMS, Providence, RI, 1998.

[15] N. Ghoussoub and Y. Guo, *On the partial differential equations of electro MEMS devices: stationary case*, SIAM J. Math. Anal. 38 (2007), 1423–1449.

[16] N. Ghoussoub and A. Moradifam, *Bessel pairs and optimal improved Hardy and Hardy-Rellich inequalities*, Math. Ann., in press (Arxive 2007).

[17] Z. Guo and J. Wei, *On a fourth order nonlinear elliptic equation with negative exponent*, SIAM J. Math. Anal. 40 (2008/09), 2034–2054.

[18] F.H. Lin and Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic acutation*, Proc. R. Soc. London Ser. A 463 (2007), 1323–1337.

[19] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, Houston J. Math. 23 (1997), 161-168.

[20] F. Mignot and J-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations 5 (1980), 791–836.

[21] A. Moradifam, *The singular extremal solutions of the bilaplacian with exponential nonlinearity*, Proc. Amer. Math. Soc., in press (2009).

[22] A. Moradifam, *On the critical dimension of a fourth order elliptic problem with negative exponent*, J. Differential Equations 248 (2010), 594–616.

[23] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations,* C. R. Acad. Sci. Paris Sèr. I Math. 330 (2000), 997–1002.

[24] P. Souplet, *The proof of the Lane-Emden conjecture in four space dimensions*, Adv. Math. 221 (2009), 1409–1427.

Chapter 6

Regularity of the extremal solutions in elliptic systems[5]

6.1 Introduction

In this short note we are interested in solutions of the elliptic system given by

$$(P)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u &= \lambda e^v & \Omega \\ -\Delta v &= \gamma e^u & \Omega \\ u &= 0 & \partial\Omega \\ v &= 0 & \partial\Omega \end{cases}$$

where λ, γ are positive parameters and where Ω is a smooth bounded domain in $\mathbb{R}^N$. In particular we are interested in the regularity of the extremal solutions associated with $(P)_{\lambda,\gamma}$, which we define more precisely later. Along the diagonal $\lambda = \gamma$ the problem $(P)_{\lambda,\gamma}$ reduces to the scalar analog of $(P)_{\lambda,\gamma}$, see below. Provided one stays sufficiently close to the diagonal we show that some basic maximum principle arguments coupled with a standard energy estimate approach (the familiar approach in the scalar case) shows the regularity of the extremal solutions in the expected dimensions.

We now recall the well studied scalar version (with general nonlinearity f) of $(P)_{\lambda,\gamma}$ given by

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) & \Omega \\ u &= 0 & \partial\Omega \end{cases}$$

where λ is a positive parameter and where Ω is a bounded domain in $\mathbb{R}^N$. See, for instance, [1], [2], [6], [7] and [8]. Here generally one assumes that f is a smooth, increasing, convex nonlinearity with $f(0) = 1$ and f superlinear at ∞, ie. $\lim_{u \to \infty} \frac{f(u)}{u} = \infty$. It is known that there is an non degenerate finite interval $\mathcal{U} = (0, \lambda^*)$ such that for all $0 < \lambda < \lambda^*$ there exists a smooth,

[5]A version of this chapter has been submitted for publication: C. Cowan *Regularity of the extremal solutions in elliptic systems*

minimal solution u_λ of $(P)_\lambda$. By minimal we mean that any other solution v of $(P)_\lambda$ satisfies $v \geq u_\lambda$ a.e. in Ω. In addition one can show that for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$. This allows one to define the **extremal solution**

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

and it can be shown that u^* is the unique weak solution of $(P)_{\lambda^*}$. Also it is known that for $\lambda > \lambda^*$ there are no weak solutions. One can also show that for each $0 < \lambda < \lambda^*$ the minimal solution u_λ is semi-stable in the sense that the principle eigenvalue of the linear operator

$$L_{\lambda, u_\lambda} := -\Delta - \lambda f'(u_\lambda),$$

over $H_0^1(\Omega)$ is nonnegative. Using the variational structure this implies that

$$\int_\Omega \lambda f'(u_\lambda) \psi^2 dx \leq \int_\Omega |\nabla \psi|^2 dx, \qquad \forall \psi \in H_0^1(\Omega).$$

One can now ask the question whether u^* is a classical solution of $(P)_{\lambda^*}$? Elliptic regularity shows this is equivalent to the boundedness of u^*. In the case where $f(u) = e^u$ one can show that u^* is bounded provided $N \leq 9$. Moreover this is optimal after one considers the fact that $u^*(x) = -2\log(|x|)$ provided Ω is the unit ball in $\mathbb{R}^N$ where $N \geq 10$. For more results concerning the regularity of the extremal solution u^* the reader should see [10], [4], [3] and [11]. We mention that vital to all the results concerning the regularity of u^* is to use the semi-stability of the minimal solutions u_λ to obtain a priori estimates and then to pass to the limit.

We now return to the system $(P)_{\lambda,\gamma}$ and we follow the work of M. Montenegro [9], where all of the following results are taken from. We also mention that he obtains many more results and also that he studies a much more general system then $(P)_{\lambda,\gamma}$. We let $\mathcal{Q} = \{(\lambda, \gamma) : \lambda, \gamma > 0\}$ and we define

$$\mathcal{U} := \{(\lambda, \gamma) \in \mathcal{Q} : \text{ there exists a smooth solution } (u, v) \text{ of } (P)_{\lambda,\gamma}\}.$$

We set $\Upsilon := \partial\mathcal{U} \cap \mathcal{Q}$. The curve Υ is well defined and separates $\mathcal{Q}$ into two connected components $\mathcal{Q}$ and $\mathcal{V}$. We omit the various properties of Υ but the interested reader should consult [9]. One point we mention is that if for $x, y \in \mathbb{R}^2$ we say $x \leq y$ provided $x_i \leq y_i$ for $i = 1, 2$ then it is easily seen, using the method of sub/supersolutions, that if $(0,0) < (\lambda_0, \gamma_0) \leq (\lambda, \gamma) \in \mathcal{U}$ then $(\lambda_0, \gamma_0) \in \mathcal{U}$. Now it can be shown that for each $(\lambda, \gamma) \in \mathcal{U}$ there exists a smooth minimal solution $(u_{\lambda,\gamma}, v_{\lambda,\gamma})$ of $(P)_{\lambda,\gamma}$ and if $(0,0) < (\lambda_1, \gamma_1) \leq (\lambda_2, \gamma_2) \in \mathcal{U}$ then

$$(u_{\lambda_1,\gamma_1}, v_{\lambda_1,\gamma_1}) \leq (u_{\lambda_2,\gamma_2}, v_{\lambda_2,\gamma_2}).$$

Now for $(\lambda^*, \gamma^*) \in \Upsilon$ there is some $0 < \sigma < \infty$ such that $\gamma^* = \sigma\lambda^*$ and we can define the extremal solution (u^*, v^*) at (λ^*, γ^*) by passing to the limit along the ray given by $\gamma = \sigma\lambda$ for $0 < \lambda < \lambda^*$. Moreover it can be shown that (u^*, v^*) is indeed a weak solution of $(P)_{\lambda^*,\gamma^*}$. We now come to the issue of stability.

Theorem 6.1. *([9]) Let $(\lambda, \gamma) \in \mathcal{U}$ and let (u, v) denote the minimal solution of $(P)_{\lambda,\gamma}$. Then (u, v) is semi-stable in the sense that there is some smooth $0 < \phi, \psi \in H_0^1(\Omega)$ and $0 \leq K$ such that*

$$-\Delta\phi = \lambda e^v \psi + K\phi, \qquad -\Delta\psi = \gamma e^u \phi + K\psi, \qquad \Omega.$$

Now one should note that $K < \lambda_1(\Omega)$. To see this one multiplies either of these equations by the first positive eigenfunction of $-\Delta$ and integrates by parts.

6.2 Main Results

Our main result is given by the following theorem.

Theorem 6.2. *Let $3 \leq N \leq 9$ and suppose that $(\lambda, \gamma) \in \Upsilon$ with*

$$\frac{N-2}{8} < \frac{\gamma}{\lambda} < \frac{8}{N-2}.$$

Then the associated extremal solution (u^*, v^*) is smooth.

Now one should note that along the diagonal the problem reduces to the scalar problem. Also by symmetry it is enough to prove the result for $0 < \gamma \leq \lambda$. We prove the above Theorem in a series of lemma's.

Lemma 6.3. *Suppose that (u, v) is a smooth solution of $(P)_{\lambda,\gamma}$ where $0 < \gamma \leq \lambda$. Then*

$$\frac{\gamma}{\lambda} u \leq v \leq u \qquad \text{a.e. in } \Omega.$$

Proof. Taking the difference of the equations in $(P)_{\lambda,\gamma}$ we have $-\Delta(u-v) = \lambda e^v - \gamma e^u$ in Ω and multiplying this by $(u-v)_-$ and integrating by parts one arrives at

$$-\int_\Omega |\nabla(u-v)_-|^2 dx = \int_\Omega (\lambda e^v - \gamma e^u)(u-v)_- dx,$$

and now note that the right hand side is nonnegative where as the left hand side is nonpositive. Hence we see that $(u-v)_- = 0$ a.e. in Ω and so $u \ge v$ a.e. in Ω. Now note that

$$-\Delta(v - \frac{\gamma}{\lambda}u) = \gamma(e^u - e^v) \ge 0 \qquad \Omega,$$

since $u \ge v$ in Ω and so $v \ge \frac{\gamma}{\lambda}u$ in Ω.

□

Lemma 6.4. *Suppose that $(\lambda, \gamma) \in \mathcal{U}$ with $0 < \gamma \le \lambda$ and we let (u, v) denote the minimal solution of $(P)_{\lambda,\gamma}$. Let K, ϕ, ψ be as in Theorem 6.1. Then*

$$\frac{\psi}{\phi} \ge \frac{\gamma}{\lambda} \qquad \textit{in } \Omega.$$

Proof. First note that

$$\begin{aligned} -\Delta(\psi - \phi) &= \gamma e^u \phi - \lambda e^v \psi + K(\psi - \phi) \\ &\ge \gamma e^v(\phi - \psi) + (\gamma - \lambda)e^v\psi + K(\psi - \phi) \end{aligned}$$

where we have used the fact that $u \ge v$ in Ω. Rearranging this we have

$$-\Delta(\psi - \phi) - K(\psi - \phi) + \gamma e^v(\psi - \phi) \ge (\gamma - \lambda)e^v\psi \qquad \Omega. \tag{6.5}$$

We now define $L := -\Delta - K$ and $H := L + \gamma e^v$. Now note that

$$\begin{aligned} H(\psi - \phi + \frac{(\lambda-\gamma)}{\gamma}\phi) &\ge L(\psi - \phi + \frac{(\lambda-\gamma)}{\gamma}\phi) + \gamma e^v(\psi - \phi) \\ &= H(\psi - \phi) + \frac{(\lambda-\gamma)}{\gamma}L(\phi) \\ &\ge (\gamma - \lambda)e^v\psi + \frac{(\lambda-\gamma)}{\lambda}\lambda e^v\psi \\ &= 0 \end{aligned}$$

in Ω. Now since H satisfies the maximum principle we see that $\psi - \phi + \frac{\lambda-\gamma}{\lambda}\phi \ge 0$ in Ω, which after re-arranging, gives the desired result.

□

Theorem 6.2 will easily follow from the following lemma.

Lemma 6.6. *Suppose $3 \le N \le 9$ and that (u_m, v_m) denotes a sequence of smooth minimal solutions to $(P)_{\lambda_m,\sigma\lambda_m}$ where $\frac{N-2}{8} < \sigma \le 1$. Then (u_m, v_m) is bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$.*

Proof. Fix $\frac{N-2}{8} < \sigma \le 1$ and for notational simplicity we drop the subscript m from u_m, v_m, ϕ_m, ψ_m and K_m. From the previous lemma we have $\frac{\psi}{\phi} \ge \sigma$. Now for any smooth positive function E one has

$$\int_\Omega \frac{-\Delta E}{E} \beta^2 dx \le \int_\Omega |\nabla \beta|^2 dx, \qquad \forall \beta \in H_0^1(\Omega),$$

see, for instance, [5].

Now note that

$$\frac{-\Delta \phi}{\phi} = \lambda e^v \frac{\psi}{\phi} + K \ge \sigma \lambda e^v \qquad \Omega.$$

Taking $E = \phi$ and $\beta = e^{tu} - 1$, where t is chosen such that $\frac{N-2}{4} < t < 2\sigma$, gives

$$\sigma \lambda \int_\Omega e^v \left(e^{tu} - 1 \right)^2 dx \le t^2 \int_\Omega e^{2tu} |\nabla u|^2 dx. \tag{6.7}$$

Now multiplying $-\Delta u = \lambda e^v$ by $e^{2tu} - 1$ and integrating by parts gives

$$2t \int_\Omega e^{2tu} |\nabla u|^2 dx = \lambda \int_\Omega e^v (e^{2tu} - 1) dx. \tag{6.8}$$

Now equating (6.7) and (6.8) gives, after some simplification,

$$\left(\frac{\sigma}{t} - \frac{1}{2} \right) \int_\Omega e^v e^{2tu} dx \le \frac{2\sigma}{t} \int_\Omega e^{tu} e^v dx.$$

Now note that since $t < 2\sigma$ the coefficient on the left is positive. Now applying Holder's inequality on the right and squaring gives

$$\left(\frac{\sigma}{t} - \frac{1}{2} \right)^2 \int_\Omega e^{2tu} e^v dx \le \frac{4\sigma^2}{t^2} \int_\Omega e^v dx,$$

and now since $u \ge v$ in Ω we see that this gives us an $L^{2t+1}(\Omega)$ bound for e^v. We now return to the sequence notation. So we have that e^{v_m} is bounded in $L^{2t+1}(\Omega)$ but note that $2t + 1 > \frac{N}{2}$ and also note that λ_m is bounded. Now since $-\Delta u_m = \lambda_m e^{v_m}$ in Ω with $u_m = 0$ on $\partial\Omega$, and since λ_m is bounded one sees, using elliptic regularity, that u_m is bounded in $L^\infty(\Omega)$. From this, and since $\sigma \lambda_m$ is bounded, we easily infer that v_m is bounded in $L^\infty(\Omega)$. □

Bibliography

[1] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa *Blow up for $u_t - \Delta u = g(u)$ revisited,* Ad. Diff. Eq. **1** (1996), 73-90.

[2] H. Brezis and L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 443–469.

[3] X. Cabré, *Regularity of minimizers of semilinear elliptic problems up to dimension four*, (preprint) (2009).

[4] X. Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal. 238 (2006), no. 2, 709–733.

[5] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements,* Commun. Pure Appl. Anal. 9 (2010), no. 1, 109–140.

[6] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variation methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rat. Mech. Anal., 58 (1975), pp.207-218.

[7] Y. Martel, *Uniqueness of weak extremal solutions of nonlinear elliptic problems*, J. Diff. Eq. **16** (1974), 103-125.

[8] F. Mignot and J-P. Puel, *Sur une classe de problemes non lineaires avec non linearite positive, croissante, convexe*, Comm. Partial Differential Equations 5 (1980), 791-836.

[9] M. Montenegro, *Minimal solutions for a class of elliptic systems*, Bull. London Math. Soc.**37** (2005) 405-416.

[10] G. Nedev, *Regularity of the extremal solution of semilinear elliptic equations,* C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), no. 11, 997–1002.

[11] D. Ye and F. Zhou, *Boundedness of the extremal solution for semilinear elliptic problems,* Commun. Contemp. Math. **4** (2002), no. 3, p. 547-558.

Chapter 7

Optimal Hardy inequalities for general elliptic operators with improvements[6]

7.1 Introduction

We begin by recalling the various Hardy inequalities. Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin and where $n \geq 3$. Then Hardy's inequality (see [11]) asserts that

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^2} dx, \tag{7.1}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\left(\frac{n-2}{2}\right)^2$ is optimal and not attained. An analogous result asserts that for any bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and $\delta(x) := dist(x, \partial\Omega)$ (the euclidean distance from x to $\partial\Omega$), there holds (see [4])

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{\delta^2} dx, \tag{7.2}$$

for all $u \in H_0^1(\Omega)$. Moreover the constant $\frac{1}{4}$ is optimal and not attained. We will refer to this inequality as Hardy's boundary inequality.

