

The space of left orderings of a group

With applications to topology

by

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Abstract

A group is left orderable if there exists a strict total ordering of its elements that is invariant under multiplication from the left. The set of all left orderings of a group comes equipped with a natural topological structure and group action, and is called the space of left orderings. This thesis investigates the topology of the space of left orderings for a given group, by analyzing those left orderings that correspond to isolated points and by characterizing the orbits of the natural group action using maps from a certain free object. Lastly, we find an application of the space of left orderings in the field of 3-manifold topology, by using compactness to show that certain fundamental groups are not left orderable.

Table of Contents

Abstract	ii
Table of Contents	iii
List of Figures	v
1 Introduction to orderable groups	1
1.1 Left orderability	1
1.2 Conrad orderability	4
1.3 Bi-orderability	7
1.4 Criteria for orderability and examples	9
1.4.1 Non-constructive criteria for orderability	9
1.4.2 Constructive criteria for orderability	11
1.4.3 Examples of left, Conrad, and bi-orderable groups	16
2 The space of left orderings	19
2.1 Definitions and elementary properties	19
2.1.1 The infinite strand braid group	25
2.2 Applications of the space of left orderings	26

Table of Contents

3	The topology of the space of left orderings	30
3.1	The case of Conradian orderings	32
3.2	Isolated points	33
3.3	Dense orderings and discrete orderings	39
3.3.1	Abelian groups	41
3.3.2	Non-abelian groups	46
3.4	The braid groups	50
4	Analysis of the natural action on the space of left orderings	56
4.1	Lattice-ordered groups	56
4.2	Free lattice ordered groups and the topology on the space of left orderings	62
4.3	Examples	73
4.3.1	The free group	73
4.3.2	Left orderable groups with all left orderings Conradian	74
4.3.3	The braid groups	77
5	3-manifolds and left orderability	79
5.1	Examples of left orderable fundamental groups	79
5.2	Geometric applications of left orderability	83
6	An application of the space of left orderings to 3-manifolds	86
6.1	The map $\psi_{G,H}$	86
6.1.1	Example: Torus knots	95
6.2	Analysis of the map $\psi_{G,H}$ for bi-orderable groups	98
	Bibliography	104

List of Figures

5.1	The figure eight knot.	82
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Chapter 1

Introduction to orderable groups

In this chapter, we cover standard notions and equivalences pertaining to orderable groups. A good reference for this material is [5], or [30] and [24] for more modern results.

1.1 Left orderability

Definition 1.1.1. *A group G is said to be left orderable if there exists a strict total ordering $<$ of the its elements such that $g < h$ implies $fg < fh$ for all f, g, h in G .*

We can associate to every left ordering $<$ of G its positive cone P , which is defined as $P = \{g \in G \mid g > 1\}$. The positive cone satisfies

1. $P \cdot P \subset P$, and
2. $P \sqcup P^{-1} \sqcup \{1\} = G$, where $A \sqcup B$ denotes the disjoint union of A and B .

Conversely, given any subset $P \subset G$ satisfying properties (1) and (2)

1.1. Left orderability

above, we can define a strict, left invariant total ordering of G according to the rule $g < h$ if and only if $g^{-1}h \in P$.

It is natural to ask whether or not there is a notion of ‘right orderability’ of a group. The idea of totally ordering the elements of a group so that the ordering is invariant under multiplication from the right is equivalent to left orderability. Observe that if $<_R$ is a right ordering of G we may define a left ordering $<_L$ of G according to the rule $g <_L h$ if and only if $g^{-1} <_R h^{-1}$. Therefore, whether we speak of left or right orderable groups is a matter of convention. As the group actions in this text are written from the left, our groups will be left orderable.

It is an easy first exercise with left orderable groups to discover that they are all torsion free. For if G is left ordered and g is an element of G satisfying $g > 1$, then successive left multiplication yields $g^2 > g$, $g^3 > g^2$, ... etc. Combining inequalities gives $1 < g < g^2 < g^3 < \dots$ so that no power of g can be equal to the identity.

Definition 1.1.2. *If G is a left ordered group with ordering $<$, a set S is said to be convex relative to the left ordering $<$ if $a, c \in S$ and $a < b < c$ implies $b \in S$.*

The case when S is a subgroup is of particular importance.

Proposition 1.1.3 ([30], Propositions 2.1.1 - 2.1.3). *Suppose that C is a subgroup of the left ordered group G with ordering $<$. Then C is convex relative to the ordering $<$ if and only if the prescription*

$$g < h \Rightarrow gC \prec hC$$

1.1. Left orderability

provides a well-defined left invariant ordering \prec of the set of left cosets G/C .

This allows us to interpret left orderings with convex subgroups as ‘lexicographic’ orderings in the following sense:

Proposition 1.1.4 ([30], Propositions 2.1.1 - 2.1.3). *Suppose that C is a convex subgroup relative to the left ordering $<$ of G , whose positive cone we denote by $P \subset G$. Set $P_C = P \cap C$, and let $P_{G/C}$ be the set of left cosets whose representatives are positive in the left ordering $<$ of G . If we denote the quotient map onto the set of left cosets by $q : G \rightarrow G/C$, then*

$$P = P_C \cup q^{-1}(P_{G/C}).$$

The previous proposition is closely related to the notion of orderability of extensions.

Proposition 1.1.5 ([30], Proposition 1.2.6). *Suppose that*

$$1 \rightarrow K \xrightarrow{i} G \xrightarrow{q} H \rightarrow 1$$

is a short exact sequence. If K and H are left orderable groups, then G is also left orderable.

One of the most common applications of Propositions 1.1.3, 1.1.4 and 1.1.5 is to ‘modify’ a given left ordering on a prescribed convex subgroup. Using the notation of Proposition 1.1.4, we can make this notion more precise: The positive cone P may be written as two disjoint parts, P_C and $q^{-1}(P_{G/C})$. If we choose any other positive cone $Q_C \subset C$ different from P_C ,

then it is easy to check that $Q_C \cup q^{-1}(P_{G/H})$ defines a new positive cone in the group G . The left ordering defined by this new positive cone differs from our original left ordering, as is easily seen by comparing the two left orderings when restricted to the subgroup C .

We may also create new convex subgroups relative to a given left ordering, by taking the intersection or union of an arbitrary family of convex subgroups.

Proposition 1.1.6 ([13]). *Suppose that the subgroups $\{C_i\}_{i \in I}$ of G are convex relative to the left ordering $<$. Then $\bigcap_{i \in I} C_i$ and $\bigcup_{i \in I} C_i$ are also convex subgroups relative to the left ordering $<$ of G .*

Finally, we introduce an important piece of terminology.

Definition 1.1.7. *A left ordering $<$ of a group G is called Archimedean if for every pair of elements $g, h \in G$, with h different from the identity, there exists an integer $n \in \mathbb{Z}$ such that $g < h^n$.*

Proposition 1.1.8 (Hölder, for a proof see [13]). *If a group G admits an Archimedean left ordering $<$, then there exists an order-preserving isomorphism $i : G \rightarrow (\mathbb{R}, +)$, so that G is isomorphic to a subgroup of the additive real numbers. Hence, G is abelian.*

1.2 Conrad orderability

A stronger notion than that of left orderability is Conrad orderability.

Definition 1.2.1 ([13]). *A group G is said to be Conrad orderable if there exists a left ordering $<$ of G such that the following implication holds: If*

1.2. Conrad orderability

g and h are positive, then there exists $n \in \mathbb{N}$ such that $g < hg^n$. If this implication holds for some left ordering $<$ of a group G , then $<$ is called a *Conradian ordering*.

It has since been observed that requiring $n = 2$ is equivalent to our definition above, so we may alternatively think of the Conradian orderings of a group G as those left orderings satisfying $g < hg^2$ for all pairs of positive elements $g, h \in G$ [40]. Saying that a group G is Conrad orderable may also be interpreted as saying that G contains a positive cone P satisfying the third condition:

3C. $g \in P$ implies $g^{-1}Pg^2 \subset P$.

The first investigation of Conradian orderings was motivated by their rich structure, in the sense that a Conradian ordering on a group G determines a family of convex subgroups of G . These convex subgroups can be indexed by elements of the group G , are ordered by inclusion, with successive subgroups in the ordering having abelian quotients.

To be more specific, we first need a definition.

Definition 1.2.2. *Suppose that $C \subset D$ are two convex subgroups in some left ordering $<$ of a group G . Then D is said to cover C , or alternatively $C \rightarrow D$ is a convex jump, if there are no convex subgroups between C and D .*

Theorem 1.2.3 (Conrad, [13]). *Let G be a group equipped with a Conradian ordering $<$. Then the following hold:*

1.2. Conrad orderability

1. Every element $g \in G$ determines a convex jump $C_g \rightarrow D_g$ with $g \in D_g$ and $g \notin C_g$, by setting

$$D_g = \bigcap_{\substack{C \text{ convex} \\ g \in C}} C$$

and

$$C_g = \bigcup_{\substack{C \text{ convex} \\ g \notin C}} C.$$

2. Every convex jump $C \rightarrow D$ in the ordering $<$ arises from an element $g \in G$ as above, that is, $C = C_g$ and $D = D_g$ for some $g \in G$.
3. For every convex jump $C \rightarrow D$, C is normal in D , and the quotient D/C is abelian. The ordering $<$ descends to a well-defined ordering on the quotient D/C (since C is convex), and this ordering is Archimedean.

In particular this theorem allows one to define, for every convex jump $C \rightarrow D$, an injective homomorphism $\tau : D/C \rightarrow (\mathbb{R}, +)$ by using Proposition 1.1.8. Often referred to as the ‘Conrad homomorphism’, these maps are essential to many of the modern dynamical approaches to left orderability [40, 41, 54]. With this theorem, we can also classify those Conradian orderings without convex subgroups.

Proposition 1.2.4. *Suppose $<$ is a Conradian left ordering of G . Then the ordering $<$ has no proper, nontrivial convex subgroups if and only if it is Archimedean.*

Proof. In one direction, it is easy to see that every Archimedean ordering is Conradian (c.f. Proposition 1.3.3). Conversely, given any non identity

element $g \in G$, there exists a convex jump $C_g \rightarrow D_g$, as defined in Theorem 1.2.3. In the case that G contains no proper, nontrivial convex subgroups, we must have $C_g = \{1\}$ and $D_g = G$. The ordering on the quotient $D_g/C_g = G$ is Archimedean by Theorem 1.2.3, and the proposition follows. \square

In light of Proposition 1.1.8, we know that Conradian ordered groups that admit no convex subgroups are abelian groups.

It should also be noted that modifying a left ordering on a convex subgroup, as explained following Proposition 1.1.5, can be done in a way that preserves the Conradian structure of a left ordering. If P defines a Conradian ordering of G relative to which C is a convex subgroup, then we may replace $P \cap C$ with any positive cone $Q \subset C$ which defines a Conradian ordering of C . Then the set $Q \cup (P \setminus (C \cap P))$ defines a positive cone in G , whose associated left ordering is Conradian.

1.3 Bi-orderability

Definition 1.3.1. *A group G is said to be bi-orderable if there exists a left ordering of G that is also invariant under multiplication from the right, so that $g < h$ implies $gf < hf$ for all $f, g, h \in G$.*

In the language of positive cones, this is equivalent to requiring that the positive cone P of a given left ordering satisfy the additional third condition

3. $gPg^{-1} = P$ for all g in G .

It is easy to see that bi-orderability is a strengthening of Conrad orderability. Indeed, a bi-ordering of a group G satisfies $g < hg^n$ for every

1.3. Bi-orderability

positive pair $g, h \in G$ by taking $n = 1$.

In the case of left orderings, we found that we were able to left order a group G if it fit into a short exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$$

with both K and H left orderable. In fact, we did not even require the set of left cosets of K to have a group structure. In the case of bi-orderings the technique of using short exact sequences to construct positive cones in groups is more restricted.

Proposition 1.3.2. *Suppose that*

$$1 \rightarrow K \xrightarrow{i} G \xrightarrow{q} H \rightarrow 1$$

is an exact sequence, and that K and H admit bi-orderings whose positive cones are P_K and P_H respectively. Then

$$P = i(P_K) \cup q^{-1}(P_H)$$

defines the positive cone of a bi-ordering of G if and only if $gP_Kg^{-1} = P_K$ for all $g \in G$.

Finally, there is an important connection between Archimedean, Conradian, and bi-orderings.

Proposition 1.3.3. *Let G be a left orderable group, and suppose that $<$ is an Archimedean ordering. Then $<$ is a bi-ordering.*

Proof. We apply Proposition 1.1.8 to conclude that G is abelian. Then, we observe that $gf < hf$ if and only if $fg < fh$, left multiplying by f^{-1} yields $g < h$, establishing right invariance. \square

This shows the underlying connection between Conrad orderability and bi-orderability: For every convex jump $C \rightarrow D$ in a Conradian ordered group G , the quotient D/C is bi-ordered by the inherited quotient ordering, because the ordering is Archimedean. These bi-orderings on successive quotients in G together do not determine a bi-ordering of G (instead merely a Conradian ordering), precisely because of the additional hypotheses required in Proposition 1.3.2 versus Proposition 1.1.4.

1.4 Criteria for orderability and examples

The primary purpose of this section is to provide a footing in the theory of orderable groups, by reviewing criteria that allow us to conclude that different groups are left, Conrad, or bi-orderable. In particular, while bi-orderable groups are Conrad orderable, and Conrad orderable groups are left orderable, we will give examples which show that neither implication is reversible.

1.4.1 Non-constructive criteria for orderability

Our first criteria is a theorem by Burns and Hale, the proof of which is non-constructive.

Theorem 1.4.1 (Burns-Hale, [8]). *A group is left orderable if and only*

1.4. Criteria for orderability and examples

if every nontrivial finitely generated subgroup maps onto a nontrivial left orderable group.

Let T be a property of groups, and suppose that every finitely generated subgroup of a group G has property T . If this implies that G itself has property T , then T is called a local property. From the theorem of Burns and Hale, we conclude that left orderability is a local property (in fact, Conrad orderability and bi-orderability are also local properties).

Theorem 1.4.2. *Suppose that every finitely generated subgroup of G is left orderable (respectively Conrad orderable, bi-orderable). Then G is left orderable (respectively Conrad orderable, bi-orderable).*

This allows for us to easily bi-order torsion free abelian groups, by first dealing with the case of \mathbb{Z}^n geometrically. Let n be any positive integer, and let H be a hyperplane in \mathbb{R}^n . We may choose H so that no element of \mathbb{Z}^n lies in H (c.f. the proof of Lemma 3.3.4). Therefore, we may create a positive cone $P \subset \mathbb{Z}^n$ by declaring the elements of \mathbb{Z}^n on one side of the hyperplane H to be P . Hence, \mathbb{Z}^n is bi-orderable, and we can apply Theorem 1.4.2 to conclude:

Corollary 1.4.3. *Every torsion free abelian group is bi-orderable.*

The idea of Burns and Hale can be strengthened to guarantee the existence of a Conradian ordering.

Definition 1.4.4. *A group G is locally indicable if every finitely generated subgroup of G maps surjectively on to \mathbb{Z} .*

Theorem 1.4.5 ([24, 40, 53]). *A group is Conrad orderable if and only if it is locally indicable.*

It was unknown for some time whether or not left orderability is equivalent to local indicability, but it is now known that local indicability is a strictly stronger notion [4]. As local indicability and Conrad orderability are very powerful restrictions on a group G and can provide great insight into its structure, it is natural to attempt to connect the weaker notion of left orderability with that of Conrad orderability. Efforts in this direction have focused on addressing the following question.

Question 1.4.6. *For what class of groups is left orderability equivalent to Conrad orderability (local indicability)?*

This question remains open at the time of writing. Modern approaches and results will be discussed in Section 2.2.

A final non-constructive criterion is as follows.

Theorem 1.4.7 ([26]). *Every torsion free one relator group is Conrad orderable.*

This criterion is sometimes quite difficult to use, as the problem of determining when a one relator group is torsion free can be quite challenging. Perhaps most useful is the following corollary:

Corollary 1.4.8. *Every left orderable one relator group is Conrad orderable.*

1.4.2 Constructive criteria for orderability

In this section we will investigate those groups which are known to be left, Conrad, or bi-orderable by virtue of the construction of a left, Conrad,

or bi-ordering of their elements. As a first class of examples, and as a useful theorem in many applications to come, we investigate groups of order-preserving automorphisms of sets.

Definition 1.4.9. *Suppose that a group G acts on a set S . The action of G on S is said to be effective if for every $g \in G$ different from the identity, there exists $x \in S$ such that $gx \neq x$.*

Theorem 1.4.10. *A group G is left orderable if and only if G acts effectively on some linearly ordered set $(\Omega, <)$ by order preserving bijections.*

Proof. One direction is trivial: If G is left orderable, then G acts on itself in an effective, order preserving manner by left multiplication.

Conversely, suppose that G acts effectively on the linearly ordered set $(\Omega, <)$. Choose some well-ordering \prec of Ω , and for each $g \in G$, let x_g be the smallest element in Ω (with respect to our chosen well ordering) such that $g(x) \neq x$. Declare $g > 1$ in G if $g(x_g) > x_g$.

We must check that this rule defines a set P satisfying $P \cdot P \subset P$, and $G = P \sqcup P^{-1} \sqcup \{1\}$. To this end, suppose that g and h are both positive, and so satisfy $g(x_g) > x_g$ and $h(x_h) > x_h$. If $x_g \prec x_h$, then $x_g = x_{gh}$ and $h(x_{gh}) = x_{gh}$ since $x_{gh} \prec x_h$, so we compute

$$gh(x_{gh}) = g(x_{gh}) = g(x_g) > x_g = x_{gh},$$

so that $gh > 1$. On the other hand, if $x_h \prec x_g$ then $x_h = x_{gh}$ and we compute

$$gh(x_{gh}) = gh(x_h) > g(x_h) = g(x_{gh}) = x_{gh},$$

so that $gh > 1$. This shows that the positive cone we have defined satisfies property (1). Property (2) is clear, we need only observe that since the action on Ω is effective the identity is the only element of G which acts trivially. \square

In the case that G is a countable group, this result can be surprisingly strengthened by taking $\Omega = \mathbb{R}$. For a proof of the following theorem, see [34].

Theorem 1.4.11. *Suppose that G is a countable group. Then the following are equivalent:*

1. G is left orderable.
2. There exists an injective homomorphism $G \rightarrow \text{Homeo}_+(\mathbb{R})$.
3. There exists an injective homomorphism $G \rightarrow \text{Homeo}_+(\mathbb{Q})$.

These criteria demonstrate that left orderability of a countable group can be interpreted geometrically, and allow for the interpretation of many properties of countable left orderable groups in the context of dynamical systems [40].

We now have our first class of examples of left orderable groups.

Corollary 1.4.12. *Every group of order preserving bijections of a linearly ordered set is left orderable.*

In order to review one of the simplest ways of constructing bi-orderings, recall the definition of the lower central series $\{G_k\}_{k>0}$ in a group G . Set

1.4. Criteria for orderability and examples

$G_1 = G$ and $G_{k+1} = [G, G_k]$, the subgroup of G generated by all commutators $ghg^{-1}h^{-1}$ where $g \in G$ and $h \in G_k$. The quotients G_k/G_{k+1} are abelian groups, and for every k the short exact sequence

$$1 \rightarrow G_k/G_{k+1} \rightarrow G/G_{k+1} \rightarrow G/G_k \rightarrow 1$$

is a central extension.

Theorem 1.4.13 ([44]). *Let G be a group. Suppose that*

$$\bigcap_{i=1}^{\infty} G_i = \{1\},$$

and that every quotient G_k/G_{k+1} is torsion free. Then G is bi-orderable.

