Fusion Algebras and Cohomology of Toroidal Orbifolds

by

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Abstract

In this thesis we exhibit an explicit non-trivial example of the twisted fusion algebra for a particular finite group. The product is defined for the group $G = (\mathbb{Z}/2)^3$ via the pairing ${}^{\theta_g(\phi)}R(G) \otimes^{\theta_h(\phi)} R(G) \to^{\theta_{gh}(\phi)} R(G)$ where $\theta : H^4(G, \mathbb{Z}) \to H^3(G, \mathbb{Z})$ is the inverse transgression map and ϕ is a carefully chosen cocycle class. We find the rank of the fusion algebra $\mathfrak{X}(G) = \sum_{g \in G} {}^{\theta_g(\phi)}R(G)$ as well as the relation between its basis elements. We also give some applications to topological gauge theories.

We next show that the twisted fusion algebra of the group $(\mathbb{Z}/p)^3$ is isomorphic to the non-twisted fusion algebra of the extraspecial p-group of order p^3 and exponent p.

The final point of my thesis is to explicitly compute the cohomology groups $H^*(X/G; \mathbb{Z})$ where X/G is a toroidal orbifold and $G = \mathbb{Z}/p$ for a prime number p. We compute the particular case where X is induced by the $\mathbb{Z}G$-module $(IG)^n$, where IG is the augmentation ideal.

Table of Contents

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Chapter 1

Introduction

In the past twenty years there has been a great influence of physical ideas on mathematics. One such example is the subject of string theory which increased the interest on some topological objects such as orbifolds, a very natural generalization of manifolds, and also on the invariants related to these objects, for example, the Euler characteristic, homotopy groups, homology groups and cohomology rings. The study of orbifolds in physics began with the introduction of the conformal field theory model on singular spaces by Dixon-Harvey-Vafa-Witten in 1985.

Orbifolds are spaces that locally look like a quotient of an open set of a vector space by the action of a group such that the isotropy group at each point is finite. Chen and Ruan [19] discovered a cohomology theory for these inspired by their ideas in quantum cohomology and orbifold string theory. The interest in K-theory is also introduced from the consideration of a D-brane charge on a smooth manifold and the notion of discrete torsion on an orbifold by Vafa. K-theory relates to equivariant theories if the orbifolds are the quotient of a smooth manifold M by a compact Lie group acting on M. Adem and Ruan [7] defined the twisted orbifold K-theory to study the resulting Chern isomorphism. In [8], Adem, Ruan and Zhang give an associative stringy product for the twisted orbifold K-theory of a compact, almost complex orbifold. This product is defined on the twisted K-theory of the inertia orbifold $\wedge\mathcal{X}$ where the twisting gerbe τ is assumed to be in the image of the inverse transgression map $H^4(B\mathcal{X}) \longrightarrow H^3(B\wedge\mathcal{X})$. If the orbifold is $\mathcal{X} = \wedge[*/G]$ and G is a group then the resulting ring ${}^\tau K_{orb}(\wedge[*/G])$ is called *fusion algebra*.

One way to approach the fusion algebra is the finite group modular data which is mainly explored in [20]. In our context, when we mention the term modular data one should understand two symmetric matrices associated to a finite group G. This modular data was originally introduced, in Lusztig's determination of the irreducible characters of the finite groups of Lie type [31], [32]. To describe the unipotent characters, he considered the modular data for some particular finite groups. The primary fields of the fusion algebra parametrize the unipotent characters associated to a given 2-sided cell in the Weyl group. Lusztig interprets this fusion algebra as the Grothendieck ring for G-equivariant vector bundles; in other words, the equivariant K-theory.

The most physical application of this modular data is in (2+1)-dimensional quantum field theories where a continuous gauge group has been spontaneously broken into a finite group [13]. Non-abelian anyons (i.e. particles whose statistics are governed by the braid group rather than the symmetric group) arise as topological objects. The effective field theory describing the long distance physics is governed by the quantum group of [23].

A set of modular data (i.e. matrices S and T) may be obtained for any choice of finite group G. Much information about a group can be recovered easily from its character table including whether it is abelian, simple, solvable, nilpotent, etc. For instance, G is simple if and only if for all irreducible $\chi \not\equiv 1$, $\chi(a) = \chi(e)$ only for $a = e$. Thus it may be expected that finite group modular data, which probably includes the character table, should provide more information about the group, i.e. be sensitive to a lot of the group-theoretic properties of G.

One way to generalize this data is to twist with a cocycle from the cohomology group. One can obtain topological (e.g. oriented knot) invariants from this twisted data, as explained in [9]. These invariants are functions of the knot group (i.e., the fundamental group of the complement of the knot). Although non-isomorphic knots can have the same invariant, these invariants can distinguish a knot from its inverse (i.e., the knot with opposite orientation), unlike the more familiar topological invariants arising from affine algebras.

Another important example of orbifolds is obtained by group actions on tori induced by the integral representations of finite groups. These orbifolds appear as examples of interest in physics. For this reason we are interested in computing the cohomology groups $H^*(X/G;\mathbb{Z})$ where X/G is a toroidal orbifold.

One can give an integral presentation $\varphi : G \to GL_n(\mathbb{Z})$ for finite group G. In this way G acts linearly on $\mathbb{R}^n$ preserving the integral lattice $\mathbb{Z}^n$, thus inducing a G-action on the torus $X_\varphi = X := \mathbb{R}^n/\mathbb{Z}^n$. The quotient $X \to X/G$ naturally has the structure of an orbifold as a global quotient, and these kind of orbifolds are usually referred to as toroidal orbifolds. The goal of this chapter is to compute the cohomology groups $H^*(X/G;\mathbb{Z})$ for the particular case where $G = \mathbb{Z}/p$ for a prime number p.

The quotients of the form X/G appear naturally in different contexts. For example, given a topological space Y, the m-th cyclic product of Y is defined to be the quotient

$$CP^m(Y) := Y^m/\mathbb{Z}/m,$$

where $\mathbb{Z}/m$ acts by cyclically permuting the product Y^m. In the particular case where the representation $\varphi : G \to GL_n(\mathbb{Z})$ induces the $\mathbb{Z}G$-module $(\mathbb{Z}G)^n$, the associated torus X is $(S^1)^p)^n$, where G acts cyclically on each $(S^1)^p$ and diagonally on the product $((S^1)^p)^n$. In this case

$$X/G = (((S^1)^p)^n)/\mathbb{Z}/p \cong ((S^1)^n)^p/\mathbb{Z}/p$$

where now $\mathbb{Z}/p$ acts cyclically on the $((S^1)^n)^p$. Therefore

$$X/G \cong CP^p((S^1)^n).$$

The homology groups of quotient spaces of the form X^m/K, where $K \subset \Sigma_m$, have long been studied. In particular, in [38] Swan formulated a method for the computation of cyclic products of topological spaces.

In Chapter 2, the stringy product on twisted orbifold K-theory in [8] is reviewed, and an example for non-trivial twistings is calculated. For a finite group G this reduces to the twisted fusion algebra. Here we exhibit an explicit non-trivial example of the twisted fusion algebra for a particular finite group. The product is defined for the group $G = (\mathbb{Z}/2)^3$ via the pairing ${}^{\theta_g(\phi)}R(G) \otimes^{\theta_h(\phi)} R(G) \to^{\theta_{gh}(\phi)} R(G)$ where $\theta : H^4(G, \mathbb{Z}) \to H^3(G, \mathbb{Z})$ is the inverse transgression map and ϕ is a carefully chosen cocycle class. We find the rank of the fusion algebra $\mathcal{X}(G) = \sum_{g \in G} {}^{\theta_g(\phi)}R(G)$ as well as the relation between its basis elements. We also provide some applications to topological gauge theories.

