Group Actions on Finite Homotopy Spheres

by

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Abstract

Recently, Grodal and Smith [7] have developed a finite algebraic model to study hG-spaces where G is a finite group. The procedure associates to each G-space X with finite $\mathbb{F}_p$ homology a perfect chain complex of functors over the orbit category. When X has the homotopy type of a sphere, this construction is particularly well behaved. The reverse construction, building an hG-space from the algebraic model, generally produces an infinite dimensional space.

In this thesis, we construct a finiteness obstruction for hG-spheres working one prime at a time. We then begin the development of a global finiteness obstruction. When G is the metacyclic group of order pq, we are able to go further and express the global finiteness obstruction in terms of dimension functions. In addition, we relate the work of tom Dieck and Petrie [19] concerning homotopy representations to the newer model of Grodal and Smith, and compute the rank of $V_w(G)$. We conclude with some new examples of finite $h\Sigma_3$-spheres.

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Introduction

The study of group actions on spheres has long been an interesting and deep pursuit. The first results in this area, known as the Spherical Space Form Problem, were due to P. Smith [15] and later J. Milnor [13]. They gave several necessary conditions for a finite group G to act freely and smoothly on some sphere. In the 1960s, R. Swan [16] achieved his celebrated result, the classification of finite groups acting freely on spheres up to homotopy. These are the groups whose cohomology is periodic. Although at first glance this problem appears very geometric, many insights into this question are algebraic in nature.

In the early 80s, tom Dieck and Petrie introduced the notion of a homotopy representation in [19]. Given a finite group G, a homotopy representation of G is a G-CW complex X which is homotopy equivalent to a sphere such that the fixed points, X^H, also have the homotopy type of a sphere for every subgroup H of G. This allows one to define a dimension function associated to X: namely $\mathrm{Dim}(X)(H) = \dim X^H + 1$. The collection of homotopy representations forms a monoid, denoted by $V^+(G)$. There are many results known about the group completed object, $V(G)$, (see, for example [19], [18], [1]) but little is known about the monoid.

There are two natural notions of what is meant by a homotopy equivalence of G-spaces: the usual notion of G-homotopy equivalence and the weaker notion of hG-homotopy equivalence. Two spaces are said to be hG-equivalent if their Borel constructions are homotopy equivalent over BG. Equivalently, a G-space X is hG-equivalent to Y if X can be connected to Y by a zig-zag of G-maps that are homotopy equivalences. Notice that this notion of 'equivariant homotopy equivalence' is weaker that the usual notion of G-homotopy equivalence.

Recently, Grodal and Smith [7] have studied a monoid structure similar to that of tom Dieck and Petrie on the collection of hG-spheres, denoted by $V_w^+(G)$. Although there is no requirement that the fixed points be spheres, we still have a dimension function defined on the set of subgroups whose order is a power of a prime number: by homotopy Smith Theory [5], we know that the homotopy fixed points have the mod-p homology of a sphere for any p-subgroup, $P < G$, where p divides the order of G. In [7], the authors show that these hG-spheres are classified via their dimension functions and that any admissible dimension function is realizable on some hG-sphere.

This thesis began as an attempt to answer the following question:

Question 1.0.1. *Given a G-space X, when does there exist a finite dimensional (respectively finite) G-space Y that is hG-equivalent to X?*

As a starting point for this investigation, we might try to answer the above question when X has the homotopy type of a sphere. In this setting, i.e. working with hG-equivalences, restricting our attention to spheres is particularly useful as homotopy Smith Theory guarantees that the P fixed points have the mod-p homology of a sphere as well. In addition, the theory of Grodal and Smith [7] gives an algebraic model of the hG-action: every hG-sphere gives rise to a finite dimensional chain complex of finitely generated projective functors over the orbit category. This chain complex is a natural place to look for a finiteness obstruction. The question is now easier, and can be stated as follows:

Question 1.0.2. *Given a G-space X, such that X has the homotopy type of a sphere, when does there exist a finite dimensional (respectively finite) G-space Y, such that X and Y are hG-equivalent?*

Since the construction of Grodal and Smith [7] produces a complex over the p-orbit category for each prime p dividing the order of G, we further reduce the problem by studying X one prime at a time. Thus the new question is one of mod-p finite replacement:

Question 1.0.3. *Given a G-space X, such that X has the homotopy type of a sphere, when does there exist a finite dimensional (respectively finite) space Y_p, such that X and Y_p are connected by a zig-zag of mod-p G-maps that are homotopy equivalences?*

In an attempt to try to answer the last two questions, it is useful to understand the results about homotopy representations. Indeed, although much has been determined virtually, it would be interesting to determine a non-virtual finiteness obstruction. Since homotopy representations are also hG-spheres, finding necessary conditions would be relevant. In addition, it is interesting to determine the rank of the groups $V(G)$ and $V_w(G)$, the group completions of $V^+(G)$ and $V_w^+(G)$ respectively. Although this has been done for $V(G)$ in [19], we give a different proof, which allows us to write down a basis for $V(G)$ in terms of dimension functions.

Theorem 3.1.3 (tom Dieck-Petrie). *The rank of $V(G)$ is equal to the number of conjugacy classes of subgroups of G that are cyclic after abelianization, i.e.*

$$\mathrm{rk}(V(G)) = |\{H \in \varphi(G) | H/H' \text{ is cyclic}\}|.$$

In any discussion involving group actions and fixed points, whether working with G-equivalences or hG-equivalences, it is often useful to consider diagrams over the orbit category. In the former case, we work over the entire orbit category, using all subgroups; in the latter case we work with a diagram over each p-orbit category, only using the p-subgroups, for each prime p dividing the order of G. With this in mind, we give a version of Wall's finiteness obstruction for CW-complexes [20], generalized to an arbitrary diagram category.

Theorem 4.1.2. *Let R be a ring, Γ a small category, $R\Gamma$-mod the category of contravariant functors $\Gamma \to R$-mod, and D a Γ-CW-complex (a diagram of CW-complexes with the shape of Γ) whose fundamental category $\Pi(\Gamma, D)$ is trivial. Moreover, suppose that D satisfies the following* *additional properties:*

i. $H_*(D) = 0$ *for all* $* > n$, *and*
ii. $H^{n+1}(D; M) = 0$ *for all* $R\Gamma$*-modules* M.

Then D *is equivalent to a finite dimensional* I*-CW-complex* K. *Furthermore, if* K *has trivial finiteness obstruction,* $\sigma(K) \in \widetilde{K}_0(R\Gamma)$, *then* K *may be assumed to be finite.*

When working with G-spaces, Γ is the full orbit category. In this setting, via a theorem of Elmendorf [6], we recover the usual equivariant finiteness obstruction. However there is no such correspondence when working with hG-equivalences. Nevertheless, when working with spheres, and at a fixed prime p, it is possible to modify the above theorem to produce a finite complex that has the same mod-p hG-homotopy type. The obstruction lies in $\widetilde{K}_0(R_p\Gamma_p)$ modulo a particular ideal. Unlike the case for CW-complexes, the homological dimension and the topological dimension are not necessarily the same.

Theorem 4.2.12. *Let* X *be a* G*-space with the homotopy type of a sphere whose* p*-fixed points are at least 3-connected. Then* X *is* hG*-equivalent to a finite dimensional complex. Furthermore, if* $\sigma_p(X) = 0$, *then there is a finite space* Y *that is* hG*-equivalent to* X *at the prime* p, *where* p *divides the order of* G. *In particular the homology of* Y *at* p*-groups agrees with that of* X.

Our goal is to classify finite hG-spheres globally. It would be even more satisfying to classify them directly, instead of with a $\widetilde{K}_0$-type obstruction. One approach to this problem is to examine conditions on the dimension functions of finite hG-spheres. Recall that an hG-sphere X has a dimension function $\mathrm{Dim}(X)(-)$, given by $\mathrm{Dim}(X)(P) = \dim X^{hP} + 1$, where P is a p-group of G. A result in this direction is the following:

Theorem 4.3.2. *Let* G *be the metacyclic group* $G = C_p \rtimes C_q$ *for distinct primes* p *and* q, *with* q *dividing* $p - 1$. *An* hG*-sphere* X *is realizable on* *a finite CW complex if and only if q divides* $\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_p)$ *where* $\mathrm{Dim}X(-)$ *is the dimension function of* X.

In [9], the authors investigate when a group can act on a finite complex with the G-homotopy type of a sphere with isotropy in some fixed family of subgroups. In that article they are only concerned with the existence of such a complex, and, since the algebraic finiteness obstructions are all torsion, a suitable join is taken to ensure that the complex is finite. In using this method, control of the dimension function is lost as the join operation is additive on dimension functions. Theorem 4.3.2 represents progress towards a more controlled process of building group actions on spheres; it determines if a particular sphere is realizable on a finite complex instead of proving that a large enough join of the sphere is finite.

As an immediate consequence of theorem 4.3.2 we see that the congruence conditions are necessary for any group G.

Corollary 4.3.3. *Let G be any finite group, and X an hG-sphere with dimension function* $\mathrm{Dim}X(-)$. *Let G contain subgroups $H \triangleleft K \triangleleft M < G$, where $M/K \cong C_q$ acts faithfully on $K/H \cong C_p$. If X is realizable on a finite complex then q divides* $\mathrm{Dim}X(H) - \mathrm{Dim}X(K)$.

In [1], the author proves that the congruence conditions given above are sufficient for a *virtual* homotopy representation to be finite. In addition, it is known (see [4]) that all p-group actions on homology spheres are realizable on *virtual* linear spheres. Thus the following conjecture is reasonable:

Conjecture 4.3.4. *Let G be any finite group, and X an hG-sphere with dimension function* $\mathrm{Dim}X(-)$. *Then X is realizable on a finite complex if and only if q^r divides* $\mathrm{Dim}X(H) - \mathrm{Dim}X(K)$ *for all chains of subgroups $K \triangleleft H \triangleleft M < G$, where $M/K \cong C_{q^r}$ acts faithfully on $K/H \cong C_p$.*

2

Background

2.1 Homotopy Representations

In this section, we recall some definitions and results of tom Dieck and Petrie on homotopy representations which will be necessary for our discussion in chapter 3. In addition, the work of tom Dieck and Petrie is a good starting point for our discussion on finiteness conditions for group actions on homotopy spheres.

The following material may be found, for example, in [19] or [18].

Definition 2.1.1. *Let G be a finite group. We make the following preliminary definitions:*

- *(a) Let $\varphi(G)$ denote the collection of all conjugacy classes of subgroups of G.*
- *(b) Let $\varphi_p(G)$ denote the collection conjugacy classes of p-groups for all primes p dividing the order of G.*
- *(c) The collection of all integer valued class functions will be denoted by $C(G)$.*

Although we introduced the concept of homotopy representation in the previous section we give a precise definition here.

Definition 2.1.2. *Let G be a finite group. A homotopy representation X of G is a finite-dimensional G-CW complex such that X^H is an $n(H)$-dimensional CW-complex which is homotopy equivalent to the $n(H)$-dimensional* *sphere $S^{n(H)}$, for each subgroup $H < G$. A homotopy representation is called finite if X is a finite G-CW complex.*

There are many examples of homotopy representations that arise quite naturally, for instance:

Example 2.1.3. *Let V be a finite dimensional real representation of a finite group G. Then the unit sphere in V, denoted by $S(V)$, is a G-homotopy representation.*

There is a natural operation on homotopy representations, namely the join. Recall that the join of two spaces X and Y, denoted by $X * Y$, is given by

$$X * Y \simeq \Sigma(X \wedge Y).$$

When X and Y are spheres, the result of the join operation is evidently a sphere as well. The collection of homotopy representations for a given group G forms a semi-group $V^+(G)$ under the join operation. The Grothendieck group of $V^+(G)$ is denoted $V(G)$.