Proof. Define a bi-ordering of G as follows. For each k , let $q_k : G_k \rightarrow G_k/G_{k+1}$ denote the quotient map. As each quotient is a torsion free abelian group, G_k/G_{k+1} is bi-orderable by Corollary 1.4.3. Thus we may fix a positive cone $P_k \subset G_k/G_{k+1}$ for every $k > 0$. Define a positive cone $P \subset G$ according to the formula:

$$P = \bigcup_{i=1}^{\infty} q_i^{-1}(P_i).$$

We must check that P satisfies the required properties. For ease of exposition, denote by $n(g)$ the unique integer n for which $g \in G_n \setminus G_{n+1}$ (such an integer exists for all g since we are assuming that $\bigcap_{i=1}^{\infty} G_i = \{1\}$). Thus $g \in G$ lies in P if and only if $q_{n(g)}(g)$ lies in $P_{n(g)}$.

1. $P \cdot P \subset P$. Given $g, h \in P$, set $n = \min\{n(g), n(h)\}$. Then $q_n(g)$ and

1.4. Criteria for orderability and examples

$q_n(h)$ are both non-negative elements in the ordered group G_n/G_{n+1} , thus their product is non-negative, i.e. $q_n(gh) \in P_n$. Hence $gh \in q_n^{-1}(P_n)$, and $gh \in P$.

2. $G = P \sqcup P^{-1} \sqcup \{1\}$. Given $g \in G$ different from the identity, observe that $g \in P$ and $g^{-1} \in P$ if and only if $q_{n(g)}(g) \in P_{n(g)}$ and $q_{n(g)}(g^{-1}) \in P_{n(g)}$ (here, we use the fact that $n(g) = n(g^{-1})$). This is not possible since $P_{n(g)}$ is a positive cone. Similarly, we find that $g \notin P$ and $g^{-1} \notin P$ is not possible, because this would imply that $q_{n(g)}(g) \notin P_{n(g)}$ and $q_{n(g)}(g)^{-1} \notin P_{n(g)}$.

3. $gPg^{-1} = P$. Let $h \in P$ be given. There are three cases to consider.

- (a) $n(g) = n(h)$. In this case, $q_{n(g)}(ghg^{-1}) = q_{n(g)}(h) \in P_{n(g)}$, since the image of $q_{n(g)}$ is abelian. Hence $ghg^{-1} \in P$.
- (b) $n(g) > n(h)$. Then $q_{n(h)}(g) = 1$, and so $q_{n(h)}(ghg^{-1}) = q_{n(h)}(h) \in P_{n(h)}$, and we conclude that $ghg^{-1} \in P$.
- (c) $n(g) < n(h)$. In this case we may use the central extension

$$1 \rightarrow G_{n(h)}/G_{n(h)+1} \rightarrow G/G_{n(h)+1} \rightarrow G/G_{n(h)} \rightarrow 1$$

to write $ghg^{-1} = hc$ for some $c \in G_{n(h)+1}$. Then c satisfies $q_{n(h)}(c) = 1$, and we calculate $q_{n(h)}(ghg^{-1}) = q_{n(h)}(hc) = q_{n(h)}(h) \in P_{n(h)}$, and hence $ghg^{-1} \in P$.

□

Bi-orderings of a group G that are constructed as in the theorem above

are called standard bi-orderings. It is well known that free groups satisfy the hypotheses of Theorem 1.4.13, namely

$$\bigcap_{i=1}^{\infty} G_i = \{1\} \text{ and } G_k/G_{k+1} \text{ is torsion free.}$$

Thus we arrive at the useful corollary:

Corollary 1.4.14. *Free groups are bi-orderable.*

1.4.3 Examples of left, Conrad, and bi-orderable groups

With some preliminary theorems behind us, we turn to the question of finding a left orderable group that is not locally indicable. The most classical method of finding a group that is left orderable, but not locally indicable, is by finding a group that is left orderable and contains a finitely generated perfect subgroup. If a group G contains a finitely generated perfect subgroup H , then G cannot be locally indicable since there is no surjective map $H \rightarrow \mathbb{Z}$. This is the method applied in [4] to show that $\widetilde{SL_2(\mathbb{R})}$ is not locally indicable, and in [53] to show that the braid groups are not locally indicable for $n > 4$ (see Section 3.4 for an explicit construction of a left ordering of the braid groups). In the case of $\widetilde{SL_2(\mathbb{R})}$, the finitely generated perfect subgroup is isomorphic to

$$\langle x, y, z \mid x^2 = y^3 = z^7 = xyz \rangle,$$

in the case of the braid groups the commutator subgroup $[B_n, B_n]$ is finitely generated and perfect for $n > 4$ [39].

1.4. Criteria for orderability and examples

This is not the only method for showing that a group is left orderable but not locally indicable. The subgroup of $\text{Aut}(\mathbb{Q})$ of functions f satisfying

1. $f(x) + 1 = f(x + 1)$, and
2. f is piecewise linear on $[0, 1]$

is also left orderable, but not locally indicable. The proof of this fact is delicate and does not rely upon the construction of a finitely generated perfect subgroup, see [61].

While it is generally difficult to find a left orderable group that is not locally indicable, it is quite easy to find examples of locally indicable groups that are not bi-orderable. As an easy example, consider the group

$$G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle.$$

This group is easily seen to be left orderable, since it fits into the short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1.$$

As G is a one relator group, it is locally indicable by Corollary 1.4.8, and so it is Conrad orderable. However, G cannot be bi-ordered, because the element $y \in G$ is conjugate to its own inverse. Supposing G were bi-ordered so that $y > 1$, we could left multiply by x and right multiply by x^{-1} to yield $1 < xyx^{-1} = y^{-1}$, a contradiction.

As a final example, recall that the braid group B_3 is defined by the presentation

$$\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

1.4. Criteria for orderability and examples

The commutator subgroup of B_3 is isomorphic to the free group on two generators, with $\sigma_2\sigma_1^{-1}$, $\sigma_1\sigma_2\sigma_1^{-2}$ as a generating set [39]. Thus, we may construct a Conradian ordering of B_3 by using the short exact sequence

$$1 \rightarrow F_2 \rightarrow B_3 \rightarrow \mathbb{Z} \rightarrow 1,$$

and a bi-ordering of the free group F_2 . However, B_3 is easily seen to be not bi-orderable, as it contains an element that is conjugate to its own inverse. It is easy to check that

$$(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_1\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1) = \sigma_2\sigma_1^{-1}.$$

Chapter 2

The space of left orderings

The idea of topologizing the set of all left orderings of a given group first appears in [58], although the results found there primarily deal with the case of countable groups. When G is a countable group, we will see that the topology on the set of all left orderings arises from a metric, and the space of left orderings has many properties in common with the Cantor set.

The introduction to the space of left orderings that we provide here does not rely on countability of the underlying group, and closely follows [21]. While the topology from this perspective can be less intuitive, elementary properties such as compactness are much easier to prove.

2.1 Definitions and elementary properties

Definition 2.1.1. *Let G be a group. The space of left orderings is denoted by*

$$LO(G) = \{P \subset G \mid P \text{ is the positive cone of a left ordering of } G\}.$$

We also define the space of Conradian orderings and bi-orderings, denoted $CO(G)$ and $BO(G)$, to be the set of all positive cones of Conradian orderings

2.1. Definitions and elementary properties

and bi-orderings respectively.

Since all bi-orderings are Conradian, and all Conradian orderings are left orderings, we have inclusions $BO(G) \subset CO(G) \subset LO(G)$.

Defined as the set of all positive cones, instead of as the set of all left orderings, there is a natural inclusion of $LO(G)$ into the power set 2^G . The power set 2^G comes equipped with a topology, a subbasis for which is given by sets of the form

$$U_g = \{S \in 2^G \mid g \in S\}$$

and

$$U'_g = \{S \in 2^G \mid g \notin S\}$$

where g ranges over all elements of G . Basic open sets in 2^G are finite intersections of the subbasic sets, and therefore are of the form

$$\{S \in 2^G \mid g_1, \dots, g_n \in S \text{ and } h_1, \dots, h_m \notin S\}$$

where $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_m\}$ are arbitrary finite sets of elements in G .

The set $LO(G)$ becomes a topological space when equipped with the subspace topology inherited from 2^G . Subbasic open sets in $LO(G)$ are therefore of the form

$$U_g = \{P \in LO(G) \mid g \in P\}$$

and

$$U'_g = \{P \in LO(G) | g \notin P\},$$

where g ranges over all elements of G . Since P is a positive cone, the decomposition $G = P \sqcup P^{-1} \sqcup \{1\}$ allows us to rewrite the sets U'_g as

$$U'_g = \{P \in LO(G) | g \notin P\} = \{P \in LO(G) | g^{-1} \in P\} = U_{g^{-1}}.$$

Thus, as a subbasis for the topology on $LO(G)$ the family of sets $\{U_g\}_{g \in G}$ suffices. A basic open set in $LO(G)$ therefore has the form $\bigcap_{i=1}^n U_{g_i}$, for some finite set of elements $\{g_1, \dots, g_n\}$ in G . The basic open set $\bigcap_{i=1}^n U_{g_i}$ contains all those positive cones whose corresponding left orderings of G satisfy $1 < g_i$ for $i = 1, \dots, n$.

Proposition 2.1.2 ([15, 21, 58]). *With the topology described above, the space $LO(G)$ is compact.*

Proof. Those sets which are not positive cones must not satisfy the two criteria: (1) $P \cdot P \subset P$, and (2) $G = P \sqcup P^{-1} \sqcup \{1\}$. For every set S which violates (1), there must exist elements $g, h \in G$ such that g and h lie in S , while gh does not. In other words, the set of all subsets violating (1) is given by the union

$$U = \bigcup_{g, h \in G} \{S \in 2^G | g, h \in S \text{ and } gh \notin S\}.$$

Similarly, if a set S is to violate condition (2) there must exist some non identity element $g \in G$ such that $g, g^{-1} \in S$ or $g, g^{-1} \notin S$, or we could also have $1 \in S$. The set of all sets satisfying one of these three conditions can

2.1. Definitions and elementary properties

be written as

$$V = \bigcup_{g \in G} \{S \in 2^G \mid g, g^{-1} \in S\} \cup \bigcup_{g \in G \setminus \{1\}} \{S \in 2^G \mid g, g^{-1} \notin S\}.$$

Both U and V are a union of basic open sets, and so $U \cup V$ is open, and thus its complement $LO(G)$ is closed. Recall that by Tychonoff's theorem, 2^G is compact, and hence the closed subset $LO(G)$ must also be compact. \square

We can show that the spaces $CO(G)$ and $BO(G)$ are compact, by checking that the conditions $gPg^{-1} = P$ and $g \in P$ implies $g^{-1}Pg^2 \subset P$ both define closed subsets of 2^G as well.

By using basic open sets, we can also see that $LO(G)$, $CO(G)$ and $BO(G)$ are Hausdorff (in fact, totally disconnected—the only connected components are singletons). Given two distinct positive cones P_1 and P_2 in a set U , suppose that they disagree on an element $g \in G$, that is, $g \in P_1$ and $g \notin P_2$. Then $P_1 \in U_g$ and $P_2 \in U_{g^{-1}}$, while $U_g \cup U_{g^{-1}}$ is the entire space, so that U_g and $U_{g^{-1}}$ disconnect U . Thus $LO(G)$, $CO(G)$ and $BO(G)$ are totally disconnected.

If the group G is countable, we have already indicated that there is a metric on the space $LO(G)$ that induces the same topology as the subbasis provided above.

Proposition 2.1.3 ([58]). *If G is countable, then $LO(G)$, $CO(G)$ and $BO(G)$ are metrizable.*

Proof. Let g_0, g_1, \dots be an enumeration of the elements of G . Given two positive cones $P_1, P_2 \in LO(G)$ (or in $CO(G)$, or $BO(G)$), suppose that the

2.1. Definitions and elementary properties

orderings determined by P_1 and P_2 agree on the elements g_0, \dots, g_{n-1} (that is, $g_i \in P_1$ if and only if $g_i \in P_2$ for all $i < n$), and disagree on g_n . In this case, define $d(P_1, P_2) = \frac{1}{2^n}$. It is easy to check that this defines a metric on the space of left, Conrad or bi-orderings. \square

In fact, we may check that the above definition yields a metric satisfying $d(P_1, P_2) \leq \max\{d(P_1, P_3), d(P_2, P_3)\}$, so that it is an ultrametric. It is a mechanical check to verify that this ultrametric defines a topology agreeing with the subspace topology inherited from 2^G , for full details see [58].

We recall a proposition from elementary topology.

Proposition 2.1.4 (E.g., [25] Corollary 2.98). *Any nonempty, compact, totally disconnected metric space without isolated points is homeomorphic to the Cantor set.*

The spaces of left, Conradian and bi-orderings therefore have many properties in common with the Cantor set when the underlying group is countable.

Proposition 2.1.5 ([58]). *Let G be a countable group, and suppose that $LO(G)$ (resp. $CO(G)$, $BO(G)$) contains no isolated points. Then $LO(G)$ (resp. $CO(G)$, $BO(G)$) is homeomorphic to the Cantor set.*

As a first attempt to better understand these spaces for a given countable group, we can therefore attempt to determine their homeomorphism types by determining whether or not the spaces $LO(G)$, $CO(G)$ or $BO(G)$ contain one-point open sets.

2.1. Definitions and elementary properties

A one-point open set corresponds to a positive P cone satisfying

$$\{P\} = \bigcap_{i=1}^n U_{g_i}$$

for some finite set of elements $\{g_1, \dots, g_n\}$ in G . Such a positive cone in the space $LO(G)$ (or $CO(G)$, or $BO(G)$) corresponds to the unique left (or Conradian, or bi-) ordering of the group G in which the elements g_1, \dots, g_n are positive. Determining the necessary conditions for the existence of such a left ordering will be the main focus of Chapter 3.

There is also a natural group action by homeomorphisms on the space of left orderings. For each element g in a group G , the action on a positive cone P is by conjugation, $P \mapsto gPg^{-1}$. In terms of left orderings of the group G , this action may be interpreted as follows: if $<$ is the left ordering of G arising from the positive cone P , and $<_g$ is the left ordering with positive cone gPg^{-1} , then $ag < bg$ if and only if $a <_g b$ for all $a, b \in G$.

The condition that distinguishes the positive cones of bi-orderings from those of left orderings is invariance under conjugation, recall that P is the positive cone of a bi-ordering if $gPg^{-1} = P$ for all $g \in G$. Therefore, the bi-orderings of a group G are exactly the fixed points of the natural action of G on $LO(G)$.

The role played by the Conradian orderings of a group with respect to this action is not yet clear, and is an active area of research. The positive cone of a Conradian ordering is required to satisfy $g^{-1}Pg^2 \subset P$ for all $g \in P$, which can almost certainly be given a clean topological or dynamical interpretation in terms of the G -action on $LO(G)$. As yet, however, no

satisfactory topological or dynamical characterization of the subspace of Conradian orderings exists (see [43] for the latest efforts in this direction).

The task of better understanding the G -action on $LO(G)$ will be the focus of Chapter 4.

2.1.1 The infinite strand braid group

One of the primary uses of the natural action on $LO(G)$ has been to disprove the existence of isolated points in $LO(G)$ by showing that a given positive cone P is an accumulation point of $Orb_G(P)$. As an example of this approach, we will investigate the infinite strand braid group B_∞ , which is defined by the presentation

$$\langle \sigma_1, \sigma_2, \sigma_3, \dots \mid \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \rangle.$$

To show that B_∞ is left orderable is quite difficult, an explicit left ordering will be described in Section 3.4, for full details see [21]. The group B_∞ is an example of a left orderable group with the property that every point in $LO(B_\infty)$ is an accumulation point of its orbit under the B_∞ -action.

Proposition 2.1.6 ([21]). *The space of left orderings of B_∞ is homeomorphic to the Cantor set, in fact every positive cone $P \in LO(B_\infty)$ is an accumulation point of its conjugates.*

Proof. Let $P \in LO(B_\infty)$, and suppose that P lies in the basic open set $U = \bigcap_{i=1}^n U_{\beta_i}$ where $\{\beta_1, \dots, \beta_n\}$ is some finite set of elements in B_∞ . We will show that U contains a conjugate of P that is different from P itself.

2.2. Applications of the space of left orderings

As the braids β_1, \dots, β_n are represented by finite words in the generators $\sigma_1, \sigma_2, \dots$, there exists $k > 0$ such that every generator σ_i for $i \geq k$ commutes with every braid β_j , for $j = 1, \dots, n$.

We claim that there exists σ_l with $l \geq k$ such that $\sigma_l P \sigma_l^{-1} \neq P$. For contradiction, suppose not, and consider the embedding $sh^k : B_\infty \rightarrow B_\infty$ given by $\sigma_i \mapsto \sigma_{i+k}$, $i > 0$. The intersection $Q = sh^k(B_\infty) \cap P$ is a positive cone that defines a left ordering of the image $sh^k(B_\infty)$. Moreover, as we are assuming that $\sigma_l P \sigma_l^{-1} = P$ for all $l \geq k$, the positive cone Q is invariant under conjugation by elements of $sh^k(B_\infty)$. Thus, Q defines a bi-ordering of $sh^k(B_\infty)$, which is a group isomorphic to B_∞ . This is not possible, as we have already observed that B_3 is not bi-orderable, and B_3 is a subgroup of B_∞ .

Therefore we may choose σ_l , with $l \geq k$ (so that σ_l commutes with the braids β_i , for $i = 1, \dots, n$) such that $\sigma_l P \sigma_l^{-1} \neq P$. Then we observe that

$$\beta_i \in P \Rightarrow \sigma_l \beta_i \sigma_l^{-1} \in \sigma_l P \sigma_l^{-1} \Rightarrow \beta_i \in \sigma_l P \sigma_l^{-1},$$

so that $\sigma_l P \sigma_l^{-1} \in U$. Since we have chosen l so that $\sigma_l P \sigma_l^{-1} \neq P$, we conclude that P is not isolated and $LO(B_\infty)$ is homeomorphic to the Cantor set by Proposition 2.1.5. □

2.2 Applications of the space of left orderings

Two open problems have recently been solved using the space of left orderings and the natural group action. We will outline their solutions and the

role played by the space of left orderings, ending with an indication of where we may find future applications of $LO(G)$.

The first question we will address is concerned with the cardinality of $LO(G)$. There exist groups for which $LO(G)$, $CO(G)$ and $BO(G)$ are finite (see [30, 54], and [5] respectively), and it is known that $BO(G)$ may be countably infinite [9]. Moreover, there are many easy cases where $LO(G)$, $CO(G)$ and $BO(G)$ are uncountable, this happens whenever there is a subset of the space that is homeomorphic to the Cantor set (which is common, as we will see). We find the following question in [35]:

Question 2.2.1. *Does there exist a group admitting countably infinitely many left orderings?*

Theorem 2.2.2 (Conjectured by Tararin [30], resolved by Linnell [33]). *No group has a countable infinity of left orderings.*

Sketch of proof. If X is a Hausdorff topological space, define X' to be the set of limit points of X . Then for each ordinal α , define $X^{(\alpha)}$ by transfinite induction as follows:

- $X^{(0)} = X$,
- $X^{(\alpha+1)} = (X^{(\alpha)})'$
- $X^{(\alpha)} = \bigcap_{\lambda < \alpha} X^{(\lambda)}$ if α is a limit ordinal.

For contradiction, we suppose that $LO(G)$ is countable. Using compactness and a theorem of Poincaré and Bendixson, it follows that if $LO(G)$ is countable then $LO(G)^{(\alpha)}$ is finite for some ordinal α , say $LO(G)^{(\alpha)} = \{P_1, \dots, P_n\}$.

2.2. Applications of the space of left orderings

Then the action of G on $LO(G)$ permutes this finite set $\{P_1, \dots, P_n\}$, since each set $X^{(\alpha)}$ is invariant under the action by homeomorphisms. Thus $Stab_G(P_i)$ is finite index in G , and bi-orderable by Lemma 3.2.3. It is known that groups admitting a finite index bi-orderable subgroup do not have countably many left orderings, [63]. \square

The next theorem outlines the most recent advancement towards answering Question 1.4.6.

Theorem 2.2.3 (Conjectured by Linnell [34], resolved by Morris [37]). *Suppose that G is an amenable group. Then G is left orderable if and only if G is locally indicable (Conrad orderable).*

Sketch of proof. If we are to check local indicability of a given amenable, left orderable group, it is sufficient to check on a finitely generated (hence countable) subgroup. Thus, we assume that G is countable.

