Global questions for evolution equations

Landau-Lifshitz flow and Dirac equation

by

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Ph.D., The University of British Columbia, 2009

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2009

Abstract

This thesis concerns the stationary solutions and their stability for some evolution equations from physics. For these equations, the basic questions regarding the solutions concern existence, uniqueness, stability and singularity formation. In this thesis, we consider two different classes of equations: the Landau-Lifshitz equations, and nonlinear Dirac equations. There are two different definitions of stationary solutions. For the Landau-Lifshitz equation, the stationary solution is time-independent, while for the Dirac equation, the stationary solution, also called solitary wave solution or ground state solution, is a solution which propagates without changing its shape.

The class of Landau-Lifshitz equations (including harmonic map heat flow and Schrödinger map equations) arises in the study of ferromagnets (and anti-ferromagnets), liquid crystals, and is also very natural from a geometric standpoint. Harmonic maps are the stationary solutions to these equations. My thesis concerns the problems of singularity formation vs. global regularity and long time asymptotics when the target space is a 2-sphere. We consider maps with some symmetry. I show that for m-equivariant maps with energy close to the harmonic map energy, the solutions to Landau-Lifshitz equations are global in time and converge to a specific family of harmonic maps for big m, while for $m = 1$, a finite time blow up solution is constructed for harmonic map heat flow. A model equation for Schrödinger map equations is also studied in my thesis. Global existence and scattering for small solutions and local well-posedness for solutions with finite energy are proved.

The existence of standing wave solutions for the nonlinear Dirac equation is studied in my thesis. I construct a branch of solutions which is a continuous curve by a perturbation method. It refines the existing results that infinitely many stationary solutions exist, but with uniqueness and continuity unknown. The ground state solutions of nonlinear Schrödinger equations yield solutions to nonlinear Dirac equations. We also show that this branch of solutions is unstable. This leads to a rigorous proof of the instability of the ground states, confirming non-rigorous results in the physical literature.

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Acknowledgements

I wish to express my sincere gratitude to my supervisors (in alphabetical order), professor Stephen Gustafson and professor Tai-Peng Tsai for their continued encouragement and invaluable suggestions during my studies at UBC. Without their inspiration and financial support, I could not finish my degree. Also I acknowledge my debt to Dr. Kyungkeun Kang and professor Dmitry Pelinovsky for their useful communications for my research.

I would like to thank the department of Mathematics at UBC. Special thanks is devoted to our department secretary Lee Yupitun.

Furthermore, I am deeply indebted to my friends at UBC. Throughout my thesis-writing period, they provided good advice to help get my thesis done and shared their experiences in many different ways.

Lastly, I express my special thanks to my parents and my husband. Thanks for their encouragement and support during these years. This thesis is particularly devoted to the birth of my daughter.

Statement of Co-Authorship

The second chapter of my thesis is co-written with my supervisors Stephen Gustafson and Tai-Peng Tsai. Inspired by their study on Schrödinger maps with energy near the harmonic map energy, we want to extend the analysis to harmonic map heat flow. Harmonic maps are the stationary solutions to Landau-Lifshitz equations, including harmonic map heat flow and Schrödinger maps. These map equations should share some common properties. For this paper I obtained the key space-time estimates for the nonlinear heat-type equation in section (2.2). In section (2.3), the technical lemmas are proved by my supervisors. The manuscript was finished by me, and my supervisors made the correction.

Chapter 5 is also a co-written with my supervisor Stephen Gustafson. The problem arose from the previous chapter on the nonlinear Dirac equation. I conjectured that the ground states are unstable. But we needed to verify it through the analysis of the linearized operator. Together with my supervisor Stephen Gustafson, we determined how to relate the eigenvalue to the linearized operator of nonlinear Schrödinger equation at its ground states (see Theorem 5.2.3). Finally I finished the manuscript.

Chapter 1

Introduction and main results

1.1 Introduction

Nonlinear evolution equations (partial differential equations with a time variable) arise throughout the sciences as descriptions of dynamics in various systems. The classical examples include nonlinear heat, wave and Schrödinger equations. Recently, evolution equations have emerged as an important tool in geometry. A prime example is the application of the Ricci flow to the Poincaré conjecture.

For the evolution equations, once the existence of a local in time solution for the Cauchy problem is established, the most basic global questions concern the existence of stationary solutions and their stability. Another important question is whether or not solutions can form singularities, which may prevent the solutions from existing for all times, and may represent a breakdown of the model.

This thesis addresses these basic questions for two classes of equations: Landau-Lifshitz equations which arise both in physics and geometry, and nonlinear Dirac equations coming from physics. These equations share a common feature: they are equations not for typically scalar valued functions, but rather for maps into manifolds and vector space respectively. Then the analysis of these equations become more challenging. For Landau-Lifshitz equations, the target space is a manifold, more precisely the 2-sphere. It is the nontrivial geometry of the target which creates the nonlinearity and provides many of the interesting and difficult features of this problem. In the Dirac case, the map is a four-dimensional complex vector $\mathbb{C}^4$. Although this is a linear space, the vector map significantly complicates the construction of stationary solutions and the analysis of their stability.

In what follows in this chapter, a brief review, including the physical and mathematical backgrounds, as well as the most recent known results related to my thesis are provided in Section 1.1. In Section 1.2, the main objectives

and methodologies in my thesis are introduced.

1.1.1 Landau-Lifshitz flow

The principle assumption of the macroscopic theory of ferromagnetism is that the state of a magnetic crystal Ω is describable by the magnetization vector m, which is a function of space and time. Magnetization m is the quantity of magnetic moment per unit volume. Suppose in a small region $dV_x \subset \Omega$, there is a number N of magnetic moments $\mu_1, \cdots, \mu_N$, then the magnetization is defined as

$$m = \frac{\sum_{j=1}^{N} \mu_j}{dV_x}.$$

It is known from quantum mechanics that the magnetic moment is proportional to the angular moment of electrons. By the momentum theorem, the rate of change of the angular momentum is equal to the torque exerted on the particle by the magnetic field H. Thus one ends up with a model which describes the precession of the magnetic moment μ_j around the field H

$$\frac{d\mu_j}{dt} = -\gamma \mu_j \times H,$$

where γ is the absolute value of the gyromagnetic ratio. By taking the volume average of both sides, one has the following continuum gyromagnetic precession model

$$\frac{\partial m}{\partial t} = -\gamma m \times H.$$

In 1935 Landau and Lifshitz proposed the Landau-Lifshitz equation [51] as a model for the precessional motion of the magnetization, in which H is replaced by the effective field H_{eff}. They arrived at the Landau-Lifshitz equation

$$\frac{\partial m}{\partial t} = -\gamma m \times H_{eff}. \tag{1.1.1}$$

The effective magnetic field H_{eff} is equal to the variational derivative of the magnetic crystal energy with respect to the vector m. In equation (1.1.1), no dissipative terms appear. Nevertheless, dissipative processes take place within dynamic magnetization processes. Landau and Lifshitz proposed to introduce an additional torque term as a dissipation that pushes magnetization in the direction of the effective field. Then, the Landau-Lifshitz (LL) equation becomes

$$\frac{\partial m}{\partial t} = -\gamma m \times H_{eff} - \frac{\lambda}{|m|} m \times (m \times H_{eff}), \tag{1.1.2}$$

where $\lambda > 0$ is a phenomenological constant characteristic of the material. The effective magnetic field H_{eff} is equal to the variational derivative of the magnetic crystal energy with respect to the vector H.

In 1955, Gilbert [34] proposed to introduce a different damping term for Landau-Lifshitz equation (1.1.1). He derived an equation which is generally referred to as the Landau-Lifshitz-Gilbert (LLG) equation:

$$\frac{\partial m}{\partial t} = -\gamma m \times H_{eff} + \frac{\lambda}{|m|} m \times \frac{\partial m}{\partial t}. \tag{1.1.3}$$

From a mathematical view point, equation (1.1.2) and (1.1.3) are very similar. The basic property of LL equation and LLG equation is that the magnitude of m is conserved since $\frac{\partial}{\partial t}|m|^2 = 0$. This implies that any magnetization motion, at a given location , will occur a sphere, which can be normalized to be the unit sphere.

In the simplest case, the energy is just the "exchange energy" $\frac{1}{2}\int_\Omega |\nabla m|^2 dx$, and we can take $H_{eff} = -\Delta m$. Let $u = m : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{S}^2 \subset \mathbb{R}^3$ with $|u| = 1$, then LL equation can be considered as a linear combination of heat and Schrödinger flow for harmonic maps

$$\partial_t u = -\alpha u \times (u \times \Delta u) + \beta u \times \Delta u, \tag{1.1.4}$$

where Δ denotes the Laplace operator in $\mathbb{R}^n, \alpha > 0$ is a Gilbert damping constant, $\beta \in \mathbb{R}$ and " $\times$ " denotes the usual vector product in $\mathbb{R}^3$. At first sight, this equation is a strongly coupled degenerate quasi-linear parabolic system, which makes it hard to analyze mathematically.

There are other equivalent forms of LL equations. Suppose there are smooth solutions, by the vector cross product formula

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

one can easily verify that equation (1.1.4) is equivalent to

$$\partial_t u = \alpha(\Delta u + |\nabla u|^2 u) + \beta u \times \Delta u. \tag{1.1.5}$$

If we let P^u denote the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_u\mathbb{S}^2 := \{\xi \in \mathbb{R}^3 \mid \xi \cdot u = 0\}$$

to $\mathbb{S}^2$ at u with

$$P^u \partial_j \xi = \partial_j \xi - (\partial_j \xi \cdot u)u = \partial_j \xi + (\partial_j u \cdot \xi)u,$$

then an equivalent equation, written in a more geometric way, is

$$\partial_t u = \alpha P^u \Delta u + \beta J^u P^u \Delta u$$

where $J^u = u\times$ is a rotation through $\pi/2$ on the tangent plane $T_u\mathbb{S}^2$.

The borderline cases for Landau-Lifshitz equation include harmonic map heat flow when $\beta = 0, \alpha = 1$:

$$\partial_t u = \Delta u + |\nabla u|^2 u, \tag{1.1.6}$$

and Schrödinger maps if $\alpha = 0, \beta = 1$:

$$\partial_t u = u \times \Delta u. \tag{1.1.7}$$

The Dirichlet energy functional of (1.1.4) given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx = \frac{1}{2}\int_\Omega \sum_{j=1}^{n}\sum_{k=1}^{3}\left|\frac{\partial u_k}{\partial x_j}\right|^2 dx$$

plays an very important role in the analysis of these flow equations. The energy identity, obtained formally by taking the scalar product with $P^u\Delta u = \Delta u + |\nabla u|^2 u$ and integrating over $\Omega \times [0,t)$,

$$E(u(t)) + \alpha\int_0^t\int_\Omega |\Delta u + |\nabla u|^2 u|^2 dxdt = E(u(0))$$

implies that for $\alpha > 0$, the energy is nonincreasing, while for $\alpha = 0$, the energy is conserved. Moreover, with respect to scaling, the energy has critical dimension $n = 2$. Rescaling the spatial variable, we have

$$E(u(\cdot)) = s^{2-n}E(u(\cdot/s)),$$

where $s > 0$. Thus the energy is invariant under the scaling if space dimension $n = 2$. This suggests that $n = 2$ is critical for the formation of singularities for these equations. Another important feature of the domain $\mathbb{R}^2$ problem is the relationship between the energy and the topology, as expressed by a lower bound for the energy:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx \geq 4\pi|deg(u)|$$

where $|deg(u)|$ is the topological degree for the map $u : \mathbb{R}^2 \sim \mathbb{S}^2 \to \mathbb{S}^2$. So space dimension 2 is particularly interesting mathematically.

These equations have been widely studied by mathematicians and physicists. One of the most interesting and challenging questions is that of singularity formation vs. global regularity: do all solutions with smooth initial data remain smooth for all time, or do they form singularities in finite time for some data? In the following, we describe some important results for the various map equations described above.

Harmonic map heat flow

Among the Landau-Lifshitz flow equations (including harmonic map heat flow and Schrödinger maps), harmonic map heat flow is the best understood and studied. Equation (1.1.6) generalizes the linear heat equation to maps into Riemannian manifolds. In geometry, it arises in the construction of harmonic maps of certain homotopy types [61]. In physics it arises in the theory of ferromagnetic materials [24–26, 51] and in the theory of liquid crystals [27]. A more general setting for equation (1.1.6) is to assume u a differentiable mapping from M, a compact two-dimensional Riemannian manifold (possibly with smooth boundary) to a compact manifold N, which is isometrically embedded in $\mathbb{R}^k$. Eells and Sampson (see [29]) used this equation to construct a harmonic map from M to N. They proved that this equation has a smooth solution defined for all time and converging to a harmonic map under the assumption that the sectional curvature of N was non-positive. This result is not true if no curvature assumptions are made on N. Struwe in [59] proved partial regularity for global weak solutions with finite initial energy ($u_0 \in H^1(M,N)$), with at most finitely many singular space-time points where nonconstant harmonic maps "separate". The small energy solutions are global. Freire [31] showed that the weak solution in [59] is unique if the energy is non-increasing along the flow.

Since space dimension 2 is energy critical, much effort has been devoted to the case where the domain is the unit disk in $\mathbb{R}^2$, the target manifold is $\mathbb{S}^2$, and for a special class of solutions:

$$u(\cdot,t) : (r,\theta) \to (\cos m\theta \sin\phi(r,t), \sin m\theta \sin\phi(r,t), \cos\phi(r,t)) \in \mathbb{S}^2 \subset \mathbb{R}^3 \tag{1.1.8}$$

for a positive integer m. (r,θ) are polar coordinates. Then the scalar function ϕ satisfies a nonlinear heat-type equation:

$$\phi_t = \phi_{rr} + \frac{1}{r}\phi_r - m^2\frac{\sin 2\phi}{2r^2}, \quad 0 < r < 1, \ \ t > 0. \tag{1.1.9}$$

One specifies initial condition $\phi(r,0) = \phi_0(r)$, and typical boundary conditions are $\phi(0,t) = 0$, and $\phi(1,t) = \phi_1 \in \mathbb{R}$. In fact the requirement that

$\sin\phi(0,t) = 0$ is necessary for solutions to have finite energy

$$E(\phi) = \pi \int_0^1 (r\phi_r^2 + \frac{\sin^2\phi}{r})dr.$$

Singularities can develop only at the origin. If singularity occurs at some space point $r = r_0 \neq 0$, then the corresponding solution $u(x,t)$ has infinitely many singular points along the circle $|x| = r_0$, which is a contradiction to Struwe's result. Using the standard notation for Lebesgue norms: we say solutions blow up at time $T < \infty$, if

$$\lim \sup_{t\to T^-} \|\nabla u(\cdot,t)\|_\infty = \infty,$$

or

$$\lim \sup_{t\to T^-} \|\Delta u(\cdot,t)\|_{L^2} = \infty,$$

while u remains bounded ($|u| = 1$). Thus, for equation (1.1.9), ϕ_r blows up while ϕ remains bounded. The issue is whether all solutions eventually converge to equilibria or singularities form in finite time for some initial data. Intuitively if we choose initial data in a topological class which does not contain any equilibria, the solutions must blow up, at least in infinite time.

For $m = 1$, global regularity is proved by Chang and Ding [17] if $|\phi_0(r)| \leq \pi, 0 \leq r \leq 1$. However the global solution in [17] may not subconverge to a harmonic map as $t \to \infty$ if $\phi_1 = \pi$, since there is no harmonic map satisfying both the boundary conditions. It must develop a singularity at $t = \infty$. In 1992 collaborating with Ye [18], they showed (again for $m = 1$) that, indeed, finite time blow-up does occur for finite energy solutions if $|\phi_1| > \pi$. A result of [11] tells us that even if $m = 1$ and $|\phi_1| < \pi$, finite time blow up is still possible if $\phi_0(r)$ rises above π for some $r \in (0,1)$.

As for $m \geq 2$, the situation may be different. Grotowski and Shatah [38] observed the difference and proved that for boundary data $|\phi_1| < 2\pi$, the solution remains regular for all time, so finite time blow up will not occur.

Those results implying the finite-time blow-up do not predict at what rate the gradient should blow up. A direct calculation shows that one parameter family of finite energy, static solutions to equation (1.1.9) are given by

$$f(r) = 2\arctan\frac{r^m}{\lambda}, \quad \lambda > 0.$$

When blow up occurs at $t = T$, an appropriate rescale will reveal a stationary solution. Let $R(t) > 0$ be the rescale function, then locally in space

$$\lim_{t\to T^-} \phi(t, rR(t)) = 2\arctan r^m.$$

In [10] the generic blow-up behavior (and blow-up rate) was analyzed via formal asymptotics by Berg, Hulfson and King, where they observed that whether or not singularities occur appears to depend on m, as well as on the initial and boundary data. Moreover, different blow up rates were given explicitly. In [1] Angenent and Hulshof rigorously prove the infinite time blow-up rate in [10] for a special setting, where they assume that the initial data can reach $\frac{\pi}{2}$ only one time.

When blow up occurs, the value of ϕ at the origin jumps to another value, the solution changes to another topological class, and the energy jumps by an integer multiple of $4\pi m$. The harmonic map heat flow splits off "bubbles", i.e. non constant harmonic maps separate at the singular points. In [57] Qing pointed out the energy loss at a singularity can be recovered as a finite sum of energies of tangent bubbles. After blow-up, the weak solutions can no longer be expected to be unique if the energy is not assumed to be decreasing. In [11] Bertsch, Passo and Van Der Hout constructed explicit examples for non-uniqueness of extensions of solutions after blow-up. These solutions are characterized by "backward bubbling" at some arbitrarily large time and all have uniformly bounded energy. The qualitative descriptions of Struwe's solutions near their singular points are also given in [65, 66].

In addition to the critical space dimension 2, different space dimensions for harmonic map heat flow are also widely studied. Finite-time blow up was proved by Coron and Ghidaglia in [20] for certain smooth initial data from $\mathbb{R}^n$ or $\mathbb{S}^n$ to $\mathbb{S}^n, n \geq 3$. Later Chen and Ding in [14] extended this result to $n \geq 3$ without the assumption of symmetries of $\mathbb{S}^n$ and the initial maps. For maps between the 3-dimensional ball and $\mathbb{S}^2$, the nonuniqueness of solutions was proved in [12, 13] for some initial data. Grotowski [37] established blow-up and convergence results for certain axially symmetric initial data.

Landau-Lifshitz flow

For Landau-Lifshitz flow equation, with the Schrödinger-type term $\beta \neq 0$, our understanding diminishes considerably. Indeed this equation is still parabolic, but maximum principle arguments are not readily applicable. Due to the extra nonlinear term $u \times \Delta u$, more estimates are required in the proof of regularity results compared to those of harmonic map heat flow.

The first existence results for weak solutions to LL equation are due to [68], see also [2, 7]. In [2], Alouges and Soyeur established a nonuniqueness result for weak solutions when $\Omega \in \mathbb{R}^3$. Their approach was based on the method introduced in [13] to prove the nonuniqueness of weak solutions to

the harmonic map heat flow. For a mapping u from a two-dimensional compact Riemannian manifold M to $\mathbb{S}^2$, partial regularity result are analogous extensions of those for Struwe's solutions in [59]. In the case of $\partial M = \emptyset$, Guo and Hong [44] proved the regularity of weak solutions for $u_0 \in H^1(M, \mathbb{S}^2)$ with the exception of at most finitely many singular space-time points; they are globally regular in the case of small initial energy. The arguments are based on the a priori uniform estimates for $\partial_t u$ and $D^2 u$. For M with smooth boundary, the similar results were obtained by Chen in [15, 16]. Moreover, the local behavior of the solution near its singularities was investigated in [16]. Instead of using the uniform estimates for $\partial_t u$ and $D^2 u$, $L^p - L^q$ and $W^{1,p}$ estimates for the linear parabolic system were used to show the partial regularity of the weak solutions.

The study of the weak solution near the singular points also follows from previous results in the harmonic map heat flow [11, 57]. For bounded domain in $\mathbb{R}^2$, Harpes [48] analyzed the geometric description of the flow at isolated singularities and showed the non-uniqueness of extensions of the flow after blow-up. But the formation of singularities is still open. Since maximum principle arguments (for example, sub-solution construction) do not apply, the blow-up argument for harmonic map heat flow may not provide a useful tool for the Landau-Lifshitz flow equation.

Schrödinger maps

In the case of $\alpha = 0$, equation (1.1.4) becomes the Schrödinger map equation (1.1.7). The analysis of Schrödinger maps becomes more difficult, even the local-in-time theory is not yet well understood. There is a great deal of recent work on the local well-posedness problems in two space dimensions ([4], [5], [23], [49], [53], [63]), see also on [50, 55] for the "modified Schrödinger map". For maps $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{S}^2$, local well-posedness is established for $\nabla u_0 \in H^k, k > n/2 + 1$ is an integer, and also the global well-posedness is proved for small data when $n \geq 2$ in [63]. Chang, Shatah and Uhlenbeck in [19] considered the global well-posedness problem for finite energy. For $u_0 \in H^1(\mathbb{R}^1)$, (1.1.7) has a unique global solution. While for $n = 2$, small energy implies global existence and uniqueness for radial and equivariant maps. For $u : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, by an m-equivariant map, we mean

$$u(r, \theta) = e^{m\theta R} v(r) \tag{1.1.10}$$

where $m \in \mathbb{Z}$ a non-zero integer,(r, θ) are polar coordinates on $\mathbb{R}^2$, and R is the matrix generating rotations around the u_3-axis. The argument in [19] is based on a generalized version of the "Hasimoto transformation". The key

observation involves the vector field $w(x,t) = u_r - \frac{m}{r} J^u R u \in T_u \mathbb{S}^2$, which lies in the tangent plane to $\mathbb{S}^2$ at u for each x, t. For an appropriate choice of an orthonormal frame on the tangent plane, the coordinates of w satisfy a kind of cubic nonlinear Schrödinger equation. Then the standard estimates (Strichartz estimates) for nonlinear Schrödinger equations can be applied to this equation.

Gustafson, Kang and Tsai in [45, 46] investigated Schrödinger flow equation for energy space initial data, i.e. the energy of the initial data is near the harmonic map energy $4\pi|m|$. Their results reveal that if the topological degree m of the map is at least four, blow-up does not occur, and the global in time solution converges (in a dispersive sense) to a fixed harmonic map for large initial data. This is the first general result for Schrödinger maps with non-small energy space initial data.

It is still open whether or not finite-time singularities can form for Schrödinger maps from $\mathbb{R}^2 \times \mathbb{R}^+$ to $\mathbb{S}^2$, partly because the class (1.1.8) is no longer preserved, as in the harmonic map heat flow. This motivated us to consider a model equation for Schrödinger maps.

1.1.2 Model equation for Schrödinger maps

Let $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$. For geometric evolution equations such as harmonic map heat flow (1.1.6) and wave maps

$$P^u u_{tt} = P^u \Delta u,$$

a special class of solutions (1.1.8) was preserved, which allows a reduction to a single scalar equation (e.g. equation (1.1.9)). However this class is not preserved by Schrödinger maps which makes the construction of singular solutions much harder. This is our motivation to introduce a model equation.

Let $\phi(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ be a radial scalar function and $m > 0$ be an integer. By analogy with (1.1.9) for the heat flow, we consider the nonlinear Schrödinger equation

$$i\phi_t + \Delta\phi - \frac{m^2}{r^2}\frac{\sin(2|\phi|)}{2|\phi|}\phi = 0, \qquad \phi(x,0) = \phi_0(x). \tag{1.1.11}$$

The nonlinearity is designed to be gauge invariant (hence the L^2 norm is preserved) and to admit a conserved energy defined by

$$E(\phi) = \pi \int_0^\infty (|\phi_r|^2 + \frac{m^2}{r^2}\sin^2|\phi|) r dr.$$

The nontrivial stationary solutions with finite energy are given by a 2-parameter class of functions

$$\Gamma_m := \{e^{i\alpha}Q(r/s)|Q(r) = 2\arctan(r^m), s > 0, \alpha \in \mathbb{R}\}.$$

Thus we comes up with two questions: whether we can draw the same conclusion as in [46] (the Schrödinger maps), i.e. harmonic maps are stable under this equation for large m, or can we show finite time blow up when $m = 1$? Moreover, since zero is a trivial static solution, do small solutions exist for all time? This equation is related to Gross-Pitaveskii equation (Nonlinear Schrödinger equation with nonzero boundary conditions, see [47]), but it is hard in some respects.

1.1.3 Nonlinear Dirac equations

In physics, the Dirac equation is a relativistic quantum mechanical wave equation and provides a description of elementary spin-$\frac{1}{2}$ particles, such as electrons. In 1928, British physicist Paul Dirac derived the linear Dirac equation from the relation of the energy E and the momentum p of a free relativistic particle

$$E^2 = c^2p^2 + m^2c^4,$$

where c is the speed of light, m is the rest mass of the electron. Quantum mechanically , $E = i\hbar\frac{d}{dt}$ is the kinetic energy and $p = -i\hbar\nabla \in \mathbb{R}^3$ is the momentum operator ($\hbar$ is Planck's constant). For the wave function $\phi(x,t)$, a relativistic equation is obtained

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = \frac{m^2c^2}{\hbar^2}\phi. \tag{1.1.12}$$

In order to derive Dirac equation, Dirac proposed to factorize the left hand side of equation (1.1.12) as follows:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = (\gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z + \frac{i}{c}\gamma^0\partial_t)(\gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z + \frac{i}{c}\gamma^0\partial_t),$$

where $\gamma^0, \gamma^j (j = 1, 2, 3)$ are to be determined. On multiplying out the right side, one found out γ^0, γ^j must satisfy

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta_{ij}I, \quad \gamma^j\gamma^0 + \gamma^0\gamma^j = 0, \quad (\gamma^0)^2 = I.$$

These conditions could be met if γ^0, γ^j are at least 4×4 matrices (see [64]) and then the wave functions have four components. The usual representation

of γ^0, γ^j is

$$\gamma^0 = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

where σ^k are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given the factorization and equation (1.1.12), one can obtain the linear Dirac equation which is first order in space and time:

$$i\partial_t \psi - D_m \psi = 0, \quad D_m = -ic\hbar \sum_{k=1}^{3} \gamma^0 \gamma^k \partial_k + mc^2 \gamma^0.$$

for $\psi : \mathbb{R}^3_x \times \mathbb{R}_t \to \mathbb{C}^4$. Without loss of generality, we always assume $c = \hbar = 1$.

The Dirac operator is symmetric and has a negative continuous spectrum which is not bounded from below, as for $-\Delta$, which makes it harder to analyze mathematically and physically. On one hand, the energy functional is strongly indefinite. On the other hand, although there is no observable electron of negative energy, the negative spectrum plays an important role in the physics.

The main interest in this thesis is to present the various results regarding the standing wave solutions for nonlinear Dirac equations. A general form of nonlinear Dirac equation is given by

$$i\partial_t \psi - D_m \psi + \gamma^0 \nabla F(\psi) = 0. \tag{1.1.13}$$

From experimental data in [58], a Lorentz-invariant interaction term $F(\psi)$ is chosen in order to find a model of the free localized electron (or on another spin-$\frac{1}{2}$ particle). So the usual assumption on F is that $F \in C^2(\mathbb{C}^4, \mathbb{R})$ and

$$F(e^{i\theta}\psi) = F(\psi) \quad \text{for all} \quad \theta.$$

As pointed out in [58], stationary wave solutions of equation (1.1.13) represent the state of a localized particle which can propagate without changing its shape

$$\psi(x, t) = e^{-i\omega t}\phi(x).$$

where ϕ is a nonzero localized function satisfying

$$D_m \phi - \omega\phi - \gamma^0 \nabla F(\phi) = 0. \tag{1.1.14}$$

Different functions F have been used to model various types of self-interaction. In [58], Rañada gave a very interesting review on the historic background of different models.

The existence of standing wave solutions has been extensively studied by different methods. In [32, 33], Finkelstein et al. proposed various models for extended particles corresponding to the fourth order self-couplings like

$$F(\psi) := a(\bar{\psi}\psi)^2 + b(\bar{\psi}\gamma^5\psi)^2, \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3, \tag{1.1.15}$$

where $\bar{\psi}\psi = (\gamma^0\psi, \psi)$ and $(\cdot,\cdot)$ is the Hermitian inner product in $\mathbb{C}^4$. This nonlinearity is a sum of a scalar and pseudoscalar terms. In these papers, they gave some numerical results of the structure of the solutions for different values of the parameters. The special case $b = 0, a > 0$, called the Soler model [60], is proposed by Soler to describe elementary fermions. The advantage of this model is that its solutions can be factorized in spherical coordinates:

$$\phi(x) = \begin{pmatrix} g(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r) & \begin{pmatrix} \cos\Psi \\ \sin\Psi e^{i\Phi} \end{pmatrix} \end{pmatrix}. \tag{1.1.16}$$

Vazquez studied the Soler model in [67] and came up with a necessary condition for the existence of localized solutions (see also [52]). In [3] the existence of infinitely many standing waves are obtained. For the generalized Soler model when

$$F(\psi) = \frac{1}{2}G(\bar{\psi}\psi), G \in C^2(\mathbb{R},\mathbb{R}), G(0) = 0. \tag{1.1.17}$$

the existence of infinity many stationary solutions have been obtained in [21] for some function G and $\omega \in (0, m)$. Since the localized solutions are separable in spherical coordinates, the nonlinear Dirac equation can be reduced to a nonautonomous planar differential system. This system can be solved by a shooting method. Later on , this result was extended to a wider class of nonlinearities by Merle [54]. In the case where G has a singularity at the origin like $G(s) = -|s|^p, 0 < p < 1$, Balabane, Cazenave and Vazquez [9] proved that for every $\omega > m$, there exists a solution of (1.1.14) such that ϕ has a compact support.

But there are other models of self couplings for which the ansatz (1.1.16) is no longer valid, for instance (1.1.15) and

$$F(\phi) = \frac{1}{2}|\bar{\phi}\phi|^{\alpha_1} + b|\bar{\phi}\gamma^5\phi|^{\alpha_2}$$

with nonzero b and $\alpha_1, \alpha_2 > 0$. Esteban and Séré in [28] studied this more general nonlinearity. They proved by a variational method, there exists an infinity of solutions under the assumption

$$G'(x) \cdot x \geq \theta G(x), \quad \theta > 1, \quad x \in \mathbb{R}$$

for $1 < \alpha_1, \alpha_2 < \frac{3}{2}$.

None of the approaches mentioned above yield a curve of solutions: the continuity of ϕ with respect to ω, even the uniqueness of ϕ was unknown. These issues are important to study the stability of the standing waves. To our knowledge, Ounaies [56] was the first one to consider the regularity of the stationary solutions. He related the solutions to (1.1.14) to those of nonlinear Schrödinger equations. The ground states of Schrödinger equations generate a branch of solutions with small parameter $\varepsilon = m - \omega$ for nonlinear Dirac equations. He claimed that for $F(s) = |s|^\alpha, 1 < \alpha < 2$, ϕ_ω is continuous w.r.t. ω when $\omega \in (m - \varepsilon_0, m)$ for some $\varepsilon_0 > 0$. For a thorough review on the linear and nonlinear Dirac equation, we refer to the work by Esteban, Lewin and Séré [30].