Associated to each homotopy representation X is a integer valued class function

$$\mathrm{Dim}(X)(-) : \varphi(G) \to \mathbb{Z},$$

whose value at H is $\dim(X^H) + 1$. Here $\dim(-)$ refers to the homological dimension of X^H.

A correction factor of plus 1 is necessary to make the join operation behave well with $\mathrm{Dim}(X)(-)$. Indeed, for two G-homotopy representations X and Y, we have

$$(\mathrm{Dim} X * Y)(H) = (\mathrm{Dim} X)(H) + (\mathrm{Dim} Y)(H).$$

One should think of the theory of homotopy representation as a 'geometric representation theory' [8]. In this analogy, the dimension function plays the role of a character equation in representation theory. Pushing this analogy further, one should think of hG-spheres, in some sense, as a homotopical version of representation theory.

In [19] and [18], the authors use the dimension function along with a degree function, whose image lives in $\mathrm{Pic}(G)$, to completely determine $V(G)$. Every homotopy representation is evidently a homotopy sphere (simply considering it as an hG-space), so exploring restrictions on the dimension functions of homotopy representations is a useful place to begin to understand similar restrictions on the dimension functions of hG-spheres.

2.2 The Orbit Category

In this section we recall several results about the orbit category Γ of a finite group G. These results are standard and can be found in many texts, for instance, [11]. In addition, we discuss what we mean by modules over the orbit category. In general, one may define the notion of modules over any category, and most of the results presented below hold for modules over any finite, ordered, E.I. category. Since we will only be concerned with the modules over the orbit category (which is, in particular, a finite ordered E.I. category) we will not present the following results in the most general setting.

Let G be a finite group and $\mathcal{F}$ a family of subgroups closed under the actions of conjugation and taking subgroups. The orbit category of G with respect to the family $\mathcal{F}$, denoted $\mathcal{O}_{\mathcal{F}}(G)$, has transitive G-sets as objects and G-maps as morphisms. More concretely, we may write the objects of the orbit category as the set

$$\mathrm{Ob}(\mathcal{O}_{\mathcal{F}}(G)) = \{G/H | H \in \mathcal{F}\} ;$$

the set morphisms between two G orbits is given by

$$\mathrm{Mor}(G/H, G/K) = \{g \in G | H^g < K\} /K.$$

In what follows, we will write $\Gamma = \mathcal{O}(G)$ for the full orbit category and Γ_p for the orbit category $\mathcal{O}_{\mathcal{F}}(G)$, where $\mathcal{F}$ is the family of subgroups of prime power order for some prime p dividing the order of G. Notice that Γ_p is a full subcategory of Γ.

The notion of length will be useful for induction arguments over the orbit category. We define it here:

Definition 2.2.1. *Let G be a finite group, and Γ the orbit category of G.*

(a) The length, denoted $l(G/H, G/K)$, between two orbits $G/H, G/K \in \Gamma$ is the maximum integer l, such that $H = H_0 < \ldots < H_l = K$ is a chain of subgroups of G.

(b) The length of an orbit category, denoted $l(\Gamma)$, is the maximum of $\{l(G/H, G/K) | G/H, G/K \in \Gamma\}$

(c) The length of an $R\Gamma$ module M, denoted $l(M)$, is the smallest integer l, such that for any chain of subgroups, $H_0 < \ldots < H_n$ in G where $M(H_i) \neq 0$ for all $0 \leq i \leq n$, we have $n \leq l$.

Let R be a commutative ring with unit. Below we use R_p to denote $\mathbb{Z}_{(p)}, \mathbb{Z}_{\hat{p}}$, or $\mathbb{Z}/(p)$, as appropriate. An $R\Gamma$-module M is a contravariant functor,

$$M : \Gamma \to R\text{−Mod},$$

from the orbit category to the category of R-modules. A morphism of $R\Gamma$-modules is then a natural transformation of functors. The category of $R\Gamma$ modules is denoted by $R\Gamma$−Mod. Since Γ is a small category and R−Mod is abelian, the category of $R\Gamma$-modules is abelian as well. We are able, therefore, to do homological algebra in the category $R\Gamma$−Mod.

For $R\Gamma$-modules, the terms exact, injective, surjective, etc. are determined object wise. For instance the sequence of $R\Gamma$-modules

$$L \to M \to N$$

is exact if and only if the sequence of $R\Gamma$-modules

$$L(x) \to M(x) \to N(x)$$

is exact for all $x \in \Gamma$.

As usual, a $R\Gamma$-module P is projective if and only if the functor

$$\mathrm{Hom}_{R\Gamma}(P, -) : R\Gamma{-}\mathrm{Mod} \to R{-}\mathrm{Mod}$$

is exact. The Yoneda Lemma immediately implies the projectivity of a collection of $R\Gamma$-modules. Indeed, if we define F_x as the *free module generated at* x where $x \in \Gamma$ by

$$F_x(y) = R\mathrm{Mor}(y, x)$$

for all $y \in \Gamma$, where $R\mathrm{Mor}(y, x)$ is the free R-module on the set of $R\Gamma$ morphisms from y to x, then

$$\mathrm{Hom}_{R\Gamma}(F_x(-), M) \cong M(x),$$

and so $F_x(-)$ is a projective $R\Gamma$-module for each $x \in \Gamma$.

We are now able to give a definition of a free $R\Gamma$-module:

Definition 2.2.2. *An $R\Gamma$-module M free if it is isomorphic to*

$$\bigoplus_{x \in \Gamma} R\mathrm{Mor}(-, x)$$

for some collection of orbits $x \in \Gamma$.

In section 4.3, we develop a global obstruction theory based on the p-local obstruction theory discussed in section 4.2. We do this by building an integral chain complex out of local complexes. Thus we will need several results about gluing p-adic chain complexes. These results hold in greater generality, nevertheless, as above, we present them in the context of $R\Gamma$-modules. The proofs are standard, and may be found, for instance, in [9].

By a gluing of a collection of p-adic chain complexes, we mean an integral chain complex whose completions recover the original p-adic chain complexes. It is possible, a priori, that the process of gluing finite dimensional p-local chain complexes might produce an infinite dimensional complex over the integers. The following proposition, however, ensures that this is not the case.

Proposition 2.2.3. *Let C be a projective chain complex of $\mathbb{Z}\Gamma$-modules which has a finite homological dimension. Suppose that $R_p \otimes_{\mathbb{Z}} C$ is chain homotopy equivalent to a finite dimensional chain complex of projectives for all primes p dividing the order of G. Then C is homotopy equivalent to a finite dimensional chain complex of projectives.*

Sketch of Proof. Since C has finite homological dimension, for sufficiently large $*$, the group $\mathrm{Ext}^*_{\mathbb{Z}\Gamma}(C, M)$ is finite abelian and killed by $|G|$. Thus after tensoring with R_p it splits into R_p-Ext groups, which, by assumption, are all zero. $\square$

Even though we are now assured to have a finite complex, the homological dimension of such a complex might be wildly distorted by the gluing process. The next proposition is a technical result that allows us to reduce the dimension of a chain complex to the homological dimension plus a gap-type term, based on the length of the category Γ. We will need such a result when we attempt to build a space from a $\mathbb{Z}\Gamma$ chain complex.

Proposition 2.2.4. *Let C be a finite free chain complex of $\mathbb{Z}\Gamma$-modules such that* $\mathrm{hdim}\ C(H) \leq \mathrm{Dim} C(H)$ *for all H. Assume that $l(H, K) \leq k$ whenever* $\mathrm{Dim}(H) = \mathrm{Dim}(K)$. *Then C is chain homotopy equivalent to a complex D which satisfies $D_i(H) = 0$ for all $i > d_H + k$.*

These results, together with a Postnikov-type argument presented in section 4.3, will allow us to construct a global finiteness obstruction from the local data given by the algebraic G-spheres described below.

2.3 Algebraic Spheres

In this section, we collect various definitions and results, due to Grodal-Smith [7], concerning hG-spheres, algebraic G-spheres, and their relationship to each other. The starting point of such a discussion is the notion of hG-equivalence.

Definition 2.3.1. *Two spaces are said to be hG-equivalent if they can be connected by a zig-zag of G-maps that are homotopy equivalences, or, equivalently, if their Borel constructions are equivalent over BG.*

Like homotopy representations, the collection of hG-spheres for a fixed group G is naturally a monoid under the join operation. To distinguish it from that of tom Dieck and Petrie, it is denoted by $V_w^+(G)$ (here the w stands for 'weak', indicating the use hG-equivalences, as opposed to G-equivalences).

Definition 2.3.2. *For a finite group G, let $V_w^+(G)$ denote the monoid, under the join operation, of G-spheres up to hG-equivalence.*

In addition, hG-spheres have dimension functions defined in a similar fashion. However, while the fixed points of a homotopy representation are *required* to have the homotopy type of spheres, from homotopy Smith Theory [5] we see that the fixed points of an hG-sphere by a p-subgroup of G *necessarily* have the mod-p homotopy type of a (p-adic) sphere. Thus the dimension function of an hG-sphere is defined only on the prime ordered subgroups of G.

Their main result, stated below, is the development of a theory which, under certain mild restrictions, assigns to a G-space X with finite $\mathbb{F}_p$ homology a finite algebraic model. The following statement is a simplified version of their more general result about G-spaces with finite $\mathbb{F}_p$ homology.

Theorem 2.3.3 (Grodal-Smith)**.** *Let G be a finite group, and X a G-space with finite $\mathbb{F}_p$ homology that is homotopy equivalent to a sphere. Consider* *the functor Φ: G-spaces $\to \mathrm{Ch}(R_p\Gamma_p)$ which associates to X associates the functor on the opposite p-orbit category Γ_p^{op} given by*

$$G/P \mapsto C_*(\mathrm{map}_G(EG \times G/P, X); \mathbb{F}_p)$$

Then Φ sends a G-space X with finite $\mathbb{F}_p$-homology that is homotopy equivalent to a sphere to a perfect complex in $\mathrm{Ch}(R_p\Gamma_p)$.

Recall that a perfect complex is a complex which is quasi-isomorphic to a finite dimensional chain complex of finitely generated (f.g.) projective modules. When X has the homotopy type of a sphere, this model is rich enough to determine X. Such a chain complex of $R_p\Gamma_p$-modules is called an algebraic G-sphere at the prime p if its homology evaluated at the orbit G/e is one dimensional.

Algebraic G-spheres are classified by their dimension functions. This is accomplished by reducing the problem to the p-orbit category, and then using homotopy Smith Theory to show that every Borel-Smith type dimension function is realizable. Recall that a function $d : \varphi(G) \to \mathbb{N}$ satisfies the Borel-Smith conditions (see, for example [2]) if for every chain of subgroups $H \triangleleft K \triangleleft M < G$ where $K/H \cong C_p$ and H is a p-group then

(a) $d(H) - d(K) \equiv 0 \bmod 2$ if p is odd or if $M/H \cong C_4$,

(b) $d(H) - d(K) \equiv 0 \bmod 4$ if M/H is isomorphic to a quaternion group,

(c) $d(H) - p \cdot d(M) = \sum d(K_i)$ if $M/H \cong C_p \times C_p$ and where the sum runs over the $p + 1$ copies of $K_i/H \cong C_p$.

Although the following result holds in greater generality, we impose several restrictions to state less a technical result which is sufficient for our purposes.