Recall that a discrete group G is amenable if and only if for every continuous action of G on a compact, Hausdorff space X , there is a G -invariant probability measure on X . Thus, the definition of amenability supplies us with a probability measure on $LO(G)$ that is invariant under the G -action. With the proper setup, an application of the Poincaré recurrence theorem [60] yields a set $A_g \subset LO(G)$ for every $g \in G$ that consists of positive cones that satisfy $g^{-1}Pg^2 \subset P$ (in fact, they satisfy a stronger condition that Morris calls ‘recurrent for the cyclic subgroup $\langle g \rangle$ ’). The conditions of the Poincaré recurrence theorem also guarantee that the complement $LO(G) \setminus A_g$ is measurable, and has measure zero.

2.2. Applications of the space of left orderings

We conclude that the union $\bigcup_{g \in G} (LO(G) \setminus A_g)$ has measure zero, since it is a countable union of sets of measure zero, and thus $\bigcap_{g \in G} A_g$ is nonempty. An element in the intersection $\bigcap_{g \in G} A_g$ satisfies $g^{-1}Pg^2 \subset P$ for all $g \in G$, and thus the corresponding ordering is Conradian. \square

Denote by *NFS* the family of all groups that contain no non-abelian free subgroups. Amenable groups do not contain non-abelian free subgroups, so every amenable group is in *NFS*. Peter Linnell has made the following stronger conjecture.

Conjecture 2.2.4. *Suppose that G is in NFS. Then G is left orderable if and only if G is Conrad orderable.*

It is thought that a proof of this conjecture may employ the space of left orderings in one of two ways. The condition of being left orderable, but not Conrad orderable places a restriction on the natural action of G on $LO(G)$, and this restriction may allow one to create a free subgroup of G . For example, a deeper understanding of the G -action and its relationship with Conradian orderings may allow for an application of the ping-pong lemma [18], or the construction of a group action on a tree so that we may apply results from Bass-Serre theory [57].

Chapter 3

The topology of the space of left orderings

As in the case of B_∞ , it is possible that a positive cone P in $LO(G)$ may be approximable by its orbit under the G -action. In this chapter, we develop conditions on a positive cone P that guarantee it is an accumulation point of its conjugates. By imposing these conditions on the positive cones of dense left orderings of G , we arrive at a dense G_δ set within a Cantor set in $LO(G)$.

Following [40], we define the Conradian soul $C_<(G)$ in a left ordered group G with ordering $<$ to be the largest convex subgroup $C \subset G$ such that the restriction of $<$ to C is a Conradian ordering. Similarly, we use the notation $B_<(G)$ to denote the largest convex subgroup $C \subset G$ such that the restriction of $<$ to C is a bi-ordering. Note that we always have $B_<(G) \subset C_<(G)$, since all bi-orderings are also Conradian orderings.

Using this notation, one of the strongest results of [40], which we will extend here to the case of uncountable groups, can be stated as follows.

Theorem 3.0.1. *Let G be a nontrivial group, and let $P \in LO(G)$ be an isolated point with associated ordering $<$ of G . Then $B_<(G)$ is abelian of rank*

one, and $C_{<}(G)$ is non-trivial and admits only finitely many left orderings.

While Theorem 3.0.1 is proven for the case of countable groups in [40], the dynamical approach used therein is entirely different than our approach, and does not seem to generalize to the case of uncountable groups.

Finally, recall that a left ordering of a group G is dense if whenever $g < h$, then there exists $f \in G$ such that $g < f < h$. If a left ordering $<$ of G is not dense, then it is discrete, meaning that in the ordering $<$ of G there is a least positive element $\epsilon > 1$. We explore the structure of $LO(G)$ by considering the cases of dense and discrete left orderings separately, and we will find the following. Recall that the rank of an abelian group A is the dimension of $\mathbb{Q} \otimes A$ considered as a vector space over \mathbb{Q} .

Theorem 3.0.2. *Let $DLO(G) \subset LO(G)$ denote the set of all dense left orderings of a countable group G , and suppose that all rank one abelian subgroups of G are isomorphic to \mathbb{Z} . Then if $DLO(G)$ is non-empty, its closure $\overline{DLO(G)}$ is homeomorphic to the Cantor set, and the set $DLO(G)$ is a G_δ set in $\overline{DLO(G)}$.*

Note that our requirement that all rank one abelian subgroups be isomorphic to \mathbb{Z} is necessary, as the conclusion of the theorem clearly fails to hold for the abelian group $G = \mathbb{Q}$.

In the case of abelian groups, our result will be slightly stronger than Theorem 3.0.2. Specifically, in the case that G is countable and abelian, we will show that $\overline{DLO(G)} = LO(G)$.

3.1 The case of Conradian orderings

We first review known results concerning Conradian orderings, and consider also the case of bi-orderings. Note that the results of this section concerning $C_{<}(G)$ appear in [40], and rely on the following difficult lemma ([40] Lemma 4.4), the bulk of which appeared first in [32], and partially in [30].

Lemma 3.1.1. *Suppose that P is the positive cone of a Conradian ordering of the group G , and that there is exactly one proper, nontrivial convex subgroup $C \subset G$. Further suppose that both C and G/C are rank one abelian groups. If P is isolated in $LO(G)$, then G is not biorderable.*

The next two theorems require the following work of Tararin ([30], Theorem 5.2.1). Recall that a group G admits a finite rational series if

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

is a finite normal series with all quotients G_{i+1}/G_i rank one abelian.

Theorem 3.1.2. *Let G be a left-ordered group.*

1. *If $LO(G)$ is finite, then G has a finite rational series.*
2. *Suppose that G has a finite rational series. Then $LO(G)$ is finite if and only if $G_i \triangleleft G$ for all i , and none of the quotients G_{i+2}/G_i are bi-orderable. Furthermore, in this case the rational series is unique, and for every left ordering of G , the convex subgroups are precisely G_0, G_1, \dots, G_n .*

Theorem 3.1.3 ([40] Proposition 4.1). *Suppose that P is the positive cone of a Conradian ordering of G . Then P is not an isolated point in the space $LO(G)$, unless $LO(G)$ is finite.*

Theorem 3.1.4. *Suppose that P is the positive cone of a bi-ordering of G . Then P is not isolated in $LO(G)$ unless G is rank 1 abelian.*

Proof. In the case that G is bi-ordered by the ordering $<$ associated to P , we have $C_{<}(G) = G$. From Theorem 3.1.3, it follows that G itself must have only finitely many left orderings if the bi-ordering $<$ is to have a positive cone that is isolated in $LO(G)$. However, by Theorem 3.1.2, we see that no group G admitting only finitely many left orders is bi-orderable, except in the case that G is rank one abelian. \square

3.2 Isolated points

We have already explicitly seen one method for approximating a given positive cone, in Section 2.1.1 the conjugation action is used to approximate a given positive cone in $LO(B_\infty)$.

We have also seen, following Proposition 1.1.5, that a left ordering of a group G may be modified on a convex subgroup C in G to create a new left ordering whose positive cone differs from the original only on C . This method of creating new left orderings can also be used as a tool for approximating a given positive cone, as outlined in the following lemmas.

Lemma 3.2.1. *Suppose $P \in LO(G)$ and that C is a convex subgroup of G . Then if $P_C = P \cap C$ is not an isolated point in $LO(C)$, P is not an isolated point in $LO(G)$.*

3.2. Isolated points

Proof. Suppose that

$$P \in \bigcap_{i=1}^m U_{g_i},$$

and suppose also that we have numbered the elements g_i so that $g_i \in C$ for $i \leq k$ (possibly $k = 0$, in the case that no g_i lies in C). Now in $LO(C)$, we have that

$$P_C \in \bigcap_{i=1}^k U_{g_i},$$

and since P_C is not an isolated point, we can choose $P'_C \in \bigcap_{i=1}^k U_{g_i}$, with $P'_C \neq P_C$.

We can now construct a positive cone $P' \neq P$ on G as follows: Given $g \in G$, $g \in P'$ if $g \in C$ and $g \in P'_C$, or if $g \notin C$ and $g \in P$.

The positive cone P' is different from P , since P and P' disagree on C , and by construction, $P' \in \bigcap_{i=1}^m U_{g_i}$. It follows that P is not isolated. \square

Lemma 3.2.2. *Suppose $P \subset G$ and that C is a normal, convex subgroup of G . Let P' denote the positive cone of the ordering inherited by the quotient G/C . If P' is not an isolated point in $LO(G/C)$, P is not an isolated point in $LO(G)$.*

The proof is routine.

Lemma 3.2.3. *Let G be a left ordered group with ordering $<$, whose positive cone we denote as P . Then the subgroup*

$$\text{Stab}_G(P) = \{g \in G : gPg^{-1} = P\}$$

is bi-ordered by the restriction of $<$ to $H = \text{Stab}_G(P)$.

3.2. Isolated points

Proof. To see that the restriction of $<$ is a bi-ordering, consider its positive cone $P_H = P \cap H$. If $g \in P_H$ and $h \in H$, then

- $hgh^{-1} \in H$ since H is a subgroup, and
- $hgh^{-1} \in P$ since, by definition, every element of H fixes the positive cone P under conjugation.

Therefore H is bi-ordered. □

The main difficulty in characterizing the Conradian soul of an isolated point in $LO(G)$ is in showing that the Conradian soul is necessarily non-trivial. If P is an isolated point in $LO(G)$ with associated ordering $<$ of G , then P is certainly not an accumulation point of its conjugates in $LO(G)$. It turns out that knowing P is not an accumulation point of its conjugates $gPg^{-1} \in LO(G)$ is enough to deduce that $B_{<}(G)$ (and hence $C_{<}(G)$) is non-trivial.

Observe that for any group G , if $1 < h < g$ in the ordering corresponding to P , then left multiplication yields $1 < h^{-1}g$, and then using the fact that h is positive, we conclude that $1 < h^{-1}gh$. Translating this observation into a topological language, we have observed that if $P \in U_g$, then $hPh^{-1} \in U_g$ for any h with $1 < h < g$. Supposing that

$$\{P\} = \bigcap_{i=1}^m U_{g_i},$$

is an isolated point, applying the above trick to the set of elements $\{g_1, \dots, g_m\}$ allows us to conclude that for any h with $1 < h < g_i$ for all $i \in \{1, \dots, m\}$,

3.2. Isolated points

we must have

$$hPh^{-1} \in \bigcap_{i=1}^m U_{g_i}.$$

However, since P is isolated, this means that $hPh^{-1} = P$, so that (in a sense soon to be made more precise) “small elements in G are bi-ordered,” as they fix the positive cone P under conjugation.

Lemma 3.2.4. *Suppose that*

$$P \in \bigcap_{i=1}^m U_{g_i},$$

where $\{g_1, \dots, g_m\}$ is some finite set of elements of G , yet no conjugates of P (different from P itself) are in this open set. Then there exists $g_i \in \{g_1, \dots, g_m\}$ such that the set

$$C_i = \{g \in G : g_i^{-k} \leq g \leq g_i^k \text{ for some } k\}$$

contains only elements of G that fix the positive cone P under conjugation, that is, $g \in C_i \Rightarrow gPg^{-1} = P$.

Proof. First, we show that there exists g_i such that all elements in the set

$$C_i^+ = \{g \in G : 1 < g \leq g_i^k \text{ for some } k\}$$

fix P under conjugation.

To this end, suppose not. Then for each g_i there exists h_i with $1 < h_i \leq g_i^{k_i}$ for some k_i , and $h_iPh_i^{-1} \neq P$. Choose $h = \min\{h_1, \dots, h_m\}$. Then for

3.2. Isolated points

each i , we have

$$h \leq g_i^{k_i} \Rightarrow 1 \leq h^{-1} g_i^{k_i} \Rightarrow 1 < h^{-1} g_i^{k_i} h,$$

and therefore $g_i^{k_i} \in hPh^{-1}$. Now since the element $g_i^{k_i}$ is positive in the order determined by the positive cone hPh^{-1} , its k_i -th root g_i is also positive. This shows that

$$hPh^{-1} \in \bigcap_{i=1}^m U_{g_i},$$

and by our choice of h , $hPh^{-1} \neq P$, a contradiction. Therefore there exists i such that all elements in C_i^+ fix P under conjugation.

To prove that all elements $g \in C_i$ fix the positive cone P , suppose that $g \in G$ satisfies $g_i^{-k} \leq g < 1$ for some k . Then $1 \leq g_i^k g < g_i^k$, so that either $g = g_i^{-k}$ or $g_i^k g \in C_i^+$.

1. In the case $g = g_i^{-k}$, then $g^{-1} \in C_i^+$ and so fixes $g^{-1}Pg = P$, which implies $gPg^{-1} = P$.
2. If $g_i^k g \in C_i^+$, then

$$g_i^k gPg^{-1} g_i^{-k} = P,$$

so that we multiply by powers of g_i from both sides and find

$$gPg^{-1} = g_i^{-k} P g_i^k = P.$$

Note that case (1) has been used to yield the final equality.

Therefore we have found g_i such that all elements in C_i fix P as claimed.

□

3.2. Isolated points

Lemma 3.2.5. *For any group G , if*

$$P \in \bigcap_{i=1}^m U_{g_i},$$

and no conjugates of P distinct from P lie in this open set, then there exists g_i such that the set

$$C_i = \{g \in G : g_i^{-k} \leq g \leq g_i^k \text{ for some } k\}$$

is a convex, bi-ordered subgroup of G .

Proof. Convexity of C_i is clear from the definition. By Lemma 3.2.4, C_i is a subset of the bi-ordered group $Stab_G(P)$, it follows that C_i is bi-ordered by the restriction ordering as well. Being bi-ordered, we can then conclude that C_i is a subgroup of G : If $1 < g \leq g_i^k$ for some k , then $g_i^{-k} \leq g^{-1} < 1$, and similarly the implication $a < b$ and $c < d \Rightarrow ac < bd$ (these implications do not hold in general for left ordered groups) shows closure under multiplication. □

Corollary 3.2.6. *Suppose that the left ordering $<$ of G has positive cone P which is not an accumulation point of its conjugates in $LO(G)$. Then both $B_{<}(G)$ and $C_{<}(G)$ are non-trivial.*

In particular, we have proven that if $<$ corresponds to an isolated point in $LO(G)$, then both $B_{<}(G)$ and $C_{<}(G)$ are non-trivial.

We are now ready to complete the proof of Theorem 3.0.1.

Proof of Theorem 3.0.1. Let P be the positive cone of a left ordering $<$ of a

group G , and suppose that P is an isolated point in $LO(G)$. We know that $B_{<}(G)$ and $C_{<}(G)$ are non-trivial by Corollary 3.2.6, so it remains to show that $B_{<}(G)$ is rank one abelian, and that $C_{<}(G)$ admits only finitely many left orderings.

Since we have assumed that P is isolated in $LO(G)$, it follows from Lemma 3.2.1 that the restriction of P to $B_{<}(G)$ must define a bi-ordering that is isolated in $LO(B_{<}(G))$. However, by Theorem 3.1.4, this is only possible in the case when $B_{<}(G)$ is rank one abelian. Similarly, It follows that the restriction of P to $C_{<}(G)$ must define a Conradian ordering that is isolated in $LO(C_{<}(G))$, which by Theorem 3.1.3 is only possible in the case that $LO(C_{<}(G))$ is finite. \square

3.3 Dense orderings and discrete orderings

In this section, we consider dense orderings of a given group. Note that the word “dense” is being used in this section in two ways: The word “dense” may refer to a property of a subset of $LO(G)$, or it may refer to a property of an ordering of a group G . Similarly, the word “discrete” is being used in this section to describe both a property of an ordering of a group, and a property of a subset of $LO(G)$. The reader should be careful of the context when these words are used.

In recent work ([12, 51]), it has proven fruitful to consider discrete and dense group orderings separately, as they reflect different structures of the underlying group. In considering the structure of $LO(G)$, dense orderings of a given group G (with minor restrictions on the group G) are in some sense

3.3. Dense orderings and discrete orderings

“generic” in $LO(G)$, in that dense orderings of G constitute a dense G_δ set inside of a Cantor set within $LO(G)$. Recall that a set U in a topological space X is a G_δ set if U can be written as a countable intersection of open sets.

Lemma 3.3.1. *Let $DLO(G) \subset LO(G)$ denote the set of dense left orderings of G . If G is countable, then $DLO(G)$ is a G_δ set.*

Proof. Observe that if $\epsilon > 1$ is the least positive element in a left ordering $<$ of G with positive cone P , then for all $g \in G$ (with $g \neq 1$ different from ϵ and ϵ^{-1}) either $g < \epsilon^{-1}$ or $\epsilon < g$. In other words, either $P \in U_{g^{-1}\epsilon^{-1}}$ or $P \in U_{\epsilon^{-1}g}$ for all $1 \neq g \in G$ different from ϵ . That is to say, if V_ϵ denotes the set of all discrete left orderings of G with least element ϵ , then we have observed that

$$V_\epsilon = \bigcap_{\substack{g \in G \\ g \neq \epsilon, 1}} (U_{g^{-1}\epsilon^{-1}} \cup U_{\epsilon^{-1}g}) \cap U_\epsilon.$$

Note that V_ϵ is closed, as it is an intersection of closed sets, and consists of those positive cones that define an ordering of G with ϵ as least positive element. Therefore, the set of dense orderings is given by

$$DLO(G) = \bigcap_{1 \neq \epsilon \in G} (LO(G) \setminus V_\epsilon),$$

a countable intersection of open sets. □

The remaining difficulty is to show that any dense ordering is an accumulation point of other dense orderings. We first consider the case of abelian groups.

3.3.1 Abelian groups

From [2], we have the following fact:

Proposition 3.3.2. *If A is a torsion-free abelian group with $\text{rank}(A) > 1$, then the space $LO(A)$ has no isolated points.*

For a given torsion-free abelian group A , we can deduce much more about the structure of $LO(A)$ by examining the set of all dense orderings of A .

Proposition 3.3.3. *Let P be any positive cone in $LO(A)$, where A is a torsion-free abelian group with $\text{rank}(A) > 1$. Then P is an accumulation point of positive cones whose associated orderings are dense orderings.*

We begin by proving a special case.

Lemma 3.3.4. *Let P be any ordering in $LO(\mathbb{Z}^k)$, where $k > 1$. Then P is an accumulation point of dense orderings.*

Proof. We follow the ideas of Sikora in [58], making modifications where necessary.

For contradiction, let $k > 1$ be the smallest k for which the claim fails.

Suppose that

$$P \in \bigcap_{i=1}^n U_{g_i},$$

with no dense orderings in this open set. Note that we may assume that none of the g_i 's are integer multiples of one another. Extend the ordering $<$ defined by P to an ordering of \mathbb{Q}^k by declaring $v_1 < v_2$ for $v_1, v_2 \in \mathbb{Q}^k$ if $nv_1 < nv_2$ whenever $nv_1, nv_2 \in \mathbb{Z}^k$. Let $H \subset \mathbb{Q}^k \otimes \mathbb{R} = \mathbb{R}^k$ be the subset

3.3. Dense orderings and discrete orderings

of elements $x \in \mathbb{R}^k$ such that every Euclidean neighbourhood of x contains both positive and negative elements. Then H is a hyperplane, and $\mathbb{R}^k \setminus H$ is divided into two open components H_- and H_+ having the property that H_+ contains only positive elements, and H_- contains only negative elements. Therefore the elements g_i lie either in H_+ or H itself.

Suppose that two or more of the elements $\{g_1, \dots, g_n\}$ lie inside H . In this case, $H \cap \mathbb{Z}^k = \mathbb{Z}^m$ for some $m > 1$ with $m < k$, and in this case the positive cone $P \cap \mathbb{Z}^m \subset \mathbb{Z}^m$ cannot be an accumulation point of dense orderings in $LO(\mathbb{Z}^m)$, for then we could change the positive cone P using Lemma 3.2.1. This change will produce a new positive cone in $LO(\mathbb{Z}^k)$ whose associated ordering of \mathbb{Z}^k is dense, because it is a dense ordering when restricted to the convex subgroup \mathbb{Z}^m . This contradicts the minimality of k .

