Schur-Positivity of Differences of Augmented Staircase Diagrams

by

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Abstract

The Schur functions $\{s_\lambda\}$ and ubiquitous Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ are instrumental in describing representation theory, symmetric functions, and even certain areas of algebraic geometry. Determining when two skew diagrams D_1, D_2 have the same skew Schur function or determining when the difference of two such skew Schur functions $s_{D_1} - s_{D_2}$ is Schur-positive reveals information about the structures corresponding to these functions.

By defining a set of staircase diagrams that we can augment with other (skew) diagrams, we discover collections of skew diagrams for which the question of Schur-positivity among each difference can be resolved. Furthermore, for certain Schur-positive differences we give explicit formulas for computing the coefficients of the Schur functions in the difference.

We extend from simple staircases to fat staircases, and carry on to diagrams called sums of fat staircases. These sums of fat staircases can also be augmented with other (skew) diagrams to obtain many instances of Schur-positivity.

We note that several of our Schur-positive differences become equalities of skew Schur functions when the number of variables is reduced. Finally, we give a factoring identity which allows one to obtain many of the non-trivial finite-variable equalities of skew Schur functions.

Contents

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Chapter 1

Introduction

1.1 Partitions, Compositions, Ferrers Diagrams, and Skew Diagrams

A *partition* λ of a positive integer n, written $\lambda \vdash n$, is a sequence of weakly decreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\sum_{i=1}^{k} \lambda_i = n$. We call each λ_i a *part* of λ, and if λ has exactly k parts we say λ is of *length* k and write $l(\lambda) = k$. When $\lambda \vdash n$ we will also write $|\lambda| = n$ and say that the *size* of λ is n.

We shall use j^r to denote the sequence $j, j, \ldots, j$ consisting of r j's. Under this notation, we shall write $\lambda = (k^{r_k}, k-1^{r_{k-1}}, \ldots, 1^{r_1})$ for the partition which has r_1 parts of size one, r_2 parts of size two, $\ldots$, and r_k parts of size k.

Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$, we let $\lambda \cup \mu$ denote the partition that consists of the parts $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ placed in weakly decreasing order.

We say $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a *composition* of n, written $\alpha \models n$, if each α_i is a positive integer and $\sum_{i=1}^{k} \alpha_i = n$. As with partitions, we call each α_i a *part* of α, write $|\alpha| = n$ for the *size* of α, and if α has exactly k parts we say α is of *length* k and write $l(\alpha) = k$. If we relax the conditions to consider sums of non-negative integers, that is, allowing some of the α_i to be zero, then we obtain the concept of a *weak composition* of n. If $\alpha \models n$ we obtain a partition of n by reordering the α_i into weakly decreasing order.

We may sometimes find it useful to treat partitions and weak compositions as vectors with non-negative integer entries. When we write the vector $z = (z_1, z_2, z_3, \ldots, z_n)$ we shall mean the infinite vector

$$z = (z_1, z_2, z_3, \ldots, z_n, 0, 0, 0, \ldots).$$

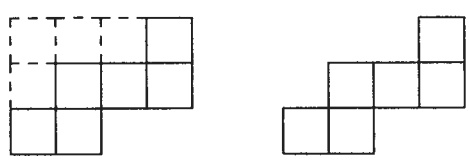
We shall only consider vectors with finitely many non-zero entries. Hence we

shall only display vectors with finite length. In this manner we may unambiguously add vectors of different lengths. Thus, we have defined addition among partitions and weak-compositions. Further, given a positive integer i, we shall let e_i denote the i-th standard basis vector. That is, the vector that has its i-th entry equal to 1 and all remaining entries equal to 0.

Given a partition λ, we can represent it via the diagram of left justified rows of boxes whose i-th row contains λ_i boxes. The diagrams of these type are called *Ferrers diagrams*, or just *diagrams* for short. We shall use the symbol λ when refering to both the partition and its Ferrers diagram.

Whenever we find a diagram D' contained in a diagram D as a subset of boxes, we say that D' is a *subdiagram* of D. Suppose partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash m$, with $m \leq n$, $k \leq j$, and $\mu_i \leq \lambda_i$ for each $i = 1, 2, \ldots, k$ are given. Then μ is a subdiagram of λ, and a particular copy of μ is found at the top-left corner of λ. We can form the *skew diagram* λ/μ by removing that copy of μ from λ. Henceforth, when we say that D is a *diagram*, it is assumed that D is either the Ferrers diagram of some partition λ, or D is the skew diagram λ/μ for some partitions λ, μ.

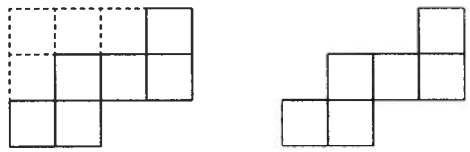
Example Here we consider the partitions $\lambda = (4, 4, 2)$ and $\mu = (3, 1)$, and form the skew diagram λ/μ.



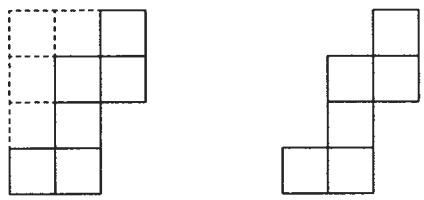
For two boxes b_1 and b_2 in a diagram D, we define a *path* from b_1 to b_2 in D to be a sequence of steps either up, down, left, or right that begins at b_1, ends at b_2, and at no time leaves the diagram D. We say that a diagram D is *connected* if for any two boxes b_1 and b_2 of D there is a path from b_1 to b_2 in D. If D is not connected we say it is *disconnected.*

Given any diagram D, the 180° *rotation* of a diagram D is denoted by D°. Further, the *conjugate* or *transpose* diagram D^t is obtained by simply transposing the array of boxes along the main diagonal. When D is the Ferrers diagram of a partition λ then the *conjugate diagram* D^t is also the Ferrers diagram of a partition called the conjugate partition and denoted λ^t. It must be noted that most authors use the notation D' and λ' for the conjugate diagram of D and conjugate partition of λ. We hope our notation causes no confusion.

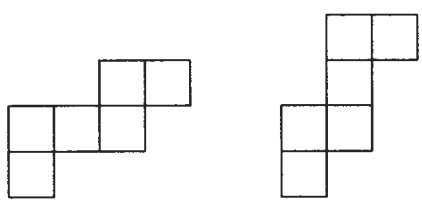
Example Here we consider the partitions $\lambda = (4,4,2)$ and $\mu = (3,1)$, and form the skew diagram λ/μ.



The conjugates, $\lambda^t = (3,3,2,2)$, $\mu^t = (2,1,1)$ and $(\lambda/\mu)^t = \lambda^t/\mu^t$ are shown below.



Finally, we see the skew diagrams $(\lambda/\mu)^\circ$ and $((\lambda/\mu)^t)^\circ = ((\lambda/\mu)^\circ)^t$.



The number of boxes that appear in a given row or a given column of a diagram is called the *length* of that row or column. A *hook* is the Ferrers diagram corresponding to a partition λ that satisfies $\lambda_i \leq 1$ for all $i > 1$. Hence a hook has at most one row of length larger than 1.

Example Here we see the hooks $\lambda = (4,1,1)$ and $\mu = (5,1)$.

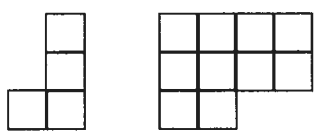
λ μ

1.2 Combining and Decomposing Diagrams

In this section we look at combining and decomposing skew diagrams in a few simple ways.

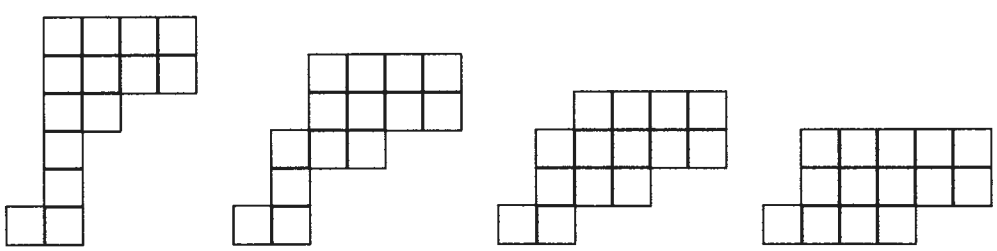
Given two diagrams D_1 and D_2 the *concatenation of* D_1 *and* D_2, denoted $D_1 \cdot D_2$, is the skew diagram obtained by placing D_1 and D_2 so that the top-right box of D_1 is immediately below the bottom-left box of D_2. Similarly, the *near-concatenation of* D_1 *and* D_2, denoted $D_1 \odot D_2$, is the skew diagram obtained by placing D_1 and D_2 so that the top-right box of D_1 is immediately left of the bottom-left box of D_2. Finally, if the last column of D_1 and first column of D_2 are both of length $\geq i$, then the *near-concatenation of depth* i *of* D_1 *and* D_2, denoted $D_1 \odot_i D_2$, is the is the skew diagram obtained by placing D_1 and D_2 so that the top-right box of D_1 is one step left and $i-1$ steps up from the bottom-left box of D_2. We note that $\odot_1 = \odot$.

Example Let D_1 be the skew diagram $(2,2,2)/(1,1)$ and D_2 be the Ferrers diagram $(4,4,2)$.



D_1 D_2

Here we show the diagrams $D_1 \cdot D_2$, $D_1 \odot D_2$, $D_1 \odot_2 D_2$, and $D_1 \odot_3 D_2$.

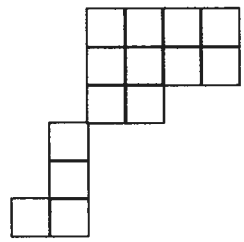


$D_1 \cdot D_2$ $D_1 \odot D_2$ $D_1 \odot_2 D_2$ $D_1 \odot_3 D_2$

Given a diagram D, the *connected components of* D are the maximally connected subdiagrams of D. Any skew diagram D decomposes into a finite number of connected components $D_1, D_2, \ldots, D_k$. Conversely, given connected diagrams $D_1, D_2, \ldots, D_k$, we define the diagram $D = \bigoplus_{i=1}^{k} D_i$, called the *direct sum of the* D_i, to be the skew diagram with connected components $D_1, D_2, \ldots, D_k$ such that the top-right box of D_i is one step left and one step down from the bottom-left box of D_{i+1} for $i = 1, 2, \ldots, k-1$. We note that $\oplus = \odot_0$.

Example With $D_1 = (2,2,2)/(1,1)$ and $D_2 = (4,4,2)$, as in the previous example, we now show the diagram $D_1 \oplus D_2$.

$D_1 \oplus D_2$



1.3 Tableaux

If D is a diagram, then a *tableau*—plural *tableaux*—$\mathcal{T}$ *of shape* D is the array obtained by filling the boxes of the D with the positive integers, where repetition is allowed. A tableau is said to be a *semistandard Young tableau* (or simply *semistandard*, for short) if each row gives a weakly increasing sequence of integers and each column gives a strictly increasing sequence of integers. We will often abbreviate "semistandard Young tableau" by SSYT and "semistandard Young tableaux" by SSYTx.

When we wish to depict a certain tableau we will either show the underlying diagram with the entries residing in the boxes of the diagram or we may simply replace the boxes with the entries, so that the entries themselves depict the underlying shape of the tableau.

The *content* of a tableau $\mathcal{T}$ is the weak composition given by

$$\nu(\mathcal{T}) = (\nu_1, \nu_2, \ldots),$$

where ν_i is the number of i's that appear in $\mathcal{T}$.

Example Here we consider a semistandard Young tableau $\mathcal{T}_1$ with shape $D_1 = (2,2,2)/(1,1)$ and content $(1,0,1,2)$, and a semistandard Young tableau $\mathcal{T}_2$ of shape $D_2 = (4,4,2)$ and content $(2,2,3,2,1)$.

$$\begin{array}{cc} & 1 \\ & 3 \\ 4 & 4 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 4 & & \end{array}$$

$\mathcal{T}_1$ $\qquad$ $\mathcal{T}_2$

We may sometimes find it useful to take the transpose of a tableau $\mathcal{T}$ so that we obtain a tableau of conjugate shape. In these cases we simply transpose the entries of $\mathcal{T}$ along with the underlying shape D. In this way, we obtain a tableau $\mathcal{T}^t$ of shape D^t with $c(\mathcal{T}^t) = c(\mathcal{T})$.

Similarly, given a tableau $\mathcal{T}$ of shape D, we may wish to focus on the entries of T that lie in some subdiagram D' of D. In this way we obtain a *subtableau* of shape D'. Also, given tableau $\mathcal{T}_1$ and $\mathcal{T}_2$ of shapes D_1 and D_2 respectively, we may form the tableaux $\mathcal{T}_1 \oplus \mathcal{T}_2$ of shape $D_1 \oplus D_2$, $\mathcal{T}_1 \cdot \mathcal{T}_2$ of shape $D_1 \cdot D_2$, $\mathcal{T}_1 \odot \mathcal{T}_2$ of shape $D_1 \odot D_2$, and $\mathcal{T}_1 \odot_i \mathcal{T}_2$ of shape $D_1 \odot_i D_2$ (when $D_1 \odot_i D_2$ is defined) in the obvious way.

Example For the SSYTx $\mathcal{T}_1$ and $\mathcal{T}_2$ in the previous example, we now create the tableaux $\mathcal{T}_1 \cdot \mathcal{T}_2$, $\mathcal{T}_1 \odot \mathcal{T}_2$, $\mathcal{T}_1 \odot_2 \mathcal{T}_2$, and $\mathcal{T}_1 \odot_3 \mathcal{T}_2$.

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 4 & & \\ 1 & & & \\ 3 & & & \\ 4 & 4 & & \end{array} \qquad \begin{array}{cccc} & 1 & 1 & 2 & 4 \\ & 2 & 3 & 3 & 5 \\ 1 & 3 & 4 & & \\ 3 & & & & \\ 4 & 4 & & & \end{array} \qquad \begin{array}{ccccc} & 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 3 & 5 \\ 3 & 3 & 4 & & \\ 4 & 4 & & & \end{array} \qquad \begin{array}{ccccc} & 1 & 1 & 1 & 2 & 4 \\ & 3 & 2 & 3 & 3 & 5 \\ 4 & 4 & 3 & 4 & & \end{array}$$

$\mathcal{T}_1 \cdot \mathcal{T}_2$ $\qquad$ $\mathcal{T}_1 \odot \mathcal{T}_2$ $\qquad$ $\mathcal{T}_1 \odot_2 \mathcal{T}_2$ $\qquad$ $\mathcal{T}_1 \odot_3 \mathcal{T}_2$

Of these tableaux, only $\mathcal{T}_1 \odot \mathcal{T}_2$ and $\mathcal{T}_1 \odot_2 \mathcal{T}_2$ are semistandard.

1.4 Symmetric Functions

In this section we consider an infinite set of variables $\mathbf{x} = (x_1, x_2, \ldots)$ and the ring of formal power series $\mathbb{C}[\mathbf{x}]$. By S_n, we mean the group of permutations of the letters $\{1, 2, \ldots, n\}$.

We can let each $\pi \in S_n$ act on elements $f(x_1, x_2, \ldots) \in \mathbb{C}[\mathbf{x}]$ by defining

$$\pi f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1}, x_{n+2}, \ldots). \tag{1.1}$$

We are interested in certain functions $f(x_1, x_2, \ldots) \in \mathbb{C}[\mathbf{x}]$ which are fixed by each $\pi \in S_n$, for every $n \in \mathbb{N}$. For instance, the *n-th power symmetric function*,

$$p_n(\mathbf{x}) = \sum_{i=1}^{\infty} x_i^n,$$

is such a function. Two other functions with this property are the the *n-th elementary symmetric function*,

$$e_n(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

and the *n-th homogeneous symmetric function*,

$$h_n(\mathbf{x}) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ by the *power symmetric function*, *elementary symmetric function*, and *homogeneous symmetric function corresponding to* λ, we shall mean

$$p_\lambda(\mathbf{x}) = \prod_{i=1}^{k} p_{\lambda_i}(\mathbf{x}),$$

$$e_\lambda(\mathbf{x}) = \prod_{i=1}^{k} e_{\lambda_i}(\mathbf{x}), \text{ and}$$

$$h_\lambda(\mathbf{x}) = \prod_{i=1}^{k} h_{\lambda_i}(\mathbf{x}).$$

Additionally, by the *monomial symmetric function corresponding to* λ we mean

$$m_\lambda(\mathbf{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

where the above sum is taken over all distinct monomial terms of the form $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$. We note that $m_{(n)}(\mathbf{x}) = p_n(\mathbf{x})$ and $m_{(1^n)}(\mathbf{x}) = e_n(\mathbf{x})$.

Each of p_λ, e_λ, h_λ, and m_λ are invariant under the action described in Equation 1.1. Further, it turns out that each of the sets $\{p_\lambda(\mathbf{x}) |\ \lambda \vdash n\}$, $\{e_\lambda(\mathbf{x}) |\ \lambda \vdash n\}$, $\{h_\lambda(\mathbf{x}) |\ \lambda \vdash n\}$, and $\{m_\lambda(\mathbf{x}) |\ \lambda \vdash n\}$ are independent and span the same space of functions; that is, they are all bases of a common space. This space, denoted Λ^n, is called the set of *homogeneous symmetric functions of degree* n and is usually defined as the span of the $m_\lambda(\mathbf{x})$ for $\lambda \vdash n$.

For any tableau $\mathcal{T}$ of content $\nu = (\nu_1, \nu_2, \ldots)$ we have the *weight* $\mathbf{x}^{\mathcal{T}}$ defined by

$$\mathbf{x}^{\mathcal{T}} = \prod_i {x_i}^{\nu_i}.$$

Given this weight function, the *Schur function corresponding to* λ is defined to be

$$s_\lambda(\mathbf{x}) = \sum_{\mathcal{T}} \mathbf{x}^{\mathcal{T}}, \tag{1.2}$$

where the sum is taken over all SSYTx $\mathcal{T}$ of shape λ.

If $|\lambda| = n$, then $s_\lambda(\mathbf{x})$ is a homogeneous symmetric function of degree n. As with the other symmetric functions, the set $\{s_\lambda(\mathbf{x}) \mid \lambda \vdash n\}$ is a basis of Λ^n. Further details can be found in [63].

Finally, given a skew diagram λ/μ, the *skew Schur function corresponding to* λ/μ is

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\mathcal{T}} \mathbf{x}^{\mathcal{T}}, \tag{1.3}$$

where the sum is taken over all SSYTx $\mathcal{T}$ of shape λ/μ. The skew Schur function $s_{\lambda/\mu}(\mathbf{x})$ is a homogeneous symmetric function of degree $n = |\lambda| - |\mu|$.

Example For a single row $\lambda = (n)$, the semistandard condition on a tableau of shape λ implies that the entries form a weakly increasing sequence. Thus we obtain

$$s_{(n)}(\mathbf{x}) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = h_n(\mathbf{x}),$$

the n-th homogeneous symmetric function.

For a single column $\mu = (1^n)$, the semistandard condition on a tableau of shape μ implies that the entries form a strictly increasing sequence. Thus we obtain

$$s_{(1^n)}(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = e_n(\mathbf{x}),$$

the n-th elementary symmetric function.

Example We wish to compute the Schur function $s_{(2,1)}(\mathbf{x})$. Thus we need to inspect each SSYT of shape $\lambda = (2,1)$. Let $\mathcal{T}$ be such a tableau and let i, j, and k be the entries of $\mathcal{T}$ as shown below.

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Since the columns of $\mathcal{T}$ must strictly increase, we require that $i < k$, and, since the rows of $\mathcal{T}$ weakly increase, we require $i \leq j$. Thus

$$\begin{aligned}
s_{(2,1)}(\mathbf{x}) &= \sum_{i \leq j,\ i < k} x_i x_j x_k \\
&= \left(\sum_j \sum_{i<k} x_i x_j x_k \right) - \sum_{j<i<k} x_i x_j x_k \\
&= \left(\sum_j x_j \right) \left(\sum_{i<k} x_i x_k \right) - \sum_{j<i<k} x_i x_j x_k \\
&= e_{(1)}(\mathbf{x}) e_{(2)}(\mathbf{x}) - e_{(3)}(\mathbf{x}).
\end{aligned}$$

We note that when $\mathbf{x}$ is taken to be a finite set of variables (for example, when setting all $x_i = 0$ for each $i > n$), then $s_\lambda(\mathbf{x})$ is often called a *Schur polynomial.*

We shall henceforth drop reference to the variables $\mathbf{x}$ and just write s_λ in place of $s_\lambda(\mathbf{x})$, and similarly with the other symmetric functions.

1.5 The Littlewood-Richardson Rule

Since the set $\{s_\lambda | \lambda \vdash n\}$ is a basis of Λ^n, any homogeneous symmetric function of degree n can be written as a unique linear combination of the s_λ. In particular, for any partitions μ and ν,

$$s_\mu s_\nu = \sum_{\lambda \vdash n} c^\lambda_{\mu\nu} s_\lambda, \tag{1.4}$$

where $n = |\mu| + |\nu|$, for some coefficients $c^\lambda_{\mu\nu}$. The coefficients $c^\lambda_{\mu\nu}$ appear in many other expressions (see Section 1.7) and are called the *Littlewood-Richardson coefficients.*

The Littlewood-Richardson coefficients also appear as the coefficients of the skew Schur function $s_{\lambda/\mu}$. That is, for any skew diagram λ/μ,

$$s_{\lambda/\mu} = \sum_{\nu \vdash n} c^\lambda_{\mu\nu} s_\nu, \tag{1.5}$$

where $n = |\lambda| - |\mu|$. In addition, the Littlewood-Richardson coefficients are non-negative integers and there is a certain collection of SSYTx that they count.

Given a tableau $\mathcal{T}$, the *reading word* of $\mathcal{T}$ is the sequence of integers obtained by reading the entries of the rows of $\mathcal{T}$ from right to left, proceeding from the top row to the bottom. We say that a sequence $r = r_1, r_2, \ldots, r_k$ is *lattice* if, for each j, when reading the sequence from left to right the number of j's that we have read is never less than the number of $j+1$'s that we have read.

Theorem 1.5.1 (Littlewood-Richardson Rule) *([43])*
For partitions λ, μ, and ν, the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$ is the number of SSYTx of shape λ/μ, content ν, with lattice reading word.

We note that the content of a tableau was defined as a weak composition, not a partition. Nevertheless, the lattice condition forces any tableau with lattice reading word to have a content that is in fact a partition.

Example Let us compute the skew Schur function $s_{(4,3,2)/(2,2)}$ by using Equation 1.5 and the Littlewood-Richardson rule. Thus we need to find all SSYTx of shape $(4,3,2)/(2,2)$ with lattice reading word, then $c^{(4,3,2)}_{(2,2)\nu}$ is the number of these tableaux with content ν.

We leave it to the reader to check that the following four tableaux are the *only* SSYTx of shape $(4,3,2)/(2,2)$ with lattice reading word.

$$\begin{array}{ccccc} & 1 & 1 & & \\ & 2 & & & \\ 1 & 1 & & & \end{array} \qquad \begin{array}{ccccc} & 1 & 1 & & \\ & 2 & & & \\ 1 & 2 & & & \end{array} \qquad \begin{array}{ccccc} & 1 & 1 & & \\ & 2 & & & \\ 1 & 3 & & & \end{array} \qquad \begin{array}{ccccc} & 1 & 1 & & \\ & 2 & & & \\ 2 & 3 & & & \end{array}$$

$\nu = (4,1)$ $\qquad$ $\nu = (3,2)$ $\qquad$ $\nu = (3,1,1)$ $\qquad$ $\nu = (2,2,1)$

Therefore $s_{(4,3,2)/(2,2)} = s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)}$.

For any $f = \sum_{\lambda \vdash n} a_\lambda s_\lambda \in \Lambda^n$, we say that f is *Schur-positive*, and write $f \geq_s 0$, if each $a_\lambda \geq 0$. Further, we say that f is *multiplicity-free* if each $a_\lambda \in \{0,1\}$. Therefore, the Littlewood-Richardson rule shows that both $s_\mu s_\nu$ and $s_{\lambda/\mu}$ are Schur-positive. For $f, g \in \Lambda^n$, we will be interested in whether or not the difference $f - g$ is Schur positive. We shall write $f \geq_s g$ whenever $f - g$ is Schur-positive. If neither $f - g$ nor $g - f$ is Schur-positive we say that f and g are *Schur-incomparable*. Further, we will find it convenient to write $D_1 \succeq_s D_2$ if $s_{D_1} \geq_s s_{D_2}$.

If we consider the relation $\succeq_s$ on the set of all Schur-equivalent classes of diagrams (i.e. $[D]_s = \{D' | s_D = s_{D'}\}$), then $\succeq_s$ defines a partial ordering. This allows us to view the Hasse diagram for the relation $\succeq_s$ on the set of these Schur-equivalent classes of skew diagrams.

1.6 Some Useful Identities for Schur Functions

In this section we list a few easy to state results that shall prove useful.

The first two results are commonly know as Pieri rules. The origins of these result can be found in [54]. To state these two results we require the following two definitions. A *row strip* is a skew diagram with no two boxes in the same column and a *column strip* is a skew diagram with no two boxes in the same row. The Pieri rules can now be stated as follows.

Theorem 1.6.1 *We have*

$$s_\nu s_{(n)} = \sum_\lambda s_\lambda,$$

where the sum is over all partitions λ such that $\lambda/(\nu)$ is a row strip of size n.

Theorem 1.6.2 *We have*

$$s_\nu s_{(1^n)} = \sum_\lambda s_\lambda,$$

where the sum is over all partitions λ such that $\lambda/(\nu)$ is a column strip of size n.

The Pieri rules have been generalized in such work as [1], [5], and [52].

We now give two results that we shall use often.

Theorem 1.6.3 *[[69], Exercise 7.56(a)] Given a skew diagram D,*

$$s_D = s_{D^\circ}. \tag{1.6}$$

A result on the Littlewood-Richardson coefficients of conjugate diagrams can be found in [30].

For products, we have the following identity.

Theorem 1.6.4 *The Schur function of any disconnected skew diagram is reducible. If $D = D_1 \oplus D_2$, then we have*

$$s_D = s_{D_1} s_{D_2}. \tag{1.7}$$

Proof Any SSYT of shape $D_1 \oplus D_2$ gives rise to a SSYT of shape D_1 and a SSYT of shape D_2 by taking the obvious subtableaux. Conversely, any SSYTx $\mathcal{T}_1$ of shape D_1 and $\mathcal{T}_2$ of shape D_2 give rise to the tableau $\mathcal{T}_1 \oplus \mathcal{T}_2$ of shape $D_1 \oplus D_2$, which is clearly semistandard. ∎

Lastly, we mention another expression for the product $s_{D_1} s_{D_2}$.

Theorem 1.6.5 *[[45], Chapter 1.5, Example 2.1 (a)]*
For skew diagrams D_1 and D_2, we have

$$s_{D_1} s_{D_2} = s_{D_1 \cdot D_2} + s_{D_1 \odot D_2}. \tag{1.8}$$

Proof Given SSYTx $\mathcal{T}_1$ of shape D_1 and $\mathcal{T}_2$ of shape D_2 we let a be the entry of the top-right box of D_1 and b be the entry of the bottom-left box of D_2. If $a \leq b$ then $\mathcal{T}_1 \odot \mathcal{T}_2$ is a SSYT, and if $a > b$ then $\mathcal{T}_1 \cdot \mathcal{T}_2$ is a SSYT, and clearly all SSYTx of shape $D_1 \odot D_2$ and $D_1 \cdot D_2$ arise in this fashion. ∎

1.7 A Brief History of Schur Functions

Schur polynomials and Schur functions have appeared in various mathematical contexts during the last two centuries. Their first definition was given in terms of certain *bialternants*. That is, as a quotient of alternants. Namely, given a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, we may obtain the *alternant*

$$a_\mu(x_1, \ldots, x_l) = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) x_1^{\mu_{\pi(1)}} \cdots x_l^{\mu_{\pi(l)}},$$

where the S_l is the group of permutations of $\{1, 2, \ldots, l\}$, and $\operatorname{sgn}(\pi)$ is the *sign* of the permutation π. Then, taking a partition λ of length l and using $\delta = (l-1, l-2, \ldots, 1, 0)$, the Schur polynomial can be defined as

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

These functions first appeared in a paper of Cauchy [15], dating 1815, in which he was able to prove that this rational function was in fact a polynomial.

The next noteworthy appearance of Schur functions is thanks to Jacobi [32] in 1841, in which the Schur functions were again defined as above and a form of the now-famous Jacobi-Trudy identity was given. This identity gives another method of computing s_λ, and can be stated as follows.

Theorem 1.7.1 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, then*

$$s_\lambda = |h_{\lambda_i - i + j}|$$

and

$$s_{\lambda^t} = |e_{\lambda_i - i + j}|$$

where λ^t is the conjugate of λ and both determinants are $l \times l$.

Throughout this time the Schur functions had not yet been given the name that we presently know them by.

Issai Schur, born in 1875, also used the bialternant definition of s_λ in his dissertation [66]. However, his interest in Schur functions arose from their connection to the study of representation theory, a topic whose origins can be traced to a collection of papers by Frobenius [20, 21, 22] dating from 1896 to 1897. Although the definition of representations did not appear until Frobenius' paper of 1897, his series of papers in 1896 introduced the character

theory of finite groups, which is a fundamental aspect of represenation theory. Schur was one of Frobenius' doctoral students, and he, together with notable others, furthered the development of representation theory and the theory of characters in his doctoral dissertation and in his subsequent work.

In the study of characters of S_n-modules, there is a bijection taking the irreducible character χ^λ to the function s_λ. It was by this type of connection between representations of the symmetric group and homogeneous symmetric functions that Schur came to study the functions s_λ, which were later called Schur functions in his honour.

For a detailed discussion of the pioneering work on representation theory, it is recommended that the reader consult Curtis [17]. This source includes a rich bibliography of several mathematicians who helped shape this field as well as provides an excellent survey of the development of representation theory. Other sources for the representation theory behind Schur functions can be found in [3] and [63]. Details on the symmetric function side can be found in [45] and [69]. More on the history of group characters can be found in [31].

Our preferred description of Schur functions is the combinatorial definition given by Equation 1.2. This description, together with Equation 1.4 and Theorem 1.5.1, allows us to work with Schur functions through purely combinatorial arguments. The equivalence between the combinatorial and the original definition of s_λ can be found in [12].

The first steps of this combinatorial description began with Kostka [36] decomposing the Schur functions into monomial symmetric functions via

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

Mitchell [50] was first to show that the *Kostka numbers* $K_{\lambda\mu}$ were non-negative. Finally Littlewood [42] posited that the *Kostka numbers* $K_{\lambda\mu}$ counted the number of SSYTx of shape λ and content μ.

The first version of the Littlewood-Richardson rule, Theorem 1.5.1, was given by Littlewood and Richardson [43]. The first complete proof of the statement was due to Schützenberger [67] in 1977. Short proofs can be found in [23], [60], and [77]. Some other objects that the Littlewood-Richardson coefficients enumerate can be found in [13], [55], and [80].

One of the early attempts to prove the Littlewood-Richardson rule led Robinson to an early form of the RSK (Robinson-Schensted-Knuth) algorithm [61]. Schensted [65] independently developed the algorithm in order to enumerate permutations by the length of their longest increasing and decreasing sequences. Knuth [35] extended this algorithm to its commonly used

form. Further extension to the RSK algorithm have been investigated and can be found in [46, 64, 79]. Further topics on the Littlewood-Richardson rule and the RSK algorithm are discussed in [72, 73, 75, 78].

Through the connection to representation theory, Equation 1.4 gives rise to the equation

$$\chi^{\mu} \cdot \chi^{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu} \chi^{\lambda},$$

describing the multiplicities of the irreducible character χ^{λ} in the product of the irreducible characters χ^{μ} and χ^{ν}, and

$$(S^{\mu} \otimes S^{\nu}) \uparrow^{S_n} = \bigoplus_{\lambda} c^{\lambda}_{\mu\nu} S^{\lambda},$$

describing the multiplicities of the Specht module S^{λ} in the induced tensor product of the Specht modules S^{μ} and S^{ν}.

In the completely different setting of enumerative geometry, the same calculations were being performed under the guise of *Schubert Calculus*, which was introduced by H. C. H. Schubert. In the cohomology ring of the Grassmanian, the *cup product* of two *Schubert classes* σ_{μ} and σ_{ν} is given by

$$\sigma_{\mu} \cup \sigma_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu\nu} \sigma_{\lambda}.$$

This was first proved by Carrell in [14]. Some studies of Littlewood-Richardson coefficients from this area include [11], [16], and [56]. Results specializing on multiplicity-free Schubert calculus can be found in [27] and [74].

Thus we see how the structure of the Schur functions plays an important role in each of these areas. Equally important is the ability to compute the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$. However, it is known that the problem of computing the Littlewood-Richardson coefficients is $\#P$-complete [51]. Therefore it is unlikely that there any efficient algorithms to compute these coefficients. An estimate on the size of the Littlewood-Richardson coefficients is given in [58]. A discussion on the many connections of Littlewood-Richardson coefficients to tableaux and the operations and algorithms on tableaux is given in [19]. A recent look at the possibility of factoring Littlewood-Richardson coefficients in special cases is researched in [33].

Other products on the Schur functions have also been studied. The Kronecker product, for instance, is examined in [2, 10, 62]

Despite the difficulty in computing Littlewood-Richardson coefficients, there is much research into Schur function and skew Schur functions. Many

recent results on determining the equivalence classes $[D]_s$ of Schur-equivalent skew Schur functions can be found in [8, 24, 26, 48, 59], and several results on the Schur-positivity of certain differences can be found in [6, 25, 34, 37, 47, 49, 57]. Each of these Schur-positive differences may be viewed as giving a set of inequalities that the corresponding Littlewood-Richardson coefficients must satisfy. In the next section we shall survey many of the known instances of Schur-positive differences and describe the Schur-positivity results that are proved in this thesis.

Instances of Schur-positivity can lead to interesting results. For instance, a Schur-positivity result is used in [18] to characterize the eigenvalues of Hermitian matrix.

However, the story does not end with Schur functions. There have been many analogues and generalisations of the Schur functions since their creation. There are the Hall-Littlewood polynomials, the Jack polynomials, and the Macdonald polynomials. Information on these polynomials can be found in [44] and [45]. There are generalizations into the realm of quasisymmetric functions. Some results in this area are discussed in [8], [29], and [39]. There are also Schur P-functions, Schur Q-functions, and k-Schur functions [71].

These new functions share many of the same properties as the original Schur functions. Also, we are interested in many of the same questions. For example, the possible equalities between a Schur and skew Schur function was given in [76] and the Schur Q-function analogue of this question is addressed in [4] and [28]. In [70], Stembridge determined which products of Schur functions was multiplicity-free and Bessenrodt [7] classified the multiplicity-free products of Schur P-functions. Another multiplicity-free result on Schur P-functions is given in [68], in which it is determined when a Schur P-function expressed in terms of regular Schur functions is multiplicity-free. Some results regarding the k-Schur functions can be found in [9] and [40].

Many new results of the classical Schur functions and of the more recent generalizations appear each year as attempts are made to better describe and understand their composition and their implications in the varied areas they appear. The breadth of open questions regarding these functions will serve to keep mathematical interest in this area open for a long time.

1.8 Overview of Schur-Positivity Results

In this section we shall outline the results that are proved in the course of this thesis. However, we begin by surveying many of the Schur-positive results that have been proved in recent years. To state many of these results,

we must make several definitions and discuss many ways of defining new partitions from given partitions. The definitions that are introduced in this section to describe these known Schur-positivity results will not be used in the remainder of this thesis.

Before we begin, we shall first define a special type of diagram that will appear in several of these results. A *ribbon* is a connected skew diagram that does not contain the diagram $(2,2)$. Similarly, a *weak ribbon* is a skew diagram that does not contain the diagram $(2,2)$.

We now begin by mentioning some of the known necessary conditions that are required for a difference of skew Schur functions to be Schur-positive.

Given a skew diagram λ/μ with r rows. Then the *k-th row overlap sequence* is defined by

$$r_k(\lambda/\mu) = (a_1, \ldots, a_{r-k+1}),$$

where a_i is the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$. Then define $\text{rows}_k(\lambda/\mu)$ to be the partition obtained by placing the values of $r_k(\lambda/\mu)$ in weakly decreasing order. Similarly $\text{cols}_k(\lambda/\mu)$ may be defined by inspecting the column overlaps of λ/μ. Also, one can define $\text{rect}_{k,l}(\lambda/\mu)$ to be the number of $k \times l$ rectangular subdiagrams contained inside λ/μ.

For partitions λ and μ, we write $\lambda \preceq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

for each $i = 1, \ldots, l(\lambda)$, where μ_i is 0 for any $i > l(\mu)$. The relation $\preceq$ defines a partial ordering on the set of partitions called the *dominance order*.

In [47], the following necessary conditions for Schur-positive differences are proved.

Theorem 1.8.1 *(Corollary 3.10, [47])*
Let λ/μ and ν/ρ be skew shapes. If $s_{\lambda/\mu} - s_{\nu/\rho} \geq_s 0$, then

1. *$\text{rows}_k(\lambda/\mu) \preceq \text{rows}_k(\nu/\rho)$ for all k,*
2. *$\text{cols}_l(\lambda/\mu) \preceq \text{cols}_l(\nu/\rho)$ for all l, and*
3. *$\text{rect}_{k,l}(\lambda/\mu) \preceq \text{rect}_{k,l}(\nu/\rho)$ for all k, l.*

Given a skew diagram λ/μ, the *base partition* of λ/μ is denoted $\mathcal{B}(\lambda/\mu)$ and is defined as the intersection of all diagrams ν such that $c^{\lambda}_{\mu\nu} \neq 0$. The *cover partition* of λ/μ is denoted by $\mathcal{C}(\lambda/\mu)$ and is the union of all diagrams

ν such that $c^{\lambda}_{\mu\nu} \neq 0$. In [25], it is remarked that $s_{\lambda/\mu} - s_{\nu/\rho} \geq_s 0$ requires that

$$\mathcal{B}(\lambda/\mu) \subseteq \mathcal{B}(\nu/\rho) \text{ and } \mathcal{C}(\nu/\rho) \subseteq \mathcal{C}(\lambda/\mu).$$

These give two additional necessary conditions for the existence of a Schur-positive difference. Alternative methods of computing the base and cover partitions are determined in [25], which gives a method of checking if these conditions are satisfied.

Having looked at some necessary conditions, we now list a variety of instances of Schur-positivity. Many of these results are phrased as differences of products of Schur functions rather than differences of skew Schur functions. However, since any product of Schur functions can be realized as a skew Schur function via Theorem 1.6.4, these can all be restated as instances of Schur-positive differences of skew Schur functions.

We begin with two Schur-positivity results discussed in [6] involving the operation $*$. Conjecture 1.8.2 was first stated in [18] and the later generalized form, Conjecture 1.8.3, was put forth in [6]. Given an ordered pair of partitions (λ, μ), a new ordered pair $(\lambda, \mu)^* = (\lambda^*, \mu^*)$ is defined by taking

$$\lambda^*_k := \lambda_k - k + \#\{l | \mu_l - l \geq \lambda_k - k\} \text{ for each } k, \text{ and}$$

$$\mu^*_l := \mu_l - l + \#\{k | \lambda_k - k > \mu_l - l\} \text{ for each } l.$$

Conjecture 1.8.2 *(Conjecture 5.1, [18])*
For partitions λ and μ we have

$$s_{\lambda^*} s_{\mu^*} - s_\lambda s_\mu \geq_s 0.$$

The definition of $*$ can also be extended to pairs of skew shapes $(\mu/\alpha, \nu/\beta)$ by taking

$$(\mu/\alpha, \nu/\beta)^* := (\mu, \nu)^*/(\alpha, \beta)^*.$$

Conjecture 1.8.3 *(Conjecture 2.9, [6])*
For any skew partitions μ/α and ν/β, if

$$(\lambda, \rho) = (\mu/\alpha, \nu/\beta)^*, \text{ then}$$

$$s_\lambda s_\rho - s_{\mu/\alpha} s_{\nu/\beta} \geq_s 0.$$

Some instances of Conjecture 1.8.2 and Conjecture 1.8.3 have been addressed in [6]. Namely, the following two results, Theorem 1.8.4 and Theorem 1.8.5, have been proved. The first describes several pairs for which these Conjectures hold, while the second shows that the bounded height case reduces to checking Conjecture 1.8.2 for a finite number of pairs.

Theorem 1.8.4 *(Theorem 3.1, [6])*
Conjecture 1.8.2 (or Conjecture 1.8.3) holds

1. *For any pair (λ, μ) of hook shapes.*

2. *For skew pairs of the form $(\mu/\alpha, \nu/\beta)$, where μ, ν, α, β are hooks with $\alpha = \beta$.*

3. *For skew pairs of the form $(0, \nu/\beta)$, with ν/β a weak ribbon.*

Theorem 1.8.5 *(Theorem 3.2, [6])*
For any positive integer p, let ν be a fixed partition with at most p parts. Then, the validity of Conjecture 1.8.2 for the infinite set of all pairs (μ, ν), with $l(\mu) \leq p$, reduces to checking the validity of the conjecture for the finite set of pairs (α, ν), with α having at most p parts, and largest part bounded as follows:

$$\alpha_1 \leq p(\nu_1 + p).$$

We shall now list several other operations on partitions that have given rise to Schur-positive differences.

Given a partition λ with each part even we define $\frac{\lambda}{2} := (\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \ldots)$.

Theorem 1.8.6 *(Conjecture 1, [37])*
For two skew shapes λ/μ and ν/ρ such that $\lambda + \nu$ and $\mu + \rho$ both have even parts, we have

$$s_{\frac{\lambda+\nu}{2}/\frac{\mu+\rho}{2}} - s_{\lambda/\mu} s_{\nu/\rho} \geq_s 0.$$

A generalization of this result may be stated as follows.

Theorem 1.8.7 *(Theorem 11, [37])*
Let λ/μ and ν/ρ be any two skew shapes. Then we have

$$s_{\lfloor\frac{\lambda+\nu}{2}\rfloor/\lfloor\frac{\mu+\rho}{2}\rfloor} s_{\lceil\frac{\lambda+\nu}{2}\rceil/\lceil\frac{\mu+\rho}{2}\rceil} - s_{\lambda/\mu} s_{\nu/\rho} \geq_s 0.$$

Given partitions λ and μ, we have the partition $\nu = \lambda \cup \mu$ obtained by listing all the parts of λ and μ together in weakly decreasing order. We then set $\mathrm{sort}_1(\lambda, \mu) = (\nu_1, \nu_3, \nu_5, \ldots)$ and $\mathrm{sort}_2(\lambda, \mu) = (\nu_2, \nu_4, \nu_6, \ldots)$.

Theorem 1.8.8 *(Conjecture 2.7, [18]) Given partitions λ and μ, we have*

$$s_{sort_1(\lambda,\mu)} s_{sort_2(\lambda,\mu)} - s_\lambda s_\mu \geq_s 0.$$

This result may be generalized to skew shapes as follows.

Theorem 1.8.9 *(Corollary 12, [37])*
Let λ/μ and ν/ρ be skew shapes. Then we have

$$s_{sort_1(\lambda,\nu)/sort_1(\mu,\rho)} s_{sort_2(\lambda,\nu)/sort_2(\mu,\rho)} - s_{\lambda/\mu} s_{\nu/\rho} \geq_s 0.$$

Given a partition λ, let $\lambda^{[i,n]} := (\lambda_i, \lambda_{i+n}, \lambda_{i+2n}, \ldots)$. In this way the parts of λ are distributed among the partitions $\lambda^{[i,n]}$ for $i = 1, \ldots, n$.

Theorem 1.8.10 *(Conjecture 6.4, [41])*
For integers $1 \leq m \leq n$ and a partition λ, we have

$$\prod_{i=1}^{n} s_{\lambda^{[i,n]}} - \prod_{i=1}^{m} s_{\lambda^{[i,m]}} \geq_s 0.$$

A generalization of this result to skew shapes can also be given.

Theorem 1.8.11 *(Theorem 13, [37])*
Let $\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(n)}/\mu^{(n)}$, be n skew shapes, let $\lambda = \bigcup \lambda^{(i)}$ be the partition obtained by the decreasing rearrangement of the parts in all $\lambda^{(i)}$, and $\mu = \bigcup \mu^{(i)}$. Then we have

$$\prod_{i=1}^{n} s_{\lambda^{[i,n]}/\mu^{[i,n]}} - \prod_{i=1}^{n} s_{\lambda^{(i)}/\mu^{(i)}} \geq_s 0.$$

Given two partitions λ and μ, we can define $\lambda \vee \mu$ and $\lambda \wedge \mu$ by

$$\lambda \vee \mu := (\max\{\lambda_1, \mu_1\}, \max\{\lambda_2, \mu_2\}, \ldots) \text{ and}$$

$$\lambda \wedge \mu := (\min\{\lambda_1, \mu_1\}, \min\{\lambda_2, \mu_2\}, \ldots).$$

Then, given skew diagrams λ/μ and ν/ρ, the operators $\vee$ and $\wedge$ can be defined on pairs of skew diagrams via

$$(\lambda/\mu) \vee (\nu/\rho) := (\lambda \vee \nu)/(\mu \vee \rho) \text{ and}$$

$$(\lambda/\mu) \wedge (\nu/\rho) := (\lambda \wedge \nu)/(\mu \wedge \rho).$$

Theorem 1.8.12 *(Conjecture 4.6, [38]) Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{(\lambda/\mu)\vee(\nu/\rho)} s_{(\lambda/\mu)\wedge(\nu/\rho)} - s_{\lambda/\mu} s_{\nu/\rho} \geq_s 0.$$

Theorem 1.8.6 was first conjectured in [53], Theorem 1.8.8 was first conjectured in [18], Theorem 1.8.10 was first conjectured in [41], and Theorem 1.8.12 was first conjectured in [38]. In [37], Theorem 1.8.12 was proved and was then used to prove each of Theorem 1.8.7, Theorem 1.8.9, and Theorem 1.8.11, which in turn imply each of Theorem 1.8.6, Theorem 1.8.8, and Theorem 1.8.10, respectively.

