INVESTIGATING STUDENTS' UNDERSTANDINGS OF PROBABILITY: A STUDY OF A GRADE 7 CLASSROOM

by

VEDA ABU-BAKARE

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in

THE FACULTY OF GRADUATE STUDIES
(Mathematics Education)

THE UNIVERSITY OF BRITISH COLUMBIA
(Vancouver)

September, 2008

Abstract

This research study probes students' understandings of various aspects of probability in a 3-week Probability unit in a Grade 7 classroom. Informing this study are the perspectives of constructivism and sociocultural theory which underpin the contemporary reform in mathematics education as codified in the NCTM standards and orient much of the teaching and learning of mathematics in today's classrooms. Elements of culturally responsive pedagogy were also adopted within the research design.

The study was carried out in an urban school where I collaborated with the teacher and students as co-teacher and researcher. As the population of this school was predominantly Aboriginal, the lessons included discussion of the tradition and significance of Aboriginal games of chance and an activity based on one of these games. Data sources included the responses in the pre- and post-tests, fieldnotes of the lessons, and audiotapes of student interviews.

The key findings of the study are that the students had some understanding of formal probability theory with strongly-held persistent alternative thinking, some of which did not fit the informal conceptions of probability noted in the literature such as the outcome approach and the gambler's fallacy. This has led to the proposal of a Personal Probability model in which the determination of a probability or a probability decision is a weighting of components such as experience, intuition and judgment, some normative thinking, and personal choice, beliefs and attitudes. Though the alternative understandings were explored in interviews and resolved to some degree, the study finds that the probability understandings of students in this study are still fragile and inconsistent. Students demonstrated marked interest in tasks that combined mathematics, culture and community.

This study presents evidence that the current prescribed learning outcomes in the elementary grades are too ambitious and best left to the higher grades. The difficulties in the teaching and learning of the subject induced by the nuances and challenges of the subject as well as the dearth of time that is needed for an adequate treatment further direct that instructional resources at this level be focused on deepening and strengthening the basic ideas.

Table of Contents

Abstract ii
Table of Contents iii
List of Tables vi
List of Figures vii
Preface viii
Acknowledgements ix
Dedication x

CHAPTER 1. INTRODUCTION 1
1.1 BACKGROUND 1
1.2 WHY TEACH PROBABILITY? 2
1.3 APPROACHES TO PROBABILITY 4
1.4 CHALLENGES IN LEARNING PROBABILITY 6
1.5 PROBLEM STATEMENT 8
CHAPTER 2. THEORETICAL FRAMEWORK AND LITERATURE REVIEW 10
2.1 THEORETICAL FRAMEWORK OF THE STUDY 10
2.1.1 Constructivism and Sociocultural Theory 11
2.1.2 Culturally Responsive Pedagogy 14
2.2 LITERATURE REVIEW 17
2.2.1 Historical Overview of Probability 17
2.2.2 Children's Probabilistic Thinking 20
2.2.3 Alternative Understandings 27
2.3 RESEARCH QUESTIONS 30
CHAPTER 3. METHODOLOGY 32
3.1 METHODOLOGICAL FRAMEWORK 32
3.2 CONTEXT OF THE STUDY 34
3.3 PROCEDURES 39
3.3.1 Pre-test 39
3.3.2 Framing the lessons 41
3.3.3 Lesson Implementation 43
3.3.4 Interviews 44
3.3.5 Post-test 46

3.3.6 The Student Participants 46
3.4 ANALYSIS OF THE DATA 47
CHAPTER 4. DATA ANALYSIS, RESULTS AND DISCUSSION 48
4.1 BASELINE OF STUDENTS' UNDERSTANDINGS OF PROBABILITY 48
4.1.1 Probability as a Ratio 49
4.1.2 Sample Space 51
4.1.3 The Meaning of Random 51
4.1.4 Independent Trials 53
4.2 UNDERSTANDINGS FROM THE INTERVIEWS AND POST-TEST 54
4.2.1 Probability as a Ratio 55
4.2.2 Sample Space 61
4.2.3 The Meaning of Random 66
4.2.4 Independent Trials 73
4.3 FINDINGS FROM THE LESSONS 76
4.4 SUMMARY OF RESULTS 80
4.5 DISCUSSION 86
CHAPTER 5. CONCLUSION, IMPLICATIONS AND RECOMMENDATIONS 92
5.1 CONCLUSION 92
5.1.1 The Personal Probability Model 93
5.1.2 Further Key Findings 97
5.2 IMPLICATIONS AND RECOMMENDATIONS 99
5.2.1 Implications and Recommendations for Teaching 100
5.2.2 Implications and Recommendations for Curriculum 102
5.3 FACTORS IN CONSIDERING UNDERSTANDING OF PROBABILITY IDEAS 105
5.3.1 Language/Vocabulary 106
5.3.2 Difficulties relating to Tasks and Settings 107
5.3.3 Cognitive Conflict 108
5.4 RECOMMENDATIONS FOR RESEARCH 109
5.5 CLOSING THOUGHT 110
REFERENCES 111
Appendix A: Prescribed Learning Outcomes (PLOs) Probability Grades 5-7 119
Appendix B: Lesson Notes for Veda Abu-Bakare 120

Appendix C: Proposed Tasks 126
Appendix D: Possible Class Activities 128
Appendix E: Pre-test 132
Appendix F: Post-test 133
Appendix G: Sample Interview Transcript 134

List of Tables

Table 4.1: Summary of Pre-test Numerical Responses (n = 24) 80
Table 4.2: Summary of Post-test Numerical Responses (n = 24)...81
Table 4.3: Summary of Pre-test Qualitative Responses (n = 24)..82
Table 4.4: Summary of Pre-test Qualitative Responses (n = 24)..83

List of Figures

Fig. 3.1: Timeline for Study as at June 5, 2008 38
Fig. 3.2: PLOs for Grade 7 Probability 41
Fig. 4.1: Pre-test Spinner Task 49
Fig. 4.2: Post-test Spinner Task 59
Fig. 4.3: Observed and Expected Activity 77
Fig. 4.4: Spinners for the Spinner Sums Activity 77
Fig. 5.1: The Personal Probability Model 94

Look to this day!

For it is life, the very life of life

In its brief course lie the verities and realities of our existence

The bliss of growth,

the glory of action,

the splendour of achievement...

...from the Sanskrit

Acknowledgements

I give thanks to all who have guided and supported me in my journey to this achievement. I especially thank my supervisors, Dr. Cynthia Nicol and Dr. Samson Nashon, for their invaluable guidance and support throughout the research process. I thank Dr. Susan Gerofsky for her insightful comments that have improved the final thesis. I thank my friends and teachers at UBC. I thank my children, my family and all my friends for their love and support. Finally I give thanks to my parents, in particular to my father by whose example and teachings I am daily guided.

This thesis is dedicated with love to my parents,
the late Mr. and Mrs. Roodal Persad of St. Augustine, Trinidad.

CHAPTER 1. INTRODUCTION

1.1 BACKGROUND

At the June 2006 CMESG (Canadian Mathematics Education Study Group) meeting, I participated in the Working Group: **Developing Links between Statistical and Probabilistic Thinking in School Mathematics Education**. The larger question of how to foster the relationship between these two types of thinking was overshadowed by the more basic questions: What factors are necessary for students' development of probabilistic thinking? and, How can understanding of these factors shape the school mathematics curriculum?

The opinions expressed were thoughtful and various, reflecting the experiences of the participants and the nuances of the subject. Some of the participants argued that there was little point in teaching probability to anyone before Grade 5 given the difficulties that adults face in understanding the subject and the difficulties teachers face in knowing and teaching probability. Others disagreed basing their view on Bruner's dictum that any subject can be taught to anyone at any level so long as it is done honestly (Bruner, 1960). A third opinion was that probability provided a rich space in which to develop problem solving skills and other aspects of mathematical proficiency. One participant expressed frustration with those who act as if statistics and probability are special subjects that only a few can understand and so best kept out of school.

As a college instructor, I know very well how my students in introductory statistics courses struggle with the topic of probability. Probability is usually taught for about two to four weeks in the middle of a 13-week semester after the beginning topic of data analysis (univariate and bivariate). While students do well in the data analysis part describing data using graphical and summary analyses, it is the case that semester after semester, year after year, our students are derailed by probability.

At the College at which I teach, the pre-requisite for general introductory statistics courses is the BC Principles of Mathematics 11 or the BC Applications of Mathematics 12 while the pre-requisite for business introductory statistics course is Calculus. Despite the exposure of students to the Statistics and Probability strand starting from Grade 5 in elementary school, teachers in college introductory statistics courses have to spend much time on the basics as students invariably come with little understanding of probability and experience great difficulty with the topic. These difficulties have as much to do with the mathematics of the topic as with the nature of what it addresses; namely uncertainty and variation. There is also a level of abstraction and sophistication that is hard to embrace even for adults. Many results in probability appear counterintuitive and require great cognitive demand.

I left the Working Group pondering the various opinions on probability learning and teaching that were articulated and thinking of the ways that the questions raised could be addressed. I began by considering why it is important that probability be taught as part of the education necessary for taking one's place in society. The questions of whether it should be taught at school and how much at what level will be considered later in this thesis.

1.2 WHY TEACH PROBABILITY?

> Phenomena having uncertain individual outcomes but a regular pattern of outcome in many repetitions are called random. "Random" is not a synonym for "haphazard", but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics. Probability is the branch of mathematics that describes randomness (Moore, 1990, p. 98).

Probability is the language of uncertainty and randomness which is an undeniable and pervasive fact of life (as the popular saying goes, only two things in life are certain, death and taxes!). Probability is part of our everyday vocabulary in that we use words relating to probability easily and fluently, such as perhaps, probably, certainly, fairly certain, on a balance of probabilities, and not likely. There are many references to probability in the media and public discourse that we have internalized and now take for granted. The weather forecasts include a probability of precipitation (there is a 60% chance of rain tomorrow), the polls tell us that two-thirds of Canadians feel that they do not have enough information in order to make a decision about the Kyoto protocol with a margin error of 4 percentage points, and the business pages describe stock market movements and geo-political risk. Other areas relating to probability include lotteries, insurance, medical testing, disease transmission, and law.

It is now generally recognized that reasoning from uncertainty is an independent and fundamentally different intellectual method that must be taught alongside all the other methods of reasoning that have been developed in the various disciplines: the various types of analysis in the social sciences, the experimental method and hypothetico-deductive thought of science, and the power of abstraction, logic and deduction of mathematics (Falk & Konold, 1992; Moore, 1990; Nisbett, Lehman, Fong & Cheng,1987). Falk and Konold (1992, p. 151) argue that "Probability is a way of thinking. It should be learned for its own sake. In this century probability has become an integral component of virtually every area of thought. We expect that understanding probability will be as important in the 21st century as mastering elementary arithmetic in the present century".

Besides the importance and relevance of the subject, a further consideration for deciding whether probability is to be taught and at what level is the nature of the subject itself.

1.3 APPROACHES TO PROBABILITY

The word, probability, refers to a multi-faceted concept that is intended to capture and convey many ideas, all relating to situations of incomplete knowledge. The language of probability and its attendant vocabulary have been developed to describe the uncertainty and randomness that have been a feature of the lives of humans ever since the first humans contemplated the skies above, the earth below and the other companions with whom they shared the world. One way that the word, probability, is used is in describing the phenomenon of long-run behaviour. We say that the probability of getting heads when a fair coin is tossed is ½ because over many tosses the fraction of heads obtained approaches ½. Of course, the use of the word 'approaches' takes us to the notion of an infinite, limiting process, which includes three abstractions - infinity, limit, and an infinite process. Also the word, 'many', is vague – how many is many? Or how many is enough in order to see the limiting behaviour needed to assign the probability of ½?

Philosophers in the 17thcentury raised strong objections to this long-run interpretation of probability on exactly this point that the long-run requires complete knowledge (Hacking, 1975). This long-run notion of probability is referred to as the **frequentist** interpretation of probability and often as **statistical** probability.

A second use of the word probability occurs in situations where there are clear conditions of symmetry. We assume that the outcomes are equiprobable (the tautology here is inescapable but not insurmountable) and assign a probability as follows:

$$\Pr(Event) = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ outcomes}.$$

This notion of probability is termed **classical** and used in situations of physical symmetry such as a fair coin, a fair die, and a fair deck of cards.

A third use of the word, probability, is in situations that are not repeatable or in which there is no appeal to a consideration of physical symmetry but are one-off situations such as in: What is the probability of life on Mars? or What is the probability that a particular flight will land safely? In these cases, probability is used to indicate our degree of belief that the event will take place or to indicate the strength of our conviction as in the amount of money we are willing to bet that the event will take place. Here the assignment of a probability does not lie with the physical reality but with an individual assessment of the likelihood that the event will take place. This approach to the assignment of a probability as a function of an individual's belief, knowledge and the amount he or she is willing to bet that the event will take place is termed **subjective**. While it may appear that this method of assignment of the probability of an event individually will lead to intractable difficulty, probabilists in this area have developed a coherent calculus of subjective probabilities, Bayesian probability and statistics, with many fruitful applications.

These three approaches to probability, **classical, frequentist** and **subjective** all lead to a quantification of the likelihood that a future event will take place. Hundreds of years after the flourish of activity in the area of gambling in the 17th century in Europe and the resulting work by noted mathematicians such as Pascal, Fermat, Huygens and the Bernoullis, Kolmogorov, in the 1930's, developed an **axiomatic** theory of probability where corresponding to an event, E, a number is assigned that is the probability of the event. Kolmogorov postulated certain axioms that these numbers must satisfy and from these he developed the modern-day theory of probability as part of measure theory.

It is to be noted that there is no difficulty once the probability of an event is assigned. How that assignment is done and what meaning and interpretation are to be made of the word, probability, are not as generally agreed upon. At the level at which we teach probability, these philosophical considerations do not impinge directly, but they shed light on impediments to the understanding of probability.

This study focuses on students' understandings of the classical and relative frequency notions of probability.

1.4 CHALLENGES IN LEARNING PROBABILITY

The understanding of probability as a school subject presents its own challenges. Garfield and Ahlgren (1988) elucidate the following difficulties in learning probability:

Limited experience with uncertainty: While life is conducted regularly amid situations relating to chance and uncertainty of future events, we all have difficulty assessing risk especially when the situations get more complex. Even a statement as simple as 'there is a 60% chance of rain tomorrow' that is regularly used in the media and which is generally accepted requires some degree of sophistication in order to articulate its meaning. Gigerenzer (1989) notes the doctor whose patients have difficulty understanding his meaning when he says there is a 30% chance of increased blood pressure with a particular medication but find it easier when he says that if he were to prescribe the drug for 10 people, about 3 would experience increased blood pressure. While there is still a need to articulate what is meant by the 'about', using frequencies rather than probabilities often helps the understanding.

Belief in determinism and a tendency to look for causes: Adults and children have a strong belief in determinism and hence have a tendency to look for and ascribe causes. We are often reluctant or unwilling to recognize and acknowledge randomness and uncertainty and often believe that we can manipulate a cause to get a particular effect.

Belief that we can manipulate cause to get a particular effect: Young children, in particular, are reluctant to accept randomness and often think that by manipulation of the situation, a particular effect can be made to happen. They have a belief that an effect can be achieved by personal will or by a mechanism, either personal or external.

We seek coherence in cognitive organization: We often have difficulty in making sense of situations that do not turn out as we expect. Kahneman & Tversky (1982) write of the conflict of 'passive expectations and conscious anticipations' and the surprise we feel when an expectation is violated.

Lack of representations of cognitive schema: Probability is an abstract concept that is not easily measured, like distance, weight and time, or easily represented, like number, shape and pattern. There are no physical representations of the concept. When we look at a die, there is nothing about the die before it is tossed that indicates that the probability of a particular face is 1/6. There is a tacit assumption of symmetry and equal weight of all sides and an agreement about meaning when we say the probability that a particular side will come up is 1/6. Even after the die is tossed, there is no way we can measure the probability of what just happened.

Further to these are the considerations of:

Complexity: Once we move past simple events, the calculations of probabilities get complex fairly quickly. As an example, tossing 2 dice gives 36 outcomes and tossing 3 dice gives 216 outcomes. The numbers get very large very quickly so that even when n is 4 or 5, the sample space is too large to be enumerated. Thus, much of the calculation in more complex situations has to be done abstractly through thinking and reasoning.

The nature of the subject matter: The subject of probability deals with uncertainty and variation which may lead to epistemological anxiety (Wilensky, 1994) and ambiguity intolerance. Epistemological anxiety, according to Wilensky, is "a feeling, often in the background, that one does not comprehend the meanings, purposes, sources or legitimacy of the mathematical objects one is manipulation and using (p. 172)". Ambiguity intolerance refers to the tendency that people have in feeling discomfort or perceiving a threat in situations of incomplete, fragmented, and insufficient knowledge.

1.5 PROBLEM STATEMENT

Returning to the questions raised in the Working Group relating to probability in the elementary grades, the various considerations above indicate that the treatment of probability in the school mathematics curriculum has to handled with caution and care and can only be successfully implemented with due research. The research literature in children's probabilistic thinking is scattered, contradictory and specific to grade, age and region. To my knowledge, few studies have been undertaken with the BC K-7 curriculum. One such study is a case study of the understandings of probability in a Grade 4/5 BC classroom (Nicolson, 2004) which found 'common misconceptions, fragile student knowledge, student-caused bias in trials, time constraints, and gaps in teacher knowledge' (p. 108) as significant impediments to the understanding of probability. The resulting dearth of data and analysis of the factors relating to the experience of both students and teachers in the elementary grades in the topic of probability calls for further research.

This thesis considers the understandings of probability of Grade 7 students before and during instruction of a unit on probability, and the instructional models to see if they enhance a deeper understanding of probability. In Chapter 2 I give the theoretical framework and literature review that underpin the study. In Chapter 3 I present the methodology employed in the study and in Chapter 4 the results of the data analyses with discussion. I conclude this thesis in Chapter 5 with a summary of the findings of the study, and an examination of the implications and recommendations for research, teaching, and curriculum.

CHAPTER 2. THEORETICAL FRAMEWORK AND LITERATURE REVIEW

In this chapter I present the underlying theoretical framework of the study, review the research literature pertaining to the problem, and state the research questions. For the theoretical framework, a discussion is provided of the two theoretical lenses that inform the study. The first is the complementary perspectives of constructivism and sociocultural theory and the second is culturally responsive pedagogy. The review of the literature is given in three parts: 1) a historical overview of the development of probability including its cultural aspects, 2) an overview of children's probabilistic thinking and 3) a consideration of alternative understandings of probability. The chapter is concluded with a rationale and statement of the research questions.

2.1 THEORETICAL FRAMEWORK OF THE STUDY

The reform movement in mathematics education in recent decades has led to a move away from direct and transmission teaching to approaches based on constructivist, discovery, inquiry-based, experiential, and problem-based philosophies of teaching and learning. There is also recognition of the influence of the social and cultural interaction of the learner and various aspects of her environment, including the other learners, the teacher, the setting or context and the materials and tasks of engagement. This provides for the first focus of the theoretical framework of the study, the complementary lenses of constructivism and sociocultural theory. A further recognition of the need to address the diversity of the classroom and the culture that learners bring with them leads to the second theoretical focus of culturally responsive pedagogy.

In the next two sections I consider each of these two foci.

2.1.1 Constructivism and Sociocultural Theory

While there is some debate in the literature regarding these two positions on how students learn, I take the view of Cobb (1994) that these are not competing but complementary in that they argue for different things – constructivism for how and what students learn and sociocultural theory for informing the conditions that enable students' learning. In asking: "Where is the mind?", Cobb questions whether cognition resides in the individual or in the interaction of the individual and the learning environment. My response is that in effective mathematics teaching and learning we have to attend to the many facets of the individual, the environment we provide and the mutual interaction of the components of the whole learning and teaching endeavour.

Constructivism takes as its starting point the position that learners bring with them a set of representations, skills and tools with which they actively construct their own understanding of the mathematics as presented to them (Davis, Maher & Noddings, 1990) and that they construct their individual knowledge actively by reacting to their environment and reorganizing their ideas and experiences as their environments change and develop (von Glasersfeld, 1990). According to von Glasersfeld, constructivism is "... a theory of knowing rather than a 'theory of knowledge' " (p. 24).

The two principles of radical constructivism as stated by von Glasersfeld (1990, p. 22-23) are:

1. Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognizing subject.
2. a. The function of cognition is adaptive in the biological sense of the term, tending towards fit or viability;
   b. Cognition serves the subject's organization of the experiential world, not the

discovery of an objective ontological reality.

I have difficulty with these principles and especially with the second, as I believe that there are mathematical truths and unmistakable mathematical objective ontological realities such as $2 + 2 = 4$ and the definition of a limit that teachers want learners to grasp, so it comes as a measure of relief when von Glasersfeld goes on to say (p. 23): "One cannot adopt the constructivist principles as an absolute truth, but only as a working hypothesis that may or may not turn out to be viable". Ernest (1998) points out that while the first principle is accepted by many workers in mathematics (my term), the second is difficult to embrace. Still, something can be salvaged because as Goldin (1990) notes, it is possible to engage in constructivist practices and not be a constructivist, a view which I endorse.

Constructivist practices place emphasis on the recognition of the learner's existing skills and abilities and the provision of opportunities for the learner to develop and construct his or her own knowledge by active methods of doing, questioning, wondering and discovering. With regard to the preconceptions that learners bring, learners do not, in general, come into classrooms with *tabula rasa* minds. They have thoughts and ideas about concepts that they have gathered from observation, reflection, and interaction with the world around them. This prior knowledge, cast as alternative conceptions or frameworks, misconceptions or theories-in-action (Driver & Erickson, 1983), is brought to bear on the processes of knowledge construction and meaning making and have to be addressed and built upon in the teaching.

Constructivist teaching practices then make room and allow for learners to put together, assemble, disassemble, reassemble, and grow their knowledge actively. Central to this is the premise of an "...emphasis on mathematical activity in a mathematical community" (Davis, Maher & Noddings, 1990, p. 3). The mathematical activity is intended to promote mathematical investigation and thinking and to build in learners a way of seeing and doing mathematics that transcends the particular topic being studied. Emphasizing the mathematical community positions the learner as part of the community of teacher and other learners thereby giving prominence to the learning that is gained from the teacher-learner and learner-learner interactions. Finally emphasizing the 'mathematical' highlights the importance of the language, tools, and practices of mathematical ways of thinking that help learners make meaning of the subject and the experience.

The sociocultural perspective underscores the social position of the learner in a community and the subsequent enculturation of the learner in that social setting (Cobb, 1994; Confrey, 1990). Sociocultural theorists base their work on Vgotsky's assertion that social learning plays a central role in the development of cognition. Vgotsky stresses culture, social factors and the role of language in affecting and shaping cognitive development, giving a secondary role to individual cognitive processes. He formulated two aspects of learning - the more knowledgeable other (person or system) who can then help the learner by superior understanding and ability and the learner's zone of proximal development that can be spanned by guidance and encouragement from a more knowledgeable other.

Sociocultural theory thus gives prominence to the social context as a means to promote mathematical development. By adapting to the social context, learners develop a system of shared and taken-for-granted meanings that are necessary for interaction and communication and that enable them to participate in the culture of the community (Wood, 2001). Further, the social environment or culture of the classroom contributes to the learners' beliefs about the nature of mathematical knowledge and the ways that it is acquired, thereby influencing the mathematics that is learned.

The emergent or social constructivist view gives weight to both the cognition and construction of the individual learner and the participation of the individual learner in a social setting and maintains a reflexive and symbiotic relationship between the two aspects (Cobb & Yackel, 1996). This coordination of perspectives of the one constituting a background for the other is not without tension as teachers carry out the balancing act of "…attending both to their students' interests and understandings, and to their mathematical heritage" (Cobb, 1994, p. 18).

2.1.2 Culturally Responsive Pedagogy

The second focus of the theoretical framework, culturally responsive pedagogy, may be considered under the umbrella of sociocultural theory but it is emphasized here because its underlying premise is that culture and cultural experiences have a direct impact on learners' cognitive development and self-efficacy. The challenges of the diversity of cultures of learners in today's classrooms and the persistent underachievement of minority students have evoked a need for pedagogy (including standards and assessment) to be culturally relevant, responsive and appropriate. Bartolome (2002) argues for a humanizing pedagogy that places at the centre the culture, language, history, and values of the learners. In this way she counters the view that improving the academic achievement of minority students is a technical issue solvable by technical solutions of the magical right "methods" and cautions against a "one size fits all" approach.

Variants of humanizing pedagogies include culturally responsive pedagogy, culturally relevant education, strategic teaching, and place-based education. Culturally responsive pedagogy is adopted in this study because it best captures the "wholistic (original spelling) and inclusive" view of engaging the hearts, minds, and bodies of students within their community and culture (Nicol, Archibald, Kelleher & Brown, 2006).

An important element of culturally responsive pedagogy is culturally responsive teaching. Gay (2002) notes the following components of culturally responsive teaching: critical cultural consciousness of teachers, culturally pluralistic classroom climates, diverse communities of learners, and multicultural curriculum and instruction. In my view, the critical cultural consciousness of teachers is vital to students' success as teachers' attitudes and beliefs about their students and their abilities greatly influence their teaching and assessment often resulting in a 'deficit view' of minority students. Villegas and Lucas (2002) characterize culturally responsive teachers as those who "... have affirming views of students from diverse backgrounds" and who " ... have a sense that they are both responsible for and capable of bringing about educational change that will make schooling more responsive to students from diverse backgrounds" (p. xiv).

A second important element of culturally responsive pedagogy is the importance of honouring the cultures and traditions of students by developing and maintaining connections to their communities. At the very least, this requires a valuing of the knowledge and wisdom that the community has amassed over its history and accomplishments. It further requires a valuing of the people who hold this wisdom, often designated as Elders, and of the significant artifacts and totems of the community. Finally it requires the observing of the proper protocols in attempts to learn about the community, its wisdom and its stories (Archibald, 2008).

