Essays on Volatility Risk Premia in Asset Pricing

by

Oliver Boguth

Diplom-Wirtschaftsmathematiker, Universitaet Ulm, 2004
M.Sc. Mathematical Finance, University of Southern California, 2004

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Business Administration)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2010

© Oliver Boguth 2010
Abstract

This thesis contains two essays. In the first essay, we investigate the impact of time varying volatility of consumption growth on the cross-section and time-series of equity returns. While many papers test consumption-based pricing models using the first moment of consumption growth, less is known about how the time-variation of consumption growth volatility affects asset prices. In a model with recursive preferences and unobservable conditional mean and volatility of consumption growth, the representative agent’s estimates of conditional moments of consumption growth affect excess returns. Empirically, we find that estimated consumption volatility is a priced source of risk, and exposure to it predicts future returns in the cross-section. Consumption volatility is also a strong predictor of aggregate quarterly excess returns in the time-series. The estimated negative price of risk together with the evidence on equity premium predictability suggest that the elasticity of intertemporal substitution of the representative agent is greater than unity, a finding that contributes to a long standing debate in the literature.

In the second essay, I present a simple model to show that if agents face binding portfolio constraints, stocks with high volatility in states of low market returns demand a premium beyond the one implied by systematic risks. Assets whose volatility positively covaries with market volatility also have high expected returns. Both effects of this idiosyncratic volatility risk premium are strongest for assets that face more binding trading restrictions. Unlike the prior empirical literature that obtains mixed results when focusing on the level of idiosyncratic volatility, I investigate the dynamic behavior of idiosyncratic volatility and find strong support for my predictions. Comovement of innovations of idiosyncratic volatility with market returns negatively predicts returns for trading restricted stocks relative to unrestricted stocks, and comovement of idiosyncratic volatility with market volatility positively predicts returns for restricted assets.
# Table of Contents

Abstract ................................................................. ii

Table of Contents ..................................................... iii

List of Tables ........................................................... vi

List of Figures ............................................................ vii

Acknowledgements ....................................................... viii

Dedication ................................................................. ix

Statement of Co-Authorship ............................................ x

1 Introduction ............................................................. 1
   1.1 Consumption Volatility Risk ........................................ 1
   1.2 Stochastic Idiosyncratic Volatility, Portfolio Constraints, and the Cross Section
       of Stock Returns .................................................. 3
   1.3 Bibliography ....................................................... 6

2 Consumption Volatility Risk .......................................... 7
   2.1 Model ................................................................. 12
       2.1.1 Consumption .................................................. 12
       2.1.2 Recursive Utility ............................................. 13
       2.1.3 Estimation ................................................... 15
       2.1.4 Implications .................................................. 17
   2.2 Cross-Sectional Return Predictability .............................. 20
       2.2.1 Data .......................................................... 20
       2.2.2 Risk Loadings ............................................... 21
       2.2.3 Portfolio Sorts .............................................. 21
| 1.1 Introduction | 13 |
| 1.2 Literature Review | 18 |
| 1.3 Methodology | 23 |
| 1.4 Empirical Analysis | 28 |
| 1.5 Conclusion | 33 |
| 1.6 Bibliography | 56 |

### Section 2: Consumption Volatility Risk Pricing

| 2.1 Introduction to Consumption Volatility | 23 |
| 2.2 Robustness | 24 |
| 2.3 Consumption Volatility Risk Pricing | 24 |
| 2.3.1 Factor Pricing with Consumption Data | 24 |
| 2.3.2 Factor Pricing with Portfolio Returns | 27 |
| 2.4 Time Series Predictability | 29 |
| 2.5 Conclusion | 31 |
| 2.6 Bibliography | 54 |

### Section 3: Stochastic Idiosyncratic Volatility, Portfolio Constraints, and the Cross Section of Stock Returns

| 3.1 Introduction to Idiosyncratic Volatility | 55 |
| 3.2 Stochastic Volatility and Pricing of Idiosyncratic Risk | 58 |
| 3.2.1 Returns on Risky Assets | 61 |
| 3.2.2 Agents and Preferences | 64 |
| 3.2.3 Equilibrium without Portfolio Holding Constraints | 66 |
| 3.2.4 Equilibrium with Portfolio Holding Constraints | 67 |
| 3.3 Empirical Analysis | 70 |
| 3.3.1 Stochastic Idiosyncratic Volatility | 70 |
| 3.3.2 Stochastic Idiosyncratic Volatility Risk | 71 |
| 3.3.2.1 Stochastic Volatility induced Skewness | 72 |
| 3.3.2.2 Stochastic Volatility induced Kurtosis | 75 |
| 3.4 Conclusion | 76 |
| 3.5 Bibliography | 96 |

### Section 4: Conclusion

| 4.1 Conclusion | 97 |
| 4.2 Bibliography | 99 |

### Appendices

<p>| A Appendix to Chapter 2 | 100 |
| A.1 Wealth-Consumption Ratio Approximation | 100 |
| A.2 Seasonal Adjustment | 101 |
| A.3 Numerical Solution | 102 |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.4 Cross-Sectional Asset Pricing Implications</td>
<td>102</td>
</tr>
<tr>
<td>A.5 Equity Premium</td>
<td>103</td>
</tr>
<tr>
<td>A.6 Learning Premium</td>
<td>104</td>
</tr>
<tr>
<td>A.7 Predictive Regression Bias</td>
<td>106</td>
</tr>
<tr>
<td>A.8 Bibliography</td>
<td>111</td>
</tr>
</tbody>
</table>
# List of Tables

2.1 Markov Model of Consumption Growth ........................................... 38  
2.2 Portfolios Formed on Risk Exposure .............................................. 39  
2.3 Characteristics of Consumption Volatility Risk Portfolios .................... 40  
2.4 Portfolios Formed on Consumption Volatility Risk and Characteristics ...... 41  
2.5 Fama-MacBeth Regressions .......................................................... 42  
2.6 Volatility Risk Pricing ............................................................... 43  
2.7 Volatility Risk Factor ............................................................... 44  
2.8 Factor Exposures to the CVR Factor .............................................. 45  
2.9 Volatility Risk Pricing Factor ..................................................... 46  
2.10 Market Predictability in the Time-Series ...................................... 47  
3.1 Parametrization for the Numerical Solution ..................................... 85  
3.2 Portfolios based on Stochastic Volatility Induced Skewness .................... 86  
3.3 Fama-French Alphas of Portfolios sorted on SVS and Holding Restrictions . 87  
3.4 Stochastic Volatility Induced Skewness and Coskewness Risk ................. 88  
3.5 Characteristics of Portfolios based on Stochastic Volatility Induced Skewness 89  
3.6 Portfolios based on Stochastic Volatility Induced Kurtosis .................... 90  
3.7 Fama-French Alphas of Portfolios sorted on SVK and Holding Restrictions . 91  
A.1 Effect of X-12-ARIMA Filter ...................................................... 107  
A.2 Wealth-Consumption Ratio ......................................................... 108  
A.3 Markov Model of Dividend Growth .............................................. 109  
A.4 Model Implications ................................................................. 110
List of Figures

2.1 Bayesian Beliefs about the Mean and Volatility State .................... 33
2.2 Wealth-Consumption Ratio for a High EIS Agent .......................... 34
2.3 Wealth-Consumption Ratio for a Low EIS Agent ........................... 35
2.4 Pricing Errors of the Consumption-Based Model .......................... 36
2.5 Pricing Errors of the Market-Based Model ................................. 37

3.1 Three Kinds of Skewness .................................................. 78
3.2 Mixture of Gaussian Distributions ........................................ 79
3.3 Timeline ................................................................. 80
3.4 Portfolio Weights in the Skewness Model ................................. 81
3.5 Stock Returns in the Skewness Model .................................... 82
3.6 Portfolio Weights in the Kurtosis Model ................................... 83
3.7 Stock Returns in the Kurtosis Model ..................................... 84
Acknowledgements

I am greatly indebted to a large number of colleagues, friends, and family who supported me during my six wonderful years in the Ph.D. program at the Sauder School.

It is impossible to overstate my gratitude to my Ph.D. supervisors Adlai Fisher and Murray Carlson. This dissertation would not have been possible without their continued encouragement, guidance, and support. I want to thank Harjoat Bhamra, Viktoria Hnatkovska, Alan Kraus, Lorenzo Garlappi, and Ron Giammarino, for their helpful advice throughout my studies and the job market.

I also thank all my friends and classmates, especially Mike Simutin and Lars Kuehn, for always being helpful and a source of inspiration and great ideas.

My mother Birgit and my father Heinz never gave up on me and made the adventure of graduate school possible. With tears in my eyes do I realize that my father will not be with us to celebrate the completion of my degree.

Finally, it was my wife Sharon and my son Markus that made my time in Vancouver truly special. Without the constant support of my family I could not have completed my thesis. I want to thank Sharon for always believing in me, and keeping up with the ups and downs inevitable in any Ph.D. program. I am looking forward to the adventures ahead with my family at my side.
Dedication

To Sharon.
Statement of Co-Authorship

The first essay of my thesis (Chapter 2) is joint work with Lars-Alexander Kuehn. My contribution to the paper includes building and refining the theoretical model and solving it numerically, developing and executing the empirical tests, as well as writing and editing of the final manuscript. The second essay (Chapter 3) was solely written by the author Oliver Boguth.
Chapter 1

Introduction

Modern asset pricing theory links expected returns on financial securities to the risk associated with holding the investments. If there was no reward for bearing risk, so the idea, rational investors would choose not to hold risky assets, and the resulting allocation could not persist in equilibrium. In a seminal contribution, Mehra and Prescott (1985) find that the reward for taking on equity risk is too high given commonly used economic models and assumptions about investor preferences. One promising solution to this equity premium puzzle builds on well documented time-variation in risk. Investors then face at least two sources of uncertainty, namely the risk of return realizations and uncertainty about the amount of risk associated with an investment.

The two essays in my thesis investigate theoretically and empirically the asset pricing implications of a volatility risk premium, a compensation required by investors if risk stochastically varies over time. The first essay takes a macroeconomic perspective and analyzes the pricing impact of stochastic consumption volatility in a general equilibrium setting with a representative agent. The second essay deviates from the assumption of a representative agent by explicitly imposing trading frictions that heterogeneously affect agents. Due to the resulting limitations to risk sharing, the non-systematic part of stock return volatility – and its stochastic time-variation – is priced in equilibrium.

1.1 Consumption Volatility Risk

The volatility of macroeconomic quantities, such as consumption and output, varies over time. While recent theoretical contributions, pioneered by Bansal and Yaron (2004), investigate the pricing impacts of time-varying consumption volatility on asset prices, there is little supporting empirical work.\footnote{Notable exceptions are Bansal et al. (2007) and Tedongap (2007).} The contribution of the first essay is twofold: First, we develop a model in which consumption volatility affects asset prices. Second, we empirically test the model on the cross-section and time-series of stock returns.
In a theoretical part, we follow the work of Bansal and Yaron (2004), Kandel and Stambaugh (1991), and Lettau et al. (2008). A representative agent with recursive Epstein and Zin (1989) preferences cares not only about shocks to current consumption growth but also about the distribution of future consumption growth rates, which is modeled by persistent states of the Markov chains for the first and second moments. The state of the economy is unobservable, similar to David (1997) and Veronesi (1999), and as a result, the agent’s estimates of the conditional first and second moments of consumption growth are priced.

If the agent prefers intertemporal risk caused by the unobservable Markov states to be resolved sooner rather than later, an asset with high (low) payoff when investors learn that future consumption growth is low (high) is a welcome insurance against future bad times and will require lower returns. Similarly, an asset that comoves positively with future consumption volatility pays off exactly when the agents learns that the future is volatile. This asset serves as insurance against uncertainty, and thus has a lower expected return.

We empirically test these predictions in two ways. We first study the relation between risk loadings and future returns at the firm level, and then estimate the market price of risk directly using a two stage estimation approach on commonly used sets of test portfolios. Following the model, we estimate a Markov chain for the first and second moments of consumption growth, and obtain the agent’s beliefs over the two moments using Bayesian updating. We obtain risk loadings with respect to innovations in the perceived conditional first and second moments of consumption growth from rolling quarterly time-series regressions of individual stock returns on consumption growth as well as innovations in beliefs for mean and volatility. Sorting stocks into portfolios based on these risk estimates, we find that loadings on innovations in the perceived expected consumption growth do not help to explain future returns, while loadings on consumption growth volatility significantly negatively forecast cross-sectional differences in returns. The return of a consumption volatility risk (CVR) portfolio, a long position in the value-weighted quintile of stocks with high volatility risk and a short position in low volatility risk, is on average $-5\%$ per year.

While the negative relation between consumption volatility risk loadings and future returns at the firm level suggests a negative price of consumption volatility risk, we also provide direct evidence from two stage regressions of excess returns on log consumption growth, changes in the perceived mean and volatility of consumption growth. Importantly, the coefficients on the innovation in the perceived consumption volatility and CVR are negative, which strongly supports the assumption that the agent prefers intertemporal risk to resolve sooner than later, a crucial component in the long-run risk framework of Bansal and Yaron (2004).
For further evidence supporting the model, we also test for time-series predictability of the aggregate equity premium. When the representative agent has an EIS greater than unity, the model predicts a high equity premium in states with low conditional mean or high conditional volatility of consumption growth. We show in a predictive regression that innovations to consumption volatility are a significant and robust predictor of one-quarter ahead equity returns. In our model, changes in consumption volatility enter the pricing kernel only because they affect the wealth-consumption ratio. Thus, one might expect that direct measures of the wealth-consumption ratio, such as $cay$ (Lettau and Ludvigson (2001)), comprise all relevant information about the volatility state. Empirically, this is not the case. Both variables are virtually uncorrelated and both remain strong and robust predictors in multivariate settings.

1.2 Stochastic Idiosyncratic Volatility, Portfolio Constraints, and the Cross Section of Stock Returns

In a frictionless market with homogenous agents, every participant will hold an identical portfolio. Given that each asset has to be held by some investor, the equilibrium allocation of each agent has to be the market. Since every investor is fully diversified, prices depend on market risk only, and asset specific volatility does not affect the asset’s value.

A premium for idiosyncratic risk is empirically very appealing as it has the potential to explain many asset pricing puzzles, such as the size and value anomalies. Levy (1978) and Merton (1987) impose exogenous market frictions to show theoretically that idiosyncratic volatility can carry a positive price of risk in equilibrium. However, empirical findings often contradict the theory.\(^2\)

In this chapter, I use a two-period model to show how trading frictions combined with stochastic volatility of the idiosyncratic component of returns can result in an idiosyncratic volatility risk premium. As intuition would suggest, assets with unexpectedly high idiosyncratic volatility in times of a high marginal rate of substitution require a return premium. The key determinant for the magnitude of that premium is the \textit{asset level} underdiversification, a measure of how well a given asset’s risk can be diversified across agents.\(^3\) I thus establish a

\(^2\)Early work by Friend and Blume (1970), Friend et al. (1978), and more recently Ang, Hodrick, Xing, and Zhang (2006, 2009) find a strong negative price of risk, while Spiegel and Wang (2005), Fu (2009), and Huang et al. (2009) confirm a positive correspondence between expected idiosyncratic volatility and future returns.

\(^3\)This notion is somewhat different from the \textit{investor level} underdiversification as in Levy (1978), which measures how well an agent can diversify risk by holding a portfolio. However, underdiversification at the investor level corresponds to underdiversification at the asset level. To see this, suppose one agent can not
direct link between the state-dependence of idiosyncratic volatility, portfolio constraints, and equity returns, and derive novel testable pricing implications.

Under standard preference assumptions, the model makes the following novel predictions for assets that face binding holding constraints: First, assets whose volatility innovations negatively comove with market returns trade at a discount and therefore have higher expected returns. Secondly, assets whose surprises in volatility are positively correlated with market volatility shocks require higher returns. In both cases, idiosyncratic volatility exceeds its expectations in states with a high marginal rate of substitution. Such assets thus contribute excessively to the volatility of the underdiversified portfolio in bad states, and investors demand compensation for this risk.

I empirically test the model predictions by decomposing idiosyncratic volatility into a conditionally expected component and a shock. I measure the skewness induced by stochastic volatility (SVS) as the covariance of assets’ volatility shocks and market returns, and the stochastic volatility induced kurtosis (SVK) as the covariance between shocks to the volatilities of the individual assets and the market volatility. I use a variety of proxies for asset level underdiversification, and find strong support for the model predictions in the restricted subset: A negative relation between SVK and future returns, and a positive relation between SVS and future returns.

hold a specific asset. In turn, the risk of that asset has to be borne by the remaining investors – asset level underdiversification.
Bibliography


Chapter 2

Consumption Volatility Risk

Many papers test consumption-based pricing models using the first moment of consumption growth, for instance, Lettau and Ludvigson (2001b), Parker and Julliard (2005) and Yogo (2006). In contrast, the impact of consumption growth volatility on asset prices has received less attention in the empirical literature. This is surprising since it is well known that the volatility of macroeconomic quantities, such as consumption and output, varies over time. The goal of this paper is to analyze the pricing implications of consumption growth volatility in the cross-section and time-series of stock returns.

This research question poses several challenges. First, a natural candidate to model consumption volatility is the ARCH model proposed by Engle (1982) and its various generalizations. Asset pricing theory, however, states that only innovations are priced and in a GARCH model the volatility has no separate innovations relative to the process for consumption growth. In particular, Restoy and Weil (2004) show that a GARCH consumption model does not give rise to a volatility risk factor in an equilibrium model with Epstein and Zin (1989) utility. Second, while consumption growth rates are observable, the conditional volatility is latent and has to be estimated from the data. Last, aggregate consumption is measured with error thereby making statistical inference more difficult (Breeden et al. (1989) and Wilcox (1992)).

Our model follows the work of Bansal and Yaron (2004) and Kandel and Stambaugh (1991) and uses the same building blocks as Lettau et al. (2008). The representative agent has recursive Epstein and Zin (1989) preferences and the conditional first and second moments of consumption growth follow independent two-state Markov chains. An important implication

4A version of this chapter will be submitted for publication. Boguth, Oliver and Kuehn, Lars-Alexander (2010) Consumption Volatility Risk.
5A notable exception is Bansal et al. (2007). Following their model, they estimate the conditional first and second moments of consumption growth as affine functions of financial data. Tedongap (2007) uses a GARCH process for consumption volatility. Other recent contributions testing the C-CAPM using realized consumption growth include Campbell (1996), Aït-Sahalia et al. (2004), Campbell and Vuolteenaho (2004), Bansal et al. (2005a), Lustig and Nieuwerburgh (2005), and Jagannathan and Wang (2007).
6For instance, see Cecchetti and Mark (1990), Kandel and Stambaugh (1990), Bonomo and Garcia (1994), Kim and Nelson (1999), or Whitelaw (2000).
of recursive preferences is that the agent cares not only about shocks to current consumption growth but also about changes to the conditional distribution of future consumption growth. In our model, these changes are driven by persistent states of the Markov chains for the first and second moments of consumption growth. We further assume that the state of the economy is unobservable and the agent uses Bayesian updating to form beliefs about the state, similar to David (1997) and Veronesi (1999). As a result, the agent’s estimates of the conditional first and second moments of consumption growth are priced.

The model has the following implication for the cross-section of returns. When the elasticity of intertemporal substitution (EIS) is greater than the inverse of the coefficient of relative risk aversion (RRA), the agent prefers intertemporal risk due to unobservable Markov states to be resolved sooner rather than later. Intuitively, consider an asset that comoves negatively with future consumption growth. Its payoff is high (low) when investors learn that future consumption growth is low (high). Investors will demand a low return from this asset as it is a welcome insurance against future bad times. Similarly, consider an asset that comoves highly with future consumption volatility. This asset has high (low) payoffs when investors learn that future consumption is (not) very volatile. This asset serves as insurance against uncertain times and thus has a lower required return. Consequently, the agent demands a positive market price of risk for shocks to expected consumption growth and a negative one for shocks to the conditional volatility of consumption growth. To provide convincing empirical evidence, we test these implications in two ways. First, we study the relation between risk loadings and future returns at the firm level. Second, we estimate the market price of risk directly using portfolios.

Following Hamilton (1989), we estimate a Markov chain for the first and second moments of consumption growth. Bayesian updating provides beliefs about the states for mean and volatility. To obtain time-varying risk loadings with respect to innovations in the perceived conditional first and second moments of consumption growth, we run rolling quarterly time-series regressions of individual stock returns on consumption growth as well as innovations in beliefs for mean and volatility. Sorting stocks into portfolios based on these risk loadings, we find that loadings on innovations in the perceived expected consumption growth do not help to explain future returns. Loadings on consumption growth volatility, however, significantly negatively forecast cross-sectional differences in returns. A consumption volatility risk factor (CVR), which is the return of holding a long position in the value-weighted quintile of stocks with high volatility risk and a short position in low volatility risk, has an average return of

\footnote{Ang et al. (2008) emphasize the use of firm level data to estimate market prices of risk because firm level data display more dispersion in betas. As a result, the estimation is more efficient.}
−5% per year. Importantly, consumption volatility risk quintiles do not display variation in average book-to-market ratios.

The negative relation between consumption volatility risk loadings and future returns at the firm level suggests a negative price of consumption volatility risk. In order to provide direct evidence, we perform two stage regressions of excess returns on log consumption growth, changes in the perceived mean and volatility of consumption and the CVR factor. Importantly, the coefficients on the innovation in the perceived consumption volatility and CVR are negative implying that the representative agent has an EIS greater than the inverse of RRA. A crucial assumption in the long-run risk framework of Bansal and Yaron (2004) is that the agent prefers intertemporal risk to resolve sooner than later. Our findings strongly support this assumption.

We also augment the market CAPM and Fama-French 3-factor model with the CVR factor. In particular, CVR shows up strongly and significant in addition to the market and the three Fama and French (1993) factors. When the CVR factor is added to specifications that already contain the value factor HML, average absolute pricing errors decline only marginally. At the same time, replacing HML with CVR does not result in larger average pricing errors. We thus conclude that HML and CVR have similar pricing implications. But in contrast to HML, the volatility risk factor has a clear economic interpretation.

Another implication of our model is the predictability of the aggregate equity premium in the time-series. In states with low conditional mean or high conditional volatility of consumption growth, the model predicts a high equity premium when the representative agent has an EIS greater than unity. We show in a predictive regression that innovations to consumption volatility are a significant and robust predictor of one-quarter ahead equity returns. A one standard deviation increase of the perceived consumption volatility results in a 1.4% rise of the quarterly equity premium, similar to the predictive power of the wealth-consumption ratio cay of Lettau and Ludvigson (2001a), the best known macroeconomic predictor of the short horizon equity premium. In our model, changes in consumption volatility enter the pricing kernel only because they affect the wealth-consumption ratio. Thus, one might expect that direct measures of the wealth-consumption ratio, such as cay, comprise all relevant information about the volatility state. Empirically, this is not the case. Both variables are virtually uncorrelated and both remain strong and robust predictors in multivariate settings.

This finding contributes to a long standing debate in the literature on the magnitude of the EIS. Early evidence suggests that the EIS is smaller than one, e.g., Hall (1988) and Campbell and Mankiw (1989). More recently, Attanasio and Weber (1993), Vissing-Jørgensen (2002) and Vissing-Jørgensen and Attanasio (2003) find the opposite. The positive relation between
consumption volatility and the equity premium provides evidence for an EIS greater than one.

In the literature, it is common to measure consumption risk by using non-durable plus service consumption. This assumption is usually justified with a felicity function which is separable across goods. With Epstein-Zin utility, however, felicity can be separable across goods, but due to the time-nonseparability of the time-aggregator, other goods still matter for asset pricing because they enter the pricing kernel via the wealth-consumption ratio. The wealth-consumption ratio can be a function, for instance, of human capital (e.g. Jagannathan and Wang (1996), Lettau and Ludvigson (2001b), and Santos and Veronesi (2006)), durable goods (e.g. Yogo (2006)) or housing consumption (e.g. Piazzesi et al. (2007)). If the wealth-consumption were observable, it would subsume all these variables.\(^8\) The contribution of this paper is to show that the conditional volatility of consumption growth is a significant determinant of the wealth-consumption ratio by documenting that it is priced in the cross-section and time-series after controlling for other factors.

Related Literature

Pindyck (1984) and Poterba and Summers (1986) are among the first to show that a decrease in prices is generally associated with an increase in future volatility, the so-called leverage or volatility feedback effect. Similarly, French et al. (1987), Campbell and Hentschel (1992) and Glosten et al. (1993) look at the relation between market returns and market volatility in the time-series. More recently, Ang et al. (2006) use a nonparametric measure of market volatility, namely the option implied volatility index (VIX), to show that innovations in aggregate market volatility carry a negative price of risk in the cross-section. Adrian and Rosenberg (2008) use a GARCH inspired model to decompose market volatility into a short- and long-run component and show how each of the two components affects the cross-section of asset prices.

All of the above papers use some measure of stock market volatility. Motivated by the long-run risk model of Bansal and Yaron (2004), several important papers study the relation between consumption volatility and prices. Notably, Bansal et al. (2005b) find that the conditional consumption volatility predicts aggregate valuation ratios. Bansal et al. (2007) estimate the long-run risk model using the cross-section of returns. Following their theory, they estimate consumption volatility as an affine function of the observable aggregate price-dividend ratio and short-term interest rate. Their Table IV indicates that consumption

---

\(^8\)One of the first papers which tries to estimate the wealth-consumption ratio is Lettau and Ludvigson (2001a). A more recent contribution is Lustig et al. (2008).
volatility plays a minor role in explaining the size and value spread relative to shocks to expected consumption growth. In contrast, we filter consumption volatility directly from consumption data without the use of financial data. We find that consumption volatility is a dominant contributor to risk premia in the cross-section.\textsuperscript{9}

Letttau et al. (2008) estimate a Markov model with learning to show that the decline in consumption volatility—also referred to as the “Great Moderation”—can explain the high observed stock market returns in the 1990s and the following decline in equity risk premia. We extend their work by studying the cross-section and time-series of returns. Parker and Julliard (2005) empirically measure a version of long-run risk as the covariance between one-period asset returns and long-horizon movements in the pricing kernel. Their ultimate consumption risk measure performs favorably in explaining the return differences of the 25 Fama-French portfolios. Similarly, Tedongap (2007) estimates conditional consumption volatility as a GARCH process and finds that value stocks covary more negatively with changes in consumption volatility over long horizons. In contrast to Tedongap (2007), we extract innovations to beliefs about consumption volatility, whereas a GARCH model does not allow that. Tedongap (2007) obtains significant results only at long horizons since GARCH models account for innovations to volatility only through realized data.