The remaining possibilities are that exactly one (or none) of the elements $\{g_1, \dots, g_n\}$ lie inside H . In this case, by slight perturbations of the hyperplane H , we can produce a new hyperplane H' containing none of the lattice points $\mathbb{Z}^k \subset \mathbb{R}^k$, and with all points g_i lying on one side of the hyperplane H' .

Specifically, suppose that

$$\mathbf{n} = (x_1, x_2, \dots, x_k)$$

is the normal vector defining H . Choose real numbers y_1, y_2, \dots, y_k that are linearly independent over \mathbb{Q} , this is possible for every $k > 0$ because \mathbb{R} is an abelian group of infinite rank. For each $i = 1, \dots, k$, we may choose a rational number q_i such that $q_i y_i$ is as near to x_i as we please, the real

3.3. Dense orderings and discrete orderings

numbers $\{q_i y_i\}_{i=1}^k$ are again linearly independent over \mathbb{Q} . Therefore we have shown that we can choose a vector

$$\mathbf{n}' = (q_1 y_1, q_2 y_2, \dots, q_k y_k)$$

arbitrarily close to \mathbf{n} such that the hyperplane H' with normal vector \mathbf{n}' satisfies $H' \cap \mathbb{Z}^k = \{0\}$.

If none of the elements $\{g_1, \dots, g_n\}$ lie inside H , then we may choose \mathbf{n}' to be sufficiently close to \mathbf{n} so that $\{g_1, \dots, g_n\}$ lie on one side of H' . This new hyperplane H' defines a new ordering P' on \mathbb{Z}^k by declaring $P' = H'_+ \cap \mathbb{Z}^k$, where H'_+ is the component of $\mathbb{R}^k \setminus H'$ containing all g_i .

If one of the elements $\{g_1, \dots, g_n\}$ lies inside H , say g_j , then we must take more care in choosing our normal vector \mathbf{n}' . In this case, we choose \mathbf{n}' to be close to \mathbf{n} , and inside of the open set

$$\{x \in \mathbb{R}^k \mid \text{the angle between } x \text{ and } g_j \text{ is less than } \frac{\pi}{2}\}.$$

Again we see that the new hyperplane H' with normal vector \mathbf{n}' defines a positive cone P' that contains all of the elements $\{g_1, \dots, g_n\}$.

To see that this ordering is dense, suppose that $\epsilon \in P'$ is the smallest element in P' . Recall that the normal vector to H' is \mathbf{n}' , and therefore the distance between a vector \mathbf{v} and the hyperplane H' is given by $|\text{proj}_{\mathbf{n}'}(\mathbf{v})|$, the length of the projection of \mathbf{v} onto \mathbf{n}' .

Since ϵ is the smallest positive element in our ordering, the vector ϵ must satisfy $\epsilon < \mathbf{v}$ for all $\mathbf{v} \in P'$, which happens if and only if $\mathbf{v} - \epsilon \in H'_+$ for all

3.3. Dense orderings and discrete orderings

\mathbf{v} in P' . Hence, for all $\mathbf{v} \in P'$ we have $|proj_{\mathbf{n}'}(\mathbf{v})| > |proj_{\mathbf{n}'}(\epsilon)|$, and so

$$0 < \varepsilon = |proj_{\mathbf{n}'}(\epsilon)| = \inf_{\mathbf{v} \in P'} \{|proj_{\mathbf{n}'}(\mathbf{v})|\}.$$

If we restrict our attention to the two-dimensional subspace of \mathbb{R}^k that is spanned by ϵ and some other vector $\mathbf{u} \in P'$, this means that all lattice points in the span are at least a distance ε from the line $H' \cap span\{\epsilon, \mathbf{u}\}$. This is not possible, as any line in \mathbb{R}^2 of irrational slope has on either side an infinite number of lattice points lying closer than any positive distance $\varepsilon > 0$ [Theorem 1.1, [45]].

□

Proof of Proposition 3.3.3. To prove the statement for an arbitrary torsion-free abelian group A with $rank(A) > 1$, we let $g_1, \dots, g_m \in A$ be any finite family of elements in a given positive cone P . We will show that there exist infinitely many positive cones with associated dense orderings on A in which all g_i are positive.

Let N be the subgroup of A generated by the elements g_1, \dots, g_m . Then $N \cong \mathbb{Z}^k$ for $k \geq 1$. Assume that $k > 1$, for if it is the case that $N \cong \mathbb{Z}$, add an additional generator g_{m+1} none of whose powers lie in N —we may do this since $rank(A) > 1$.

By Lemma 3.3.4, N admits infinitely many dense orderings in which all of g_1, \dots, g_m are positive, each constructed by perturbations of the hyperplane associated to the restriction ordering whose positive cone is $P_N = N \cap P$. Fix a positive cone P'_N with a dense associated ordering of N , with $P'_N \neq P_N$.

3.3. Dense orderings and discrete orderings

We may extend P'_N to a distinct ordering Q on the isolator of N

$$I(N) = \{g \in A : g^l \in N \text{ for some } l\}$$

by declaring $g \in Q$ iff $g^l \in P'_N$ for some l .

Observe that the ordering of $I(N)$ with positive cone Q is dense, for suppose not, say Q had least element ϵ . Then $\epsilon \in P'$ is not possible since P' is a dense ordering, so let $l > 1$ be the least positive integer such that $\epsilon^l \in P'$. By density of P' , we may then choose $g \in P' \subset Q$ with $1 < g < \epsilon^l$. Since the only positive elements less than ϵ^l are $\epsilon, \epsilon^2, \dots, \epsilon^{l-1}$, we have that $g = \epsilon^i$ for $i < l$. This contradicts our choice of l .

Now $I(N)$ is normal, and the quotient $A/I(N)$ is torsion-free abelian, so we may order the quotient. Using any ordering on the quotient, we can extend the dense ordering of $I(N)$ with positive cone Q to give a dense ordering of A with the required properties. \square

Therefore, when A is an abelian group with $\text{rank}(A) > 1$, we know that the closure of the set of dense orderings in $LO(A)$ is the entire space $LO(A)$. Thus, Proposition 3.3.3 and Lemma 3.3.1 together give us the following theorem.

Theorem 3.3.5. *Suppose that A is a torsion free countable abelian group with $\text{rank}(A) > 1$. Then $LO(A)$ is a Cantor set, and the set $DLO(A)$ of all dense left orderings of A is a dense G_δ set within $LO(A)$.*

Note that the case of discrete orderings must necessarily be different than this, for there exist abelian groups admitting no discrete orderings: divisible

torsion free abelian groups are such an example.

Question 3.3.6. *Let A be a torsion free abelian group with $\text{rank}(A) > 1$. What is the closure of the set of the discrete orderings in $LO(A)$?*

3.3.2 Non-abelian groups

Our results concerning dense orderings of abelian groups generalize to some non-abelian groups.

Proposition 3.3.7. *Let G be any group in which all rank one abelian subgroups are isomorphic to \mathbb{Z} . If $P \in LO(G)$ corresponds to a dense left ordering $<$ of G , then P is an accumulation point of positive cones whose associated left orderings are dense orderings.*

Proof. Let $U = \bigcap_{i=1}^m U_{g_i}$ be an open set in $LO(G)$ containing P , the positive cone of a dense left ordering $<$ of G .

If U contains any conjugates of P (different from P itself), then we are done, so suppose that no conjugate orderings lie in U . Then by proposition 3.2.5, G contains a convex, bi-ordered subgroup C of the form

$$C = C_i = \{g \in G : g_i^{-k} \leq g \leq g_i^k \text{ for some } k\},$$

where $g_i \in \{g_1, \dots, g_m\}$. Denote by C' the intersection of all non-trivial convex subgroups of C . There are now two cases to consider.

1. $C' \neq \{1\}$. In this case, since C' is bi-ordered and contains no nontrivial convex subgroups, we can use a theorem of Conrad (Theorem 1.2.3) which tells us the order must be Archimedean, and so C' must be

3.3. Dense orderings and discrete orderings

abelian. From our assumption on G , if $\text{rank}(C') = 1$, we have $C' \cong \mathbb{Z}$, meaning our ordering is discrete. Therefore $\text{rank}(C') > 1$.

Now the restriction ordering on C' with positive cone $P \cap C'$ is a dense ordering, and we know from Theorem 3.3.3 that every dense ordering in $LO(C')$ is an accumulation point of other dense orderings. Therefore we may change the positive cone P as in the proof of Lemma 3.2.1, creating a new positive cone P' containing all g_i , and corresponding to a dense ordering of G .

2. $C' = \{1\}$. In this case, C must have infinitely many convex subgroups whose intersection is trivial. Therefore, we may choose a convex subgroup K , that is non-trivial and contains no g_i . Define the positive cone of the “flipped ordering” of K to be $(P^{-1} \cap K) = P_K^{-1}$. Then we define a new positive cone $P' \subset G$, with $P' \in U$, by setting $P' = P_K^{-1} \cup (P \cap (G \setminus K))$. Again, the new ordering $<'$ of K with positive cone P' is dense, and so the ordering we have defined on G is dense.

□

In the case of an abelian group A , the closure of the set of dense orderings was the entire space $LO(A)$, which is known to be homeomorphic to the Cantor set when A is countable. In the non-abelian case, Theorem 3.0.2 gives us a similar result.

Proof of Theorem 3.0.2. Let G be any countable group with all rank one abelian subgroups isomorphic to \mathbb{Z} . Then since G is countable, $LO(G)$ is

3.3. Dense orderings and discrete orderings

metrizable, as is the space $\overline{DLO(G)} \subset LO(G)$. Proposition 3.3.7 shows that the set $\overline{DLO(G)}$ contains no isolated points, and since it is closed, it is compact. Therefore $\overline{DLO(G)}$ is a compact, metrizable, totally disconnected perfect space, and so is homeomorphic to the Cantor set [25]. By Lemma 3.3.1, the set $DLO(G)$ is also a dense G_δ set within $\overline{DLO(G)}$. \square

With the restriction that all rank one abelian subgroups of G be isomorphic to \mathbb{Z} , it also follows readily that any isolated point in $LO(G)$ must correspond to a discrete left ordering of G . This can be seen by appealing to either Theorem 3.0.2 (which is stronger than what we need), or by appealing to Theorem 3.0.1, and remarking that the smallest convex subgroup in the Conradian soul of an isolated left ordering must be a rank one abelian group.

Next we turn our attention to discrete orderings, and observe conditions under which a discrete ordering of G is not an isolated point in $LO(G)$. We no longer need the restriction that all rank one abelian subgroups be isomorphic to \mathbb{Z} .

Lemma 3.3.8. *Let G be an arbitrary nontrivial group. Suppose that $P \subset G$ is the positive cone of a discrete left ordering $<$ with least positive element ϵ . Then if $geg^{-1} > 1$ for all $g \in G$ and*

$$P \in \bigcap_{i=1}^m U_{g_i}$$

contains no conjugates of P , there exists g_i which is not a power of ϵ such

3.3. Dense orderings and discrete orderings

that

$$C_i = \{g \in G : g_i^{-k} \leq g \leq g_i^k \text{ for some } k\}$$

is a convex, bi-ordered subgroup which properly contains the convex subgroup $\langle \epsilon \rangle$.

Proof. Suppose that $U = \bigcap_{i=1}^m U_{g_i}$ contains P , but no conjugates of P . If no g_i is equal to a power of ϵ , then we are done, as we may apply proposition 3.2.5.

On the other hand, suppose that some g_i is a power of ϵ , say $g_1 = \epsilon^l$. Then the condition $g\epsilon g^{-1} > 1$ for all $g \in G$ guarantees that the open set U_ϵ contains every conjugate of P . Therefore, if

$$\left(\bigcap_{i=2}^m U_{g_i} \right) \cap U_{\epsilon^l}$$

contains no conjugates of P , neither does the open set $\bigcap_{i=2}^m U_{g_i}$. Continuing to eliminate powers of ϵ in this way, we can eventually find an open set $\bigcap_{i=r}^m U_{g_i}$ containing no conjugates of P , and with no g_i equal to a power of ϵ . From here we may apply Proposition 3.2.5. □

Theorem 3.3.9. *Let G be a group, and P the positive cone of a discrete left ordering $<$ with least positive element ϵ . If $g\epsilon g^{-1} \in P$ for all $g \in G$, then P is not isolated in $LO(G)$.*

Proof. We proceed very similarly to the proof of Theorem 3.0.2. Let $U = \bigcap_{i=1}^m U_{g_i}$ be an open set in $LO(G)$ containing P . If U contains any conjugates of P distinct from P , then we are done, so by Lemma 3.3.8, we may suppose

that there exists convex subgroup C properly containing $\langle \epsilon \rangle$, which is bi-ordered by the restriction of P .

Note that the convex subgroup C is not rank one abelian: Suppose that $\text{rank}(C) = 1$. As the containment $\langle \epsilon \rangle \subset C$ is proper, we can choose $c \in C$ with $C \neq 1$, that is not a power of ϵ . If we then assume that C is rank one abelian, we arrive at $\epsilon^k = c^l$ for some integers k, l , we may take k and l to be relatively prime. Choose integers s, t such that $sk + tl = 1$, and observe that

$$\epsilon = \epsilon^{sk+tl} = (\epsilon^k)^s \cdot \epsilon^{tl} = (c^l)^s \cdot \epsilon^{tl} = (c^s \epsilon^t)^l.$$

Therefore ϵ may be written as a power of some (necessarily positive) element, contradicting the fact that ϵ is the least positive element.

Thus, by Theorem 3.1.4, we know that the restriction of P to the subgroup C is not isolated in $LO(C)$, and it follows from Lemma 3.2.1 that P is not isolated in $LO(G)$. \square

3.4 The braid groups

As a sample application of these results, we turn our focus to the braid groups. It is known that the space of left orders $LO(B_n)$ is *not* homeomorphic to the Cantor set for $n \geq 2$. We begin by defining the Dehornoy left ordering of the braid groups (also known as the ‘standard’ ordering), whose positive cone we shall denote P_D [20, 21]. Recall that for each integer $n \geq 2$, the Artin braid group B_n is the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, subject

3.4. The braid groups

to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1, \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1.$$

Definition 3.4.1. *Let w be a word in the generators $\sigma_1, \dots, \sigma_{n-1}$. Then w is said to be: i -positive if the generator σ_i occurs in w with only positive exponents, i -negative if σ_i occurs with only negative exponents, and i -neutral if σ_i does not occur in w .*

It is shown in [20] that every nontrivial braid $\beta \in B_n$ admits a representative word that is i -positive or i -negative for some $i = 1, \dots, n - 1$. If a braid admits a i -positive (resp. i -negative) representative word, then it is called an i -positive braid (resp. i -negative braid). We may then define the positive cone of the Dehornoy ordering as

Definition 3.4.2. *The positive cone $P_D \subset B_n$ of the Dehornoy ordering is the set*

$$P_D = \{\beta \in B_n : \beta \text{ is } i\text{-positive for some } i \leq n - 1\}.$$

There is also a second positive cone of interest, discovered by the authors of [23], which we shall denote by P_{DD} . Denote by $P_i \subset B_n$ the set of all i -positive braids. Note that the set of all i -negative braids is simply P_i^{-1} .

Definition 3.4.3. *The positive cone $P_{DD} \subset B_n$ is the set*

$$P_{DD} = P_1 \cup P_2^{-1} \cup \dots \cup P_{n-1}^{(-1)^n}.$$

That either of these notions defines a positive cone in B_n is difficult to

3.4. The braid groups

show, as it is not clear that no braid is both i -positive and i -negative. This was the main idea introduced to braid theorists in Dehornoy's seminal paper [20].

The positive cone P_{DD} was originally defined in light of the following property:

Proposition 3.4.4 (Dubrovina, Dubrovin [23]). *The positive cone P_{DD} is generated as a semigroup by the braids*

$$y_1 = \sigma_1 \cdots \sigma_{n-1}, y_2 = (\sigma_2 \cdots \sigma_{n-1})^{-1}, y_3 = \sigma_3 \cdots \sigma_{n-1}, \dots, y_{n-1} = \sigma_{n-1}^{(-1)^n}.$$

Note that for two positive cones P and Q , if $P \subset Q$ then necessarily $P = Q$. Therefore

Corollary 3.4.5. *The positive cone P_{DD} is an isolated point in $LO(B_n)$, in particular,*

$$\{P_{DD}\} = \bigcap_{i=1}^{n-1} U_{y_i}.$$

Knowing that $LO(B_n)$ has isolated points for $n \geq 2$, it makes sense to ask the question: Is the standard ordering P_D an isolated point in $LO(B_n)$? This question is answered in [21], using a very explicit calculation. That P_D is not isolated, however, was originally proven in [40], though the techniques are different than those used here.

First, we begin with a proposition which establishes a very important property of the ordering P_D . Recall the Garside monoid $B_n^+ \subset B_n$ is the monoid generated by the elements $\sigma_1, \dots, \sigma_{n-1}$.

Proposition 3.4.6. *Let $\beta \in B_n$ and $\alpha \in B_n^+$ be given. If $\alpha \neq 1$, then*

3.4. The braid groups

$\beta\alpha\beta^{-1} \in P_D$.

This property of the Dehornoy ordering is referred to as the subword property, or property **S** [21].

Next, we must know that the Dehornoy ordering is discrete [12].

Proposition 3.4.7. *The Dehornoy ordering of B_n is discrete, with smallest positive element σ_{n-1} .*

Proof. Suppose that $\beta > 1$. There are two possibilities: First, if $\beta = \sigma_{n-1}^k$ is a power of σ_{n-1} , then $k \geq 1$ and so $\sigma_{n-1} \leq \sigma_{n-1}^k = \beta$. On the other hand, if β is not a power of σ_{n-1} , then β must be i -positive for some $i < n - 1$. In this case, the braid $\sigma_{n-1}^{-1}\beta$ must also be i -positive, and hence $1 < \sigma_{n-1}^{-1}\beta$ and so $\sigma_{n-1} < \beta$ as claimed. \square

These two propositions together show us that P_D satisfies the hypotheses of Theorem 3.3.9. If we can additionally show that P_D has no biorderable convex subgroups properly containing $\langle \sigma_{n-1} \rangle$, then we can conclude that P_D is an accumulation point of its orbit under the B_n -action on $LO(B_n)$.

Recall the natural inclusions $B_m \subset B_n$ whenever $m \leq n$ which takes $\sigma_i \in B_m$ to $\sigma_i \in B_n$. A useful operation is the shift homomorphism $sh: B_m \rightarrow B_n$, $m < n$ defined by $sh(\sigma_i) = \sigma_{i+1}$. This is clearly injective and order-preserving. The shift may be iterated, and we note that $sh^r(B_{n-r})$ is just the subgroup $\langle \sigma_{r+1}, \dots, \sigma_{n-1} \rangle$ of B_n , or in other words, the subgroup of all elements which are i -neutral for all $i \leq r$.

Lemma 3.4.8. *The subgroups $sh^r(B_{n-r})$, $r > 0$, are the only convex subgroups of B_n under the left ordering defined by P_D .*

3.4. The braid groups

Proof. Set $H_r = sh^r(B_{n-r})$, and let C be a convex subgroup in the Dehornoy ordering. Choose i to be the smallest integer such that C contains an i -positive braid. Then clearly $C \subset H_{i-1}$, our aim is to show the opposite inclusion, which establishes the lemma.

Let $\beta \in C$ be an i -positive braid. The braid $\sigma_j^{-1}\beta$ is i -positive for $j > i$, so that $1 < \sigma_j < \beta \Rightarrow \sigma_j \in C$, and so $H_i \subset C$. Considering the generator σ_i , we write $\beta = w_1\sigma_iw_2$, where w_1 is an empty or i -neutral word, and w_2 is an empty, i -neutral, or i -positive word. We claim that this implies $\sigma_i \in C$.