Recently Hardy inequalities involving more general distance functions than the distance to the origin or distance to the boundary have been studied (see [3]). Suppose Ω is a domain in $\mathbb{R}^n$ and M a piecewise smooth surface of co-dimension k, $k = 1, ..., n$. In case $k = n$ we adopt the convention that M is a point, say, the origin. Set $d(x) := dist(x, M)$. Suppose $k \neq 2$ and

[6] A version of this chapter has been accepted for publication: C. Cowan: *Optimal Hardy inequalities for general elliptic operators with improvements*, Commun. Pure Appl. Anal. 9 (2010), no. 1, 109–140.

$-\Delta d^{2-k} \geq 0$ in $\Omega\backslash M$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{(k-2)^2}{4} \int_\Omega \frac{u^2}{d^2} dx, \tag{7.3}$$

for all $u \in H_0^1(\Omega\backslash M)$. We comment that the above inequalities all have L^p analogs.

In the last few years improved versions of the above inequalities have been obtained, in the sense that non-negative terms are added to the right hand sides of the inequalities; see [6], [4], [3], [5],[8],[9],[16]. One common type of improvement for the above Hardy inequalities are the so called potentials; we call $0 \leq V(x)$, defined in Ω, a potential for (7.1) provided

$$\int_\Omega |\nabla u|^2 dx - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx \geq \int_\Omega V(x) u^2 dx, \qquad u \in H_0^1(\Omega).$$

Most of the results in this direction are explicit examples of potentials V where, in the best results, V is an infinite series involving complicated inductively defined functions. Very recently Ghoussoub and Moradifam [10] gave the following necessary and sufficient conditions for a radial function $V(x) = v(|x|)$ to be a potential in the case of Hardy's inequality (7.1) on a radial domain Ω:
V is a potential if and only if there exists a positive function $y(r)$ which solves $y'' + \frac{y'}{r} + vy = 0$ in $(0, \sup_{x\in\Omega} |x|)$.

In another direction people have considered Hardy inequalities for operators more general than the Laplacian. One case of this is the results obtained by Adimurthi and A. Sekar [1]:
Suppose Ω is a smooth domain in $\mathbb{R}^n$ which contains the origin, $A(x) = ((a^{i,j}(x)))$ denotes a symmetric, uniformly positive definite matrix with suitably smooth coefficients and for $\xi \in \mathbb{R}^n$ we define $|\xi|_A^2 := |\xi|_{A(x)}^2 := A(x)\xi \cdot \xi$. Now suppose E is a solution of $\mathcal{L}_{A,p}(E) := -div\left(|\nabla E|_A^{p-2} A \nabla E\right) = \delta_0$ in Ω with $E = 0$ on $\partial\Omega$ where δ_0 is the Dirac mass at 0. Then for all $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx \geq 0.$$

Improvements of this inequality were also obtained and they posed the following question: Is $\left(\frac{p-1}{p}\right)^p$ optimal?
We show this is the case, even for a more general inequality.

7.1.1 Outline and approach

Our approach will be similar to the one taken by Adimurthi and A. Sekar but we mostly concentrate on the quadratic case ($p = 2$) and for this we define $\mathcal{L}_A(E) := -div(A\nabla E)$.

We now motivate our main inequality. Suppose E is a smooth positive function defined in Ω. Let $u \in C_c^\infty(\Omega)$ and set $v := E^{\frac{-1}{2}}u$. Then a calculation shows that

$$|\nabla u|_A^2 - \frac{|\nabla E|_A^2}{4E^2}u^2 = E|\nabla v|_A^2 + \frac{A\nabla E \cdot \nabla(v^2)}{2}, \qquad \text{in } \Omega$$

and after integrating this over Ω we obtain

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx = \int_\Omega E|\nabla v|_A^2 dx + \frac{1}{2}\int_\Omega \frac{u^2}{E}\mathcal{L}_A(E)dx. \quad (7.4)$$

If we further assume that $\mathcal{L}_A(E) \geq 0$ in Ω then

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx, \qquad u \in H_0^1(\Omega). \quad (7.5)$$

From this we see that the optimal constant $C(E)$

$$C(E) := \inf\left\{ \frac{\int_\Omega |\nabla u|_A^2}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2}dx : \; u \in H_0^1(\Omega)\backslash\{0\} \right\} \geq \frac{1}{4}.$$

It is possible to show that for all non-zero $u \in H_0^1(\Omega)$ we have

$$\int_\Omega E|\nabla v|_A^2 dx > 0,$$

where v is defined as above. Using this and (7.4) one sees that if $C(E) = \frac{1}{4}$ then $C(E)$ is not attained and hence if $C(E)$ is attained then $C(E) > \frac{1}{4}$. This shows that $\frac{|\nabla E|_A^2}{E^2}$ needs to be singular if we want $C(E) = \frac{1}{4}$. In fact one can show that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega, \frac{|\nabla E|_A^2}{E^2}dx)$ if $\frac{|\nabla E|_A^2}{E^2} \in L^p(\Omega)$ for some $p > \frac{n}{2}$ and so one could then apply standard compactness arguments to show that $C(E)$ is attained. We are only interested in the case where $C(E) = \frac{1}{4}$ and hence we need to ensure $\frac{|\nabla E|_A^2}{E^2}$ is singular and this can be done in two obvious ways. This naturally leads one to consider the following two classes of functions E (weights).

Definition 7.6. *Suppose* $0 < E$ *in* Ω *and* $\mathcal{L}_A(E)$ *is a nonnegative nonzero finite measure in* Ω *denoted by* μ.
1) *If in addition* $E \in H_0^1(\Omega)$ *then we call* E *a boundary weight on* Ω.
2) *If in addition* $E \in C^\infty(\overline{\Omega}\backslash K)$ *where* $K \subset \Omega$ *denotes the support of* μ, $E = \infty$ *on* K *and* $dim_{box}(K) < n-2$ *(see below) then we call* E *an interior weight on* Ω.

Given a compact subset K of $\mathbb{R}^n$ we define the box-counting dimension (entropy dimension) of K by

$$dim_{box}(K) := n - \lim_{r \searrow 0} \frac{\log(\mathcal{H}^n(K_r))}{\log(r)}$$

provided this limit exists and where $K_r := \{x \in \Omega : dist(x, K) < r\}$ and $\mathcal{H}^\alpha$ is the α- dimensional Hausdorff measure.

Remark 7.1.1. *It is possible to show that* $C_c^{0,1}(\Omega\backslash K)$ *is dense in* $W_0^{1,p}(\Omega)$ *provided* K *is compact and* $dim_{box}(K) < n-p$ *(use appropriate Lipschitz cut off functions).*

From here on μ will denote the measure $\mathcal{L}_A(E)$ and in the case where E is an interior weight on Ω, K will denote the support of μ.

We now list the main results.
We show that if E is either an interior or a boundary weight in Ω then we have the following inequality:

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq 0, \qquad u \in H_0^1(\Omega) \tag{7.7}$$

with optimal constant which is not attained.
In the case that E is a boundary weight on Ω we obtain

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \qquad u \in H_0^1(\Omega). \tag{7.8}$$

Moreover $\frac{1}{2}$ is optimal (once one fixes $\frac{1}{4}$) and is not attained.

Using the methods developed in [10] we obtain necessary and sufficient conditions on $0 \leq V(x)$ to be a potential for (7.7) in the case where E is an interior weight. We show that the following are equivalent:
1) For all $u \in H_0^1(\Omega)$

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega V(x) u^2 dx.$$

2) There exists some $0 < \theta \in C^2(\Omega \backslash K)$ such that

$$\frac{-\mathcal{L}_A(\theta)}{\theta} + \frac{|\nabla E|_A^2}{4E^2} + V \leq 0 \qquad \text{in } \Omega \backslash K. \tag{7.9}$$

If we further assume that $E = \gamma \geq 0$ (constant) on $\partial\Omega$ and if we are only interested in potentials of the form $V(x) = f(E)|\nabla E|_A^2$ then we can replace 2) with

2') There exists some $0 < h \in C^2(\gamma, \infty)$ such that

$$h''(t) + \left(f(t) + \frac{1}{4t^2} \right) h(t) \leq 0, \qquad \text{in } (\gamma, \infty). \tag{7.10}$$

In practice this ode classification is more useful because of the shear abundance of ode results in the literature.

We obtain weighted versions of (7.7) (respectively (7.8)) in the case that E is an interior weight (respectively boundary weight) on Ω which can be viewed as generalized versions of the Cafferelli-Kohn-Nirenberg inequality. To be more precise we obtain:

Suppose E is an interior weight on Ω and $t \neq \frac{1}{2}$. Then

$$\int_\Omega E^{2t} |\nabla u|_A^2 dx \geq (t - \frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx, \tag{7.11}$$

for all $u \in C_c^{0,1}(\Omega \backslash K)$. Moreover the constant is optimal and not attained in the naturally induced function space.

Suppose E is a boundary weight on Ω and $0 \neq t < \frac{1}{2}$. Then (7.11) holds for all $u \in C_c^\infty(\Omega)$ and is not attained in the naturally induced function space. Similarly

$$\int_\Omega E^{2t} |\nabla u|_A^2 dx - (t - \frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx \geq (\frac{1}{2} - t) \int_\Omega E^{2t-1} u^2 d\mu, \tag{7.12}$$

for all $u \in C_c^\infty(\Omega)$. Moreover the constant on the right is optimal and not attained in the natural function space. In addition we show that the class of potentials for (7.11) is given by $\{E^{2t} V : V \text{ is a potential for } (7.7)\}$.

We also examine generalized Hardy inequalities which are valid for functions $u \in H^1(\Omega)$. Suppose E a positive function with $\mathcal{L}_A(E) + E$ a nonnegative nonzero finite measure denoted by μ, $E = \infty$ on the K (as before K denotes the support of μ) and where we assume that E satisfies a Neumann boundary condition. Then

$$\int_\Omega |\nabla u|_A^2 dx + \frac{1}{2} \int_\Omega u^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx, \qquad u \in H^1(\Omega). \tag{7.13}$$

Moreover these constants are optimal (in the sense that if one is fixed then the other is optimal). Improvements of (7.13) are also obtained. Assuming the same conditions on E we show that for $0 \leq V$ we have

$$\int_\Omega |\nabla u|_A^2 + \frac{1}{2}\int_\Omega u^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega V(x) u^2 dx, \qquad u \in H^1(\Omega)$$

if and only if there exists some $0 < \theta \in C^\infty(\overline{\Omega}\backslash K)$ such that

$$-\mathcal{L}_A(\theta) - \frac{\theta}{2} + \frac{|\nabla E|_A^2}{4E^2}\theta + V\theta \leq 0 \qquad \text{in } \Omega\backslash K, \tag{7.14}$$

with $A\nabla\theta \cdot \nu = 0$ on $\partial\Omega$.
Weighted version of (7.13) are established. Assuming the same conditions on E we show that for $t \neq \frac{1}{2}$ we have

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx + \frac{1}{2}\int_\Omega E^{2t} u^2 dx \geq \left(t - \frac{1}{2}\right)^2 E^{2t-2}|\nabla E|_A^2 u^2 dx,$$

for all $u \in C_c^\infty(\overline{\Omega}\backslash K)$. Moreover the constants are optimal and not obtained in the naturally induced function space.

We establish optimal Hardy inequalities which are valid on exterior and annular domains. Suppose Ω is a exterior domain in $\mathbb{R}^n$, $E > 0$ in $\mathbb{R}^n$, $\lim_{|x|\to\infty} E = 0$, $-\Delta E = \mu$ in $\mathbb{R}^n$ where μ is a nonzero nonnegative finite measure with compact support K. In addition we assume that $dist(K, \Omega) > 0$ and $\partial_\nu E \geq 0$ on $\partial\Omega$. We define $D^1(\Omega \cup \partial\Omega)$ to be the completion of $C_c^\infty(\Omega \cup \partial\Omega)$ with respect to the norm $\|\nabla u\|_{L^2(\Omega)}$. Then

(i) For all $u \in D^1(\Omega \cup \partial\Omega)$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx. \tag{7.15}$$

Moreover the constant is optimal and not attained.

(ii) For all $u \in D^1(\Omega \cup \partial\Omega)$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2 \partial_\nu E}{E} dS(x). \tag{7.16}$$

Now suppose $\Omega = \Omega_2\backslash\overline{\Omega_1}$ where $\Omega_1 \subset\subset \Omega_2$ are both connected and Ω is connected. Suppose $0 < E$ in Ω_2 and $-\Delta E = \mu$ in Ω_2 where μ is a nonnegative nonzero finite measure compactly supported in Ω_1. In addition we assume that $E = 0$ on $\partial\Omega_2$ and $\partial_\nu E \leq 0$ on $\partial\Omega_1$. Then (7.15) is optimal and not attained over $H_0^1(\Omega \cup \partial\Omega_1) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_2\}$.

Optimal non-quadratic Hardy inequalities are also obtained in both the interior and boundary cases.

7.1.2 Examples

We now look at various examples of Hardy inequalities (and applications of) which can be obtained after making suitable choices of weights E and matrices A. In most of the examples we will take A to be the identity matrix.

1. **Hardy's inequality:** Let Ω denote a domain in $\mathbb{R}^n$ $(n \geq 3)$ which contains the origin and set $E(x) := |x|^{2-n}$. Then $-\Delta E = c\delta_0$ where $c > 0$ and δ_0 is the Dirac mass at 0. Also $\frac{|\nabla E|^2}{4E^2} = \left(\frac{n-2}{2}\right)^2 \frac{1}{|x|^2}$ and so (7.7) gives the Hardy's inequality.

2. **Hardy's inequality in dimension two:** Now suppose Ω is a domain in $\mathbb{R}^2$ which contains the origin. Put $E(x) := -\log(R^{-1}|x|)$ where $R := \sup_\Omega |x|$. Then $-\Delta E = c\delta_0$ where $c > 0$ and putting E into (7.7) gives
$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2 \log^2(R^{-1}|x|)} dx, \qquad u \in C_c^\infty(\Omega).$$

3. **Hardy's boundary inequality:** Let Ω denote a bounded convex set in $\mathbb{R}^n$ and set $E(x) := \delta(x) := dist(x, \partial\Omega)$. Since Ω is convex one can show δ is concave and hence $-\Delta\delta \geq 0$ in Ω. Putting E into (7.8) gives an improved version of (7.2).

4. **Hardy's boundary inequality in the unit ball:** Let B denote the unit ball in $\mathbb{R}^n$ and set $E(x) := 1 - |x|$. Putting E into (7.8) gives
$$\int_B |\nabla u|^2 dx \geq \frac{1}{4} \int_B \frac{u^2}{(1-|x|)^2} dx + \frac{n-1}{2} \int_B \frac{u^2}{|x|(1-|x|)} dx,$$
for all $u \in C_c^\infty(B)$.

5. **Intermediate case:** Set $E(x) := d(x)^{2-k}$ where d and k are as in (7.3). Since $-\Delta E \geq 0$ we obtain (7.3) after subbing E into (7.7).

6. **Hardy's boundary inequality in the half space:** Let $\mathbb{R}^n_+$ denote the half space and set $E(x) := dist(x, \mathbb{R}^n_+) = x_n$. Then putting E into (7.7) gives
$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx, \qquad u \in C_c^\infty(\mathbb{R}^n_+).$$

Maz'ja (see [12]) obtained the following improvement

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} dx \geq \frac{1}{16}\int_{\mathbb{R}^n_+} \frac{u^2}{(x_n^2 + x_{n-1}^2)^{\frac{1}{2}} x_n} dx,$$

for all $u \in C_c^\infty(\mathbb{R}^n_+)$. One might ask whether we can take a more symmetrical potential in the improvement, say something like $V(x) = f(x_n)$ where f is strictly positive. Using our ode classification of potentials we will see that this is not possible.

7. **Hardy's inequality valid for $u \in H^1(\Omega)$:** Let B denote the unit ball in $\mathbb{R}^3$ and set $E(x) := |x|^{-1} e^{|x|}$. Then a computation shows that

$$-\Delta E + E = 4\pi^2 \delta_0 \qquad \text{in } B$$

and where $\partial_\nu E = 0$ on ∂B. Here δ_0 is the Dirac mass at 0. Putting E into (7.13) we see that

$$\int_B |\nabla u|^2 dx + \frac{1}{2}\int_B u^2 dx \geq \frac{1}{4}\int_B \frac{(1-|x|)^2}{|x|^2} u^2 dx, \qquad u \in H^1(B).$$

Also the constants are optimal (in the sense mentioned in (7.13)) and are not attained.