In the original long-run risk calibration, Bansal and Yaron (2004) find a large premium for the first moment of consumption growth, and only a minor effect of consumption volatility. Calvet and Fisher (2007) study the asset pricing implications of multi-fractal Markov switching in a recursive preference model at the aggregate level. They look at the relative importance of both channels, and find the volatility premium to be more important than the drift premium. Beeler and Campbell (2009) show that long-run risk model calibrations without a large volatility premium make several predictions that are difficult to reconcile with the empirical facts. Extending the long-run risk model to accommodate generalized disappointment aversion, Bonomo et al. (2010) show that persistence of consumption volatility is most important.

Drechsler and Yaron (2008) extend the long-run risk model to include jumps in consumption growth and volatility. Their model generates a variance premium and return predictability which are consistent with the data. Bansal and Shaliastovich (2008) find evidence that measures of investors’ uncertainty about their estimate of future growth contain information about large moves in returns at frequencies of about 18 months. They explain this regularity

\textsuperscript{9}Jacobs and Wang (2004) and Balduzzi and Yao (2007) use survey data to estimate the variability of idiosyncratic consumption across households. They find that exposure to idiosyncratic consumption risk bears a negative risk premium for the 25 Fama-French portfolios.
with a recursive-utility based model in which investors learn about latent expected consumption growth from signals with time-varying precision. Bollerslev et al. (2008) study the asset pricing implication when the variance of stochastic volatility is stochastic.\footnote{Other papers building on the long-run risk framework of Bansal and Yaron (2004) include Bhamra et al. (2007), Hansen et al. (2008) and Bansal et al. (2009).}

The remainder of the paper is organized as follows: In Section 2.1, we derive the asset pricing implication of a recursive preference model where the agent does not observe the state of the economy. This section motivates our empirical analysis of Sections 2.2-2.4. In Section 2.2, we test whether loadings on consumption growth and its conditional moments forecast returns in the cross-section. We form portfolios and run Fama-MacBeth regressions based on consumption volatility loadings. In Section 2.3, we test whether consumption growth and its conditional moments as well as the CVR factor are priced risk factors. Section 2.4 contains time-series predictability tests and Section 2.5 concludes. The appendix contains derivations and additional results.

2.1 Model

In this section, we derive the asset pricing implications of a model where the representative agent has recursive preferences and the state of the economy is unobservable. In our model, future consumption growth is influenced by time-variation in its conditional mean and volatility and the agent’s beliefs about the aggregate state enter the pricing kernel through the wealth-consumption ratio.

2.1.1 Consumption

We assume that the conditional first and second moments of consumption growth follow a Markov chain. Specifically, log consumption growth, $\Delta c_{t+1}$, follows

$$\Delta c_{t+1} = \mu_t + \sigma_t \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(0, 1) \tag{2.1}$$

where $\mu_t$ denotes its conditional expectation and $\sigma_t$ its conditional standard deviation. For tractability in the empirical estimation, we assume two states for the mean and two for the volatility which are denoted by $\mu_t \in \{\mu_l, \mu_h\}$ and $\sigma_t \in \{\sigma_l, \sigma_h\}$. The conditional first and second moments of consumption growth follow Markov chains with transition matrices $P^\mu$.
and $P^σ$, respectively, given by

$$P^μ = \begin{bmatrix} p^μ_{ll} & 1 - p^μ_{ll} \\ 1 - p^μ_{hh} & p^μ_{hh} \end{bmatrix} \quad P^σ = \begin{bmatrix} p^σ_{ll} & 1 - p^σ_{ll} \\ 1 - p^σ_{hh} & p^σ_{hh} \end{bmatrix}$$  \hspace{1cm} (2.2)$$

The assumption of two states each for drift and volatility results in a total of four states, \{(μl, σl), (μl, σh), (μh, σl), (μh, σh)\}, denoted by $s_t \in \{1, \ldots, 4\}$. The joint transition matrix over those four states, $P$, depends on $P^μ$, $P^σ$, as well as the correlations between switches in mean and volatility.

In contrast to Bansal and Yaron (2004) and Kandel and Stambaugh (1991), we assume that the representative agent does not observe the state of the economy. Instead, she must infer it from observable consumption data as in Lettau et al. (2008). This assumption ensures that the empirical exercise is in line with the model. The inference at date $t$ about the underlying state is captured by the posterior probability of being in each state based on the available data $Y_t$. We denote by $ξ_{t+1|t}$ the date-$t$ prior belief vector about tomorrow’s states

$$ξ_{t+1|t} = P' \frac{ξ_{t|t-1} \odot η_t}{1(ξ_{t|t-1} \odot η_t)}$$  \hspace{1cm} (2.3)$$

where

$$η_t = \begin{bmatrix} f(Δc_t|μ_{t-1} = μ_l, σ_{t-1} = σ_l, Y_{t-1}) \\ f(Δc_t|μ_{t-1} = μ_l, σ_{t-1} = σ_h, Y_{t-1}) \\ f(Δc_t|μ_{t-1} = μ_h, σ_{t-1} = σ_l, Y_{t-1}) \\ f(Δc_t|μ_{t-1} = μ_h, σ_{t-1} = σ_h, Y_{t-1}) \end{bmatrix}$$

is a vector of Gaussian likelihood functions.

### 2.1.2 Recursive Utility

The representative agent maximizes recursive utility over consumption following Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989)

$$U_t = \left\{ (1 - β)C_t^σ + β \left( E_t[U_{t+1}^{1-γ}] \right)^{ρ/(1-γ)} \right\}^{1/ρ}$$  \hspace{1cm} (2.4)$$

where $C_t$ denotes consumption, $β \in (0, 1)$ the rate of time preference, $ρ = 1 - 1/ψ$ and $ψ$ the elasticity of intertemporal substitution (EIS), and $γ$ relative risk aversion (RRA). Implicit in the utility function (2.4) is a constant elasticity of substitution time and risk aggregator.

Epstein-Zin preferences provide a separation between the EIS and RRA. These two con-
cepts are inversely related when the agent has power utility. Intuitively, the EIS measures the agent’s willingness to postpone consumption over time, a notion well-defined under certainty. Relative risk aversion measures the agent’s aversion to atemporal risk across states.

We know from Epstein and Zin (1989) that the Euler equation for an arbitrary return \( R_{i,t+1} \) can be stated as

\[
\mathbb{E}_t \left[ \beta^\theta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{PC_{t+1} + 1}{PC_t} \right)^{-(1-\theta)} R_{i,t+1} \right] = 1 \tag{2.5}
\]

where \( \theta = \frac{1-\gamma}{1-1/\psi} \) and \( PC_t = P_t/C_t \) denotes the wealth-consumption ratio. For the empirical exercise, it is useful to study the log-linearized pricing kernel. Intuitively, the first (stochastic) term in the pricing kernel is consumption growth, \( C_{t+1}/C_t \), and the second one the growth rate of the wealth-consumption ratio, \( PC_{t+1}/PC_t \). Hence, a log-linear approximation of the pricing kernel implicit in (2.5) is given by

\[
m_{t+1} \approx k - \gamma \Delta c_{t+1} - (1 - \theta) \Delta pc_{t+1} \tag{2.6}
\]

where small letters are logs of capital letters and \( \Delta \) denotes first differences.\(^{11} \) The log pricing kernel (2.6) implies that excess returns are determined as covariance between returns and log consumption growth as well as the covariance between returns and changes of the log wealth-consumption ratio

\[
\mathbb{E}_t [R^e_{i,t+1}] \approx \gamma \text{Cov}_t (R_{i,t+1}, \Delta c_{t+1}) + (1 - \theta) \text{Cov}_t (R_{i,t+1}, \Delta pc_{t+1}) \tag{2.7}
\]

In an endowment model which is solely driven by i.i.d. shocks, the wealth-consumption ratio is constant. In our model, however, the first and second moments of consumption growth follow a Markov chain. The unobservability of the Markov state implies that the agent’s prior probabilities characterize the state of the economy. Consequently, the wealth-consumption ratio is a function of the agent’s beliefs, i.e., \( PC_t = PC(\xi_{t+1}|t) \).

Since one of the beliefs is redundant, the wealth consumption ratio can be expressed as

\[^{11}\text{More precisely, a log-linear approximation is given by}
\]

\[
m_{t+1} \approx (\theta \ln \beta - (1 - \theta)k_0) - \gamma \Delta c_{t+1} - (1 - \theta)(pc_{t+1} - k_1pc_t)
\]

where \( pc_t = \ln(P_t/C_t) \) denotes the log wealth-consumption ratio and \( k_0, k_1 \) are constants. The value of \( k_1 \) is given by \( k_1 = PC/(PC - 1) > 1 \), where \( PC \) is the mean wealth-consumption ratio. Lustig et al. (2008) estimate the unconditional quarterly wealth-consumption ratio to be close to 351 implying that \( k_1 = 1.003 \). Consequently, the log pricing kernel can be closely approximated by the log growth rate of the wealth-consumption ratio.
a function of any set of three beliefs. The univariate effects of changing beliefs about state $i$ while holding beliefs about two other states constant can locally be approximated. We show in the appendix that, holding beliefs over two of the states constant, changes of the log wealth-consumption ratio are

$$\Delta pc_{t+1} \approx \Delta \xi_{t+2|t+1}(i) \left( \theta - \frac{PC_{\xi(i)=1,\xi_{-i}}^\theta - PC_{\xi(i)=0,\xi_{-i}}^\theta}{\sum_{j \neq i} PC_{\xi(i)=1,\xi_{-i}}^\theta} \right)$$

(2.8)

where $PC_{\xi(i),\xi_{-i}}$ denotes the wealth-consumption ratio when the posterior belief about state $i$ is $\xi(i)$ and the remaining beliefs are $\xi_{-i}$. Equation (2.8) illustrates that changes in the log wealth-consumption ratio are locally proportional to changes in beliefs. From an empirical asset pricing perspective, this finding implies that changes in beliefs are priced in the time-series and cross-section since they affect the wealth-consumption ratio, according to Equation (2.7).

### 2.1.3 Estimation

To estimate the model, we obtain data on quarterly per capita real consumption expenditures from the Bureau of Economic Analysis as the sum of nondurables and services. The data is seasonally adjusted using the X-12-ARIMA filter. Ferson and Harvey (1992) analyze the impact of using filtered consumption data on asset pricing tests. In the appendix, we provide evidence that Markov states in the raw data survive the filter and can be identified from seasonally adjusted data.\footnote{The BEA stopped providing seasonally unadjusted quarterly data in 2005.} In accordance with the observation that consumption behavior in the United States in the years following World War II is systematically different from later years, we restrict our time-series from the first quarter of 1955 until the fourth quarter of 2008. The choice of 1955 provides sufficient consumption observations before the beginning of the portfolio analysis in 1964.

To keep the model parsimonious, we impose that mean and volatility states switch independently. Thus, the joint transition matrix is the product of the marginal transition probabilities for mean and volatility states and the 16-element matrix can be fully characterized by 4 parameters. Importantly, the assumption of independent switching probabilities does not imply that the beliefs about mean and volatility states are independent. Our specification follows Kandel and Stambaugh (1991), Kim and Nelson (1999), and Lettau et al. (2008).

The resulting parameter estimates of the Markov chain are reported in Table 2.1, Panels
A and B. Expected consumption growth is always positive and about twice as large in the high state relative to the low state ($\mu_l = 0.37\%$, $\mu_h = 0.78\%$ quarterly). State conditional consumption volatilities are $\sigma_l = 0.21\%$ and $\sigma_h = 0.48\%$. The probability of remaining in a given regime for the mean is 0.93 in the low state and 0.89 in the high state. The volatility regimes are somewhat more persistent, with probabilities of 0.95 and 0.96, respectively. Our estimates differ slightly from the ones presented by Lettau et al. (2008), who estimate volatility in both states to be more persistent (0.991 and 0.994). While there are small differences in the consumption measure, the inclusion of the crisis year 2008 in our analysis results in a much lower persistence of the volatility regimes.

The independence assumption greatly reduces the number of parameters to be estimated and thus improves estimation precision. This constraint, however, is not a significant restriction of consumption data. We also estimate a model that allows the Markov chains for mean and volatility of consumption growth to be dependent. This unrestricted model has 8 additional parameters in the joint transition matrix. Yet the likelihood improves only marginally relative to the restricted model with independent Markov chains. A likelihood ratio test cannot be rejected at any significance level.\(^\text{13}\)

With independent Markov chains for mean and volatility, it is convenient to focus on beliefs about the first and second moments of consumption growth rather than the four states needed to describe the dependent model. We define the prior belief that the mean state is high tomorrow by

$$b_{\mu,t} = P(\mu_{t+1} = \mu_h | F_t) = \xi_{t+1|t} (3) + \xi_{t+1|t} (4) (2.9)$$

and the prior belief that the volatility state is high tomorrow by

$$b_{\sigma,t} = P(\sigma_{t+1} = \sigma_h | F_t) = \xi_{t+1|t} (2) + \xi_{t+1|t} (4) (2.10)$$

conditional on the current information set $F_t$. Following Equation (2.8), we obtain the approximate univariate effects of changing beliefs about the mean (volatility) state while holding the volatility (mean) state constant by

$$\Delta p_{\mu,t+1} \approx \Delta b_{\mu,t+1} \left( 1 - b_{\mu,t} PC_{\mu=\mu_h,\sigma}^\theta - PC_{\mu=\mu_l,\sigma}^\theta \right) (2.11)$$

\(^\text{13}\)The likelihood ratio test statistic is 2.15 which is $\chi^2$-distributed with 8 degrees of freedom. The corresponding critical value at 10% is 13.36.
where $PC_{\mu,\sigma}$ denotes the wealth-consumption ratio when expected consumption growth is $\mu$ and the consumption volatility is $\sigma$. Analogously, given a constant mean, changes of the log wealth-consumption ratio are

$$
\Delta pc_{t+1} \approx b_{\sigma,t} \left( \frac{PC^{\theta}_{\mu,\sigma=\sigma_h} - PC^{\theta}_{\mu,\sigma=\sigma_l}}{\theta b_{\sigma,t} PC^{\theta}_{\mu,\sigma=\sigma_h} + (1 - b_{\sigma,t}) PC^{\theta}_{\mu,\sigma=\sigma_l}} \right). 
$$

(2.12)

Figure 2.1 shows the filtered beliefs for the regimes. Panel A depicts the belief dynamics for mean consumption growth $b_{\mu,t}$ and Panel B for the standard deviation $b_{\sigma,t}$. These graphs visually confirm that the mean regimes are less persistent than the volatility states. In particular, the parameter estimates for the Markov chain imply that mean states last for 3.1 years whereas volatility states last for 5.7 years on average. Further, a decline in consumption volatility from the 1990s onwards, as pointed out by Kim and Nelson (1999), is easily observable. The 2008 recession demonstrates that this shift was not permanent.

2.1.4 Implications

Based on the parameter estimates for consumption data in Table 2.1, we solve the model numerically to study its properties. In the following, we are interested in how the perception about the conditional moments of consumption growth affect the wealth-consumption ratio. To this end, we define the perceived first and second moments of consumption growth as belief weighted averages

$$
\hat{\mu}_t = b_{\mu,t}{\mu}_h + (1 - b_{\mu,t}){\mu}_l \\
\hat{\sigma}_t = b_{\sigma,t}\sigma_h + (1 - b_{\sigma,t})\sigma_l
$$

(2.13)

and the corresponding changes in the perceived moments as

$$
\Delta \hat{\mu}_t = \hat{\mu}_t - \hat{\mu}_{t-1} \\
\Delta \hat{\sigma}_t = \hat{\sigma}_t - \hat{\sigma}_{t-1}
$$

(2.14)

In Figures 2.2 and 2.3, we plot the wealth-consumption ratio as a function of the perceived conditional first $\hat{\mu}_t$ (left graph) and second $\hat{\sigma}_t$ (right graph) moments of consumption growth when the agent has a high EIS of 1.5 (Figure 2.2) and a low EIS of 0.5 (Figure 2.3). We further calibrate the model to a quarterly rate of time preference, $\beta$, of 0.995 and relative risk aversion, $\gamma$, of 30. Risk aversion of 30 seems unrealistically high. This section, however, is meant to yield qualitative guidance and not quantitative results. Figure 2.2 illustrates

14More details on the solution procedure are contained in the appendix. There we also report model implied moments for stock returns and risk-free rate.
that the wealth-consumption ratio is increasing in the perceived mean and decreasing in the perceived volatility of consumption growth when the EIS equals 1.5. The opposite is true when the EIS equals 0.5 as in Figure 2.3.

To gain a better understanding of the economics, it is convenient to recall the Gordon growth model. Under the assumption that discount and growth rates are constant, the Gordon growth model states that the wealth-consumption ratio is negatively related to the risk-free rate \( r_f \) and risk premium \( r_E \) and positively to the growth rate \( g \), i.e., \( PC = 1/(r_f + r_E - g) \). The sign change in the slope of the wealth-consumption ratio with respect to expected consumption growth is driven by two opposing effects. On the one hand, a higher perception about the growth rate increases the wealth-consumption ratio as in the Gordon growth model. On the other hand, in equilibrium, an increase in expected consumption growth also raises the risk-free rate since the riskless asset becomes less attractive relative to the risky asset. This second effect lowers the wealth-consumption ratio. When the EIS is greater than unity, the first effect (intertemporal substitution effect) dominates the second effect (wealth effect). As a result, the demand for the risky asset and thus the wealth-consumption ratio rises with the perceived expected growth rate of consumption.

Similarly, the sign change in the slope of the wealth-consumption ratio with respect to expected consumption growth volatility (Figure 2.2 versus 2.3) is also driven by two opposing effects. On the one hand, a higher perceived conditional consumption volatility increases the risk premium which lowers the wealth-consumption ratio as in the Gordon growth model. On the other hand, in equilibrium, an increase in expected consumption growth volatility also reduces the risk-free rate since the riskless asset becomes more attractive relative to the risky asset. This effect increases the wealth-consumption ratio. If \( \gamma > 1 \), the first effect dominates the second one when the EIS is greater than one.

In order to test the model in the cross-section of returns, it is convenient to restate the fundamental asset pricing equation (2.7) in terms of betas:

\[
E_t[R_{i,t+1}^e] \approx \beta_{c,t}^i \lambda_{c,t} + \beta_{\mu,t}^i \lambda_{\mu,t} + \beta_{\sigma,t}^i \lambda_{\sigma,t} \tag{2.15}
\]

where \( \beta_{c,t}^i, \beta_{\mu,t}^i, \beta_{\sigma,t}^i \) denote risk loadings of asset \( i \) at date \( t \) with respect to consumption growth, and changes in the conditional first and second moments of consumption growth and

\[15\]To derive Equation (2.15), we have to assume that the log wealth-consumption is approximately affine in the perceived first and second moments of consumption growth implying that \( \Delta pc_t \approx \lambda \Delta \mu_t + B \Delta \sigma_t \). In the appendix we show that this approximation works well. In particular, we run time-series regressions of the log wealth-consumption ratio on the perceived conditional mean and volatility of consumption growth using simulated data. Even with risk aversion as high as 30, the regression \( R^2 \) exceeds 99%.
\( \lambda_{c,t}, \lambda_{\mu,t}, \lambda_{\sigma,t} \) are the respective market prices of risk given by

\[
\begin{align*}
\lambda_{c,t} &= \gamma \text{Var}_t(\Delta c_{t+1}) \\
\lambda_{\mu,t} &= A(1 - \theta) \text{Var}_t(\Delta \hat{\mu}_{t+1}) \\
\lambda_{\sigma,t} &= B(1 - \theta) \text{Var}_t(\Delta \hat{\sigma}_{t+1})
\end{align*}
\] (2.16)

where \( A \) and \( B \) are the sensitivities of the wealth-consumption ratio with respect to changes in the conditional first and second moments of consumption growth. The main cross-sectional implications of the model are the following. Assuming that the EIS is greater than the inverse of relative risk aversion \( (\psi > 1/\gamma) \), the agent requires lower expected excess returns for stocks which load less (low betas) on expected consumption growth and more (high betas) on consumption growth volatility.

Even though the sign switch of the sensitivity coefficients \( A \) and \( B \) occurs at unity, as explained above, the market prices of the conditional growth rate and volatility of consumption growth switch sign when the EIS equals the inverse of the coefficient of relative risk aversion \( (\psi = 1/\gamma) \). When the EIS is greater than the inverse of relative risk aversion \( (\psi > 1/\gamma) \), the agent prefers intertemporal risk due to the unobservable Markov states to be resolved sooner rather than later. Consequently, she dislikes negative shocks to expected consumption growth and requires a positive market price of risk. At the same time, she likes negative shocks to the conditional volatility of consumption growth and requires a negative market price of risk.

Intuitively, assets, which comove negatively with future consumption growth, have high payoffs when investors learn that future consumption growth is low. These assets thus provide insurance against future bad times. Similarly, assets, which comove highly with future consumption volatility, have high payoffs when investors learn that future consumption is very volatile. These assets serve as insurance against uncertain times. Consequently, investors require higher compensation for holding stocks which load strongly (high beta) on expected consumption growth and less compensation for stocks which load strongly (high beta) on consumption growth volatility.

These implications do not necessarily follow from an equilibrium model where the conditional consumption volatility follows a GARCH process. In a GARCH model, the conditional volatility is a function of lagged volatility and lagged squared residuals of the consumption process. Thus, a GARCH process is not driven by separate innovations relative to the consumption process. Consequently, Restoy (1991) and Restoy and Weil (2004) have shown that a GARCH consumption model does not give rise to a priced risk factor in a log-linearized approximation to an equilibrium model.\(^{16}\) Specifically, Equation (4.5) in Restoy and Weil (2004)

\(^{16}\) In empirical tests of equilibrium models, GARCH-inspired processes have been used by Adrian and Rosenberg (2008) and Tedongap (2007) to motivate additional factors in the cross-section.
states that the covariance of any stock with the wealth-consumption ratio is proportional to its covariance with consumption growth. Volatility, which affects the wealth-consumption ratio, therefore can have pricing implications as it determines the loading on the consumption growth factor, but it does not give rise to a second priced risk factor. Restoy and Weil continue to say on p. 44: “This is an important result because it embodies the fundamental insight that, for our AR(1)-GARCH(1,1) process, returns are only able to predict future conditional means of consumption growth but carry no information about the future conditional variances.”

To find evidence regarding the magnitude of the representative agent’s EIS, we perform three empirical exercises. First, we estimate time-varying risk loadings on the conditional first and second moments of consumption growth at the firm level and form portfolios based on these loadings. If the agent is not indifferent to intertemporal risk, we expect to find systematic return differences across portfolios. Second, we estimate the market prices directly using portfolios. Both exercises are closely related and we expect findings to be consistent. Third, we run time-series regressions of future excess returns on the perceived first and second moments of consumption growth. The last exercise provides a test whether the EIS is smaller or greater than unity because this relation depends only on the sensitivity of the wealth-consumption ratio with respect to consumption growth moments.

2.2 Cross-Sectional Return Predictability

The goal of this section is to demonstrate that loadings on the estimated conditional consumption volatility forecast returns. To this end, we first run quarterly time-series regressions to obtain loadings on risk factors. Next, we test using both Fama-MacBeth regressions and portfolio sorts whether these risk loadings forecast returns. Our main finding is that future returns are strongly and robustly predicted by exposure to innovations in consumption volatility, while exposure to consumption growth and changes in expected consumption growth do not help to predict the cross-section of asset prices.

2.2.1 Data

Our sample consists of all common stocks (shrcd = 10 or 11) on CRSP that are traded on the NYSE or AMEX (exchcd = 1 or 2). While the results are generally robust to the inclusion of NASDAQ stocks, this restriction mitigates concerns that only a small fraction of total market capitalization has a large impact on the portfolio analysis. To obtain valid risk measurements for a given quarter, the asset is required to have at least 60 months of prior data and at least 16
out of 20 valid quarterly returns. Since we use size and book-to-market ratio as characteristics, we require market capitalization to be available in December that occurs 7 to 18 months prior to the test month as well as book value of equity from Compustat in the corresponding year. The choice of the long delay is motivated by the portfolio formation strategies in Fama and French (1992), who want to ensure that the variables are publicly available when they are used in the study. Due to limited availability of book values in earlier years, we begin the empirical exercise in January 1964. The first time-series regression to estimate risk loadings thus covers the time span from 1959 to 1963. We end our analysis in December 2008.

2.2.2 Risk Loadings

Our first set of empirical results is based on time-series regressions of individual securities onto log consumption growth and the perceived conditional mean and volatility of consumption growth. In particular, for each security, we estimate factor loadings in each quarter $t^*$ using the previous 20 quarterly observations from

$$R_t^i - R_t^f = \alpha_i^t + \beta_{c,t}^i \Delta c_t + \beta_{\mu,t}^i \Delta \hat{\mu}_t + \beta_{\sigma,t}^i \Delta \hat{\sigma}_t + \epsilon_t^i$$

(2.17)

where $R_t^f$ denotes the risk-free rate and $\Delta c_t$ consumption growth for $t \in \{t^* - 19, t^*\}$. Further, $\Delta \hat{\mu}_t$ and $\Delta \hat{\sigma}_t$ are changes in the perceived conditional moments of consumption growth as defined in Equation (2.14).

The estimated parameters from Equation (2.17) allow us to evaluate the cross-sectional predictive power of these loadings in two different ways. First, we form portfolios based on the estimated risk exposures and analyze their properties in the time-series. Second, we use cross-sectional regressions as in Fama and MacBeth (1973) to investigate whether the factor loadings help to predict cross-sectional variation in returns.

2.2.3 Portfolio Sorts

We now investigate the predictive power of the estimated loadings from model (2.17) by forming portfolios. This approach has an important advantage relative to Fama-McBeth regressions where a potential error-in-variable problem leads to underestimated standard errors. In contrast, statistical inference based on portfolios is conservative. When variables are measured with noise, the portfolio assignment will be less accurate as some stocks are sorted into the wrong group. Under the assumption of cross-sectional predictive power, this leads to smaller return differences across portfolios. Since the statistical inference is based
solely on portfolio returns, the measurement error ultimately leads to a decrease in statistical significance.