In Chapter 3, we study the twisted fusion algebra for the finite group $(Z/p)^3$. One way to calculate the coefficients in the fusion algebra is to use conformal field theory. The modular data in conformal field theory provides the Verlinde formula, which gives the fusion coefficients for fusion algebras. The modular data associated to finite groups is introduced in the work of Coste, Gannon and Ruelle [20]. They also consider the data arising from a cohomological twist, and write down the modular data for a general twist in terms of quantities associated directly with the finite group. We utilise the results from the conformal field theory, namely finite group modular data, to prove this algebra is isomorphic to the non-twisted fusion algebra of extraspecial p-group with exponent p and of order p^3.

Chapter 4 is devoted to computing the cohomology of orbit space X/G where $G = \mathbb{Z}/p$ and $X = ((S^1)^{p-1})^s$ and the action is induced by the action on the $\mathbb{Z}G$-module $H^1(X) = L = IG^s$, where IG is the augmentation ideal of $\mathbb{Z}G$.

Chapter 2

A Twisted Fusion Algebra

Inspired by the Chen-Ruan cohomology for orbifolds, it has been shown by Adem, Ruan, Zhang [8] that there is an internal product in twisted orbifold K-theory ${}^{\alpha}K_{orb}(\mathcal{X})$. The information determining this stringy product lies in $H^4(B\mathcal{X},\mathbb{Z})$ instead of $H^3(B\mathcal{X},\mathbb{Z})$: Given a class $\phi \in H^4(B\mathcal{X},\mathbb{Z})$, it induces a class $\theta(\phi) \in H^3(B \wedge \mathcal{X},\mathbb{Z})$ where $\wedge\mathcal{X}$ is the inertia stack. As a result, one can define a twisted K-theory on ${}^{\theta(\phi)}K(\wedge\mathcal{X})$. The map θ can be thought of the inverse of the classical transgression map.

The construction of this internal product is motivated by the so-called Pontryagin product on $K_G(G)$ for a finite group G. Indeed, if the orbifold is $\mathcal{X} = \wedge[*/G]$, one obtains the same product for the orbifold K-theory in the untwisted case. There is also an explicit calculation of the inverse transgression map θ for the cohomology of finite groups (see [8]). Using these results, we exhibit a non-trivial product structure in the case of $G = (\mathbb{Z}/2)^3$. We use an integral cohomology class $\phi \in H^4(G,\mathbb{Z})$ such that under the inverse transgression it maps non-trivially for every twisted sector, yielding a product structure on the algebra ${}^{\theta(\phi)}K_G(G) = \mathcal{X}(G) = \sum_{g\in G} {}^{\theta_g(\phi)}R(G)$ defined via the pairing ${}^{\theta_g(\phi)}R(G) \otimes^{\theta_h(\phi)} R(G) \rightarrow^{\theta_{gh}(\phi)} R(G)$. In this chapter, we derive the relations between the basis elements of the algebra $\mathcal{X}(G)$, and we prove the uniqueness of this product in this particular case. $G = (\mathbb{Z}/2)^3$ is indeed the abelian group of smallest rank such that it has non-trivial transgressions (see [8], section 5).

These twisted rings have also been worked out in the conformal field theory literature. In [35], the modular invariant (i.e. S and T matrices) of this group $G = (\mathbb{Z}/2)^3$ is calculated. As a result, one can calculate the relations between basis elements of $\mathcal{X}(G)$ using the Verlinde formula. Moreover, the same example is considered in [15], where a decomposition formula for twisted K-theory is given and the product is calculated after tensoring by rational numbers.

There is also a physical counterpart of this theory. In our case this map is the inverse transgression map, which is actually the map coming from the correspondence between the Chern-Simons action and the Wess-Zumino terms that arise in connecting a specific three-dimensional quantum field theory to its related two-dimensional quantum field theory. One can see that the Chern-Simons theory associates to each group element $g_i \in G$ a 2-cocycle β_i of the stabilizer group N_{g_i}, which is $G = (\mathbb{Z}/2)^3$ in our abelian case. We use the formulations in [25] to calculate the partition function $Z(S^1 \times S^1 \times S^1)$. It is also worth mentioning that the algebra $\mathcal{X}(G)$ corresponds to a fusion algebra in this physical context.

In this chapter, we first introduce projective representation and its basic properties. We next give some preliminaries and the definition of our fusion algebra. In the third section, we calculate the rank and the uniqueness of this algebra as well as the relation between the basis elements which are the projective representations. Finally, we present the application to topological gauge theories by using the formulation in [25].

2.1 Projective representations

For the entirety of this section, G is a finite group. More information on projective representations can be found in [30].

Definition 2.1.1. *Let V be a complex vector space. A* projective representation *of G is a map $\rho : G \to GL(V)$ such that there exists $\alpha : G \times G \to \mathbb{C}^*$ with*

- $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ *for all x, $y \in G$, and*
- $\rho(1) = I$.

One can easily see that α is a 2-cocycle of G. The associativity of multiplication in $GL(V)$ combined with the first condition gives the cocycle condition, and $\alpha(g, 1) = \alpha(1, g) = 1$ for all $g \in G$ by the second condition.

Any projective representation associated to α is called an α-twisted representation of G. As in the case of complex representations of G, we have a notion of isomorphism between two representations.

Definition 2.1.2. *Two α-twisted representations $\rho_i : G \to GL(V_i)$, $i = 1, 2$, are called* linearly equivalent *if there is a vector space isomorphism $f : V_1 \to V_2$ with*

$$\rho_2(g) = f\rho_1(g)f^{-1} \text{ for all } g \in G.$$

Clearly, the direct sum of two α-twisted representations is an α-twisted representation. Thus we can form the monoid of isomorphism classes of α-twisted representations of G. $R^\alpha(G)$ is the associated Grothendieck group. Note that if α is the trivial cocycle, then $R^\alpha(G)$ is $R(G)$, the complex representation ring of G.

It is not hard to see that the tensor product of two α-twisted representations is no longer an α-twisted representation. Note however that if α and β are two cocycles, then the tensor product of an α-twisted representation and a β-twisted representation is an $(\alpha+\beta)$-twisted representation. This can be extended to a pairing

$$R^{\alpha}(G) \otimes R^{\beta}(G) \to R^{\alpha+\beta}(G).$$

In order to study $R^{\alpha}(G)$, we introduce the α-twisted group algebra $\mathbb{C}^{\alpha}G$. We denote by $\mathbb{C}^{\alpha}G$ the vector space over $\mathbb{C}$ with basis $\{\bar{g}\}_{g\in G}$ and with product $\bar{g}\bar{h} = \alpha(g,h)\overline{gh}$ extended by linearity. This makes $\mathbb{C}^{\alpha}G$ into a $\mathbb{C}$-algebra with $\bar{1}$ as the identity.

Definition 2.1.3. *If α and β are cocycles of G, then we say that $\mathbb{C}^{\alpha}G$ and $\mathbb{C}^{\beta}G$ are equivalent if there exists a $\mathbb{C}$-algebra isomorphism $\phi : \mathbb{C}^{\alpha}G \to \mathbb{C}^{\beta}G$ and a mapping $t : G \to \mathbb{C}^*$ such that $\phi(\bar{g}) = t(g)\tilde{g}$ for all $g \in G$, where $\bar{g}$ and $\tilde{g}$ are the bases for the two twisted group algebras.*

This defines an equivalence relation on such twisted algebras, and we have the following result.