8. **$H^1(\Omega)$ Hardy inequalities in exterior domains:** Let Ω denote an exterior domain in $\mathbb{R}^n$ wiith $n \geq 3$, $0 \notin \overline{\Omega}$ and such that $\nu(x) \cdot x \geq 0$ for all $x \in \partial\Omega$. Setting $E := |x|^{2-n}$ in (7.15) we obtain

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx,$$

for all $u \in C_c^\infty(\Omega \cup \partial\Omega)$. Moreover the constant is optimal and not attained in the naturally induced function space.

9. **Hardy's inequality in a annular domain:** Assume that $0 \in \Omega_1 \subset\subset B_R \subset \mathbb{R}^2$ where Ω_1 is connected and B_R is the open ball centered at 0 with radius R. In addition we assume that $x \cdot \nu(x) \geq 0$ on $\partial\Omega_1$ where ν is the outward pointing normal. Define $\Omega := B_R \backslash \overline{\Omega_1}$, which we assume is connected, and set $E(x) := -\log(R^{-1}|x|)$. Then by the above mentioned results on annular domains one has

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{|x|^2 \log^2(R^{-1}|x|)} dx,$$

for all $u \in H_0^1(\Omega \cup \partial\Omega_1) := \{u \in H^1(\Omega) : u\big|_{\partial B_R} = 0\}$. Moreover the constant is optimal and not attained.

10. Suppose $E > 0$ in Ω, let $f : (0,\infty) \to (0,\infty)$ and set $\tilde{E} := f(E)$. Putting $\tilde{E}$ into (7.4) for E gives

$$\begin{aligned}\int_\Omega |\nabla u|_A^2 dx \quad &\geq \quad \int_\Omega |\nabla E|_A^2 \left(\frac{f'(E)^2}{4f(E)^2} - \frac{f''(E)}{2f(E)} \right) u^2 dx \\ &\quad + \frac{1}{2} \int_\Omega \frac{f'(E)\mathcal{L}_A(E)}{f(E)} u^2 dx,\end{aligned}$$

for all $u \in C_c^\infty(\Omega)$. An important example will be when $f(E) := E^t$ where $0 < t < 1$; in fact we will use $E(x) := \delta(x)^t$ ($\delta(x) := dist(x, \partial\Omega)$) to show that if one drops the requirement that μ is a *finite* measure (and just assumes μ a locally finite measure) (7.7) need not be optimal.

11. **Eigenvalue bound:** Let Ω be a bounded subset of $\mathbb{R}^n$ and $E > 0$, $\mathcal{L}_A(E) \geq 0$ in Ω with $|\nabla E|_A^2 = 1$ a.e. in Ω. Let $\lambda_A(\Omega)$ denote the first eigenvalue of $\mathcal{L}_A$ in $H_0^1(\Omega)$. Then $\lambda_A(\Omega)\|E\|_{L^\infty}^2 \geq \frac{\pi^2}{4}$. To show this one puts $f(z) := \sin^2(\frac{\pi z}{2\|E\|_{L^\infty}})$ into the above result and drops the term involving the measure.

12. Suppose E is an interior weight on Ω with $E = 1$ on $\partial\Omega$. Then by using the above result with $f(E) := (\log(E))^{\frac{1}{2}}$ one obtains the inequality

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{16} \int_\Omega \frac{3 + 4\log(E)}{E^2 \log^2(E)} |\nabla E|_A^2 u^2 dx, \qquad u \in H_0^1(\Omega).$$

Taking instead $f(E) := E\log(E)$ gives

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{\log^2(E) + 1}{E^2 \log^2(E)} u^2 dx, \qquad u \in H_0^1(\Omega).$$

13. **Poincare's inequality in an unbounded slab:** In general

$$\int_\Omega |\nabla u|^2 dx \geq C \int_\Omega u^2 dx, \qquad u \in C_c^\infty(\Omega)$$

does not hold for unbounded domains. It is known that for certain unbounded domains the inequality does in fact hold. One example would be $\Omega := \{x \in \mathbb{R}^n : 0 < x_n < \pi\}$. We now use (7.4) to show a slightly stronger result. Put $E(x) := \sin(x_n)$ into (7.4) and drop a term to arrive at

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_\Omega \frac{u(x)^2}{\tan^2(x_n)} dx + \frac{1}{2} \int_\Omega u(x)^2 dx, \qquad u \in C_c^\infty(\Omega).$$

14. **Hardy's boundary inequality in a cone:** Put $\Omega := (0,\infty)\times(0,\infty)$ and $E(x) := dist(x,\partial\Omega) = \min\{x_1,x_2\}$. Then $-\Delta E = \sqrt{2}\sigma$ where σ is the measure associated with the line $\Gamma := \{x : x_2 = x_1\}$. Putting E into (7.4) gives

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{(\min\{x_1,x_2\})^2} dx + \frac{1}{\sqrt{2}}\int_\Gamma \frac{u^2}{\min\{x_1,x_2\}} d\sigma,$$

for all $u \in C_c^\infty(\Omega)$.

15. Suppose $-\Delta\phi = 1$ in Ω with $\phi = 0$ on $\partial\Omega$. Define $E := e^{t\phi} - 1$. Then $-\Delta E = te^{t\phi}(1 - t|\nabla\phi|^2)$ which is non-negative for sufficiently small $t > 0$. Then E is a boundary weight and hence putting E into (7.8) gives

$$\int_\Omega |\nabla u|^2 dx \geq \frac{t^2}{4}\int_\Omega \frac{e^{2t\phi}|\nabla\phi|^2}{(e^{t\phi}-1)^2} u^2 dx + \frac{t}{2}\int_\Omega \frac{e^{t\phi}(1-t|\nabla\phi|^2)}{e^{t\phi}-1} u^2 dx,$$

for all $u \in H_0^1(\Omega)$, which is optimal. Sending $t \searrow 0$ recovers (7.8) with $E = \phi$.

16. **Trace theorem:** Let Ω denote a domain in $\mathbb{R}^n$ where $n \geq 3$ and such that $B \subset\subset \Omega$ (here B is the unit ball). Define

$$E(x) := \begin{cases} 1 & |x| < 1 \\ \frac{1}{|x|^{n-2}} & |x| > 1. \end{cases}$$

A computation shows that $-\Delta E = c\sigma$ where $c > 0$ and where σ is the surface measure associated with ∂B. Putting this E into (7.8) and dropping a couple of terms gives

$$\int_\Omega |\nabla u|^2 dx \geq \frac{c}{2}\int_{\partial B} u^2 d\sigma, \qquad u \in C_c^\infty(\Omega).$$

17. **Regularity:** Suppose $E \in L^\infty_{loc}(\Omega)$ is a positive solution to $\mathcal{L}_A(E) = \mu$ in Ω where μ is locally finite measure. Then using (7.7) we see that $E \in H^1_{loc}(\Omega)$.

18. **Baouendi-Grushin operator:** Here we mention that various operators can be put into the form we are interested in. Suppose Ω is an open subset of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^k$ and $\xi \in \Omega$ is written $\xi = (x,y)$ using the above decomposition of $\mathbb{R}^N$. For $\gamma > 0$ one defines the

vector field $\nabla_\gamma := (\nabla_x, |x|^\gamma \nabla_y)$ and the Baouendi-Grushin operator $\mathcal{L}_A := -\Delta_x - |x|^{2\gamma}\Delta_y$. Take

$$A(\xi) := \begin{pmatrix} I_n & 0 \\ 0 & |x|^{2\gamma} I_k \end{pmatrix}$$

where I_n, I_k are the identity matrices of size n and k. Then $|\nabla_\gamma E|^2 = |\nabla E|_A^2$ and $-div(A\nabla E) = \mathcal{L}_A(E)$.

7.2 Main Results

Throughout this article we shall assume that Ω is a bounded connected domain in $\mathbb{R}^n$ (unless otherwise mentioned) with smooth boundary and $A(x) = ((a^{i,j}(x)))$ is a $n \times n$ symmetric, uniformly positive definite matrix with $a^{i,j} \in C^\infty(\overline{\Omega})$ and for $\xi \in \mathbb{R}^n$ we define $|\xi|_A^2 := |\xi|_{A(x)}^2 := A(x)\xi \cdot \xi$.

If E is an interior weight or a boundary weight on Ω we have, by the strong maximum principle (see [15]), E bounded away from zero on compact subsets of Ω.

The following theorem gives the main inequalities. In addition we consider a slight generalization of the case where E is a boundary weight on Ω.

Theorem 7.17. *(i) Suppose E is either an interior or a boundary weight on Ω. Then*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq 0, \tag{7.18}$$

for all $u \in H_0^1(\Omega)$. Moreover $\frac{1}{4}$ is optimal and not attained.
(ii) Suppose E is a boundary weight on Ω. Then

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \tag{7.19}$$

for all $u \in H_0^1(\Omega)$. Moreover $\frac{1}{2}$ is optimal (once one fixes $\frac{1}{4}$) and is not attained.
(iii) Suppose $E \in C^\infty(\overline{\Omega})$ with $E > 0$, $\mathcal{L}_A(E) \geq 0$ in Ω and $\Gamma := \{x \in \partial\Omega : E(x) = 0\}$ contains $B(x_0, r) \cap \partial\Omega$ for some $x_0 \in \partial\Omega$ and $r > 0$. Then (7.18) is optimal.

Remark 7.2.1. *One can consider more general functions E. Most of the results (including the above one) concerning interior weights on Ω can be* *generalized to the case where $\mathcal{L}_A(E) = \mu + h$, here μ is again a nonnegative nonzero finite measure and h is a suitably smooth non-negative function.*

We begin by justifying (7.4).

Lemma 7.20. *(i) Suppose E is an interior weight on Ω. Then*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega E|\nabla v|_A^2 dx, \tag{7.21}$$

for all $u \in C_c^{0,1}(\Omega \backslash K)$ and where $v := E^{\frac{-1}{2}} u$.
(ii) Suppose E is a boundary weight on Ω. Then

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega E|\nabla v|_A^2 dx + \frac{1}{2}\int_\Omega \frac{u^2}{E} d\mu, \tag{7.22}$$

for all $u \in H_0^1(\Omega)$ and $v := E^{\frac{-1}{2}} u$.

Proof. (i) Since E is smooth away from K and noting the supports of both u and v the integration by parts used in obtaining (7.4) is valid.
(ii) Now suppose E in a boundary weight. Extend E to all of $\mathbb{R}^n$ by setting $E = 0$ outside of $\overline{\Omega}$ and let E_ε denote the ε mollification of E. Let $u \in C_c^\infty(\Omega)$, $v_\varepsilon := E_\varepsilon^{\frac{-1}{2}} u$ and define $F_\varepsilon := \mathcal{L}_A(E_\varepsilon)$. Now one easily obtains (7.4) but with E and v replaced with $E_\varepsilon, v_\varepsilon$. Standard arguments show that $uE_\varepsilon^{-1} \to uE^{-1}$ in $H_0^1(\Omega)$, $|\nabla E_\varepsilon|_A^2 E_\varepsilon^{-2} \to |\nabla E|_A^2 E^{-2}, E_\varepsilon |\nabla v_\varepsilon|_A^2 \to E|\nabla v|_A^2$ a.e. in Ω and $uF_\varepsilon \rightharpoonup u\mu$ in $H^{-1}(\Omega)$. Using these results along with Fatou's lemma allows us to pass to the limit.

□

Remark 7.2.2. *When we prove our various Hardy inequalities, which all stem from (7.4) we will generally drop the term*

$$\int_\Omega E|\nabla\left(\frac{u}{\sqrt{E}}\right)|_A^2 dx.$$

To show the given inequality does not attain we will generally just not drop this term. This term is positive for non-zero u provided u is not a multiple of $\sqrt{E}$. Since $\sqrt{E} \notin H_0^1(\Omega)$ this will not be an issue. In theorem 7.27 this will be a concern.

As usual we will need an ample supply of test functions for best constant calculations. The next lemma provides this. When E is an interior weight we let g denote a solution to $\mathcal{L}_A(g) = 0$ in Ω with $g = E$ on $\partial\Omega$.

Lemma 7.23. *Suppose E is an interior weight on Ω and $0<\gamma:=\min_{\partial\Omega} E$. Then*
(i) $u_t := E^t - g^t \in H^1_0(\Omega)$ *for* $0<t<\frac{1}{2}$.
(ii) *Define* $I(t) := \int_\Omega |\nabla E|_A^2 E^{2t-2}dx$. *Then* $I(t)$ *is finite for* $t<\frac{1}{2}$ *and* $I(t)\to\infty$ *as* $t\nearrow \frac{1}{2}$.
(ii) *Suppose* $E=\gamma>0$ *on* $\partial\Omega$. *Define* $v_{t,\tau} := E^t\log^\tau(\gamma^{-1}E)$ *and*

$$J_t(\tau) := \int_\Omega E^{2t-2}|\nabla E|_A^2 \log^{2\tau-2}(\gamma^{-1}E)dx.$$

Then $v_{t,\tau}\in H^1_0(\Omega)$ *for* $0<t<\frac{1}{2}$ *and* $\tau>\frac{1}{2}$. *Moreover for each* $0<t<\frac{1}{2}$ *we have* $J_t(\tau)\to\infty$ *as* $\tau\searrow\frac{1}{2}$.

Proof. We prove the results up to some unjustified integration by parts; which can be justified by regularizing the measure, integrating by parts and passing to limits.
(i), (ii) Fix $0<t<\frac{1}{2}$ and then note that $|\nabla u_t|^2 \le CE^{2t-2}|\nabla E|_A^2 + Cg^{2t-2}|\nabla g|_A^2$ where C is some uniform constant. The term involving g is harmless. Now multiply $\mathcal{L}_A(E)=\mu$ by E^{2t-1} and integrate over Ω to obtain

$$\begin{aligned}(1-2t)\int_\Omega E^{2t-2}|\nabla E|_A^2dx &= -\int_{\partial\Omega} g^{2t-1}(A\nabla E)\cdot\nu\\ &= \varepsilon(t) - \int_{\partial\Omega}(A\nabla E)\cdot\nu\\ &= \varepsilon(t) - \int_\Omega div(A\nabla E)dx\\ &= \varepsilon(t)+\mu(\Omega),\end{aligned}$$

where $\varepsilon(t)\to 0$ as $t\nearrow\frac{1}{2}$. Note $\int_\Omega E^{2t-1}d\mu = 0$ since $t<\frac{1}{2}$ and $E=\infty$ on K. From this we see that $I(t)=\int_\Omega|\nabla E|_A^2E^{2t-2}dx<\infty$ and so $u_t\in H^1_0(\Omega)$. We also see that $\lim_{t\nearrow\frac{1}{2}} I(t)=\infty$.
(iii) Take $0<t<\frac{1}{2}$, $\tau>\frac{1}{2}$ and $v_{t,\tau}$ defined as above. One easily sees that $v_{t,\tau}$ is continuous near $\partial\Omega$ and vanishes on $\partial\Omega$. So to show $v_{t,\tau}\in H^1_0(\Omega)$ it is sufficient to show

$$w_1 := E^{2t-2}|\nabla E|^2\log^{2\tau}(\gamma^{-1}E),\; w_2 := E^{2t-2}|\nabla E|^2\log^{2\tau-2}(\gamma^{-1}E)\in L^1(\Omega).$$