At the end of each quarter, we sort all stocks in our sample into portfolios based on their estimated risk loadings from the time-series regression (2.17). Table 2.2 reports the average returns of equally-weighted (EW) and value-weighted (VW) quintiles as well as a long-short strategy that each month invests $1 into quintile 5 (high risk) and sells $1 of quintile 1 (low risk).

In Panel A, portfolios are formed based on loadings with respect to consumption growth, $\beta_{i,t}^\mu$. Consistent with prior research (e.g., Mankiw and Shapiro (1986) and Lettau and Ludvigson (2001b)), an asset’s contemporaneous short horizon loading on consumption growth does not help to generate a return differential across portfolio for either weighting scheme. A similar result follows by forming portfolios based on changes in beliefs about expected consumption growth, $\beta_{i,t}^\mu$ (Panel B). In contrast, exposure to consumption volatility risk, $\beta_{i,t}^\sigma$, predicts future returns strongly and negatively (Panel C). Stocks that comove highly with changes in consumption volatility underperform their peers in the future. An equally-weighted strategy results in a return of the long-short portfolio of $-0.19\%$ monthly. The value weighted return is even larger (in absolute value) with $-0.43\%$ per month or in excess of $-5\%$ annually. In Panel D, we repeat the analysis but we control for market returns in the time-series estimation of risk loadings. By comparing Panels C and D, we observe that all point estimates are nearly identical but the $t$-statistics on the zero cost portfolio are now larger. By including the market return, consumption volatility risk loadings have a purely cross-sectional interpretation since the market controls for time-series variation not captured by consumption growth. Consequently, standard errors are smaller. We focus on this specification in the remaining cross-sectional analysis.

What do these findings mean? The novel implications of our model are that beliefs about mean and volatility states of consumption growth are priced sources of risk. As a result, exposure to these sources should be associated with a spread in future returns. The sign of the risk premium associated with each of these two factors depends on preference parameters. In the case where the EIS is greater than the inverse of RRA, the model predicts that returns are positively related to $\beta_{i,t}^\mu$ and negatively to $\beta_{i,t}^\sigma$. We do not find convincing evidence that exposure to fluctuations in expected consumption growth predicts returns but exposure to fluctuations in consumption volatility does so negatively. This finding is consistent with the model only if the agent dislikes intertemporal risk and the EIS is greater than the inverse of RRA.
2.2.4 Robustness

Cross-sectional differences in returns might not be surprising if consumption volatility betas covary with other variables known to predict returns. Crucially, Table 2.3 shows that this is not the case for the firm characteristics size and book-to-market. In Panel A, we again report average returns for each consumption volatility exposure quintile and its average beta. Panel B reports firm characteristics for each portfolio. Since market capitalization is non-stationary, and the value characteristic varies dramatically over time, we compute size- and value deciles for each stock at each month and take the average over these deciles within each portfolio. The table reports time-series means of portfolio characteristics. For market equity, we observe that the two extreme quintiles are composed of somewhat smaller than average stocks. This effect often shows up when ranking stocks by a covariance measure. Returns of small stocks are on average more volatile and risk estimates are therefore more likely to be very large or very small. However, there is no difference in size rank between quintiles 1 and 5. Most importantly, there is no variation in the book-to-market ratio across portfolios. Thus, consumption risk portfolios do not load on firm characteristics which are known to predict future returns.

A number of so-called anomalies are confined to small subsets of stocks, often just to small companies or illiquid stocks (e.g. Fama and French (2008), Avramov et al. (2007)). In Table 2.4, stocks are independently sorted into three portfolios based on $\beta_i^{\text{vol},t}$ and into two portfolios based on market capitalization (Panel A) or book-to-market ratio (Panel B). The number of portfolios for each variable follows Fama and French (1993) and trades off the desire to obtain sufficient dispersion along each dimension while keeping the number of stocks in each portfolio large enough to minimize idiosyncratic risk. The bivariate sort in Panel A shows that consumption volatility risk is consistently present and strong for both equal and value-weighted strategies with return differences ranging from $-0.09\%$ to $-0.23\%$ monthly. The effect is stronger for big than for small companies since returns of smaller stocks have a larger idiosyncratic component and, thus, the risk estimates from the first stage regression are less precise. With these findings, there is no reason to believe that the predictive power of consumption volatility risk is associated with possible mispricing or slow information diffusion in small stocks. Similarly, Panel B confirms that consumption volatility risk is also present within book-to-market groups.

As an alternative to portfolio sorts, we also perform Fama-MacBeth regressions by cross-sectionally regressing monthly returns of each asset onto its latest available risk loadings as well as size and value characteristics. The explanatory variables are normalized each quarter.
so they are centered around zero with unit variance. Each set of three monthly regressions in one quarter will share the same predictor variables. For example, the returns in each of the months April, May, and June are regressed onto the risk loadings estimated from the window ending in the first quarter of the same year. We are interested whether the factor loadings have any predictive power for the cross-sectional variation of returns.

The results of the Fama-MacBeth regressions are presented in Table 2.5. Model specifications I-III present univariate effects of each risk loading. Confirming previous findings, the average coefficients on consumption growth betas, $\beta_{c,t}^i$, as well as expected consumption growth betas, $\beta_{\mu,t}^i$, are small and insignificant. Exposure to consumption volatility risk, however, as measured by $\beta_{\sigma,t}^i$, shows up strongly negative and significant. Specification IV is the full model. Now, both the loading on consumption growth and consumption volatility risk are significant. In regression V, we add two characteristics known to predict stock returns, namely, the market capitalization ($ME_i^t$) and the ratio of book value of equity to market value ($BM_i^t$), to confirm that the predictive power of consumption volatility is not already captured by these predictors. The absolute value of the point estimate is slightly reduced by the addition of the two characteristics, but it remains significant.

### 2.3 Consumption Volatility Risk Pricing

Building on the findings of the previous section, we now investigate the pricing implications of beliefs about consumption moments cross-sectionally. We find that changes in beliefs about consumption volatility carry a negative price of risk, while changes in beliefs about the mean state do not contribute to explaining the cross-section of returns. Alternatively, we also form a long-short portfolio based on consumption volatility risk (CVR) and demonstrate that it shows up strongly and significantly as a priced factor in cross-sectional regressions. While the CVR portfolio only modestly correlates with the value factor HML, both factors are substitutes in the pricing relation. This evidence provides an economic interpretation for the risk associated with the HML factor.

#### 2.3.1 Factor Pricing with Consumption Data

Equation (2.15) states that, in a log-linear approximation, expected excess returns depend on consumption growth and changes of the perceived conditional first and second moments of consumption growth. We evaluate the performance of our model in two stages. First, for
each test asset, we obtain risk loadings from the time-series regression

$$R^i_t - R^f_t = \alpha^i + \beta^i_c \Delta c_t + \beta^i_\mu \Delta \hat{\mu}_t + \beta^i_\sigma \Delta \hat{\sigma}_t + \epsilon^i_t$$  \hspace{1cm} (2.18)$$

In the second stage, we estimate the prices of risk by a cross-sectional regression of returns onto the loadings from the first stage.

Results from the second stage regression are summarized in Table 2.6. For each factor, the table reports point estimates for the prices of risk and associated $t$-statistics, which are adjusted for estimation error in the first stage as proposed by Shanken (1992) and are robust to heteroscedasticity and autocorrelation as in Newey and West (1987) with 4 quarterly lags. In addition, the following regression statistics are shown: The second stage $R^2$, mean absolute pricing error (MAPE) and the model $J$-test ($\chi^2$ statistic) with its associated $p$-value (in percent). Return observations are at a quarterly frequency and the factors used are log consumption growth ($\Delta c_t$), changes in beliefs about the conditional mean of consumption growth ($\Delta \hat{\mu}_t$), as well as changes in beliefs about consumption growth volatility ($\Delta \hat{\sigma}_t$). A fourth factor, which is the return of a long-short portfolio that buys assets with high consumption volatility risk and sells assets with low consumption volatility risk, is also considered and denoted by CVR.

As test assets, we use the 25 Fama-French portfolios in Panel A which have been shown to challenge the single factor CAPM. Lewellen et al. (2008) criticize the use of only those 25 portfolios as test assets since they exhibit a strong factor structure. Following their suggestions, we also expand the set of assets. In Panel B, we add the 5 value-weighted consumption volatility risk portfolios (Table 2.2, Panel D) as test assets. To broaden the scope beyond equity pricing, we also consider the 6 CRSP bond return portfolios with maturities of 1, 2, 3, 4, 5, and 10 years in addition to the 5 volatility risk portfolios in Panel C.

Regression I in each panel shows results for the standard consumption CAPM. Confirming prior research, the market price of consumption risk in Panels A and B is insignificant and low $R^2$s indicate that the C-CAPM performs poorly in pricing the set of test assets. When bond returns are included as test assets (Panel C), the C-CAPM performs better because there is a large spread in returns and betas between asset classes. Regression II in each panel reports the full three factor model (2.18). Similar to our previous findings, the market price of expected consumption growth, $\Delta \hat{\mu}_t$, is insignificant. In contrast, consumption volatility risk, $\Delta \hat{\sigma}_t$, is a priced factor in the cross-section independent of the test assets. Importantly, the price of volatility risk is negative which is consistent with our portfolio sort results.

Alternatively, we form a consumption volatility risk (CVR) portfolio as a proxy for $\Delta \hat{\sigma}_t$ to
reduce measurement error in consumption volatility.\textsuperscript{17} The CVR factor is a zero investment strategy that is long in the value-weighted quintile with the highest exposure and short in the value-weighted quintile with the lowest exposure to innovations in beliefs about consumption volatility as measured by $\beta_{\sigma,t}^i$ in Table 2.2, Panel D. We do not form a factor based on loadings on expected consumption growth, $\Delta\hat{\mu}_t$, since the spread between the high and low quintile is on average close to zero. Consequently, theory predicts that its market price of risk should be zero too.

Regression III in each panel of Table 2.6 shows a significantly negative price of risk for the CVR factor, while beliefs about the mean consumption growth continue to be insignificant. Replacing the estimated consumption volatility with a traded portfolio results in drastic improvements in second stage $R^2$.

The predictions of our theory in Section 2.1 depend on the preference parameters of the representative agent. While prior research often finds a negative price of risk for market volatility (Ang et al. (2006), Adrian and Rosenberg (2008)), only a general equilibrium consumption-based model allows us to draw conclusions about preference parameters. The estimated prices of risk for both $\Delta\hat{\sigma}_t$ and its mimicking CVR portfolio are significantly negative, thus suggesting an EIS greater than the inverse of RRA for the representative agent.

Figure 2.4 displays average second stage pricing errors of quarterly excess returns of the 25 Fama-French portfolios (black dots) and 5 volatility risk portfolios (red stars), as in Table 2.6, Panel B. Each graph plots average quarterly excess returns against the model predicted excess returns for a given set of explanatory variables. If the model correctly prices assets and there are no errors induced from estimation or small sample size, all asset returns should line up exactly on the diagonal line.

The first graph depicts the consumption CAPM (Regression I). Visually, this graph confirms that the consumption CAPM does not perform well in pricing the 30 test portfolios. While the portfolios vary drastically in their average realized returns, the model predicted returns are all very close together, resulting in a narrow cloud. In the second graph, we present the full model (Regression II) and in the third graph, we substitute $\Delta\hat{\sigma}_t$ by its mimicking factor CVR. Both graphs confirm that in the full specifications pricing errors are small and loadings on risk factors successfully explain average excess returns.

\textsuperscript{17}While researchers often treat $\Delta c_t$ as observable, the consumption time-series actually is measured with significant noise (Breeden et al. (1989) and Wilcox (1992)). Moreover, both $\Delta\hat{\mu}_t$ and $\Delta\hat{\sigma}_t$ are estimates and themselves depend on the imposed model for consumption growth dynamics.
2.3.2 Factor Pricing with Portfolio Returns

To relate the pricing implications of consumption volatility risk to the existing literature, we now study market based rather than consumption based models. Even though CVR is independent of the book-to-market characteristic and comoves only modestly with HML, we find that substituting HML with CVR in the Fama-French three factor model results in similar pricing and leaves pricing errors unaffected.

Summary statistics for the CVR portfolio are given in Table 2.7. The CVR portfolio has a mean return of $-0.44\%$ and a standard deviation of $3.40\%$ per month. Its standard deviation is lower than the market volatility, but comparable to the ones of the Fama-French factors. The monthly Sharpe ratio (in absolute value) of 0.13 is larger in magnitude than the Sharpe ratio of size factor SMB (0.08) and close to the Sharpe ratio of value factor HML (0.15). The correlation matrix of the pricing factors (Panel B) shows that the CVR portfolio returns are uncorrelated with the market. The correlations with the SMB and HML factors are moderate at 17% and $-26\%$, respectively, even though the CVR portfolio is neutral with respect to size and book-to-market characteristics (see Panel B of Table 2.3). To put these correlations in perspective, we note that all the pairwise correlations between the Fama-French factors are larger. Parameter estimates from a time-series regression of the CVR factor onto the other factors are reported in Panel C. The CAPM (regression II) does not explain the returns of the CVR portfolio. In regression III, the Fama-French factors attenuate the estimated intercept $\hat{\alpha}$ towards zero, but it remains large and significant. This reduction is solely driven by HML and both the market and SMB have insignificant coefficients. The three factors only explain about 8% of the variation in the CVR factor.

Table 2.8 reports factor loadings from regressions of excess returns on the market excess return $(R_{M,t})$ and the CVR factor

$$R_i^t - R_f^t = \alpha_i^t + \beta_{M}^i R_{M,t} + \beta_{CVR}^i CVR_t + \epsilon_i^t. \tag{2.19}$$

Panel A reports estimated coefficients for the five value-weighted book-to-market portfolios, and Panel B for our five volatility risk sorted portfolios. We observe that the loadings of the value portfolios on the volatility risk factor decrease from growth to value portfolios. A low risk exposure is consistent with high expected returns for value stocks since the price of CVR risk is negative. Loadings on the CVR factor therefore suggest a risk based explanation of the value anomaly.

For the volatility risk portfolios, the loading on the CVR factor increases monotonically from $-0.56$ to 0.44. This finding can be interpreted as evidence that volatility exposure is
a systematic source of risk because the portfolios move together. Cochrane (2001) points out that comovement would not be expected if the return differentials are explained by characteristics.

To relate the pricing implications of CVR to existing factors, Table 2.9 shows estimated prices of risks for the market excess return (MKT), size (SMB), value (HML), and consumption volatility (CVR) risk factors, and associated t-statistics, which are adjusted for estimation error in the first stage as proposed by Shanken (1992) and are robust to heteroscedasticity and autocorrelation as in Newey and West (1987) with 12 monthly lags. We also report second stage $R^2$, mean absolute pricing error (MAPE) and the model J-test ($\chi^2$ statistic) with its associated p-value (in percent). The test assets considered are the 25 Fama-French portfolios augmented with our five consumption volatility risk portfolios.

Regressions I and III show the results for the benchmark models, the market CAPM (I) and the Fama-French model (III). The CAPM does a very poor job in explaining the cross-section of returns. The point estimate for the market risk premium is negative and the regression $R^2$ is less than 7%. The three factor model reduces the pricing errors significantly and yields an $R^2$ of 76%. The estimated market risk premium remains negative and the model is still rejected as indicated by the high $\chi^2$ statistic.

The remaining regressions show various combinations of the benchmark factors with CVR. In all specifications, the estimates for the price of a unit CVR risk are significant and negative, ranging from $-0.45\%$ to $-0.56\%$ monthly. These estimates are remarkably close to the mean return of the CVR factor of $-0.44\%$. In regression III, the factors are the market portfolio and CVR. This specification yields improvements over the one factor market model. Interestingly, although CVR is based on consumption data, a three factor model based on the market, SMB and CVR (regression IV) generates an identical mean absolute pricing error to the Fama-French model. Augmenting the Fama-French three-factor model with CVR as a fourth factor (regression V) leads to a marginal improvement in the model’s ability to price the cross-section. In summary, replacing HML with CVR does not deteriorate the model’s performance, while including both CVR and HML as factors improve the model fit only slightly. Hence, HML and CVR are substitutes in cross-sectional pricing for our test assets. In contrast to HML, however, the consumption volatility risk portfolio has a clear economic interpretation.

Adrian and Rosenberg (2008) perform a similar analysis. They decompose stock market volatility into two components, which differ in persistence, and estimate them with a GARCH inspired model. In contrast, our CVR portfolio is based on a Markov model for low-frequency consumption data. Interestingly, their short-run volatility component has similar
pricing implications to CVR, whereas their long-run component performs worse than CVR. However, the persistence of their short-run volatility component is 0.327 for daily data while our consumption volatility regimes last on average for several years. The CVR factor thus has a much different and macroeconomically more meaningful interpretation.

Figure 2.5 replicates Figure 2.4 for market based pricing models. The 25 size-value portfolios are represented by black dots, and the five volatility risk portfolios by red stars. The top left graph depicts the CAPM (regression I in Table 2.9). The remaining graphs show the CAPM augmented with the volatility risk factor (top right graph, regression II), the Fama-French three factor model (bottom left graph, regression III), and a three factor model that uses CVR instead of HML (bottom right graph, regression IV). Visually, these graphs confirm that simply adding CVR to the market factor improves the model fit. At first sight, both the Fama-French model and the three factor CVR model seem to price the 30 portfolios well. Upon closer inspection, however, the Fama-French model does not succeed in generating a spread in predicted excess returns of the five volatility risk portfolios as indicated by the narrow cloud of red stars. In contrast, the three-factor CVR model works well for both size-value portfolios and consumption volatility risk portfolios.

2.4 Time Series Predictability

In the previous sections, we have shown that loadings on consumption growth volatility predict returns cross-sectionally and that consumption growth volatility is a priced risk factor. The model also predicts that the first and second moments of consumption growth forecast aggregate returns in the time-series. As explained in Section 2.1, the model implies a negative relation between expected returns and expected consumption growth and a positive relation between expected returns and consumption growth volatility when the EIS is greater than unity. Noting that the wealth-consumption ratio is inversely related to expected returns, this effect can be seen in Figure 2.2. The opposite holds when the EIS is smaller than unity (see Figure 2.3).

Table 2.10 reports forecasting regressions of quarterly excess market returns onto lagged variables. The data ranges from 1955 through 2008. Predictors are the dividend yield (DY), payout ratio (DE), the term spread (TS), aggregate book-to-market ratio (BM), the consumption-wealth ratio of Lettau and Ludvigson (2001a) (cay), and changes in beliefs about the first (Δμt) and second moments (Δσt) of consumption growth.18 All of those have

---

18 We thank Amit Goyal and Martin Lettau for making their data available.
been shown to predict stock returns at various horizons.\textsuperscript{19} Lettau and Ludvigson (2001a) use the household budget constraint to motivate the variable \textit{cay} and show that it works exceptionally well at short horizon forecasts.

Regressions I-III show the benchmark results of multivariate predictive regressions. The four standard predictor variables jointly result in an $R^2$ of about 5%. In regressions IV and V, we study the predictive power of our two consumption state variables. Similar to the cross-sectional results in the previous sections, we find that beliefs about expected consumption growth do not predict stock returns, while changes in beliefs about the volatility state show up economically and statistically significant and yield a regression $R^2$ of 2.6% in a univariate regression. The $R^2$ of \textit{cay} in the univariate regression II is somewhat larger at 4%. The economic impact of consumption volatility risk is large. A one standard deviation increase in $\Delta \hat{\sigma}_t$ results in an increase in the expected risk premium of 1.4% quarterly.\textsuperscript{20} The economic impact is similar to \textit{cay} (1.6% quarterly).

Regressions VI to VIII demonstrate that the marginal impact of $\Delta \hat{\sigma}_t$ remains strong and significant even after controlling for all other predictors, including \textit{cay}. Moreover, the coefficients on consumption volatility are virtually unaffected by the inclusion of other predictors, indicating that its forecasting ability is orthogonal to existing variables. The predictive $R^2$ exceeds 9% in the multivariate setting with all predictor variables.

The observation that consumption volatility and \textit{cay} are orthogonal is surprising. In our model, changes in consumption volatility enter the pricing kernel only because they affect the wealth-consumption ratio. Thus, one might expect that direct measures of the wealth-consumption ratio, such as \textit{cay}, comprise all relevant information about the volatility state. Our findings therefore suggest that \textit{cay} is an imperfect measure of the wealth-consumption ratio.

It is well known that parameter estimates and $t$-statistics are potentially biased in predictive regressions, for instance, when the predictor variable is persistent and its innovations are correlated with future returns, as discussed in Stambaugh (1999), Lewellen (2004), Boudoukh et al. (2006) and Ang and Bekaert (2007). Especially when price ratios are used as predictors, this bias shows up strongly. For the variable $\Delta \hat{\sigma}_t$, this bias is less of a concern since it is not a price scaled variable. The appendix shows this bias is immaterial in our setup.

We acknowledge that the predictive results presented have limitations. First, they are in sample results. Second, there is a look-ahead bias in $\Delta \hat{\sigma}_t$. In estimating the Markov chain\textsuperscript{19} See, for example, Fama and Schwert (1977), Rozeff (1984), Keim and Stambaugh (1986), Campbell (1987), Campbell and Shiller (1988), Lamont (1998), Campbell and Thompson (2008), and Goyal and Welch (2008).\textsuperscript{20} Note that $\Delta \sigma_t$ has a standard deviation of around 0.00035.
for consumption growth, beliefs are updated according to Bayes’ rule and therefore are not forward looking. The parameter estimates, however, are obtained by maximum likelihood employing the full sample. This is similar to the critique by Brennan and Xia (2005), who point out that estimating $cay$ over the entire sample induces a look ahead bias and a simple linear time trend would work as well as $cay$. Their criticism does not apply to our results since we use changes in beliefs as predictor which do not have a trend. Third, aggregate consumption data is not publicly available at the end of a quarter. Instead, initial estimates are published within the following month and they are subject to revisions for up to three years. Hence, we cannot conclude that it is possible to implement our predictability results in practice. Yet we succeed in identifying a new source of aggregate risk.

2.5 Conclusion

When consumption growth is not i.i.d. over time and the representative household has recursive preferences, the wealth-consumption ratio is time-varying and enters the pricing kernel as a second factor (Epstein and Zin (1989), Weil (1989)). We follow Lettau et al. (2008), who generalize Bansal and Yaron (2004) to account for the latent nature of the conditional first and second moments of consumption growth. In the model, we identify innovations in beliefs about the conditional mean and volatility of consumption growth as two state variables that affect the wealth-consumption ratio and thus asset prices.

To test these predictions, we estimate a Markov model with two states for the conditional mean and two states for the conditional volatility of consumption growth, as in Kandel and Stambaugh (1991) and Lettau et al. (2008). Using the estimated beliefs from the Markov model, we empirically test the pricing implications for the cross-section and time-series of stock returns. In the cross-section, we first show that firm level loadings on changes in beliefs about consumption volatility significantly forecast returns, while loadings on changes in beliefs about expected consumption growth do not. A negative relation between betas and future returns indicates a negative price of consumption volatility risk. This is confirmed in cross-sectional pricing tests, where both consumption volatility and its mimicking portfolios are negatively priced sources of risk. In the context of our model, these findings suggest an EIS greater than the inverse of the RRA for the representative agent.

In time-series tests, we find that shocks to beliefs about the volatility state forecast the equity premium. In a univariate regression, changes about perceived consumption volatility achieve an $R^2$ of 2.6%. A one standard deviation increase in perceived volatility is followed by an increase of the equity returns of 1.4% quarterly. The economic impact is comparable to the
one generated by \textit{cay} of Lettau and Ludvigson (2001a). Surprisingly, \textit{cay}, a direct measure of the wealth-consumption ratio, does not subsume the predictive power of consumption volatility. The positive coefficient of consumption volatility in the predictive regressions indicates that the representative agent has elasticity of intertemporal substitution greater than unity.
Figure 2.1: Bayesian Beliefs about the Mean and Volatility State

This figure displays the estimated Bayesian belief processes for being in the high expected growth rate state (top figure) and high volatility state (bottom figure). The estimation procedure follows Hamilton (1994). We use quarterly per capita real consumption expenditure for non-durable goods and services for the years 1955.Q1-2008.Q4.
This figure shows the wealth-consumption ratio as a function of the perceived conditional first $\hat{\mu}_t$ (left graph) and second $\hat{\sigma}_t$ (right graph) moments of consumption growth. The dynamics of the underlying consumption process are summarized in Table 2.1. The representative agent has an EIS of 1.5, RRA of 30 and quarterly rate of time preference of 0.995.
Figure 2.3: Wealth-Consumption Ratio for a Low EIS Agent

This figure shows the wealth-consumption ratio as a function of the perceived conditional first $\hat{\mu}_t$ (left graph) and second $\hat{\sigma}_t$ (right graph) moments of consumption growth. The dynamics of the underlying consumption process are summarized in Table 2.1. The representative agent has an EIS of 0.5, RRA of 30 and quarterly rate of time preference of 0.995.
Figure 2.4: Pricing Errors of the Consumption-Based Model

This figure depicts average quarterly excess returns of the 25 Fama-French portfolios (black dots) and 5 volatility risk portfolios (red stars) against model predicted excess returns. The first graph represents the standard consumption CAPM and the second graph the full model with consumption growth ($\Delta c_t$) as well as changes in the perceived first ($\Delta \hat{\mu}_t$) and second ($\Delta \hat{\sigma}_t$) moments of consumption growth as explanatory factors. In the third graph, we replace beliefs about the volatility state with the CVR factor in the full model.
Figure 2.5: Pricing Errors of the Market-Based Model

This figure depicts average monthly excess returns of the 25 Fama-French portfolios (black dots) and 5 volatility risk portfolios (red stars) against model predicted excess returns. The top-left graph represents the standard CAPM and the bottom-left graph the Fama-French three factor model. In the top-right graph, we add the CVR factor to the CAPM and, in the bottom-right graph, we replace the HML factor with the CVR in the Fama-French model.
Table 2.1: Markov Model of Consumption Growth

This table reports parameter estimates of the Markov model for log consumption growth

\[ \Delta c_{t+1} = \mu_t + \sigma_t \epsilon_{t+1} \quad \epsilon_{t+1} \sim N(0,1) \]

where \( \mu_t \in \{\mu_l, \mu_h\} \) and \( \sigma_t \in \{\sigma_l, \sigma_h\} \) follow independent Markov processes with transition matrices \( P^\mu \) and \( P^\sigma \), respectively,

\[ P^\mu = \begin{bmatrix} p_{\mu l l}^\mu & 1 - p_{\mu l l}^\mu \\ 1 - p_{\mu h h}^\mu & p_{\mu h h}^\mu \end{bmatrix} \quad P^\sigma = \begin{bmatrix} p_{\sigma l l}^\sigma & 1 - p_{\sigma l l}^\sigma \\ 1 - p_{\sigma h h}^\sigma & p_{\sigma h h}^\sigma \end{bmatrix} \]

The estimation procedure follows Hamilton (1994). We use quarterly per capita real consumption expenditure for non-durable goods and services for the years 1955.Q1-2008.Q4. Standard errors are reported in parentheses.