Lemma 2.1.4. *There is an equivalence of twisted group algebras, $\mathbb{C}^{\alpha}G \simeq \mathbb{C}^{\beta}G$, if and only if α is cohomologous to β. In fact, $\alpha \mapsto \mathbb{C}^{\alpha}G$ induces a bijective correspondence between $H^2(G,\mathbb{C}^*)$ and the set of equivalence classes of twisted group algebras of G over $\mathbb{C}$.*

In [30], it is proved that these twisted group algebras play the same role in determing $R^{\alpha}(G)$ that $\mathbb{C}G$ plays in determining $R(G)$:

Theorem 2.1.5. *There is a bijective correspondence between α-twisted representations of G and $\mathbb{C}^{\alpha}G$-modules. This correspondence preserves sums* *and bijectively maps linearly equivalent representations into isomorphic modules.*

2.2 The twisted fusion product for finite groups

In this section, we review a special case for the product in twisted orbifold K-theory which is formalised by Adem, Ruan and Zhang in [8]. We consider the inertia orbifold $\wedge[*/G]$ where G is a finite group. In this case, the untwisted orbifold K-theory of $\wedge[*/G]$ is simply $K_G(G)$, which is additively isomorphic to $\sum_{(g)} R(Z_G(g))$, where $Z_G(g)$ denotes the centraliser of g in G, and the sum is taken over conjugacy classes. The product in $K_G(G)$ is defined as follows. An equivariant vector bundle over G can be thought of as a collection of finite dimensional vector spaces V_g with a G-module structure on $\sum_{g\in G} V_g$ such that $gV_h = V_{ghg^{-1}}$. The product is defined as

$$(V \star W)_g = \bigoplus_{g_1 g_2 = g} V_{g_1} \otimes W_{g_2}.$$

One can give an alternative definition. We first define the maps $e_1 : G \times G \to G$, $e_2 : G \times G \to G$ and $e_{12} : G \times G \to G$ as $e_1(g,h) = g$, $e_2(g,h) = h$ and $e_{12}(g,h) = gh$ respectively, which are G-equivariant up to the conjugation action. If α, β are elements in $K_G(G)$ the product is defined as

$$\alpha \star \beta = e_{12*}(e_1^*(\alpha) e_2^*(\beta)).$$

We now need to review the inverse transgression map for finite groups to extend the latter definition to twisted K-theory. In order to define the product in twisted K-theory, Adem, Ruan and Zhang [8] define a map to match up the levels which appear in the twistings. This cochain map θ is called inverse transgression map, and it induces the homomorphism

$$\theta_* : H^{k+1}(B\mathcal{G}, \mathbb{Z}) \to H^k(B \wedge \mathcal{G}, \mathbb{Z}).$$

If the orbifold $\mathcal{G}$ is $[*/G]$, where G is a finite group, the inverse transgression has a classical interpretation in terms of the shuffle product. Recall that $\wedge[*/G]$ is equivalent to $\bigsqcup_{(g)}[*/Z_G(g)]$ (see [8]). Hence we would like to focus on the map $\theta_g : C^k(G, U(1)) \to C^{k-1}(Z_G(g), U(1))$. If G is a finite group then the cochain complex $C^*(G, U(1))$ is in fact equal to $Hom_G(B_*(G), U(1))$, where $B_*(G)$ is the bar resolution for G (see [18], page 18). If t is the generator of $\mathbb{Z}$, the shuffle product is $B_k(Z_G(g)) \otimes B_1(\mathbb{Z}) \to B_{k+1}(G)$ given by

$$[g_1|g_2|\dots|g_k] \star [t^i] = \sum_\sigma \sigma[g_1|g_2|\dots|g_k|g_{k+1}],$$

where $g_{k+1} = g^i$, σ ranges over all $(k,1)$–shuffles and

$$\sigma[g_1|g_2|\dots|g_{k+1}] = (-1)^{sgn(\sigma)}[g_{\sigma(1)}|g_{\sigma(2)}|\dots|g_{\sigma(k+1)}]. \qquad (2.1)$$

A $(k,1)$–shuffle is an element σin the symmetric group S_{k+1} such that $\sigma(i) < \sigma(j)$ for $1 \leq i < j \leq k$.

We can dualize this using integral coefficients. Given a cocycle $\phi \in C^{k+1}(G, \mathbb{Z})$, one can see that the inverse transgression $\theta_g(\phi) \in C^k(Z_G(G), \mathbb{Z})$ can be defined as

$$\theta_g(\phi)([g_1|g_2|\dots|g_k]) = \phi([g_1|g_2|\dots|g_k] \star [g]) \qquad (2.2)$$

where $g_1, g_2, \dots, g_k \in G$. Hence it induces a map in integral cohomology.

We can now induce the inverse transgression map for $H^*(G, \mathbf{F}_2)$ for $G = (\mathbb{Z}/2)^3$ using Bockstein homomorphism. We want to find a non-trivial cocycle in the image of the inverse transgression map. Notice that $H^*(G, \mathbf{F}_2)$ is a polynomial algebra on three degree one generators x, y and z. In general, for an elementary abelian 2-group, the modulo 2 reduction map for $k > 0$ is a monomorphism $H^k(G, \mathbb{Z}) \to H^k(G, \mathbf{F}_2)$ which is the kernel of the Bockstein homomorphism $Sq^1 : H^k(G, \mathbf{F}_2) \to H^{k+1}(G, \mathbf{F}_2)$. In order to get nontrivial cocycle in the image of the inverse transgression map, we choose $\alpha = Sq^1(xyz) = x^2yz + xy^2z + xyz^2$, which represents a non-square element in $H^4(G, \mathbb{Z})$. The following lemma is proved in [8] by analyzing the multiplication map in the cohomology.

Lemma 2.2.1. *Let $g = x^a y^b z^c$ be an element in $G = (\mathbb{Z}/2)^3$, where we are writing in terms of the standard basis (identified with its dual). Let us consider $\alpha = Sq^1(xyz) = x^2yz + xy^2z + xyz^2$ which represents a non-square element in $H^4(G, \mathbb{Z})$. Then*

$$\theta_g^*(\alpha) = a(y^2z + z^2y) + b(x^2z + xz^2) + c(x^2y + xy^2),$$

and so $\theta_g^*(\alpha)$ is non-zero on every component except the one corresponding to the trivial element in G.

Proof. See [8], Lemma 5.2.

This implies that for all g, $h \in G$, $\theta_g^* + \theta_h^* = \theta_{gh}^*$ in the cohomology up to coboundaries. This also implies that the correspondence $g \mapsto \theta_g(\alpha)$ defines a homomorphism. In the case of $G = (\mathbb{Z}/2)^3$, we have the isomorphism $\theta_?(\alpha) : G \to H^3(G, \mathbb{Z}) = G$.

We now define the product as follows:

Definition 2.2.2. *Let τ be a 2-cocycle for the orbifold defined by the conjugation action of a finite group G on itself which is in the image of the inverse transgression. The product on ${}^{\tau}K_G(G)$ is defined by the following formula: if α, $\beta \in {}^{\tau} K_G(G)$, then*

$$\alpha \star \beta = e_{12*}(e_1^*(\alpha)e_2^*(\beta)).$$

If $\tau = \theta(\phi)$ then we have the following formula proved in [8]:

$$e_1^*\tau + e_2^*\tau = e_{12}^*\tau$$

up to coboundary. Hence the product $e_1(\alpha)e_2(\beta)$ lies in

$${}^{e_1^*\tau + e_2^*\tau}K_G(G) = {}^{e_{12}^*\tau} K_G(G).$$

Applying e_{12*}, this is mapped to ${}^{\tau}K_G(G)$, which gives the product in the twisted K-theory.

Using the identification $\theta_g^* + \theta_h^* = \theta_{gh}^*$, the following product is defined on the algebra

$${}^{\theta(\phi)}K_G(G) = \mathfrak{X}(G) = \sum_{g \in G} {}^{\theta_g(\phi)}R(G)$$

via the pairing

$${}^{\theta_g(\phi)}R(G) \otimes {}^{\theta_h(\phi)}R(G) \to {}^{\theta_{gh}(\phi)}R(G).$$

In the next section, we investigate the properties of this algebra while calculating its rank and the relations between the irreducible projective representation.