These functions are only singular near K and $\partial\Omega$. Now set $W_\tau := E^{2t-2}|\nabla E|^2\log^{2\tau-2}(\gamma^{-1}E)$ and so $w_2=W_\tau$ and $w_1=W_{\tau+1}$. Now suppose $t'\in(t,\frac{1}{2})$ and so

$$W_{\tau+1} = E^{2t'-2}|\nabla E|^2\frac{\log^{2\tau}(\gamma^{-1}E)}{E^{2t'-2t}} \le CE^{2t'-2}|\nabla E|^2 \qquad \text{near } K\ ,$$

and so $w_1 = W_{\tau+1} \in L^1(K_\varepsilon)$ where K_ε is a small neighborhood of K. Now note that w_2 is better behaved than w_1 near K and so we also have $w_2 \in L^1(K_\varepsilon)$.
Define $\Omega_\varepsilon := \{x \in \Omega : E(x) < \gamma + \varepsilon\}$ and take $\varepsilon > 0$ sufficiently small such that $K \subset \Omega \backslash \Omega_{2\varepsilon}$. Now using the co-area formula we have

$$\begin{aligned} \int_{\Omega_\varepsilon} E^{2t-2} |\nabla E|^2 \log^{2\tau-2}(\gamma^{-1} E) dx &\leq \sup_{\Omega_\varepsilon} |\nabla E| \Lambda \\ &\leq C \int_1^{1+\frac{\varepsilon}{\gamma}} s^{2t-2} \log^{2\tau-2}(s) ds, \end{aligned}$$

where

$$\Lambda := \int_{\Omega_\varepsilon} E^{2t-2} \log^{2\tau-2}(\gamma^{-1} E) |\nabla E| dx,$$

which is finite for $\tau > \frac{1}{2}$. So we see that $w_2 \in L^1(\Omega_\varepsilon)$ for sufficiently small $\varepsilon > 0$ and noting that w_1 is better behaved near $\partial\Omega$ than w_2 we have the same for w_1. Combining these results we see that $v_{t,\tau} \in H_0^1(\Omega)$.
Fix $0 < t < \frac{1}{2}$ and $\tau > \frac{1}{2}$. By Hopf's lemma we have $|\nabla E(x)|$ bounded away from zero on Ω_ε for $\varepsilon > 0$ sufficiently small; fix $\varepsilon > 0$ sufficiently small. Then

$$\begin{aligned} J_t(\tau) &\geq C \int_{\Omega_\varepsilon} E^{2t-2} \log^{2\tau-2}(\gamma^{-1} E) |\nabla E| dx \\ &\geq \tilde{C} \int_1^{1+\frac{\varepsilon}{\gamma}} s^{2t-2} \log^{2\tau-2}(s) ds, \end{aligned}$$

and a computation shows the last integral becomes unbounded as $\tau \searrow \frac{1}{2}$.
□

Proof of theorem 7.17: (i) Using lemma 7.20 and, in the case where E is a interior weight on Ω, the fact that $C_c^{0,1}(\Omega \backslash K)$ is dense in $H_0^1(\Omega)$ we obtain (7.18). We now show the constant is optimal. Suppose E is an interior weight on Ω and define $E_\varepsilon := \varepsilon + E$, $g_\varepsilon := \varepsilon + g$ where $\varepsilon > 0$. Define $I_\varepsilon(t) := \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} dx$. As in the proof of lemma 7.23 one can show that for each $\varepsilon > 0$ $\lim_{t \nearrow \frac{1}{2}} I_\varepsilon(t) = \infty$. We use $u_{t,\varepsilon} := E_\varepsilon^t - g_\varepsilon^t$ as test functions. Let $0 < t < \frac{1}{2}$ and $\varepsilon > 0$. Then

$$Q_{t,\varepsilon} := \frac{\int_\Omega |\nabla u_{t,\varepsilon}|_A^2 dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u_{t,\varepsilon}^2 dx} \leq \frac{t^2 I_\varepsilon(t) + C_0 + C_1 \sqrt{I_\varepsilon(t)}}{I_\varepsilon(t) - C_2 I_\varepsilon(\frac{t}{2}) - C_3 I_\varepsilon(0)},$$

where the constants C_k possibly depend on ε. From this we see that $\lim_{t \nearrow \frac{1}{2}} Q_{t,\varepsilon} = \frac{1}{4}$ after recalling $Q_{t,\varepsilon} \geq \frac{1}{4}$. Now fix $\varepsilon > 0$ and let $u \in C_c^\infty(\Omega)$ be non-zero. Then a simple computation shows

$$\frac{\int_\Omega |\nabla u|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx} \leq \frac{\int_\Omega |\nabla u|_A^2 dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx},$$

which, when combined with the above facts, gives the desired best constant result. To see $\frac{1}{4}$ is not attained use (7.21).
Now suppose E is a boundary weight on Ω, $\varepsilon > 0$ and $t > \frac{1}{2}$. Define $f_\varepsilon(z) := z^{2t-1} - \varepsilon^{2t-1}$ for $z > \varepsilon$ and 0 otherwise. Using $f_\varepsilon(E) \in H_0^1(\Omega)$ as a test function in the pde associated with E one obtains, after sending $\varepsilon \searrow 0$,

$$(2t-1) \int_\Omega E^{2t-2} |\nabla E|_A^2 dx = \int_\Omega E^{2t-1} d\mu, \tag{7.24}$$

which shows that $E^t \in H_0^1(\Omega)$ for $\frac{1}{2} < t \leq 1$. To see $\frac{1}{4}$ is optimal in (7.18) use E^t (as $t \searrow \frac{1}{2}$) as a minimizing sequence.
(ii) Suppose E is a boundary weight on Ω. Let $\frac{1}{2} < t < 1$ and so $E^t \in H_0^1(\Omega)$. Using (7.24) we have

$$\frac{\int_\Omega |\nabla E^t|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} (E^t)^2}{\int_\Omega \frac{(E^t)^2}{E} d\mu} = \frac{t}{2} + \frac{1}{4},$$

which shows that $\frac{1}{2}$ is optimal.
(iii) Suppose E is as in the hypothesis. The only issue is whether $\frac{1}{4}$ is optimal. Without loss of generality assume that $0 \in \partial\Omega$ and $B(0, 2R) \cap \partial\Omega \subset \Gamma$. Suppose $0 < r < R$ and define

$$\phi(x) := \begin{cases} 1 & x \in \Omega(r) \\ \frac{R-|x|}{R-r} & x \in \Omega(R) \backslash \Omega(r) \\ 0 & x \in \Omega \backslash \Omega(R), \end{cases}$$

where $\Omega(r) := B(0, r) \cap \Omega$. Define $u_t := E^t \phi$ which can be shown to be an element of $H_0^1(\Omega)$ for $t > \frac{1}{2}$. One uses u_t as $t \searrow \frac{1}{2}$ as a minimizing sequence along with arguments similar to the above to show $\frac{1}{4}$ is optimal.

□

The following example shows that if we just assume that $0 < E \in H_0^1(\Omega)$ with $\mathcal{L}_A(E)$ a locally finite measure then (7.18) need not be optimal.

Example 7.25. *Take Ω a bounded convex domain in $\mathbb{R}^n$ and set $\delta(x) := dist(x, \partial\Omega)$. Fix $\frac{1}{2} < t < 1$ and set $E := \delta^t \in H_0^1(\Omega)$. Then*

$$\frac{|\nabla E|^2}{E^2} = \frac{t^2}{\delta^2}, \qquad \mu := -\Delta E = t(1-t)\delta^{t-2} + t\delta^{t-1}(-\Delta\delta) \geq 0 \qquad \text{in } \Omega,$$

and so putting E into (7.18) gives

$$\int_\Omega |\nabla u|^2 dx \geq \frac{t^2}{4} \int_\Omega \frac{u^2}{\delta^2} dx,$$

for $u \in H_0^1(\Omega)$. This shows that (7.18) was not optimal. This apparent failure of theorem 7.17 is due to the fact μ not a finite measure; use the co-area formula to show $\delta^{t-2} \notin L^1(\Omega)$.

We now give an alternate way to view best constants in (7.19). Define $\mathcal{C}$ to be the set of $(\beta, \alpha) \in \mathbb{R}^2$ such that

$$\int_\Omega |\nabla u|_A^2 dx \geq \alpha \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \beta \int_\Omega \frac{u^2}{E} d\mu, \qquad u \in H_0^1(\Omega). \tag{7.26}$$

Theorem 7.27. *Suppose $E \in L^\infty(\Omega)$ is a boundary weight on Ω. Then*

$$\mathcal{C} = \left\{(\beta, \alpha) : \beta > \frac{1}{2}, \alpha \leq \beta - \beta^2\right\} \cup \left(-\infty, \frac{1}{2}\right] \times \left(-\infty, \frac{1}{4}\right] =: \mathcal{C}'.$$

Moreover (7.26) does attain on $\Gamma := \{(\tau, \tau - \tau^2) : \tau > \frac{1}{2}\} \subset \partial\mathcal{C}$ and does not attain on $\partial\mathcal{C} \backslash \Gamma$.

Proof. Using similar arguments to the above one can show that $E^t \in H_0^1(\Omega)$ for all $t > \frac{1}{2}$. Suppose $(\beta, \alpha) \in \mathcal{C}$. If $\beta > \frac{1}{2}$ then testing (7.26) on $u := E^\beta$ shows that $\alpha \leq \beta - \beta^2$. If $\beta \leq \frac{1}{2}$ then testing (7.26) on $u := E^t$ and sending $t \searrow \frac{1}{2}$ shows that $\alpha \leq \frac{1}{4}$. Now for the other inclusion.

Fix $t \geq 1$ and put $E_2 := E^t$. Then we have

$$\frac{|\nabla E_2|_A^2}{E_2^2} = t^2 \frac{|\nabla E|_A^2}{E^2}, \qquad \frac{\mathcal{L}_A(E_2)}{E_2} = t(1-t)\frac{|\nabla E|_A^2}{E^2} + t\frac{\mathcal{L}_A(E)}{E}.$$

Putting $E = E_2$ into (7.8) we obtain

$$\int_\Omega |\nabla u|_A^2 dx \geq (\frac{t}{2} - \frac{t^2}{4}) \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \frac{t}{2} \int_\Omega \frac{u^2}{E} d\mu, \tag{7.28}$$

and so we see that $(\frac{t}{2}, \frac{t}{2} - \frac{t^2}{4}) \in \mathcal{C}$ for all $t \geq 1$. From this we see that the curve $\alpha = \beta - \beta^2$ for $\beta \geq \frac{1}{2}$ is contained in $\mathcal{C}$. It is straightforward to see the remaining portion of $\partial \mathcal{C}'$ is contained in $\mathcal{C}$.
To see the inequality does not attain when $(\beta, \alpha) \in \partial \mathcal{C} \backslash \Gamma$ use the fact that (7.7) does not attain in H_0^1 and the fact that $\mu \geq 0$. To see the inequality does attain on the remaining portion of $\partial \mathcal{C}$ note that (7.28) attains at $u := E^{\frac{t}{2}} \in H_0^1(\Omega)$ for $t > 1$. □

We now give a result relating to the first eigenvalue of $\mathcal{L}_A$ on subdomains of Ω. Suppose $(E, \lambda_A(\Omega))$ is the first eigenpair (with $E > 0$) of $\mathcal{L}_A$ on $H_0^1(\Omega)$ and for $B \subset \Omega$ we let $\lambda_A(B)$ denote the first eigenvalue of $\mathcal{L}_A$ on $H_0^1(B)$.

Corollary 7.29. *Let E be as above. For $B \subset \Omega$ we set*

$$\underline{\alpha}(B) := \inf_B \frac{|\nabla E|_A^2}{E^2}, \qquad \overline{\alpha}(B) := \sup_B \frac{|\nabla E|_A^2}{E^2}.$$

(i) *If $\underline{\alpha}(B) > \lambda_A(\Omega)$ then*

$$4\lambda_A(B) \geq \frac{(\underline{\alpha}(B) + \lambda_A(\Omega))^2}{\underline{\alpha}(B)}.$$

(ii) *If $\lambda_A(\Omega) > \overline{\alpha}(B)$ then*

$$4\lambda_A(B) \geq \frac{(\overline{\alpha}(B) + \lambda_A(\Omega))^2}{\overline{\alpha}(B)}.$$

Proof. Let $B \subset \Omega$ and let $u \in C_c^\infty(B)$ with $\int_B u^2 = 1$. Using (7.28) gives

$$2 \int_B |\nabla u|_A^2 dx \geq (t - \frac{t^2}{2}) \inf_B \frac{|\nabla E|_A^2}{E^2} + \lambda_A(\Omega) t,$$

for $0 < t < 2$. If $t > 2$ then we get the same expression but with the infimum replaced with supremum. Now take the infimum over u and in case (i) set $t := 1 + \frac{\lambda_A(\Omega)}{\underline{\alpha}(B)} < 2$ and in case (ii) set $t := 1 + \frac{\lambda_A(\Omega)}{\overline{\alpha}(B)} > 2$ to see the result. □

7.2.1 Weighted versions

We now examine weighted versions of the above inequalities which, as mentioned earlier, can be seen as analogs of Cafferelli-Kohn-Nirenberg inequalities. We now introduce the spaces we work in.

Definition 7.30. *For $t \in \mathbb{R}$ we define $\|u\|_t^2 := \int_\Omega E^{2t} |\nabla u|_A^2 dx$.*
Suppose E is an interior weight on Ω. We define X_t to be the completion of $C_c^{0,1}(\Omega \backslash K)$ with respect to $\| \cdot \|_t$. In the case that E is a boundary weight on Ω we define X_t to be the completion of $C_c^{0,1}(\Omega)$ with respect to the same norm.

Remark 7.2.3. *One should note that if E is an interior weight on Ω and $t > \frac{1}{2}$ then X_t does not contain $C_c^\infty(\Omega)$. To see this use (7.32) to see that if $C_c^\infty(\Omega) \subset X_t$ then $E^t \in H_{loc}^1(\Omega)$ which we know to be false. For $t < \frac{1}{2}$ we do have $C_c^\infty(\Omega) \subset X_t$.*

Theorem 7.31. *Suppose $t \neq \frac{1}{2}$ and E an interior weight on Ω. Then*

$$\int_\Omega E^{2t} |\nabla u|_A^2 dx \geq (t - \frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx, \tag{7.32}$$

for all $u \in X_t$. Moreover the constant is optimal and not attained.

Proof. Let $t \neq 0, \frac{1}{2}$, $u \in C_c^{0,1}(\Omega \backslash K)$ and define $w := E^t u \in C_c^{0,1}(\Omega \backslash K)$. Put w into

$$\int_\Omega |\nabla w|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} w^2 dx,$$

and re-group to obtain (7.32). We now show the constant is optimal. Let $v_m \in C_c^{0,1}(\Omega \backslash K)$ be such that

$$D_m := \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}.$$

Define $u_m := E^{-t} v_m \in X_t$. A computation shows that

$$\frac{\int_\Omega E^{2t} |\nabla u_m|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2 dx} = D_m + t^2 - t,$$

and since $D_m \to \frac{1}{4}$ we see that $(t - \frac{1}{2})^2$ is optimal.
For the case $\gamma := \min_{\partial\Omega} E > 0$ we can show the constant is not obtained by using later results on improvements. If $\gamma = 0$ we then sub w into (7.4) instead of (7.18) and hold onto the extra term

$$\int_\Omega E |\nabla (E^{t-\frac{1}{2}} u)|_A^2 dx$$

to see the optimal constant is not attained.