A basic question about standing waves is their stability, which has been studied for nonlinear Klein-Gordon equations and nonlinear Schrödinger equations. Grillakis, Shatah and Strauss [35, 36] proved a general orbital stability and instability condition in a very general setting, which can be applied to traveling waves of nonlinear PDEs such as Klein-Gordon, and Schrödinger and wave equations. Their assumptions allow the second variation operator to have only one simple negative eigenvalue, a kernel of dimension one and the rest of the spectrum to be positive and bounded away from zero. But this method cannot be applied to the Dirac operator directly. Contrary to $-\Delta$, the Dirac operator $D_m = -i\gamma^0 \sum_{j=1}^3 \gamma^j \partial_j + \gamma^0 m$ is not bounded from below. However there are some partial results about the application of this method to the Dirac equation. Bogolubsky in [6] requires the positivity of the second variation of the energy functional as a necessary condition for stability. Werle [69], Strauss and Vázquez in [62] claim that the solitary waves are unstable, if the energy functional does not have a local minimum at the solitary waves.

1.2 Objectives

In the following I describe the results and the methods used in my thesis for the above equations.

1.2.1 Global well posedness and blow up for Landau-Lifshitz flow

For the Landau-Lifshitz flow equation (1.1.4) where $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$, the well posedness vs. blowup is studied in my thesis [39, 40]. A good starting point to analyze the flow equation is to assume some symmetry. For maps u with equivariant symmetry $u = e^{m\theta R}v(r)$, the energy $E(u)$ has a minimal energy $4\pi|m|$, which is attainable by a two-parameter family of harmonic maps:

$$H^{s,\alpha} = e^{(m\theta+\alpha)R}h(r/s), \quad \alpha \in \mathbb{R},\ s > 0$$
$$h(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix} = \begin{pmatrix} \frac{2}{r^m+r^{-m}} \\ 0 \\ \frac{r^m-r^{-m}}{r^m+r^{-m}} \end{pmatrix}.$$

The harmonic maps are static solutions of the evolution equation. The natural question is to consider the stability of the harmonic maps under the Landau-Lifshitz flow.

Our first result concerns m-equivariant maps with energy near the minimal energy $4\pi|m|$,

$$E(u_0) = 4\pi|m| + \delta_0^2,\ 0 < \delta_0 \ll 1.$$

We have shown that there is no finite time blowup for $|m| \geq 4$. Furthermore the solutions converge to a specific family of harmonic maps in the space-time norm sense. Hence we say that the harmonic maps are *asymptotically stable* under the Landau-Lifshitz flow. This result is a rigorous verification of [10] where the authors showed no singularity formation in finite time by formal asymptotic analysis. Our main ingredients involved in the proof are the usage of two different coordinate systems. In an appropriate orthonormal frame on the tangent plane, the coordinates of the tangent vector field $u_r - \frac{m}{r}J^uRu$ satisfy a nonlinear heat-Schrödinger type equation. It leaves us with an equation with small L^2 initial data. This equation is also coupled to a 2-dimensional dynamical system describing the dynamics of the scaling parameter $s(t)$ and rotation parameter$\alpha(t)$ of a nearby harmonic map $H^{(s(t),\alpha(t))}$. A careful choice of these parameters must be made at each time to allow estimates. The key to prove convergence of the solutions is the space-time estimates for the linear operator of the nonlinear heat-Schrödinger type equation. This can be done because of the energy inequality, the positivity of the energy and the assumption of radial functions.

For harmonic map heat flow, i.e. $\beta = 0$, in the subclass of m-equivariant maps (see (1.1.8)), we proved that for $m = 1$, finite time singularities do occur for some initial data close to the energy of harmonic maps. This result is an adaptation of the blow up result in [18] for a disk domain D^2 in $\mathbb{R}^2$.

1.2.2 Well-posedness and scattering of a model equation for Schrödinger maps

Recall the model equation (1.1.11)

$$i\phi_t = -\Delta\phi + \frac{m^2}{r^2}\frac{\sin 2|\phi|}{2|\phi|}\phi, \quad \phi(0,r) = \phi_0(r)$$

where $\phi(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ is a radial scalar function and $m > 0$ is an integer. We are interested in finite energy solutions which leads to $\sin|\phi_0(x)| = 0$ both at $|x| = 0$ and $|x| = \infty$. Therefore this yields the following boundary conditions

$$\begin{aligned} \phi_0 \in \Sigma := \{&\phi_0 : [0,\infty) \to \mathbb{C}, E(\phi_0) < \infty, \\ &\lim_{|x|\to 0} \phi_0(x) = k_1\pi, \lim_{|x|\to\infty} \phi_0(x) = k_2\pi\}, \quad k_1, k_2 \in \mathbb{Z}. \end{aligned} \tag{1.2.1}$$

By examining the energy $E(\phi)$, we find that for solutions in the class with $\phi_0(0) = 0, \phi_0(\infty) = \pi, E(\phi)$ has a minimal lower bound $4\pi|m|$. Harmonic maps are stationary solutions of (1.1.11). Moreover equation (1.1.11) possesses constant solutions $k\pi (k \in \mathbb{Z})$ with finite energy. I consider the well-posedness for solutions in the form $\phi(r,t) = \eta(r,t) + S$ for either $S = 0$ or $S = Q(r) = 2\arctan(r^m)$. In [43] I proved that for any

$$\eta_0 \in X := \{\phi : [0,\infty) \to \mathbb{C} | \phi_r \in L^2(rdr), \frac{\phi}{r} \in L^2(rdr)\},$$

there exists a maximal time interval I containing 0, such that the model equation (1.1.11) has a unique solution in the class $\phi(r,t) = \eta(r,t) + S$, satisfying

$$\eta \in C(I;X) \cap L^q_{loc}(I;X^r)$$

where

$$X^p := \{u : [0,\infty) \to \mathbb{C} | u_r \in L^p(rdr), \frac{u}{r} \in L^p(rdr)\}$$

and $2 < q < \infty, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Moreover, if $S = 0$ and $\|\phi_0\|_X \leq \delta$ for sufficiently small δ, then the solution to equation (1.1.11) is defined for all time. The approach is to linearize equation (1.1.11) at the stationary solutions. Treating the linearized operator as $-\Delta_r + \frac{m^2}{r^2}$ plus some perturbation, then the Strichartz estimates can be applied to yield the results.

1.2.3 Solitary wave solutions for a class of nonlinear Dirac equations

We consider a class of nonlinear Dirac equations in [41]

$$i\partial_t\psi + i\sum_{j=1}^{3}\gamma^j\partial_j\psi - m\psi + |\bar{\psi}\psi|^\theta\psi = 0. \tag{1.2.2}$$

Under the spherical coordinates ansatz (1.1.16), the equation for solitary waves can be reduced to a nonautonomous planar differential system for (f, g). A rescaling argument reveals that the solutions to this system is generated by the ground states of nonlinear Schrödinger equations

$$-\frac{\Delta v}{2m} + v - |v|^{2\theta}v = 0. \tag{1.2.3}$$

It is well known that for $\theta \in (0, 2)$, equation (1.2.3) admits a unique solution called the ground state $Q(x)$ which is smooth,positive, decreases monotonically as a functions of $|x|$ and decays exponentially at infinity.

We prove that when $1 \leq \theta < 2$, there exists $\varepsilon_0 > 0$ such that for $\omega \in (m - \varepsilon_0, m)$, there exists a solution $\psi(t, x) = e^{-i\omega t}\phi_\omega(x)$ and the mapping from ω to ϕ_ω is continuous. This result is different from that in [56] where Ounaies claimed it for $0 < \theta < 1$. But with the restriction $0 < \theta < 1$, we are unable to verify the Lipshitz continuity of the nonlinearities. Thus the contraction mapping theorem is not readily applied to construct solutions.

1.2.4 Instability of standing waves for the nonlinear Dirac equations

The stability problem of the standing wave solutions for the nonlinear Dirac equation with scalar self-interaction is considered in [42]. It is shown that the branch of standing waves constructed in [41] is unstable.

The question of stability is related to the eigenvalues of the linearized operator. To show the instability, the main goal is to show the linearized operator has an eigenvalue with positive real part. Since the linearized operator is four-by-four matrix, it is important to block-diagonalize it as in [22]. It turns out that the eigenvalue is related to that of the cubic, focusing and radial nonlinear Schrödinger equations. We use formal expansion of the eigenvalue and eigenfunction to show that there exists such an eigenvalue λ with $\operatorname{Re}\lambda$ positive. Then we use the "Lyapounov-Schmidt reduction" method from bifurcation theory to verify it rigorously.

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Chapter 2

Global existence and blow up for Landau-Lifshitz flow

As we introduced in the first chapter, we have obtained global result for the Landau-Lifshitz equation, including harmonic map heat flow. The following content is concentrated on harmonic map heat flow. The proof of global regularity and asymptotic stability results for Landau-Lifshitz equation is similar and is given in [11] (see the remark (*) in Section 2.2).

2.1 Introduction and main results

The harmonic map heat flow we consider is given by the equation

$$u_t = \Delta u + |\nabla u|^2 u, \qquad u(x,0) = u_0(x) \tag{2.1.1}$$

where $u(\cdot,t) : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{S}^2$, $\mathbb{S}^2$ is the 2-sphere

$$\mathbb{S}^2 := \{u = (u_1, u_2, u_3) \mid |u| = 1\} \subset \mathbb{R}^3,$$

Δ denotes the Laplace operator in $\mathbb{R}^n$, $|\nabla u|^2 = \sum_{j=1}^n \sum_{i=1}^3 (\frac{\partial u_i}{\partial x_j})^2$. This equation written in a more geometric way is

$$u_t = \sum_{j=1}^n D_j \partial_j u = P^u \Delta u$$

where P^u denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_u\mathbb{S}^2 := \{\xi \in \mathbb{R}^3 \mid \xi \cdot u = 0\}$$

to $\mathbb{S}^2$ at u, $\partial_j = \frac{\partial}{\partial x_j}$ is the usual partial derivative, and D_j is the covariant derivative, acting on vector fields $\xi(x) \in T_{u(x)}\mathbb{S}^2$:

$$D_j\xi := P^u \partial_j \xi = \partial_j \xi - (\partial_j \xi \cdot u)u = \partial_j \xi + (\partial_j u \cdot \xi)u.$$

A version of this chapter has been published. Guan, M., Gustafson, S. and Tsai, T.-P. Global existence and blow-up for harmonic map heat flow, J. Diff. Equations (2009) 246, 1–20.

Equation (2.1.1) is the gradient flow for the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx.$$

Static solutions of Equation (2.1.1) are harmonic maps from Ω to $\mathbb{S}^2$. Equation (2.1.1) is a particular case of the harmonic map heat flow between Riemannian manifolds introduced by Eells and Sampson ([8]). On the other hand, Equation (2.1.1) is a borderline case of the Landau-Lifshitz-Gilbert equations which model isotropic ferromagnetic spin systems:

$$u_t = aP\Delta u + bu \times P\Delta u, \quad a \geq 0. \tag{2.1.2}$$

(see [14, 16]). The harmonic map heat flow corresponds to the case $a = 1, b = 0$.

In this chapter, we consider space dimension $n = 2$, which makes the energy $E(u)$ invariant under scaling, and so is in some sense a borderline case for the interesting question of singularity formation vs. global regularity: do all solutions with smooth initial data remain smooth for all time, or do they form singularities in finite time for some data?

Let us recall some of the important results for $n = 2$. Struwe in [19] proved that weak solutions to Equation (2.1.1) exist globally for finite-energy initial data, and are smooth except for at most finitely many singular space-time points where non-constant harmonic maps "separate". Also, solutions are global for small initial energy. Freire showed that the weak solution is unique if the energy is non-increasing along the flow ([9]). Much effort has been devoted to the case where the domain is the unit disk in $\mathbb{R}^2$, and for a special class of solutions:

$$u(\cdot, t) : (r, \theta) \to (\cos m\theta \sin\phi(r,t), \sin m\theta \sin\phi(r,t), \cos\phi(r,t)) \tag{2.1.3}$$

for a positive integer m, usually $m = 1$. (r, θ) are polar coordinates. Then ϕ satisfies the equation:

$$\phi_t = \phi_{rr} + \frac{1}{r}\phi_r - m^2\frac{\sin 2\phi}{2r^2}, \quad 0 < r < 1, \ \ t > 0. \tag{2.1.4}$$

One specifies initial conditions $\phi(r, 0) = \phi_0(r)$, and typical boundary conditions are $\phi(0, t) = 0$, and $\phi(1, t) = \phi_1 \in \mathbb{R}$. For $m = 1$, global regularity is proved if $|\phi(0, r)| \leq \pi$ in [4]. However even if the flow exists for all time, it may develop singularities at $T = \infty$, so that it fails to converge asymptotically. In [5] the authors showed (again for $m = 1$) that, indeed, finite time

blow-up *does* occur for finite energy solutions, if $\phi_1 > \pi$. A result of [2] tells us that even if $m = 1$ and $|\phi_1| < \pi$, finite time blow up is still possible if $\phi(0, r)$ rises above π for some $r \in (0, 1)$. Recently, the generic blow-up behavior (and blow-up rate) was analyzed via formal asymptotics in [1], where they observed that whether or not singularities occur appears to depend on the degree m, as well as on the initial and boundary data. One of the purposes of our study is to provide a rigorous proof of this observation.

We note that finite-time singularities may also form in the harmonic map heat flow when $n \geq 3$ ([3, 7]).

It is worth remarking that the blow-up vs. global smoothness question is also currently studied for both the wave and Schrödinger "analogues" of the harmonic map heat flow. The possibility of finite-time blow-up for the energy-space critical ($n = 2$) *wave maps* was established recently ([15, 18]), while the problem remains open for $n = 2$ *Schrödinger maps* (the $a = 0$ case of (2.1.2)), though a partial answer was given in [13]: in contrast to the wave map case, high-degree equivariant (see next section for the definition) Schrödinger maps with near-harmonic energy are globally smooth.

The main goal of the present paper is *to address the global regularity vs. finite-time blowup question for a larger class of maps than* (2.1.3), *and for the problem on the plane, rather than a disk.* This means that the evolution is no longer described by a single, simple nonlinear heat equation like (2.1.4), but rather by a more complex *system.* In particular, maximum principles are no longer available (at least directly).

To be more precise, we consider m-equivariant maps $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$ with $m \in \mathbb{Z}$ a non-zero integer. An *m-equivariant map* $u : \mathbb{R}^2 \to \mathbb{S}^2$ is of the form

$$u(r, \theta) = e^{m\theta R} v(r)$$

where (r, θ) are polar coordinates on $\mathbb{R}^2$, $v : [0, \infty) \to \mathbb{S}^2$, and R is the matrix generating rotations around the u_3-axis:

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{\alpha R} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In what follows, we will take $m > 0$ (the $m < 0$ cases are equivalent, by a simple transformation).

If u is m-equivariant, we have $|\nabla u|^2 = |u_r|^2 + r^{-2}|u_\theta|^2 = |v_r|^2 + \frac{m^2}{r^2}|Rv|^2$ and so

$$E(u) = \pi \int_0^\infty \left(|v_r|^2 + \frac{m^2}{r^2}(v_1^2 + v_2^2) \right) r dr.$$

For finite energy $E(u)$, it is necessary to have $v(0), v(\infty) = \pm\hat{k}$, where $\hat{k} = (0,0,1)^T$ (see [10] Section 2.2 for details). We fix $v(0) = -\hat{k}$ and denote by Σ_m the class of m-equivariant maps with $v(\infty) = \hat{k}$:

$$\Sigma_m = \left\{ u : \mathbb{R}^2 \to \mathbb{S}^2 \mid \quad u = e^{m\theta R} v(r),\ \mathcal{E}(u) < \infty,\ v(0) = -\hat{k},\ v(\infty) = \hat{k} \right\}.$$

We measure distances between maps in Σ_m in the energy norm

$$\|u - \tilde{u}\|_{\dot{H}^1} = \|\nabla(u - \tilde{u})\|_{L^2}.$$

The class Σ_m contains (3.1.1) as a special case (up to a trivial reflection $u_3 \to -u_3$, and ignoring boundary conditions).

For u m-equivariant, the energy $E(u)$ can be rewritten as follows:

$$E(u) = \pi \int_0^\infty \left(|v_r|^2 + \frac{m^2}{r^2} |J^v R v|^2 \right) r dr = \pi \int_0^\infty |v_r - \frac{|m|}{r} J^v R v|^2 \, r dr + E_{min}$$

where $J^v := v\times$ is a $\pi/2$ rotation on $T_v\mathbb{S}^2$, and

$$E_{min} = 2\pi \int_0^\infty v_r \cdot \frac{|m|}{r} J^v R v \, r dr = 2\pi |m| \int_0^\infty (v_3)_r dr = 2\pi |m| [v_3(\infty) + 1]$$

(using $v_1^2 + v_2^2 + v_3^2 = 1$). The number E_{min}, which depends only on the boundary conditions, is in fact 4π times the absolute value of the *degree* of the map u, considered as a map from $\mathbb{S}^2$ to itself by compactifying the domain $\mathbb{R}^2$ (via stereographic projection); the degree is defined, for example, by integrating the pullback by u of the volume form on $\mathbb{S}^2$. It provides a lower bound for the energy of an m-equivariant map, $E(u) \geq E_{min}$, and this lower bound is attained if and only if

$$v_r = \frac{|m|}{r} J^v R v. \tag{2.1.5}$$

If $v(\infty) = -\hat{k}$, the minimal energy is $E_{min} = 0$ and is attained by the constant map, $u \equiv -\hat{k}$. On the other hand, if $u \in \Sigma_m$ so that $v(\infty) = \hat{k}$, the minimal energy is

$$E(u) \geq E_{min} = 4\pi |m|$$

and is attained by the 2-parameter family of harmonic maps

$$\mathcal{O}_m := \left\{ e^{m\theta R} h^{s,\alpha}(r) \mid s > 0,\ \alpha \in [0, 2\pi) \right\}$$

where

$$h^{s,\alpha}(r) := e^{\alpha R} h(r/s),$$

and

$$h(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix}, \quad h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, \quad h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}.$$

We record for later use that $h(r)$ satisfying (2.1.5) means

$$(h_1)_r = -\frac{m}{r} h_1 h_3, \quad (h_3)_r = \frac{m}{r} h_1^2.$$

So $\mathcal{O}_m$ is the orbit of the single harmonic map $e^{m\theta R} h(r)$ under the symmetries of the energy $\mathcal{E}$ which preserve equivariance: scaling, and rotation. Explicitly,

$$e^{m\theta R} h^{s,\alpha}(r) = \begin{pmatrix} \cos(m\theta + \alpha) h_1(r/s) \\ \sin(m\theta + \alpha) h_1(r/s) \\ h_3(r/s) \end{pmatrix}.$$

We begin with the energy-space local-in-time theory:

Theorem 2.1.1 *Let $m \geq 1$. There exist $\delta > 0$ and $C > 0$ such that if $u_0 \in \Sigma_m$ and $\mathcal{E}(u_0) = 4\pi m + \delta_0^2$ for some $\delta_0 \leq \delta$, then the following hold:*

(a) there exists $T = T(u_0) > 0$ and a unique solution $u(t) \in C([0,T); \Sigma_m)$ to Equation (2.1.1). $E(u(t))$ is non-increasing for $t \in [0,T)$.

(b) there exist $s(t) \in C([0,T); (0,\infty))$ and $\alpha(t) \in C([0,T); \mathbb{R})$ so that

$$\left\| u(x,t) - e^{(m\theta + \alpha(t))R} h(r/s(t)) \right\|_{\dot{H}^1(\mathbb{R}^2)} < C\delta_0, \quad \forall t \in (0,T). \quad (2.1.6)$$

(c) Suppose $T < \infty$. Then T is the maximal existence time of the solution $u(t) \in C([0,T); \Sigma_m)$ if and only if

$$\liminf_{t \to T^-} s(t) = 0. \quad (2.1.7)$$

Remark 2.1.2 Statement (b) of Theorem 2.1.1 implies the *orbital stability* of harmonic maps (at least up to the possible blow-up time) under the heat flow. If the initial data u_0 is close to $\mathcal{O}_m$ in $\dot{H}^1$, then solutions of equation (2.1.1) will stay close to the harmonic maps in $\dot{H}^1$ (though not necessarily in $\dot{H}^2$). Statement (c) can be viewed as a characterization of blow-up for energy near E_{min}: solutions blow-up if and only if the $\dot{H}^1$-nearest harmonic map "collapses" (i.e., its length-scale goes to zero). Here $s(t)$ and $\alpha(t)$ are determined by finding, at each time t, the harmonic map which is $\dot{H}^1$- closest to $u(t)$.

A theorem identical to Theorem 2.1.1 is established in [12] for the (more delicate) corresponding Schrödinger flow problem. The proof there uses the same geometric representation and decomposition of the solution used in the present paper, and indeed we show here that the same estimates (and more) hold for the linearized problem (Section 2.2) and the nonlinear terms (Section 2.3), and so the proof carries over with no significant alteration. For this reason, and since the full energy space local well-posedness (without symmetry or energy restrictions) is already well-understood for the heat flow (in particular, Struwe [19]), we will not provide the details. We remark that the blow-up characterization (2.1.7) corresponds to the "separation" of a harmonic map at a singularity in [19].

The next theorem, our main result, shows that when the degree is at least 4, singularities do *not* form, and we can describe precisely the asymptotic behavior:

Theorem 2.1.3 *Let $m \geq 4$. There exists $\delta_1 \in (0, \delta)$ and $C > 0$ such that if $\delta_0 \leq \delta_1$, then the existence time $T = T(u_0)$ in Theorem 2.1.1 can be taken to be $T = \infty$. One also has*

$$\|\nabla(u(x,t) - e^{m\theta R} h^{(s(t),\alpha(t))})\|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \leq C\delta_0. \tag{2.1.8}$$

Moreover there exist α_∞ and positive s_∞ such that

$$(s(t), \alpha(t)) \to (s_\infty, \alpha_\infty) \text{ as } t \to \infty. \tag{2.1.9}$$

Remark 2.1.4

1. Not only is the solution global, but (2.1.8) and (2.1.9) show that $u(\cdot, t)$ converges to a fixed harmonic map as $t \to \infty$ (at least in a time-averaged sense) – in particular, this gives *asymptotic stability* of the harmonic maps for $m \geq 4$.

2. For the cases $m = 2, 3$, we conjecture that solutions are still global, but this is presently beyond the reach of our methods. The technical reason is that we need $r^2 h_1(r) \in L^2(rdr)$, which requires $m > 3$.

The final theorem shows that when $m = 1$, finite-time blow-up *does* occur within our class of solutions.

Theorem 2.1.5 *If $m = 1$, for any $\delta > 0$, there exists $u_0 \in \Sigma_1$ with $0 < E(u_0) - 4\pi \leq \delta^2$ such that the corresponding solution of the harmonic map heat flow blows up in finite time, in the sense that $\|\nabla u(\cdot, t)\|_{L_x^\infty} \to \infty$.*

Remark 2.1.6 Our result that blowup occurs for degree one, but not for higher degree, is consistent with the formal asymptotic analysis of [1].

The paper is organized as follows. In Section 2 we derive, from the harmonic map heat flow equation, a related nonlinear heat equation, by a choice of frame on the tangent space. We also establish space-time estimates (including "endpoint"-type estimates) and weighted estimates for the linear operator which comes from the perturbation about the harmonic maps. Even though the potential appearing in the linear operator behaves like $1/|x|^2$ both at the origin and as $|x| \to \infty$, we can treat it by an energy inequality to avoid the difficulty. In Section 3, we obtain explicit equations for the parameters $(s(t), \alpha(t))$ by a choice of suitable orthogonality condition which only works for $m \geq 3$. On the basis of the space-time estimates obtained in Section 2, we give the proof of Theorem 2.1.3. In Section 4, we construct an example to show that finite time singularities really occur for energy close to the harmonic map energy when $m = 1$, proving Theorem 2.1.5.

Throughout the paper, the letter C is used to denote a generic constant, the value of which may change from line to line.

2.2 Derived nonlinear heat equation

In this section we derive a nonlinear heat equation associated to the harmonic map heat flow. We use the technique introduced in [6], obtaining an equation for the coordinates of the tangent vector field $v_r - \frac{m}{r} J^v Rv$ with respect to a certain orthonormal frame.

Under the m-equivariance assumption that the solution to equation (2.1.1) has the form $u(x,t) = e^{m\theta R} v(r,t)$, v satisfies the evolution equation:

$$v_t = (D_r^v + \frac{1}{r} - \frac{m v_3}{r})(v_r - \frac{m}{r} J^v Rv)$$

where, recall, D_r^v is the covariant derivative, acting on vector fields tangent to $\mathbb{S}^2$ at v. Let $e \in T_v\mathbb{S}^2$ be a unit tangent vector field parallel transported along the curve $v(\cdot, t)$:

$$D_r^v e = 0.$$

Then $\{e, J^v e\}$ is an orthonormal frame on $T_v\mathbb{S}^2$. Let $q(r,t) = q_1(r,t) + iq_2(r,t)$ be the complex coordinates of the vector field $v_r - \frac{m}{r} J^v Rv \in T_v\mathbb{S}^2$ in this basis :

$$v_r - \frac{m}{r} J^v Rv = q_1 e + q_2 J^v e.$$

We sometimes write $qe = q_1 e + q_2 J^v e$ for convenience. Define

$$J^v R v = \nu_1 e + \nu_2 J^v e, \quad \nu = \nu_1 + i\nu_2.$$

Now it is a straightforward matter to show that the complex function $q(r,t)$ solves the following nonlinear heat equation with a non-local nonlinearity

$$q_t = \Delta_r q - \frac{(1-mv_3)^2}{r^2} q - \frac{m(v_3)_r}{r} q - qN(q) \tag{2.2.1}$$

where

$$N(q) = \int_r^\infty \frac{1}{r'} Q(r') dr', \quad Q := i \operatorname{Im}\left(r\bar{q} q_r + m\bar{\nu}\left[q_r + \frac{1-mv_3}{r} q \right] \right).$$

We will use equation (2.2.1) to obtain estimates on q.

Given an m-equivariant map $u(x,t) \in \Sigma_m$ with $E(u_0) - 4\pi m < \delta_0^2$, we would like to write the solution $u = e^{m\theta R} v(r,t)$ with:

$$v(r,t) = e^{\alpha(t)R}(h(r/s(t)) + \xi(r/s(t),t)) \tag{2.2.2}$$

where $\xi(r/s(t),t)$ is a perturbation. Using the explicit orthonormal basis of $T_h\mathbb{S}^2$,

$$\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad J^h \hat{\jmath} = \begin{pmatrix} -h_3 \\ 0 \\ h_1 \end{pmatrix},$$

it is convenient to decompose the perturbation term ξ into components tangential and normal to $\mathbb{S}^2$ at $h(\rho)$:

$$\xi(\rho,t) = z_1(\rho,t)\hat{\jmath} + z_2(\rho,t) J^h \hat{\jmath} + \gamma(\rho,t) h \tag{2.2.3}$$

for $\rho = r/s(t)$. This decomposition defines the complex-valued function $z := z_1 + iz_2$. Using $v_3(\rho,t) = h_3(\rho) + \xi_3(\rho,t)$, we find $(v_3)_r = \frac{m}{r} h_1^2(\rho) + (\xi_3(\rho,t))_r$, which we substitute into (2.2.1) to obtain

$$\begin{aligned} q_t = &-Hq - 2m\frac{h_3(\rho) - h_3(r)}{r^2} q \\ &- \frac{2m^2 h_3 \xi_3 + m^2 \xi_3^2 - 2m\xi_3}{r^2} q - \frac{m(\xi_3)_r}{r} q - qN(q), \end{aligned} \tag{2.2.4}$$

where H is the operator

$$H = -\Delta + V(r), \qquad V(r) = \frac{1 + m^2 - 2mh_3(r)}{r^2}.$$

So $q(r,t)$ satisfies a nonlinear heat equation with linear operator H. Now the difficulty comes from the singular potential $V(r)$ which behaves like $const./r^2$ as $r \to 0$ and as $r \to \infty$ (with different constants). To some extent, H is like the heat operator with an inverse-square potential which is studied in [21]. But their arguments only work for dimension $n \geq 3$ because of the lack of Hardy inequality in dimension 2. In this chapter we need to obtain space-time estimates for e^{-tH}, the one-parameter semigroup generated by $-H$. We know that for the free heat operator $e^{t\Delta}$, the following inequalities hold ([10]):

$$\begin{aligned} \|e^{t\Delta}\phi\|_{L_t^r L_x^p} &\leq C\|\phi\|_{L^2}, \\ \|\int_0^t e^{(t-s)\Delta} f(s)ds\|_{L_t^r L_x^p} &\leq C\|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}} \end{aligned} \tag{2.2.5}$$

where (r,p) is an "admissible pair" – i.e., $1/r + 1/p = 1/2$, $(\tilde{r}', \tilde{p}')$ is the conjugate exponent pair of another admissible pair $(\tilde{r}, \tilde{p})$, excluding the case $r = \tilde{r} = 2$. But in a following Lemma, we prove that not only do estimates like (2.2.5) hold for the operator H, but the "endpoint" version ($r = \tilde{r} = 2$) also holds for radial functions ϕ and f.

A preliminary lemma ensures that H is self adjoint.

Lemma 2.2.1 *([17] pp.161) Let $V(r)$ be a continuous radial potential on $\mathbb{R}^n \backslash \{0\}$ satisfying*

$$V(r) + \frac{(n-1)(n-3)}{4r^2} \geq \frac{3}{4r^2}.$$

Then $-\Delta + V(r)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n \backslash \{0\})$.

On the basis of Lemma 2.2.1, we have the following:

Lemma 2.2.2 *The operator H extends to a positive self-adjoint operator on a domain $D(H)$ with $C_0^\infty(R^2 \backslash \{0\}) \subset D(H) \subset L^2(\mathbb{R}^2)$, hence $-H$ is the infinitesimal generator of a contraction semigroup $\{e^{-tH}\}_{t \geq 0}$ on $L^2(\mathbb{R}^2)$. Furthermore, $D(H) \subset L^\infty(\mathbb{R}^2)$.*

Proof. We first consider $m \geq 2$. Since $V(r) \geq (1-m)^2/r^2$, by Lemma 2.2.1, H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \backslash \{0\})$, and so its closure (which we still denote by H) can be uniquely extended to a self-adjoint operator on a dense domain $D(H) \subset L^2(\mathbb{R}^2)$. When $m = 1$, the operator H is simplified as

$$H = -\Delta_r + \frac{4}{r^2} - \frac{4}{1+r^2}.$$

By the above argument $\tilde{H} = -\Delta + \frac{4}{r^2}$ extends to a self-adjoint operator on $L^2(\mathbb{R}^2)$, and under the bounded perturbation $\frac{4}{1+r^2}$ it remains self-adjoint by the Kato perturbation theory ([17]).

So, H generates a semigroup e^{-tH} on $L^2(\mathbb{R}^2)$. The non-negativity of V immediately implies $H \geq 0$, and so e^{-tH} is a contraction semigroup on $L^2(\mathbb{R}^2)$.