<table>
<thead>
<tr>
<th>Panel A: Parameter Estimates (%)</th>
<th>( \mu_l )</th>
<th>( \mu_h )</th>
<th>( \sigma_l )</th>
<th>( \sigma_h )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3668</td>
<td>0.7800</td>
<td>0.2112</td>
<td>0.4772</td>
</tr>
<tr>
<td></td>
<td>(0.0376)</td>
<td>(0.0600)</td>
<td>(0.0279)</td>
<td>(0.0532)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Marginal Transition Probabilities</th>
<th>( p_{\mu l}^\mu )</th>
<th>( p_{\mu h}^\mu )</th>
<th>( p_{\sigma l}^\mu )</th>
<th>( p_{\sigma h}^\mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.9304</td>
<td>0.8929</td>
<td>0.9452</td>
<td>0.9617</td>
</tr>
<tr>
<td></td>
<td>(0.0353)</td>
<td>(0.0597)</td>
<td>(0.0400)</td>
<td>(0.0458)</td>
</tr>
</tbody>
</table>
Table 2.2: Portfolios Formed on Risk Exposure

This table reports average equally-weighted (EW) and value-weighted (VW) monthly returns (%) of portfolios based on time-varying loadings from rolling time-series regressions (2.17). Specifically, we regress individual returns on log consumption growth ($\beta_{i,c,t}$) and changes in the perceived first ($\beta_{i,\mu,t}$) and second moments ($\beta_{i,\sigma,t}$) of consumption growth using 20 quarterly observations. In Panel D, we also control for fluctuations in the market return when estimating risk loadings. The $t$-statistics reported in parentheses are based on Newey-West adjusted standard errors using 12 lags. The sample period is from January 1964 to December 2008.

| Panel A: Univariate Sorts Based on $\beta_{i,c,t}$ |
|-------|-------|-------|-------|-------|
| Low   | Med   | High  | High - Low |
| EW 1.09 | 1.16 | 1.15 | 1.17 | 1.25 | 0.16 |
|       (4.01) | (5.19) | (5.09) | (4.68) | (4.16) | (1.16) |
| VW 0.84 | 0.89 | 0.84 | 0.88 | 0.91 | 0.07 |
|       (3.62) | (4.50) | (4.25) | (4.53) | (3.69) | (0.37) |

| Panel B: Univariate Sorts Based on $\beta_{i,\mu,t}$ |
|-------|-------|-------|-------|-------|
| Low   | Med   | High  | High - Low |
| EW 1.22 | 1.17 | 1.14 | 1.14 | 1.15 | -0.06 |
|       (4.42) | (5.13) | (5.03) | (4.82) | (3.86) | (-0.57) |
| VW 0.95 | 0.90 | 0.81 | 0.86 | 0.91 | -0.03 |
|       (4.32) | (4.72) | (4.08) | (4.17) | (3.38) | (-0.22) |

| Panel C: Univariate Sorts Based on $\beta_{i,\sigma,t}$ |
|-------|-------|-------|-------|-------|
| Low   | Med   | High  | High - Low |
| EW 1.28 | 1.16 | 1.15 | 1.16 | 1.07 | -0.19 |
|       (4.26) | (4.77) | (5.02) | (5.14) | (4.00) | (-1.68) |
| VW 1.08 | 0.90 | 0.85 | 0.89 | 0.65 | -0.43 |
|       (4.26) | (4.42) | (4.30) | (4.62) | (2.73) | (-2.61) |

| Panel D: Univariate Sorts Based on $\beta_{i,\sigma,t}$ - Controlling for $R_M$ |
|-------|-------|-------|-------|-------|
| Low   | Med   | High  | High - Low |
| EW 1.30 | 1.19 | 1.17 | 1.08 | 1.08 | -0.21 |
|       (4.46) | (5.16) | (5.17) | (4.71) | (3.79) | (-2.09) |
| VW 1.08 | 0.93 | 0.82 | 0.89 | 0.64 | -0.44 |
|       (4.42) | (4.93) | (4.21) | (4.52) | (2.71) | (-3.10) |
Table 2.3: Characteristics of Consumption Volatility Risk Portfolios

This table reports characteristics and risk measures for quintile portfolios based on consumption volatility loadings (as in Table 2.2, Panel D). Panel A shows average returns and average consumption volatility betas ($\beta_{\sigma}$). Panel B reports the average value and mean decile rank for size (ME) and book-to-market (BM) characteristics of each portfolio.

<table>
<thead>
<tr>
<th>Panel A: Univariate Sorts Based on $\beta_{\sigma,t}$</th>
<th>Low</th>
<th>Med</th>
<th>High</th>
<th>High - Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>Return</td>
<td>1.08</td>
<td>0.93</td>
<td>0.82</td>
<td>0.64</td>
</tr>
<tr>
<td>$\beta_{\sigma}/100$</td>
<td>-2.69</td>
<td>-0.87</td>
<td>-0.03</td>
<td>0.84</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Characteristics of Sorts Based on $\beta_{\sigma,t}$</th>
<th>Low</th>
<th>Med</th>
<th>High</th>
<th>High - Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>ME</td>
<td>1428.21</td>
<td>2418.57</td>
<td>2388.74</td>
<td>1742.64</td>
</tr>
<tr>
<td>ME Rank</td>
<td>4.25</td>
<td>5.43</td>
<td>5.64</td>
<td>5.28</td>
</tr>
<tr>
<td>BM</td>
<td>0.91</td>
<td>0.92</td>
<td>0.91</td>
<td>0.91</td>
</tr>
<tr>
<td>BM Rank</td>
<td>4.87</td>
<td>5.02</td>
<td>5.04</td>
<td>4.99</td>
</tr>
</tbody>
</table>
Table 2.4: Portfolios Formed on Consumption Volatility Risk and Characteristics

This table reports average equally-weighted and value-weighted monthly returns (%) of independent double sorts based on consumption volatility loadings ($\beta_{c,t}$) and market capitalizations in Panel A and based on consumption volatility loadings and book-to-market ratios in Panel B. The $t$-statistics reported in parentheses are based on Newey-West adjusted standard errors using 12 lags.

### Panel A: Portfolios Formed on Consumption Volatility Risk and Market Capitalization

<table>
<thead>
<tr>
<th></th>
<th>Equally-Weighted Returns</th>
<th>Value-Weighted Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Med</td>
</tr>
<tr>
<td>Small</td>
<td>1.36</td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td>(4.43)</td>
<td>(4.84)</td>
</tr>
<tr>
<td>Big</td>
<td>1.05</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>(4.61)</td>
<td>(5.15)</td>
</tr>
<tr>
<td>S - B</td>
<td>-0.30</td>
<td>-0.31</td>
</tr>
<tr>
<td></td>
<td>(-2.02)</td>
<td>(-2.27)</td>
</tr>
</tbody>
</table>

### Panel B: Portfolios Formed on Consumption Volatility Risk and Book-to-Market Ratio

<table>
<thead>
<tr>
<th></th>
<th>Equally-Weighted Returns</th>
<th>Value-Weighted Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Med</td>
</tr>
<tr>
<td>Low BM</td>
<td>1.06</td>
<td>1.01</td>
</tr>
<tr>
<td></td>
<td>(3.90)</td>
<td>(4.45)</td>
</tr>
<tr>
<td>High BM</td>
<td>1.44</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td>(5.42)</td>
<td>(5.60)</td>
</tr>
<tr>
<td>H - L</td>
<td>0.38</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>(3.99)</td>
<td>(3.50)</td>
</tr>
</tbody>
</table>
Table 2.5: Fama-MacBeth Regressions

This table reports cross-sectional regressions of monthly returns on lagged estimated risk loadings and characteristics. Time-varying risk loadings are obtained from 5-year rolling time-series regressions of individual excess returns on the market excess return, log consumption growth, and changes in the perceived conditional mean and volatility of consumption growth using quarterly data. In the cross-section, we regress monthly future returns onto the loadings of log consumption growth ($\beta_{c,t}^i$), changes in the perceived conditional mean ($\beta_{\mu,t}^i$) and volatility ($\beta_{\sigma,t}^i$) of consumption growth as well as market capitalization ($ME_t^i$) and book-to-market ratio ($BM_t^i$). Both characteristics are measured in December which is 7 to 18 months prior to the test month. All explanatory variables are normalized so they are centered around zero with unit variance. We report time-series averages of the second stage coefficients. The $t$-statistics in parentheses are based on Newey-West adjusted standard errors using 12 lags.

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{c,t}^i$</th>
<th>$\beta_{\mu,t}^i$</th>
<th>$\beta_{\sigma,t}^i$</th>
<th>$ME_t^i$</th>
<th>$BM_t^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.97)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>-0.02</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-0.54)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>-0.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-2.22)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>0.16</td>
<td>0.02</td>
<td>-0.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.03)</td>
<td>(0.39)</td>
<td>(-1.62)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>-0.06</td>
<td>-0.07</td>
<td>0.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.79)</td>
<td>(-1.43)</td>
<td>(3.98)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2.6: Volatility Risk Pricing

This table reports market prices of risk from two stage regressions where $\Delta c_t$ denotes log consumption growth, $\Delta \hat{\mu}_t$ and $\Delta \hat{\sigma}_t$ are changes in filtered beliefs about the first and second moments of consumption growth. The CVR factor is the return of holding a long position in the value-weighted quintile of stocks with high volatility risk ($\beta_i^{\sigma,t}$) and a short position in low volatility risk, as reported in Panel D of Table 2.2. In Panel A, the test assets are the value-weighted 25 Fama-French value and size portfolios, in Panel B, we add the 5 value-weighted consumption volatility risk portfolios as in Panel D of Table 2.2, and in Panel C, we add 6 bond portfolios from CRSP. The data covers January 1964 to December 2008. The $t$-statistics are corrected for estimation error in the first stage as proposed by Shanken (1992) and are Newey-West adjusted using 4 lags. For each specification, we report the $R^2$, mean absolute pricing error (MAPE) in parentheses, regression $J$-statistic ($\chi^2$) with the associated $p$-value (in %).

<table>
<thead>
<tr>
<th>Panel A: 25 Value-Size Portfolios</th>
<th>Const.</th>
<th>$\Delta c$</th>
<th>$\Delta \hat{\mu}$</th>
<th>$\Delta \hat{\sigma}$</th>
<th>CVR</th>
<th>$R^2$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1.54</td>
<td>0.10</td>
<td>2.54</td>
<td>92.44</td>
<td>(2.01)</td>
<td>(0.62)</td>
<td>(0.00)</td>
</tr>
<tr>
<td></td>
<td>(2.01)</td>
<td>(0.52)</td>
<td>(0.62)</td>
<td>(0.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>1.33</td>
<td>0.25</td>
<td>-0.02</td>
<td>-0.03</td>
<td>9.91</td>
<td>46.18</td>
<td>(0.19)</td>
</tr>
<tr>
<td></td>
<td>(1.31)</td>
<td>(1.29)</td>
<td>(-0.57)</td>
<td>(-2.26)</td>
<td>(0.60)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>0.79</td>
<td>0.01</td>
<td>0.03</td>
<td>-6.49</td>
<td>75.47</td>
<td>28.13</td>
<td>(17.13)</td>
</tr>
<tr>
<td></td>
<td>(0.64)</td>
<td>(0.03)</td>
<td>(0.94)</td>
<td>(-2.46)</td>
<td>(0.31)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: 25 Value-Size Portfolios &amp; 5 CVR Portfolios</th>
<th>Const.</th>
<th>$\Delta c$</th>
<th>$\Delta \hat{\mu}$</th>
<th>$\Delta \hat{\sigma}$</th>
<th>CVR</th>
<th>$R^2$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1.13</td>
<td>0.16</td>
<td>6.11</td>
<td>122.33</td>
<td>(1.46)</td>
<td>(0.60)</td>
<td>(0.00)</td>
</tr>
<tr>
<td></td>
<td>(1.46)</td>
<td>(0.83)</td>
<td>(0.60)</td>
<td>(0.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>0.94</td>
<td>0.30</td>
<td>-0.02</td>
<td>-0.04</td>
<td>13.83</td>
<td>46.62</td>
<td>(1.09)</td>
</tr>
<tr>
<td></td>
<td>(0.79)</td>
<td>(1.35)</td>
<td>(-0.49)</td>
<td>(-2.24)</td>
<td>(0.58)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>1.06</td>
<td>0.06</td>
<td>0.02</td>
<td>-2.84</td>
<td>42.35</td>
<td>75.00</td>
<td>(0.00)</td>
</tr>
<tr>
<td></td>
<td>(1.25)</td>
<td>(0.42)</td>
<td>(0.80)</td>
<td>(-3.99)</td>
<td>(0.46)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: 25 Value-Size Portfolios &amp; 5 CVR Portfolios &amp; 6 Bond Portfolios</th>
<th>Const.</th>
<th>$\Delta c$</th>
<th>$\Delta \hat{\mu}$</th>
<th>$\Delta \hat{\sigma}$</th>
<th>CVR</th>
<th>$R^2$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.69</td>
<td>0.25</td>
<td>42.56</td>
<td>146.40</td>
<td>(2.84)</td>
<td>(0.52)</td>
<td>(0.00)</td>
</tr>
<tr>
<td></td>
<td>(2.84)</td>
<td>(1.87)</td>
<td>(0.52)</td>
<td>(0.00)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>0.87</td>
<td>0.29</td>
<td>-0.01</td>
<td>-0.04</td>
<td>47.05</td>
<td>67.11</td>
<td>(0.04)</td>
</tr>
<tr>
<td></td>
<td>(1.77)</td>
<td>(1.86)</td>
<td>(-0.38)</td>
<td>(-2.30)</td>
<td>(0.52)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>0.43</td>
<td>0.24</td>
<td>0.02</td>
<td>-2.90</td>
<td>63.12</td>
<td>98.87</td>
<td>(0.00)</td>
</tr>
<tr>
<td></td>
<td>(1.18)</td>
<td>(1.91)</td>
<td>(1.01)</td>
<td>(-4.07)</td>
<td>(0.41)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 2.7: Volatility Risk Factor

This table provides descriptive statistics of the volatility risk (CVR) portfolio. The CVR portfolio is the return of holding a long position in the value-weighted quintile of stocks with high volatility risk ($\beta_{i,t}$) and a short position in low volatility risk, as reported in Panel D of Table 2.2. Panel A shows the mean, standard deviation and the Sharpe Ratio of the CVR portfolio as well as the three Fama-French factors. Panel B presents the correlation matrix of the factor returns. Panel C reports parameter estimates from time-series regressions of the CVR portfolio on the market and the three Fama-French factors with Newey-West adjusted standard errors using 12 lags.

### Panel A: Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>CVR</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (%)</td>
<td>-0.44</td>
<td>0.37</td>
<td>0.26</td>
<td>0.43</td>
</tr>
<tr>
<td>Std. Dev. (%)</td>
<td>3.40</td>
<td>4.47</td>
<td>3.20</td>
<td>2.90</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>-0.13</td>
<td>0.08</td>
<td>0.08</td>
<td>0.15</td>
</tr>
</tbody>
</table>

### Panel B: Correlations

<table>
<thead>
<tr>
<th></th>
<th>CVR</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKT</td>
<td>0.09</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMB</td>
<td>0.17</td>
<td>0.30</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>HML</td>
<td>-0.26</td>
<td>-0.38</td>
<td>-0.26</td>
<td>1</td>
</tr>
</tbody>
</table>

### Panel C: Time-Series Regressions

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$ (%)</th>
<th>$\beta_{MKT}$</th>
<th>$\beta_{SMB}$</th>
<th>$\beta_{HML}$</th>
<th>$R^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>-0.44</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-3.10)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>-0.47</td>
<td>0.07</td>
<td></td>
<td>0.86</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-3.21)</td>
<td>(1.27)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>-0.34</td>
<td>-0.03</td>
<td>0.12</td>
<td>-0.29</td>
<td>7.92</td>
</tr>
<tr>
<td></td>
<td>(-2.35)</td>
<td>(-0.49)</td>
<td>(0.87)</td>
<td>(-2.51)</td>
<td></td>
</tr>
</tbody>
</table>
This table reports factor loadings of the 5 value portfolios (Panel A) and 5 volatility risk portfolios (Panel B) with the market return (MKT) and consumption volatility risk factor (CVR). The CVR portfolio is the return of holding a long position in the value-weighted quintile of stocks with high volatility risk ($\hat{\beta}_{i,t}$) and a short position in low volatility risk, as reported in Panel D of Table 2.2. The data starts in January 1964 and ends in December 2008. The $t$-statistics reported in parentheses are based on Newey-West adjusted standard errors using 12 lags.

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>Med</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: Value Portfolios</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$ (%)</td>
<td>-0.09</td>
<td>0.02</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>(-1.28)</td>
<td>(0.30)</td>
<td>(1.17)</td>
</tr>
<tr>
<td>$\beta_{MKT}$</td>
<td>1.06</td>
<td>0.99</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>(67.89)</td>
<td>(38.02)</td>
<td>(31.86)</td>
</tr>
<tr>
<td>$\beta_{CVR}$</td>
<td>-0.01</td>
<td>-0.09</td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td>(-0.41)</td>
<td>(-2.02)</td>
<td>(-1.31)</td>
</tr>
<tr>
<td>Panel B: Volatility Risk Portfolios</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$ (%)</td>
<td>-0.02</td>
<td>0.03</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>(-0.38)</td>
<td>(0.36)</td>
<td>(-0.28)</td>
</tr>
<tr>
<td>$\beta_{MKT}$</td>
<td>1.08</td>
<td>0.89</td>
<td>0.88</td>
</tr>
<tr>
<td></td>
<td>(40.13)</td>
<td>(30.30)</td>
<td>(37.25)</td>
</tr>
<tr>
<td>$\beta_{CVR}$</td>
<td>-0.56</td>
<td>-0.26</td>
<td>-0.11</td>
</tr>
<tr>
<td></td>
<td>(-16.84)</td>
<td>(-4.94)</td>
<td>(-2.21)</td>
</tr>
</tbody>
</table>
Table 2.9: Volatility Risk Pricing Factor

This table reports market prices of risk from two stage regressions where MKT denotes the market excess return, and SMB and HML are the Fama-French size and value factors. The CVR factor is the return of holding a long position in the value-weighted quintile of stocks with high volatility risk ($\beta_{i,t}$) and a short position in low volatility risk, as reported in Panel D of Table 2.2. Test assets are the value-weighted 25 Fama-French size and book-to-market portfolios as well as the 5 value-weighted consumption volatility risk portfolios as in Panel D of Table 2.2. The $t$-statistics are corrected for estimation error in the first stage as proposed by Shanken (1992) and are Newey-West adjusted with 12 lags to account for heteroskedasticity and autocorrelation. For each specification, we report the $R^2$, mean absolute pricing error (MAPE) in parentheses, regression J-statistic ($\chi^2$) with the associated $p$-value (in %). The data covers January 1964 to December 2008.

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>MKT</th>
<th>SMB</th>
<th>HML</th>
<th>CVR</th>
<th>$R^2$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(t-stat)</td>
<td>(t-stat)</td>
<td>(t-stat)</td>
<td>(t-stat)</td>
<td>(t-stat)</td>
<td>MAPE</td>
<td>p-val</td>
</tr>
<tr>
<td>I</td>
<td>1.00</td>
<td>-0.39</td>
<td></td>
<td></td>
<td></td>
<td>6.73</td>
<td>112.31</td>
</tr>
<tr>
<td></td>
<td>(2.47)</td>
<td>(-0.85)</td>
<td></td>
<td></td>
<td></td>
<td>(0.20)</td>
<td>(0.00)</td>
</tr>
<tr>
<td>II</td>
<td>0.60</td>
<td>-0.03</td>
<td>-0.52</td>
<td></td>
<td></td>
<td>15.09</td>
<td>99.02</td>
</tr>
<tr>
<td></td>
<td>(1.60)</td>
<td>(-0.07)</td>
<td></td>
<td></td>
<td></td>
<td>(3.30)</td>
<td>(0.20)</td>
</tr>
<tr>
<td>III</td>
<td>1.01</td>
<td>-0.62</td>
<td>0.21</td>
<td>0.48</td>
<td></td>
<td>76.05</td>
<td>82.75</td>
</tr>
<tr>
<td></td>
<td>(3.21)</td>
<td>(-1.68)</td>
<td>(1.40)</td>
<td>(3.13)</td>
<td></td>
<td>(0.09)</td>
<td>(0.00)</td>
</tr>
<tr>
<td>IV</td>
<td>1.75</td>
<td>-1.32</td>
<td>0.20</td>
<td></td>
<td>-0.56</td>
<td>74.26</td>
<td>68.59</td>
</tr>
<tr>
<td></td>
<td>(3.94)</td>
<td>(-2.84)</td>
<td>(1.30)</td>
<td></td>
<td>(-3.51)</td>
<td>(0.09)</td>
<td>(0.00)</td>
</tr>
<tr>
<td>V</td>
<td>1.00</td>
<td>-0.62</td>
<td>0.22</td>
<td>0.46</td>
<td>-0.45</td>
<td>80.06</td>
<td>69.69</td>
</tr>
<tr>
<td></td>
<td>(3.16)</td>
<td>(-1.67)</td>
<td>(1.42)</td>
<td>(2.98)</td>
<td>(-3.07)</td>
<td>(0.08)</td>
<td>(0.00)</td>
</tr>
</tbody>
</table>
Table 2.10: Market Predictability in the Time-Series

This table reports time-series regressions of the market excess return on lagged predictor variables. The market return is the value-weighted CRSP index less the 90 day T-Bill rate. Predictor variables are the dividend yield (DY), payout ratio (DE), the term spread (TS), aggregate book-to-market ratio (BM), the consumption-wealth ratio of Lettau and Ludvigson (2001a) \(cay\), and changes in beliefs about the first \(\Delta \hat{\mu}_t\) and second moments \(\Delta \hat{\sigma}_t\) of consumption growth. The sample period includes the first quarter of 1955 until the fourth quarter of 2008. \(t\)-statistics are reported in parentheses and based on Newey-West adjusted standard errors using 4 lags.

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>DY</th>
<th>DE</th>
<th>TS</th>
<th>BM</th>
<th>cay</th>
<th>(\Delta \hat{\mu})</th>
<th>(\Delta \hat{\sigma})</th>
<th>(R^2(%))</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.01</td>
<td>3.83</td>
<td>-0.09</td>
<td>0.80</td>
<td>-0.12</td>
<td>1.13</td>
<td>(2.42)</td>
<td>(3.23)</td>
<td>5.10</td>
</tr>
<tr>
<td>II</td>
<td>0.01</td>
<td>1.13</td>
<td>(2.42)</td>
<td>(3.23)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>0.02</td>
<td>1.51</td>
<td>-0.11</td>
<td>0.60</td>
<td>-0.02</td>
<td>1.05</td>
<td>(0.57)</td>
<td>(0.83)</td>
<td>(1.39)</td>
</tr>
<tr>
<td>IV</td>
<td>0.01</td>
<td>2.79</td>
<td>37.48</td>
<td>2.64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>0.01</td>
<td>(0.29)</td>
<td>(2.32)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>-0.02</td>
<td>3.67</td>
<td>-0.08</td>
<td>0.86</td>
<td>-0.12</td>
<td>38.33</td>
<td>(2.53)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>0.01</td>
<td>1.15</td>
<td>38.84</td>
<td>6.78</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VIII</td>
<td>0.02</td>
<td>1.27</td>
<td>-0.10</td>
<td>0.66</td>
<td>-0.01</td>
<td>1.08</td>
<td>(0.50)</td>
<td>(0.74)</td>
<td>(1.50)</td>
</tr>
</tbody>
</table>
Bibliography


Chapter 3

Stochastic Idiosyncratic Volatility, Portfolio Constraints, and the Cross Section of Stock Returns

An idiosyncratic risk premium of an appropriate magnitude has the potential to explain many asset pricing anomalies. For instance, small stocks have higher idiosyncratic volatility than large stocks, and value stocks have higher idiosyncratic volatility than growth stocks. In standard pricing models with perfect capital markets, however, the stock-specific component of risk does not affect asset returns since the representative agent will hold a well diversified portfolio. Levy (1978), Merton (1987), and Malkiel and Xu (2006) impose exogenous market frictions to show theoretically that idiosyncratic volatility can carry a positive price of risk in equilibrium. While the theoretical argument is intuitive and requires only few realistic assumptions, the empirical evidence on the pricing implications is mixed: Spiegel and Wang (2005), Fu (2009), and Huang et al. (2009) confirm a positive correspondence between expected idiosyncratic volatility and future returns, while Friend and Blume (1970), Friend et al. (1978), and more recently Ang, Hodrick, Xing, and Zhang (2006, 2009) use alternative time periods and estimation methods to provide strong evidence of a negative relation.

I extend the previous literature by allowing the volatility of the idiosyncratic component of returns to stochastically depend on the state of the economy. Time-variation in stock return volatility is empirically well-documented and its importance for option pricing has long been recognized. In contrast, equity prices are unaffected by asset-specific volatility, even if stochastic, under the common assumption of a representative agent.

---

21A version of this chapter will be submitted for publication. Boguth, Oliver (2010) Stochastic Idiosyncratic Volatility, Portfolio Constraints, and the Cross-Section of Stock Returns.

22Other papers that empirically tested the cross-sectional pricing implications of idiosyncratic volatility include Lintner (1965), Friend and Westerfield (1981), Tinic and West (1986), and Lehmann (1990).