While there is much work being done in various countries in improving the learning of mathematics by placing the subject in a cultural context with the aim of making it more relevant and meaningful, I drew inspiration in this area from studies in North America that have focused on American Indian, Inuit and Aboriginal groups in Alaska, the mainland US, and BC (Lipka et al., 2005, Nelson-Barber & Estrin, 1995; Nicol, Archibald, Kelleher & Brown, 2006; Tambe, Carroll, Mitchell, Lopez, Horsch, & St. John, 2007; White, 2003). These studies aim to connect local knowledge and school knowledge in ways that are grounded in the local culture and language.

The classroom in which this research study was carried out had a majority of students of First Nations descent. As I participated in this study as both co-teacher and researcher, I have been guided by the 4 R's of Respect, Relevance, Reciprocity and Responsibility as presented by Kirkness & Barnhardt (1991) and later by Archibald (2008) as Indigenous educators working to honour the traditions and cultures of Aboriginal peoples.

As I planned and conducted the study, these theoretical perspectives framed my method of instruction and choice of activities as well as my conduct as a researcher. I was keenly aware of the need to find out what prior knowledge and beliefs about probability that the students hold, the need to value them as individual beings with their own experiences and cultural backgrounds, the need to engage them individually and as a class by presenting interesting activities, and the need to create a collaboration of co-teachers and co-participants.

2.2 LITERATURE REVIEW

2.2.1 Historical Overview of Probability

Standard textbook accounts of probability cast the beginning of the theory of probability from the mid-seventeenth century in Europe in the exchange between the mathematicians, Pascal and Fermat, relating to a problem posed by a gambler, Chevalier de Méré, on the division of stakes in an interrupted game of chance. However, it is evident from historical texts that probability and related ideas of luck, fate, hazard, and destiny have been a part of the lives of humans from earliest times.

Intuition and activity relating to chance, randomness and variation have been with us since earliest civilizations. In the paragraphs below, the historical beginnings of probability have been drawn from Bennett (1983), David (1993), and Hacking (1975, 1990).

The first indication is evidence of astragali (ankle bones of hoofed animals such as deer, calf, sheep or goat) and tali (heel bones) 40,000 years ago in pre-historic sites in Mesopotamia, Egypt, the Indus Valley, Greece, and the Roman Empire. Gaming boards were found in ancient Babylonia around 2700 BC, at the palace of Knossos in Crete around 2400-2100 BC, and at later Babylonian and Assyrian sites. These boards were similar to backgammon boards found in Egyptian tombs and to boards in games such as Hounds and Jackals and Snakes and Ladders with pieces or 'men' made of ivory but no accompanying dice.

The earliest die excavated was dated around 2700 BC in ancient Mesopotamia in Northern Iraq. Dice in classical times were made of crystal, ivory, sandstone, ironstone, wood, marble, clay and other materials. Another example of a situation in which chance played a part was the throwing stick made of wood or ivory by the ancient Britons, the Greeks, the Romans, Egyptians and Maya Indians of North America.

There is other widespread evidence of chance games in North America predating Columbus. The American Indians played games with 2-faced dice made of bone, wood, beaver or wood-chuck teeth, walnut shells, hickory staves, crows' claws and stones. The Inuit used 6-sided ivory and wooden dice and the Papago Indians used bison astragali as dice.

Besides leisure games, a second important use of chance in antiquity was in divination or seeking divine direction. The use of an objective chance mechanism eliminated human tendencies to judgment so that the will of the gods could be clearly manifested. Pebbles, arrows, astragali and dice were thrown in temples to probe divine will. Besides divination by priests, people regularly threw astragali, tali and dice to make decisions in their daily lives about love, career and business dealings. However, even though they used chance mechanisms and randomizers, the ancients believed that the gods controlled the outcome.

Chance was further used to ensure fairness and to avoid dissension much as it is used today in lotteries and bingo. It was used in times of war to improve morale and fight boredom and in times of famine as a distraction from hunger.

If we now fast-forward to the standard textbook beginning, we have on the one hand gamblers and mathematicians working out probabilities when dice are thrown and on the other, philosophers grappling with the concepts of chance and necessity. Is the outcome unknowable or is it that we are ignorant of the hidden cause? Are all events determined? Is what we see as unpredictability merely our ignorance?

During the Middle Ages in Europe, the Church held sway – all events flowed from the Creator. The English philosopher Thomas Hobbes (1588-1679) believed that all events were predetermined by God or they happened by causes outside of them that were determined by God. No element of chance was permitted. Hobbes believed that it was ignorance of causes that prevented people from seeing the necessity of events and resulted in their attribution to chance. However determinism had to give way to the theory and distribution of errors, random and unexpected in measurement and observation, which was developed by scientists such as Galileo, Simpson, Bernoulli, Laplace and Gauss. Gauss is now credited with the discovery of the normal distribution.

As the development of probability gained momentum from gambling games and the involvement of mathematicians in the 17^{th} and 18^{th} centuries, ideas of statistics were being developed by Galton and his studies of the inheritance of traits from generation to generation, by Fisher and his application of sampling and the design of experiments to agriculture, and in actuarial work relating to censuses, mortality rates, insurance and annuities. While there was still the notion of physical laws, order, design, and determinism in the universe, by the beginning of the 20^{th} century the three main notions of probability that we use today, the classical probability of equally likely outcomes, the long-run frequency notion and the subjective notion of probability, had been formulated. In addressing why it took so long to formulate a theory of probability, Ian Hacking (1975) argues that the nature of probability is essentially dual – on the one hand, aleatory, concerned with chances and contingencies and on the other, epistemological, concerned with assessing varying degrees of belief of propositions, and that a long incubation was required for addressing that duality.

There are two main themes in the research literature in probability understandings, the development of probability ideas and the judgment and biases that people hold in reasoning under uncertainty. Many studies (including this one) address both themes making it hard to tease apart the two strands. I will review the theme of the development of probability ideas in children in 2.2.2 Children's Probabilistic Thinking and the theme of judgment and bias in the second in 2.2.3 Alternative Understandings.

2.2.2 Children's Probabilistic Thinking

Three phases have been marked out in the area of research relating to children's ideas of probability (Jones, 2005): the Piagetian period, the post-Piagetian period, and the Contemporary period.

The seminal work in the area of how the concept of chance develops in children was carried out by Piaget and Inhelder in the 1950s and published in English in 1975 as *The Origin of the Idea of Chance in Children* as part of a number of studies on the cognitive development in children of concepts such as number, space and proportional reasoning. Piaget and Inhelder engaged children in tasks involving random mixtures, distributions (centered and uniform), random drawings, and combinations and permutations. From these results they delineated three stages in the development of the idea of chance:

1. Preoperational (Four to seven or eight years)

Here the child is unable to differentiate the **possible** from the **necessary** (that which has a cause). There is no reference to operations and no idea of chance or deduction but only an intuition of real or imaginary regularity. When faced with a random mixture of objects the child is intuitively sure that the objects will return to a regular ordering.

2. Concrete Operational (Seven or eight years to eleven or twelve years)

At this stage the operations of logic and arithmetic begin to appear and there is some understanding of the difference between necessary and possible events but the child has no systematic approach to generating a list of possibilities due to a lack of combinatorial skills or mathematical maturity.

3. Formal Operational (Eleven or twelve years and older)

It is only in this stage that a judgment of probability becomes organized and the evolution of the idea of chance is achieved. The child begins to understand the ideas of experimental and theoretical probability and to have some facility with listing all possible outcomes and other combinatorial analyses.

Piaget and Inhelder concluded that chance is gradually discovered and it is only by constantly making reference to the operations of logic and arithmetic in a parallel development with these operations that children eventually come to an understanding of chance and probability,

These findings were challenged by Fischbein and other researchers in the post-Piagetian period. Fischbein (1975) found evidence of probabilistic thinking after instruction as early as the third grade while others criticized the narrow tasks and settings of the Piagetian experiments and demonstrated that elements of probabilistic thinking were evident in children even as young as preschoolers (Davies, 1965; Goldberg, 1966; Yost et al., 1962).

In a more recent study of children in grades 3 and 4, Jones et al. (1997) developed four levels of probabilistic reasoning (subjective, transitional, informal quantitative, and numerical) for each of four probability ideas of sample space, experimental probability, theoretical probability (including probability comparisons), and independence. Jones and colleagues do not suggest that children demonstrate an ordered progression through the levels but that the framework presents broad guidelines to inform instruction in probability.

Because of the variation in emphasis (probability ideas, misconceptions, combinations of probability ideas and misconceptions, age groups, tasks, and environments) of the various studies of children's probabilistic thinking in the post-Piagetian and Contemporary periods, for the purposes of this study, the research is best considered with respect to the core ideas of probability: Uncertainty and randomness, Probability as a ratio, Sample space, and Theoretical and Experimental Probability. Two further considerations are added: Language of probability and Cultural aspects.

2.2.2.1 Uncertainty and Randomness

The age at which children understand the idea of uncertainty varies by study but is generally agreed to be between 4 and 10 years with the inconsistencies in the results of the studies due to the differences in the tasks used (Acredolo, O'Connor, Banks & Horobin, 1989; Byrnes & Beilin, 1991; Falk & Wilkening, 1988; Horvath & Lehrer, 1998; Kuzmak & Gelman, 1986). It has been found that children simultaneously hold ideas of chance and a deterministic outlook with the deterministic stance often persisting with age (Jones, Langrall, Thornton & Mogill, 1997; Konold, 1991; Metz, 1998a, 1998b; Shaughnessy, 1992). Other studies show that children in the elementary grades have superstitious beliefs, think they can will or influence chance outcomes, and often ascribe outcomes to external, uncontrollable forces (Falk, 1983; Fischbein, Nello & Marino 1991; Green, 1983; Piaget & Inhelder, 1951/1975).

In theorizing children's understanding of probability, Fischbein (1975, 1987) proposed the concept of **intuitions** which are cognitive beliefs or acquisitions that come spontaneously from and are self-evident to the believer. He asserted that all people have intuitions with regard to concepts such as number, pattern and probability which are immediate, holistic, adaptable, and obvious to the believer. A **primary intuition** is one that a person has from his or her own experience without the benefit of instructional intervention. A **secondary intuition** is a restructured cognitive belief that a person has acquired after instruction and experience in a particular cultural community. Fischbein also noted that replacing a primary intuition by a secondary one is not a gradual process, but one that takes place as a whole, or all at once in a flash of discovery or insight. Fischbein, Pampu & Minzat (1967) found that pre-school children performed well in probability tasks but older children were less successful. They argued that the pre-schoolers' thinking was intuitive and untutored and that the teaching process in science and mathematics 'orients the child toward a deterministic interpretation of phenomena, in the sense of looking for and explaining in terms of clear-cut, certain, and univocal relations'(p. 169). Greer (2001) terms this a 'cultural bias of deterministic thinking' (p. 20).

2.2.2.2 Probability as a Ratio

The research on probability as ratio is contradictory. Falk, Falk & Levin (1980) and Acredolo et al. (1989) claim that children as young as age six have some understanding of probability as ratio of the number of favourable cases to the total number of cases using problems that involved changes in the numbers of favourable and total cases. Green (1983, 1988) found that 11-16 year-olds in the UK had fragile understanding of probability. Shaughnessy (1992) writes that Green 'paints a bleak picture of the stochastics situation in England' (p. 479). Green based his results on a paper-and-pencil multiple-choice survey instrument while Falk et al. used binary choice tasks with two colours, one of them the payoff colour. The child had to choose which of the two colours to draw (urns) or land on (spinners and roulette wheels) for a reward. Acredolo et al. used a sliding scale that children used in order to indicate the likelihood of an event. Because of the variety of tasks in these studies (paper-and-pencil survey instruments and tasks with manipulatives offering binary choice and some discrimination) it is difficult to gauge the level of children's understanding of the concept of probability as a ratio. The question of how these tasks may be amended will be taken up later in the thesis.

Besides spontaneous responses there have been studies on the effect of instruction on probability ideas. Fischbein, Pampu & Manzat (1970) in a study of three groups of children aged 5, 9 and 12 years found that after a short program of instruction the 9 and 12 year olds were able to correctly use comparison of quantitative ratios in a binary-choice task of choosing which bag offered the best chance of getting a particular colour.

2.2.2.3 Sample Space

The concept of sample space is key in understanding probability because correctly working out the probability of an event requires recognizing what an outcome of the experiment is and knowing how many there are including distinguishing between outcomes. An example that eluded even Laplace is the sample space of the experiment of 2 coins being tossed, {HH, HT, TH, TT}. A misguided notion of this sample space is that there are 3 outcomes, two heads, two tails and one of each. Also a layer of complexity is added in working out the sample space as the number of outcomes multiplies quickly as we move from one-stage to two-stage and then three-stage experiments.

The research in this area is also contradictory. Piaget and Inhelder (1951/1975) claim that by age 7 children can list the outcomes in a simple one-stage experiment but later researchers found that older children could not perform this task (Jones, 1974; Borovcnik & Bentz, 1991). Piaget and Inhelder claimed that combinatorics skills in 2-stage experiments were not acquired until around 11 years but English (1993) found that 7- and 8-year-olds had combinatorics skills in concrete problems. Polaki (2002), in a study of Basotho children ages 9-10 in Lesotho, assessed two versions of an instructional program, one using small experimental data sets and the other computer-generated large experimental data sets and found improved understanding of the notion of sample space in an overall assessment of probability ideas before and after instruction.

2.2.2.4 Theoretical and Experimental Probability

While these terms do not sit well with me from my view of effective teaching and learning of probability, they are the ones that are used in the literature, in textbooks and teaching materials, and in learning outcomes in the curriculum. I address their use and my unease with the terms in the Discussion in Chapter 4.

Metz (1998a, 1998b) in a study involving three groups, undergraduates, children in grade 3 and children in kindergarten, found that both children and adults have difficulties in probability determination which are related to knowing and assessing relative magnitude, the ability to compare numbers, and an understanding of part-part and part-whole relations. Metz (1998a) also comments that '... the unschooled primary-grade children's conceptions fail to reflect the mathematician's framing of both randomness and probability over large numbers of events. ... Thus instruction will presumably need to address the subtle idea of phenomena whose individual outcomes may be undetermined but that reveal patterns of outcomes over the long haul' (p. 169).

2.2.2.5 Language of Probability

Fischbein, Nello, & Marino (1991), in a study of 618 pupils in elementary and junior high school in Italy with the aim of obtaining a better understanding of some probabilistic intuitive obstacles, concluded that the role of language was pivotal and that "the concept of certitude is much more complex than expected while that of the possible develops earlier! " (p. 527, their emphasis). They further noted that the 9-14-year-olds in their study "do not have in mind a clear definition of the terms 'possible', 'impossible' and 'certain' ... and confuse 'rare' with 'impossible'." (p. 547).

2.2.2.6 Cultural aspects

There are a few studies on probability understanding carried out in countries other than the ones referred to above. These include Cyprus, Greece, Russia (Langrall & Mooney, 2005), and China (Li & Pereira-Mendoza, 2002) and are noted here for evidence of work done outside North America and Europe. These studies have mostly replicated the North American studies of frameworks for probability reasoning similar to Jones et al. (1997) in the various countries.

Amir and Williams (1999) investigated the cultural influences (beliefs, language, experience) on probabilistic thinking. They studied two groups of 11-12-year-old pupils in two inner-city schools in Manchester, UK, one whose parents were European (English) and the other whose parents were Asian or African (mainly Asian). The Asian group was considered to be more religious and to have stronger ties to family and live in a more cohesive community. Amir and Williams found that many children held superstitious beliefs that affected their understanding of probability (such as some numbers were believed luckier than others), that the Asian (mostly Muslim) group had more attributions to God than the English (mostly Christian) group and that while numerical reasoning was similar in both groups, the significant difference in verbal reasoning impacted success in probability tasks. This underscores the importance of language in designing and implementing tasks.

2.2.3 Alternative Understandings

A parallel development in the research on probabilistic thinking deals with the difficulties that people have and inconsistencies that arise when reasoning about probability. Reviews of this research are given in Konold (1989), Konold et al. (1993), Shaughnessy (1992, 2003) and Jones, Langrall & Mooney (2007). The heuristics and biases used include **representativeness** (Kahneman and Tversky, 1972), **availability** (Kahneman & Tversky, 1973), the **conjunction fallacy** (Kahneman & Tversky, 1982), the **outcome approach** (Konold, 1991), the **most likely-least likely switch** (Konold et al., 1993), **equiprobability bias** (LeCoutre, 1992), and the **gambler's fallacy**.

Representativeness occurs when the outcome of HTHTTHT is considered more likely in 6 tosses of a fair coin than HHHHHH, say. A determination of the likelihood of an outcome is influenced by how representative the outcome appears to be. In 6 tosses of a fair coin, a head is as likely as a tail on each toss so a string with 3 heads and 3 tails is considered more likely than a string of 6 heads. Related to this, an outcome such as HTHTHT is not considered as likely since it is described as "too regular" or "not random enough" (Shaughnessy, 1977).

Availability occurs when the likelihood of an event such as a person having a heart attack is determined by the ease with which similar cases can be recalled. This results in an estimate that is biased by personal experience.

A third heuristic is the **conjunction fallacy** which occurs when a description of certain qualities is provided of person and one is asked to rank the probability that the person has a particular occupation. This heuristic has elements of representativeness because one is often influenced by one's beliefs about the likely traits of people in a particular occupation without due consideration of the base rates of people holding those occupations. Kahneman and Tversky (1982) give an example of subjects being presented with the following description: "Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations" (p. 126) and then asked: "Which of the following statements about Linda is more probable: A. Linda is a bank teller or B. Linda is a bank teller and a feminist?" (p. 126). More than three-quarters of an undergraduate class who had not taken a statistics course responded that B was more probable.

The outcome approach was proposed by Konold (1989) in order to explain the non-normative thinking displayed in a study of 16 undergraduates. The outcome-oriented individual interprets a probe of which is the more likely outcome as a request for predicting the outcome of the next trial. For example, in a coin task where a coin is flipped 4 times in a row with a result of 4 heads, to a probe of which outcome is more likely on the fifth flip, a person reasoning according to the outcome approach may answer: 'Any of them could happen' or 'You can't really predict it'. To further understand this behaviour, Konold and colleagues (1993) varied the coin task by giving various strings of the outcomes of the 4 trials such as HTHT and asking which is the least likely outcome in addition to which is the most likely outcome on the fifth flip. They found that students used the outcome approach when the question contained 'most likely' but switched to the representativeness heuristic when the question contained 'least likely'. Konold termed this inconsistency the **most likely-least likely switch.**

The **equiprobability bias** or the **50-50 approach** occurs when a person assigns equal probabilities to two outcomes of an experiment (either I get an A on the course or not). LeCoutre (1992) in a study of 600 subjects recast two probability problems using dice and urns into more concrete contexts where the chance element was 'masked' and found that in these situations the subjects used the appropriate cognitive models. However in the context of dice and urns of chips, the usual randomizers in school mathematics, the equiprobability bias persisted in groups that had varying degrees of exposure to probability (none, a little, and a thorough background in probability).

The **gambler's fallacy** (also called **the law of small numbers** or the **negative recency** effect) is the common belief that a string of outcomes of a random process will in the short run correct itself so as to reflect the expected results. So, for example, in tossing a fair coin repeatedly, according to this belief, after a string of 6 heads, a tail is considered to be more likely than a head because the commonly-held misconception is that tails are 'due'.

Much of this research on heuristics and biases has been conducted on adults (university and college students) but some studies carried out with children indicate that they have similar misconceptions. Fischbein and Gazit (1984) studied the effects of a teaching program in probability devised for junior high school pupils (ages 13-15 years) and found the 13-year-olds were unable to use the ideas in the program, 60-70% of the 14-year-olds and 80-90% of the 15-year-olds were able to understand most of the concepts of probability presented in the program. They found that the misconceptions of representativeness and the positive recency effect and superstitious beliefs were ameliorated by the instruction.

Fischbein & Schnarch (1997) investigated the evolution with age of probabilistic, intuitively based misconceptions. Five groups, each approximately 20 students, were investigated, grade 5 (ages 10-11), grade 7 (12-13), grade 9 (14-15), grade 11 (16-17) and college students who were prospective teachers specializing in mathematics. Using a questionnaire with 7 probability problems that addressed representativeness, negative and positive recency effects, compound and simple events, the conjunction fallacy, effect of sample size, availability and the time axis effect, they found that the results were contrary to their initial hypothesis of the stabilizing of intuitions after the emergence of formal reasoning (about age 12) and that: 'In the end then, the picture was rather complex: some misconceptions diminished with age, one was stable and some gained greater influence' (p. 103).

This last description is indicative of the results of the research literature on student understandings of probability in general and sets the stage for the research questions.

2.3 RESEARCH QUESTIONS

Probability and statistics are a relatively recent addition to the school mathematics curriculum in North America and Europe (in Europe they are referred to as Stochastics). Following calls in the 1980s for inclusion of these subjects due to their importance and relevance in preparing citizens to participate in society, these two topics constitute the latest strand in school mathematics curricula that have previously been devoted to topics such as number, patterns, relations, shape, and space. This has led to much research devoted to the teaching and learning of probability and statistics along the themes of frameworks for children's probabilistic thinking, misconceptions in probabilistic reasoning of students, response to instruction in probability and statistics, teachers' understandings of probability, and strategies for teaching probability and statistics including the use of computer environments (Jones, 2005; Jones, Langrall & Mooney, 2007; Kapadia & Borovcnik, 1991; Shaughnessy, 1992).

While the strand of Probability and Statistics in school curricula is now well-established (NCTM, 2000; BC MOE, 2007), the questions of how these topics are best learned and taught are still open (Kilpatrick, 2001). The wide range of difficulties and challenges that students face in basic probability courses despite exposure to the topic for many years in elementary and secondary school points to the importance of finding out the nature of the learning and teaching of probability at the foundational levels in order to develop an understanding of where the problems might lie. Further the research literature has been found to be wanting in focus and to be lacking in knowledge of the situation as it obtains in probability in the elementary grades in BC. This study will contribute to this knowledge by addressing the following:

RQ1: What understandings of probability do Grade 7 students demonstrate before instruction in probability?

RQ2: How are these understandings affected during and after an instructional unit in probability?

RQ3: What instructional models can be proposed to enhance deeper understandings of probability?

In the next chapter, I present the methodological framework and describe the context in which the study was carried out. I then outline the various components of the procedures followed in the study.

CHAPTER 3. METHODOLOGY

In this chapter, I present the methodological framework of the study and provide details of the context of the study and the research procedures used.

3.1 METHODOLOGICAL FRAMEWORK

The purpose of this study was to explore the extent of Grade 7 students' understanding of probability, that is, to see what probability ideas and beliefs they hold and how they respond to exposure to some probability instruction. Students' understanding of any subject matter is not easily observed, identified or measured and is subject to the interpretation of the researcher. A further complication arises from the particular challenges of the subject matter earlier described in Chapter 1. My intent was to capture a rich description of the ideas and the reasonings, the interactions and reactions of the students with probability notions from their frames of reference. I did not frame any hypotheses but I was open to examining the phenomenon of understanding in all its complexity. Hence this study was located in the qualitative interpretive paradigm (Guba & Lincoln, 1994) and carried out as a phenomenological case study (Stake, 1995, 1998; Merriam 1998).

I approached this study as qualitative and interpretive because as Denzin and Lincoln (1994) write, "… qualitative researchers study things in their natural settings attempting to make sense, or interpret, phenomena in terms of the meanings people bring to them". Hence I wanted to study the understandings of probability from the students' point of view and to place any activities in the natural setting of the classroom and one-on-one interviews with the students.

The study was characterized as phenomenological because I was studying the phenomenon of the students' understanding and meaning making of various probability ideas. I adopted the phenomenological perspective with its emphasis on *verstehen,* the interpretive understanding of human interaction, (Bogdan & Biklen, 2007) in order to make meaning of events and interactions of the students, the classroom teacher and the researcher in course of the research. Psathas (1973) states that "[p]henomenological inquiry begins with silence" which underlines my position of approaching the research with no assumptions and no prejudgments of the students and their knowledge and beliefs about probability.

I saw the study as having elements of a case study according to Merriam (1998)'s definition of a case study as "an intensive holistic description and analysis of a bounded phenomenon such as a program, an institution, a person, a process, or social unit" (p. 16). I chose the classroom as a case because I was interested in what obtains at that school level and age group in the educational system with regard to the teaching and learning of probability. This aligns with Stake's description that "[c]ase study is the study of the particularity and complexity of a single case, coming to understand its activity within important circumstances" (1995, p. xi). My focus was to observe and interpret the extent and quality of probabilistic thinking at the Grade 7 level and to work with the teacher in helping students gain a thorough understanding of the content.

This research was carried out collaboratively with all participants, namely, the researcher, the teacher, and the students in the classroom. My role as researcher was more than observer as I collaborated with the classroom teacher in developing and carrying out the instructional program. The students collaborated in the research by their participation and engagement in the activities.

The opportunity for research presented by the classroom, the students, the teacher and the time of the school year enabled me to plan the study along three main components - paper-and-pencil instruments to assess understanding at the beginning and end of the study, whole-class lessons, and individual interviews. The principal methods employed were observation, note-taking, and interviewing. I tried to make observation more than just an objective recording because as researcher, I had to interpret events that took place in order to understand the responses and behaviour (Wiersma, 1991). I used flexible and open-ended interviews in order to further probe the thinking of the students in the study. For the interviews I was aware of the importance of building relationships, "listening beyond", and being "critically aware" during the interview (Measor, 1995). I gave much thought and attention to the influence of the tasks and the task environment (Jones, 1974) and a number of related tasks were used in order to retain reliable information on the students' thinking and understanding.