2.3 Calculations

2.3.1 2-cocycles in G with values in $U(1)$

We will assume that $G = (\mathbb{Z}/2)^3$ for the remainder of this chapter. We recall that, for a finite dimensional complex vector space, a mapping $\rho : G \to GL(V)$ is called a *projective representation* of G if there exists a $U(1)$-valued 2-cocycle $\alpha \in Z^2(G; U(1))$ such that $\rho(x)\rho(y) = \alpha(x,y)\rho(x,y)$ for all $x, y \in G$ and $\rho(1) = Id_V$. Hence in order to compute ${}^{\theta_g(\phi)}R(G)$ we first need to find the 2-cocycles in $H^2(G, U(1))$ corresponding to $\theta_g(\phi)$ in $H^3(G,\mathbb{Z})$ where both cohomology groups are isomorphic to G. For this purpose, we consider the isomorphism

$$H^2(G, U(1)) \to H^3(G, \mathbb{Z})$$

induced by the natural coefficient sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$. As $H^3(G,\mathbb{Z}) \cong G$, we need to find the eight non-cohomologous 2-cocycles in $H^2(G, U(1))$ corresponding to each $\theta_g(\phi)$ for all $g \in G$.

We now determine the relations in order to obtain the 2-cocycles in $C^2(G, U(1))$. Any 2-cocycle β in $C^2(G, U(1))$ should satisfy:

$$\delta\beta = 1.$$

By the boundary formula of the bar resolution of G, we derive:

$$\beta(g_2, g_3)\beta(g_1g_2, g_3)^{-1}\beta(g_1, g_2g_3)\beta(g_1, g_2)^{-1} = 1$$

for all $g_i \in G$, $i = 1, 2, 3$. Some interesting relations result when we plug in

$g_1 = g_3 = g$ and $g_2 = 1$ into this formula and we obtain

$$\beta(g, 1) = \beta(1, g). \tag{2.3}$$

Moreover, for $g_1 = g_2 = g$, we have

$$\beta(1, g_3)\beta(g, g) = \beta(g, g_3)\beta(g, gg_3). \tag{2.4}$$

As β is defining a projective representation, say ρ, it should satisfy $\rho(1)\rho(g) = \beta(1, g)\rho(g)$. This implies $\beta(1, g) = 1$ for all $g \in G$. Now we consider the following tables for 2-cocycles $\beta_i : G \times G \to U(1)$ which satisfies the identities (3.3) and (2.4). We call these cocycles fundamental cocycles.We choose β_1 as the trivial co-cycle. Here, x_i, y_i and z_i's are in $U(1)$, and they will be determined later.

β_2	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	1	x_1	x_2	x_1	x_2	x_3	x_3
g_3	1	x_1	-1	x_4	$-x_1$	x_6	$-x_4$	$-x_6$
g_4	1	x_2	$-x_4$	1	x_5	x_2	$-x_4$	x_5
g_5	1	x_1	$-x_1$	$-x_5$	-1	x_8	$-x_8$	x_5
g_6	1	x_2	$-x_6$	x_2	$-x_8$	1	$-x_8$	$-x_6$
g_7	1	x_3	x_4	x_4	x_8	x_8	1	x_3
g_8	1	x_3	x_6	$-x_5$	$-x_5$	x_6	x_3	1

β_3	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	y_1	y_2	$-y_1$	$-y_2$	y_3	$-y_3$
g_3	1	y_1	1	y_4	y_1	y_6	y_4	y_6
g_4	1	$-y_2$	y_4	1	y_5	$-y_2$	y_4	y_5
g_5	1	$-y_1$	y_1	$-y_5$	-1	y_8	$-y_8$	y_5
g_6	1	y_2	y_6	y_2	$-y_8$	1	$-y_8$	y_6
g_7	1	$-y_3$	y_4	y_4	y_8	y_8	1	$-y_3$
g_8	1	y_3	y_6	$-y_5$	$-y_5$	y_6	y_3	1

β_4	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	z_1	z_2	$-z_1$	$-z_2$	z_3	$-z_3$
g_3	1	$-z_1$	1	z_4	z_1	z_6	z_4	z_6
g_4	1	z_2	z_4	-1	z_5	$-z_2$	$-z_4$	$-z_5$
g_5	1	z_1	z_1	z_5	1	z_8	z_8	z_5
g_6	1	$-z_2$	z_6	$-z_2$	$-z_8$	1	$-z_8$	z_6
g_7	1	$-z_3$	z_4	$-z_4$	$-z_8$	z_8	-1	z_3
g_8	1	z_3	z_6	$-z_5$	z_5	$-z_6$	$-z_3$	-1

As the multiplication of two cocycles gives us another cocycle, one can construct five more cocycles, one of which is the trivial cocycle. We recall that for $\beta \in Z^2(G; U(1))$, an element $g \in G$ is called β-regular if $\beta(g, x) = \beta(x, g)$ for all $x \in C_G(g)$ (see [30], page 107). Thus all of the 2-cocycles have two

β-regular elements, one of which is 1, the other one is different for each cocycle. For example, one can immediately see that the β-regular elements for β_2, β_3 and β_4 are g_2, g_3 and g_4, respectively.

On the other hand, from the boundary formula we note that a 2-coboundary should satisfy $\beta(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$ where σ is in $C^1(G, \mathbb{Z})$. As G is abelian, we have

$$\beta(g_i, g_j) = \beta(g_j, g_i)$$

for all g_i and g_j in G. This implies that all of these eight cocycles represent different cohomology classes, as multiplication with a coboundary does not change the β-regular elements. Thus, we have following proposition.

Proposition 2.3.1. *The rank of* ${}^{\theta_g(\phi)}R(G)$ *is* 2 *if* $\theta_g(\phi)$ *is nontrivial.*

Proof. A basic result of projective representations states that ${}^{\alpha}R(G)$ is a free abelian group of rank equal to the number of distinct α-regular conjugacy classes of G (see [30], theorem 6.7). So, the ranks of ${}^{\beta_i}R(G)$ is 2 for any non-trivial β_i.

On the other hand, we obtained eight non-cohomologous cocycles which should correspond to $\theta_g(\phi)$'s because $H^2(G, U(1))$ and $H^3(G, \mathbb{Z})$ are isomorphic to G. The result follows. □

We can therefore conclude:

Corollary 2.3.2. *The rank of* $\mathfrak{X}(G)$ *is equal to* 22*.*

Proof. The ranks of ${}^{\beta_i}R(G)$ are 2 for any non-trivial β_i accounting for 14 and ${}^{beta_1}R(G)$ has rank 8.

2.3.2 The projective representations

In order to compute the irreducible projective representations of G, it is helpful to determine the x_i, y_i and z_i's. From the boundary formula, we have the following relations in β_2:

$$-1 = x_1 x_3 x_4 x_5$$

$$-1 = x_2 x_3 x_5 x_8$$

$$1 = x_6 x_8 x_3 x_1.$$

By a routine calculation, one can check that the other relations depend on these three relations. We can choose $x_1 = x_2 = x_3 = x_4 = -x_5 = x_6 = x_7 = x_8 = 1$ that obviously satisfy these relations. Similarly, we find y_i's and z_i's. The other cocycles are computed by multiplying β_2, β_3 and β_4. We will later show that the choice of x_i, y_j and z_k from the set $\{\pm 1\}$ does not change our representations. Here are our eight cocycles.