The final part of the lemma, the L^∞ estimate, is more delicate. Let $\varphi \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})$. We have

$$\int_{\mathbb{R}^2} \{|\nabla\varphi|^2 + V|\varphi|^2\} = (\varphi, H\varphi) \leq \|\varphi\|_{L^2}\|H\varphi\|_{L^2}$$

and since for a fixed disk D_R centered at the origin

$$|x|^2 V(|x|) \geq 1 \text{ on } D_R, \text{ and } V(|x|) \text{ bounded on } D_R^c, \tag{2.2.6}$$

we conclude

$$\|\nabla\varphi\|_{L^2}^2 + \left\|\frac{\varphi}{|x|}\right\|_{L^2}^2 \leq C(\|\varphi\|_{L^2}^2 + \|H\varphi\|_{L^2}^2). \tag{2.2.7}$$

Now multiply $H\varphi = -\Delta\varphi + V\varphi$ by $\varphi/|x|$ and integrate by parts to obtain

$$\begin{aligned}\left\|\frac{\varphi}{|x|}\right\|_{L^2}\|H\varphi\|_{L^2} &\geq \int_0^{2\pi}\int_0^\infty (-\varphi\varphi_{rr} - \varphi\varphi_r/r - \varphi\varphi_{\theta\theta}/r^2 + V\varphi^2)drd\theta \\ &= \int_0^{2\pi}\int_0^\infty (\varphi_r^2 + \varphi_\theta^2/r^2 + (r^2V - \frac{1}{2})\varphi^2/r^2)drd\theta\end{aligned}$$

and so by (2.2.6) again, and (2.2.7),

$$\||x|^{-1/2}\nabla\varphi\|_{L^2}^2 + \||x|^{-3/2}\varphi\|_{L^2}^2 \leq C(\|\varphi\|_{L^2}^2 + \|H\varphi\|_{L^2}^2). \tag{2.2.8}$$

Fix $p \in (1, 4/3)$, and set $q := \frac{2p}{2-p} \in (2, 4)$. Using (2.2.8), we have

$$\begin{aligned}\|\Delta\varphi\|_{L^p(D_R)} &\leq \|H\varphi\|_{L^p(D_R)} + \|V\varphi\|_{L^p(D_R)} \\ &\leq C(\|H\varphi\|_{L^2(D_R)} + \||x|^{3/2}V\|_{L^q(D_R)}\||x|^{-3/2}\varphi\|_{L^2(D_R)}) \\ &\leq C(\|H\varphi\|_{L^2} + \|\varphi\|_{L^2})\end{aligned}$$

which combined with

$$\|\Delta\varphi\|_{L^2(D_R^c)} \leq \|H\varphi\|_{L^2} + \|V\|_{L^\infty(D_R^c)}\|\varphi\|_{L^2} \leq C(\|H\varphi\|_{L^2} + \|\varphi\|_{L^2})$$

and a Sobolev inequality, yields the required estimate

$$\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{W^{2,p}(D_R)\cap H^2(D_R^c)} \leq C(\|H\varphi\|_{L^2} + \|\varphi\|_{L^2}). \tag{2.2.9}$$

Remark 2.2.3 The Kato perturbation theory for self-adjoint operators is not applicable here, since $V(r)$ is too singular at the origin.

Next we establish some properties of the semigroup e^{-tH} satisfied by the well-known semigroup $e^{t\Delta}$.

Lemma 2.2.4 *Let $\{e^{-tH}\}_{t\geq 0}$ be the semigroup generated by the operator H in $L^2(\mathbb{R}^2)$. Let $1 \leq a \leq b \leq \infty$. For $\varphi \in L^a(\mathbb{R}^2)$,*

$$\left\|e^{-tH}\varphi\right\|_{L^b} \leq C\, t^{-(1/a-1/b)} \left\|\varphi\right\|_{L^a} \quad \textit{for all } t > 0$$

(here $e^{-tH}\varphi$ can be defined by density).

Proof. If $a = b = 2$, the statement just follows from the fact that $\{e^{-tH}\}_{t\geq 0}$ is a contraction semigroup on $L^2(\mathbb{R}^2)$. Now let $\varphi \in C_0^\infty(\mathbb{R}^2\backslash\{0\}) \subset D(H)$. Then $u(t) := e^{-tH}\varphi \in D(H)$. Thus $Hu(t) = He^{-tH}\varphi = e^{-tH}H\varphi$, and so $\|Hu(t)\|_{L^2} \leq \|H\varphi\|_{L^2}$. Using (2.2.9), we find

$$\|u(t)\|_{L^\infty} \leq C(\|Hu(t)\|_{L^2} + \|u(t)\|_{L^2}) \leq C(\|H\varphi\|_{L^2} + \|\varphi\|_{L^2}),$$

so there exists $M > 0$ such that $\sup_{x,t} |u(x,t)| \leq M$. Now we want to apply the maximum principle in $\mathbb{R}^2$. We first assume $\varphi \geq 0$, so that $e^{-tH}\varphi \geq 0$ a.e. for each $t > 0$, since the semigroup e^{-tH} is positivity preserving (see pp. 246 in [17]). Since $u(t) = e^{-tH}\varphi$ solves

$$u_t = \Delta u - V(x)u,$$

and $V(x) \geq 0$, given any $0 \leq \varphi \in C_0^\infty(\mathbb{R}^2\backslash\{0\})$, we have $u_t \leq \Delta u$, which means $e^{-tH}\varphi$ is a subsolution to the heat equation. Since $\sup_{x,t} |u(x,t)| \leq M$, it follows by the maximum principle that

$$0 \leq e^{-tH}\varphi \leq e^{t\Delta}\varphi$$

for all $t > 0$, and hence

$$\left\|e^{-tH}\varphi\right\|_{L^b(\mathbb{R}^2)} \leq \left\|e^{t\Delta}\varphi\right\|_{L^b(\mathbb{R}^2)} \leq C\, t^{-(1/a-1/b)} \left\|\varphi\right\|_{L^a(\mathbb{R}^2)}.$$

For general $\varphi \not\geq 0$, we can rewrite $\varphi = \varphi^+ - \varphi^-$ where $\varphi^+ = \max\{\varphi, 0\} \geq 0, \varphi^- = \max\{-\varphi, 0\} \geq 0$ and reach the same conclusion since $|e^{-tH}\varphi| \leq e^{t\Delta}|\varphi|$. The proof of the inequality for $\varphi \in L^a$ follows by the density of $C_0^\infty(\mathbb{R}^2\backslash\{0\})$ in L^a.

Recall that the potential is singular at the origin. Fortunately, precisely because of its inverse-square form and positivity, it yields "endpoint"-type

space-time estimates. Consider the mixed space-time Lebesgue norms: for an interval $I \subset \mathbb{R}^+$,

$$\|f\|^r_{L^r_t L^p_x(\mathbb{R}^2 \times I)} := \int_I \left(\int_{\mathbb{R}^2} |f(x,t)|^p dx \right)^{r/p} dt.$$

Theorem 2.2.5 *Let the exponent pairs (r,p) and $(\tilde{r},\tilde{p})$ be "admissible" (i.e. $1/r + 1/p = 1/2$), but exclude the case $r = \tilde{r} = 2$. Then we have*

$$\|e^{-tH}\varphi\|_{L^r_t L^p_x} + \| \int_0^t e^{-(t-s)H} f(s) ds\|_{L^r_t L^p_x} \leq C(\|\varphi\|_{L^2} + \|f\|_{L^{\tilde{r}'}_t L^{\tilde{p}'}_x})$$

for all functions $\varphi, f(\cdot,t)$ on $\mathbb{R}^2$. If, in addition, $m \geq 2$, and $\varphi, f(\cdot,t)$ are radial functions, then the estimate holds also in the "endpoint" case $r = \tilde{r} = 2$, and we have weighted estimates

$$\|\frac{1}{|x|} e^{-tH}\varphi\|_{L^2_t L^2_x} + \|(e^{-tH}\varphi)_r\|_{L^2_t L^2_x} \leq C\|\varphi\|_{L^2}$$

$$\|\frac{1}{|x|} \int_0^t e^{-(t-s)H} f(s) ds\|_{L^2_t L^2_x} + \|(\int_0^t e^{-(t-s)H} f(s) ds)_r\|_{L^2_t L^2_x} \leq C\|f\|_{L^{\tilde{r}'}_t L^{\tilde{p}'}_x}.$$

Remark 2.2.6 It is easy to check that the above "endpoint" estimate *fails* for the free heat operator $e^{t\Delta}$, even for radial functions (see [20] for the Schrödinger case).

Proof. Let's first prove the non-endpoint estimates. For the homogeneous estimate, we establish a more general result, following [10]:

$$\|e^{-tH}\varphi\|_{L^r_t L^p_x} \leq C\|\varphi\|_{L^a}$$

with $\frac{1}{r} + \frac{1}{p} = \frac{1}{a}$, $r \geq a > 1$. For fixed $p \in (1,\infty]$, define $\Gamma(t)\varphi = \|e^{-tH}\varphi\|_{L^p}$. By Lemma 2.2.4

$$\Gamma(t)\varphi \leq Ct^{-(1-1/p)}\|\varphi\|_{L^1}.$$

So Γ is of weak type $(1, \frac{p}{p-1})$. On the other hand,

$$\|e^{-tH}\varphi\|_{L^\infty_t L^p_x} \leq C\|\varphi\|_{L^p},$$

so Γ is of strong type (p,∞). By the Marcinkiewicz interpolation theorem, Γ is of strong type (a,r) with $\frac{1}{r} = \frac{1}{a} - \frac{1}{p}$, $r \geq a > 1$, and

$$\|e^{-tH}\varphi\|_{L^r_t L^p} \leq C\|\varphi\|_{L^a}.$$

Now we turn to the nonhomogeneous estimate. For $1 \le \tilde{p}' \le p \le \infty$, we have

$$\| \int_0^t e^{-(t-s)H} f(s) ds \|_{L_x^p} \le \int_0^t (t-s)^{-(1/\tilde{p}'-1/p)} \|f(s)\|_{L_x^{\tilde{p}'}} ds,$$

so by the Hardy-Littlewood inequality,

$$\| \int_0^t e^{-(t-s)H} f(s) ds \|_{L_t^r L_x^p} \le C \|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}$$

with $\frac{1}{r} + \frac{1}{p} + \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}} = 1$, provided $0 < 1/\tilde{r} + 1/r < 1$. In particular if $(r,p), (\tilde{r},\tilde{p})$ are admissible (excepting $r = \tilde{r} = 2$), we obtain the desired space-time estimate.

Now we will prove the "endpoint" case $(r,p) = (\tilde{r},\tilde{p}) = (2,\infty)$ for radial functions, if $m \ge 2$. Our method relies on the energy inequality. Let $\varphi, f(s)$ be radial functions and $u(x,t) = e^{-tH}\varphi$. By the imbedding inequality

$$\|u\|_{L^\infty} \le C(\|u_r\|_{L^2} + \|u/r\|_{L^2}) \tag{2.2.10}$$

for radial functions in two-dimensions (see [10]), we have

$$\|u\|_{L_t^2 L_x^\infty} \le C(\|u_r\|_{L_t^2 L_x^2} + \|\frac{u}{r}\|_{L_t^2 L_x^2}). \tag{2.2.11}$$

Since in two dimensions the Hardy inequality does not hold, we change variable

$$v(x,t) := e^{i\theta} u(r,t)$$

so that we have

$$\|u_r\|_{L_t^2 L_x^2} + \|\frac{u}{r}\|_{L_t^2 L_x^2} \le C \|\nabla v\|_{L_t^2 L_x^2} \tag{2.2.12}$$

since $|\nabla v|^2 = |u_r|^2 + |\frac{u}{r}|^2$. Now $v(x,t)$ solves the equation:

$$v_t = \Delta v - \frac{m^2 - 2mh_3}{r^2} v \tag{2.2.13}$$

with initial data $e^{i\theta}\varphi$. Multiplying equation (2.2.13) by $\bar{v}$, taking the real part and using $m^2 - 2mh_3 > 0$ (since $m \ge 2$), then integrating over space and time, we arrive at

$$\|v\|_{L_t^\infty L_x^2}^2 + \|\nabla v\|_{L_t^2 L_x^2}^2 \le \|v(0)\|_{L^2}^2 = 2\pi \|\varphi\|_{L^2}^2.$$

Note that using (2.2.11)-(2.2.12) gives us $\|e^{-tH}\varphi\|_{L_t^2L_x^\infty} \le C\|\varphi\|_{L^2}$ (which in any case is covered by the above argument). Similarly we get:

$$\|\frac{e^{-tH}\varphi}{r}\|_{L_t^2L_x^2}+\|(e^{-tH}\varphi)_r\|_{L_t^2L_x^2} = \|\frac{u}{r}\|_{L_t^2L_x^2}+\|u_r\|_{L_t^2L_x^2} \le C\|\nabla v\|_{L_t^2L_x^2} \le C\|\varphi\|_{L^2}.$$

Now let $(\tilde{r},\tilde{p}) = (2,\infty)$. If $w(r,t) := \int_0^t e^{-(t-s)H}f(s)ds$, then $w(r,0) = 0$. By the above arguments we get, for $v = e^{i\theta}w$,

$$\begin{aligned}\|w\|^2_{L_t^2L_x^\infty} \le C\|\nabla v\|^2_{L_t^2L_x^2} \le C\int\int_{\mathbb{R}^2}|fv|dxds \le \varepsilon\|v\|^2_{L_t^2L_x^\infty} + C(\varepsilon)\|f\|^2_{L_t^2L_x^1}\\ = \varepsilon\|w\|^2_{L_t^2L_x^\infty} + C(\varepsilon)\|f\|^2_{L_t^2L_x^1}.\end{aligned}$$

Taking $\varepsilon = 1/2$, we obtain $\|w\|_{L_t^2L_x^\infty} \le C\|f\|_{L_t^2L_x^1}$, the "endpoint" estimate we were seeking. We also get the weighted estimate, since

$$\begin{aligned}&\|\frac{w}{r}\|^2_{L_t^2L_x^2} + \|w_r\|^2_{L_t^2L_x^2} \le \|\nabla v\|^2_{L_t^2L_x^2} \le \int\int_{\mathbb{R}^2}|fv|dxds\\ &\le C\|v\|_{L_t^rL_x^p}\|f\|_{L_t^{\tilde{r}'}L_x^{\tilde{p}'}} \le C\|f\|^2_{L_t^{\tilde{r}'}L_x^{\tilde{p}'}}.\end{aligned}$$

As a direct result of this proof, we have:

Corollary 2.2.7 *Theorem 2.2.5 also holds for e^{-tH} where H is any operator of the form*

$$H = -\Delta + \frac{a(r)}{r^2}, \quad 1 \le a(r) \le C.$$

Remark (*). A theorem similar to Theorem 2.2.5 is obtained for the linearized operator of Landau-Lifshitz equation. Let $\alpha = \beta = 1$. A Schrödinger-heat type equation similar to equation (2.2.1) is obtained, where the linear part is

$$q_t = (1+i)(\Delta - V(r))q. \tag{2.2.14}$$

We can space-time estimates for the linear evolution operator $e^{-(1+i)tH}$ without using maximum principle.

Theorem 2.2.8 *Let the exponent pairs (r,p) and $(\tilde{r},\tilde{p})$ be "admissible" (i.e. $1/r + 1/p = 1/2$), including the case $r = \tilde{r} = 2$. Then we have*

$$\|e^{-t(1+i)H}\varphi\|_{L_t^rL_x^p} + \|\int_0^t e^{-(t-s)(1+i)H}f(s)ds\|_{L_t^rL_x^p} \le C(\|\varphi\|_{L^2} + \|f\|_{L_t^{\tilde{r}'}L_x^{\tilde{p}'}})$$

for $m \geq 2$, and radial functions $\varphi, f(\cdot,t)$. Moreover the weighted estimates hold

$$\|\frac{1}{|x|}e^{-t(1+i)H}\varphi\|_{L^2_tL^2_x} + \|(e^{-t(1+i)H}\varphi)_r\|_{L^2_tL^2_x} \leq C\|\varphi\|_{L^2}$$

$$\|\frac{1}{|x|}\int_0^t e^{-(t-s)(1+i)H}f(s)ds\|_{L^2_tL^2_x} + \|(\int_0^t e^{-(t-s)(1+i)H}f(s)ds)_r\|_{L^2_tL^2_x} \leq C\|f\|_{L^{\tilde{r}'}_tL^{\tilde{p}'}_x}.$$

Proof. The goal is to estimate $u(x,t)$ a solution of the linear inhomogeneous initial value problem

$$u_t + (1+i)Hu = f, \quad w(x,0) = \phi(|x|)$$

First we prove the basic L^2 estimate. Multiplying this equation by $\bar{u}$, taking the real part, and integrating in space and time, then we obtain the following estimate

$$\|u\|_{L^\infty_tL^2_x} + \|\nabla u\|_{L^2_tL^2_x} \leq C(\|\phi\|_{L^2} + \|f\|_{L^1_tL^2_x}).$$

Then we can obtain the weighted space-time estimates. Since the hardy inequality is not true in dimension 2, by using the change of functions

$$w(x,t) = e^{i\theta}u(x,t),$$

then

$$|\nabla w| = |\frac{u}{r}| + |u_r|$$

and $w(x,t)$ solves the equation

$$w_t + (1+i)(-\Delta + \tilde{V})w = fe^{i\theta}.$$

where $\tilde{V} = \frac{m^2-2mh_3}{r^2} \geq 0$ for $m \geq 2$. By embedding theorem, if radial function u,

$$\|u\|_{L^\infty} \leq \|u_r\|_{L^2_x} + \|\frac{u}{r}\|_{L^2_x} = \|\nabla w\|_{L^2_x}$$

Therefore

$$\|u\|_{L^2_tL^\infty_x} + \|u_r\|_{L^2_xL^2_t} + \|\frac{u}{r}\|_{L^2_xL^2_t} \leq C\|\nabla w\|_{L^2_xL^2_t}.$$

Similarly multiplying the equation for w by $\bar{w}$, taking the real part, and integrating in space and time (using the positivity of $\tilde{V}$),

$$\|\nabla w\|_{L^2_tL^2_x} \leq \|\phi\|_{L^2} + \|wf\|_{L^1_tL^1_x}.$$

Then

$$\|u\|_{L_t^2 L_x^\infty} + \|u_r\|_{L_x^2 L_t^2} + \|\frac{u}{r}\|_{L_x^2 L_t^2} \leq C\|\phi\|_{L^2} + \varepsilon\|u\|_{L_t^2 L_x^\infty} + C(\varepsilon)\|f\|_{L_t^2 L_x^1}.$$

If ε is small, we arrive at the endpoint estimate

$$\|u\|_{L_t^2 L_x^\infty} + \|u_r\|_{L_x^2 L_t^2} + \|\frac{u}{r}\|_{L_x^2 L_t^2} \leq C(\|\phi\|_{L^2} + \|f\|_{L_t^2 L_x^1}).$$

By Hölder inequality and interpolation yields all the desired estimates. □

2.3 Proof of the main theorem

In this section we prove Theorem 2.1.3 which gives the global well-posedness for the equivariant harmonic map heat-flow with near-harmonic energy when the degree m is at least four. This is done through the study of a coupled system of ODEs for the parameters $s(t)$ and $\alpha(t)$, and a nonlinear heat-type PDE for the deviation of the solution from the harmonic maps. In particular, we will show that the length-scales $s(t)$ of certain "nearby" harmonic maps, stay bounded away from zero, and, in fact, converge as $t \to \infty$.

For an m-equivariant solution $u(x,t) \in \Sigma_m$ of (2.1.1), with initial energy

$$E(u_0) = 4\pi m + \delta_0^2, \qquad \delta_0 \ll 1$$

the decomposition (2.2.2)-(2.2.3) of u into a harmonic map with time-varying parameters, and a controlled correction, is established in [13]:

Lemma 2.3.1 *[13] If $m \geq 3$ and δ_0 is sufficiently small, then for any map $u \in \Sigma_m$ with $\mathcal{E}(u) \leq 4\pi m + \delta_0^2$, there exists $s > 0$, $\alpha \in \mathbb{R}$, and a complex function $z = z_1 + iz_2$, such that*

$$u(r,\theta) = e^{(m\theta+\alpha)R}[(1+\gamma(\rho))h(\rho) + z_1(\rho)\hat{\jmath} + z_2(\rho)J^h\hat{\jmath}], \quad \rho := r/s,$$

with z satisfying

$$(z, h_1)_{L^2(\rho d\rho)} = 0$$

and

$$\|z\|_X^2 \leq C(\mathcal{E}(u_0) - 4\pi m).$$

Here $X := \{z : [0,\infty) \to \mathbb{C} \mid z_\rho \in L^2(\rho d\rho),\ \frac{z}{\rho} \in L^2(\rho d\rho)\}$ with the norm

$$\|z\|_X^2 := \int_0^\infty (|z_\rho|^2 + \frac{|z|^2}{\rho^2})\rho d\rho$$

is the natural space for z, corresponding to the energy space $\dot{H}^1$ for $u(x)$. As above, for $u \in \Sigma_m$, we define

$$\begin{aligned} v(r) &= e^{-m\theta R}u(x) = (1+\gamma(\rho))h(\rho) + z_1(\rho)\hat{\mathfrak{j}} + z_2(\rho)J^h\hat{\mathfrak{j}} \\ &=: h(\rho) + \xi(\rho). \end{aligned}$$

The pointwise constraint $|v| = 1$ gives $1 = |z|^2 + (1+\gamma)^2$ and since (as we shall prove) ξ remains pointwise small,

$$\gamma = \sqrt{1-|z|^2} - 1 \leq 0, \quad |\gamma| \leq C|z|^2, \quad |\gamma_\rho| \leq C|z_\rho z|.$$

Making the decomposition given by Lemma 2.3.1 for $u(x,t)$, at each time t, yields the complex function $z(\rho,t)$, and time-varying parameters $s(t)$, $\alpha(t)$, which together give a full description of the solution map $u(x,t)$.

Since we will use the estimates of the previous section to estimate $q(r,t)$ rather than $z(\rho,t)$, we need to know that z can be controlled by q. This follows from the lemmas below. For convenience, we introduce spaces X_p, $2 \leq p < \infty$, with the norm:

$$\|z\|_{X_p}^p := \int_0^\infty (|z_\rho|^p + \frac{|z|^p}{\rho^p})\rho d\rho$$

so that $X = X_2$.

Lemma 2.3.2 *[13] Let $2 \leq p < \infty$. If $(z, h_1)_{L^2} = 0$ and $m > 3$, then*

$$\begin{aligned} \|z\|_{X_p} &\leq C\|L_0 z\|_{L^p} \\ \||z_\rho|/\rho + |z|/\rho^2\|_{L^2} &\leq C\|L_0 z/\rho\|_{L^2} \end{aligned}$$

where L_0 is the operator

$$L_0 := (\partial_\rho + \frac{m}{\rho}h_3(\rho)) = h_1(\rho)\partial_\rho \frac{1}{h_1(\rho)}. \tag{2.3.1}$$

Since, modulo nonlinear terms, $(L_0 z)(\rho) \approx sq(s\rho)$, we also have:

Lemma 2.3.3 *[13] Under the assumptions of Lemma 2.3.2, if $\|z\|_X \ll 1$, then*

$$\begin{aligned} \|z\|_{X_p(\rho d\rho)} &\leq Cs^{1-2/p}\|q\|_{L^p(rdr)}, \\ \|\frac{z_\rho}{\rho}\|_{L^2(\rho d\rho)} + \|\frac{z}{\rho^2}\|_{L^2(\rho d\rho)} &\leq Cs\left\|\frac{q}{r}\right\|_{L^2(rdr)}. \end{aligned}$$

So the original map $u(x,t)$ can be fully described by the function $z(\rho,t)$ and the parameters $s(t)$ and $\alpha(t)$, while $z(\rho,t)$ can be controlled by $q(r,t)$. So we are going to derive the equation for $z(\rho,t)$ in order to estimate $s(t)$ and $\alpha(t)$, and then use Lemma 2.3.3 and estimates on the equation for q to complete the proof of Theorem 2.1.3.

Rewriting equation (2.1.1) in terms of the vector $v(r,t)$ yields

$$v_t = M_r v + (|v_r|^2 + \frac{m^2}{r^2}|Rv|^2)v, \quad M_r := \partial_r^2 + \frac{1}{r}\partial_r + \frac{m^2}{r^2}R^2. \tag{2.3.2}$$

Inserting the decomposition

$$v(r,t) = e^{\alpha(t)R}[h(\rho) + \xi(\rho,t)], \qquad \rho = r/s(t)$$

into (2.3.2), we arrive at

$$s^2([\dot{\alpha}R - s^{-1}\dot{s}\rho\partial_\rho](h+\xi) + \xi_t) = (M_\rho + |\partial_\rho(h+\xi)|^2 + \frac{m^2}{\rho^2}|R(h+\xi)|^2)(h+\xi). \tag{2.3.3}$$

Now using the further decomposition of the perturbation ξ into tangent and normal components, $\xi = z_1\hat{\mathfrak{j}} + z_2 J^h\hat{\mathfrak{j}} + \gamma h$, we find, after routine (though somewhat involved) computations, that the tangential components of (2.3.3) yield our desired equation for z:

$$s^2 z_t = -Nz + (ims\dot{s} - s^2\dot{\alpha})h_1 + F_1 + F_2 \tag{2.3.4}$$

where N denotes the differential operator

$$N := -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1 - 2h_1^2) \;=\; L_0^* L_0 \tag{2.3.5}$$

(here L_0^* is the adjoint of L_0 in $L^2(\rho d\rho)$). We will not write the nonlinear terms F_1 and F_2 explicitly, but only give the necessary estimates (we are omitting many of these details since very similar calculations are presented in [13]):

$$\begin{aligned} &\exists\, C_0 > 0 \text{ such that } \|z\|_X \le C_0 \implies \\ &|F_1(\rho,t) + s\dot{s}\rho z_\rho| \le C(|s^2\dot{\alpha}| + |s\dot{s}|)|z|, \\ &|F_2(\rho,t)| \le C(h_1 + |z|)(|z_\rho|^2 + |z|^2/\rho^2). \end{aligned} \tag{2.3.6}$$

From (2.3.4), we see that the *linearized* equation (setting $s(t) \equiv 1$ for now) for $z(\rho,t)$ is

$$\partial_t z = -Nz. \tag{2.3.7}$$

We would like to impose some orthogonality condition on z which ensures: (a) solutions of (2.3.7) decay; (b) certain norms of $z(\rho,t)$ are controlled by norms of $q(r,t)$. Since N is self-adjoint in $L^2(\rho d\rho)$, and from (2.3.5)- (2.3.1) we see that $\ker(N) = span\{h_1\}$, it is natural to impose

$$(z,h_1)_{L^2} = \int_0^\infty z(\rho)h_1(\rho)\rho d\rho \equiv 0. \tag{2.3.8}$$

Lemma 2.3.3 then gives the desired control of z by q.

Now we may explain the source of our restriction $m \geq 4$. Firstly, in the energy space we have $z \in X$, and in general $z \notin L^2$. But we have

$$|(z,h_1)_{L^2}| \leq \|\frac{z}{\rho}\|_{L^2}\|\rho h_1\|_{L^2} \leq \|z\|_X\|\rho h_1\|_{L^2}$$

and so to make sense of the condition (2.3.8), we require $z/\rho \in L^2$, which leads to the restriction $m \geq 3$. The further restriction $m > 3$ is needed for (seemingly) technical reasons in the second estimate of Lemma 2.3.2 (and hence also in Lemma 2.3.3).

The next step is to estimate the parameter velocities:

Lemma 2.3.4 *If z satisfies* (2.3.4), $\|z\|_X \ll 1$, *and* $(z,h_1)_{L^2} \equiv 0$, *then*

$$|s\dot{s}| + |s^2\dot{\alpha}| \leq C\left(\left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2 + \left\|\frac{z}{\rho^2}\right\|_{L^2}^2\right).$$

Proof. Differentiating (2.3.8) with respect to t, and using (2.3.4), yields:

$$(s^2\dot{\alpha} - ims\dot{s})\|h_1\|_{L^2}^2 = (h_1, F_1 + F_2)$$

and by the estimates (2.3.6), we obtain

$$\begin{aligned}
|s\dot{s}| + |s^2\dot{\alpha}| &\leq C(|(h_1,F_1)| + |(h_1,F_2)|) \\
&\leq C(|s\dot{s}| + |s^2\dot{\alpha}|)|(h_1, \rho z_\rho + |z|)| + C\int h_1(h_1+|z|)(|z_\rho|^2 + |z|^2/\rho^2) \\
&\leq C\|z\|_X(|s\dot{s}| + |s^2\dot{\alpha}|) + C(\|z_\rho/\rho\|_{L^2}^2 + \|z/\rho^2\|_{L^2}^2)
\end{aligned} \tag{2.3.9}$$

where we used

- $\|z\|_{L^\infty} \leq C\|z\|_X$ (i.e. (2.2.10))
- $|(h_1, \rho z_\rho)| = |\rho(\partial_\rho + \frac{1}{\rho})(\rho h_1), z/\rho)| \leq C\|\rho h_1\|_{L^2}\|z/\rho\|_{L^2} \leq C\|z\|_X$ (using $m \geq 3$)

- $\|\rho^2 h_1\|_{L^\infty} \leq C$ (true for $m \geq 2$).

Absorbing the first term on the r.h.s. of (2.3.9) completes the proof of Lemma 2.3.4.

The next step is to estimate the function $q(r,t)$. Without loss of generality, we will rescale the solution so that $s_0 := s(0) = 1$. Let $q(r,t)$ be the corresponding complex-valued function derived from the map $u(x,t)$ in Section 2.2. We use Lemma 2.3.3, together with Lemma 2.2.5, to prove the following estimate for q:

Lemma 2.3.5 *For $\sigma > 0$, set $I := [0,\sigma)$, $Q := \mathbb{R}^2 \times I$ and define the spacetime norm*

$$\|q\|_Y := \|q\|_{(L_t^2 L_x^\infty \cap L_t^\infty L_x^2)(Q)} + \left\|\frac{q}{r}\right\|_{L_t^2 L_x^2(Q)} + \|q_r\|_{L_t^2 L_x^2(Q)}.$$

If $\|q_0\|_{L^2} = \delta_0/\sqrt{\pi}$ is sufficiently small, we have

$$\|q\|_Y \leq C\left(\|q_0\|_{L_x^2} + \|s^{-1}\|_{L_t^\infty([0,\sigma])}\|s-1\|_{L_t^\infty([0,\sigma])}\|q\|_Y + \|q\|_Y^2 + \|q\|_Y^3\right).$$

Proof. Note that

$$\begin{aligned}\|q\|^2_{L^2(rdr)} &= \|v_r - \frac{m}{r}J^v Rv\|^2_{L^2(rdr)} = \frac{1}{\pi}(\mathcal{E}(u) - 4\pi m)\\ &\leq \frac{1}{\pi}(\mathcal{E}(u_0) - 4\pi m) = \delta_0^2/\pi\end{aligned}$$

can be taken small. We start by rewriting equation (2.2.4) in integral form:

$$q(t) = e^{-tH}q_0 + \int_0^t e^{-(t-s)H}(F(q(s)) + 2m\frac{h_3(r) - h_3(\rho)}{r^2}q(s))ds \quad (2.3.10)$$

for $0 \leq t \leq \sigma$ where

$$F(q) := -\frac{2m^2 h_3\xi_3 + m^2\xi_3^2 - 2m\xi_3}{r^2}q - \frac{m(\xi_3)_r}{r}q - qN(q) \;=: I + II + III. \quad (2.3.11)$$

Due to Lemma 2.2.5 (including the endpoint case) we have

$$\|q\|_Y \leq C(\|q_0\|_{L^2} + \|\frac{h_3(r) - h_3(\rho)}{r^2}q\|_{L_t^2 L_x^1} + \|F(q)\|_{L_t^{4/3}L_x^{4/3}}). \quad (2.3.12)$$

Note

$$\begin{aligned}\|\frac{h_3(r)-h_3(\rho)}{r^2}q\|_{L_t^2L_x^1} &\le C\|q\|_{L_t^2L_x^\infty}\|\frac{h_3(r)-h_3(r/s)}{r^2}\|_{L_t^\infty L_x^1} \\ &\le C\|q\|_{L_t^2L_x^\infty}\|s^{-1}\|_{L_t^\infty}\|s-1\|_{L_t^\infty}\end{aligned} \tag{2.3.13}$$

where the last estimate comes from expressing $h_3(r)-h_3(r/s)$ as the integral of its derivative with respect to s.