As carried out, this study fits the definition of case study given by Yin (1984) as "an empirical inquiry that investigates a contemporary phenomenon within the real life context … in which multiple sources of evidence are used" (p. 23).

3.2 CONTEXT OF THE STUDY

The study was undertaken at an elementary school that was part of TEAMS-Learning, a collaborative community-led action research project guided by my supervisor, Dr. Cynthia Nicol, and her colleagues. I was fortunate to have one of the Grade 7 teachers, Ms. Thomson (pseudonym), give her consent to my carrying out a study in her classroom in the month of June, 2008. Hence my Ethics approval for this research study is under the umbrella of the Ethics approval for the TEAMS-Learning project.

The TEAMS-Learning (Transformative Education for Aboriginal Mathematics and Science-Learning) project is a collaboration of Aboriginal and non-Aboriginal educators who seek to find ways to improve the success of Aboriginal students in mathematics. Slightly less than half of Aboriginal students in their final year of high-school graduate and the proportion who succeed in mathematics is much lower. For example, at Vancouver's Britannia Secondary School, one-third of the student population is Aboriginal but not one Aboriginal student has successfully completed Principles of Math 12 (UBC Department of Science website, 2008). Nicol, Archibald, Kelleher, and Brown (2006) state that, "TEAM[S]-Learning aims to improve student achievement in math and to develop a critical mass of Aboriginal students who are successful in math. We are working with teachers to adapt and develop materials and approaches to teaching that flexibly and meaningfully incorporate Aboriginal cultural values, experiences, and practices".

The TEAMS-Learning Project works with students, teachers, parents, elders and other community members to find culturally responsive ways of teaching that "honour student thinking and emotions, respect and build upon community values and views, and prepare students to be successfully with mathematics in a range of contexts and open possibilities for future study or careers". The Project is carried out as a collaboration of school districts in BC and the University of British Columbia.

The school in which the study was conducted is an Inner City project school that receives additional funding for literacy, social responsibility and community involvement. Over half of the student population is of First Nations ancestry, coming from a variety of nations, including several West Coast nations. The next largest group of students is made up of children of new immigrants from places such as China, Vietnam, the Philippines, and the Middle East, who require support for language development. About one-third of the student population is designated as having special needs.

Apart from a South Asian Learning Resources teacher and a School Secretary of Aboriginal descent, the principal, teachers and other authority figures in the school are mostly Caucasian. My first visit to the school was marked by the feeling that the composition of the school was an unsettling reflection of the political make-up of the society which appeared to not bode well for the success of strategies that aim to remove the effects of colonization. I am reminded of my own experience in attending a secondary school in Trinidad that was set up by the Canadian Mission, an alliance of churches in Canada including the United Church of Canada. In my early years, I had a few Caucasian teachers but there was a steady replacement of a local principal and local teachers. I have no doubt that an important part of my self-esteem was founded in seeing people who looked like me in positions of authority and I can only wonder at the effects of being schooled in this context.

However, there is a significant attempt to have representations of Indigeneity in the school and its surroundings. As one enters the school, there are many indications of a First Nations orientation: a big banner proclaiming the Indigenous name of the school alongside its 'regular' name, two totem poles in the hall, many works of Indigenous art on the walls including ones signed by the famed Haida artist, Bill Reid, and a small longhouse on the school grounds. I was pleased one morning to hear the announcements read out as the 'Raven's Call'.

Ms. Thomson's Grade 7 class had 24 students with 8 girls and 16 boys, all between 12 and 13 years old. The two Grade 7 classes in the school were "platooned", namely the strong students from both classes were put together for the Math class she taught. Ms. Thomson's classroom was nicely arranged with desks in 'islands' of four or five in one half of the room and the other half carpeted with couches for a discussion area near the white board and teacher's desk. On the walls there were many examples of student posters and other art work and on the window sill, a row of plants.

As I was the researcher and outsider to the school, the classroom teacher, and the class, it was important to ensure that the research relationship was based on respect and appreciation for their gift of allowing me into their space and schedule. Ms. Thomson and I built our relationship before the study by having short conversations as her schedule would allow and having preliminary visits by me to get to know her and the class. I had three preliminary class visits. It was a happy coincidence that in two of them, the Family Support worker played the guitar and led a singalong. Ms. Thomson shared with me the class book of songs and explained that it was a regular part of their schedule. I was captivated by the singing and playing but more so by his chatting with the students in between the songs as he asked for requests. The last session I attended was poignant as he alluded to the opportunities and challenges that awaited them as they make the transition from elementary to high school.

Ms. Thomson and I sat down on May 21 and worked out a mutually agreeable schedule for June. There would be two Getting-to- know- the- class visits, three Lesson times (the first one she would teach and the remaining two I would teach), an opportunity for a pre-test and a post-test, and times for interviewing the students with audio recording. Later in the study there was a possibility of two more Lesson times but because of time and scheduling constraints there was only an opportunity for one more so that in all there were four Lesson times (Ms. Thomson taught one and I taught the other three). Figure 3.1 gives the schedule of pre-test, lessons, interviews and post-test as at June 5.

The pre- and post-tests were designed by me and administered by Ms. Thomson as her schedule permitted. Each had 6 questions and was intended to take between 20 and 30 minutes. Both tests were checked by Ms. Thomson to ensure that she considered the questions within the students' capabilities. Ms. Thomson and I agreed that the interviews would be no more than 20 minutes each and that she would assist in selecting students to be interviewed to provide a range of ability and responses. She shared the materials (textbook and other teacher resources) that she was using in Math and I contributed the ideas that I was considering for the activities and tasks in the study.

Date	**Task**
May 21 Wednesday	Preliminary Class visit 1h 15 mins. Ms. Thomson read a chapter of novel. Short meeting setting up times for lessons and interviews
June 3 Tuesday	Gave Ms. Thomson pretest. Short meeting
June 4 Wednesday	Ms. Thomson returned the completed pre-test. 1st Getting to know Class visit 1h 15 mins: Ms. Thomson read a chapter of novel, class discussion.
June 5 Thursday	2nd Getting to know Class visit 1h 15 mins: Youth and Family worker led singalong. Class worked on Art projects.
June 9 Monday	1st class time period 40 mins Ms. Thomson teaches
June 11 Wednesday	2nd class time period 40 mins Veda teaches
June 16 Monday	3rd class time period 1 hour Veda teaches
June 17 Tuesday	4th class time period 1h 20 mins Ms. Thomson teaches PLO D4 sample space Interviews 1h 15 mins Veda
June 19 Thursday	5th class time period 1h 20 mins Veda teaches PLO D5 Independence Interviews 1h 15 mins Veda
June 25 Wednesday	Interviews 1 hr Veda Post-test

Figure 3.1 Timeline for Study as at June 5, 2008

For the two Getting-to-know-the-class visits, in the first, Ms. Thomson led a Literature Circle. She read a chapter of a novel that she had been reading to the class in installments and engaged them in discussion. She used the pair-share technique and then had them come back to a whole-class discussion about what parallels they saw in their life and what they thought would happen next. Ms. Thomson shared with me her love of literature which was evident in her conduct of the Literature Circle. Times certainly have changed – the novel she was reading to her students was *The Uglies* in the science-fiction genre (I think). I remember my teacher reading *Lamb's Tales from Shakespeare* to me at that age. In the second visit, the students were working on their Art projects and I had an opportunity to mingle and chat with them about their art.

3.3 PROCEDURES

The instruments used in this study were developed from a review of the research literature as given in Chapter 2 and my own knowledge and experience of teaching mathematics and statistics in general and probability in particular. It was also important to me that I address the diversity of the students and that I find ways not only to teach probability but to do so by placing it in a cultural context that would resonate with the students. This was in keeping with my overall philosophy of teaching that I address my students as 'whole' beings and that I try my best to engage and motivate them. Also as this study was not cast as a pre-test-intervention-post-test study the various components were designed from an exploratory and investigative point of view.

3.3.1 Pre-test

I chose to give a pre-test as a way of finding out what the students already knew with a view to designing appropriate instruction and activities. It was a way of establishing a base line of their knowledge and beliefs about probability. As noted in Chapter 2, the research literature in this area is in two parts – the cognitive development literature and the judgment and bias literature. I developed the questions in the pre-test in order to address these two parts. I titled it Beginning Probability with the intent of getting a snapshot of the students' understanding before exposure to lessons in probability. There were six items in the pre-test, the first two similar to the probability questions for this level from textbook materials and the remaining four were modeled on items common in the research literature that examine biases such as the gambler's fallacy, the 'spinner mystique' (Jones, 1974) and the lottery task.

3.3.2 Framing the lessons

For Grade 7 Probability, there are three Prescribed Learning Outcomes (PLOs) which are given in Figure 3.2; a description of the mathematical processes indicated in the square brackets is given in Appendix A.

General Outcome: Use experimental or theoretical probabilities to represent and solve problems involving uncertainty	
D4 express probabilities as ratios, fractions, and percents [C, CN, R, T, V]	Determine the probability of a given outcome occurring for a given probability experiment , and express it as a ratio, fraction and percent Provide an example of an event with a probability of 0 or 0% (impossible) and an event with a probability of 1 or 100% (certain)
D5 identify the sample space (where the combined sample space has 36 or fewer elements) for a probability experiment involving two independent events [C, ME. PS]	Provide an example of two independent events, such as spinning a four section spinner and an eight-sided die, tossing a coin and rolling a twelve-sided die, tossing two coins, rolling two dice and explain why the are independent Identify the sample space (all possible outcome) for each of two independent event using a tree diagram, table, or another graphic organizer
D6 conduct a probability experiment to compare the theoretical probability (determined using a tree diagram, table or another graphic organizer) and experimental probability of two independent events [C, PS. R, T]	Determine the theoretical probability of a given outcome involving two independent events Conduct a probability experiment for an outcome involving tow independent events, with and without technology, to compare the experimental probability to the theoretical probability Solve a given probability problem involving two independent events

Fig. 3.2 PLOs for Grade 7 Probability

When Ms. Thomson and I worked out the schedule, she noted that the students already knew D4 *Express probabilities as ratios, fractions, and percents*. She proposed that she would cover D5 *Identify the sample space (where the combined sample space has 36 or fewer elements) for a probability experiment involving two independent events* and that I would cover D6

Conduct a probability experiment to compare the theoretical probability (determined using a tree diagram, table or another graphic organizer) and experimental probability of two independent events. As I contemplated the lessons, I realized that I had to find culturally responsive ways to address the diversity of the students while conveying the basic concepts of probability. Also as this was my first time teaching Grade 7, I wanted to be sure that my lessons were pitched at the right level and so made sure to discuss my lesson ideas and send my lessons notes to Ms. Thomson and my supervisor, Dr. Nicol, before the class.

In order to prepare for my lessons, I read children's books on probability (Cushman, 1991; Linn, 1972; Razzell & Watts, 1967), books on Aboriginal games and gaming (Carlson, 1994; Gabriel, 1996) and looked at websites on Aboriginal games and other probability activities. I also consulted with Dr. Nicol who helped me to see the teaching from the students' point of view and who provided manipulables that I could use in the classroom. As I struggled to represent the subject matter in ways that I had not considered previously and to meet the challenges of making what I saw as clear also clear to individuals with whom I had no previous experience, I began to realize the enormity and the complexity of the endeavour.

I further discussed with a Statistics colleague at Langara the various possible activities that I was considering for the lessons. I remember being carried away with the different activities and hearing the constant refrain: Yes, the activity is fun but what probability idea are you trying to teach? While probability units in texts and other materials are generally taught in the elementary grades using randomizers such as dice and spinners, I wanted to find a way that would make the subject come alive for the students. From Ms. Thomson's lesson I saw the level at which the topic was pitched and planned accordingly.

The first lesson was an introduction to the language of probability and an exercise in sample space. In the next lesson I explored the idea of 'observed' and 'expected' using a manipulative with balls being dropped from a height and making their way down tracks, and in the third lesson I included an Aboriginal chance game that led to the idea of the mean or expected number. In the final lesson I explored the notions of sample space and distribution with the sum of the numbers from two spinners.

3.3.3 Lesson Implementation

I approached the lessons with a little trepidation and much worry. I have been teaching Mathematics and Statistics for over 20 years at the college and university level with two short stints at high-school very early in my teaching career. Despite my extensive teaching experience, I have no formal teacher training – this experience in the MA program is my first exposure to a Faculty of Education, all my studies having been in a Faculty of Science and Departments of Mathematics and Statistics. I had never faced an elementary class before and I worried about my lack of training and experience in probability at this level. Still I was guided by a good understanding of the nuances and complexities of the topic of probability and by a clear idea of what I wanted to get across in the lessons.

As I pondered the lessons I was aware of many things that obtain in today's classrooms – the diversity of the students' backgrounds from various cultural communities, the sophistication of today's students given their exposure to so many forms of media, the vast amount of knowledge and stimuli available to them, the need to engage them, the need to cover the outcomes as prescribed in the provincial mathematics curriculum. How was I going to keep all of that in my consciousness and manage to teach them a little probability? I began to wonder how teachers manage to walk into classrooms and teach at all … .

I considered many activities - the Cereal Box problem, Coin on a Grid (variation of Buffon's needle experiment), simulations, activities with the usual randomizers of coins, dice spinners, activities that usually appear in teachers' resources that accompany the textbook, activities in books that present activities to foster an 'hands-on' approach to the subject such as Burns (1995), notably the Game of Pig, and math in literature.

Because of the many students of First Nations ancestry, I searched for and studied many of the Aboriginal games of chance and their significance and tried to decide which ones could be adapted to the classroom and could be used to convey more of the significance and tradition rather than just as an example of chances and odds. At the time, I was also reading articles on culturally responsive pedagogy in the TEAMS-Learning project and was mindful of the 4 R's of Respect, Relevance, Reciprocity and Responsibility (Kirkness and Barnhardt, 1991). I had also heard an Aboriginal Mathematics professor, Edward Doolittle, speak on 'Mathematics as Medicine (Doolittle, 2006). I was heedful of his observation: "My feeling is that Indigenous students who are presented with such oversimplifications feel that their culture has been appropriated by a powerful force for the purpose of leading them away from the culture" (p. 20). So I wanted to avoid oversimplifying and token references.

My lesson notes and the activities that were carried out are given in Appendix B.

3.3.4 Interviews

The interviews were designed to be further explorations of the students' understanding of the basic probability ideas. I wanted to follow up on some of the written responses in the pre-test as a way of validating what I had understood of their grasp of the ideas. I wanted to see what their ideas were and to see how they would respond to attempts of gentle leading on my part.

I had given some thought to the structure of the interview - some initial interaction in order to put the student at ease, some general probes on the student's feeling for Math in general and probability in particular, some tasks designed to probe understanding of various aspects of probability and a final thank you for the participation, the time spent and the willingness to engage. For the initial interaction, I developed the following script prior to the interviews:

Interview script

1. How are you? How is school? Looking forward to grad?

2. How about Math... how are you finding it?

3. Let's talk about probability.... I am trying to find out how students understand probability, what they like, what's hard for them and so on... anything that strikes you?

4. Let's talk about the Beginning probability test that you did. Do you remember it? What about this question...

5. Let's try out some things, shall we? Here I would play a small game or pose a small problem or so...

By the time I carried out the interviews I had visited the class a number of times and had acquired a fair knowledge of the students. I had put names to faces and had noted the participation and overall bearing of the students in various activities. I knew which ones were lively and put their hands up in class discussions, which ones were quiet but attentive, and which ones appeared disengaged (though I knew this was by no means an indication of lack of understanding or interest). So I followed Steps 1-4 fairly closely but I found that in Step 5 I did not follow the same sequence of tasks each time or even use the same phrasing because I had some degree of familiarity with the students and adjusted my manner as the situation presented itself. Despite the recommendation of being careful to have the same interview script for each interviewee (Goldin, 2000), I found that with each student the interview unfolded differently.

What are we asking of subjects when we undertake task-based research? Important factors that have to be taken into account when designing tasks to probe understanding are context, framing, presentation, and wording. The levels of the task and the subjects' content knowledge have also to be considered and care has to be taken to ensure the subjects' understanding of the task. Some of the tasks that I considered using in the interviews are given in Appendix C.

3.3.5 Post-test

A post-test was given in order to see if there were any qualitative changes in the students' understanding of probability ideas. The post-test was similar to the pre-test in that I repeated the mix of items of the textbook type questions and judgment and bias questions used in the literature. Variations of most of the questions were included such as the spinner task and the lottery task in order to see if there was any change in the thinking. The difficulty in the sample space question was increased from one-stage to two-stage. Overall the post-test tested similar notions of the pre-test but at a deeper level.

3.3.6 The Student Participants

Because of the various factors beyond the control of the researcher such as scheduling and absences, the 24 students who wrote the pre-test were not the same as the participants in each the lessons, and not the same as the 24 who wrote the post-test. There was a core of 19 students who wrote both the pre- and the post- tests and of these 19, ten were interviewed. These ten either volunteered or were recommended by Ms. Thomson. The students will be identified by S#;

S1... S19 indicate the 19 students who took both the pre- and post-tests;

S20...S24 indicate the 5 students who took the pre-test but not the post-test;

S25...S29 indicate the 5 students who took the post-test but not the pre-test.

The ten students who were interview were among the S1...S19 students and will be identified accordingly.

3.4 ANALYSIS OF THE DATA

At the end of the study, I had collected the written responses from the pre- and post- tests, my notes and observations from the lessons, and audiotapes of the interviews. The data consists of the student responses from the pre- and post-tests, the transcripts of the interview audiotapes, and my notes of the lessons.

I began by scoring the pre-test (mostly as a reflex action after decades of teaching) but quickly moved on to analyzing the written responses of the pre-test for the degree with which they displayed normative thinking and the strategies used in the thinking in order to establish a baseline of understanding with respect to the four probability ideas. I studied the transcripts of the interviews for further evidence of understanding along these four probability ideas. As well, the written responses of the post-test were studied for any evidence of shifts in thinking.

The results will be presented and discussed in Chapter 4 with conclusions and recommendations in Chapter 5.

CHAPTER 4. DATA ANALYSIS, RESULTS AND DISCUSSION

This chapter has two parts. In the first part I present the data analysis and results relating to the understandings of various probability ideas. I begin by establishing a baseline of the understandings displayed from the written responses of the pre-test organized along four main probability constructs: Probability as a ratio, Sample space, Meaning of random, and Independent trials. The first three have been chosen because they are core basic ideas of probability and while I consider the fourth intermediate, it is included because it is part of the probability curriculum for this grade. I then follow up with an analysis of the strategies used in the thinking as evidenced in the interviews and post-test followed along the same 4 constructs. In the second part, I summarize the results and discuss some considerations arising from the results.

4.1 BASELINE OF STUDENTS' UNDERSTANDINGS OF PROBABILITY

At the outset of the study, it was important to establish a baseline of the knowledge and thinking about probability that the students at this level already held. This was done by the administering of a pre-test that had a mix of numerical and qualitative questions covering various probability ideas. As indicated earlier in Chapter 3 these questions were taken from the teacher resource materials and from the literature.

The baseline of the students' understanding of probability as seen in the results of the pre-test will be organized around 4 main ideas: Probability as a Ratio, Sample Space, The Meaning of Random, and Independent Trials.

4.1.1 Probability as a Ratio

A baseline understanding of the concept of probability of a ratio of the number of favourable cases to the total number of cases was established using the responses to the first two questions of the pre-test.

Beginning Probability June 4, 2008

1. A bag contains 5 blue marbles, 4 red marbles and 6 white marbles. A marble is chosen from the bag.

a) Find the probability that the marble is blue. ________

b) Find the probability that the marble is blue or white. ________

2. Ten chips numbered 1 to 10 are placed in a bag. One chip is drawn from the bag.

a) Find the probability that the number on the chip is less than 5. ________

b) Find the probability that the number on the chip is 12. ________

Fifteen of the 24 students who wrote the pre-test could correctly find the probability of a simple event but only nine could successfully move to the next level of finding the probability of the union of two simple events. Twenty of the 24 students were able to identify an event that had zero probability, namely an impossible event, but many had difficulty in counting the set of favourable outcomes such as the set of numbers that are less than five. In general students had difficulty in deciding whether five is included or not.

Another layer of difficulty relating to finding the probability of event as a ratio of two areas was presented in the spinner question in Fig. 4.1.

3. Suppose you are playing a game with spinners. You win a point if the spinner lands on the dots. Here are 2 spinners. Which one would you choose to play with? Why?

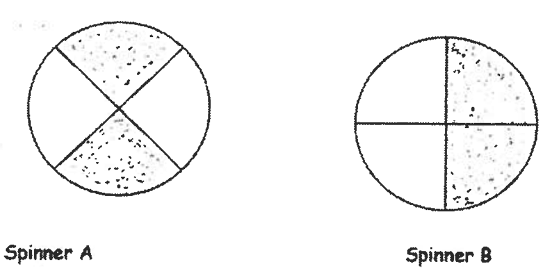


Figure 4.1 Pre-test Spinner Task

Only three of the 24 students correctly identified that the chances were the same for each spinner so either one would be acceptable *(S11: It doesn't matter because on both spinners you have a 50-60 chance of winning, S5: I would play with either as they each have the same probability of landing on dots)*. Two students did not attempt the question and eighteen chose one over the other for various reasons. Sample responses include:

> *S23: Spinner A because there is more variety of space rather than both on one side.*
>
> *S6: Spinner A because the dots are the same on each half no matter which way you cut it. You would have a better chance of landing on the dots on Spinner A than on Spinner B.*
>
> *S24: Spinner B because there are more dots on one whole side. More of a chance to win and more area.*
>
> *S4: Spinner B because 2 halves added together side by side seems like a whole piece and I think then I would have a better chance of winning.*

Ten students chose A because of the 'spread out' and 'not bunched up' notion and 9 chose B because they saw their chance of landing on dots as greater due to the dots being 'all on one side'. One student, S17, likened the situation to soccer: *It's like a soccer game, I feel my chances are better spread out than all bunched together'.* These responses indicated that students have some idea of the chances involved (albeit incorrectly evaluated) but are arriving at different conclusions based on how they see the dotted areas, the distinction of contiguity and non-contiguity serving as a distraction. Student S9 referred to the tensions in the question with elements of normative thinking:

> *S9: I don't really know because there is a 50% I could win on either. But if I had to choose I would choose spinner B because if it lands on the right hand side ever it would get me a point.*

While the spinner task proved to be difficult for the students, the basic idea of the probability of a simple event was grasped by many of the students.

4.1.2 Sample Space

The notion of sample space was probed in the pre-test in the Die instead of Coin question.

> 4. Suppose that in a game with two teams, the team that gets to start first is usually determined by a coin toss. Suppose that instead of a coin we had a die. How could we use the die to decide which team starts?

Just over half (fourteen of the 24) of the students correctly explained how a correspondence can be set up with the six outcomes of a die and the two outcomes of head and tail of a coin. The commonest method was odd-even but two indicated that any subset of three outcomes will do. Nine students used different schemes such as:

> *S10: I've got two ways. See who gets the highest number. Or whatever team gets a 5.*
>
> *S8: One team could be going first if the team rolled and the dots were 2 or 1, or the other team rolled the die and got 5 or 6 and if 3 was rolled the die would be rolled again.*

This shows that a majority of the students had a good understanding of the sample space of simple one-stage experiments.

4.1.3 The Meaning of Random

The understanding of this notion was probed in the pre-test using a random draw task of the Lotto 6/49 (drawing six numbers at random from the numbers 1,..., 49).

> 6. Suppose your friend is planning to buy a Lotto 6/49 ticket and picks the numbers {1, 2, 3, 4, 5, 6}. You plan to use numbers from the Quick Pick machine. Which of you has a better chance of winning? Explain.

Again only three of the 24 students indicated correctly that both have equal chances of winning (*S12: Well both of you have the exact same chance that you would win because all you are doing is changing* *your numbers, it doesn't make a difference in your probability of winning*). More than half of the students (fifteen) did not respond to this question.

For the ones who responded, a consideration of what is random was based on personal experience and beliefs and other external factors. There was some consideration of chance but often incorrect probability determinations. Examples include:

> *S19: Your friend because the Quick Pick machine would probably choose incorrect numbers for you.*
>
> *S2: My friend because the computer might give you the number that might not want you to win. Because the point is to guess the number by yourself.*
>
> *S10: I don't ever know what numbers to pick but if the last available number is even then that has better chance of winning.*

There was much consideration given to the facts that the numbers 1, 2, 3 4, 5, 6 were low and in a row which proved to a distraction similar to contiguity and non-contiguity in the spinner question. For the students, random numbers indicated numbers that would be 'spaced out' and 'not clumped together', and higher as 'in the thirties and forties'. Similar thinking was used in the spinner task where a higher probability of landing on dots was ascribed to the spinner that appeared 'spread out'.

> *S22: You are because the Lotto draws are elaborate numbers and the quick pick machine will give you numbers that are high and elaborate.*
>
> *S6: You because you are more spaced out and she more clumped together and the numbers are usually random numbers.*

Further, the numbers 1, 2, 3 4, 5, 6 were seen to be too simple:

> *S2: Yeah, because 1, 2, 3, 4, 5, 6, like nobody would do that because it is just 1, 2, 3. 4, 5, 6, it would be too easy...*

Subjective thinking is evident as the students could not accept the numbers {1, 2, 3, 4, 5, 6} as a winning combination:

> *S4: Me, because the winning numbers couldn't be just 1, 2, 3, 4, 5, 6 because the winning numbers, too, are drawn randomly and it would be one in a million chances that the number would be those.*

Overall the nuances of the word, 'random', are overshadowed by other considerations.

4.1.4 Independent Trials

To probe this notion in the pre-test, I used a task that I call the 5th flip task. This task has been well-studied in the literature and tests the gambler's fallacy that after a long string of successes, a failure is more likely in order to 'balance out' the proportion of successes and failures as in coin-tossing.

> 5. Suppose that you are flipping a coin repeatedly and that you get 4 heads in a row. On the 5th flip is a head more likely than a tail? Explain.