First, we note that the the braid represented by the word σ_iw_2 lies in C , as w_1 contains only $\sigma_{i+1}, \dots, \sigma_{n-1}$, all of which are in C . If w_2 is empty, the claim is proven, if w_2 is i -neutral, then we may right multiply by appropriate σ_j or σ_j^{-1} for $j > i$ to arrive at $\sigma_i \in C$, and again the claim is proven. Lastly, if w_2 is i -positive, then we get:

$$1 < w_2 \Rightarrow 1 < \sigma_i < \sigma_iw_2 \in C,$$

and the claim follows from convexity of C . □

Since all convex subgroups are isomorphic to a shifted copy of the braid groups, we conclude that

Corollary 3.4.9. *No subgroup that is convex under the ordering P_D is bi-orderable, except for the subgroup $\langle \sigma_{n-1} \rangle$.*

Of course, this relies upon the fact that the braid groups are not bi-orderable for $n > 2$, as we have already seen.

Theorem 3.4.10. *For every $n > 2$, the positive cone P_D in B_n is an*

3.4. The braid groups

accumulation point of its conjugates in $LO(B_n)$.

Proof. Apply Corollary 3.4.9 and Lemma 3.3.8.

□

Chapter 4

Analysis of the natural action on the space of left orderings

In this chapter we will use the category of lattice ordered groups to investigate the topological properties of the orbits in $LO(G)$ under the G -action. In particular, we are able to characterize those points lying in the closure of a given orbit by using a free object in the category of lattice ordered groups. Appealing to known results in the field of lattice ordered groups, this characterization yields a proof that the space of left orderings of the non abelian free group contains a dense orbit.

4.1 Lattice-ordered groups

A group G is said to be lattice ordered, referred to as an l -group, if there exists a partial ordering $<$ of the elements of G satisfying:

1. $g < h$ implies $fg < fh$ and $gf < hf$ for all f, g, h in G , and
2. the ordering $<$ admits a lattice structure, that is, every finite set has a least upper bound and a greatest lower bound.

4.1. Lattice-ordered groups

As is standard, we denote the greatest lower bound and the least upper bound of g_1, \dots, g_n by $\bigwedge_{i=1}^n g_i$ and $\bigvee_{i=1}^n g_i$ respectively. Recall that a lattice L is distributive if the following identities hold for all x, y, z in L :

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

We will show that all lattice ordered groups must necessarily be distributive lattices, by beginning with the following lemma.

Lemma 4.1.1. [17] *Let L be a lattice-ordered group, and let $x, y \in L$ be given. Then:*

1. $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$, and
2. $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$.

Proof. We prove (1), the proof of (2) is dual. Observe that $x \vee y \geq x$ and $x \vee y \geq y$, and so $(x \vee y)^{-1} \leq x^{-1}$ and $(x \vee y)^{-1} \leq y^{-1}$. Hence $(x \vee y)^{-1} \leq x^{-1} \wedge y^{-1}$.

On the other hand, we have $x^{-1} \wedge y^{-1} \leq x^{-1}$ and $x^{-1} \wedge y^{-1} \leq y^{-1}$, so that $(x^{-1} \wedge y^{-1})^{-1} \geq x$ and $(x^{-1} \wedge y^{-1})^{-1} \geq y$. Therefore $(x^{-1} \wedge y^{-1})^{-1} \geq x \vee y$, and so $x^{-1} \wedge y^{-1} \leq (x \vee y)^{-1}$.

It follows that $x^{-1} \wedge y^{-1} = (x \vee y)^{-1}$. □

Proposition 4.1.2. [17] *Let L be a lattice ordered group. Then L is a distributive lattice.*

4.1. Lattice-ordered groups

Proof. We use Lemma 4.1.1 without reference. Let x, y, z in L be given, and write $g = x \vee y$. Then $1 \leq gx^{-1}$, and so $z \leq gx^{-1}z$, and hence

$$z \wedge g \leq gx^{-1}z \wedge g = (gx^{-1})(z \wedge x).$$

Similarly, we find that

$$z \wedge g \leq gy^{-1}z \wedge g = (gy^{-1})(z \wedge y).$$

Right multiplying the inequality

$$(z \wedge x) \leq z \wedge g \leq (gx^{-1})(z \wedge x)$$

by $(z \wedge x)^{-1}$, we find $1 \leq (z \wedge g)(z \wedge x)^{-1} \leq gx^{-1}$. We also find $1 \leq (z \wedge g)(z \wedge y)^{-1} \leq gy^{-1}$.

However, we may compute

$$1 = g(x \vee y)^{-1} = g(x^{-1} \wedge y^{-1}) = gx^{-1} \wedge gy^{-1},$$

and thus

$$\begin{aligned} 1 &= (z \wedge g)(z \wedge y)^{-1} \wedge (z \wedge g)(z \wedge x)^{-1} \\ &= (z \wedge g)((z \wedge y)^{-1} \wedge (z \wedge x)^{-1}) \\ &= (z \wedge g)((z \wedge y) \vee (z \wedge x))^{-1}. \end{aligned}$$

The dual is proved similarly. □

A homomorphism h of lattice ordered groups from L_1 to L_2 (often called an l -homomorphism) is a map $h : L_1 \rightarrow L_2$ that is simultaneously a group homomorphism and a morphism of lattices, so that h respects the partial ordering $<$, as well as distributing over all finite meets and joins. The following two examples are standard constructions which are of great importance in what follows.

First, given a totally ordered set Ω with ordering $<$, the group of all order preserving automorphisms of Ω forms a lattice ordered group, which we denote by $Aut(\Omega, <)$. The lattice ordering \prec of $Aut(\Omega, <)$ is defined pointwise: for any f, g in $Aut(\Omega, <)$, we declare $f \prec g$ if $f(x) < g(x)$ for all $x \in \Omega$.

As a second example, let $\{L_i\}_{i \in I}$, be an arbitrary collection of lattice ordered groups, with L_i having the lattice ordering \prec_i . We can form a new lattice ordered group L by setting

$$L = \prod_{i \in I} L_i$$

and for any x, y in L we declare $x \prec y$ if $\pi_i(x) \prec_i \pi_i(y)$ for all $i \in I$. Here, $\pi_i : L \rightarrow L_i$ is projection onto the i -th factor in the product.

We are now ready to introduce the main object of concern in this chapter. A lattice ordered group $F(G)$ is said to be the free lattice ordered group over G if $F(G)$ satisfies:

1. There exists an injective homomorphism $i : G \rightarrow F(G)$ such that $i(G)$ generates $F(G)$ as an l -group, so that no proper l -subgroup of $F(G)$ contains the set $i(G)$.

4.1. Lattice-ordered groups

2. For any lattice ordered group L and any homomorphism of groups $\phi : G \rightarrow L$ there exists a unique l -homomorphism $\bar{\phi} : F(G) \rightarrow L$ such that $\bar{\phi} \circ i = \phi$.

Obviously any such group is unique up to l -isomorphism.

We present a construction of the free lattice ordered group $F(G)$ due to Conrad [14], in the case that G is a left orderable group.

For each positive cone $P \in LO(G)$, denote the natural lattice ordering of $Aut(G, <_P)$ by $<_P$, defined as in the previous discussion. We may embed the group G into $Aut(G, <_P)$ by sending every $g \in G$ to the order-preserving automorphism of the totally ordered set $(G, <_P)$ defined by left multiplication by g . Denote this order preserving automorphism by g_P , so that $g_P \in Aut(G, <_P)$ has action $g_P(h) = gh$.

Define a map

$$i : G \rightarrow \prod_{P \in LO(G)} Aut(G, <_P),$$

according to the rule $\pi_P(i(g)) = g_P$, where

$$\pi_P : \prod_{Q \in LO(G)} Aut(G, <_Q) \rightarrow Aut(G, <_P)$$

is projection. Thus, on each factor in the product, $g \in G$ acts by left multiplication.

Denote by $F(G)$ the smallest lattice ordered subgroup of

$$\prod_{P \in LO(G)} Aut(G, <_P)$$

containing the set $i(G)$. Then $F(G)$, together with the map i defined above, is the free lattice ordered group over the left orderable group G . The lattice ordering of $F(G)$ will be denoted by \prec , and is the restriction of the canonical product ordering to the lattice ordered subgroup $F(G)$.

Essential in showing that this construction produces a group satisfying the required universal property is the following proposition, due to Conrad (though if we assume the construction of $F(G)$ above yields a group with the desired universal property, the proposition is obvious) [14].

Proposition 4.1.3. *Let G be a left orderable group, and suppose that x is any non-identity element of $F(G)$. Then there exists $P \in LO(G)$ such that $\pi_P(x) \neq 1$.*

From this point forward we will simplify our notation by writing $g \in F(G)$ in place of $i(g) \in F(G)$, for every element g of G . Thus, every element of $F(G)$ can be (non-uniquely) written in the form $\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}$ for suitable $g_{ij} \in G$. The map $\pi_P : F(G) \rightarrow \text{Aut}(G, <_P)$ sends such an element to the order-preserving map whose action on an element $h \in G$ is defined according to the rule:

$$\pi_P\left(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}\right)(h) = \max_{i \in I} \min_{j \in J} \{g_{ij}h\}.$$

Here, the *max* and *min* are taken relative to the total ordering $<_P$ of G .

Corollary 4.1.4. *Let $x \in F(G)$ be any element satisfying $\pi_P(x)(1) = 1$ for all P in $LO(G)$. Then $x = 1$.*

Proof. Suppose that x is a non-identity element of $F(G)$, from Conrad's proposition there exists P such that $\pi_P(x)(h) \neq h$ for some h in G . Set

4.2. Free lattice ordered groups and the topology on the space of left orderings

$Q = hPh^{-1}$, we will show that $\pi_Q(x)(1) \neq 1$.

We may write $x = \bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}$ for some elements $g_{ij} \in G$, so that the condition $\pi_P(x)(h) \neq h$ becomes

$$\pi_P\left(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}\right)(h) = \max_{i \in I}^P \min_{j \in J}^P \{g_{ij}h\} \neq h,$$

and thus $\max_{i \in I}^P \min_{j \in J}^P \{h^{-1}g_{ij}h\} \neq 1$. Here we have added a superscript P to indicate that the maxima and minima are relative to the left ordering of G with positive cone P .

Recall that in the left ordering $<_h$ of G defined by the positive cone $Q = hPh^{-1}$, we have $g <_h f$ if and only if $h^{-1}gh < h^{-1}fh$. Thus we may rewrite the inequality $\max_{i \in I}^P \min_{j \in J}^P \{h^{-1}g_{ij}h\} \neq 1$ as $\max_{i \in I}^Q \min_{j \in J}^Q \{g_{ij}\} \neq 1$, where the maxima and minima are now taken relative to the left ordering defined by the positive cone Q . It follows that $\pi_Q(x)(1) \neq 1$. \square

4.2 Free lattice ordered groups and the topology on the space of left orderings

In this section, we establish several connections between the topology of $LO(G)$ and the structure of the group $F(G)$. We begin with a generalization of a known result, which was originally proven in the case of $F(F_n)$, the free lattice ordered group over the free group on n generators [1]. We observe, however, that the same result holds for any left orderable group G .

Recall that an element x in a lattice ordered group L is said to be a basic element if the set $\{y \in L \mid 1 \leq y \leq x\}$ is totally ordered by the restriction of

4.2. Free lattice ordered groups and the topology on the space of left orderings

the lattice ordering.

Lemma 4.2.1. [1] *Suppose that $1 \prec x$ is an element of $F(G)$. Then x is a basic element if and only if there exists a unique left ordering $<_P$ of G such that $\pi_P(x)(1) >_P 1$.*

Proof. Suppose that $<_P$ is the unique left ordering of G for which $\pi_P(x)(1) >_P 1$, and suppose that y_1, y_2 are two distinct elements of $F(G)$ that satisfy $1 \prec y_i \prec x$. Without loss of generality, we may assume that $\pi_P(y_1)(1) \leq_P \pi_P(y_2)(1)$, and hence $1 \leq_P \pi_P(y_1^{-1}y_2)(1)$. Therefore, considering the element $y_1^{-1}y_2 \wedge 1 \in F(G)$, we compute that

$$\pi_P(y_1^{-1}y_2 \wedge 1)(1) = \min\{\pi_P(y_1^{-1}y_2)(1), 1\} = 1.$$

Now in any left ordering $<_Q$ with $Q \neq P$, we have $1 \leq_Q \pi_Q(y_i)(1) \leq_Q \pi_Q(x)(1) \leq_Q 1$, where the final inequality follows from our assumption that $<_P$ is the unique left ordering with $\pi_P(x)(1) >_P 1$.

Therefore $\pi_Q(y_1^{-1}y_2 \wedge 1)(1) = 1$ is true for all those positive cones Q different from P and in the case $Q = P$. It follows that $y_1^{-1}y_2 \wedge 1 = 1$ by Corollary 4.1.4, and hence $1 \prec y_1^{-1}y_2$, and $y_1 \prec y_2$ as desired.

On the other hand, suppose that x is a (positive) basic element, and suppose that P and Q are distinct positive cones such that $\pi_P(x)(1) >_P 1$ and $\pi_Q(x)(1) >_Q 1$. Choose an element h of G such that $h >_P 1$ and $h^{-1} >_Q 1$. Then the elements $y_1 = (x \wedge h) \vee 1$ and $y_2 = (x \wedge h^{-1}) \vee 1$ satisfy $1 \prec y_i \prec x$, yet are not comparable in the partial ordering \prec of $F(G)$. This

4.2. Free lattice ordered groups and the topology on the space of left orderings

follows from computing

$$\pi_P(y_1)(1) = \max\{\min\{\pi_P(x)(1), h\}, 1\} >_P 1,$$

and

$$\pi_P(y_2)(1) = \max\{\min\{\pi_P(x)(1), h^{-1}\}, 1\} = 1,$$

while $\pi_Q(y_1)(1) = 1$ and $\pi_Q(y_2)(1) >_Q 1$.

□

Theorem 4.2.2. [1] *Let G be a left orderable group. Then $F(G)$ contains a basic element if and only if $LO(G)$ contains an isolated point.*

Proof. Suppose that

$$\{P\} = U_{g_1} \cap \cdots \cap U_{g_n}$$

is an isolated point in $LO(G)$. Then P is the unique left ordering for which $1 <_P g_j$ for all j , and is therefore the unique ordering for which $\pi_P(g_j)(1) >_P 1$ for all j . Therefore, $<_P$ is the unique left ordering for which $\pi_P(\bigwedge_{j=1}^n g_j)(1) >_P 1$. By definition, a basic element must be positive, and so we consider the element $(\bigwedge_{j=1}^n g_j) \vee 1$. The left ordering $<_P$ is the unique left ordering for which $\pi_P((\bigwedge_{j=1}^n g_j) \vee 1)(1) >_P 1$, and it now follows that $(\bigwedge_{j=1}^n g_j) \vee 1$ is a basic element of $F(G)$, by Lemma 4.2.1.

Conversely, if $\bigvee_{i=1}^m \bigwedge_{j=1}^n g_{ij}$ is a basic element in $F(G)$, then there is a unique left ordering $<_P$ of G such that $\pi_P(\bigvee_{i=1}^m \bigwedge_{j=1}^n g_{ij})(1) >_P 1$. Therefore, for some index i , $\pi_P(\bigwedge_{j=1}^n g_{ij})(1) = \min_{j=1}^n \{g_{ij}\} >_P 1$, and $<_P$ is the unique left ordering of G for which this inequality holds, for our chosen index

4.2. Free lattice ordered groups and the topology on the space of left orderings

i. In other words, $<_P$ is the unique left ordering in which all the elements g_{ij} are positive for our chosen index i , so that

$$\{P\} = \bigcap_{j \in J} U_{g_{ij}}$$

is an isolated point in $LO(G)$. □

Recall that for any left-orderable group G , the space $LO(G)$ comes equipped with a G -action by conjugation, which is an action by homeomorphisms. Set

$$Orb_G(P) = \{gPg^{-1} \mid g \in G\},$$

and we denote the closure of each such orbit by $\overline{Orb_G(P)}$. The group action on the space of left orderings has been used to great success in investigating the structure of the space of left orderings, see [11, 40, 42, 43]. However, in most applications, one often asks if a positive cone P is an accumulation point of its own conjugates in order to show that P is not an isolated point. More useful would be an answer to the question: When is Q an accumulation point of the conjugates of some different positive cone P , that is, when is $Q \in \overline{Orb_G(P)}$? We find that this question is equivalent to an algebraic question about the free lattice-ordered group $F(G)$.

Theorem 4.2.3. *Let G be a left orderable group, and let $P, Q \in LO(G)$ be given. Then $Q \in \overline{Orb_G(P)}$ if and only if $\ker(\pi_P) \subset \ker(\pi_Q)$.*

Proof. Suppose that $\ker(\pi_P) \subset \ker(\pi_Q)$, and that Q lies in the basic open set $\bigcap_{j=1}^n U_{g_j}$. We must show that some conjugate of P lies in this open set as well.

4.2. Free lattice ordered groups and the topology on the space of left orderings

Consider the element $(\bigwedge_{j=1}^n g_j) \vee 1$ in $F(G)$. As $Q \in \bigcap_{j=1}^n U_{g_j}$, we know that $g_j >_Q 1$ for all j , and hence we find that

$$\pi_Q((\bigwedge_{j=1}^n g_j) \vee 1)(1) = \max\{\min\{g_j\}, 1\} = \min\{g_j\} >_Q 1.$$

Therefore, $(\bigwedge_{j=1}^n g_j) \vee 1$ is not in the kernel of the map π_Q , and so from our assumption it is not in the kernel of the map π_P . Thus, there exists $h \in G$ such that $\pi_P((\bigwedge_{j=1}^n g_j) \vee 1)(h) \neq h$, and we compute

$$\pi_P((\bigwedge_{j=1}^n g_j) \vee 1)(h) = \max\{\min\{g_j h\}, h\} >_P h,$$

so that $g_j h >_P h$ for $j = 1, \dots, n$. Therefore, $h^{-1} g_j h >_P 1$, and hence $h^{-1} g_j h \in P$ for all j . This is equivalent to $g_j \in h P h^{-1}$ for all j , or $h P h^{-1} \in \bigcap_{j=1}^n U_{g_j}$, so that Q is an accumulation point of the orbit of P .

On the other hand, suppose that $Q \in \overline{Orb_G(P)}$, and let $\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}$ be any element of $F(G)$ such that $\pi_Q(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}) \neq 1$. There are two cases to consider, in order to show that $\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij} \notin \ker(\pi_P)$.

Case 1. There exists i such that $\pi_Q(\bigwedge_{j \in J} g_{ij})(h) >_Q h$ for some $h \in G$. Then $\min_j g_{ij} h >_Q h$, and therefore $h^{-1} g_{ij} h \in Q$ for all j , hence $Q \in \bigcap_{j \in J} U_{h^{-1} g_{ij} h}$. By assumption, we can choose $f \in G$ such that $f P f^{-1} \in \bigcap_{j \in J} U_{h^{-1} g_{ij} h}$, so that $h^{-1} g_{ij} h \in f P f^{-1}$ for all j . In other words, $f^{-1} h^{-1} g_{ij} h f >_P 1$ for all j , so that $g_{ij} h f >_P h f$ for all j .

Now, we may compute

$$\pi_P(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij})(h f) >_P \pi_P(\bigwedge_{j \in J} g_{ij})(h f) = \min_j \{g_{ij} h f\} >_P h f.$$

4.2. Free lattice ordered groups and the topology on the space of left orderings

We conclude that $\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij} \notin \ker(\pi_P)$.

Case 2. For all i , $\pi_Q(\bigwedge_{j \in J} g_{ij})(h) \leq_Q h$ for all $h \in G$. Then in particular, we may choose $h \in G$ such that $\pi_Q(\bigwedge_{j \in J} g_{ij})(h) <_Q h$ for every $i \in I$ (strict inequality), since the image of $\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij}$ must act nontrivially on $(G, <_Q)$.