□

Theorem 7.33. *(i) Suppose $0 \neq t < \frac{1}{2}$ and E is a boundary weight on Ω. Then*

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx \geq 0, \tag{7.34}$$

for all $u \in X_t$. Moreover the constant is optimal and not attained.
(ii) Suppose $0 \neq t < \frac{1}{2}$ and E is a boundary weight on Ω. Then

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx \geq (\frac{1}{2}-t) \int_\Omega E^{2t-1} u^2 d\mu, \tag{7.35}$$

for all $u \in X_t$. Moreover the constant on the right is optimal and not attained.
(iii) Suppose $t > \frac{1}{2}$ and $E \in L^\infty(\Omega)$ is a boundary weight on Ω. Then

$$\inf\left\{ \frac{\int_\Omega E^{2t}|\nabla u|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx} : \quad u \in X_t \backslash \{0\} \right\} = 0.$$

Proof. We first prove (7.35) for $u \in C_c^{0,1}(\Omega)$ which then gives us (7.34) for the same class of u's. Suppose $0 \neq t < \frac{1}{2}$ and E is a boundary weight on Ω. We now use the notation introduced in the proof of lemma 7.20; namely E_ε is the standard mollification of E and $F_\varepsilon := \mathcal{L}_A(E_\varepsilon)$. Recall that for any $u \in C_c^{0,1}(\Omega)$ we have $uF_\varepsilon \to u\mu$ in $H^{-1}(\Omega)$ and that we have

$$\int_\Omega |\nabla v|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} v^2 dx + \frac{1}{2} \int_\Omega \frac{v^2}{E_\varepsilon} F_\varepsilon dx,$$

for all $v \in H_0^1(\Omega)$. Now let $u \in C_c^{0,1}(\Omega)$ and set $v := E_\varepsilon^t u \in C_c^{0,1}(\Omega)$. Putting v into the above gives

$$\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx \geq (t-\frac{1}{2})^2 \int_\Omega |\nabla E_\varepsilon|_A^2 E_\varepsilon^{2t-2} u^2 dx + (\frac{1}{2}-t) \int_\Omega E_\varepsilon^{2t-1} u^2 F_\varepsilon dx. \tag{7.36}$$

Now since $E_\varepsilon^{2t} \to E^{2t}$ in $L^1_{loc}(\Omega)$ we have

$$\int_\Omega E_\varepsilon^{2t}|\nabla u|_A^2 dx \to \int_\Omega E^{2t}|\nabla u|_A^2 dx,$$

and using similar ideas from the proof of lemma 7.20 one can show that

$$\int_\Omega E_\varepsilon^{2t-1} u^2 F_\varepsilon dx \to \int_\Omega E^{2t-1} u^2 d\mu.$$

So using these results, sending $\varepsilon \searrow 0$ in (7.36) and after an application of Fatou's lemma we arrive at (7.35) for $u \in C_c^{0,1}(\Omega)$.
Now we show the constants are optimal. Recalling the proof of theorem 7.17 there exists $v_m \in C_c^\infty(\Omega)$ such that

$$D_m := \frac{\int_\Omega |\nabla v_m|_A^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}, \qquad F_m := \frac{\int_\Omega |\nabla v_m|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}{\int_\Omega \frac{v_m^2}{E} d\mu} \to \frac{1}{2}.$$

Define $u_m := E^{-t} v_m$ which one easily sees is an element of X_t. Then

$$\Phi_m := \frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx}{\int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2 dx} = D_m + t^2 - 2t\frac{\int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx}, \quad \text{and}$$

$$\begin{aligned}\Psi_m \quad &:= \quad \frac{\int_\Omega E^{2t}|\nabla u_m|_A^2 dx - (t-\frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u_m^2 dx}{\int_\Omega E^{2t-1} u_m^2 d\mu} \\ &= \quad F_m + \frac{t\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx - 2t\int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx}{\int_\Omega \frac{v_m^2}{E} d\mu}.\end{aligned}$$

Using $E_\varepsilon, F_\varepsilon$ as defined above one can show, using similar methods, that

$$2\int_\Omega E^{-1} v_m \nabla v_m \cdot A\nabla E dx = \int_\Omega \frac{v_m^2}{E} d\mu + \int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx. \tag{7.37}$$

So from this we see that

$$\Phi_m = D_m + t^2 - t - t\frac{\int_\Omega \frac{v_m^2}{E} d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx},$$

and noting that

$$\frac{\int_\Omega \frac{v_m^2}{E} d\mu}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} = \frac{D_m - \frac{1}{4}}{F_m} \to 0,$$

we see that (7.34) is optimal. Similarly one sees using (7.37) that $\Psi_m = F_m - t$ and hence (7.35) is optimal.
To show the constants are not obtained we as usual hold on to the extra term that we dropped in the above calculations. Since $\int_\Omega E^{-1}|\nabla E|_A^2 dx = \infty$ one can show this extra term is positive for $u \in X_t \backslash \{0\}$.
(iii) Now take $t > \frac{1}{2}$ and E a boundary weight on Ω. For $\varepsilon, \tau > 0$ but small define

$$u_{\varepsilon,\tau}(x) := \begin{cases} 0 & E < \varepsilon \\ E^\tau - \varepsilon^\tau & E > \varepsilon. \end{cases}$$

Then $u_{\varepsilon,\tau} \in X_t$. Now use the sequence u_m where $u_m := u_{\varepsilon_m,\tau_m}$ to see desired result where $\varepsilon_m := m^{-m}$ and $\tau_m := m^{-1}$.

□

7.2.2 More general weighted inequalities

We now investigate the possibility of inequalities of the form

$$\int_\Omega W(x)|\nabla u|_A^2 dx \ge \int_\Omega U(x) u^2 dx, \qquad u \in C_c^{0,1}(\Omega\backslash K).$$

Theorem 7.38. *Suppose E is an interior weight on Ω with $\gamma := min_{\partial\Omega} E$ and $0 < f \in C^\infty(\gamma,\infty)$. Then*

$$\int_\Omega f(E)^2 |\nabla u|_A^2 dx \ge \int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E) \right) u^2 dx, \qquad (7.39)$$

for all $u \in C_c^{0,1}(\Omega\backslash K)$. In addition this is optimal (in the sense that the optimal constant is 1) if $\liminf_{z\to\infty} f''(z) > 0$ or if $\lim_{z\to\infty} \frac{z^2 f''(z)}{f(z)} = 0$.

Proof. Let $u \in C_c^{0,1}(\Omega\backslash K)$ and define $w := f(E)u \in C_c^{0,1}(\Omega\backslash K)$. Putting w into (7.18), integrating by parts and re-grouping gives (7.39). Let $v_m \in C_c^{0,1}(\Omega\backslash K)$ be such that

$$D_m := \frac{\int_\Omega |\nabla v_m|^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2} v_m^2 dx} \to \frac{1}{4}.$$

Without loss of generality we can assume the supports of v_m concentrate on K. Define $u_m := \frac{v_m}{f(E)} \in C_c^{0,1}(\Omega\backslash K)$. Then a computation shows that

$$\begin{aligned} Q_m \quad &:= \quad \frac{\int_\Omega f(E)^2 |\nabla u_m|_A^2 dx}{\int_\Omega |\nabla E|_A^2 \left(\frac{f(E)^2}{4E^2} + f(E) f''(E) \right) u_m^2 dx} \\ &= \quad \frac{\int_\Omega |\nabla v_m|_A^2 dx + \int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)^2} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{4E^2} v_m^2 dx + \int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)^2} v_m^2 dx}. \end{aligned}$$

Now suppose $\liminf_{z\to\infty} f''(z) > 0$. Then using the monotonicity of $x \mapsto \frac{\alpha+x}{\beta+x}$, where α and β are positive constants, shows $Q_m \to 1$. Now suppose

$\lim_{z\to\infty} \frac{z^2 f''(z)}{f(z)} = 0$. Using this and the fact that the v_m's support concentrates on K one easily sees that

$$\frac{\int_\Omega \frac{|\nabla E|_A^2 f''(E)}{f(E)^2} v_m^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{4E^2} v_m^2 dx} \to 0.$$

Using this one sees that $Q_m \to 1$.

□

7.2.3 Improvements

We now investigate the possibility of improving (7.18) in the sense of potentials. The method we employ was first used by Ghoussoub and Moradifam (see [10]). We now define precisely what we mean by a potential. Suppose E is an interior weight on Ω and $0 \le V \in C^\infty(\Omega \backslash K)$ (recall K is the support of μ). We say V is a potential for E provided

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \ge \int_\Omega V(x) u^2 dx, \tag{7.40}$$

for all $u \in H_0^1(\Omega)$. We analogously define a potential V for the case that E is a boundary weight on Ω except we restrict our attention to $0 \le V \in C^\infty(\Omega)$. The next theorem gives necessary and sufficient conditions for V to be a potential of E in terms of solvability of a singular linear equation. For the necessary direction we will need to assume some conditions on Ω.
(B1) Suppose E is an interior weight on Ω. We assume that that there exists a sequence $(\Omega_m)_m$ of non-empty subdomains of Ω which are connected, have a smooth boundary, $\Omega_m \subset\subset \Omega\backslash K$, $\Omega_m \subset\subset \Omega_{m+1}$ and $\Omega\backslash K = \cup_m \Omega_m$.
(B2) Suppose E is a boundary weight on Ω. We assume that there exists a sequence $(\Omega_m)_m$ of non-empty subdomains of Ω which are connected, have a smooth boundary, $\Omega_m \subset\subset \Omega_{m+1}$ and $\Omega = \cup_m \Omega_m$.

Theorem 7.41. ***(interior improvements)*** *Suppose E is an interior weight on Ω and $0 \le V \in C^\infty(\Omega\backslash K)$.*

(i) Suppose there exists some $0 < \phi \in C^2(\Omega\backslash K)$ such that

$$-\mathcal{L}_A(\phi) + \frac{A\nabla E \cdot \nabla \phi}{E} + V\phi \le 0 \qquad \text{in } \Omega\backslash K. \tag{7.42}$$

Then V is a potential for E. After the change of variables $\theta := E^{\frac{1}{2}}\phi$ one sees that it is sufficient to find a $0 < \theta \in C^2(\Omega\backslash K)$ such that

$$\frac{-\mathcal{L}_A(\theta)}{\theta} + \frac{|\nabla E|_A^2}{4E^2} + V \le 0 \qquad \text{in } \Omega\backslash K. \tag{7.43}$$

(ii) Suppose V is a potential for E and Ω satisfies (B1). Then there exists some $0 < \theta \in C^\infty(\backslash K)$ which satisfies (7.43).

It is important to note that the above theorem can be used (in theory) for best constant calculations; without the need for constructing appropriate minimizing sequences. To see this suppose $0 \le V$ is a potential for the interior weight E and let $C(V) > 0$ denote the associated best constant, ie

$$C(V) := \inf \left\{ \frac{\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx}{\int_\Omega V u^2 dx} \ : \ u \in H_0^1(\Omega)\backslash\{0\} \right\}.$$

Then one sees that

$$\begin{aligned} C(V) \ &= \ \sup \Big\{ c > 0 : \exists 0 < \theta \in C^2(\Omega\backslash K) \ \ s.t. \\ & \qquad \frac{-\mathcal{L}_A(\theta)}{\theta} + \frac{|\nabla E|_A^2}{4E^2} + cV \le 0 \qquad \text{in } \Omega\backslash K \Big\}. \end{aligned}$$

After theorem 7.53, which is analogous result to the above theorem but phrased in terms of solvability of a linear ode, this remark on best constants will be of more importance because of the shear magnitude of results concerning solvability of ode's.

Theorem 7.44. ***(boundary improvements)*** *Suppose E is a boundary weight on Ω and $0 \le V \in C^\infty(\Omega)$.*

(i) Suppose $E \in C^{0,1}(\overline{\Omega})$, V is a potential for E and Ω satisfies (B2). Then there exists some $0 < \theta \in C^{1,\alpha}(\Omega)$ for all $\alpha < 1$ such that

$$\frac{-\mathcal{L}_A(\theta)}{\theta} + \frac{|\nabla E|_A^2}{4E^2} + V \le 0 \qquad \text{in } \Omega. \tag{7.45}$$

(ii) Suppose there exists some $0 < \phi \in C^2(\Omega)$ such that

$$\frac{-\mathcal{L}_A(\phi)}{\phi} + \frac{A\nabla E \cdot \nabla \phi}{E\phi} - \frac{\mu}{2E} + V \le 0 \qquad \text{in } \Omega. \tag{7.46}$$

Then V is a potential for E.

Remark 7.2.4. *Note that putting $\theta := E^{\frac{1}{2}}\phi$ into (7.46) gives, at least formally, (7.45). Also one can replace μ by the absolutely continuous part of μ in (7.46).*

Proof of theorem 7.41. (i) Suppose $V \in C^\infty(\Omega\backslash K)$ is non-negative and there exists some $0 < \phi \in C^2(\Omega\backslash K)$ which solves (7.42). Let $u \in C_c^{0,1}(\Omega\backslash K)$ and define $v := E^{\frac{-1}{2}} u$ so by lemma 7.20 we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx = \int_\Omega E|\nabla v|_A^2 dx.$$

Now define $\psi \in C_c^{0,1}(\Omega\backslash K)$ by $v := \phi\psi$. A calculation shows that

$$E|\nabla v|_A^2 = E\psi^2|\nabla\phi|_A^2 + E\phi^2|\nabla\psi|_A^2 + 2E\phi\psi A\nabla\phi\cdot\nabla\psi, \tag{7.47}$$

and integrating, by parts, the last term over Ω we obtain

$$\begin{aligned}\int_\Omega \psi^2 E|\nabla\phi|_A^2 dx \quad &+ \quad 2\int_\Omega \phi\psi E A\nabla\phi\cdot\nabla\psi dx \\ &= \int_\Omega \psi^2\left(\mathcal{L}_A(\phi)\phi E - \phi A\nabla E\cdot\nabla\phi\right) dx \\ &= \int_\Omega u^2 \left(\frac{\mathcal{L}_A(\phi) - \frac{A\nabla E\cdot\nabla\phi}{E}}{\phi}\right) dx \\ &=: Q,\end{aligned}$$

but by (7.42) $Q \geq \int_\Omega V(x)u^2 dx$ and so we see

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega E\phi^2|\nabla\psi|_A^2 dx + \int_\Omega V u^2 dx,$$

for all $u \in C_c^{0,1}(\Omega\backslash K)$. Now since $C_c^{0,1}(\Omega\backslash K)$ is dense in $H_0^1(\Omega)$ and using Fatou's lemma one can show (7.40) holds for all $u \in H_0^1(\Omega)$.

(ii) Now suppose $V \in C^\infty(\Omega\backslash K)$ is a potential for E and $(\Omega_m)_m$ is the sequence of domains from assumption (B1). Define the elliptic operator P by

$$P(u) := \mathcal{L}_A(u) - \frac{|\nabla E|_A^2}{4E^2} u - Vu.$$

Using a standard constrained minimization argument along with the strong maximum principle there exists some $0 < \theta_m \in H_0^1(\Omega_m)$ such that

$$\begin{aligned} P(\theta_m) &= \lambda_m\theta_m \qquad \text{in } \Omega_m \\ \theta_m &= 0 \qquad \text{on } \partial\Omega_m, \end{aligned} \tag{7.48}$$

where $0 \leq \lambda_m$, ie. (θ_m, λ_m) is the first eigenpair of P in $H_0^1(\Omega_m)$. Since $H_0^1(\Omega_m) \subset H_0^1(\Omega_{m+1})$ we see that λ_m is decreasing and hence there exists some $0 \leq \lambda$ such that $\lambda_m \searrow \lambda$. Let $x_0 \in \cap_m \Omega_m$ and suitably scale θ_m such that $\theta_m(x_0) = 1$ for all m. Now fix k and let $m > k+1$. Then

$$P(\theta_m) - \lambda_m \theta_m = 0 \qquad \text{in } \Omega_{k+1},$$

and we now apply Harnacks inequality to the operator $P - \lambda_m$ to see there exists some C_k such that

$$\sup_{\Omega_k}(\theta_m) \leq C_k \inf_{\Omega_k}(\theta_m) \leq C_k.$$

So we see that (θ_m) is bounded in $L^\infty_{loc}(\Omega \backslash K)$. Now applying elliptic regularity theory and a bootstrap argument one sees that $(\theta_m)_{m>k+1}$ is bounded in $C^{1,\alpha}(\Omega_k)$ for $\alpha < 1$ and after applying a diagonal argument one sees that there exists some non-zero $0 \leq \theta \in C^{1,\alpha}(\Omega \backslash K)$ such that $\theta_m \to \theta$ in $C^{1,\alpha}(\Omega_k)$ for all k. Using this convergence one can pass to the limit in (7.48) to see that $P(\theta) = \lambda\theta$ in $\Omega \backslash K$ and after applying the strong maximum principle on Ω_m one sees that $\theta > 0$ in $\Omega \backslash K$. Now applying regularity theory one sees that $\theta \in C^\infty(\Omega \backslash K)$.

□

Proof of theorem 7.44. (i) The proof is essentially unchanged from the proof of theorem 7.41.
(ii) Again the proof is the same as in theorem 7.41 except now the measure μ does not drop out. □

The next theorem gives some explicit examples of potentials.

Theorem 7.49. *(i) Suppose E is an interior weight on Ω, $0 < \gamma := \min_{\partial\Omega} E$ and $0 < f \in C^2((\gamma, \infty))$. Then for all $u \in C_c^{0,1}(\Omega \backslash K)$ we have*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega \frac{|\nabla E|_A^2}{f(E)} \left(-f''(E) - \frac{f'(E)}{E} \right) u^2 dx.$$

In particular by taking $f(E) := \sqrt{\log(\gamma^{-1}E)}$ we obtain

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2 \log^2(\gamma^{-1}E)} u^2 dx, \qquad (7.50)$$

for all $u \in H_0^1(\Omega)$. Now suppose $0 < \gamma = E$ on $\partial\Omega$. Then $\frac{1}{4}$ (on the right hand side of (7.50)) is optimal.

(ii) Suppose $E \in L^\infty(\Omega)$ is a boundary weight. Then

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2 \log^2\left(\frac{E}{e\|E\|_{L^\infty}}\right)} u^2 dx, \qquad (7.51)$$

for all $u \in H_0^1(\Omega)$.