Again three students corectly responded that the chances of a head and a tail are the same (*S14: You still have an equal amount of chance to get either because nothing has changed except for the number of times you have won and that doesn't influence the chances of getting heads or tails*). Three students did not respond to the question, ten expressed the gambler's fallacy (*S7: Tail because the head will eventually lose*) and seven considered the outcome of a coin-tossing experiment as something that can be manipulated as in a flipping strategy (*S8: Tails because you were lucky to get 4 heads in a row. Heads because you might have found some flipping strategy*). There was a mix of thinking strategies that will be taken up later in the Chapter in Section 4.2.4,

Overall the results of the pre-test show a range of understandings. While the three students who correctly answered the spinner, Lotto 6/49 and 5th flip questions demonstrated an excellent grasp of the basic probability ideas, a greater part of the class gave non-normative responses indicating a shaky understanding of the basic ideas and a susceptibility to well-known misconceptions such as the 'spinner mystique' (Jones, 1974) and the gambler's fallacy. A further conclusion is that these students do not have a clear and resilient idea of the meaning of random, drawn at random, and independent trials.

In the next section I examine the understandings and the strategies employed in the thinking as they emerge from the interviews and the post-test as I followed up on the pre-test questions and engaged the students in similar tasks. The analysis will be given along the 4 constructs above.

4.2 UNDERSTANDINGS FROM THE INTERVIEWS AND POST-TEST

The understandings displayed continued to be a mix of standard and non-standard reasoning with a pre-dominance of non-standard reasoning. In order to probe further into the non-normative thinking of the students in the different situations of uncertainty, an analysis of the reasoning strategies used was carried out. This proved to be difficult since in many cases multiple strategies were used in a single response. Also, the strategies used varied across tasks and settings (pre-test, interview, post-test) and often the same strategy was used to arrive at opposite conclusions. In order to tease these out further, I first consider the strategies that have been identified in the literature.

In general, children and adults when faced with situations of uncertainty reason using the notions of chance and probability as given by **formal knowledge of probability theory** or are guided by natural assessments, intuitions and informal heuristics which often result in **biases or misconceptions** such as availability, anchoring and representativeness (identified in Chapter 2).

An alternative model, the **outcome approach,** has been advanced by Konold (1989, 1993) in order to explain responses to probability questions as a request to make a prediction of the next trial. Features of a problem that may induce people who usually begin normatively in simpler situations to resort to the outcome approach are the conditions of elementary events not being equally likely and the repeatability of trials. The analysis of the thinking that follows will be organized according to the 4 probability constructs above and will make reference to these strategies.

4.2.1 Probability as a Ratio

While the responses in the pre-test indicated that many students can find the probability of an elementary event in a single-stage experiment, I wanted to see how stable this knowledge was. In the interviews I used a variation of a well-known binary-model task (Shaughnessy, 1992) where a choice is required (two bags of candy, two urns of marbles, two spinners, etc.). Instead of real bags of candy, I just asked the students to imagine the scenario as I drew a box |________| and wrote in the numbers and colours. For the most part the set-up was as follows: One bag contains 2 red and 3 blue candies (bag A). Another contains 6 red and 9 blue candies (bag B). You can choose one candy from a bag and you want a red candy. Which bag would you choose from? Why? Because I wanted any mention of chance to come from the student, I did not phrase the question as: In which bag would you have a better chance of getting a red candy? Sometimes I posed the task slightly differently varying the composition of the bags but keeping the proportion the same (for example, a bag contains 20 red and 30 blue candies). I let the interaction lead my choice of numbers. If the student seemed comfortable with the smaller numbers I would step it up a little to explore the ideas of ratio and proportion.

While four of the ten students interviewed gave the normative response according to formal probability theory that both bags offered the same chance and it would not matter which you choose, the nonstandard reasoning demonstrated in this task was persistent and hard to overcome. Four students expressed a personal choice, recognizing the bags are the same but choosing the bag with fewer candies in total, and two students made the error of comparing absolute numbers. This misconception has been noted by Green (1983) in his study of 3000 pupils aged 11-16 years and Shaughnessy (1992) among college students.

Often, students chose one bag, then reconsidered and reversed their answer. As an example, with S16, I use bags of 2 red and 4 blue candies (A) and 10 red and 20 blue candies (B). He first chooses B because it has 10 red candies but when I point that it also has more blue candies, he reconsiders. After a discussion of proportions and an agreement that the chance is the same in each bag, he indicates a preference for bag B with the reason that it just has more. When I ask about whether the proportions are not important, he ponders a long time and then goes back to thinking through it again considering one bag and then the other (*S16: If you pick a candy in this bag (B) there will be a lot of blue candies surrounding the reds so you might pick any blue candy but in this one (A) it has a smaller amount so you can pick a red one but you can still get a blue one*). He continues to compare the one with the other even after we establish the chances and I ask: *So the one-third, one-third, does that influence you in any way?* He continues his train of thought (*S16: Maybe the top one (A). I just think the top one has more chance cos in this one, it'll be like 30 candies and you get a handful of some of each, in this one it'll just be like only 6 candies and you can just pick one*). In the end, he reluctantly decides on A. This going back and forth and hesitation is a frequent phenomenon.

Another example of personal choice for the bag with fewer candies despite a recognition of the same chance is strongly stated by S19 in the following excerpt:

I: Suppose there is a bag of candies with 2 red and 3 blue candies (A) and another bag that has 20 red and 30 blue candies (B) and you wanted a red candy, which bag would you choose?

S19: I'd choose the one that had 2 red.

I: You'd choose this one, A, tell me why.

S19: Ohhh, it's kind of the same, but I prefer this one, I don't know how to explain it. I know this one has 20% chance.

I: 20% chance to get a red?

S19: I have 40% for a red and 60% for a blue. They are the same but I will choose A because I have less to choose from.

I: But....

S19: I always choose, if I had some random choice from a bag, I'd have to wear gloves so I couldn't feel anything and I am blindfolded and I ask how many of what are there I always try and get the one that has the least amount like 2 bags one with 10 cats and 5 dogs and one has 2 cats and 1 dog, I'd go for the one that has 1 dog in it.

I: Okay, but, your chances, your chance here in A for getting the red is what?

S19: 40%

I: And the chance here in B is...

S19: 40%

I: So how does the number make a difference if your chances are the same?

S19: It is, I just like choosing things with the low numbers.

I: You think that will help you get a better outcome?

S19: Yes

Student S19 has a notion that having less to choose from somehow increases his chances. It is interesting that he brings in an example of cats and dogs (I had to pause to mentally picture a bag containing 10 cats and 5 dogs) but it is noteworthy that he has the proportions right. In the end, it is a personal choice: *I just like choosing things with the low numbers.*

For an exploration of the idea of probability as a ratio of two areas, I followed up on the spinner question in the interviews and post-test. In interview, two students who had previously chosen one or the other in the pre-test now gave a normative response and one who had appealed to a causal deterministic analysis of a spinning strategy (*S1: spinner B because you just have to keep spinning it hard enough to land on the dotted half*) also gave a normative response. Five students maintained their answers in the pre-test and gave spirited defenses of their position by expressing a personal choice. The following excerpt is representative of the reasoning:

> *I: ... But here there are two (sectors) and here there are two (sectors)...*
>
> *S22: Yeah (not very convincingly). Well, I think that mathematically they are the same but I prefer this one because it is not lop-sided and I know there's a 50-50 chance but in this one there is only one space between the dots.*
>
> *I: It's interesting that you say mathematically they are the same. So are there times when the reasoning tells you one thing but you still feel something else?*
>
> *S22: Yes.*
>
> *I: What would convince you that those are actually the same?*
>
> *S22: Well, I guess they are the same but I just think in my mind, go with this one (A) because it will probably land on dots on that one.*

In the above excerpt, there is an appeal to correct thinking using formal probability theory and an appeal to the configuration ('not lopsided', although one can argue that both spinners show symmetry albeit along different lines) but then there is a conclusion that 'it will probably land on dots' on A.

Another strategy for choosing a spinner besides the appeal to 'spread out' and to symmetry ('not lopsided'), is a consideration of the contiguity of the dotted sectors in spinner B which leads some students to choose B. This argument is often conflated with an incorrect assessment of the chance for each outcome of white and dotted. Examples of the arguments here are: *S8 pre-test: Spinner B because there are more dots on one whole side. More of a chance to win and more area.*, and *S4, pre-test: Spinner B, I would choose this spinner, because there are no white spaces between the dotted pieces of the circle, and I think that will land me a higher probability to win).*

The spinner question on the post-test was a slight variation of the one in the pre-test. There are 8 sectors in each circle instead of 4 and while I have changed where the dots appear, there are dots on 4 sectors of each, non-contiguous in A and contiguous in B.

3. Suppose you are playing a game with spinners. You win a point if the spinner lands on the dots. Here are 2 spinners. Which one would you choose to play with? Why?

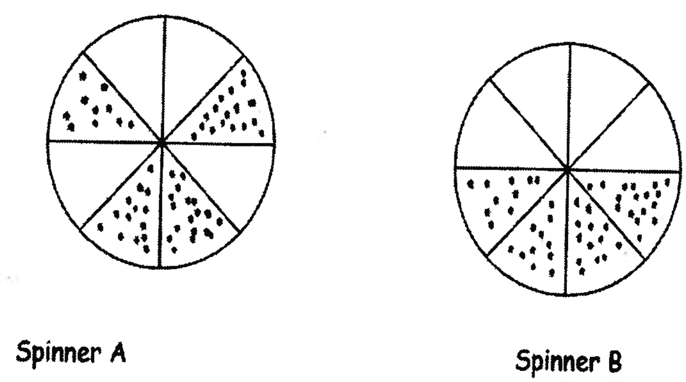


Fig. 4.2 Post-test Spinner Task

By the post-test, five students who had answered A or B in the pre-test gave a normative response with a normative explanation but many students persisted in their thinking with one switch from B to A. One student responded, '*I wouldn't play with spinner B just because it is so simple*'. Others combined the notion of 'spread out' with being random and having greater chance as in:

S13: A. The dotted areas are more spread out, meaning that there are more chances than the other one, since they're not bunched up,

S17: I would choose spinner A because I like my chances spread out. So I would end up having two spots on each side.

S10: I choose B because you can be more definite with a win or a loss. The other one is separated and more random'.

The final excerpt (from an interview) on this task underscores the fact that thinking under uncertainty produces its own anxieties and lack of confidence and that students often go back and forth between positions unable to come to a conclusion.

I: So the question says which spinner would you choose to play with and why?

S16 points to Spinner A

I: But you just said that this one is 50% and that one is 50%

S16: I just like this one

I: But they have the same chance, right?

S16: Yeah

I: Is there anything more special about this one that you like?

S16: It's like more spread out, it's not like grouped together like this one

I: Is there something that would make the needle, the pointer, if there were equal chance, what would make it different?

S16: I think like this one (pointing to B), it is more fair so if it like slows down here you have like more chance of landing here. It has more space.....after much thought... You have more chance to win with B than A.

I: So you first said A, are you changing your mind?

S16: Yeah,... shrugs.

Clearly this student is thinking this through but cannot reconcile the various aspects of the problem.

4.2.2 Sample Space

As many students correctly understood the notion of sample for a one-stage experiment in the pre-test, the understanding of sample space in harder situations was probed during interviews using a task with 2 coins and in a variation with 2 dice. In the task, I show the student two pennies, an American penny and a Canadian penny and pose the following game: The coins are tossed at the same time and if both sides showing are the same I get a point and if different, the student gets a point. We take turns tossing the 2 coins and noting the outcome and then I ask the question: If we were to play this game many times who would be leading?

Normative thinking for this task requires an enumeration of the sample space when two coins are tossed. Many students take a little time to do this but after one or two incorrect statements they very soon come to the realization that the chances for same and different are equal.

In the following excerpt, Student S22 first begins with a 50-50 chance of two sides being the same for one coin but as I gently lead him through outcomes and chances, he comes around to the sample space of 4 outcomes, an assignment of the probabilities (I have to use an analogy with a die) and finally to the conclusion I am looking for.

I: We have two coins and we are going to toss the two coins. If it's the same, you get a point and if it's different I get a point. After about ten tries who do you think will be leading?

S22: You.

I: Why?

S22: Because it's a 50-50 chance of coins, two sides. It's harder to get, I don't know... With a coin, if you have two, it's a less chance that it's going to be both the same? Cos it would land on different each time. The more coins there are the more chance each time they are not the same.

I: That's a good insight about more coins. But with two coins do you know what your probability is and what my probability is? You know if you only had one coin, it would be H or T, yes?

S22: Yes.

I: If you had two coins, do you know what that might be? What would be the possibilities?

S22: Head or tail.

I: But we have two coins...

S22: Oh, both of them would be heads and heads or tails and tails

I: What else?

S22: Um, tails and heads

I: Yes, and,...? .... One more?

S22: Heads and tails

I: So there are 4 outcomes, (gets him to show the 4). So do you know what probability we would assign to each of us?

S22: No.

I: No? You know how when you have a die and you have 1, 2, 3, 4, 5, 6 and you say the probabilities are all 1/6. You say you have a 1/6 probability of each of those. So what do you think?

S22: So the total possibility is four so I guess each one is $\frac{1}{4}$.

I: Oh, nice, you are doing well. So $\frac{1}{4}$. $\frac{1}{4}$. $\frac{1}{4}$, $\frac{1}{4}$. *So would I be winning?*

S22: Well, because there are two ways for you and two ways for me. So it's even.

Many of the students eventually come to an understanding of the sample space and the probabilities in both cases of two coins and two dice. They may initially begin incorrectly but after some consideration and coaching they succeed at this task. The following excerpt with S19 shows this path:

> *I: So let's try this, we toss these two coins. If they're the same I get a point and if they're different you get a point. Okay, let's try it out a few times... (We try this out and find we get alternate points). Now if we were to do this lots and lots of times who do you think would be leading, you or me?*
>
> *S19: I think that I would be leading*
>
> *I: Really?*
>
> *S19: Because it is not definite that I would be winning but I think I would have a slightly more likely chance because it would be harder to get two of the same even though it's random, it's kind of more likely to get two different ones, I think it is more likely to get different ones because it's going up in the air and it's staying there longer and stuff and then when it lands like this, it could like spin, and it could go anyway...*

Here the student is thinking this through but does not quite yet have the sample space worked out correctly.

> *I: So what are the chances for you and what are the chances for me?*
>
> *S19: Well, if they were both tails, it's... ...(keeps turning over the coins). I have 4 possibilities and you would have 2 possibilities.*
>
> *I: Really, tell me what the possibilities are ...*
>
> *S19: This heads and this tails, this tails and this heads ....*
>
> *I: So that's two possibilities for you. How many do I have?*
>
> *S19: You have two.*
>
> *I: Right, so you have two and I have two, so the chances would be...*
>
> *S19: 50-50.*
>
> *I: So it means that if we were to play this lots and lots of times, what does it mean?*
>
> *S19: Most likely it would be equal.*

I was not so successful with one student. For the other side of the coin (so to speak) in the following excerpt with S18, the chances stated are correct of ½ and ½, but the reasoning ('*Well, it's two people and it's randomly*') is not. The 'ah-huh' on my side is an abbreviation for not wanting to push too hard and contradict him but to lead him on a little. I pose a variation with two dice instead of two coins and he gets the chances right again. But when I ask him about the chances with the coins he repeats the earlier argument. It is not clear that he sees the outcomes and the connection between the outcomes and the two people or perhaps he does and he has not articulated it in a way that I can see that he has the connection.

I: So let's try a few things. Here are two coins. If they're the same when tossed I get a point and if they are different you get a point. (We try this out a few times). So you're leading 2 to 1. Now if we were to play this game many many times who do you think would be leading?

S18: Me if I were to go first

I: Because you went first?

S18: Yes, it looks like the same again, if you were to do it over again it would be the same...

I: Ah-huh, but can you think of same and different, who has the better chance, can you work out the probabilities? What's the chance for you and what's the chance for me?

S18: 50-50

I: How did you get 50-50?

S18: Well, it's two people and it's randomly

I: Yeah, but say I have two dice, a red and green and if it's same I get a point and if it's different you get a point, is it also 50-50 in this situation?

S18: No

I: Why?

S18: Because, umm, there's more numbers and the probability...

I: So what is the chance for you in the dice situation? And what's the chance for me in the dice situation?

S18: Mine is one-sixth and yours is five-sixth

I: So for the coins do you know why it's half and half?

S18: Because there's two coins and two sides to each coin and two people...

The understanding of sample space of a two-stage experiment was further probed in the post-test using another common task which will be referred to '5 and 6 or two sixes'. It was stated as follows:

In tossing 2 dice (one red and one green), are you more likely to get a 5 and 6 or to get two sixes. Explain your thinking.

Nine of the 24 students correctly stated that a 5 and 6 are more likely. Four students did not attempt this question and eleven stated that the chance was 50-50 (the equiprobability bias). Of the nine students, only two gave a correct statement of the probabilities involved. Sample faulty explanations include:

S13: 5 and 6 because it is not very likely that you'll get doubles.

S10: 5 and 6 because everything is more likely to random. But if you get 6 and 6, it would be too lucky. But in my opinion, I think people would get 5 and 6 because you're dealing with 2 separate numbers.

The range of understandings of sample space in this task indicate that many students have difficulty in enumerating the outcomes of a two-stage experiment and assigning correct probabilities to the outcomes.

4.2.3 The Meaning of Random

To many of the students, 'random' is a word that is so elemental and so taken-for-granted that it needs little explanation. When I asked in the interviews what random means, as in six numbers are drawn randomly in the Lotto 6/49, many shrugged and were silent. Other responses include:

S8: Just picked out in no specific order or pattern

S9: It's just random because you can choose numbers yourself

S4: Just by random, any number can come up

The last response indicates thinking similar to that of the outcome approach of speaking to a prediction of the outcome on the next trial.

The Lotto 6/49 task was chosen to explore the notions of 'random' and 'drawn randomly' in both the pre- and post-tests. In the post-test, I posed a variation of the question in the pre-test that would remove the considerations that proved to be distractors (binary choice, low numbers in a row). I added another layer of complexity with 4 people instead of two and put in numbers that were higher and not in a row. This time, I hoped to remove the distraction of sets of actual numbers but many students inspected the numbers closely and considered them for spacing and for where they fell in the sequence from 1 to 49. The question on the post-test was as follows:

7. Each of four friends, Amy, Bob, Cathy and David plans to buy a Lotto 6/49 ticket. Amy chooses the numbers {10, 14, 19, 28, 33, 41}. Bob chooses {12, 15, 20, 29, 34, 42} Cathy chooses {15, 25, 35, 45, 46, 48} and David chooses {14, 24, 34, 44, 45, 47}. Who has the best chance of winning and why? Explain your thinking.

Responses of the type that appeal to extraneous factors include:

S10: Either Amy or Bob since everybody else have a lot numbers with the second digit the same. Amy or Bob have more random.

S2: Amy or Bob, because both them have random numbers. David and Cathy have the same numbers that keep repeating and that seems to be too easy.

S18: I think David is better because his numbers are closer together so if one of the numbers is barely off, at least another number will be close to that number.

S13: I think Amy has a greater chance because she stuck with a lot of lower numbers so maybe she knows that lower numbers have a greater chance of being picked.

The determination of the chances involved vary in the responses including those of the students who correctly answer that they each have the same chance of winning. Some students think the chance is ¼ (this is an example of a misconception, namely the equiprobability bias, of considering 4 outcomes and distributing the total probability of 1 equally among the 4 outcomes). Others think that that the chance of winning is 6/49 and one student responded that '*everybody will have about a 28 percent chance because the numbers are randomized and you're not really sure what number it is*'.

The big idea that I am trying to elicit in this question is that to say that six numbers are drawn randomly means that every combination of 6 numbers has exactly the same chance of being drawn. I know that that language is much too sophisticated for students at this level to express but I was trying to dislodge the notion that some sets of 6 numbers are more likely than others or that some sets of six numbers can never be drawn. The hard notion to accept that *each* set of 6 numbers has the same extremely small chance of one in fourteen million of being drawn.

As I have come to expect I get a lively argument from S19 and I think that I can safely use that language with him in following up with the pre-test version of this task.

I: Let's look at the Lotto 6/49 question

S19: It's so random

I: So random, what does random mean to you?

S19: Means well, really, to me it means that if something is random, you can't really control it at all. Whatever it supposed to be, that's what it will be, it can't change.

This is a sophisticated understanding about the objectivity and the lack of manipulation in an objective random mechanism.

I: So you have numbers 1 to 49 and you have to choose 6. How many possibilities would there be?

S19: Lots

I: Good. Yes, but the thing is, are some more likely than others?

S19: Yes, sort of...

I: Sort of, why?

S19: Because the machine has every combination. And we do have a good chance of getting that, I also took into fact how hard it is to win the Lotto 6/49 so many possibilities that choosing the numbers for yourself is a better option than getting the machine to pick them for you because as far as you know the machine could only pick like, could only pick random numbers.

I: But the idea about random is that every combination of numbers has the same chance, that's the whole idea of random, random means that every combination has the same chance including this one {1, 2, 3 4, 5, 6}.

S19: Yeah

I: So what do you think of your answer to this question in the pre-test? (He had answered the friend in this question.)

S19: Still the friend because I wouldn't rely on a machine and I wouldn't be wasting however much money on a machine that most likely would be wrong. The one right number out of hundreds of different combinations, it could be 1, 2, 3 4, 5, 6 just like, 43, 44, 45, 46, 47, 48, 49.

I: Do you see picking 43 to 49 the same as picking 1, 2, 3, 4, 5, 6?

S19: Yes.

I: So do you agree that any combination is as likely as any other combination?

S19: Ummmm, it is as likely but I still wouldn't rely on it...

Student S19 has an excellent insight about the meaning of random in that the outcome cannot be controlled by a person and that it can't be changed. He also recognizes that there are 'lots' of possibilities for the 6 numbers but thinks that some are more likely than others 'sort of'. Even as I advance the idea that all combinations are equally likely, I don't get the agreement that I am expecting that the chance will be the same. Student S19 still thinks that the friend who picks her own numbers has a better chance because he personally would not rely on a machine. Here S19 is making a determination of chances using subjective probability by considering the amount that he is willing to bet that a particular event will take place (*'I wouldn't be wasting however much money on a machine that most likely would be wrong'*). He cites the sequence from 43 to 49 (which is 7 numbers but I do not want to interrupt the flow of the conversation) and agrees that it is likely as the sequence from 1 to 6 but when I follow that up by asking in general whether any combination is as likely as any other combination, his response shows that while he sees it in the abstract, practically he wouldn't bet on it. The thinking displayed here from start to finish is a perfect illustration of a combination of many strategies rolled into one.

As a final consideration of the issues in this task, one factor that the students mention in their reasons for thinking that {1, 2, 3, 4, 5, 6} cannot be a winning combination is their experience. They have seen winning numbers and the ones they have seen are not the sequence 1 to 6. Their experience is a key component of their decision and it cannot be denied. The chance of any combination of 6 numbers chosen is one in 14 million so practically every combination is rare even though some combination is selected. At this level it is hard to reconcile these seemingly dissonant ideas. With S6 in the following excerpt, I attempt to address this but with little success.

> *S6: Because there are 49 numbers and they are drawn randomly and there are usually some up in the forties and thirties.*
>
> *I: Ah, usually, by what you have seen.... But if you had a machine that did nothing, nothing but just threw up six numbers at a time, do you think this would occur any more times than that one?.... I think you're thinking that if you look at what the winning numbers, you kind of see that some of the numbers are higher*
>
> *S6: Yes...*
>
> *I: But if you were to think over a long, long period of time and you just had a machine and all it did was just throw six numbers all the time, would you see some more often than others?*
>
> *S6: Ummm, ummm, shrugs slightly*
>
> *I: Yes, it's hard to imagine, but you see that's the thing about probability, you can't really see it in a small number of trials, can you? Like when you toss a coin six times you can't really see all those possibilities but you know that's what they are. (Buzzer for lunch) All right, thank you so much.*

Student S6 does not really buy my argument of 'a long, long period'. She appears to be thinking but her response is that of a small shrug and while I go on about imagining and possibilities the buzzer goes and we are out of time.

Another probe of the meaning of random consisted of exploring the everyday notion of probability of rain tomorrow in the interviews. As stated above, many of the students take this notion for granted and find it hard to articulate. In interview with student S2, there is evidence of the equiprobability bias in this understanding as well as some individual thinking:

> *I: So in general if they say on the radio or TV, there's a 60% chance of rain tomorrow, what does that mean to you?*

S2: It might or it might not.

I: Does it mean that it will rain 60% of the day or does it mean it will rain on 60% of the area? What does that mean, 60% chance of rain?

S2: It might rain.

I: So would you take your umbrella?

S2: For me, if I go outside and look at the sky, I will decide...

I: The numbers don't help you?

S2: They don't help much because if I watch the TV, it might not be true.

The normative explanation of '60% percent chance of rain' tomorrow is that it will rain in about 60% of days in a sequence of days that are like tomorrow. Of course this normative explanation is hardly satisfactory and definitely not the answer I am expecting. It underscores the difficulty with probability and the confidence one places on a process that gives you information about one future event based on a series of past events. Student S2 is right in trusting her own instincts and the evidence from looking at the sky but it is desirable that students have some idea of the meaning of the quantitative way in which the uncertainty of the situation is captured. Student S18 captures the notions well with correct language and a matter-of-fact approach.

I: So when it says on the radio or the TV that there's a 60% chance of rain tomorrow, what does that mean to you?

S18: A more chance of it raining and a 40% chance of not raining.

I: Does it matter whether it is 60% or 80% or 90%. What do those numbers mean to you?

S18: It's closer to 100% which makes it more likely.

I: So when will you take your umbrella?

S18: Well, if it's over 50%, I'd say I would take it, 50% is either way, and more than 50% it is more likely.