23Overall stock volatility can be decomposed into systematic and idiosyncratic elements. Evidence for time-variation in the idiosyncratic volatility is provided by Ang et al. (2009), Fu (2000), and Huang et al. (2009). Early work on option pricing with stochastic volatility of stock returns includes Hull and White (1987), Johnson and Shanno (1987), and Wiggins (1987).
In this paper, I show how a non-zero correlation between the volatility of the idiosyncratic component of returns and the pricing kernel combined with trading frictions can explain why not only the level of idiosyncratic volatility, but also its dynamics, are priced in equilibrium. In a two-period model, assets with unexpectedly high idiosyncratic volatility in times of high marginal rates of substitution require a return premium.

Portfolio constraints at the investor level are a key component of the model: The inability of investors to hold a well diversified portfolio is crucial for non-systematic components of returns to be priced. These portfolio constraints, in turn, imply that some assets will be less widely held than in an unconstrained economy. This asset level underdiversification is important for pricing as it measures how well a given asset’s risk can be diversified across agents. The premium required for idiosyncratic components of returns is proportional to the asset-level underdiversification.

I thus establish a direct link between the state-dependence of idiosyncratic volatility, portfolio constraints, and equity returns, and derive novel testable pricing implications. The empirical evidence supports the model’s main predictions. The findings suggest that trading frictions of individual investors can aggregate to impact stock prices, which previous empirical research on idiosyncratic volatility has not been able to consistently show.

Many pricing models such as the CAPM require that stock-specific return innovations, or regression residuals, be uncorrelated to all systematic events such as the factor returns. In general, however, they are not required to be independent. No restrictions are placed on the variance of residuals and how it interacts with the pricing kernel. In particular, the underlying assumption for representative agent equilibrium pricing models is that all idiosyncratic risk – independent of its volatility – can be diversified away. Finite variances are thus sufficient to ensure compliance with the CAPM while at the same time allowing for the desired volatility interactions.\textsuperscript{24}

Under the CAPM, individual stock returns can exhibit non-trivial unconditional skewness even when idiosyncratic shocks conditional on the market return realization are normally distributed: Higher than expected volatility in times of high returns leads to a fatter right tail, or positive ex-ante skewness, while a negative covariance between shocks to asset volatility and returns leads to negative skewness. It is important to distinguish skewness that arises from state-dependent volatility and an asymmetric distribution of residuals conditional on the market return, a point discussed theoretically by Mitton and Vorkink (2007) and empirically

\textsuperscript{24}The argument is related to the factor structure assumed for stock returns. While early APT formulations assume a strict factor structure, extensions are compatible with heteroscedasticity for an arbitrary number of assets provided these average out in random large portfolios. For example, Chamberlain and Rothschild (1983) generalize Ross (1976). See also Grinblatt and Titman (1985).
tested by Boyer et al. (2009).

Similarly, when surprises in idiosyncratic volatility of a particular asset positively comove with surprises in market volatility, holding this asset contributes a large amount of idiosyncratic risk in times of high systematic risk. Even if the return distribution conditional on market return realizations is gaussian, it can exhibit higher probability in both tails than implied by the normal distribution, or excess kurtosis.

Using my model where stock returns follow a mixture of Gaussian distributions with stochastic volatility, I numerically solve for a general equilibrium in a two-period economy with three risky assets and agents that are homogenous in their preferences, but face heterogeneous trading frictions in the risky assets. The stochastic volatility interacts with market returns and market return volatility to generate non-trivial idiosyncratic third and fourth moments. The market incompleteness caused by the trading frictions ensures that these non-systematic third and fourth moments cannot be perfectly diversified, and that there are risk premia associated with them.

Under common preference assumptions, the model makes the following novel predictions for assets that face binding holding constraints: First, assets whose volatility innovations negatively comove with market returns trade at a discount and therefore have higher expected returns. The monotonic negative relation between this stochastic volatility induced skewness in my general equilibrium setting contrasts with Chabi-Yo (2009), who finds that bearing “idiosyncratic coskewness” risk earns an inverted U-shaped reward in a partial equilibrium. Secondly, assets whose surprises in volatility are positively correlated with the market volatility shocks require higher returns. In both cases, idiosyncratic volatility exceeds its expectations in states with a high marginal rate of substitution. Such assets thus contribute excessively to the volatility of the underdiversified portfolio in bad states, and investors demand compensation for this risk.

To test the model implications empirically, I estimate an $ARMA(p, q), 0 \leq p, q \leq 3$ model on the logarithm of monthly realized idiosyncratic volatility at the firm level to separate shocks to the idiosyncratic volatility process from the conditionally expected component. I measure the skewness induced by stochastic volatility (SVS) as the covariance of assets’ volatility shocks and factor returns, and the stochastic volatility induced kurtosis (SVK) as the covariance between shocks to the volatilities of the individual asset and the factor.

Sorting all stocks into portfolios based on the SVS, assets with a higher SVS exhibit higher raw returns, apparently contradicting the intuition provided by the model. While appropriate risk adjusting can partially explain the puzzling return pattern, it is important to understand that the theoretical predictions apply to assets for which underdiversification
is significant. I use as proxies two variables that are related to holding restrictions: First, if underdiversification arises because investors are informed only about a subset of available securities, as in Merton (1987), market capitalization is a reasonable proxy for limited knowledge about firms. Fewer investors are informed about small companies, and therefore the investor base for small companies is limited. Secondly, I follow Cooper et al. (1985) and Amihud et al. (1997) and use the Amivest price impact metric, defined as the average dollar volume divided by the average absolute return, to isolate assets that investors might optimally choose not to hold if the transaction costs exceed the diversification benefits. When focusing attention on size and price impact subgroups, the impact of SVS switches sign for the group that faces more holding restrictions, and assets with a low stochastic volatility induced skewness have higher returns. This is consistent with the predictions of the theory.

Similarly, sorting all assets into portfolios based on the SVK results in a strong negative difference in raw returns, which can be explained by standard factor adjusting using the Fama-French model. After splitting up the sample in two, however, the return difference between high and low SVK within the more constrained subgroups is significantly positive, again in support of the theoretical predictions.

The remainder of the paper is organized as follows: Section 3.1 provides an overview over the types of skewness observable in stock returns and discusses related literature. Section 3.2 develops and solves a two-period equilibrium model with stochastic volatility and priced idiosyncratic risk components. Section 3.3 describes the empirical methodology and presents the findings. Section 3.4 concludes.

### 3.1 Types of Skewness and Related Literature

This paper builds on two strands of literature that previously have developed largely independently: Higher moments in asset pricing and the implication of idiosyncratic risk.

To motivate the equilibrium pricing mechanism, consider a portfolio choice problem of an investor who invests a fraction \(1 - \alpha\) of his wealth in an existing portfolio \(P\), and the remainder, \(\alpha\), in a given asset \(A\). It is straightforward to compute the changes in the distribution of the portfolio returns as described by the central moments \(m_k^i \equiv \mathbb{E} \left[ (R_i - \mathbb{E}(R_i))^k \right] \):

\[
\begin{align*}
\frac{\Delta m_2^P}{m_2^P} &= o(\alpha) \left( \frac{\text{Cov}(R_A, R_P)}{m_2^P} - 1 \right) + o(\alpha^2) \frac{m_4^A}{m_2^P} - 1 \\
\frac{\Delta m_4^P}{m_4^P} &= o(\alpha) \left( \frac{\text{Cov}(R_A^2, R_P^2)}{m_4^P} - 1 \right) + o(\alpha^2) \left( \frac{\text{Cov}(R_A^3, R_P^3)}{m_4^P} - 1 \right) + o(\alpha^3) \left( \frac{m_4^A}{m_4^P} - 1 \right) \\
\frac{\Delta m_4^P}{m_4^P} &\approx o(\alpha) \left( \frac{\text{Cov}(R_A^2, R_P^2)}{m_4^P} - 1 \right) + o(\alpha^2) \left( \frac{\text{Cov}(R_A^3, R_P^3)}{m_4^P} - 1 \right) + o(\alpha^3) \left( \frac{\text{Cov}(R_A^3, R_P)}{m_4^P} - 1 \right)
\end{align*}
\] (3.1)
where $o(\cdot)$ denotes the order of the coefficients. Note that the last equation - the fourth central moment - is an approximation of third order, and a term with coefficient $o(\alpha^4)$ is omitted. The first order terms in Equation (3.1) correspond to the familiar expressions for the systematic risks in the multi-moment CAPM:

$$
\beta_2 = \frac{\text{Cov}(R_A, R_P)}{m_2^P} \quad \beta_3 = \frac{\text{Cov}(R_A, R_P^2)}{m_3^P} \quad \beta_4 = \frac{\text{Cov}(R_A, R_P^3)}{m_4^P}
$$

(3.2)

In addition to the well-known covariance beta, $\beta_2$, Rubinstein (1973), Kraus and Litzenberger (1976), and Harvey and Siddique (2000) show that if returns and investor preferences have not-trivial third moments, coskewness risk as measured by $\beta_3$ will be an important determinant of portfolio choice and can be negatively priced in general equilibrium. Similarly, using the restriction of decreasing absolute prudence by Kimball (1993), Dittmar (2002) and Guidolin and Timmermann (2008) show that cokurtosis, $\beta_4$, carries a positive price of risk. All these systematic risk factors measure the contribution of holdings of an individual assets to the corresponding moment of the overall portfolio to a first order approximation. They are motivated by the observation that any position in an individual asset within a well-diversified portfolio should be small and higher order terms are negligible.

The argument that only co-moments are priced, however, can break down even in general equilibrium. Idiosyncratic volatility, denoted $m_A^4$, can affect equilibrium pricing if perfect diversification is not feasible due to trading restrictions, as shown by Levy (1978), Merton (1987), and Malkiel and Xu (2006). While the theoretical predictions are clear, there is considerable disagreement about the empirical pricing implications of idiosyncratic volatility. Ang, Hodrick, Xing, and Zhang (2006, 2009) find a significant negative reward for idiosyncratic realized volatility. In contrast, Fu (2009) and Huang et al. (2009) find it to be overwhelmingly positive using EGARCH specifications. The difference has been attributed to short term reversal, which plays a role in the realized volatility measures, but does not impact the EGARCH specification.

Considerable less research has been done on idiosyncratic higher moments. To clearly understand the different components, Figure 3.1 illustrates the three kinds of skewness. The graph simulates asset returns conditional on market returns ranging between -25% and 25%. The black line indicates the CAPM implied relation with a beta equal to unity. In the first graph, the asset exhibits negative coskewness. The returns are simulated from $R_i = a + R_M - bR_M^2 + \sigma_i \varepsilon, \quad \varepsilon \sim N(0,1)$. Importantly, conditional on the market return, the expected return of the asset does not equal the CAPM implied return. Therefore even a portfolio that is well diversified in assets with similar coskewness will exhibit a fat tail and
the equilibrium pricing implications of this coskewness are well understood. The second plot shows the distribution of asset returns conditional on the market return when the volatility of asset returns covaries with the realized market return. In particular, returns are simulated from \( R_i = R_M + \sigma_i(R_M)\varepsilon \), where \( \sigma_i(R_M) = \sigma_i^{0.25-R_M^{0.25}} \) and \( \varepsilon \sim N(0,1) \). Conditional on the market return, residuals are normally distributed with mean zero. This risk can be diversified away, but a large portfolio is required. In the third graph, returns are simulated from \( R_i = R_M + \sigma_i\eta \), where \( \eta \sim Neg.Skew(0,1) \). This is true stock-specific skewness and can be diversified away even in a relatively small portfolio.

Mitton and Vorkink (2007) show in a general equilibrium setting that heterogeneous skewness preferences can lead to under-diversification relative to the homogeneous investor benchmark. Given standard preference assumptions, they show that idiosyncratic skewness, the third order effect, is negatively related to expected returns, and provide some supporting evidence from household level asset allocation. Other theoretical justification for priced idiosyncratic skewness is provided by Barberis and Huang (2008), who show that cumulative prospect theory preferences and the associated subjective probability weighting are sufficient to generate lower average returns for securities that are perceived to be skewed. Similarly, Brunnermeier and Parker (2005) and Brunnermeier et al. (2007) solve an endogenous probabilities model and obtain comparable asset pricing predictions. A direct test of these theoretical predictions for asset returns is done in Boyer et al. (2009), who investigate the relationship between expected daily idiosyncratic skewness over 60 months windows and stock returns. Using a variety of lagged variables including lagged estimates of idiosyncratic skewness, they empirically confirm the negative relation between expected idiosyncratic skewness and stock returns.

All of the above papers on idiosyncratic skewness, both theoretical and empirical, center on the third order skewness effect in Equation (3.1), denoted by \( o(\alpha^3) \). The potentially more important second order effect involving the expression \( Cov(R^2_A, R_P) \), which is the focus of this paper, is ignored or –as in Mitton and Vorkink (2007)– explicitly assumed to be zero. A notable exception is Chabi-Yo (2009), who uses pricing kernels to analytically derive the price of this idiosyncratic coskewness risk. In a partial equilibrium setting, he finds that the price of risk switches signs. In particular, for negative values of idiosyncratic coskewness the price of risk is positive, and for positive values negative, resulting in an inverted U-shaped relation between expected returns and his idiosyncratic coskewness measure. In contrast, I solve for the pricing impact numerically in a general equilibrium setting. Importantly, I find the price of this type of skewness risk to be globally negative and it does not change sign.

In empirical tests, Chabi-Yo (2009) and Chabi-Yo and Yang (2009) directly estimate the
idiosyncratic coskewness on daily returns over 12 months, and confirm the inverted U-shaped relation. They do not model the stochastic volatility explicitly, and their findings thus build on the assumption that idiosyncratic volatility is constant over 12 months intervals. This assumption is at odds with the large literature that finds strong predictability and large variation in return volatility, documented for example in Andersen et al. (2001). Even in a setting with constant idiosyncratic volatility, however, the absolute value of this estimated idiosyncratic coskewness is by construction highly correlated with the level of idiosyncratic volatility itself. The inverted U-shaped relation thus comes at no surprise given the results in Ang, Hodrick, Xing, and Zhang (2006, 2009). In contrast to Chabi-Yo (2009) and Chabi-Yo and Yang (2009), I model returns as a mixture of normal distributions, and empirically carefully separate expected idiosyncratic volatility from its stochastic shocks. The resulting measure is embodied in the idiosyncratic coskewness, but it is unrelated to idiosyncratic volatility. The restriction to this component allows me to separate the effects of second and third moments. I further estimate the model at a monthly return frequency, which is more appropriate for investment horizons greater than one day. I find a negative relation between this stochastic volatility induced skewness and future stock returns for assets that are more likely to face trading restrictions.

3.2 Stochastic Volatility and Pricing of Idiosyncratic Risk

In a simple general equilibrium economy with stochastic idiosyncratic volatility, I show how correlations between the shocks to asset volatility and market returns as well as market return volatility affect asset prices when investors face trading frictions. The asset pricing results extend those of Kraus and Litzenberger (1976) and Dittmar (2002) who find the priced components of higher moments in representative agent settings to be systematic coskewness and cokurtosis. The pricing implications of idiosyncratic skewness in an economy with heterogeneous preferences has be the focus of Mitton and Vorkink (2007), while Barberis and Huang (2008), Brunnermeier and Parker (2005), and Brunnermeier et al. (2007) look at subjectively perceived skewness induced by excessive probability weighting in the tails of the distribution. My model uses more traditional preference assumptions, restrictions to asset holdings, and puts a stronger structure on the return distribution to isolate the impact of stochastic volatility, rather than the third-order idiosyncratic skewness.
3.2.1 Returns on Risky Assets

Excess returns on the $K$ risky assets, $R^e$, are driven by a common factor $f$ and subject to factor-unrelated shocks $\varepsilon$. The key deviation of my model relative to standard assumptions is the existence of an unobservable state vector $X$ that jointly drives returns and variances, and thus generates state-dependent volatility. Importantly, conditional on the state vector, returns follow a standard factor model with normally distributed innovations

$$R^e = \mu^e + fX b + \varepsilon X$$

(3.3)

with

$$\left[ \begin{array}{c} fX \\ \varepsilon X \end{array} \right] \sim \mathcal{N} \left( \left[ \begin{array}{c} \mu_X \\ 0_K \end{array} \right], \left[ \begin{array}{cc} \sigma^2_X & 0_K' \\ 0_K & \Sigma_X \end{array} \right] \right)$$

(3.4)

This specification allows the idiosyncratic volatility to be correlated with both the systematic shocks $f$ and the systematic volatility. Crucially, this model does not exhibit idiosyncratic skewness caused by asymmetrically distributed residuals, which has been the focus of recent literature (e.g. Mitton and Vorkink (2007)). Confounding effects from skewed residuals are thus eliminated, and it is possible to isolate the effects due only to stochastic return volatility.

Conditional on the state vector, all returns are normally distributed. Before the realization of the state variables, however, returns are drawn from mixtures of gaussian distributions. Marron and Wand (1992) point out that mixtures of normal distributions can be used to closely approximate numerous other distributions, including skewed distributions and distributions with excess kurtosis. Moments of the mixed distribution can easily be characterized as a function of the mean, variance, and correlations of the gaussian distributions in the underlying states. Praetz (1972) and Blattberg and Gonedes (1974) were among the first to model stock returns using student-$t$ distributions, a particular finite variance-mixtures of normals. Mandelbrot and Taylor (1967), Clark (1973) and Andersen (1996) model stock returns as gaussian distributions whose variance explicitly depends on the number of transactions or trading volume. More recently, Andersen et al. (2001) find support that the distribution of the returns of individual stocks in the Dow Jones Industrial Average can be well approximated by a continuous variance mixture of normals.

Figure 3.2 illustrates how correlations between both the stochastic mean and variance of the factor shock as well as the variance-covariance-matrix of the residuals $\Sigma_X$ can generate a return distribution with non-trivial third and fourth moments. The two plots of Panel A on the left show how a mixture of two normal distributions can result in ex-ante skewness. For simplicity, assume the underlying factor return can take two values. Conditional on a low
factor return, the asset volatility is high, and vice versa. The bottom graph shows the asset return distribution before the realization of the factor return is known. This distribution exhibits negative skewness. A similar intuition underlies Panel B, where the factor return can take three values, and asset volatility conditional on low or high factor realizations is high, while asset volatility in the medium state is low. The resulting distribution exhibits excess kurtosis.\textsuperscript{25} My model deviates from this simple intuition in two ways. The underlying expected factor return in the model is continuous, and an additional shock adds noise to the state-conditioned factor return.

Figure 3.3 shows the timing of events and clarifies the different stages of conditioning information. At \( t = 0 \), investors make their portfolio decisions and trade based on the unconditional distribution of returns. Then, the vector of state variables \( \mathbf{X} \) is realized and determines the mean and variance of the normal distributions from which factor and stock returns are to be drawn. Importantly, no trading is allowed at this time. At \( t = 1 \), all uncertainty is resolved. Factor and asset returns are realized, in turn determining investors final wealth.

Additional assumptions are required to specify the precise impact of the state vector on the return generating process. The states are described by the vector \( \mathbf{X} = [X_1, X_2, X_3]' \) of independent standard normals, \( X \sim \mathcal{N}(0, I_3) \), and conditional on \( \mathbf{X} \), the factor distribution is given by

\[
f_X \sim \mathcal{N}\left(\delta \mu X_1, \sigma^2 + \delta \sigma X_2\right),
\]

where \( \sigma^2 \) is the expected conditional factor variance, and \( \delta \mu \) and \( \delta \sigma \) denote the sensitivities of the mean factor returns and the factor variance with respect to the state variables \( X_1 \) and \( X_2 \).

The factor-unrelated shocks \( \varepsilon \) have the conditional distribution

\[
\varepsilon_X \sim \mathcal{N}(0, \Sigma + \Delta \Sigma X_1 + \Delta \Sigma X_2 + \Delta \Sigma X_3)
\]

where \( \Sigma \) is the expected variance-covariance matrix, and \( \Delta \Sigma k, k = 1, 2, 3 \), denote the sensitivities of the covariance matrix with respect to the corresponding state variables. For simplicity, I assume that all components of the residual covariance are diagonal matrices. The factor exposure thus accounts for all asset correlations, and stochastic volatility affects

\textsuperscript{25}Stochastic stock return volatility is sufficient to generate a fat-tailed return distribution (see, for example, Stein and Stein (1991)). The resulting kurtosis, however, will enter the portfolio choice problem as a fourth order term only, and its effect can –given reasonable magnitudes and a relatively small number of assets– be largely diversified away. The joint state-dependence of stochastic volatility and factor returns results in second-order effects for portfolio choice, and consequently is more relevant for partially diversified portfolios.
asset specific volatility only and not the covariance structure.

The state variable $X_1$ drives the expected factor return as well as asset specific variances, which results in stochastic volatility induced skewness. In contrast, $X_2$ jointly determines factor volatility and asset specific variances, resulting in stochastic volatility induced kurtosis. $X_3$ only affects the residual volatility, and serves the sole purpose of providing a benchmark case where idiosyncratic shocks unconditionally are fat-tailed, but do not covary with the pricing kernel.

Since the determinants of the factor distribution, $X_1$ and $X_2$, are independent, conditional factor mean and volatility will be independent and the factor thus is unconditionally symmetrically distributed. In particular, if $\delta_\sigma = 0$, the factor will be normally distributed with mean zero and variance $\sigma^2 + \delta_\mu^2$. For $\delta_\sigma \neq 0$, mean and variance are unchanged, but the stochastic volatility causes excess kurtosis.

By folding the factor volatility into the residual covariance matrix, the conditional return generating process can now conveniently be rewritten as

$$R^e = \mu^e + \delta_\mu b X_1 + e_X \quad (3.7)$$

where $e_X$ is normally distributed with

$$E[e_X] = 0 \quad \text{and} \quad E[e_X e_X'] = S + \sum_{k=1}^3 S_k X_k, \quad (3.8)$$

and the components of the variance are

$$S = \sigma^2 bb' + \Sigma, \quad S_1 = \Delta_X^1, \quad S_2 = \delta_\sigma bb' + \Delta_X^2, \quad \text{and} \quad S_3 = \Delta_X^3. \quad (3.9)$$

Equations (3.5) and (3.6), together with the assumption of normally distributed state variables, imply that a negative realization of the stochastic volatility is possible. A derivation that avoids negative variances – by modeling the log of volatility linear in $X$ – is feasible but results in more involved first order condition. Choosing the parameters allows to bound the probability of negative variance realizations below any desired level.

### 3.2.2 Agents and Preferences

Each agent’s preferences are assumed to be approximated by a polynomial function

$$EU(W) \approx EW - \frac{1}{2\phi_2} m_2(W) + \frac{1}{3\phi_3} m_3(W) - \frac{1}{12\phi_4} m_4(W), \quad (3.10)$$
where \( m_k(W) = \mathbb{E}(W - \mathbb{E}W)^k \) denotes the central moments of the distribution of wealth \( W \), and \( m_4^*(W) \equiv m_4(W) - 3m_2^2(W) \) is the kurtosis in excess of the level expected if wealth was to follow a normal distribution. The parameter \( \phi_2 \) is the coefficient of risk tolerance, and \( \phi_3 \) and \( \phi_4 \) similarly measure the tolerance for third and fourth moments. Positive values for \( \phi_3 \) indicate the typically assumed preference for skewness, while positive values of \( \phi_4 \) indicate aversion to fourth moments.

The choice of polynomial approximations to utility functions is consistent with the majority of prior literature that involves higher moments. While this class of preferences has several well documented shortcomings, most prominently parameter restrictions and compact support of payoffs necessary to ensure non-satiation, Levy and Markowitz (1979) show that a quadratic approximation to utility functions performs well, and Hlawitschka (1994) provides empirical evidence that finite-order Taylor-series approximations to utility functions may provide excellent approximations to expected utility – whether the infinite Taylor-series is convergent or not.\textsuperscript{26} Preferences for the fourth moment in Equation (3.10) are defined with respect to excess kurtosis, a specification choice that permits a clean separation of second and fourth moments.

In addition to the risky assets described above, a risk-free bond pays an interest rate \( r_f \). Agent \( j \) is endowed with an initial wealth \( W_{0,j} \) and chooses her investments in the risky assets, \( \omega_j \), to maximize her expected utility

\[
\max_{\omega_j} \mathbb{E}[U(W)]
\]  

subject to a budget constraint

\[
W = \omega_j' (\iota + R) + (W_{0,j} - \omega_j' \iota) (1 + r_f),
\]

where \( \iota \) denotes a vector of ones, as well as an exogenous holding constraints

\[
\omega_j (i) = 0 \quad \text{for given } i, j.
\]

While the budget constraint is the standard for an endowment economy where all income is derived from asset returns, the holding constraint commands for some discussion. The limitations to risk sharing I consider are simple restrictions for some investors to take positions

\textsuperscript{26}While the normally distributed returns do not satisfy the compact support requirement of the utility function, all parameterizations considered in this chapter are chosen to ensure non-satiation and risk aversion over at least a \( \pm 3 \) standard deviation range.
in a subset of assets. This is consistent with the approaches by Merton (1987) and Malkiel and Xu (2006), who use the same frictions to generate a price for idiosyncratic risk. In these models, the marginal utility of each agent will price the assets within her diversification boundaries, but to the econometrician, who observes only the market portfolio and is unable to identify the trading frictions in the economy – and thus implicitly assumes a representative agent –, idiosyncratic risk appears to be priced. Trading restrictions of this kind can be motivated either by information or by trading frictions. Alternative approaches that result in comparable pricing implications generate underdiversification from heterogeneous preferences (Mitton and Vorkink (2007)) or endogenize the decision to invest given varying familiarity towards assets (Boyle et al. (2009)). For the theoretical predictions in this paper, the mechanism that generates idiosyncratic risk to be priced is irrelevant. But in contrast to alternative approaches, the choice of holding constraints generates clear empirically testable predictions. Importantly, the empirical support for apparent underdiversification of individual investors is substantial. Evidence is provided, for example, in Blume and Friend (1975), Odean (1999), Vissing-Jorgensen (2002), and Goetzmann and Kumar (2008).

Investors facing the holding constraints (3.13) are exogenously restricted from optimal risk sharing, and are thus facing an incomplete market. Agents in this economy have an incentive to overcome or reduce the market incompleteness. This can be done either directly by eliminating the holding constraints, or through financial innovation. The introduction of derivative securities or allowing to trade the state variables $X$ directly will reduce or eliminate the impact of the holding constraints since mimicking portfolios can be traded. The holding constraints are a vital component of the model, and thus any financial innovation or additional securities would potentially alter the model predictions.