We have seen many operations that construct examples of Schur-positive differences. A seperate problem to investigate involves finding families of skew diagrams for which all instances of Schur-positive differences within these families can be determined.

One result of this type occurs in [49], in which the collection of multiplicity-free ribbons is inspected. Using results of [27], multiplicity-free ribbons are classified as those ribbons with at most two rows of length greater than one and at most two columns of length greater than one. Then, among the collection of multiplicity-free ribbons of a given size, the Hasse diagram which describes all Schur-positive differences and Schur-incomparabilities among these diagrams is explicitly described. Moreover, the Hasse diagram is essentially a product of two chains.

We now begin to inspect another collection of ribbons for which the question of Schur-positivity is completely answered. It is not difficult to see that the skew diagram λ/μ is uniquely determined by the row overaps $\text{rows}_1(\lambda/\mu)$ and $\text{rows}_2(\lambda/\mu)$, thus we may identify the skew Schur function using the overlap notation

$$s_{\lambda/\mu} = \{\text{rows}_1(\lambda/\mu)|\text{rows}_2(\lambda/\mu)\}.$$

In the case when λ/μ is a ribbon we have

$$s_{\lambda/\mu} = \{\alpha|1^{l(\lambda)-1}\},$$

where α is the composition given by the row lengths of λ/μ. In this situation we use the notation

$$r_\alpha := s_{\lambda/\mu} = \{\alpha|1^{l(\lambda)-1}\}$$

and call r_α a *ribbon Schur function.*

In [8], it was shown that the collection $\{r_\lambda\}_{\lambda\vdash n}$ forms a basis of Λ^n. In [34], the following theorem was proved, which identifies the Schur-positive differences among this collection.

Theorem 1.8.13 *(Theorem 3.3 , [34])*

Let λ and μ be partitions of n, then

$$r_\mu - r_\lambda \geq_s 0$$

if and only if $\mu \preceq \lambda$ and $l(\lambda) = l(\mu)$.

More Schur-positive differences were discovered in [34], and are easiest to state in terms of this overlap notation. We begin by considering the following hypothesis on compositions σ and τ.

Hypothesis 1.8.14 *Let σ and τ be compositions such that $l(\sigma) = s \geq 0$ and $l(\tau) = t \geq 0$, and let $\bar{\sigma}$ and $\bar{\tau}$ be sequences of non-negative integers that satisfy the following conditions:*

1. *The lengths of $\bar{\sigma}$ and $\bar{\tau}$ are s and t respectively;*
2. *$\bar{\sigma}_s = 1$ when $s > 0$;*
3. *$\bar{\tau}_1 = 1$ when $t > 0$;*
4. *$\bar{\sigma}_i \leq \min\{\sigma_i, \sigma_{i+1}\}$ for $1 \leq i < s$, and $\bar{\tau}_i \leq \min\{\tau_i, \tau_{i-1}\}$ for $1 < i \leq t$.*

Under this hypothesis, [34] proves that the following differences of skew Schur functions are Schur-positive.

Theorem 1.8.15 *(Theorem 3.1, [34])*
Assume σ, τ, $\bar{\sigma}$, and $\bar{\tau}$ satisfy Hypothesis 1.8.14. If $a \geq b \geq 2$ then we have

$$\{\sigma, a, b, \tau|\bar{\sigma}, 1, \bar{\tau}\} - \{\sigma, a+1, b-1, \tau|\bar{\sigma}, 1, \bar{\tau}\} \geq_s 0 \text{ and}$$
$$\{\sigma, a^n, \tau|\bar{\sigma}, 1^{n-1}, \bar{\tau}\} - \{\sigma, a+1, a^{n-2}, a-1, \tau|\bar{\sigma}, 1^{n-1}, \bar{\tau}\} \geq_s 0.$$

We now turn to describing the results and the flow of this thesis.

We begin in Chapter 2 by defining *staircases* and *fat staircases*. We then describe how we augment these staircases with a skew diagram called the *foundation* by attaching the skew diagram to the staircase via a parameter k. The augmented diagrams of this type are called *staircases with bad foundations* and will remain the focus for nearly all of this thesis.

In Section 2.3 we define what it means for a diagram to be a *sum of fat staircases* and we prove Theorem 2.3.1 that, for each sum of fat staircases D, provides a collection of Schur-positivity results. Namely, we obtain an instance of Schur-positivity for each choice of the foundation. The remainder

of Section 2.3 attempts to determine conditions for a diagram D to be a sum of fat staircases.

In Chapter 3 we fix a fat staircase and a value $k \geq 0$ and, for a given hook length, consider all possible hook foundations. We determine the structure of the entire Hasse diagram, which describes which pairs of the staircases with hook foundations have Schur-positive difference and which are Schur-incomparable. In Section 3.1 we describe the Hasse diagram for values $0 \leq k \leq 1$ via Theorem 3.1.1, Theorem 3.1.2, Theorem 3.1.3, Theorem 3.1.4, and Theorem 3.1.5. In Section 3.3 we extend these results to describe the Hasse diagram for values $k > 1$ by proving Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.4, Theorem 3.3.5, and Theorem 3.3.6. In Section 3.2 the expression for each Schur-positive difference for values $0 \leq k \leq 1$ are computed by Theorem 3.2.2 and Theorem 3.2.3.

In Chapter 4 we inspect the case of differences of fat staircases with bad foundations in which the foundations of the two diagrams are transposes of each other. In Theorem 4.1.1 and Theorem 4.2.1, we show that the difference is Schur-positive when one foundation is taken as either a single row or any two row partition. Also, when reducing to finitely many variables, we determine the largest number of variables for which this difference is zero.

Most of the results of the earlier chapters are extended in Chapter 5 to yield results for fat staircases with complementary foundations. In particular, for a given fat staircase and hook length, Theorem 5.2.1 describes the Hasse diagram for the collection of fat staircases with hook complement foundations and the Schur-positive differences is calculated by Theorem 5.2.2 and Theorem 5.2.3. Theorem 5.4.3 gives a collection of Schur-positive results from each sum of fat staircases D. Further, Theorem 5.3.1 and Theorem 5.3.2 allow us to construct many examples of sums of fat staircases. Finally, in Chapter 6 we once again consider reducing Schur functions to finitely many variables and prove several equalities of skew Schur functions that arise in this setting.

The majority of our results make use of Lemma 2.2.2 and Lemma 2.2.3, which are proved in Section 2.2, and allow us to prove these results of Schur-positivity by focusing solely on fillings of the foundation. These lemma provide much simplification while considering these skew Schur functions.

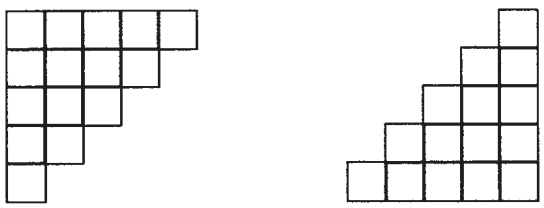
Chapter 2

Staircases with Bad Foundations

2.1 Regular Staircases

A Ferrers diagram is a *staircase* if it is the Ferrers diagram of a partition of the form $\lambda = (n, n-1, n-2, \ldots, 2, 1)$ or if it is the 180° rotation of such a diagram. Both these diagrams are referred to as *staircases of length* n and will be denoted by δ_n and Δ_n respectively. For what follows we will also define $\delta_{-1} = \delta_0 = \emptyset$.

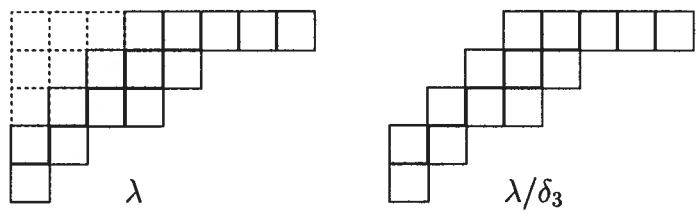
Example Here we see the two staircases of length 5:



δ_5 Δ_5

If $\delta_n \subseteq \lambda$ as Ferrers diagrams then we say that λ *contains a staircase of size* n. In these cases we may create the skew diagram λ/δ_n.

Example Here we see that the partition $\lambda = (8, 5, 4, 2, 1)$ contains a staircase of size 3 and the corresponding skew diagram λ/δ_3.



In this example λ actually contains a staircase of size 4 and a staircase of size 5, but the skew diagrams obtained by removing these staircases are not connected.

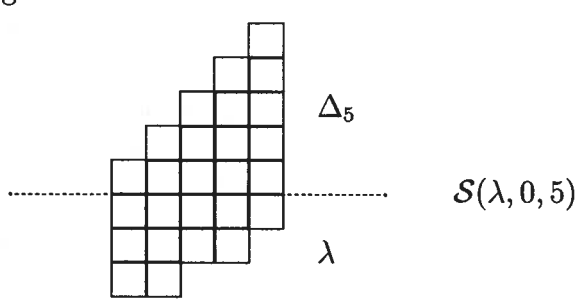
Definition Given $n \geq 1$, $k \geq 0$, and a partition λ containing a staircase δ_{k-1} such that skew diagram λ/δ_{k-1} is connected and $k \leq \lambda_1 \leq n+k$, then we can create a skew diagram called a *staircase with a bad foundation*, denoted $\mathcal{S}(\lambda, k, n)$, by placing λ/δ_{k-1} immediately below Δ_n such that the rows of the two diagrams overlap in precisely $\lambda_1 - k$ positions. We call the subdiagram λ/δ_{k-1} the *foundation* of $\mathcal{S}(\lambda, k, n)$.

We use the terminology "bad foundation" since if we imagine the diagram $\mathcal{S}(\lambda, k, n)$ being pulled downward by a gravitational force then, for most choices of the foundation λ, the foundation would fail to support $\mathcal{S}(\lambda, k, n)$ properly, causing the diagram to tip over to the right.

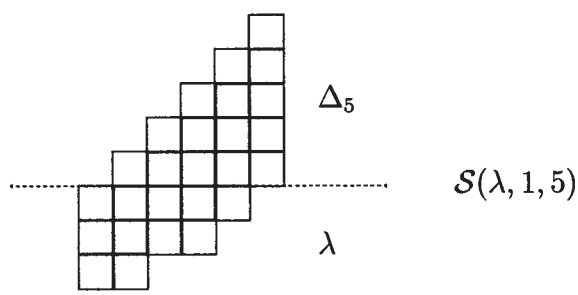
In the staircase with bad foundation $\mathcal{S}(\lambda, k, n)$, k is the distance to the left of Δ_n that λ extends before the copy of δ_{k-1} is removed. The requirement $k \leq \lambda_1$, guarantees that the top row of the skew diagram λ/δ_{k-1} is non-empty. The diagram $\mathcal{S}(\lambda, k, n)$ is connected if $k < \lambda_1$ and has two connected components, λ/δ_{k-1} and Δ_n, if $k = \lambda_1$. We also note that the skew diagram $\mathcal{S}(\lambda, k, n)$ can be written in the form μ/δ_{n+k-1}, where μ is a partition.

Example

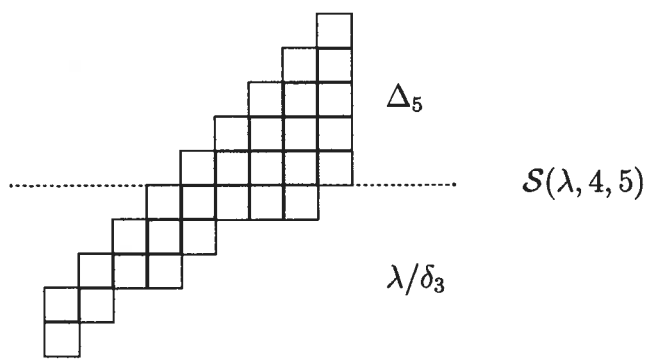
For $k = 0$ we have $\delta_{k-1} = \delta_{-1} = \emptyset$, and Δ_n and λ overlap in λ_1 places. Thus $\mathcal{S}(\lambda, k, n)$ consists of the foundation λ left-justified with the staircase Δ_n. For example, with $n = 5$, $k = 0$, and $\lambda = (5, 4, 2)$ we have the following staircase with bad foundation:



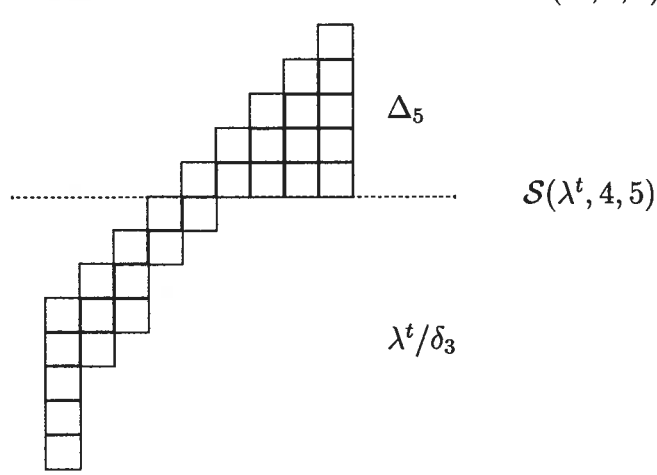
For $k = 1$ we have $\delta_{k-1} = \delta_0 = \emptyset$, and Δ_n and λ overlap in $\lambda_1 - 1$ places. Thus $\mathcal{S}(\lambda, k, n)$ is the diagram $\mathcal{S}(\lambda, 0, n)$ with its foundation λ shifted one to the left. For example, with $n = 5$, $k = 1$, and $\lambda = (5, 4, 2)$ we have the following staircase with bad foundation:



Example Here we show the staircase with a bad foundation $\mathcal{S}(\lambda, 4, 5)$, where $\lambda = (8, 5, 4, 2, 1)$.



In this instance the conjugate of λ, $\lambda^t = (5, 4, 3, 3, 2, 1, 1, 1)$, also gives rise to a staircase with a bad foundation $\mathcal{S}(\lambda^t, 4, 5)$.



One of the advantages in computing the skew Schur functions of staircases with bad foundations is that, when using the Littlewood-Richarson rule, the Δ_n portion of the diagram can be filled in only one way. This can be seen algebraically from Theorem 1.6.3 since $s_{\Delta_n} = s_{\Delta_n^\circ} = s_{\delta_n}$, where s_{δ_n} is a Schur function. The unique filling of Δ_n obeying the semistandard conditions and the lattice condition is easily found to be the filling shown below.

$$\begin{array}{ccc|c|ccc}
 & & & & & & 1 \\
 & & & & & 1 & 2 \\
 & & & & 1 & 2 & 3 \\
 & & & & & \vdots & \\
 & & 1 & & n-4 & n-3 & n-2 \\
 & 1 & 2 & \cdots & n-3 & n-2 & n-1 \\
1 & 2 & 3 & & n-2 & n-1 & n
\end{array}$$

Another advantage in computing the skew Schur function of a staircase with bad foundation is the following fact, which arises from the structure of the unique content, $\nu = \delta_n = (n, n-1, \ldots, 2, 1)$, just mentioned.

Lemma 2.1.1 *Let $\mathcal{S}(\lambda, k, n)$ be a staircase with bad foundation and $\mathcal{T}$ be a SSYT of shape $\mathcal{S}(\lambda, k, n)$ whose reading word is lattice. Then the entries in the first row of the foundation λ/δ_{k-1} of $\mathcal{T}$ form a strictly increasing sequence.*

Proof Let $\mathcal{T}$ be a SSYT of shape $\mathcal{S}(\lambda, k, n)$ whose reading word is lattice. As discussed previously, the Littlewood-Richardson rule allows only one way of filling the Δ_n portion of the shape $\mathcal{S}(\lambda, k, n)$. The content of this filling of Δ_n is $(n, n-1, \ldots, 2, 1)$.

Let R be the first row of the foundation, and suppose that R contains the value j twice. Then $j \neq 1$ since the second value in R is immediately below a value that is greater than or equal to 1 and the columns of $\mathcal{T}$ strictly increase. Also, if $j > 1$, then when reading $\mathcal{T}$ the lattice condition will be violated once we have read both j's. Therefore no value j can be repeated in R, and hence the first row of the foundation of $\mathcal{T}$ contains no repeated values. ∎

2.2 Fat Staircases

In the previous section we saw that in creating any SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word, the following two properties held. Namely,

1. the entries of Δ_n are uniquely determined, and

2. the entries in the first row of λ/δ_{k-1} are distinct.

In this section we look into what other shapes besides the staircase Δ_n can we append a foundation, so that we still have the analogous properties.

Instead of Δ_n, we begin by looking at a general skew diagram $D = \rho/\mu$. Also, instead of removing δ_{k-1} from λ, we consider removing an arbitrary partition $\kappa \subseteq \lambda$. Then we want to append the diagram λ/κ to the bottom of D. Namely, for a given k, we place λ/κ immediately below D such that the rows of the two diagrams overlap in precisely $\lambda_1 - \kappa_1 - k$ positions. Then k is the distance to the left that the first row of λ/κ extends from the last row of D. We represent this new skew diagram by $\mathcal{S}(\lambda, \kappa, D; k)$. Now given $\mathcal{S}(\lambda, \kappa, D; k)$, we are interested in what conditions will guarantee that, when creating any SSYT $\mathcal{S}(\lambda, \kappa, D; k)$ with lattice reading word, the filling requires that

1′. the entries D are uniquely determined, and

2′. the entries in the first row of λ/κ are distinct.

We now look at which skew diagrams $\mathcal{S}(\lambda, \kappa, D; k)$ satisfy these properties.

Suppose a given skew diagram $\mathcal{S}(\lambda, \kappa, D; k)$ satisfies **1′** and **2′**. Then, by **1′**, any SSYT $\mathcal{S}(\lambda, \kappa, D; k)$ with lattice reading word has the same content for the subtableau of shape D. Let ν be this unique content. Since any SSYT of shape D with lattice reading word can be extended to a SSYT of shape $\mathcal{S}(\lambda, \kappa, D; k)$ with lattice reading word, **1′** implies that there is only one SSYT of shape D with lattice reading word, and this tableau has content ν. Therefore $s_D = s_\nu$, where ν is a partition by the Littlewood-Richardson rule. We now make use of the following result.

Theorem 2.2.1 *[76, Theorem 2.1]*
For partitions λ, μ, ν

$$s_{\lambda/\mu} = s_\nu \text{ if and only if } \lambda/\mu = \nu \text{ or } \nu^\circ.$$

Hence, if $\mathcal{S}(\lambda, \kappa, D; k)$ satisfies **1'**, then $D = \nu$ or $D = \nu^\circ$ where ν is a partition.

We now inspect property **2'**. The proof of property **2** (i.e. Lemma 2.1.1) relied on the the fact that the first row of λ/δ_{k-1} extended at most 1 box to the left of the staircase Δ_n and the fact that the content of Δ_n was the partition δ_n, which is the sequence of consecutively decreasing integers $(n, n-1, \ldots, 2, 1)$. In fact, if the first row of λ/κ extends at most 1 box to the left of the diagram D, where $D = \nu$ or $D = \nu^\circ$ for ν equal to *any sequence of consecutively decreasing integers* ending in 1 with repetitions allowed, then the same method proves that **2'** holds. That is, all partitions ν given by sequences of the form $(n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})$, where each $\alpha_i \geq 1$. This partition arises as the content of the tableaux of shape $D = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})$ and $D = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})^\circ$. We shall denote these two diagrams by

$$\delta_\alpha = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})$$

and

$$\Delta_\alpha = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})^\circ,$$

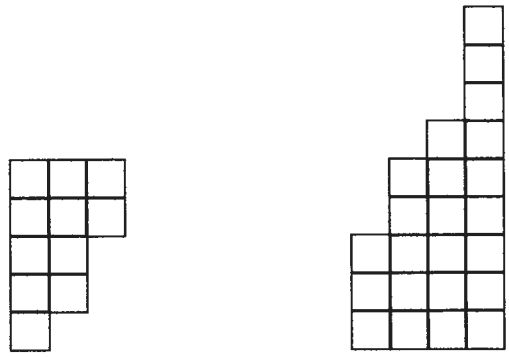
where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a composition. This composition α has $|\alpha| = \sum_{i=1}^n \alpha_i = j$ where $j = l(\delta_\alpha) = l(\Delta_\alpha)$ is the *length* of these diagrams.

We have just showed that for diagrams $D = \delta_\alpha$ and $D = \Delta_\alpha$ and $0 \leq k \leq 1$ we have that $\mathcal{S}(\lambda, \kappa, D; k)$ satisfies **1'** and **2'**. Conversely, if either $k > 1$ or if either $D = \nu$ or $D = \nu^\circ$, where $\nu_{i-1} - \nu_i > 1$ for some i, then $\mathcal{S}(\lambda, \kappa, D; k)$ cannot satisfy **2'**. This follows since the value 1 can be placed in the first row of λ/κ a total of k times and the value i can be placed in the first row of λ/κ a total of $\nu_{i-1} - \nu_i$ times. Therefore $D = \delta_\alpha$ and $D = \Delta_\alpha$ are precisely the diagrams in which we are interested.

We call a skew diagram D a *fat staircase* if $D = \delta_\alpha$ or $D = \Delta_\alpha$ for some composition α. The numbers α_i count the number of rows of D with i boxes, for each i. Using this notation the regular staircases may be expressed as $\delta_n = \delta_{(1^n)}$ and $\Delta_n = \Delta_{(1^n)}$, respectively. We note that the unique semistandard filling of Δ_α (δ_α, respectively) has $|\alpha|$ as its largest entry.

We choose the terminology "fat staircase" for describing these staircases with repeated rows in analogy to the usage of "fat hook" in [70] in which a *fat hook* is defined as a diagram of the form $(\lambda_1^{l_1}, \lambda_2^{l_2})$.

Example Here we see the fat staircases $\delta_{(1,2,2)}$ and $\Delta_{(3,1,2,3)}$.



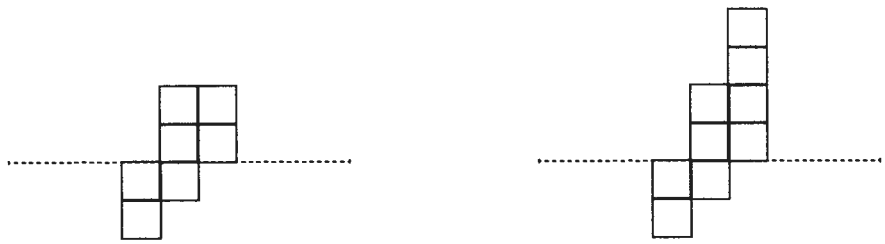
$\delta_{(1,2,2)}$ $\Delta_{(3,1,2,3)}$

Just as the diagram $\mathcal{S}(\lambda, k, n)$ was of the form μ/δ_{n+k-1}, where δ_{n+k-1} is a staircase, it would be nice if $\mathcal{S}(\lambda, \kappa, \Delta_\alpha; k)$ was of the form μ/δ_γ, where δ_γ is a fat staircase. This requires that $0 \leq k \leq 1$ and $\kappa = \delta_\beta$ for some composition β. Thus the diagrams of this form can be written as $\mathcal{S}(\lambda, \delta_\beta, \Delta_\alpha; k)$ for $0 \leq k \leq 1$. We shall be interested in the more general case of a foundation λ/μ and $k \geq 0$. Therefore we make the following definition and use the following simplified notation.

Given a composition α, $k \geq 0$, and partitions λ, μ with $\lambda_1 - \mu_1 - k \leq l(\alpha)$ we now define $\mathcal{S}(\lambda, \mu, \alpha^\triangleleft; k)$ to be the diagram obtained by placing λ/μ immediately below Δ_α such that the rows of the two diagrams overlap in precisely $\lambda_1 - \mu_1 - k$ positions. We call $\mathcal{S}(\lambda, \mu, \alpha^\triangleleft; k)$ a *fat staircase with bad foundation.* The subdiagram λ/μ is called the *foundation* of $\mathcal{S}(\lambda, \mu, \alpha^\triangleleft; k)$.

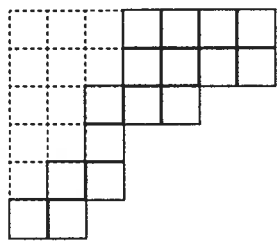
If $\mu = \delta_\beta$ for a composition β, then we write $\mathcal{S}(\lambda, \beta^\triangleleft, \alpha^\triangleleft; k)$. When we do not wish to cut out any diagram μ, we shall write $\mu = \emptyset$ and simply use $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ in place of $\mathcal{S}(\lambda, \mu, \alpha^\triangleleft; k)$.

The reason we write $\mathcal{S}(\lambda, \beta^\triangleleft, \alpha^\triangleleft; k)$ instead of writing $\mathcal{S}(\lambda, \beta, \alpha; k)$ is to avoid confusion in the cases when the compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ are weakly decreasing, and could be misinterpreted as representing diagrams of partitions. For example, consider the diagrams $\mathcal{S}((2,1),(2,2);1)$ and $\mathcal{S}((2,1),(2,2)^\triangleleft;1)$ shown below. The first uses the diagram $D = (2,2)$, whereas the second uses the diagram $D = \Delta_{(2,2)}$.

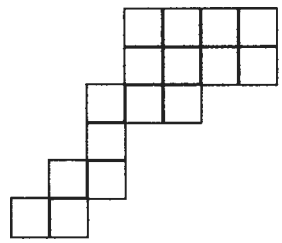


$\mathcal{S}((2,1),(2,2);1)$ $\mathcal{S}((2,1),(2,2)^\triangleleft;1)$

Example Here we show that the partition λ contains the fat staircase δ_β for $\lambda = (7,7,5,3,3,2)$ and $\beta = (1,2,2)$. We also display the skew diagram λ/δ_β.

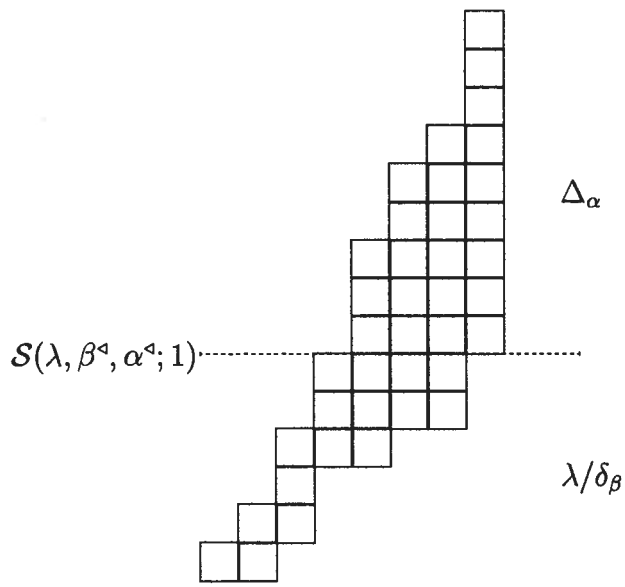


$\delta_{(1,2,2)} \subseteq \lambda$



$\lambda/\delta_{(1,2,2)}$

Now we show the fat staircase with bad foundation $\mathcal{S}(\lambda, \beta^{\triangleleft}, \alpha^{\triangleleft}; 1)$ for $\lambda = (7,7,5,3,3,2)$, $\beta = (1,2,2)$, and $\alpha = (3,1,2,3)$.



We have seen that the fat staircases with bad foundations $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ satisfy **1′** and **2′** when $0 \leq k \leq 1$. Moreover, we have the following result when considering any $k \geq 0$.

Lemma 2.2.2 *Let $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ be a fat staircase with bad foundation for some $k \geq 0$ and $\mathcal{T}$ be a SSYT of shape $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ whose reading word is lattice. If $\alpha = (\alpha_1, \ldots, \alpha_n)$, then the entries in the first row of the foundation of $\mathcal{T}$ consist of values taken from the set*

$$R_{\alpha,k} = \left\{ 1 + \sum_{i=1}^{j} \alpha_{n+1-i} \;\middle|\; j = 1, 2, \ldots, n \right\} \cup \begin{cases} \{1\} & \text{if } k > 0 \\ \emptyset & \text{if } k = 0. \end{cases}$$

Furthermore, the value 1 can occur at most k times and the rest of the values can appear at most once.

Proof Let R be the first row of the foundation of $\mathcal{T}$ and $t \in R$.

Since $\mathcal{T}$ is a SSYT, the columns strictly increase. Thus $t = 1$ is allowed if and only if $k \geq 1$ since it is precisely in that case that the first value in R is not below an entry of Δ_α. Furthermore, since there are only k boxes from the first row of the foundation of $\mathcal{T}$ that extend out from Δ_α, there can be at most k 1's in R.

If $t > 1$ then, when reading the row R from right to left, the lattice condition implies that there is at least one more $t-1$ in Δ_α than there are t's in Δ_α. Since the content of Δ_α is $(n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 1^{\alpha_1})$, the only instances when this occurs are when $t = 1 + \sum_{i=1\ldots j} \alpha_{n+1-j}$ for $j = 1, 2, \ldots, n$. Therefore every entry of R is an element of $R_{\alpha,k}$. Further, if a value $t > 1$ appeared twice in R, then the lattice condition would be violated. Hence each $t \in R_{\alpha,k}$, $t \neq 1$, can appear at most once in R. ∎

The next result tells us when we may obtain a SSYT of shape $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ with lattice reading word from a SSYT of shape $(\lambda/\mu) \oplus \Delta_\alpha$ with lattice reading word, where $D_1 \oplus D_2$ denotes the disjoint union of the diagrams D_1 and D_2.

Lemma 2.2.3 *Let α be a composition, λ and μ be partitions, and $k \geq 0$ such that $\lambda_1 - \mu_1 - k \leq l(\alpha)$. If T is a SSYT of shape $\lambda/\mu \oplus \Delta_\alpha$ with lattice reading word such that there are at most k 1's in the first row of λ/μ, then the tableau of shape $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ obtained from T by shifting the foundation λ/μ to the right is also a SSYT with lattice reading word.*

Proof Let T be a SSYT of shape $\lambda/\mu \oplus \Delta_\alpha$ with lattice reading word and let T'_k be the tableau of shape $\mathcal{S}(\lambda, \mu, \alpha^{\triangleleft}; k)$ obtained from T by shifting the foundation λ to the right. Since shifting λ/μ to the right does not affect the order in which the entries are read, T'_k has a lattice reading word. Also, the rows of T'_k weakly increase since they are the same as the rows of T. Further, to check that the columns of T'_k strictly increase, we need only check that they strictly increase at the positions where the two subdiagrams Δ_α and λ/μ are joined.

Let R denote the first row of λ/μ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. As in the proof of Lemma 2.2.2, the lattice condition on T implies that the entries of R consist of values of $R_{\alpha,k}$. Further, the value 1 can occur at most k times and the rest of the values of R are distinct. Let q be the number of times 1 appears in R, so that $k \geq q$. Further, let $r_1 \leq r_2 \leq \cdots \leq r_{\lambda_1}$ be the entries of R.

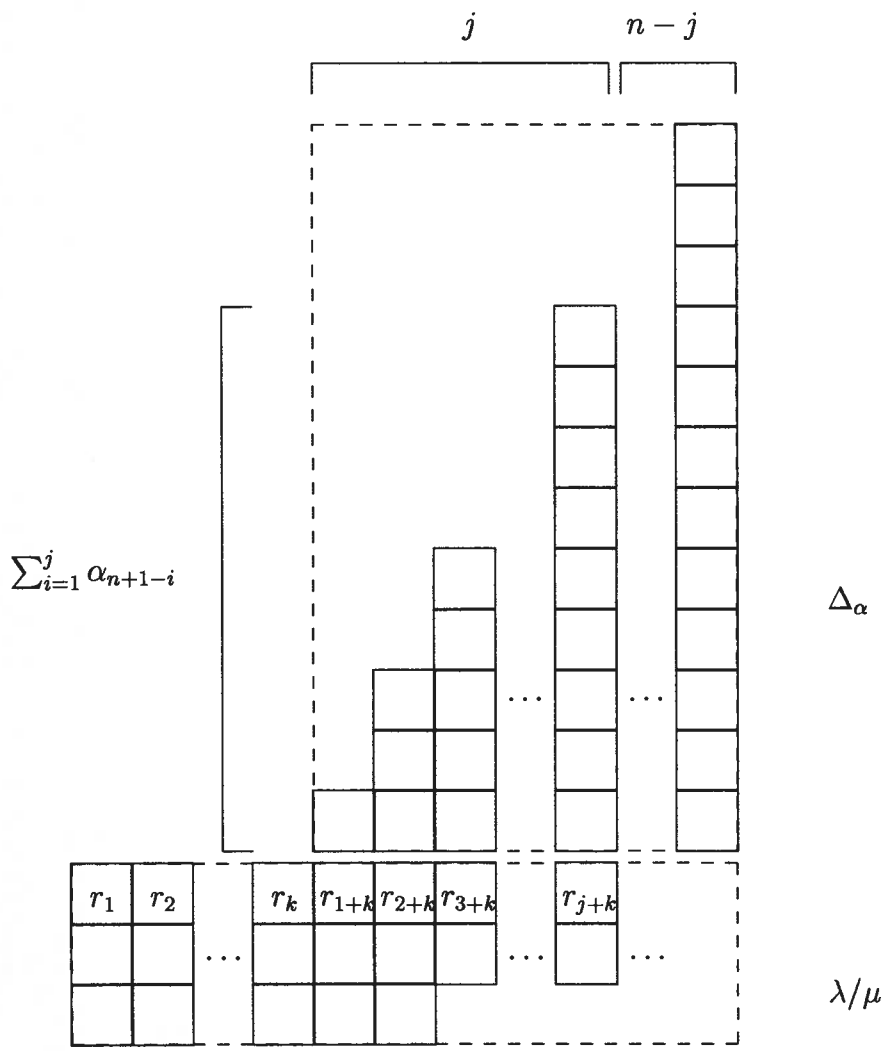
Consider the case $k \geq 1$. Since $r_1 = r_2 = \cdots = r_q = 1$, we have $r_q = \min(R_{\alpha,k})$ and for each $1 \leq j \leq n$ we have

$$r_{j+q} \geq \text{ the } (j+1)\text{-th smallest value of } R_{\alpha,k} = 1 + \sum_{i=1}^{j} \alpha_{n+1-i}.$$

Since $k \geq q$, for each $1 \leq j \leq n$ we have

$$r_{j+k} \geq r_{j+q} \geq 1 + \sum_{i=1}^{j} \alpha_{n+1-i}.$$

As illustrated in the diagram below, the entry r_{j+k} is beneath $\sum_{i=1}^{j} \alpha_{n+1-i}$ boxes. From the unique filling of Δ_α, the entry of Δ_α directly above r_{j+k} is $\sum_{i=1}^{j} \alpha_{n+1-i}$.



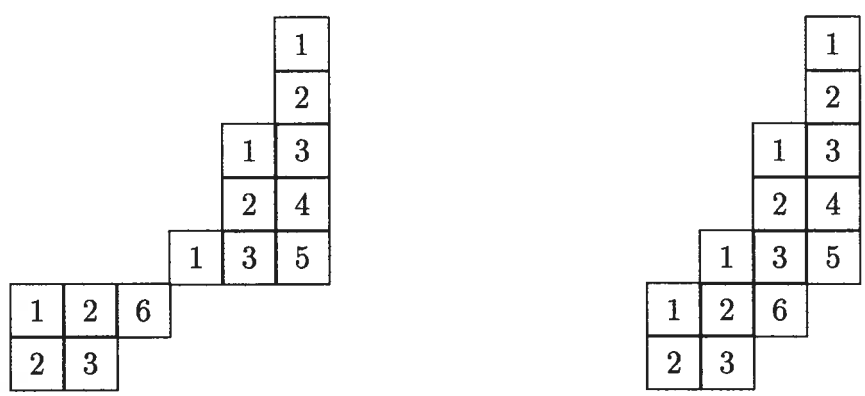
Thus the columns strictly increase. Therefore T_1' is a SSYT with lattice reading word, as desired.

Now consider the case when $k = 0$. Then for each $1 \leq j \leq n$ we have

$$r_j \geq j\text{-th smallest value of } R_{\alpha,k} \geq 1 + \sum_{i=1}^{j} \alpha_{n+1-i}.$$

Also, the entry r_j is beneath precisely $\sum_{i=1}^{j} \alpha_{n+1-i}$ boxes, so the entry of Δ_α directly above r_j is $\sum_{i=1}^{j} \alpha_{n+1-i}$. Thus the columns strictly increase. Therefore T_k' is a SSYT with lattice reading word, as desired. ∎

Example Let $\alpha = (2,2,1)$, $\beta = \emptyset$, $\lambda = (3,2)$, and $k = 1$. Consider the SSYT T of shape $\lambda \oplus \Delta_\alpha$ with lattice reading word and only one 1 in the first row of λ shown on the left. This gives rise to the SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; 1)$ with lattice reading word shown on the right.



2.3 Sums of Fat Staircases

Suppose a diagram $D = \rho/\kappa$ is such that

$$s_D = \sum_{\nu=\text{fat staircase}} c^{\rho}_{\kappa\nu} s_\nu.$$

That is, for every ν that is not a fat staircase, we have $c^{\rho}_{\kappa\nu} = 0$. When this happens we say that D is a *sum of fat staircases.* For each fat staircase ν we

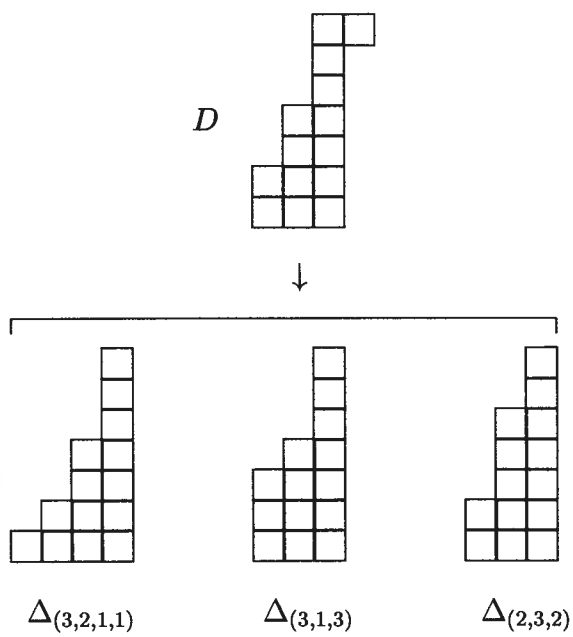
let $\alpha(\nu)$ be the composition such that $\delta_{\alpha(\nu)} = \nu$. That is, $\alpha(\nu)_i$ is the number of parts of ν equal to i. Then using these compositions we can rewrite s_D as

$$s_D = \sum_{\nu=\text{fat staircase}} c^{\rho}_{\kappa\nu} s_{\Delta_{\alpha(\nu)}}.$$

Example Let $\rho = (4,3,3,3,3,3,3)$, $\kappa = (2,2,2,1,1)$, and $D = \rho/\kappa$. Then

$$\begin{aligned} s_D &= s_{(4,3,2,2,1,1,1)} + s_{(3,3,3,2,1,1,1)} + s_{(3,3,2,2,2,1,1)} \\ &= s_{\Delta_{(3,2,1,1)}} + s_{\Delta_{(3,1,3)}} + s_{\Delta_{(2,3,2)}}. \end{aligned}$$

Thus, when computing the Schur function of D, we are interested in the following diagrams.



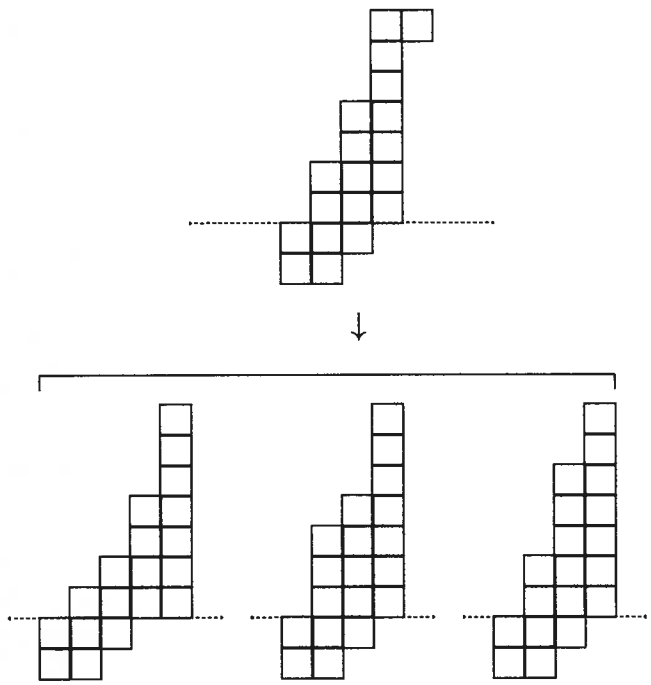
For a sum of fat staircases D and a value $k \geq 0$, we consider the diagram $\mathcal{S}(\lambda, \mu, D; k)$. By assumption, we have

$$s_D = \sum_{\nu=\text{fat staircase}} c^{\rho}_{\kappa\nu} s_{\Delta_{\alpha(\nu)}}.$$

When we compute the skew Schur function $s_{\mathcal{S}(\lambda,\mu,D;k)}$ we notice that, since the subdiagram D is strictly above the foundation λ/δ_β, the contents that arise from the fillings of D are precisely the fat staircases $\delta_{\alpha(\nu)}$, for each ν with $[s_\nu](s_D) \neq 0$. For each of these fillings of D, we then need to extend them by

filling the foundation λ/μ such that the resulting tableaux are SSYTx with lattice reading words.

Thus, when computing the Schur function of $\mathcal{S}(\lambda, \mu, D; k)$, we are interested in the possible fillings of the following diagrams.



The next result summarizes the relationship between these fillings.

Theorem 2.3.1 *Suppose a diagram $D = \rho/\kappa$ is such that*

$$s_D = \sum_{\nu = \textit{fat staircase}} c^{\rho}_{\kappa\nu} s_\nu.$$

Then for each partition λ and μ with $\mu \subseteq \lambda$, and for each $k \geq 0$ we have

$$s_{\mathcal{S}(\lambda,\mu,D;k)} \leq_s \sum_{\nu} c^{\rho}_{\kappa\nu} s_{\mathcal{S}(\lambda,\mu,\alpha(\nu)^{\triangleleft};k)}.$$

Proof Since D is a sum of fat staircases, the content of each SSYT of shape D with lattice reading word is some fat staircase ν. To prove the identity, we consider the map that takes a SSYT T of shape $\mathcal{S}(\lambda, \mu, D; k)$ with lattice reading word and $c(D) = \nu$ to the tableau T' of shape $\mathcal{S}(\lambda, \mu, \alpha(\nu)^{\triangleleft}; k)$ obtained by filling the copy of $\Delta_{\alpha(\nu)}$ with content ν and filling the copy of λ/μ identically to its filling in T.

We claim that each tableau T' is also a SSYT with lattice reading word. Given such a tableau T', consider the corresponding tableau $\mathcal{T}'$ of shape $\lambda/\mu \bigoplus \Delta_{\alpha(\nu)}$. Then $\mathcal{T}'$ is a SSYT since both of the subtableaux of shape λ/μ and $\Delta_{\alpha(\nu)}$ are semistandard. Further $\mathcal{T}'$ has a lattice reading word since checking the lattice condition within the subtableau of shape $\Delta_{\alpha(\nu)}$ is trivial, and then, by the construction of T', the lattice condition after this point simply becomes identical to the lattice condition for T. Therefore $\mathcal{T}'$ is a SSYT with lattice reading word. Further, the first row of the subtableau λ/μ of $\mathcal{T}'$ contains at most k 1's since T was a SSYT. Hence, by Lemma 2.2.3, T' is also a SSYT with lattice reading word, exactly as claimed.