β_2	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	1	1	1	1	1	1	1
g_3	1	1	-1	1	-1	1	-1	-1
g_4	1	1	-1	1	-1	1	-1	-1
g_5	1	1	-1	1	-1	1	-1	-1
g_6	1	1	-1	1	-1	1	-1	-1
g_7	1	1	1	1	1	1	1	1
g_8	1	1	1	1	1	1	1	1

β_3	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	1	1	-1	-1	1	-1
g_3	1	1	1	1	1	1	1	1
g_4	1	-1	1	1	-1	-1	1	-1
g_5	1	-1	1	1	-1	-1	1	-1
g_6	1	1	1	1	1	1	1	1
g_7	1	-1	1	1	-1	-1	1	-1
g_8	1	1	1	1	1	1	1	1

β_4	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	1	1	-1	-1	1	-1
g_3	1	-1	1	1	-1	-1	1	-1
g_4	1	1	1	-1	1	-1	-1	-1
g_5	1	1	1	1	1	1	1	1
g_6	1	1	1	1	1	1	1	1
g_7	1	-1	1	-1	-1	1	-1	1
g_8	1	1	1	-1	1	-1	-1	-1

β_5	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	1	1	-1	-1	1	-1
g_3	1	1	-1	1	-1	1	-1	-1
g_4	1	-1	-1	1	1	-1	-1	1
g_5	1	-1	-1	1	1	-1	-1	1
g_6	1	1	-1	1	-1	1	-1	-1
g_7	1	-1	1	1	-1	-1	1	-1
g_8	1	1	1	1	1	1	1	1

β_6	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	1	1	-1	-1	1	-1
g_3	1	-1	-1	1	1	-1	-1	1
g_4	1	1	-1	-1	-1	-1	1	1
g_5	1	1	-1	1	-1	1	-1	-1
g_6	1	-1	-1	-1	1	1	1	-1
g_7	1	-1	1	-1	-1	1	-1	1
g_8	1	1	1	-1	1	-1	-1	-1

β_7	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	1	1	1	1	1	1	1
g_3	1	-1	1	1	-1	-1	1	-1
g_4	1	-1	1	-1	-1	1	-1	1
g_5	1	-1	1	1	-1	-1	1	-1
g_6	1	-1	1	-1	-1	1	-1	1
g_7	1	1	1	-1	1	-1	-1	-1
g_8	1	1	1	-1	1	-1	-1	-1

β_8	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	1	1	1	1	1	1	1
g_3	1	-1	-1	1	1	-1	-1	1
g_4	1	-1	-1	-1	1	1	1	-1
g_5	1	-1	-1	1	1	-1	-1	1
g_6	1	-1	-1	-1	1	1	1	-1
g_7	1	1	1	-1	1	-1	-1	-1
g_8	1	1	1	-1	1	-1	-1	-1

By considering the 2-cocycles that we obtained, it is obvious that there is no 1-dimensional projective representation whenever the 2-cocycle is not trivial. For the trivial cocycle, we have eight 1-dimensional representations which are just the irreducible linear representations of G. For the other cases, we find two 2-dimensional irreducible representations for each of the 2-cocycles (see [30], Theorem 6.7). Let ρ_1^i and ρ_2^i be the irreducible representations corresponding to the cocycle β_i. It is also enough to give the matrices corresponding to the generators of G. Without loss of generality, we assume $g_2 = (1,0,0)$, $g_3 = (0,1,0)$, $g_4 = (0,0,1)$, $g_5 = (1,1,0)$, $g_6 = (1,0,1)$, $g_7 = (0,1,1)$, $g_8 = (1,1,1)$. Here, ${}^{\beta_i}R(G)$ is the projective representation for 2-cocycle β_i. We give only the matrices corresponding to three elements the rest can be obtained from these.

${}^{\beta_2}R(G)$:

$$\rho_1^2 : g_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_2^2 : g_2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

${}^{\beta_3}R(G)$:

$$\rho_1^3 : g_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_2^3 : g_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

${}^{\beta_4}R(G)$:

$$\rho_1^4 : g_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_5 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho_2^4 : g_3 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} g_5 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

${}^{\beta_5}R(G)$:

$$\rho_1^5 : g_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_5 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_2^5 : g_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_4 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} g_4 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

${}^{\beta_6}R(G)$:

$$\rho_1^6 : g_2 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_2^6 : g_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

${}^{\beta_7}R(G)$:

$$\rho_1^7 : g_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho_2^7 : g_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} g_3 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

${}^{\beta_8}R(G)$:

$$\rho_1^8 : g_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_5 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho_2^8 : g_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} g_5 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g_6 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The projective representations ρ_1 and ρ_2 are not linearly isomorphic to each other. Indeed, for the 2-cocycles β_2, β_3, β_5 and β_6, there exist g_i such that

$$\rho_1(g_i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while

$$\rho_2(g_i) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

. Thus there is no $M \in GL_2(\mathbb{Z})$ such that $M\rho_1(g_i)M^{-1} = \rho_2(g_i)$. The other representations of β_4, β_7 and β_8 are obtained from the book of Karpilovski [30] (see page 120 and 124 Theorem 7.1 and 7.2). Thus none of them are linearly isomorphic to each other.

One question that needs to be answered is whether these representations depend on the choice of x_i, y_i and z_i. One can check by calculation that the representations only depend on the values of $\beta_i(g,g)$. More precisely, $\rho(g)\rho(g)$ should be equal to $\beta(g,g)$ for all $g \in G$. For example,

$$\rho(g_3)\rho(g_8) = \beta(g_3,g_8)\rho(g_7)$$

$$\Leftrightarrow \rho(g_3)\beta(g_2,g_7)\rho(g_2)\rho(g_7) = \beta(g_3,g_8)\beta(g_2,g_4)\rho(g_2)\rho(g_4)$$

$$\Leftrightarrow \rho(g_3)\beta(g_2,g_7)\rho(g_2)\beta(g_3,g_4)\rho(g_3)\rho(g_4) = \beta(g_3,g_8)\beta(g_2,g_4)\rho(g_2)\rho(g_4)$$

$$\Leftrightarrow \rho(g_3)\rho(g_3)\beta(g_2,g_7)\beta(g_3,g_4) = \beta(g_3,g_8)\beta(g_2,g_4)I$$

which is true if and only if $\rho(g_3)\rho(g_3) = \beta(g_3,g_3)I$ by the relations we get from the boundary formulas. The other elements can be checked in a similar manner.

Thus we found the basis of our algebra. We will show that this product is unique up to coboundary. First we prove the following result.

Proposition 2.3.3. *If ϕ and ϕ' are cohomologous cocycles in $H^4(G,\mathbb{Z})$ then the fusion algebras corresponding to these cocycles are isomorphic.*

Proof. If ϕ and ϕ' are cohomologous cocycles then $\theta_g(\phi)$ and $\theta_g(\phi')$ represent the same cohomology class in $H^3(G,\mathbb{Z})$. In order to compute the 2-cocycle corresponding to $\theta_g(\phi) \in H^3(G,\mathbb{Z}) \cong G$, we will use the isomorphism induced from the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1.$$

Thus $\theta_g(\phi)$ is mapped to a certain class of 2-cocycles in $H^2(G,U(1)) \cong G$. As we found eight non-cohomologous 2-cocycles in G, it is enough to check how the representations change if we multiply our fundamental 2-cocycles by a 2-coboundary. This is a basic result of projective representation theory (see page 72 in [30]). After multiplying our fundamental 2-cocycles by a 2-coboundary, the new projective representation of this cocycle becomes linearly isomorphic to the former one. The result follows from the above argument. □

2.3.3 The relations.

Now we are able to calculate the relation of this basis using the pairing ${}^{\theta(\phi)_g}R(G) \otimes {}^{\theta(\phi)_h}R(G) \to {}^{\theta(\phi)_{gh}}R(G)$. The calculations are nothing but solving linear equations. Namely one should prove that $\rho_i^k \otimes \rho_i^l$'s are linearly isomorphic to a sum of some basis elements. Here ρ_i^1 denotes the irreducible regular representations of G for $i = 1, 2, \ldots, 8$.