Next we use Lemma 2.3.3 to estimate $\|F(q)\|_{L_t^{4/3}L_x^{4/3}(Q)}$ term by term. Note that the estimate of $F(q) = I + II + III$ has some overlap with the nonlinear estimates appearing in [9], and so we only give brief computations. Recall $v_3(r,t) = h_3(\rho,t) + \xi_3(\rho,t)$ and $\xi_3 = z_2h_1 + \gamma h_3$ for $\rho = r/s(t)$, hence

$$\begin{aligned}\|I\|_{L_x^{4/3}} &\le C(s^{-3/4}\left\|\frac{z}{\rho}\right\|_{L^8}\|q\|_{L^8} + s^{-1}\left\|\frac{z}{\rho}\right\|_{L^4}^2(1+\|z\|_{L^\infty})\|q\|_{L^4}) \\ &\le C\left(\|q\|_{L_x^8}^2 + \|q\|_{L_x^4}^3(1+\|q\|_{L_x^2})\right).\end{aligned}$$

So we obtain

$$\|I\|_{L_t^{4/3}L_x^{4/3}} \le C\left(\|q\|_{L_t^{8/3}L_x^8}^2 + \|q\|_{L_t^4L_x^4}^3\right) \le C\left(\|q\|_Y^2 + \|q\|_Y^3\right). \tag{2.3.14}$$

Next we estimate $\|II\|_{L_t^{4/3}L_x^{4/3}} := \|(m(\xi_3)_r/r)q\|_{L_t^{4/3}L_x^{4/3}}$. Compute

$$\frac{1}{r}(\xi_3)_r = \frac{1}{sr}\left[h_1(z_2)_\rho - \frac{mh_1h_3z_2}{\rho} + \gamma_\rho h_3 + \frac{m\gamma h_1^2}{\rho}\right].$$

Again using Lemma 3.3, we arrive at

$$\|II\|_{L_t^{4/3}L_x^{4/3}} \le C\,\|q\|_{L_t^4L_x^4}^2(1+\|q\|_{L_t^\infty L_x^2}) \le C(\|q\|_Y^2 + \|q\|_Y^3). \tag{2.3.15}$$

Finally, using the Hardy inequality

$$\left(\int_0^\infty\left(\int_x^\infty f(y)dy\right)^p x^{\rho-1}dx\right)^{1/p} \le p/\rho\left(\int_0^\infty (yf(y))^p y^{\rho-1}dy\right)^{1/p}$$

for $f \ge 0$, in the case $p = \rho = 2$, we find

$$\|N(q)\|_{L_x^2} \le C\|Q\|_{L_x^2} \le C(\|q\|_{L_x^4}^2 + \|\nu\|_{L_x^\infty}\|q/r\|_{L_x^2})$$

and since $|\nu| \le 1$, we obtain

$$\begin{aligned}\|III\|_{L_t^{4/3}L_x^{4/3}} &\le \|q\|_{L_t^4L_x^4}\|N(q)\|_{L_t^2L_x^2} \le C(\|q\|_{L_t^4L_x^4}^3 + \|q/r\|_{L_t^2L_x^2}^2) \\ &\le C\left(\|q\|_Y^2 + \|q\|_Y^3\right).\end{aligned} \tag{2.3.16}$$

Combining (2.3.12)- (2.3.16), completes the proof of Lemma 2.3.5.

Now we are ready to finish the proof of Theorem 2.1.3.

Completion of the proof of Theorem 2.1.3. From Lemma 2.3.1, we have

$$u_0 = e^{(m\theta+\alpha_0)R}[(1+\gamma_0(r/s_0))h(r/s_0) + (z_0)_1(r/s_0)\hat{\jmath} + (z_0)_2(r/s_0)J^h\hat{\jmath}]$$

with $(z_0, h_1)_{L^2} = 0$ and $\|z_0\|_X \leq C\delta_0$. Let $\tilde{u}(r,t) := u(s_0 r, s_0^2 t)$. Then $\tilde{u}$ is also a solution to the heat flow equation (2.1.1) with initial data

$$\tilde{u}(r,0) = e^{(m\theta+\alpha_0)R}[(1+\gamma_0(r))h(r) + (z_0)_1(r)\hat{\jmath} + (z_0)_2(r)J^h\hat{\jmath}],$$

and time-dependent decomposition

$$\tilde{u} = e^{(m\theta+\alpha(t))R}[(1+\gamma(r/s(t)))h(r/s(t)) + z_1(r/s(t),t)\hat{\jmath} + z_2(r/s(t),t)J^h\hat{\jmath}]$$

with $s(t) \in C([0,T);\mathbb{R}^+)$, $\alpha(t) \in C([0,T);\mathbb{R})$, $s(0) = 1$, and $\alpha(0) = \alpha_0$. If δ_0 is sufficiently small, the estimates of Lemma 2.3.4 and Lemma 2.3.5 together yield

$$\|s^{-1}\dot{s}\|_{L^1_t} + \|\dot{\alpha}\|_{L^1_t} \leq C\|q\|_Y^2 \leq C\delta_0^2,$$

and in particular that $s(t) \geq C_0 > 0$. Hence the solution extends to $T = \infty$, and as $t \to \infty$, $s(t) \to s_\infty > 0$, and $\alpha(t) \to \alpha_\infty$. Furthermore, for any pair (r,p) with $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, $2 \leq r \leq \infty$,

$$\|\nabla(\tilde{u} - e^{m\theta R}h^{s(t),\alpha(t)})\|_{L^r_t L^p_x} \leq Cs^{\frac{2}{p}-1}\|z\|_{L^r_t X_p} \leq C\|q\|_{L^r_t L^p_x} \leq C\|q\|_Y \leq C\delta_0.$$

Finally, undoing the rescaling $u(r,t) = \tilde{u}(r/s_0, t/s_0^2)$ completes the proof of Theorem 2.1.3.

2.4 Finite time blow up

In this section we give the proof of Theorem 2.1.5 by constructing a 1-equivariant finite-time blow-up solution in $\mathbb{R}^2$ with near-harmonic energy. The proof is a variant of that of [5], adapted to the plane $\mathbb{R}^2$.

One special subclass of 1-equivariant solutions is given by

$$u(\cdot,t) : (r,\theta) \to (\cos\theta\sin\phi(r,t), \sin\theta\sin\phi(r,t), \cos\phi(r,t)), \tag{2.4.1}$$

where (r, θ) are the polar coordinates on the plane, and $\phi(r,t)$ is the angle with the u_3-axis. If u solves the harmonic map heat flow, then, as is easily checked, $\phi(r,t)$ satisfies

$$\begin{aligned} &\phi_t = \phi_{rr} + \frac{1}{r}\phi_r - \frac{\sin 2\phi}{2r^2}, \quad 0 < r < \infty, \ \ t > 0, \\ &\phi(r,0) = \phi_0(r). \end{aligned} \tag{2.4.2}$$

The energy of u can be written in terms of ϕ:

$$E(u(t)) = e(\phi) := \pi \int_0^\infty (\phi_r^2 + \frac{\sin^2\phi}{r^2}) r dr.$$

In order to have a degree-1 solution with finite energy, we impose the boundary conditions

$$\phi(0,t) = 0, \quad \lim_{r\to\infty} \phi(r,t) = \pi. \tag{2.4.3}$$

We take C^1 initial data:

$$\phi_0 \in C^1([0,\infty)), \quad \lim_{r\to 0} \phi_0(r) = 0, \quad \lim_{r\to\infty} \phi_0(r) = \pi. \tag{2.4.4}$$

Local existence of a classical solution is straightforward: there is $T > 0$ such that (2.4.2)–(2.4.4) admits a unique solution

$$\phi(r,t) \in C([0,T]; C^1([0,\infty))) \cap C^\infty((0,\infty) \times (0,T))$$

(one way to see this is to solve the full harmonic map heat-flow with the initial data corresponding to ϕ_0, locally in time in classical spaces, use the fact that the form (2.4.1) is preserved ([4]), and then recover $\phi(r,t)$).

Next we establish an extension to $\mathbb{R}^2$ of the comparison principle of [4] for the unit disk (where they observed that although equation (2.4.2) is singular at $r = 0$, the maximum principle may still be applied).

Lemma 2.4.1 *Let $\phi_1, \phi_2 \in BC([0,\infty) \times [0,T]) \cap C^2((0,\infty) \times (0,T))$ be solutions of the problem (2.4.2)– (2.4.4) with initial data ϕ_{01}, ϕ_{02}. If $\phi_{01} \le \phi_{02}$, then*

$$\phi_1(r,t) \le \phi_2(r,t), \quad (r,t) \in [0,\infty) \times [0,T].$$

Proof. Let $\varphi := \phi_2 - \phi_1$. Then φ satisfies

$$\varphi_t = \varphi_{rr} + \frac{1}{r}\varphi_r + p(r,t)\varphi, \tag{2.4.5}$$

where

$$p(r,t) := -\frac{\sin 2\phi_2 - \sin 2\phi_1}{2r^2(\phi_2 - \phi_1)} = -\frac{1}{r^2}\cos(\phi_1 + \phi_2)\frac{\sin(\phi_2 - \phi_1)}{\phi_2 - \phi_1}.$$

Fix $T_1 \in (0,T)$. Since $\phi_1(0,t) = \phi_2(0,t) = 0$, there exists $\delta > 0$ such that ϕ_1, $\phi_2 \in [-\pi/8, \pi/8]$ in $(0,\delta) \times (0,T_1]$ (continuity of ϕ_1 and ϕ_2), and hence $p < 0$ on this set. Therefore, there is $K \geq 0$ such that $p \leq K$ on $(0,\infty) \times (0,T_1)$. Setting $v(r,t) := e^{-(K+1)t}\varphi(r,t)$, (5.1.5) yields

$$v_t = v_{rr} + \frac{1}{r}v_r + (p(r,t) - K - 1)v.$$

And setting $w^\epsilon(r,t) := v(r,t) + \epsilon(T_2 - t)^{-1}e^{r^2/(4(T_2-t))}$ for some $T_2 > T_1$, we find

$$w_t^\epsilon = w_{rr}^\epsilon + \frac{1}{r}w_r^\epsilon + (p(r,t) - K - 1)(w^\epsilon - \epsilon(T_2 - t)^{-1}e^{r^2/(4(T_2-t))}). \quad (2.4.6)$$

Now suppose $\inf_{[0,\infty)\times[0,T_1]} w^\epsilon < 0$. By the boundary conditions and boundeness of φ, this implies $w^\epsilon(r,t) = \inf_{[0,\infty)\times[0,T_1]} w^\epsilon < 0$ for some $r > 0$, $t \in (0,T_1]$. Hence $w_t^\epsilon(r,t) \leq 0$, $w_r^\epsilon(r,t) = 0$, and $w_{rr}^\epsilon(r,t) \geq 0$. This contradicts (2.4.6). So we have $w^\epsilon \geq 0$, and sending $\epsilon \to 0$, we recover $\varphi \geq 0$ in $[0,\infty) \times [0,T_1]$. Since $T_1 < T$ was arbitrary, we are done.

The proof of Theorem 2.1.5 is a combination of methods from [5] (subsolution construction and comparison principle) and [2] (use of comparison principle on a subdomain). The following Lemma is proved in [5]:

Lemma 2.4.2 *[5] Let $\sigma, \lambda_0, \mu \in \mathbb{R}^+$ and $\nu \in (0,1)$. Let $\lambda(t)$ be the solution of*

$$\lambda' = -\sigma\lambda^\nu, \quad t > 0, \quad \lambda(0) = \lambda_0.$$

If $T_\lambda := \sup\{t > 0, \lambda(t) > 0\}$ (the "blow-up time of f"), then the function

$$f(r,t) := \arccos\left(\frac{\lambda(t)^2 - r^2}{\lambda(t)^2 + r^2}\right) + \arccos\left(\frac{\mu^2 - r^{2(1+\nu)}}{\mu^2 + r^{2(1+\nu)}}\right), \quad (r,t) \in (0,1)\times(0,T_\lambda)$$

satisfies the following properties:

(i) $f \in C^\infty([0,1] \times [0,T_\lambda))$;

(ii) $\lim_{r\to 0} f(r,t) = 0$ for $t \in [0,T_\lambda)$, $\lim_{r\to 0} f(r,T_\lambda) = \pi$;

(iii) there exists $\overline{\mu} > 0$ such that for every $\mu > \overline{\mu}$ we can find $\overline{\sigma}(\mu,\nu)$ such that

$$f_t \leq f_{rr} + \frac{1}{r}f_r - \frac{\sin 2f}{2r^2}, \quad (r,t) \in (0,1) \times (0,T_\lambda).$$

for all $\sigma \leq \overline{\sigma}$.

Proof of Theorem 2.1.5. Given any small $\delta > 0$, let the initial data $\underline{\phi_0}(r)$ to equation (2.4.2) be of the form

$$\underline{\phi_0}(r) = \arccos\left(\frac{s_0^2 - r^2}{s_0^2 + r^2}\right) + \frac{\delta}{10}\zeta(r)$$

where $\zeta(r)$ is a non-negative C^∞ function supported in $[\frac{7}{8}, \frac{9}{8}]$. We can ensure $\pi < \underline{\phi_0}(1)$ by choosing $s_0 = s_0(\delta)$ small enough (depending on δ) so that $\pi - \arccos\left(\frac{s_0^2-1}{s_0^2+1}\right) \geq \pi - \delta$, and choosing $\zeta(1) > 10$ (thus ζ can be chosen independent of δ). It is then easy to check that $e(\underline{\phi_0}) \leq 4\pi + C\delta^2$. Moreover, $\zeta(r)$ and s_0 can be chosen such that $e(\underline{\phi_0})_{(7/8<r<\infty)} \leq \delta^2/10$, which means that the energy is concentrated in a neighborhood of the origin.

Now let $\underline{\phi}(r,t)$ be the unique classical solution with initial data $\underline{\phi_0}(r)$, and let T be its maximal existence time. Since $\underline{\phi_0}(1) > \pi$, by continuity there exists $T^* \in (0,T)$ and $\gamma > \pi$ such that $\underline{\phi}(1,t) \geq \gamma$ for $0 \leq t \leq T^*$. Let $h(r) = \arccos\left(\frac{\mu^2 - r^{2(1+\nu)}}{\mu^2 + r^{2(1+\nu)}}\right)$. Choose μ sufficiently large to ensure $h(1) = \arccos\left(\frac{\mu^2-1}{\mu^2+1}\right) \leq \gamma - \pi$. Let $f(r,t)$ be the function from Lemma 2.4.2, and choose σ small enough so that property *(iii)* if Lemma 2.4.2 holds. And finally, choose λ_0 small enough so that $T_\lambda \leq T^*$, and so that the energy of $f(r,0)$ on $[0,1]$ is $\leq 4\pi + C\delta^2$. So $f(r,t)$ is a subsolution for the problem

$$\begin{aligned} &g_t = g_{rr} + \frac{1}{r}g_r - \frac{\sin 2g}{2r^2}, \quad 0 < r < 1 \\ &g(0,t) = 0 \\ &g(1,t) = \gamma. \end{aligned} \tag{2.4.7}$$

Now select a smooth, bounded function $\phi_0(r)$ satisfying $\phi_0(0) = 0$, and

$$\begin{aligned} &\phi_0(r) \geq \underline{\phi_0}(r), \quad r \in [0,\infty), \\ &\phi_0(r) \geq f(r,0), \quad r \in [0,1], \\ &e(\phi_0) \leq 4\pi + C\delta^2 \end{aligned}$$

(the last property can be achieved since it holds for both $\phi_0(r)$ and $f(r,0)$, the latter on $[0,1]$). Let T_0 be the maximal existence time of the classical solution $\phi(r,t)$ with initial data $\phi_0(r)$. By the comparison principle Lemma 4.1, $\phi(r,t) \geq \underline{\phi}(r,t)$ for $(r,t) \in (0,\infty) \times (0, \min\{T_0, T\})$, and in particular, $\phi(1,t) > \underline{\phi}(1,t) \geq \gamma$ for $0 \leq t \leq T^*$. So $\phi(r,t)$ is a supersolution for problem (2.4.7), while $f(r,t)$ is a subsolution. So by the maximum principle for this problem on the disk ([4]), for $0 \leq r \leq 1$, we have $\phi(r,t) \geq f(r,t)$. Hence $\phi(r,t)$ blows up (in the C^1 sense) at or before time T_λ.

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Chapter 3

Well-posedness and scattering for a model equation for Schrödinger maps

3.1 Introduction and main results

Some geometric evolution equations from $\mathbb{R}^d \times \mathbb{R}$ to $\mathbb{S}^2$ (the unit 2-sphere): harmonic map heat flow (HMHF)

$$\mathbf{U}_t = P\Delta\mathbf{U}$$

([1, 2, 5–7, 10]), Schrödinger maps (SM)

$$\mathbf{U}_t = JP\Delta\mathbf{U}$$

([4, 8, 9]) and wave maps (WM)

$$P\mathbf{U}_{tt} = P\Delta\mathbf{U}$$

([12, 13]) in critical dimension $d = 2$ have been studied (here P denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane of $\mathbb{S}^2$ and $J = \mathbf{U}\times$). For HMHF and WM equations, much effort has been devoted to a subclass of m-equivariant maps $\mathbf{U}(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{S}^2$:

$$\mathbf{U}(\cdot,t) : (r,\theta) \to (\cos m\theta \sin u(r,t), \sin m\theta \sin u(r,t), \cos u(r,t)) \qquad (3.1.1)$$

A version of this chapter will be submitted for publication, Guan, M. Well-posedness and scattering for a model equation for Schrödinger maps.

where $u(r,t)$ stands for the longitudinal angle. Then the map equations are reduced to scalar PDEs for u as

$$u_t(\text{or } u_{tt}) = \Delta u - \frac{m^2}{2r^2}\sin 2u. \tag{3.1.2}$$

However this subclass is not preserved by Schrödinger maps which makes the construction of singular solutions for Schrödinger flow much harder. This is our motivation to study for this model equation:

$$iu_t + \Delta u - \frac{m^2}{r^2}\frac{\sin(2|u|)}{2|u|}u = 0, \qquad u(x,0) = u_0(x). \tag{3.1.3}$$

where $u(x,t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ is radial and m is a nonzero integer. Without loss of generality, we assume $m > 0$ in this context. Equation (3.1.3) is a nonlinear Schrödinger equation with smooth, but spatially varying nonlinearity. It is invariant under "gauge" rotation: $u \to e^{i\alpha}u, \alpha \in \mathbb{R}$. This equation is a natural Schrödinger analogue of equation (3.1.2).

We observe that an energy is formally preserved by equation (3.1.3):

$$\frac{d}{dt}E(u(t)) = 0,$$

where the energy is defined by

$$E(u) = \pi\int_0^\infty (|u_r|^2 + \frac{m^2}{r^2}\sin^2|u|)rdr.$$

Finite energy solutions require $\sin|u_0(x)| = 0$ both at $|x| = 0$ and $|x| = \infty$. Therefore this yields the following boundary conditions

$$\begin{aligned} u_0 \in \Sigma := \{&u_0 : [0,\infty) \to \mathbb{C}, E(u_0) < \infty, \\ &\lim_{|x|\to 0} u_0(x) = k_1\pi, \lim_{|x|\to\infty} u_0(x) = k_2\pi, \quad k_1, k_2 \in \mathbb{Z}\}. \end{aligned} \tag{3.1.4}$$

Equation (3.1.3) is energy critical in space dimension 2 in the sense that the energy is invariant under the scaling since

$$E(u(\cdot/s)) = E(u(\cdot)).$$

Throughout this chapter, we are interested in the global existence and long time behavior of the solutions.

Rewriting the energy $E(u)$, we find out for solutions in the class $u_0(0) = 0, u_0(\infty) = \pi, E(u)$ has a minimal lower bound $4\pi m$

$$\begin{aligned} E(u) =& \pi \int_0^\infty (|u_r|^2 + \frac{m^2}{r^2}\sin^2|u|) r dr \\ =& \pi \int_0^\infty |u_r - \frac{m}{r}\frac{\sin|u|}{|u|}u|^2 r dr + 2\pi m \operatorname{Re}\int_0^\infty \frac{\sin|u|}{|u|}\bar{u}u_r dr \end{aligned}$$

where

$$\begin{aligned} 2\pi m \operatorname{Re}\int_0^\infty \frac{\sin|u|}{|u|}\bar{u}u_r dr &= \pi m \int_0^\infty \frac{\sin|u|}{|u|}\frac{d}{dr}(|u|^2) dr \\ &= -2\pi m \int_0^\infty \frac{d}{dr}\cos(|u|) dr = 4\pi m. \end{aligned}$$

Therefore

$$E(u) \geq 4\pi m.$$

The minimal energy is attainable when

$$u_r - \frac{m}{r}\frac{\sin|u|}{|u|}u = 0. \tag{3.1.5}$$

The solution to equation (3.1.5) is obtained by the 2-parameter family of harmonic maps:

$$\Gamma_m := \{e^{i\alpha}Q(r/s) | Q(r) = 2\arctan(r^m), \alpha \in \mathbb{R}, s > 0\}.$$

These harmonic maps are the stationary solutions of (3.1.3). Thus there are two natural questions: whether we can draw the same conclusion as in [9], the Schrödinger flow case, i.e. harmonic maps are stable under this equation for large m; and whether we can construct singular solutions when $m = 1$? Moreover equation (3.1.3) possesses constant solutions $k\pi (k \in \mathbb{Z})$ with finite energy. It is also natural to consider the stability of these static solutions. Compared to equation (3.1.2), the analysis of equation (3.1.3) is more complicated. It turns out to be related to the Gross-Pitaveskii equation(NLS with nonzero boundary conditions, see [11]). Let $u(r,t) = S(r) + \eta(r,t)$, where $S(r)$ can be taken as either $k\pi$ or $Q(r)$, equation (3.1.3) becomes

$$\begin{aligned} & i\eta_t = -\Delta\eta + \frac{m^2}{r^2}\cos 2S(r)\operatorname{Re}\eta + i\frac{m^2}{r^2}\frac{\sin 2S(r)}{2S(r)}\operatorname{Im}\eta + \frac{m^2}{r^2}N(\eta), \\ & \eta(r,0) = \eta_0(r), \quad \eta_0(0) = \eta_0(\infty) = 0. \end{aligned} \tag{3.1.6}$$

where $N(\eta)$ represents quadratic and higher order terms of η :

$$N(\eta) = \frac{\sin 2|\eta + S|}{2|\eta + S|}(\eta + S) - \frac{\sin 2S}{2} - \cos 2S \operatorname{Re}\eta - i\frac{\sin 2S}{2S}\operatorname{Im}\eta.$$

The convenient way to study equation (3.1.6) is to write the linear operator as a matrix operator acting on $(\operatorname{Re}\eta, \operatorname{Im}\eta)^T$:

$$\eta_t = \mathcal{L}\eta + nonlinear\ terms \tag{3.1.7}$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \tag{3.1.8}$$

$$L_+ = -\Delta + \frac{m^2}{r^2}\cos 2S, \quad L_- = -\Delta + \frac{m^2}{r^2}\frac{\sin 2S}{2S}.$$

The operators L_+, L_- are self-adjoint with continuous spectrum $[0, \infty)$. For the non-self-adjoint operator $\mathcal{L}$, the imaginary axis is the continuous spectrum. Dispersive estimates for this kind of operator were obtained under various decay assumptions on the potential and the assumption that zero is neither an eigenvalue nor a resonance of $\mathcal{L}$. But unfortunately these assumptions do not hold for $\mathcal{L}$ if $S = Q(r)$. Since on one hand $Q(r/s)$ is the stationary solution to equation (3.1.3), then

$$\frac{\partial}{\partial s}\left(\Delta Q(r/s) - \frac{m^2}{r^2}\sin 2Q(r/s)\right)|_{s=1} = 0.$$

This yields

$$L_+ \sin Q = 0.$$

Thus L_+ has $\sin Q$ as the unique ground state if $m \geq 2(\sin Q \in L^2)$ and as a resonance if $m = 1(\|\sin Q\|_{L^2} = \infty)$. On the other hand, for the stationary solution $e^{i\alpha}Q(r)$,

$$\frac{\partial}{\partial \alpha}\left(\Delta(e^{i\alpha}Q(r)) - \frac{m^2}{2r^2}\sin 2Q(r)e^{i\alpha}\right) = 0$$

gives a resonance Q for L_-,

$$L_-Q = -\Delta Q + \frac{m^2}{2r^2}\sin 2Q = 0$$

since $Q \notin L^2$ for $m \geq 1$. So it is unknown at this point how to get the dispersive estimates for the evolution operator $e^{t\mathcal{L}}$.

Because of the difficulties described above, in this chapter we will only consider finding local in time solutions with finite energy and global solutions with small energy.

When $E(u_0)$ is small, we must have that $u_0(0) = u_0(\infty) = 0$ and then zero is the static solution. Rewriting equation (3.1.3), we have

$$iu_t + \Delta u - \frac{m^2}{r^2}u - \frac{m^2}{r^2}(\frac{\sin|2u|}{|2u|} - 1)u = 0. \tag{3.1.9}$$

This equation resembles cubic NLS with inverse square potential. The boundedness (smallness) of

$$\int_0^\infty (|u_r|^2 + \frac{m^2}{r^2}|u|^2)rdr$$

implies boundedness (smallness) of $E(u)$. Hence the natural space for u is $u \in \dot{H}^1$ and $\frac{u}{r} \in L^2$. This is different from usual cubic NLS equations where $u \in L^2$ or $u \in H^1$.

Let us introduce some Banach spaces. Define $X := \{u : [0,\infty) \to \mathbb{C} | u_r \in L^2(rdr), \frac{u}{r} \in L^2(rdr)\}$, with the norm

$$\|u\|_X^2 := \int_0^\infty \left\{ |u_r|^2 + m^2\frac{|u|^2}{r^2} \right\} rdr.$$

For $2 \le p < \infty$, also define $X^p := \{u : [0,\infty) \to \mathbb{C} | u_r \in L^p(rdr), \frac{u}{r} \in L^p(rdr)\}$, with the norm

$$\|u\|_{X^p}^p := \int_0^\infty \left\{ |u_r|^p + m^p\frac{|u|^p}{r^p} \right\} rdr.$$

So $X = X_2$. For radial complex-valued function $u(r), v(r)$, define the following inner product,

$$(u,v)_{L^2} = \int_0^\infty \bar{u}vrdr$$

and

$$\langle u,v\rangle_X = \int_0^\infty \left(\bar{u}_r(r)v_r(r) + \frac{m^2}{r^2}\bar{u}(r)v(r) \right) rdr.$$

For an interval $I \subset \mathbb{R}$, define space-time norms:

$$\|u\|_{L_t^q(I,X_x^p)}^q := \int_I \left(\int_0^\infty (|u_r|^p + m^p\frac{|u|^p}{r^p})rdr \right)^{q/p} dt.$$

In this chapter, we will use the following Strichartz estimate and Sobolev-type embedding theorem.

Lemma 3.1.1 *(Strichartz estimate [3]) In space dimension 2, we say that a pair of exponents (q,r) is admissible if $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}, 2\le r<\infty$. Then we have*

(i)

$$\|e^{it\Delta}\phi\|_{L_t^q L_x^r}\le C\|\phi\|_{L^2}$$

(ii)

$$\left\|\int_0^t e^{i(t-\tau)\Delta}g(\cdot,\tau)d\tau\right\|_{L_t^q L_x^r}\le C\|g\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}$$

where $\tilde{q}',\tilde{r}'$ are the conjugate exponents of admissible pairs $(\tilde{q},\tilde{r})$.

Lemma 3.1.2 *(Sobolev-type Embedding [8]) Suppose $f\in H^1_{loc}(\mathbb{R}^2)$ is radial and $f_r, f/r\in L^2(\mathbb{R}^2), \lim_{r\to\infty}f(r)=0$. Then*

$$\|f\|^2_{L^\infty(\mathbb{R}^2)}\le C\left(\int_0^\infty(|f_r|^2+m^2\frac{|f|^2}{r^2})rdr\right).$$

One can find this Lemma in [8]. Here we provide a very elementary proof. For any radial function $f\in C_0^\infty(\mathbb{R}^2)$,

$$f^2(r)=-2\int_r^\infty f(s)f'(s)ds=-2\int_r^\infty\frac{f}{s}f'(s)sds.$$

Thus by Hölder inequality

$$\|f\|^2_{L^\infty(\mathbb{R}^2)}\le C\left(\int_0^\infty(|f_r|^2+m^2\frac{|f|^2}{r^2})rdr\right).$$

The lemma follows by approximating $f\in X$ by functions in $C_0^\infty\cap X$. Now we are in a position to state our theorems.

Theorem 3.1.3 (Local wellposedness) *Let $m\ge 1$. Let either $S=0$ or $S=Q(r)=2\arctan(r^m)$. For any $\eta_0\in X$, there exists maximal time interval $I=(-T_{\min},T_{\max})$ containing 0, such that the integral equation of* (3.1.6) *has a unique solution in the class satisfying*

$$\eta(r,t)=u(r,t)-S(r)\in C(I;X)\cap L^q_{loc}(I;X^r)$$

where $2<q<\infty$ and $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$.

Moreover, the function $u(r,t)=S(r)+\eta(r,t)$ solves equation (3.1.3) *in a distribution sense and $u(t)\in C(I,\Sigma)$ and $E(u(t))=E(u_0)$ for all time $t\in I$.*

We know from Lemma 3.2.1 in the next section, $E(u)$ is finite if $\eta \in X$. The energy is not necessarily near the harmonic map energy. For maps with energy close to the harmonic map energy $E(u) = 4\pi m + \delta^2, 0 < \delta \ll 1$, the local existence can be extended to global solutions if we have a space-time estimate for the linear operator $\mathcal{L}$. We hope to develop the global existence theory in forthcoming research.

Theorem 3.1.4 (Global wellposedness and scattering) *For $m \geq 1$, there exists $\varepsilon_0 > 0$ such that when $u_0 \in \Sigma, k_1 = k_2 = 0$ and*

$$\|u_0\|_X \leq \varepsilon_0,$$

then the solution u to equation (3.1.3) *given in Theorem 3.1.3 is defined for all time*

$$u \in C(\mathbb{R}; X) \cap L^q(\mathbb{R}; X^r).$$

Furthermore, there exist unique functions $u^+, u^- \in X$ such that

$$\|e^{-it(\Delta - \frac{m^2}{r^2})}u(t) - u^{\pm}\|_X \to 0, \quad as \quad t \to \pm\infty.$$

This result is consistent with Schrödinger flow in [4] (small energy implies global wellposeness). Since $u(t)$ is radial, the operator $-\Delta + \frac{m^2}{r^2}$ acting on $u(t)$ is like $-\Delta$ acting on function of the form $v(x,t) = e^{im\theta}u(r,t)$. It suffices to prove that v is global and scatters in X.