Thus, for each SSYT T of shape $\mathcal{S}(\lambda, \mu, D; k)$ with lattice reading word and $c(D) = \nu$ we obtain a SSYT T' of shape $\mathcal{S}(\lambda, \mu, \alpha(\nu)^{\triangleleft}; k)$. Now fix an image T' and suppose that $T_1 \neq T_2$ are both SSYT of shape $\mathcal{S}(\lambda, \mu, D; k)$ with lattice reading word such that $T_1' = T' = T_2'$. Let $\mathcal{S}(\lambda, \mu, \alpha(\nu)^{\triangleleft}; k)$ be the shape of $T_1' = T' = T_2'$. Then the filling of the foundation λ/μ in T_1 is the same as the filling of the foundation λ/μ in T_2, and the content of the subdiagram D is ν for both T_1 and T_2. Since there are only $c^{\rho}_{\kappa\nu}$ SSYT of shape D with lattice reading word and content ν, there are at most $c^{\rho}_{\kappa\nu}$ distinct SSYT of shape $\mathcal{S}(\lambda, \mu, D; k)$ with lattice reading word that could map to T'. This implies that

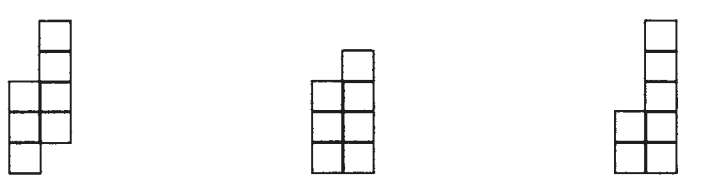
$$s_{\mathcal{S}(\lambda,\mu,D;k)} \leq_s \sum_{\nu} c^{\rho}_{\kappa\nu} s_{\mathcal{S}(\lambda,\mu,\alpha(\nu)^{\triangleleft};k)},$$

as we desired. ∎

Example Let $D = (2,2,2,2,1)/(1,1)$, $\lambda = (2,2)$, $\mu = \emptyset$, and $k = 1$. Then

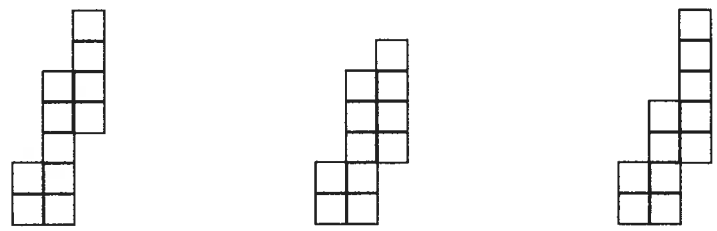
$$s_D = s_{(2,2,2,1)} + s_{(2,2,1,1,1)} = s_{\Delta_{(1,3)}} + s_{\Delta_{(3,2)}}$$

is a sum of fat staircases. Here we display D, $\Delta_{(1,3)}$, and $\Delta_{(3,2)}$.



D $\qquad$ $\Delta_{(1,3)}$ $\qquad$ $\Delta_{(3,2)}$

We are now interested in the diagrams $\mathcal{S}(\lambda, D; 1)$, $\mathcal{S}(\lambda, (1,3)^{\triangleleft}; 1)$, and $\mathcal{S}(\lambda, (3,2)^{\triangleleft}; 1)$ for $\lambda = (2,2)$.



$\mathcal{S}(\lambda, D; 1)$ $\mathcal{S}(\lambda, (1,3)^{\triangleleft}; 1)$ $\mathcal{S}(\lambda, (3,2)^{\triangleleft}; 1)$

We can compute that

$$\begin{aligned} s_{\mathcal{S}(\lambda,D;1)} &= s_{(3,3,2,2,1)} + s_{(3,3,2,1,1,1)} + s_{(3,3,1,1,1,1,1)} + s_{(3,2,2,2,2)} \\ &+ s_{(3,2,2,2,1,1)} + s_{(3,2,2,1,1,1,1)} + s_{(2,2,2,2,2,1)} + s_{(2,2,2,2,1,1,1)} \end{aligned}$$

and

$$\begin{aligned} s_{\mathcal{S}(\lambda,(1,3)^{\triangleleft};1)} + s_{\mathcal{S}(\lambda,(3,2)^{\triangleleft};1)} &= 2s_{(3,3,2,2,1)} + 2s_{(3,3,2,1,1,1)} + s_{(3,3,1,1,1,1,1)} \\ &+ s_{(3,2,2,2,2)} + 2s_{(3,2,2,2,1,1)} + s_{(3,2,2,1,1,1,1)} \\ &+ s_{(2,2,2,2,2,1)} + s_{(2,2,2,2,1,1,1)}, \end{aligned}$$

so we see that

$$s_{\mathcal{S}(\lambda,D;1)} \leq_s s_{\mathcal{S}(\lambda,(1,3)^{\triangleleft};1)} + s_{\mathcal{S}(\lambda,(3,2)^{\triangleleft};1)},$$

as predicted by Theorem 2.3.1.

We now proceed to examine which skew diagrams D are sums of fat staircases. The following results are concerned with adding or removing columns to a skew diagram on the left or right side of the diagram. We recall that $D_1 \odot_i D_2$ denotes the near-concatenation of D_1 and D_2 of depth i. That is, if the last column of D_1 and first column of D_2 are both of length greater than or equal to i, then $D_1 \odot_i D_2$ is the is the skew diagram obtained by placing D_1 and D_2 so that the top-right box of D_1 is one step left and $i-1$ steps up from the bottom-left box of D_2. In other words, the last column of D_1 and the first column of D_2 overlap in exactly i boxes.

Lemma 2.3.2 *Let c be a column and D be a skew diagram that is not a sum of fat staircases and let $D_1 = D \odot_i c$ and $D_2 = c \odot_i D$ be obtained from D by the addition of a single column c. Then neither D_1 nor D_2 is a sum of fat staircases.*

Proof We shall begin by proving that D_2 is not a sum of fat staircases. Since D is not a sum of fat staircases there is a SSYT T of shape D and lattice reading word whose content is ν, where ν is not a fat staircase. That is, $\nu_j \geq \nu_{j+1} + 2$ for some j.

Let n be the length of the column c. We create a tableau T' of shape D_2 and lattice reading word by filling c with the numbers $1, 2, \ldots, n$ and by filling the rest of D_2 as D is filled in the tableau T. Then T' is semistandard and has a lattice reading word. Further, $c(T') = \nu + (1^n)$. There are three cases to consider. If $j < n$, comparing the j-th and $(j+1)$-th entries of this content gives

$$c(T')_j = \nu_j + 1 \geq \nu_{j+1} + 3 = c(T')_{j+1} + 2.$$

If $j = n$, comparing the j-th and $(j+1)$-th entries of this content gives

$$c(T')_j = \nu_j + 1 \geq \nu_{j+1} + 3 = c(T')_{j+1} + 3.$$

And if $j > n$, comparing the j-th and $(j+1)$-th entries of this content gives

$$c(T')_j = \nu_j \geq \nu_{j+1} + 2 = c(T')_{j+1} + 2.$$

In all cases we see that $c(T')$ is not a fat staircase. Hence D_2 is not a sum of fat staircases.

Then, since $s_{D^\circ} = s_D$ is also not a sum of fat staircases, the above shows that $c \odot_i D^\circ$ is not a sum of fat staircases. Now, since

$$s_{D_1} = s_{D \odot_i c} = s_{(D \odot_i c)^\circ} = s_{c \odot_i D^\circ},$$

we find that D_1 is not a sum of fat staircases. ∎

Corollary 2.3.3 *If D is not a sum of fat staircases and D' is obtained from D by the addition any number of columns, then D' is not a sum of fat staircases.*

Proof Let D' be a diagram obtained from D by adding, in order, the columns $c_1, c_2, \ldots, c_n$. Let $D = D_0$ and, for each $i = 1, \ldots, n$, let D_i be the subdiagram of D' consisting of D and the columns $c_1, \ldots, c_i$. Then using Lemma 2.3.2 repeatedly we find that D_i is not a sum of fat staircases for each $i = 1, \ldots, n$. Since $D_n = D'$, we are done. ∎

Corollary 2.3.4 *If D is a sum of fat staircases and D' is a connected subdiagram of D obtained by removing columns, then D' is a sum of fat staircases.*

Proof This is precisely the contrapositive of Corollary 2.3.3 with the roles of D and D' reversed. ∎

Therefore every diagram D that is a sum of fat staircases can be viewed as a column extension of a smaller diagram D' that is also a sum of fat staircases. Knowing this, for a given diagram D that is a sum of fat staircases, we now consider what length of columns can be added to D and how much can a column overlap with D if we wish the new diagram to also be a sum of fat staircases. Towards this end, we have the following result.

Lemma 2.3.5 *Let c be a column, $D = \rho/\mu$ be a sum of fat staircases, and D' be given by either $D' = c \odot_i D$ or $D' = D \odot_i c$. If D' is also a sum of fat staircases, then we have $l(c) + 1 \notin R_{\alpha(\nu),1}$ and $i + 1 \notin R_{\alpha(\nu),1}$ for each ν with $c^{\rho}_{\mu\nu} \neq 0$.*

Proof We shall consider the case $D' = c \odot_i D$. As in the proof of Lemma 2.3.2, the second case $D' = D \odot_i c$ can then be obtained by rotating diagrams by 180°.

Suppose $l(c) + 1 \in R_{\alpha(\nu),1}$ for some ν with $c^{\rho}_{\mu\nu} \neq 0$. Since $c^{\rho}_{\mu\nu} \neq 0$, there is a SSYT T of shape D with lattice reading word and content ν. Now we create a tableau T' of shape D' by filling c with the values $1, 2, \ldots l(c)$ and filling D as in the tableau T. Then T' is clearly a SSYT with lattice reading word. Further, $c(T') = \nu + (1^{l(c)})$. Since $l(c) + 1 \in R_{\alpha(\nu),1}$, we have $l(c) = \sum_{j=1}^{k} \alpha(\nu)_{n+1-j}$ for some k, where $\nu = (n^{\alpha(\nu)_n}, n-1^{\alpha(\nu)_{n-1}}, \ldots, 1^{\alpha(\nu)_1})$. Thus

$$\begin{aligned} c(T') &= \nu + (1^{l(c)}) \\ &= (n^{\alpha(\nu)_n}, n-1^{\alpha(\nu)_{n-1}}, \ldots, n+1-k^{\alpha(\nu)_{n+1-k}}, n-k^{\alpha(\nu)_{n-k}}, \ldots, 1^{\alpha(\nu)_1}) \\ &\quad +(1^{\alpha(\nu)_n+\alpha(\nu)_{n-1}+\cdots+\alpha(\nu)_{n+1-k}}) \\ &= (n+1^{\alpha(\nu)_n}, n^{\alpha(\nu)_{n-1}}, \ldots, n+2-k^{\alpha(\nu)_{n+1-k}}, n-k^{\alpha(\nu)_{n-k}}, \ldots, 1^{\alpha(\nu)_1}). \end{aligned}$$

Thus $c(T')$ is not a fat staircase, and so D' is not a sum of fat staircases.

Therefore, if D' is a sum of fat staircases, then we require that $l(c) + 1 \notin R_{\alpha(\nu),1}$ for each ν with $c^{\rho}_{\mu\nu} \neq 0$.

Similarly, suppose that $i+1 \in R_{\alpha(\nu),1}$ for some ν with $c^{\rho}_{\mu\nu} \neq 0$. Since $c^{\rho}_{\mu\nu} \neq 0$, there is a SSYT T of shape D with lattice reading word and content ν. We create a tableau T' of shape D' by filling c with the values $1, 2, \ldots, i, l(\nu) + 1, l(\nu) + 2, \ldots, l(\nu) + l(c) - i$ and filling D as in the tableau T. Again, T' is a SSYT with lattice reading word. Further, $c(T') = \nu + (1^i) + (0^{l(\nu)}, 1^{l(c)-i})$.

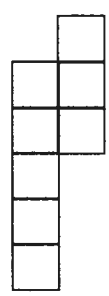
Since $i+1 \in R_{\alpha(\nu),1}$, we have $i = \sum_{j=1}^{k} \alpha(\nu)_{n+1-j}$ for some k, where $\nu = (n^{\alpha(\nu)_n}, n-1^{\alpha(\nu)_{n-1}}, \ldots, 1^{\alpha(\nu)_1})$. Thus

$$\begin{aligned} c(T') &= \nu + (1^i) + (0^{l(\nu)}, 1^{l(c)-i}) \\ &= (n^{\alpha(\nu)_n}, n-1^{\alpha(\nu)_{n-1}}, \ldots, n+1-k^{\alpha(\nu)_{n+1-k}}, n-k^{\alpha(\nu)_{n-k}}, \ldots, 1^{\alpha(\nu)_1}) \\ &\quad +(1^{\alpha(\nu)_n+\alpha(\nu)_{n-1}+\cdots+\alpha(\nu)_{n+1-k}}) + (0^{l(\nu)}, 1^{l(c)-i}) \\ &= (n+1^{\alpha(\nu)_n}, n^{\alpha(\nu)_{n-1}}, \ldots, n+2-k^{\alpha(\nu)_{n+1-k}}, n-k^{\alpha(\nu)_{n-k}}, \ldots, \\ &\qquad\qquad 2^{\alpha(\nu)_2}, 1^{\alpha(\nu)_1+l(c)-i}). \end{aligned}$$

Thus $c(T')$ is not a fat staircase, and so D' is not a sum of fat staircases.

Therefore, if D' is a sum of fat staircases, then we require that $i+1 \notin R_{\alpha(\nu),1}$ for each ν with $c^{\rho}_{\mu\nu} \neq 0$. ∎

Example Consider the diagram $D = (2,2,2,1,1,1)/(1)$, seen below.



We have

$$s_D = s_{(2,2,2,1,1)} + s_{(2,2,1,1,1,1)} = s_{\Delta_{(2,3)}} + s_{\Delta_{(4,2)}}.$$

Therefore, when adding a column, we must avoid lengths and overlaps that are one more that an element of

$$R_{(2,3),1} \cup R_{(4,2),1} = \{1,4,6\} \cup \{1,3,7\},$$

if we wish our new diagram to also be a sum of fat staircases.

For instance, if we add a column of length 6 to the diagram, then since $6+1 \in R_{(2,3),1} \cup R_{(4,2),1}$ the resulting diagram is not a sum of fat staircases. We show a tableau that demonstrates this when the overlap $i = 4$.

		1
	1	2
1	2	3
2	4	
3	5	
4	6	
5		
6		

Since the content of this tableau is $\nu = (3,3,2,2,2,2)$, the new diagram $D' = (1^6) \odot_4 D$ is not a sum of fat staircases.

It is important to note that although the conditions $l(c)+1 \notin R_{\alpha(\nu),1}$ and $i+1 \notin R_{\alpha(\nu),1}$ are necessary in Lemma 2.3.5, they are by no means sufficient conditions.

Theorem 2.3.6 *If D is a sum of fat staircases then the columns of D have distinct lengths.*

Proof Let D be a sum of fat staircases. Suppose D has n columns $c_1, \ldots, c_n$ and, for each j, let m_j be the number of columns of length j.

We create a tableau T of shape D by filling each column c_i with the entries $1, 2, \ldots, l(c_i)$. Then T is a SSYT with lattice reading word and content $\nu = \sum_{i=1}^{n}(1^{l(c_i)})$. Thus, for each j, ν_j is the number of columns of D of length greater than or equal to j. Therefore we have

$$\nu_j = \nu_{j-1} + m_j, \tag{2.1}$$

for each j. Since D is a sum of fat staircases, ν is a fat staircase. Therefore Equation 2.1 implies that $0 \leq m_j \leq 1$ for each j. In other words, the columns of D have distinct lengths. ∎

Example Consider the diagram $D = (5,4,4,3,3,3,2,1)/(3,2,2,1,1)$, which has two columns of length 3. Filling each column c of D with the entries $1, 2, 3, \ldots, l(c)$ gives the following tableau of content $\nu = (5,4,4,2,1)$.

			1	1
		1	2	
		2	3	
	1	3		
	2	4		
1	3	5		
2	4			
3				

Since ν is not a fat staircase, D is not a sum of fat staircases.

We finish this chapter by proving the converse of Theorem 2.3.6 in the case of a diagram with two columns.

Lemma 2.3.7 *If D is a connected diagram with two columns and these columns have distinct lengths, then D is a sum of fat staircases.*

Proof Let D be a connected diagram with two columns of distinct lengths. Let T be a SSYT of shape D with lattice reading word and let ν be the content of T. Since there are only two columns in T and the entries of each column strictly increases we have $\nu_i \leq 2$ for each i.

Hence, if D is not a sum of fat staircases then there must be a SSYT T of shape D and lattice reading word with content $\nu = (2^n)$, for some n. Since the columns of T strictly increase, this implies that both columns are of length n, contrary to assumption. Therefore D is a sum of fat staircases. ∎

Chapter 3

Hook Foundations

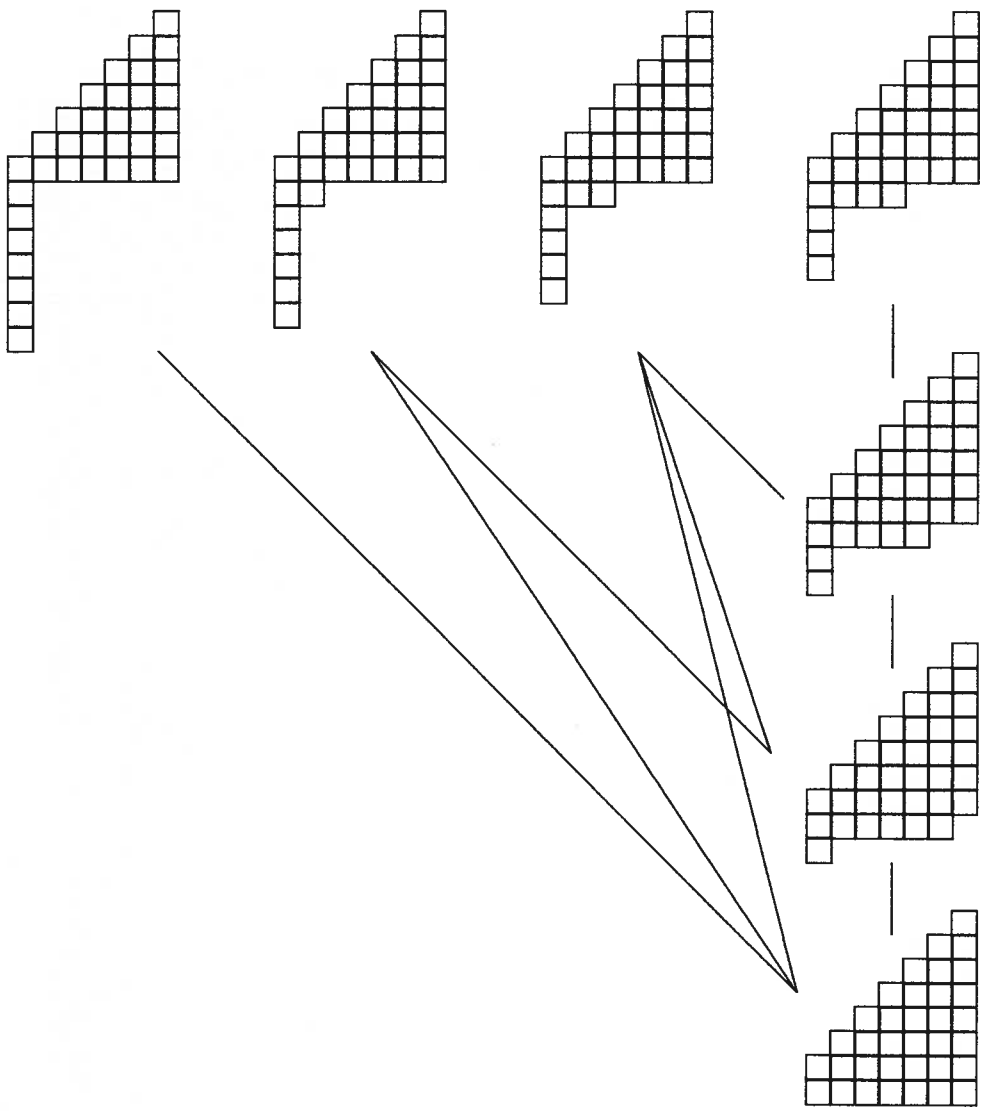
In this chapter we consider fat staircases with bad foundations using two simple families for the foundations. In Section 3.1 we shall look at $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ for hook diagrams λ. We shall be able to completely describe which pairs of these fat staircases with hook foundations D_1, D_2 satisfy $s_{D_1} - s_{D_2} \geq_s 0$ and which pairs are Schur-incomparable. We also give a Schur-positivity result using a diagram D that is a sum of fat staircases.

In Section 3.2 we will give an explicit formula for computing the Schur-positive differences of fat staircases with hook foundations that were discussed in Section 3.1.

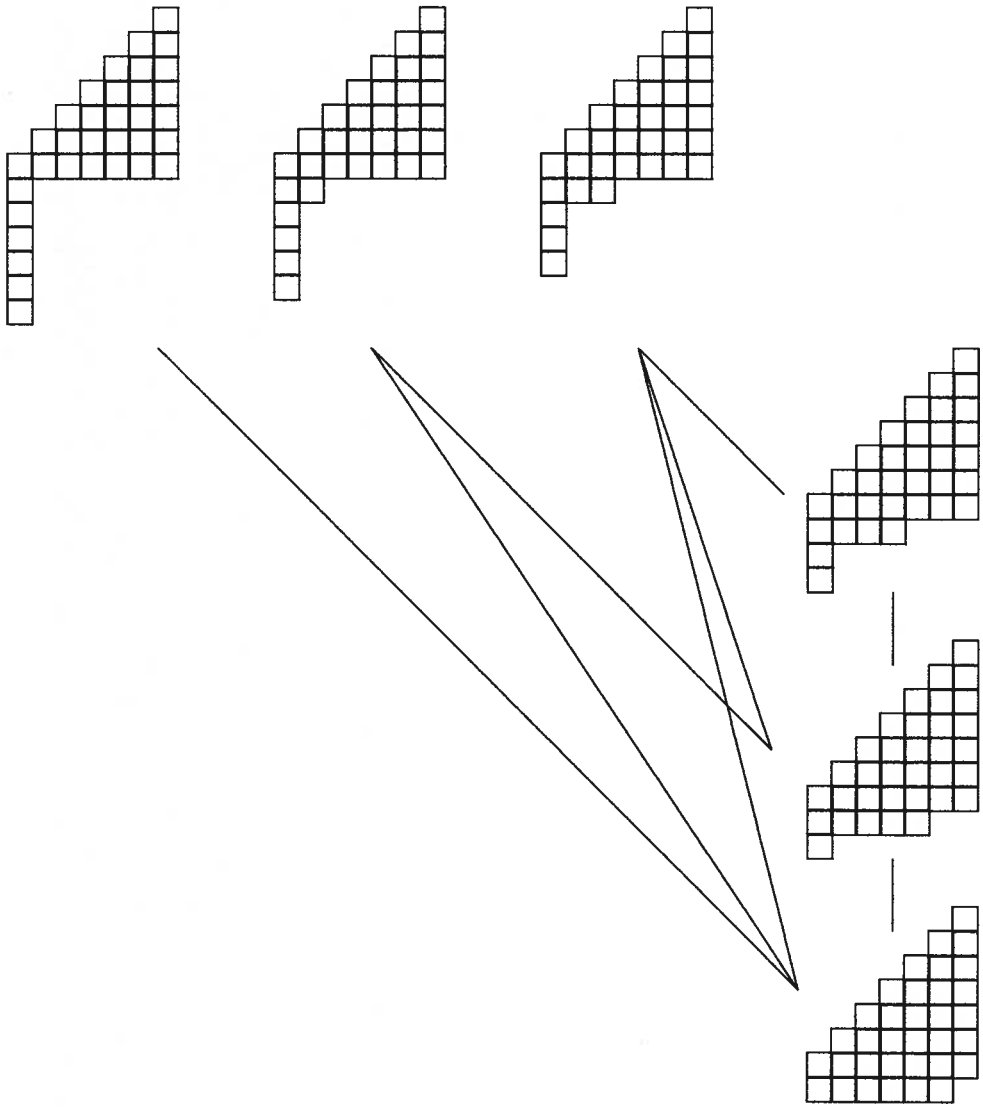
3.1 Schur Comparability / Incomparability

Recall from the introduction, that we write $D_1 \succeq_s D_2$ whenever $s_{D_1} - s_{D_2} \geq_s 0$, and if we consider the relation $\succeq_s$ on the set of all Schur-equivalent classes of diagrams (i.e. $[D]_s = \{D' | s_D = s_{D'}\}$), then $\succeq_s$ defines a partial ordering. This allows us to view the Hasse diagram for the relation $\succeq_s$ on the set of these Schur-equivalent classes. For the sake of convenience, we write D in place of $[D]_s$.

Example Here we show the Hasse diagram for $\succeq_s$ on the collection of staircases with bad foundations $\mathcal{S}(\lambda, 0, 7)$ (or $\mathcal{S}(\lambda, (1^7)^{\triangleleft}; 0)$ in the fat staircase notation), for λ varying over all hooks of size 7. A line drawn from a diagram D_1 to a diagram D_2 in an upwards direction indicates that $s_{D_1} - s_{D_2} \geq_s 0$. We note that the diagrams along the top are all Schur-incomparable. That is, they form an anti-chain with regards to $\succeq_s$. Also, the diagrams along the right are all comparable. That is, they form a chain with regards to $\succeq_s$.



Now, if we consider hooks λ of size 6, we will obtain the following Hasse diagram. Again, the diagrams along the top are all Schur-incomparable and the diagrams along the right are all comparable.



When working with hooks, we shall find it convenient to describe each hook by its *arm length* and *leg length*. Hence, we let μ be the hook $(\mu_a, 1^{\mu_l - 1})$ and λ be the hook $(\lambda_a, 1^{\lambda_l - 1})$. In this chapter we shall use a fixed fat staircase Δ_α, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Thus n is the length of the bottom row of the staircase Δ_α.

The following results summarise all the $\succeq_s$ relationships between diagrams of the form $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ when λ is a hook of fixed size $h \leq n+k$ and $0 \leq k \leq 1$. The restriction $h \leq n+k$ is needed to guarantee that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ is a skew diagram for every hook λ of size h.

For each pair of hooks λ, μ with $\lambda_a, \mu_a \leq \lceil \frac{h}{2} \rceil$, Theorem 3.1.1 and Theorem 3.1.2 each prove one side of the Schur-incomparability of this pair, thus describing the antichain structure displayed along the top of the Hasse diagrams in the previous examples.

For each pair of hooks λ, μ with $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$, Theorem 3.1.3 shows that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. This describes the chain structure displayed along the right of the Hasse diagrams in the previous examples.

Finally, for each pair of hooks λ, μ with $\lambda_a, \mu_l < \lceil \frac{h}{2} \rceil$, Theorem 3.1.4 and Theorem 3.1.5 show that $\lambda_a \geq \mu_l$ if and only if $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. This describes the relationships between the diagrams displayed on the right with the diagrams displayed along the top in the previous Hasse diagrams.

Before we begin, we review some facts regarding any SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word. Any such tableau will have Δ_α filled with content $\delta_\alpha = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1})$. This content led us to define the set

$$R_{\alpha,k} = \left\{ 1 + \sum_{i=1}^{j} \alpha_{n+1-i} \;\middle|\; j = 1-k, 2-k, \ldots, n \right\}$$

of size $n+k$ and largest element $|\alpha| + 1$. This set gives the values of the foundation λ which can be read immediately after reading Δ_α. If we read some other value immediately after reading Δ_α, then the lattice condition will be violated. Hence, when reading the entries of the foundation, in order to read some value $x \notin R_{\alpha,k}$ without violating the lattice condition, we must first read some value $r \in R_{\alpha,k}$ and read each of $r+1, r+2, \ldots, x-1$ before reading x.

Let us finally begin. We start by looking at the antichain structure.

Theorem 3.1.1 *Let λ and μ be distinct hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lambda_a < \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$ and let $0 \leq k \leq 1$. Then we have $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) \not\succeq_s \mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$.*

Proof We shall show that there exists a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word such that there is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word having the same content. This is sufficient to prove the theorem.

Since $\lambda_a < \mu_a$ and $|\lambda| = |\mu|$, we have $\lambda_l > \mu_l$. Let $r_1 < r_2 < \cdots < r_{n+k}$ be the values of $R_{\alpha,k}$. We can create a SSYT of shape λ by filling the boxes of λ as follows.

$$\begin{array}{ccccc} r_1 & r_2 & \cdots & r_{\lambda_a-1} & |\alpha|+1 \\ |\alpha|+2 & & & & \\ |\alpha|+3 & & & & \\ \vdots & & & & \\ |\alpha|+\lambda_l & & & & \end{array}$$

It is easy to check that the resulting tableau of shape $\lambda \oplus \Delta_\alpha$ has lattice reading word since each of the entries in the first row of λ are from $R_{\alpha,k}$. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word, where λ is filled as shown above.

Since $\mu_l < \lambda_l$, we have $l(\mathcal{S}(\mu, \alpha^{\triangleleft}; k)) = |\alpha| + \mu_l < |\alpha| + \lambda_l$. Therefore no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word can contain the entry $|\alpha| + \lambda_l$. Thus no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word can have the same content as $\mathcal{T}$. Hence, it follows that $s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} \not\geq_s 0$. ∎

Theorem 3.1.2 *Let λ and μ be distinct hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lambda_a < \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$ and let $0 \leq k \leq 1$. Then we have $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \not\geq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof Let $r_1 < r_2 < \cdots < r_{n+k}$ be the values of $R_{\alpha,k}$. We can create a SSYT of shape μ by filling the boxes of μ as follows.

$$\begin{array}{cccc} r_1 & r_2 & \cdots & r_{\mu_a} \\ r_{\mu_a+1} & & & \\ r_{\mu_a+2} & & & \\ \vdots & & & \\ r_h & & & \end{array}$$

Since we are using distinct values of $R_{\alpha,k}$, it is easy to check that the resulting tableau of shape $\mu \oplus \Delta_\alpha$ has lattice reading word. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word, where μ is filled as shown above.

We now wish to count all SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ (shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, respectively) with content $\nu = c(\mathcal{T})$. Since Δ_α has a unique way of being filled, we must find all semistandard fillings of μ (λ, resp.) with the values $r_1, r_2, \ldots, r_h$. Since $r_1 < r_2 < \cdots < r_h$, the value r_1 must appear in the top-left corner of μ (λ, resp.). Further, once we choose $\mu_a - 1$ ($\lambda_a - 1$, resp.)

of the values $r_2 < r_3 < \cdots < r_h$ to appear in the first row of μ (first row of λ, resp.), then the remaining r's must appear in the first column and the order of all these values is uniquely determined by the semistandard conditions.

Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{h-1}{\mu_a - 1}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa}_{\rho\nu} = \binom{h-1}{\lambda_a - 1}.$$

Since $\lambda_a < \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$, we have $h + 1 > \lambda_a + \mu_a$. Therefore we have $h - \lambda_a > \mu_a - 1$ and we obtain

$$h - \lambda_a - i > \mu_a - 1 - i, \tag{3.1}$$

for each i.

Therefore

$$\begin{aligned} c^{\kappa'}_{\rho'\nu} &= \frac{(h-1)!}{(h-\mu_a)!(\mu_a - 1)!} \\ &= \frac{(h-1)!}{(h-\lambda_a)!(\lambda_a - 1)!} \times \prod_{i=0}^{\mu_a - \lambda_a - 1} \frac{h - \lambda_a - i}{\mu_a - 1 - i} \\ &= c^{\kappa}_{\rho\nu} \times \prod_{i=0}^{\mu_a - \lambda_a - 1} \frac{h - \lambda_a - i}{\mu_a - 1 - i} \\ &> c^{\kappa}_{\rho\nu}, \end{aligned}$$

where we have used Equation 3.10 in the final step. Since we have found a content ν with $c^{\kappa'}_{\rho'\nu} > c^{\kappa}_{\rho\nu}$, we can conclude that $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \not\geq_s 0$. ∎

We now depart from looking at the hooks λ, μ satisfying $\lambda_a < \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$. Instead, we turn to the hooks λ, μ satisfying $\left\lceil \frac{h}{2} \right\rceil \leq \lambda_a < \mu_a$. The following theorem shows that we have the chain stucture that was evident in the examples.

Theorem 3.1.3 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$ and let $0 \leq k \leq 1$. Then we have $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof To prove the result, we shall consider any content ν such that a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word exists. First we shall show that there is also a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word. Then, letting $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ and $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ for partitions κ, κ', ρ, and ρ', we shall show that the Littlewood-Richardson coefficients for these two diagrams and this content satisfy

$$c^{\kappa}_{\rho\nu} \geq c^{\kappa'}_{\rho'\nu}.$$

Having shown that this inequality holds for any content ν for which a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word exists, this will imply that $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \geq_s 0$.

Let ν be a content such that there is a SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word. By Lemma 2.2.2 we know that the first row of μ contains a strictly increasing sequence $a_1 < a_2 < \cdots < a_{\mu_a}$ where each $a_i \in R_{\alpha,k}$. Also, since the columns of $\mathcal{T}_1$ strictly increase, the first column of λ contains a strictly increasing sequence $a'_1 < a'_2 < \cdots < a'_{\mu_l}$, where $a'_1 = a_1$. Since Δ_α can only be filled in one way (in both shapes $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ and $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$), in order to obtain a tableau of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ and content ν we only need to show how to place the values of $\{a_1, a_2, \ldots, a_{\mu_a}\} \cup \{a'_2, a'_3, \ldots, a'_{\mu_l}\}$ in λ.

If $l(\nu) > |\alpha| + 1$ for this particular content ν then there are entries of $\mathcal{T}_1$ greater than $|\alpha| + 1$. Since Δ_α has content δ_α, the lattice condition implies that $|\alpha| + 2$ appears in μ. Since each $a_i \in R_{\alpha,k}$, we have $a_i \leq |\alpha| + 1$ for each i and so $a'_j = |\alpha| + 2$ for some $j > 1$. The lattice condition and the fact that the column strictly increases gives that

$$\begin{aligned} a'_j &= |\alpha| + 2 \\ a'_{j+1} &= |\alpha| + 3 \\ &\vdots \\ a'_{\mu_l} &= |\alpha| + \mu_l - j + 2. \end{aligned}$$

Again, by the lattice condition, it is clear that any SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν must also have these values a'_j, a'_{j+1},

$\ldots, a'_{\mu_l}$ as the last $\mu_l - j + 1$ entries of the first column of λ. Since $\lambda_a < \mu_a$ gives $\mu_l < \lambda_l$, we have $\mu_l - j + 1 < \lambda_l - j + 1$. This gives,

$$\mu_l - j + 1 \leq \lambda_l, \tag{3.2}$$

so these entries do fit in this column. Note that if $l(\nu) \leq |\alpha| + 1$, then this sequence of values $a'_j, a'_{j+1}, \ldots, a'_{\mu_l}$ is empty and we do not have to worry about placing any entries larger than $|\alpha| + 1$.

Let M be the multiset $\{a_1, a_2, \ldots, a_{\mu_a}\} \cup \{a'_2, a'_3, \ldots, a'_{j-1}\}$. Then M is the remaining entries that we still need to place in λ to obtain a tableau of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ and content ν. We have $|M| = \mu_a + j - 2$ and $\max(M) = |\alpha| + 1$. Let $R = \{a_1, a_2, \ldots, a_{\mu_a}\} \cap \{a'_1, a'_2, \ldots, a'_{j-1}\}$. We note that $R = \{a_1, a_2, \ldots, a_{\mu_a}\} \cap \{a'_1, a'_2, \ldots, a'_{\mu_l}\}$ since Lemma 2.2.2 shows $a_{\mu_a} \in R_{\alpha,k}$, which implies $a_{\mu_a} < |\alpha| + 2 = a'_j$. Thus *$R$ is the set of values that appear in both the first row and the first column of μ.*

Since the values of R all appear in the first row of μ, Lemma 2.2.2 gives that $R \subseteq R_{\alpha,k}$. For any SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, Lemma 2.2.2 shows that the values in the first row of λ are distinct, so, when creating a filling of λ, the values in R must also appear in both the first row of λ and the first column of λ.

Consider $A = \{a_1, a_2, \ldots, a_{\mu_a}\} - R$ and $A' = \{a'_1, a'_2, \ldots, a'_{j-1}\} - R$. Since we know that the values of R must appear in both the first row of λ and first column of λ, $A \cup A'$ contains the remaining values of M that need to be placed in λ. In other words, *$A \cup A'$ is the set of all values $\leq |\alpha| + 1$ that can appear in exactly one of the first row of λ or the first column of λ.*

Since $\lambda_a \geq \left\lceil \frac{h}{2} \right\rceil$ we have $\lambda_a \geq \lambda_l$. Thus we have $|R| \leq \mu_l < \lambda_l \leq \lambda_a$. Now because

$$|R| \leq \lambda_a, \tag{3.3}$$

we can extend the values of R to an increasing sequence $b_1 < b_2 < \cdots < b_{\lambda_a}$ by choosing $\lambda_a - |R|$ additional values from $(A \cup A') \cap R_{\alpha,k}$. There are enough values to choose from since there are

$$\mu_a - |R| \geq \lambda_a - |R| \tag{3.4}$$

values of $(A \cup A') \cap R_{\alpha,k}$ present in the first row of μ. The sequence of b_i's is strictly increasing since $(A \cup A') \cap R = \emptyset$.

Now $M - \{b_1, \ldots, b_{\lambda_a}\} \subseteq M - R$ contains $w = |M| - \lambda_a = \mu_a + j - 2 - \lambda_a$ distinct values, each no greater than $|\alpha| + 1$. That is, they are an increasing sequence $c_1 < c_2 < \cdots < c_w$, where $c_w \leq |\alpha| + 1$. We have

$$\lambda_l = \mu_a + \mu_l - \lambda_a$$

$$\begin{aligned} &= 1 + (\mu_a + j - 2 - \lambda_a) + (\mu_l - j + 1) \\ &= 1 + w + (\mu_l - j + 1), \end{aligned}$$

so we may fill λ as shown below.

$$\begin{array}{cccc} b_1 & b_2 & \cdots & b_{\lambda_a} \\ c_1 & & & \\ c_2 & & & \\ \vdots & & & \\ c_w & & & \\ a'_j & & & \\ a'_{j+1} & & & \\ \vdots & & & \\ a'_{\mu_l} & & & \end{array}$$

By the definitions of R and the sequence of b_i, it is clear that $b_1 = a_1 = a'_1$. Thus, $a_1 < c_1$. Since the sequence of c_i is strictly increasing and since $c_w \leq |\alpha| + 1 < |\alpha| + 2 = a'_j$ we have

$$b_1 < c_1 < c_2 < \cdots < c_w < a'_j < \cdots < a'_{\mu_l}.$$

That is, the first column of λ is increasing. We also had

$$b_1 < b_2 < \cdots < b_{\mu_a},$$

so the first row of λ is increasing. Hence this filling gives us a SSYT T of shape $\lambda \oplus \Delta_\alpha$ and content ν.

We now check that T has a lattice reading word so that we may apply Lemma 2.2.3 to obtain the desired tableau $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$. Suppose that T does not have a lattice reading word. Then, when reading the foundation λ of T, we must reach a point where the lattice condition failed. Let x be the value that, when read, caused the lattice condition to fail. The lattice condition could not have failed when reading the first row of λ since the values in the first row of λ were distinct values chosen from $R_{\alpha,k}$. Therefore *this* x appears somewhere in the first column of λ. We consider the the two cases $x \in R_{\alpha,k}$ and $x \notin R_{\alpha,k}$.

Consider the first case, $x \in R_{\alpha,k}$. If a value of $R_{\alpha,k}$ appears only once in the foundation then reading this value cannot violate the lattice condition. Thus, since the lattice condition failed at this x, this x cannot be the first

time that x was read in λ. Since the columns strictly increase, the previous x must have appeared in the first row of λ and, since the values in the first row are distinct, this is the only other x in λ. Since the content of λ is the same as the content of μ, these two x's appear in μ as well. Using the fact that $\mathcal{T}_1$ has a lattice reading word, together with the content of Δ_α, we find that the value $x-1$ appeared in μ. Thus the value $x-1$ also appears in λ. Now, since the rows and columns of λ strictly increase, either the $x-1$ appears in the first column of λ above the entry x, or the $x-1$ appears in the first row of λ left of the entry x. In either case the $x-1$ is read before the second x is read and the lattice condition will not fail when reading this second x, contrary to our assumption.

We now look at the second case, where $x \notin R_{\alpha,k}$. Again, the x we are interested in appears in the first column of λ. There cannot be a second x in λ since the column strictly increases and $x \notin R_{\alpha,k}$ implies that no other x was placed in the first row of λ. As before, x must have appeared in μ and, in particular, it also appears somewhere in the first column. Since $\mathcal{T}_1$ has a lattice reading word we must read a sequence of values t, $t+1$, $t+2$, ..., $x-2$, $x-1$ in μ, where $t \in R_{\alpha,k}$, before we read the x, and we may assume that none of the values $t+1$, $t+2$, ..., $x-2$, $x-1$ are from $R_{\alpha,k}$. Hence each of the values t, $t+1$, ..., $x-2$, $x-1$ also appear in λ. None of the values $t+1$, $t+2$, ..., $x-1$ can appear in the first row of λ since the first row was chosen from $R_{\alpha,k}$. That is, each value $t+1, t+2, \ldots, x-1$ appears in the first column of λ. Also, since the rows and columns of λ strictly increase, either the t appears either in the first column of λ above the entry $t+1$, or the t appears in the first row of λ. In either case the entire sequence t, $t+1$, ..., $x-2$, $x-1$ is read before the x is read in λ and the lattice condition does not fail at x, contradicting our assumption.

Since T has a lattice reading word, we can now apply Lemma 2.2.3 to obtain the SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. Therefore from any SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν we can create a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν.

Let $c^{\kappa}_{\rho\nu}$ be the number of SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, and $c^{\kappa'}_{\rho'\nu}$ be the number of SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. We shall show that the sets R and $A \cup A'$ that were described above are completely determined by the content ν. That is, without starting with a specific tableau, but only starting with the desired content of a SSYT of some fat staircase with hook foundation with lattice reading word, we show how to determine R, the set of values that

must appear in both the first row and first column of the foundation, and determine $A \cup A'$, the set of values less than or equal to $|\alpha|+1$ that can only appear in one of the first row or the first column of the foundation.

Since Δ_α is uniquely filled, from ν we can determine the content of the foundation μ (λ, respectively) needed to create a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ ($\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, resp.) with lattice reading word and content ν. From the content of the foundation we can determine the values $a'_j, a'_{j+1}, \ldots$ greater than $|\alpha|+1$. Since Lemma 2.2.2 shows that the entries in the first row of the foundation strictly increase, any value that appears twice in the foundation must appear in both the first row of μ (λ, resp.) and first column of μ (λ, resp.). These values, together with the minimum value in the content of the foundation, give the set R. Then $A \cup A'$ is the values in the foundation that are less than or equal to $|\alpha|+1$ but are not in R. After determining the remaining entries of the first row of μ (first row of λ, resp.), the rest of the foundation is uniquely determined.

Now, to actually form a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ ($\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, resp.) with lattice reading word and content ν, we only need to choose the remaining $\mu_a - |R|$ ($\lambda_a - |R|$, resp.) values from the set $(A \cup A') \cap R_{\alpha,k}$ to place in the first row of μ (first row of λ, resp.). Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content ν is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\mu_a - |R|}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content ν is given by

$$c^{\kappa}_{\rho\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\lambda_a - |R|}.$$

Since $\lambda_a \geq \lambda_l > \mu_l \geq j-1$, we have

$$0 \geq j - 1 - \lambda_a. \tag{3.5}$$

Thus, for each i we have

$$\begin{aligned}
\mu_a - |R| - i &\geq \mu_a - |R| - i + (j - 1 - \lambda_a) \\
&\geq (\mu_a - |R|) + (j - 1 - |R|) - (\lambda_a - |R|) - i \\
&\geq |A| + |A'| - (\lambda_a - |R|) - i \\
&\geq |(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i.
\end{aligned}$$

That is,

$$\mu_a - |R| - i \geq |(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i, \tag{3.6}$$

for each i.

Therefore

$$\begin{aligned}
c^{\kappa}_{\rho\nu} &= \frac{|(A \cup A') \cap R_{\alpha,k}|!}{(|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|))!(\lambda_a - |R|)!} \\
&= \frac{|(A \cup A') \cap R_{\alpha,k}|!}{(|(A \cup A') \cap R_{\alpha,k}| - (\mu_a - |R|))!(\mu_a - |R|)!} \\
&\quad \times \prod_{i=0}^{\mu_a - \lambda_a - 1} \frac{\mu_a - |R| - i}{|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i} \\
&= c^{\kappa'}_{\rho'\nu} \times \prod_{i=0}^{\mu_a - \lambda_a - 1} \frac{\mu_a - |R| - i}{|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i} \\
&\geq c^{\kappa'}_{\rho'\nu},
\end{aligned}$$

where we have used Equation 3.6 in the final step. Since this inequality holds for all contents ν for which there was a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, we have $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \succeq_s 0$. ∎

Finally, we turn to the hooks λ, μ satisfying $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$.

Theorem 3.1.4 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$ and let $0 \leq k \leq 1$. If $\lambda_a \geq \mu_l$ then $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof We are given that $\lambda_a \geq \mu_l$ and we wish to show that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. We claim that proof of Theorem 3.1.3, with a few equations verified under the current hypotheses, also proves this theorem.

Given the SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, in order to create the SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν we needed to check that Equation 3.2, Equation 3.3 and Equation 3.4 held. Namely, we required that $\mu_l - j + 1 \leq \lambda_l$, $|R| \leq \lambda_a$, and $\mu_a - |R| \geq \lambda_a - |R|$. Since the first two equations were satisfied, we were able to fit the required values into the first row and first column of λ. Further, since $\mu_a - |R| \geq \lambda_a - |R|$, there were enough values to fill the first row of λ, and therefore we could construct the tableau with all the desired properties.