Now with h as above, we observe that for every index i there exists an index j such that $g_{ij}h <_Q h$. Thus, for every i , we may choose h_i from the set of elements $\{g_{ij}\}_{j \in J}$, so that h_i satisfies $h_i h <_Q h$. Then each h_i satisfies $h^{-1}h_i h <_Q 1$, so that $Q \in \bigcap_{i \in I} U_{h^{-1}h_i^{-1}h}$. We may therefore choose $f \in G$ so that $fP f^{-1} \in \bigcap_{i \in I} U_{h^{-1}h_i^{-1}h}$, in other words, $f^{-1}h^{-1}h_i^{-1}h f >_P 1$ for all $i \in I$. It follows that $f^{-1}h^{-1}h_i h f <_P 1$ and so $h_i h f <_P h f$ for all $i \in I$.

Thus, we find that for every $i \in I$,

$$\pi_P(\bigwedge_{j \in J} g_{ij})(h f) = \min_j \{g_{ij} h f\} \leq_P h_i h f <_P h f,$$

where the inequality $\min_j \{g_{ij} h f\} \leq_P h_i h f$ follows from the fact that h_i lies in the set $\{g_{ij}\}_{j \in J}$.

Thus, when we take a (finite) maximum over all i , we compute that $\pi_P(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij})(h f) = \min_j \{g_{ij} h f\}$ for some i , and hence

$$\pi_P(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij})(h f) = \min_j \{g_{ij} h f\} \leq_P h_i h f <_P h f.$$

It follows that $\pi_P(\bigvee_{i \in I} \bigwedge_{j \in J} g_{ij})$ is nontrivial, and the claim is proven. \square

Corollary 4.2.4. *For a given positive cone P in a left orderable group G , $\overline{\text{Orb}_G(P)} = \text{LO}(G)$ if and only if $\pi_P : F(G) \rightarrow \text{Aut}(G, <_P)$ is injective.*

4.2. Free lattice ordered groups and the topology on the space of left orderings

Proof. It is clear that if $\pi_P : F(G) \rightarrow \text{Aut}(G, <_P)$ has trivial kernel, then $\overline{\text{Orb}_G(P)} = LO(G)$ by Theorem 4.2.3.

For the other direction, suppose that $\overline{\text{Orb}_G(P)} = LO(G)$. From Theorem 4.2.3 we deduce the containment $\ker(\pi_P) \subset \bigcap_{Q \in LO(G)} \ker(\pi_Q)$. However, from Proposition 4.1.3 we find that $\bigcap_{Q \in LO(G)} \ker(\pi_Q) = \{1\}$, so that $\ker(\pi_P) = \{1\}$. \square

Thus, for a given left orderable group G , injectivity of the map $\pi_P : F(G) \rightarrow \text{Aut}(G, <_P)$ for some $P \in LO(G)$ tells us a great deal about the structure of $LO(G)$.

Proposition 4.2.5. *Let G be a left orderable group, and suppose that there exists $P \in LO(G)$ such that π_P is injective. Then $LO(G)$ contains no isolated points.*

Proof. If the map π_P is injective, then we know that we may write $LO(G) = \overline{\text{Orb}_G(P)}$, and so only those points in $\text{Orb}_G(P)$ itself are possibly isolated in $LO(G)$, which can only happen if P itself is isolated.

Supposing that P is an isolated point, it follows that P^{-1} is also an isolated point in $LO(G)$, and hence $P^{-1} \in \text{Orb}_G(P)$; so we may write $P^{-1} = gPg^{-1}$ for some $g \in G$, with g different from the identity. This is impossible, for supposing $g \in P$ yields (upon conjugation by g) $g \in gPg^{-1} = P^{-1}$. Similarly, $g \in P^{-1}$ is impossible, so that P is not isolated. \square

From Theorem 4.2.3, we now have a bijection between certain normal subgroups of $F(G)$ and certain closed subsets of $LO(G)$. Specifically, if K is the kernel of the map π_P , we can associate to K the closed set $\overline{\text{Orb}_G(P)}$.

4.2. Free lattice ordered groups and the topology on the space of left orderings

Note that if π_Q is some other map with kernel K , then $\overline{Orb_G(Q)} = \overline{Orb_G(P)}$, so that the closed set associated to K is well-defined. Inclusion of kernels $ker(\pi_P) \subset ker(\pi_Q)$ yields a reverse inclusion of associated subsets, $\overline{Orb_G(Q)} \subset \overline{Orb_G(P)}$.

It is well known that the space $LO(G)$ is compact (Theorem 2.1.2), and compactness has thus far proven to be one of the most useful properties of $LO(G)$. In [32], compactness is the key ingredient in showing that no group has countably infinitely many left orderings (which has been proven again recently in [43]), and in [37], compactness is used to show that an amenable group is left orderable if and only if it is locally indicable. In our present setting, compactness of the space of left orderings yields the following:

If G is a left orderable group, let S_G denote the set of all normal subgroups of $F(G)$ that occur as the kernel of some map $\pi_P : F(G) \rightarrow Aut(G, <_P)$, where P ranges over all positive cones in $LO(G)$. The set S_G is partially ordered by inclusion.

Proposition 4.2.6. *Every chain in S_G has an upper bound. In particular, S_G has a maximal element.*

Proof. Let T be a subset of $LO(G)$ such that $\{ker(\pi_P)\}_{P \in T}$ is a totally ordered subset of S_G . Observe that $\overline{Orb_G(P)} \subset \overline{Orb_G(Q)}$ if and only if $ker(\pi_Q) \subset ker(\pi_P)$, and thus $\{\overline{Orb_G(P)}\}_{P \in T}$ is a nested collection of closed subsets of $LO(G)$. In particular, this nested collection of sets has the finite intersection property. For if P_1, \dots, P_n is some finite subset of T , upon renumbering if necessary, we may assume that $\overline{Orb_G(P_1)} \subset \dots \subset \overline{Orb_G(P_n)}$, from which it is obvious that P_1 is contained in their intersection.

4.2. Free lattice ordered groups and the topology on the space of left orderings

Thus, the intersection

$$\bigcap_{P \in T} \overline{Orb_G(P)}$$

is nonempty, as $LO(G)$ is compact. Choosing any positive cone R from this intersection yields a closed set $\overline{Orb_G(R)}$ that lies in $\overline{Orb_G(P)}$ for every P in T , and hence $\ker(\pi_P) \subset \ker(\pi_R)$ for every P in T . It now follows from Zorn's lemma that S_G contains a maximal element. \square

The following is a standard definition in the theory of dynamical systems.

Definition 4.2.7. *A nonempty set U in $LO(G)$ is said to be a minimal invariant set if U is closed and G -invariant, and for every closed G -invariant set V in $LO(G)$, $U \cap V \neq \emptyset$ implies $U \subset V$.*

The equivalence of (1) and (2) in the following proposition is a standard result from the theory of dynamical systems ([19], pp. 69-70).

Proposition 4.2.8. *For any nonempty closed subset U of $LO(G)$, the following are equivalent:*

1. U is a minimal invariant set
2. for every $P \in U$, $U = \overline{Orb_G(P)}$
3. $U = \overline{Orb_G(P)}$ for some $P \in LO(G)$ whose kernel is maximal in S_G .

Proof. (1) if and only if (2). Suppose that U is a minimal invariant set, and let $P \in U$ be given. Then $\overline{Orb_G(P)} \subset U$, since U is closed and G -invariant. Since U is small, this implies $U \subset \overline{Orb_G(P)}$, and so $U = \overline{Orb_G(P)}$. Conversely, suppose that (2) is satisfied and let V be some other closed, G -invariant set such that $U \cap V$ is nonempty. Choose $Q \in U \cap V$, and observe

4.2. Free lattice ordered groups and the topology on the space of left orderings

that $U = \overline{Orb_G(Q)} \subset V$, since V is closed and G -invariant. Therefore U is a minimal invariant set.

(2) if and only if (3). Suppose property (2) holds, and let $P \in U$ be given, and suppose that $\ker(\pi_P) \subset \ker(\pi_Q)$ for some $Q \in LO(G)$. Then by Theorem 4.2.3, $Q \in \overline{Orb_G(P)} = U$, and hence, by condition (2), $\overline{Orb_G(Q)} = U = \overline{Orb_G(P)}$. It follows that $\ker(\pi_P) = \ker(\pi_Q)$ is maximal. Conversely, suppose (3) and let $P \in U$ be given. Then for any other Q in $U = \overline{Orb_G(P)}$, we have $\ker(\pi_P) \subset \ker(\pi_Q)$ by Theorem 4.2.3. Since $\ker(\pi_P)$ is maximal, this gives $\ker(\pi_Q) = \ker(\pi_P)$ and (2) follows. \square

We can now see that Proposition 4.2.6 mirrors a standard proof of the existence of minimal invariant sets, see for example [19] Theorem 3.12. It is also clear from the above characterization that $\ker(\pi_P)$ is maximal if P is the positive cone of a bi-ordering (so $\overline{Orb_G(P)} = \{P\}$), or if $<_P$ is a bi-ordering when restricted to some finite index subgroup $H \subset G$ (for then $\overline{Orb_G(P)}$ is finite).

We may apply the notion of minimal invariant sets to provide yet another proof that no left orderable group has countably infinitely many left orderings.

Proposition 4.2.9. *Let G be a left orderable group, and let U be a minimal invariant subset of $LO(G)$. Then U is finite, or uncountable.*

Proof. Suppose that U is not finite. If U is infinite and contains no isolated points, then U is uncountable, as U is a compact Hausdorff space [Theorem 2-80, [25]]. Thus, suppose that U contains an isolated point, and we will

4.2. Free lattice ordered groups and the topology on the space of left orderings

arrive at a contradiction. Note that in this context, “isolated point” means isolated in U , not isolated in $LO(G)$.

Choose P in U . By (2) of Proposition 4.2.8, $U = \overline{Orb_G(P)}$, and it follows that P itself must be an isolated point (and hence every point in $Orb_G(P)$ is isolated). Since U is compact, the set of conjugates $Orb_G(P)$ must accumulate on some $Q \in U$, moreover, Q does not lie in $Orb_G(P)$, since Q is not isolated. We again apply (2) of Proposition 4.2.8 to find that $U = \overline{Orb_G(Q)}$, and it follows that P cannot be an isolated point, a contradiction. \square

Corollary 4.2.10. *For any group G , $LO(G)$ is either finite or uncountable.*

Proof. Given any left orderable group G , by Proposition 4.2.6, there exists $P \in LO(G)$ for which $\ker(\pi_P)$ is maximal in S_G . Correspondingly, the set $\overline{Orb_G(P)}$ is a minimal invariant set, and hence it is either finite, or uncountable.

Assuming $\overline{Orb_G(P)}$ is finite, we must have that $Orb_G(P)$ is finite, and hence the stabilizer $Stab_G(P)$ is a finite index subgroup of G that is bi-ordered by the restriction of the left ordering $<_P$. It follows that G is locally indicable [53], and hence has uncountably many left orderings, by [63]. \square

It should be noted that this proof is very similar to the proof of Peter Linnell, given in [32]. The crucial difference in our proof is that some difficult topological arguments have been replaced by an application of Zorn’s lemma.

It does not appear that a clean topological statement will characterize precisely those closed sets that occur as $\overline{Orb_G(P)}$ for some ordering whose

4.3. Examples

associated kernel is minimal in S_G . We do, however, have the following observation.

Proposition 4.2.11. *Let G be a left orderable group, and let P in $LO(G)$ be an isolated point. Then $\ker(\pi_P)$ is minimal in S_G .*

Proof. Suppose that $\ker(\pi_Q) \subset \ker(\pi_P)$ for some $Q \in LO(G)$. Then $P \in \overline{Orb_G(Q)}$ by Theorem 4.2.3, but P is isolated, so $P \in Orb_G(Q)$. Therefore Q is conjugate to P , and so $Q \in \overline{Orb_G(P)}$, and $\ker(\pi_P) \subset \ker(\pi_Q)$ by Theorem 4.2.3. Thus $\ker(\pi_Q)$ is equal to $\ker(\pi_P)$, so that $\ker(\pi_P)$ is minimal. \square

Not every minimal kernel in S_G corresponds to an isolated point in $LO(G)$. In the next section, we will see that with $G = F_n$, the free group on n generators, S_{F_n} contains a minimal kernel ([28, 29]) that does not correspond to an isolated point.

4.3 Examples

4.3.1 The free group

It now follows easily from work of Kopytov that $LO(F_n)$ contains a dense orbit under the conjugation action by F_n . This case appears to be the first known example of a left ordering (of any group) whose orbit is dense in the space of left orderings.

Corollary 4.3.1. *Let F_n denote the free group on $n > 1$ generators. There exists P such that $\overline{Orb_{F_n}(P)} = LO(F_n)$.*

Proof. There exists a left ordering of F_n with positive cone P such that $\pi_P : F(G) \rightarrow \text{Aut}(G, <_P)$ is injective ([28, 29]), and we apply Corollary 4.2.4. \square

We may apply Proposition 4.2.5 and Theorem 4.2.2 to arrive at a new proof of the following corollary (see [36, 40] for alternate proofs).

Corollary 4.3.2. *The space $LO(F_n)$ has no isolated points for $n > 1$. Equivalently, $F(F_n)$ has no basic elements for $n > 1$.*

Recall that the Conradian soul of a left ordering is the largest convex subgroup on which the restriction ordering is Conradian [40]. It was shown in [36] and [22] that the construction of Kopytov can be improved so that the map π_P is injective, and the left ordering $<_P$ of F_n has no convex subgroups. Consequently, the Conradian soul of the ordering $<_P$ must be trivial, and so P is an accumulation point of its own conjugates in $LO(F_n)$ [11, 40].

It is also worth noting that McCleary's construction in [36] of a faithful $o-2$ transitive action of $F(F_n)$ on some linearly ordered set is much stronger than is needed to conclude that $F(F_n)$ has no basic elements; yet [36] appears to contain the first proof of this fact appearing in the literature.

4.3.2 Left orderable groups with all left orderings

Conradian

In the case that all left orderings of a given left orderable group are Conradian, we may highlight two cases of interest. First we will observe that no finitely generated group G , all of whose left orderings are Conradian, can have a dense orbit in $LO(G)$. Second, we will show that if we allow the

4.3. Examples

group to be infinitely generated, then it may be the case that every orbit in $LO(G)$ is dense. Recall that, for example, every left ordering of a torsion free locally nilpotent group is Conradian [3, 5].

An element g in a left ordered group G with ordering $<$ is cofinal if for every h in G , there exists $n \in \mathbb{Z}$ such that $g^{-n} < h < g^n$.

Proposition 4.3.3. *Suppose that G is a left orderable group, and suppose that g is cofinal in $<_P$, for some $P \in LO(G)$. Then P^{-1} is not in $\overline{Orb_G(P)}$.*

Proof. Suppose that $g >_P 1$ is cofinal, and let $1 <_P h \in G$ be given. Choose an integer n so that $h <_P g^n$, and observe that $1 <_P h^{-1}g^n$, and hence $1 <_P h^{-1}g^n h$, since h is positive. Therefore $1 <_P h^{-1}gh$, and so $h <_P gh$. On the other hand, if h is negative, we similarly conclude that $h <_P gh$, and thus $\pi_P(g \wedge 1)(h) = h$ for all $h \in G$, so that $g \wedge 1 \in \ker(\pi_P)$. On the other hand, in the reverse ordering with positive cone P^{-1} , we find $h >_{P^{-1}} gh$ for all $h \in G$, so that $\pi_{P^{-1}}(g \wedge 1)(h) <_{P^{-1}} h$ for all $h \in G$, so that $g \wedge 1 \neq 1$ in $F(G)$. Thus the map π_P is not injective, and in particular, $\ker(\pi_P)$ is not contained in $\ker(\pi_{P^{-1}})$. By Theorem 4.2.3, P^{-1} is not in $\overline{Orb_G(P)}$. \square

Corollary 4.3.4. *Let G be a finitely generated group, all of whose left orderings are Conradian. Then $LO(G)$ does not contain a dense orbit.*

Proof. Let P be any positive cone in a finitely generated group G . Since the associated left ordering $<_P$ is Conradian, there exists a convex subgroup $C \subset G$ such that G/C is abelian, and the induced ordering on G/C is Archimedean. Every element in $G \setminus C$ is then cofinal in the left ordering $<_P$, and the claim follows. \square

4.3. Examples

Next, we consider the group

$$T_\infty = \langle x_i, i \in \mathbb{N} : x_{i+1}x_i x_{i+1} = x_i^{-1}, x_i x_j = x_j x_i \text{ for } |i - j| > 1 \rangle.$$

Note that each subgroup $T_n = \langle x_1, \dots, x_n \rangle$ has only 2^n left orderings, as it is one of the Tararin groups, as described in [30], Theorem 5.2.1. Moreover, the convex subgroups of any left ordering of T_n are precisely $T_i \subset T_n$ for $i \leq n$, and so the convex subgroups of T_∞ are exactly T_i for $i \in \mathbb{N}$, for any left ordering of T_∞ . The orderings of T_∞ are all Conradian, with convex jumps $T_{i+1}/T_i \cong \mathbb{Z}$. Given any positive cone $P \in LO(T_\infty)$, it is therefore determined by the signs of the generators x_i , which we record in a sequence

$$\varepsilon = (\pm 1, \pm 1, \dots)$$

writing $+1$ in the i -th position if x_i is in P , and -1 otherwise. We then write $P = P_\varepsilon$. With this notation, we observe that $x_{i+1}P_\varepsilon x_{i+1}^{-1} = P_{\varepsilon'}$, where ε' differs from ε only in the sign of the i -th entry (this follows from the defining relations of the group T_∞ , and corresponds to the idea of “flipping” the ordering on the i -th convex jump).

Proposition 4.3.5. *Let $P \in LO(T_\infty)$ be any positive cone. Then $LO(T_\infty) = \overline{Orb_{T_\infty}(P)}$.*

Proof. Let P_{ε_1} and P_{ε_2} be two positive cones in $LO(T_\infty)$. It is enough to show that for every $n \in \mathbb{N}$, there exists $g \in T_\infty$ such that $gP_{\varepsilon_1}g^{-1} \cap T_n = P_{\varepsilon_2} \cap T_n$, so that the associated left orderings agree upon restriction to the subgroup T_n .

4.3. Examples

The proof is a simple induction. First, note that there exists a conjugate of P_{ε_1} that agrees with P_{ε_2} upon restriction to the subgroup $T_1 = \langle x_1 \rangle$: if ε_1 and ε_2 agree in the first entry, then P_{ε_1} and P_{ε_2} agree on T_1 , whereas if ε_1 and ε_2 disagree in the first entry, then $x_2 P_{\varepsilon_1} x_2^{-1}$ and P_{ε_2} agree on T_1 .

For induction, suppose that Q is a conjugate of P_{ε_1} with associated sequence ε'_1 first differing from ε_2 in the n -th entry, so that $Q \cap T_{n-1} = P_{\varepsilon_2} \cap T_{n-1}$. Then $x_{n+1} Q x_{n+1}^{-1}$ will have associated sequence ε''_1 that agrees with ε_2 in the sign of the n -th entry. Thus the conjugate $x_{n+1} Q x_{n+1}^{-1}$ of P_{ε_1} agrees with P_{ε_2} upon restriction to the subgroup T_n , and the result follows by induction. \square

4.3.3 The braid groups

The braid groups $B_n, n > 2$ also provide an interesting class of examples, as their spaces of left orderings are known to contain isolated points [11, 23, 40].

Proposition 4.3.6. *For $n > 2$, the space $LO(B_n)$ contains no dense orbit.*

Proof. This follows immediately from Proposition 4.2.5 and Corollary 4.2.4, in light of the fact that $LO(B_n)$ contains isolated points. \square

As an alternative way of proving Proposition 4.3.6, recall that the center of the braid group B_n , for $n > 2$, is infinite cyclic with generator Δ_n^2 . Here,

$$\Delta_k := (\sigma_{k-1} \sigma_{k-2} \cdots \sigma_1)(\sigma_{k-1} \sigma_{k-2} \cdots \sigma_2) \cdots (\sigma_{k-1} \sigma_{k-2})(\sigma_{k-1})$$

is the Garside half-twist. It is well known that Δ_n^2 is cofinal in any left ordering of B_n [21], from which it follows by Proposition 4.3.3 that $LO(B_n)$

4.3. *Examples*

contains no dense orbit.