Theorem 2.3.4 (Grodal-Smith). *Let X and Y be G-spheres that are at least 2-connected. Then X and Y are hG-equivalent if and only if their associated algebraic G-spheres, $\Phi(X)$ and $\Phi(Y)$ are quasi-isomorphic. After* *group completion, there is a one-to-one correspondence between G-spheres up to hG-equivalence and algebraic G-spheres with monotonic dimension functions. Furthermore, every Borel-Smith dimension function is realized, possibly more than one way in low dimensions.*

The natural place to look for finiteness obstructions of a space X, turns out to be its algebraic model, the chain complex $C_*(X;\mathbb{Z})$: one looks for obstructions to the complex $C_*(X;\mathbb{Z})$ being free. It would therefore seem that, in our attempt to find finiteness conditions on hG-spheres, we should look at the obstructions to freeness of the algebraic analogue of hG-spheres, namely, algebraic G-spheres. The main difference between the two ideas is that by algebraic G-sphere we mean a family of p-adic chain complexes, one for each prime p dividing the order of G, instead of a single chain complex $C_*(X;\mathbb{Z})$.

3

The Rank of $V(G)$ and $V_w(G)$

We have seen that for a fixed group G, the collection of G-homotopy representations or hG-spheres forms a monoid, $V^+(G)$ or $V_w^+(G)$, under the operation of join. As preliminary question, one may ask for the rank of these monoids. We instead compute the rank of the group completed objects. These ranks may be computed, indirectly, by determining the rank of $\mathrm{Dim}(V(G))$.

3.1 The Rank Computation

In this section, we recall a result of tom Dieck and Petrie which computes the rank of $V(G)$. A new proof of the lower bound is given; this is an improvement over the old proof as it is straightforward to write down a rational basis for $V(G)$ using the new method of proof. As a corollary, this result also computes the rank of $V_w(G)$.

We will use the dimension functions of homotopy representations to compute the rank of $V(G)$. Recall that each homotopy representation has an associated dimension function. We can use these to construct a natural map from $V(G)$ to $C(G)$.

Lemma 3.1.1 (tom Dieck-Petrie)**.** *The associated mapping* $\mathrm{Dim} : V(G) \to C(G)$ *is a homomorphism and its image,* $\mathrm{Dim}(V(G))$*, is torsion free. Furthermore, the kernel of this mapping is torsion, and so the rank of* $V(G)$ *is the same as the rank of* $\mathrm{Dim}V(G)$*.*

Remark 3.1.2. *It is worth mentioning that there is no kernel when working with hG-spheres.*

As with linear representations, we have the notion of an induced homotopy representation. The dimension function of an induced representation behaves as expected. Indeed, if X is a G-homotopy representation, and H is a subgroup of G then

$$\mathrm{Dim}\mathrm{Ind}_H^G(X)(K) = \sum_{K_i} \mathrm{Dim}(X)(K_i),$$

where the action of K on G/H decomposes into orbits as $\coprod K/K_i$.

In view of lemma 3.1.1, determining the rank of $\mathrm{Dim}V(G)$ is an interesting problem. This will be the main result of this section.

Theorem 3.1.3 (tom Dieck-Petrie)**.** *The rank of* $\mathrm{Dim}V(G)$ *is equal to the number of conjugacy classes of subgroups of* G *that are cyclic after abelianization, i.e.*

$$\mathrm{rk}(\mathrm{Dim}V(G)) = |\{H \in \varphi(G) | H/H' \text{ is cyclic}\}|.$$

This theorem is the result of the following lemmas presented below. First, we show that for every subgroup of G whose abelianization is cyclic, we can produce a certain homotopy representation. This collection turns out to be linearly independent. Next, using the so called Borel conditions, which give restrictions on dimension functions of homotopy representations, we give an upper bound for the rank of $\mathrm{Dim}(V(G))$.

Lemma 3.1.4 (tom Dieck-Petrie)**.** *Let* G *be a finite group with* G/G' *cyclic. Then there exists a homotopy representation* X *of* G *such that* $\mathrm{Dim}(X)(H) = 0$ *if and only if* $H \neq G$.

Proof. Construct the appropriate Brieskorn Variety. □

Lemma 3.1.5 (tom Dieck-Petrie)**.** *The corank of* $\mathrm{Dim}V(G) \otimes \mathbb{Q}$ *in* $C(G)$ *is at least the cardinality of the set* $\{H \in \varphi(G) | H/H' \text{ is not cyclic}\}$

Proof. If H/H' is not cyclic, then there exists a normal subgroup K of H such that $H/K \cong C_p \times C_p$ for some prime p. Then the Borel condition lifts to H to give a linear relation

$$l_H = e_H + \sum_{K<H} a_K e_K$$

where

$$l_H(\mathrm{Dim}(X)) = 0$$

for all $X \in V(G)$. Here e_H is the evaluation map at H and $a_K \in \mathbb{Q}$. Thus each such H corresponds to a relation in $C(G)$, and so the corank of $\mathrm{Dim}V(G)\otimes\mathbb{Q}$ in $C(G)$ is at most the size of $\{H \in \varphi(G)|H/H' \text{ is not cyclic}\}$.

□

We are now ready to prove the above theorem:

Proof of Theorem 3.1.3. By Lemma 3.1.5 we have an upper bound on the rank of $\mathrm{Dim}V(G)\otimes\mathbb{Q}$. To show that this is indeed the rank, tom Dieck and Petrie [19] used the module structure of $V(G)$ over $A(G)$. However this can be done directly.

Let $A \subset \varphi(G)$ be the subset of $\varphi(G)$ containing all conjugacy classes (H) such that H/H' is cyclic. For each $H \in A$, construct a homotopy representation X_H as in Proposition 3.1.4 and consider the collection of corresponding dimension functions after inducing them up to G:

$$\left\{\mathrm{DimInd}_H^G(X_H)\right\}_{H\in A}.$$

This set is linearly independent in $\mathrm{Dim}V(G)\otimes\mathbb{Q}$.

Consider the matrix (a_{ij}) with

$$a_{ij} = \mathrm{DimInd}_{H_i}^G X_{H_i}(H_j)$$

with $H_i \in A$ and $H_j \in \varphi(G)$. Notice that (a_{ij}) is upper-triangular as the first nonzero entry of $\mathrm{DimInd}_{H_i}^G(X_{H_i})(-)$ occurs at H_i. Thus the rank is exactly the size of $\{H|H/H' \text{ is cyclic}\}$.

□

Since Lemma 3.1.5 is concerned only with dimension functions, and each homotopy representation gives rise to an element of $V_w(G)$, we have the following version of Theorem 3.1.3:

Theorem 3.1.6. *The rank of* $\mathrm{Dim}V_w(G)$ *is given by:*

$$\mathrm{rk}(\mathrm{Dim}V_w(G)) = |\left\{H \in \varphi_p(G)|H/H' \text{ is cyclic}\right\}|.$$

In addition, this result gives a comparison of the work of tom Dieck-Petrie with that of Grodal-Smith. The projection map below, simply considers a dimension function on all subgroups as a dimension function on subgroups of prime power order.

Corollary 3.1.7. *The natural projection from* $\mathrm{Dim}V(G)$ *to* $\mathrm{Dim}V_w(G)$ *is a surjection after tensoring with* $\mathbb{Q}$*.*

The above mapping is not, in general, surjective before tensoring with $\mathbb{Q}$. As an example, consider the symmetric group on three letters. The dimension function $(0, 0, 2)$ is not in the image of the above projection map without rationalizing.

3.2 Examples

In this section, we use the results above to compute a basis for $V(G)$ for several different groups. We compare these to the more natural, but more difficult to compute, basis of non-virtual representations. Although a basis for Σ_3 was constructed in [19], the method used was ad hoc, and does not easily generalize to larger groups. The current method gives one basis element for each subgroup of G whose abelianization is cyclic.

As the group Σ_3 will be discussed in more detail in chapter 4, we provide a basis here:

Example 3.2.1. *Consider the symmetric group on three letters, $G = \Sigma_3$. A basis for* $\mathrm{Dim}V(G)$*, computed by inducing up from subgroups whose abelianization is cyclic:*

	e	C_2	C_3	Σ_3
$\mathrm{DimInd}_e^G(X_e)(-)$	6	3	2	1
$\mathrm{DimInd}_{C_2}^G(X_{C_2})(-)$	0	1	0	1
$\mathrm{DimInd}_{C_3}^G(X_{C_3})(-)$	0	0	4	2
$\mathrm{DimInd}_{\Sigma_3}^G(X_{\Sigma_3})(-)$	0	0	0	2

The above basis may be rewritten more familiarly as follows:

	e	C_2	C_3	Σ_3
$\mathrm{Dim}1(-)$	1	1	1	1
$\mathrm{Dim}X(-)$	1	0	1	0
$\mathrm{Dim}Y(-)$	2	1	0	0
$\mathrm{Dim}Z(-)$	4	0	0	0.

A more complicated example:

Example 3.2.2. *Let $G = A_4$, the alternating group on four letters. Then $\varphi(G) = \{e, C_2, C_3, V_4, A_4\}$, and the only subgroup that does not have a cyclic abelianization is V_4. Thus the rank of $V(G)$ is 4. A basis for* $\mathrm{Dim}V(G) \otimes \mathbb{Q}$

as constructed in Theorem 3.1.3 is presented in the table below.

	e	C_2	C_3	V_4	A_4
$\mathrm{DimInd}_e^G(X_e)(-)$	12	6	4	3	1
$\mathrm{DimInd}_{C_2}^G(X_{C_2})(-)$	0	2	0	3	1
$\mathrm{DimInd}_{C_3}^G(X_{C_3})(-)$	0	0	2	0	2
$\mathrm{DimInd}_{A_4}^G(X_{A_4})(-)$	0	0	0	0	2

This basis is $\mathbb{Z}$-equivalent to

	e	C_2	C_3	V_4	A_4
$\mathrm{Dim}(1)$	1	1	1	1	1
$\mathrm{Dim}(X)$	2	2	0	2	0
$\mathrm{Dim}(Y)$	3	1	1	0	0
$\mathrm{Dim}(Z)$	6	2	0	0	0

where $1, X$ and Y are linear.

4

Finiteness Conditions

4.1 Finiteness Conditions for G-spaces

It is often useful to consider a space X with a group action as a diagram of fixed points. In other words, we think of X as a functor over some orbit category Γ. When using the usual notion of G-homotopy equivalence, our preferred viewpoint is to consider a G-space X as a diagram over $\Gamma = \mathcal{O}(G)$. With this in mind, we begin with a result about finiteness conditions for diagrams, which is a generalization of Wall's classical result. The proof follows the original, found in [20]. Before we state the theorem, we introduce a definition:

Definition 4.1.1. *Let R be a ring, Γ a small category, $R\Gamma$-mod the category of contravariant functors $\Gamma \to R$-mod, and D a Γ-CW-complex (a diagram of CW-complexes with the shape of Γ) . Define the fundamental category of the Γ-CW-complex D, denoted by $\Pi(\Gamma, D)$, as the homotopy colimit of the contravariant functor*

$$\Gamma \to \{\text{groupoids}\}$$

defined object-wise via

$$\gamma \to \Pi(D(\gamma)).$$

See [11] for further explanation and results concerning the fundamental category. We may now introduce the following theorem:

Theorem 4.1.2. *Let R be a ring, Γ a small category, $R\Gamma$-mod the category of contravariant functors $\Gamma \to R$-mod, and D a Γ-CW-complex whose fun-*

damental category $\Pi(\Gamma, D)$ is trivial. Moreover, suppose that D satisfies the following additional properties:

i. $H_*(D) = 0$ for all $* > n$, and
ii. $H^{n+1}(D; M) = 0$ for all $R\Gamma$-modules M.

Then D is equivalent to a finite dimensional complex Γ-CW-complex K. Furthermore, if K has trivial finiteness obstruction $\sigma(K) \in \widetilde{K}_0(R\Gamma)$, then K may be assumed to be finite.

Before proceeding with the proof, we begin with the following lemma, which allows us to recognize the cofiber of the $(n-1)$-skeleton approximation map as a projective module.