I: So say you are tossing a die and you are hoping to get a six and you get something other than your prediction, what does that say to you?

S18: It's obvious that you are not definitely going to be right, there are six possible outcomes.

It is to be noted that Student 18 clearly demonstrates a normative understanding of the meaning of a probability in this situation, but in the sample space task with 2 coins in interview his understanding was not so clearly articulated. This attests to the fact that task and settings play a role in probing for probability understandings.

The final excerpt with this task shows a student, S19, who has an extremely healthy dose of skepticism and is included here to show the other end of the spectrum.

I: Say you hear on the radio that there's a 60% percent chance that it will rain tomorrow. What does that mean to you?

S19: Nothing. 'Cos I was reading this science book and it says every year British Columbia, Canada has the same chance as Washington, DC of having an earthquake which is 13% and I said that to the teacher and she said, oh they are pinpointing it to the exact number...

I: Where would they get 13%? Do you know how they would get 13%?

S19: No...

I: The way the weather forecasters do it, they round the number so they might say 30% chance, 60% chance, 80% chance, so you're saying when they say 60% chance it doesn't mean anything to you? What if they said there's an 80% chance that it will rain tomorrow?

S19: It still wouldn't mean anything unless of course they said 100% of it being clear. If they said 40% of it being clear I'll just wait for some other day for it to be clear because there is still that 60% chance for it to be raining. I would only believe something, like the weather forecast, I would only believe it if it was a hundred percent. Like they'd say, it's going to rain today and (sarcastically) like five minutes later the sun is shining.

I: So it doesn't really help unless it's a hundred percent?

S19: Yeah.

On rereading the transcript I realized that I have missed an excellent opportunity given to me by Student S19 to explore how a figure like 13% for the probability of an earthquake is obtained. At the time, I had to weigh all the factors of time and tasks allotted for the interview and the length of a possible explanation and to decide which way the interview would unfold as I wanted the students to do most of the speaking in our conversations. This reminded me of the need in all good instruction and interaction of keeping my wits about me. Also S19's insistence on a hundred percent (*I would only believe it if it was a hundred percent)* is reminiscent of a student quoted in Falk and Konold (1992, p. 155): "I don't believe in probability, because even if there is a 20% chance, it could happen. Even 1%, it could happen. I don't believe in probability". These are examples of ambiguity intolerance where a person decides to simply discount any attempts to come to terms with or to make meaning of situations of partial information.

4.2.4 Independent Trials

The well-known and much-used 5th flip task was chosen to explore the notion of independent trials. The task on the pre-test was stated as follows:

> 5. Suppose that you are flipping a coin repeatedly and that you get 4 heads in a row. On the 5th flip is a head more likely than a tail? Explain.

The wording of this task has to be carefully noted as variations of this task have been used in the literature in probing related probability ideas. The classic gambler's fallacy is that a tail is more likely than a head on the 5th flip. Often, there is a consideration of the string of heads that has already happened as seen in *S17: I would think you would get tails because with already 4 times in a row the chances get lower every time you flip*) as well as a consideration of outcomes that come after as in *S7: Tails, because the head will eventually lose.* This response is quite sophisticated because of her use of 'eventually' indicating the sequence of events that unfold. Fischbein & Schnarch (1997) note that both positive and negative recency effects are given as responses to this question, meaning that arguments are given for each of head (positive recency) and tail (negative recency).

The strategies used in the responses in this study included normative thinking, causal deterministic thinking (the outcome can be manipulated, in this case with a flipping strategy), an appeal to a 50-50 chance, the outcome approach, and combinations of two or more of these. Again it was difficult to separate these as can be seen from the following examples:

Causal and 50-50 chance

S11 (pre-test): It is a 50-50 chance because there are two sides on a coin and each side is supposed to have an equal weight so it just depends how you flip the coin.

Causal, Positive Recency, and Outcome Approach

S18 (interview):I think I would get a head because it's probably the way you are flipping it and if you already have it four times, it's likely that it is not a coincidence. You're likely to get another one, probably because of the way you are tossing it.

I: Ah-huh, but what if it were completely random each time, you don't think it is possible to get four heads in a row?

S18: If it was completely random, it could be either way.

Outcome Approach

S3 (pre-test): You could get any of them because they are random.

S9 (pre-test):Well I suppose it would land on a tails since I've only seen someone get heads 5 times or higher in a row. And I've learned that the two would balance out if you did it over 100 times.

This response suggests an understanding of getting 5 heads in a row as a rare event as well as the idea that in the long run with a fair coin the proportions of heads and tails approach ½.

Sometimes the reasoning presented contains elements of both normative and nonstandard reasoning as in *S5 (pre-test): Tails, I think. If you got 4 heads in a row that's pretty lucky, sooner or later there's going to be a tail. It's 50-50'.* The 50-50 refers to the chances of head and tail for each flip but the 'sooner or later there's going to be a tail' refers to the gambler's fallacy. A variation of this is *S1 (pre-test): Tail because the first chance was 50-50 and after 4 heads in a row it lowers the possibility of getting another head.*

It is to be noted that the wording of the task in this study says 'which are you *more likely* to get (emphasis added)? Students often hear that as 'which will you get?' and thus reason according to the outcome approach. The variety of responses to this question suggests that these kinds of questions contain nuances that do not quite yet make an impression on the students at this level.

While the 5^{th} flip task was chosen to probe for understanding of independent trials, it is clear that many of the students do not have the realization that the key factor is here is that the notion of independent trials, that is, that the outcome of any trial is independent of the outcome of any other trial. This key factor eludes many people when they consider this task, even the ones who have had some exposure to probability. I probed a related notion by posing the following 'In a row or all at once' task on the post-test:

> Suppose that in playing with dice, you want to obtain three sixes. Do you have a better chance of getting three sixes by tossing a single die three times in a row or by tossing three dice at the same time? Explain.

Just under half of the students (eleven) answered correctly that it does not matter which procedure is used but they often had an incorrect appeal to formal probability theory as in *S14: No, all you are doing by tossing the three dice at the same time is speeding up the process of obtaining your* *data. There is still an even chance of getting it if you roll one die three times.* Others expressed a personal reflection:

> *S29: A single die three times. In my mind it's more precise and seems to have a better chance of getting three in a row.*
>
> *S7: All at the same time because you would have the same power over all the dice, so there's a good chance they would out the same number.*
>
> *S2: I think 3 times in a row. Because if we tossed a dice and it landed on six, we could try repeating the same movement to get six, again, but if we threw it all together and at the same time, it would be in random numbers.*

It can be seen from this that many students do not hold a deep understanding of independence of trials.

4.3 FINDINGS FROM THE LESSONS

The instructional unit of four lessons was intended to give the students some exposure to the basic ideas and to engage them in probability activities that would be interest them and stimulate their thinking. The topics covered in the unit were the language of probability (such as probable, possible and certain), sample space for a two-stage experiment, the idea of 'observed' and 'expected', and the average.

The idea of 'observed' and 'expected' are not ones that are commonly used in this level but to my mind they are critical. This will be explored further in Chapter 5. In this lesson I used the BinStat manipulative where marbles are dropped from a height so that they run along tracks that branch off either to the left or to the right. This left-right action was simulated using a die with the outcomes of the trials recorded on an activity sheet.

If you get an ODD number go LEFT Even number go RIGHT

Toss the die 3 times and note what you get. Trace the path and note where you land. So for example, if you get OEO you will end up at B. Do this 7 more times (eight times in all) and note how many times you end up at each of A, B, C, D.

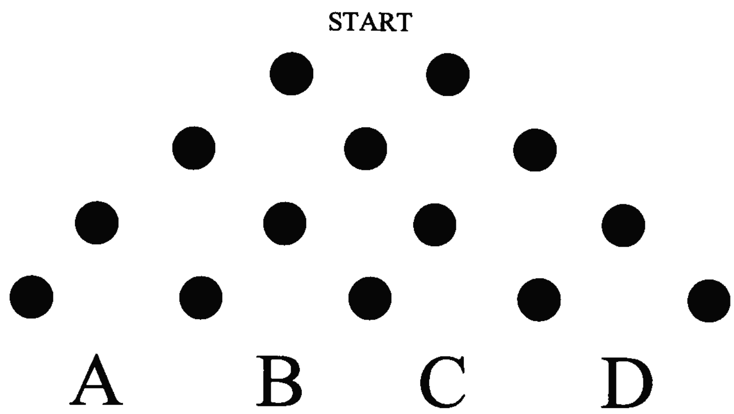


Fig. 4.3 Observed and Expected Activity

The notion of sample space in a two-stage experiment was explored in an activity with spinner sums and included a working out of the distribution of the sums with the corresponding probabilities.

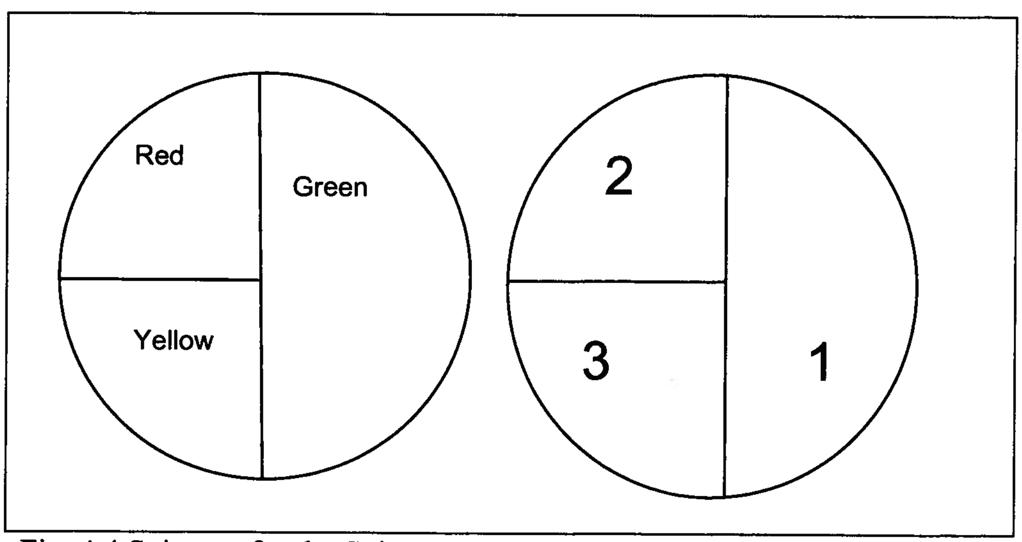


Fig. 4.4 Spinners for the Spinner Sums Activity

An important consideration for me as researcher and co-teacher was the inclusion in the lessons of an activity that had a cultural component through which I could highlight a probability idea. In the second lesson, I introduced a hiding game that was common to many of the Aboriginal groups. It was also a game that I played as a child in Trinidad and I expect is common in many other countries.

I began the activity by speaking about the tradition and significance of Aboriginal gambling games and the occasions at which they played such as traditional ceremonies, redistribution of property, and diversions while awaiting the outcome of a hunt. I explained from my reading that there were mainly two types of games – a tossing game of 'dice' in a basket or a hiding game. In the variation of the game I presented, the game is played in pairs; one person with both hands behind the back, hides a seed or a counter in one hand and presents two fists to the other player. The other player then points to the hand in which he or she thinks it is hidden. The first player continues hiding until the other player finds it.

For the seed or counter in the game, in a gesture of reciprocity, I used seeds that I had collected and brought from Mayaro beach, one of the beaches on the south-east coast of Trinidad. I shared with the students that collecting these seeds (also called donkeys' eyes because of the characteristic two black bands across the brown seeds) was a common beachcombing activity for this beach much as collecting shells. I also shared the story I was told as a child that these seeds are washed up on the beach by ocean currents from the Orinoco river and that while I knew as a child that Trinidad is an island on the edge of South America and the Orinoco river starts in South America, every time I picked up a seed and held it in my hand, I was struck by the wonder of a seed being washed up all the way to my island from South America.

In each pair, each player would play the game 5 times as the hider and keep a record of how many tries it took for the seed or counter to be found. This game was taken up by the class with great enthusiasm and pleasure. As the results were collected and put on the board, it was evident that the number of tries ranged from 1 to 4. There was much discussion of strategies in being a hider and a guesser and in being able to read and thus outwit one's opponent. Everyone joined in as the data was put into a frequency table and then displayed as a line diagram. There was more discussion on how the data could be summarized. The average number of tries turned out to be 1.41 which prompted further discussion on the meaningfulness of such a number when the number of tries is a whole number. A comparison was made to the statistic of 2.2 children per Canadian family.

The lessons were well-received with enthusiasm and generated much interest and discussion [about mathematics or about culture or about both?].

4.4 SUMMARY OF RESULTS

An analysis of the pre- and post-test responses indicates a good grasp of the basic concepts of probability demonstrated by the top half of the class. The two tests were comparable, with a mix of numerical and qualitative questions that covered basic concepts such as probability of an event as a ratio/fraction/proportion, geometric probability as the ratio of two areas (illustrated by the spinner task with contiguous and non-contiguous dotted sectors), the meaning of random and randomly drawn, misconceptions such as the gambler's fallacy, and sample-space in one- and two-stage experiments. Table 4.1 gives a summary of the pre-test numerical responses.

	Probability of drawing a red marble from a bag of red, white and blue marbles	**Probability of drawing a blue or white marble from a bag of red, white and blue marbles**	**Prob. of drawing a chip with number less than 5 from a bag with chips numbered 1-10**	**Prob. of drawing a chip with number 12 from a bag with chips numbered 1-10**
No response	0	0	0	0
Non-normative response	7 (29%)	9 (38%)	14 (58%)	3 (13%)
Non-normative response; Elements of normative thinking in response	2 (8%)	6 (25%)	0	1 (4%)
Normative response	15 (63%)	9 (38%)	10 (42%)	20 (83%)

Table 4.1 Summary of Pre-test Numerical Responses (n = 24)

A comparable summary for the post-test numerical questions is given in Table 4.2. The results here reinforce the conclusion of solid normative knowledge for the top half of the class.

	Probability of drawing a marble that is not blue from a bag of red, white and blue marbles	**Probability of drawing a blue or white marble from a bag of red, white and blue marbles**	**Listing the sample space when 2-sided counter (red, green) is tossed twice**	**Probability of getting green both times when 2-sided counter (red, green) is tossed twice**	**Expected number of raisons in sample of trail mix**	**Expected number of seeds (sunflower and pumpkin) in a sample of trail mix**
No response	0	0	0	0	1(4%)	3 (13%)
Non-normative response	6 (25%)	9(38%)	7 (29%)	11(46%)	10 (42%)	8 (33%)
Non-normative response; Elements of normative thinking in response	0	1(4%)	1(4%)	0	0	0
Normative response	18 (67%)	14 (53%)	16 (67%)	13 (54%)	13(54%)	13 (54%)

Table 4.2 Summary of Post-test Numerical Responses (n = 24)

A comparison of the percentages for pre- and post-tests numerical questions showed that the students performed well in the numerical aspects of probability with some variation in tasks but generally at the same level over both tests.

For the qualitative questions which probed understanding of concepts such as the meaning of random and misconceptions that have been cited in the literature, the students did not fare so well. The summaries for the qualitative questions in the pre- and post-test (including sample responses) are given in Tables 4.3 and 4.4. The sample responses were chosen as illustrative of the types of thinking displayed.

	Spinners	Die instead of coin	5^{th} Flip	Lotto 6/49
No response	2 (8%) Blank, ?	1 (4%) Blank, ?	3 (13%) Blank, ?	5 (21%) Blank, ?
Non-normative response	17 (71%) Spinner A because there is more variety of space. Rather than both on one side	9 (38%) I've got two ways See who gets the highest number. Or whatever team gets a 5.	10 (42%) Tail because the head will eventually lose	10 (42%)
Non-normative response; elements of normative thinking in response	1 (4%) Spinner A because it's equal, there's a fair chance. If spinner B was used and was slowing down on the white it is farther away to the dots than it would be on spinner A. More probability on Spinner A.	0	7 (29%) Tails because you're lucky to get 4 heads in a row. Heads because you might have found some flipping strategy	2 (8%) Me, I choose random numbers (when they show the 6/49 numbers, the numbers are always random numbers)
Normative response; faulty explanation	1 (4%) I don't really know because there is a 50% chance I could win on either. But if I had to choose I would choose spinner B because if it lands on the right hand side ever it would get me a point	0	1 (4%) It is a 50-50 chance because there are two sides on a coin and each side is supposed to have an equal weigh so it just depends how you flip the coin.	4 (17%) None of you because you both have a 6 out of 49 chance of winning unless the quick pick machine cheats you
Normative response; normative explanation	3 (13%) It doesn't matter because on both spinners you have a 50-50 chance of winning	14 (58%) If it lands on an odd number it's one of the teams. If it lands on evens, it's the other team	3 (13%) The fact that you had 4 heads in a row means nothing , so that means you always have a1:2 chance of getting a head or a tail.	3 (13%) Well both of you have the exact same chance that you would win because all you are doing is changing your numbers, it doesn't make a difference in your probability of winning.

Table 4.3 Summary of Pre-test Qualitative Responses (n = 24)

	Spinners	In a row or all at once	5 and 6	Lotto6/49
No response	0	2 (8%) BLANK	4 (17%) BLANK	6 (25%) BLANK
Non-normative response	14 (58%) A. The dotted areas are more spread out, meaning that there are more chances than the other one, since they're not bunched up	11 (46%) A single die as you'll get to roll 3 times as opposed to rolling 3 once	11 (46%) It's a 50/50 chance	6 (25%) Cathy because the numbers are pretty big and random
Non-normative response, Elements of normative thinking in response	3 (13%) A because it is more spread out so you have a better chance of landing on the dots (there is an even chance for both of them but I like this one better)	0	0	0
Normative response; Faulty explanation	1 (4%) I would play with both of them cause they are both are 4/8 and that means they are equal. But if I could spin it as hard or light I would choose spinner B.	7 (29%) It does not matter, they are both 11/6 chances and about 16% for each	6 (24%) I'd say 5 and 6 because having 2 of the same number is even more since it's getting 2 same numbers of 12 possibilities	7 (29%) They each have a ¼ chance because you can't ever know what the numbers are
Normative response; No explanation	0	0	1 (4%) 5 and 6	1 (4%) All are equal
Normative response; Normative explanation	6 (25%) These 2 spinner's probability of landing on the dots are the same, so I'm fine with both of them	4 (17%) It doesn't matter how you throw them, the outcome would still be numbers so neither is more likely to get 3 sixes	2 (8%) The 5/6 because there is a higher probability because 6/6 can only be acquired with both dice 6s and 5/6 can be achieved with green die 5 and die 6 , or the red die 5 and the green die 6	4 (17%) I don't think any of them have a better chance because it is a random draw and they all have the same probability

Table 4.4 Summary of Post-test Qualitative Responses (n = 24)

From these summaries, the normative thinking displayed by about one- third to one-half of the class followed standard formal probability theory and showed that these students were not susceptible to any of the misconceptions such as the gambler's fallacy. However, a majority of the class gave non-normative responses on the qualitative questions. The percentages of non-normative responses have improved somewhat from pre- to post-test but since they account for more than half the class, it was important to further examine the strategies used.

An analysis of the thinking as given by the framework of Jones et al. (1997, 1999) across the four levels of Subjective, Transitional, Informal Quantitative, and Numerical proved unsatisfactory as the class showed evidence of each level across tasks and across settings. For the pre- and post-tests, I began by considering the responses to a particular question and then tried to code them according to one of the four levels. This proved difficult as the responses often contained more than one level of thinking and it was hard to say that a particular response was one level or the other. It was also difficult to say that the thinking was demonstrated by an individual was at one level or the other as the performance was uneven across questions and tasks. The constructs considered in the framework of Jones and colleagues were Sample Space, Experimental probability of an Event, Theoretical Probability of an Event, Probability Comparisons, Conditional Probability and Independence. The last two of these were not considered as the study was designed to probe more basic ideas of probability and it was unrealistic to cover these big ideas when the basic ideas were not well understood or held. Still, it was difficult to match the responses to the levels across the constructs as the responses in this study were not clean or clear-cut.

Overall it was evident that the students had clear and well- formed (in their minds) ideas of probability and they demonstrated an interest and a willingness to engage with the subject area. Many were quick to consider the material that was presented in the unit and it was clear from the interviews that there was a shift in their understandings in that they had acquired more knowledge of probability and had greater understanding that helped them to clear up some of the alternative understandings that they had previously held. One student indicated that on the pre-test she did not know how to calculate a probability but by the time of interview she could carry out the numerical considerations of counting favourable outcomes and total outcomes. Others indicated their growing understanding of the consideration of equal chances and its consequence for choice in a binary task.

In summary, the thinking that was displayed across the four probability constructs considered is consonant with that of the studies in the literature with evidence of degrees of knowledge of formal probability theory and levels of probabilistic reasoning.

1. The students have a fair to excellent idea of the probability of a simple event as a ratio of the number of favourable cases to the number of total cases but experience difficulty in compound events such as the union of two simple events. The students also experience difficulty in delineating the set of favourable cases and the set of total cases when the situation is no longer simple.

2. The students have a fair to excellent knowledge of probability comparisons. Many have difficulty in making a discrimination of the effect or non-effect of equal chances in binary choice situations. This difficulty may be due to the features of the task itself, a consideration which will be taken up later in the Discussion section of this Chapter.

3. The students can distinguish the elements of a sample space in one-stage experiments but the difficulties increase as the experiment becomes more complex.

4. The students have a fragile understanding of the meaning of the words 'random' and 'randomly drawn' and do not fully realize the consequences of what being randomly drawn means.

5. The students used a variety of strategies in their thinking, often with multiple strategies in one response. These included some recognition of the randomness involved, aspects of deterministic thinking in that the outcome can be controlled by external means including themselves as agents, and well-known misconceptions and biases. There were also many references to personal choice and consideration.

6. The students held a number of beliefs and pre-conceptions based on their intuition and experience and displayed a natural curiosity, interest and excitement toward learning and knowledge that augur well for their future development in all subjects.

4.5 DISCUSSION

The range of understandings across probability constructs and across the students in this study very nearly spanned the spectrum from formal probability theory to non-standard reasoning covering a tangle of beliefs and judgments, and well-known misconceptions. Understanding is difficult to see, measure, and assess and the imperfect instruments of a paper-and-pencil pre- and post-test written under 'test' conditions only scratch the surface. The interviews helped considerably in probing the understandings and provided an opportunity to engage with the students on a deeper level. However the conditions of the study, namely the timing at the end of a crowded school year, the scheduling constraints, and the lack of emphasis on the strand of Probability and Statistics only served to limit the time and attention that was spent on the topic of Probability. In the discussion that follows I focus on the challenges faced in the endeavour to discern and to make meaning of the students' understandings of probability.

As I carried out this study and engaged with the students, it was abundantly clear that probability was **not so much a mathematical** concept as a **psychological** one. Fischbein, Nello and Marino (1991) note that: "The psychological aspects of the concept of probability seem to be much more complex, in many respects, than it is usually considered" (p. 547). It was evident in this study that for some students any consideration regarding the probability of an event or a probability-related choice in a task involved feelings and emotions that resulted from the difficulties of 'epistemological anxiety' (Wilensky, 1994) in addressing situations of insufficient and fragmented knowledge. Sometimes these feelings and emotions were played out as hesitation, second-guessing, and vacillation between possible answers. In other cases there was an emotional tendency to reject situations of ambiguity.

Besides the psychological aspect of probability, the nature of the probability tasks and scenarios used at this level presents a challenge. While the applications of probability are indeed important and useful in many branches of knowledge and in everyday situations of managing risk and making decisions, the tasks and scenarios used in probing understanding at this level often use the simplest randomizers (dice, coins, spinners, and bags of marbles) and hence appear to be contrived and artificial. They may even be considered misleading as will be shown in the discussion that follows.

Further, probability is itself a many-faceted concept, the development of which required a long incubation. Many of the tasks by which researchers probe understanding of probability at this level (from previous studies and this study) presuppose an understanding of formal probability theory that is not apparent to an 'untutored mind'. The fundamental notion of probability is deceptively simple but elusive. Metz (1998b) found that 3rd graders and undergraduates had the same difficulties when faced with probability tasks.

One of the difficulties in the notion of the probability of an event is highlighted in the bags of candy task. To be fair to the student we have to consider what we are asking the student to do in deciding between the two bags when the probability of the target event (red candy) is the same in both bags. The student has to calculate the probabilities of each, note that they are the same and then, most importantly, decide **if that consideration of equal probability makes a difference**. Saying that the probability of getting a red candy is 40% means that if we were to draw a candy many times with replacement, then the proportion of draws on which we get a red candy will be close to 40%, will vary around 40% and will approach 0.4 as the number of draws increases. This is the Law of Large Numbers but it applies to a long sequence of draws with the **actual** or **observed** number of successes varying around the **expected** number of successes over the long run. To again be fair to the student, when one is faced with the immediacy of a choice and a single trial, the Law of Large Numbers is helpful only in so far as to lead one to the conclusion that the chances are the same and the proportion of time that you will get a red candy in the long run is about 40%. Asking for a choice in a single trial is then perhaps not a fair question if we are trying to elicit knowledge of the long-run notion of probability. For the student, it is most likely that is it simply a matter of 'taking one's chances' in that one trial. By posing this task as is commonly done and as I have used it, I have added to the difficulty by not making explicit the tacit consideration of the one trial versus the long run. While some students experienced no difficulty and saw right away that the changes in the numbers of target candies and total candies did not affect the probability of the event and hence that neither bag offered an advantage, this degree of sophisticated thinking required here was hard for many to accept.