Chapter 5

3-manifolds and left orderability

As left orderability of a countable group is equivalent to an action on the real line by order-preserving homeomorphisms, it is not surprising that groups arising in a geometric context behave exceptionally with respect to left orderability. The class of groups that occur as the fundamental groups of 3-manifolds exhibit such exceptional behaviour with respect to left orderability. In turn, left (or bi-) orderability of the fundamental group of a manifold M can be connected to the underlying geometry of M .

5.1 Examples of 3-manifolds with left orderable fundamental group

An example of such behaviour is the following variation of Theorem 1.4.1. Recall that a 3-manifold M is P^2 -irreducible if M is irreducible and does not contain any two-sided, properly embedded projective planes.

Theorem 5.1.1 (See [7], attributed to Howie and Short [27]). *Suppose that M is a compact, connected, P^2 -irreducible 3-manifold and that $\pi_1(M)$*

5.1. Examples of left orderable fundamental groups

is nontrivial. Then $\pi_1(M)$ is left orderable if and only if there exists a surjective homomorphism $h : \pi_1(M) \rightarrow L$ onto a nontrivial left orderable group L .

Sketch of proof. We will show that $\pi_1(M)$ is left orderable by showing that every finitely generated subgroup surjects onto a nontrivial left orderable group. This will allow us to apply the Burns-Hale Theorem (Theorem 1.4.1).

Thus, suppose that $h : \pi_1(M) \rightarrow L$ is a homomorphism onto a nontrivial left orderable group, and let H be a finitely generated subgroup of $\pi_1(M)$. If H is finite index, then the image $h(H)$ is finite index in L , and so $h(H)$ is a nontrivial left orderable group. Hence, the restriction of h to the subgroup H provides us with a homomorphism onto a nontrivial left orderable group in this case.

In the case that H is infinite index, we consider the covering space \widetilde{M} of M corresponding to the subgroup H . As H is infinite index in $\pi_1(M)$, the cover \widetilde{M} is noncompact. Here, we apply the compact core theorem of Scott [56] to produce a compact submanifold $C \subset \widetilde{M}$ (necessarily with boundary) such that $\pi_1(C) = H$. Since M is P^2 -irreducible, the covering space \widetilde{M} is P^2 -irreducible. We can use P^2 -irreducibility to show that we may assume the boundary of C in \widetilde{M} contains no spheres or projective planes. The analysis concludes by using an Euler characteristic argument to find that $H_1(C)$ is infinite, and thus surjects onto the integers. We now have a map $H \cong \pi_1(C) \xrightarrow{ab} H_1(C) \rightarrow \mathbb{Z}$ onto the left orderable group \mathbb{Z} . This allows us to apply the Burns-Hale theorem to conclude that $\pi_1(M)$ is left orderable. \square

An important corollary follows from this proof. Observe that the finitely

generated subgroups of infinite index in $\pi_1(M)$ must all surject onto the integers. Thus if we impose a condition that guarantees that the finitely generated subgroups of finite index map onto the integers, we will be able to conclude that $\pi_1(M)$ is locally indicable, and hence Conrad orderable. Thus, we require that $H_1(M)$ is infinite.

Corollary 5.1.2. *Suppose that M is a compact, connected, P^2 -irreducible 3-manifold and that $H_1(M)$ is infinite. Then $\pi_1(M)$ is Conrad orderable.*

Calegari and Dunfield have also succeeded in left ordering the fundamental groups of a large class of 3-manifolds whose homology is finite. See [10] for background on taut foliations.

Theorem 5.1.3 ([10]). *Let M be an irreducible, atoroidal \mathbb{Q} -homology 3-sphere which admits a transversely orientable taut foliation. Then the commutator subgroup $[\pi_1(M), \pi_1(M)]$ is left orderable, and hence the covering space \widetilde{M} of M corresponding to $[\pi_1(M), \pi_1(M)]$ has left orderable fundamental group. Moreover, if M is compact, then \widetilde{M} is also compact since the cover is finite-sheeted.*

Finally, it has been observed very recently that left orderability of fundamental groups behaves in an interesting way with respect to Dehn surgery. We recall the basics of Dehn surgery, see [55] for full details.

Let M be a 3-manifold with torus boundary, and let m, l be a choice of meridian and longitude basis for $\pi_1(\partial M)$. Let $T \cong S^1 \times D^2$ be a torus, and $h : S^1 \times D^2 \rightarrow \partial M$ be the homeomorphism that sends ∂D^2 to a curve representing the class $pm + ql \in \pi_1(\partial M)$. Denote by $M_{p/q}$ the manifold obtained by gluing M and T along their boundaries, using the homeomorphism h .

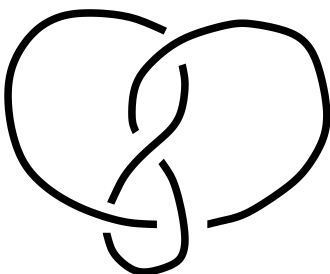


Figure 5.1: The figure eight knot.

The manifold $M_{p/q}$ is said to be the manifold obtained from Dehn surgery on M with surgery coefficient p/q .

In the case that M is a knot or link complement in S^3 , the boundary ∂M admits a preferred meridian and longitude basis for $\pi_1(\partial M)$, and so the manifold $M_{p/q}$ is determined by the pair of (relatively prime) numbers p, q . Recall that the figure eight knot complement is the complement in S^3 of the embedding of the torus depicted in Figure 5.1.

Theorem 5.1.4 ([7]). *Let M be the figure eight knot complement, and let $p/q \in (-4, 4)$ be given. Then there exists a finite index subgroup of $\pi_1(M_{p/q})$ which is left orderable.*

It has also recently been announced that this theorem may be strengthened, to show that $\pi_1(M_{p/q})$ is left orderable whenever $p/q \in (-4, 4)$ [6]. We will further investigate the connection between Dehn surgery and left orderability of the fundamental group $\pi_1(M_{p/q})$ in Chapter 6.

5.2 Geometric applications of left orderability

At present, there are several known geometric consequences of left orderability of the fundamental group of a given 3-manifold. We recall the definition of an \mathbb{R} -covered foliation. Let \mathcal{F} be a codimension one foliation of the 3-manifold M , and let $p : \widetilde{M} \rightarrow M$ be the universal covering space. Denote by $\widetilde{\mathcal{F}}$ the pullback foliation to the universal cover. If we identify every leaf of $\widetilde{\mathcal{F}}$ to a point, the resulting topological space is denoted by $\widetilde{M}/\widetilde{\mathcal{F}}$, and is a (not necessarily Hausdorff) 1-dimensional manifold. In the case that $\widetilde{M}/\widetilde{\mathcal{F}}$ is homeomorphic to \mathbb{R} , the foliation \mathcal{F} of M is said to be \mathbb{R} -covered. There is a natural action of $\pi_1(M)$ on \widetilde{M} by deck transformations, which descends to an action on the leaf space $\widetilde{M}/\widetilde{\mathcal{F}}$. It is from this action, and Theorem 1.4.11 that we arrive at the following.

Theorem 5.2.1 ([7]). *Let M be a compact, connected P^2 -irreducible 3-manifold. If $\pi_1(M)$ is not left orderable, then M does not admit a transversely oriented \mathbb{R} -covered foliation.*

We may also use orderability as an obstruction to the existence of a map of nonzero degree between closed, oriented manifolds.

Theorem 5.2.2 ([7]). *Let M and N be closed, oriented 3-manifolds and suppose that M is prime. Suppose that $\pi_1(M)$ is not left orderable, and that $\pi_1(N)$ is left orderable and is nontrivial. Then every continuous map $h : M \rightarrow N$ must be of degree zero.*

There has also been a very recent discovery that connects non-left orderability of $\pi_1(M)$ for a given 3-manifold M to its Heegaard-Floer homology

groups (see [46, 47] for definitions). Roughly speaking, a manifold M is called an L -space if it is a rational homology 3-sphere whose Heegaard-Floer homology groups are as simple as possible [49].

Theorem 5.2.3 ([52], [6], [62]). *Suppose that M is a closed, connected, orientable Seifert fibered 3-manifold. Then M is an L -space if and only if $\pi_1(M)$ is not left orderable.*

Remarkable is that the evidence accumulated thus far seems to point towards a stronger version of this theorem which may cover the case of hyperbolic 3-manifolds. For example, from [16] we have the following infinite families of 3-manifolds, all of whose fundamental groups are shown to be not left orderable.

Theorem 5.2.4 ([16]). *Denote by $M^n(L)$ the n -fold branched cyclic cover of S^3 , branched along the link $L \subset S^3$, with $n > 1$. The fundamental group $\pi_1(M^n(L))$ is not left orderable for the following (oriented) links L :*

1. $L = T_{(2', 2k)}$ is the torus link of type $(2, 2k)$ where the strings are given anti-parallel orientation, and where n is arbitrary.
2. $L = P(n_1, n_2, \dots, n_k)$ is the pretzel link of type (n_1, n_2, \dots, n_k) , $k > 2$, where either $n_1, n_2, \dots, n_k > 0$ or $n_1 = n_2 = \dots = n_{k-1} = 2$, $n_k = -1$ and $k > 3$, with $n = 2$.
3. $L = L_{[2k, 2m]}$ is the 2-bridge knot of type $p/q = 2m + \frac{1}{2k} = [2k, 2m]$ where $k, m > 0$ and n is arbitrary.
4. $L = L_{[n_1, 1, n_3]}$ is the 2-bridge knot of type $p/q = n_3 + \frac{1}{1 + \frac{1}{n_1}}$, where

5.2. Geometric applications of left orderability

n_1 and n_3 are odd, positive integers. The degree of the covering is $n = 2, 3$.

Many of these manifolds are hyperbolic, see [16] for details. The main theorem of [52] is the following:

Theorem 5.2.5. *All of those spaces listed above are L -spaces.*

Boyer, Watson and Cameron have also recently announced that the 2-fold branched cover of S^3 , branched along an alternating link, also has non-left orderable fundamental group [6]. As above, these manifolds are known to be L -spaces [48].

Chapter 6

An application of the space of left orderings to 3-manifolds

Our goal in this chapter will be to use the topology of $LO(G)$ in order to establish non-left orderability of the fundamental groups of manifolds obtained from Dehn surgery.

6.1 The map $\psi_{G,H}$

Let H be a subgroup of G . We may define a map $r : LO(G) \rightarrow LO(H)$ by restricting a given left ordering of G to the subgroup H . In terms of positive cones, the map r acts according to the formula $r(P) = P \cap H$. Note that if H is a subgroup of G , then for every $h \in H$ the notation U_h can ambiguously refer to an open set in $LO(G)$ and an open set in $LO(H)$. We therefore introduce the notation U_h^H and U_h^G to distinguish between these two sets.

Proposition 6.1.1. *The map r is continuous.*

6.1. The map $\psi_{G,H}$

Proof. We will show that the inverse image of any basic open set is a basic open set. Let $U_h^H \subset LO(H)$ be given, where h is a nonidentity element in H . Then

$$r^{-1}(U_h^H) = \{P \in LO(G) | h \in P \cap H\} = \{P \in LO(G) | h \in P\} = U_h^G.$$

It follows that r is continuous. □

We now wish to consider the case when $H = \mathbb{Z} \times \mathbb{Z}$, for which we need a structure theorem, due to Sikora [58]. Denote by \mathbb{R}_{\lceil} the real line with the topology whose basis consists of sets of the form $(a, b]$, and define \mathbb{R}_{\lfloor} analogously. These topologies on \mathbb{R} descend to two different topologies on $S^1 \subset \mathbb{R}^2$, via the map $z \mapsto e^{2\pi iz}$, yielding quotient topologies via $\mathbb{R}_{\lceil}/\mathbb{Z} \rightarrow S_{\lceil}^1$ and $\mathbb{R}_{\lfloor}/\mathbb{Z} \rightarrow S_{\lfloor}^1$.

A point $(p, q) \in S^1$ is said to be rational if $p/q \in \mathbb{Q}$. Let X denote the union of S_{\lfloor}^1 and S_{\lceil}^1 with corresponding irrational points identified. Define a map $\phi : X \rightarrow LO(\mathbb{Z}^2)$ as follows. Given any point $x \in S_{\lfloor}^1$, ϕ will map x to the positive cone in \mathbb{Z}^2 (which we have identified with the integer points in \mathbb{R}^2) consisting of all those pairs (p, q) such that the counter-clockwise oriented angle between the vector x and (p, q) lies in the interval $(0, \pi]$. Similarly given $x \in S_{\lceil}^1$, ϕ will map x to the positive cone of elements (p, q) whose oriented angle with x lies in the interval $[0, \pi)$. Note that this assignment respects the equivalence relation used in defining X : If x is irrational, then our definition of $\phi(x)$ is well-defined regardless of whether we consider x to be a point in S_{\lfloor}^1 or S_{\lceil}^1 , as no points of $\mathbb{Z} \times \mathbb{Z}$ lie on the line determined by x .

Theorem 6.1.2. [58] *The map ϕ is a homeomorphism.*

We use the notation

$$\widehat{(p, q)} = \left(\frac{p}{\|(p, q)\|}, \frac{q}{\|(p, q)\|} \right)$$

to denote rational points in any of S^1 , S^1_{\emptyset} or S^1_{\square} , for relatively prime integers $p, q \in \mathbb{Z}$.

In the case that $x \in S^1$ is a rational point we will also consider $x \in S^1_{\emptyset}$ (resp. S^1_{\square}) as lying in the space X , as the rational points in $x \in S^1_{\emptyset}$ (resp. S^1_{\square}) unambiguously correspond to unique points in X via the quotient map $S^1_{\emptyset} \sqcup S^1_{\square} \rightarrow X$.

In the following theorem, we write the group operation in \mathbb{Z}^2 multiplicatively.

Lemma 6.1.3. *Let $P \in LO(\mathbb{Z}^2)$ be a positive cone, and let $\phi : X \rightarrow LO(\mathbb{Z}^2)$ be defined as above. Suppose that \mathbb{Z}^2 has generators m, l , and that p and q are relatively prime integers. The positive cone P defines an ordering of \mathbb{Z}^2 relative to which*

1. *the subgroup $\langle m^p l^q \rangle$ is convex, and*
2. *$m^p l^q$ is positive*

if and only if

$$\phi^{-1}(P) = \widehat{(p, q)}$$

considered as a point in S^1_{\square} , or

$$\phi^{-1}(P) = -\widehat{(p, q)}$$

6.1. The map $\psi_{G,H}$

considered as a point in S_{\square}^1 .

Proof. Let x denote the point

$$\widehat{(p, q)} \in S_{\square}^1.$$

We will show that $\phi(x)$ defines a bi-ordering of $\mathbb{Z} \times \mathbb{Z}$ for which the subgroup $\langle m^{pl^q} \rangle$ is convex, and the element m^{pl^q} is positive.

By definition, the positive cone $\phi(x)$ corresponds to all those elements $m^a l^b$ such that the oriented angle between the vector (a, b) and x lies in the interval $[0, \pi)$. Observe that (p, q) has oriented angle zero with the vector x , therefore m^{pl^q} lies in $\phi(x)$, and property (2) above is satisfied.

Next, let $h \in \mathbb{Z}^2$ be an element in the positive cone $\phi(x)$ not of the form $(m^{pl^q})^n$, and suppose that $h = m^c l^d$. Since p and q are relatively prime, m^{pl^q} is not a power of h , so the oriented angle between (c, d) and x lies in the open interval $(0, \pi)$. As the vectors x and (p, q) are parallel, any point of the form

$$(c, d) + n(p, q) = (c + np, d + nq)$$

also has an oriented angle with x lying in the interval $(0, \pi)$. Therefore any element in \mathbb{Z}^2 of the form $m^{c+np} l^{d+nq}$ lies in the positive cone $\phi(x)$, and hence the bi-ordering defined by $\phi(x)$ satisfies

$$1 < m^{c+np} l^{d+nq} = (m^{pl^q})^n m^c l^d \Rightarrow (m^{pl^q})^{-n} < m^c l^d.$$

Thus, every element $h \in \phi(x)$ that is not a power of m^{pl^q} must be greater than all powers of m^{pl^q} . This shows that property (1) holds for the ordering

6.1. The map $\psi_{G,H}$

of \mathbb{Z}^2 defined by the positive cone $\phi(x)$.

Next, consider the case

$$y = -\widehat{(p, q)} \in S_{\emptyset}^1.$$

Here, the positive cone $\phi(y)$ is defined as all those elements $m^a l^b$ such that the oriented angle between the vector (a, b) and y lies in the interval $(0, \pi]$. Observe that the oriented angle between y and (p, q) is π , so that $m^p l^q \in \phi(y)$, and $\phi(y)$ satisfies property (1). The proof of property (2) in this case is similar to the case of $\phi(x)$.

To prove the converse, we must show that any positive cone that defines an ordering of $\mathbb{Z} \times \mathbb{Z}$ satisfying properties (1) and (2) must be mapped by ϕ^{-1} to one of the two points of the required form. Observe that there are exactly two positive cones in $LO(\mathbb{Z}^2)$ which define an ordering satisfying properties (1) and (2), namely those positive cones arising from the short exact sequence

$$1 \rightarrow \langle m^p l^q \rangle \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 1,$$

where $\langle m^p l^q \rangle \cong \mathbb{Z}$ is ordered so that $m^p l^q$ is positive. Call these two positive cones P_1 and P_2 .

We have already proven that the positive cones $\phi(x)$ and $\phi(y)$ above correspond to orderings satisfying properties (1) and (2), and it is easy to see that $\phi(x)$ and $\phi(y)$ are distinct (for example, $m^{-p} l^q$ lies in $\phi(x)$ but not in $\phi(y)$). Therefore the sets $\{P_1, P_2\}$ and $\{\phi(x), \phi(y)\}$ must be equal, so that P_1 and P_2 are mapped by ϕ^{-1} to the points $y \in S_{\emptyset}^1$ and $x \in S_{\emptyset}^1$, which are of the required form.

□

Let $f : X \rightarrow S^1$ denote the map obtained by identifying the rational points of S^1_{\emptyset} with the rational points of S^1_{\emptyset} . The map f is continuous, since the topologies on S^1_{\emptyset} and S^1_{\emptyset} are both refinements of the standard topology on S^1 .

Let G be a left orderable group with subgroup H isomorphic to $\mathbb{Z} \times \mathbb{Z}$. We have seen that there exists a continuous map $\psi_{G,H} : LO(G) \rightarrow S^1$, which is the composition of the maps

$$LO(G) \xrightarrow{r} LO(H) \xrightarrow{\phi^{-1}} X \xrightarrow{f} S^1.$$

Proposition 6.1.4. *Let G be a left orderable group with subgroup $H \cong \mathbb{Z} \times \mathbb{Z}$ having basis m and l , and let (p, q) be any relatively prime pair of integers. With the map $\psi_{G,H} : LO(G) \rightarrow S^1 \subset \mathbb{R}^2$ defined as above, the points $\pm(\widehat{p, q}) \in S^1$ lie in the image of $\psi_{G,H}$ if and only if there exists a left ordering $<$ of G satisfying:*

1. $1 < m^{pl^q}$, and
2. $(m^{pl^q})^n < h$ for all positive integers n and all positive $h \in H$ not of the form $(m^{pl^q})^k$.

Proof. Suppose that $\pm(\widehat{p, q}) \in S^1$ both lie in the image of $\psi_{G,H}$, and let P be any positive cone in $\psi_{G,H}^{-1}(\widehat{(p, q)})$, we will show that either P or P^{-1} has the desired properties.

Considering $\phi^{-1}(r(P))$, we see that $\phi^{-1}(r(P))$ lies in the set $f^{-1}(x)$ and

6.1. The map $\psi_{G,H}$

is therefore equal to either

$$\widehat{(p, q)} \in S_{\emptyset}^1 \text{ or } \widehat{(p, q)} \in S_{\square}^1 \text{ in } X,$$

we will deal with these two cases separately.