Lemma 4.1.3. *Let D be as above. We may assume that D is a I-CW-complex with $(n-1)$-skeleton D_{n-1}. Then the cofiber of the inclusion map $D_{n-1} \to D$ has projective homology.*

Proof. Let D be as above. Let C be the cofiber of the inclusion map. Then we have the following equalities:

$$H_n(C) = H_n(D, D_{n-1}) = C_n(D)/B_n(D),$$

where C_n and B_n are the n-chains and n-boundaries respectively. To show that $H_n(C)$ is projective, we will show that it is a direct summand of the free module $C_n(D)$. Begin by considering the following diagram:

$$\begin{tikzcd}
C_{n+2} \arrow[d, "d"] & & \\
C_{n+1} \arrow[r, "c"] \arrow[d, "d"] & B_n \arrow[r, "j"] & C_n. \\
C_n \arrow[ur, dashed, "\exists s"'] & &
\end{tikzcd}$$

Since $C_* = C_*(D)$ is a chain complex, $d^2 = 0$. Furthermore, we may identify the composition jc as $jc = d$. Thus we see that

$$0 = d^2 = jcd,$$

and so $cd = 0$. This implies that c is a cocycle. Notice that $B_n(D)$ is an $R\Gamma$-module and $\Pi(\Gamma, D)$ is trivial, so $H^{n+1}(D; B_n(D)) = 0$ by assumption. Therefore, since the cohomology vanishes, c is a coboundary as well. Hence there exists a map $s : C_n \to B_n$ such that $c = sd = sjc$. The map c is surjective, and so we may conclude that the composition sj is an identity. Thus the sequence

$$\begin{tikzcd} B_n(D) \arrow[r, shift right, "j"'] & C_n(D) \arrow[l, shift right, "s"'] \arrow[r] & H_n(D/D_{n-1}) \end{tikzcd}$$

splits and so $H_n(D/D_{n-1}) = H_n(C)$ is a direct summand of the free module $C_n(D)$, and hence is projective. $\square$

We are now ready to prove theorem 4.1.2.

Proof of Theorem 4.1.2. Let C be the cofiber of the inclusion map $j : D_{n-1} \to D$. Denote its homology, which is projective by the previous lemma, by $P = H_n(C)$. Let Q be a projective $R\Gamma$ module such that the sum, $F = P \oplus Q$, is free. Write

$$\begin{aligned} F' &= P \oplus Q \oplus P \oplus \dots \\ &\cong \bigoplus F \\ &\cong \bigoplus_{i \in I} (F^i)^{s_i}, \end{aligned}$$

where F^i is the free functor generated at $i \in I$. First, wedge on the free cell $S^{n-1} \times F^i$, for each generator of our free module F'. Denote the resulting diagram by

$$K = D_{n-1} \vee \bigvee_{F'} (S^{n-1} \times F^i).$$

Next, trivially extend the map j to K by mapping the wedge of free cells to a point (the constant diagram). Write C_K for the cofiber of modified map $j \vee * : K \to D$. The above diagrams may be arranged as follows:

$$\begin{tikzcd}
D_{n-1} \arrow[r, "j"] \arrow[d] & D \arrow[r] \arrow[d, equal] & C \arrow[d] \\
K \arrow[r, "j\vee *"] \arrow[d] & D \arrow[r] \arrow[d] & C_K \\
\bigvee_{F'}(S^{n-1} \times F^i) \arrow[r] & * &
\end{tikzcd}$$

Since everything in sight is a cofiber sequence, the cofiber of the 3^{rd} vertical map is homotopy equivalent to the cofiber of the 3^{rd} horizontal map, which is clearly,

$$\Sigma(\vee_{F'}(S^{n-1} \times F^i)).$$

Thus the map $C \hookrightarrow C_K$, gives rise to the cofiber sequence

$$C \to C_K \to \Sigma(\vee_{F'}(S^{n-1} \times F^i)),$$

which induces a long exact sequence in homology which splits (since K dominates D_{n-1}) and so

$$H_n(C_K) \cong H_n(C) \oplus F' = P \oplus F' \cong F'$$

is free. Now attach n-cells

$$\coprod_{F'} D^n \times F^i$$

to K via the attaching maps

$$F' \cong H_n(C_K) \cong \pi_n(C_K) \xrightarrow{\partial} \pi_{n-1}(K).$$

Denote by L the result of attaching these cells to K. Finally, consider the cofibration given by the map $C_K \to C_L$:

$$C_K \to C_L \to C_{K,L}.$$

It induces a long exact sequence in homology which was constructed such that

(a) $H_n(C_{K,L}) = 0$ for all $i \neq n$, and

(b) $H_n(C_K) = 0$ for all $i \neq n$.

In addition, in dimension n, we have the exact sequence

$$0 \longrightarrow H_n(C_{L,K}) \xrightarrow{\cong} H_n(C_K) \longrightarrow 0.$$

Thus $H_n(C_L) = 0$ for all $i \geq 0$ and hence D is equivalent to a finite dimensional I-CW-complex.

We may define the finiteness obstruction of K by $\sigma(K) = (^{-}1)^n H_n(C)$ Where, as above, C is the cofiber of the inclusion of the $n-1$ skeleton. If $\sigma(K)$ vanishes, then $H_n(C)$ is stably free. That is there is a f.g. free module F such that $H_n(C) \oplus F = F'$ is free. Since F' is a f.g. free module, we only need to attach finitely many cells in the above argument.

□

4.2 Local Finiteness Conditions for hG-spheres

In this section, we begin an investigation into finiteness obstructions for hG-spheres. When working with the notion of hG-homotopy equivalence, the appropriate diagrams to consider are ones over the p-orbit categories of G. By the work of [7] discussed in section 2.3 we know that two G-spheres over the p-orbit categories are equivalent if and only if they have the same dimension function. As a first step to determining when an hG-sphere X is equivalent to a finite complex, we investigate when X has a finite Γ_p-replacement. By a finite Γ_p-replacement, we mean a finite diagram

Y (possibly over a larger category), whose mod-p homology agrees with X at each p-subgroup of G.

At first glance, it would appear that the finiteness obstruction given in section 4.1, applied to Γ_p, would provide the correct answer to question 1.0.3 of the introduction. However, as the following example demonstrates, this is far too restrictive. Instead, theorem 4.1.2 applied to Γ_p, provides an obstruction to which G-spheres are realizable on a finite complex with isotropy in $\mathcal{F}_p$.

Example 4.2.1. *Consider the $R_3\mathcal{O}_3^{op}(\Sigma_3)$ space consisting of two points, with trivial action. Its reduced chain complex consists of the trivial functor in dimension zero. The diagram category has the following shape:*

$$\begin{tikzcd} \Sigma_3/C_3 \arrow[loop left, "C_2"] \arrow[r, shift left, "C_2"] \arrow[r, shift right] & \Sigma_3/e \arrow[loop right, "\Sigma_3"] \end{tikzcd}$$

There are two free functors, namely

$$F_{\Sigma_3/C_3}(\gamma) = \begin{cases} R_3C_2 & \text{if } \gamma = \Sigma_3/C_3 \\ R_3C_2 & \text{if } \gamma = \Sigma_3/e \end{cases}$$

and

$$F_{\Sigma_3/e}(\gamma) = \begin{cases} 0 & \text{if } \gamma = \Sigma_3/C_3 \\ R_3\Sigma_3 & \text{if } \gamma = \Sigma_3/e \end{cases}$$

Notice that the free resolution of the trivial functor is not finite dimensional. Indeed, the resolution evaluated at the orbit Σ_3/e is a resolution of R_3 by R_3C_2-modules and $R_3\Sigma_3$-modules, and such a resolution is infinite dimensional since both C_2 and Σ_3 are 2-periodic at the prime 3.

More generally, when G is not a p-group, the trivial functor does not have a finite resolution by free modules over the p-orbit category. It is worth noting, however, that such a complex does have a finite projective resolution:

Remark 4.2.2. *Recall that even though the trivial functor over the p-orbit category may not have a finite resolution of free modules, it always has a* *finite resolution by projectives, a result first proved in [10]. This is also verified by the work of [7]: simply apply theorem 2.3.3, as stated in the introduction, to the sphere S^0.*

It is certainly reasonable to ask that S^0 be a finite Γ_p-replacement for the hG-sphere corresponding to the trivial functor in the previous example. The reason the obstruction theory given in section 4.1 does not allow for such a replacement is that as a complex over the p-orbit category, its non-p-group isotropy is empty. In what follows we relax this condition, allowing for arbitrary isotropy at non-p-groups, in order to avoid this example. We are therefore led to consider not only modules that are free over $R_p\Gamma_p$, but modules that arise as restrictions of free modules over $R_p\Gamma$ (the latter clearly including the former). With this in mind, we make the following definition.

Definition 4.2.3. *An $R_p\Gamma_p$ module M is said to be pseudo-free if it is the restriction of a free module on the full orbit category $R_p\Gamma$. That is to say*

$$M(-) = \mathrm{Res}^{\Gamma}_{\Gamma_p} F_x(-),$$

where $F_x(y) = R_p\mathrm{Mor}_\Gamma(y, x)$ for some $x \in \Gamma$.

Pseudo-free modules have the following properties:

Lemma 4.2.4. *Let M be a pseudo-free $R_p\Gamma_p$-module. Then*

(a) $M(G/e) = R_p\mathrm{Mor}(G/e, G/D) = R_p[G/D]$, for some $D < G$,

(b) $M(G/e)$ is a p-permutation module, and

(c) M has finite projective dimension.

Proof. Part (a) follows from the fact that the Γ_p is a full subcategory of Γ, and part (b) from the definition of p-permutation module. Part (c) follows from the more general result, 'Rim's Theorem for Orbit Categories' and Corollary 3.12 of [9]. One shows that $\mathrm{Res}^G_P(R_p\mathrm{Mor}(-, G/D))$ has finite projective dimension if a Sylow p-subgroup of D is an object of Γ_p. The proof relies on induction on the length of the module M, with the base case established by the classical result of Rim. □

Our motivating example suggests that we should consider *extensions* of $R_p\Gamma_p$-modules instead of $R_p\Gamma_p$-modules themselves. Although we will not place any requirements on the non-p-group isotropy, it appears that we will still need to keep track of it. In order to accomplish this, we need a way of extending modules to the full orbit category. There are two natural ways of extending functors along the inclusion $\Gamma_p \hookrightarrow \Gamma$, based on the left and right Kan extensions. We recall the definitions of the left and right Kan extensions. The following definition can be found in [12].

Definition 4.2.5. *Let ι be an inclusion of categories $\iota : \mathcal{C} \to \mathcal{D}$ and a F a functor $F : \mathcal{C} \to \mathcal{E}$. If $\mathcal{C}$ is small and $\mathcal{E}$ is cocomplete, then there exists a left Kan extension* $\mathrm{Lan}_\iota(F)$ *of F along ι defined at each object $d \in \mathcal{D}$ by*

$$\mathrm{Lan}_\iota(F)(d) = \mathrm{colim}_{\iota\downarrow d} F(c),$$

where the colimit is taken over the comma category $(\iota \downarrow d)$. The right Kan extension, denoted $\mathrm{Ran}_\iota(-)$*, is defined dually.*

The left Kan extension behaves like an inclusion. More concretely, the functor $\mathrm{Lan}(-)$ takes an $R_p\Gamma_p$ module M and extends it to $R_p\Gamma$ by assigning a value of zero to the non-p-subgroups. More interestingly, the right Kan extension is not a simple inclusion. However the functor $\mathrm{Ran}(-)$ on the category of $R_p\Gamma_p$ modules adds norm-type elements which is not desirable for our purposes. It would be more useful to have a functor that 'recovered', in some sense, a free functor on the entire orbit category from its restriction to the p-orbit category.