Notation. In this chapter we use the notation $A \lesssim B$ whenever there exists some constant $C > 0$ so that $A \leq CB$. Similarly, we use $A \simeq B$ if $A \lesssim B \lesssim A$. $A \ll B$ means that for some constant $c > 0$, which may be choosen arbitrarily small, $A \leq cB$. The letter C is used to denote a generic constant unless specified, the value of which may change from line to line. The proofs of Theorem 3.1.3 and Theorem 3.1.4 are given in section 3.2 and section 3.3 respectively.

3.2 Local wellposedness

In this section, we aim to get local in time solutions for η equation. Let $S = 0$ or $S = Q = 2\arctan(r^m)$. Substitution $u = S + \eta$ into equation (3.1.3) yields

$$\begin{aligned} i\eta_t = &-\Delta\eta + \frac{m^2}{r^2}\cos 2S \operatorname{Re}\eta + i\frac{m^2}{r^2}\frac{\sin 2S}{2S}\operatorname{Im}\eta + \frac{m^2}{r^2}N(\eta), \\ &\eta(r,0) = \eta_0 = u_0 - S(r) \end{aligned} \tag{3.2.1}$$

where

$$N(\eta) = \frac{\sin 2|\eta + S|}{2|\eta + S|}(\eta + S) - \frac{\sin 2S}{2} - \cos 2S \operatorname{Re}\eta - i\frac{\sin 2}{2S}\operatorname{Im}\eta$$

are nonlinear terms. Let $\eta = \eta_1 + \eta_2$, and

$$f(\eta_1, \eta_2) = \frac{\sin 2|S + \eta|}{2|S + \eta|}(S + \eta),$$

then using Taylor expansion for the function $f(\eta_1, \eta_2)$, we have

$$\begin{aligned} |N(\eta)| &\leq (|f_{xx}(\xi_1, \xi_2)\eta_1^2| + |f_{yy}(\xi_1, \xi_2)\eta_2^2| + |f_{xy}(\xi_1, \xi_2)\eta_1\eta_2|) \\ &\leq C(S|\eta|^2 + |\eta|^5). \end{aligned}$$

where ξ_1 is between 0 and η_1, ξ_2 is between 0 and η_2. It is worth remarking that on the right hand, the role of $S(r)$ is very important since we need $S(r)/r$ to be bounded (seen the reason from the estimate (3.2.13)). Because of this, for the constant solution $S(r) = \pi$, even the local existence of the solutions to equation (3.1.6) is hard to obtain. In order to prove local existence, we need to have dispersive estimates for the linear operator near the origin. As $r \to 0, \mathcal{L}$ is basically like $-\Delta + \frac{m^2}{r^2}$ since

$$\lim_{r\to 0} \cos 2S(r) = 1, \quad \lim_{r\to 0} \frac{\sin 2S(r)}{2S(r)} = 1.$$

We treat the operator $-\Delta + \frac{m^2}{r^2}$ as the linear operator and the differences $\frac{m^2}{r^2}(\cos 2S - 1), \frac{m^2}{r^2}(\frac{\sin 2S}{2S} - 1)$ as the perturbations. Then the natural space for η is X. In fact, $\eta \in X$ implies $E(u)$ is finite in the following lemma.

Lemma 3.2.1 *Suppose* $\|\eta\|_X^2 = \int_0^\infty (|\eta_r|^2 + \frac{m^2}{r^2}|\eta|^2) r dr < \infty, \eta(0,t) = \eta(\infty, t) = 0$ *and* $u(r,t) = S(r) + \eta(r,t)$, *then* $u \in \Sigma$.

Proof. If $S = Q$, rewriting the energy as

$$E(u) = 4\pi m + \int_0^\infty (|\eta_r|^2 + 2Q_r Re\eta_r + \frac{m^2}{r^2}(\sin^2|Q + \eta| - \sin^2 Q)) r dr.$$

Using integration by parts

$$\int_0^\infty 2Q_r \operatorname{Re}\eta_r r dr = -\int_0^\infty \frac{m^2}{r^2}\sin 2Q \operatorname{Re}\eta r dr. \tag{3.2.2}$$

For the difference of the squares, writing the first few terms in the Taylor series for $\sin^2|(Q+\eta)|$, we get

$$\begin{aligned}\sin^2|Q+\eta| - \sin^2 Q &= \sin 2Q \operatorname{Re}\eta + \cos 2Q(\operatorname{Re}\eta)^2 \\ &\quad + \frac{\sin 2Q}{2Q}(\operatorname{Im}\eta)^2 + R(\eta).\end{aligned} \tag{3.2.3}$$

where $|R(\eta)| \le C|\eta|^3$. Using (3.2.2), (3.2.3), the energy can be estimated as

$$E(u) \le 4\pi m + \int_0^\infty (|\eta_r|^2 + \frac{m^2}{r^2}\cos 2Q|\operatorname{Re}\eta|^2 + \frac{m^2}{r^2}\frac{\sin 2Q}{2Q}|\operatorname{Im}\eta|^2 + C\frac{m^2}{r^2}|\eta|^3)rdr.$$

To show $E(u)$ is finite, it suffices to prove that the integral on the right hand is finite. Since $\|\eta\|_X < \infty$, we claim that

$$\int_0^\infty (|\eta_r|^2 + \frac{m^2}{r^2}\cos 2Q|\operatorname{Re}\eta|^2 + \frac{m^2}{r^2}\frac{\sin 2Q}{2Q}|\operatorname{Im}\eta|^2)rdr \le C\|\eta\|_X^2.$$

The claim is true since

$$\frac{m^2}{r^2}\cos 2Q \sim \begin{cases} \frac{m^2}{r^2} & \text{as } r \to 0 \\ \frac{m^2}{r^2} & \text{as } r \to \infty \end{cases}$$

and

$$\frac{m^2}{r^2}\frac{\sin 2Q}{2Q} \sim \begin{cases} \frac{m^2}{r^2} & \text{as } r \to 0 \\ -\frac{m^2}{r^{m+2}} & \text{as } r \to \infty \end{cases}$$

From Lemma 3.1.2, $\|\eta\|_{L^\infty} \le C\|\eta\|_X$, we have

$$\|\frac{m^2}{r^2}|\eta|^3\| \le C\|\eta\|_{L^\infty}\|\eta\|_X^2 \le C\|\eta\|_X^3.$$

Therefore

$$E(u) \le 4\pi m + C(\|\eta\|_X^2 + \|\eta\|_X^3) < \infty$$

and u satisfies the boundary conditions $u_0(0) = 0, u_0(\infty) = \pi$.

If $S = 0$, then $u = \eta$ and u has zero boundary conditions. Using $|\sin\eta| \le C|\eta|$, we have

$$E(u) = E(\eta) = \int_0^\infty (|\eta_r|^2 + \frac{m^2}{r^2}|\sin\eta|^2)rdr \le C\|\eta\|_X^2.$$

The proof is complete. □

Now we are ready to prove Theorem 3.1.3. The operator $-\Delta + \frac{m^2}{r^2}$ is basically $-\Delta$ conjugated by $e^{im\theta}$ when acting on radial functions. By changing of variable from η to ξ with $\xi(x,t) = e^{im\theta}\eta(r,t)$, we get $|\nabla\xi|^p \sim |\eta_r|^p + \frac{m^p}{r^p}|\eta|^p, 2 \le p < \infty$ and $\frac{|\xi|}{|x|} = \frac{|\eta|}{r}$. Thus $\eta \in X$ implies $\xi \in \dot{H}^1(\mathbb{R}^2)$ and $\frac{\xi}{|x|} \in L^2(\mathbb{R}^2)$. For convenience, let us define two notations

$$\tilde{X} := \{v : \mathbb{R}^2 \to \mathbb{C} | \nabla v \in L^2, |v/r| \in L^2\} \tag{3.2.4}$$

and

$$\tilde{X}^p = \{v : \mathbb{R}^2 \to \mathbb{C} | \nabla v \in L^p, |v/r| \in L^p, 2 < p < \infty\}. \tag{3.2.5}$$

Proof of local well-posedness. Let $S = Q(r)$ in equation (3.1.6). The proof of $S = 0$ is similar. Since $\xi = e^{im\theta}\eta$, then ξ solves the equation

$$\begin{aligned} i\xi_t = -\Delta\xi + \frac{m^2}{r^2}(\cos 2Q - 1)(\operatorname{Re}\eta)e^{im\theta} + i\frac{m^2}{r^2}(\frac{\sin 2Q}{2Q} - 1)(\operatorname{Im}\eta)e^{im\theta} \\ + \frac{m^2}{r^2}N(\eta)e^{im\theta}, \quad \xi_0 = e^{im\theta}\eta_0. \end{aligned} \tag{3.2.6}$$

We use this equation to construct a contraction mapping. By Duhamel's formula, it is enough to find solutions to the integral equation

$$\xi(t) = e^{it\Delta}\xi_0 + i\int_0^t e^{i(t-s)\Delta}F(\eta(s))ds \tag{3.2.7}$$

where

$$F(\eta) = \frac{m^2}{r^2}(\cos 2Q - 1)\operatorname{Re}\eta e^{im\theta} + i\frac{m^2}{r^2}(\frac{\sin 2Q}{2Q} - 1)\operatorname{Im}\eta e^{im\theta} + \frac{m^2}{r^2}N(\eta)e^{im\theta}.$$

Define the solution map by

$$\mathcal{M}(\xi)(x,t) = e^{it\Delta}\xi_0 + i\int_0^t e^{i(t-s)\Delta}F(\eta(s))ds.$$

We want to find fixed point of the map $\mathcal{M}$ in the set

$$D := \{\xi(x,t) = \eta(r,t)e^{im\theta} | \xi \in L_t^\infty(I;\tilde{X}) \cap L_t^4(I;\tilde{X}^4); \|\xi\|_{L_t^4\tilde{X}^4} \le 3\delta\} \tag{3.2.8}$$

for $\delta > 0$ to be specified later. Let $I = (-T, T) \in \mathbb{R}$ with $0 \in I$. Suppose that for some $\delta > 0, \xi_0 \in \dot{H}^1, \frac{\xi_0}{r} \in L^2$ satisfies

$$\|e^{it\Delta}\nabla\xi_0\|_{L_t^4(I,L_x^4)} < \delta. \tag{3.2.9}$$

Then

$$\|\mathcal{M}(\xi)\|_{L_t^\infty(I,\tilde{X})\cap L_t^4(I,\tilde{X}_x^4)} < 2\delta + C\|\nabla F\|_{L_t^{4/3}(I,L_x^{4/3})} \tag{3.2.10}$$

where we have used Strichartz estimate Lemma 3.1.1 and $\|\frac{\xi}{r}\|_{L_x^p} \leq C\|\nabla\xi\|_{L_x^p}$.

Then we estimate $\|\nabla F\|_{L_t^{4/3}L_x^{4/3}}$ term by term. Let $f_1(r) = \frac{m^2}{r^2}(\cos 2Q - 1), f_2(r) = \frac{m^2}{r^2}(\frac{\sin 2Q}{2Q} - 1)$. Both functions are smooth. Since

$$|\nabla(f_1(r)\operatorname{Re}\eta e^{im\theta})| \leq C(|f_1||\nabla\xi| + |f_{1r}||\xi|),$$

By using of Hölder inequality in space and time, we have

$$\begin{aligned}\|\nabla(f_1\operatorname{Re}\eta e^{im\theta})\|_{L_t^{4/3}L_x^{4/3}} &\leq CT^{1/2}(\|\nabla\xi\|_{L_t^4L_x^4} + \|\frac{\xi}{r}\|_{L_t^4L_x^4}) \\ &= CT^{1/2}\|\xi\|_{L_t^4\tilde{X}_x^4}.\end{aligned} \tag{3.2.11}$$

Similarly we get

$$\|\nabla(f_2\operatorname{Im}\eta e^{im\theta})\|_{L_t^{4/3}L_x^{4/3}} \leq CT^{1/2}\|\xi\|_{L_t^4\tilde{X}_x^4}. \tag{3.2.12}$$

It is left to estimate $\|\nabla\frac{m^2}{r^2}N(\eta)\|_{L_t^{4/3}L_x^{4/3}}$. Recall that $|N(\eta)| \leq C(Q|\eta|^2+|\eta|^5)$ then

$$\left|\nabla\frac{m^2}{r^2}N(\eta)\right| \lesssim |\frac{Q(r)}{r^2}\xi\nabla\xi| + |(\frac{Q(r)}{r^2})_r\xi^2| + |\xi^4\nabla\xi|,$$

so that

$$\|\nabla\frac{m^2}{r^2}N(\eta)\|_{L_t^{4/3}L_x^{4/3}} \leq C(T^{1/4}\|\xi\|^2_{L_t^4\tilde{X}^4} + \|\xi\|^3_{L_t^4\tilde{X}^4}) \tag{3.2.13}$$

where we have used

$$\frac{Q}{r} \in L^\infty \quad \text{and} \quad \frac{Q}{r} \in L^4.$$

The combinations of (3.2.11), (3.2.12), (3.2.13) and (3.2.10) give

$$\begin{aligned}\|\mathcal{M}(\xi)\|_{L_t^\infty\tilde{X}\cap L_t^4\tilde{X}_x^4} &< 2\delta + C(T^{1/2}\|\xi\|_{L_t^4\tilde{X}_x^4} \\ &\quad + T^{1/4}\|\xi\|^2_{L_t^4\tilde{X}_x^4} + \|\xi\|^3_{L_t^4\tilde{X}_x^4}) \\ &< 2\delta + C(T^{\frac{1}{4}}\delta + T^{\frac{1}{2}}\delta^2 + \delta^3) < 3\delta\end{aligned}$$

if δ and T are small enough such that $C(T^{\frac{1}{2}}+T^{\frac{1}{4}}\delta+\delta^2) \leq 1/2$. Then choosing the time interval I sufficiently small, we can ensure equation (3.2.9). Hence M maps to itself.

Then we need to prove the contraction under the metric

$$d(\xi_1, \xi_2) = \|\xi_1 - \xi_2\|_{L_t^\infty \tilde{X} \cap L_t^4 \tilde{X}_x^4}.$$

We will use the two inequalities

$$||\xi_1|^2 - |\xi_2|^2| \leq C(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|$$

and

$$||\xi_1|^2\xi_1 - |\xi_2|^2\xi_2| \leq C(|\xi_1|^2 + |\xi_2|^2)|\xi_1 - \xi_2|.$$

Let T be small enough. Then by Strichartz estimates

$$\begin{aligned}
d(\mathcal{M}(\xi_1), \mathcal{M}(\xi_2)) &\leq C\|\nabla(F(\xi_1) - F(\xi_2))\|_{L_t^{4/3}L_x^{4/3}} \\
&\leq C(T^{1/2}\|\nabla\xi_1 - \nabla\xi_2\|_{L_t^4 L_x^4} \\
&+ T^{1/4}(\|\nabla\xi_1\| + \|\nabla\xi_2\|)_{L_t^4 L_x^4}\|\nabla\xi_1 - \nabla\xi_2\|_{L_t^4 L_x^4} \\
&+ (\|\nabla\xi_1\|^2 + \|\nabla\xi_1\|^2)_{L_t^4 L_x^4}\|\nabla\xi_1 - \nabla\xi_2\|_{L_t^4 L_x^4}) \\
&\leq C(T^{1/2} + T1/4\delta + \delta^2)d(\xi_1, \xi_2) \\
&\leq \frac{1}{2}d(\xi_1, \xi_2)
\end{aligned}$$

if δ, T are chosen sufficiently small. By the fixed point theorem, we get a solution on $(-T, T)$. The regularity property of the solution follows from Strichartz estimates.

It remains to establish the blowup alternative. We show the blow up alternative by contradiction. Suppose $T_{\max} < \infty$ and $\|\xi\|_{L^4((0,T_{\max});\tilde{X}^4)} < \infty$. Let $0 \leq t \leq t + \tau < T_{\max}$. It follows that

$$e^{it\Delta}\nabla\xi(\tau) = \nabla\xi(t+\tau) - i\int_0^\tau e^{i(t-t')\Delta}\nabla F(\xi(\tau+t'))dt'.$$

Strichartz estimate and $\|\frac{\xi}{r}\|_{L_x^4} \leq \|\nabla\xi\|_{L_x^4}$ yield

$$\begin{aligned}
\|e^{it\Delta}\xi(\tau)\|_{L^4((0,T_{\max}-t),\tilde{X}^4)} &\leq 2\|e^{it\Delta}\nabla\xi(\tau)\|_{L^4((0,T_{\max}-t),L^4)} \\
&\leq 2\|\nabla\xi\|_{L^4((t,T_{\max}),L^4)} + C(\tau^{1/2} \\
&+ \tau^{1/4}\|\nabla\xi\|_{L^4((t,T_{\max}),L^4)} \\
&+ \|\nabla\xi\|^2_{L^4((t,T_{\max}),L^4)})\|\nabla\xi\|_{L^4((t,T_{\max}),L^4)}.
\end{aligned}$$

Therefore for t sufficiently close to $T_{\max}(\tau \to 0)$ such that

$$C(\tau^{1/2} + \tau^{1/2}\|\nabla\xi\|_{L^4((t,T_{\max}),L^4)} + \|\nabla\xi\|^2_{L^4((t,T_{\max}),L^4)}) \leq \frac{1}{2}$$

and

$$\|\nabla\xi\|_{L^4((t,T_{\max}),L^4)} \leq \frac{\delta}{4},$$

we obtain

$$\|\nabla e^{it\Delta}\xi(t)\|_{L^4((0,T_{\max}-t),L^4)} \leq \frac{5\delta}{8} < \delta.$$

By the existence theorem, ξ can be extended past $T_{\max}$ which is a contraction. This shows

$$\|\xi\|_{L^4((0,T_{\max}),\tilde{X}^4)} = \infty.$$

The local well-posedness theory for η is a direct result of $\eta(r,t) = e^{-im\theta}\xi(x,t)$.

The energy $E(u(t))(t \in I)$ is conserved since

$$E(u(t)) = 4\pi m + \int_0^\infty (|\eta_r|^2 + 2Q_r Re\eta_r + \frac{m^2}{r^2}(\sin^2|(Q+\eta)| - \sin^2 Q))rdr.$$

This integral is a conserved quantity which follows formally from multiplying equation (3.2.1) by $\bar{\eta}_t$, integrating over $\mathbb{R}^2$, and taking the real part. This can be rigorously justified following along the line of [3]. The proof is complete.

3.3 Global small solutions and scattering states

In this section we prove Theorem 3.1.4. We first show that for small energy solutions, the local solutions and be extended to global solutions in time, then construct the scattering states and the wave operators.

Recall equation (3.1.9)

$$iu_t + \Delta u - \frac{m^2}{r^2}u + \frac{m^2}{r^2}(\frac{\sin|2u|}{|2u|} - 1)u = 0, \quad u(x,0) = u_0(x).$$

There exists ε_0 such that if $\|u\|_X \leq \varepsilon_0$, then $|u| \leq C\varepsilon_0$, $E(u) \leq C\varepsilon_0$ and

$$|G(u)| = |\frac{\sin|2u|}{|2u|} - 1| \leq C|u|^2.$$

We show that the corresponding maximal solution given by Theorem 3.1.3 is global in time, i.e. $T_{\min} = T_{\max} = \infty$.

Let $v(x,t) = e^{im\theta}u(r,t)$, then v solves the equation

$$iv_t + \Delta v + \frac{m^2}{r^2}G(u)v = 0, \quad v_0(x) = u_0 e^{im\theta} \tag{3.3.1}$$

By Duhamel's formula,

$$v(t) = e^{it\Delta}v_0 + i\int_0^t e^{i(t-s)\Delta}\frac{m^2}{r^2}G(u)v ds, \quad t \in I.$$

For $0 < t < T_{\max}$, we define

$$g(t) = \|v\|_{L^\infty((0,t),\tilde{X})} + \|v\|_{L^4((0,t),\tilde{X}^4)}$$

It follows from Strichartz estimates

$$\begin{aligned} g(t) &\le C\|\nabla v_0\|_{L^2} + \|\nabla\frac{m^2}{r^2}G(u)v\|_{L_t^{4/3}L_x^{4/3}} \\ &\le C\|v_0\|_X + \|\frac{v}{r}\|^3_{L_t^4L_x^4} + \|\frac{v}{r}\|^2_{L_t^4L_x^4}\|\nabla v\|_{L_t^4L_x^4} \\ &\le C\|v_0\|_X + Cg(t)^3. \end{aligned}$$

If ε_0 is sufficiently small such that

$$(2C)^3\varepsilon_0^2 < 1,$$

then

$$g(t) \le 2C\|v_0\|_X \quad \text{for all} \quad 0 < t < T_{\max}.$$

Letting $t \to T_{\max}$, we obtain in particular that

$$\|v\|_{L^4((0,T_{\max}),\tilde{X}^4)} < \infty,$$

so that $T_{\max} = \infty$ by the blow up alternative. This implies $g(t)$ is bounded as $t \to \infty$ and $v \in L^q((0,\infty),\tilde{X}^r)$.

Next we construct the scattering states. Let $w(t) = e^{-it\Delta}v(t)$, we have

$$w(t) = v_0 + i\int_0^t e^{-is\Delta}\frac{m^2}{r^2}G(u(s))v(s)ds.$$

Therefore for $0 < t < \tau$,

$$w(t) - w(\tau) = i\int_\tau^t e^{-is\Delta}\frac{m^2}{r^2}G(u(s))v(s)ds.$$

If follows from $e^{-is\Delta}$ is unitary

$$\|w(t)-w(\tau)\|_{\tilde{X}} \lesssim \int_\tau^t \|e^{-is\Delta}\frac{m^2}{r^2}|v(s)|^2 v(s)\|_{\dot{H}^1} ds$$
$$\lesssim \|\frac{v}{r}\|^2_{L^4((t,\tau);L^4)}\|\nabla v\|_{L^4((t,\tau);L^4)} + \|\frac{v}{r}\|^3_{L^4((t,\tau);L^4)}$$

By the global existence for v in $L_t^4\tilde{X}^4$,

$$\|w(t)-w(\tau)\|_{\tilde{X}} \to 0 \qquad \text{as} \quad \tau, t \to \infty.$$

Therefore there exists $v^+ \in \tilde{X}$ such that

$$\|e^{-it\Delta}v(t) - v^+\|_{\tilde{X}} \to 0 \qquad \text{as} \quad t \to \infty.$$

One can show as well that there exists $v^- \in \tilde{X}$ such that

$$\|e^{-it\Delta}v(t) - v^-\|_{\tilde{X}} \to 0 \quad \text{as} \quad t \to -\infty.$$

We now construct the wave operators for v, which completes the proof of Theorem 3.1.4.

Lemma 3.3.1 *(1) For every $v^+ \in \tilde{X}$, there exists a unique $\psi \in \tilde{X}$ such that the solution $v \in C(\mathbb{R}, \tilde{X})$ of equation* (3.3.1) *satisfies*

$$\|e^{-it\Delta}v(t) - v^+\|_{\tilde{X}} \to 0 \quad \textit{as} \quad t \to \infty.$$

(2) For every $v^- \in \tilde{X}$, there exists a unique $\psi \in \tilde{X}$ such that the solution $v \in C(\mathbb{R}, \tilde{X})$ of equation (3.3.1) *satisfies*

$$\|e^{-it\Delta}v(t) - v^-\|_{\tilde{X}} \to 0 \quad \textit{as} \quad t \to -\infty.$$

Proof. We only prove *(1)*, the proof of *(2)* is similar. Consider $T > 0$, by Duhamel's formula on $[T, \infty)$, v satisfies the equation

$$v(t) = e^{it\Delta}v^+ - i\int_t^\infty e^{i(t-s)\Delta}\frac{m^2}{r^2}G(u(s))v(s)ds \tag{3.3.2}$$

for $t \geq T$. Now the idea is to solve equation (3.3.2) by the fixed-point theorem. Taking admissible pair $(q, r) = (4, 4)$ and applying Strichartz estimate, we get

$$\|e^{it\Delta}v^+\|_{L^4\tilde{X}^4} \leq C\|v^+\|_{\tilde{X}}.$$

Let $I_T = [T, \infty)$ and

$$C_T = \|e^{it\Delta}v^+\|_{L^4(I_T, \tilde{X}^4)}.$$

So $C_T \to 0$ as $T \to \infty$. Define the set

$$D = \{v \in L^4(I_T, \tilde{X}^4) \cap L^\infty(I_T, \tilde{X}) : \|v\|_{L^4(I_T,\tilde{X}^4)} \leq 2C_T\}$$

and $\mathcal{T}(v)$ by

$$\mathcal{T}(v)(t) = i\int_t^\infty e^{i(t-s)\Delta}\frac{m^2}{r^2}G(u(s))v(s)ds.$$

Then

$$\begin{aligned}&\|\mathcal{T}(v)\|_{L^\infty(I_T,\tilde{X})} + \|\mathcal{T}(v)\|_{L^4(I_T,\tilde{X}^4)}\\ &\leq C(\|v/r\|^3_{L^4(I_T,L^4)} + \|\nabla v\|^3_{L^4(I_T,L^4)}) \leq C(2C_T)^3.\end{aligned}$$

We see if T is sufficiently large, such that $C(2C_T)^3 \leq C_T$, then

$$\|\mathcal{T}(v)\|_{L^\infty(I_T,\tilde{X})} + \|\mathcal{T}(v)\|_{L^4(I_T,\tilde{X}^4)} \leq C_T.$$

Thus the solution map $\mathcal{M}$ defined by

$$\mathcal{M}(v)(t) = e^{it\Delta}v^+ + \mathcal{T}(v)(t) \quad \text{for} \quad t \geq T$$

maps D to itself if T is large enough. We can easily verify that

$$d(\mathcal{M}v_1, \mathcal{M}v_2) \leq \frac{1}{2}d(v_1, v_2), \quad v_1, v_2 \in D.$$

So $\mathcal{M}$ has a fixed point $v \in D$ satisfying equation (3.3.2) on $[T, \infty)$. Let $\phi = v(T) \in \tilde{X}$, and then

$$v(t+T) = e^{it\Delta}\phi(T) + i\int_0^t e^{i(t-s)\Delta}\frac{m^2}{r^2}G(u(s+T))v(s+T)ds.$$

Therefore v is the solution of the equation (3.3.1) with $v(T) = \phi$. By the global existence solution of equation (3.3.1), we know $v(0) \in \tilde{X}$ is well defined and

$$e^{-it\Delta}v(t) - v^+ = -i\int_t^\infty e^{is\Delta}\frac{m^2}{r^2}G(u(s))v(s)ds.$$

Since $v \in D$, then

$$\|e^{-it\Delta}v(t) - v^+\|_{\tilde{X}} \lesssim \|v\|^3_{L^4((t,\infty)\tilde{X}^4)} \to 0 \quad \text{as} \quad t \to \infty.$$

therefore, $v(0) = \psi$ satisfies the conclusion of the lemma.

Finally, we show uniqueness. let $\psi_1, \psi_2 \in \tilde{X}$ and v_1, v_2 be the corresponding solutions of (3.3.1) satisfying

$$\|e^{-it\Delta} v_j(t) - v^+\|_{\tilde{X}} \to 0$$

as $t \to \infty$ for $j = 1, 2$. It follows by the above arguments that v_j is a solution of equation (3.3.2), and satisfies $v_j \in L^4(\mathbb{R}, \tilde{X}^4)$. By the routine argument we obtain $v_1(t) = v_2(t)$ for t sufficiently large. The uniqueness of the Cauchy problem at finite time gives $\psi_1 = \psi_2$. The proof of Theorem 3.1.4 is complete.

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Chapter 4

Solitary wave solutions for a class of nonlinear Dirac equations

4.1 Introduction

A class of nonlinear Dirac equations for elementary spin-$\frac{1}{2}$ particles (such as electrons) is of the form

$$i\sum_{j=0}^{3}\gamma^j\partial_j\psi - m\psi + F(\bar{\psi}\psi)\psi = 0. \tag{4.1.1}$$

Here $F : \mathbb{R} \to \mathbb{R}$ models the nonlinear interaction. $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ is a four-component wave function, and m is a positive number. $\partial_j = \partial/\partial x_j$, and γ^j are the 4×4 Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

where σ^k are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define

$$\bar{\psi} = \gamma^0\psi, \quad \bar{\psi}\psi = (\gamma^0\psi, \psi) = \sum_{i=1}^{2}(\psi_i, \psi_i) - \sum_{i=3}^{4}(\psi_i, \psi_i)$$

A version of this chapter has been submitted for publication. Guan, M. Solitary wave solutions for a class of nonlinear Dirac equations.

where $(\cdot,\cdot)$ is the Hermitian inner product in $\mathbb{C}^1$.

Throughout this chapter we are interested in the case

$$F(s) = |s|^\theta, \quad 0 < \theta < \infty. \tag{4.1.2}$$

The local and global existence problems for nonlinearity as above have been considered in [5, 8]. For us, we seek standing waves (or solitary wave solutions, or ground states solutions of (4.1.1)) of the form

$$\psi(x_0, x) = e^{-i\omega t}\phi(x)$$

where $x_0 = t, x = (x_1, x_2, x_3)$. It follows that $\phi : \mathbb{R}^3 \to \mathbb{C}^4$ solves the equation

$$i\sum_{j=1}^{3}\gamma^j\partial_j\phi - m\phi + \omega\gamma^0\phi + F(\bar{\phi}\phi)\phi = 0. \tag{4.1.3}$$

Different functions F have been used to model various types of self couplings. Stationary states of the nonlinear Dirac field with the scalar fourth order self coupling (corresponding to $F(s) = s$) were first considered by Soler [11, 12] proposing them as a model of extended fermions. Subsequently, existence of stationary states under certain hypotheses on F was studied by Balabane [1], Cazenave and Vazquez [3] and Merle[6], where by shooting method they established the existence of infinitely many localized solutions for every $0 < \omega < m$. Esteban and Séré in [4], by a variational method, proved the existence of an infinity of solutions in a more general case for nonlinearity

$$F(\phi) = \frac{1}{2}(|\bar{\phi}\phi|^{\alpha_1} + b|\bar{\phi}\gamma^5\phi|^{\alpha_2}), \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$$

for $0 < \alpha_1, \alpha_2 < \frac{1}{2}$. Vazquez [15] prove the existence of localized solutions obtained as a Klein-Gordon limit for the nonlinear Dirac equation ($F(s) = s$). A summary of different models with numerical and theoretical developments is described by Ranada [10].

None of the approaches mentioned above yield a curve of solutions: the continuity of ϕ with respect to ω, and the uniqueness of ϕ was unknown. Our purpose is to give some positive answers to these open problems. These issues are important to study the stability of the standing waves, a question we will address in future work.