Proof. (i) Let E be an interior weight on Ω, $\gamma := \min_{\partial\Omega} E > 0$ and suppose $0 < f \in C^2((\gamma,\infty))$. Put $\phi := f(E)$ into (7.42) to obtain the result. Now take $f(E) := \sqrt{\log(\gamma^{-1}E)}$ to obtain (7.50) for all $u \in C_c^{0,1}(\Omega\backslash K)$ and extend to all of $H_0^1(\Omega)$ by density and by Fatou's lemma. We now show $\frac{1}{4}$ is optimal.
Fix $0 < t < \frac{1}{2}$ and for $\tau > \frac{1}{2}$ define $u_\tau := E^t \log^\tau(\gamma^{-1}E)$. By lemma 7.23 $u_\tau \in H_0^1(\Omega)$. A computation shows that

$$\frac{\int_\Omega |\nabla u_\tau|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u_\tau^2 dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2 \log^2(E\gamma^{-1})} u_\tau^2 dx}$$

is equal to

$$(t^2 - \frac{1}{4})\frac{J_t(\tau+1)}{J_t(\tau)} + \tau^2 + 2t\tau\frac{J_t(\tau+1/2)}{J_t(\tau)},$$

where $J_t(\tau)$ is defined in lemma 7.23. Sending $\tau \searrow \frac{1}{2}$ and using results from lemma 7.23 we see $\frac{1}{4}$ is optimal.
(ii) Suppose $E \in L^\infty(\Omega)$ is a boundary weight on Ω. Here we use the notation from the proof of lemma 7.20; $E_\varepsilon := \eta_\varepsilon * E$, $F_\varepsilon := \mathcal{L}_A(E_\varepsilon)$. Let $0 < f \in C^2((0, \|E\|_{L^\infty}])$. Then starting at (7.22) for E_ε and decomposing v as usual one arrives at

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \quad \geq \quad M + N \qquad (7.52)$$

where

$$M := \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{f(E_\varepsilon)}\left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{E_\varepsilon}\right) u^2 dx,$$

$$N := \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon}\right) u^2 F_\varepsilon dx$$

for all $u \in C_c^\infty(\Omega)$ after using methods similar to the proof of (i). Now take $f(z) := \sqrt{-\log(\frac{z}{e\|E\|_{L^\infty}})}$ and let $u \in C_c^\infty(\Omega)$. Then one has

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E_\varepsilon^2 \log^2(\frac{E_\varepsilon}{e\|E\|_{L^\infty}})} u^2 dx + I_\varepsilon,$$

where

$$I_\varepsilon := \frac{1}{2}\int_\Omega \frac{u^2}{E_\varepsilon^2}\left(1+\frac{1}{\log(\frac{E_\varepsilon}{e\|E\|_{L^\infty}})}\right)F_\varepsilon dx.$$

Using methods similar to ones used in the proof of lemma 7.20 one easily sees that $\lim_{\varepsilon\searrow 0} I_\varepsilon \geq 0$. Using this and standard results on convolutions and Fatou's lemma we obtain the desired inequality for $u \in C_c^\infty(\Omega)$ and we then extend to all of $H_0^1(\Omega)$.

□

We now obtain a more useful (than (7.43)) necessary and sufficient condition for V to be a potential for E; at least in the case where E is an interior weight on Ω and $E = \gamma \geq 0$ on $\partial\Omega$. As in theorem 7.41 we assume some geometrical properties of Ω.

Theorem 7.53. ***(Interior improvements using ode methods)*** *Suppose E is an interior weight on Ω, $E = \gamma \geq 0$ on $\partial\Omega$, $0 \leq f \in C^\infty(\gamma,\infty)$ and $\Omega_t := \{x \in \Omega : \gamma + \frac{1}{t} < E(x) < t\}$ is connected for sufficiently large t. Then the following are equivalent:*
(i) *For all $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx \geq \int_\Omega f(E)|\nabla E|_A^2 u^2 dx. \tag{7.54}$$

(ii) *There exists some $0 < h \in C^2(\gamma,\infty)$ such that*

$$h''(t) + \left(f(t) + \frac{1}{4t^2}\right)h(t) \leq 0, \tag{7.55}$$

in (γ,∞).

Proof. Let E be an interior weight on Ω, $E = \gamma \geq 0$ on $\partial\Omega$ and $0 \leq f \in C^\infty(\gamma,\infty)$.
(ii) ⇒ (i)
Setting $\theta := h(E)$ and using (ii) along with theorem 7.41 gives (i).
(i) ⇒ (ii).
The proof will be similar to theorem 7.41 (ii). Let $\gamma < t_m \nearrow \infty$ and define $\Omega_m := \{x \in \Omega : \gamma + \frac{1}{t_m} < E(x) < t_m\}$. By hypothesis we can take Ω_m to be connected and non-empty for each m. Now define $H_{0,E}^1(\Omega_m) := \{\phi \in H_0^1(\Omega_m) : \phi$ is constant on level sets of $E\}$ and set

$$F(\phi) := \frac{1}{2}\int_{\Omega_m} |\nabla\phi|_A^2 dx, \quad J(\phi) := \frac{1}{2}\int_{\Omega_m} |\nabla E|_A^2 (f(E) + \frac{1}{4E^2})\phi^2 dx,$$

$$M_m := \{\phi \in H^1_{0,E}(\Omega_m) : J(\phi) = 2^{-1}\}.$$

Standard methods show the existence of $0 < \phi_m \in H^1_{0,E}(\Omega_m)$ such that $\lambda_m := \inf_{M_m} F = F(\phi_m)$ and hence $\mathcal{L}_A(\phi_m) = \lambda_m |\nabla E|^2_A (f(E) + \frac{1}{4E^2})\phi_m$ in Ω_m with $\phi_m = 0$ on $\partial\Omega_m$. Since $H^1_{0,E}(\Omega_m) \subset H^1_{0,E}(\Omega_{m+1})$ one sees that λ_m is decreasing and from (7.54) one sees that $\lambda_m \geq 1$ and hence there exists some $\lambda \geq 1$ such that $\lambda_m \searrow \lambda$. By suitably scaling ϕ_m as before and after an application of Harnacks inequality we can assume that $\phi_m \to \phi$ in $C^{1,\alpha}_{loc}(\Omega \backslash K)$ where $\phi \geq 0$ is nonzero and constant on level sets of E. Passing to the limit shows that

$$\mathcal{L}_A(\phi) = \lambda |\nabla E|^2_A \left(f(E) + \frac{1}{4E^2} \right) \phi \qquad \text{in } \Omega \backslash K,$$

and a strong maximum principle argument shows that $\phi > 0$ in $\Omega \backslash K$. Since ϕ constant on level sets of E we have $\phi = h(E)$ for some $0 < h$ in (γ, ∞) and since ϕ smooth on $\Omega \backslash K$ we see that h is smooth on (γ, ∞). Writing the equation for ϕ in terms of h gives

$$-h''(E)|\nabla E|^2_A = \lambda h(E) \left(f(E) + \frac{1}{4E^2} \right) |\nabla E|^2_A \qquad \text{in } \Omega \backslash K,$$

and using Hopfs lemma we can cancel the gradients.

□

Using the vast knowledge of ode's one can use the above theorem to obtain various results concerning potentials of the form $V(x) = |\nabla E|^2_A f(E)$. We don't exploit this fact other than to look at one result.

Corollary 7.56. *Suppose E is an interior potential on Ω and $E = 0$ on $\partial\Omega$. Then there no $0 < f \in C(0, \infty)$ such that*

$$\int_\Omega |\nabla u|^2_A dx - \frac{1}{4} \int_\Omega \frac{|\nabla E|^2_A}{E^2} u^2 dx \geq \int_\Omega f(E) |\nabla E|^2_A u^2 dx, \qquad u \in H^1_0(\Omega).$$

Proof. Suppose there is such a function f. Using the proof of theorem 7.53 one sees that there is some $0 < h \in C^2(0, \infty)$ such that

$$h''(t) + \lambda \left(f(t) + \frac{1}{4t^2} \right) h(t) = 0,$$

in $(0, \infty)$ where $\lambda \geq 1$. Now set $h(t) = \sqrt{t} y(t)$ to see that

$$0 = y''(t) + \frac{y'(t)}{t} + y(t) \left(\lambda f(t) + \frac{\lambda - 1}{4t^2} \right),$$

in $(0, \infty)$ and $y(t) > 0$. But oscillation theory from ordinary differential equations shows this is impossible.

$\square$

Other than some regularity issues this ode approach extends immediately to the case where E is a boundary weight in Ω. Using this corollary (but in the boundary case) one can show the result mentioned in the examples section regarding improvements of Hardy's boundary inequality in the half space; the regularity is not an issue in this example since $\delta(x) := dist(x, \partial\mathbb{R}^n_+) = x_n$ is smooth.

We now present a result obtained by Avkhadiev and Wirths (see [2]). Given a domain Ω in $\mathbb{R}^n$ we say it has finite inradius if $\delta(x) := dist(x, \partial\Omega)$ is bounded in Ω. We let λ_0 (Lambs constant) denote the first positive zero of $J_0(t) - 2tJ_1(t)$ where J_n is the Bessel function of order n. Numerically one sees that $\lambda_0 = 0.940....$ Now for their result.

Theorem 7.57. *(Avkhadiev, Wirths) Suppose Ω is a convex domain in $\mathbb{R}^n$ with finite inradius. Then*

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2} dx + \frac{\lambda_0^2}{\|\delta\|^2_{L^\infty}} \int_\Omega u^2 dx, \qquad u \in H^1_0(\Omega)$$

is optimal.

This extends a result of H. Brezis and M. Marcus (see [4]) which said that if Ω is a convex subset of $\mathbb{R}^n$ then

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4}\int_\Omega \frac{u^2}{\delta^2} dx + \frac{1}{4diam^2(\Omega)} \int_\Omega u^2 dx, \qquad u \in H^1_0(\Omega)$$

where $diam(\Omega)$ denotes the diameter of Ω. Note that there are unbounded convex domains with infinite diameter but finite inradius; for example take a cylinder.

We establish a generalized version of this result. Suppose μ is a nonnegative nonzero locally finite measure in Ω (possibly unbounded) and $0 < E \in L^\infty(\Omega)$ is a solution to

$$\begin{aligned} \mathcal{L}_A(E) &= \mu \qquad \text{in } \Omega \\ |\nabla E|_A &= 1 \qquad \text{a.e. in } \Omega \\ E &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We then have the following theorem.

Theorem 7.58. *Suppose E is as above. Then*

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{u^2}{E^2} dx + \frac{\lambda_0^2}{\|E\|_{L^\infty}^2} \int_\Omega u^2 dx,$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Let E be as above. Now extend E to all of $\mathbb{R}^n$ by setting $E = 0$ on $\mathbb{R}^n \backslash \overline{\Omega}$, let E_ε denote the ε mollification of E and $F_\varepsilon := \mathcal{L}_A(E_\varepsilon)$. Returning to the proof of theorem 7.49 (ii) we have

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \geq \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{f(E_\varepsilon)} \left(-f''(E_\varepsilon) - \frac{f'(E_\varepsilon)}{E_\varepsilon} \right) u^2 dx + I_\varepsilon,$$

where

$$I_\varepsilon := \int_\Omega \left(\frac{f'(E_\varepsilon)}{f(E_\varepsilon)} + \frac{1}{2E_\varepsilon} \right) u^2 F_\varepsilon dx,$$

for $u \in C_c^\infty(\Omega)$ and $0 < f \in C^2((0, \|E\|_{L^\infty}])$. Now set $\lambda := \frac{\lambda_0^2}{\|E\|_{L^\infty}^2}$ where λ_0 is Lambs constant and define $f(t) := J_0(\sqrt{\lambda} t)$. It is possible to show that

$$f(t) > 0, \qquad \frac{1}{f(t)} \left(-f''(t) - \frac{f'(t)}{t} \right) = \lambda, \qquad l(t) := \frac{f'(t)}{f(t)} + \frac{1}{2t} \geq 0$$

in $(0, \|E\|_{L^\infty})$. Fixing $u \in C_c^\infty(\Omega)$ and subbing this f into the above gives

$$\int_\Omega |\nabla u|_A^2 dx - \frac{1}{4} \int_\Omega \frac{|\nabla E_\varepsilon|_A^2}{E_\varepsilon^2} u^2 dx \geq \frac{\lambda_0^2}{\|E\|_{L^\infty}^2} \int_\Omega |\nabla E_\varepsilon|_A^2 u^2 dx + I_\varepsilon,$$

after noting that $\|E_\varepsilon\|_{L^\infty} \leq \|E\|_{L^\infty}$ and where $I_\varepsilon := \int_\Omega l(E_\varepsilon) u^2 F_\varepsilon dx$. It is possible to show that $l \in C^\infty((0, \|E\|_{L^\infty}])$. A standard argument shows that $l(E_\varepsilon)u \to l(E)u$ in $H_0^1(\Omega)$ and $uF_\varepsilon dx \rightharpoonup u\mu$ in $H^{-1}(\Omega)$ and hence one can conclude that $\liminf_{\varepsilon \searrow 0} I_\varepsilon \geq 0$. Passing to the limit (as $\varepsilon \searrow 0$) in the remaining integrals gives the desired result. $\square$

We now look at improvements of the weighted generalized Hardy inequalities. The next theorem allows us to transfer our knowledge of improvements from the non-weighted case to the weighted case, at least in the case that E is an interior weight.

Theorem 7.59. ***(Weighted interior improvements)*** *Suppose E is an interior weight on Ω and $0 \leq V \in C^\infty(\Omega \backslash K)$. Then the following are equivalent:*
(i) *For all $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|_A^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx + \int_\Omega V u^2 dx. \tag{7.60}$$

(ii) *For all $t \neq \frac{1}{2}$ and $u \in X_t$*

$$\int_\Omega E^{2t} |\nabla u|_A^2 dx \geq (t - \frac{1}{2})^2 \int_\Omega |\nabla E|_A^2 E^{2t-2} u^2 dx + \int_\Omega V E^{2t} u^2 dx. \tag{7.61}$$

(iii) *For all $u \in X_{\frac{1}{2}}$*

$$\int_\Omega E |\nabla u|_A^2 dx \geq \int_\Omega V E u^2 dx. \tag{7.62}$$

Using similar arguments one can obtain a version of theorem 7.59 for the case when E is a boundary weight on Ω; we omit the details since the results is not as clean.

Proof. Let E be an interior weight on Ω and $0 \leq V \in C^\infty(\Omega \backslash K)$.
(i) $\Rightarrow$ (ii)
Suppose (i) holds, $t < \frac{1}{2}$, $u \in C_c^{0,1}(\Omega \backslash K)$ and define $v := E^t u \in C_c^{0,1}(\Omega \backslash K)$. Then putting v into (7.60) and performing some integration by parts gives (7.61).
(ii) $\Rightarrow$ (iii)
Suppose (ii) holds. Let $u \in C_c^{0,1}(\Omega \backslash K)$ which is an element of X_t for all t. Now using (7.61) for this u and sending $t \nearrow \frac{1}{2}$ gives (7.62).
(iii) $\Rightarrow$(i)
Suppose (iii) holds, $u \in C_c^{0,1}(\Omega \backslash K)$ and $v := E^{\frac{-1}{2}} u \in C_c^{0,1}(\Omega \backslash K)$. Putting v into (7.62) and integrating by parts gives (7.60) for all $u \in C_c^{0,1}(\Omega \backslash K)$.
□

7.2.4 Hardy inequalities valid for $u \in H^1(\Omega)$

Let K be a compact subset of Ω with $dim_{box}(K) < n - 2$. Standard arguments show that $C_c^{0,1}(\overline{\Omega} \backslash K)$ is dense in $H^1(\Omega)$.

Definition 7.63. *We say E is a Neumann interior weight on Ω provided: there exists some compact $K \subset \Omega$, $dim_{box}(K) < n - 2$, $E \in C^\infty(\overline{\Omega} \backslash K)$,*

$\inf_\Omega E > 0$, $\mathcal{L}_A(E) + E$ is a nonnegative nonzero measure μ whose support is K, $E = \infty$ on K and $A\nabla E \cdot \nu = 0$ on $\partial\Omega$ where $\nu(x)$ denotes the outward normal vector at $x \in \partial\Omega$.

Theorem 7.64. *Suppose E is a Neumann interior weight on Ω. Then*

(i) *For $u \in C_c^{0,1}(\overline{\Omega}\backslash K)$ and $v := E^{\frac{-1}{2}}u$ we have*

$$\int_\Omega |\nabla u|_A^2 dx + \frac{1}{2}\int_\Omega u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \int_\Omega E|\nabla v|_A^2 dx. \tag{7.65}$$

(ii)

$$\int_\Omega |\nabla u|_A^2 dx + \frac{1}{2}\int_\Omega u^2 dx \geq \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx, \tag{7.66}$$

holds for all $u \in H^1(\Omega)$. Moreover $\frac{1}{4}$ and $\frac{1}{2}$ are optimal in the sense that if one fixes $\frac{1}{4}$ then you can do no better than $\frac{1}{2}$ and vice-versa. Also the inequality is not attained.