The Brauer construction [17] accomplishes this by dividing out the norm elements of the right Kan extension, but, in the process, it kills all of the non-p group values of $\mathrm{Ran}(M)(-)$. A better approach is to first factor the $R_p\Gamma_p$-module valued functor M_x as the free $R_p\Gamma_p$ module on the set valued functor $\mathrm{Mor}(-, x)$. We then extend $\mathrm{Mor}(-, x)$ to the category Γ, using the right Kan extension, and take the free R_p-module on the result.

Definition 4.2.6. *Let M_x be the pseudo-free $R_p\Gamma_p$-module generated at $x \in$*

Γ. *That is to say*

$$M_x(y) = R_p\mathrm{Mor}(y, x).$$

Extend M_x to Γ by taking the free R_p-module on the right Kan extension:

$$E(M_x)(y) = R_p(\mathrm{Ran}_{\Gamma_p}^{\Gamma}(\mathrm{Mor}(y, x))).$$

As discussed, there are many ways to extend modules along the inclusion $\Gamma_p \hookrightarrow \Gamma$, however the free module on the right Kan extension has evident advantages.

Lemma 4.2.7. *The extension $E : R_p\Gamma_p \to R_p\Gamma$ is an honest extension, that is to say $E(M)(G/P) = M(G/P)$ for all p-subgroups P of G. In addition, E sends pseudo-free modules to free modules. In other words, $E(\mathrm{Res}_{\Gamma_P}^{\Gamma} F) = F$ where F is a free $R_p\Gamma$-module.*

Proof. The first statement follows from the fact that the inclusion of Γ_p into Γ is an inclusion of a full subcategory. The second statement can be broken into three cases:

Let F be a free Γ-functor at G/D, where D is a p-group. Since E is an honest extension, $E(\mathrm{Res}_{\Gamma_p}^{\Gamma} F)(-) = F(-)$ on p-groups. Since

$$\mathrm{Mor}(G/H, G/K) = \emptyset$$

if $K^g \nleq H$ for all $g \in G$, the value of $E(\mathrm{Res}_{\Gamma_p}^{\Gamma} F)(-)$ is zero elsewhere. Hence $E(\mathrm{Res}_{\Gamma_p}^{\Gamma} F) = F$.

Now suppose that F is free at G/D, a where D is a q-group, q a prime distinct from p. Then $\mathrm{Res}_{\Gamma_p}^{\Gamma}(F)$ is the module

$$\mathrm{Res}_{\Gamma_p}^{\Gamma} F(G/N) = \begin{cases} G/D & \text{if } N = e \\ 0 & \text{otherwise.} \end{cases}$$

It is only necessary to compute the extension at subgroups of D, as $E(\mathrm{Res}_{\Gamma_p}^{\Gamma} F)$ is clearly zero elsewhere. Let $Q \leq D$. The extension at G/Q is computed by taking the free R_p-module on the limit of the arrow category

$$\begin{tikzcd} G/Q \arrow[r, shift left=2, "G/Q"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & G/e \arrow[loop right, "G"] \end{tikzcd} .$$

Notice that this limit may be computed by finding the isotropy of one of the $|G/Q|$ arrows, this isotropy subgroup is Q, and then computing the fixed points of said arrow's target under the action of Q. Thus the value of $E(\mathrm{Res}^{\Gamma}_{\Gamma_p} F)(-)$ evaluated at Q is given by the following:

$$E(\mathrm{Res}^{\Gamma}_{\Gamma_p} F)(Q) = R_p((G/D)^Q) = R_p \{g \in G | Q^g \leq D\} / D = F(G/Q)$$

as desired.

Finally, for composite subgroups, we proceed in a similar fashion. Let F be free at a composite subgroup D of G. When computing the value at a subgroup Q of D, we first note that we only need to consider the arrows from the orbit G/Q to the trivial orbit G/e, since the group action moves all of the other arrows here. The argument is then the same as the p' case. $\square$

The above lemma suggests that pseudo-free modules will play an important role in our theory. It will therefore be necessary to determine when a complex is equivalent to a complex of pseudo-free modules. As such we introduce the following obstruction.

Definition 4.2.8. *Let X be a finite dimensional chain complex of projective $R_p\Gamma_p$ modules whose top dimension is n. Define the pseudo-free finiteness obstruction of X, denoted by*

$$\sigma_p(X) \in \widetilde{K}_0(R_p\Gamma_p)/I,$$

as the alternating sum

$$\sigma_p(X) = \sum_{i=0}^{n} ({}^{-}1)^i [X_i].$$

Generators of the ideal I are given as the alternating sum of the projective resolution of restrictions of free modules. More concretely, a generator $i \in I$ has the following form:

$$i = \sum (^{-}1)^n [P_n(\mathrm{Res}(F))],$$

where $P_*(\mathrm{Res}(F))$ is a projective resolution of the restriction of the free $R_p\Gamma$-module F.

It is necessary to use projective resolutions of elements in the image, since the functor

$$\mathrm{Res} : R_p\Gamma{-}\mathrm{mod} \to R_p\Gamma_p{-}\mathrm{mod}$$

does not preserve projectives; however, as can be seen from lemma 4.2.4, this definition still makes sense because the restriction functor takes free modules to modules of finite projective dimension, ensuring that the alternating sum in the above definition is well defined.

Notice that this obstruction will vanish if the usual obstruction vanishes, but also vanishes on our motivating example. We collect various facts about σ_p in the following proposition:

Proposition 4.2.9. *Let X and Y be a finite dimensional chain complexes of projective $R_p\Gamma_p$-modules.*

- *(a) If X and Y are chain homotopy equivalent, then $\sigma_p(X) = \sigma_p(Y)$.*
- *(b) If $\sigma_p(X) = 0$ then X is chain homotopy equivalent to a finite dimensional chain complex of f.g. pseudo-free $R_p\Gamma_p$-modules.*
- *(c) If $\sigma_p(X) = 0$ then there is a finite dimensional chain complex of f.g. free $R_p\Gamma$-modules Y, such that $\mathrm{Res}^{\Gamma}_{\Gamma_p}(Y) = X$.*

Proof. To prove part (a), note that the cofiber C, of the map $X \xrightarrow{\simeq} Y$ is contractible. Its chain complex is therefore split, and hence $\sigma_p(C) = 0$. Since σ_p is additive, we are done.

For part (b), assume that $X_i = 0$ for $i > n$ where n is a positive integer. Then after adding complexes of the form $P \xrightarrow{1} P$, where P is a projective $R_p\Gamma_p$-module, we may assume that X has the following form:

$$0 \to Q \xrightarrow{f_n} F_{n-1} \to \ldots \to F_0 \to 0,$$

where each F_i (respectively Q) is a free (respectively projective) $R_p\Gamma_p$-module. Since σ_p is invariant under chain homotopy, we have $0 = \sigma_p(X) \cong \sigma_p(Q)$, thus Q is equivalent to some pseudo-free $R_p\Gamma$ module.

We now have a finite dimensional chain complex of pseudo-free $R_p\Gamma_p$-modules which we would like to extend to $R_p\Gamma$. However, the extension of Q to the free $R_p\Gamma$-module $E(Q)$ is not compatible with the map f_n. The extension would be non-empty at p'-subgroups, and therefore, necessarily map to zero. We fix this by wedging on an additional copy of Q.

Since Q is projective, there is a projective module P, such that $P \oplus Q \cong F$ is free. Thus, without altering the homology, we may add the complex $P \xrightarrow{1} P$ in dimension n:

$$0 \to P \to \begin{matrix} Q \\ \oplus \\ P \end{matrix} \xrightarrow{f_n} F_{n-1} \to \ldots \to F_0 \to 0.$$

After writing $P \oplus Q$ as F and adding the complex $Q \xrightarrow{1} Q$ in dimension n, we have

$$0 \to \begin{matrix} P \\ \oplus \\ Q \end{matrix} \to \begin{matrix} F \\ \oplus \\ Q \end{matrix} \xrightarrow{f_n \oplus 0} F_{n-1} \to \ldots \to F_0 \to 0.$$

The above complex is equivalent to the following complex:

$$0 \to F \to \begin{matrix} F \\ \oplus \\ Q \end{matrix} \xrightarrow{f_n \oplus 0} F_{n-1} \to \ldots \to F_0 \to 0.$$

This is now a complex of free $R_p\Gamma_p$-modules, except for Q in dimension n

which is pseudo-free, and maps to zero. It is this zero map that will allow us to extend the complex to $R_p\Gamma$.

For part (c), extend the modified complex obtained above, from $\mathrm{Ch}(R_p\Gamma_p)$ to $\mathrm{Ch}(R_p\Gamma)$, using $\mathrm{Lan}(-)$ except at the pseudo-free module Q in dimension n. Extend Q via the extension $E(-)$, to the free module $E(Q)(-)$. This is possible because the attaching map of $E(Q)(-)$ is trivial. Call the extended complex Y. Since $\mathrm{Lan}(-)$ clearly preserves free modules and $E(Q)(-)$ is free by lemma 4.2.7, Y is a finite dimensional chain complex of f.g. free $R_p\Gamma$-modules whose restriction to $R_p\Gamma_p$ is X. In particular X and Y have the same homology at p-groups. □

Remark 4.2.10. *It is worth mentioning that extensions to the full orbit category are, in general, not unique. There are several pseudo free complexes with the same p-group homology. In fact the non-p group isotropy may not even be a sphere. It is this non-uniqueness property that makes the global finiteness obstruction, discussed in the next section, difficult.*

The following example demonstrates the non-uniqueness of such extensions.

Example 4.2.11. *Let $G = \Sigma_3$. The following two chain complexes, in fact algebraic Σ_3-spheres, of free $R_3\Gamma$-modules have the same homology for prime ordered subgroups at the prime $p = 3$, i.e. the cyclic subgroup C_3 and the trivial subgroup e. They are clearly not equal as complexes over $R_3\Gamma$, but they are both equivalent after restriction to $R_3\Gamma_3$ to the complex whose dimension function is $(0, 4)$.*

The first $R_3\Gamma$-sphere has dimension function $(0,0,2,4)$,

$$
\begin{tikzcd}
{[3]} & & & & R_3[\Sigma_3] \arrow[d, "f_3"'] \\
{[2]} & & & & R_3[\Sigma_3] \oplus R_3[\Sigma_3] \arrow[d, "f_2"'] \\
{[1]} & & & R_3 \arrow[d, "0"'] & R_3[\Sigma_3] \oplus R_3[\Sigma_3/C_2] \arrow[d, "f_1 \oplus 0"'] \\
{[0]} & & & R_3 \arrow[d, "1"'] & R_3[\Sigma_3/C_2] \arrow[d, "\epsilon"'] \\
{[{}^{-}1]} & R_3 & R_3 & R_3 & R_3 \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{tikzcd}
$$

and the second $R_3\Gamma$-sphere has dimension function $(0,0,0,4)$

$$
\begin{tikzcd}
{[3]} & & & & R_3[\Sigma_3] \arrow[d, "g_3"'] \\
{[2]} & & & & R_3[\Sigma_3] \oplus R_3[\Sigma_3] \arrow[d, "g_2"'] \\
{[1]} & & & & R_3[\Sigma_3] \oplus R_3[\Sigma_3] \arrow[d, "g_1"'] \\
{[0]} & & & & R_3[\Sigma_3] \arrow[d, "\epsilon"'] \\
{[{}^{-}1]} & R_3 & R_3 & R_3 & R_3 \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{tikzcd}
$$

Notice that after restricting to the 3-orbit category, both are equivalent to the *complex of projective $R_3\Gamma_3$-modules which has dimension function* $(0, 4)$*.*

$$\begin{tikzcd}
{[3]} & & R_3P_+ \arrow[d, "p_3"'] \\
{[2]} & & R_3P_- \arrow[d, "p_2"'] \\
{[1]} & & R_3P_- \arrow[d, "p_1"'] \\
{[0]} & & R_3P_+ \arrow[d, "p_0"'] \\
{[^{-}1]} & R_3 & R_3 \\
& \Sigma_3/C_3 & \Sigma_3/e
\end{tikzcd}$$

If fact any Borel-Smith dimension function $(0, 0, m, 4)$*, where* $0 \leq m \leq 4$*, has such a restriction.*

Let us restate our work up to this point. Given an hG-sphere X we would like to determine if there is a mod-p equivalent space Y that is finite dimensional or finite. So we look at the algebraic G-sphere associated to X. It is a finite dimensional chain complex of projectives. We modify it so that it is a finite dimensional chain complex of f.g. free modules except, possibly, in the penultimate dimension. Based on example 4.2.1, we know it is not enough to produce a chain complex over $R_p\Gamma_p$, so we must extend our chain complex to $R_p\Gamma$. Now that we have a finite dimensional chain complex of f.g. free $R_p\Gamma$ modules we can use a modification of a lemma of Pamuk, found in [14], to build a space whose chain complex has the same homology at the p-groups. As our current chain complex has empty homology at non-p-groups, we will need to suspend our complex to fix this.