### 3.2.3 Equilibrium without Portfolio Holding Constraints

While both the covariance structure and the processes for stochastic volatility are exogenously imposed, the mean return vector $\mu^e$ will be determined in equilibrium by equating the market value of assets, $s$, to aggregate demand.

The first order condition for each agent is given by

$$
\mu^e = \frac{1}{\phi_2} \left( S + (\delta^2 + \sigma^2)bb' \right) \omega_j - \frac{1}{\phi_3} \delta_\mu \left( \omega_j'S_1\omega_j^b + 2\omega_j'bS_1b\omega_j \right) + \frac{1}{\phi_4} \sum_{i=1}^3 \left( \omega_j'S_i\omega_jS_i\omega_j \right) \quad (3.14)
$$

The equilibrium is characterized by portfolio holdings $\omega_j$, $j = 1, 2, 3$ and the expected excess return vector $\mu^e$ such that, given $\mu^e$, $\omega_j$ solves the first order condition (3.14) subject
to the budget constraint (3.12) and the holding constraints (3.13) for each investor \( j = 1, 2, 3 \), and the asset market clears, i.e. \( \sum_{j=1}^{3} \omega_j = s \).

Under the assumed preferences, the demand for risky assets is independent of \( W_{0,j} \) and \( r_f \). The zero-net supply condition for the bond links \( r_f \) to \( W_{0,j} \). This, however, is irrelevant for the portfolio choice in the risky assets.

If all investors can freely choose their portfolio holdings, the economy reduces to the case of a representative agent discussed in Kraus and Litzenberger (1976) and Dittmar (2002). We define the excess market return as the market capitalization weighted average return, \( R_{eM} = s' R^e \). Setting aggregate demand \( \omega = \sum \omega_j \) equal to aggregate supply of each asset in Equation (3.14) yields the well known pricing relation

\[
\mu^e = \frac{1}{\phi_2} Cov (R, R_{eM}^e) - \frac{1}{\phi_3} Cov (R, R_{eM}^{e2}) + \frac{1}{\phi_4} Cov^* (R, R_{eM}^{e3}),
\]

where \( Cov^* (R, R_{eM}^{e3}) \equiv Cov (R, R_{eM}^{e3}) - 3 m_2 (R_{eM}^e) Cov (R, R_{eM}^e) \) denotes the excess cokurtosis. The excess returns are driven by a total of three risk premia: Covariance, coskewness, and cokurtosis.

The market return in this benchmark case is given by

\[
\mu^e = \frac{1}{\phi_2} m_2 (R_{eM}^e) - \frac{1}{\phi_3} m_3 (R_{eM}^e) + \frac{1}{\phi_4} m_4^* (R_{eM}^e).
\]

As has been pointed out in previous literature, the pricing equation (3.15) depends on the three preference parameters. Imposing the expression for market returns (3.16) allows to substitute only one parameter. Obtaining a preference-independent solution for the four-moment model requires two additional aggregate restrictions. These could, for example, be given by the returns of portfolios that have unit loading on the coskewness and cokurtosis factors respectively.

### 3.2.4 Equilibrium with Portfolio Holding Constraints

In the general case with portfolio holding constraints, an analytical solution to the problem is not available and I resort to numerical methods. In particular, for the remainder of the section, I assume there are three investors \( j = 1, 2, 3 \) and three risky assets \( i = 1, 2, 3 \) with equal market capitalization \( s = [1/3, 1/3, 1/3]' \). While investor 1 can choose his allocation freely, investor 2 and 3 are restricted from holding assets 2 and 3, respectively.

\[\text{The sum of all market capitalizations is standardized to unity.}\]
I further assume the preferences and asset return parameters given in Table 3.1. The risk tolerance of $\phi_2 = 1.5$ is high but within generally accepted bounds. Since the aggregate demand will predominantly depend on the assumed risk aversion, while the aggregate supply is given exogenously, the main impact of risk aversion for the purposes of this paper is to standardize returns. The literature provides less guidelines about reasonable magnitudes of preferences for skewness and kurtosis. I specify $\phi_3 = 0.15$, and $\phi_4 = 0.015$, which results in somewhat stronger skewness preference and kurtosis aversion compared to general utility specifications such as power utility or exponential utility.

The level of preference parameters plays a minor role for the qualitative effects on returns as it can be offset by changes in the total supply of risky assets the economy. More important are the rates of substitution between volatility, skewness, and kurtosis. Performing a Taylor expansion of power (constant relative risk aversion) or exponential (constant absolute risk aversion) utility function shows that risk aversion between 10 and 20 is necessary to generate the ratios $\frac{\phi_2}{\phi_3} \approx \frac{\phi_3}{\phi_4} \approx 10^{28}$.

Panel B of Table 3.1 illustrates the parametrization of the factor returns. Depending on the quantity of interest, variation in the factor return originates either from $\delta_{\mu}$ (SVS) or from $\sigma$ (SVK). In both cases, the volatility of factor returns is identical. The different variance attribution is a modeling choice that enables both effects to be independently strong while keeping the overall variance of the factor return reasonable. The choice of $\delta_{\sigma} = 0.4\sigma^2$ bounds the probability of negative variance realizations below 0.6%.

As shown in Panel C of Table 3.1, all assets have equal factor exposure $b_i$, and equal residual volatility $\sigma_i^2$. While there is considerable empirical evidence of a positive cross sectional relation between idiosyncratic volatility and skewness (see, for example, Boyer et al. (2009)), keeping all assets identical allows to identify the impact of stochastic volatility without being tainted by other effects. $\Delta_{\Sigma 3} = 0_{3,3}$ indicates that all stock specific stochastic volatility in the economy is correlated with the pricing kernel.

### 3.2.4.1 Stochastic Volatility and Portfolio Skewness

To illustrate the implications of nontrivial correlation between assets’ volatility shocks and the realized market return on asset returns, I now numerically solve for the equilibrium for a particular parametrization of the constrained portfolio choice problem.

The state variable $X_1$ jointly drive asset-specific volatility (through $\Delta_{\Sigma 1}$), and the factor

---

28 In unreported results, I solve the model with alternative utility specifications. In both the power and exponential case, the quantitative impact of the higher moment effects is smaller relative to the pricing impact of idiosyncratic variance, but qualitatively all results remain.
return (through $\delta_\mu$). If both sensitivity parameters are different from zero, asset volatility is correlated with factor return realizations, and the assets exhibit stochastic volatility induced skewness. I start with the baseline model where $\Delta \Sigma_1 = 0_{3,3}$ is the zero matrix, and I vary the idiosyncratic volatility of asset 2 by changing the corresponding entry. In particular, I vary $\Delta \Sigma_1(2, 2) = 0.4k\sigma_i^2$ for $k = -1, \ldots, 1$. Given the choice for $\delta_\mu$, the covariance between factor-unrelated volatility of asset 2 and the factor returns thus changes from $-0.4\sigma_i^2\delta_\mu$ to $0.4\sigma_i^2\delta_\mu$. Again, since the the elements of the variance-covariance matrix in my model are normally distributed, a positive definite matrix is not guaranteed. The multiplier of 0.4 ensures that the matrix remains positive definite at least 99.4% of the times.

The sole purpose of asset 3 in this economy is to offset any effect asset 2 has on the moments of aggregate market returns, and I specify $\Delta \Sigma_1(3, 3) = -\Delta \Sigma_1(2, 2)$. This allows a clear separation between idiosyncratic and systematic skewness, since the market skewness is constant across different values of $k$.

Figure 3.4 shows the portfolio weights of the three agents across a changing covariance between stock volatility and the factor realization. Going from left to right, asset 2 will first have a large volatility when market return realizations are low - as indicated by the negative covariance - and thus delivers idiosyncratic risk in bad times. The restrictions of portfolio holdings are represented by the zero holding of agent 2 in asset 2, and of agent 3 in asset 3. To clear the market, agent 1 has to take larger positions in those assets, his holdings of the unrestricted asset 1 are correspondingly smaller. Initially, as asset 2 can be considered a bad asset, agent 1 portfolio is slightly tilted toward this asset, while agent 3 holds less. This is the result of market clearing and the fact that agent 1 has better diversification options than agent 3. Overall, there is little endogenous variation in portfolio holdings as $\Delta \Sigma_1(2, 2)$ changes from $-0.4\sigma_i^2$ to $0.4\sigma_i^2$.

The effects of stochastic volatility induced skewness on expected returns, as illustrated in Figure 3.5, is much bigger. The figure plots expected returns against $\Delta \Sigma_1(2, 2)$, the sensitivity of idiosyncratic volatility shocks to factor realizations. The top graph depicts raw returns, the second graph CAPM risk-adjusted returns, and the bottom graph shows 4-moment model risk adjusted returns. Investors initially demand a higher expected return of asset 2. Toward the center of the graph, where $\Delta \Sigma_1(2, 2) = 0$, there is not stochastic volatility in the market. Assets 2 and 3 require higher returns than asset 1 because of the exogenously imposed limits to diversification, as in Merton (1987). Toward the right of the graphs, asset 2 has little idiosyncratic risk in bad times, and is thus a welcome addition to the portfolio of the underdiversified investor. Importantly, the expected idiosyncratic variance of all assets is constant across the graphs, and only the stochastic realizations differ. The magnitude of
the return impact is unaffected by CAPM risk adjusting, while the more appropriate four-moment risk adjusting in the bottom panel attenuates the effect of changing sensitivity. This reduction reflects the equilibrium nature of the model: Skewness that is idiosyncratic with respect to the exogenous factor is partially accounted for as systematic coskewness risk using the endogenous market return. Nevertheless, the effects of stochastic idiosyncratic volatility on asset skewness and its associated pricing implications remain qualitatively unchanged.

3.2.4.2 Stochastic Volatility and Portfolio Kurtosis

In this section, I repeat the previous exercise under the assumption that stochastic idiosyncratic volatility is related to the volatility of the pricing factor. In the model, $\Delta \Sigma_2$ loads on the same stochastic state variable as the factor volatility, and thus generates the desired effect. In particular, I change $\Delta \Sigma_2(2, 2) = 0.4 k \sigma_i^2$ from $k = -1, ..., 1$. $\Delta \Sigma_2(3, 3)$ again is chosen to keep the moments of the market portfolio unaffected.

Figure 3.6 shows the portfolio weights of the three investors, and they are approximately constant as the covariance between shocks to idiosyncratic volatility and shocks to factor volatility changes from negative to positive. The expected asset returns, raw or risk-adjusted, in Figure 3.7, show the predicted pattern: Initially, the volatility of asset 2 will covary negatively with the factor volatility, implying that asset 2 contributes little idiosyncratic risk in times of high systematic risk. The required return on the asset is lower than what would be expected given the level of its idiosyncratic risk. If the volatility of asset 2 comoves positively with factor volatility, it makes the portfolio of the underdiversified investor riskier in risky times, and the required returns for holding this asset are higher.

The main predictions of this model are that for assets that face holding restrictions, such as assets where information is more costly to obtain or assets with larger trading frictions, the covariation between shocks to idiosyncratic volatility and factor realizations should be negatively related to returns, and the covariation between shocks to idiosyncratic volatility and factor volatility positively.

3.3 Empirical Analysis

The goal of this paper is to examine whether underdiversified investors are compensated for being exposed to idiosyncratic higher order effects caused by stochastic volatility in stock returns. From a theoretical perspective, underdiversified investors demand a premium for holding assets whose stochastic volatility shocks negatively covary with (unexpected) market returns or positively with shocks to the market volatility. Since volatility, and consequently
shocks to volatility, is unobservable, I follow the existing literature and estimate monthly idiosyncratic volatility as the residual root mean squared error from regressions of daily returns onto factors in one-month windows. An ARMA model on the monthly realized volatility estimates is used to identify innovations to the volatility process.

I first describe the volatility estimation procedure, and then compute standardized covariances between stochastic shocks to the volatility process and the realized market at the firm level as measure of the idiosyncratic skewness induced by stochastic volatility, and covariances between shocks to idiosyncratic volatility and market volatility as a measure of the second order effect of idiosyncratic kurtosis.

3.3.1 Stochastic Idiosyncratic Volatility

Extracting stochastic shocks to volatility at the firm level is not straightforward. Estimating true stochastic volatility models at the stock level is not advisable: Models with easily computable likelihood functions are not sufficiently general to be robustly estimated for the entire universe of assets. Alternative approaches based on Monte Carlo Markov Chains or methods of moments estimation are computationally too expensive. I obtain stochastic volatility shocks by imposing a time-series structure on the monthly realized volatility.

Following Ang, Hodrick, Xing, and Zhang (2006, 2009), I measure the idiosyncratic risk of an individual stock as follows. In every month, the daily excess returns of individual stocks, $R_{i,\tau}$, are regressed on either the market excess return $R_{M,\tau}$ or the 3 Fama and French (1993) factors that additionally account for risk associated with small stocks ($SMB$) and value stocks ($HML$).

\begin{equation}
R_{i,\tau} = \alpha_t + \beta_{M,t} R_{M,\tau} + \beta_{S,t} SMB_{\tau} + \beta_{H,t} HML_{\tau} + \epsilon_{i,\tau}
\end{equation}

In the above, $\tau$ denotes days within month $t$. Returns are from CRSP, and I include all common stocks traded on the NYSE, Amex, or Nasdaq from 1926 to 2008. Daily factor data from 1963 to 2008 as well as the historical book values are obtained from Ken French’s website.\textsuperscript{29} The daily factors before 1963 are constructed as described in Fama and French (1993). I restrict estimation to asset-months with at least 15 valid return observations. The realized variance of stock $i$ is defined as the root mean squared error of residuals in Equation (3.17):

\begin{equation}
RV_{i,t} = \sqrt{\mathbb{E}(\epsilon_{i,\tau}^2)} \text{ for } \tau \in t.
\end{equation}

\textsuperscript{29}I thank Ken French for making the data available.
In order to extract shocks to volatility, it is necessary to specify a model of conditional expectations of the realized variance. The implicit assumption underlying Ang, Hodrick, Xing, and Zhang (2006, 2009) is that $RV_t$ follows a martingale, i.e. $\mathbb{E}(RV_{t+1}) = RV_t$. Subsequent research provides strong empirical evidence against this martingale assumption. In particular, Jiang and Lee (2006) and Fu (2009) estimate positive autocorrelations significantly below unity. In accordance to these findings, following Huang et al. (2009), I specify expected idiosyncratic volatility for each asset as the fitted value of an $ARMA(p,q)$ model of the logarithm of realized volatility,

$$
\log(RV_{i,t}) = c + \delta_{i,t}^{RV} + \sum_{k=1}^{p} \varphi_k \log(RV_{i,t-k}) + \sum_{k=1}^{q} \theta_k \delta_{i,t-k}^{RV}
$$

(3.19)

For each asset, I choose the combination of $0 \leq p, q \leq 3$ that minimizes Akaike’s information criterion, corrected for small sample size (AICC). In this formulation, $\delta_{i,t}^{RV}$ can be interpreted as the shock to idiosyncratic volatility at time $t$.\(^{30}\)

The empirical procedure to extract shocks to the market volatility follows similar steps, except that intercept is the sole explanatory variable in the first regression. Since the literature has found market returns to be largely unpredictable (see, for example, the comprehensive study of Goyal and Welch (2008)), I do not explicitly model the expected market return, but rather assume it to be a constant.\(^{31}\)

### 3.3.2 Stochastic Idiosyncratic Volatility Risk

As the previous section has shown, stochastic idiosyncratic volatility can affect expected returns if its shocks covary with market return realizations or with innovations in market

\(^{30}\)As robustness checks, I also used the assumption of constant expected idiosyncratic volatility, i.e. $left(RV_{i,t} = c + \delta_{i,t}^{RV}$. In a similar fashion, I estimated a GARCH model on monthly returns with constant factor loadings and interpreted the innovations to the conditional volatility process as idiosyncratic volatility shocks. The findings for both cases are comparable to the ones presented here.

\(^{31}\)There is, of course, also a large literature defending market return predictability. The $R^2$ of out-of-sample predictive regressions, however, is mostly below 1% at a monthly frequency. While this certainly is economically meaningful, as pointed out by Campbell and Thompson (2008) and Rapach et al. (2009), it will have a negligible impact on the estimated shocks.
volatility. I directly measure skewness and kurtosis induced by stochastic volatility as

\[
\beta_{SVS}^i = \frac{\text{Cov}(\delta_{RVi,t}, R_{Mt})}{RV_{i,t}} \quad (3.20)
\]

\[
\beta_{SVK}^i = \frac{\text{Cov}(\delta_{RVi,t}, \delta_{RV_{Mt}})}{RV_{i,t}} \quad (3.21)
\]

The standardization by \( RV_{i,t} \) accounts for the empirical observations that idiosyncratic skewness and volatility are highly correlated, as shown in Chen et al. (2001) and Boyer et al. (2009). A similar standardization has been used in Harvey and Siddique (2000) to ensure that estimates for coskewness are unrelated to idiosyncratic volatility.

To obtain the main result of this paper, I estimate the stochastic volatility risks as in Equation (3.20) using rolling estimates of 60 monthly observations at the individual firm level. At the end of each month, I sort all assets with valid risk measures into quintiles based on their estimated \( \beta_{SVS}^i \) and \( \beta_{SVK}^i \), and hold the resulting portfolios for one month.

### 3.3.2.1 Stochastic Volatility induced Skewness

Table 3.2 shows raw- as well as risk-adjusted returns of value-weighted quintile portfolios based on the stochastic volatility skewness, \( \beta_{SVS}^i \). In Panel A, the dynamics of idiosyncratic risk are estimated using CAPM residuals, and in Panel B using Fama-French three factor residuals. All \( t \)-statistics are robust to heteroscedasticity and autocorrelation as in Newey and West (1987) with 6 monthly lags. The findings of both methods are generally very consistent, and I focus discussion on Panel A. The raw returns are monotonically increasing from 0.62% monthly for the portfolio with low \( \beta_{SVS}^i \) to 0.77% monthly for the portfolio with high \( \beta_{SVS}^i \). The difference of 15 basis points a month is economically and statistically significant. Risk adjusting both with market returns and the 3 Fama-French factors has a negligible impact on the pattern in returns. For example, the Fama-French alphas increase from \(-0.01\%\) to \(0.14\%\) monthly for an unexplained return on the difference portfolio of 15 basis points.

This finding seems to contradict the theoretical motivation that a higher covariance between shocks to stochastic volatility and factor realizations should be accompanied by lower expected returns. However, the theoretical predictions apply to assets for which there are more holding restrictions. Since underdiversification in assets is not observable, I use as proxies two variables that are related to holding restrictions. First, if underdiversification arises because investors are informed only about a subset of available securities, as in Merton
(1987), the market capitalization is a reasonable proxy for limited knowledge about firms. Fewer investors are informed about small companies and therefore are not able to hold positions in them, while information about large cap firms is generally easily available and less costly to obtain. Alternatively, investor underdiversification can arise endogenously if trading frictions are important. Investors will optimally choose not to take any positions in the subset of assets for which the cost of trading outweighs the diversification benefits. I use the Amivest price impact measure to proxy for the cost of trading.\footnote{The Amivest price impact measure is defined as the average daily dollar volume divided by the average of absolute returns of daily data in the calendar year preceding the month of analysis. It is as such inversely related to the Amihud (2002) illiquidity measure.}

I investigate the relationship between returns of SVS portfolios and the two proxies for holding restrictions in Table 3.3. In addition to the five SVS groups, assets are sorted independently into two size groups or two Amivest price impact groups. In Panel A, realized volatility is again obtained from CAPM residuals, and in Panel B from the Fama-French residuals. The table reports Fama-French alphas with robust \( t \)-statistics. The results show that for more restricted assets, the effect of stochastic volatility induced skewness is opposite that for more liquid assets: The point return difference for illiquid stocks switches signs and is consistent with the predictions of the theory, at \(-0.10\%\) for small stocks, and \(-0.08\%\) for high price-impact stocks. The difference of differences is \(0.24\%\) monthly for small versus big market capitalization, and \(0.22\%\) monthly for the price impact measure. Both values are economically and statistically significant.

**Why is the Overall Effect Positive?**

The question remains why the overall impact of comovement between shocks to volatility and market return realizations is positive. I show that it is related to Kraus and Litzenberger (1976) systematic coskewness. Unfortunately, a simple interpretation of alpha as abnormal performance is lost when the squared market return is used as an explanatory variable because the square of market returns is a payoff and not a return. As such it impossible to cleanly interpret the intercept as long as the price of coskewness risk is unknown.\footnote{See Boguth et al. (2009) for a detailed discussion.} It is thus necessary to use cross-sectional methods to estimate the price of coskewness risk.

Table 3.4 shows estimated prices of risk from two-stage OLS regressions using the univariate quintile portfolios from Table 3.6 as test assets (Panel A), as well as first stage coskewness exposures (Panel B). The estimated price of risk is negative at \(-0.19\%\) monthly, and coskewness risk can thus explain parts of the overall positive relation between \(\beta^{SVS}\) and
future returns. In particular, the difference in coskewness betas is $-0.28$ across the quintile portfolios. The product of risk loadings and the price of risk, \((-0.28)(-0.19\%) = 0.05\%\), indicates the coskewness is responsible for about 5 basis points of the monthly return difference. The negative estimated price of risk is consistent with standard assumptions on skewness-preferences.

The coskewness betas of the portfolios sorted on size and $\beta^{SVS}$ (Fama-French residuals) presented in Panel C are decreasing across SVS groups in both subsamples. Within the small stock group, however, the decrease is much more pronounced than among big stocks. Coskewness risk is thus unable to explain the opposing return patterns in both groups. As such, the estimated difference in differences in Table 3.3 are a lower bound on the true risk adjusted differences if coskewness risk is priced. For portfolios sorted on Amivest price impact and $\beta^{SVS}$, the difference in coskewness betas is nearly zero for high price impact stocks, and $-0.39$ for low price impact stocks. The coskewness model could therefore explain about \((-0.4)(-0.19\%) = 0.08\%\) of the return difference.

Table 3.5 reports average characteristics of quintile sorts based on the stochastic volatility induced skewness, SVS. The results indicate that $\beta^{SVS}$ is mostly unrelated to the characteristics considered. Both idiosyncratic volatility measures, the average standard deviation over the last 60 months as well as the conditional idiosyncratic standard deviation implied by the $ARMA(p,q)$ model, are highest for moderate $\beta^{SVS}$ and somewhat lower in the two extreme portfolios. The difference between the idiosyncratic volatility measures of the high and low skewness risk portfolios is small. Stocks with low $\beta^{SVS}$ are slightly larger and have lower trading price impacts on average, and there are no differences across portfolios in book-to-market ratios or prior returns. Importantly, none of the characteristics considered is able to generate the observed return patterns.

3.3.2.2 Stochastic Volatility induced Kurtosis

The analysis of quintile portfolios based on $\beta^{SVK}$ in Table 3.6 provides additional support for the model’s main implications. Panel A again shows the results for idiosyncratic volatility with respect to the market return, and Panel B with respect to the three Fama-French factors. Raw returns are monotonically decreasing in $\beta^{SVK}$, from 0.94% monthly for the quintile with the lowest covariance of shocks to stock volatility with shocks to market volatility to 0.61% for the quintile with the highest covariance. This implies that assets with high risk in risky times trade at a premium and have lower expected returns. The difference is a large $-0.33$%.

\[^{34}\]The estimated price of $-0.19\%$ is consistent with estimates in the prior literature. Harvey and Siddique (2000) report a slightly larger value of $-0.30\%$ monthly.
monthly.

While this finding seems very puzzling, standard risk adjustment methods are able to explain the difference in returns. The CAPM reduces the effect to \(-0.15\%\), and the Fama-French model eliminates all abnormal returns.

The subgroup analysis based on holding restrictions in Table 3.7 shows expected pattern: Opposite to the findings for SVS, returns in the more restricted subgroup of small stocks are positively related to SVK, while the effect for large stocks is insignificantly negative. In this case, the difference of 20 basis points for the CAPM residuals, and 17 basis points for the Fama-French residuals is large but the statistically support is less strong. The general finding is much stronger when the Amivest measure is used as proxy for holding restrictions. For stocks with high price impact measures, returns are increasing in \(\beta_i^{SVK}\), while there is no pattern in the subgroup of low price impact. The difference of the differences is large at about \(-30\) basis points per month.

### 3.4 Conclusion

Assets exhibit skewness if state-dependent idiosyncratic volatility is correlated with market returns, even if the distribution of residuals conditional on the realized market return is normal. Similarly, a positive covariance between shocks to idiosyncratic volatility and the stochastic realizations of market volatility results in excess kurtosis.

I develop a two-period equilibrium model in which first and second moments of market returns and idiosyncratic volatility are driven by a common state variable. An exogenous restriction to diversification ensures that non-systematic risks can have a pricing impact. Using standard preference assumptions, the model predicts a negative relation between stochastic volatility induced skewness and expected returns, while stochastic volatility induced kurtosis is positively related to returns. Both effects are strongest for assets that face larger trading restrictions.

I empirically test the predictions using individual stock data. I carefully separate idiosyncratic volatility into a conditionally expected component and shocks using an ARMA specification. Sorting stocks into portfolios based on the historical covariance between shocks to their idiosyncratic volatility and market returns results in a surprising positive relation to returns. This positive relation, however, switches signs for the subset of assets that face more holding restrictions as proxied for by size or the Amivest price impact measure, as predicted by the model. Similarly, portfolios based on the covariance between shocks to idiosyncratic volatility and shocks to market volatility do not show any abnormal return pattern over-
all. When assets are grouped into low and high trading restrictions, the relationship turns significantly positive for the more restricted subgroups.

There is convincing evidence on time variation in idiosyncratic volatility, provided for example by Ang et al. (2009), Fu (2009), and Huang et al. (2009). This paper presents a first contribution to better understand the asset pricing implications of stochastic idiosyncratic volatility.
Figure 3.1: Three Kinds of Skewness

This figure illustrates the three kinds of skewness. All three graphs plot random realizations of asset returns against market returns ranging between -25% and 25%. The black line indicates the CAPM implied relation with a beta equal to unity, and the residual volatility is $\sigma_i = 0.1$ in all cases.