One can again view the best constants in a different manner which is analogous to theorem 7.27; we omit the details.

Proof. Let E be a Neumann interior weight on Ω.
(i) Let $u \in C_c^{0,1}(\overline{\Omega}\backslash K)$ and define $v := E^{\frac{-1}{2}}u$. Then

$$|\nabla u|_A^2 = E|\nabla v|_A^2 + \frac{|\nabla E|_A^2}{4E^2}u^2 + v\nabla v \cdot A\nabla E,$$

and integrating this over Ω gives

$$\int_\Omega |\nabla u|_A^2 dx + \frac{1}{2}\int_\Omega u^2 dx = \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2}u^2 dx + \int_\Omega E|\nabla v|_A^2 dx. \tag{7.67}$$

(ii) Using (i) and the fact that $C_c^{0,1}(\overline{\Omega}\backslash K)$ is dense in $H^1(\Omega)$ one obtains (7.66) for all $u \in H^1(\Omega)$.
We now show the constants are optimal. We first show that $E^t \in H^1(\Omega)$ for $0 < t < \frac{1}{2}$.
As in the proof of lemma 7.23 the following calculations are only formal but they can be justified as hinted at there; by first regularizing the measure, obtaining approximate solutions and passing to the limit. Fix $0 < t < \frac{1}{2}$ and multiply $\mathcal{L}_A(E) + E = \mu$ by E^{2t-1} and integrate over Ω using integration by

parts and the fact that $E=\infty$ on K along with the boundary conditions of E to see that

$$\int_\Omega E^{2t}dx = (1-2t)\int_\Omega E^{2t-2}|\nabla E|_A^2 dx, \tag{7.68}$$

which shows that $E^t \in H^1(\Omega)$ for $0<t<\frac{1}{2}$. To show the constants are optimal we will use as a minimizing sequence E^t as $t \nearrow \frac{1}{2}$. A computation shows

$$\frac{\int_\Omega |\nabla E^t|_A^2 dx + \frac{1}{2}\int_\Omega E^{2t}dx}{\int_\Omega \frac{|\nabla E|_A^2}{E^2}E^{2t}dx} = t^2 + \frac{1}{2} - t,$$

and we see that $\frac{1}{4}$ is optimal. One similarly shows $\frac{1}{2}$ is optimal.
To show the inequality does not attain we, as usual, just hold on to the extra term that we dropped in the above calculations. This term is positive for non-zero $u \in H^1(\Omega)$ provided $E^{\frac{1}{2}} \notin H^1(\Omega)$ which is the case after one considers (7.68).

□

We now examine weighted versions of (7.66). Suppose E is a Neumann interior weight on Ω and as usual we let K denote the support of μ. For $t \neq \frac{1}{2}$ and $u \in C_c^{0,1}(\overline{\Omega}\backslash K)$ we define

$$\|u\|_t^2 := \begin{cases} \int_\Omega E^{2t}|\nabla u|^2 dx + \int_\Omega E^{2t}u^2 dx & t<\frac{1}{2} \\ \int_\Omega E^{2t}|\nabla u|^2 dx & t>\frac{1}{2}, \end{cases}$$

and we let Y_t denote the completion of $C_c^{0,1}(\overline{\Omega}\backslash K)$ with respect to this norm. We then have the following theorem.

Theorem 7.69. *Suppose E is a Neumann interior weight on Ω and $t \neq \frac{1}{2}$. Then*

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx + \left(\frac{1}{2}-t\right)\int_\Omega E^{2t}u^2 dx \geq \left(t-\frac{1}{2}\right)^2 \int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx,$$

for all $u \in Y_t$. Moreover the constants are optimal and not attained.

Note in particular that for $t>\frac{1}{2}$ one only has a gradient term on the left hand side and so we can conclude that $C^\infty(\overline{\Omega})$ is not contained in Y_t for $t>\frac{1}{2}$.

Proof. Suppose E is a Neumann interior weight on Ω, $t \neq \frac{1}{2}$ and let $u \in C_c^{0,1}(\overline{\Omega}\backslash K)$. Putting $E^t u$ into (7.65) gives

$$\int_\Omega E^{2t}|\nabla u|_A^2 dx + \left(\frac{1}{2} - t\right)\int_\Omega E^{2t} u^2 dx \quad \geq \quad \left(t - \frac{1}{2}\right)^2 \int_\Omega E^{2t-2}|\nabla E|_A^2 u^2 dx + \int_\Omega E|\nabla w|_A^2 dx$$

where $w := E^{t-\frac{1}{2}}u$. To show the constants are optimal one takes the same approach as in theorem 7.31. We now show the optimal constants are not obtained. Suppose we have equality for some nonzero $u \in Y_t$. Then it is easily seen that $\sqrt{E} \in H^1(\Omega)$ which we know is not the case.

□

We now examine improvements of (7.66).

Theorem 7.70. *Suppose E is a Neumann interior weight on Ω. Then*
(i) Suppose $V \in C^\infty(\Omega\backslash K)$ and there exists some $0 < \phi \in C^2(\Omega\backslash K) \cap C^1(\overline{\Omega}\backslash K)$ such that

$$-\mathcal{L}_A(\phi) + \frac{A\nabla E \cdot \nabla\phi}{E} + V\phi \leq 0 \qquad \text{in } \Omega\backslash K, \tag{7.71}$$

with $A\nabla\phi \cdot \nu \geq 0$ on $\partial\Omega$. Then

$$\int_\Omega |\nabla u|_A^2 + \frac{1}{2}\int_\Omega u^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega V(x) u^2 dx,$$

for all $u \in H^1(\Omega)$.
(ii) Suppose $0 \leq V \in C^\infty(\overline{\Omega}\backslash K)$ is such that

$$\int_\Omega |\nabla u|_A^2 + \frac{1}{2}\int_\Omega u^2 dx - \frac{1}{4}\int_\Omega \frac{|\nabla E|_A^2}{E^2} u^2 dx \geq \int_\Omega V(x) u^2 dx,$$

holds for all $u \in H^1(\Omega)$. In addition we assume that $\{x \in \Omega : E(x) < t\}$ is connected for sufficiently large t. Then there exists some $0 < \theta \in C^\infty(\overline{\Omega}\backslash K)$ such that

$$-\mathcal{L}_A(\theta) - \frac{\theta}{2} + \frac{|\nabla E|_A^2}{4E^2}\theta + V\theta \leq 0 \qquad \text{in } \Omega\backslash K, \tag{7.72}$$

with $A\nabla\theta \cdot \nu = 0$ on $\partial\Omega$.

Note that one can go from (7.71) to (7.72) by using the change of variables $\theta = \phi E^{\frac{1}{2}}$ in the case that $A\nabla\phi \cdot \nu = 0$ on $\partial\Omega$.

Proof. The proof is similar to the proof of theorem 7.41.

□

Remark 7.2.5. *One can obtain an analogous version of theorem 7.59 for the case where E is an interior weight on Ω satisfying a Neumann boundary condition.*

7.2.5 $H^1(\Omega)$ inequalities for exterior and annular domains

In this section we obtain optimal Hardy inequalities which are valid on exterior and annular domains. Moreover these inequalities will be valid for functions u which are nonzero on various portions of the boundary. For simplicity we only consider the case where $A(x)$ is the identity matrix and hence $\mathcal{L}_A = -\Delta$; the results immediately generalize to the case where $A(x)$ is not the identity matrix. We first examine the exterior domain case.

Condition (Ext.): We suppose that $E > 0$ in $\mathbb{R}^n$, $-\Delta E$ is a nonnegative nonzero finite measure (which we denote by μ) with compact support K and we let Ω denote a connected exterior domain in $\mathbb{R}^n$ with $dist(K, \Omega) > 0$. In addition we assume that the compliment of Ω denoted by Ω^c is connected, $\lim_{|x|\to\infty} E = 0$ and $\partial_\nu E \geq 0$ on $\partial\Omega$.

We will work in the following function space. Let $D^1(\Omega \cup \partial\Omega)$ denote the completion of $C_c^\infty(\Omega \cup \partial\Omega)$ with respect to the norm $\|\nabla u\|_{L^2(\Omega)}$. Note we don't require u to be zero on the boundary of $\partial\Omega$. We then have the following theorem.

Theorem 7.73. *Suppose E, μ, K, Ω are as in condition (Ext.). Then*
(i) For all $u \in D^1(\Omega \cup \partial\Omega)$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx. \tag{7.74}$$

Moreover the constant is optimal and not attained.
(ii) For all $u \in D^1(\Omega \cup \partial\Omega)$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx + \frac{1}{2} \int_{\partial\Omega} \frac{u^2 \partial_\nu E}{E} dS(x). \tag{7.75}$$

Proof. Let $u \in C_c^\infty(\Omega \cup \partial\Omega)$ and set $v := E^{\frac{-1}{2}} u$. Then as before we have

$$|\nabla u|^2 - \frac{|\nabla E|^2 u^2}{4E^2} = E|\nabla v|^2 + v\nabla v \cdot \nabla E, \qquad \text{in } \Omega. \tag{7.76}$$

Integrating the last term by parts gives

$$\int_\Omega v\nabla v \cdot \nabla E dx = \frac{1}{2}\int_{\partial\Omega} \frac{u^2 \partial_\nu E}{E} dS(x).$$

We obtain (7.75) by integrating (7.76) over Ω and since $\partial_\nu E \geq 0$ on $\partial\Omega$ we obtain (7.74). We now show the constant is optimal. For big R we set $\Omega_R := \Omega \cap B_R$ where B_R is the ball centered at 0 with radius R. Let $\frac{1}{2} < t < 1$ and multiply $-\Delta E = \mu$ by E^{2t-1} and integrate over Ω_R to obtain

$$(2t-1)\int_{\Omega_R} E^{2t-2}|\nabla E|^2 dx = \int_{\partial\Omega} \partial_\nu E E^{2t-1} dS(x) + \int_{\partial B_R} \partial_\nu E E^{2t-1} dS(x).$$

Using a Newtonian potential argument one can show that as $R \to \infty$ the surface integral over the ball B_R goes to zero. So using this one sees that

$$(2t-1)\int_{\Omega} E^{2t-2}|\nabla E|^2 dx = \int_{\partial\Omega} \partial_\nu E E^{2t-1} dS(x), \tag{7.77}$$

and so $\int_\Omega |\nabla E^t|^2 dx < \infty$. With this along with a standard cut-off function argument one sees that $E^t \in D^1(\Omega \cup \partial\Omega)$. Now one uses E^t as $t \searrow \frac{1}{2}$ as a minimizing sequence to show that $\frac{1}{4}$ is optimal. We now show the constant is not attained. Now assume that $x_0 \in \partial\Omega$ is such that $E(x_0) = \min_{\partial\Omega} E$. Then by Hopf's lemma $\partial_\nu E(x_0) > 0$ and so using this along with continuity and (7.77) one sees that $E^{\frac{1}{2}} \notin D^1(\Omega \cup \partial\Omega)$. Now to finish the proof it will be sufficient to show that

$$\int_\Omega E|\nabla v|^2 dx > 0$$

for all nonzero $u \in D^1(\Omega \cup \partial\Omega)$. The only nonzero u's for which this integral is zero are multiples of $E^{\frac{1}{2}}$ which are not in $D^1(\Omega \cup \partial\Omega)$.

□

Example 7.78. *Take Ω a exterior domain in $\mathbb{R}^n$ where $n \geq 3$, $0 \notin \overline{\Omega}$, and such that $\nu(x) \cdot x \leq 0$ on $\partial\Omega$ where $\nu(x)$ is the outward pointing normal. Define $E(x) := |x|^{2-n}$ and use theorem 7.73 to see that*

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx, \tag{7.79}$$

for all $u \in D^1(\Omega \cup \partial\Omega)$. Moreover the constant is optimal and not attained. In fact using (ii) from the same theorem shows we can add the following nonnegative term to the right hand side of (7.79):

$$\frac{(n-2)}{2}\int_{\partial\Omega} \frac{u^2(-x\cdot\nu)}{|x|^2} dS(x).$$

We now examine the annular domain case.

Condition (Annul.): We assume that $\Omega_1 \subset\subset \Omega_2$ are two bounded connected domains in $\mathbb{R}^n$ with smooth boundaries and $\Omega := \Omega_2 \backslash \overline{\Omega_1}$ is connected. In addition we assume that $E > 0$ in Ω_2 with $-\Delta E = \mu$ in Ω_2 where μ is a nonnegative nonzero finite measure supported on $K \subset \Omega_1$. We also assume that $\partial_\nu E \leq 0$ on $\partial\Omega_1$.

We then have the following theorem.

Theorem 7.80. *Suppose Ω, K, E are as in condition (Annul.). Then*
(i) For all $u \in H^1(\Omega)$ with $u = 0$ on $\partial\Omega_2$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx. \tag{7.81}$$

Moreover the constant is optimal and not attained if we assume that $E = 0$ on $\partial\Omega_2$.
(ii) For all $u \in H^1(\Omega)$ with $u = 0$ on $\partial\Omega_2$ we have

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|\nabla E|^2}{E^2} u^2 dx + \frac{1}{2} \int_{\partial\Omega} \frac{u^2 \partial_\nu E}{E} dS(x). \tag{7.82}$$

Proof. The proof of (7.81) and (7.82) is similar to the previous theorem so we omit the details. We now show the constant is optimal. Let $H_0^1(\Omega \cup \partial\Omega_1)$ denote $\{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_2\}$. Again we multiply $-\Delta E = \mu$ by E^{2t-1} for $\frac{1}{2} < t < 1$ and integrate over Ω to obtain

$$(2t-1) \int_\Omega E^{2t-2} |\nabla E|^2 dx = -\int_{\partial\Omega_1} \partial_\nu E E^{2t-1} dS(x),$$

which shows that $E^t \in H_0^1(\Omega \cup \partial\Omega_1)$. From this one obtains

$$\lim_{t \searrow \frac{1}{2}} (2t-1) \int_\Omega E^{2t-2} |\nabla E|^2 dx = \mu(\Omega_1) > 0,$$

which shows that $E^{\frac{1}{2}} \notin H_0^1(\Omega \cup \partial\Omega_1)$. To see the constant is optimal one uses the same minimizing sequence as in the previous theorem. To see the constant is not attained one uses the fact that $E^{\frac{1}{2}} \notin H_0^1(\Omega \cup \partial\Omega_1)$.

□

Remark 7.2.6. *These inequalities have analogous weighted versions and using the methods developed earlier one easily obtains results concerning improvements. We leave this for the reader to develop.*

7.2.6 The non-quadratic case

For $1 < p \leq n$ we define $\mathcal{L}_{A,p}(E) := -div(|\nabla E|_A^{p-2} A \nabla E)$. As mentioned earlier Adimurthi and Sekar [1] obtained generalized Hardy inequalities of the form

$$\int_\Omega |\nabla u|_A^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E} |u|^p dx \geq 0, \tag{7.83}$$

where $u \in W_0^{1,p}(\Omega)$. There approach (as their title suggests) was to look at functions E which solve

$$\begin{aligned} \mathcal{L}_{A,p}(E) &= \delta_0 \qquad \text{in } \Omega \\ E &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $0 \in \Omega$ and where δ_0 is again the Dirac mass at 0.
They posed the question (see [1]) as to whether $(\frac{p-1}{p})^p$ is optimal in (7.83)? The next theorem shows this is the case (at least for $1 < p < n$); infact we show the result for a more general case.

Interior case

Suppose μ is a nonnegative nonzero finite measure supported on $K \subset \Omega$, $dim_{box}(K) < n - p$ (and hence $C_c^{0,1}(\Omega \backslash K)$ is dense in $W_0^{1,p}(\Omega)$) and $0 < E$ is a solution of

$$\mathcal{L}_{A,p}(E) = \mu \qquad \text{in } \Omega. \tag{7.84}$$

By regularity theory (see [7], [14]) there is some $0 < \sigma < 1$ such that $E \in C^{1,\sigma}(\Omega \backslash K)$ and by the maximum principle (see [15]) $E > 0$ in $\Omega \backslash K$. Now if we assume that $\mu = \delta_0$, as was the case in the question posed in [1], then one can show $E(0) = \infty$.