In order to show Equation 3.2 holds for the assumptions of this theorem, we note that $\mu_l \leq \left\lceil \frac{h}{2} \right\rceil \leq \lambda_l$. In order to show Equation 3.3 holds for the assumptions of this theorem, we note that $|R| \leq \mu_l \leq \lambda_a$. In order to show Equation 3.4 holds for the assumptions of this theorem, we note that $\mu_a \geq \left\lceil \frac{h}{2} \right\rceil \geq \lambda_a$. Therefore we can create a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν whenever there exists a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν.

Next, the proof of Theorem 3.1.3 checked that, for each of these contents ν, the number of SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν is greater than or equal to the number of SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. To prove this, we first required that Equation 3.5 held. Namely, we required that $\lambda_a \geq j - 1$. We used this equation to show that Equation 3.6 held for each i, which gave us the desired inequality for the Littlewood-Richardson numbers. In order to show Equation 3.5 holds for the assumptions of this theorem, we note that $\lambda_a \geq \mu_l \geq j - 1$. Therefore the inequality for the Littlewood-Richardson numbers holds here as well, which proves that

$$s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \geq_s 0. \quad \blacksquare$$

Theorem 3.1.5 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$ and let $0 \leq k \leq 1$. If $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$, then $\lambda_a \geq \mu_l$.*

Proof Towards a contradiction, suppose $\lambda_a < \mu_l$.

As before, we let $r_1 < r_2 < \cdots < r_{n+k}$ be the values of $R_{\alpha,k}$. We can create a SSYT of shape μ by filling the boxes of μ as follows.

$$\begin{array}{cccc} r_1 & r_2 & \cdots & r_{\mu_a} \\ r_{\mu_a+1} & & & \\ r_{\mu_a+2} & & & \\ \vdots & & & \\ r_h & & & \end{array}$$

Since we are using distinct values of $R_{\alpha,k}$, it is easy to check that the resulting tableau of shape $\mu \oplus \Delta_\alpha$ has lattice reading word. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word, where μ is filled as shown above.

We now wish to count all SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ (shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, respectively) with content $\nu = c(\mathcal{T})$. Since Δ_α has a unique way of being filled, we must find all semistandard fillings of μ (λ, resp.) with the values $r_1, r_2, \ldots, r_h$. Since $r_1 < r_2 < \cdots < r_h$, the value r_1 must appear in the top-left corner of μ (λ, resp.). Further, once we choose $\mu_l - 1$ ($\lambda_a - 1$, resp.) of the values $r_2 < r_3 < \cdots < r_h$ to appear in the first column of μ (first row of λ, resp.), then the remaining r's must appear in the first row of μ (first column of λ, resp.) and the order of all these values is uniquely determined by the semistandard conditions.

Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{h-1}{\mu_l - 1}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa}_{\rho\nu} = \binom{h-1}{\lambda_a - 1}.$$

Since $\lambda_a < \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$, we have $h + 1 > \lambda_a + \mu_l$. Therefore we have $h - \lambda_a > \mu_l - 1$ and we obtain

$$h - \lambda_a - i > \mu_l - 1 - i, \tag{3.7}$$

for each i.

Therefore

$$\begin{aligned} c^{\kappa'}_{\rho'\nu} &= \frac{(h-1)!}{(h-\mu_l)!(\mu_l - 1)!} \\ &= \frac{(h-1)!}{(h-\lambda_a)!(\lambda_a - 1)!} \times \prod_{i=0}^{\mu_l - \lambda_a - 1} \frac{h - \lambda_a - i}{\mu_l - 1 - i} \\ &= c^{\kappa}_{\rho\nu} \times \prod_{i=0}^{\mu_l - \lambda_a - 1} \frac{h - \lambda_a - i}{\mu_l - 1 - i} \\ &> c^{\kappa}_{\rho\nu}, \end{aligned}$$

where we have used Equation 3.7 in the final step. Since we have found a content ν with $c^{\kappa'}_{\rho'\nu} > c^{\kappa}_{\rho\nu}$, we can conclude that $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \not\geq_s 0$, which is a contradiction. Therefore we have $\lambda_a \geq \mu_l$. ∎

We finish this section with a result for the difference $s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)}$, where D is a sum of fat staircases and λ is a single row.

Theorem 3.1.6 *Let $D = \rho/\mu$ be such that*

$$s_D = \sum_{\nu = \textit{fat staircase}} c^{\rho}_{\mu\nu} s_\nu,$$

and λ be a single row. Then we have

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)} \geq_s \sum_{\nu} c^{\rho}_{\mu\nu} \left(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)} \right) \geq_s 0.$$

Proof Applying Theorem 2.3.1 shows that

$$s_{\mathcal{S}(\lambda,D;1)} \leq_s \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)},$$

so we have

$$-s_{\mathcal{S}(\lambda,D;1)} \geq_s -\sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}. \tag{3.8}$$

We now intend to show that

$$s_{\mathcal{S}(\lambda^t,D;1)} = \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)}. \tag{3.9}$$

Then, adding Equation 3.8 and Equation 3.9 will give that

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)} \geq_s \sum_{\nu} c^{\rho}_{\mu\nu} \left(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\right).$$

To prove Equation 3.9, we first note that Theorem 2.3.1 shows that

$$s_{\mathcal{S}(\lambda^t,D;1)} \leq_s \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)}.$$

We now recall how the proof of Theorem 2.3.1 proceeded. The proof looked at the map that took a SSYT T of shape $\mathcal{S}(\lambda^t, D; 1)$ with lattice reading word and $c(D) = \nu$ to the tableau T' of shape $\mathcal{S}(\lambda^t, \alpha(\nu)^{\triangleleft}; 1)$ whose foundation λ^t was filled identically as in T. Then for each such image T' the proof showed that

1. T' was a SSYT with lattice reading word and $c(T') = c(T)$, and
2. there are at most $c^{\rho}_{\mu\nu}$ distinct SSYT of shape $\mathcal{S}(\lambda^t, D; 1)$ that could map to T'.

Further, the $c^{\rho}_{\mu\nu}$ possible tableau of shape $\mathcal{S}(\lambda^t, D; 1)$ mentioned in 2 were the tableaux of shape $\mathcal{S}(\lambda^t, D; 1)$ where the filling of λ^t was identical to the filling of λ^t in T' and the filling of D was one of the $c^{\rho}_{\mu\nu}$ semistandard fillings of D with lattice reading word and content ν. We now check that each of these preimages of T' *is* a SSYT of shape $\mathcal{S}(\lambda^t, D; 1)$ with lattice reading word.

Let $\mathcal{T}$ be a preimage of T'. Then the subtableau of $\mathcal{T}$ of shape D is semistandard and has lattice reading word since, as just mentioned, it is one of the $c^{\rho}_{\mu\nu}$ semistandard fillings of D with lattice reading word and content ν. We also have that the foundation λ^t of $\mathcal{T}$ is semistandard since it is filled

indentically to the foundation of T', which was semistandard. Further, since λ^t is a single column and $k = 1$, the foundation of $\mathcal{S}(\lambda^t, D; 1)$ does not appear directly below any box of D. Therefore $\mathcal{T}$ is a SSYT. Also, $\mathcal{T}$ has a lattice reading word since D has a lattice reading word and, since we have $c(D) = c(\Delta_{\alpha(\nu)})$, after reading D the lattice restrictions on $\mathcal{T}$ are identical to the lattice restrictions on T', which has a lattice reading word.

Hence, we have shown that each of these $c^{\rho}_{\mu\nu}$ preimages of T' is a SSYT of shape $\mathcal{S}(\lambda^t, D; 1)$ with lattice reading word. This proves Equation 3.9. That is,

$$s_{\mathcal{S}(\lambda^t,D;1)} = \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)}.$$

We may now combine Equation 3.8 and Equation 3.9 to obtain

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)} \geq_s \sum_{\nu} c^{\rho}_{\mu\nu}\big(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\big).$$

Now, using Theorem 3.1.4 on each pair $s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)}$, $s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}$ gives

$$s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)} \geq_s 0,$$

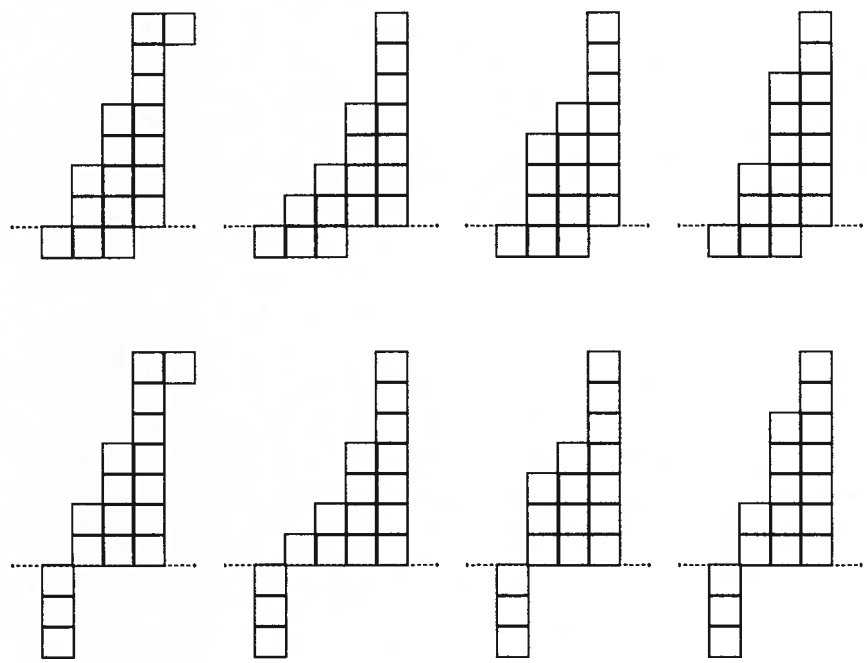
for each fat staircase $\alpha(\nu)$. Therefore, we now have that

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)} \geq_s \sum_{\nu} c^{\rho}_{\mu\nu}\big(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\big) \geq_s 0. \quad \blacksquare$$

Example Once again we consider the example with $\rho = (4,3,3,3,3,3,3)$, $\mu = (2,2,2,1,1)$, and $D = \rho/\mu$. We have previous seen that D is a sum of fat staircases. Namely, we have

$$\begin{aligned} s_D &= s_{(4,3,2,2,1,1,1)} + s_{(3,3,3,2,1,1,1)} + s_{(3,3,2,2,2,1,1)} \\ &= s_{\Delta_{(3,2,1,1)}} + s_{\Delta_{(3,1,3)}} + s_{\Delta_{(2,3,2)}}. \end{aligned}$$

We now wish to apply Theorem 3.1.6 to this diagram D using $\lambda = (3)$. We are interested in appending λ to D and each of $\Delta_{(3,2,1,1)}$, $\Delta_{(3,1,3)}$, and $\Delta_{(2,3,2)}$ and also in appending λ^t to D and each of $\Delta_{(3,2,1,1)}$, $\Delta_{(3,1,3)}$, and $\Delta_{(2,3,2)}$. Therefore, we are interested in the following diagrams.



Theorem 3.1.6 states that

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)} \geq_s \sum_{\nu} c^{\rho}_{\mu\nu}\big(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\big) \geq_s 0.$$

In fact if we compute both

$$\begin{aligned}\sum_{\nu} c^{\rho}_{\mu\nu}\big(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\big) &= 1\big(s_{\mathcal{S}(\lambda^t,(3,2,1,1)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,(3,2,1,1)^{\triangleleft};1)}\big)\\ &+ 1\big(s_{\mathcal{S}(\lambda^t,(3,1,3)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,(3,1,3)^{\triangleleft};1)}\big)\\ &+ 1\big(s_{\mathcal{S}(\lambda^t,(2,3,2)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,(2,3,2)^{\triangleleft};1)}\big)\end{aligned}$$

and

$$s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)},$$

then the difference is given by

$$\begin{aligned}&\big(s_{\mathcal{S}(\lambda^t,D;1)} - s_{\mathcal{S}(\lambda,D;1)}\big) - \left(\sum_{\nu} c^{\rho}_{\mu\nu}\big(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\big)\right)\\ =\ & s_{(5,4,3,2,1,1,1)} + s_{(5,4,2,2,2,1,1)} + s_{(5,4,2,2,1,1,1,1)}.\end{aligned}$$

Further, one can also check that

$$\sum_{\nu} c^{\rho}_{\mu\nu}\left(s_{\mathcal{S}(\lambda^t,\alpha(\nu)^{\triangleleft};1)} - s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};1)}\right) \geq_s 0.$$

3.2 Expressions for Schur-Positive Differences

We now wish extend the proof of Theorem 3.1.3 to give an exact formula for the difference $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)}$. In order to state this result we shall find it helpful to think of weak compositions as vectors with non-negative integer entries. When we write the vector $z = (z_1, z_2, z_3, \ldots, z_n)$ we shall mean the infinite vector $z = (z_1, z_2, z_3, \ldots, z_n, 0, 0, 0, \ldots)$. We shall only consider vectors with finitely many non-zero entries and hence we shall only display vectors with finite length, but in this way we unambiguously may add vectors of different lengths.

Given a positive integer i, we let e_i denote the i-th standard basis vector. That is, the vector that has its i-th entry equal to 1 and all remaining entries equal to 0. Further, for A being a finite set of positive integers, we let $e_A = \sum_{a\in A} e_a$ be the indicator vector of A. Given $C \subseteq A$, C is called a *maximal interval* of A if C is of the form $C = \{c, c+1, c+2, \ldots, c+j\}$ where $c-1, c+j+1 \notin A$. We say that two maximal intervals A_1, A_2 of A are disjoint if $A_1 \cap A_2 = \emptyset$. It is clear that any finite set A of non-negative integers can be decomposed into a finite collection of *maximal disjoint intervals* $\{A_1, A_2, \ldots, A_m\}$, and that this collection is unique. Finally, when we write $\min(A)$ we mean the minimum of the set A in the usual sense. That is, $\min(A)$ is the smallest number in A.

We begin with the following lemma.

Lemma 3.2.1 *Let Δ_α a fat staircase, $0 \leq k \leq 1$, and $h \geq 1$. Then*

$$\nu(B, C) = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|})$$

is a partition for each pair of sets B, C satisfying

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha|+1\}$, $C \subseteq B \cap R_{\alpha,k} - \min(B)$,
- $|\alpha|+1 \in B$ *if* $|B|+|C| \leq h-1$,
- *if* $B = \bigcup_{j=1}^{p} B_j$ *where the* B_j *are the maximal disjoint intervals of* B, *then* $\min(B_j) \in R_{\alpha,k}$ *for each* j, *and*
- *if* $C = \bigcup_{j=1}^{m} C_j$ *where the* C_j *are the maximal disjoint intervals of* C, *then* $\min(C_j) - 1 \in B - C$ *for each* j.

Proof To begin with, δ_α is a partition. We shall think of δ_α as being the content of Δ_α and show that we can read the remaining h values of $\nu(B,C)$ while maintaining the lattice condition. This will prove the result.

Adding the content e_B corresponds to reading the values of $B = \bigcup_{j=1}^{p} B_j$. Since each B_j is a set of consecutively increasing numbers and each $\min(B_j) \in R_{\alpha,k}$, the lattice condition is maintained while reading each of these maximal disjoint intervals. Therefore the lattice condition is maintained while reading all of B.

Next, adding the content e_C corresponds to reading the values of $C = \bigcup_{j=1}^{m} C_j$. Since $C \subseteq B$, this is the second time we are reading each of these values. The fact that each value is in $R_{\alpha,k}$ allowed us to read all these values the first time. Now, since each C_j is a set of consecutively increasing numbers and we have previously read a $\min(C_j) - 1 \in B - C$, the lattice condition is maintained while reading each of these maximal disjoint intervals. Therefore the lattice condition is maintained while reading all of C this time. We note that we have not yet added any values greater than $|\alpha| + 1$.

Finally, adding $(0^{|\alpha|+1}, 1^{h-|B|-|C|})$ corresponds to reading the sequence $|\alpha|+2, |\alpha|+3, \ldots, |\alpha|+h-|B|-|C|+1$. This sequence is non-empty only if $|B| + |C| < h$, and in this case we have $|\alpha| + 1 \in B$. Since $|\alpha| + 1 \in R_{\alpha,k}$ the lattice condition is maintained while reading these values.

Since the reading word is lattice, this implies that

$$\nu(B,C) = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|})$$

is a partition. ∎

We can now state a formula for the differences discussed in Theorem 3.1.3.

Theorem 3.2.2 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$ and let $0 \leq k \leq 1$. Then*

$$s_{\mathcal{S}(\lambda,\alpha^q;k)} - s_{\mathcal{S}(\mu,\alpha^q;k)} = \sum_{B,C} a_{B,C} s_{\nu(B,C)}$$

where the coefficients $a_{B,C}$ are given by

$$a_{B,C} = \binom{|(B-C) \cap R_{\alpha,k}| - 1}{\lambda_a - |C| - 1} - \binom{|(B-C) \cap R_{\alpha,k}| - 1}{\mu_a - |C| - 1},$$

the partitions $\nu(B,C)$ that arise are given by

$$\nu(B,C) = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|}),$$

and the sum is over all sets B, C such that

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha|+1\}$, $C \subseteq B \cap R_{\alpha,k} - \min(B)$,
- $|C| + 1 \leq \lambda_l$,
- $\lambda_a \leq |B \cap R_{\alpha,k}|$,
- $|B| + |C| \leq h$, *and* $|\alpha| + 1 \in B$ *if* $|B| + |C| \leq h - 1$,
- *if* $B = \bigcup_{j=1}^{p} B_j$ *where the* B_j *are the maximal disjoint intervals of* B, *then* $\min(B_j) \in R_{\alpha,k}$ *for each* j, *and*
- *if* $C = \bigcup_{j=1}^{m} C_j$ *where the* C_j *are the maximal disjoint intervals of* C, *then* $\min(C_j) - 1 \in B - C$ *for each* j.

Proof From the proof of Theorem 3.1.3 we saw that, for a given content ν, a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν exists whenever a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν exists. Furthermore, when such tableaux exist, we saw that the number of SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content ν is given by

$$c^{\kappa}_{\rho\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\lambda_a - |R|}$$

and the number of SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content ν is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\mu_a - |R|},$$

where, for the given content ν, R was the set of values that must appear in both the first row and first column of the foundation of each diagram, and $A \cup A'$ are the values less than or equal to $|\alpha| + 1$ that must also appear in the foundation of each diagram but are not in R. We recall that all but one element of R appear twice in the foundation. Namely, all but the smallest value of R, which is necessarily the top-left entry of each foundation.

Let ν be such that s_ν appears in the difference with non-zero coefficient. We consider the following two cases.

Case 1: There is a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ and content ν with lattice reading word.

Case 2: There is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ and content ν with lattice reading word.

Case 1 will give rise to the terms s_ν where there is a SSYT with content ν and lattice reading word of both shapes $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ and $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$. **Case 2** will give rise to the terms s_ν where there is a SSYT with content ν and lattice reading word of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ but not of shape $\mathcal{S}(\mu, \alpha^\triangleleft; k)$.

We will see that in each case there exists sets B and C with the desired properties such that $\nu(B, C) = \nu$ and $a_{B,C}$ gives the correct coefficients in the difference. Further, we will see that in **Case 1** we also have the properties that

$$\mu_l \geq |C| + 1 \text{ and } \mu_a \leq |B \cap R_{\alpha,k}|,$$

and in **Case 2** we have the property that either

$$\mu_l < |C| + 1 \text{ or } \mu_a > |B \cap R_{\alpha,k}|.$$

Therefore, in each case, we shall show that for each term s_ν in the difference there are sets B and C with the properties listed in the statement of the theorem together with these extra properties. Then, conversely, in each case, we shall show that for each pair of sets B and C with the properties listed in the statement of the theorem together with these extra properties on μ we will obtain a term $s_{\nu(B,C)}$ in the difference.

Case 1:

Since there is a SSYT of shape $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ with content ν and lattice reading word, Theorem 3.1.3 implies that there is also SSYT of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ with lattice reading word and content ν. Since the content of Δ_α is fixed, the contents of the foundations are equal. If we let C be the values less than or equal to $|\alpha| + 1$ that appear twice in the foundation we obtain $|R| = |C| + 1$. Further, if we let B be the set of *all* values less than or equal to $|\alpha| + 1$ that appear in the foundation, we have $B - R = A \cup A'$. Since Lemma 2.2.2 shows that the first row of each foundation strictly increases, each value of C must appear in both the first row and the first column of each foundation.

Thus for each content ν we obtain a pair B, C. We now show that the sets B and C have the properties listed in the statement of the theorem together with the two extra properties of this case.

By the definition of B we have $B \subseteq \{1, 2, \ldots, |\alpha| + 1\}$. When $k = 0$ every box in the foundation occurs below some box of Δ_α. Hence, since the columns strictly increase, the value 1 cannot appear in the foundation for $k = 0$. Thus $B \subseteq \{2, 3, \ldots, |\alpha| + 1\}$ for $k = 0$. Therefore, for $0 \leq k \leq 1$, we have

$$B \subseteq \{2 - k, 3 - k, \ldots, |\alpha| + 1\}.$$

Since the multiset of values $B \cup C$ all appear in each foundation, we have

$$|B| + |C| \leq h.$$

Further, if $|B| + |C| \leq h - 1$, then there are entries in each foundation that do not belong to either B or C. By the definition of B and C, these entries must be greater than or equal to $|\alpha|+2$ and then the lattice condition implies that $|\alpha| + 1$ must appear in the foundation. That is,

$$\textit{if } |B| + |C| \leq h - 1 \textit{ then } |\alpha| + 1 \in B.$$

Since the values of R appear in the first column of λ and appear in the first column of μ we have $|R| \leq \lambda_l, \mu_l$ which, since $|R| = |C| + 1$, gives

$$|C| + 1 \leq \lambda_l$$

and

$$|C| + 1 \leq \mu_l.$$

Since Lemma 2.2.2 shows that each entry of the first row of λ and the each entry of first row of μ is in $R_{\alpha,k}$ we have

$$|B \cap R_{\alpha,k}| \geq \lambda_a$$

and

$$|B \cap R_{\alpha,k}| \geq \mu_a.$$

Decompose $C = \bigcup_{j=1}^{m} C_j$ for disjoint C_j. Then, by the definition of C, for each C_j, $\min(C_j) \in C$ appears twice in the foundation but $\min(C_j) - 1 \notin C$ does not. The lattice condition implies that $\min(C_j) - 1$ appears in each foundation. Thus we have

$$\min(C_j) - 1 \in B - C \text{ for each } j.$$

The smallest entry in the foundation is $b = \min(B)$, and, as previously mentioned, this value appears in the top-left corner of each foundation. Since both the first row and first column strictly increase, b occurs only once in each foundation. Hence $b \notin C$. Since every value of C appears in the first row of the foundation, Lemma 2.2.2 shows that $C \subseteq R_{\alpha,k}$. Also, the definition of B and C implies $C \subseteq B$. Thus we obtain

$$C \subseteq B \cap R_{\alpha,k} - \min(B).$$

Now decompose $B = \bigcup_{j=1}^{p} B_j$ for disjoint B_j. Since $\min(B_j) - 1 \notin B$ for each j, each value $\min(B_j) - 1$ does not appear in either foundation. Hence, the lattice condition requires that

$$\min(B_j) \in R_{\alpha,k} \text{ for each } j.$$

Therefore for each content ν we obtain B and C with the desired properties. Also, $h - |B| - |C|$ denotes the number of values $\geq |\alpha| + 2$ that appear in the foundation, and the lattice condition shows that these must be the values $|\alpha| + 2$, $|\alpha| + 3$, ..., $|\alpha| + h - |B| - |C| + 1$. Altogether, the entries of Δ_α, B, C, and the values $|\alpha| + 2$, $|\alpha| + 3$, ..., $|\alpha| + h - |B| - |C| + 1$ make up the entire content ν. Hence the content ν can be expressed as

$$\nu = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|}) = \nu(B, C).$$

Conversely, for any sets B and C with the properties in the statement of the theorem together with the two extra properties, we show that there exists a SSYT of shape $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ with lattice reading word and content

$$\nu(B, C) = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|}).$$

To this end we begin by creating a filling of μ with content $e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|})$. The steps of creating the tableau are the same as in the proof of Theorem 3.1.3, but are now phrased in terms of the sets B and C. First, we place the $h-|B|-|C|$ values $|\alpha|+2, |\alpha|+3, \ldots, |\alpha|+h-|B|-|C|+1$ at the bottom of the first column of μ. This adds $(0^{|\alpha|+1}, 1^{h-|B|-|C|})$ to our content. We then place $\min(B)$ in the top-left box of μ and place the values of C in both the first row and first column of μ, which we can do since $\mu_a > \mu_l \geq |C|+1$. Then we choose $\mu_a - |C| - 1$ values from $(B - C - \min(B)) \cap R_{\alpha,k}$, which is possible since $|B \cap R_{\alpha,k}| \geq \mu_a$ and $C \cup \{\min(B)\} \subseteq R_{\alpha,k}$, to fill the remaining $\mu_a - |C| - 1$ boxes in the first row of μ. We place the remaining values of $B - C - \min(B)$ in increasing order in the remaining boxes of the first column of μ. This gives a semistandard filling of μ. Moreover, we have placed each entry of C twice and each entry of $B - C$ once. This adds $2e_C + e_{B-C} = e_B + e_C$ to the content. Together with the unique filling of Δ_α, this gives a SSYT T of shape $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ and content $\nu(B, C)$.

We now show that T has a lattice reading word. Suppose not. Then, when reading the foundation μ of T, we must reach a point where the lattice condition failed. Let x be the entry that, when read, caused the lattice condition to fail. The lattice condition could not have failed when reading the first row of μ since the values in the first row of μ were distinct values chosen from $R_{\alpha,k}$. Therefore this x appears somewhere in the first column of μ. We consider the the two cases $x \in R_{\alpha,k}$ and $x \notin R_{\alpha,k}$.

Consider the first case, $x \in R_{\alpha,k}$. Since the lattice condition failed at this x, this x cannot be the first x to appear in μ. Thus there was a previous x in the first row of μ. Hence $x \in C = \bigcup_{j=1}^{m} C_j$. So $x \in C_j$, for some j. Let

$t = \min(C_j)$. Then $t-1 \in B - C$ appears exactly once in μ and each of the values $t, t+1, \ldots, x \in C$ appear twice in μ. Since the first row and first column of μ strictly increase, we must read the $t-1$ before reading the second copy of any of the values $t, t+1, \ldots, x$, but then the lattice condition does not fail at x, contrary to our assumption.

Now consider the second case, $x \notin R_{\alpha,k}$. Since $x \notin R_{\alpha,k}$, Lemma 2.2.2 implies that there is no x in the first row. Since the tableau is semistandard this implies that there is only one x in the tableau. If $x \leq |\alpha| + 1$, then $x \in B = \bigcup_{i=1}^{p} B_i$. Thus $x \in B_i$ for some i. Let $t = \min(B_i) \in R_{\alpha,k}$. Then each of the values value $t, t+1, \ldots, x-1$ also appears in the tableau and, since the first row and first column of μ strictly increase, they are read before x. Thus the lattice condition does not fail at x, contrary to assumption. If $x > |\alpha| + 1$, then $|B| + |C| \leq h - 1$. Therefore $|\alpha| + 1 \in B$. Further, the values in μ greater than $|\alpha| + 1$ are precisely the $h - |B| - |C|$ values $|\alpha|+2, |\alpha|+3, \ldots, |\alpha|+h-|B|-|C|+1$, and these all appear at the bottom of the first column of μ. Since the $|\alpha| + 1$ is read prior to these, the lattice condition could not fail at x, contrary to assumption.

Therefore, in **Case 1**, the contents ν for which there exists a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word are enumerated precisely by the contents $\nu(B, C)$, where B and C satisfy the properties listed in the statement of the theorem together with the two extra properties. Theorem 3.1.3 showed that, for these ν, there is also a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word. Furthermore, for $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ and $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$, we have that

$$\begin{aligned} c^{\kappa'}_{\rho'\nu(B,C)} &= \binom{|(A \cup A') \cap R_{\alpha,k}|}{\mu_a - |R|} \\ &= \binom{|(B - R) \cap R_{\alpha,k}|}{\mu_a - |C| - 1} \\ &= \binom{|(B - C) \cap R_{\alpha,k}| - 1}{\mu_a - |C| - 1} \end{aligned}$$

and

$$\begin{aligned} c^{\kappa}_{\rho\nu(B,C)} &= \binom{|(A \cup A') \cap R_{\alpha,k}|}{\lambda_a - |R|} \\ &= \binom{|(B - R) \cap R_{\alpha,k}|}{\lambda_a - |C| - 1} \\ &= \binom{|(B - C) \cap R_{\alpha,k}| - 1}{\lambda_a - |C| - 1}. \end{aligned}$$

This shows that $a_{B,C}$ are the correct coeffiecients for the terms $s_{\nu(B,C)}$ that arise in **Case 1**.

Case 2:

In this case there is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. However, we are assuming that content ν is such that the Schur function s_ν appears in the difference with non-zero coefficient. Therefore there is a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν.

Following the same reasoning as in **Case 1** we find that there exists sets B and C such that

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha|+1\}$, $C \subseteq B \cap R_{\alpha,k} - \min(B)$,
- $|C| + 1 \leq \lambda_l$,
- $\lambda_a \leq |B \cap R_{\alpha,k}|$,
- $|B| + |C| \leq h$, and $|\alpha| + 1 \in B$ if $|B| + |C| \leq h - 1$,
- if $B = \bigcup_{j=1}^{p} B_j$ where the B_j are the maximal disjoint intervals of B, then $\min(B_j) \in R_{\alpha,k}$ for each j, and
- if $C = \bigcup_{j=1}^{m} C_j$ where the C_j are the maximal disjoint intervals of C, then $\min(C_j) - 1 \in B - C$ for each j,

Furthermore, given such sets, the only two ways there could not exist a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content $\nu(B, C)$ is

- there was not enough room in one of the first row or first column of μ to fit all of C and $\min(B)$, or
- after placing $\min(B)$ and the values of C in the first row of μ, there were not enough elements of $(B - C - \min(B)) \cap R_{\alpha,k}$ to fill the rest of the first row of μ.

Since $\mu_l < \mu_a$, the first case implies that $\mu_l < |C| + 1$. Since $\min(B) \cup C \subseteq R_{\alpha,k} \cap B$, the second case implies that

$$|B \cap R_{\alpha,k}| < \mu_a.$$

Therefore all the desired properties are satisfied.

Conversely, if we are given any sets B and C satisfying the properties listed in the statement of the theorem together with either $\mu_l < |C| + 1$ or $|B \cap R_{\alpha,k}| < \mu_a$, then by following the same method as in **Case 1** we can show

there is a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content $\nu(B, C)$. Moreover, since either $\mu_l < |C| + 1$ or $|B \cap R_{\alpha,k}| < \mu_a$, either there is not enough room in the first column of μ to fit all of C and $\min(B)$ or Lemma 2.2.2 shows that there are not enough values of B to fill the first row of μ while satisfying the conditions for a SSYT with lattice reading word. That is, there is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content $\nu(B, C)$. Therefore the contents ν for which there exists a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word but there is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word are enumerated precisely by the contents $\nu(B, C)$, where B and C satisfy the desired properties of this case.

Again, for $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ and $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$, we have that

$$\begin{aligned} c^{\kappa'}_{\rho'\nu(B,C)} &= \binom{|(A \cup A') \cap R_{\alpha,k}|}{\mu_a - |R|} \\ &= \binom{|(B - C) \cap R_{\alpha,k}| - 1}{\mu_a - |C| - 1} \end{aligned}$$

and

$$\begin{aligned} c^{\kappa}_{\rho\nu(B,C)} &= \binom{|(A \cup A') \cap R_{\alpha,k}|}{\lambda_a - |R|} \\ &= \binom{|(B - C) \cap R_{\alpha,k}| - 1}{\lambda_a - |C| - 1}, \end{aligned}$$

but now, since $\mu_a > |B \cap R_{\alpha,k}|$ where $C \cup \{\min(B)\} \subseteq B \cap R_{\alpha,k}$, we have $\mu_a - |C| - 1 > |(B - C) \cap R_{\alpha,k}| - 1$, and so

$$c^{\kappa'}_{\rho'\nu(B,C)} = \binom{|(B - C) \cap R_{\alpha,k}| - 1}{\mu_a - |C| - 1} = 0.$$

This shows that $a_{B,C}$ are the correct coefficients for the terms $s_{\nu(B,C)}$ that arise in **Case 2**.

Thus the formula given for $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)}$ is correct. ∎

We now give the formula for the Schur-positive differences discussed in Theorem 3.1.4.

Theorem 3.2.3 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lambda_a, \mu_l < \lceil \frac{h}{2} \rceil$, and let $0 \leq k \leq 1$. If $\lambda_a \geq \mu_l$, then*

$$s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} = \sum_{B,C} a_{B,C} s_{\nu(B,C)}$$

where the coefficients $a_{B,C}$ are given by

$$a_{B,C} = \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1},$$

the partitions $\nu(B,C)$ that arise are given by

$$\nu(B,C) = \delta_\alpha + e_B + e_C + (0^{|\alpha|+1}, 1^{h-|B|-|C|}),$$

and the sum is over all sets B, C such that

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha|+1\}$, $C \subseteq B \cap R_{\alpha,k} - \min(B)$,
- $|C| + 1 \leq \lambda_a \leq |B \cap R_{\alpha,k}|$,
- $|B| + |C| \leq h$, *and* $|\alpha| + 1 \in B$ *if* $|B| + |C| \leq h - 1$,
- *if* $B = \bigcup_{j=1}^{p} B_j$ *where the* B_j *are the maximal disjoint intervals of* B, *then* $\min(B_j) \in R_{\alpha,k}$ *for each* j, *and*
- *if* $C = \bigcup_{j=1}^{m} C_j$ *where the* C_j *are the maximal disjoint intervals of* C, *then* $\min(C_j) - 1 \in B - C$ *for each* j.

Proof The proof follows as in the proof of Theorem 3.2.2. The only condition on B and C that is different in the statement of this theorem is that we now insist that $|C| + 1 \leq \lambda_a$ instead of $|C| + 1 \leq \lambda_l$. This is because $\lambda_a < \lambda_l$ and all the values of $R = C \cup \min(B)$ must appear in the first row of λ.

For each ν that does appear in the difference, there is one of the two following cases.

Case 1: There is a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ and content ν with lattice reading word.

Case 2: There is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ and content ν with lattice reading word.

As before, in each case there exists sets B and C with the desired properties such that $\nu(B,C) = \nu$. This time we can see that in **Case 1** we must also have the properties that

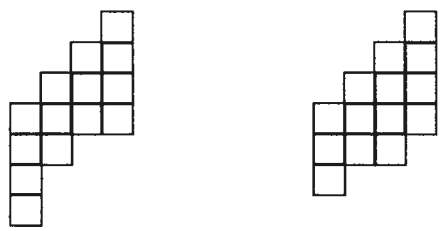
$$\mu_a \geq |C| + 1 \text{ and } \mu_a \leq |B \cap R_{\alpha,k}|,$$

and in **Case 2** we must also have that either

$$\mu_a < |C| + 1 \text{ or } \mu_a > |B \cap R_{\alpha,k}|.$$

Then the proof that, in each of **Case 1** and **Case 2**, the sets B and C with these desired properties enumerate all partitions ν that arise in the difference follows exactly as in the proof of Theorem 3.2.2 using Theorem 3.1.4 in place of Theorem 3.1.3 when appropriate. Also, it once again easy to see that, in each case, the numbers $a_{B,C}$ are the correct coefficients of $\nu(B,C)$ in the difference. ∎

Example Let $\alpha = (1,1,1,1)$, $\lambda = (2,1,1)$, $\mu = (3,1)$ and $k = 0$. We are interested in the following two diagrams.



$\mathcal{S}(\lambda, \alpha^{\triangleleft}; 0)$ $\qquad$ $\mathcal{S}(\mu, \alpha^{\triangleleft}; 0)$

For this α, k, we have $R_{\alpha,k} = \{2,3,4,5\}$. Thus Theorem 3.2.3 gives

$$s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};0)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};0)} = \sum_{B,C} a_{B,C} s_{\nu(B,C)},$$

where

- $B \subseteq \{2,3,4,5\}$, $C \subseteq \{3,4,5\}$,
- $|C| + 1 \leq 2 \leq |B|$,
- $|B| + |C| \leq 4$, and $5 \in B$ if $|B| + |C| \leq 3$,
- if $B = \bigcup_{j=1}^{p} B_j$ where the B_j are the maximal disjoint intervals of B, then $\min(B_j) \in R_{\alpha,k}$ for each j, and
- if $C = \bigcup_{j=1}^{m} C_j$ where the C_j are the maximal disjoint intervals of C, then $\min(C_j) - 1 \in B - C$ for each j.

The condition $|C|+1 \leq 2$ gives $|C| \leq 1$, and this implies that C is empty, or C is a singleton. If $C = \{c\}$ is a singleton then $|B| + |C| \leq 4$ gives that $|B| \leq 3$. Since $C \subseteq B$ and $\min(C) - 1 \in B$, we have $\{c-1, c\} \subseteq B$. Also, if we assume $|B| = 2$ then $|B| + |C| \leq 3$, so we require that $5 \in B$, which implies that $C = \{5\}$. Hence, for singleton sets C, we have either

1. $C = \{3\}$ or $C = \{4\}$ and $|B| = 3$, or

2. $C = \{5\}$ and $|B| \leq 3$.

If $C = \{3\}$, then $\{2,3\} \subseteq B$, so the possibilities for B are the sets $\{2,3,4\}$ and $\{2,3,5\}$. If $C = \{4\}$, then $\{3,4\} \subseteq B$, so the possibilities for B are the sets $\{2,3,4\}$ and $\{3,4,5\}$. In each of these four cases we can compute

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{2-1}{2-1-1} - \binom{2-1}{3-1-1} \\
&= \binom{1}{0} - \binom{1}{1} \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

If $C = \{5\}$, then $\{4,5\} \subseteq B$, so the possibilities for B are the sets $\{4,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$. When B is either $\{2,4,5\}$ or $\{3,4,5\}$ we once again obtain

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{2-1}{2-1-1} - \binom{2-1}{3-1-1} \\
&= \binom{1}{0} - \binom{1}{1} \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

When $B = \{4,5\}$ we have

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{1-1}{2-1-1} - \binom{1-1}{3-1-1} \\
&= \binom{0}{0} - \binom{0}{1} \\
&= 1 - 0 \\
&= 1.
\end{aligned}$$

This completes the cases when C is a singleton. The only other cases occur when $C = \emptyset$. In this case we have $2 \leq |B| \leq 4$ and $5 \in B$ if $|B| \leq 3$. Therefore the possible sets B are $\{2,5\}$, $\{3,5\}$, $\{4,5\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$, and $\{2,3,4,5\}$. When B is one of $\{2,5\}$, $\{3,5\}$, or $\{4,5\}$ we have

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{2-1}{2-0-1} - \binom{2-1}{3-0-1} \\
&= \binom{1}{1} - \binom{1}{2} \\
&= 1-0 \\
&= 1,
\end{aligned}$$

when B is one of $\{2,3,5\}$, $\{2,4,5\}$, or $\{3,4,5\}$ we have

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{3-1}{2-0-1} - \binom{3-1}{3-0-1} \\
&= \binom{2}{1} - \binom{2}{2} \\
&= 2-1 \\
&= 1,
\end{aligned}$$

and when B is $\{2,3,4,5\}$ we have

$$\begin{aligned}
a_{B,C} &= \binom{|(B-C)\cap R_{\alpha,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha,k}|-1}{\mu_a - |C| - 1} \\
&= \binom{4-1}{2-0-1} - \binom{4-1}{3-0-1} \\
&= \binom{3}{1} - \binom{3}{2} \\
&= 3-3 \\
&= 0.
\end{aligned}$$

Thus we have

$$\begin{aligned}
s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};0)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};0)} &= \sum_{B,C} a_{B,C} s_{\nu(B,C)} \\
&= 1s_{\nu(\{4,5\},\{5\})} + 1s_{\nu(\{2,5\},\emptyset)} + 1s_{\nu(\{3,5\},\emptyset)} + 1s_{\nu(\{4,5\},\emptyset)} \\
&\quad +1s_{\nu(\{2,3,5\},\emptyset)} + 1s_{\nu(\{2,4,5\},\emptyset)} + 1s_{\nu(\{3,4,5\},\emptyset)} \\
&= s_{(4,3,2,2,2,1)} + s_{(4,4,2,1,1,1,1)} + s_{(4,3,3,1,1,1,1)} + s_{(4,3,2,2,1,1,1)} \\
&\quad +s_{(4,4,3,1,1,1)} + s_{(4,4,2,2,1,1)} + s_{(4,3,3,2,1,1)},
\end{aligned}$$

where we have omitted the details of computing $\nu(B,C)$ for each B,C. For an example of this, we have

$$\begin{aligned}
\nu(\{4,5\},\{5\}) &= \delta_{(1,1,1,1)} + \sum_{b\in\{4,5\}} e_b + \sum_{c\in\{5\}} e_c + (0^5, 1^{4-2-1}) \\
&= (4,3,2,1) + (0,0,0,1,1) + (0,0,0,0,1) + (0,0,0,0,0,1) \\
&= (4,3,2,2,2,1).
\end{aligned}$$

3.3 Hasse Diagrams for $k > 1$

In Sections 3.1 and 3.2, we required that $0 \leq k \leq 1$. For these two values the Hasse diagram we obtained in Section 3.1 was identical and we could explicitly compute the Schur-positive differences in the diagram. As a byproduct, we could then express D as a skew shape of the form $D = \mu/\delta_\gamma$. That is, the partition removed from μ was a fat staircase. This fact will become necessary for our results in Chapter 5. However, we can generalize the hook foundation results of this chapter by relaxing the restriction on k.

We return to the case of fat staircases with hook foundations. As before, we let μ be the hook $(\mu_a, 1^{\mu_l - 1})$, λ be the hook $(\lambda_a, 1^{\lambda_l - 1})$ and Δ_α be a fat staircase, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Now we consider any k satisfying $0 \leq k \leq h$, where h is the size of the hooks under consideration.