A second level of difficulty is the hidden layer of complexity which occurs in the spinner task which has a deceptively simple appearance. The difficulties that are evident in the spinner task are generally considered to be related to the distraction of contiguous and non-contiguous dotted sectors (Green, 1983; Jones et al., 1997; Nicolson, 2004; this study). However, these difficulties are further compounded by the fact that the question while still probing for an understanding of probability as a ratio has an increased level of complexity from the questions that have discrete settings such as the bags of candy task in that the student is being asked to consider probability as a **ratio of two areas** as opposed to a **ratio of two counts (which are whole numbers)**. While it may seem as only a matter of discriminating between areas that are 'all in one piece' and areas that are separated or 'spread out', there is a subtle but important jump in the level of complexity of the task in that the situation is no longer discrete as in the bags of candies task but now continuous over an area. If one tries to argue using the ratio of favourable cases to the total number of cases, then one has to consider the possible positions of the needle. The pointer of the needle can stop at any of infinitely many positions along the circumference of the circle. Further this infinity is not countable but that of the continuum which requires a greater level of abstraction. This task requires that we consider the probability as the **ratio of the lengths of two intervals** (or **two areas**). Students are mostly unaware of this distinction of discrete and continuous and it is evident that an appeal to equal chances or an appeal to probability as a ratio of the number of favourable cases to the number of total cases is insufficient to help them come to a response that is deemed correct by formal probability theory.

A third area of difficulty concerns the tacit understanding among those who know formal probability theory of what distinguishes one outcome from another in sample space considerations. For example, in enumerating the sample space for tossing 2 coins, the understanding that HT and TH are two different outcomes does not come naturally. Children and adults new to probability often see this as one outcome of a head and a tail. This occurred many times in the course of the study and the correct enumeration had to be pointed out each time. Similar discussion arises in enumerating the sample space in tossing two dice in that students see two sixes as one outcome and a 5 and a 6 as another and then assigning the same probability to each resulting in the equiprobability bias.

Finally, there is a difficulty in abstracting the underlying mathematical structure from the task embodiment. The common randomizers of coins, dice and spinners are the simplest embodiment of equiprobable outcomes but they present difficulties in that they are deceptively so. The probability of an event is a mathematical realization of an observable phenomenon, the relative frequency of the event, and the terminology of **theoretical** and **experimental probability** in the textbooks and PLOs adds to the confusion. My unease with these terms lies in the fact there is only one probability for an event but the relative frequencies that estimate that probability change according to the number of trials of the experiment. Hence I do not agree with the juxtaposition of the two words, ‘experimental’ and ‘probability’. I consider it misleading to speak of a probability associated with an experiment since the concept of a probability is defined with respect to an event. As an example, consider the probability of getting a six in a single toss of a fair die. This probability is agreed upon by an appeal to symmetry to be 1/6. If we were to carry out an experiment of tossing the die 100 times, we may get, perhaps, 18 sixes in the 100 tosses giving a relative frequency of 0.18. If we were to carry out the experiment 200 times we may get 35 sixes giving a relative frequency of 0.175. I consider it inaccurate to call each of the 0.18 and the 0.175 an experimental probability as it obscures the idea of ‘the’ probability of the event.

It is to be noted that the difficulties that students experience in probability are not insurmountable and can be addressed but this requires more time and attention to the basics of the subject than is currently given.

In the next and final Chapter, I state my conclusions from the study and give recommendations for research, teaching and curriculum.

CHAPTER 5. CONCLUSION, IMPLICATIONS AND RECOMMENDATIONS

In this chapter, I reflect on the research questions and propose a model to explain the thinking displayed that could not be explained by the strategies that have been given in the literature. I discuss the implications and recommendations of the findings for teaching and curriculum and highlight some important considerations that arose in the course of my probing for understanding in this area. I conclude the thesis with recommendations for research.

5.1 CONCLUSION

The purpose of this research was to see for myself how a subject as nuanced and sophisticated as probability is negotiated at the elementary school level especially as it proves to be difficult at the college level. The results of the study as presented in the previous chapter point to compelling evidence that Grade 7 students while interested in and familiar to an extent with notions of probability experience much difficulty and confusion with probability and probability ideas.

At the beginning of my study, my research questions were:

RQ1: What understandings of probability do Grade 7 students demonstrate before instruction in probability?

RQ2: How are these understandings affected during and after an instructional unit in probability?

RQ3: What instructional models can be proposed to enhance deeper understandings of probability?

The first research question was answered in the baseline of understandings that was established from the written responses of the pre-test. It was found that students were interested in the subject and were eager to engage in the challenges of probability. They demonstrated fair to excellent normative understandings across the basic probability constructs of probability as a ration and sample space and demonstrated fragile understanding of the concepts of random, drawn randomly, and independent trials, often with a causal deterministic outlook and holding alternative understandings. These findings are consistent with those of earlier studies such as Green (1983), Jones et al. (1997), and Shaughnessy (1992).

For the second question, it was found on the whole that exposure to probability ideas during the instruction improved their understandings of the basic ideas but that alternative understandings persisted to a large extent. The improvement as seen in the interviews and post-test was not uniform across individuals and tasks and it was difficult to separate the mix of strategies used in reasoning about situations that involved uncertainty. A qualitative analysis of the strategies used in the responses from the various data sources suggests that the strategies outlined in the literature do not fully account for the kinds of non-standard thinking displayed in the study. I propose a model of **personal probability** that explains the predominantly subjective thinking which overrides other factors including elements of normative thinking.

5.1.1 The Personal Probability Model

The analysis of strategies in Chapter 4 revealed that the alternative thinking demonstrated by students could not be neatly explained by the strategies in the informal conceptions and misconceptions literature. A Personal Probability model is proposed in order to describe the subjective thinking displayed in the responses pertaining to uncertainty that do not adhere to formal probability theory or to the already-noted heuristics and biases. This model is used in situations where a determination of probability or a decision based on such determination must be made. It is also used when probabilities are to be calculated and compared as in choice situations.

This model does not address a specific bias or fallacy and does not suggest a particular approach such as the outcome approach. It is a consequence of the recognition that students at this level when faced with chance situations draw on many different considerations in order to arrive at a response. These considerations include: Some normative thinking about probability, Experience, Intuition and Judgment, Expectation, and Personality factors.

Fig. 5.1 The Personal Probability Model

Experience: In assessing a probability or a probability situation, the person draws on his or her own life experience, interactions with others, everyday experiences, and exposure to the media about the situation under consideration. The person is deeply influenced by the knowledge and meaning making he or she has gained by his or her own senses, by what has been seen, heard, and understood.

Intuition and Judgment: The person is guided by degrees of intuition and natural intuitive assessments. He or she may operate from cultural beliefs, hunches, maxims and other popular wisdom such as proverbs and sayings. While the person recognizes the chance aspect of the situation, he or she may hold different beliefs about other aspects such as a causal or deterministic outlook. Further the person may use analogies and similar structures to make sense of the situation.

Expectation: The person has some idea of a possible pay-off and the value with which the pay-off is held to the self and to others whom he or she deems important. This possible pay-off and its attached value influence a probability determination or a decision relating to probability.

Some consideration of normative thinking about probability: The person has some idea of formal probability theory and can make a determination of a probability in a simple situation using one of the 3 approaches to probability including a subjective probability estimate. There is also some understanding of the more complex situations but this falls off as the degree of complexity increases.

Personality factors: The person is guided by personality factors such as a conservative attitude or a propensity for risk and makes a decision based on personal choice and beliefs. There is also some appeal to feelings and emotions that arise out of the difficulty of dealing with ambiguity and uncertainty.

A person using the Personal Probability model in chance situations holds all these aspects even when conflicting or inconsistent, gives a personal weighting usually with personality factors paramount and arrives at an answer that may not be correct but definitely fits with that person's view of the world and his or her place in it. The thinking does not unfold in a linear fashion but frequently goes back and forth, winding and looping back, touching on various considerations, first taking up one and then discarding it for another, all in search of an answer that fits with the understanding and sense making of the situation.

In the case of the students in this study, it was clear that much of the alternative thinking around uncertainty and probability can be explained by the model and the various factors of the model. Besides the examples of personal choice, reflection and consideration given in the presentation of results in Chapter 4, this model is evident in many of the students' responses across tasks. Further examples are:

The 5th flip task – Experience and Some normative thinking: '*Tail. There is a 90% chance that you can get tails at 5th flip (I flipped my coin 3 times and 4th one was different).*

The 5 and a 6 or two sixes task – Experience and Intuition and Judgment: '*5 and 6 because everything is more likely to be random. But if you get 6 and 6, it would be too lucky. But in my opinion, I think people would get 5 and 6 because I always get random numbers when I toss dice'.*

The 'In a row or all at once' (Do you have a better chance of getting three sixes by tossing a single die three times in a row or by tossing three dice at the same time?) task – Intuition and Judgment: '*I think that rolling 1 die at a time would be easier but it really depends on if you have a way of rolling dice or not*' and '*A single die three times. In my mind it's more precise and seems to have a better chance of getting three in a row'.*

The spinner task – Some normative thinking and Personality factors: '*I know they have the same chances, I just like this one better'.*

5.1.2 Further Key Findings

The key findings of the study are that the students have some understanding of formal probability theory with strongly-held persistent misconceptions as outlined in the literature such as the outcome approach and the gambler's fallacy. There was a prevalence of subjective thinking as personal preference displayed across basic probability constructs. Though these alternative understandings were explored in interviews and resolved to some degree, the study provides evidence that the probability understandings of students in the elementary grades are still fragile and inconsistent.

For the third research question, the instructional model dictated by the perspective of culturally responsive pedagogy of attending to the culture and the community of the students proved to be invaluable to the success of the instruction. Any attempt to understand how children reason must focus on the children, their background and their world view. The awareness of the diversity of today's classrooms has resulted in educational initiatives that recognize the components of the child's world of home, school and community and places the child at the centre of that triad. Due recognition and the respect of cultures, their capital and values have resulted in culturally responsive education that emphasizes continuing attention to the role these play in the development of motivation and self-efficacy in children towards the larger goal of preparation for taking their place as citizens in a wider world.

Culturally responsive initiatives in mathematics and science education include Math in a Cultural Context developed with Alaska Yup'ik elders and teachers (Lipka et al., 2005), various research institutes such as the National Center for Culturally Responsive Education Systems with a focus on native students in the US (Nelson-Barber & Estrin, 1995; Tambe, Carroll, Mitchell, Lopez, Horsch & St. John, 2007), and TEAM-Learning in rural and urban schools in BC (Nicol et al., 2006). Other innovations include Civil's (1998) bridging of in-school mathematics and out-of-school mathematics and Moll and colleagues' (1992) work on 'strategically' connecting homes and classrooms in order to access 'children's funds of knowledge'.

It was clear from the instructional experience that students respond well to activities that are engaging and relevant to their interests, and with which they can identify. This type of culturally responsive activity lays the foundation for continued interest in learning in general and in learning mathematics in particular. Teachers and other players in the educational endeavour can ensure that mathematics is indeed for all (Davis, 2001; Gates & Vistro-Yo, 2003), by responding to all aspects of the children who make up today's classrooms and teaching in culturally responsive ways.

Marker (2006) sounds a deeper warning of the complexity of the limits of the culturally responsive endeavour: "It is futile to promote cultural responsiveness in the classroom for indigenous students without a genuinely disruptive approach to the underlying Cartesian beliefs in dominant educational assumptions" (p.506). Marker's comment, while a little unsettling to anyone who has taken to heart the aims and ideals of culturally responsive pedagogy, is a political one and speaks to the underlying social and political tensions in an educational system that reflect the inequities and power imbalances of the society as whole. The question of how a history of oppression can be righted is complex and mostly political in that educators working individually and in teams can make some difference by means such as culturally responsive pedagogy but political will has to be demonstrated across the larger society in its educational and other social institutions.

I understand Marker's use of 'Cartesian' in the sense of Descartes' pronouncement of "I think; therefore I am" and his "underlying Cartesian beliefs in the dominant educational assumptions" as referring to the existent beliefs at the heart of schools as social and moral institutions. Marker is addressing the complex and knotty question of what pedagogy is best for Indigenous students, a question that is reminiscent to me of the debate in Toronto of the establishment of Afrocentric schools or even the debate of whether girls do better in mathematics in same-gender classrooms. As for the establishment of Afrocentric schools, there are Black educators on either side of the debate. I do not advocate the separation of schools because as a product of multicultural schooling in Trinidad, I have seen and experienced the virtues of recognizing, honouring, and learning from all cultures including one's own. From this study it is clear that students are more interested and engaged when they can see expressions of themselves in all aspects of their education including the school and the experiences and environment it provides.

While Marker's statement merits further thinking and exploration of its implications and consequences intended and unintended, the impressive efforts and gains made by notable North American educators in this area cannot be denied.

5.2 IMPLICATIONS AND RECOMMENDATIONS

As I see teaching and curriculum as inextricably linked I begin with the implications and recommendations for teaching. Next, in the implications and recommendations for curriculum, I propose a curriculum for Grade 7 Probability and continue with recommendations for teaching the important ideas in the proposed curriculum. I then consider some important factors relating to discerning and assessing understanding of probability and conclude the thesis with recommendations for further research.

5.2.1 Implications and Recommendations for Teaching

The findings of this study indicate that effective teaching of probability at this grade level requires more time and attention than is presently given. There are several factors that have to be addressed. The first is that the wealth of alternative preconceptions of probability and reasoning strategies under uncertainty as seen in this study has to be attended to before the ideas of formal probability theory can take hold. The students in this study demonstrated considerable resilience in their conceptions and reasoning strategies. Teachers of probability have to be able to recognize and address these alternative conceptions and strategies even as they attempt to advance the ideas of formal probability theory as more coherent and more logical. There is considerable pedagogic value in this approach as it is only by addressing and building on students' preconception can a space be created for students to shape and reshape their knowledge.

A second factor is the way that the teaching of probability unfolds at this level. Probability, if taught at all, is usually relegated to the last few weeks of the school year and is often taught as an abstract isolated topic. I recommend that the teaching of probability be integrated with the teaching of the other strands of mathematics from the beginning and throughout the year and that the topic be introduced in the context of data analysis so that the natural variation that occurs in data from random phenomena can be studied and used to hone students' perceptions of randomness and probability. One example might be the setting up of a station in the classroom where data can be collected periodically on the weather or the outcome of a sports activity or where data can be generated by a randomizer such as a fortune wheel or the simulation of a random walk. In this way, students can become more familiar with random phenomena and begin to use the tools of data analysis such as the calculation of percentages, relative frequencies, and averages in order to think about probability ideas such as probability as a ratio and the notion of sample space.

A third factor is that an adequate treatment of probability in the elementary grades requires a curriculum that has realistic learning outcomes appropriate to the level (this will be taken up in the next section). It also requires teacher preparedness to teach the curriculum. Shulman (1986) outlines three categories of content knowledge for teaching: subject matter content knowledge, pedagogical content knowledge and curricular knowledge. In the strand of data analysis and probability, teachers are often under-prepared in these areas resulting in a lack of confidence or enthusiasm for the strand. This consideration in combination with time constraints often results in the topics being left out altogether in the elementary grades. Further as Wood (2001) notes with regard to teaching mathematics, "[a]lthough it is important in teaching to know the content of the mathematics, what is more important is know the points at which children's current conceptions or theories about how mathematics works conflict with mathematical thought" (p. 114).

A final factor is the special nature of the subject. Fischbein and Schnarch (1997) state that "[p]robability does not consist of mere technical information and procedures leading to solutions. Rather, it requires a way of thinking that is genuinely different from that required by most school mathematics. In learning probability, students must create new intuitions. Instruction can lead students to actively experience the conflicts between their primary intuitive schemata and the particular types of reasoning specific to stochastic situations. If students can learn to analyze the causes of these conflicts and mistakes, they may be able to overcome them and attain a genuine probabilistic way of thinking" (p. 104). Teachers of probability have to carefully stake a path through the thicket of 'conflicts' and 'mistakes' in leading their students to a 'genuine probabilistic way of thinking'. The discussion of the challenges in Chapter 1 of this thesis combined with the evidence of the results and findings in this study on students' understandings indicate that achievement of a 'genuine probabilistic way of thinking' can be accomplished but only by adequate time and attention to addressing instruction in the basics of probability.

As I ponder the implications for teaching after having carried out this study, I must state that teaching probability per se does not take special skills. The subject poses its own challenges but it is of a piece with teaching mathematics and teaching in general. I agree with Confrey (1990) as she states her intent in teaching beyond the 'mathematical structures': " ... [W]hen I teach mathematics I am not teaching students about the mathematical structures which underlie objects in the world; I am teaching them how to develop their cognition, how to see the world through a set of quantitative lenses which I believe provide a powerful way of making sense of the world, how to reflect on those lenses to create more and more powerful lenses and how to appreciate the role these lenses play in the development of their culture. **I am trying to teach them to use one tool of the intellect, mathematics** (my emphasis)" (p. 110).

5.2.2 Implications and Recommendations for Curriculum

The findings of this study indicate that the Prescribed Learning Outcomes (PLOs) in probability at the elementary level are too ambitious, that students are best served by deferring probability until high school, and at the very least that the first exposure to probability in the elementary grades be at Grade 7. Bruner's claim that 'anyone can be taught anything so long as it is done honestly' has to be re-examined in the light of the special nature of the subject of probability. Despite the claims in the research literature, from my experience the acquisition of some areas of mathematics unfolds developmentally. I believe that a problem can only be addressed within the constraints of the situation in which it is posed so that after this study I see it as idealistic to expect that at the school level there is enough time, effort, and required mathematical proficiency that will ensure that students acquire an understanding of probability that will be useful in dealing with randomness and variation in their lives.

The special nature of the concept of probability further dictates that any introduction to the subject must address the philosophical underpinnings that lie at the heart of the difficulties in considerations of pedagogy and curriculum in this area. While there is an axiomatic theory of probability, the fact that there are three approaches to the assignment of the probability of an event (classical, frequentist, and subjective) with a variety of situations in which they can be used adds a level of complexity to the determination and interpretation of the probability of an event. Any introduction to the subject must begin with a discussion of the multi-faceted nature of the concept, the long history of the development of the concept, and the variety of interpretations that can be made depending on the situations. Further it is crucial that such discussion be placed in context of everyday and relevant situations so as to strengthen the understandings of the notions of chance, probability, variation, and randomness.

Because of the difficulties experienced by the students of this study, I recommend that the probability ideas in Grade 7 be kept to the basics with a rudimentary treatment of independence and the omission of conditional probability. David Moore, the eminent statistician who has led the reform movement in statistics education, has given a core of probability basics (Moore, 1990, p. 120-121) and regards the ideas of independence and conditional probability intermediate (p. 122). As a deep understanding of the basics requires much time and attention, the following basic ideas are to be recommended for Probability in Grade 7.

Proposed Probability in Grade 7

Vocabulary

Certain, uncertain, sure, unsure, possible, impossible, probable, improbable, odds (later), chance, dicey, 50-50 chance, even chance, 'invented language' (Jones et al., 1997) such as 1 in 4 chance

The range (continuum) of probability from 0 to 1, expressed as a fraction, decimal or percentage

Events, rare events, frequent events

Outcome, collection of all possible outcomes, actual or observed outcome versus the predicted or expected outcome

Probability Ideas

Frequency, relative frequency, probability, distribution of outcomes, comparison of probabilities, sample space of one- and two-stage experiments

The Vocabulary topics are important in that they clarify the **language** of probability. Many probability terms are part of today's vernacular and it is important that they be place in the continuum from 0 to 1 or 0 to 100 percent.

The key notions for me are that of distinguishing the **actual** or **observed** outcome from the **predicted** or **expected** and the idea of a **distribution** of the range of possible outcomes, some more likely than others. Further to these are the notions of **frequency** and **relative frequency** which capture the idea of **estimates** of the probability of an event, the estimates becoming more accurate as the number of trials increases. While it is important for students to play games and carry out chance experiments in order to appreciate randomness, variation and chance outcomes, it is important that the students keep in mind that the sequence of outcomes and resulting **relative frequencies** only point to the probability which is an idealization. This is the basis to my discomfort with the terms, theoretical and experimental probability, which seem to be entrenched in Math for Elementary Teachers textbooks, and Math textbooks, Teacher Resources and Curriculum documents at this level.

It is also important to realize that chance experiments are intrinsically different from physical experiments. The POE (Predict, Observe, and Explain) method of conceptual change theory (White and Gunstone, 1992) does not apply. Take the example of a physical experiment where two buckets of sand are suspended, one at each end of a rope from a pulley. Students are asked to predict what would happen if two teaspoonfuls of sand are added to one of the buckets. Then the action is carried out, students observe the resulting effect and are then encouraged to explain what has happened and to reconcile their prediction with the explanation. Such an experiment is deterministic with the outcome determined by physical forces and as such the outcome will be mostly the same with each repetition. The outcomes are generally different in repetitions of chance experiments and as such a particular outcome only gives information about that **realization** of the experiment. Thinking about a prediction is useful in so far as it makes one think of the set of possible outcomes. Instruction during chance experiments should focus not on prediction but on **observing the outcome**, **recording the sequence**, **making a distribution (with graphical display)** and then **making an inference about the probability of the event.**

5.3 FACTORS IN CONSIDERING UNDERSTANDING OF PROBABILITY IDEAS

From my experience of carrying out this study and subsequent reflections, there are a number of factors relating to the discerning and assessing of the understanding of probability ideas that are important to list as they will be valuable to future researchers in this area.

5.3.1 Language/Vocabulary

In conducting this study I found myself constantly aware of the language and vocabulary of English and Mathematics that I was accustomed to using with my college students and of the need to address the children at their level. The level of English was mostly appropriate as many of the students in the study had excellent English vocabularies and speech patterns. However mathematics as a language has its own vocabulary where certain concepts are considered axiomatic or elementary at one level but have to be explained at another. This was especially the case in speaking about probability, especially with phrasing and description. I often felt as if I were speaking a foreign language and found myself a few times looking to the teacher somewhat as to a translator. Whenever I caught a glimpse of a student looking at me with cocked head and furrowed brow, I had to stop and think about how to rephrase or recast a mathematical statement. I am again reminded of Bruner's claim that anything can be taught to anyone at any level provided that it is done honestly. While Bruner has since had some afterthoughts about his use of the word 'honestly', my experience of this study prompts me to add '*and consciously*'.

With probability, the response elicited is often more than a function of the way the question is asked. Konold's outcome approach is based on precisely this observation, namely, that the questions asks one thing, Which is more likely, but students hear and respond to another thing, what will you get? I suggest that we avoid asking the students to make a prediction and instead ask: What outcomes are likely? or, What are the possible outcomes? This will focus the attention from giving a prediction or giving a single outcome and will lead to a consideration of the **distribution** of outcomes. This idea of distribution captures all the necessary information about a chance situation – the distribution gives the set of possible outcomes and the probabilities with which they occur.

In order to probe understanding, a construct that is hard to physically see or measure, it is important to develop tools that provide insights into understanding. Multiple–choice items tell us little as do questions that require a yes-no response. Questions that require an answer of one or more sentences provide a little more insight. Interviewing is to be recommended for a fuller exploration of a student's thinking 'to trace intuitive barriers and to provoke thought' (Borovcnik & Peard, 1996, p. 276). The interviews in this study were particularly enjoyable for me as I found many of the students enthusiastic and thoughtful. I was delightfully surprised by their original insights and thinking and found myself challenged to uncover ways to counter their arguments.

5.3.2 Difficulties relating to Tasks and Settings

Steinbring (1991) recommends the development of a task system as an important part of instruction which includes as variables elements such as the tasks, teaching method, and means of representation and activity (p. 156). This is ideal in cases where there is dedicated time for probability that will allow for a full treatment of the topic at that level.

The tasks in this study were chosen from the ones used in the literature and sometimes modified slightly. In two cases I used a binary model asking for a choice of two spinners or two bags of candy. The question I asked was, Which would you choose? In both cases the situations offered equal chance for each of the two alternatives. Hence the answer from normative thinking according to formal probability theory (and the answer that I was expecting) is, Either since both are equally likely. In interview, some students responded that while they knew that there were equal chances they thought that they had to choose. They are rightly thinking that they are being asked to make a one-off decision. This indicates that choice of task and wording can prompt responses that do not give a true picture of a student's thinking and understanding.

5.3.3 Cognitive Conflict

The results of this study show that students do not experience cognitive conflict in their thinking of probability. They had no difficulty with their positions. I, on the other hand, kept thinking, but this is wrong because it did not conform to normative thinking according to formal probability theory. I consistently found myself trying to argue and explain (I have to come to realize that being a teacher for many years induces an automatic reflex of feeling compelled to correct and explain which I have to fight against). This is similar to the finding of Konold and his colleagues (1993) that 'what may appear to be instructor to be a contradiction may nevertheless induce little or no conflict in the student' (p. 412). Many of the students demonstrated an often unshakable belief in their own thinking and clearly felt that it made sense within their rationalization of the scheme of things. Konold and his colleagues (1993) further note that "there is no simple story about how students reason about chance" and that "students bring a variety of beliefs and perspectives about chance" (p. 413). They recommend that "teachers of probability would do better to minimize, or perhaps delay their role as a producer of cognitive conflict, adopting instead an ethnographer's frame, trying to understand the language and practices of a foreign culture" (p. 413).

5.4 RECOMMENDATIONS FOR RESEARCH

This explorative and descriptive study of Grade 7 students' understandings of probability stands in the field of research by contributing new knowledge of students' understandings in a particular context and place. Research at different grade levels and in different contexts and regions will add to the knowledge base of how children learn and understand probability.

It is also recommended that the tasks that have been be used in the literature and in this study be scrutinized and modified to more accurately gauge understanding in this difficult subject so that improved curriculum and instruction can be designed. Many of the tasks used in the literature and in this study are variants and modifications of the ones used by Piaget and Inhelder in the 1950s. This is not because researchers have not been thoughtful and creative but only that there are a limited number of examples of simple probability situations. Hence the tasks have involved simple randomizers such as dice, coins, spinners, and bags of marbles. One promising avenue of research is the use of tasks that 'mask' the chance aspect such as used in LeCoutre (1992) where she used same-sized cards with various shapes drawn on and asked such questions as if two cards were drawn, are you more likely to construct a house (a triangle and a rectangle) or a rhombus (two triangles). Posing the question in this way requires greater attention to be paid to looking at the various outcomes in order to enumerate the sample space and to work out probabilities.