We are now ready to introduce the main theorem of this section.

Theorem 4.2.12. *Let X be an hG-sphere whose p-fixed points are at least*

3-connected. Then X is hG-equivalent to a finite dimensional complex. Furthermore, if $\sigma_p(C_*(X)) = 0$, then there is a finite space Y whose mod-p homology at p-groups agrees with that of X, that is to say X has a finite Γ_p replacement.

Proof. As discussed above, if the obstruction $\sigma_p(C_*(X))$ vanishes, then we may construct a finite dimensional chain complex of f.g. free modules over the full orbit category, Γ, whose homology at p-groups agrees with that of X. The proof then follows the standard argument of building a space from a chain complex of free modules, with the exception of how to attach free cells. That result is presented as a lemma below. Once we show that we can attach free cells, we proceed by induction.

The induction begins with the chain complex for the classifying space, $E\Gamma$, of the category Γ. Such a space always exists: see, for example, [9]. We require that the p-fixed points of X are at least 3-connected so that we may replace the first few stages of our free chain complex with the chains on $E\Gamma$. Finally, after constructing the space Y, we note that the dimension function of Y is the same (on p-groups) as the dimension function of X. Thus, by [7], they have the same mod-p hG-homotopy type. □

The following lemma is adapted from [9] or [14]. In those cases, the authors were concerned with attaching $\mathcal{F}$-cells over the orbit category $\mathcal{O}_{\mathcal{F}}(G)$. In the present version we will be attaching $\mathcal{F}_p$-cells over the entire orbit category $\mathcal{O}(G)$.

Lemma 4.2.13. *Let $X^{(-)}$ be a finite G-CW-complex. Suppose that we are given a free $R_p\Gamma$ module $F(-)$ and a $R_p\Gamma$ module homomorphism $\varphi : F(-) \to H_n(X^{(-)}; R_p)$, for some $n \geq 2$. Assume further that*

(a) if $R_p[G/P](-)$ is a summand of $F(-)$, then $X^{(G/P)}$ is $(n-1)$ connected for every p-group P of G; and

(b) if $R_p[G/D](-)$ is a summand of $F(-)$, for any non-p group D, then the restriction of φ to $R_p[G/D](-)$ is trivial, i.e. $\varphi|_{R_p[G/D](-)} = 0$.

Then by attaching $(n+1)$-cells to X, we can obtain a G-CW-complex Y such that

$$H_i(X^{(-)}; R_p) \cong H_i(Y^{(-)}; R_p), \quad \text{for} \quad i \neq n, n+1,$$

and

$$H_{n+1}(X^{(-)}; R_p) \hookrightarrow H_{n+1}(Y^{(-)}; R_p) \to F \to H_n(X^{(-)}; R_p) \twoheadrightarrow H_n(Y^{(-)}; R_p)$$

is exact for all p-groups P.

Proof. We attach a wedge of Γ-spheres, with the appropriate actions; for the free p-cells corresponding to $R_p[G/P]$, we attach them to X^P using the homotopy groups of X^P and then extend them equivariantly, for the free cells generated by non-p groups, we simply wedge them on to X via the zero map. Specifically, we do the following procedure:

Let Z be a wedge of n-dimensional Γ-spheres such that $H_n(Z; R_p) = F$. We construct a map $Z \to X$ that will realize φ one summand at a time. For summands of the form $R_p[G/P]$, where P is a p-group, our assumptions on the connectivity of X^P imply that

$$H_n(X^P; R_p) \cong \pi_n(X^P).$$

Therefore elements in the image, $\varphi(R_p[G/P](P))$, correspond to maps $e_{gP} : S^n \to X^P$. Extend these maps equivariantly to a map $e_P : S^n \times G/P \to X$. For the non-$p$-group cells, $R_p[G/D](-)$, define $e_D : S^n \times G/D \to X$ to be zero. In this way we construct a map $e : Z \to X$ that realizes φ on homology. Let Y be the cofiber of $e : Z \to X$. Such a construction satisfies the exact sequence given above. $\square$

4.3 Global Finiteness Conditions for hG-spheres

In the previous section, we constructed a finite replacement for X at a prime p. In general, there may exist many replacements whose homology differs at non p-groups. In order to construct a global finiteness replacement, we must ensure that the homology of extensions from different primes is somehow compatible. In this section, we begin an investigation into a global finiteness condition. For hG-spheres, the compatibility is verified via the p-dimension functions. This allows one to determine the finiteness of an hG-sphere based on its dimension function, something easy to verify, instead of a $\widetilde{K}_0$ obstruction.

The problem of globally realizing a given hG-sphere on a finite complex can be broken down into steps, seen below in the following diagram:

Diagram 4.3.1.

$$\begin{tikzcd}
\bigoplus_{p||G|} \mathrm{Ch}(R_p\Gamma_p) \arrow[r] \arrow[d, bend right, "i"'] & \mathrm{Ch}(\mathbb{Z}\Gamma) \arrow[r] & \mathcal{CW}_* \\
\bigoplus_{p||G|} \mathrm{Ch}(R_p\Gamma) \arrow[u, bend right, "r"'] \arrow[ur, "g"'] & &
\end{tikzcd}$$

Given a collection of finite dimensional chain complexes, $\{C^{(p)}\}_{p||G|}$, of projective $R_p\Gamma_p$-modules representing X, we first extend these complexes from the p-orbit category, Γ_p, to the full orbit category Γ. Call these modified complexes $\widetilde{C}^{(p)}$. Such an extension is a necessary step, as seen in the previous section, since there are examples of finite (even linear) hG-spheres whose 'finiteness obstruction' over $R_p\Gamma_p$ is nontrivial.

Next, we show that if the collection $\left\{\widetilde{C}^{(p)}\right\}_{p||G|}$ satisfies a compatibility condition, which can be verified by the dimension functions of each complex, we can use a Postnikov tower to glue them together. This resulting complex,

C, is a finite dimensional chain complex of projective $\mathbb{Z}\Gamma$-modules. This complex has the property that $C \otimes R_p = \widetilde{C}^{(p)}$ for all primes p dividing the order of G. Furthermore, the dimension functions the original complexes, $\{C^{(p)}\}_{p||G|}$, will also satisfy this compatibility condition.

We then determine when the obstruction, which lives in $\widetilde{K}_0(\mathbb{Z}\Gamma)$, to C being homotopy equivalent to a finite dimensional chain complex of f.g. free modules vanishes on the image of g. When this is the case, C is homotopy equivalent to a finite dimensional chain complex of f.g. free $\mathbb{Z}\Gamma$-modules. Finally, we verify that the chain complex C satisfies the conditions of [14], in order to construct a finite complex $X \in V_w^+(G)$ that realizes C.

When G is the group $C_p \rtimes C_q$, where p and q are distinct primes such that $q|p-1$ and C_q acts by conjugation on C_p, the above discussion is summarized by the following theorem, which will we will prove below.

Theorem 4.3.2. *Let $G = C_p \rtimes C_q$ where p and q are distinct primes such that $q|p-1$. An hG-sphere X is realizable on a finite complex if and only if q divides $\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_p)$ where $\mathrm{Dim}X(-)$ is the dimension function of X. When $q = 2$, all hG-spheres are realizable on a finite complex.*

This result immediately gives conditions on larger groups containing cyclic subgroups acting on quotients:

Corollary 4.3.3. *Let G be any finite group, and X an hG-sphere with dimension function $\mathrm{Dim}(X)$. Let G contain subgroups $H \triangleleft K \triangleleft M < G$, where $M/K \cong C_q$ acts faithfully on $K/H \cong C_p$. If X is realizable on a finite complex then q divides $\mathrm{Dim}X(H) - \mathrm{Dim}X(K)$.*

The above corollary suggests the following conjectural answer to question 1.0.2 posed in the introduction:

Conjecture 4.3.4. *Let G be any finite group, and X an hG-sphere with dimension function $\mathrm{Dim}(X)$. Then X is realizable on a finite complex if and only if q^r divides $\mathrm{Dim}X(H) - \mathrm{Dim}X(K)$ for all chains of subgroups $H \triangleleft K \triangleleft M < G$, where $M/K \cong C_{q^r}$ acts faithfully on $K/H \cong C_p$.*

The sufficiency of such congruencies is indicated by the classification *virtual* homotopy representations. Recall the result of Bauer [1], which concludes that the homotopy representations that are realizable as a difference of finite homotopy representations are those whose dimension functions satisfy the congruencies given above.

Theorem 4.3.2 will be proved as a consequence of the following lemmas. The proof of the first lemma will be deferred to section 4.5.

Lemma 4.3.5. *In diagram 4.3.1, the map r is onto when G is the group of order pq^r where faithful C_{q^r} acts faithfully by conjugation on C_p. Consequently, every dimension function defined on the primes orbit category has a (non-unique) extension to the full orbit category.*

In the next proposition, we examine the numerical condition needed to insure that a given pair of $R_l\Gamma_l$ chain complexes, $l \in \{p, q\}$, lifts via the map g to a chain complex of $\mathbb{Z}\Gamma$-modules.

Proposition 4.3.6. *When G is the metacyclic group of order pq, a pair of chain complexes, $C^{(p)}$ and $C^{(q)}$, lifts to a chain complex over $\mathbb{Z}\Gamma$ if and only if $2q$ divides $\mathrm{Dim}X(G/e) - \mathrm{Dim}(G/C_p)$.*

Before we prove the proposition, we recall the following definition and an associated lemma.

Definition 4.3.7. *A family of chain complexes $\left\{C^{(l)}\right\}_l$ of $R_l\Gamma$-modules is said to be compatible if there exists a finitely generated $\mathbb{Z}\Gamma$ module H_* such that $H_i \otimes \mathbb{Z}_l = H_i(C^{(l)})$.*

We may now ask for conditions on the compatibility of chain complexes. The following lemma does exactly that.