We shall summarise all the $\succeq_s$ relationships between diagrams of the form $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ when λ is a hook of fixed size $h \leq n + k$ and $0 \leq k \leq h$. The restriction $h \leq n + k$ is needed to guarantee that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ is a skew diagram for every hook λ of size h. We impose the restriction $k \leq h$ since, for $k \geq h$, every diagram of the form $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, when λ is a hook of fixed size h, is disconnected. Thus, for each $k \geq h$, the skew Schur function of

these diagrams factor as

$$s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} = s_\lambda s_{\Delta_\alpha}.$$

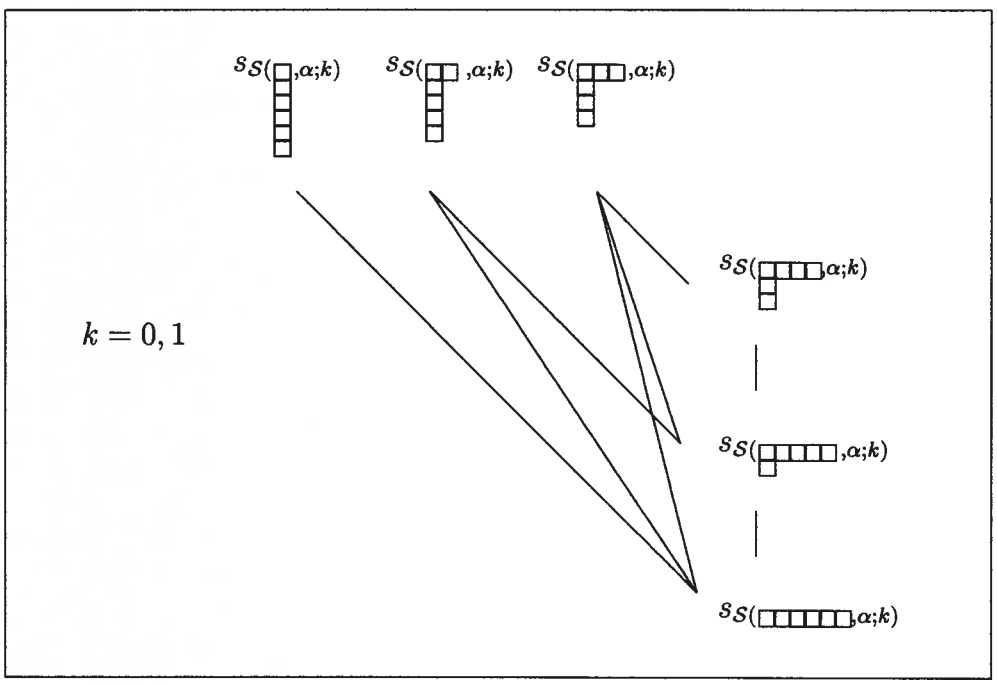
In particular

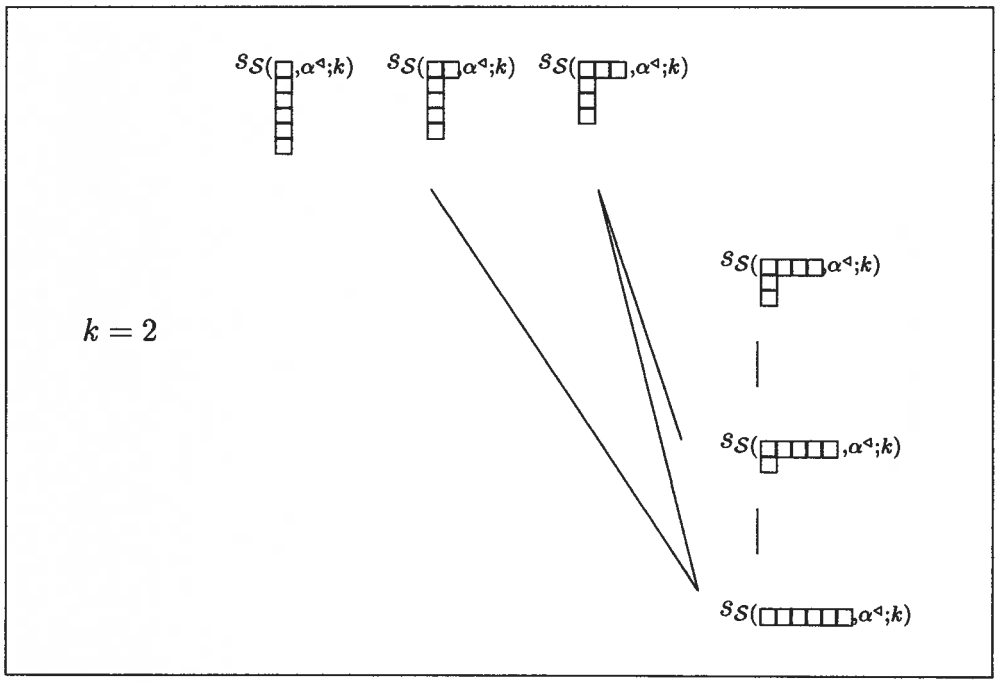
$$s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} = s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};h)}$$

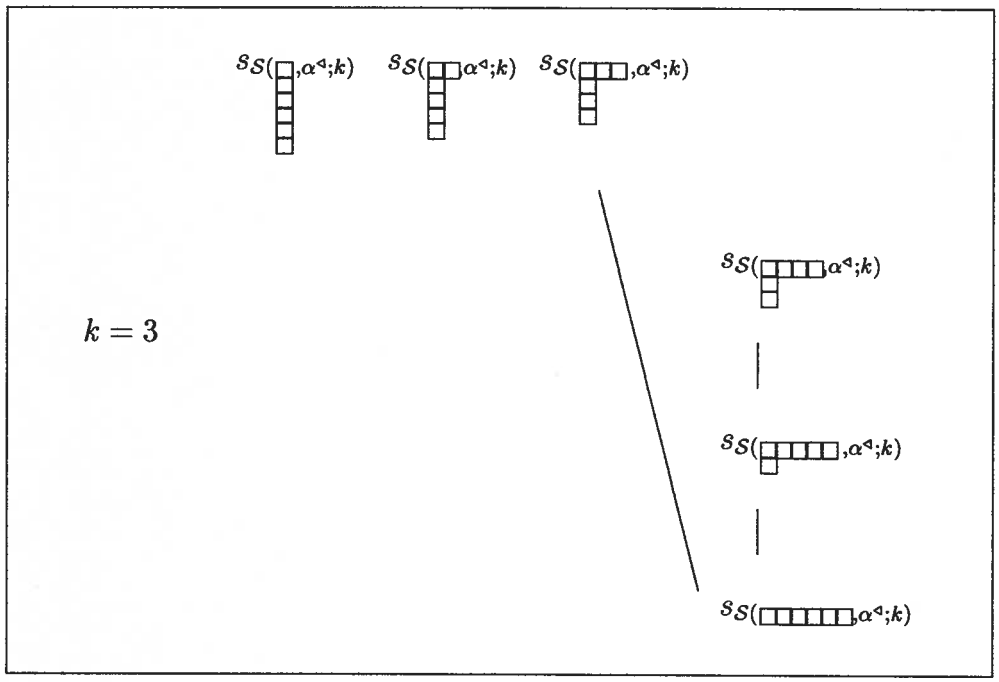
for all $k \geq h$, so there is no change among the differences for $k \geq h$. Furthermore, we shall see that for $k \geq h$, none of the differences is Schur-positive.

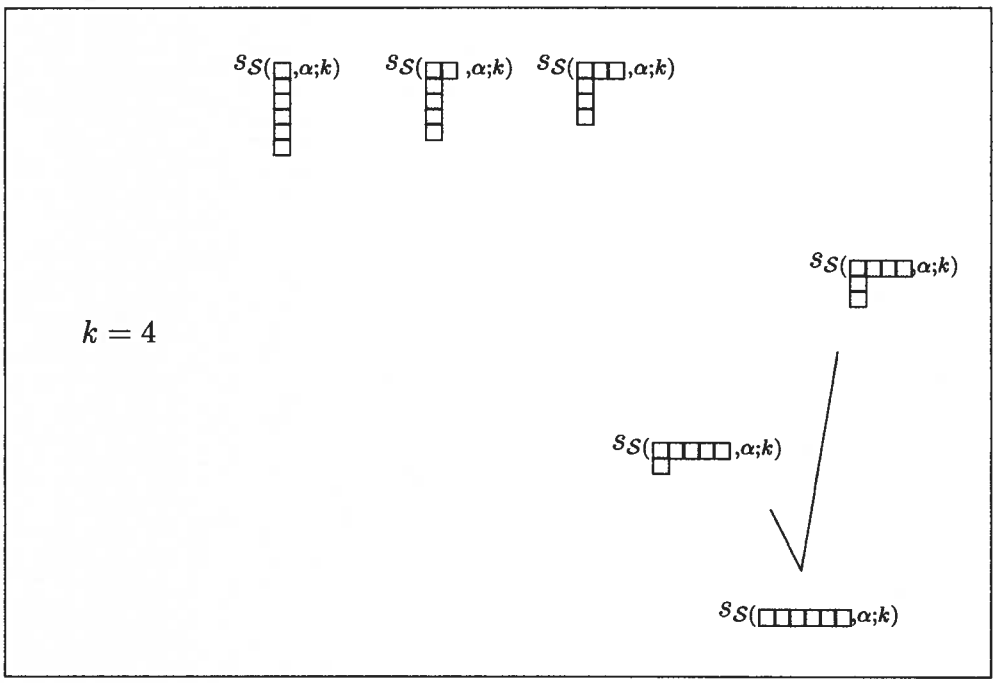
Let us begin with the following example.

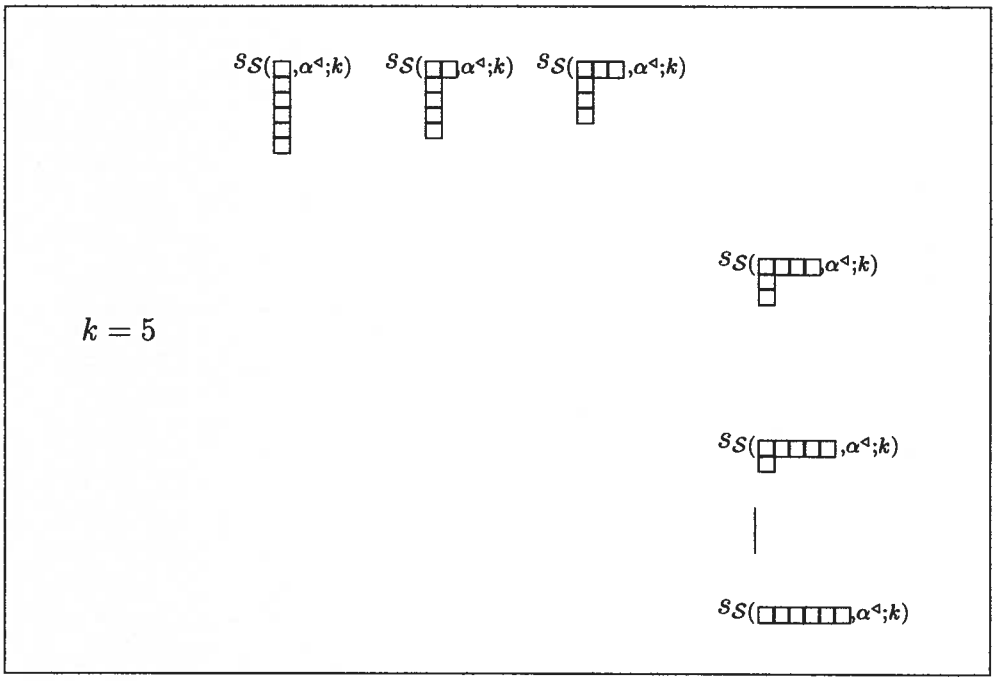
Example For each $0 \leq k \leq 6$ we show the Hasse diagrams for $\succeq_s$ on the collection of staircases with bad foundations $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ for some $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $n \geq 6$, and λ varying over all hooks of size 6.

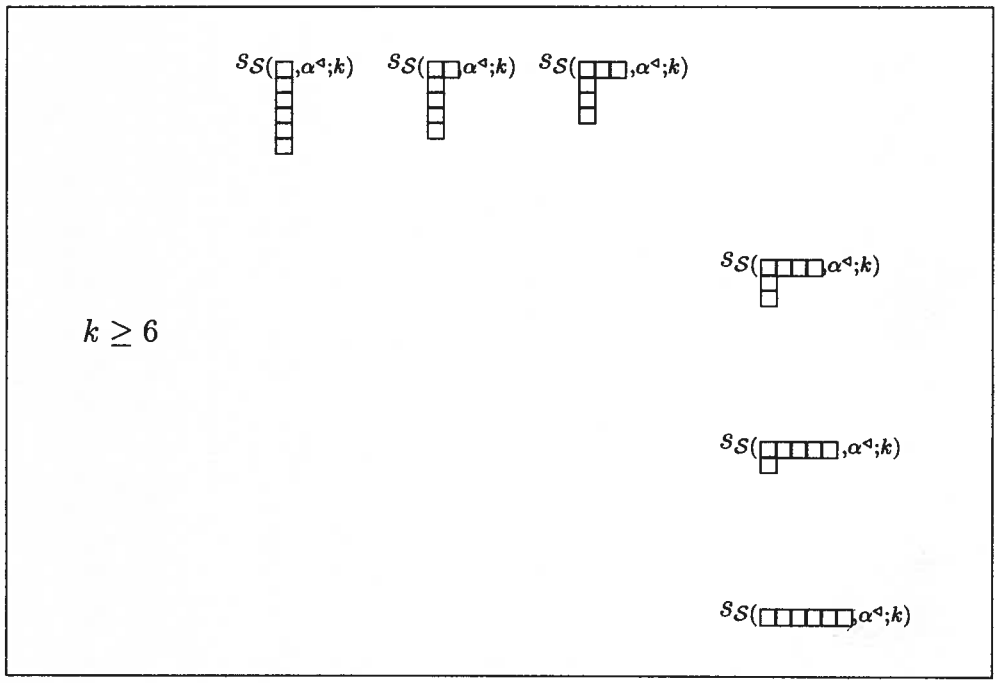



$k = 2$


$k = 3$


$s_{\mathcal{S}}(\ ,\alpha;k)$
$s_{\mathcal{S}}(\ ,\alpha;k)$
$s_{\mathcal{S}}(\ ,\alpha;k)$
$s_{\mathcal{S}}(\ ,\alpha;k)$
$k = 4$
$s_{\mathcal{S}}(\ ,\alpha;k)$
$s_{\mathcal{S}}(\ ,\alpha;k)$


$s_{\mathcal{S}}(\ ,\alpha^{\triangleleft};k)$
$s_{\mathcal{S}}(\ \alpha^{\triangleleft};k)$
$s_{\mathcal{S}}(\ ,\alpha^{\triangleleft};k)$
$s_{\mathcal{S}}(\ \alpha^{\triangleleft};k)$
$k = 5$
$s_{\mathcal{S}}(\ ,\alpha^{\triangleleft};k)$
$s_{\mathcal{S}}(\ ,\alpha^{\triangleleft};k)$



For $0 \leq k \leq 1$ we have the same structure of the Hasse diagrams that was discussed in Section 3.1. As k increases, fewer of the $\succeq_s$ relations remain satisfied. Finally, when $k \geq 6$, there are no Schur-positive differences among these diagrams. We note that the chain structure that was prevalent among the diagrams on the right for $0 \leq k \leq 1$ also lost its structure as k increased. In particular, when $k = 4$, each of the three diagrams on the right still was comparable to at least one of the others, but the three no longer formed a chain.

The following results summarise all the $\succeq_s$ relationships between diagrams of the form $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ when λ is a hook of fixed size $h \leq n+k$ and $0 \leq k \leq h$.

For each pair of hooks λ, μ with $\lambda_a, \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$, Theorem 3.3.1 and Theorem 3.3.2 each prove one side of the Schur-incomparability of this pair, thus describing the antichain structure displayed along the top of the Hasse diagrams in the previous example.

For each pair of hooks λ, μ with $\left\lceil \frac{h}{2} \right\rceil \leq \lambda_a < \mu_a$, Theorem 3.3.3 and Theorem 3.3.4 shows that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ if and only if $\lambda_a \geq \mu_l + k - 1$. This describes the relations among those diagrams displayed on the right of the Hasse diagrams in the previous example.

Finally, for each pair of hooks λ, μ with $\lambda_a, \mu_l < \left\lceil \frac{h}{2} \right\rceil$, Theorem 3.3.5 and Theorem 3.3.6 shows that when $1 \leq k \leq h$ we have $\lambda_a \geq \mu_l + k - 1$ if and

only if $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$, and when $k = 0$ we have $\lambda_a \geq \mu_l$ if and only if $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. This describes the relationships between the diagrams displayed on the right with the diagrams displayed along the top in the previous Hasse diagrams.

Let us finally begin. We start by looking at the antichain structure.

Theorem 3.3.1 *Let λ and μ be distinct hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lambda_a < \mu_a \leq \lceil \frac{h}{2} \rceil$, and let $0 \leq k \leq h$. Then we have $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) \not\succeq_s \mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$.*

Proof We shall show that there exists a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word such that there is no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word having the same content. This is sufficient to prove the theorem.

Since $\lambda_a < \mu_a$ and $|\lambda| = |\mu|$, we have $\lambda_l > \mu_l$. Let $r_1 = r_2 = \cdots = r_k = 1$ and let $r_{1+k} < r_{2+k} < \cdots < r_{n+k}$ be the values of $R_{\alpha,k}$ greater than 1. We can create a SSYT of shape λ by filling the boxes of λ as follows.

$$\begin{array}{cccccccc}
r_1 & \cdots & r_k & r_{1+k} & r_{2+k} & \cdots & r_{\lambda_a - 1} & |\alpha| + 1 \\
|\alpha| + 2 & & & & & & & \\
|\alpha| + 3 & & & & & & & \\
\vdots & & & & & & & \\
|\alpha| + \lambda_l & & & & & & &
\end{array}$$

Using the unique filling of Δ_α, it is easy to check that the resulting tableau of shape $\lambda \oplus \Delta_\alpha$ has lattice reading word since each of the entries in the first row of λ are from $R_{\alpha,k}$ and the entry 1 appears k times. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word, where λ is filled as shown above.

Since $\mu_l < \lambda_l$, we have $l(\mathcal{S}(\mu, \alpha^{\triangleleft}; k)) = |\alpha| + \mu_l < |\alpha| + \lambda_l$. Therefore no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word can contain the entry $|\alpha| + \lambda_l$. Thus no SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word can have the same content as $\mathcal{T}$. Hence, it follows that $s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} \not\succeq_s 0$. ∎

Theorem 3.3.2 *Let λ and μ be distinct hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lambda_a < \mu_a \leq \lceil \frac{h}{2} \rceil$ and let $0 \leq k \leq h$. Then we have $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \not\succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof The proof for $k = 0$ was proved in Theorem 3.1.2. Thus we consider the case when $k > 0$.

We let $r_1 = r_2 = \cdots = r_k = 1$ and let $r_{1+k} < r_{2+k} < \cdots < r_{n+k}$ be the values of $R_{\alpha,k}$ greater than 1, where we note $n + k \geq h$. We can create a SSYT of shape μ by filling the boxes of μ as follows.

$$\begin{array}{ccccccc} r_1 & \cdots & r_k & r_{1+k} & r_{2+k} & \cdots & r_{\mu_a} \\ r_{\mu_a+1} & & & & & & \\ r_{\mu_a+2} & & & & & & \\ \vdots & & & & & & \\ r_h & & & & & & \end{array}$$

Since $r_1 = r_2 = \cdots = r_k = 1$ and $r_{1+k} < r_{2+k} < \cdots < r_h$ are distinct values of $R_{\alpha,k}$, it is easy to check that the resulting tableau of shape $\mu \oplus \Delta_\alpha$ has lattice reading word. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word, where μ is filled as shown above.

We now wish to count all SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ (shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, respectively) with content $\nu = c(\mathcal{T})$. Since Δ_α has a unique way of being filled, we must find all semistandard fillings of μ (λ, resp.) with the values $r_1, r_2, \ldots, r_h$. Since $r_1 = r_2 = \cdots = r_k = 1$ and $r_{1+k} < r_{2+k} < \cdots < r_h$, the values $r_1, r_2, \ldots, r_k$ must appear as the first k values of the first row of μ (λ, resp.). Further, once we choose $\mu_a - k$ ($\lambda_a - k$, resp.) of the values $r_{1+k}, r_{2+k}, \ldots, r_h$ to appear in the first row of μ (first row of λ, resp.), then the remaining r's must appear in the first column and the order of all these values is uniquely determined by the semistandard conditions.

Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{h-k}{\mu_a - k}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa}_{\rho\nu} = \binom{h-k}{\lambda_a - k}.$$

Since $\lambda_a < \mu_a \leq \left\lceil \frac{h}{2} \right\rceil$, we have $\lambda_a + \mu_a < h+1 \leq h+k$. Therefore we have $h - \lambda_a > \mu_a - k$ and we obtain

$$h - \lambda_a - i > \mu_a - k - i, \tag{3.10}$$

for each i.

Therefore

$$c^{\kappa'}_{\rho'\nu} = \frac{(h-k)!}{(h-\mu_a)!(\mu_a - k)!}$$

$$
\begin{aligned}
&= \frac{(h-k)!}{(h-\lambda_a)!(\lambda_a-k)!} \times \prod_{i=0}^{\mu_a-\lambda_a-1} \frac{h-\lambda_a-i}{\mu_a-k-i} \\
&= c^{\kappa}_{\rho\nu} \times \prod_{i=0}^{\mu_a-\lambda_a-1} \frac{h-\lambda_a-i}{\mu_a-k-i} \\
&> c^{\kappa}_{\rho\nu},
\end{aligned}
$$

where we have used Equation 3.10 in the final step. Therefore $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \not\geq_s 0$. ∎

We now depart from looking at the hooks λ, μ satisfying $\lambda_a < \mu_a \leq \lceil \frac{h}{2} \rceil$. Instead, we turn to the hooks λ, μ satisfying $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$. The following theorems describe the relations among the diagrams we displayed on the right in our examples.

Theorem 3.3.3 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$, and let $0 \leq k \leq h$. If $\lambda_a \geq \mu_l + k - 1$ then $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof To prove the result, we shall consider any content ν such that a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word exists. First we shall show that there is also a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word. Then, letting $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ and $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ for partitions κ, κ', ρ, and ρ', we shall show that the Littlewood-Richardson coefficients for these two diagrams and this content satisfy

$$c^{\kappa}_{\rho\nu} \geq c^{\kappa'}_{\rho'\nu}.$$

Having shown that this inequality holds for any content ν for which a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word exists, this will imply that $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \geq_s 0$.

Let ν be a content such that there is a SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with content ν and lattice reading word. By Lemma 2.2.2 we know that the first row of μ contains at most k 1's. Let q be the number of 1's in the first row of μ. In the case that $q = 0$, then we may follow the proof of Theorem 3.1.3 to show that the inequality holds on the Littlewood-Richardson coefficients on these contents. Thus we consider the case $q \geq 1$. We write $a_1 = a_2 = \cdots = a_q = 1$. Then the rest of the first row of μ consists of a strictly increasing sequence $a_{q+1} < a_{q+2} < \cdots < a_{\mu_a}$ where each $a_{q+i} \in R_{\alpha,k}$ with $a_{q+i} > 1$. Also, since the columns of $\mathcal{T}_1$ strictly increase, the first column of μ contains a strictly increasing sequence $a'_1 < a'_2 < \cdots < a'_{\mu_l}$, where $a'_1 = a_1 =$

1. Since Δ_α can only be filled in one way (in both shapes $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ and $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$) and the values $a_1, \ldots, a_q$ must be placed in the first q positions in the first row of either foundation, in order to obtain a tableau of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ and content ν we only need to show how to place the values of $\{a_{q+1}, a_{q+2}, \ldots, a_{\mu_a}\} \cup \{a'_2, a'_3, \ldots, a'_{\mu_l}\}$ in λ.

If $l(\nu) > |\alpha| + 1$ for this particular content ν then there are entries of $\mathcal{T}_1$ greater than $|\alpha| + 1$. Since Δ_α has content δ_α, the lattice condition implies that $|\alpha| + 2$ appears in μ. Since each $a_i \in R_{\alpha,k}$, we have $a_i \leq |\alpha| + 1$ for each i and so $a'_j = |\alpha| + 2$ for some j. The lattice condition and the fact that the column strictly increases gives that

$$\begin{aligned} a'_j &= |\alpha| + 2 \\ a'_{j+1} &= |\alpha| + 3 \\ &\vdots \\ a'_{\mu_l} &= |\alpha| + \mu_l - j + 2. \end{aligned}$$

Again, by the lattice condition, it is clear that any SSYT of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ with lattice reading word and content ν must also have these values a'_j, a'_{j+1}, $\ldots$, a'_{μ_l} as the last $\mu_l - j + 1$ entries of the first column of λ. Since $\lambda_a < \mu_a$ gives $\mu_l < \lambda_l$, we have $\mu_l - j + 1 < \lambda_l - j + 1$. Since $j \geq 2$ this gives,

$$\mu_l - j + 1 \leq \lambda_l, \tag{3.11}$$

so these entries do fit in this column. Note that if $l(\nu) \leq |\alpha| + 1$, then this sequence of values $a'_j, a'_{j+1}, \ldots, a'_{\mu_l}$ is empty and we do not have to worry about placing any entries larger than $|\alpha| + 1$ into λ. (In such a case we may consider $j = \mu_l + 1$.)

Let M be the multiset $\{a_{q+1}, a_{q+2}, \ldots, a_{\mu_a}\} \cup \{a'_2, a'_3, \ldots, a'_{j-1}\}$. Then M is the remaining entries that we still need to place in λ to obtain a tableau of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ and content ν. We have $|M| = \mu_a + j - 2 - q$ and $\max(M) = |\alpha| + 1$. Let $R = \{a_{q+1}, a_{q+2}, \ldots, a_{\mu_a}\} \cap \{a'_2, a'_3, \ldots, a'_{j-1}\}$. We note that $R = \{a_{q+1}, a_{q+2}, \ldots, a_{\mu_a}\} \cap \{a'_2, a'_3, \ldots, a'_{\mu_l}\}$ since Lemma 2.2.2 shows $a_{\mu_a} \in R_{\alpha,k}$, which implies $a_{\mu_a} < |\alpha| + 2 = a'_j$. Thus *$R$ is the set of values (except 1) that appear in both the first row and the first column of μ.*

Since the values of R all appear in the first row of μ, Lemma 2.2.2 gives that $R \subseteq R_{\alpha,k}$. For any SSYT of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ with lattice reading word and content ν, Lemma 2.2.2 shows that, besides 1, the values in the first row

of λ are distinct, so, when creating a filling of λ, the values in R must also appear in both the first row of λ and the first column of λ.

Consider $A = \{a_{q+1}, a_{q+2}, \ldots, a_{\mu_a}\} - R$ and $A' = \{a'_2, a'_3, \ldots, a'_{j-1}\} - R$. Since we know that the values of R must appear in both the first row of λ and first column of λ, $A \cup A'$ contains the remaining values of M that need to be placed in λ. In other words, $A \cup A'$ *is the set of all values* $\leq |\alpha| + 1$ *that can appear in exactly one of the first row of* λ *or the first column of* λ.

We wish to show that

$$|R| \leq \lambda_a - q \tag{3.12}$$

holds. If $q = 0$, then $|R| \leq j \leq \mu_l + 1 \leq \lambda_l \leq \lambda_a = \lambda_a - q$. Now, when $q \geq 1$ the top-left entry of μ is 1 which is not in R, hence we have $|R| \leq \mu_l - 1 \leq \lambda_a - k \leq \lambda_a - q$, where we have used the fact that $\lambda_a \geq \mu_l + k - 1$.

Now, because Equation 3.12 holds, we can extend the values of R to an increasing sequence $b_{q+1} < b_{q+2} < \cdots < b_{\lambda_a}$ by choosing $\lambda_a - |R| - q$ additional values from $(A \cup A') \cap R_{\alpha,k}$. There are enough values to choose from since there are

$$\mu_a - |R| - q \geq \lambda_a - |R| - q \tag{3.13}$$

values of $(A \cup A') \cap R_{\alpha,k}$ present in the first row of μ. The sequence of b_i's is strictly increasing since $(A \cup A') \cap R = \emptyset$.

Now $M - \{b_{q+1}, \ldots, b_{\lambda_a}\} \subseteq M - R$ contains $w = |M| - (\lambda_a - q) = \mu_a + j - 2 - \lambda_a$ distinct values, each no greater than $|\alpha| + 1$. That is, they are an increasing sequence $c_1 < c_2 < \cdots < c_w$, where $c_w \leq |\alpha| + 1$ and $c_1 > 1$. We have

$$\begin{aligned} \lambda_l &= \mu_a + \mu_l - \lambda_a \\ &= 1 + (\mu_a + j - 2 - \lambda_a) + (\mu_l - j + 1) \\ &= 1 + w + (\mu_l - j + 1), \end{aligned}$$

so, letting $b_1 = b_2 = \cdots = b_q = 1$, we may fill λ as shown below.

$$\begin{array}{cccc} b_1 & b_2 & \cdots & b_{\lambda_a} \\ c_1 & & & \\ c_2 & & & \\ \vdots & & & \\ c_w & & & \\ a'_j & & & \\ a'_{j+1} & & & \\ \vdots & & & \\ a'_{\mu_l} & & & \end{array}$$

Since the sequence of c_i is strictly increasing and since $c_1 > 1$ and $c_w \leq |\alpha| + 1 < |\alpha| + 2 = a'_j$ we have

$$b_1 < c_1 < c_2 < \cdots < c_w < a'_j < \cdots < a'_{\mu_l}.$$

That is, the first column of λ is increasing. We also have

$$b_1 = b_2 = \cdots = b_q = 1$$

and

$$b_{q+1} < b_{q+2} < \cdots < b_{\mu_a},$$

so the first row of λ is weakly increasing with $q \leq k$ 1's. Hence this filling gives us a SSYT T of shape $\lambda \oplus \Delta_\alpha$ and content ν.

We now check that T has a lattice reading word so that we may apply Lemma 2.2.3 to obtain the desired tableau $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$. Suppose that T does not have a lattice reading word. Then, when reading the foundation λ of T, we must reach a point where the lattice condition failed. Let x be the value that, when read, caused the lattice condition to fail. The lattice condition could not have failed when reading the first row of λ since the lattice condition places no restriction on the number of 1's and the remaining values in the first row of λ were distinct values chosen from $R_{\alpha,k}$. Therefore $x > 1$ and *this* x which violated the lattice condition appears somewhere in the first column of λ. We inspect the the two cases $x \in R_{\alpha,k}$ and $x \notin R_{\alpha,k}$.

Consider the first case, $x \in R_{\alpha,k}$. If a value of $R_{\alpha,k}$ appears only once in the foundation then reading this value cannot violate the lattice condition. Thus, since the lattice condition failed at this x, this x cannot be the first time that x was read in λ. Since the columns strictly increase, the previous x must have appeared in the first row of λ and, since the values in the first row are distinct, this is the only other x in λ. Since the content of λ is the same as the content of μ, these two x's appear in μ as well. Using the fact that $\mathcal{T}_1$ has a lattice reading word, together with the content of Δ_α, we find that the value $x - 1$ appeared in μ. Thus the value $x - 1$ also appears in λ. Now, since both the rows and columns of λ must weakly increase, either the $x - 1$ appears in the first column of λ above the entry x, or the $x - 1$ appears in the first row of λ left of the entry x. In either case the $x - 1$ is read before the second x is read and the lattice condition will not fail when reading this second x, contrary to our assumption.

We now look at the second case, where $x \notin R_{\alpha,k}$. Again, the x we are interested in appears in the first column of λ. There cannot be a second x in λ since the column strictly increases and $x \notin R_{\alpha,k}$ implies that no other x

was placed in the first row of λ. As before, x must have appeared in μ and, in particular, it also appears somewhere in the first column. Since $\mathcal{T}_1$ has a lattice reading word we must read a sequence of values t, $t+1$, $t+2$, ..., $x-2$, $x-1$ in μ, where $t \in R_{\alpha,k}$, before we read the x, and we may assume that none of the values $t+1$, $t+2$, ..., $x-2$, $x-1$ are from $R_{\alpha,k}$. Hence each of the values t, $t+1$, ..., $x-2$, $x-1$ also appear in λ. None of the values $t+1$, $t+2$, ..., $x-1$ can appear in the first row of λ since the first row was chosen from $R_{\alpha,k}$. That is, each value $t+1, t+2, \ldots, x-1$ appears in the first column of λ. Also, since both the rows and columns of λ must weakly increase, either the t appears in the first column of λ above the entry $t+1$, or the t appears in the first row of λ. In either case the entire sequence t, $t+1$, ..., $x-2$, $x-1$ is read before the x is read in λ and the lattice condition does not fail at x, contradicting our assumption.

Since T has a lattice reading word, we can now apply Lemma 2.2.3 to obtain the SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. Therefore from any SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν we can create a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν.

Let $c^{\kappa}_{\rho\nu}$ be the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, and $c^{\kappa'}_{\rho'\nu}$ be the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. We shall show that the sets R and $A \cup A'$ that were described above are completely determined by the content ν. That is, without starting with a specific tableau, but only starting with the desired content of a SSYT of some fat staircase with hook foundation with lattice reading word, we show how to determine q, the number of 1's in the foundation; R, the set of values larger than 1 that must appear in both the first row and first column of the foundation; and $A \cup A'$, the set of values less than or equal to $|\alpha| + 1$ that can only appear in one of the first row or the first column of the foundation.

Since Δ_α is uniquely filled, from ν we can determine the content of the foundation μ (λ, respectively) needed to create a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ ($\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, resp.) with lattice reading word and content ν. From the content of the foundation we can determine q, the number of 1's in the foundation, and the values $a'_j, a'_{j+1}, \ldots$ greater than $|\alpha| + 1$. Since Lemma 2.2.2 shows that the entries greater than 1 that appear in the first row of the foundation strictly increase, any value greater than 1 that appears twice in the foundation must appear in both the first row of μ (λ, resp.) and first column of μ (λ, resp.). These values give the set R. Then $A \cup A'$ is the set of values in the foundation that are both greater than 1 and less than or equal to $|\alpha| + 1$, but

are not in R. The first row of μ (first row of λ, resp.) must contain q 1's and the values in R. After we determine the remaining entries of the first row of μ (first row of λ, resp.), the rest of the foundation is uniquely determined.

Now, to actually form a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ ($\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, resp.) with lattice reading word and content ν, we only need to choose the remaining $\mu_a - q - |R|$ ($\lambda_a - q - |R|$, resp.) values from the set $(A \cup A') \cap R_{\alpha,k}$ to place in the first row of μ (first row of λ, resp.). Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content ν is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\mu_a - q - |R|}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content ν is given by

$$c^{\kappa}_{\rho\nu} = \binom{|(A \cup A') \cap R_{\alpha,k}|}{\lambda_a - q - |R|}.$$

Since $\lambda_a \geq \lambda_l > \mu_l \geq j - 1$, we have

$$0 \geq j - 1 - \lambda_a. \tag{3.14}$$

Thus, for each i we have

$$\begin{aligned}
\mu_a - q - |R| - i &\geq \mu_a - q - |R| - i + (j - 1 - \lambda_a) \\
&\geq (\mu_a - |R|) + (j - 1 - |R|) - (\lambda_a - |R|) - i - q \\
&\geq |A| + |A'| - (\lambda_a - |R|) - i - q \\
&\geq |(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i - q.
\end{aligned}$$

That is,

$$\mu_a - q - |R| - i \geq |(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i - q, \tag{3.15}$$

for each i.

Therefore

$$\begin{aligned}
c^{\kappa}_{\rho\nu} &= \frac{|(A \cup A') \cap R_{\alpha,k}|!}{(|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - q - |R|))!(\lambda_a - q - |R|)!} \\
&= \frac{|(A \cup A') \cap R_{\alpha,k}|!}{(|(A \cup A') \cap R_{\alpha,k}| - (\mu_a - q - |R|))!(\mu_a - q - |R|)!} \\
&\quad \times \prod_{i=0}^{\mu_a - \lambda_a - 1} \frac{\mu_a - q - |R| - i}{|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i - q}
\end{aligned}$$

$$
\begin{aligned}
&= c^{\kappa'}_{\rho'\nu} \times \prod_{i=0}^{\mu_a-\lambda_a-1} \frac{\mu_a - q - |R| - i}{|(A \cup A') \cap R_{\alpha,k}| - (\lambda_a - |R|) - i - q} \\
&\geq c^{\kappa'}_{\rho'\nu},
\end{aligned}
$$

where we have used Equation 3.15 in the final step. Since this inequality holds for all contents ν for which there was a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, we have $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \geq_s 0$. ∎

Theorem 3.3.4 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$, and let $0 \leq k \leq h$. If $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$, then $\lambda_a \geq \mu_l + k - 1$.*

Proof Since $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$, where $|\lambda| = |\mu| = h$, we have

$$\mu_l < \lambda_l \leq \left\lceil \frac{h}{2} \right\rceil \leq \lambda_a < \mu_a.$$

In the case $k = 0$ we only need to show that $\lambda_a \geq \mu_l - 1$, which is true since $\lambda_a \geq \lceil \frac{h}{2} \rceil$ and $\mu_l \leq \lceil \frac{h}{2} \rceil$.

We turn to the case $1 \leq k \leq h$. Towards a contradiction, suppose $\lambda_a < k + \mu_l - 1$.

As before, we let $r_1 = r_2 = \cdots = r_k = 1$ and $r_{k+1} < r_{k+2} < \cdots < r_{k+n}$ be the values of $R_{\alpha,k}$ greater than 1. We can create a SSYT of shape μ by filling the boxes of μ as follows.

$$
\begin{array}{ccccccc}
r_1 & r_2 & \cdots & r_k & r_{k+1} & \cdots & r_{\mu_a} \\
r_{\mu_a+1} & & & & & & \\
r_{\mu_a+2} & & & & & & \\
\vdots & & & & & & \\
r_h & & & & & &
\end{array}
$$

Since we are using k 1's followed by distinct values of $R_{\alpha,k}$, it is easy to check that the resulting tableau of shape $\mu \oplus \Delta_\alpha$ has a lattice reading word. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word, where μ is filled as shown above.

We now wish to count all SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ (shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, respectively) with content $\nu = c(\mathcal{T})$. Since Δ_α has a unique way of being filled, we must find all semistandard fillings of μ (λ, resp.) with the values $r_1, r_2, \ldots, r_h$. Since $r_1 = r_2 = \cdots = r_k = 1$, the values r_1, r_2, ..., r_k must appear in the first k positions of the first row of μ (λ, resp.). Further, once we choose $\mu_l - 1$ ($\lambda_a - k$, resp.) of the values $r_{k+1} < r_{k+2} < \cdots < r_h$ to

appear in the first column of μ (first row of λ, resp.), then the remaining r's must appear in the first row of μ (first column of λ, resp.) and the order of all these values is uniquely determined by the semistandard conditions.

Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{h-k}{\mu_l - 1}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa}_{\rho\nu} = \binom{h-k}{\lambda_a - k}.$$

Since $\mu_l < \lambda_l \leq \left\lceil \frac{h}{2} \right\rceil \leq \lambda_a < \mu_a$, where $|\lambda| = h$, we have

$$\mu_l + \lambda_a < \lambda_l + \lambda_a = h + 1.$$

This gives $h - \lambda_a > \mu_l - 1$ and we obtain

$$h - \lambda_a - i > \mu_l - 1 - i, \tag{3.16}$$

for each i.

Therefore

$$\begin{aligned} c^{\kappa'}_{\rho'\nu} &= \frac{(h-k)!}{(h-k-\mu_l+1)!(\mu_l-1)!} \\ &= \frac{(h-k)!}{(h-\lambda_a)!(\lambda_a-k)!} \times \prod_{i=0}^{\mu_l-\lambda_a+k-2} \frac{h-\lambda_a-i}{\mu_l-1-i} \\ &= c^{\kappa}_{\rho\nu} \times \prod_{i=0}^{\mu_l-\lambda_a+k-2} \frac{h-\lambda_a-i}{\mu_l-1-i} \\ &> c^{\kappa}_{\rho\nu}, \end{aligned}$$

where we have used Equation 3.16 in the final step. Therefore $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \not\geq_s 0$, which is a contradiction. Therefore we have $\lambda_a \geq \mu_l + k - 1$. ∎

Finally, we turn to the hooks λ, μ satisfying $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$.

Theorem 3.3.5 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k$ and $\lambda_a, \mu_l \leq \lceil \frac{h}{2} \rceil$. If $1 \leq k \leq h$ and $\lambda_a \geq \mu_l + k - 1$ then $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. If $k = 0$ and $\lambda_a \geq \mu_l$ then $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$.*

Proof The case $k = 0$ was proved in Theorem 3.1.4.

For $1 \leq k \leq h$ we are given that $\lambda_a \geq \mu_l + k - 1$ and we wish to show that $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$. We claim that proof of Theorem 3.3.3, with a few equations verified under the current hypotheses, also proves this theorem.

Given the SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν, in order to create the SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν we first needed to check that Equation 3.11, Equation 3.12 and Equation 3.13 held. Namely, we required that $\mu_l - j + 1 \leq \lambda_l$, $|R| \leq \lambda_a - q$, and $\mu_a - |R| - q \geq \lambda_a - |R| - q$. Since the first two equations were satisfied, we were able to fit the required values into the first row and first column of λ. Further, since $\mu_a - |R| - q \geq \lambda_a - |R| - q$, there were enough values to fill the first row of λ, and therefore we could construct the tableau with all the desired properties.

In order to show Equation 3.11 holds for the assumptions of this theorem, we note that $\mu_l \leq \lceil \frac{h}{2} \rceil \leq \lambda_l$. In order to show Equation 3.12 holds for the assumptions of this theorem, we note that

$$|R| \leq j - 2 \leq \mu_l - 1 \leq \lambda_a - k \leq \lambda_a - q.$$

In order to show Equation 3.13 holds for the assumptions of this theorem, we note that $\mu_a \geq \lceil \frac{h}{2} \rceil \geq \lambda_a$. Therefore we can create a SSYT of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν whenever there exists a SSYT of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν.

Next, the proof of Theorem 3.3.3 checked that, for each of these contents ν, the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν is greater than or equal to the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word and content ν. To prove this, we first required that Equation 3.14 held. Namely, we required that $\lambda_a \geq j - 1$. We used this equation to show that Equation 3.15 held for each i, which gave us the desired inequality for the Littlewood-Richardson numbers. In order to show Equation 3.14 holds for the assumptions of this theorem, we note that

$$\lambda_a \geq \mu_l + k - 1 \geq j - 1 + k - 1 \geq j - 1.$$

Therefore the inequality for the Littlewood-Richardson numbers holds here as well, which proves that

$$s_{\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)} - s_{\mathcal{S}(\mu, \alpha^{\triangleleft}; k)} \geq_s 0. \quad \blacksquare$$

Theorem 3.3.6 *Let λ and μ be distinct hooks with $|\lambda| = |\mu| = h \leq n+k$ and $\lambda_a, \mu_l \leq \lceil \frac{h}{2} \rceil$. If $1 \leq k \leq h$ and $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$, then $\lambda_a \geq k+\mu_l-1$. If $k = 0$ and $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{\triangleleft}; k)$, then $\lambda_a \geq \mu_l$.*

Proof The case $k = 0$ was proved in Theorem 3.1.5.

We turn to the case $1 \leq k \leq h$. Towards a contradiction, suppose $\lambda_a < k + \mu_l - 1$.

As before, we let $r_1 = r_2 = \cdots = r_k = 1$ and $r_{k+1} < r_{k+2} < \cdots < r_{k+n}$ be the values of $R_{\alpha,k}$ greater than 1. We can create a SSYT of shape μ by filling the boxes of μ as follows.

$$\begin{array}{ccccccc} r_1 & r_2 & \cdots & r_k & r_{k+1} & \cdots & r_{\mu_a} \\ r_{\mu_a+1} & & & & & & \\ r_{\mu_a+2} & & & & & & \\ \vdots & & & & & & \\ r_h & & & & & & \end{array}$$

Since we are using k 1's followed by distinct values of $R_{\alpha,k}$, it is easy to check that the resulting tableau of shape $\mu \oplus \Delta_\alpha$ has a lattice reading word. Thus Lemma 2.2.3 provides us with a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ with lattice reading word, where μ is filled as shown above.

We now wish to count all SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k)$ (shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k)$, respectively) with content $\nu = c(\mathcal{T})$. Since Δ_α has a unique way of being filled, we must find all semistandard fillings of μ (λ, resp.) with the values $r_1, r_2, \ldots, r_h$. Since $r_1 = r_2 = \cdots = r_k = 1$, the values r_1, r_2, ... , r_k must appear in the first k positions of the first row of μ (λ, resp.). Further, once we choose $\mu_l - 1$ ($\lambda_a - k$, resp.) of the values $r_{k+1} < r_{k+2} < \cdots < r_h$ to appear in the first column of μ (first row of λ, resp.), then the remaining r's must appear in the first row of μ (first column of λ, resp.) and the order of all these values is uniquely determined by the semistandard conditions.

Therefore the number of SSYTx of shape $\mathcal{S}(\mu, \alpha^{\triangleleft}; k) = \kappa'/\rho'$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa'}_{\rho'\nu} = \binom{h-k}{\mu_l - 1}$$

and the number of SSYTx of shape $\mathcal{S}(\lambda, \alpha^{\triangleleft}; k) = \kappa/\rho$ with lattice reading word and content $\nu = c(\mathcal{T})$ is given by

$$c^{\kappa}_{\rho\nu} = \binom{h-k}{\lambda_a - k}.$$

Since $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$, we have $h+1 \geq \lambda_a + \mu_l$. If we have $h+1 = \lambda_a + \mu_l$, then this implies that h is odd and $\lambda_a = \mu_l = \left\lceil \frac{h}{2} \right\rceil = \frac{h+1}{2}$. Using the fact that $|\lambda| = |\mu| = h$ implies $\lambda_l = \mu_a = \left\lceil \frac{h}{2} \right\rceil = \frac{h+1}{2}$ as well. Therefore $\lambda = \mu$. However, we are only interested in distinct hooks λ and μ. Thus, among distinct pairs λ and μ, we cannot have $h+1 = \lambda_a + \mu_l$.

Thus for $\lambda \neq \mu$, with $\lambda_a, \mu_l \leq \left\lceil \frac{h}{2} \right\rceil$, we have $h+1 > \lambda_a + \mu_l$. This gives $h - \lambda_a > \mu_l - 1$ and we obtain

$$h - \lambda_a - i > \mu_l - 1 - i, \tag{3.17}$$

for each i.

Therefore

$$\begin{aligned}
c_{\rho'\nu}^{\kappa'} &= \frac{(h-k)!}{(h-k-\mu_l+1)!(\mu_l-1)!} \\
&= \frac{(h-k)!}{(h-\lambda_a)!(\lambda_a-k)!} \times \prod_{i=0}^{\mu_l-\lambda_a+k-2} \frac{h-\lambda_a-i}{\mu_l-1-i} \\
&= c_{\rho\nu}^{\kappa} \times \prod_{i=0}^{\mu_l-\lambda_a+k-2} \frac{h-\lambda_a-i}{\mu_l-1-i} \\
&> c_{\rho\nu}^{\kappa},
\end{aligned}$$

where we have used Equation 3.17 in the final step. Therefore $s_{\mathcal{S}(\lambda,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{\triangleleft};k)} \not\geq_s 0$, which is a contradiction. Therefore we have $\lambda_a \geq \mu_l + k - 1$. ∎

Chapter 4

Differences of Transposed Foundations

In this chapter we shall only work with the regular staircase Δ_n. We shall use the notation $\mathcal{S}(\lambda, k, n)$ for a staircase with bad foundation as defined in Section 2.1. We shall look at differences of skew Schur functions of the form $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$, where λ is restricted to be a partition with one part or two parts. In both cases the difference is seen to be Schur-positive. Further, we find a simple expression for the difference in the case when λ has a single part, from which we can see that the difference is multiplicity-free.

4.1 Single Part Partitions

The Schur-positivity and formula stated in the following result are a special case of Theorem 3.2.3. We see fit to re-examime this special case since the formula for the difference is easy to state and compute and the difference is seen to be multiplicity-free.