Another consideration is that of the difficulty of being a co-teacher and researcher. Each of these roles requires different approaches and duties. Being a co-teacher often meant that I was unable to capture many of the interactions during lessons and hence not able to attend to the work of being a researcher. I enjoyed the co-teaching but was worried that I had missed the opportunities of listening, observing and interpreting from the vantage point of researcher. Polaki (2002) notes that in his study, he as researcher, led the instructional sessions in the presence of a witness who provided feedback on the extent of development and growth in probabilistic thinking. Such an arrangement of having an observer present in the classroom will help in painting a more complete picture of students' understandings.

Finally research in this area of mathematics and at this level is often difficult to do because of the realities of a school-year packed with competing activities and a curriculum that is demanding and difficult to teach, making it very likely that teachers leave this strand to the end of the year or leave it out altogether. Hence it is a challenge to find a classroom in which to carry out such research and researchers often end up carrying out the study in their own classrooms or providing much of the instruction in other classrooms. This is a serious limitation at this level that can only be overcome by long-term planning and collaboration.

5.5 CLOSING THOUGHT

The journey through this study has been challenging and gratifying. I am privileged to have had the opportunity to collaborate with Ms. Thomson and her students. I am appreciative of their welcoming reception and their enthusiastic engagement in the study. I have taught probability and statistics for many years and I can gratefully say that after having carried out this study I am now seeing my own classes and attending to my students with new eyes and ears.

REFERENCES

Acredolo, C., O'Connor, J., Banks, L., & Horobin, K. (1989). Children's ability to make probability estimates: Skills revealed through application of Anderson's functional measurement methodology. *Child Development, 60,* 933-945.

Amir, G. & Williams, J. (1999). Cultural influences on children's probabilistic thinking. *Journal of mathematical behavior, 18(1),*85-107.

Archibald, J. (2008). *Indigenous storywork: Educating heart, mind, body and soul.* Pacific Educational Press, Vancouver, BC.

Bartolome, L. (2002). Beyond the methods fetish: Towards a humanizing pedagogy. In A. Darder, M. Baltodano, R. Torres (Eds.), *The Critical Pedagogy Reader,* pp. 408-429. Routledge, UK.

Bennett, D. (1983). *Randomness.* Harvard University Press, Cambridge, MA.

Bogdan, R. & Biklen, S. (2007). *Qualitative research for education: An introduction to theories and methods.* Pearson Education Inc., Boston, MA.

Borovcnik, M. & Bentz, H. (1991). Empirical research in understanding probability. In R. Kapadia & M. Borovcnick (Eds.), *Chance encounters: Probability in education*, pp. 73-105. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Borovcnik, M. & Peard, R. (1996). Probability. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International Handbook of Mathematics Education*, pp. 239-287. Kluwer Academic Publishers, Dordrecht, The Netherlands.

British Columbia Ministry of Education (2007). *Integrated Resource Packages, Mathematics, K – 7.* Victoria, BC.

Bruner, J. (1960). *The process of education.* Harvard University Press, Cambridge, MA.

Burns, M. (1995). *Math by all means: Probabilty Grades 3-4.* Math Solutions, Sausalito, CA.

Byrnes, J. & Beilin, H. (1991). The cognitive basis of uncertainty. *Human Development, 34,*189-203.

Carlson, L. (1994). *More than moccasins: A kid's activity guide to traditional North American Life.* Chicago Review Press, Chicago.

Civil, M. (1998). Bridging in-school mathematics and out-of-school mathematics: A reflection. AERA Annual Meetings, San Diego, CA.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23(7),*13-20.

Cobb, P. & Yackel, E. (1996). Constructivist, emergent and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31 (3/4),* 175-190.

Confrey, J. (1990). What constructivism implies for teaching. In R. Davis, C. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics.* Journal for Research in Mathematics Education, Monograph No. 4, pp. 107-124, National Council of Teachers of Mathematics, Reston, VA.

Cushman, J. (1991). *Do you wanna bet? Your chance to find out about probability.* Clarion Books, New York.

David, F. (1998). *Games, gods and gambling: A history of probability and statistical ideas.* Dover Publications, UK.

Davies, C. M. (1965). Development of the probability concept in children. *Child Development, 36(3),* p. 779-788.

Davis, B. (2001). Why teach mathematics to all students? *For the Learning of Mathematics, 21(1),* 17-24.

Davis, R., Maher, C. & Noddings, N. (Eds.) (1990). *Constructivist views on the teaching and learning of mathematics.* Journal for Research in Mathematics Education, Monograph No. 4, National Council of Teachers of Mathematics, Reston, VA.

Denzin, N. & Lincoln, Y. (Eds.) (1994). *Handbook of Qualitative Research.* Sage, Thousand Oaks, CA.

Driver, R. & Erickson, G. (1983). Theories-in-action: some theoretical and empirical issues in the study of students' conceptual frameworks in science. *Studies in Science Education, 10,* 37-60.

English, L. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal for Research in Mathematics Educaton, 22,* 255-273.

Ernest, P. (1998). The epistemological basis of qualitative reseracn in mathematics education: A postmodern perspective. In A. Teppo (Ed.), *Qualitative research methods in mathematics education*, pp. 22-39. Journal for Research in Mathematics Education, Monograph Number 9, National Council of Teachers of Mathematics, Reston, VA.

Falk, R. (1983). Children's choice behaviour in probabilistic situations. *Proceedings of the First International Conference on Teaching Statistics* (Vol. 2, pp. 714-726). University of Sheffield, Sheffield, England.

Falk, R., Falk, R. & Levin, I. (1980). A potential for learning probability in young children. *Educational Studies in Mathematics, 11,* 181-204.

Falk, R. & Konold, C. (1992). The psychology of learning probability. In Gordon, F. & Gordon, F. (Eds.), *Statistics for the Twenty-First Century*, pp. 151-164. MAA Notes, Number 26, The Mathematical Association of America, New York.

Falk, R. & Wilkening, F. (1988). Children's construction of fair chances: Adjusting probabilities. *Developmental Pyschology, 34(6)*, 1340-1357.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children.* D. Reidel Publishing Company, Dordrecht, The Netherlands.

Fischbein, E. (1987). *Intuition in science and mathematics.* D. Reidel Publishing Company, Dordrecht, The Netherlands.

Fischbein, E. & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15*, 1-24.

Fischbein, E., Nello, M. & Marino, M. (1991). Factors affecting probabilistic judgments in children in adolescence. *Educational Studies in Mathematics, 22,* 523-549.

Fischbein, E., Pampu, I. & Manzat, I. (1967). The child's intuition of probability. In E. Fischbein (1975), *The intuitive sources of probabilistic thinking in children*, Appendix II, pp. 156-174. D. Reidel Publishing Company, Dordrecht, The Netherlands.

Fischbein, E., Pampu, I. & Manzat, I. (1970). Comparison of ratios and the chance concept in children. *Child Development, 41(2)*, 377-389.

Fischbein, E. & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Educational Studies in Mathematics, 22*, 523-549.

Gabriel, K. (1996). *Gambler way: Indian gaming in mythology, history and archaeology in North America.* Johnson Printing, Boulder, Colorado.

Garfield, J. & Ahlgren, A. (1988). Difficulties in learning basic concepts of probability and statistics: Implications for research. *Journal for Research in Mathematics Education, 19,* 44-63.

Gates, P. & Vistro-Yo, C. (2003). *Is* mathematics for all? In A. Bishop, M. Clements, C. Keitel, J. Kilpatrick & F. Leung (Eds.), *Second International Mathematics handbook*, pp. 57-73. Lawrence Erlbaum, Mahwah, NJ.

Gay, G. (2002). Culturally responsive teaching in special education for ethnically diverse students: Setting the stage. *Qualitative Studies in Education, 15(6),* 613-629.

Gigerenzer, G., Swijtink, Z., Porter, T., Daston, L., Beatty, J. & Kruger, L. (1989). *The Empire of Chance: How probability changed science and everyday life.* Cambridge University Press, Cambridge, UK.

Goldberg, E. (1996). Probability judgment by preschool children: Tasks, condition and performance. *Child Development, 37,* 157-167.

Goldin, G. (1990). Epistemology, Constructivism, and Discovery Learning Mathematics. In R. Davis, C. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics.* Journal for Research in Mathematics Education, Monograph No. 4, pp. 31-37, National Council of Teachers of Mathematics, Reston, VA.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education,* pp. 517-545. Lawrence Erlbaum, Mahwah, NJ.

Green, D. (1983). A survey of probability concepts in 3000 pupils aged 11-16 years. In D. Grey, P. Holmes, V. Barnett & G. Constable (Eds.), *Proceedings of the First international Conference on Teaching of Statistics*, pp. 766-783. Teaching Statistics Trust, Sheffield, UK.

Green, D. (1988). Children's understanding of randomness: Report of a survey of 1600 children aged 7-11 years. In R. Davidson & J. Swift (Eds.), *Proceedings of the Second international Conference on teaching Statistics*, pp. 287-201. University of Victoria, Victoria, BC.

Greer, B. (2001). Understanding probabilistic thinking: The legacy of Efraim Fischbein. *Educational Studies in Mathematics, 45,* 15-33.

Guba, E. & Lincoln, Y. (1994). Competing paradigms in qualitative research. In N. Denzin &Y. Lincoln (Eds.), *Handbook of qualitative research,* Sage, Thousand Oaks, CA.

Hacking, I. (1975). *The emergence of probability: A philosophical study of early ideas about probability, induction and statistical inference.* Cambridge University Press, Cambridge, UK.

Hacking I. (1990). *The taming of chance.* Cambridge University Press, Cambridge, UK.

Horvath, J. & Lehrer, R. (1998). A model-based perspective on the development of children's understanding of chance and uncertainty. In S. P. Lajoie (Ed.), *Reflections in statistics: Learning, teaching, and assessment in Grade K-12,* pp. 121-148. Lawrence Erlbaum, Mahwah, NJ.

Kuzmak, S., & Gelman, R. (1986). Young children's understanding of random phenomena. *Child Development, 57,* 559-566.

Jones, G. (Ed.). (2005). *Exploring Probability in School: Challenges for Teaching and Learning.* Springer Mathematics Education Library, New York.

Jones, G., Langrall, C. & Mooney, E. (2007). Research in Probability. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning,* National Council of Teachers of Mathematics, Information Age Publishing, Charlotte, NC.

Jones, G., Langrall, C., Thornton, C. & Mogill, A. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics, 32,* 101-125.

Jones, G., Langrall, C., Thornton, C. & Mogill, A. (1999). Students' probabilistic thinking in instruction. *Journal for Research in Mathematics Education, 30,* 487-519.

Jones, G., Thornton, C., Langrall, C. & Tarr, J. (1999). Understanding students' probabilistic reasoning. In L. Stiff & F. Curcio (Eds.), *Developing mathematical reasoning in grades K-12: 1999 Yearbook,* pp. 146-155. National Council of Teachers of Mathematics, Reston, VA.

Kahneman, D. & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology, 3,* 430-454.

Kahneman, D. & Tversky, A. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive Psychology, 5,* 207-232.

Kahneman, D. &Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases.* Cambridge University Press, Cambridge, New York.

Kapadia, R. & Borovcnik, M. (1991). *Chance encounters: Probability in education.* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Kilpatrick, J., Swafford, J. & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* National Academy Press, Washington, DC.

Kirkness, V. J. & Barnhardt, R. (1991). First Nations and higher education: The four R's, Respect, Relevance, Reciprocity, and Responsibility. *Journal of American Indian Education, 30(3),* 1-15.

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6,* 59-98.

Konold, C. (1991). Understanding students' beliefs about probability. In E. von Glasersfeld (Ed.), *Constructivism in mathematics education*, pp. 139-156. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education, 24,* 392-414.

LeCoutre, V. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics, 23,* 557-568.

Li, J. & Pereira-Mendoza, L. (2002). Misconceptions in Probability. International Conference on Teaching Statistics 6 (online), http://icots6.haifa.ac.il/icots6.html. Retrieved May 27, 2008.

Linn, C. (1972). *Probability.* Fitzhenry & Whiteside Limited, Toronto.

Lipka, J., Hogan, M., Webster, J., Yanez, E., Adams, B., Clark, S., & Lacy, D. (2005). Math in a cultural context: Two case studies of a successful culturally based math project. *Anthropology and Education Quarterly, 36(4),* 367-385.

Marker, M. (2006). After the Makah whale hunt: Indigenous knowledge and limits to multicultural discourse. *Urban Education, 41(2),* 482-505.

Measor L. (1985). Interviewing: A strategy in qualitative research. In Burgess, R. (Ed.), *Strategies of educational research: Qualitative methods.* The Falmer Press, London.

Merriam, S. (1998). *Qualitative research and case study applications in education.* Jossey-Bass, San Francisco, CA.

Metz, K. (1998a). Emergent ideas of chance and probability in primary-grade children. In S. Lajoie (Ed.), R*eflections on statistics: Learning, teaching, and assessment in grades K-12,* pp.149-174. Lawrence Erlbaum, Mahwah, NJ.

Metz, K. (1998b). Emergent understanding and attribution of randomness: Comparative analysis of reasoning of primary grade children and undergraduates. *Cognition and Instruction, 16,* 285-365.

Moll, L., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice, 31(2),* 132-141.

Moore, D. (1990). Uncertainty. In L. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy.* Mathematical Association of America. New York.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* Reston, VA.

Nelson-Barber, S. & Estrin, E. (1995). *Culturally responsive mathematics and science education for Native students.* Regional Educational Laboratory Network, California.

Nicol, C., Archibald, J., Kelleher, H., & Brown, L. (2006). Transformative education for aboriginal mathematics learning: A community-based action research project. *BC Educational Leadership Research* (online), June 2006.

Nicolson, C. (2004). *Probabilistic thinking in a Grade 4/5 class*. Unpublished MA thesis, UBC.

Nisbett, R., Fong, G., Lehman, D. & Cheng, P. (1987). Teaching reasoning. *Science 238*, 625-631.

Polaki, M. (2002). Assessing and tracing the development of Basotho elementary students' growth in probabilistic thinking. International Conference on Teaching Statistics 6 (online). http://icots6.haifa.ac.il/icots6.html, retrieved May 27, 2008.

Psathas, G. (Ed.). (1973). *Phenomenological sociology*. Wiley, New York.

Piaget, J. & Inhelder, B. (1951/1975). *The origin of the idea of chance in children* (L. Leake Jr., P. Burrell, & H. D. Fischbein, Trans.). Norton, New York.

Razzell, A. & Watts, K. (1967). *Probability: The science of chance*. Doubleday & Company, New York.

Shaughnessy, J. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8,* 285-316.

Shaughnessy, J. (1992). Research in probability and statistics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and* learning, pp. 465-494. Macmillan, New York.

Shaughnessy, J. (2003). Research on students' understandings of probability. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *The Educational Researcher, 15(2)*, 4-14.

Stake, R. (1995). *The art of case study research*. Sage, Thousand Oaks, CA.

Stake, R. (2005). Qualitative case studies. In N. Denzin & Y. Lincoln (Eds.), *The Sage handbook of qualitative research*. Sage, Thousand Oaks, CA.

Steinbring, H. (1991). The theoretical nature of probability in the classroom. In R. Kapadia & M. Borovcnik (Eds.), *Chance encounters: Probability in education*, pp. 135-167. Kluwer Academic Publishers, The Netherlands.

Tambe, P., Carroll, B., Mitchell, H., Lopez, L., Horsch, E., & St. John, M. (2007). *How do you get the numbers to dance? Effective education practices in mathematics for Native American learners: A conference summary*. Inverness Research Associates, Inverness, California, http://www.inverness-research.org/abstracts/ab2007-09_Rpt-nativeamconf.html. Retrieved May 7, 2008.

Tobin, K. (2000). Interpretive Research in Science Education. In Kelly, A. & Lesh, R. (Eds.), H*andbook of research design in mathematics and science education*, pp. 487-512. Lawrence Erlbaum, Mahwah, NJ.

UBC Department of Science. (2008). http://www.science.ubc.ca/support/giving/math. Accessed July 21, 2008.

Villegas, A. & Lucas, T. (2002). *Educating culturally responsive teachers: A coherent approach.* State University of New York Press, Albany, NY.

White, D. (2003). Promoting productive mathematical classroom discourse with diverse students. *Journal of Mathematical Behavior, 22*, 37-53.

White, R. & Gunstone, R. (1992). *Probing understanding*. The Falmer Press, London.

Wilensky, U. (1997). What is normal anyway? Therapy for epistemological anxiety. *Educational Studies in Mathematics 33(2),* 171-202.

Wood, T. (2001). Teaching differently: Creating opportunities for learning mathematics. *Theory into Practice, 40(2),*110-116.

Yin, R. (1994). *Case study research: Design and methods.* Sage, Thousand Oaks, CA.

Yost, P., Siegel, A., & Andrews, J. (1962). Nonverbal probability judgments by young children. *Child Development, 33*, p. 769-780.

Appendix A: Prescribed Learning Outcomes (PLOs) Probability Grades 5-7

Grade 5
Likelihood of a single outcome

Grade 6
Experimental and theoretical probability

Grade 7
Ratios, fractions & percents to express probabilities
Two independent events
Tree diagrams for two independent events

PLOs General outcome: Use experimental or theoretical probabilities to represent and solve problems involving uncertainty

D4 express probabilities as ratios, fractions, and percents [C, CN, R, T, V]
D5 identify the sample space (where the combined sample space has 36 or fewer elements) for a probability experiment involving two independent events [C, ME, PS]
D6 conduct a probability experiment to compare the theoretical probability (determined using a tree, diagram, table or another graphic organizer) and experimental probability of two independent events [C, PS, R, T]

7 Mathematical processes:
[C] Communication - Communicate in order to learn and express their understanding

[CN] Connections - Connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines

[ME] Mental Mathematics and Estimation - Demonstrate fluency with mental mathematics and estimation

[PS] Problem Solving - Develop and apply new mathematical knowledge through problem solving

[R] Reasoning - Develop mathematical reasoning

[T] Technology - Select and use technologies as tools for learning and solving problems

[V] Visualization -Develop visualization skills to assist in processing information, making connections, and solving problems.

Appendix B: Lesson Notes for Veda Abu-Bakare

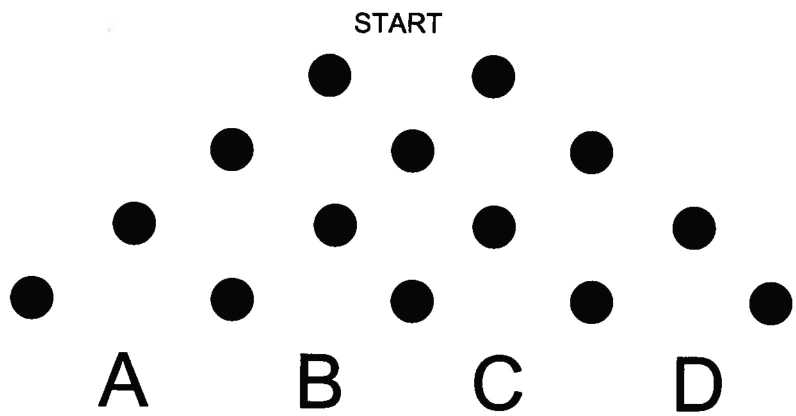
Lesson 1

First Lesson June 11   940 am-1020 am Veda

Hello everyone. Thank you for having me in your class today. I am very happy to be here and to spend some time doing Mathematics with you. Just to tell you a little about myself - I come from the Caribbean near South America. I was born in the island of Trinidad and attended elementary and high- school there. I came to Canada with my family when I was quite young – some of you have a similar experience of coming to Canada from elsewhere.
I teach Mathematics and Statistics at Langara College – you may have heard of it or someone you know goes to Langara. Some of you when you finish high-school may attend Langara. Besides being a teacher I am also a student, like you. I attend UBC where I am a graduate student working with Dr. Cynthia Nicol. You will remember her – this is all part of a project that is exploring how we might improve our teaching of Math. I am particularly interested in stats and probability. On Monday Ms. Thomson introduced you to the probability spectrum and you did an activity with 10-sided double dice.
Today I have brought a demonstration of an activity that you may have seen at carnivals or fairs. In this activity, balls are dropped from the top and they make their way down randomly and collect in the channels. Can we make any prediction about how the balls will fall?
Notice that at each node there are 2 possibilities. I have put a simpler version of the game on a piece of paper and we are going to model the path of the balls using a die. How are we going to model the 2 possibilities using the die? Elicit Answer: Odd and Even. Let's decide now: Odd go left and Even go right. This activity has many rows and because of time we will only consider 4 rows.
Action: Hand out activity sheet and die. Give instructions. Have them pool their results for each table. Then one person from each table (about 5 students per table) adds their results to table on the board. Elicit the expected proportions,
1 3 3 1 and the probabilities, 1/8, 3/8, 3/8, 1/8.

Name________________________ Date: June 11, 2008

If you get an ODD number go LEFT Even number go RIGHT
Toss the die 3 times and note what you get. Trace the path and note where you land. So for example, if you get OEO you will end up at B. Do this 7 more times (eight times in all) and note how many times you end up at each of A, B, C, D.



	A	B	C	D
Actual n = 8 balls				
Expected n = 8 balls				
Probability				

Questions for later: What if we had 5 rows? How many outcomes would there be? What would the probabilities be? Can you extend this to 6 rows? This activity leads to a famous triangle, Pascal's triangle and a well-known distribution called the Binomial distribution.

Lesson 2

Lesson Monday June 16, 2008 1- 2 pm Veda teaches

Reflections from Wednesday's activity. The activity went well but I think too ambitious on the theory and the fractions, distribution of probability, etc. So time to get back to the basics...

A. Beginning Activity: Talk about probability. What are we trying to capture when we talk about probability? What are some everyday experiences with probability? Weather, 50-50 chance with a coin, die a one in six chance. Probe the ways we describe probability, fraction, decimal, 2 out 3 chance, 2:3

Hold up bag with gold and silver coins. How can we know the chance of getting a gold coin? a silver coin? Want to elicit the proportion of favourable outcomes over the total number. What about the chance of a copper coin? Reinforce idea of fraction, decimal, percent,...

Hold up bag with red, blue, green chips. Repeat activity above. This time, ask about the chance of getting a blue or green chip, etc...

Ask if any questions about probability so far?

So let's talk about predictions. Go back to bag with red, blue, green chips (note how many of each, 20, 20, 20, say). Ask a student to make a prediction and then choose. Do this for 6 students, each time writing the prediction and the outcome. Stand so that the data are hidden and ask if you were to dip into the bag and take 6 chips what do you think you would get? Note the predictions and compare with the data. Elicit: If random draws, would expect to see the same proportion in the 6 chips, 2 of each colour. Reinforce the idea if we do this again we get different actual proportions but after many, many times of doing this, the fraction of each colour would settle down to around 1/3.

B. Talk about aboriginal gambling games and their significant aspects, ceremonial, healing, spiritual aspects. Games were always considered a sacred activity and closely related to the stories and rituals. Talk about the times of the year they were played (when summer came, the children threw their winter toys in the river), about the groups that played them (some games only played by women and girls, some by chiefs and warriors), about the purposes (for redistribution of food and resources as a kind of trade rather than by raiding or warfare, for passing the time in waiting for the outcome of a hunt or fishing expedition).

Aboriginal games were of two kinds. One was played in a basket placed in a hole dug in the ground cushioned by a piece of bark. In the basket were 2-sided dice made from bones or sticks that were patterned on one side. The basket represented the moon and the dice the stars and the earth. Sometimes up to 13 dice were used and these were tossed in the air and points were accumulated for the various combinations of patterned and blank sides. Mention that they kept track in their heads – no small feat because of the different point values for certain combinations.

The other kind of games was the hiding games. In one style of this game the dealer shuffles ten wooden chips with one marked differently from the others, and hides them under two piles of wood shavings. Opposing teams would guess which pile contained the different chip. The game could also be played in a circle with two opposing teams in semicircles and many rounds of guessing with opposing members going back and forth. There's a very simple version that I played as a girl in Trinidad and I am sure you have played it as well. Two people play, one person hides the counter with both hands behind the back and then presents two fists, with the counter hidden in one fist. The other person guesses which fist it's in. If it is not found then the hider hides it again and the guesser tries again. We are going to play this game and note how many tries it takes for the guesser to find the counter.

Activity: At your desk, find someone to play with. Take turns being the hider and the guesser and write down the number of tries until there is a correct guess. Each person plays the game three times and note how many tries the game takes. Each person will have three numbers. Compare your numbers with those at your table. Think about how you would like to represent those numbers. What do the numbers tell you?

Bring them back to the carpet to talk about the play and the strategies used. Could you outwit your opponent? Try to make sense of the numbers and make a display. What is most interesting for you about the numbers? If you were to tell someone about the game, what would you report?