Let us start with $\rho_i^1 \otimes \rho_i^j$ which is linearly isomorphic to ρ_k^j for some $k \in \{1, 2\}$ where $j \neq 1$ because $\rho_i^1 \otimes \rho_i^j$ should be 2 dimensional β_j representation. Here is the results of these multiplications:

$\otimes$	ρ_1^1	ρ_2^1	ρ_3^1	ρ_4^1	ρ_5^1	ρ_6^1	ρ_7^1	ρ_8^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_1^1	ρ_1^1	ρ_2^1	ρ_3^1	ρ_4^1	ρ_5^1	ρ_6^1	ρ_7^1	ρ_8^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_2^1	ρ_2^1	ρ_1^1	ρ_5^1	ρ_6^1	ρ_3^1	ρ_4^1	ρ_8^1	ρ_7^1	ρ_2^2	ρ_1^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_2^5	ρ_1^5	ρ_2^6	ρ_1^6	ρ_1^7	ρ_2^7	ρ_2^8	ρ_1^8
ρ_3^1	ρ_3^1	ρ_5^1	ρ_1^1	ρ_7^1	ρ_2^1	ρ_8^1	ρ_4^1	ρ_6^1	ρ_1^2	ρ_2^2	ρ_2^3	ρ_1^3	ρ_1^4	ρ_2^4	ρ_2^5	ρ_1^5	ρ_1^6	ρ_2^6	ρ_2^7	ρ_1^7	ρ_2^8	ρ_1^8
ρ_4^1	ρ_4^1	ρ_6^1	ρ_7^1	ρ_1^1	ρ_8^1	ρ_2^1	ρ_3^1	ρ_5^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_2^4	ρ_1^4	ρ_1^5	ρ_2^5	ρ_2^6	ρ_1^6	ρ_2^7	ρ_1^7	ρ_2^8	ρ_1^8
ρ_5^1	ρ_5^1	ρ_3^1	ρ_2^1	ρ_8^1	ρ_1^1	ρ_7^1	ρ_6^1	ρ_4^1	ρ_2^2	ρ_1^2	ρ_2^3	ρ_1^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_2^6	ρ_1^6	ρ_2^7	ρ_1^7	ρ_1^8	ρ_2^8
ρ_6^1	ρ_6^1	ρ_4^1	ρ_8^1	ρ_2^1	ρ_7^1	ρ_1^1	ρ_5^1	ρ_3^1	ρ_2^2	ρ_1^2	ρ_1^3	ρ_2^3	ρ_2^4	ρ_1^4	ρ_2^5	ρ_1^5	ρ_1^6	ρ_2^6	ρ_2^7	ρ_1^7	ρ_1^8	ρ_2^8
ρ_7^1	ρ_7^1	ρ_8^1	ρ_4^1	ρ_3^1	ρ_6^1	ρ_5^1	ρ_1^1	ρ_2^1	ρ_1^2	ρ_2^2	ρ_2^3	ρ_1^3	ρ_2^4	ρ_1^4	ρ_2^5	ρ_1^5	ρ_2^6	ρ_1^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_8^1	ρ_8^1	ρ_7^1	ρ_6^1	ρ_5^1	ρ_4^1	ρ_3^1	ρ_2^1	ρ_1^1	ρ_2^2	ρ_1^2	ρ_2^3	ρ_1^3	ρ_2^4	ρ_1^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_2^8	ρ_1^8

Another type of multiplication is $\rho_i^j \otimes \rho_i^j$, which is linearly isomorphic to a sum of four irreducible regular representations ρ_k^1 as $\beta_j(g,h)^2 = 1$ for all j. We calculate all of these using the associativity of our algebra $\mathfrak{X}(G)$ and investigating the eigenvalues of the matrices. Of course, one can also calculate them by defining the linear isomorphism explicitly. As $\rho_1^i(g_j) = -\rho_2^i(g_j)$ for three g_j's in the definitions of representations, we have $\rho_1^i \otimes \rho_1^i = \rho_2^i \otimes \rho_2^i$ as well as $\rho_2^i \otimes \rho_1^i = \rho_1^i \otimes \rho_2^i$.

$\otimes$	ρ_1^2	ρ_2^2
ρ_1^2	$\rho_1^1 + \rho_3^1 + \rho_4^1 + \rho_7^1$	$\rho_2^1 + \rho_5^1 + \rho_6^1 + \rho_8^1$
ρ_2^2	$\rho_2^1 + \rho_5^1 + \rho_6^1 + \rho_8^1$	$\rho_1^1 + \rho_3^1 + \rho_4^1 + \rho_7^1$

$\otimes$	ρ_1^3	ρ_2^3
ρ_1^3	$\rho_1^1 + \rho_2^1 + \rho_4^1 + \rho_6^1$	$\rho_3^1 + \rho_5^1 + \rho_7^1 + \rho_8^1$
ρ_2^3	$\rho_3^1 + \rho_5^1 + \rho_7^1 + \rho_8^1$	$\rho_1^1 + \rho_2^1 + \rho_4^1 + \rho_6^1$

$\otimes$	ρ_1^4	ρ_2^4
ρ_1^4	$\rho_4^1 + \rho_6^1 + \rho_7^1 + \rho_8^1$	$\rho_1^1 + \rho_2^1 + \rho_3^1 + \rho_5^1$
ρ_2^4	$\rho_1^1 + \rho_2^1 + \rho_3^1 + \rho_5^1$	$\rho_4^1 + \rho_6^1 + \rho_7^1 + \rho_8^1$

$\otimes$	ρ_1^5	ρ_2^5
ρ_1^5	$\rho_1^1 + \rho_5^1 + \rho_6^1 + \rho_8^1$	$\rho_2^1 + \rho_3^1 + \rho_4^1 + \rho_7^1$
ρ_2^5	$\rho_2^1 + \rho_3^1 + \rho_4^1 + \rho_7^1$	$\rho_1^1 + \rho_5^1 + \rho_6^1 + \rho_8^1$

$\otimes$	ρ_1^6	ρ_2^6
ρ_1^6	$\rho_1^1 + \rho_3^1 + \rho_6^1 + \rho_8^1$	$\rho_2^1 + \rho_4^1 + \rho_5^1 + \rho_7^1$
ρ_2^6	$\rho_2^1 + \rho_4^1 + \rho_5^1 + \rho_7^1$	$\rho_1^1 + \rho_3^1 + \rho_6^1 + \rho_8^1$

$\otimes$	ρ_1^7	ρ_2^7
ρ_1^7	$\rho_3^1+\rho_4^1+\rho_5^1+\rho_6^1$	$\rho_1^1+\rho_2^1+\rho_7^1+\rho_8^1$
ρ_2^7	$\rho_1^1+\rho_2^1+\rho_7^1+\rho_8^1$	$\rho_3^1+\rho_4^1+\rho_5^1+\rho_6^1$

$\otimes$	ρ_1^8	ρ_2^8
ρ_1^8	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_8^1$	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_7^1$
ρ_2^8	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_7^1$	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_8^1$

The last type of multiplication that we have to consider is $\rho_i^j\otimes\rho_m^n$, where distinct i, m are in $\{1,2\}$ and j and n are in $\{2,3,\ldots,8\}$. As $\rho_i^j\otimes\rho_m^n$ is four-dimensional, it should be linearly isomorphic to $2\rho_1^l$, $2\rho_2^l$ or $\rho_1^l+\rho_2^l$. None of $-2\rho_1^l$, $-2\rho_2^l$ nor $-\rho_1^l-\rho_2^l$ are possible as they are not β_l representations as indicated by the list of our representations in the previous section. The representations $2\rho_1^l$ and $2\rho_2^l$ are also impossible by the following associativity argument.

Suppose $\rho_i^j\otimes\rho_m^n=2\rho_1^l$ by the table above we can always find ρ_k^1 such that $\rho_k^1\otimes\rho_i^j=\rho_i^j$ and $\rho_k^1\otimes\rho_1^l=\rho_2^l$. This gives a contradiction if we multiply each side of $\rho_i^j\otimes\rho_m^n=2\rho_1^l$ by ρ_k^1.

We can conclude $\rho_i^j\otimes\rho_m^n=\rho_1^l+\rho_2^l$. We have finished calculating all the relations.