Following [11], we study solutions which are separable in spherical coordinates,

$$\phi(x) = \begin{pmatrix} g(r) & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ if(r) & \begin{pmatrix} \cos\Psi \\ \sin\Psi e^{i\Phi} \end{pmatrix} \end{pmatrix}$$

where $r = |x|, (\Psi, \Phi)$ are the angular parameters and f, g are radial functions. Equation (4.1.3) is then reduced to a nonautonomous planar differential system in the r variable

$$\begin{aligned} f' + \frac{2}{r} f &= (|g^2 - f^2|^\theta - (m - \omega))g \\ g' &= (|g^2 - f^2|^\theta - (m + \omega))f. \end{aligned} \tag{4.1.4}$$

Ounaies in [9] studied the existence of solutions for equation (4.1.3) using a perturbation method. Let $\varepsilon = m - \omega$. By a rescaling argument, (4.1.4) can be transformed into a perturbed system

$$\begin{aligned} &u' + \frac{2}{r}u - |v|^{2\theta}v + v - (|v^2 - \varepsilon u^2|^\theta - |v|^{2\theta})v = 0, \\ &v' + 2mu - \varepsilon(1 + |v^2 - \varepsilon u^2|^\theta)u = 0 \end{aligned} \tag{4.1.5}$$

If $\varepsilon = 0$, (4.1.5) can be related to the nonlinear Schrödinger equation

$$-\frac{\Delta v}{2m} + v - |v|^{2\theta}v = 0, \qquad u = -\frac{v'}{2m}. \tag{4.1.6}$$

It is well known that for $\theta \in (0, 2)$, the first equation in (4.1.6) admits a unique positive, radially and symmetric solution called the ground state $Q(x)$ which is smooth, decreases monotonically as a function of $|x|$ and decays exponential at infinity(see [13] and references therein). Let $U_0 = (Q, -\frac{1}{2m}Q')$, then we want to continue U_0 to yield a branch of bound states with parameter ε for (4.1.5) by contraction mapping theorem.

Ounaies carried out this analysis for $0 < \theta < 1$ and he claimed that the nonlinearities in (4.1.5) are continuously differentiable. But with the restriction $0 < \theta < 1$ we are unable to verify it. The term $|v^2 - \varepsilon u^2|^\theta$ has a cancelation cone when $v = \pm\sqrt{\varepsilon}u$. Along this cone, the first derivative of $|v^2 - \varepsilon u^2|^\theta$ is unbounded for $0 < \theta < 1$. But Ounaies' argument may go through for $\theta \geq 1$, which gives us the motivation of the current research. However we can not work in the natural Sobolev space $H^1(\mathbb{R}^3, \mathbb{R}^2)$. Since $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we lose regularity. To overcome these difficulties, we

want to consider equation (4.1.5) in the Sobolev space $W^{1,p}(\mathbb{R}^3, \mathbb{R}^2), p > 2$ and $\theta \geq 1$.

To state the main result, we introduce the following notations. For any $1 \leq p \leq \infty, L_r^p = L_r^p(\mathbb{R}^3)$ denotes the Lebesgue space for radial functions on $\mathbb{R}^3$. $W_r^{1,p} = W_r^{1,p}(\mathbb{R}^3)$ denotes the Sobolev space for radial functions on $\mathbb{R}^3$. Let $X_r^p = W_r^{1,p} \times W_r^{1,p}, Y_r^p = L_r^p \times L_r^p$. Unless specified, the constant C is generic and may vary from line to line. In this chapter, we assume that $m = \frac{1}{2}$, since after a rescaling $\psi(x) = (2m)^{\frac{1}{2\theta}}\Psi(2mx)$, equation (4.1.1) becomes

$$i\sum_{j=0}^{3}\gamma^j\partial_j\Psi - \frac{1}{2}\Psi + F(\bar{\Psi}\Psi)\Psi = 0.$$

We prove the following results:

Theorem 4.1.1 *Let $\varepsilon = m - \omega$. For $1 \leq \theta < 2$ there exists $\varepsilon_0 = \varepsilon_0(\theta) > 0$ and a unique solution of* (4.1.4) $(f, g)(\varepsilon) \in C((0, \varepsilon_0), W_r^{1,4}(\mathbb{R}^3, \mathbb{R}^2))$ *satisfying*

$$f(r) = \varepsilon^{\frac{\theta+1}{2\theta}}(-Q'(\sqrt{\varepsilon}r) + e_2(\sqrt{\varepsilon}r))$$
$$g(r) = \varepsilon^{\frac{1}{2\theta}}(Q(\sqrt{\varepsilon}r) + e_1(\sqrt{\varepsilon}r))$$

with

$$\|e_j\|_{W_r^{1,2}} \leq C\varepsilon \quad \textit{for some} \quad C = C(\theta) > 0, j = 1, 2.$$

Remark: The necessary condition $|\omega| \leq m$ must be satisfied in order to guarantee the existence of localized states for the nonlinear Dirac equation (see [15], [7]).

The solutions constructed in Theorem 4.1.1 have more regularity. In fact, they are classical solutions and have exponential decay at infinity.

Theorem 4.1.2 *There exists $C = C(\varepsilon) > 0, \sigma = \sigma(\varepsilon) > 0$ such that*

$$|e_j(r)| + |\partial_r e_j(r)| \leq Ce^{-\sigma r} \quad j = 1, 2.$$

Moreover, the solutions (f, g) in Theorem 4.1.1 are classical solutions

$$f, g \in \bigcap_{2 \leq p < +\infty} W_r^{2,p}.$$

Remark. From the physical view point, the nonlinear Dirac equation with $F(s) = s$ (Soler model) is the most interesting. In fact, Theorem 4.1.1, Theorem 4.1.2 are both true for the Soler model. In fact, from (4.1.5) one can find out that $(v^2 - \varepsilon u^2) - v^2 = -\varepsilon u^2$ which is Lipschitz continuous. An adaption of the proofs of the above theorems will yield:

Theorem 4.1.3 *For the Soler model $F(s) = s$, there is a localized solution of equation* (4.1.3) *satisfying Theorem 4.1.1 and Theorem 4.1.2.*

Next we proceed as follows. In section 2, we introduce several preliminary lemmas. In section 3, we give the proof of Theorem 4.1.1, Theorem 4.1.2.

4.2 Preliminary lemmas

We list several lemmas which will be used in Section 3.

Lemma 4.2.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = |t|^{2\theta}t, \theta > 0$, then*

$$\left|g(a+\sigma) - g(a) - (2\theta+1)|a|^{2\theta}\sigma\right| \leq (C_1|a|^{2\theta-1} + C_2|\sigma|^{2\theta-1})|\sigma|^2$$

where C_1, C_2 depends on θ and $C_1 = 0$ if $0 < \theta \leq \frac{1}{2}$.

Proof. We may assume that $a > 0$ in our proof. It is trivial if $\sigma = 0$. So we assume that $\sigma \neq 0$. If $a < 2|\sigma|$, then $|a+\sigma| < 3|\sigma|$ and

$$\begin{aligned}
&\left|g(a+\sigma) - g(a) - (2\theta+1)a^{2\theta}\sigma\right| \\
&\leq |g(a+\sigma)| + |g(a)| + (2\theta+1)|a^{2\theta}\sigma| \\
&< C_1|\sigma|^{2\theta+1}.
\end{aligned}$$

If $a \geq 2|\sigma|$, then

$$a + \sigma \geq 2|\sigma| + \sigma \geq |\sigma| > 0,$$

so that

$$g(a+\sigma) = (a+\sigma)^{2\theta+1}.$$

Taylor's theorem gives

$$g(a+\sigma) - g(a) - (2\theta+1)a^{2\theta}\sigma = \frac{1}{2}g''(\xi)\sigma^2$$

where ξ is between $a+\sigma$ and a. Since $g''(\xi) = 2\theta(2\theta+1)\xi^{2\theta-1}$, if $2\theta-1<0$, then

$$|g''(\xi)| \le C|\sigma|^{2\theta-1}.$$

If $2\theta-1>0$, we have

$$|g''(\xi)| \le C\max\{(a+\sigma)^{2\theta-1}, a^{2\theta-1}\} \le C(|a|^{2\theta-1}+|\sigma|^{2\theta-1}).$$

Hence we prove the lemma. □

Lemma 4.2.2 *For any $a, b \in \mathbb{R}, \theta > 0$, we have*

$$\Big||a-b|^\theta - |a|^\theta\Big| \le C_1|a|^{\theta-1}|b| + C_2|b|^\theta$$

where C_1, C_2 depends on θ and $C_1 = 0$ if $0<\theta\le 1$.

Proof. The proof is basically similar to that of the lemma as above. It is trivial if $b=0$. So we may assume that $b\ne 0$ and $a>0$. If $a<2|b|$, then

$$\Big||a-b|^\theta - |a|^\theta\Big| \le C(|a|^\theta + |b|^\theta) < C|b|^\theta.$$

On the other hand, if $a \ge 2|b|$, then $|a-b| \ge a-|b| \ge |b|$. So by using the mean value theorem

$$\Big||a-b|^\theta - |a|^\theta\Big| = \theta|t|^{\theta-1}|b|$$

where t is between $a-b$ and a. If $\theta-1>0$, then

$$|t|^{\theta-1} \le C(|a|^{\theta-1}+|b|^{\theta-1}),$$

hence

$$\Big||a-b|^\theta - |a|^\theta\Big| \le C_1|a|^{\theta-1}|b| + |b|^\theta.$$

If $\theta-1<0$, then $|t|^{\theta-1} \le C|b|^{\theta-1}$, so that we conclude

$$\Big||a-b|^\theta - |a|^\theta\Big| \le C|b|^\theta.$$

The proof is complete. □

Lemma 4.2.3 *For any $a, b, c \in \mathbb{R}$, if $1 \le \theta < 2$, then*

$$\Big||a+b+c|^\theta - |a+b|^\theta - |a+c|^\theta + |a|^\theta\Big| \le C(|c|^{\theta-1}+|b|^{\theta-1})|b|,$$

where C depends on θ.

Remark. This inequality is symmetric about b, c, so the right hand side can be equivalently replaced by $C(|c|^{\theta-1} + |b|^{\theta-1})|c|$. Without loss of generality, we assume that $|b| \geq |c|$ in the following.
Proof. For simplicity, let

$$L = |a+b+c|^\theta - |a+b|^\theta - |a+c|^\theta + |a|^\theta.$$

It is trivial for $\theta = 1$, since if $|a| \geq 5|b|$ then $L = 0$. If $|a| \leq 5|b|$,

$$|L| \leq C(|b| + |c|).$$

So next we consider $\theta > 1$. If $|a| \leq 5|b|$, by triangle inequality and Lemma 4.2.2, we have

$$\begin{aligned} |L| &\leq C(|a+c|^{\theta-1} + |a|^{\theta-1} + |b|^{\theta-1})|b| \\ &\leq C(|c|^{\theta-1} + |b|^{\theta-1})|b|. \end{aligned}$$

If $|a| \geq 5|b|$, by using Taylor's theorem

$$|L| = C|(a + t_1 b + t_2 c)|^{\theta-2}|bc|.$$

where $t_1, t_2 \in (0,1)$ and

$$|(a + t_1 b + t_2 c)| \geq |a| - 2|b| - |c| \geq |c|.$$

So if $1 < \theta < 2$, we have

$$|L| \leq C|c|^{\theta-1}|b|.$$

The proof is complete. □

Lemma 4.2.4 *Let $2 \leq p \leq \infty, f : \mathbb{R}^3 \to \mathbb{R}$ be radial and bounded. Suppose $f_r + \frac{2}{r}f \in L^p_{loc}, \frac{2}{r}f \in L^p_{loc}$. If $f_r + \frac{2}{r}f \in L^p$, then $\frac{f}{r} \in L^p$ and*

$$\|\frac{f}{r}\|_{L^p} \leq C\|\partial_r f + \frac{2}{r}f\|_{L^p}$$

Proof. We begin with $p = \infty$. Using integration by parts

$$r^2 f(r) = \int_0^r (\partial_\rho f + \frac{2}{\rho}f)\rho^2 d\rho. \tag{4.2.1}$$

Hence

$$|r^2 f| \leq \|f_r + \frac{2}{r}f\|_{L^\infty} \int_0^r s^2 ds = \frac{r^3}{3}\|f_r + \frac{2}{r}f\|_{L^\infty}$$

which gives

$$\|\frac{f}{r}\|_{L^\infty} \leq C\|\partial_r f + \frac{2}{r}f\|_{L^\infty}.$$

Next let us consider $p = 2$. Let $0 < r_1 < r_2 < \infty$. Denote $D = \{x \in \mathbb{R}^3, 0 < r_1 < |x| < r_2\}$ and

$$I = 2\pi^2 \int_{r_1}^{r_2} (f_r + \frac{2}{r}f)\frac{f}{r}r^2 dr.$$

By Hölder inequality,

$$I \leq C\|\frac{f}{r}\|_{L^2(D)}\|f_r + \frac{2}{r}f\|_{L^2(D)}.$$

On the other hand, we have

$$I = \frac{3}{2}\|\frac{f}{r}\|^2_{L^2(D)} + \pi^2(r_2 f^2(r_2) - r_1 f^2(r_1)).$$

Since $r_2 f^2(r_2) > 0$ we have

$$\|\frac{f}{r}\|^2_{L^2(D)} \leq C\left(\|\frac{f}{r}\|_{L^2(D)}\|(f_r + \frac{2}{r}f)\|_{L^2(D)} + r_1 f^2(r_1)\right).$$

Let $r_2 \to \infty, r_1 \to 0$, we obtain

$$\|\frac{f}{r}\|_{L^2} \leq C\|\partial_r + \frac{2}{r}f\|_{L^2}.$$

The intermediate case $2 < p < \infty$ is a direct result of interpolation . □

4.3 Proof of the main theorems

Similar to [9], we use a rescaling argument to transform (4.1.4) into a perturbed system. Let $\varepsilon = m - \omega$ (remember $m = \frac{1}{2}$). The first step is to introduce the new variables

$$f(r) = \varepsilon^{\frac{\theta+1}{2\theta}} u(\sqrt{\varepsilon}r), \quad g(r) = \varepsilon^{\frac{1}{2\theta}} v(\sqrt{\varepsilon}r)$$

where (f, g) are the solutions of (4.1.4). Then (u, v) solve

$$\begin{aligned} &u' + \frac{2}{r}u - |v|^{2\theta}v + v - (|v^2 - \varepsilon u^2|^\theta - |v|^{2\theta})v = 0, \\ &v' + u - \varepsilon(1 + |v^2 - \varepsilon u^2|^\theta)u = 0. \end{aligned} \tag{4.3.1}$$

Our goal is to solve (4.3.1) near $\varepsilon = 0$. If $\varepsilon = 0$, (4.3.1) becomes

$$\begin{aligned} &u' + \frac{2}{r}u - |v|^{2\theta}v + v = 0 \\ &v' + u = 0. \end{aligned} \tag{4.3.2}$$

This yields the elliptic equation

$$-\Delta v + v = |v|^{2\theta}v, \quad u = -v' \tag{4.3.3}$$

It is well known that for $0 < \theta < 2$, there exists a unique positive radial solution $Q(x) = Q(|x|)$ of the first equation in (4.3.3) which is smooth and exponentially decaying. This solution called a nonlinear ground state. Therefore $U_0 = (-Q', Q)$ is the unique solution to (4.3.3) under the condition that v is real and positive. We want to ensure that the ground state solutions U_0 can be continued to yield a branch of solutions of (4.3.1).

Let

$$v(r) = Q(r) + e_1(r), \quad u(r) = -Q'(r) + e_2(r).$$

Substitution into (4.3.1) gives rise to

$$\begin{aligned} &e_2'(r) + \frac{2}{r}e_2(r) + e_1 - (2\theta+1)Q^{2\theta}e_1 = K_1(\varepsilon, e_1, e_2) \\ &e_1'(r) + e_2(r) = K_2(\varepsilon, e_1, e_2) \end{aligned} \tag{4.3.4}$$

where

$$\begin{aligned} K_1(\varepsilon, e_1, e_2) &= |Q + e_1|^{2\theta}(Q + e_1) - (2\theta+1)Q^{2\theta}e_1 - Q^{2\theta+1} \\ &\quad + (|v^2 - \varepsilon u^2|^\theta - v^{2\theta})v \\ K_2(\varepsilon, e_1, e_2) &= \varepsilon(1 + |v^2 - \varepsilon u^2|^\theta)u. \end{aligned}$$

Define L the first order linear differential operator $L : X_r^p \to Y_r^p$ by

$$L\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 - (2\theta+1)Q^{2\theta} & \partial_r + \frac{2}{r} \\ \partial_r & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Then we aim to solve the equation

$$Le = K(\varepsilon, e) \tag{4.3.5}$$

where $e = (e_1, e_2)^T, K(\varepsilon, e) = (K_1, K_2)^T(\varepsilon, e)$. Let $I = (0, \sigma), \sigma > 0$. We say $e(\varepsilon)$ is a weak X^p-solution to equation (4.3.5) if e satisfies

$$e = L^{-1}K(\varepsilon, e) \tag{4.3.6}$$

for a.e. $\varepsilon \in I$. L is indeed invertible as we learn from the following lemma.

Lemma 4.3.1 *Let $0 < \theta < 2$, the linear differential operator*

$$L = \begin{pmatrix} 1-(2\theta+1)Q^{2\theta} & \partial_r + \frac{2}{r} \\ \partial_r & 1 \end{pmatrix}$$

is an isomorphism from X_r^p onto Y_r^p for $2 \le p \le \infty$.

Proof. First we prove that L is one to one. Suppose that there exist radial functions $e_1, e_2 \in W_r^{1,p}$ such that

$$L\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 0.$$

Then

$$-\Delta_r e_1 + e_1 - (2\theta+1)Q^{2\theta}e_1 = 0, \quad e_2 = -e_1'. \tag{4.3.7}$$

It is well known (see, eg. [14]) that $e_1 = 0$ is the unique solution in H^1.

Next we prove that L is onto. Indeed L is a sum of an isomorphism and a relatively compact perturbation:

$$L = \begin{pmatrix} 1 & \partial_r + \frac{2}{r} \\ \partial_r & 1 \end{pmatrix} + \begin{pmatrix} -(2\theta+1)Q(r)^{2\theta} & 0 \\ 0 & 0 \end{pmatrix} = \tilde{L} + M.$$

M is relatively compact because of the exponentially decay of the ground state at infinity. So we only need to prove that $\tilde{L}$ is an isomorphism from X_r^p to Y_r^p, i.e. for any $(\phi_1, \phi_2) \in L_r^p \times L_r^p$, there exist $(e_1, e_2) \in W_r^{1,p} \times W_r^{1,p}$ such that

$$\tilde{L}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

It is equivalent to solve

$$\begin{aligned} e_1 + (\partial_r + \frac{2}{r})e_2 &= \phi_1 \\ \partial_r e_1 + e_2 &= \phi_2 \end{aligned} \tag{4.3.8}$$

and show that $e_1, e_2 \in W_r^{1,p}$. By eliminating e_2 we know that e_1 satisfies

$$(-\Delta_r + 1)e_1 = \phi_1 - (\partial_r + \frac{2}{r})\phi_2. \tag{4.3.9}$$

Define $G(x) = (4\pi)^{-1}|x|^{-1}e^{-|x|}$. (4.3.9) has the solution

$$\begin{aligned} e_1 &= G(x) * \left(\phi_1 - (\partial_r + \frac{2}{r})\phi_2\right) \\ &= G(x) * \phi_1 + \partial_r G(x) * \phi_2. \end{aligned}$$

Here we have used the property of convolution and the fact $(\partial_r + \frac{2}{r})^* f(r) = -\partial_r f(r)$ in $\mathbb{R}^3$. By Young's inequality and $G, \partial_r G \in L^1(\mathbb{R}^3)$, we have

$$\|e_1\|_{L^p} \leq \|G\|_{L^1}\|\phi_1\|_{L^p} + \|\partial_r G\|_{L^1}\|\phi_2\|_{L^p_r}$$

which implies

$$e_1 \in L^p_r.$$

Similarly e_2 satisfies

$$(-\Delta_r + 1 + \frac{2}{r^2})e_2 = \phi_2 - \partial_r \phi_1.$$

Let $H(x) = \frac{x_3}{|x|}G(x)$, then

$$\begin{aligned} e_2 &= H(x) * \phi_2 - H(x) * (\partial_r \phi_1) \\ &= H(x) * \phi_2 + (\partial_r H + \frac{2}{r}H) * \phi_1 \in L^p \end{aligned}$$

since $H, (\partial_r + \frac{2}{r})H \in L^1(\mathbb{R}^3)$.

To improve the regularities of e_1, e_2, we go back to (4.3.8). Since

$$\partial_r e_1 = \phi_1 - e_2 \in L^p_r,$$

we have $e_1 \in W^{1,p}_r$. Regarding the regularity of e_2, we know that

$$(\partial_r + \frac{2}{r})e_2 = \phi_1 - e_1 \in L^p_r.$$

By Lemma 4.2.4

$$\begin{aligned} \|\partial_r e_2\|_{L^p} &\leq C(\|\frac{e_2}{r}\|_{L^p_r} + \|(\partial_r + \frac{2}{r})e_2\|_{L^p_r}) \\ &\leq C\|(\partial_r + \frac{2}{r})e_2\|_{L^p_r} = C\|\phi_1 - e_1\|_{L^p_r}. \end{aligned}$$

Hence we have $e_2 \in W^{1,p}_r$. □

Now we are ready to construct solutions of (4.3.6) by using the contraction mapping theorem.

Proof of Theorem 4.1.1. To prove Theorem 4.1.1, we prove there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, there is a unique solution to equation (4.3.6)

$$e = L^{-1}K(\varepsilon, e)$$

in a small ball in X_r^4. First we must ensure that $K(\varepsilon, e)$ is well defined in Y_r^p if $e \in X_r^p$. Recall that

$$\begin{aligned} K_1(\varepsilon, e_1, e_2) &= |Q + e_1|^{2\theta}(Q + e_1) - (2\theta + 1)Q^{2\theta}e_1 - Q^{2\theta+1} \\ &\quad + (|v^2 - \varepsilon u^2|^\theta - v^{2\theta})v \\ K_2(\varepsilon, e_1, e_2) &= \varepsilon(1 + |v^2 - \varepsilon u^2|^\theta)u. \end{aligned}$$

Let us consider K_1, the estimate for K_2 is similar. Since

$$|K_1(\varepsilon, e)| \leq C_{\varepsilon,\theta}(|v|^{2\theta+1} + |u|^{2\theta+1})$$

where $C_{\varepsilon,\theta}$ is a real constant depending on ε, θ, it suffices to show that $(|v|^{2\theta+1} + |u|^{2\theta+1}) \in L^p$. By Sobolev's embedding

$$W^{1,p}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$$

for any q if $p > 3$. We choose $p = 4$ in the following. The same argument is available for K_2. From Lemma 4.3.1, we know that $L^{-1}K \in X_r^p$.

Fix δ, to be chosen later. Consider the set

$$\Omega = \{e \in X_r^4; \|e\|_{X_r^4} \leq \delta\},$$

and suppose $e \in \Omega$. We know that

$$\|L^{-1}K(\varepsilon, e)\|_{X_r^4} \leq C(\|K_1(\varepsilon, e)\|_{L_r^4} + \|K_2(\varepsilon, e)\|_{L_r^4}).$$

Let $K_1(\varepsilon, e) = K_1^n(\varepsilon, e) + K_1^s(\varepsilon, e)$ where

$$K_1^n(\varepsilon, e) = |Q + e_1|^{2\theta}(Q + e_1) - (2\theta + 1)Q^{2\theta}e_1 - Q^{2\theta+1}$$

and

$$K_1^s(\varepsilon, e) = \left(|(Q + e_1)^2 - \varepsilon(-Q' + e_2)^2|^\theta - |(Q + e_1)|^{2\theta}\right)(Q + e_1).$$

Thus

$$\|K_1\|_{L_r^4} \leq \|K_1^s\|_{L_r^4} + \|K_1^n\|_{L_r^4}.$$

For $\|K_1^s\|_{L_r^4}$, let $a = v^2 = (Q + e_1)^2, b = \varepsilon u^2 = \varepsilon(-Q' + e_2)^2$ in Lemma 4.2.2, then

$$\begin{aligned} \|K_1^s\|_{L^4} &\leq C_\theta \varepsilon \left\| |Q + e_1|^{2\theta-1}| - Q' + e_2|^2 + | - Q' + e_2|^{2\theta}|Q + e_1| \right\|_{L_r^4} \\ &\leq C_\theta \varepsilon(\|Q\|_{W_r^{1,4}}^{2\theta+1} + \|e\|_{X_r^4}^{2\theta+1}) \\ &\leq C_\theta \varepsilon(\|Q\|_{W_r^{1,4}}^{2\theta+1} + \delta) \leq \delta/4 \end{aligned}$$

if $\delta \leq 1$ and ε is small enough such that

$$C_\theta \varepsilon(\|Q\|^{2\theta+1}_{W^{1,p}_r} + \delta) \leq \frac{\delta}{4}.$$

For $\|K_1^n\|_{L^4}$, let $a = Q(r), \sigma = e_1$ in Lemma 4.2.1, then

$$\begin{aligned}\|K_1^n\|_{L^4_r} &\leq C_\theta(\|Q^{2\theta-1}e_1^2\|_{L^4_r} + \|e_1^{2\theta+1}\|_{L^4_r}) \\ &\leq C_\theta(\|e_1\|^2_{W^{1,4}_r} + \|e_1\|^{2\theta+1}_{W^{1,4}_r}) \\ &\leq C_\theta(\delta^2 + \delta^{2\theta+1}) \leq 2C_\theta\delta^2 \leq \delta/4\end{aligned}$$

if $\delta \leq \frac{1}{8C_\theta}$. A similar argument can be applied to K_2 (with similar condition on ε, δ) to obtain that

$$\|K_2(\varepsilon, e)\|_{L^p_r} \leq \frac{\delta}{4}.$$

Hence we obtain

$$L^{-1}K(\varepsilon, e) \in \Omega.$$

Next we want to show that for any $e, f \in \Omega$, and δ, ε as above,

$$\|L^{-1}(K(\varepsilon, e) - K(\varepsilon, f))\|_{Y^p_r} \leq \frac{3}{4}\|e - f\|_{X^p_r},$$

i.e. $L^{-1}K$ is a contraction mapping. We have

$$|K(\varepsilon, e) - K(\varepsilon, f)| \leq |K_1^n(e) - K_1^n(f)| + |K_1^s(e) - K_1^s(f)| + |K_2(e) - K_2(f)|.$$

We compute the r.h.s. term by term. After rewriting $K_1^n(e) - K_1^n(f)$,

$$\begin{aligned}|K_1^n(e) - K_1^n(f)| \leq &\left||Q + e_1|^{2\theta}(Q + e_1) - |Q + f_1|^{2\theta}(Q + f_1) - (2\theta + 1)|Q + f_1|^{2\theta}(e_1 - f_1)\right| \\ &+ (2\theta + 1)\left||Q + f_1|^{2\theta}(e_1 - f_1) - Q^{2\theta}(e_1 - f_1)\right| = D_1^n + D_2^n.\end{aligned}$$

For D_1^n, let $a = Q + f_1, \sigma = e_1 - f_1$ and by use of Lemma 4.2.1, then

$$D_1^n \leq C(|Q + f_1|^{2\theta-1} + |e_1 - f_1|^{2\theta-1})|e_1 - f_1|^2.$$

By Sobolev embedding and Hölder inequality, we have

$$\begin{aligned}\|D_1^n\|_{L^4_r} &\leq C_\theta(\|e_1 - f_1\|^2_{W^{1,4}_r} + \|e_1 - f_1\|^{2\theta+1}_{W^{1,4}_r}) \\ &\leq C_\theta(\delta + \delta^{2\theta})\|e_1 - f_1\|_{W^{1,4}_r} \leq \frac{1}{8}\|e_1 - f_1\|_{W^{1,4}_r}\end{aligned}$$

if $\delta < \frac{1}{16C_\theta}$. Using Lemma 4.2.2, we find

$$|D_2^n| \leq C_\theta(Q^{2\theta-2} + |2Qf_1 + f_1^2|^{\theta-1})|2Qf_1 + f_1^2||e_1 - f_1|.$$

Hence

$$\|D_2^n\|_{L_r^p} \leq C_\theta\delta\|e_1 - f_1\|_{W_r^{1,p}} \leq \frac{1}{8}\|e_1 - f_1\|_{W_r^{1,p}}.$$

Then let us study $K_1^s(e) - K_1^s(f)$:

$$\begin{aligned} K_1^s(e) - K_1^s(f) = &\left(|(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2|^\theta\right)(e_1 - f_1) \\ &+ (|(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2|^\theta)(Q+f_1) \\ &- (|(Q+f_1)^2 - \varepsilon(-Q'+f_2)^2|^\theta - |(Q+f_1)^2|^\theta)(Q+f_1). \end{aligned}$$

Notice that the first line in the r.h.s. is easy to estimate since

$$\begin{aligned} &\left\|\left(|(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2|^\theta\right)(e_1 - f_1)\right\|_{L_r^p} \\ &\leq C_\theta\varepsilon\left\|(|Q+e_1|^{2\theta-2}| - Q'+e_2|^2 + |-Q'+e_2|^{2\theta})(e_1 - f_1)\right\|_{L_r^p} \\ &\leq \frac{1}{8}\|e_1 - f_1\|_{W_r^{1,p}} \end{aligned}$$

for ε sufficiently small. For the second and the third line, let us define

$$\begin{aligned} E(e,f) = &(|(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2|^\theta)(Q+f_1) \\ &- (|(Q+f_1)^2 - \varepsilon(-Q'+f_2)^2|^\theta - |(Q+f_1)^2|^\theta)(Q+f_1). \end{aligned}$$

We discuss the contractive property for two different situations $\theta > 1$ and $\theta = 1$ separately. For $\theta > 1$, we use Lemma 4.2.3. Set $a = (Q+f_1)^2, b = (Q+e_1)^2 - (Q+f_1)^2, c = -\varepsilon(Q+f_2)^2$ (notice that b, c can be taken sufficiently small), and rewrite $E(e,f)$ to get

$$\begin{aligned} |E(e,f)| \leq &\left||a+b+c|^\theta - |a+b|^\theta - |a+c|^\theta + |a|^\theta\right|\sqrt{|a|} \\ &+ \left||(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2 - \varepsilon(-Q'+f_2)^2|^\theta\right|\sqrt{|a|} \\ &\leq C_\theta(|b|^{\theta-1} + |c|^{\theta-1})|b|\sqrt{|a|} \\ &+ \left||(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2|^\theta - |(Q+e_1)^2 - \varepsilon(-Q'+f_2)^2|^\theta\right|\sqrt{|a|}. \end{aligned} \tag{4.3.10}$$

where for the last line, we applied Lemma 4.2.2. We obtain

$$\|E(e,f)\|_{L_r^4} \leq C_\theta(\varepsilon^{\theta-1} + \delta^{\theta-1})\|e - f\|_{W_r^{1,4}} \leq \frac{1}{8}\|e - f\|_{W_r^{1,4}}$$

for ε, δ sufficiently small. Hence we have for $1 < \theta < 2$,

$$\|K_1(e) - K_1(f)\|_{L^4_r} \leq \frac{1}{2}\|e - f\|_{W^{1,4}}.$$

Next we prove that $E(e, f)$ is contractive for $\theta = 1$ directly. Lemma 4.2.3 can not be used since $|b|^{\theta-1} = |c|^{\theta-1} = 1$. In (4.3.10), if $|a| \geq \max\{5|b|, 5|c|\}$, then

$$|a + b + c| - |a + b| - |a + c| + |a| = 0.$$

Thus

$$\|E(e, f)\|_{L^4_r} \leq C_\theta \varepsilon \|e - f\|_{W^{1,4}_r} \leq \frac{1}{4}\|e - f\|_{W^{1,4}_r}.$$

Hence we only need to consider $E(e, f)$ if $|a|$ is small, i.e. if $|a| \leq 5\max\{5|b|, 5|c|\}$,

$$|a + b + c| - |a + b| - |a + c| + |a| \leq C(|b| + |c|).$$

Simply assume that $|c| \leq |b|$, we have

$$\|E(e, f)\|_{L^p} \leq C_\theta \delta^{1/2}\|e - f\|_{W^{1,4}} \leq \frac{1}{4}\|e - f\|_{W^{1,4}}.$$

Therefore if $\theta = 1$,

$$\|K_1(\varepsilon, e) - K_1(\varepsilon, f)\|_{L^4_r} \leq \frac{1}{2}\|e - f\|_{W^{1,4}_r}.$$

Similarly, we can prove that

$$\|K_2(e) - K_2(f)\|_{Y^4_r} \leq \frac{1}{4}\|e - f\|_{W^{1,p}_r}.$$

Note we can satisfy all the condition above by choosing $\delta = C_\theta \varepsilon$ and taking ε sufficiently small. Then the contraction mapping theorem implies $L^{-1}K$ has a unique fixed point $e(\varepsilon) \in \Omega$ which is a weak solution of equation (4.3.5). The continuity w.r.t. ε follows from the continuity w.r.t ε of the map $L^{-1}K$ and its contractibility. This completes the proof of Theorem 4.1.1. □

Let us see why a solution of equation (4.3.5) which is in $X^{1,4}_r$ has more regularity. This is done by using a standard bootstrap argument and the following standard lemma:

Lemma 4.3.2 *(see [2]) Let $F : \mathbb{C} \to \mathbb{C}$ satisfy $F(0) = 0$, and assume that there exists $\alpha \geq 0$ such that*

$$|F(v) - F(u)| \leq C(|v|^\alpha + |u|^\alpha)|v - u| \quad \textit{for all} \quad u, v \in \mathbb{C}.$$

Let

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}, \quad 1 \leq p, q, r \leq \infty.$$

It follows that if $u \in L^p, \nabla u \in L^q$, *then* $\nabla F(u) \in L^r$ *and*

$$\|\nabla F(u)\|_{L^r} \leq C\|u\|_{L^p}^{\alpha}\|\nabla u\|_{L^q}.$$

Proof of Theorem 4.1.2. First we can prove that

$$e_1, e_2 \in \bigcap_{4 \leq p < \infty} W_r^{2,p}.$$

Recall that e_1, e_2 satisfy

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = L^{-1} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

and L is an isomorphism from $X_p^r \to Y_p^r$. We know that

$$|K_1| \leq C_{\varepsilon,\theta}(|v|^{2\theta+1} + |u|^{2\theta+1})$$

and

$$|K_2| \leq \varepsilon(|u| + |v|^{2\theta+1} + |u|^{2\theta+1})$$

Since $e_1, e_2 \in W_r^{1,4}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, then $K_1 \times K_2 \in L_r^p \times L_r^p$ for $p \geq 4$. By Lemma 4.3.1, we have

$$(e_1, e_2) \in \bigcap_{4 \leq p < +\infty} W_r^{1,p} \times W_r^{1,p}.$$

Next from Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.3

$$|K(e_1, e_2) - K(f_1, f_2)| \leq C(|Q + Q'|^{2\theta} + |e_1|^{2\theta} + |e_2|^{2\theta} + |f_1|^{2\theta} + |f_2|^{2\theta}) \\ (|e_1 - f_1| + |e_2 - f_2|).$$

So by Lemma 4.3.2,

$$\nabla K_1 \times \nabla K_2 \in \bigcap_{4 \leq p < \infty} L_r^{\frac{p}{2\theta}} \times L_r^{\frac{p}{2\theta}}.$$

This gives that

$$(e_1, e_2) \in \bigcap_{4 \leq p < \infty} W_r^{2,p} \times W_r^{2,p}$$

and

$$\|e\|_{W_r^{2,p}\times W_r^{2,p}} \leq C\varepsilon.$$

Going back to equation (4.3.5), we know that $e_1, e_2 \in W_r^{3,p} \subset C^2$. So (f, g) are classical solutions.