In the first plot, the asset exhibits negative coskewness. Returns are simulated from $R_i = a + R_M - bR_M^2 + \sigma_i \varepsilon$, $\varepsilon \sim \mathcal{N}(0, 1)$, and the expected return of the asset conditional on the market return realization is represented by the red line. The second graph shows the distribution of asset returns conditional on the market return when the volatility of asset returns negatively covaries with the realized market return. Returns are simulated from $R_i = R_M + \sigma_i(R_M) \varepsilon$, where $\sigma_i(R_M) = \sigma_i \frac{0.25 - R_M}{0.25}$ and $\varepsilon \sim \mathcal{N}(0, 1)$. The third plot shows asset returns are simulated from $R_i = R_M + \sigma_i \eta$, where $\eta$ is drawn from a negatively skewed distribution with mean zero and unit variance.
Figure 3.2: Mixture of Gaussian Distributions

This figure illustrates the effects of mixing gaussian distributions with different means and variances. Panel A on the left shows how a mixture of two conditional normal distributions can result in unconditional skewness, and Panel B depicts how three normal distributions can interact to generate excess kurtosis. In Panel A, the factor return can have the values $-0.06$ and $0.06$, and the asset’s standard deviation is $0.14$ in the low state and $0.06$ in the high state. In Panel B, the factor can take the values $-0.1$, $0$, or $0.1$, and the asset standard deviations are $0.12$ in the low and high state, and $0.06$ in the medium state.
Figure 3.3: Timeline

This figure shows the timing of events in the model described in Section 3.2. At $t = 0$, investors make their portfolio decisions and trade based on the unconditional distribution of returns. Then, the vector of state variables $X$ is realized and determines the mean and variance of the normal distribution from which the factor and the stock returns are to be drawn. No trading is allowed at this time. At $t = 1$, all uncertainty is resolved. Factor and asset returns are realized, determining investors final wealth.
Figure 3.4: Portfolio Weights in the Skewness Model

This figure shows the portfolio weights of the three investors as a function of the stochastic volatility induced skewness of asset two, $\Delta \Sigma_1(2, 2)$, as in Section 3.2.4.1. The model parameters are given in Table 3.1.
Figure 3.5: Stock Returns in the Skewness Model

This figure plots the asset returns (top), abnormal returns with respect to the CAPM (middle), as well as abnormal returns with respect to the 4-moment CAPM (bottom) of the three assets as a function of the stochastic volatility induced skewness of asset two, $\Delta_{\Sigma_1}(2, 2)$, as in Section 3.2.4.1. The model parameters are given in Table 3.1.
Figure 3.6: Portfolio Weights in the Kurtosis Model

This figure shows the portfolio weights of the three investors as a function of the stochastic volatility induced kurtosis of asset two, $\Delta \Sigma_2(2, 2)$, as in Section 3.2.4.2. The model parameters are given in Table 3.1.
Figure 3.7: Stock Returns in the Kurtosis Model

This figure plots the asset returns (top), abnormal returns with respect to the CAPM (middle), as well as abnormal returns with respect to the 4-moment CAPM (bottom) of the three assets as a function of the stochastic volatility induced kurtosis of asset two, $\Delta \Sigma^2(2,2)$, as in Section 3.2.4.2. The model parameters are given in Table 3.1.
Table 3.1: Parametrization for the Numerical Solution

This table summarized the parameters used in the numerical solution of the equilibrium with holding constraints in sections 3.2.4.1 (SVS) and 3.2.4.2 (SVK). Investors maximize their expected utility

$$EU(W) = EW - \frac{1}{2\phi_2} m_2(W) + \frac{1}{3\phi_3} m_3(W) - \frac{1}{12\phi_4} m_4^*(W)$$

subject to a budget constraint and the holding constraints $\omega_j(i) = 0$ for $i = j = 2$ and $i = j = 3$. Asset returns follow

$$R^e = \mu_e + \delta \mu b X_i + e_X$$
$$e_X \sim \mathcal{N}(0, S + \sum_{k=1}^{3} S_k X_k)$$

where the components of the state conditioned variance-covariance matrix are

$$S = \sigma^2 bb' + \Sigma, \ S_1 = \Delta \Sigma_1, \ S_2 = \delta \sigma bb' + \Delta \Sigma_2, \ and \ S_3 = \Delta \Sigma_3.$$
Table 3.2: Portfolios based on Stochastic Volatility Induced Skewness

This table reports average raw as well as risk adjusted monthly returns (%) of portfolios based on time-varying estimates of \( \beta^{SVS} = \text{Cov}\left(\delta_{i,t}^{RV}, R_{M,t}\right) / RV_{i,t} \), where \( \delta_{i,t}^{RV} \) are the shocks to and \( RV_{i,t} \) is the level of idiosyncratic volatility as described in Section 3.3.1. Specifically, I estimate \( \beta^{SVS} \) based on the previous 60 monthly observations, and hold the portfolios for one month. In Panel A, idiosyncratic volatility is defined with respect to the CAPM, and in Panel B with respect to the Fama-French three factor model. The sample period for the estimation starts in July 1926, portfolios are analyzed from July 1931 to August 2008, and the t-statistics reported in parenthesis are based on Newey-West adjusted standard errors using 6 lags.

<table>
<thead>
<tr>
<th>Panel A: CAPM Residuals</th>
<th>Panel B: Fama-French Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Low} ) 2 3 4 High ( \text{H - L} )</td>
<td>( \text{Low} ) 2 3 4 High ( \text{H - L} )</td>
</tr>
<tr>
<td>I. Raw Returns (% monthly)</td>
<td></td>
</tr>
<tr>
<td>0.62 (3.42)</td>
<td>0.61 (3.38)</td>
</tr>
<tr>
<td>0.67 (3.48)</td>
<td>0.68 (3.46)</td>
</tr>
<tr>
<td>0.77 (3.32)</td>
<td>0.71 (3.45)</td>
</tr>
<tr>
<td>0.77 (4.34)</td>
<td>0.71 (3.59)</td>
</tr>
<tr>
<td>0.15 (2.30)</td>
<td>0.77 (4.36)</td>
</tr>
<tr>
<td>II. CAPM</td>
<td></td>
</tr>
<tr>
<td>( \alpha ) 0.01 (0.22)</td>
<td>-0.01 (-0.17)</td>
</tr>
<tr>
<td>-0.03 (-0.66)</td>
<td>-0.03 (-0.68)</td>
</tr>
<tr>
<td>-0.03 (-0.63)</td>
<td>-0.01 (-0.12)</td>
</tr>
<tr>
<td>0.07 (1.54)</td>
<td>0.01 (0.27)</td>
</tr>
<tr>
<td>0.15 (3.46)</td>
<td>0.15 (3.86)</td>
</tr>
<tr>
<td>0.14 (2.07)</td>
<td>0.16 (2.43)</td>
</tr>
<tr>
<td>( \beta_{M} ) 0.94 (41.98)</td>
<td>0.95 (41.52)</td>
</tr>
<tr>
<td>1.08 (35.73)</td>
<td>1.09 (35.09)</td>
</tr>
<tr>
<td>1.08 (76.04)</td>
<td>1.10 (48.41)</td>
</tr>
<tr>
<td>1.07 (66.20)</td>
<td>1.06 (71.36)</td>
</tr>
<tr>
<td>0.96 (58.64)</td>
<td>0.94 (49.30)</td>
</tr>
<tr>
<td>0.02 (0.65)</td>
<td>0.00 (-0.10)</td>
</tr>
<tr>
<td>III. Fama-French 3-Factor Model</td>
<td></td>
</tr>
<tr>
<td>( \alpha ) -0.01 (-0.18)</td>
<td>-0.03 (-0.03)</td>
</tr>
<tr>
<td>-0.06 (-1.37)</td>
<td>-0.05 (-0.05)</td>
</tr>
<tr>
<td>-0.03 (-0.71)</td>
<td>-0.03 (-0.12)</td>
</tr>
<tr>
<td>0.05 (0.96)</td>
<td>0.00 (0.27)</td>
</tr>
<tr>
<td>0.14 (3.27)</td>
<td>0.15 (3.69)</td>
</tr>
<tr>
<td>0.15 (2.17)</td>
<td>0.18 (2.60)</td>
</tr>
<tr>
<td>( \beta_{M} ) 0.96 (76.56)</td>
<td>0.96 (70.63)</td>
</tr>
<tr>
<td>1.07 (42.94)</td>
<td>1.08 (41.50)</td>
</tr>
<tr>
<td>1.07 (71.39)</td>
<td>1.08 (63.47)</td>
</tr>
<tr>
<td>1.05 (81.28)</td>
<td>1.06 (95.48)</td>
</tr>
<tr>
<td>0.97 (55.23)</td>
<td>1.06 (47.10)</td>
</tr>
<tr>
<td>0.01 (0.43)</td>
<td>(-0.52)</td>
</tr>
<tr>
<td>( \beta_{SMB} ) -0.15 (-5.80)</td>
<td>-0.14 (-3.54)</td>
</tr>
<tr>
<td>-0.01 (-1.34)</td>
<td>0.05 (2.01)</td>
</tr>
<tr>
<td>0.04 (1.53)</td>
<td>0.05 (-0.37)</td>
</tr>
<tr>
<td>0.06 (2.09)</td>
<td>-0.03 (-1.31)</td>
</tr>
<tr>
<td>-0.05 (-2.34)</td>
<td>0.11 (4.27)</td>
</tr>
<tr>
<td>0.09 (3.44)</td>
<td>0.12 (5.09)</td>
</tr>
<tr>
<td>-0.01 (-0.54)</td>
<td>0.08 (2.88)</td>
</tr>
<tr>
<td>0.05 (1.84)</td>
<td>0.04 (1.36)</td>
</tr>
<tr>
<td>0.05 (1.54)</td>
<td>0.04 (1.54)</td>
</tr>
<tr>
<td>-0.06 (-1.35)</td>
<td>0.03 (0.79)</td>
</tr>
<tr>
<td>( \beta_{HML} ) 0.11 (4.21)</td>
<td>(5.09)</td>
</tr>
<tr>
<td>0.09 (3.44)</td>
<td>(2.88)</td>
</tr>
<tr>
<td>-0.01 (-0.54)</td>
<td>(1.36)</td>
</tr>
<tr>
<td>0.05 (1.84)</td>
<td>(1.54)</td>
</tr>
<tr>
<td>0.05 (1.54)</td>
<td>(0.79)</td>
</tr>
<tr>
<td>-0.06 (-1.35)</td>
<td>-0.09 (-2.13)</td>
</tr>
</tbody>
</table>
Table 3.3: Fama-French Alphas of Portfolios sorted on SVS and Holding Restrictions

This table reports monthly Fama-French Alphas (%) of independent double sorts based on $\beta_{SVS}$ and market capitalizations (top panel) as well as double sorts based on $\beta_{SVS}$ and the Amivest price impact measure (bottom panel). $\beta_{SVS}$ is estimated using the previous 60 monthly observations of shocks to stock volatility of individual stocks and market return realizations. Market Capitalization is measured in the December which is 7 to 18 months prior to the test month. The Amivest price impact is defined as the average of the ratio of dollar volume divided by absolute value of return in the calendar year preceding the month of analysis. The sample period for the estimation starts in July 1926, portfolios are analyzed from July 1931 to August 2008, and the t-statistics reported in parenthesis are based on Newey-West adjusted standard errors using 6 lags.

<table>
<thead>
<tr>
<th>Panel A: CAPM Residuals</th>
<th>Panel B: Fama-French Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I. Sorts on Market Capitalization and SVS</strong></td>
<td><strong>II. Sorts on Amivest Price Impact and SVS</strong></td>
</tr>
<tr>
<td><strong>Low</strong></td>
<td>2</td>
</tr>
<tr>
<td>Small</td>
<td>0.13</td>
</tr>
<tr>
<td>Big</td>
<td>-0.02</td>
</tr>
<tr>
<td>S - B</td>
<td>-0.14</td>
</tr>
<tr>
<td>Difference in Differences</td>
<td>0.24</td>
</tr>
<tr>
<td><strong>Low</strong></td>
<td>2</td>
</tr>
<tr>
<td>High</td>
<td>0.29</td>
</tr>
<tr>
<td>Low</td>
<td>-0.02</td>
</tr>
<tr>
<td>H - L</td>
<td>-0.31</td>
</tr>
<tr>
<td>Difference in Differences</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Table 3.4: Stochastic Volatility Induced Skewness and Coskewness Risk

This table reports estimated prices of risk for coskewness (Panel A) as well as the associated first stage risk loadings (Panel B). Test assets are the quintile portfolios based on $\beta^{SVS}$ as in Table 3.2, Panel B. The remaining panels report the coskewness risk loading of the portfolios sorted on SVS and size (Panel C) and Amivest price impact (Panel D), as in Table 3.3, Panel B. The sample period for the estimation starts in July 1926, portfolios are analyzed from July 1931 to August 2008.

<table>
<thead>
<tr>
<th>Panel A: Prices of Risk (% monthly)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
</tr>
<tr>
<td>0.32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: First Stage Coskewness Risk Loadings</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{R_M}$</td>
</tr>
<tr>
<td>Low</td>
</tr>
<tr>
<td>0.13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: Coskewness Risk of Size / SVS Portfolios</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
</tr>
<tr>
<td>Low</td>
</tr>
<tr>
<td>0.89</td>
</tr>
<tr>
<td>Big</td>
</tr>
<tr>
<td>Low</td>
</tr>
<tr>
<td>0.21</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D: Coskewness Risk of Amivest / SVS Portfolios</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
</tr>
<tr>
<td>Low</td>
</tr>
<tr>
<td>0.51</td>
</tr>
<tr>
<td>Low</td>
</tr>
<tr>
<td>0.24</td>
</tr>
</tbody>
</table>
Table 3.5: Characteristics of Portfolios based on Stochastic Volatility Induced Skewness

This table reports average characteristics of portfolios based on time-varying estimates of $\beta^{SVS}$. The characteristics shown are $\beta^{SVS}$, the average idiosyncratic volatility in the previous 60 month period, the conditional idiosyncratic volatility as implied by the $ARMA(p,q)$ model, as well as percentiles of market capitalization, Amivest Price Impact, the ratio of book value to market value, and the runup in returns over the previous 12 and 36 months. The sample period for the estimation starts in July 1926, and portfolios are analyzed from July 1931 to August 2008.

<table>
<thead>
<tr>
<th></th>
<th>Low</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{SVS}$</td>
<td>-0.23</td>
<td>-0.09</td>
<td>-0.03</td>
<td>0.03</td>
<td>0.13</td>
</tr>
<tr>
<td>Idio. Std. Dev. (avg.)</td>
<td>1.50</td>
<td>2.05</td>
<td>2.32</td>
<td>2.33</td>
<td>1.88</td>
</tr>
<tr>
<td>Idio. Std. Dev. (cond.)</td>
<td>1.72</td>
<td>2.38</td>
<td>2.73</td>
<td>2.78</td>
<td>2.26</td>
</tr>
<tr>
<td>ME Percentile</td>
<td>66.84</td>
<td>54.53</td>
<td>48.76</td>
<td>48.31</td>
<td>55.89</td>
</tr>
<tr>
<td>Amivest Percentile</td>
<td>62.42</td>
<td>51.85</td>
<td>46.90</td>
<td>46.90</td>
<td>53.30</td>
</tr>
<tr>
<td>B/M Percentile</td>
<td>51.02</td>
<td>50.60</td>
<td>52.37</td>
<td>54.19</td>
<td>55.84</td>
</tr>
<tr>
<td>MOM12 Percentile</td>
<td>51.07</td>
<td>50.16</td>
<td>50.00</td>
<td>49.95</td>
<td>51.49</td>
</tr>
<tr>
<td>MOM36 Percentile</td>
<td>52.32</td>
<td>49.98</td>
<td>49.29</td>
<td>48.94</td>
<td>51.18</td>
</tr>
</tbody>
</table>
Table 3.6: Portfolios based on Stochastic Volatility Induced Kurtosis

This table reports average raw as well as risk adjusted monthly returns (%) of portfolios based on time-varying estimates of \( \beta_{SVK} = \text{Cov} \left( \delta_{RV_i,t}^{RV}, \delta_{RV_M,t}^{RV} \right) / RV_{i,t}, \) where \( \delta_{RV_i,t}^{RV} \) are the shocks to and \( RV_{i,t} \) is the level of idiosyncratic volatility as described in Section 3.3.1, and \( \delta_{RV_M,t}^{RV} \) are shocks to the market volatility. Specifically, I estimate \( \beta_{SVK} \) based on the previous 60 monthly observations, and hold the portfolios for one month. In Panel A, idiosyncratic volatility is defined with respect to the CAPM, and in Panel B with respect to the Fama-French three factor model. The sample period for the estimation starts in July 1926, portfolios are analyzed from July 1931 to August 2008, and the t-statistics reported in parenthesis are based on Newey-West adjusted standard errors using 6 lags.

<table>
<thead>
<tr>
<th></th>
<th>Panel A: CAPM Residuals</th>
<th>Panel B: Fama-French Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>2</td>
</tr>
<tr>
<td>I. Raw Returns (% monthly)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.94</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>(4.03)</td>
<td>(3.72)</td>
</tr>
<tr>
<td>II. CAPM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.18</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>(2.22)</td>
<td>(1.15)</td>
</tr>
<tr>
<td>( \beta_M )</td>
<td>1.17</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>(36.60)</td>
<td>(50.01)</td>
</tr>
<tr>
<td>III. Fama-French 3-Factor Model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.06</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(0.84)</td>
<td>(0.46)</td>
</tr>
<tr>
<td>( \beta_M )</td>
<td>1.08</td>
<td>1.14</td>
</tr>
<tr>
<td></td>
<td>(52.13)</td>
<td>(51.59)</td>
</tr>
<tr>
<td>( \beta_{SMB} )</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>(6.26)</td>
<td>(6.12)</td>
</tr>
<tr>
<td>( \beta_{HML} )</td>
<td>0.23</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(4.83)</td>
<td>(0.89)</td>
</tr>
</tbody>
</table>
Table 3.7: Fama-French Alphas of Portfolios sorted on SVK and Holding Restrictions

This table reports monthly Fama-French Alphas (%) of independent double sorts based on $\beta^{SVK}$ and market capitalizations (top panel) as well as double sorts based on $\beta^{SVK}$ and the Amivest price impact measure (bottom panel). $\beta^{SVK}$ is estimated using the previous 60 monthly observations of shocks to stock volatility of individual stocks and shocks to market volatility. Market Capitalization is measured in the December which is 7 to 18 months prior to the test month. The Amivest price impact is defined as the average of the ratio of dollar volume divided by absolute value of return in the calendar year preceding the month of analysis. The sample period for the estimation starts in July 1926, portfolios are analyzed from July 1931 to August 2008, and the t-statistics reported in parenthesis are based on Newey-West adjusted standard errors using 6 lags.

<table>
<thead>
<tr>
<th></th>
<th>Panel A: CAPM Residuals</th>
<th></th>
<th></th>
<th></th>
<th>Panel B: Fama-French Residuals</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>High</td>
<td>H - L</td>
<td>Low</td>
<td>2</td>
</tr>
<tr>
<td>I. Sorts on Market Capitalization and SVK</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small</td>
<td>-0.04</td>
<td>-0.21</td>
<td>-0.03</td>
<td>0.02</td>
<td>0.12</td>
<td>0.16</td>
<td>-0.02</td>
<td>-0.21</td>
</tr>
<tr>
<td></td>
<td>(-0.49)</td>
<td>(-3.15)</td>
<td>(-0.42)</td>
<td>(2.01)</td>
<td>(1.40)</td>
<td></td>
<td>(-0.20)</td>
<td>(-3.26)</td>
</tr>
<tr>
<td>Big</td>
<td>0.07</td>
<td>0.01</td>
<td>0.00</td>
<td>0.02</td>
<td>0.03</td>
<td>-0.04</td>
<td>0.08</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(1.24)</td>
<td>(0.21)</td>
<td>(-0.12)</td>
<td>(0.41)</td>
<td>(0.65)</td>
<td>(-0.04)</td>
<td>(1.51)</td>
<td>(-0.19)</td>
</tr>
<tr>
<td>S - B</td>
<td>0.11</td>
<td>0.22</td>
<td>0.02</td>
<td>0.00</td>
<td>-0.09</td>
<td></td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>(1.22)</td>
<td>(2.71)</td>
<td>(0.30)</td>
<td>(-0.04)</td>
<td>(-1.50)</td>
<td></td>
<td>(1.10)</td>
<td>(2.63)</td>
</tr>
<tr>
<td>Difference in Differences</td>
<td>-0.20</td>
<td>(-0.83)</td>
<td></td>
<td></td>
<td>-0.17</td>
<td>(-0.83)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.69)</td>
<td>(-2.21)</td>
<td></td>
<td></td>
<td>(-1.49)</td>
<td>(-2.21)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II. Sorts on Amivest Price Impact and SVK</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>0.16</td>
<td>-0.11</td>
<td>0.05</td>
<td>0.19</td>
<td>0.40</td>
<td>0.25</td>
<td>0.14</td>
<td>-0.13</td>
</tr>
<tr>
<td></td>
<td>(1.50)</td>
<td>(-1.48)</td>
<td>(0.69)</td>
<td>(2.54)</td>
<td>(5.59)</td>
<td>(1.91)</td>
<td>(1.34)</td>
<td>(-1.56)</td>
</tr>
<tr>
<td>Low</td>
<td>0.07</td>
<td>-0.01</td>
<td>-0.05</td>
<td>-0.01</td>
<td>0.04</td>
<td>-0.03</td>
<td>0.09</td>
<td>-0.04</td>
</tr>
<tr>
<td></td>
<td>(1.25)</td>
<td>(-0.23)</td>
<td>(-1.27)</td>
<td>(0.89)</td>
<td>(-0.39)</td>
<td></td>
<td>(1.51)</td>
<td>(-0.93)</td>
</tr>
<tr>
<td>H - L</td>
<td>-0.08</td>
<td>0.10</td>
<td>-0.11</td>
<td>-0.20</td>
<td>-0.36</td>
<td></td>
<td>-0.05</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>(-0.83)</td>
<td>(1.12)</td>
<td>(-1.20)</td>
<td>(-2.51)</td>
<td>(-5.48)</td>
<td></td>
<td>(-0.47)</td>
<td>(0.88)</td>
</tr>
<tr>
<td>Difference in Differences</td>
<td>-0.28</td>
<td>(-0.83)</td>
<td></td>
<td></td>
<td>-0.30</td>
<td>(-0.83)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-2.21)</td>
<td>(-2.27)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Bibliography


95


Chapter 4

Conclusion

In this thesis, I examine theoretically and empirically the asset pricing implications of a volatility risk premium - a compensation required by investors if risk stochastically varies over time - in two different settings. The first essay studies the effects of persistent stochastic shocks to macroeconomic volatility on asset prices. The focus in this chapter is on the volatility of consumption growth, as in the Bansal and Yaron (2004) long-run-risks framework. In the second essay, I develop a model with a risk premium for the stock-specific component of volatility. I deviate from the common assumption of a representative agent by imposing portfolio constraints, which result in pricing of idiosyncratic volatility.

My first essay builds on the insight that when the volatility of consumption growth varies over time and the representative household has recursive preferences, the wealth-consumption ratio will not be constant and enters the pricing kernel as a second factor (Epstein and Zin (1989), Weil (1989)). In a model following Bansal and Yaron (2004) and Lettau et al. (2008), we identify innovations in beliefs about the conditional mean and volatility of consumption growth as important variables that affect the wealth-consumption ratio and thus asset prices. Using the estimated beliefs from the Markov model, we empirically test the pricing implications for the cross-section and time-series of stock returns. We find a negative price of consumption volatility risk at both the firm level as well as using standard sets of test portfolios, while the conditional mean of consumption growth has no significant effect. In the context of our model, these findings suggest an elasticity of intertemporal substitution greater than the inverse of the relative risk aversion for the representative agent. In time-series tests, we find that shocks to beliefs about the volatility state forecast the equity premium positively, thus indicating that the elasticity of intertemporal substitution is greater than unity.

The findings in this essay present a direct empirical test of long-run volatility risk, but there are important questions that we do not answer in this chapter. First, we derive how consumption volatility enters the pricing kernel, and thus how assets with different exposure to consumption volatility require different equilibrium returns. It would be equally interesting, and in many ways more fundamental, to understand the reasons why assets have
different exposures to consumption volatility. Second, the original contribution on long-run risks by Bansal and Yaron (2004) and subsequent empirical work focuses on persistent time variation in the mean consumption growth rate as theory suggests that the effect should be much stronger than the impact of consumption volatility. We confirm this in the numerical solution to the model, but empirically find no support for a priced first moment. This suggests that either the empirical process of consumption volatility assumed in this chapter is too simplistic, or the theoretical model needs to be changed to put more emphasis on second moments. Both these questions are inherently interested and are left for future research.

My second essay presents a first contribution to better understand the asset pricing implications of stochastic idiosyncratic volatility. It builds on stochastic time variation in idiosyncratic volatility documented in previous research, and derives conditions under which there is an idiosyncratic volatility risk premium.

For any non-systematic part of returns to affect pricing, it is necessary to deviate from the common assumption of a representative agent. In this essay, this is done by exogenously preventing investors from holding a well-diversified portfolio. A key insight is that asset returns exhibit one fat tale, or skewness, if state-dependent idiosyncratic volatility is correlated with market returns, even if the distribution of residuals conditional on the realized market return is normal. Similarly, a positive covariance between shocks to idiosyncratic volatility and the stochastic realizations of market volatility results in two fat tails, or excess kurtosis. Agents with commonly accepted skewness preference and kurtosis aversion will require lower returns in exchange for stochastic volatility induced skewness, and higher returns if they face stochastic volatility induced kurtosis.

The theoretical setup of the second essay follows an old literature by Levy (1978) and Merton (1987) that imposes trading frictions to generate investor heterogeneity, which in turn results in pricing of stock-specific risks. The trading frictions are a very particular case as heterogeneity can be modeled in many different ways, for example through endowments or preferences (Mitton and Vorkink (2007), Chabi-Yo (2009)). While the main empirical predictions for the pricing of idiosyncratic moments might be similar across different modeling choices, each approach will have specific implications that can be used to differentiate the models. We can thus use the pricing of idiosyncratic risk to identify strengths and weaknesses of heterogeneous agent models, which is beyond the scope of this thesis and left for future research.