2.4 Topological gauge theories

In [25], Dijkgraaf and Witten show that three dimensional Chern-Simons gauge theories with a compact gauge group can be classified by the integer cohomology group $H^4(BG,\mathbb{Z})$. Wess-Zumino interactions of such groups G are classified by $H^3(G,\mathbb{Z})$. The relation between three dimensional sigma models involves a certain natural map of $H^4(BG,\mathbb{Z})$ to $H^3(G,\mathbb{Z})$ which is the inverse transgression map defined in the second section. Our calculations provide an example of three-dimensional topological theories with finite gauge group. In this context, our algebra $\mathcal{X}(G)$ is a fusion algebra. In QFT (Quantum Field Theory) one can associate to a $(d+1)$-dimensional manifold M a certain number $Z(M)$, the *partition function*. A detailed discussion is provided in [25].

We consider the partition function of the 3-torus $S^1 \times S^1 \times S^1$. If g, h and k are three commuting gauge fields, the partition function is evaluated to give

$$Z(S^1 \times S^1 \times S^1) = \frac{1}{|G|} \sum_{g,h,k \in G} W(g,h,k),$$

where $[g,h] = [h,k] = [k,g] = 1$. We define W as

$$W(g,h,k) = \frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)}.$$

for $\alpha \in H^3(BG, U(1))$.

The Chern-Simons theory associates to each group element $g_i \in G$ a 2-cocycle β_i, which we calculated above. Again, by the result of Witten and Dijkgraaf [25], we may express β_i in terms of a 3-cocycle $\alpha \in H^3(G, U(1))$:

$$\beta_i(h_1,h_2) = \frac{\alpha(g_i,h_1,h_2)\alpha(h_1,h_2,g_i)}{\alpha(h_1,g_i,h_2)}.$$

This may also be obtained by the formula for inverse transgression map (2.2):

$$\beta_i(h_1,h_2) = \theta_{g_i}(\alpha)([h_1|h_2]) = \alpha([h_1|h_2] \star [g_i]) = \frac{\alpha([g_i|h_1|h_2])\alpha([h_1|h_2|g_i])}{\alpha([h_1|g_i|h_2])}.$$

Note that the shuffle product (2.1) is defined via additive notation.

Thus the action W can be written in terms of 2-cocycles:

$$W(g_i, h, k) = \beta_i(h, k)\beta_i(k, h)^{-1}.$$

We define $\epsilon_g(h) = \beta(g, h)\beta(h, g)^{-1}$ for fixed g which is a one dimensional representation of G. Thus an element g is β-regular if and only if $\epsilon_g = 1$. This implies

$$r(G, \beta) = \frac{1}{|G|} \sum_{g,h \in G} \beta(g, h)\beta(h, g)^{-1}$$

where $r(G; \beta)$ denotes the number of irreducible projective representations of ${}^{\beta}R(G)$. Our 2-cocycles defined in the second section satisfies this condition.

Comparing all these results we obtain the following result for the partition function of the 3-torus where $G = (\mathbb{Z}/2)^3$:

Proposition 2.4.1.

$$Z(S^1 \times S^1 \times S^1) = \sum_i r(G; \beta_i) = 22.$$

Using our representations, we find that the basis elements ν_α of Hilbert space corresponds to 3-torus in QFT. These basis elements are given in [25] as

$$\nu_\alpha(g_i, h) = Tr\rho_i(h).$$

In this context, our algebra $\mathcal{X}(G)$ can be regarded as the smallest twisted non-trivial fusion algebra for abelian groups.

Chapter 3

The Fusion Algebra of an Extraspecial p-group

3.1 Introduction

In this chapter, we give an application of the finite group modular data which is mainly explored in [20]. This modular data was originally introduced, in Lusztig's determination of the irreducible characters of the finite groups of Lie type [31], [32]. To describe the unipotent characters, he considered the modular data for some particular finite groups. The primary fields of the fusion algebra parametrize the unipotent characters associated to a given 2-sided cell in the Weyl group. Lusztig interprets this fusion algebra as the Grothendieck ring for G-equivariant vector bundles; in other words, the equivariant K-theory.

The most physical application of this modular data is in (2+1)-dimensional quantum field theories where a continuous gauge group has been spontaneously broken into a finite group [13]. Non-abelian anyons (i.e. particles whose statistics are governed by the braid group rather than the symmetric group) arise as topological objects. The effective field theory describing the long distance physics is governed by the quantum group of [23].

A set of modular data (i.e. matrices S and T) may be obtained for any choice of finite group G. Much information about a group can be recovered easily from its character table including whether it is abelian, simple, solvable, nilpotent, etc. For instance, G is simple if and only if for all irreducible $\chi \not\equiv 1$, $\chi(a) = \chi(e)$ only for $a = e$. Thus it may be expected that finite group modular data, which probably includes the character table, should provide more information about the group, i.e. be sensitive to a lot of the group-theoretic properties of G.

One way to generalize this data is to twist with a cocycle from the cohomology group. One can obtain topological (e.g. oriented knot) invariants from this twisted data, as explained in [9]. These invariants are functions of the knot group (i.e., the fundamental group of the complement of the knot). Although non-isomorphic knots can have the same invariant, these invariants can distinguish a knot from its inverse (i.e., the knot with opposite orientation), unlike the more familiar topological invariants arising from affine algebras.

In this chapter, we consider the internal product in twisted orbifold K-theory ${}^{\alpha}K_{orb}(\mathfrak{X})$ formalised by Adem, Ruan and Zhang [8] employing results from the associated finite group modular data. Our aim is to exhibit that the non-trivial product structure for ${}^{\alpha}K_G(G)$ for $G = (\mathbb{Z}/p)^3$ is the same as the product structure for $K_H(H)$ where H is the extraspecial p-group of order p^3 and exponent p. The product structure in these theories is the same as in the fusion algebras associated to the same finite group.

We first give some preliminaries and the definition of finite group modular data, then mention the results of Coste, Gannon and Ruelle [20] who calculated the S matrix for $(\mathbb{Z}/p)^3$ with the particular twist. After this, we calculate the S matrix of the extraspecial p-group and show that it gives the same fusion algebra as the twisted case of $(\mathbb{Z}/p)^3$.

3.2 Finite group modular data

In this section, we concentrate on the modular data associated with any finite group G. We first fix a set R of representatives of each conjugacy class of G. The identity e of G is in R, and more generally the center $Z(G)$ of G is a subset of R. For any $a \in G$, let K_a denote the conjugacy class of a in G and $C_G(a)$ be the centralizer of a in G. We have $|G| = |K_a||C_G(a)|$.

The primary fields of the G modular data are labeled by pairs (a, χ), where $a \in R$, and where χ is an irreducible character of $C_G(a)$. We will write Φ for the set of these pairs. We define the S matrix and T matrix as

$$S_{(a,\chi),(b,\chi')} = \frac{1}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \chi(gbg^{-1})^* \chi'(g^{-1}ag)^*$$

$$= \frac{1}{|G|} \sum_{g \in K_a, h \in K_b \cap C_G(g)} \chi(xhx^{-1})^* \chi'(ygy^{-1})^*,$$

$$T_{(a,\chi)(a',\chi')} = \delta_{a,a'} \delta_{\chi,\chi'} \frac{\chi(a)}{\chi(e)}$$

where $G(a,b) = \{g \in G | agbg^{-1} = gbg^{-1}a\}$ and where x and y are any solutions to $g = x^{-1}ax$ and $h = y^{-1}by$. If $G(a,b)$ is empty then the sum is equal to zero.

The matrix S is symmetric and unitary, and gives rise (via the Verlinde formula) to non-negative integer fusion coefficients $N_{(a,\chi_1)(b,\chi_2)}^{(c,\chi_3)}$, where

$$N_{ab}^{c} = \sum_{d \in \Phi} \frac{S_{ad} S_{bd} S_{cd}^*}{S_{0d}}.$$

One way to generalize the group data is by introducing some twisting. The twisting of the modular data is described in [20], where the explicit expressions for the modular matrix S appear.