Theorem 4.1.1 *Let n be a positive integer, $0 \leq k \leq 1$, and λ be a partition consisting of a single part such that $\lambda_1 \leq k + n$. Then*

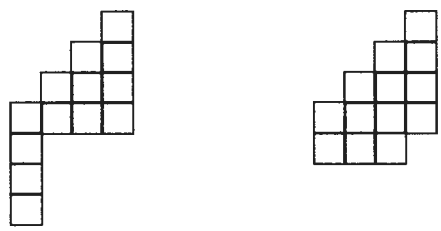
1. $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} \geq_s 0$,
2. $s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$, *and*
3. $s_{\mathcal{S}(\lambda^t,k,n)} \neq_{n+2} s_{\mathcal{S}(\lambda,k,n)}$ *if* $\lambda \neq \lambda^t$.

Furthermore, we have

$$s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} = \sum_{\substack{A \subseteq \{2-k, 3-k, \ldots, n\} \\ |A| \leq \lambda_1 - 2}} s_{\nu(A)},$$

where $\nu(A) = \delta_n + e_A + (0^n, 1^{\lambda_1 - |A|})$. *In particular,* $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ *is multiplicity-free.*

Example Let $n = 4$, $\lambda = (3)$, and $k = 0$. We are interested in the following two diagrams.



$\mathcal{S}(\lambda^t, 0, 4)$ $\qquad$ $\mathcal{S}(\lambda, 0, 4)$

Theorem 4.1.1 gives

$$\begin{aligned} s_{\mathcal{S}(\lambda^t,0,4)} - s_{\mathcal{S}(\lambda,0,4)} &= \sum_{\substack{A \subseteq \{2,3,4\} \\ |A| \leq 1}} s_{\nu(A)} \\ &= s_{\nu(\emptyset)} + s_{\nu(\{2\})} + s_{\nu(\{3\})} + s_{\nu(\{4\})} \\ &= s_{(4,3,2,1,1,1,1)} + s_{(4,4,2,1,1,1)} + s_{(4,3,3,1,1,1)} + s_{(4,3,2,2,1,1)}, \end{aligned}$$

where we have omitted the details of computing $\nu(A)$ for the various A. For an example of this computation, we have

$$\begin{aligned} \nu(\{2\}) &= \delta_4 + \sum_{a \in \{2\}} e_a + (0^4, 1^{3-1}) \\ &= (4,3,2,1) + (0,1,0,0) + (0,0,0,0,1,1) \\ &= (4,4,2,1,1,1). \end{aligned}$$

Proof (of Theorem 4.1.1)

1.

To prove the Schur-positivity of $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ we show that any SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word gives rise to a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word and the same content. We then show that we can recover $\mathcal{T}_1$ from $\mathcal{T}_2$.

Consider any SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word. Lemma 2.1.1 implies that the foundation λ contains a strictly increasing sequence $a_1 < a_2 < \cdots < a_{\lambda_1}$ where $a_{\lambda_1} \leq n + 1$ by the lattice condition.

Since $a_1 < a_2 < \cdots < a_{\lambda_1}$, we can create a SSYT of shape λ^t by transposing the entries of the foundation λ. Thus we have a SSYT of shape $\lambda \oplus \Delta_n$. Suppose the tableau did not have a lattice reading word. Then, since the content of Δ_n was $(n, n-1, \ldots, 3, 2, 1)$ and $a_i \leq n+1$ for each i, we would require two of the a_i to be equal. Since the a_i are distinct, this tableau does have a lattice reading word. Then, we may apply Lemma 2.2.3, we obtain a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word. Furthermore, we may recover $\mathcal{T}_1$ from $\mathcal{T}_2$ by transposing the entries of the foundation λ^t of $\mathcal{T}_2$.

This completes the proof that $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} \geq_s 0$.

2.

As discussed in the introduction, using the Littlewood-Richardson rule to compute the skew Schur functions in $n+1$ variables reduces the usual Littlewood-Richardson rule to considering only those fillings that use numbers from the set $\{1, 2, \ldots, n+1\}$. We now show that we can reverse the process described in the proof of **1** when we restrict to fillings using these numbers. So we take any SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word whose entries are from the set $\{1, 2, \ldots, n+1\}$. Therefore, the foundation λ^t consists of a strictly increasing sequence $a_1 < a_2 < \cdots < a_{\lambda_1}$ and $a_{\lambda_1} \leq n+1$.

As before, we can check that the tableau of shape $\lambda \oplus \Delta_n$ has lattice reading word. If not, then since the content of Δ_n was $(n, n-1, \ldots, 3, 2, 1)$ and $a_i \leq n+1$ for each i, we would require two of the a_i to be equal. Since the a_i are distinct, this tableau does have a lattice reading word. We may now use Theorem 2.2.3 to obtain a SSYT $\mathcal{T}_1$ of shape $s_{\mathcal{S}(\lambda,k,n)}$.

Also, we can recover the tableau $\mathcal{T}_2$ from $\mathcal{T}_1$ by transposing the entries of the foundation λ. This shows that $s_{\mathcal{S}(\lambda,k,n)} - s_{\mathcal{S}(\lambda^t,k,n)} \geq_s 0$ as functions in $n+1$ variables. This and **1** completes the proof that $s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$.

3.

If $\lambda \neq \lambda^t$ then $\lambda_1 \geq 2$. Therefore λ^t consists of a column of length λ_1, where $2 \leq \lambda_1 \leq n+k$. We shall fill the last two spaces of the column with $n+1$ and $n+2$, and then fill the remaining $\lambda_1 - 2$ spaces with $2-k, 3-k, \ldots, \lambda_1 - 1 - k$. That is, we fill the column with the entries $2-k, 3-k, \ldots, \lambda_1 - 1 - k, n+1, n+2$. This filling gives rise to a SSYT of shape $\mathcal{S}(\lambda^t, k, n)$ with a lattice reading word. Further, no SSYT of shape $\mathcal{S}(\lambda, k, n)$ with the same content can have a lattice reading word since the lattice condition implies that the $n+1$ lies to the right of the $n+2$ and this violates the fact that the row weakly increases.

This completes the proof of **1**, **2**, and **3**. We now show that the formula

stated for $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ is correct. From **2** we know that the only terms s_ν with non-zero coefficient that appear in the difference have $l(\nu) \geq n+2$.

Let ν be any partition with $l(\nu) \geq n+2$. No SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word and content ν exists since the lattice condition requires that both the values $n+1$ and $n+2$ appear in λ, but the lattice condition also requires the $n+1$ appears to the right of the $n+2$ and this violates the fact that the row weakly increases.

Any SSYT of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word and content ν must contain both the values $n+1$ and $n+2$ in λ^t. These appear one above the other in λ^t. The lattice condition implies that any values below the entry $n+2$ in λ^t, assuming there are any, must be the values $n+3$, $n+4$, ..., until the end of the column is reached. The values of λ^t above the entry $n+1$ form some set $A \subseteq \{1, 2, \ldots, n\}$. In fact, for $k = 0$ we must have $A \subseteq \{2, 3, \ldots, n\}$ since there is an entry 1 directly above λ^t. Hence, in either case, $A \subseteq \{2-k, 3-k, \ldots, n\}$. The order that these values appear in λ^t is uniquely determined since the column must strictly increase. Since there are only λ_1 entries in λ^t and we know that both $n+1$, $n+2 \in \lambda^t$, we have $|A| \leq \lambda_1 - 2$.

Furthermore, any $A \subseteq \{2-k, 3-k, \ldots, n\}$ with $|A| \leq \lambda_1 - 2$ gives rise to a unique SSYT of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word that contains A in its foundation. Namely, we fill the first $|A|$ boxes of λ^t with the values of A and the remaining $\lambda_1 - |A|$ boxes of λ^t with $n+1, n+2, \ldots, n+\lambda_1-|A|$. The content of this filling is $\nu(A) = \delta_n + e_A + (0^n, 1^{\lambda_1 - |A|})$. Since $l(\nu(A)) \geq n+2$, there is no SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word and content $\nu(A)$. Since all SSYT of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word and content ν, for $l(\nu) \geq n+2$, arise in this manner, we have

$$s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} = \sum_{\substack{A \subset \{2-k, 3-k, \ldots, n\} \\ |A| \leq \lambda_1 - 2}} s_{\nu(A)},$$

as claimed.

Now, if we have two sets A and A', with $\nu(A) = \nu(A')$, then looking at the first n rows of these partitions gives that $\delta_n + e_A = \delta_n + e_{A'}$. Hence $e_A = e_{A'}$, so we obtain $A = A'$. This shows that no distinct subsets contribute to the same term s_ν. Therefore the difference is multiplicity-free. ∎

We note that for λ a partition with one part, the difference $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ will still be multiplicity-free if we use any fat staircase Δ_α, since this only adds more restrictions to which values can appear in the foundation. However, since the expression for the difference in the case of fat staircases is more complex, we choose not to examine the matter here.

4.2 Two Part Partitions

We now turn to the case of partitions with exactly two parts. In Theorem 4.2.1 the condition $\lambda_1 > 1$ is present to prevent λ from being a single column. We wish to exclude this case since the previous section already inspected the difference when the foundations were a row and column, respectively.

Theorem 4.2.1 *Let n be a positive integer, $0 \leq k \leq 1$, and λ is a partition with two rows such that $1 < \lambda_1 \leq k + n$. Then*

1. $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} \geq_s 0$,
2. $s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$, *and*
3. $s_{\mathcal{S}(\lambda^t,k,n)} \neq_{n+2} s_{\mathcal{S}(\lambda,k,n)}$ *if* $\lambda \neq \lambda^t$.

Proof

1.

Consider any SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word, where $0 \leq k \leq 1$. Then the entries of the foundation λ have the following relations.

$$\begin{array}{ccccccccc} a_1 & < & a_2 & < & \cdots & < & a_{\lambda_2} & < & \cdots < a_{\lambda_1} \\ \wedge & & \wedge & & \cdots & & \wedge & & \\ b_1 & \leq & b_2 & \leq & \cdots & \leq & b_{\lambda_2} & & \end{array}$$

The fact that $a_1 < a_2 < \cdots < a_{\lambda_1}$ follows from Lemma 2.1.1, and the rest of the relations follow from the definition of a SSYT. We also have the condition $a_1 > 1$ when $k = 0$.

The relations $a_1 < a_2 < \cdots < a_{\lambda_1}$ show that there is no repeated value in the first row. Further, if there is a repeated value j in the second row, then $j > 1$ and the lattice condition shows that a $j - 1$ occured in the first row, and that another j could not have occured in the first row. Therefore, if there is a repeated value in the foundation of $\mathcal{T}_1$, then it only appears twice. We shall create the desired tableau $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ in four steps that we shall describe presently.

In **Step 1** we will transpose the foundation of $\mathcal{T}_1$, giving us a tableau T of shape λ^t, where T is not necessarily semistandard. In **Step 2** we will check whether or not $n + 2$ appears in our tableau T. If it does, we will swap its position so that it will be read after reading an $n + 1$. This gives

us a tableau T' of shape λ^t which still may not be semistandard, but will not violate the lattice condition when reading the value $n+2$. In **Step 3** we will fix any places in T' where a column is not strictly increasing by rotating certain blocks of entries. After completing this step, we will have a SSYT T'' of shape λ^t. In **Step 4** we will append T'' to Δ_n, creating a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$. During **Step 4**, we will have shown that $\mathcal{T}_2$ has a lattice reading word. Finally, we check $\mathcal{T}_1$ can be recovered from $\mathcal{T}_2$.

Step 1: *Transpose Foundation* ($\mathcal{T}_1 \to T$)

Let us consider the tableau T that is obtained by transposing the entries of the foundation of $\mathcal{T}_1$. Then T is a tableau of shape λ^t.

$$\begin{array}{ccc}
a_1 & < & b_1 \\
\wedge & & \wedge \\
\vdots & \vdots & \vdots \\
\wedge & & \wedge \\
a_{\lambda_2-1} & < & b_{\lambda_2-1} \\
\wedge & & \wedge \\
a_{\lambda_2} & < & b_{\lambda_2} \\
\wedge & & \\
\vdots & & \\
\wedge & & \\
a_{\lambda_1} & &
\end{array}$$

Step 2: *Fix Potential $n+2$ Lattice Problems* ($T \to T'$)

The tableau T cannot be immediately extended to a tableau of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word if there was both a $n+1$ and a $n+2$ in T. Since $\mathcal{T}_1$ had a lattice reading word, the tableau T can have at most one $n+2$, which must occur at the bottom of the second column. Further, having an $n+2$ requires an $n+1$ in the first column of T, which must also appear at the bottom. We split into three cases and define a tableau T' of shape λ^t in each case. In each case we display the resulting tablueau T'.

1. If $a_{\lambda_1} \leq n+1$ and $b_{\lambda_2} < n+2$ then $T' = T$.
2. If $a_{\lambda_1} = n+1$ and $b_{\lambda_2} = n+2$, then
   (a) If $\lambda_1 > \lambda_2$ then $T' = T$ with a_{λ_1} and b_{λ_2} swapped.
   (b) If $\lambda_1 = \lambda_2$ then $T' = T$ with a_{λ_2} and b_{λ_2-1} swapped.

$$
\begin{array}{ccc}
 & 1. & \\
a_1 & < & b_1 \\
\wedge & & \rotatebox{90}{$\leq$} \\
\vdots & \vdots & \vdots \\
\wedge & & \rotatebox{90}{$\leq$} \\
a_{\lambda_2-1} & < & b_{\lambda_2-1} \\
\wedge & & \rotatebox{90}{$\leq$} \\
a_{\lambda_2} & < & b_{\lambda_2} \\
\wedge & & \\
\vdots & & \\
\wedge & & \\
a_{\lambda_1} & &
\end{array}
\qquad
\begin{array}{ccc}
 & 2.(a) & \\
a_1 & < & b_1 \\
\wedge & & \rotatebox{90}{$\leq$} \\
\vdots & \vdots & \vdots \\
\wedge & & \rotatebox{90}{$\leq$} \\
a_{\lambda_2-1} & < & b_{\lambda_2-1} \\
\wedge & & \rotatebox{90}{$\leq$} \\
a_{\lambda_2} & < & n+1 \\
\wedge & & \\
\vdots & & \\
\wedge & & \\
n+2 & &
\end{array}
\qquad
\begin{array}{ccc}
 & 2.(b) & \\
a_1 & < & b_1 \\
\wedge & & \rotatebox{90}{$\leq$} \\
\vdots & \vdots & \vdots \\
\wedge & & \rotatebox{90}{$\leq$} \\
a_{\lambda_2-1} & < & n+1 \\
\wedge & & \wedge \\
b_{\lambda_2-1} & < & n+2
\end{array}
$$

The lattice condition implies that there was only one $n+2$ in the second column and only one $n+1$ in the first column. Thus, the inequalities shown above are accurate. Now, when reading any of these resulting tableaux T', we shall always read an $n+1$ before we read the $n+2$.

Step 3: *Fix Strictly Increasing Problems* ($T' \to T''$)

We first recall that each entry in the second column of T' is at least 2. Further, if we had the entry 2 repeated in the second column of T', then the semistandard condition on $\mathcal{T}_1$ would have required two of the a_i equal to 1. However, we know that the a_i are distinct, hence we cannot have the entry 2 being repeated in the second column of T'.

Let j be such that $3 \leq j \leq n+1$ and suppose that the pair of entries $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$ appear in the second column of T'. That is, one j appears immediately below another j. Since $\mathcal{T}_1$ has a lattice reading word, a $j-1$ must have appeared in the first column of T'. Since the first row of $\mathcal{T}_1$ and hence the first column of T' strictly increases there can only be one $j-1$ in this column. We now consider which positions in this column of T' could this entry $j-1$ appear. There are two cases to consider.

1. The $j-1$ appears directly to the left of the bottom j.

2. The $j-1$ appears somewhere below the box directly to the left of the bottom j.

The $j-1$ cannot occur above the left of the bottom j else the element x directly to the left of the bottom j would satisfy $j-1 < x < j$.

In this step we shall work from the top to the bottom of the second column of T'. We begin with j being the first pair of repeated elements in the second column, and proceed downwards. We now show how we fix these strictly increasing problems in each of the two cases mentioned above.

Case I: *The $j-1$ appears directly to the left of the bottom j.*

Starting from the top j, we search up the second column. Suppose the entry above j is the value $j-1$. We have already mentioned that there is another $j-1$ in the first column of T'. Since there are two $j-1$'s in T', there are also two $j-1$'s in λ, and since $\mathcal{T}_1$ had a lattice reading word, there must be a $j-2$ in λ. Hence, there is a $j-2$ in T' as well. If this $j-2$ appeared in the second column of T', then it must appear directly above the $j-1$. If this is the case, then in $\mathcal{T}_1$ we wouldn't read this $j-2$ until after we had read both $j-1$'s. Hence if this $j-2$ is in the second column then there is also a $j-2$ in the first column, and it must appear directly above the $j-1$ in the first column.

There are now two $j-2$'s in T'. Using the lattice condition on $\mathcal{T}_1$, this implies there is at least one $j-3$ in T'. Just as with the $j-2$ previously mentioned, if this $j-3$ appears in the second column of T' then we will find that there is a $j-3$ in the first column of T', which appears directly above the $j-2$. Continuing in this manner, we have a sequence $j-1, j-2, \ldots, j-m$ above the top j in the second column and corresponding sequence $j-1, j-2, \ldots, j-m$ in the first column, and the lattice condition of $\mathcal{T}_1$ requires a $j-m-1$ to appear in T'.

Since the tableau is finite, the above procedure must terminate for some value $m = i$ and we find that the element $j-i-1$ *does not* occur in the second column. Thus the $j-i-1$ appears in the first column, directly above the $j-i$. The entries $j-1, j-2, \ldots, j-i, j-i-1$ in the first column and the entries $j, j, j-1, j-2, \ldots, j-i$ in the second column define a *block* of entries. We highlight the block in the diagram below. Consider rotating this block of entries clockwise as shown.

$$
\begin{array}{ccc}
* & \leq & y \\
\wedge & & \wedge \\
x & < & \boldsymbol{j-i} \\
\wedge & & \wedge \\
\boldsymbol{j-i-1} & < & \boldsymbol{j-i+1} \\
\wedge & & \wedge \\
\boldsymbol{j-i} & < & \boldsymbol{j-i+2} \\
\wedge & & \wedge \\
\vdots & \vdots & \vdots \\
\wedge & & \wedge \\
\boldsymbol{j-3} & < & \boldsymbol{j-1} \\
\wedge & & \wedge \\
\boldsymbol{j-2} & < & \boldsymbol{j} \\
\wedge & & \| \\
\boldsymbol{j-1} & < & \boldsymbol{j} \\
\wedge & & \wedge \\
z & < & w
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
* & \leq & y \\
\wedge & & \wedge \\
x & < & \boldsymbol{j-i-1} \\
\wedge & & \wedge \\
\boldsymbol{j-i} & = & \boldsymbol{j-i} \\
\wedge & & \wedge \\
\boldsymbol{j-i+1} & = & \boldsymbol{j-i+1} \\
\wedge & & \wedge \\
\vdots & \vdots & \vdots \\
\wedge & & \wedge \\
\boldsymbol{j-2} & = & \boldsymbol{j-2} \\
\wedge & & \wedge \\
\boldsymbol{j-1} & = & \boldsymbol{j-1} \\
\wedge & & \wedge \\
\boldsymbol{j} & = & \boldsymbol{j} \\
\wedge & & \wedge \\
z & < & w
\end{array}
$$

This *block rotation* fixes the strictly increasing problem caused by the j's in the second column of T'. We make a few comments to justify the accuracy of the relations displayed in the above diagrams.

First, a note on the relation $* \leq y$ that appears at the top of each diagram. We are fixing these $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$ problems in T' from top to bottom. Initially all rows of T' strictly increase since the columns of T were strictly increasing. Yet, after we perform one or more of these block rotations all rows contained in a given block, except the top row, will now display equality. However, we will soon show that any two blocks are disjoint (see p. 101), hence all other row equalities appear strictly above the current block that is displayed.

Initially, the entries of the first column of T' were strictly increasing. This gives that $x < j - i - 1$ and $j - 1 < z$. The first inequality gives $x < j - i$. Also, there are can only be two j's in the diagram, we have that $z \neq j$. Therefore the second inequality gives that $j < z$. This shows that the block rotation preserves the strictly increasing nature of the first column.

Since we are working from top to bottom, we know that the second column strictly increases above the j's. This gives $y < j - i$ and $j < w$. Since we assumed that $j-i-1$ did not appear in the second column, the first inequality gives $y < j - i - 1$. Therefore the strictly increasing segment of the second column has been extended downwards.

Again, since we are working from top to bottom and the blocks are disjoint, the rows strictly increase for each row that the current block intersects. After rotating this block it is evident that we obtain row equalities for each

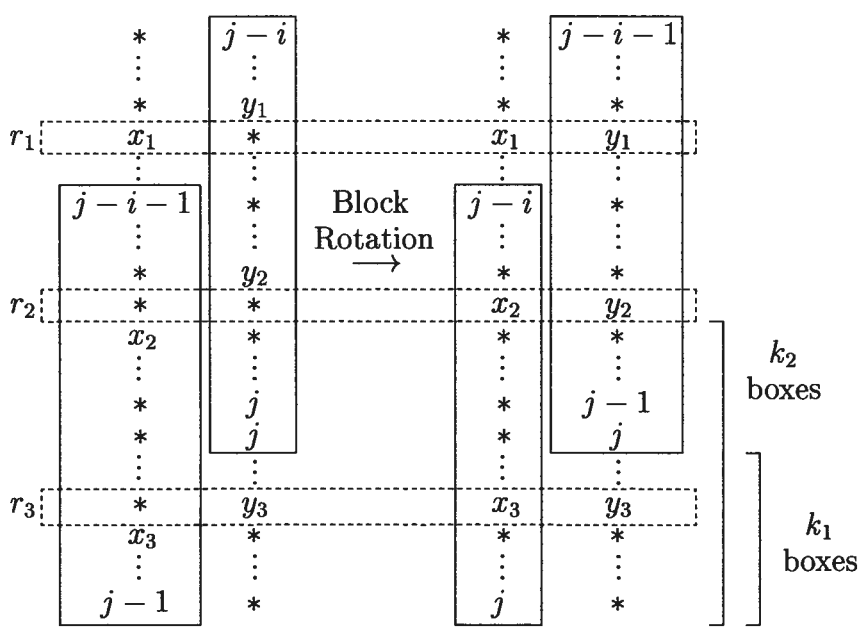
of these rows except the top-most.

Thus we have seen how to resolve repeated elements in the second column that falls into the first case. We now turn to the second case.

Case II: *The $j-1$ appears somewhere below the box immediately left of the bottom j.*

We define the block of entries exactly as in **Case I**. The only difference in this case is that the block of entries consists of a sub-block in the second column and a lower sub-block in the first column. These two sub-blocks can be row-disjoint from one another. We still must check that we obtain a SSYT after we rotate the block. The strict inequality down the columns follows for the same reasons as in the first case. The rows were strictly increasing to begin with since the columns of $\mathcal{T}_1$ strictly increase. We need to check that, after rotating, the rows still weakly increase. There are three cases to check, each of which is displayed in the following diagram.

1. The row of interest r_1 is above the sub-block in the first column.
2. The row of interest r_2 is within the sub-block in both columns.
3. The row of interest r_3 is below the sub-block in the second column.



Thus, in the left-hand diagram, the value x_3 may in fact be the bottom-most entry of the sub-block in the first column. That is, $x_3 = j-1$ is a case considered by the various possible positions of r_3.

For the rows r_1, r_2, and r_3 shown, we wish to show $x_i \leq y_i$ for $i = 1, 2, 3$. This will show that the rows weakly increase after the block rotation. Since the columns strictly increase we have

$$x_1 < j - i - 1 < j - i \leq y_1, \text{ and}$$

$$x_3 \leq j - 1 < j < y_3.$$

Further, with k_1 and k_2 as shown, we have $x_2 = j - k_2$ and $y_2 = j - k_2 + k_1$, hence

$$x_2 \leq x_2 + k_1 = j - k_2 + k_1 = y_2.$$

Therefore the rows of the resulting tableau weakly increase.

We can now check that no two block rotations move the same elements. If we have performed a block rotation to fix a $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$ problem then there cannot be two $j+1$'s below the new position of the j's, since, if there were, then in $\mathcal{T}_1$ we would have read two $j+1$'s in the foundation λ before reading any j's, which would violate the lattice condition. Since the blocks are formed by extending consecutively increasing sequences, this shows that the next block could not overlap this block. Thus any two blocks are disjoint.

This process of block rotation removes the pairs of repeating elements in the second column while maintaining the strict inequalities of the two columns up to that point and never violates the weakly increasing restrictions on the rows. By working top to bottom and repeating this process for each pair of repeated elements in the second column we shall produce a SSYT T'' of shape λ^t.

Step 4: *Extend from shape λ^t to a SSYT of shape $\mathcal{S}(\lambda^t, k, n)$ ($T'' \to \mathcal{T}_2$)*

Using T'' and the unique semistandard filling of Δ_n with lattice reading word, we obtain a SSYT T_2 of shape $\lambda^t \oplus \Delta_n$.

We also claim that T_2 has a lattice reading word. In each case there is at most one $n+2$ in T_2. If an $n+2$ does appear, then it appeared at the end of the second column of T, and none of the block rotations in **Step 3** moved its position. Further, since $\mathcal{T}_1$ was lattice, we know that an $n+1$ must have appeared at the end of the first column of T. If $\lambda_1 > \lambda_2$ then Case 2(a) placed the $n+2$ at the bottom of the first column and placed the $n+1$ at the bottom of the second column. The only way this $n+1$ could have moved in **Step 3** was if there was another $n+1$ above it. In either case, an $n+1$ remains at the end of the second column, so we read this $n+1$ before reading the $n+2$. Similarly, if $\lambda_1 = \lambda_2$ then Case 2(b) placed the $n+2$ at the bottom

of the second column and placed the $n+1$ directly above it. Again, the only way this $n+1$ could have moved in **Step 3** was if there was another $n+1$ above it, and in either case, an $n+1$ remains above the $n+2$, so we read this $n+1$ before the $n+2$. Therefore the single $n+2$ that occurs does not violate the lattice condition.

Nevertheless, suppose that the reading word of T_2 is not lattice. Let $j+1$ be the entry that, when read, violated the lattice condition. We know that the content after reading Δ_n is $(n, n-1, \ldots, 3, 2, 1)$, so we must have read at least two more $j+1$'s than j's since we finished reading Δ_n. Since there are only two column for these two $j+1$'s to appear in, there can only be two $j+1$'s. Thus we must have read exactly two $j+1$'s and no j's since we finished reading Δ_n. Let the values of the columns be $x_1, x_2, \ldots, x_{\lambda_1}$ and $y_1, y_2, \ldots, y_{\lambda_2}$ respectively. Thus $x_i = j+1$ and $y_k = j+1$ for some $i \geq k$.

We know $j \notin \{x_i, x_{i+1}, \ldots, x_{\lambda_1}\} \cup \{y_k, y_{k+1}, \ldots, y_{\lambda_2}\}$ since the columns strictly increase. But we also know $j \notin \{x_1, x_2, \ldots, x_i\} \cup \{y_1, y_2, \ldots, y_i\}$ since by assumption we had not read any j's when reaching the second $j+1$. Thus

$$j \notin \{x_1, x_2, \ldots, x_i\} \cup \{y_1, y_2, \ldots, y_i\} \cup \{x_i, x_{i+1}, \ldots, x_{\lambda_1}\} \cup \{y_k, y_{k+1}, \ldots, y_{\lambda_2}\},$$

which is to say $j \notin \{x_1, x_2, \ldots, x_{\lambda_1}, y_1, y_2, \ldots, y_{\lambda_2}\}$. This is a contradiction since $\{x_1, x_2, \ldots, x_{\lambda_1}, y_1, y_2, \ldots, y_{\lambda_2}\} = \{a_1, a_2, \ldots, a_{\lambda_1}, b_1, b_2, \ldots, b_{\lambda_2}\}$ and the fact that there are two $j+1$'s in the foundation of $\mathcal{T}_1$ implies there was at least one j as well, else $\mathcal{T}_1$ could not have a lattice reading word. Therefore the SSYT T_2 has a lattice reading word.

Thus we may apply Lemma 2.2.3 to obtain a SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word.

Check: *Can we recover $\mathcal{T}_1$?*

We claim that $\mathcal{T}_1$ can be recovered from $\mathcal{T}_2$. **Step 4** can be undone by passing back to the foundation of $\mathcal{T}_2$. The swapping performed in **Step 2** is easily detected and reversed. Undoing **Step 1** requires a mere transposition of diagrams. So to prove the claim it remains to check that the block rotations in **Step 3** are reversible. Since we have noted that no two block rotations move a common element, it is enough to check that each of the two cases are reversible. Hence we look at reversing both cases after fixing a $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$ problem.

If we made such a manipulation in **Case I** of **Step 3** (i.e. $j-1$ *was* immediately left of the bottom j), then the foundation of $\mathcal{T}_2$ *now* has a row containing $[\,j \quad j\,]$. By transposing the foundation we obtain a tableau of shape λ with a column $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$. We now search to the left of this column until we come upon a nonrepeating column $\left[\begin{smallmatrix} x \\ j-i-1 \end{smallmatrix}\right]$ where $x < j-i-1$. This

determines the block of entries we wish to rotate. We then rotate the entries clockwise to undo the original block rotation.

$$
\begin{array}{ccccccccccccc}
* & < & x & < & \boldsymbol{j-i} & < & \cdots & < & \boldsymbol{j-1} & < & \boldsymbol{j} & < & * \\
\wedge & & \wedge & & \| & & \cdots & & \| & & \| & & \wedge \\
y & < & \boldsymbol{j-i-1} & < & \boldsymbol{j-i} & < & \cdots & < & \boldsymbol{j-1} & < & \boldsymbol{j} & < & *
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ccccccccccccc}
* & < & x & < & \boldsymbol{j-i-1} & < & \cdots & < & \boldsymbol{j-2} & < & \boldsymbol{j-1} & < & * \\
\wedge & & \wedge & & \wedge & & \cdots & & \wedge & & \wedge & & \wedge \\
y & < & \boldsymbol{j-i} & < & \boldsymbol{j-i+1} & < & \cdots & < & \boldsymbol{j} & = & \boldsymbol{j} & < & *
\end{array}
$$

If we made such a manipulation in the second case (i.e. $j-1$ was below the left of the bottom j), then upon transposing the foundation of $\mathcal{T}_2$ and reading its entries we come across two $j-i$'s before reading the $j-i-1$. This determines the smallest element $j-i-1$ in the block we wish to form. The $j-i-1$ necessarily appears in the second row, immediately to the left of the second $j-i$ that we have just read. The remaining elements of the block are the sequence of consecutive integers $j-i, j-i+1, \ldots, j$ to its right and the copy of this sequence in the first row. The consecutive sequence cannot be further extended to the right, otherwise, in the foundation of $\mathcal{T}_1$, we would have read two $j+1$'s before reading either of the j's. Therefore the correct endpoint j for this block of entries can be correctly determined. Again we rotate the elements in the block clockwise to recover the original. Therefore we can undo the block rotations of **Step 3**. Hence $\mathcal{T}_1$ can be recovered from $\mathcal{T}_2$. This completes the proof that $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)} \geq_s 0$.

An example of block rotations is shown after the remainder of the proof.

2.

We now show that we can reverse the process described in the proof of **1** when we restrict to fillings using the numbers $\{1, 2, \ldots, n+1\}$. If we take any SSYT $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word then the foundation λ^t consists of two strictly increasing columns.

$$\begin{array}{ccc} a_1 & \leq & b_1 \\ \wedge & & \wedge \\ \vdots & \vdots & \vdots \\ \wedge & & \wedge \\ a_{\lambda_2-1} & \leq & b_{\lambda_2-1} \\ \wedge & & \wedge \\ a_{\lambda_2} & \leq & b_{\lambda_2} \\ \wedge & & \\ \vdots & & \\ \wedge & & \\ a_{\lambda_1} & & \end{array}$$

Since the columns strictly increase we may see a pair of equal elements $a_i = b_i$ in λ^t, but never three equal values. We shall create the desired tableau $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ using three steps that we shall describe presently.

In **Step A** we will transpose the foundation of $\mathcal{T}_2$, giving us a tableau T of shape λ, which is not necessarily semistandard. In **Step B** we will fix any places in T where the lattice condition would fail if we appended T to Δ_n by rotating certain blocks of entries. After completing this step, we will have a SSYT T' of shape λ. In **Step C** we will append T' to Δ_n, creating a SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$. During **Step C**, we will have shown that $\mathcal{T}_1$ has a lattice reading word. Finally, we show that $\mathcal{T}_2$ can be recovered from $\mathcal{T}_1$.

Step A: *Transpose Foundation* ($\mathcal{T}_2 \to T$)

Let us consider the tableau T that is obtained by transposing the entries of the foundation of $\mathcal{T}_2$. Then T is a tableau of shape λ.

$$\begin{array}{ccccccccc} a_1 & < & a_2 & < & \cdots & < & a_{\lambda_2} & < & \cdots < a_{\lambda_1} \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \cdots & & \rotatebox{90}{$\leq$} & & \\ b_1 & < & b_2 & < & \cdots & < & b_{\lambda_2} & & \end{array}$$

Step B: *Fix Future Lattice Problems* ($T \to T'$)

In this step we move certain entries of T so that the resulting tableau of shape λ can be extended to a tableau $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ whose reading word is lattice. Given T, the only problems that may arise is that, when reading the entries of T, we come to a point where we have read two j's before reading any $(j-1)$'s. We call this a *future lattice problem.* We look at how to remedy these problems.

If there are two 2's in T, then there are the correpsonding two 2's in $\mathcal{T}_2$. Further since $\mathcal{S}(\lambda^t, k, n)$ has a lattice reading word we must read a 1 in

$\mathcal{T}_2$ before reading the second 2. By the semistandard conditions, we must have $a_1 = 1$. Since at most one 2 can appear in the first row of T, this 1 is read before the second 2 is read in T. Therefore $j = 2$ cannot give rise to a future lattice problem. Also, since the lattice condition places no restriction on reading 1's, $j = 1$ also cannot give rise to a future lattice problem.

Let $j \geq 3$ be the first value that we have read twice without reading any $j-1$. In this case we say that we have a j, j problem. Since $\mathcal{T}_2$ has a lattice reading word, a $j-1$ must have appeared somewhere in T. The only place we have not read is to the left of the j in the second row. Since the rows of T weakly increase (in fact strictly increase, to begin with), the $j-1$ must appear immediately to the left of the j in the bottom row. We now search to the right of both j's and find the largest consecutive sequence $j, j+1, \ldots, j+i$ that appears in both rows. These sequences together with the entry $j-1$ define a *block*. There are two cases to consider:

1. The two j's are in the same column.

2. The top j appears to the right of the bottom j.

We note that the top j cannot occur to the left of the bottom j, otherwise the entry m above the bottom j satisfies $j < m \leq j$. The two cases are displayed below.

$$\begin{array}{cc|ccccc|c}
 & x & j & j+1 \cdots & & \cdots j+i-1 & j+i & z \\
\hline
y & j-1 & j & j+1 \cdots & & \cdots j+i-1 & j+i & w
\end{array}$$

or

$$\begin{array}{cc|ccccc|c}
 & x & j & j+1 \cdots & & \cdots j+i-1 & j+i & z \\
\hline
y & j-1 & j & j+1 \cdots & & \cdots j+i-1 & j+i & w
\end{array}$$

After rotating the entries of either block clockwise the columns will strictly increase. We inspect the relations of the rows in both of the two cases simultaneously. We may do this since the entries of the rows are the same in each case. We display the rotation in the second case. The entries of the block are rotated one position clockwise as shown.

$$
\begin{array}{cccc}
 & x & \boxed{j \quad j+1 \cdots \qquad\qquad \cdots j+i-1 \quad j+i} & z \\
y & \boxed{j-1 \quad j \quad j+1 \cdots \qquad\qquad \cdots j+i-1 \quad j+i} & & w \\
 & & \downarrow & \\
 & x & \boxed{j-1 \quad j \quad \cdots \qquad\qquad \cdots j+i-2 \quad j+i-1} & z \\
y & \boxed{j \quad j+1 \quad j+2 \cdots \qquad\qquad \cdots j+i \quad j+i} & & w
\end{array}
$$

Since both rows strictly increase to begin with, we have $x < j$, $y < j-1$, $j+i < z$, and $j+i < w$. Therefore we have the inequalities $y < j$, $j+i-1 < z$ and $j+i < w$. Further, since there was no $j-1$ in the first row, we have $x < j-1$. Hence the only pair of equal values in a row that is created by this block rotation is the pair $j+i, j+i$ in the bottom-right of the block.

Initially, the columns of T were weakly increasing. Now consider any column of T with a pair of equal values, say $\left[\begin{smallmatrix} m \\ m \end{smallmatrix}\right]$. Since the reading word for $\mathcal{T}_2$ is lattice, there is a $m-1$ in T. This $m-1$ must occur in the column immediately to the left of the $\left[\begin{smallmatrix} m \\ m \end{smallmatrix}\right]$. If this $m-1$ appears in the first row of T, then since the colums weakly increase and there are only two m's in T we find that a second $m-1$ appears immediately below it. Thus we obtain a column $\left[\begin{smallmatrix} m-1 \\ m-1 \end{smallmatrix}\right]$. We continue extending this block of equal valued columns to the left. It terminates when we reach some column $\left[\begin{smallmatrix} j \\ j \end{smallmatrix}\right]$ where a $j-1$ appears immediately left of this column in the second row, but not in the first row.

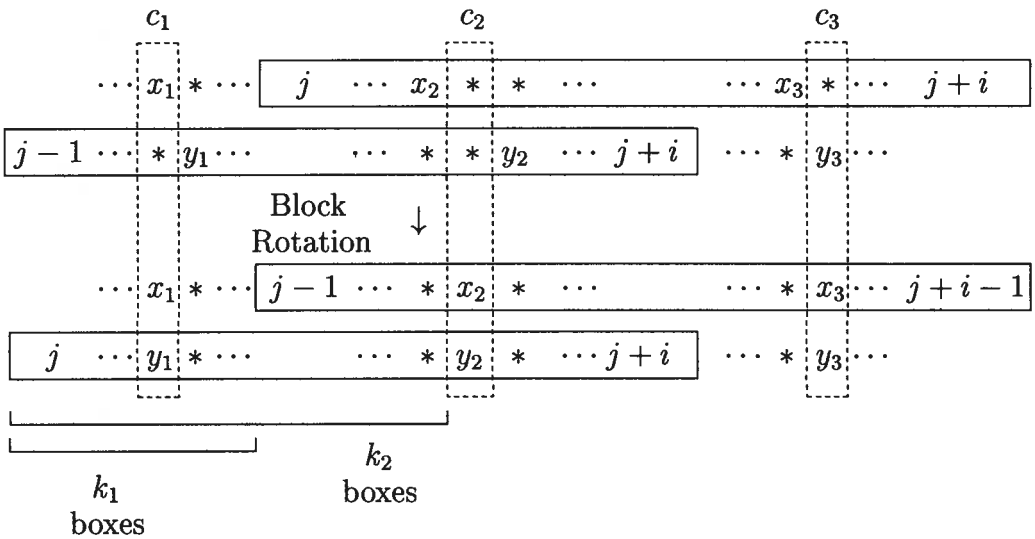
$$
\begin{array}{ccccccccc}
x & < & \boldsymbol{j} & < & \cdots & < & \boldsymbol{m-1} & < & \boldsymbol{m} \\
\wedge & & \| & & \cdots & & \| & & \| \\
\boldsymbol{j-1} & < & \boldsymbol{j} & < & \cdots & < & \boldsymbol{m-1} & < & \boldsymbol{m}
\end{array}
$$

This gives rise to a future lattice problem. Thus all columns in T that did not strictly increase are contained in some block. We now check that the columns of each block strictly increase after performing the block rotation. There are three cases to consider.

1. The column of interest c_1 is left of the sub-block in the first row.

2. The column of interest c_2 is within the sub-block in both rows.

3. The column of interest c_3 is right of the sub-block in the second row.



For the columns c_1, c_2, and c_3 we wish to show that $x_i < y_i$ for $i = 1, 2, 3$. Since the rows were strictly increasing to begin with we have

$$x_1 < j \leq y_1, \text{ and}$$

$$x_3 < j + i < y_3.$$

Further, for k_1 and k_2 shown, we have $x_2 = j + k_2 - k_1 - 1$ and $y_2 = j + k_2$ hence

$$x_2 \leq x_2 + k_1 = j + k_2 - 1 < y_2.$$

It can be seen that no two of these block rotations performed in **Step B** can move the same elements. If we have done a manipulation to fix a j, j problem then there cannot be a second $j - 1$ immediately left of the bottom-left element of this block since, otherwise, before rotating that block the row had the value $j - 1$ repeated occuring on the bottom-left of the block. However, we know that the rows were strictly increasing to begin with and the only repeated values we have introduced were on the bottom-right of each block we have rotated. So since we are working right to left, this repeated value $j - 1, j - 1$ could not occur. Therefore, after rotating this block, there is no $j - 1$ in the bottom row, and hence the next block's pair of consecutively increasing sequences cannot enter this block. Hence the two blocks are disjoint.

We continue this process of reading values and fixing future lattice problems by rotating blocks until there are no more future lattice problems left in the tableau. This results in a SSYT T' of shape λ.

Step C: *Extend from shape λ to SSYT of shape $\mathcal{S}(\lambda, k, n)$ ($T' \to \mathcal{T}_1$)*

Using T' and the unique semistandard filling of Δ_n with lattice reading word, we obtain a SSYT T_1 of shape $\lambda \oplus \Delta_n$. We check that T_1 is a SSYT with a lattice reading word. Since we removed all future lattice problems, T' has the property that, when read, the number of $j+1$'s is always at most one more than the number of j's. Thus the SSYT T_1 does have a lattice reading word. Therefore, applying Lemma 2.2.3, we obtain a SSYT $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word.

Check: *Can we recover $\mathcal{T}_2$?*

Finally, we must show that we can recover $\mathcal{T}_2$ from $\mathcal{T}_1$. Undoing **Step A** and **Step C** are trivial. Thus we need only show that the block rotations in **Step B** can be reversed. Since any two block rotations are disjoint, it suffices to check that a single block rotation can be undone. We have seen that a block rotation creates a tableau with a repeated value $j+i, j+i$ in the bottom-right of the block. Transposing the tableau, there is a repeated value in the second column and so the block rotation described in **Step 3** applies and it results in the transpose of the original tableau. Therefore these block rotations can be reversed, and we can recover $\mathcal{T}_2$ from $\mathcal{T}_1$.

Thus we have shown that $s_{\mathcal{S}(\lambda,k,n)} - s_{\mathcal{S}(\lambda^t,k,n)} \geq_s 0$ as functions in $n+1$ variables. This and **1** completes the proof that $s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$.

3.

We now wish to show that the two skew Schur functions are distinct when there is at least $n+2$ variables. Hence we consider fillings using the values $\{1, 2, \ldots, n+2\}$.

Let $l(\lambda) = 2$ and $\lambda \neq \lambda^t$. The hypotheses of the theorem required $\lambda_1 > 1$, so that λ was not a single column. If $\lambda_1 = 2$, then λ is either the shape $(2,1)$ or $(2,2)$, which both have $\lambda^t = \lambda$.

Thus we need only consider $\lambda_1 \geq 3$. We split into three cases depending on the size of λ_2. Namely,

1. $\lambda_2 = 1$,

2. $\lambda_2 = 2$, and

3. $\lambda_2 \geq 3$.

In each case we wish to create a SSYT $\mathcal{T}$ of shape $\mathcal{S}(\lambda^t, k, n)$ with lattice reading word and content ν such that no SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word and content ν exists. Here we show the fillings of the foundation of the required tableau $\mathcal{T}$ for each of the above three cases.

1.		2.		3.	
2	$n+1$	2	$n+1$	2	3
3		3	$n+2$	3	4
$\vdots$		$\vdots$		$\vdots$	$\vdots$
$\lambda_1 - 2$		$\lambda_1 - 2$		$\lambda_2 - 2$	$\lambda_2 - 1$
n		n		$\lambda_2 - 1$	n
$n+1$		$n+1$		λ_2	$n+1$
$n+2$		$n+2$		$\lambda_2 + 1$	$n+2$
				$\vdots$	
				$\lambda_1 - 2$	
				$\lambda_1 - 1$	
				$n+1$	
				$n+2$	

If $\lambda_1 = 3$ in case 1 or 2, then the first column consists of the three entries $n, n+1, n+2$, and if $\lambda_1 = 3$ in case 3, then the first column consists of $2, n+1, n+2$. Similarly, if $\lambda_2 = 3$ in case 3, then the second column consists of $n, n+1, n+2$.

In each case it is clear that $\mathcal{T}$ is a SSYT with a lattice reading word. We now check that no SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word and the same content can exist. Any SSYT of shape $\mathcal{S}(\lambda, k, n)$ with lattice reading word can only have a single $n+1$ in the first row of λ and a single $n+2$ in the second row of λ. Further, no $n+2$ can appear in the first row of λ. Thus, in Case 1, the $n+2$ must appear in the second row of λ, but then both $n+1$'s would have to appear in the first row, which is impossible. In Case 2 and Case 3, both $n+2$'s would have to fit in the second row of λ, which is impossible. Therefore $s_{\mathcal{S}(\lambda^t,k,n)} \neq_{n+2} s_{\mathcal{S}(\lambda,k,n)}$. ∎

Example

Here we illustrate a concrete example of **Step 3** and reversing **Step 3** in the proof of Theorem 4.2.1. Let us begin with the foundation of $\mathcal{S}((7,6), 0, n)$, where $n \geq 11$.