Moreover we show that e_1, e_2 have exponential decay at infinity. We know e_1, e_2 are classical solutions and $|e_1|, |e_2| \leq C\varepsilon$ by Sobolev's embedding theorem. Taking derivatives in (4.3.2) and after tedious computations we find

$$\begin{cases} e_1'' - e_1 = \delta_1(r)e_1 + \delta_2(r)e_1' + \delta_3(r)Q & \text{for } r \text{ large} \\ e_2'' - e_2 = \sigma_1(r)e_2 + \sigma_2(r)e_2' + \sigma_3(r)Q & \text{for } r \text{ large} \end{cases} \tag{4.3.11}$$

where $\sigma_i, \delta_i \in W^{2,p}$ and $|\sigma_1|, |\delta_1| \leq C\varepsilon (i = 1, 2, 3)$ for r large.

We conclude that there exist constants $r_0, \nu(\varepsilon), C(\varepsilon)$ positive such that

$$|e_1(r)| + |e_2(r)| \leq Ce^{-\nu r} \text{ for } r \geq r_0. \tag{4.3.12}$$

We prove it by an application of the maximum principle. Without loss of generality, suppose $e_1(r_0) = 2\varepsilon$ (r_0 is sufficiently large). Let

$$h(r) = e^{-\nu(r-r_0)} + \beta e^{\nu(r-r_0)}$$

where $\beta > 0$ is arbitrary and $0 < \nu < 1$ is to be determined later. If $g = e_1 - h$, then g satisfies

$$g'' = (1 + \delta_1)g + \delta_2 g' + (1 - \nu^2 + \delta_1)h + \delta_2 h' + \delta_3 Q$$

Since $h' = \nu(-e^{-\nu(r-r_0)} + \beta e^{\nu(r-r_0)}) \leq \nu h$ and $Q \leq h$, then

$$g'' \geq (1 + \delta_1)g + \delta_2 g' + (1 - \nu^2 + \delta_1 + |\delta_3|)h \tag{4.3.13}$$

with $g(r_0) = e_1(r_0) - (1 + \beta) < 0, g(\infty) < 0$. Thus we claim that

$$g(r) \leq 0 \quad \text{for} \quad r \geq r_0,$$

if ν is small enough such that

$$1 - \nu^2 + \delta_1 + |\delta_3| \geq 0.$$

If the claim is not true, then $g(r)$ obtains maximum at $r = r_1$ and $g(r_1) > 0$. Thus $g''(r_1) < 0, g'(r_1) = 0$. But this contradicts with equation (4.3.13) since

the right hand side of (4.3.13) is positive evaluated at $r = r_1$. Therefore the claim is true if $\nu \leq \sqrt{1 - C\varepsilon}$ and then

$$e_1(r) \leq h(r) \quad \text{if } r \text{ is large enough.}$$

Then similarly we can show that

$$e_1(r) \geq -h(r) \quad \text{if } r \text{ is large enough.}$$

Thus

$$|e_1(r)| \leq h(r) = e^{-\nu(r-r_0)} + \beta e^{\nu(r-r_0)}.$$

Letting $\beta \to 0$, we have

$$|e_1(r)| \leq Ce^{-\nu r}.$$

for r large enough. The exponential decay estimate for e_2 can be obtained in a similar way. Once we have (5.2.5), it is obvious that $|\partial_r e_j(r)| \leq Ce^{-\nu r}$ and $e_j \in H^2$. This completes the proof of Theorem 4.1.2. □

Remark. For $0 < \theta < 1$, our method does not work since Lemma 4.2.3 is not valid. Let us consider a special example. Suppose $e_2 = f_2 = 0$, then

$$E(e_1, f_1) = (|(Q+e_1)^2 - \varepsilon(Q')^2|^\theta - |Q+e_1|^{2\theta} - |(Q+f_1)^2 - \varepsilon(Q')^2|^\theta + |Q+f_1|^{2\theta})(Q+f_1)$$

We want to know whether or not the following inequality is true

$$|E(e_1(r), f_1(r))| \leq \frac{1}{4}|e_1(r) - f_1(r)|, \quad r \in (0, \infty) \tag{4.3.14}$$

if ε small enough. Letting r_0 large enough and $s = \varepsilon^\alpha, \alpha > 0$ to be determined later, we assume that

$$\begin{aligned} Q(r_0) + e_1(r_0) &= \sqrt{\varepsilon}|Q'(r_0)|(1+s), \\ Q(r_0) + f_1(r_0) &= \sqrt{\varepsilon}|Q'(r_0)|. \end{aligned}$$

Then under this ansatz,

$$\begin{aligned} |E(e_1(r_0), f_1(r_0))| &= [(s^2 + 2s)^\theta - ((1+s)^{2\theta} - 1)]h^{2\theta+1} = g(s)h^{2\theta+1}, \\ |e_1(r_0) - f_1(r_0)| &= sh \end{aligned}$$

where $h = \sqrt{\varepsilon}|Q'(r_0)|$. Then

$$|E(e_1(r_0), f_1(r_0))| = \frac{g(s)}{s} h^{2\theta} |e_1(r_0) - f_1(r_0)|.$$

We claim that if $\alpha > \frac{\theta}{1-\theta}$, then

$$\frac{g(s)}{s}h^{2\theta} \gg \frac{1}{2}, \quad \text{as} \quad \varepsilon \to 0.$$

In fact, we have

$$g(s) \geq Cs^{\theta}$$

since

$$(s^2 + 2s)^{\theta} \geq Cs^{\theta}$$

and

$$|(1+s)^{2\theta} - 1| \leq C(s + s^{2\theta}) \ll Cs^{\theta}.$$

So

$$\frac{g(s)}{s}h^{2\theta} \geq Cs^{\theta-1}h^{2\theta} = C|Q'(r_0)|^{2\theta}\varepsilon^{\theta+\alpha(\theta-1)} \gg \frac{1}{2}, \quad \text{as} \quad \varepsilon \to 0$$

since $\theta + \alpha(\theta - 1) < 0$. The claim is proved and consequently, (4.3.14) does not hold for every $r \in (0, \infty)$.

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Chapter 5

Instability of solitary waves for nonlinear Dirac equations

5.1 Introduction

In this chapter we show the instability of solitary waves, or standing waves for the nonlinear Dirac equation

$$i\gamma^0\partial_t\psi + i\sum_{j=1}^{3}\gamma^j\partial_j\psi - m\psi + (\bar{\psi}\psi)\psi = 0 \tag{5.1.1}$$

where $\psi(x,t) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{C}^4, m > 0$ and

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1,2,3,$$

σ^k are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define

$$\bar{\psi} = \gamma^0\psi, \quad \bar{\psi}\psi = (\gamma^0\psi, \psi) = \sum_{i=1}^{2}(\psi_i, \psi_i) - \sum_{i=3}^{4}(\psi_i, \psi_i)$$

where $(\cdot,\cdot)$ is the Hermitian inner product in $\mathbb{C}^1$.

By solitary wave, or standing wave we mean a solution of the form

$$\psi(x,t) = e^{-i\omega t}\phi(x) \tag{5.1.2}$$

with ω a real parameter and $\phi_\omega(x)$ decays as $|x| \to \infty$. The existence of such solutions are extensively studied in [2, 4, 5, 9–12] for a large variety of nonlinearities. While by shooting method [2, 4, 10] and variational method [5],

A version of this chapter will be submitted for publication. Guan, M. and Gustafson, S. Instability of solitary waves for nonlinear Dirac equations.

the authors proved there exists infinity many localized solutions for every $\omega \in (0, m)$. Guan in [9] refined the result in [11] to show that for nonlinearities with the form $|\bar{\psi}\psi|^{\theta}\psi, 1 \leq \theta < 2$, or $(\bar{\psi}\psi)\psi$, there is a continuous curve of solitary waves ϕ_ω for $\omega \in (m - \varepsilon_0, m)$ with some $\varepsilon_0 > 0$. These are of the form

$$\phi_\omega = \begin{pmatrix} g(r) \\ 0 \\ if(r)\cos\Psi \\ if(r)\sin\Psi e^{i\Phi} \end{pmatrix}, \tag{5.1.3}$$

where f, g are real functions, and r, Ψ, Φ are standard spherical coordinates in $\mathbb{R}^3$.

We left an open problem about the stability of the solitary waves constructed in [9]. Stability of a solitary waves means that any initial datum sufficiently near this state gives rise to a solution which exists and remains near that state (at least up to symmetries) for later times. Otherwise it is unstable. This issue has been extensively studied for nonlinear Klein-Gordon equations and nonlinear Schrödinger equations. Grillakis, Shatah and Strauss [7, 8] proved a general orbital stability and instability condition in a very general setting, which can be applied to traveling waves of nonlinear PDEs such as Klein-Gordon, and Schrödinger and wave equations. Let $E(\phi)$ be the energy functional, and $L(\phi)$ be the charge related to the symmetry that gives the solitary wave dynamics (in the case of solitary waves of the form (5.1.2), $L(\phi) = \int |\phi|^2 dx$). Their assumptions allow the second variation operator $H_\omega = E''(\phi_\omega) - \omega L''(\phi_\omega)$ to have only one simple negative eigenvalue, a kernel of dimension one and the rest of the spectrum to be positive and bounded away from zero. Then sharp instability condition is

$$d''(\omega) < 0 \text{ where } d(\omega) = E(\phi_\omega) - \omega L(\phi_\omega). \tag{5.1.4}$$

Shatah and Strauss also came up with this as the instability criterion in [13]. But this method cannot be applied to Dirac operator directly. Contrary to $-\Delta$, the Dirac operator $D_m = -i\sum_{j=1}^{3}\gamma^0\gamma^j\partial_j + m\gamma^0$ is not bounded from below. The spectrum of D_m is $\sigma(D_m) = (-\infty, m] \cap [m, +\infty)$.

However there are some partial results about the application of this method to Dirac equations. Bogolubsky in [1] required the positivity of the second variation of the energy functional as a necessary condition for stability, just like (5.1.4). Werle [16], Strauss and Vázquez in [14] claimed that the ground states are unstable if they exist, since the energy functional does not have a local minimum at the ground states. The condition (5.1.4) suggests that the ground states in [9] are unstable. Recall for $\varepsilon = m - \omega > 0$,

the ground states for the problem with nonlinearity $|\bar{\psi}\psi|^{\theta}\psi$have the ansatz

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} \varepsilon^{\frac{1}{2\theta}}\phi_1^{\varepsilon}(\sqrt{\varepsilon}x) \\ \varepsilon^{\frac{1}{2\theta}+\frac{1}{2}}\phi_2^{\varepsilon}(\sqrt{\varepsilon}x) \end{pmatrix} \tag{5.1.5}$$

where

$$\phi_1^{\varepsilon} = \begin{pmatrix} Q \\ 0 \end{pmatrix} + O(\varepsilon), \quad \phi_2^{\varepsilon} = -\frac{iQ'}{2m}\begin{pmatrix} \cos\Psi \\ \sin\Psi e^{i\Phi} \end{pmatrix} + O(\varepsilon),$$

and $Q(x) = Q(|x|)$ satisfies the nonlinear elliptic equation

$$-\Delta_r Q + Q = Q^{2\theta}Q, \quad 1 \le \theta < 2.$$

Q is smooth and positive, decreases monotonically, and decays exponentially at infinity (see [6] and the references therein). We have

$$\|\phi_1\|_{L^2}^2 = C\varepsilon^{\frac{1}{\theta}-\frac{3}{2}} + O(\varepsilon^{\frac{1}{\theta}-\frac{1}{2}}), \quad \|\phi_2\|_{L^2}^2 = O(\varepsilon^{\frac{1}{\theta}-\frac{1}{2}}).$$

Since $d'(\omega) = -L(\phi) = -(\|\phi_1\|_{L^2}^2 + \|\phi_2\|_{L^2}^2)$, then if $\theta > \frac{2}{3}$, we have

$$d''(\omega) = -\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 < 0, \quad \text{as} \quad \varepsilon \to 0^+.$$

But this is not a rigorous proof. The question of stability is related to the eigenvalues of the linearized operator. In general the stability analysis of nonlinear Dirac equations is harder than that of nonlinear Schrödinger equations.

For equation (5.1.1), the solitary waves have the form

$$\psi(x,t) = e^{-i\omega t}\begin{pmatrix} g(r,t) \\ 0 \\ if(r,t)\cos\Psi \\ if(r,t)\sin\Psi e^{i\Phi} \end{pmatrix}.$$

Here $f(r,t), g(r,t)$ are $\mathbb{C}$-valued functions. Then equation (5.1.1) is reduced to a coupled system

$$\begin{cases} \omega g + i\partial_t g = \partial_r f + \frac{2f}{r} - (|g|^2 - |f|^2)g + mg, \\ -\omega f - i\partial_t f = \partial_r g - (|g|^2 - |f|^2)f + mf. \end{cases} \tag{5.1.6}$$

In [9], we used rescaling and perturbation arguments. Let $\varepsilon = m - \omega, \rho = \sqrt{\varepsilon}r$. Introducing two new functions (u,v) such that $f(\rho,t) = \varepsilon u(\sqrt{\varepsilon}r,t), g(\rho,t) = \varepsilon^{1/2}v(\sqrt{\varepsilon}r,t)$, then (u,v) solve the coupled system

$$\begin{cases} -i\partial_t u = \partial_\rho v + u - \varepsilon(1 + |v|^2 - \varepsilon|u|^2)u, \\ i\partial_t v = \varepsilon(\partial_\rho u + \frac{2u}{\rho} - (|v|^2 - \varepsilon|u|^2)v + v). \end{cases} \tag{5.1.7}$$

Where without loss of generality, we take $m = \frac{1}{2}$. The ground states solutions $(u_0(\rho), v_0(\rho))$ satisfy

$$\begin{cases} \partial_\rho u_0 + \frac{2u_0}{\rho} - (v_0^2 - \varepsilon u_0^2)v_0 + v_0 = 0, \\ \partial_\rho v_0 + 2mu_0 - \varepsilon(1 + v_0^2 - \varepsilon u_0^2)u_0 = 0. \end{cases} \tag{5.1.8}$$

When ε is small enough, then

$$v_0 = Q + e_2, \quad u_0 = -Q' + e_1,$$

$$\|e_1\|_{H^1}, \|e_2\|_{H^1} \leq C\varepsilon,$$

and $Q(x) = Q(|x|)$ is the ground state of the cubic, focusing and radial nonlinear Schrödinger equation. Linearizing the system (5.1.7) around (u_0, v_0), we obtain the following linearized operator

$$A_\varepsilon = \begin{pmatrix} 0 & -1 + \varepsilon(1 + v_0^2 - \varepsilon u_0^2) & 0 & -\partial_r \\ 1 - \varepsilon(1 + v_0^2 - 3\varepsilon u_0^2) & 0 & \partial_r - 2\varepsilon u_0 v_0 & 0 \\ 0 & \varepsilon(\partial_r + \frac{2}{r}) & 0 & \varepsilon(1 - v_0^2 + \varepsilon u_0^2) \\ -\varepsilon(\partial_r + \frac{2}{r} + 2\varepsilon u_0 v_0) & 0 & -\varepsilon(1 - 3v_0^2 + \varepsilon u_0^2) & 0 \end{pmatrix}.$$

To prove the instability, it needs to show that A_ε has an eigenvalue $\lambda = \lambda(\varepsilon)$ with $\operatorname{Re}\lambda > 0$. Then we can say that solitary wave is linearly unstable. Furthermore solitary wave can be proved to be nonlinearly unstable if we can verify that A_ε satisfies Theorem 6.1 in [8]

$$\|e^{tA_\varepsilon}\| \leq C_\varepsilon e^{\mu t} \quad \text{for some} \quad 0 < \mu < 2\operatorname{Re}\lambda.$$

The idea to find $\lambda(\varepsilon)$ with positive real part is first to formally expand eigenvalues and eigenfunctions as a power series of ε. In order to proceed with this expansion, it is useful to block-diagonalize A_ε into A^ε with

$$A^\varepsilon = \begin{pmatrix} 0_{2\times 2} & A_-^\varepsilon \\ -A_+^\varepsilon & 0_{2\times 2} \end{pmatrix},$$

where $A_-^\varepsilon, A_+^\varepsilon$ are two-by-two matrices. The formal expansion implies that the eigenvalue λ is related to the eigenvalues of the cubic, focusing, radial NLS with the linearized operator around its ground states

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \tag{5.1.9}$$

where

$$L_- = -\Delta + 1 - Q^2, \; L_+ = -\Delta + 1 - 3Q^2,$$

which is known to be unstable [15]. Notice that A_ε is not self-adjoint, so the formal expansion of the eigenvalue and eigenfunction can not easily be used to conclude that there exist such an eigenvalue. Rather, we will verify the result by the "Lyapounov-Schmidt reduction" method from bifurcation theory.

Before introducing the main result of this chapter, let us recall the main theorem in [9]: the nonlinear Dirac equation with nonlinearities $|\bar{\psi}\psi|^\theta\psi, 1 \leq \theta < 2$ or $(\bar{\psi}\psi)\psi$, admits the solitary waves of the form (5.1.3) with (f, g) satisfying

Lemma 5.1.1 *[9] Let $\varepsilon = m - \omega$. There exists $\varepsilon_0 = \varepsilon_0(\theta) > 0$ and a unique solution $(f, g)(\varepsilon) \in C((0, \varepsilon_0), W_r^{1,4}(\mathbb{R}^3, \mathbb{R}^2))$ satisfying*

$$f(r) = \varepsilon^{\frac{\theta+1}{2\theta}}(-Q'(\sqrt{\varepsilon}r) + e_2(\sqrt{\varepsilon}r))$$
$$g(r) = \varepsilon^{\frac{1}{2\theta}}(Q(\sqrt{\varepsilon}r) + e_1(\sqrt{\varepsilon}r))$$

with

$$\|e_j\|_{W_r^{1,2}} \leq C\varepsilon \quad \text{for some} \quad C = C(\theta) > 0, j = 1, 2,$$

and e_1, e_2 have exponential decay as $|x| \to \infty$.

An important Lemma addressing the instability of the ground states Q of nonlinear Schrödinger equation is as follows. Let $Q = Q(|x|), x \in \mathbb{R}^n$ satisfy the nonlinear elliptic equation

$$-\Delta_r Q + Q = Q^{p-1}Q,$$

where Q is smooth and positive, decreases monotonically, and decays exponentially at infinity, then

Lemma 5.1.2 *[6] Suppose λ is an eigenvalue of $\mathcal{L}$ (5.1.9) with corresponding eigenfunction U*

$$\mathcal{L}U = \lambda U,$$

and let $\mu = -\lambda^2$. If $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$, then μ is real and the lowest eigenvalue μ are

$$\mu_1 < 0, \mu_2 = \cdots = \mu_{n+1} = 0, \mu_{n+2} > 0.$$

Since $\mu_1 < 0$, the first pair eigenvalues of $\mathcal{L}, (\lambda_1)^\pm = \pm\sqrt{-\mu_1}$. Therefore we say the ground state is linearly unstable. In this chapter, the space dimension $n = 3$.

Now we are ready to state the main theorem in this chapter.

Theorem 5.1.3 *For the nonlinear Dirac equation* (5.1.1), *the ground states in Lemma 1.1 are linearly unstable for ε sufficiently small, i.e. the linearized operator A_ε has an eigenvalue λ with positive real part.*

In this chapter, we only consider Dirac equations with nonlinearity $(\bar{\psi}\psi)\psi$, but in fact this theorem is still true for $|\bar{\psi}\psi|^\theta\psi, 1 \leq \theta < 2$. The readers can carry out the same argument to prove it.

In this chapter, we use the following notations. $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ denotes a column vector in $\mathbb{C}^4$. Sometimes we also write the column vector U as $(u_1, u_2, u_3, u_4)^T$. We denote by (U, V) the Hermitian product of two vectors U, V in $\mathbb{C}^4$. The usual Hermitian product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is denoted by

$$(f, g)_{L^2} = \int_{\mathbb{R}^3} (f(x), g(x)) d^3x.$$

5.2 Spectral analysis for the linearized operator

In this section, we present a block-diagonal representation of the linearized operator A_ε. Then a formal expansion of eigenvalues and the bifurcation theory tell us that $\lambda = \varepsilon\sqrt{-\mu_1} + o(\varepsilon)$ is one of the eigenvalue of A_ε.

To study the stability of the ground state solution, one linearizes equation (5.1.7) at (u_0, v_0). Let $u(\rho, t) = u_0(\rho) + e(\rho, t), v(\rho, t) = v_0(\rho) + h(\rho, t), e, h$ are $\mathbb{C}$-valued functions, then the perturbation (e, h) satisfy the system

$$\begin{cases} -i\dfrac{de}{dt} = \partial_\rho h + e - \varepsilon(1 + |v_0 + h|^2 - \varepsilon|u_0 + e|^2)e - \\ \qquad \varepsilon((|v_0 + h|^2 - \varepsilon|u_0 + e|^2) - (v_0^2 - \varepsilon u_0^2))u_0, \\ i\dfrac{dh}{dt} = \varepsilon\partial_\rho e + \varepsilon\dfrac{2e}{\rho} + \varepsilon h - \varepsilon(|v_0 + h|^2 - \varepsilon|u_0 + e|^2)h - \\ \qquad \varepsilon(|v_0 + h|^2 - \varepsilon|u_0 + e|^2)v_0 + \varepsilon((v_0)^2 - \varepsilon(u_0)^2)v_0. \end{cases} \tag{5.2.1}$$

A convenient way to study equation (5.2.1) is to write functions e, h in their the real and imaginary parts. Let

$$U = (\operatorname{Re} e, \operatorname{Im} e, \operatorname{Re} h, \operatorname{Im} h)^T,$$

then equation (5.2.1) becomes a system with a four-by-four matrix operator

$$\frac{dU}{dt} = A_\varepsilon U + nonlinear \quad terms \tag{5.2.2}$$

where

$$A_\varepsilon = \begin{pmatrix} 0 & -1+\varepsilon(1+v_0^2-\varepsilon u_0^2) & 0 & -\partial_\rho \\ 1-\varepsilon(1+v_0^2-3\varepsilon u_0^2) & 0 & \partial_\rho - 2\varepsilon u_0 v_0 & 0 \\ 0 & \varepsilon(\partial_\rho + \frac{2}{\rho}) & 0 & \varepsilon(1-v_0^2+\varepsilon u_0^2) \\ -\varepsilon(\partial_\rho + \frac{2}{\rho} + 2\varepsilon u_0 v_0) & 0 & -\varepsilon(1-3v_0^2+\varepsilon u_0^2) & 0 \end{pmatrix}.$$

A_ε is a first order, non-self-adjoint operator. From the phase invariance of the dynamics, we know zero is an eigenvalue and

$$A_\varepsilon \begin{pmatrix} 0 \\ u_0 \\ 0 \\ v_0 \end{pmatrix} = 0.$$

Because of its connection to the stability problem, our interest is in the spectrum of this non-self-adjoint operator. The analysis of the spectrum $\sigma(A_\varepsilon)$, especially of the eigenvalues is hard because of its four-by-four format. For such kind of operator A_ε, the block-diagonalization is often used in fast numerical computations of eigenvalues with the Chebyshev interpolation algorithm [3]. We adopt this method to reduce A_ε to a new operator which is more convenient to study.

Theorem 5.2.1 *There exists an orthogonal similarity transformation M, such that $M^{-1} = M^T = M$, where*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that block-diagonalizes the operator A_ε,

$$M^{-1} A_\varepsilon M = \begin{pmatrix} 0 & A_-^\varepsilon \\ -A_+^\varepsilon & 0 \end{pmatrix} = A^\varepsilon,$$

where

$$A_-^\varepsilon = \begin{pmatrix} -1+\varepsilon(1+v_0^2-\varepsilon u_0^2) & -\partial_\rho \\ \varepsilon(\partial_\rho + \frac{2}{\rho}) & \varepsilon(1-v_0^2+\varepsilon u_0^2) \end{pmatrix}$$

and

$$A_+^\varepsilon = \begin{pmatrix} -1+\varepsilon(1+v_0^2-3\varepsilon u_0^2) & -\partial_\rho + 2\varepsilon u_0 v_0 \\ \varepsilon(\partial_\rho + \frac{2}{\rho} + 2\varepsilon u_0 v_0) & \varepsilon(1-3v_0^2+\varepsilon u_0^2) \end{pmatrix}.$$

Moreover, A^ε and A_ε has the same spectrum, $\sigma(A^\varepsilon) = \sigma(A_\varepsilon)$.

Proof. M is nonsingular since $|M| = 1$. We can apply the similarity transformation to A_ε to yield A^ε. The eigenvalue problem

$$A_\varepsilon U = \lambda U, \quad U \in \mathbb{C}^4,$$

is equivalent to

$$MA^\varepsilon M^{-1}U = A_\varepsilon U = \lambda U.$$

Since M is nonsingular, then

$$A^\varepsilon V = \lambda V, \quad V = M^{-1}U.$$

Then we conclude that $\sigma(A_\varepsilon) = \sigma(A^\varepsilon)$. □

Because of Theorem 5.2.1, we change from studying to studying $\sigma(A_\varepsilon)$ to $\sigma(A^\varepsilon)$. For the continuous spectrum $\sigma_c(A^\varepsilon)$, by Weyl's lemma we can drop the localized terms v_0, u_0 since they have exponential decay at infinity. A^ε is a sum of the operator $\tilde{A}^\varepsilon$ and a relatively compact perturbation,

$$\tilde{A}^\varepsilon = \begin{pmatrix} 0_{2\times 2} & -iA' \\ iA' & 0_{2\times 2} \end{pmatrix}$$

with

$$A' = \begin{pmatrix} i(1-\varepsilon) & i\partial_\rho \\ -i\varepsilon(\partial_\rho + \frac{2}{\rho}) & -i\varepsilon \end{pmatrix}.$$

The following theorem identifies the continuous spectrum of A^ε covering almost the entire imaginary axis.