Lemma 4.3.8. *When G is the metacyclic group of order pq, a pair of chain complexes, $C^{(p)}$ and $C^{(q)}$, is compatible if and only if $2q$ divides $\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_p)$.*

Proof. Let $n = \mathrm{Dim}X(G/e)$ and $m = \mathrm{Dim}X(G/C_p)$. Since the Weil group of C_p, given by $W_G(C_P) = N_G(C_p)/C_p = C_q$, acts on X^{C_p}, it acts on the homology of X^{C_P} as well. Since X is a sphere, $C^{(p)}(-)$ is a chain complex starting with the one dimensional $R_p\Gamma$ module $R_p^\omega(-)$ for some root of unity ω. Here $R_p^\omega(-)$ is given by $R_p^\omega(\gamma) = R_p^\omega$ for all $\gamma \in \Gamma$, and R_p^ω denotes the module R_p together with the C_q action of multiplication by ω. Notice that there are q such projective $R_p\Gamma$ modules corresponding to the splitting of the free $R_p\Gamma$ module

$$F_{G/C_p}(\gamma) = \begin{cases} R_pC_q & \text{if } \gamma = C_p, e \\ 0 & \text{else} \end{cases}$$

into projectives.

The finitely generated module H_* exists if and only if the Weil group of C_p acts trivially on both $H_n(C^{(p)}(G/e); R_p) = R_p$ and $H_m(C^{(p)}(G/C_p); R_p) = R_p$. Thus both $C^{(p)}(G/e)$ and $C^{(p)}(G/C_p)$ are resolutions of R_p^ω, which end with R_p. This can occur if and only if $2q$, the period of C_q, divides $\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_p)$. $\square$

Proposition 4.3.6 then follows from a standard Postnikov tower argument (given in [9]): it can be shown that a compatible family of p-local chain complexes lifts to an integral chain complex. The argument involving Postnikov towers given here is due to Dold [3]. We recall the proof for completeness:

Proof of Proposition 4.3.6. We construct the $\mathbb{Z}\Gamma$ chain complex C by induction. Let $C_0 = P(H_0)$, where $P(H_0)$ is a projective resolution of H_0. If H_i is trivial, then let $C_i = C_{i-1}$. Otherwise note that

$$\begin{aligned} \mathrm{Ext}^{i+1}_{\mathbb{Z}\Gamma}(C_{i-1}, H_i) &\cong \bigoplus_{p||G|} \mathrm{Ext}^{i+1}_{R_p\Gamma}(C_{i-1} \otimes R_p, H_i \otimes R_p) \\ &= \bigoplus_{p||G|} \mathrm{Ext}^{i+1}_{R_p\Gamma}(C^{(p)}_{i-1}, H^{(p)}_i) \end{aligned}$$

holds whenever $i + 1 > l(C_{i-1}) + \mathrm{hdim}(C_{i-1})$. This condition is satisfied trivially when G has order pq. Thus the Postnikov invariants of C_i are determined by its p-adic Postnikov invariants. $\square$

In order to utilize the theory of finiteness obstructions over $\mathbb{Z}\Gamma$, some preliminaries are in order. For instance, it is not obvious that the complex C constructed from the Postnikov tower approximation has finite homological dimension. However, recall that proposition 2.2.3 states that an integral Γ chain complex is finite dimensional if its p-completions are finite dimensional for all $p||G|$. Since our starting complexes $C^{(p)}$ and $C^{(q)}$ were finite dimensional , it is indeed the case that our integral complex C is equivalent to a finite dimensional chain complex as well.

Since we have now shown that our gluing operation results in a finite complex, it makes sense to talk about a finiteness obstruction (as the alternating sum will be finite) on the image of g. The proof of the following lemma will be postponed until section 4.5.

Lemma 4.3.9. *For the metacyclic group of order pq, the finiteness obstruction vanishes on the image of the map g in diagram 4.3.1.*

Proof. See Section 4.5. $\square$

Thus we may assume that C is a finite dimensional chain complex of f.g. free modules. To construct a finite CW-complex X from the chain complex C we appeal to the following theorem of Pamuk [14]:

Theorem 4.3.10 (Pamuk)**.** *Let C be a finite dimensional chain complex of f.g. free $\mathbb{Z}\Gamma$-modules. Suppose that C is an $\underline{n}$-Moore complex such that $\dim(C(H)) \geq 3$ for all $H < G$. Suppose further that $C_i(H) = 0$ for all $i > \dim(C(H)) + 1$, and all H. Then there is a finite G-CW-complex Z such that $C(Z^{(-)}, \mathbb{Z})$ is chain homotopy equivalent to C as chain complexes of $\mathbb{Z}\Gamma$-modules.*

In order to apply the above theorem to our chain complex C, we must first modify it so that it satisfies the necessary gap hypothesis. In order to do so, we use proposition 2.2.4. Notice that when G is the metacyclic group of order pq, and $\mathrm{Dim}(-)$ is any non-trivial dimension function, $l(H,K)$ is at most one whenever $\mathrm{Dim}(H) = \mathrm{Dim}(K)$. Thus by using proposition 2.2.4, we may assume that C is equivalent to a complex that satisfies the various gap hypotheses. Finally, as in the previous section, notice that the dimension function of the new space Z, is the same as the dimension function of our original hG-sphere X, and so Z and X have the same hG-homotopy type.

We now give an example of a complex over the p-orbit category, which cannot be extended to a finite complex over the full orbit category.

Example 4.3.11. *Let $G = C_7 \rtimes C_3$ be the metacyclic group of order 21. Let X be the following algebraic sphere over the category $R_7\Gamma_7$:*

$$\begin{array}{ll} G/C_7 & R_7 \\ & \\ G/e & R_7 \xleftarrow{p_1} R_7P_1 \xleftarrow{p_2} R_7^{\omega}P_2, \end{array}$$

where P_i is the projective resolution of R_7, and ω is a 3^{rd} root of unity in R_7. The dimension function of X is $(0,2)$, and since the difference

$$\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_7)$$

is not congruent to 0 modulo 3, by theorem 4.3.2, X cannot be extended to a finite integral complex. Explicitly, notice that the of action of G on $H_2(X(G/e); R_7)$ is non-trivial, and therefore is not compatible with a (necessarily trivial) G-action on $H_2(Y(G/e); R_3)$, where Y any $R_3\Gamma_3$-complex with the homology of a sphere and dimension function $(*, 2)$.

Moreover, X does not even have a finite Γ_7-replacement. Indeed its finiteness obstruction

$$\sigma_7(X) = R_7^{\omega}P_2$$

is non-trivial in $K_0(R_7\Gamma_7)/I$. Notice, however, that the 3-fold tensor of X has dimension function $(0,6)$ and will satisfy the congruence condition, and $\sigma_7(X^{\otimes 3}) = 0$ as well.

4.4 Examples of Finite $h\Sigma_3$-spheres

In this section, we give a complete picture of the monoids $V^+(\Sigma_3)$ and $V_w^+(\Sigma_3)$. Notice that all of the obstructions for a given $h\Sigma_3$-sphere to be hG-equivalent to a finite complex vanish. The congruence condition,

$$\mathrm{Dim}X(\Sigma_3/C_3) - \mathrm{Dim}X(\Sigma_3/e) \equiv 0 \bmod 2,$$

is not a new restriction: indeed, it simply a restatement of a Borel-Smith condition.

Nonetheless, it is of interest to exhibit several families of finite (non-linear, non-free, non-virtual) Σ_3-spheres. These spheres, together with the linear ones, and Swan's free Σ_3 sphere [16], generate $V_w^+(\Sigma_3)$. Recall that $V^+(\Sigma_3)$ is defined on the full orbit category. As a result, to complete the picture for $V^+(\Sigma_3)$, we include an additional nontrivial (non smooth) Σ_3-sphere. This sphere is, after reduction, equivalent to the trivial sphere.

Remark 4.4.1. *These results go further than tom Dieck and Petrie in [19] and Mackrodt [13]. In those papers the authors computed the structure of the group completed object $V(\Sigma_3)$. That is to say they only verified that the homotopy representations given below existed virtually.*

It is much easier to produce the required chain complexes over $\bigoplus_{p||G|} \mathbb{Z}_{\hat{p}}\Gamma$ and then glue them together via a Postnikov tower, than to build them over $\mathbb{Z}\Gamma$ directly. The complexes presented below are the result of such a procedure. We will use the following presentation of Σ_3:

$$\Sigma_3 = \left\{\sigma, \tau | \sigma^3 = \tau^2 = e, \tau\sigma\tau = \sigma^2\right\}$$

Example 4.4.2. *The chain complex given below has dimension function* $(0, 0, 4, 4)$.

$$\begin{tikzcd}
{[4]} & & & & \mathbb{Z}[\Sigma_3] \arrow[d, "f_4"'] \\
{[3]} & & & \mathbb{Z} \arrow[d, "0"'] & \mathbb{Z}[\Sigma_3/C_2] \oplus \mathbb{Z}[\Sigma_3] \arrow[d, "0 \oplus f_3"'] \\
{[2]} & & & {} & \mathbb{Z}[\Sigma_3] \arrow[d, "f_2"'] \\
{[1]} & & & & \mathbb{Z}[\Sigma_3] \arrow[d, "f_1"'] \\
{[0]} & & & \mathbb{Z} \arrow[d, "1"'] & \mathbb{Z}[\Sigma_3/C_2] \arrow[d, "\epsilon"'] \\
{[{}^{-}1]} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{tikzcd}$$

The maps f_i, $1 \leq i \leq 4$ *are given by*

$$\begin{aligned}
f_1(e) &= \bar{1} - \bar{2} \\
f_2(e) &= e - \sigma\tau - \sigma^2\tau \\
f_3(e) &= e - \sigma^2 + \sigma^2\tau - \tau \\
f_4(e) &= (e + \sigma + \sigma\tau, \bar{1}).
\end{aligned}$$

Example 4.4.3. *The chain complex given below has dimension function*

$(0, 1, 3, 3)$.

$$\begin{tikzcd}
{[3]} & & & & \mathbb{Z}[\Sigma_3] \arrow[d, "f_3"'] \\
{[2]} & & & \mathbb{Z} \arrow[d, "0"'] & \mathbb{Z}[\Sigma_3/C_2] \oplus \mathbb{Z}[\Sigma_3] \arrow[d, "0 \oplus f_2"'] \\
{[1]} & & & {} & \mathbb{Z}[\Sigma_3] \arrow[d, "f_1"'] \\
{[0]} & & \mathbb{Z}[C_2] \arrow[d, "\epsilon"] & \mathbb{Z} \arrow[d, "\epsilon"] & \mathbb{Z}[C_2] \oplus \mathbb{Z}[\Sigma_3/C_2] \arrow[d, "\epsilon"] \\
{[-1]} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{tikzcd}$$

The maps f_i, $1 \leq i \leq 3$ are given by

$$\begin{aligned}
f_1(e) &= (e, -\bar{1}) \\
f_2(e) &= e - \sigma + \tau - \sigma^2\tau \\
f_3(e) &= (e + \sigma\tau + \sigma^2\tau, 1).
\end{aligned}$$

The above complexes are enough to generate all of $V_w^+(\Sigma_3)$ (see remarks at the end of the section), but we need one more to generate $V^+(\Sigma_3)$. Notice that this is an example of a nontrivial sphere that is locally trivial.

Example 4.4.4. *The chain complex given below has dimension function*

$(0, 2, 2, 2)$.

$$
\begin{tikzcd}
{[2]} & & & & \mathbb{Z}[\Sigma_3] \arrow[d, "f_2"'] \\
{[1]} & & \mathbb{Z}[C_2] \arrow[d, "\bar{N}"'] & \mathbb{Z} \arrow[d, "0"'] & \mathbb{Z}[C_2] \oplus \mathbb{Z}[\Sigma_3/C_2] \oplus \mathbb{Z}[\Sigma_3] \arrow[d, "f_1"'] \\
{[0]} & & \mathbb{Z}[C_2] \arrow[d, "\epsilon"'] & \mathbb{Z} \arrow[d, "1"'] & \mathbb{Z}[C_2] \oplus \mathbb{Z}[\Sigma_3/C_2] \arrow[d, "\epsilon"'] \\
{[-1]} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
 & \Sigma_3/\Sigma_3 & \Sigma_3/C_3 & \Sigma_3/C_2 & \Sigma_3/e
\end{tikzcd}
$$

The maps f_i, $1 \leq i \leq 3$ are given by

$$
\begin{aligned}
f_1(e) &= (e, -\bar{1}) \\
f_2(e) &= e - \sigma + \tau - \sigma^2\tau.
\end{aligned}
$$

Since each of these chain complexes is free, (and $l(\Gamma_p) < 2$) we may realize each of them on a finite dimensional CW-complex.