2	4	5	6	9	10	12
6	7	7	8	11	11	

We now tranpose and perform **Step 3** twice. For convenience, in each step we highlight the block that is about to be rotated.

$$\begin{array}{cc} 2 & \mathbf{6} \\ 4 & \mathbf{7} \\ \mathbf{5} & \mathbf{7} \\ \mathbf{6} & 8 \\ 9 & 11 \\ 10 & 11 \\ 12 & \end{array} \quad \rightarrow \quad \begin{array}{cc} 2 & 5 \\ 4 & 6 \\ 6 & 7 \\ 7 & 8 \\ 9 & \mathbf{11} \\ \mathbf{10} & \mathbf{11} \\ 12 & \end{array} \quad \rightarrow \quad \begin{array}{cc} 2 & 5 \\ 4 & 6 \\ 6 & 7 \\ 7 & 8 \\ 9 & 10 \\ 11 & 11 \\ 12 & \end{array}$$

To reverse **Step 3** to return to the original tableau we transpose and then either follow the rules for rotating discussed in the **Check** section of the proof of part **1** of Theorem 4.2.1 or, equivalently, follow the the rules for rotating discussed in the **Step B** section of the proof of part **2** of Theorem 4.2.1.

$$\begin{array}{ccccccc} 2 & 4 & 6 & 7 & 9 & \mathbf{11} & 12 \\ 5 & 6 & 7 & 8 & \mathbf{10} & \mathbf{11} & \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccccc} 2 & 4 & \mathbf{6} & \mathbf{7} & 9 & 10 & 12 \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & 11 & 11 & \end{array}$$

$$\downarrow$$

$$\begin{array}{ccccccc} 2 & 4 & 5 & 6 & 9 & 10 & 12 \\ 6 & 7 & 7 & 8 & 11 & 11 & \end{array}$$

Chapter 5

Complements in a Rectangle

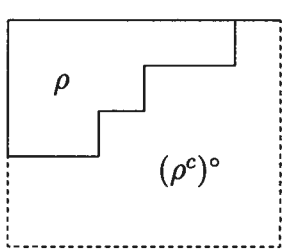
In this chapter we show how to extend the results of the previous chapters to fat staircases with bad foundations where the foundations are the complements of the foundations we previously inspected.

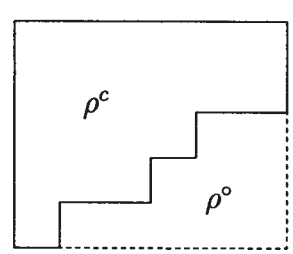
Throughout this chapter, when considering a composition α we shall let $n = l(\alpha)$. That is, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. With this convention, the diagram δ_α (Δ_α, respectively) has width n and length $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

In this chapter we shall be particularly concerned with the size of diagrams. Thus for a diagram D we let $l(D)$ denote the *length* of the diagram and $w(D)$ denote the *width* of the diagram. That is, $(w(D)^{l(D)})$ is the smallest rectangle that contains D.

5.1 Preliminaries

Given a partition ρ contained in the $a \times b$ rectangle (b^a) we define the *complementary partition* ρ^c *in the rectangle* (b^a) by $\rho^c = ((b^a)/\rho)^\circ$. That is, ρ^c is the complement of ρ in (b^a) rotated by 180°. It is easy to see that this definition does define a partition. We display the relevant diagrams below for clarity.





For what follows, we shall make use of the following fact that can be found in [34].

Theorem 5.1.1 *Let ρ be a partition contained in the $a \times b$ rectangle (b^a), $\kappa \subset \rho$ be a second partition. Then the skew diagram ρ/κ satisfies*

$$s_{\rho/\kappa} = \sum_{\nu \subseteq (b^a)} c^{\nu}_{\kappa\rho^c} s_{\nu^c},$$

where $c^{\nu}_{\kappa\rho^c}$ are the Littlewood-Richardson coefficients.

Given a symmetric function $f = \sum_\nu a_\nu s_\nu$ we define the *truncated complement of f in the rectangle (b^a)* as

$$c(f) = \sum_{\nu \subseteq (b^a)} a_\nu s_{\nu^c}. \tag{5.1}$$

The rectangle being used should be clear from the context if it is not specifically mentioned.

We note that one effect of the restriction $\nu \subseteq (b^a)$ is that in passing from f to $c(f)$, we are reducing computations to a variables. This follows since any term s_ν from f with $l(\nu) > a$ is effectively being set to zero, which is precisely what happens when reducing f to a variables.

This will not cause us any difficulties, since for each skew Schur function $s_{\rho/\kappa}$ that we wish to compute, we will always choose an appropriate rectangle (b^a) with $\rho/\kappa \subseteq (b^a)$ and, when computing

$$s_{\rho/\kappa} = \sum_{\nu} c^{\rho}_{\kappa\nu} s_\nu,$$

it is easy to see that each ν must satisfy $\nu \subseteq (b^a)$. For instance, we can check that each ν in the above sum satisfies $l(\nu) \leq l(\rho/\kappa)$ since the Littlewood-Richardson rule implies that, for each SSYT T of shape ρ/κ and content ν with lattice reading word, the entries of the i-th row of T are $\leq i$ for each $i = 1, 2, \ldots, l(\rho/\kappa)$. Therefore $s_{\rho/\kappa}$ is completely determined by using only $l(\rho/\kappa)$ variables. Also, each ν in the above sum satisfies $w(\nu) \leq w(\rho/\kappa)$. If not, then $w(\nu) > w(\rho/\kappa)$. Then in particular $\nu_1 > w(\rho/\kappa)$, which implies that there is a SSYT of shape ρ/κ and content ν with lattice reading word using ν_1 1's. However, there is no way to fit $\nu_1 > w(\rho/\kappa)$ 1's in the shape ρ/κ while maintaining the semistandard conditions.

We may now restate Theorem 5.1.1 as follows.

Corollary 5.1.2 *Let ρ be a partition contained in the $a \times b$ rectangle (b^a), and $\kappa \subset \rho$ be a second partition. Then the skew diagram ρ/κ satisfies*

$$s_{\rho/\kappa} = c(s_\kappa s_{\rho^c}).$$

Proof From the definition of the Littlewood-Richardson numbers, we have

$$s_\kappa s_{\rho^c} = \sum_\nu c^\nu_{\kappa\rho^c} s_\nu.$$

Hence

$$c(s_\kappa s_{\rho^c}) = c\left(\sum_\nu c^\nu_{\kappa\rho^c} s_\nu\right) = \sum_{\nu \subseteq (b^a)} c^\nu_{\kappa\rho^c} s_{\nu^c}.$$

By Theorem 5.1.1, this is just $s_{\rho/\kappa}$, so we are done. ∎

One must be careful in working with the expression $c(s_\kappa s_{\rho^c})$. We must truncate any term s_ν we obtain from the product with $\nu \not\subseteq (b^a)$. Thus we must be mindful of the dimensions of the rectangle we are working in.

We are interested in extending the results of the previous chapters to differences of skew Schur functions $s_{\rho/\kappa}$ for more types of diagrams ρ/κ. In particular, we shall see how to generalize to fat staircases with bad foundations where the foundations are either complements of hooks or complements of two-row diagrams. Since $s_{\rho/\kappa} = c(s_\kappa s_{\rho^c})$, we shall begin by inspecting the symmetric function $c(s_\kappa s_{\rho^c})$.

Lemma 5.1.3 *For a partition λ, composition α, and $0 \leq k \leq 1$ we have*

$$c(s_\lambda s_{\Delta_\alpha}) = c(s_{\mathcal{S}(\lambda,\alpha^\triangleleft;k)}),$$

for any complementation in a rectangle of width $w = n + k$.

We note that in general $s_\lambda s_{\Delta_\alpha} \neq s_{\mathcal{S}(\lambda,\alpha^\triangleleft;k)}$. For instance, we know that $s_\lambda s_{\Delta_\alpha} = s_{\lambda\oplus\Delta_\alpha}$, and if λ has more than one column, then there exists SSYT of shape $\lambda \oplus \Delta_\alpha$ with lattice reading word for which the first row of λ has repeated 1's, but no SSYT of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ with lattice reading word can have the same content by Lemma 2.2.2.

Proof (of Lemma 5.1.3)

Let the rectangle be (w^l), say. We begin by comparing $c(s_{\lambda\oplus\Delta_\alpha})$ and $c(s_{\mathcal{S}(\lambda,\alpha^\triangleleft;k)})$.

Consider a content ν that contributes to $c(s_{\lambda\oplus\Delta_\alpha})$. Then $\nu \subseteq (w^l)$ and there is a SSYT $\mathcal{T}$ of shape $\lambda \oplus \Delta_\alpha$ and content ν^c with lattice reading

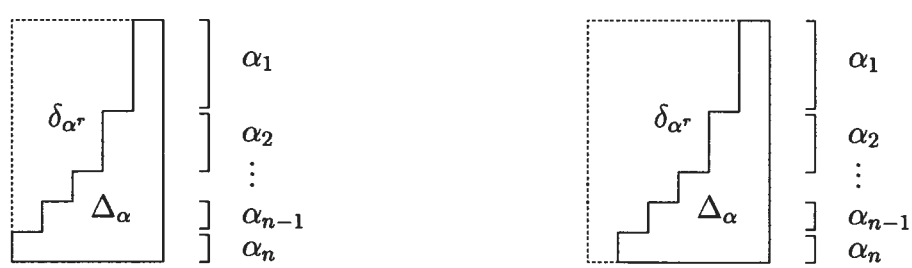
word. Since ν^c is contained in a rectangle of width w, $\mathcal{T}$ contains at most $w = n + k$ 1's. We know that exactly n 1's appear in the copy of Δ_α. Thus the copy of λ contains at most k 1's. Therefore, by Lemma 2.2.3, we can obtain a SSYT $\mathcal{T}'$ of shape $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ with lattice reading word of content ν^c by simply filling the entries of λ in $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$ identically to the filling of λ in $\mathcal{T}$. This correspondence, $\mathcal{T} \mapsto \mathcal{T}'$ gives a bijection. That is, $[s_{\nu^c}](s_{\lambda \oplus \Delta_\alpha}) = [s_{\nu^c}](s_{\mathcal{S}(\lambda, \alpha^\triangleleft; k)})$ for all $\nu^c \subseteq (w^l)$. This gives $c(s_{\lambda \oplus \Delta_\alpha}) = c(s_{\mathcal{S}(\lambda, \alpha^\triangleleft; k)})$.

We also have $s_{\lambda \oplus \Delta_\alpha} = s_\lambda s_{\Delta_\alpha}$, and so $c(s_{\lambda \oplus \Delta_\alpha}) = c(s_\lambda s_{\Delta_\alpha})$. Thus we have $c(s_\lambda s_{\Delta_\alpha}) = c(s_{\mathcal{S}(\lambda, \alpha^\triangleleft; k)})$, as desired. ∎

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a width $w = n + k$, where $0 \leq k \leq 1$, we let

$$\alpha^r = \begin{cases} (\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1) & \text{if} \quad k = 1 \\ (\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_2, \alpha_1) & \text{if} \quad k = 0 \end{cases}$$

denote the *reverse composition.* With this definition we have ${\delta_\alpha}^c = \delta_{\alpha^r}$, where the complement is performed in the rectangle $(w^{|\alpha|})$. We illustrate the two cases $k = 0$ and $k = 1$ below.



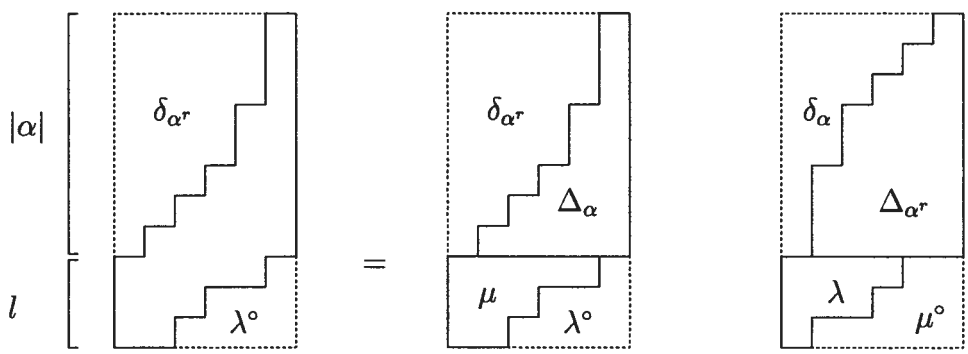
$k = 0$ $\qquad$ $k = 1$

Thus we have $l(\delta_{\alpha^r}) \leq l(\delta_\alpha)$ and $w(\delta_{\alpha^r}) \leq w(\delta_\alpha)$. In particular, we have $|\alpha| = |\alpha^r| + (1 - k)\alpha_n$.

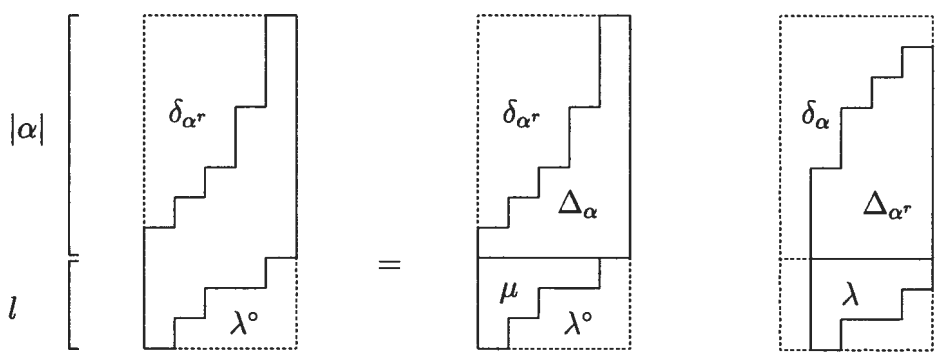
Theorem 5.1.4 *Let α be a composition, $w = n + k$ where $0 \leq k \leq 1$, $l \geq 1$, and ρ be a partition with $|\alpha| + l$ parts such that $(w^{|\alpha|}) \subset \rho \subset (w^{|\alpha|+l})$, $\mu = (\rho_{|\alpha|+1}, \rho_{|\alpha|+2}, \ldots, \rho_{|\alpha|+l})$, and $\lambda = \rho^c$ be the complement of ρ in $(w^{|\alpha|+l})$. Then $\rho/\delta_{\alpha^r} = \mathcal{S}(\mu, \alpha^\triangleleft; k)$ and*

$$s_{\mathcal{S}(\mu, \alpha^\triangleleft; k)} = c(s_{\mathcal{S}(\lambda, \alpha^{r\triangleleft}; k)}).$$

Proof We are interested in the following diagrams, each contained in the rectangle (w^l). The first set of diagrams illustrates the case $k = 1$ and the second set illustrates the case $k = 0$.



ρ/δ_{α^r} $\quad$ $\mathcal{S}(\mu, \alpha^\triangleleft; 1)$ $\quad$ $\mathcal{S}(\lambda, \alpha^{r\triangleleft}; 1)$



ρ/δ_{α^r} $\quad$ $\mathcal{S}(\mu, \alpha^\triangleleft; 0)$ $\quad$ $\mathcal{S}(\lambda, \alpha^{r\triangleleft}; 0)$

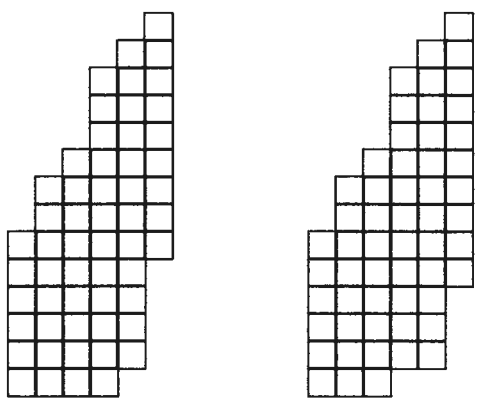
It is clear from the definition of μ that we have $\mathcal{S}(\mu, \alpha^\triangleleft; k) = \rho/\delta_{\alpha^r}$. Therefore we obtain

$$\begin{aligned} s_{\mathcal{S}(\mu,\alpha^\triangleleft;k)} &= s_{\rho/\delta_{\alpha^r}} \\ &= c(s_\lambda s_{\delta_{\alpha^r}}) \text{ by Corollary 5.1.2} \\ &= c(s_\lambda s_{\Delta_{\alpha^r}}) \\ &= c(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)}) \text{ by Lemma 5.1.3,} \end{aligned}$$

which is what we wanted to prove. ∎

5.2 Staircases with Hook Complement Foundations

We now consider two hooks λ, μ both contained in a rectangle (w^l) and let λ^c and μ^c denote their complements in this rectangle. We call these *hook complements.* For a fat staircase Δ_α, we now inspect when the difference $s_{\mathcal{S}(\lambda^c,\alpha^\triangleleft;k)} - s_{\mathcal{S}(\mu^c,\alpha^\triangleleft;k)}$ is Schur-positive. Thus we are interested in the differences of skew Schur functions for pairs of diagrams such as the pair displayed below.



Our first result, Theorem 5.2.1 states that we obtain the same Hasse diagram for fat staircases with hook complement foundations as was obtained for fat staircases with hook foundations.

Recall that for partitions λ and μ we define $\lambda \cup \mu$ to be the partition consisting of parts $\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots$ placed in weakly decreasing order. We shall be interested in the case when the first partition is the rectangle (w^n) and the second partition κ has $w(\kappa) \leq n$. In this case $(w^n) \cup \kappa = (w^n, \kappa_1, \kappa_2, \ldots)$ is obtained by merely appending κ to the rectangle (w^n).

Theorem 5.2.1 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n + k = w$ and let $0 \leq k \leq 1$. Then $\mathcal{S}(\lambda^c, \alpha^\triangleleft; k) \succeq_s \mathcal{S}(\mu^c, \alpha^\triangleleft; k)$ if and only if $\mathcal{S}(\lambda, \alpha^{r\triangleleft}; k) \succeq_s \mathcal{S}(\mu, \alpha^{r\triangleleft}; k)$.*

Proof We wish to apply Theorem 5.1.4 to both diagrams. To this end, we need to choose appropriate partitions ρ. We let $\rho(\lambda^c) := (w^{|\alpha|}) \cup \lambda^c$ and $\rho(\mu^c) := (w^{|\alpha|}) \cup \mu^c$ Then we have $(w^{|\alpha|}) \subset \rho(\lambda^c), \rho(\mu^c) \subset (w^{|\alpha|+l})$ so we may apply Theorem 5.1.4 to both $\rho(\lambda^c)/\delta_{\alpha^r} = \mathcal{S}(\lambda^c, \alpha^\triangleleft; k)$ and $\rho(\mu^c)/\delta_{\alpha^r} = \mathcal{S}(\mu^c, \alpha^\triangleleft; k)$. This gives

$$s_{\mathcal{S}(\lambda^c,\alpha^\triangleleft;k)} - s_{\mathcal{S}(\mu^c,\alpha^\triangleleft;k)} \quad = \quad c(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)}) - c(s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)})$$

$$= \quad c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big),$$

where these complements are performed in the rectangle $(w^{|\alpha|+l})$.

Now, if $\mathcal{S}(\lambda,\alpha^{r\triangleleft};k) \succeq_s \mathcal{S}(\mu,\alpha^{r\triangleleft};k)$, then the above equation shows that $\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k) \succeq_s \mathcal{S}(\mu^c,\alpha^{\triangleleft};k)$ as well.

For the converse direction, suppose that $\mathcal{S}(\lambda,\alpha^{r\triangleleft};k) \not\succeq_s \mathcal{S}(\mu,\alpha^{r\triangleleft};k)$. Thus, by assumption, the difference $s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}$ is not Schur-positive. However, we need to verify that the truncated version $c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big)$ is also not Schur-positive.

In Section 3.1, we saw that the only cases where the difference was not Schur-positive among diagrams of this type were those in cases covered by Theorem 3.1.1, Theorem 3.1.2, and Theorem 3.1.5. Thus λ and μ must satisfy the hypotheses of one of these three theorems. In each of these three theorems, by inspecting a particular term s_ν in the difference, it was proved that the difference was not Schur-positive. We need only check that for each theorem the partition ν satisfies $\nu \subseteq (w^{|\alpha|+l})$.

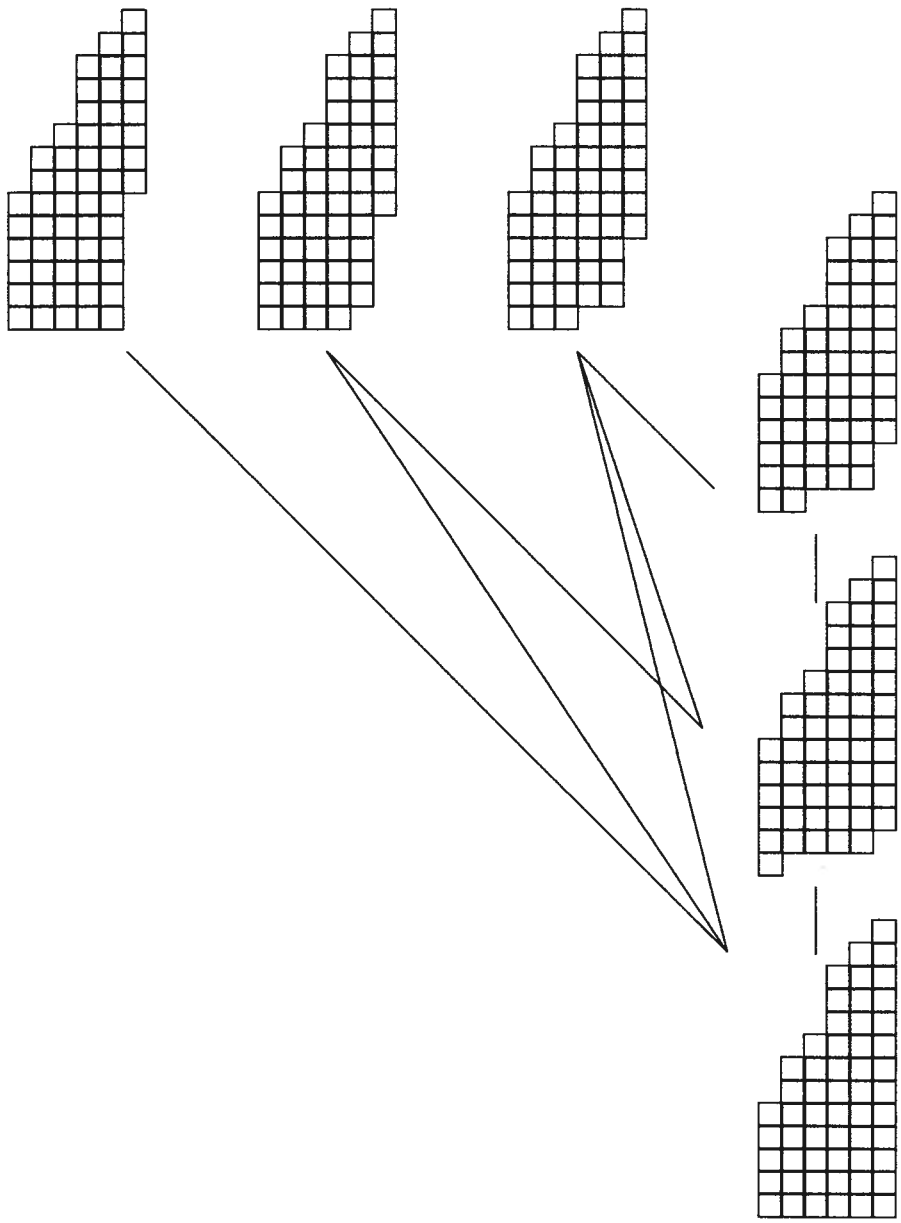
In both Theorem 3.1.2 and Theorem 3.1.5 we used $\nu = \delta_{\alpha^r} + \sum_{i=1}^{h} e_{r_i}$, where $r_1 < r_2 < \cdots$ are the values of $R_{\alpha^r,k}$. We have $w(\delta_{\alpha^r}) \leq w(\delta_\alpha) = n$. Further, since the r_i are distinct, adding the terms $\sum_{i=1}^{h} e_{r_i}$ to δ_{α^r} can only increase the width by 1, and this only happens when $r_1 = 1$ which implies that $k = 1$. Thus, in either case, $w(\nu) \leq n+k = w$. Also, since $l(\delta_{\alpha^r}) \leq l(\delta_\alpha) = |\alpha|$ and each $r_i \leq |\alpha|+1$, we have $l(\nu) \leq |\alpha|+1 \leq |\alpha|+l$. Therefore ν is contained in the rectangle $(w^{|\alpha|+l})$.

In Theorem 3.1.1 we used $\nu = \delta_{\alpha^r} + \sum_{i=1}^{\lambda_a-1} e_{r_i} + (0^{|\alpha^r|}, 1^{\lambda_l})$, where $r_1 < r_2 < \cdots$ are the values of $R_{\alpha^r,k}$. As in the previous case we find that $w(\nu) \leq n+k = w$. For the length of ν we have $l(\nu) = |\alpha^r| + \lambda_l \leq |\alpha| + l$, since $|\alpha^r| \leq |\alpha|$ and $\lambda \subseteq (w^l)$. Therefore ν is contained in the rectangle $(w^{|\alpha|+l})$.

Thus, in each case ν is contained in the rectangle $(w^{|\alpha|+l})$. Therefore the term s_ν in the difference $s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}$, is also in the difference $c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big)$. Since this term has a negative coefficient, it shows that $c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big)$, and hence $s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)}$ is not Schur-positive. That is, $\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k) \not\succeq_s \mathcal{S}(\mu^c,\alpha^{\triangleleft};k)$. This completes the converse direction.

Therefore we have shown that $\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k) \succeq_s \mathcal{S}(\mu^c,\alpha^{\triangleleft};k)$ if and only if $\mathcal{S}(\lambda,\alpha^{r\triangleleft};k) \succeq_s \mathcal{S}(\mu,\alpha^{r\triangleleft};k)$. ∎

Example Here we see the Hasse diagram obtained by considering all diagrams of the form $\mathcal{S}(\lambda^c, \alpha^{\triangleleft}; k)$ where $\alpha = (1,1,3,1,2)$, $k = 1$, and λ is a hook of size 6, where λ^c is computed in the rectangle (6^6).



We now wish to calculate the difference $s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)}$, for hook diagrams λ and μ, in the cases when this difference is Schur-positive. First we look at the case when $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$, which was explored in Theorem 3.2.2. As before, we let λ^c and μ^c denote the complements in (w^l).

Theorem 5.2.2 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n+k = w$ and $\lceil \frac{h}{2} \rceil \leq \lambda_a < \mu_a$ and let $0 \leq k \leq 1$. Then*

$$s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)} = \sum_{B,C} a^r_{B,C} s_{\eta(B,C)}$$

where the coefficients $a^r_{B,C}$ are given by

$$a^r_{B,C} = \binom{|(B-C)\cap R_{\alpha^r,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha^r,k}|-1}{\mu_a - |C| - 1},$$

the partitions $\eta(B,C)$ that arise are given by

$$\begin{aligned}\eta(B,C) &= (n+k^{l-1+|\lambda|+|B|+|C|+(1-k)\alpha_{n+k}}, n+k-1^{|\lambda|-|B|-|C|+\alpha_{n+k-1}}, \\ &\quad n+k-2^{\alpha_{n+k-2}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}) \\ &\quad - \sum_{b\in B} e_{|\alpha|+1+l-b} - \sum_{c\in C} e_{|\alpha|+1+l-c},\end{aligned}$$

and the sum is over all sets B, C such that

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha^r|+1\}$, $C \subseteq B \cap R_{\alpha^r,k} - \min(B)$,
- $|C|+1 \leq \lambda_l$,
- $\lambda_a \leq |B \cap R_{\alpha^r,k}|$,
- $|B|+|C| \leq h$, *and* $|\alpha^r|+1 \in B$ *if* $|B|+|C| \leq h-1$,
- *if* $B = \bigcup_{j=1}^{p} B_j$ *where the* B_j *are the maximal disjoint intervals of* B, *then* $\min(B_j) \in R_{\alpha^r,k}$ *for each* j, *and*
- *if* $C = \bigcup_{j=1}^{m} C_j$ *where the* C_j *are the maximal disjoint intervals of* C, *then* $\min(C_j) - 1 \in B - C$ *for each* j.

We note that the term α_{n+k} in $\eta(B,C)$ is not defined for $k=1$, but this causes no problem in computing the above expression since the only time the term α_{n+k} appears is as part of the term $(1-k)\alpha_{n+k}$, which is zero for $k=1$.

Proof (of Theorem 5.2.2)

We wish to apply Theorem 5.1.4 to both diagrams. To this end, we let $\rho(\lambda^c) := (w^{|\alpha|}) \cup \lambda^c$ and $\rho(\mu^c) := (w^{|\alpha|}) \cup \mu^c$ Then we have $(w^{|\alpha|}) \subset$

$\rho(\lambda^c), \rho(\mu^c) \subset (w^{|\alpha|+l})$ so we may apply Theorem 5.1.4 to both $\rho(\lambda^c)/\delta_{\alpha^r} = \mathcal{S}(\lambda^c, \alpha^{\triangleleft}; k)$ and $\rho(\mu^c)/\delta_{\alpha^r} = \mathcal{S}(\mu^c, \alpha^{\triangleleft}; k)$. This gives

$$\begin{aligned} s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)} &= c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)}\big) - c\big(s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big) \\ &= c\big(s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}\big), \end{aligned}$$

where these complements are performed in the rectangle $(w^{|\alpha|+l})$.

Since $s_{\mathcal{S}(\lambda,\alpha^{r\triangleleft};k)} - s_{\mathcal{S}(\mu,\alpha^{r\triangleleft};k)}$ is just Theorem 3.2.2 with α replaced by α^r, we have

$$s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)} = c\left(\sum_{B,C} a^r_{B,C} s_{\nu(B,C)}\right).$$

That is,

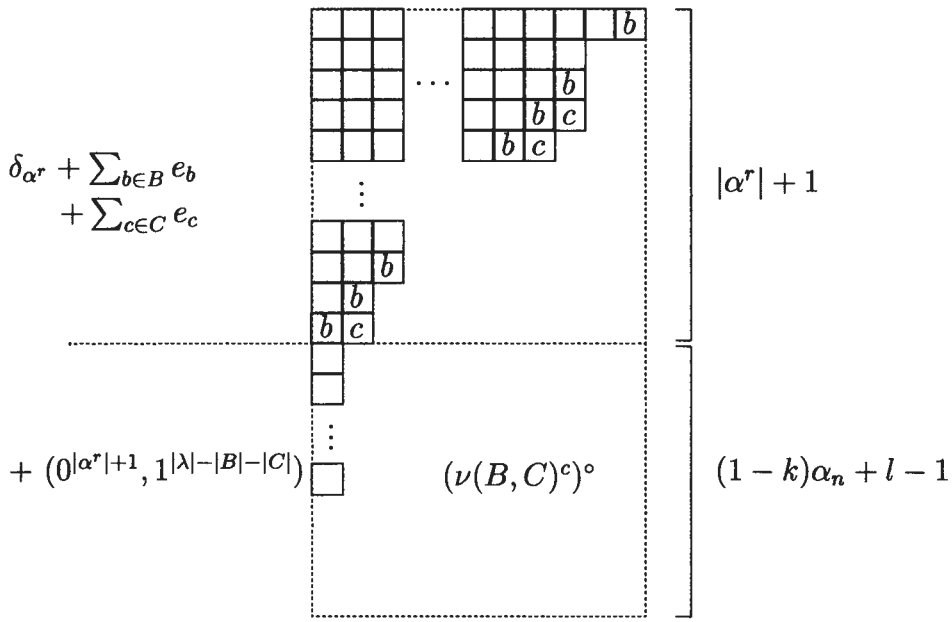
$$s_{\mathcal{S}(\lambda^c,\alpha^{\triangleleft};k)} - s_{\mathcal{S}(\mu^c,\alpha^{\triangleleft};k)} = \sum_{B,C} a^r_{B,C} s_{\nu(B,C)^c}, \tag{5.2}$$

where $\nu(B,C) = \delta_{\alpha^r} + \sum_{b\in B} e_b + \sum_{c\in C} e_c + (0^{|\alpha^r|+1}, 1^{|\lambda|-|B|-|C|})$ and and the sum is over all sets B, C such that

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha^r|+1\}$, $C \subseteq B \cap R_{\alpha^r,k} - \min(B)$,
- $|C| + 1 \leq \lambda_l$,
- $\lambda_a \leq |B \cap R_{\alpha^r,k}|$,
- $|B| + |C| \leq h$, and $|\alpha^r| + 1 \in B$ if $|B| + |C| \leq h - 1$,
- if $B = \bigcup_{j=1}^{p} B_j$ where the B_j are the maximal disjoint intervals of B, then $\min(B_j) \in R_{\alpha^r,k}$ for each j, and
- if $C = \bigcup_{j=1}^{m} C_j$ where the C_j are the maximal disjoint intervals of C, then $\min(C_j) - 1 \in B - C$ for each j.

For sets B and C satisfying the above properties, we display the partition $\nu(B,C)$ in the diagram below. We split the rectangle $(w^{|\alpha|+l}) = (w^{|\alpha^r|+(1-k)\alpha_n+l})$ into sections of length $|\alpha^r|+1$ and $(1-k)\alpha_n + l - 1$. The length $|\alpha^r|+1$ is a convenient choice since the sets B and C only contribute to the first $|\alpha^r|+1$ parts of $\nu(B,C)$. In the depiction of $\nu(B,C)$, the staircase in the top-left is Δ_{α^r}, which arises from the filling of δ_{α^r}. The boxes $\boxed{b}$ and $\boxed{c}$ denote the boxes that arise in $\nu(B,C)$ from filling the foundation with the terms of $\sum_{b\in B} e_b$ and $\sum_{c\in C} e_c$. The column at the bottom represents the values from the term $(0^{|\alpha^r|+1}, 1^{|\lambda|-|B|-|C|})$.

Since this diagram represents $\nu(B,C)$, the blank space in the bottom-right corner of the diagram represents $(\nu(B,C)^c)^\circ$.



Consider the case when $k = 1$. Then these two rectangles are of size $(n + 1^{|\alpha|+1})$ and $(n + 1^{l-1})$, respectively. Rotating the blank space in the $(n + 1^{l-1})$ rectangle gives the partition

$$(n + 1^{l-1-|\lambda|+|B|+|C|}, n^{|\lambda|-|B|-|C|}).$$

Rotating the blank space in the $(n + 1^{|\alpha|+1})$ rectangle gives the partition

$$\delta_\alpha - \sum_{b\in B} e_{|\alpha|+2-b} - \sum_{c\in C} e_{|\alpha|+2-c}.$$

Thus we find

$$\begin{aligned}\nu(B,C)^c &= (n + 1^{l-1-|\lambda|+|B|+|C|}, n^{|\lambda|-|B|-|C|}) \\ &\quad \cup (\delta_\alpha - \sum_{b\in B} e_{|\alpha|+2-b} - \sum_{c\in C} e_{|\alpha|+2-c}).\end{aligned}$$

Hence

$$\begin{aligned}\nu(B,C)^c &= (n + 1^{l-1+|\lambda|+|B|+|C|}, n^{|\lambda|-|B|-|C|+\alpha_n}, n-1^{\alpha_{n-1}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}) \\ &\quad - \sum_{b\in B} e_{|\alpha|+1+l-b} - \sum_{c\in C} e_{|\alpha|+1+l-c}.\end{aligned}$$

Now consider the case when $k = 0$. Then these two rectangles are of size $(n^{|\alpha|-\alpha_n+1})$ and (n^{α_n+l-1}), respectively. Rotating the blank space in the (n^{α_n+l-1}) rectangle gives the partition

$$(n^{\alpha_n+l-1-|\lambda|+|B|+|C|}, n-1^{|\lambda|-|B|-|C|}).$$

Rotating the blank space in the $(n^{|\alpha|-\alpha_n+1})$ rectangle gives the partition

$$\delta_{(\alpha_1,\ldots,\alpha_{n-1})} - \sum_{b\in B} e_{|\alpha|+2-b} - \sum_{c\in C} e_{|\alpha|+2-c}.$$

Thus we find

$$\begin{aligned}\nu(B,C)^c &= (n^{\alpha_n+l-1-|\lambda|+|B|+|C|}, n-1^{|\lambda|-|B|-|C|}) \\ &\quad \cup (\delta_{(\alpha_1,\ldots,\alpha_{n-1})} - \sum_{b\in B} e_{|\alpha|+2-b} - \sum_{c\in C} e_{|\alpha|+2-c}).\end{aligned}$$

Hence

$$\begin{aligned}\nu(B,C)^c &= (n^{l-1+|\lambda|+|B|+|C|+\alpha_n}, n-1^{|\lambda|-|B|-|C|+\alpha_{n-1}}, n-2^{\alpha_{n-2}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}) \\ &\quad - \sum_{b\in B} e_{|\alpha|+1+l-b} - \sum_{c\in C} e_{|\alpha|+1+l-c}.\end{aligned}$$

For both $k = 1$ and $k = 0$ we can rewrite these expressions as

$$\begin{aligned}\nu(B,C)^c &= (n+k^{l-1+|\lambda|+|B|+|C|+(1-k)\alpha_{n+k}}, n+k-1^{|\lambda|-|B|-|C|+\alpha_{n+k-1}}, \\ &\qquad n+k-2^{\alpha_{n+k-2}}, \ldots, 2^{\alpha_2}, 1^{\alpha_1}) \\ &\quad - \sum_{b\in B} e_{|\alpha|+1+l-b} - \sum_{c\in C} e_{|\alpha|+1+l-c} \\ &= \eta(B,C).\end{aligned}$$

Substituting this expression into Equation 5.2 gives the formula stated for $s_{\mathcal{S}(\lambda^c,\alpha^\triangleleft;k)} - s_{\mathcal{S}(\mu^c,\alpha^\triangleleft;k)}$. ∎

We now look at the case when $\lambda_a, \mu_l < \lceil \frac{h}{2} \rceil$, which was explored in Theorem 3.2.3.

Theorem 5.2.3 *Let λ and μ be hooks with $|\lambda| = |\mu| = h \leq n+k = w$ and $\lambda_a, \mu_l < \lceil \frac{h}{2} \rceil$, and let $0 \leq k \leq 1$. If $\lambda_a \geq \mu_l$, then*

$$s_{\mathcal{S}(\lambda^c,\alpha^\triangleleft;k)} - s_{\mathcal{S}(\mu^c,\alpha^\triangleleft;k)} = \sum_{B,C} a^{\tau}_{B,C} s_{\eta(B,C)}$$

where the coefficients $a^r_{B,C}$ *are given by*

$$a^r_{B,C} = \binom{|(B-C)\cap R_{\alpha^r,k}|-1}{\lambda_a - |C| - 1} - \binom{|(B-C)\cap R_{\alpha^r,k}|-1}{\mu_a - |C| - 1},$$

the partitions $\eta(B,C)$ *that arise are given by*

$$\begin{aligned}\eta(B,C) &= (n+k^{l-1+|\lambda|+|B|+|C|+(1-k)\alpha_{n+k}}, n+k-1^{|\lambda|-|B|-|C|+\alpha_{n+k-1}},\\ &\quad n+k-2^{\alpha_{n+k-2}},\ldots,2^{\alpha_2},1^{\alpha_1})\\ &\quad -\sum_{b\in B} e_{|\alpha|+1+l-b} - \sum_{c\in C} e_{|\alpha|+1+l-c},\end{aligned}$$

and the sum is over all sets B, C *such that*

- $B \subseteq \{2-k, 3-k, \ldots, |\alpha^r|+1\}$, $C \subseteq B \cap R_{\alpha^r,k} - \min(B)$,
- $|C|+1 \le \lambda_a \le |B \cap R_{\alpha^r,k}|$,
- $|B|+|C| \le h$, *and* $|\alpha^r|+1 \in B$ *if* $|B|+|C| \le h-1$,
- *if* $B = \bigcup_{j=1}^p B_j$ *where the* B_j *are the maximal disjoint intervals of* B, *then* $\min(B_j) \in R_{\alpha^r,k}$ *for each* j, *and*
- *if* $C = \bigcup_{j=1}^m C_j$ *where the* C_j *are the maximal disjoint intervals of* C, *then* $\min(C_j) - 1 \in B - C$ *for each* j.

Proof The proof of this theorem proceeds identically to the proof of Theorem 5.2.2 but uses Theorem 3.2.3 in place of Theorem 3.2.2. ∎

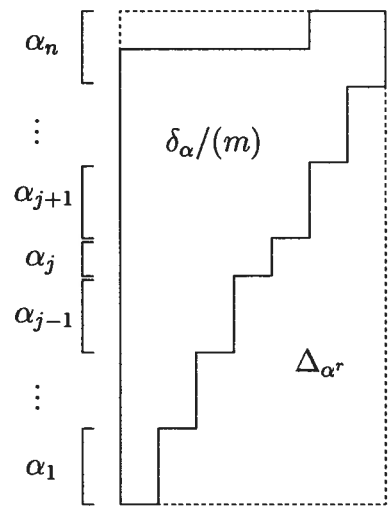
5.3 Sums of Fat Staircases

By using complements in a rectangle we now describe when a diagram of the form δ_α/λ is a sum of fat staircases in the cases when λ is either a single row or a single column.

Theorem 5.3.1 *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be a composition. Then for* $1 \le m < n$ *the skew diagram* $\delta_\alpha/(m)$ *is a sum of fat staircases if and only if* $\alpha_j > 1$ *for each* $j = 1, 2, \ldots, n-1$.

Theorem 5.3.2 *Let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be a composition. Then for* $1 \le m < |\alpha|$ *the skew diagram* $\delta_\alpha/(1^m)$ *is a sum of fat staircases if and only if there is no* j *such that* $\alpha_j \le m \le |\alpha| - \alpha_{j+1}$.

Proof (of Theorem 5.3.1) The diagram $\delta_\alpha/(m)$ is contained in the rectangle $(n^{|\alpha|})$. Using Corollary 5.1.2 we obtain $s_{\delta_\alpha/(m)} = c(s_{(m)}s_{\delta_{\alpha^r}})$. We note that $\alpha^r = (\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1)$ since the width of the rectangle is n.



We can compute the product $s_{(m)}s_{\delta_{\alpha^r}}$ using Theorem 1.6.1 (Pieri's rule). Thus we have

$$s_{(m)}s_{\delta_{\alpha^r}} = \sum_\lambda s_\lambda,$$

where the sum is over all partitions λ such that $\lambda/\delta_{\alpha^r}$ is a row strip with m boxes. That is, the sum is over all partitions λ such that the skew diagram $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same column.

Since we have

$$\begin{aligned} s_{\delta_\alpha/(m)} &= c(s_{(m)}s_{\delta_{\alpha^r}}) \\ &= c\left(\sum_\lambda s_\lambda\right), \end{aligned}$$

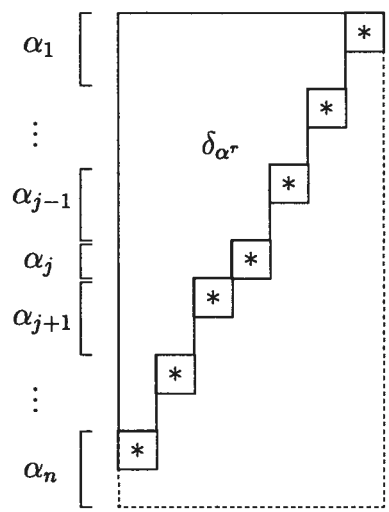
we obtain

$$s_{\delta_\alpha/(m)} = \sum_{\lambda \subseteq (n^{|\alpha|})} s_{\lambda^c}, \tag{5.3}$$

where the sum is over all partitions $\lambda \subseteq (n^{|\alpha|})$ such that the skew diagram $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same column.

Suppose there is a j, where $1 \leq j \leq n-1$, such that $\alpha_j = 1$. We intend to show that $\delta_\alpha/(m)$ is not a sum of fat staircases. Namely, we will show that there is a term s_ν in Equation 5.3 where ν is not a fat staircase.

Since δ_{α^r} is a fat staircase and we are constrained by $\lambda \subseteq (n^{|\alpha|})$, there are only n possible positions to place the m boxes. Namely, there is one position in the r-th row for each $r \in R_{\alpha^r,1}$. These n possible positions are shown in the diagram below.



Since we must place $m \geq 1$ boxes, where $m < n$, we choose to place a box at the end of the row of length α_j and place the other $m-1 \leq n-2$ boxes in any of the other possible positions *except for the position at the end of the first row of length* α_{j+1}. This gives a partition $\lambda \subseteq (n^{|\alpha|})$ such that $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same column. Further, by the choice of placement of these boxes we see that λ is not a fat staircase. Therefore λ^c is also not a fat staircase. Since this term arises in Equation 5.3, this shows that $\delta_\alpha/(m)$ is not a sum of fat staircases. This completes the first half of the proof.

Now suppose that $\alpha_j > 1$ for each $j = 1, 2, \ldots, n-1$. We intend to show that $\delta_\alpha/(m)$ is a sum of fat staircases.