\footnote{In the chapter, we show that two characteristics – market capitalization and book-to-market ratio – are unrelated to consumption volatility exposure.}

\footnote{See, for example, Ang et al. (2009), Fu (2009), and Huang et al. (2009).}
Bibliography


Appendix A

Appendix to Chapter 2

A.1 Wealth-Consumption Ratio Approximation

For the pricing of the return on the consumption claim, Euler equation (2.5) simplifies to

\[ PC_t^{\theta} = \mathbb{E}_t \left[ \beta^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} (PC_{t+1} + 1)^{\theta} \right] \]  
(A.1)

Based on the law of iterated expectations, equation (A.1) can be written as

\[ PC_t^{\theta} = \sum_{i=1}^{4} \xi_{t+1|t}(i) PC_{t,i}^{\theta} \]  
(A.2)

where \( \xi_{t+1|t}(i) \) is \( i \)-the element of \( \xi_{t+1|t} \) and

\[ PC_{t,i}^{\theta} = \mathbb{E} \left[ \beta^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} (PC_{t+1} + 1)^{\theta} \left| s_{t+1} = i, \xi_{t+1|t} \right. \right] \]  
(A.3)

Equation (A.2) says that the agent forms a belief-weighted average of the state- and belief-conditioned wealth-consumption ratios (A.3).

Given Equation (A.2), the local univariate approximation (2.8) of the wealth-consumption
The consumption data used in this paper are quarterly, per capita, real consumption of nondurable goods and services, seasonally adjusted at annual rates. Ferson and Harvey (1992) investigate the asset pricing implications of consumption growth rates obtained from data that are seasonally adjusted with the X-12-ARIMA filter. Ex-ante, the impact of the filter on latent volatility regimes is not obvious.

To measure the impact of the X-12-ARIMA filter, we simulate 300 time series of 200 quarterly log consumption growth rates generated by the Markov model estimated in Table 2.1. We then perturb every fourth quarter data point by +5% and every first quarter data point by -5%, which approximately generates the seasonality observed in the discontinued series for seasonally unadjusted quarterly consumption data. We transform the seasonally perturbed series into consumption levels and apply the X-12-ARIMA filter to get seasonally adjusted consumption data. We lastly estimate the four-state Markov model of Section 2.1 on both the original and the seasonally adjusted data.

Table A.1 shows summary statistics of the estimated Markov chain parameters. We observe that the seasonal adjustment has negligible influence on the estimated states and state-transitions. Moreover, the median correlation between beliefs over states estimated from the original and the filtered data is very high (0.98 for the mean, 0.90 for the standard deviation state). We conclude that the Markov model is robust to the X-12-ARIMA filter.

\[ \Delta p_{c_t+1} = \frac{1}{\theta} \ln \left( \frac{b_t \Delta b_{t+1} + (1 - b_t) \Delta b_{t+1}}{b_t \theta + (1 - b_t) \theta} \right) \]

\[ = \frac{1}{\theta} \ln \left( \frac{(b_t + \Delta b_{t+1}) + (1 - (b_t + \Delta b_{t+1})) \theta}{b_t \theta + (1 - b_t) \theta} \right) \]

\[ = \frac{1}{\theta} \ln \left( \frac{b_t \theta + (1 - b_t) \theta}{b_t \theta + (1 - b_t) \theta} \right) \]

\[ = \frac{1}{\theta} \ln \left( 1 + \frac{\Delta b_{t+1} (PC_{t+1,1} - PC_{t+1,1})}{b_t \theta + (1 - b_t) \theta} \right) \]

\[ \approx \frac{1}{\theta} \Delta b_{t+1} \frac{PC_{1} - PC_{2}}{b_t \theta + (1 - b_t) \theta} \]

### A.2 Seasonal Adjustment

The consumption data used in this paper are quarterly, per capita, real consumption of nondurable goods and services, seasonally adjusted at annual rates. Ferson and Harvey (1992) investigate the asset pricing implications of consumption growth rates obtained from data that are seasonally adjusted with the X-12-ARIMA filter. Ex-ante, the impact of the filter on latent volatility regimes is not obvious.

To measure the impact of the X-12-ARIMA filter, we simulate 300 time series of 200 quarterly log consumption growth rates generated by the Markov model estimated in Table 2.1. We then perturb every fourth quarter data point by +5% and every first quarter data point by -5%, which approximately generates the seasonality observed in the discontinued series for seasonally unadjusted quarterly consumption data. We transform the seasonally perturbed series into consumption levels and apply the X-12-ARIMA filter to get seasonally adjusted consumption data. We lastly estimate the four-state Markov model of Section 2.1 on both the original and the seasonally adjusted data.

Table A.1 shows summary statistics of the estimated Markov chain parameters. We observe that the seasonal adjustment has negligible influence on the estimated states and state-transitions. Moreover, the median correlation between beliefs over states estimated from the original and the filtered data is very high (0.98 for the mean, 0.90 for the standard deviation state). We conclude that the Markov model is robust to the X-12-ARIMA filter.

---

37X-12-ARIMA is a seasonal adjustment program developed at the U.S. Census Bureau. The program is based on the Bureau’s earlier X-11 program and the X-11-ARIMA program developed at Statistics Canada.
A.3 Numerical Solution

Using Equation (A.2), the wealth-consumption ratio, \( PC_t = PC(\xi_{t+1|t}) \), solves the following functional equation

\[
PC(\xi_{t+1|t}) = \left( \sum_{i=1}^{4} \xi_{t+1|t}(i) \mathbb{E} \left[ \beta^\theta (PC(\xi_{t+2|t+1}) + 1)^\theta (e^{\mu_{i,t+1}} + \sigma_{i,t+1}^2)^{1-\gamma} \mid s_{t+1} = i \right] \right)^{1/\theta}
\]

where \( \xi_{t+1|t}(i) \) is the element of \( \xi_{t+1|t} \). We solve this equation as a fixed-point in the wealth-consumption ratio. The grid for the belief state-vector has increments of size 0.025 and the expectation is approximated using Gauss-Hermite quadrature with 10 nodes. Three-dimensional linear interpolation is used between grid points.

A.4 Cross-Sectional Asset Pricing Implications

For our empirical exercise, we assume that the log wealth-consumption ratio is approximately affine in in the perceived first and second moments of consumption growth

\[
\log(\text{pc}_t) = k + A\hat{\mu}_t + B\hat{\sigma}_t
\]

This step provides a more meaningful economic interpretation for mean and volatility states. In Table A.2, we confirm the quality of this approximation based on simulations of the model. We simulate 300 economies for 100 years at quarterly frequency. In all three panels, the representative agent has an EIS of 1.5 and a rate of time preference of 0.995. The coefficient of relative risk aversion (RRA) increases from 10 (Panel A) to 20 (Panel B) and 30 (Panel C). In the first regression of each panel, we regress the log wealth-consumption ratio, \( \text{pc}_t \), on the prior probabilities of being in a given state, \( \xi_{t+1|t}(i) \), \( i = 1, 2, 3 \). In the second regression, we regress the log wealth-consumption ratio, \( \text{pc}_t \), on the perceived first, \( \hat{\mu}_t \), and second moment, \( \hat{\sigma}_t \), of consumption growth. We report the (across simulation) average regression coefficient and regression \( R^2 \).

Equation (A.2) states that variations in the wealth-consumption ratio depend on the beliefs about four states, three of which are linearly independent. In an exact implementation of the model, the wealth-consumption ratio is thus a nonlinear function of three variables. The first regression of each panel confirms that the log wealth-consumption ratio is approximately affine in the prior probabilities about the state with regression \( R^2 \) exceeding 99%.

The second regression of each panel confirms that the log wealth-consumption ratio is
approximately affine in in the perceived first and second moments of consumption growth. This approximation captures most variation of changes in the wealth-consumption ratio with regression $R^2$ exceeding 99%. Intuitively, the third prior probability captures the perceived comovement between the Markov chains for mean and volatility. However, since these two Markov chains are independent by assumption, the third prior probability is redundant.

In order to test the model in the cross-section of returns, it is convenient to restate the fundamental asset pricing equation (2.5) in terms of betas

$$
\mathbb{E}_t[R_{i,t+1}] \approx -\text{Cov}_t(R_{i,t+1}, m_{t+1})
= \gamma \text{Cov}_t(R_{i,t+1}, \Delta c_{t+1}) + (1 - \theta) \text{Cov}_t(R_{i,t+1}, \Delta p_{c,t+1})
= \gamma \text{Cov}_t(R_{i,t+1}, \Delta c_{t+1}) + (1 - \theta) A \text{Cov}_t(R_{i,t+1}, \Delta \hat{\mu}_{t+1}) + (1 - \theta) B \text{Cov}_t(R_{i,t+1}, \Delta \hat{\sigma}_{t+1})
= \beta_{c,t}^i \lambda_{c,t} + \beta_{\mu,t}^i \lambda_{\mu,t} + \beta_{\sigma,t}^i \lambda_{\sigma,t}
$$

with

$$
\beta_{c,t} = \frac{\text{Cov}_t(R_{i,t+1}, \Delta c_{t+1})}{\text{Var}_t(\Delta c_{t+1})}
\beta_{\mu,t} = \frac{\text{Cov}_t(R_{i,t+1}, \Delta \hat{\mu}_{t+1})}{\text{Var}_t(\Delta \hat{\mu}_{t+1})}
\beta_{\sigma,t} = \frac{\text{Cov}_t(R_{i,t+1}, \Delta \hat{\sigma}_{t+1})}{\text{Var}_t(\Delta \hat{\sigma}_{t+1})}
$$

and

$$
\lambda_{c,t} = \gamma \text{Var}_t(\Delta c_{t+1})
\lambda_{\mu,t} = A(1 - \theta) \text{Var}_t(\Delta \hat{\mu}_{t+1})
\lambda_{\sigma,t} = B(1 - \theta) \text{Var}_t(\Delta \hat{\sigma}_{t+1})
$$

where $\beta_{c,t}^i, \beta_{\mu,t}^i, \beta_{\sigma,t}^i$ denote risk loadings of asset $i$ at date $t$ with respect to consumption growth and the conditional first and second moments of consumption growth, $\lambda_{c,t}, \lambda_{\mu,t}, \lambda_{\sigma,t}$ are the respective market prices of risk.

### A.5 Equity Premium

To quantify the equity premium generated by our model, we first have to specify a process for dividend growth. A common approach is to postulate a levered consumption process for dividends such as $D = C^\lambda$. The Markov switching model allows a more general approach by fitting a Markov model for the conditional first and second moments of dividend growth. Specifically, we assume that log dividend growth follows

$$
\Delta d_{t+1} = \mu_t^d + \sigma_t^d \epsilon_{t+1}
\epsilon_{t+1} \sim \mathcal{N}(0, 1)
$$
where $\mu^d_t \in \{\mu^d_l, \mu^d_h\}$ and $\sigma^d_t \in \{\sigma^d_l, \sigma^d_h\}$ follow the same Markov process as consumption. Consequently, we do not re-estimate the transition matrix of the Markov process but use the estimates reported in Table 2.1. We compute quarterly dividends for the period 1955-2008 using the value-weighted CRSP index with and without distributions. Parameter estimates are summarized in Table A.3.

In Table A.4, we report statistics about the risky and risk-free asset. We simulate 300 economies for 100 years at quarterly frequency. In all three panels, the representative agent has an EIS of 1.5 and rate of time preference of 0.995. The coefficient of relative risk aversion increases from 10 (Panel A) to 20 (Panel B) and 30 (Panel C). We report the average excess return, $E[R^e]$, the standard deviation of stock returns, $\sigma[R]$, the average risk-free rate, $E[R^f]$, and the standard deviation of the risk-free rate, $\sigma[R^f]$. In the last row of each panel, we also report moments of the the Markov switching model without learning where the agent knows the state of the economy.

In the specification with RRA of 10, the model generates an annual risk premium of 1%, stock return volatility of 6%, an average risk-free rate of 3% and risk-free rate volatility of 0.3%. This poor performance is not surprising since the Markov chain is not very persistent compared to the specification of Bansal and Yaron (2004). For RRA of 30, the model generates a risk premium of 3.6%.

### A.6 Learning Premium

The Euler equation for the return on wealth is given by

$$1 = \beta^\theta E_t \left[ \left( \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]$$

Without learning, it follows that

$$\beta^\theta E_t \left[ \left( \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] = \beta^\theta E_t \left[ \left( \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta} \right] E_t \left[ \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]$$

since the Markov switches are independent from the normal shocks to consumption growth. In a model with learning, however, the wealth-consumption ratio is correlated with consumption growth, i.e.,

$$\text{Cov}_t \left( \frac{PC_{t+1} + 1}{PC_t}, \frac{C_{t+1}}{C_t} \right) \neq 0$$
since the agent cannot differentiate whether the change in consumption growth comes from the Markov chain or the normal innovation. We call this covariance the learning premium.

To quantify the learning premium, it is convenient to log linearize returns and the pricing kernel. In a log-linear world, the expected excess return on wealth has to satisfy

\[
E_t[R_{w,t+1}] \approx -Cov_t(r_{w,t+1}, m_{t+1})
\]

where the log return on wealth can be approximated by

\[
r_{w,t+1} \approx \Delta pc_{t+1} + \Delta c_{t+1}
\]

and the log pricing kernel by

\[
m_{t+1} \approx (\theta \ln \beta - (1 - \theta)(k_0 - k_1 z_t)) - \gamma \Delta c_{t+1} - (1 - \theta)\Delta pc_{t+1}
\]

By substituting these two approximation into the expression for the expected excess return on wealth, one obtains

\[
E_t[R_{w,t+1}] \approx -Cov_t(r_{w,t+1}, m_{t+1}) \approx -Cov_t(\Delta pc_{t+1} + \Delta c_{t+1}, -\gamma \Delta c_{t+1} - (1 - \theta)\Delta pc_{t+1}) = \gamma Var_t(\Delta c_{t+1}) + (1 - \theta)Var_t(\Delta pc_{t+1}) + (1 - \theta + \gamma)Cov_t(\Delta c_{t+1}, \Delta pc_{t+1})
\]

Hence, the risk premium in the full model has three components: a short-run, a long-run and a learning premium. The short-run component arises in a model with i.i.d. consumption and power utility. For the long-run component to be non-zero, the model has to contain persistent shocks and the agent has to care about the temporal resolution of risk, i.e., \( \theta \neq 1 \). The learning premium arises because shocks to consumption growth also lead the agent to update her beliefs about states.

Table A.4 can be used to quantify the importance of the learning premium. In the last row of each panel, we also report moments of the the Markov switching model without learning where the agent knows the state of the economy. The difference between the mean excess return generated by the full model and the model without learning is the learning premium. Holding the EIS fixed at 1.5, for RRA of 10 (Panel A), the learning premium is only 7 basis points; for RRA of 20 (Panel B), the learning premium increases to 44 basis points; and for RRA of 30 (Panel C), the learning premium reaches 88 basis points. So the fraction of the
total excess return coming from learning increases from 7% to 19% to 24%.

A.7 Predictive Regression Bias

It is well known that parameter estimates and $t$-statistics are potentially biased in predictive regressions. Hodrick (1992) shows that using overlapping observations leads to biased inference. More importantly, when the predictor variable is persistent and its innovations are correlated with future returns, Stambaugh (1999), Lewellen (2004), Boudoukh et al. (2006) and Ang and Bekaert (2007) show that standard econometric techniques can be misleading. When price ratios are used as predictors, this bias shows up strongly and conventional tests will reject the null hypothesis too frequently. To gain a better understanding, consider the following setup

$$
{r}_t = \alpha + \beta {x}_{t-1} + \epsilon_r^t \\
{x}_t = \phi + \rho {x}_{t-1} + \epsilon_x^t
$$

where $r_t$ denotes returns and $x_t$ a predictor variable. Lewellen (2004) shows that $\beta$ estimates are biased by $\gamma(\hat{\rho} - \rho)$ where $\gamma = \text{Cov}(\epsilon_r^t, \epsilon_x^t)/\text{Var}(\epsilon_x^t)$ when $\epsilon_r^t$ is correlated with $x_t$. When the dividend yield is used as predictor, for instance, an increase in price leads to a positive realized return as well as a decrease in the dividend yield. Consequently, $\epsilon_r^t$ is correlated with $x_t$. Lewellen (2004) reports an auto-correlation of 0.997 and $\text{Corr}(\epsilon_r^t, \epsilon_x^t) = -0.96$ for the dividend yield as predictor, invaliding standard estimates and tests. For the variable $\Delta \hat{\sigma}_t$, this bias is less of a concern since it is not a price scaled variable. For our one period forecasts, we estimate $\text{Corr}(\Delta \hat{\sigma}_t, \Delta \hat{\sigma}_{t-1}) = 0.006$ and $\text{Corr}(\Delta \hat{\sigma}_t, \epsilon_t) = 0.032$, which is too small to bias statistical inference.
Table A.1: Effect of X-12-ARIMA Filter

We simulate 300 time-series of 200 quarterly log consumption growth rates generated from the Markov model estimated in Table 2.1. We then perturb every fourth quarter data point by +5% and every first quarter data point by -5%, which approximately generates the seasonality observed in the discontinued series for seasonally unadjusted quarterly consumption data. We transform the seasonally perturbed series into consumption levels and apply the X-12-ARIMA filter to get seasonally adjusted consumption data. We lastly estimate the four-state Markov model of Section 2.1 on both the original and the seasonally adjusted data.

<table>
<thead>
<tr>
<th>Panel A: Summary Statistics for estimates of undisturbed and X-12 data</th>
<th>( \mu_l )</th>
<th>( \mu_h )</th>
<th>( \sigma_l )</th>
<th>( \sigma_h )</th>
<th>( p_{\mu}^{ll} )</th>
<th>( p_{\mu}^{lh} )</th>
<th>( p_{\sigma}^{ll} )</th>
<th>( p_{\sigma}^{lh} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.36</td>
<td>0.79</td>
<td>0.20</td>
<td>0.49</td>
<td>92.26</td>
<td>93.83</td>
<td>92.39</td>
<td>93.77</td>
</tr>
<tr>
<td>Mean X-12</td>
<td>0.37</td>
<td>0.80</td>
<td>0.19</td>
<td>0.47</td>
<td>91.94</td>
<td>93.52</td>
<td>90.36</td>
<td>89.96</td>
</tr>
<tr>
<td>Median</td>
<td>0.36</td>
<td>0.79</td>
<td>0.20</td>
<td>0.49</td>
<td>94.11</td>
<td>95.67</td>
<td>94.76</td>
<td>95.95</td>
</tr>
<tr>
<td>Median X-12</td>
<td>0.36</td>
<td>0.79</td>
<td>0.18</td>
<td>0.46</td>
<td>93.38</td>
<td>94.90</td>
<td>93.64</td>
<td>93.47</td>
</tr>
<tr>
<td>SD</td>
<td>0.08</td>
<td>0.06</td>
<td>0.04</td>
<td>0.06</td>
<td>8.18</td>
<td>7.98</td>
<td>9.07</td>
<td>6.91</td>
</tr>
<tr>
<td>SD X-12</td>
<td>0.08</td>
<td>0.05</td>
<td>0.04</td>
<td>0.10</td>
<td>6.38</td>
<td>5.09</td>
<td>12.40</td>
<td>11.70</td>
</tr>
<tr>
<td>5th Percent</td>
<td>0.28</td>
<td>0.71</td>
<td>0.15</td>
<td>0.41</td>
<td>77.67</td>
<td>85.49</td>
<td>78.57</td>
<td>79.93</td>
</tr>
<tr>
<td>5th Percent X-12</td>
<td>0.27</td>
<td>0.72</td>
<td>0.13</td>
<td>0.38</td>
<td>79.36</td>
<td>84.83</td>
<td>66.38</td>
<td>67.92</td>
</tr>
<tr>
<td>95th Percent</td>
<td>0.47</td>
<td>0.87</td>
<td>0.27</td>
<td>0.57</td>
<td>98.30</td>
<td>98.91</td>
<td>98.75</td>
<td>99.19</td>
</tr>
<tr>
<td>95th Percent X-12</td>
<td>0.48</td>
<td>0.88</td>
<td>0.24</td>
<td>0.57</td>
<td>97.89</td>
<td>98.66</td>
<td>98.60</td>
<td>99.01</td>
</tr>
</tbody>
</table>

Panel B: Median Correlations of beliefs from undisturbed and X-12 data

\[ \rho_\mu = 0.98 \quad \rho_\sigma = 0.90 \]
We simulate 300 economies for 100 years at quarterly frequency. In all three panels, the representative agent has an EIS of 1.5 and rate of time preference of 0.995. The coefficient of relative risk aversion (RRA) increases from 10 (Panel A) to 20 (Panel B) and 30 (Panel C). In the first regression of each panel, we regress the log wealth-consumption ratio, $pc_t$, on the prior probabilities of being in a given state, $\xi_{t+1,i}(i)$, $i = 1, 2, 3$. In the second regression, we regress the log wealth-consumption ratio, $pc_t$, on the perceived first, $\hat{\mu}_t$, and second moment, $\hat{\sigma}_t$, of consumption growth. We report the average regression coefficient and average $R^2$.

<table>
<thead>
<tr>
<th>Const.</th>
<th>$\xi(1)$</th>
<th>$\xi(2)$</th>
<th>$\xi(3)$</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: RRA=10, EIS=1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.7172</td>
<td>-0.0056</td>
<td>-0.0074</td>
<td>0.0009</td>
<td>0.0009</td>
<td>0.9973</td>
<td></td>
</tr>
<tr>
<td>5.7115</td>
<td></td>
<td></td>
<td></td>
<td>0.0069</td>
<td>-0.0015</td>
<td>0.9948</td>
</tr>
<tr>
<td>Panel B: RRA=20, EIS=1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.6922</td>
<td>-0.0050</td>
<td>-0.0071</td>
<td>0.0012</td>
<td>0.0012</td>
<td>0.9957</td>
<td></td>
</tr>
<tr>
<td>5.6870</td>
<td></td>
<td></td>
<td></td>
<td>0.0067</td>
<td>-0.0018</td>
<td>0.9933</td>
</tr>
<tr>
<td>Panel C: RRA=30, EIS=1.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.6696</td>
<td>-0.0044</td>
<td>-0.0068</td>
<td>0.0016</td>
<td>0.0016</td>
<td>0.9937</td>
<td></td>
</tr>
<tr>
<td>5.6650</td>
<td></td>
<td></td>
<td></td>
<td>0.0064</td>
<td>-0.0021</td>
<td>0.9914</td>
</tr>
</tbody>
</table>
Table A.3: Markov Model of Dividend Growth

This table reports parameter estimates of the Markov model for log dividend growth

\[ \Delta d_{t+1} = \mu_d^d + \sigma_d^d \epsilon_{t+1} \quad \epsilon_t \sim N(0, 1) \]

where \( \mu_d^d \in \{ \mu_{d_l}^d, \mu_{d_h}^d \} \) and \( \sigma_d^d \in \{ \sigma_{d_l}^d, \sigma_{d_h}^d \} \) follow independent Markov processes with transition matrices \( P_{\mu} \) and \( P_{\sigma} \), respectively. The consumption and dividend process follow the same Markov switching process as reported in Table I. We compute quarterly dividends for the period 1955-2008 using the value-weighted CRSP index with and without distributions. Standard errors are reported in parentheses.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{d_l}^d )</td>
<td>-0.4549 (0.1654)</td>
</tr>
<tr>
<td>( \mu_{d_h}^d )</td>
<td>1.4378 (0.2091)</td>
</tr>
<tr>
<td>( \sigma_{d_l}^d )</td>
<td>1.1747 (0.0838)</td>
</tr>
<tr>
<td>( \sigma_{d_h}^d )</td>
<td>3.4576 (0.4980)</td>
</tr>
</tbody>
</table>
Table A.4: Model Implications

We simulate 300 economies for 100 years at quarterly frequency. In all three panels, the representative agent has an EIS of 1.5 and rate of time preference of 0.995. The coefficient of relative risk aversion (RRA) increases from 10 (Panel A) to 20 (Panel B) and 30 (Panel C). We report the average excess return, $\mathbb{E}[R^e]$, the standard deviation of stock returns, $\sigma[R^e]$, the average risk-free rate, $\mathbb{E}[R^f]$, and the standard deviation of the risk-free rate, $\sigma[R^f]$. In the last row of each panel, we also report moments of the Markov switching model without learning where the agent knows the state of the economy.

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{E}[R^e]$</th>
<th>$\sigma[R^e]$</th>
<th>$\mathbb{E}[R^f]$</th>
<th>$\sigma[R^f]$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: RRA=10, EIS=1.5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0105</td>
<td>0.0599</td>
<td>0.0315</td>
<td>0.0032</td>
</tr>
<tr>
<td>25%</td>
<td>0.0058</td>
<td>0.0563</td>
<td>0.0307</td>
<td>0.0030</td>
</tr>
<tr>
<td>75%</td>
<td>0.0150</td>
<td>0.0634</td>
<td>0.0323</td>
<td>0.0034</td>
</tr>
<tr>
<td>No Learning</td>
<td>0.0098</td>
<td>0.0719</td>
<td>0.0336</td>
<td>0.0027</td>
</tr>
<tr>
<td><strong>Panel B: RRA=20, EIS=1.5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0235</td>
<td>0.0560</td>
<td>0.0274</td>
<td>0.0053</td>
</tr>
<tr>
<td>25%</td>
<td>0.0182</td>
<td>0.0523</td>
<td>0.0264</td>
<td>0.0049</td>
</tr>
<tr>
<td>75%</td>
<td>0.0285</td>
<td>0.0594</td>
<td>0.0286</td>
<td>0.0058</td>
</tr>
<tr>
<td>No Learning</td>
<td>0.0191</td>
<td>0.0700</td>
<td>0.0329</td>
<td>0.0026</td>
</tr>
<tr>
<td><strong>Panel C: RRA=30, EIS=1.5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.0361</td>
<td>0.0530</td>
<td>0.0241</td>
<td>0.0071</td>
</tr>
<tr>
<td>25%</td>
<td>0.0309</td>
<td>0.0492</td>
<td>0.0225</td>
<td>0.0064</td>
</tr>
<tr>
<td>75%</td>
<td>0.0413</td>
<td>0.0566</td>
<td>0.0255</td>
<td>0.0079</td>
</tr>
<tr>
<td>No Learning</td>
<td>0.0273</td>
<td>0.0685</td>
<td>0.0322</td>
<td>0.0026</td>
</tr>
</tbody>
</table>
Bibliography