Theorem 7.85. *Suppose E is as above but we don't assume that $E = \infty$ on K.*
(i) Then

$$\int_\Omega |\nabla u|_A^2 dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx, \tag{7.86}$$

for all $u \in W_0^{1,p}(\Omega)$.
(ii) Suppose $E = \infty$ on K and $E = \gamma$ on $\partial\Omega$ where γ is a non-negative constant. Then the constant in (7.86) is optimal.

Proof. (i) Let $u \in C_c^{0,1}(\Omega \backslash K)$. Then $\nabla E^{1-p} = (1-p)E^{-p}\nabla E$ and dotting both sides with $|\nabla E|_A^{p-2} A\nabla E|u|^p$ and integrating over Ω gives

$$\begin{aligned}(1-p)\int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx &= \int_\Omega \nabla E^{1-p} \cdot \left(|\nabla E|_A^{p-2} A\nabla E|u|^p\right) dx \\ &= \int_\Omega E^{1-p}|u|^p d\mu \\ &\quad - \int_\Omega E^{1-p}|\nabla E|_A^{p-2} A\nabla E \cdot p|u|^{p-2}u\nabla u dx \\ &= -\int_\Omega E^{1-p}|\nabla E|_A^{p-2} A\nabla E \cdot p|u|^{p-2}u\nabla u dx,\end{aligned}$$

where we used the divergence theorem and also the fact that $u = 0$ on K. Now using the Cauchy-Schwarz inequality on the inner product induced by $A(x)$ we see that

$$\frac{p-1}{p}\int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx \leq \int_\Omega \frac{|\nabla E|_A^{p-1}|u|^{p-1}}{E^{p-1}}|\nabla u|_A dx,$$

and we now apply Holder's inequality on the right after recalling $(p-1)p' = p$ where p' is the conjugate of p. Now use density to extend to all of $W_0^{1,p}(\Omega)$.
(ii) We first consider the case $\gamma > 0$. We begin by showing that $u_t := E^t - \gamma^t \in W_0^{1.p}(\Omega)$ for $0 < t < \frac{p-1}{p}$. Fix $0 < t < \frac{p}{p-1}$ and multiply (7.84) by E^{tp-p+1} and integrate over Ω to get

$$\begin{aligned}0 = \int_\Omega E^{tp-p+1} d\mu &= (tp-p+1)\int_\Omega |\nabla E|_A^2 E^{tp-p} dx \\ &\quad -\gamma^{tp-p+1}\int_{\partial\Omega} |\nabla E|_A^{p-2} A\nabla E \cdot \nu d\mathcal{H}^{n-1} \\ &= (tp-p+1)\int_\Omega |\nabla E|_A^2 E^{tp-p} dx \\ &\quad -\gamma^{tp-p+1}\int_\Omega div(|\nabla E|_A^{p-2} A\nabla E) dx \\ &= (tp-p+1)\int_\Omega |\nabla E|_A^2 E^{tp-p} dx + \gamma^{tp-p+1}\mu(\Omega),\end{aligned}$$

where the first integral is zero since $E = \infty$ on K and $tp - p + 1 < 0$. Re-arranging this we arrive at

$$\int_\Omega |\nabla E^t|_A^p dx = \frac{\mu(\Omega)\gamma^{tp-p+1}t^p}{p - tp - 1},$$

from which we see that $E^t \in W^{1,p}(\Omega)$ for $0 < t < \frac{p-1}{p}$ and we also see that

$$\lim_{t \nearrow \frac{p-1}{p}} \int_\Omega |\nabla E|_A^p E^{tp-p} dx = \infty.$$

Put t as above and set $u_t := E^t - \gamma^t \in W_0^{1,p}(\Omega)$. By the binomial theorem we have

$$(1+x)^p = \sum_{m=0}^{\infty} (p,m) x^m,$$

for all $|x| \leq 1$ where (p,m) are the binomial coefficients. One should note that (p,m) is eventually alternating and since we have convergence at $x = -1$ we see that $\sum_m (p,m)(-1)^m$ converges; which shows that $\sum_m |(p,m)| < \infty$. Now we have

$$\begin{aligned} |u_t|^p &= E^{tp} \big| 1 - \frac{\gamma^t}{E^t} \big| \\ &= E^{tp} \sum_{m=0}^{\infty} (p,m) \frac{(-1)^m \gamma^{tm}}{E^{tm}}, \end{aligned}$$

and we define

$$Q_t := \frac{\int_\Omega \frac{|\nabla E|_A^p}{E^p} |u_t|^p dx}{\int_\Omega |\nabla u_t|_A^p dx}.$$

So

$$Q_t - \frac{1}{t^p} = \frac{\int_\Omega |\nabla E|_A^p E^{tp-p} \left(\sum_{m=1}^{\infty} (p,m)(-1)^m \frac{\gamma^{tm}}{E^{tm}} \right) dx}{t^p \int_\Omega |\nabla E|_A^p E^{tp-p} dx},$$

and so

$$\begin{aligned} \big| Q_t - \frac{1}{t^p} \big| &\leq \frac{1}{t^p} \sum_{m=1}^{\infty} \big| (p,m) \big| \frac{\int_\Omega |\nabla E|_A^p E^{tp-p} \gamma^{tm} E^{-tm} dx}{\int_\Omega |\nabla E|_A^p E^{tp-p} dx} \\ &= \frac{1}{t^p} \sum_{m=1}^{\infty} \big| (p,m) \big| \frac{p - tp - 1}{p - tp - 1 + tm} \\ &\leq \frac{p - tp - 1}{t^{p+1}} \sum_{m=1}^{\infty} \frac{\big| (p,m) \big|}{m} \\ &=: \frac{p - tp - 1}{t^{p+1}} C_p, \end{aligned}$$

and so we see that

$$\lim_{t \nearrow \frac{p-1}{p}} \left| Q_t - \frac{1}{t^p} \right| = 0,$$

which shows the constant in (7.86) is optimal.
Now we handle the case $\gamma = 0$. Let $\mathcal{L}_{A,p}(E) = \mu$ in Ω and $E = 0$ on $\partial\Omega$ and define $E_\varepsilon := \varepsilon + E$ where $\varepsilon > 0$. Then $\mathcal{L}_{A,p}(E_\varepsilon) = \mu$ in Ω and $E_\varepsilon = \varepsilon$ on $\partial\Omega$. For $u \in C_c^\infty(\Omega)$ non-zero we have, after some simple algebra,

$$\frac{\int_\Omega |\nabla u|_A^p dx}{\int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx} \leq \frac{\int_\Omega |\nabla u|_A^p dx}{\int_\Omega \frac{|\nabla E_\varepsilon|_A^p}{E_\varepsilon^p} |u|^p dx},$$

which shows the constant is optimal in the case of $\gamma = 0$.

□

Boundary case

Analogously to the quadratic case we will be interested in the validity of (7.86) when E is a solution to

$$\begin{aligned} \mathcal{L}_{A,p}(E) &= \mu \qquad \text{in } \Omega, \\ E &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

where μ is a nonnegative nonzero finite measure and where we impose some added regularity restrictions to E or μ. Recall in the quadratic case we added the condition that $E \in H_0^1(\Omega)$. For simplicity we will assume that μ is smooth; say $d\mu = f dx$ where $0 \leq f \in C^\infty(\overline{\Omega})$ is non-zero. One can show that $E \in C^{1,\sigma}(\overline{\Omega})$ for some $0 < \sigma < 1$.

Theorem 7.87. *Suppose E is a positive solution to $\mathcal{L}_{A,p}(E) = \mu$ in Ω where μ is as above.*
(i) Then

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx + \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{E^{p-1}} d\mu \leq \int_\Omega |\nabla u|_A^p dx, \tag{7.88}$$

for all $u \in W_0^{1,p}(\Omega)$. Since μ is a measure we have

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p} |u|^p dx \leq \int_\Omega |\nabla u|_A^p dx, \tag{7.89}$$

for all $u \in W_0^{1,p}(\Omega)$.
(ii) Suppose $E = 0$ on $\partial\Omega$. Then (7.89) is optimal.

(iii) Suppose $E = 0$ on $\partial\Omega$. If one fixes the optimal constant from part (ii) then the other constant is also optimal in (7.88) ie.

$$\inf\left\{ \frac{\int_\Omega |\nabla u|_A^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx}{\int_\Omega \frac{|u|^p}{E^{p-1}} d\mu} : u \in W_0^{1,p}(\Omega) \right\} = \left(\frac{p-1}{p}\right)^{p-1}.$$

Proof. (i) Suppose E is a positive solution to $\mathcal{L}_{A,p}(E) = \mu$ in Ω and let $u \in C_c^\infty(\Omega)$. From the proof of theorem 7.85 we have

$$\begin{aligned}(p-1)\int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx + \int_\Omega \frac{|u|^p}{E^{p-1}} d\mu &= p\int_\Omega \frac{|\nabla E|_A^{p-2}}{E^{p-1}} A\nabla E \cdot \nabla u |u|^{p-2} u dx \\ &\leq p\left(\int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx\right)^{\frac{1}{p'}} \||\nabla u|_A\|_{L^p}.\end{aligned}$$

Now let q denote p' and

$$B := \int_\Omega \frac{|\nabla E|_A^p}{E^p}|u|^p dx, \qquad C := \int_\Omega \frac{|u|^p}{E^{p-1}} d\mu, \qquad D := \int_\Omega |\nabla u|_A^p dx.$$

Using Young's inequality with $t > 0$ we arrive at

$$\frac{(p-1)}{p}B + \frac{C}{p} \leq B^{\frac{1}{q}} D^{\frac{1}{p}} \leq tB + C(t)D,$$

where

$$C(t) := p^{-1} q^{\frac{-p}{q}} t^{\frac{-p}{q}},$$

and so

$$\frac{1}{C(t)}\left(\frac{p-1}{p} - t\right)B + \frac{1}{pC(t)}C \leq D,$$

for all $t > 0$. Picking $t = q^2$ gives the desired result.

(ii) Let $t > \frac{p-1}{p}$, multiply $\mathcal{L}_{A,p}(E) = \mu$ by E^{tp-p+1} and integrate over Ω to obtain

$$\int_\Omega E^{tp-p+1} d\mu = (tp - p + 1)\int_\Omega |\nabla E|_A^p E^{tp-p} dx, \tag{7.90}$$

which shows that $E^t \in W_0^{1,p}(\Omega)$ for $p > \frac{p-1}{p}$. If one uses as a minimizing sequence $u_t := E^t$ and sends $t \searrow \frac{p-1}{p}$ they immediately see that (7.89) is optimal.

(iii) Again one uses $u_t := E^t$ and sends $t \searrow \frac{p}{p-1}$. The result is immediate after using (7.90).

□

An important example is when $A(x)$ is the identity matrix and $E(x) = \delta(x) := dist(x, \partial\Omega)$ so $|\nabla \delta| = 1$ a.e.. Then $\mathcal{L}_{A,p}(\delta) = -div(|\nabla\delta|^{p-2}\delta) = -\Delta\delta =: \mu$ which is non-negative if we further assume that Ω is convex. In this case we have the L^p analog of (7.2) (provided one relaxes the regularity assumption on the measure μ):

Corollary 7.91. *Suppose Ω is convex and $\delta(x) := dist(x, \partial\Omega)$. Then for $1 < p < \infty$ and $u \in W_0^{1,p}(\Omega)$ we have*

$$\int_\Omega |\nabla u|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\delta^p} dx,$$

$$\int_\Omega |\nabla u|^p dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\delta^p} dx \geq \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{\delta^{p-1}} d\mu,$$

where $d\mu := -\Delta\delta dx$. Moreover all constants are optimal.

The first inequality is due to [13].

Bibliography

[1] Adimurthi, Anusha Sekar, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proceedings of the Royal Society of Edinburgh, 136A, 1111-1130, 2006

[2] F.G. Avkhadiev, K.J. Wirths, *Unified Poincare and Hardy inequalities with sharp constants for convex domains*, Math. Mech. 87, No. 8-9, 632-642 (2007)

[3] B. Barbatis, S. Filippas, A. Tertikas, *Series expansions for L^p Hardy inequalities*, (August 20, 2007) preprint

[4] H. Brezis, M. Marcus, *Hardy's inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1-2, 217–237 (1998).

[5] H. Brezis, M. Marcus, I.Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal. 171 (2000), 177-191

[6] H. Brezis, J.L. Vazquez, *Blowup Solutions of some nonlinear elliptic problems*, Revista Mat. Univ. Complutense Madrid 10 (1997), 443-469

[7] E. DiBenedetto. *C^1 local regularity of weak solutions of degenerate elliptic equations*, Nonlin. Analysis 7 (1983), 827850

[8] S. Filippas, A. Tertika, *Optimizing improved Hardy inequalities*, J.Funct. Anal. 192 (2002), n0.1, 186-233

[9] J.Fleckinger, E.M. Harrell II, F. Thelin, *Boundary behaviour and estimates for solutions of equations containing the p-Laplacian*, Electron. J. Differential Equations 38 (1999), 1-19

[10] N. Ghoussoub, A. Moradifam, *On the best possible remaining term in the Hardy inequality*, Proc. Natl. Acad. Sci. USA 105 (2008), no. 37, 13746–13751.

[11] B. Opic, A. Kufner, *Hardy Type Inequalities*, Pitman Research Notes in Mathematics, Vol. 219, Longman, New York, 1990

[12] V.G. Maz'ja *Sobolev Spaces*, Berlin, Springer-Verlag

[13] M. Marcus, V.J. Mizel, Y. Pinchover, *On the best constant for Hardy's inequality in* $\mathbb{R}^n$, Transactions of the American Mathematical Society, Vol. 350, No. 8 (aug., 1998), pp 3237-3255

[14] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eqns 51 (1984), 473484

[15] J.L. Vazuez, *A Strong Maximum Principle for some Quasilinear Elliptic Equations*, Appl. Math. Optim. 12:191-202 (1984)

[16] J.L. Vazquez, E. Zuazua *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential*, J. Funct. Anal. 173 (2000), 103-153

Chapter 8

Conclusion

Chapters 2, 4, 5 and 6 all involve obtaining results concerning the regularity of the extremal solutions.

In Chapter 2 the main issues is that the equation is nonvariational and hence the stability does not immediately allow the use of arbitrary test functions which can be used to obtain estimates. This is overcome through the use of a general Hardy inequality which allows the use of arbitrary test functions and then one proceeds as usual.

In Chapter 3 we proved various curious results which were noticed numerically by various authors.

In Chapter 4 we used the approach developed in [1] to show the extremal solution is bounded in low dimensions. This is not a complete straightforward extension of the case considered in [1] where $f(u) = e^u$, since in their case $f'(u) = f(u)$, which greatly streamlines the approach. To complete the picture one needs to show that the extremal solution is singular for $N \geq 9$. Our approach showed this without needing to resort to a computer assisted proof.

In Chapter 5 we obtained many results concerning the regularity of the extremal solution of fourth order problems on general domains. We mention that our results are vast improvements over the known results but, one expects they are not optimal after considering the known results on radial domains. The key ideas for our results were to test the stability on $\psi := \Delta u$ where u is a stable solution and then to obtain a pointwise lower bound on $-\Delta u$, which allows one to obtain estimates. Extending these results to optimal results, we suspect, will require a new idea.

In Chapter 6 we obtained results concerning the regularity of the extremal solutions in the elliptic system given by $-\Delta u = \lambda e^v$, $-\Delta v = \gamma e^u$. The main goal of this chapter was to show, at least in this special case, that the standard approach for the scalar case can be modified to handle the systems case.

In Chapter 7 we examined very general Hardy inequalities and showed many known Hardy inequalities can be thought of a special cases of this general Hardy inequality. In addition improvements and weighted versions were obtained. We believe this approach to Hardy inequalities is a somewhat unifying approach to the various Hardy inequalities.

8.1 Future directions

The results from Chapter 5 need to be improved since they are not optimal. As mentioned above we suspect this will require a new idea.

The results from Chapter 6 needs to be extended to the case where the nonlinearities are not equal.

Most of the equations examined in this thesis can be viewed as stationary solutions of various parabolic or hyperbolic models coming from certain physical models. These fourth order parabolic and hyperbolic models have not received much attention and this is possible future work.

Many of the equations examined here also have various quasi linear versions and many of the results obtained here can possibly extended. This needs to be investigated.

Bibliography

[1] J. Davila, L. Dupaigne, I. Guerra and M. Montenegro, *Stable solutions for the bilaplacian with exponential nonlinearity*, SIAM J. Math. Anal. 39 (2007), 565-592.