By shifting the module in dimension 4 (resp. 3) we can produce chain complexes with dimension functions $(0, 4n, 4n)$ (resp. $(1, 3 + 4n, 3 + 4n)$). These dimension functions, together with the linear and free dimension functions generate all of $V^{+}_{w,f}(G)$ modulo obvious low dimensional requirements. For instance, it is not possible to realize (0,0,2), except as a virtual sphere.

4.5 Proofs of Lemmas 4.3.5 and 4.3.9

In this section we present two proofs of 4.3.5 and a proof of 4.3.9.

We have two versions of the proof that the restriction map is surjective.

The first version is for the group of order pq; it is a constructive proof. The second version works for order pq^r.

Lemma 4.3.5. *In diagram 4.3.1, the map r is onto when G is the metacyclic group of order pq. Consequently, every dimension function defined on the primes orbit category has a (non-unique) extension to the full orbit category.*

Proof. We show, by construction, that the following dimension functions exist in $\mathbb{Z}_{\hat{l}}\Gamma$ for $l = p, q$:

G	C_p	C_q	e
1	1	1	1
0	2	0	2
0	0	2	2
0	0	0	2
0	2	2	2.

One notes that after restricting from Γ to $\Gamma_{\mathcal{F}_{\text{primes}}}$ (i.e. deleting the first column) these dimension functions generate $\mathrm{Dim}(V_w^+(G))$, and so the map r is onto. (The last dimension function is not needed to describe $V_w(G)$ but is needed for $V^+(G)$.)

All chain complexes are written over the orbits listed as follows:

$$G/G \quad G/C_p \quad G/C_q \quad G/e.$$

The first two dimension functions above are given by the following projective chain complexes, where $R = \mathbb{Z}_{\hat{l}}$:

$$R \qquad R \qquad R \qquad R$$

$$\begin{tikzcd} & RC_q \arrow[d, "N"] & & RC_q \arrow[d, "N"] \\ & RC_q \arrow[d, "\epsilon"] & & RC_q \arrow[d, "\epsilon"] \\ R & R & R & R \end{tikzcd}$$

When $l = p$, the second two dimension functions are realized by

$$\begin{tikzcd}
 & & & R[G/C_q] \arrow[d, "(0,1)"] \\
 & & R \arrow[d, "0"] & R[G/C_q] \oplus R_\omega[G/C_q] \arrow[d, "0 \oplus p_0"] \\
 & & R \arrow[d, "1"] & R[G/C_q]^\epsilon \arrow[d] \\
R & R & R & R
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & R_\omega[G/C_q] \arrow[d, "p_0"] \\
 & & & R[G/C_q] \arrow[d, "\epsilon"] \\
R & R & R & R
\end{tikzcd}$$

where ω is a p^{th} root of unity, and p_0 is the projective cover of $R[G/C_q]$.

When $l = q$, the second two dimension functions are realized by

$$\begin{tikzcd}
 & & & Q \arrow[d, "(0,1,0)"] \\
 & & R \arrow[d, "0"] & R \oplus Q \oplus Q \arrow[d, "(0,0) \oplus (0,0) \oplus (0,1)"] \\
 & & R \arrow[d, "1"] & R \oplus Q \arrow[d, "1 \oplus 0"] \\
R & R & R & R
\end{tikzcd}$$

and

$$
\begin{tikzcd}
 & & & R[C_q] \arrow[d, "N"] \\
 & & & R[C_q] \arrow[d, "\epsilon"] \\
R & R & R & R
\end{tikzcd}
$$

where the $\mathbb{Z}_{\hat{q}}\Gamma$-module Q is the nontrivial summand of $\mathbb{Z}_{\hat{q}}[G/C_q]$. The last dimension function is given as

$$
\begin{tikzcd}
 & & & R[G/C_q] \arrow[d, "{(0,1,0)}"] \\
 & R \arrow[d, "0"] & R \arrow[d, "0"] & R \oplus R[G/C_q] \oplus R[G/C_q] \arrow[d, "{(0,0)\oplus(0,0)\oplus(\epsilon,1)}"] \\
 & R \arrow[d, "1"] & R \arrow[d, "1"] & R \oplus R[G/C_q] \arrow[d, "\epsilon"] \\
R & R & R & R
\end{tikzcd}
$$

when $l = p$ and

$$
\begin{tikzcd}
 & & & Q \oplus Q \arrow[d] \\
 & R[C_q] \arrow[d, "N"] & R \arrow[d, "0"] & R[C_q] \oplus R[G/C_q] \oplus Q \oplus Q \arrow[d] \\
 & R[C_q] \arrow[d, "\epsilon"] & R \arrow[d, "1"] & R[C_q] \oplus R[G/C_q] \arrow[d, "\epsilon"] \\
R & R & R & R
\end{tikzcd}
$$

when $l = q$. □

Lemma 4.3.5. *In diagram 4.3.1, the map r is onto when G is the metacyclic group of order pq^r with C_q^r acting faithfully on C_p. Consequently, every dimension function defined on the primes orbit category has a (non-unique) extension to the full orbit category.*

Proof. We show that every 'elementary' dimension function $\mathrm{Dim}(-)$ is realizable on a finite dimensional chain complex of projectives. By an elementary dimension function, we mean a function $\mathrm{Dim}(-)$, where $\mathrm{Dim}(H) \in \{0, 2\}$ for all subgroups H of G, for all possible monotonic combinations. These allowable dimension functions are of the following two types:

Type I:

$$\mathrm{Dim}(H) = \begin{cases} 2 & \text{if } H < C_{q^s} \\ 2 & \text{if } H = C_p \\ 0 & \text{otherwise} \end{cases}$$

Type II:

$$\mathrm{Dim}(H) = \begin{cases} 2 & \text{if } H < C_{q^s} \\ 0 & \text{if } H = C_p \\ 0 & \text{otherwise} \end{cases}$$

for some integer s, where $0 \leq s \leq r$.

Since $C_p \rtimes C_{q^s}$ is normal in G with cyclic quotient, all Type I dimension functions exist (even integrally). Note that Type II dimension functions are trivial extensions over $\mathbb{Z}_{\hat{q}}$. Over $\mathbb{Z}_{\hat{p}}$, more work is needed. First extend the 'elementary' dimension function from $\mathbb{Z}_{\hat{p}}\Gamma_p$ to $\mathbb{Z}_{\hat{p}}\Gamma$, via a left Kan extension.

We modify this complex to have the appropriate homology for the non-trivial q-subgroups using the same argument as proposition 6.8 in [9]:

Use the following two exact sequences:

$$0 \to M \to H_0 \to \hat{H}_0 \to 0$$
$$0 \to H_2 \to \hat{H}_2 \to M \to 0$$

where H_i is the original homology, $\hat{H}_i$ is the modified homology, and M is the $\mathbb{Z}_{\hat{p}}\Gamma$-module given by

$$M(H) = \begin{cases} \mathbb{Z}_{\hat{p}} & \text{if } H < C_{q^s} \\ \mathbb{Z}_{\hat{p}}\bar{p} & \text{if } H = 1 \\ 0 & \text{otherwise} \end{cases}$$

We need only verify that M is projective. Indeed, M is a summand of the free module $\mathbb{Z}_{\hat{p}}\mathrm{Mor}(-, G/C_{q^s})$. $\square$

Remark 4.5.1. *This argument also shows that the resulting complex will be free: since two copies of M are added in dimensions of opposite parity, the modified complex still has trivial obstruction. This shows that all dimension functions over $R_p\Gamma$ are realizable on finite dimensional complexes of f.g. free modules.*

For q, we only have that every dimension function exists as a finite complex of projectives over $R_q\Gamma$ (just include them from $R_q\Gamma_q$).

Lemma 4.3.9. *In diagram 4.3.1, the finiteness obstruction vanishes on the image of g.*

Proof. We show that if $\mathrm{Dim}X(-)$ is a dimension function such that q divides $\mathrm{Dim}X(G/e) - \mathrm{Dim}X(G/C_p)$, we can construct *unreduced* chain complexes $C^{(l)}$ over $\mathbb{Z}_{\hat{l}}$ such that the *unreduced* finiteness obstruction, $\sigma(C^{(l)}) \in K_0(\mathbb{Z}_{\hat{l}}\Gamma)$ vanishes. Since

$$\sigma(C) \otimes \mathbb{Z}_{\hat{l}} = \sigma(C^{(l)})$$

we are done.

Below, we present a pair of chain complexes over $\mathbb{Z}_{\hat{p}}\Gamma$ and $\mathbb{Z}_{\hat{q}}\Gamma$ that have dimension function $(0, 2i, 2q)$. The dimension of the chain complex is given on the left hand side of the page, and the complexes are written over the orbits

$$G/C_p \quad G/e \quad \text{and} \quad G/C_q \quad G/e,$$

respectively. Notice that these dimension functions, for $0 \leq i \leq q$, together with the dimension functions induced from normal subgroups, generate all of the dimension functions that satisfy the above congruence. Furthermore, notice that these complexes are constructed in a manner that that ensures their finiteness obstruction vanishes.

$$
\begin{tikzcd}[row sep=large]
{[2q]} & & \mathbb{Z}_{\hat{p}}[G/C_q] \arrow[d] & & \mathbb{Z}_{\hat{q}}[C_q] \arrow[d] \\
& & \mathbb{Z}_{\hat{p}_{\omega}^{n-1}}[G/C_q] \arrow[d] & & \mathbb{Z}_{\hat{q}}[C_q] \arrow[d] \\
& & \vdots \arrow[d] & & \vdots \arrow[d] \\
& & \mathbb{Z}_{\hat{p}_{\omega}^{i+1}}[G/C_q] \arrow[d] & & \mathbb{Z}_{\hat{q}}[C_q] \arrow[d] \\
& & \mathbb{Z}_{\hat{p}}[G/C_q] \oplus \mathbb{Z}_{\hat{p}_{\omega}^{i+1}}[G/C_q] \arrow[d] & & \mathbb{Z}_{\hat{q}}[C_q] \oplus \mathbb{Z}_{\hat{q}}Q \arrow[d] \\
{[2i]} & \mathbb{Z}_{\hat{p}} \arrow[d] & \mathbb{Z}_{\hat{p}}[G/C_q] \oplus \mathbb{Z}_{\hat{p}_{\omega^i}}[G/C_q] \arrow[d] & \mathbb{Z}_{\hat{q}} \arrow[d, "0"'] & \mathbb{Z}_{\hat{q}}[G/C_q] \arrow[d, "0"'] \\
& 0 & \vdots \arrow[d] & 0 & \vdots \arrow[d] \\
& & \mathbb{Z}_{\hat{p}_{\omega^2}}[G/C_q] \arrow[d] & & 0 \arrow[d] \\
& & \mathbb{Z}_{\hat{p}_{\omega}}[G/C_q] \arrow[d] & & 0 \arrow[d] \\
& & \mathbb{Z}_{\hat{p}_{\omega}}[G/C_q] \arrow[d] & & \mathbb{Z}_{\hat{q}}Q \arrow[d] \\
{[0]} & \mathbb{Z}_{\hat{p}} & \mathbb{Z}_{\hat{p}}[G/C_q] & \mathbb{Z}_{\hat{q}} & \mathbb{Z}_{\hat{q}}[G/C_q]
\end{tikzcd}
$$

□

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