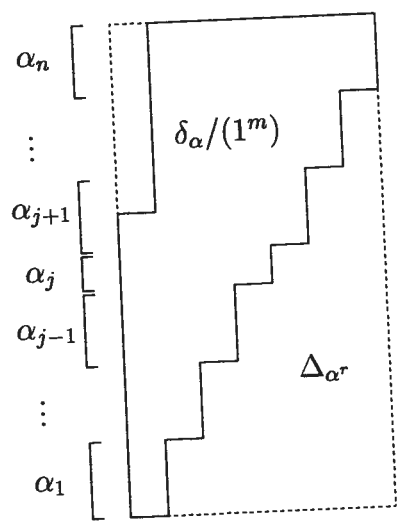
As in the previous case, any term in Equation 5.3 arises from placing the m boxes among those n possible positions. Since $\alpha_j > 1$ for each $j = 1, 2, \ldots, n-1$, it is clear that each of these placements results in a fat staircase. Therefore $\delta_\alpha/(m)$ is a sum of fat staircases, as claimed. ∎

Example Consider the skew diagram $\delta_{(2,2,2)}/(2) = (3,3,2,2,1,1)/(2)$. Since

$\alpha = (2,2,2)$ satisfies the hypotheses of Theorem 5.3.1, $s_{\delta_{(2,2,2)}/(2)}$ is a sum of fat staircases. In particular, we have

$$\begin{aligned} s_{\delta_{(2,2,2)}/(2)} &= s_{(3,3,2,1,1)} + s_{(3,2,2,2,1)} + s_{(3,2,2,1,1,1)} \\ &= s_{\delta_{(2,1,2)}} + s_{\delta_{(1,3,1)}} + s_{\delta_{(3,2,1)}}. \end{aligned}$$

Proof (of Theorem 5.3.2) Again, the $\delta_\alpha/(1^m)$ is contained in the rectangle $(n^{|\alpha|})$. Using Corollary 5.1.2 we obtain $s_{\delta_\alpha/(1^m)} = c(s_{(1^m)} s_{\delta_{\alpha^r}})$.



We can compute the product $s_{(1^m)} s_{\delta_{\alpha^r}}$ using Theorem 1.6.2 (Pieri's rule). Thus we have

$$s_{(1^m)} s_{\delta_{\alpha^r}} = \sum_\lambda s_\lambda,$$

where the sum is over all partitions λ such that $\lambda/\delta_{\alpha^r}$ is a column strip with m boxes. That is, the sum over all partitions λ such that the skew diagram $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same row.

Since

$$\begin{aligned} s_{\delta_\alpha/(1^m)} &= c(s_{(1^m)} s_{\delta_{\alpha^r}}) \\ &= c\left(\sum_\lambda s_\lambda\right), \end{aligned}$$

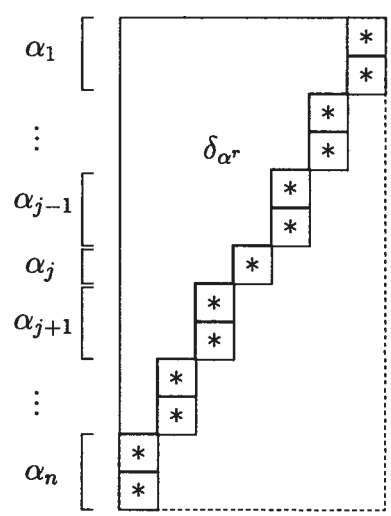
we obtain

$$s_{\delta_\alpha/(1^m)} = \sum_{\lambda \subseteq (n^{|\alpha|})} s_{\lambda^c}, \tag{5.4}$$

where the sum is over all partitions $\lambda \subseteq (n^{|\alpha|})$ such that the skew diagram $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same row.

Suppose there is a j such that $\alpha_j \leq m \leq |\alpha| - \alpha_{j+1}$. Then, necessarily, $1 \leq j \leq n-1$. We intend to show that $\delta_\alpha/(1^m)$ is not a sum of fat staircases. Namely, we will show that there is a term s_ν in Equation 5.4 where ν is not a fat staircase.

Since δ_{α^r} is a fat staircase and we are constrained by $\lambda \subseteq (n^{|\alpha|})$, the only possible positions to place the m boxes are among the $|\alpha|$ positions shown below. For the remainder of the proof we shall refer to the columns of this column strip of possible positions as *possible columns.* We note that when placing boxes in these possible columns, we must start from the top in order for the resulting diagram to be weakly decreasing in row lengths.



Since we must place $m \geq \alpha_j$ boxes, where $m \leq |\alpha| - \alpha_{j+1}$, we choose to place a box at the end of each row of length α_j and place the other $m - \alpha_j \leq |\alpha| - \alpha_{j+1} - \alpha_j$ boxes in any of the other possible positions *except for the positions at the end of the rows of length* α_{j+1}. This gives a partition $\lambda \subseteq (n^{|\alpha|})$ such that $\lambda/\delta_{\alpha^r}$ has m boxes, no two of which are in the same row. Further, by the choice of placement of these boxes we see that λ is not a fat staircase. Therefore λ^c is also not a fat staircase. Since this term arises in Equation 5.4, this shows that $\delta_\alpha/(1^m)$ is not a sum of fat staircases. This completes the first half of the proof.

Now suppose that there is no j such that $\alpha_j \leq m \leq |\alpha| - \alpha_{j+1}$. We intend to show that $\delta_\alpha/(1^m)$ is a sum of fat staircases.

As in the previous case, any term in Equation 5.4 arises from placing the m boxes among those $|\alpha|$ possible positions. Since no j satifies $\alpha_j \leq m \leq |\alpha| - \alpha_{j+1}$, there is no way to place these m boxes so that a particular possible column is filled yet the possible column immediately to the left is empty. Since such a placement is the *only* way to obtain a partition that is not a fat staircase, this shows that $\delta_\alpha/(1^m)$ is a sum of fat staircases. ∎

Example Consider the skew diagram given by

$$\delta_{(3,3,3)}/(1,1) = (3,3,3,2,2,2,1,1,1)/(1,1).$$

Since $\alpha = (3,3,3)$ satisfies the hypotheses of Theorem 5.3.2, $s_{\delta_{(3,3,3)}/(1,1)}$ is a sum of fat staircases. In particular, we have

$$\begin{aligned} s_{\delta_{(3,3,3)}/(1,1)} &= s_{(3,3,3,2,2,2,1)} + s_{(3,3,3,2,2,1,1,1)} + s_{(3,3,3,2,1,1,1,1,1)} + s_{(3,3,2,2,2,2,1,1)} \\ &\quad + s_{(3,3,2,2,2,1,1,1,1)} + s_{(3,2,2,2,2,2,1,1,1)} \\ &= s_{\delta_{(1,3,3)}} + s_{\delta_{(3,2,3)}} + s_{\delta_{(5,1,3)}} + s_{\delta_{(2,4,2)}} + s_{\delta_{(4,3,2)}} + s_{\delta_{(3,5,1)}}. \end{aligned}$$

5.4 Miscellaneous Results

In Theorem 4.1.1 we saw that when λ had only one part the difference $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ was multiplicity-free. The same can now be said of the difference $s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)}$, where the complementation is performed in a rectangle (w^l) for $w = n + k$.

Theorem 5.4.1 *Let λ be a partition with $l(\lambda) = 1$ and $\lambda_1 \leq l$, for some l, and $0 \leq k \leq 1$. Then*

$$s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)} = \sum_{\substack{A \subseteq \{2-k, 3-k, \ldots, n+k-1\} \\ |A| \leq \lambda_1 - 2}} s_{\bar{\nu}(A)},$$

where

$$\begin{aligned} \bar{\nu}(A) &= (n + k^{l+1-k-\lambda_1+|A|}, n+k-1^{\lambda_1-|A|+1}, n+k-2, n+k-3, \ldots, 2, 1) \\ &\quad - \sum_{a \in A} e_{l+1+n-a}. \end{aligned}$$

In particular, $s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)}$ is multiplicity-free.

Proof We wish to apply Theorem 5.1.4 to both diagrams. Since we are working with regular staircases, we have $\alpha = (1^n)$, which gives $\alpha^r = (1^{n-1+k})$.

We let $\rho(\lambda^{tc}) := (w^n) \cup \lambda^{tc}$ and $\rho(\lambda^c) := (w^n) \cup \lambda^c$. Then we have $(w^n) \subset \rho(\lambda^{tc}), \rho(\lambda^c) \subset (w^{n+l})$ so we may apply Theorem 5.1.4 to both of $\rho(\lambda^{tc})/\delta_{n-1+k} = \mathcal{S}(\lambda^{tc}, k, n)$ and $\rho(\lambda^c)/\delta_{n-1+k} = \mathcal{S}(\lambda^c, k, n)$. This gives

$$\begin{aligned} s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)} &= c\big(s_{\mathcal{S}(\lambda^t,k,n-1+k)}\big) - c\big(s_{\mathcal{S}(\lambda,k,n-1+k)}\big) \\ &= c\big(s_{\mathcal{S}(\lambda^t,k,n-1+k)} - s_{\mathcal{S}(\lambda,k,n-1+k)}\big) \\ &= c\left(\sum_{\substack{A \subseteq \{2-k,3-k,\ldots,n-1+k\} \\ |A| \leq \lambda_1 - 2}} s_{\nu(A)}\right), \end{aligned}$$

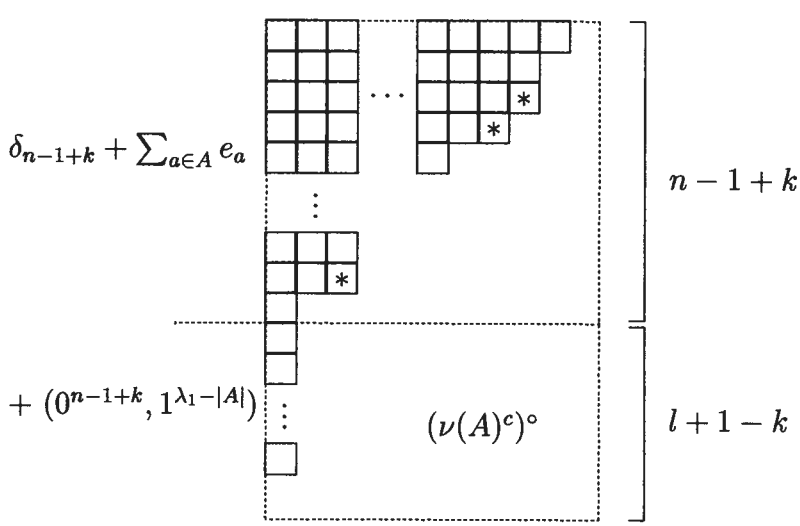
where the last step follows from Theorem 4.1.1.

Therefore

$$s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)} = \sum_{\substack{A \subseteq \{2-k,3-k,\ldots,n-1+k\} \\ |A| \leq \lambda_1 - 2}} s_{\nu(A)^c}, \tag{5.5}$$

where $\nu(A) = \delta_{n-1+k} + \sum_{a \in A} e_a + (0^{n-1+k}, 1^{\lambda_1 - |A|})$. This shows that the difference is multiplicity-free. We now wish to obtain an expression for the partitions $\nu(A)^c$.

For such A, we display $\nu(A)$ in the diagram below. We split the rectangle (w^{n+l}) into sections of length $n-1+k$ and $l+1-k$. The length $n-1+k$ is a convenient choice since the set A only contributes to the first $n-1+k$ parts of $\nu(A)$. In the depiction of $\nu(A)$, $\boxed{*}$ denotes a box that appears in $\nu(A)$ from a term of $\sum_{a \in A} e_a$. The blank space in the bottom-right corner of the diagram represents $(\nu(A)^c)^\circ$.

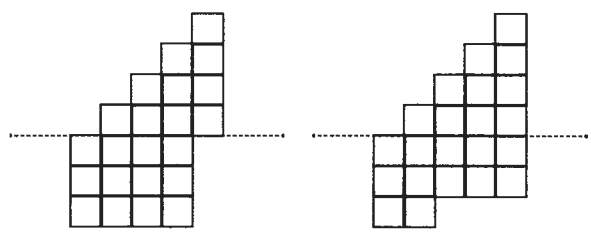


Thus we find

$$\begin{aligned}
\nu(A)^c &= \left(w^{l+1-k-\lambda_1+|A|}, w-1^{\lambda_1-|A|}\right) \cup \left(\delta_{n-1+k} - \sum_{a\in A} e_{n+k-a}\right) \\
&= \left(n+k^{l+1-k-\lambda_1+|A|}, n+k-1^{\lambda_1-|A|+1}, n+k-2, n+k-3, \ldots, 1\right) \\
&- \sum_{a\in A} e_{l+1+n-a}.
\end{aligned}$$

Substituting this expression into Equation 5.5 gives the formula stated for $s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)}$. ∎

Example Let $n = 4$, $l = 3$, $k = 1$, and $\lambda = (3)$. Then $\lambda^c = (5,5,2)$ and $\lambda^{tc} = (4,4,4)$. We are interested in the following diagrams.



$\mathcal{S}(\lambda^{tc},1,4)$ $\qquad$ $\mathcal{S}(\lambda^c,1,4)$

Theorem 5.4.1 gives

$$s_{\mathcal{S}(\lambda^{tc},1,4)} - s_{\mathcal{S}(\lambda^c,1,4)} = \sum_{\substack{A\subseteq\{1,2,3,4\}\\ |A|\leq 1}} s_{\bar{\nu}(A)}.$$

For $A = \emptyset$ we have

$$\begin{aligned}
\bar{\nu}(A) &= (5^{3+1-1-3+0}, 4^{3-0+1}, 3, 2, 1) \\
&= (4,4,4,4,3,2,1)
\end{aligned}$$

and for $A = \{a\}$, for each $1 \leq a \leq 4$, we have

$$\begin{aligned}
\bar{\nu}(A) &= (5^{3+1-1-3+1}, 4^{3-1+1}, 3, 2, 1) - \sum_{a\in A} e_{3+1+4-a} \\
&= (5,4,4,4,3,2,1) - e_{8-a}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
s_{\mathcal{S}(\lambda^{tc},1,4)} - s_{\mathcal{S}(\lambda^c,1,4)} &= s_{(4,4,4,4,3,2,1)} + s_{(5,4,4,4,3,2)} + s_{(5,4,4,4,3,1,1)} + s_{(5,4,4,4,2,2,1)} \\
&+ s_{(5,4,4,3,3,2,1)}.
\end{aligned}$$

We now inspect the analogue of Theorem 4.2.1, which looked at the difference $s_{\mathcal{S}(\lambda^t,k,n)} - s_{\mathcal{S}(\lambda,k,n)}$ for a partition λ with two parts. As before, we use the rectangle (w^l), where $w = n + k$.

Theorem 5.4.2 *Let λ be a partition such that $\lambda, \lambda^t \subseteq (w^l)$ for some l, and let $0 \leq k \leq 1$. If $l(\lambda) = 2$ and $\lambda_1 > 1$ then*

$$s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)} \geq_s 0.$$

Proof We wish to apply Theorem 5.1.4 to both diagrams. Since we are working with regular staircases, we have $\alpha = (1^n)$, which gives $\alpha^r = (1^{n-1+k})$.

We let $\rho(\lambda^{tc}) := (w^n) \cup \lambda^{tc}$ and $\rho(\lambda^c) := (w^n) \cup \lambda^c$. Then we have $(w^n) \subset \rho(\lambda^{tc}), \rho(\lambda^c) \subset (w^{n+l})$ so we may apply Theorem 5.1.4 to both of $\rho(\lambda^{tc})/\delta_{n-1+k} = \mathcal{S}(\lambda^{tc}, k, n)$ and $\rho(\lambda^c)/\delta_{n-1+k} = \mathcal{S}(\lambda^c, k, n)$. This gives

$$\begin{aligned} s_{\mathcal{S}(\lambda^{tc},k,n)} - s_{\mathcal{S}(\lambda^c,k,n)} &= c\big(s_{\mathcal{S}(\lambda^t,k,n-1+k)}\big) - c\big(s_{\mathcal{S}(\lambda,k,n-1+k)}\big) \\ &= c\big(s_{\mathcal{S}(\lambda^t,k,n-1+k)} - s_{\mathcal{S}(\lambda,k,n-1+k)}\big), \end{aligned}$$

where these complements are performed in the rectangle (w^{n+l}).

Theorem 4.2.1 shows that $s_{\mathcal{S}(\lambda^t,k,n-1+k)} - s_{\mathcal{S}(\lambda,k,n-1+k)} \geq_s 0$, so the difference is Schur-positive. ∎

Finally, using Theorem 2.3.1, we can obtain the following result.

Theorem 5.4.3 *Suppose a diagram $D = \rho/\mu$ is such that*

$$s_D = \sum_{\nu = \textit{fat staircase}} c^{\rho}_{\mu\nu} s_\nu.$$

Given a partition λ and $0 \leq k \leq 1$, let κ and θ be the partitions such that $\mathcal{S}(\lambda, D; k) = \kappa/\theta$. Then

$$c(s_\theta s_{\kappa^c}) \leq_s \sum_\nu c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda,\alpha(\nu);k)},$$

where κ^c is the complement of κ in the rectangle $(w(\kappa/\theta)^{l(D)+l(\lambda)})$.

Proof From Corollary 5.1.2 we have

$$s_{\kappa/\theta} = c(s_\theta s_{\kappa^c}).$$

From Theorem 2.3.1 we have

$$s_{\mathcal{S}(\lambda,D;k)} \leq_s \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};k)}.$$

Therefore

$$\begin{aligned} c(s_\theta s_{\kappa^c}) &= s_{\kappa/\theta} \\ &= s_{\mathcal{S}(\lambda,D;k)} \\ &\leq_s \sum_{\nu} c^{\rho}_{\mu\nu} s_{\mathcal{S}(\lambda,\alpha(\nu)^{\triangleleft};k)}. \quad \blacksquare \end{aligned}$$

Chapter 6

Reduction to Finite Variables

In this chapter we are concerned with equalities of skew Schur functions in finitely many variables. In the first section we begin by reiterating the finite variable equalities for staircases with bad foundations that were proved in Chapter 4. These equalities involved differences of staircases with transposed foundations. We extend these results to the case of transposed hook foundations, and make a conjecture for the general case. In the second section we prove a generalization of the factorization $s_{D_1 \oplus D_2} = s_{D_1} s_{D_2}$ when using finitely many variables.

6.1 Differences of Transposed Foundations

We return to the notation of Section 2.1. In Theorem 4.1.1 and Theorem 4.2.1 we proved that for each n and $0 \leq k \leq 1$, we have

$$s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$$

when λ has either one or two parts.

We can also prove this equality in the case when λ is a hook.

Theorem 6.1.1 *For each n, $0 \leq k \leq 1$, and hook λ with $|\lambda| \leq n+k$, we have*

$$s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}.$$

Proof To prove $s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$, we show that, when considering fillings with the entries taken from the set $\{1, 2, \ldots, n+1\}$, there is an bijection between the set of SSYTx of shape $\mathcal{S}(\lambda, k, n)$ with content ν and lattice reading word and the set of SSYTx of shape $\mathcal{S}(\lambda^t, k, n)$ with content ν and lattice reading word. As in Theorem 4.1.1, we simply consider the map that transposes the foundation of a given tableau.

Given any tableau $\mathcal{T}_1$ of shape $\mathcal{S}(\lambda, k, n)$ and content ν, this map gives us a tableau $\mathcal{T}_2$ of shape $\mathcal{S}(\lambda^t, k, n)$ and content ν. Now suppose that $\mathcal{T}_1$ is a SSYT with lattice reading word. The first row of λ strictly increases by Lemma 2.1.1 and the first column of λ strictly increases since the tableau is semistandard. Therefore in $\mathcal{T}_2$, the foundation λ^t is semistandard. In fact, the first row and the first column of λ^t strictly increase.

Now suppose that the reading word of $\mathcal{T}_2$ was not lattice. Then, when reading the foundation λ^t of $\mathcal{T}_2$, we must come to a point where we have read two $j+1$'s but no j. Since both the first row and first column of λ^t strictly increase, one of the $j+1$'s appears in the first row and the other $j+1$ appears in the first column. Since $\mathcal{T}_1$ has a lattice reading word and contains these two $j+1$'s in its foundation λ, a j must appear somewhere in λ. Thus λ^t contains a j as well. Since λ^t is semistandard, either the j appears to the left of the $j+1$ in the first row of λ^t or the j appears above the $j+1$ in the first column of λ^t. Either way, this j is read before reading the second $j+1$, contradicting our assumption.

Therefore the reading word of $\lambda^t \oplus \Delta_n$ is lattice. Since λ^t is semistandard and the entries of the each foundation are contained in $\{2-k, 3-k, \ldots, n+1\}$, Lemma 2.2.3 now shows that $\mathcal{T}_2$ is a SSYT with lattice reading word.

Thus when restricted to fillings with entries that are taken from the set $\{1, 2, \ldots, n+1\}$, we have a map from the set of SSYTx of shape $\mathcal{S}(\lambda, k, n)$ with content ν and lattice reading word to the set SSYTx of shape $\mathcal{S}(\lambda^t, k, n)$ with content ν and lattice reading word. This map is clearly its own inverse. Thus we obtain a bijection. ∎

Since we have proven the finite variable equality

$$s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}$$

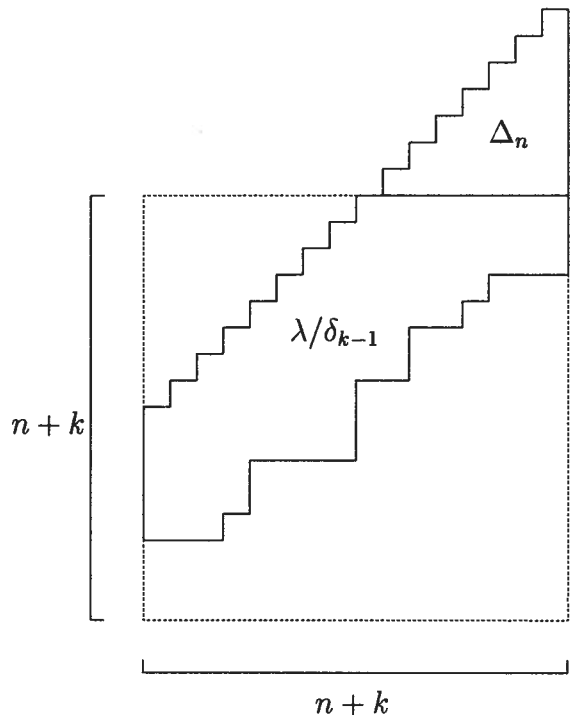
for these simple shapes λ, one might ask for which other partitions λ is this equality true.

We make the following conjecture.

Conjecture 6.1.2 *For each $n \geq 1$, $k \geq 0$, and partition λ such that $\mathcal{S}(\lambda, k, n)$ and $\mathcal{S}(\lambda^t, k, n)$ are connected skew diagrams we have*

$$s_{\mathcal{S}(\lambda^t,k,n)} =_{n+1} s_{\mathcal{S}(\lambda,k,n)}.$$

For each pair n, k, there are many partitions λ such that both $\mathcal{S}(\lambda, k, n)$ and $\mathcal{S}(\lambda^t, k, n)$ are defined. Namely, for all partitions λ such that $\delta_{k-1} \subseteq \lambda \subseteq ((n+k)^{n+k})$.



Both will be connected skew diagrams so long as $\delta_{k+1} \subseteq \lambda \subseteq ((n+k)^{n+k})$.

Computer-assisted calculations have shown that Conjecture 6.1.2 holds for the following values of the pair n, k, when searching through all pairs of connected skew shapes $\mathcal{S}(\lambda, k, n)$, $\mathcal{S}(\lambda^t, k, n)$.

- $n = 2$, $0 \leq k \leq 5$,
- $n = 3$, $0 \leq k \leq 4$, and
- $n = 4$, $0 \leq k \leq 1$.

When $n = 1$, the conjecture gives an equality in two variables. In this case it is not hard to check that the only pairs of connected skew shapes $\mathcal{S}(\lambda, k, n)$, $\mathcal{S}(\lambda^t, k, n)$ with *nonzero* skew Schur functions in two variables have $\lambda = \lambda^t$, so the conjecture does hold for $n = 1$.

As n and k increase, the number of shapes $\mathcal{S}(\lambda, k, n)$ grows quickly. This search space can be reduced by only considering the partitions λ such that λ is contained in δ_{n+k}, since otherwise both skew diagrams $\mathcal{S}(\lambda, k, n)$ and $\mathcal{S}(\lambda^t, k, n)$ contain a column of length $n+2$, which implies that both skew Schur functions equal 0 in $n+1$ variables. Nevertheless, as n and k increase the average size of the shapes increase, which makes it very time-consuming to compute each skew Schur function $s_{\mathcal{S}(\lambda,k,n)}$. This makes computationally

verifying Conjecture 6.1.2 for a given pair n, k difficult for larger values of n and k.

6.2 Factoring in Finite Variables

In this section we are interested in generalizing the factorization result

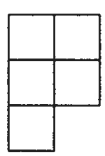
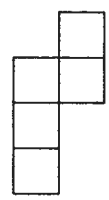
$$s_{D_1 \oplus D_2} = s_{D_1} s_{D_2},$$

when working with finitely many variables.

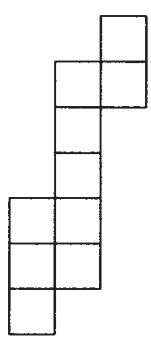
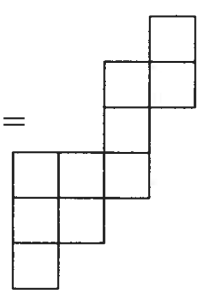
We begin by inspecting Theorem 1.6.5, which stated that for skew diagrams D_1, D_2, we have

$$s_{D_1} s_{D_2} = s_{D_1 \cdot D_2} + s_{D_1 \odot D_2}.$$

Example Suppose we have the following two skew diagrams.

$D_1 =$  $D_2 =$ 

The concatenation and near-concatenation are given by the following.

$D_1 \cdot D_2 =$  $D_1 \odot D_2 =$ 

From Theorem 1.6.5 we can conclude that $s_{D_1} s_{D_2} = s_{D_1 \cdot D_2} + s_{D_1 \odot D_2}$.

We note that the diagram $D_1 \cdot D_2$ contains a column of length 5, whereas each of D_1, D_2, and $D_1 \odot D_2$ only contain columns of length 3 or less. Thus we have $s_{D_1 \cdot D_2} =_4 0$ and

$$s_{D_1 \odot D_2} =_4 s_{D_1} s_{D_2}$$

gives a non-trivial factorization of $s_{D_1 \odot D_2}$ in four variables.

In order to state the next few results, we find it convenient to make the following notational convention. For a pair of skew diagrams D_1 and D_2, we shall let l_1 denote the length of the right-most column of D_1, and let l_2 denote the length of the left-most column of D_2.

Corollary 6.2.1 *If $l_1 + l_2 \geq n + 1$, then we have $s_{D_1 \odot D_2} =_n s_{D_1} s_{D_2}$.*

Proof Using Theorem 1.6.5 gives

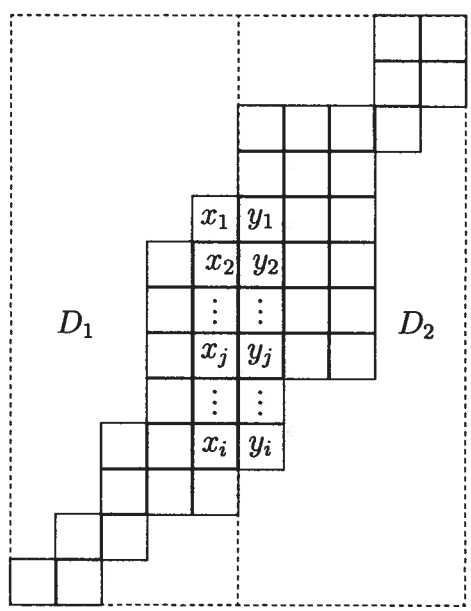
$$s_{D_1} s_{D_2} = s_{D_1 \cdot D_2} + s_{D_1 \odot D_2}.$$

However, if $l_1 + l_2 \geq n + 1$, then $D_1 \cdot D_2$ contains a column of length $> n$, which shows that $s_{D_1 \cdot D_2} =_n 0$. Therefore $s_{D_1 \odot D_2} =_n s_{D_1} s_{D_2}$, as desired. ∎

We now generalize to our main result for factoring skew Schur functions in finitely many variables. We consider skew diagrams D_1 and D_2 as before, but now assume that the overlap $i \leq \min\{l_1, l_2\}$, so that $D_1 \odot_i D_2$ is a skew diagram.

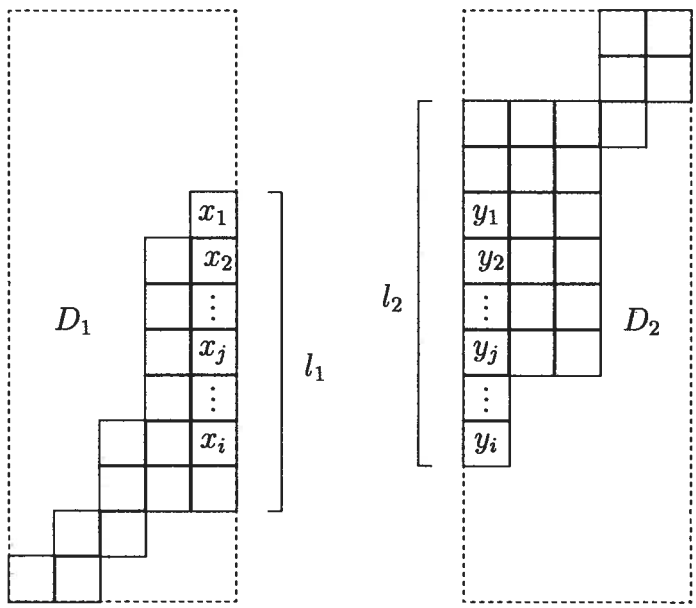
Theorem 6.2.2 *If $l_1 + l_2 \geq n + i$, then we have $s_{D_1 \odot_i D_2} =_n s_{D_1} s_{D_2}$.*

Proof We are only interested in fillings using the values $\{1, 2, \ldots, n\}$. We let $x_1, x_2, \ldots, x_i$ denote the first i entries of the right-most column of D_1 and $y_1, y_2, \ldots, y_i$ denote the last i entries of the left-most column of D_2. Then any SSYT of shape $D_1 \odot_i D_2$ consists of a SSYT of shape D_1 and a SSYT of shape D_2 such that $x_j \leq y_j$ for each $j = 1, 2, \ldots, i$.



Conversely, suppose we have a SSYT of shape D_1 and a SSYT of shape D_2. We intend to show that $x_j \leq y_j$ for each $j = 1, 2, \ldots, i$.

We claim that $x_j \leq n - l_1 + j$ for each $j = 1, 2, \ldots, i$. If not, then $x_j > n - l_1 + j$ for some $1 \leq j \leq i$. Since there are $l_1 - j$ boxes below x_j, and since the columns of the tableau strictly increase we find that the last entry in this column is larger than n. Since this is impossible, we have $x_j \leq n - l_1 + j$ for each $j = 1, 2, \ldots, i$, as claimed.



We also claim that $y_j \geq l_2 - i + j$ for each $j = 1, 2, \ldots, i$. If not, then $y_j < l_2 - i + j$ for some $1 \leq j \leq i$. Since there are $l_2 - i + j - 1$ boxes above y_j, and since the columns of the tableau strictly increase we find that the first entry in this column is less than 1. Since this is impossible, we have $y_j \geq l_2 - i + j$ for each $j = 1, 2, \ldots, i$, as claimed.

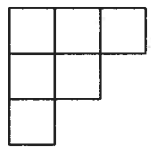
By assumption, $n - l_1 \leq l_2 - i$. Thus, for any SSYT of shape D_1 and any SSYT of shape D_2 we have

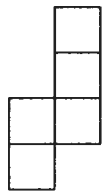
$$x_j \leq n - l_1 + j \leq l_2 - i + j \leq y_j$$

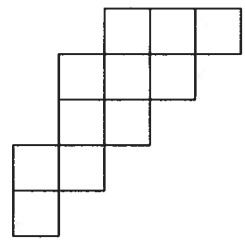
for each $j = 1, 2, \ldots, i$. Therefore this gives a SSYT of shape $D_1 \odot_i D_2$.

Therefore $s_{D_1 \odot_i D_2} =_n s_{D_1} s_{D_2}$, as desired. ∎

Example With the following skew diagrams D_1 and D_2 we have $l_1 + l_2 = 3 + 3 = 6 \geq 4 + 2 = n + i$. Thus, Theorem 6.2.2 gives $s_{D_1 \odot_2 D_2} =_4 s_{D_1} s_{D_2}$.

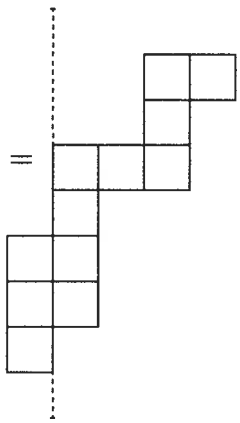
$D_2 =$ 

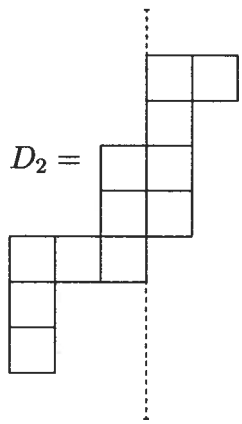
$D_1 =$ 

$D_1 \odot_2 D_2 =$ 

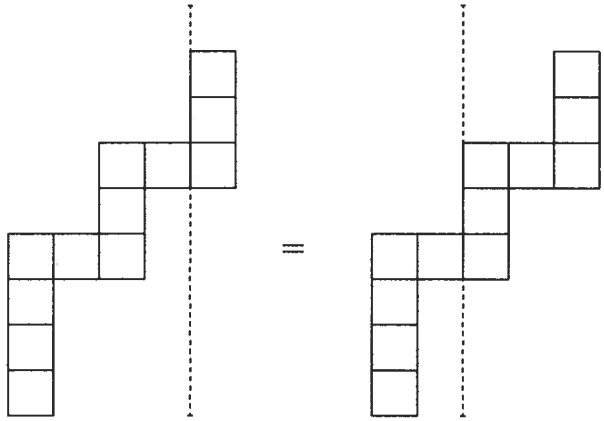
When using Theorem 6.2.2 to check a pair of diagrams for equality, one hopes that the pair shares the same collection of factors. Even when this is not the case, we may sometimes delve further into the products that arise in order to check for equality.

Example We wish to show that the following two diagrams give the same skew Schur function in $n = 5$ variables. We have indicated the breaks where Theorem 6.2.2 applies.

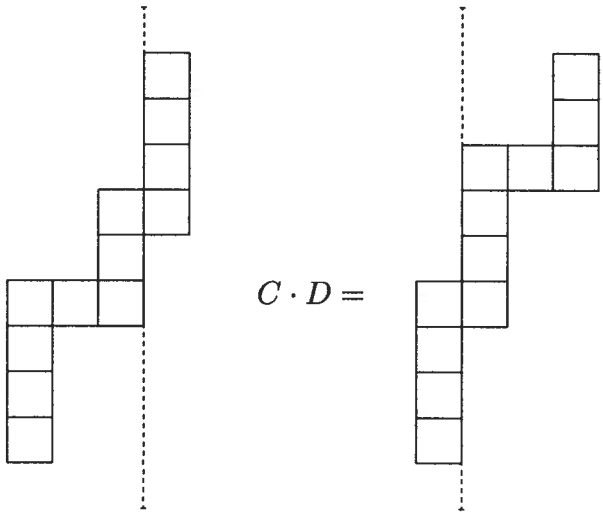
$D_1 =$ 

$D_2 =$ 

We let $A = (4,3,3,1,1,1)/(2,2)$, $B = (1,1,1)$, $C = (2,1,1,1)$, and $D = (3,3,3,1,1)/(2,2)$, so that $D_1 = B \odot_2 A$ and $D_2 = D \odot_2 C$. Theorem 6.2.2 shows that $s_{D_1} =_5 s_A s_B$ and $s_{D_2} =_5 s_C s_D$. We note that $A \odot B = (5,5,5,3,3,1,1,1)/(4,4,2,2) = C \odot D$, as is illustrated below.

$A \odot B =$  $= C \odot D$

Further, we find that $s_{A \cdot B} =_5 s_{(3,3,3,1,1,1)/(2,2)} s_{(1,1,1,1)} =_5 s_{C \cdot D}$ from applying Theorem 6.2.2 to both $A \cdot B$ and $C \cdot D$.

$A \cdot B =$  $C \cdot D =$

Thus we obtain

$$
\begin{aligned}
s_{D_1} &=_5 s_A s_B \\
&=_5 s_{A \cdot B} + s_{A \odot B} \\
&=_5 s_{(3,3,3,1,1,1)/(2,2)} s_{(1,1,1,1)} + s_{(5,5,5,3,3,1,1,1)/(4,4,2,2)} \\
&=_5 s_{C \cdot D} + s_{C \odot D} \\
&=_5 s_C s_D \\
&=_5 s_{D_2}.
\end{aligned}
$$

We have the following result to aid us when we are inspecting an equality of the form $s_{A\odot B} = s_{B\odot A}$.

Lemma 6.2.3 *For skew diagrams A and B, we have*

$$s_{A\odot B} = s_{B\odot A} \text{ if and only if } s_{A\cdot B} = s_{B\cdot A}.$$

Proof Using Theorem 1.6.5 and the fact that $s_A s_B = s_B s_A$, the result immediately follows. ∎

For the purposes of the factoring that results from Theorem 6.2.2, it is generally easier to work with diagrams that have longer columns. Therefore, with respect to Theorem 6.2.2, it is usually more convenient to inspect the pair $A \cdot B$, $B \cdot A$ in place of the pair $A \odot B$, $B \odot A$.

In the statement of the next lemma, we let a_l and a_r denote the lengths of the left-most and right-most column of A, respectively. Further, we let $A - a_l$ denote the diagram obtained by removing the left-most column from A and we let $A - a_r$ denote the diagram obtained by removing the right-most column from A. We also make the same notational conventions for the diagram B.

Lemma 6.2.4 *Let A and B be skew diagrams with $a_r + b_l = n$ and $b_r + a_l = n$ and $s_{A-a_r} s_{B-b_l} =_n s_{B-b_r} s_{A-a_l}$, then we have $s_{A\odot B} =_n s_{B\odot A}$.*

Proof Using Theorem 6.2.2 and our assumption that

$$s_{A-a_r} s_{B-b_l} =_n s_{B-b_r} s_{A-a_l},$$

we have

$$\begin{aligned} s_{A\cdot B} &=_n s_{A-a_r} s_{(1^n)} s_{B-b_l} \\ &=_n s_{B-b_r} s_{(1^n)} s_{A-a_l} \\ &=_n s_{B\cdot A} \end{aligned}$$

Thus Lemma 6.2.3 shows that $s_{A\odot B} =_n s_{B\odot A}$. ∎

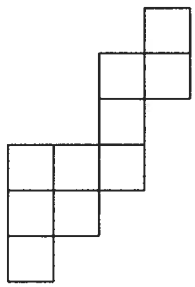
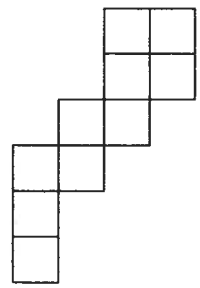
The simplest application of the previous lemma is the following corollary.

Corollary 6.2.5 *Let $n = a + b$ and $1 \leq i_1 < i_2 \leq \min\{a, b\}$, then*

$$s_{((1^a)\odot_{i_1}(1^b))\odot((1^a)\odot_{i_2}(1^b))} =_n s_{((1^a)\odot_{i_2}(1^b))\odot((1^a)\odot_{i_1}(1^b))}.$$

Proof Take $A = (1^a) \odot_{i_1} (1^b)$ and $B = (1^a) \odot_{i_2} (1^b)$ in Lemma 6.2.4. ∎

Example Taking $a = 3$, $b = 2$, $i_1 = 2$, and $i_2 = 1$ in Corollary 6.2.5 we find that $s_{D_1 \odot D_2} =_5 s_{D_2 \odot D_1}$ where D_1 and D_2 are as shown below.

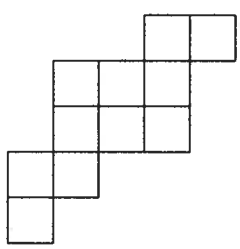
$D_1 \odot D_2 =$  $D_2 \odot D_1 =$ 

Although the techniques in this section may be used to explain many of the finite variable equalities of skew Schur functions, they seem inadaquate when it comes to proving the following conjectured equalites.

Conjecture 6.2.6 *For each* $n \geq 4$,

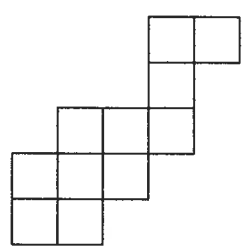
$$s_{((1^2)\odot(1^{n-1}))\odot_2((1^2)\odot_2(1^{n-1}))\odot(1)} =_n s_{((1^2)\odot_2(1^{n-1}))\odot_2((1^2)\odot(1^{n-1}))\odot(1)}.$$

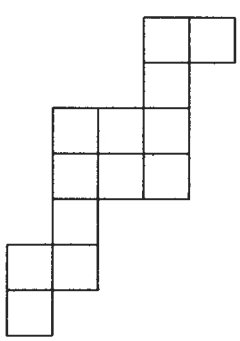
Example Here we see the pairs for $n = 4$ and $n = 5$.



$\longleftrightarrow$

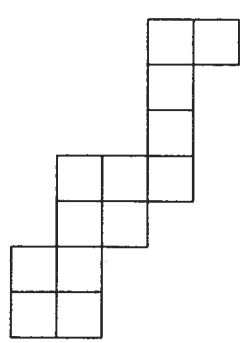
$n = 4$





$\longleftrightarrow$

$n = 5$



This conjecture has been verified for the values $4 \leq n \leq 11$.

We finish this chapter with the following result, which gives the factorization of any skew diagram in $n = 2$ variables.

Theorem 6.2.7 *Consider a connected skew diagram D. If D has a column of length larger than two, then $s_D =_2 0$. Otherwise, if j is the number of columns of length two in D, then $s_D =_2 (s_{(1,1)})^j \prod_k (s_{(k)})^{e_k}$, where the e_k are nonnegative integers, only finite number of which are nonzero.*

Proof Clearly, if D has a column of length larger than two, then $s_D =_2 0$.

Suppose that all columns of D have length less than or equal to 2, and let j be the number of columns of length equal to 2.

If two columns of length two are adjacent, then their overlap is at most $i = 2$ and Theorem 6.2.2 shows that we can factor the skew diagram by splitting the diagram between these columns. Further, whenever a column of length two is adjacent to a column of length one, then their overlap is at most $i = 1$ and Theorem 6.2.2 shows that we can factor the skew diagram by splitting the diagram between these columns.

Through these factorizations, we can collect all the columns of length two into the term $(s_{(1,1)})^j$. The only remaining diagrams that could not be factored are rows. These give rise to a product of the form $\prod_k (s_{(k)})^{e_k}$. ∎

Chapter 7

Conclusion

The aim of this thesis was to determine instances of Schur-positivity.

In Section 2.3 we proved Theorem 2.3.1, which provided a collection of Schur-positivity results for each sum of fat staircases D. In fact we obtained a Schur-positivity result by augmenting D with *any* skew diagram λ/μ. The remainder of Section 2.3 sought to determine necessary conditions for a diagram to be a sum of fat staircases. Theorem 5.3.1 and Theorem 5.3.2 gave necessary and sufficient conditions for certain simple skew diagrams to be sums of fat staircases. Namely, these results inspected skew diagrams of the form $\delta_\alpha/(m)$ and $\delta_\alpha/(1^m)$.

In Section 3.3 we showed that for each composition $\alpha = (\alpha_1, \ldots, \alpha_n)$, and each h and k with $h \leq n + k$ and $0 \leq k \leq h$, we can determine the entire Hasse diagram which describes which pairs of skew diagrams of the form $\mathcal{S}(\lambda, \alpha^\triangleleft; k)$, $\mathcal{S}(\mu, \alpha^\triangleleft; k)$ have Schur-positive difference and which are Schur-incomparable, where λ and μ are hooks of size h. Expressions for certain Schur-positive differences were given in Section 3.2.

Chapter 4 looked at some special cases of differences of fat staircases with bad foundations in which the foundations of the two diagrams were transposes of each other. Equality in finite variables for these diagrams was also discussed, which led to Conjecture 6.1.2.

Most of the results of the earlier chapters were extended in Chapter 5 to yield results for the fat staircases with complementary foundations.

Many of the proofs in this thesis relied heavily on Lemma 2.2.2 and Lemma 2.2.3. In fact, nearly every result from Chapter 2 to Chapter 5 used at least one of these two lemmas. The defining structure of the fat staircases was made precisely so that we could develop these results. Unfortunately these fat staircases are the only partitions for which Lemma 2.2.2 and Lemma 2.2.3 hold. As we saw with the sums of fat staircases, some generalizations can still be reached with these techniques.

However, there is still a vast amount of questions remaining about Schur-positivity and Schur-equality. There is significant interest in the resolution of these and many other related problems.

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