Lesson 3

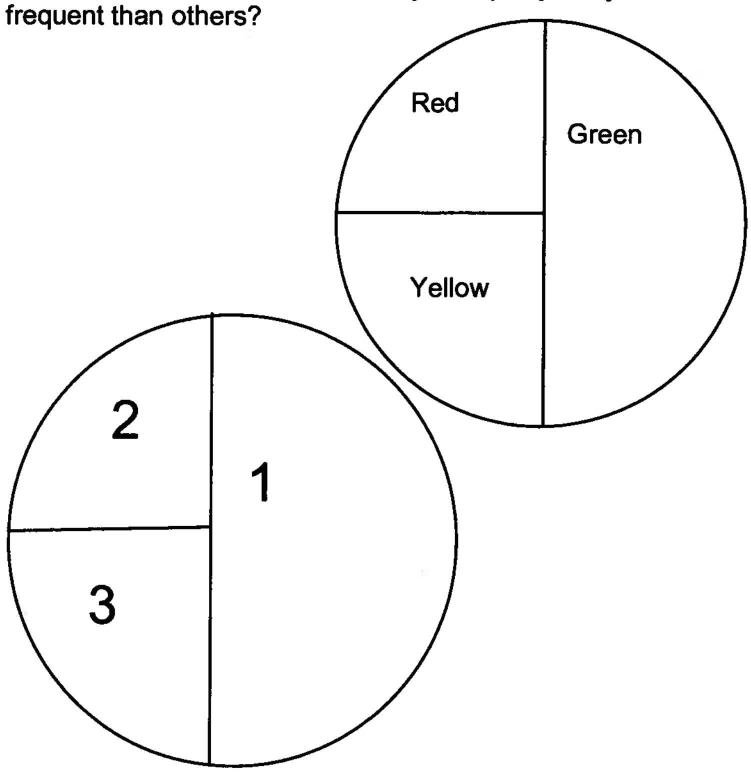
Lesson June 19 9-1020am Veda teaches

Reflection: Aim to build up understandings of probability and to develop the language of probability. Hope to put in ideas of sample space and tree diagrams.

A. Recall the things we have been doing involving uncertainty – tossing coins, dice, rolling marbles down tracks. Point out that these are examples of **random** generators in that we are generating chance data as opposed to something that will happen **deterministic**.
Point out in these experiments we have **possible outcomes**. Consider some experiments and list the possible outcomes. Do some simple ones like tossing a coin, tossing a die, drawing a chip from a bag with chips of 3 colours. Now let's look at what happens with 2 coins. Elicit the sample space (HH, HT, TH, TT}. Now let's look at 2 dice. Elicit 36 possible outcomes. What are they? How would we write them down? Is there a systematic way? Now recall the double dice that Miss Thomson had. How many possible outcomes? Can we write them down?

B. Another random generator that you may have seen is a spinner. Where have you seen this? Draw some spinners, first with 2 sectors and elicit half-half for the probabilities, trying out the diameter in different places. Step it up to 3 sectors and then to 4 sectors. Emphasize that it has to do with areas. Address whether contiguous or non-contiguous makes a difference.

Discuss the Spinner Sum Activity. The spinner has 3 sections, Red, Yellow, and Green. Talk about the probabilities. We are going to make this harder by putting in numbers 1, 2,3 and looking at what happens when we spin two spinners and add the numbers we get. What are the possible outcomes for the sum? Are they all equally likely? Are some of the sums going to be more frequent than others?



Let's try it out and see what we get. The idea is to spin the two spinners and add them up to get a sum. We have only 10 spinners so each table will get 2 spinners and you will take turns spinning. Think about the possible sums and which one will be most frequent. Think about why your results are coming out the way they are…

Then bring them back for a discussion of the probabilities. What do you think would have happened? Were you surprised? Can you explain this?

Name: June 19, 2008

Spinner Sums Activity

1. What are the possible sums when you spin the two spinners?

2. Take your turn spinning the 2 spinners and write down the sum. Do this 4 more times so that you have 5 numbers. What do you observe? Which sum is the most frequent? Which is the least frequent?

3. Write down the sums for each person at your table. Organize the data so that you can describe it to someone. What do you observe?

4. Can you make a picture of the data of all the sums from the persons at your table?

Appendix C: Proposed Tasks

Tasks to probe understanding of theoretical probability

1) A counter, red on one side, blue on the other

Probe: Jane and Louisa are sharing candies. There is one left over. They agree to decide who gets the last candy by each choosing a colour and then flipping a counter. Is this a good strategy? Why? Is it fair?

Further probe: Jane and Louis have a bag of 6 candies. What would be a good strategy for Winner Take All? How many flips would you agree to? Why?

Further probe: In deciding hockey tournaments, the winner is the team that gets 3 wins out of 5. What do you think of that strategy? How much of the result would you say is due to chance? Are some teams just lucky?

2) Bags with varying proportions of red and blue candies

Probe: Bags with proportions 1:4, 2:4, 1:3, 2:6. If you want a red candy, which bag would you choose from? Which is the most favourable? Which is the least favourable?

Further probe: If you want a red candy, can you put the bags in order from least favourable to most favourable?

Probe: One bag 3:5 and one bag 30:50. If you want a red candy which bag would you choose from? Does it matter?

Tasks to probe of understanding of relative frequency notion of probability

1) Spinners

a) Here are 4 spinners each with red or blue

Spinner I	1:1 contiguous
Spinner 2	1:1 non-contiguous
Spinner 3	2:1 contiguous
Spinner 4	2: 1 non-contiguous

We will play a game First Home. You have a Red counter and I have Blue one. Board with path from START to HOME marked out in 10 steps. We place our counters on START. We take turns spinning the spinner. If it stops on Red I move 1 space, If it stops on blue, you move 1 space. Which spinner do you want to play with? Why?

b) Spinners with numbers (positive, negative and zero) – move the number of spaces ahead if positive, fall back the number of spaces, if negative.

c) Probing the notion of the continuum. If you wanted to be certain to win, what would your spinner look like? Can you arrange the spinners from least favourable to most favourable to you?

Tasks to probe understanding of sample space

1) Two counters, red on one side, blue on the other
Game: Two students, one chooses Same, the other Different. They both flip their counters. Same gets a point if both colours are the same, Different if colours different. At the end of 10 flips, who do you think is leading? Why?

2) Who is right?
Probe: Jane and Louisa are having a disagreement. They have to work out the chance of there being 2 girls in a family of two children. Jane says the answer is 1/3 and Louisa says that it is ¼. Who is right? What assumptions are they making?

Tasks to probe misconceptions
1) Representativeness
Probe: Jane and Louisa are having a disagreement. Jane says that in flipping a coin 6 times it is more likely to get HTTHTH than 6 heads in a row or 6 tails in a row. Do you agree? Why?

Probe: Jane and Louisa are picking numbers for a Lotto 5/49 ticket. Jane picks { 5, 10, 15, 20, 25, 30} because it is easy to remember. Louisa says that the numbers are not random enough and her choice of {3, 17, 29, 33, 40, 43} has a better chance of winning. Do you agree with either of them? Why? Which numbers would you choose? Why? How would you choose your numbers? Do you have a strategy?

2) Gambler's fallacy
Probe: Jane is flipping a coin. She is amazed to get 6 heads in a row. She says, Watch, I am sure to get a tail on the next flip. Do you agree? Why?

Probe: Jane is flipping a coin. She is amazed to get 6 heads in a row. She says, Watch, I am sure to get a heads on the next flip. Do you agree? Why?

Probe: A friend that you trust tells you that she has been spinning a coin and has had 6 heads in a row. What would you say to her?

Appendix D: Possible Class Activities
May, 2008

Class Activities

1 Experimental and theoretical probability

1.1 Activity with cuboocathedron
Pages 8 – 11 from *Probability* by Charles F. Linn.

1.2 Coin on a grid (text, page 365)
In the text, a 5 by 5 grid of 3-cm squares is used with a dime being tossed onto the grid. Students are to estimate the probability of the dime landing completely in a square. If we assume that a dime is 18 mm in diameter (more accurately 18.03 mm), then there is a square of side 12 mm in which the centre of the dime can fall and be completely in the square. So the theoretical probability is (12^2)/(30^2) = .16.

1.3 Buffon Needle Problem
This is covered in the Enrichment section of the Teacher's Edition on page 366 using toothpicks cut 3cm long and parallel lines drawn 6 cm apart (horizontally, say). Students toss the toothpick 200 times onto the paper, each time noting whether it crosses the line. The experimental probability, P, that the toothpick crosses the line is then calculated with a reference to observe the value of 1/P.

The connection to pi is as follows: If the length of the needle is L1 and the spacing is of length L2 then the probability of a hit, P, works out to (2/pi) (L1/L2). In the text, L1 = 3 and L2 = 6 so P is 1/pi and 1/P would be close to pi. But a good approximation would take hundreds of tries.

After the class activity, a demonstration of the applet on the web for this problem would be nice.

2. Simulations

Many applets on the Web but this one is fairly straight-forward. Here the length of the needle is the same as the length of the spacing.

http://www.math.csusb.edu/faculty/stanton/m262/buffon/buffon.html

2.1 Cereal Box Problem
Also called the Coupon collector problem. The WHEATY cereal company puts a counter in each box. Each counter has one of the letters W, H, E, A, T, Y on it. You have to collect all 6 in order to get a prize. How many boxes do you have to buy on average to get all 6?

How can this be simulated using a die? Using random numbers?

2.2 Random Walk Problem
Students work in threes, one to toss the coin and call out the outcome, one to move, and one to record.
1. Mark out a line on the floor and mark off from the left, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5. One student stands at 0. One student flips a coin and the student at 0 moves one step to the right if heads and one step to the left if tails.
2. Flip the coin 4 more times with student moving according to the flips and position noted after 5 flips.
3. Repeat 9 more times. What is the average distance moved after 5 flips?
4. Think of a way to diagram the path for each of your 5 flips (use different colours).

5. Pool results with others and find the average distance.

3. Independent Events and tree diagrams

3.1. Drawing 2 counters from a bag with red and blue counters with and without replacement.
3.2. Three Coins and three spinners

4. Math in Literature

4.1. Talk about Lewis Carroll, page 372, Text. Read another of his poems, Jabberwocky or Tweedledum and Tweedledee.
4.2 Show the 3 probability books (Grades 3-5).

5. Harder Activities

5.1 Strings and Loops
From Long C., DeTemple, D., and Millman, R. (2007) *Mathematical Reasoning for Elementary teachers.* Pearson. Pages 611, 652 – 653.

6. Games from the Aboriginal people of North America

http://mathcentral.uregina.ca/RR/database/RR.09.00/treptau1/

From http://mathcentral.uregina.ca/RR/database/RR.09.00/treptau1/game6.html

Games of Throw sticks

Origin:
Apaches in the Southwest

History:
During large celebrations that would last about four days, nations would get together and would feast, dance and play games. Many of these games involved gambling and setting large wagers against neighbouring tribes.

Materials:
40 stones - circle; 3 sticks; 2 feathers (shells, sticks, etc.)- place markers

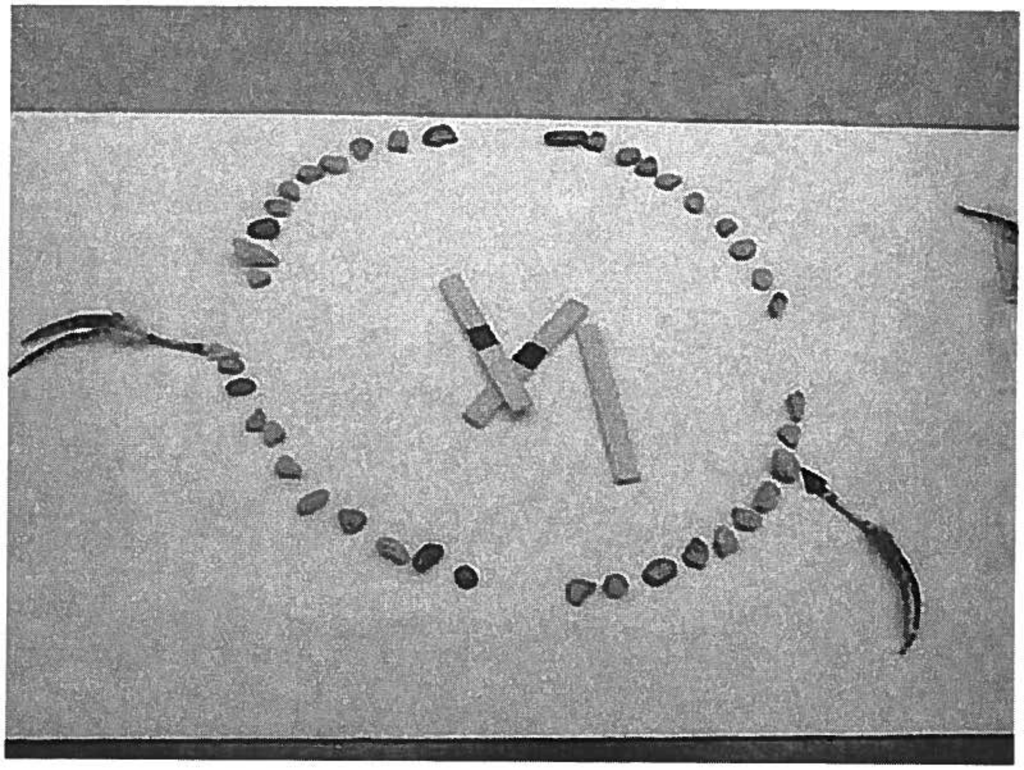
Players:
2 players

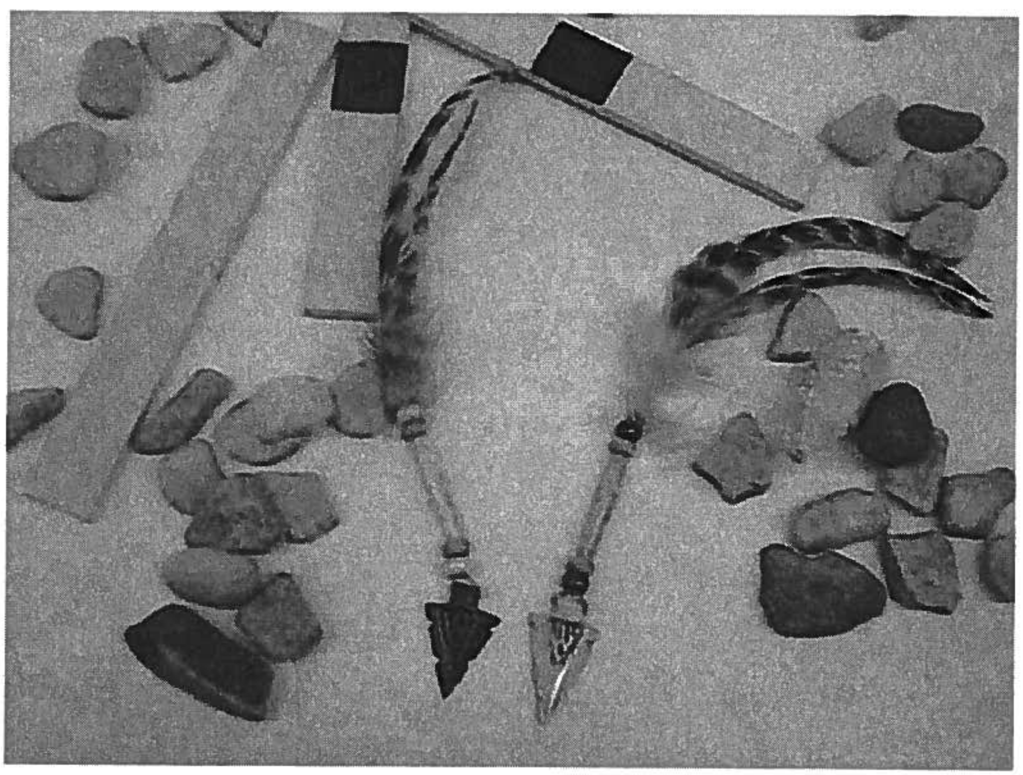
Setup:
Arrange the 40 stones in a large circle laying them in 4 groups of 10. Place the 2 markers on opposite sides. The three stick dice should be decorated the same on one side and blank on the other.

To Play:
Toss the 3 sticks down in the center of the circle. Move the place marker according to the points received around the stones. Each stone counts as one point. If a person's marker lands on or passes another's marker, the person passed over must go to the starting point.
To Score:
Different combinations will score different point values. The first player to move around the circle past all 40 stones wins the game.
3 blank sides up
10 points
2 blank sides, 1 painted side up
1 point
1 blank side, 2 painted sides up
3 points
3 painted sides up
5 points

Math Content:
Probability, patterns and relations, numbers and operations (place value), data management.

Source:
Carlson L. (1994). *More Than Moccasins*, Chicago Il: Chicago Review Press. ISBN 1-55652-213-4.





Appendix E: Pre-test

Beginning Probability June 4, 2008

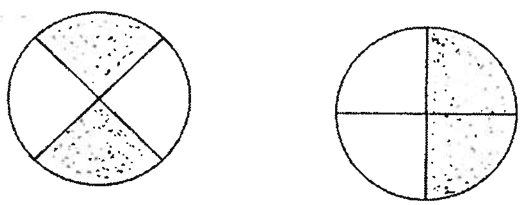
1. A bag contains 5 blue marbles, 4 red marbles and 6 white marbles. A marble is chosen from the bag.
a) Find the probability that the marble is blue. __________

b) Find the probability that the marble is blue or white. __________

2. Ten chips numbered 1 to 10 are placed in a bag. One chip is drawn from the bag.
a) Find the probability that the number on the chip is less than 5. _________

b) Find the probability that the number on the chip is 12. _________

3. Suppose you are playing a game with spinners. You win a point if the spinner lands on the dots. Here are 2 spinners. Which one would you choose to play with? Why?



Spinner A Spinner B

4. Suppose that in a game with two teams, the team that gets to start first is usually determined by a coin toss. Suppose that instead of a coin we had a die. How could we use the die to decide which team starts?

5. Suppose that you are flipping a coin repeatedly and that you get 4 heads in a row. On the 5th flip is a head more likely than a tail? Explain.

6. Suppose your friend is planning to buy a Lotto 6/49 ticket and picks the numbers {1, 2, 3, 4, 5, 6}. You plan to use numbers from the Quick Pick machine. Which of you has a better chance of winning? Explain.

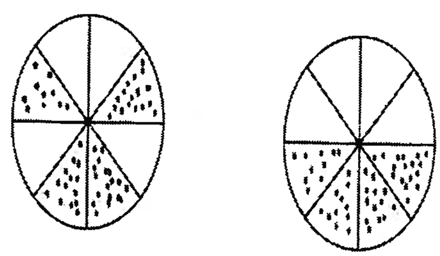
Appendix F: Post-test

Probability Checkup (written June 25, 2008)

1. A bag contains 3 blue marbles, 4 red marbles and 3 white marbles. A marble is chosen from the bag.
a) Find the probability (as a fraction) that the marble is not blue.
b) Find the probability (as a percent) that the marbles is blue or white.

2. A counter, which is red on one side and green on the other, is tossed twice.
a) List all possible outcomes
b) Find the probability (as a percent) of getting green both times.

3. Suppose you are playing a game with spinners. You win a point if the spinner lands on the dots. Here re 2 spinners. Which one would you choose to play with? Why?



Spinner A   Spinner B

4. Suppose that in a sample of trail mix there are 10 sunflower seeds, 5 pumpkin seeds, 5 raisins and 10 peanuts.
a) How may raisins would you expect in a sample of 120?
b) How many seeds (sunflower and pumpkin) would you expect in a sample of 120?

5. Suppose that in playing with dice, you want to obtain three sixes. Do you have a better chance of getting three sixes by tossing a single die three times in a row or by tossing three dice at the same time? Explain your thinking...

6. In tossing 2 dice (one red and one green), are you more likely to get a 5 and a6 or to get two sixes? Explain your thinking...

7. Each of four friends, Amy, Bob, Cathy and David plans to buy a Lotto 6/49 ticket. Amy chooses the numbers {10, 14, 19, 28, 33, 41}. Bob chooses {12, 15, 20, 29 , 34, 42} Cathy chooses {15, 25, 35, 45, 46, 48} and David chooses {14 24, 34, 44, 45, 47}. Who has the best chance of winning and why? Explain your thinking.

Appendix G: Sample Interview Transcript

1130 am Thursday June 19 2008

Veda	Taylor (pseudonym)
Hello Taylor.	
	Is this like a test?
No, it's more like a talk… I just want to see how you are feeling about Math, tell me about how you are feeling about Math in general….	
	It could be fun, it kinda depends on how you teach it and the circumstances…
Yeah, but is it one of your strong subjects?	
	I think yeah
You might want to do something with Math later on?	
	Maybe
What high school are you going to go to?	
	Britannia
Good, what I like about Britannia is that it is a community school with so much going on…	
	And not too many people so you don't get left out…
Oh I see, I see… I heard you on the Raven's c all today, really nice	
	Oh, thank you.
And probability, is there anything about probability that you are finding interesting or hard or…	
	Mmmmm, no
No? do you find it different from the other kind of math you do?	
	It just kind of seems like less work but more kind of taking the time to think about something.
Yes, it really is an abstract concept…	
	Yes…
And the thing about probability is that it is hard to demonstrate in the short-run	
	mmm-hmmm
Like you could measure distance…	
	It could be used way more in real life than two times two….
Yes, you did really nicely on this question (Question 1). In #2a, find the probability that the number is less than 5… you have numbers 1, 2, 3 up to 10, how many numbers are less than 5	
	Four
Four, yes, that's right… Now 0 over 10 when you leave your answer like that, what does it mean…	

	I wasn't really paying attention…
So 0 over 10 is how many percent?	
	Zero?
Zero, yes, that's right. What about the spinner question?	
	This question? It's kind of like what we had I was thinking…
In this question you said that you'd play with B because there are more dots .. and more area… Would you still have that answer you think?	
	Well it is more chance wise and I was thinking more like a general area and not spread out as much. It's either you win something or you lose something here it's like you're gonna lose, you're gonna lose (*pointing to A*), here (*pointing to B*) it's more spread out and if you were like smart enough you could spin out just so you could land here
But if the spinner is totally random, if it is random what does it mean?	
	Well then, it is probably equal…
So would there be a difference between the two spinners?	
	No…
Not really, so would it make a difference which one you chose	
	Not really
Do you have a personal preference?	
	But I think when things are grouped together, it's kind of easier to pick something out…
Ok, we'll come back to this but let's talk about question 6 (the lotto 6/49 question). You wrote 'both people would have the same chance' and that's exactly right. But this piece here ' Because the reasons balance each other out….	
	Yes.
So something has just occurred to me. Let's go back to spinner B and suppose it was like a clock face, 12 1, 2, 3, … up to 11. So suppose you could bet on the numbers 1, 2,3, 4, 5,6 or 11, 12, 1, and 7, 6, 5. So is there a difference between the two sets of numbers?	
	For sure if it lands on one of these, it's for sure it is on one side. If it lands on one side, it's going to be that side or not that side but this is the top and .....
You know, this is proving to be the most resistant question because even though people are seeing it abstractly both one half, they still think A has more chance. I thought the numbers might convince you…	
	But it's numbers and how many numbers are in one piece?
So in B, I am taking half the numbers, 1, 2,3 4, 5, 6 and in A, I am taking half the numbers still	

	Yes, but if the spinner did land on this side, these are closer together…
Okay, what happened to my pennies, let's play a game. So we are going to throw this up and if they are the same, I'm going to get a point and if they are different, you're going to get a point.	
	I win.
Yes, they are different so you get a point.	
	No that's an American and a Canadian
So they were different?	
	No, that's a tail and that's a tail …
So they were both tails, so I won, okay, so now my turn	
	You win again
There you go, try it	
	I win
So, if we were to play this game a hundred times, who do you think would be leading, you or me?	
	Well, it could be pretty even…
Yeah? So what chance do I have and what chance do you have?	
	50%
Because?	
	Because there's only two, for you it's the same and for me it's different but it's either one of them is heads and one of them is tails or one of them is tails and one of them is heads or the other way around but we're not saying like they have a specific order..
So who would be leading, would you have a bigger chance than I have?	
	I think I would but that's only from experience with die
But your chance and my chance are they the same?	
	Practically…
Practically the same? As opposed to theoretically the same? Theoretically what would they be?	
	Anytime you can just drop them… Some people just have a way of throwing them so if they figure out a strategy that they always win… it's like that seed game, you figure out a strategy and you have to try and make the other guy guess the wrong one and it isn't exactly all on his side, kinda like that.
Ah-huh. So let's try something else. Suppose I have a bag with 2 red and 3 blue candies (A). Suppose I have another bag with 20 red candies and 30 blue candies (B). And you wanted a red candy. Which one would you prefer to draw from?	
	The last one… they are practically both the same, you just timesed it by 10.
I did, so would one bag give you a better chance than the other?	

	Not really.
What if I had a bag like this, with 6 red and	
9 blue?	
	It think it 's times 3
So have the chances changed?	
	Not really?
So you agree the chance is the same in all?	
	Yes
If you had to choose and you wanted a red candy, which one would you choose?	
	Probably the lesser one
The lesser one, because?	
	It's just like simpler and like, I don't know, this stuff makes you think...
That's exactly right because you have to think about your chances...	
	I am going to try something...*plays with the dice, makes a guess a few times and notes what he gets*..
When you make a guess and it turns out to be something else what does that make you think?	
	Cheaters! *laughs*
But you see probability is not about what happens on the next turn. It's all in the behaviour...	
	Because it would be like photographic memories, photographic numbers...
So let's look at the question of using a die instead of a coin toss..	
	I messed up on that, I meant 1, 2, 3 and then 4, 5, 6
Or? Or odds and evens, sure,	
	Yeah.
Ok, now this one here you have a coin and you get 4 heads in a row...	
	I said two things that could happen.
You had Tails because you were luck you get 4 heads in a row. Heads because you might have found some flipping strategy. But if you weren't trying to influence the outcome of the toss, which one would be more likely? Would anyone be more likely?	
	I think it would be the last one...
It's not that we asking you say what would happen on the next toss...	
	I am not psychic...
But the idea is... this is something that gamblers do all the time, they think they haven't got tails in a while so they think they are due for tails, what do you think about that....	
	Those gamblers,,,,, well you watch poker and sometimes they'd fold and I have seen people fold repetitively because they have gotten really bad cards so anything could really happen because the deck is set in a certain way and you

	shuffle it and don't know anything so it's all in the deck and you get what you get
So if you have 10 heads in a row does it make tails more likely?	
	No, and it doesn't make heads more likely because it could mean many other reasons…
Lovely. Thank you for your insights.	