Theorem 5.2.2 *The spectrum of A'*

$$\sigma(A') = i[1-\varepsilon, \infty) \cup i(-\infty, -\varepsilon], 0 < \varepsilon \ll 1.$$

Thus,

$$\sigma_c(A^\varepsilon) = \sigma_c(\tilde{A}^\varepsilon) = \{ir, r \in \mathbb{R}, |r| \geq \varepsilon\}.$$

Proof. The eigenvalue problem $\tilde{A}^\varepsilon U = \mu U$ is equivalent to the block-diagonalized eigenvalue problems

$$(A')^2 U_1 = \lambda U_1, \quad (A')^2 U_2 = \lambda U_2, \quad \lambda = \mu^2 \tag{5.2.3}$$

with $U_1, U_2 \in \mathbb{C}^2$. Therefore it suffices to find the spectrum of A'. Suppose $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in D(A')$ are the eigenfunctions with eigenvalue λ such that $A'\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = \lambda\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, then

$$\begin{cases} i(1-\varepsilon)u + i\partial_\rho v = \lambda u, \\ -i\varepsilon(\partial_\rho u + \frac{2}{\rho}u) - i\varepsilon v = \lambda v. \end{cases} \tag{5.2.4}$$

Multiplying the second equation by $\lambda - i(1-\varepsilon)$ and using the first equation, we have

$$\Delta v = \frac{(\lambda + i\varepsilon)(\lambda - i(1-\varepsilon))}{\varepsilon} v.$$

Since Δ has only continuous spectrum $(-\infty, 0]$, let

$$\frac{(\lambda + i\varepsilon)(\lambda - i(1-\varepsilon))}{\varepsilon} = -\nu^2,$$

then we have

$$\lambda = \frac{i(1-2\varepsilon) \pm i\sqrt{1+4\varepsilon\nu^2}}{2}.$$

We claim that the spectrum of A' is on the imaginary axis and

$$\sigma(A') = i[1-\varepsilon, \infty) \cup i(-\infty, -\varepsilon].$$

This is the first part of the theorem. Define

$$\Sigma = i[1-\varepsilon, \infty) \cup i(-\infty, -\varepsilon].$$

On one hand, if $\lambda \notin \Sigma$, it implies that

$$\beta = \frac{(\lambda + i\varepsilon)(\lambda - i(1-\varepsilon))}{\varepsilon} \notin (-\infty, 0).$$

For any $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in L^2 \times L^2(\rho^2 d\rho)$, let

$$v = (-\Delta_r + \beta)^{-1} \left(i(\partial_\rho + \frac{2}{\rho}) f_1 - (\lambda - i(1-\varepsilon) f_2) \right),$$

then $v \in H^1$ by the standard regularity argument. Since $\lambda \neq i(1-\varepsilon)$, then

$$u = \frac{1}{\lambda - i(1-\varepsilon)} (i\partial_\rho v - f_1) \in L^2,$$

and u, v satisfy

$$(A' - \lambda I)\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) = f.$$

Hence $(A' - \lambda I)^{-1}$ is invertible, which is a contradiction. Thus $\sigma(A') \subseteq \Sigma$.

On the other hand, if $\lambda \in \Sigma$, we want to show that $\lambda \in \sigma(A')$. We are seeking $\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right) \in L^2 \times L^2$ satisfy the following

a)$\|\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right)\|_{L^2\times L^2} = 1$,

b)$\|(A'-\lambda)\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right)\|_{L^2\times L^2} \to 0$ as $j \to \infty$,

c)$\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right) \rightharpoonup 0$ weakly as $j \to \infty$.

Since $\lambda \in \Sigma$, then $\beta \in (-\infty, 0] = \sigma(\Delta)$. By Weyl's lemma, there exists $\phi_j \in H^2$ with $\|\phi_j\|_{L^2} = 1$ such that

$$\|(\Delta - \beta)\phi_j\|_{L^2} \to 0 \quad \text{and} \quad \phi_j \rightharpoonup 0 \quad \text{as} \quad j \to \infty.$$

Let $v_j = \frac{1}{\alpha}\phi_j, \alpha = \sqrt{1 + \frac{|\lambda+i\varepsilon|}{|\lambda - i(1-\varepsilon)|}}$ and $u_j = \frac{i\partial_\rho v_j}{\lambda - i(1-\varepsilon)} \in H^1(\lambda \neq i(1-\varepsilon))$, then

$$\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right) \in D(A')$$

and

$$\|\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right)\|^2_{L^2\times L^2} = \frac{1}{\alpha^2}(\|\phi_j\|_{L^2} + \frac{1}{|\lambda - i(1-\varepsilon)|^2}\|\partial_\rho\phi_j\|^2_{L^2}).$$

Since

$$\begin{aligned} \|\partial_\rho\phi_j\|^2_{L^2} &= (\phi_j, -\Delta\phi_j) \\ &= (\phi_j, (-\Delta + \beta)\phi_j) + (\phi_j, -\beta v_j) \\ &\to |\beta| \end{aligned}$$

Hence

$$\|\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right)\|^2_{L^2\times L^2} \to \frac{1}{\alpha^2}(1 + \frac{|\lambda + i\varepsilon|}{|\lambda - i(1-\varepsilon)|}) = 1.$$

Moreover, we have

$$\|(A'-\lambda)\left(\begin{smallmatrix} u_j \\ v_j \end{smallmatrix}\right)\| \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore $\Sigma \subset \sigma(A')$ and the claim is proved. By relation (5.2.3), we know

$$\mu \in i[\varepsilon, \infty) \cap i(-\infty, -\varepsilon].$$

The continuous spectrum of A^ε covers almost the pure imaginary axis. □

In order to prove the instability, we will show A^ε has at least one eigenvalue with positive real part.

Theorem 5.2.3 *For ε sufficiently small, the operator A^ε admits an eigenvalue of the form*

$$\lambda_\varepsilon = \varepsilon\nu + O(\varepsilon^2)$$

and eigenfunction

$$U_\varepsilon = (-\eta_{01}', \eta_{01}, -\eta_{02}', \eta_{02})^T + o(1).$$

where $\nu = \sqrt{-\mu_1}$ is taken from Lemma 5.1.2, and η_{01}, η_{02} satisfy

$$(\mathcal{L} - \nu)\begin{pmatrix} \eta_{01} \\ \eta_{02} \end{pmatrix} = 0.$$

Proof. Recall

$$A^\varepsilon = \begin{pmatrix} 0 & A_-^\varepsilon \\ -A_+^\varepsilon & 0 \end{pmatrix}.$$

To find the eigenvalue λ_ε such that

$$A^\varepsilon U_\varepsilon = \lambda_\varepsilon U_\varepsilon, \quad U_\varepsilon \in \mathbb{C}^4, \tag{5.2.5}$$

we first proceed with a formal expansion of eigenvalue and eigenfunction. A^ε can be written as a power series of ε,

$$A^\varepsilon = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{5.2.6}$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & -1 & -\partial_\rho \\ 0 & 0 & 0 & 0 \\ 1 & \partial_\rho & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 1+Q^2 & 0 \\ 0 & 0 & \partial_\rho + \frac{2}{\rho} & 1-Q^2 \\ -1-Q^2 & 2QQ' & 0 & 0 \\ -\partial_\rho - \frac{2}{\rho} & -1+3Q^2 & 0 & 0 \end{pmatrix}.$$

But A^ε is not a standard perturbation of $A_0 = A^\varepsilon|_{\varepsilon=0}$: the kernel of A_0 is infinite dimensional and the essential spectrum is dramatically different from that of A^ε. Hence A^ε is a singular perturbation of A_0. It turns out there is a more subtle relation with the linearized operator around the ground state of cubic, focusing and radial NLS

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

where

$$L_- = -\Delta + 1 - Q^2,\ L_+ = -\Delta + 1 - 3Q^2.$$

It is known that $\mathcal{L}$ is unstable (see, e.g. [6, 15]). From lemma 5.1.2, there exist $\nu = \sqrt{-\mu_1}$ with $\operatorname{Re}\nu > 0$ and $\eta_0 = \begin{pmatrix} \eta_{01} \\ \eta_{02} \end{pmatrix} \in \mathbb{C}^2$ such that

$$(\mathcal{L} - \nu)\eta_0 = 0. \tag{5.2.7}$$

Formally expanding $\lambda_\varepsilon, U_\varepsilon$

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \tag{5.2.8}$$

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots \tag{5.2.9}$$

Substituting into equation (5.2.5), and upon collecting the powers of ε, the following sequences of problems are obtained

$$(A_0 - \lambda_0)U_0 = 0, \tag{5.2.10}$$

$$(A_0 - \lambda_0)U_1 + (A_1 - \lambda_1)U_0 = 0, \tag{5.2.11}$$

$$(A_0 - \lambda_0)U_2 + (A_1 - \lambda_1)U_1 + (A_2 - \lambda_2)U_0 = 0. \tag{5.2.12}$$

We will determine λ_0, λ_1 from those sequences of equations. Let $U_0 = (U_{01}, U_{02}, U_{03}, U_{04})^T$, from equation (5.2.10), we have

$$\begin{aligned} \lambda_0 U_{01} + U_{03} + U_{04}' &= 0, \\ \lambda_0 U_{02} &= 0, \\ U_{01} + U_{02}' - \lambda_0 U_{03} &= 0, \\ \lambda_0 U_{04} &= 0. \end{aligned}$$

Then λ_0 can be taken as either 0 or $\pm i$. But we will ignore $\lambda_0 = \pm i$ and concentrate on $\lambda_0 = 0$. Then $U_{01} = -U_{02}', U_{03} = -U_{04}'$. We take

$$U_0 = (-\xi', \xi, -\eta', \eta)^T$$

for some functions $\xi, \eta \in H^1(\rho^2 d\rho)$ to be determined later. From equation (5.2.11), let $U_1 = (U_{11}, U_{12}, U_{13}, U_{14})^T$, we obtain

$$\begin{aligned} (1 + Q^2)\eta + \lambda_1\partial_\rho\xi &= U_{13} + \partial_\rho U_{14}, \\ -\Delta_\rho\eta + (1 - Q^2)\eta &= \lambda_1\xi, \\ (1 + Q^2)\partial_\rho\xi + 2QQ'\xi + \lambda_1\partial_\rho\eta &= -U_{11} - \partial_\rho U_{12}, \\ -\Delta_\rho\xi + (1 - 3Q^2)\xi &= -\lambda_1\eta. \end{aligned}$$

From the second and fourth components, we have

$$\begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mathcal{L} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda_1 \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

It allows us to choose from (5.2.7)

$$\lambda_1 = \nu, \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \eta_0.$$

For the eigenvalue problem (5.2.5), the leading order term of eigenvalue is $\varepsilon\nu$ with $\operatorname{Re}\nu > 0$, with leading order eigenfunction $(-\eta_{01}', \eta_{01}, -\eta_{02}', \eta_{02})^T$. We will not push the perturbation theory further since the eigenvalue is determined by the leading order. Next a rigorous proof shows indeed there exists a small eigenvalue

$$\lambda_\varepsilon = \varepsilon\nu + O(\varepsilon^2)$$

and the corresponding eigenfunction

$$U = (-\eta_{01}', \eta_{01}, -\eta_{02}', \eta_{02})^T + o(1).$$

The proof depends on the "Lyapunov-Schmidt reduction" method from bifurcation theory. Let $A^\varepsilon = A_0 + \varepsilon\hat{A}, \lambda_\varepsilon = \varepsilon\hat{\lambda}$, and $A^\varepsilon U = \lambda_\varepsilon U$. Then it is equivalent to solve

$$A_0 U = \varepsilon(\hat{\lambda} - \hat{A})U. \tag{5.2.13}$$

As we mentioned before, A_0 is a very degenerate operator. It has an infinite dimensional kernel and cokernel and produces only the first and third nonzero components. The idea to solve equation (5.2.13) is to project equation (5.2.13) into the range space and complementary space of N, where

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The projections split equation (5.2.13) into two equations:

$$NA_0 U = \varepsilon N(\hat{\lambda} - \hat{A})U, \tag{5.2.14}$$

and

$$0 = \bar{N}A_0 U = \varepsilon\bar{N}(\hat{\lambda} - \hat{A})U \tag{5.2.15}$$

where

$$\bar{N} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We will first solve equation (5.2.14). Let $U = (u_1, u_2, u_3, u_4)^T$, then (5.2.14) can be reduced to

$$J\begin{pmatrix} u_1 + u_2' \\ u_3 + u_4' \end{pmatrix} = \begin{pmatrix} -u_3 - u_4' \\ u_1 + u_2' \end{pmatrix} = \varepsilon N(\hat{\lambda} - \hat{A})U$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

J is invertible and $J^{-1} = -J$, then we get

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = -\begin{pmatrix} u_2' \\ u_4' \end{pmatrix} + \varepsilon JN(\hat{A} - \hat{\lambda})U. \tag{5.2.16}$$

Now we use this equation to write u_1, u_3 in terms of u_2, u_4, ε and $\hat{\lambda}$. Since

$$NU = \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} := \alpha, \quad \bar{N}U = \begin{pmatrix} u_2 \\ u_4 \end{pmatrix} := \beta$$

and

$$U = N^T NU + \bar{N}^T \bar{N}U,$$

then equation (5.2.16) can be written as

$$(I - \varepsilon JN(\hat{A} - \hat{\lambda})N^T)\alpha = (-\partial_\rho + \varepsilon JN(\hat{A} - \hat{\lambda})\bar{N}^T)\beta.$$

Now it is ready to solve α in terms of β and $\varepsilon, \hat{\lambda}$. The operator $JN(\hat{A} - \hat{\lambda})N^T$ is bounded because of the exponential decays of u_0, v_0. So for small enough ε,

$$\|\varepsilon JN(\hat{A} - \hat{\lambda})N^T\| < 1,$$

and the inverse of $(I - \varepsilon JN(\hat{A} - \hat{\lambda})N^T)$ exists,

$$(I - \varepsilon JN(\hat{A} - \hat{\lambda})N^T)^{-1} = \sum_{k=0}^{\infty} \varepsilon^k \left(JN(\hat{A} - \hat{\lambda})N^T\right)^k.$$

Let $B_{\varepsilon,\hat{\lambda}} = \sum_{k=0}^{\infty} \varepsilon^k \left(JN(\hat{A} - \hat{\lambda})N^T\right)^k (-\partial_\rho + \varepsilon JN(\hat{A} - \hat{\lambda})\bar{N}^T)$, we have

$$\alpha = B_{\varepsilon,\hat{\lambda}}\beta,$$

with $B_{0,\hat{\lambda}} = -\partial_\rho$. This is the solution for equation (5.2.14). Now plugging $\mathbf{U} = (N^T\alpha + \bar{N}^T\beta) = (\bar{N}^T + N^T B_{\varepsilon,\hat{\lambda}})\beta$ into equation (5.2.15), we have

$$\bar{N}(\hat{A} - \hat{\lambda})(\bar{N}^T + N^T B_{\varepsilon,\hat{\lambda}})\beta = 0. \tag{5.2.17}$$

Let us identify the leading order,

$$\hat{A} = A_1 + \varepsilon\tilde{A}, \quad B_{\varepsilon,\hat{\lambda}} = -\partial_\rho + \varepsilon\tilde{B},$$

Then equation (5.2.17) becomes

$$\begin{aligned}\bar{N}(A_1 - \hat{\lambda})(\bar{N}^T - \partial_\rho N^T)\beta &= \varepsilon\bar{N}\left((\hat{\lambda} - \hat{A})N^T\tilde{B} - \tilde{A}(\bar{N}^T + N^T B)\right)\beta \\ &:= \tilde{N}_{\hat{\lambda},\varepsilon}\beta.\end{aligned} \tag{5.2.18}$$

For this equation, after computation, the l.h.s. can be simplified as

$$l.h.s.(5.2.18) = (\mathcal{L} - \hat{\lambda})\beta.$$

Recall that

$$(\mathcal{L} - \nu)\eta_0 = 0,$$

hence we can take

$$\hat{\lambda} = \nu + \varepsilon\tilde{\lambda}, \quad \beta = \eta_0 + \varepsilon\beta_1.$$

$(\tilde{\lambda}, \beta_1)$ solve the equation

$$(\mathcal{L} - \nu)\beta_1 = (\tilde{N}_{\hat{\lambda},\varepsilon} + \tilde{\lambda})(\eta_0 + \varepsilon\beta_1). \tag{5.2.19}$$

To solve $(\tilde{\lambda}, \beta_1)$, we use a solvability condition. Since

$$ker(\mathcal{L} - \nu)^* = \begin{pmatrix} \eta_{02} \\ \eta_{01} \end{pmatrix} = \eta_0^*.$$

Equation (5.2.19) is solvable iff

$$\left(\eta_0^*, (\tilde{N}_{\hat{\lambda},\varepsilon} + \tilde{\lambda})(\eta_0 + \varepsilon)\right) = 0, \tag{5.2.20}$$

$(\cdot,\cdot)$ is a standard inner product in $L^2(\mathbb{C}^2)$. Denote by Q the L^2-orthogonal projection onto η_0^*, then

$$\beta_1 = (\mathcal{L} - \nu)^{-1}\bar{Q}(\tilde{N}_{\hat{\lambda},\varepsilon} + \tilde{\lambda})(\eta_0 + \varepsilon\beta_1)$$

where $\bar{Q} = I - Q$. Therefore

$$\beta_1\left(I - \varepsilon(\mathcal{L} - \nu)^{-1}\bar{Q}(\tilde{N}_{\hat{\lambda},\varepsilon} + \tilde{\lambda})\right) = (\mathcal{L} - \nu)^{-1}\bar{Q}(\tilde{N}_{\hat{\lambda},\varepsilon} + \tilde{\lambda})\eta_0.$$

It yields a function β_1 in terms of $\varepsilon, \tilde{\lambda}$, and$\beta_1 = \beta_1(\varepsilon, \tilde{\lambda})$, since $(\mathcal{L}-\nu)^{-1}\bar{Q}(\tilde{N}_{\hat{\lambda},\varepsilon}+\tilde{\lambda})$ is bounded, for sufficiently small ε,

$$||\varepsilon(\mathcal{L}-\nu)^{-1}\bar{Q}(\tilde{N}_{\hat{\lambda},\varepsilon}+\tilde{\lambda})|| < 1.$$

Next we use the solvability condition (5.2.20) to obtain $\tilde{\lambda}$ by implicit function theorem. Let

$$G(\varepsilon, \tilde{\lambda}) = \left(\eta_0^*, (\tilde{N}_{\hat{\lambda},\varepsilon}+\tilde{\lambda})(\eta_0+\varepsilon\beta_1(\varepsilon, \tilde{\lambda})\right).$$

We first show that there exists λ_2 such that $G(0, \lambda_2) = 0$. After a series of computations, we find

$$G(0, \tilde{\lambda}) = \tilde{\lambda}(\eta_0^*, \eta_0) + (\eta_0^*, \tilde{N}_{\hat{\lambda},0}\eta_0).$$

We claim that $(\eta_0^*, \eta_0) > 0$ since

$$(\eta_0^*, \eta_0) = 2\,\mathrm{Re}(\eta_{01}, \eta_{02}) = 2(\mathrm{Re}\,\frac{1}{\nu})(L_-\eta_{02}, \eta_{02}).$$

L_- is nonnegative and Q is the ground state [6], $L_-Q = 0$, since $\eta_{02} \neq CQ$, we have

$$(L_-\eta_{02}, \eta_{02}) > 0.$$

Therefore, if we choose

$$\lambda_2 = -(\eta_0^*, \eta_0)^{-1}(\eta_0^*, \tilde{N}_{\hat{\lambda},0}\eta_0).$$

then $G(0, \lambda_2) = 0$. Finally if $\frac{\partial G}{\partial \tilde{\lambda}}(0, \lambda_2) \neq 0$, we conclude by implicit function theorem, for ε small, there exists $\tilde{\lambda} = \tilde{\lambda}(\varepsilon)$ with $\tilde{\lambda}(0) = \lambda_2$. In fact,

$$\frac{\partial G}{\partial \tilde{\lambda}}(0, \lambda_2) = (\eta_0^*, \eta_0) \neq 0$$

Therefore the eigenvalue problem

$$(A^\varepsilon - \lambda^\varepsilon)U = 0$$

has an eigenvalue of the form

$$\lambda^\varepsilon = \varepsilon\nu + \varepsilon^2\lambda_2, \quad \lambda_2 = (\eta_0^*, \eta_0)^{-1}(\eta_0^*, \tilde{N}_{\hat{\lambda},0}\eta_0)$$

and eigenfunction

$$\begin{aligned} U &= (\bar{N}^T + N^T B_{\varepsilon,\hat{\lambda}})(\eta_0 + \varepsilon\beta_1) \\ &= (-\eta_{01}', \eta_{01}, -\eta_{02}', \eta_{02})^T + o(1). \end{aligned}$$

The formal asymptotics expansion is verified. □

The linearly instability is a direct result from Theorem 5.2.3 and Theorem 5.2.1.

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Chapter 6

Conclusions

6.1 Summary

This thesis focuses on the stability analysis of localized solutions Landau-Lifshitz flow and nonlinear Dirac equations. There are two major contributions to the field of study. For Landau-Lifshitz flow including harmonic map heat flow, most of the current research is focused on the compact 2-dimensional manifold, or the bounded domain, and for a special class of maps. My thesis studies a bigger class of maps in the whole plane. Thus the flow equations can not be described by a single equation for a scalar function, but a more complicated coupled system. In my thesis, a rigorous proof of global existence for equivariant near-minimal energy higher degree Landau-Lifshitz flow is given, and finite time blow-up for degree $m = 1$ in $\mathbb{R}^2$ is proved as well. For the nonlinear Dirac equations, my research is among the few to consider the uniqueness and continuity of the stationary solutions. A linearized Dirac operator around the ground states is explicitly given to analyze its spectrum. It turns out an unstable eigenvalue is related to that of the linearized operator of the Nonlinear Schrödinger equations around the ground states.

In Chapter 2, we studied Landau-Lifshitz flow with equivariant symmetry. The main achievement of this chapter is to use two different "coordinate systems" in the energy space of maps to study the flow equation. Compared with the single equation for a scalar function (2.1.4), these two different coordinate systems yield more complicated equations. One is using the "generalized hasimoto transform" (see [5, 8–10]), which will remove the harmonic map component, and produce an equation (2.2.4). In this equation the double end points space-time estimates for H with a critical-decay potential $1/r^2$ are obtained, due to the energy inequality and the positivity of potential. Usually the double end points estimates do not hold even for $-\Delta$ on radial functions. For Schrödinger maps [9], the weighted space-time estimates are used to replace the double endpoints estimates. The other decomposes the map into a nearby harmonic map plus a perturbation. This system (2.3.3) can be used to tracking the time-varying parameters $s(t), \alpha(t)$

of the nearest harmonic maps. But the restriction $m \geq 4$ must be placed to make an appropriate choice of $s(t), \alpha(t)$. Finally we relate the two coordinate systems to show that time-varying length-scale $s(t)$ has a limit away from 0. For $m = 2, 3$ we conjecture that the solution is still global. For $m = 1$, we restricted ourselves to study (2.1.4). As we learn from [3, 4], in the disc domain, the borderline for singularity formation is that the boundary condition $\phi(1, t) = \pi$. For the domain $\mathbb{R}^2$, the requirement $\phi(\infty, t) = \pi$ is a necessary condition for finite energy solutions. Hence a special choice of the initial data must be made to construct finite time blow-up solution. In Chapter 2, maximum principle and comparison theorem are developed as adaptations of disk in $\mathbb{R}^2$ to the whole plane.

In Chapter 3, a model equation for Schrödinger maps is studied. Since the subclass (3.1.1) is not preserved by Schrödinger maps, the construction of singular solutions becomes harder. The main goal to study equation (3.1.3) is to understand more about the Schrödinger maps when $m = 1$. But compared to equation (3.1.2), this hyperbolic equation is more complicated. We are looking for solutions with the form $u(r, t) = Q(r) + \eta(r, t)$ where Q is the stationary solution. The linearized operator acting on the real and imaginary parts of η is non-self-adjoint, and the continuous spectrum is the whole imaginary axis. When one studies matrix operators, one usually assumes that zero is neither an eigenvalue nor a resonance. But unfortunately, $\sin Q$ is the unique ground state if $m \geq 2$ and is a resonance if $m = 1$. The dispersive estimate for the linearized operator is difficult to obtain. We are only able to deal with local existence for finite energy, and global existence for small energy by the usual Strichartz estimates.

In Chapter 4, the uniqueness and continuity of the solitary wave solutions for a class of nonlinear Dirac equations are proved. Although there are many researches on the existence of solitons for Dirac equation, none of them have proved that the solution is a continuous curve. It is Ouanies [17] who first uses the perturbation method to relate the ground states to those of nonlinear Schrödinger equations. One can find the application of this approach in [16]. By a symmetry argument, the nonlinear Dirac equation is reduced to a simpler system. In order to show that this system is solvable and admit a unique solution, the linear operator is proved to be isomorphism, and the nonlinear terms are Lipschitz continuous. The properties of the ground states Q of the nonlinear Schrödinger equations play an very important role in the analysis. Moreover, we show that the solutions have exponential decay. So they are classical solutions. There is one discrepancy between our result and that in [17]. But it leaves an interesting problem for the future research. Can we construct a unique

solution for $0 < \theta < 1$?

In Chapter 5, we continue to study the nonlinear Dirac equations. Since we prove that the map from the parameter ω to the solitary waves is continuous in Chapter 4, it is natural to consider the stability. The current research either is devoted to a numerical study of stability, or uses energy minimization as a stability criterion. However, these criteria are not necessary rigorous. Mathematically, the question of stability is related to the eigenvalues of the linearized operator. The goal of this chapter is to give a rigorous proof of the instability. The linearized operator is a four-by-four matrix with a parameter ε. Most of the spectrum lies on the imaginary axis. We want to show that the linearized operator has an eigenvalue with positive real part. To do that, the first step is to formally expand eigenvalues and eigenfunctions as a power series of ε. It turns out that the eigenvalue is related to that of the linearized operator of nonlinear Schrödinger equations. The latter is known to be unstable. We then verify the result by the "Lyapunov-Schmidt reduction" method. Indeed our result is called linear instability. Although linear instability always implies nonlinear instability, it needs to be verified. To prove the nonlinear instability, according to [6], Theorem 6.1 in [6] must be satisfied. This will be discussed in details in our future research.

6.2 Future research

6.2.1 Blow up for 1-equivariant harmonic map heat flow

In Chapter 2, we proved that if the degree of the map $m = 1$, then the solution with initial energy close to the harmonic map energy may blow up in finite time. The approaches we used are maximum principle and the comparison principle. From [1], we know blow up is generic for $m = 1$ with the symmetry class The maximum principle is widely used to study the harmonic map heat flow both for global existence [7] and blow up [4].

The finite-time blow-up for the energy-space critical ($n = 2$) wave maps was studied in [2, 13, 15, 18] in the subclass (3.1.1). Blow-up for higher degree ($m \geq 4$) turns out to be generic [18]. When the degree $m = 1$, the nongeneric blow up behavior was given in [15]. This is consistent with the harmonic map heat flow [8], but the methodologies are totally different. Our proof of finite time blow up greatly relies on maximum principle. The possible alternative technique to study 1-equivariant harmonic map heat flow is provided in [15]. Suppose that $u(r,t) = Q(r/s(t)) + v(r,t)$, where u is the polar angle on the sphere, $Q(r) = 2\arctan r$ is the ground state harmonic

maps, $s(t) = t^\nu$, and $v(r,t)$ is the error with local energy going to zero as $t \to 0$. The scheme is to first find an approximate solution $Q(r/s(t))+u^e(r,t)$. $u^e(r,t)$ can be obtained by a finite sequence of approximation near the origin ($r = t^\nu$). But this process does not lead to an actual solution, since we keep losing time derivatives which leads to worse and worse implicit constants. So we turn to construct a parametrix for the heat equation by passing to coordinates (R,τ) where $R = \frac{r}{s(t)}, \tau = \frac{\tau^{-(2\nu-1)}}{2\nu-1}$. Then it gives us an equation with zero initial value. We want to solve the equation by the contraction mapping theorem.

6.2.2 Construction of unique ground states for nonlinear Dirac equation when $0 < \theta < 1$

In Chapter 4, we constructed unique ground states for Dirac equation with $1 \le \theta < 2$, but our method does not work for $0 < \theta < 1$ since an important technical lemma is used

$$\Big||a+b+c|^\theta - |a+b|^\theta - |a+c|^\theta + |a|^\theta\Big| \le C(|c|^{\theta-1} + |b|^{\theta-1})|b|,$$

for any $a, b, c \in \mathbb{R}$. This lemma is true only for $\theta \ge 1$. However we didn't exclude the possibility of existence and uniqueness of ground states at least for $0 < \theta < 1$. For any $\phi_1, \phi_2 \in \mathbb{C}^4$, we know

$$\Big||(\bar{\phi}_1\phi_1)|^\theta\phi_1 - |(\bar{\phi}_2\phi_2)|^\theta\phi_2\Big| \le C(|\phi_1|^{2\theta} + |\phi_2|^{2\theta})|\phi_1 - \phi_2|$$

So the nonlinear term of the Dirac equation is Lipschitz continuous for all $\theta > 0$.

To solve equation (4.3.4) for $0 < \theta < 1$, one can consider to study this equation in some weighted function space. Using the positivity of Q and the boundedness of Q'/Q, we define new functions such as $f_1 = \frac{e_1}{Q}, f_2 = -\frac{e_2}{Q'}$. Then a first order differential system for (f_1, f_2) is obtained. The idea to solve this system is the contraction mapping theorem. One advantage changing functions from (e_1, e_2) to (f_1, f_2) is that the K_1, K_2 can be simplified such as:

$$\begin{aligned} |v^2 - \varepsilon u^2| &= |(Q+e_1)^2 - \varepsilon(-Q'+e_2)^2| \\ &= Q^2\left|(1+f_1)^2 - \varepsilon(\frac{Q'}{Q})^2(1+f_2)^2\right|. \end{aligned}$$

For sufficiently small ε and $f_1, f_2 \in X$ (X is a function space to be determined), $(1+f_1)^2 - \varepsilon(\frac{Q'}{Q})^2(1+f_2)^2$ is pointwise positive. Hence we can avoid

using the technical lemma. Then we left to show that linear operator is an isomorphism. Through studying this problem, a long term goal is to prove the asymptotic stability of ground state solutions.

6.2.3 Asymptotic stability for the model equation for Schrödinger maps

For the model equation (3.1.3), we have obtained the linear operator as an matrix

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \tag{6.2.1}$$

$$L_+ = -\Delta + \frac{m^2}{r^2}\cos 2Q, \quad L_- = -\Delta + \frac{m^2}{r^2}\frac{\sin 2Q}{2Q}.$$

To prove the asymptotic stability of the stationary solution, the first thing is to prove the space-time estimates (Strichartz estimates) for $e^{t\mathcal{L}}$. This operator is non-self-adjoint and the continuous spectrum is the whole imaginary axis. When studying this kind of operator, one always assumes that zero is neither an eigenvalue nor a resonance. But this assumption does not apply to $\mathcal{L}$. We know that L_+ has $\sin Q$ as the unique ground state if $m \geq 2$ and as a resonance if $m = 1$. Also notice that L_- has Q as a resonance

$$L_- Q = -\Delta Q + \frac{m^2}{2r^2}\sin 2Q = 0.$$

but for $m \geq 1$

$$\|Q\|_{L^2} = \infty.$$

It may be a good starting point to study the resolvent $(\mathcal{L}-\lambda)^{-1}$ for $\lambda \notin \sigma(\mathcal{L})$. The dispersive estimates through resolvent analysis was also given in [14].

6.2.4 Nonlinear instability of the ground states

In Chapter 5, we prove that zero is linearly unstable, i.e. A^ϵ has an eigenvalue with positive real part. Although linearly instability always implies nonlinear instability, we still need to verify it. In [6], the theorem on unstable zero solution is given.

$$\frac{dx}{dt} = Ax + f(x), \quad x(t) \in X \tag{6.2.2}$$

where X is an Hilbert space. let $f : X \to X$ be a locally Lipschitz mapping such that $||f(x)|| \leq k||x||^2$ for $||x|| \leq l$, for some constants $k > 0$ and $l > 0$.

Let A be a linear operator which generates a strong continuous semigroup $exp(tA)$ on X. Assume that A has an eigenvalue λ with $\operatorname{Re}\lambda > 0$ and that

$$\|e^{tA}\| \leq be^{\mu t} \text{ for some } 0 < \mu < 2\operatorname{Re}\lambda \tag{6.2.3}$$

Then the zero solutions is (nonlinearly) unstable for (6.2.2). Thus we need to check equation (6.2.3) is satisfied for A^{ε}.

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