If $\phi^{-1}(r(P)) = \widehat{(p, q)} \in S_{\emptyset}^1$, by Lemma 6.1.3 the positive cone P defines a left ordering satisfying properties (1) and (2) when restricted to H . This is sufficient to guarantee that P defines a left ordering of G satisfying properties (1) and (2), so we are done.

In the case that $\phi^{-1}(r(P)) = \widehat{(p, q)} \in S_{\square}^1$, it follows that

$$\phi^{-1}(r(P^{-1})) = -\widehat{(p, q)} \in S_{\square}^1,$$

and by an application of Lemma 6.1.3 we conclude that the positive cone P^{-1} defines a left ordering of G satisfying properties (1) and (2).

Conversely, suppose that $<$ is a left ordering of G satisfying properties (1) and (2), with associated positive cone $P \in LO(G)$. In this case, $r(P)$ defines an ordering of H in which m^{pl^q} is positive, and relative to which the subgroup $\langle m^{pl^q} \rangle$ is convex (in H). Thus we may apply Lemma 6.1.3 to conclude that the points $\pm\widehat{(p, q)}$ lie in the image of $\psi_{G,H}$.

□

Proposition 6.1.5. *Let M^3 be a P^2 -irreducible 3-manifold with torus boundary, set $G = \pi_1(M^3)$ and $H = \pi_1(\partial M^3)$, and let $\{m, l\}$ be a meridian and longitude basis of $\pi_1(\partial M^3)$. Let $M_{p/q}^3$ denote the 3-manifold obtained by p/q Dehn surgery on M^3 , where p and q are relatively prime. If*

6.1. The map $\psi_{G,H}$

1. $\pi_1(M_{p/q}^3)$ is left orderable, and
2. $\pi_1(\partial M^3)$ is not killed in the quotient $\pi_1(M_{p/q}^3)$,

then the points $\pm(\widehat{p,q})$ lie in the image of $\psi_{G,H}$.

Proof. Suppose that $\pi_1(M_{p/q}^3)$ is left orderable. Denote by $\langle\langle m^{pl^q} \rangle\rangle$ the normal closure of m^{pl^q} in G . From the short exact sequence

$$1 \rightarrow \langle\langle m^{pl^q} \rangle\rangle \rightarrow G \rightarrow \pi_1(M_{p/q}^3) \rightarrow 1$$

we may left order G so that the subgroup $\langle\langle m^{pl^q} \rangle\rangle$ is convex, and m^{pl^q} is positive. Here, we are applying Theorem 5.1.1 to conclude that the kernel $\langle\langle m^{pl^q} \rangle\rangle$ may be left ordered.

Observe that in any ordering of $H \cong \mathbb{Z} \times \mathbb{Z}$, if C is convex then either

1. $C = \{0\}$,
2. $C \cong \mathbb{Z}$, or
3. $C = H$.

In our situation, the subgroup $\langle m, l \rangle \cap \langle\langle m^{pl^q} \rangle\rangle$ is nontrivial and convex in the restriction ordering of H , and so there are two possibilities:

1. $\langle m, l \rangle \cap \langle\langle m^{pl^q} \rangle\rangle = H$, or
2. $\langle m, l \rangle \cap \langle\langle m^{pl^q} \rangle\rangle$ is infinite cyclic, generated by m^{pl^q} .

Case (1) is not possible, as we have assumed that $\pi_1(\partial M^3)$ is not killed in the quotient $\pi_1(M_{p/q}^3)$. Therefore, we are in case (2), and so the restriction

6.1. The map $\psi_{G,H}$

ordering to H satisfies (1) $m^{pl^q} > 1$, and (2) $\langle m^{pl^q} \rangle$ is convex. We now apply Proposition 6.1.4 to conclude that the points $\pm(\widehat{p, q})$ lie in the image of $\psi_{G,H}$.

□

Proposition 6.1.6. *Let M^3 be an irreducible 3-manifold with torus boundary, and let $\{m, l\}$ be a meridian and longitude basis of $\pi_1(\partial M^3)$. Suppose further that $\pi_1(\partial M^3)$ is not killed by the quotient map $\pi_1(M^3) \rightarrow \pi_1(M_{p/q}^3)$, for every surgery slope $p/q \in \mathbb{Q}$. If there is no left ordering of $\pi_1(M^3)$ such that $1 < m \ll l$, then there exists $N \in \mathbb{R}$ such that $\pi_1(M_{p/q}^3)$ is not left orderable whenever $p/q > N$.*

Proof. With $G = \pi_1(M^3)$, and $H = \pi_1(\partial M)$, we consider the map $\psi_{G,H}$. If there exists no left ordering of $\pi_1(M^3)$ such that $1 < m \ll l$, then by Proposition 6.1.4, the points $\pm(1, 0)$ do not lie in the image of $\psi_{G,H}$.

Since $LO(\pi_1(M^3))$ is compact, its image under the map $\psi_{G,H}$ is compact in S^1 and hence closed. As $\pm(1, 0)$ are not contained in the closed set $\psi_{G,H}(LO(\pi_1(M^3)))$, there must be open sets U_1 about $(1, 0)$ and U_2 about $(-1, 0)$ in S^1 such that no point in $U_1 \cup U_2$ lies in the image of the map $\psi_{G,H}$. This proves the claim: Points of the form $\widehat{(p, q)}$ in S^1 that are in the neighbourhood $U_1 \cup U_2$ of $\{(1, 0), (-1, 0)\} \subset S^1$ cannot lie in the image of the map $\psi_{G,H}$, and so $\pi_1(M_{p/q}^3)$ cannot be left orderable by Proposition 6.1.5.

□

The condition on M^3 above, namely that $\pi_1(\partial M^3)$ is not killed by the quotient map $\pi_1(M^3) \rightarrow \pi_1(M_{p/q}^3)$, is satisfied by all knot complements. For if $\pi_1(\partial M^3)$ is sent to $\{1\}$ under the quotient map $\pi_1(M^3) \rightarrow \pi_1(M_{p/q}^3)$ for

some knot complement M^3 , then

$$\pi_1(M_{p/q}^3) \cong \pi_1(M^3) / \langle\langle m, l \rangle\rangle \cong \{1\}.$$

This cannot happen for nontrivial Dehn surgeries on a nontrivial knot, since all nontrivial knots satisfy property P :

Theorem 6.1.7 ([31]). *If $K \subset S^3$ is a nontrivial knot in S^3 , and $M_{p/q}^3$ is the 3-manifold obtained by p/q Dehn surgery on $M^3 = S^3 \setminus K$ with $q \neq 0$, then $\pi_1(M_{p/q}^3)$ must be nontrivial.*

In particular, this means that we may apply Proposition 6.1.6 to knot complements. If we start with the complement of a hyperbolic knot (a knot whose complement admits a complete Riemannian metric of constant negative curvature), then Thurston's hyperbolic Dehn surgery theorem guarantees that only finitely many Dehn surgeries yield non-hyperbolic 3-manifolds. Thus Proposition 6.1.6 yields infinitely many hyperbolic 3-manifolds with non-left orderable fundamental group, if we can find one knot complement that satisfies the necessary hypotheses.

6.1.1 Example: Torus knots

Recall that for an (r, s) torus knot, the fundamental group of the complement has presentation $\langle a, b \mid a^r = b^s \rangle$, where r and s are relatively prime. With this presentation, the meridian corresponds to the element $m = a^i b^j$, where i and j are relatively prime integers such that $is + jr = 1$, and the longitude is given by $l = m^{-sr} a^r$. An element g in a left ordered group G with ordering

$<$ is called cofinal if

$$G = \{h \in G \mid g^{-n} < h < g^n \text{ for some } n \in \mathbb{Z}\}.$$

Lemma 6.1.8. *In every left ordering of the group $G = \langle a, b \mid a^r = b^s \rangle$, the central element a^r is cofinal.*

Proof. Suppose that $<$ is a left ordering of G , and let

$$C = \{x \in G \mid \text{there exists } n \in \mathbb{Z} \text{ such that } a^{-rn} < x < a^{rn}\}.$$

We check that C is a subgroup of G .

Given $x, y \in C$, choose integers n, m such that $a^{-rn} < x < a^{rn}$ and $a^{-rm} < y < a^{rm}$. Then $x < a^{rn} \Rightarrow yx < ya^{rn} = a^{rn}y$, while $y < a^{rm} \Rightarrow a^{rn}y < a^{r(n+m)}$; combining these two inequalities gives $xy < a^{r(n+m)}$. Similarly, we may compute that $a^{-r(n+m)}$ is a lower bound for xy , so that C is closed under multiplication.

Next, we observe that C is closed under taking inverses, for if $a^{-rn} < x < a^{rn}$, then

$$x < a^{rn} \Rightarrow 1 < x^{-1}a^{rn} \Rightarrow a^{-rn} < a^{-rn}x^{-1}a^{rn} = x^{-1},$$

so that a^{-rn} is a lower bound for x^{-1} . Similarly we show that a^{rn} is an upper bound for x^{-1} , so that C is closed under taking inverses.

It is easy to see from the definition that C is convex in the given left ordering, so that C is a convex subgroup of G .

Finally, we may observe that both a and b lie in C , since a and b are

6.1. The map $\psi_{G,H}$

clearly both bounded by powers of a in any left ordering of G . This shows that the generators of G are contained in C , and hence G is contained in C . It follows that $G = C$ and the element a^r is confinal in the left ordering $<$. □

Proposition 6.1.9. *If M^3 is the complement of an (r, s) torus knot, then there exists an open neighbourhood $U \subset \mathbb{R}$ of $rs \in \mathbb{R}$ such that for every surgery slope p/q in U , the group $\pi_1(M_{p/q}^3)$ is not left orderable.*

Proof. Let $G = \pi_1(M^3) = \langle a, b | a^r = b^s \rangle$, and let $H = \pi_1(\partial M^3)$. From Lemma 6.1.8, the powers of $m^{rsl} = a^r$ are not bounded above by any element of the group G in any left ordering, so by Proposition 6.1.4 the points $\pm(\widehat{rs}, 1)$ are not in the image of $\psi_{G,H}$. Since the image of $\psi_{G,H}$ is compact, and hence closed, it follows that there is an open set U containing $\pm(\widehat{rs}, 1) \in S^1$ such that no points in the image of $\psi_{G,H}$ lie inside of U . By Proposition 6.1.5, this means that the group $\pi_1(M_{p/q}^3)$ is not left orderable whenever $p/q \in U$. □

Thus, Proposition 6.1.9 yields infinitely many examples of 3-manifolds with non-left orderable fundamental group, all of which are known to be Seifert fibered or the connect sum of two lens spaces.

Theorem 6.1.10 ([38]). *Suppose that M is the complement of a torus knot in S^3 . For every surgery coefficient $p/q \in \mathbb{Q}$, the manifold $M_{p/q}$ is either a Seifert fibered space, or a connect sum of two lens spaces.*

6.2 Analysis of the map $\psi_{G,H}$ for bi-orderable groups

The goal of this section is to prove the following theorem, and discuss its consequences when applied to Dehn surgery on knots. Recall that a subgroup H of a group G is isolated if for every $g \in G$ the following implication holds: If there exists a nonzero integer $k \in \mathbb{Z}$, such that $g^k \in H$, then $g \in H$.

Theorem 6.2.1. *Suppose that G is a bi-orderable group, and that $H \cong \mathbb{Z} \times \mathbb{Z}$ is an isolated subgroup of G . Then the map $\psi_{G,H} : LO(G) \rightarrow S^1$ is surjective.*

The following theorem is essentially an application of Theorem 1.4.10. Below, we are considering a bi-ordered group G , together with the action of G on itself by conjugation. This action is necessarily order preserving if G is equipped with a bi-ordering. While the proof below is essentially a special case of the proof of Theorem 1.4.10, we fully cover the details of this special case.

Theorem 6.2.2. *Let G be a bi-orderable group with bi-ordering $<$, and let A be any nontrivial subgroup of G . Denote by $C_G(A)$ the centralizer of A in G . Then there exists a left ordering of G in which the centralizer $C_G(A)$ is convex.*

Proof. Let \prec be any well-ordering of A , and for any $g \notin C_G(A)$, set

$$a_g = \min_{\prec} \{a \in A : g^{-1}ag \neq a\}.$$

6.2. Analysis of the map $\psi_{G,H}$ for bi-orderable groups

Define a positive cone $P \subset G$ according to the rule: $g \in P$ if $g \notin C_G(A)$ and $g^{-1}a_gg > a_g$, or if $g \in C_G(A)$ and $g > 1$. We check that P satisfies the required properties.

1. $P \cdot P \subset P$. Let $g, h \in P$. There are three cases to consider.

(a) Suppose $g, h \in C_G(A)$. Then $gh \in C_G(A)$, and $gh > 1$, so $gh \in P$.

(b) Suppose that $g \in C_G(A)$ and $h \notin C_g(A)$.

Consider the product hg . Observe that $a_{hg} = a_h$, for suppose not, say $a_{hg} \prec a_h$. Then both h and g commute with a_{hg} , while hg does not, a contradiction. Similarly, if $a_h \prec a_{hg}$, then hg commutes with a_h , and g commutes with a_h , while h does not. This is again a contradiction, as we compute

$$a_h = (hg)^{-1}a_hhg = g^{-1}h^{-1}a_hhg > g^{-1}a_hg = a_h,$$

where we have used that $h^{-1}a_hh > a_h$, and $>$ is a bi-ordering. Thus $a_{hg} = a_h$ implies $(hg)^{-1}a_hhg > a_{hg}$, so that $hg \in P$.

Next, consider the product gh . Then $gh \notin C_G(A)$, and $a_{gh} = a_h$, for suppose that $a_{gh} \prec a_h$. Then h commutes with a_{gh} , and g commutes with a_{gh} . Therefore gh commutes with a_{gh} , a contradiction. We arrive at a similar contradiction if $a_h \prec a_{gh}$: then gh commutes with a_h , as does g , while h does not. We have already dealt with a similar case above, showing a contradiction,

and so we conclude that $a_h = a_{gh}$. Then we compute that

$$(gh)^{-1}a_{gh}gh = h^{-1}g^{-1}a_hgh = h^{-1}a_hh > a_h = a_{gh},$$

so that $gh \in P$.

- (c) Suppose that $g, h \notin C_G(A)$, and consider the product gh . Then $a_{gh} = \min_{<} \{a_g, a_h\}$, for suppose not, say $\min_{<} \{a_g, a_h\} = a_h < a_{gh}$. Then gh commutes with a_{gh} , and so does g , while h does not. We have already dealt with such a situation above, showing a contradiction. Similarly, $a_{gh} < a_h$ reduces to a case previously considered, and we arrive at a contradiction, concluding that $a_{gh} = \min_{<} \{a_g, a_h\} = a_h$. Then using the fact that $a_h < a_g$, so g commutes with a_h , we compute

$$(gh)^{-1}a_{gh}gh = h^{-1}g^{-1}a_hgh = h^{-1}a_hh > a_h = a_{gh},$$

so that $gh \in P$. The case of $gh \in P$ when $\min_{<} \{a_g, a_h\} = a_g$ is nearly identical.

Lastly, we must consider the case when $a_g = a_h$. In this case, we have $a_{gh} = a_g = a_h$, and it follows that

$$(gh)^{-1}a_{gh}(gh) = (gh)^{-1}a_g(gh) > h^{-1}a_g h = h^{-1}a_h h > a_h = a_{gh}.$$

2. We must show that $P \sqcup P^{-1} \sqcup \{1\} = G$. This follows immediately from the fact that $a_g = a_{g^{-1}}$ for any $g \notin C_G(A)$, and the observation that $g^{-1}a_g g > a_g \Rightarrow ga_{g^{-1}}g^{-1} < a_{g^{-1}}$, since $<$ is a bi-ordering of G .

Lastly, we must show that $C_G(A)$ is convex, so suppose that $g \in C_G(A)$, and that $h \notin C_G(A)$ is positive, and consider the element $g^{-1}h$. Observe that $g^{-1}h \notin C_G(A)$, and that $a_{g^{-1}h} = a_h$, by the arguments used in case 1. (b). Therefore we compute

$$(g^{-1}h)^{-1}a_{g^{-1}h}g^{-1}h = h^{-1}ga_hg^{-1}h > a_h = a_{g^{-1}h},$$

so that $g^{-1}h \in P$ and $C_G(A)$ is convex. \square

Theorem 6.2.3. *Suppose that A is a nontrivial isolated abelian subgroup in G , and that G is a bi-orderable group. Then there exists a left ordering of G in which the subgroup A is convex.*

Proof. We will construct an ordering of $C_G(A)$ in which A is convex. The constructed ordering can be extended to an ordering of the whole group G , by using Theorem 6.2.2 to construct a left ordering of G in which $C_G(A)$ is convex.

By [[5], Theorem 2.2.4], if G is bi-orderable, then so is $G/Z(G)$, where $Z(G)$ denotes the center of the group G . Therefore we can construct a bi-ordering of $C_G(A)$ in which $Z(C_G(A))$ is convex. In the abelian group $Z(C_G(A))$ the subgroup A is isolated, and so the quotient by A is a torsion free abelian group, and hence bi-orderable. Therefore we can bi-order the convex subgroup $Z(C_G(A))$ lexicographically using the short exact sequence

$$1 \rightarrow A \rightarrow Z(C_G(A)) \rightarrow Z(C_G(A))/A \rightarrow 1,$$

thus producing a bi-ordering of $C_G(A)$ with A as a convex subgroup. \square

Proof of Theorem 6.2.1. Recall that $\psi_{G,H}$ is the composition

$$LO(G) \xrightarrow{r} LO(H) \xrightarrow{\phi^{-1}} X \xrightarrow{f} S^1,$$

where ϕ^{-1} and f are both surjective. Thus, to show surjectivity of $\psi_{G,H}$ it is enough to show surjectivity of the restriction map r .

To this end, let G be a bi-orderable group, H an isolated subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and let $P \in LO(H)$ be any positive cone. By Theorem 6.2.3 there exists a left ordering of G relative to which H is a convex subgroup, so the set of left cosets G/H can be ordered in a way that is preserved by multiplication from the left. Following Proposition 1.1.4, let $q : G \rightarrow G/H$ denote the quotient map onto the set of left cosets, and let $P_{G/H}$ denote those cosets which are greater than the identity coset in the ordering of G/H . Then $Q = P \cup q^{-1}(P_{G/H})$ defines a positive cone in $LO(G)$ satisfying $r(Q) = Q \cap H = P$, so that the restriction map is surjective. Hence, $\psi_{G,H}$ is a composition of surjective maps, and is itself surjective. \square

In order to apply this theorem to Dehn surgery on knots, we recall the following.

Theorem 6.2.4 ([50]). *If K is a fibered knot in S^3 and all the roots of its Alexander polynomial are real and positive, then $\pi_1(S^3 \setminus K)$ is bi-orderable.*

In particular, this means that the complement of the figure eight knot has bi-orderable fundamental group, as the Alexander polynomial of the figure eight knot is $\Delta(t) = t^2 - 3t + 1$, which has real, positive roots $\frac{3 \pm \sqrt{5}}{2}$. Finally, we need the following result concerning peripheral elements of knot

groups.

Theorem 6.2.5 ([59]). *If K is a hyperbolic knot, the group $\pi_1(\partial(S^3 \setminus K))$ is isolated in $\pi_1(S^3 \setminus K)$.*

It is well known that the figure eight knot is hyperbolic, so the fundamental group of its boundary is isolated. Therefore, we are in a position to apply Theorem 6.2.1 to the complement of the figure eight knot, and we conclude the following.

Theorem 6.2.6. *Suppose that K is the figure eight knot. If $G = \pi_1(S^3 \setminus K)$ and $H = \pi_1(\partial(S^3 \setminus K))$, then the map $\psi_{G,H}$ is surjective.*

In particular, this shows that we will never be able to apply Proposition 6.1.5 (as in the case of torus knots), to show that non-left orderable fundamental groups may be obtained by Dehn surgery on the complement of the figure eight knot. This is in line with the results observed in Theorem 5.1.4.

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