The primary fields Φ^α in the model twisted by a given 3-cocycle α consist of all pairs (g,χ) where $g \in R$ and χ is a β_g twisted irreducible character of $C_G(g)$, where β_g is defined by

$$\beta_g(a,b) = \alpha(g,a,b)\alpha(a,a^{-1}ga,b)^{-1}\alpha(a,b,(ab)^{-1}gab)$$

from a normalized element α of $H^3(G,U(1))$. The S matrix is calculated as

$$S^\alpha_{(a,\chi)(b,\chi')} = \frac{1}{|G|}\sum_{g\in K_a, g'\in K_b\cap C_G(g)} \left(\frac{\beta_g(g',x^{-1})\beta_{g'}(g,y^{-1})}{\beta_g(x^{-1},h)\beta_{g'}(y^{-1},h')}\right)^* \chi(h)^*\chi'(h')^*,$$

$$= \frac{1}{|G|}\sum_{g\in K_a, g'\in K_b\cap C_G(g)} \left(\frac{\beta_a(x,g')\beta_a(xg',x^{-1})\beta_b(y,g)\beta_b(yg,y^{-1})}{\beta_a(x,x^{-1})\beta_b(y,y^{-1})}\right)^* \chi(h)^*\chi'(h')^*,$$

where $g = x^{-1}ax = y^{-1}h'y$, $g' = y^{-1}by = x^{-1}hx$, $h \in C_G(a)$, $h' \in C_G(b)$.

In order to compute the T matrix, in [20]the 1-cochain $\epsilon_a : C_G(a) \to U(1)$ is introduced. It is determined by the following equalities:

$$\epsilon_a(e) = 1,$$

$$\beta_a(h,g) = \epsilon_a(h)\epsilon_a(g)\epsilon_a(hg)^{-1},$$

$$\epsilon_{x^{-1}ax}(x^{-1}hx) = \frac{\beta_a(x,x^{-1}hx)}{\beta_a(h,x)}\epsilon_a(h),$$

for all g, $h \in C_G(a)$ and $x \in G$. Then the T matrix is

$$T^\alpha_{(a,\chi)(a',\chi')} = \delta_{a,a'}\delta_{\chi,\chi'}\frac{\chi(a)}{\chi(e)}\epsilon_a(a).$$

3.3 Twisted example $G = (\mathbb{Z}/p)^3$

The simplest non-cyclic group is $G = (\mathbb{Z}/n)^2$ for a positive integer n, but it does not lead to something new (see [8]). The cohomology group

$$H^3((\mathbb{Z}/n)^2, U(1)) = (\mathbb{Z}/n)^3$$

has three generators, but all 3-cocycles give β's which are all coboundaries. Thus all twistings are cohomologically trivial.

The more interesting case of $G = (\mathbb{Z}/n)^3$ is given in [20]. Here, we recall that work in order to compare it with our calculation in the next section:

The cohomology $H^3(G, U(1)) = (\mathbb{Z}/n)^7$ is generated by the following cocycles:

$$\alpha_I^{(j)}(a,b,c) = \exp\left\{2i\pi a_j(b_j + c_j - b_j + c_j)/n^2\right\}, \qquad 1 \le j \le 3,$$

$$\alpha_{II}^{(jk)}(a,b,c) = \exp\left\{2i\pi a_j(b_k + c_k - b_k + c_k)/n^2\right\}, \qquad 1 \le j < k \le 3,$$

$$\alpha_{III}(a,b,c) = \exp\left\{2i\pi a_1 b_2 c_3/n\right\},$$

where the group elements are the triplets $a = (a_1, a_2, a_3)$. The cocycles which involve a non-trivial power of α_{III} define non-trivial twistings. The 2-cocycles are of the form

$$\beta_a(b,c) = \exp\left\{2i\pi q(a_1 b_2 c_3 - b_1 a_2 c_3 + b_1 c_2 a_3)/n\right\}.$$

Given a, we wish to count the number of classes b (elements here) which are β_a-regular, i.e., which satisfy $\beta_a(b,c) = \beta_a(c,b)$ for all c. Taking $c = (1,0,0), (0,1,0)$ and $(0,0,1)$, the β_a-regular elements b are those which satisfy

$$a_2b_3 - a_3b_2 \equiv a_1b_3 - a_3b_1 \equiv a_1b_2 - a_2b_1 \equiv 0 \pmod{f},$$

where $f = n/\gcd(q,n)$. The number of solutions $(b_1, b_2, b_3) \in (\mathbb{Z}/n)^3$ to this system is equal to $n^3[\gcd(a_1,a_2,a_3,f)/f]^2$.

It remains to sum those numbers for all a to obtain the number of primaries:

$$|\Phi^\alpha| = \frac{n^6}{f^3} \prod_{\substack{p|f \\ p \text{ prime}}} \Big[(p^{k_p} - 1)(1 + p^{-1} + p^{-2}) + 1\Big].$$

We now consider the case when n is an odd prime number p, and when the 3-cocycle is α_{III}.

When G is abelian, all factors in the formula for S^α that involve the cocycles drop out, and one is left with the simple expression:

$$S^\alpha_{(a,\tilde{\chi}),(b,\tilde{\chi}')} = \frac{1}{|G|}\chi^*(b)\,\chi'^*(a),$$

where χ and χ' are respectively β_a and β_b-projective characters, for the cocycles given above with $q = 1$.

It remains to compute the projective characters. One then finds p distinct, irreducible β_a-projective representations of dimension p if a is not the identity, while there are of course p^3 representations of dimension 1 if $a = e$. Depending on the value of $a = (a_1, a_2, a_3)$, the characters are given in the following table, where it is implicit that the element $g = (g_1, g_2, g_3)$ must be β_a-regular for the character not to vanish. In the first three cases, the character label u runs over $\mathbb{Z}/p$, and in the last column, $\vec{u}$ takes all values in $(\mathbb{Z}/p)^3$.

	$a_1 \neq 0$	$a_1 = 0,\ a_2 \neq 0$	$a_1 = a_2 = 0,\ a_3 \neq 0$	$a_1 = a_2 = a_3 = 0$
$\chi(g)$	$p\,\xi_p^{a_1^{-1}ug_1 - a_1^{-1}a_2a_3g_1^2/2}$	$p\,\xi_p^{a_2^{-1}ug_2}$	$p\,\xi_p^{a_3^{-1}ug_3}$	$\xi_p^{\vec{u}\cdot\vec{g}}$

If for instance a_2 is also invertible, then $a_1^{-1}g_1 = a_2^{-1}g_2$, so that the first character value is also equal to $\tilde{\chi}(g) = p\,\xi_p^{a_2^{-1}ug_2 - a_1a_2^{-1}a_3g_2^2/2}$. The primary fields are thus $(e, \chi_{\vec{u}})$ and $(a, \tilde{\chi}_u)$, for a total of

$$|\Phi^\alpha| = p^3 + (p^3 - 1)p = p^4 + p^3 - p.$$

The matrices S^α and T^α are now straightforward to establish. Taking the condition of β-regularity into account, one finds that $S^\alpha_{(a,\chi_u)(b,\chi_{u'})}$ is almost block-diagonal:

$$\frac{1}{p}\begin{pmatrix} \frac{1}{p^2} & \frac{1}{p}\xi_p^{-u_1b_1-u_2b_2-u_3b_3} & \frac{1}{p}\xi_p^{-u_2b_2-u_3b_3} & \frac{1}{p}\xi_p^{-u_3b_3} \\ \frac{1}{p}\xi_p^{-u'_1a_1-u'_2a_2-u'_3a_3} & \begin{matrix}\xi_p^{-ua_1^{-1}b_1-u'b_1^{-1}a_1+(a_2a_3b_1+b_2b_3a_1)/2} \\ \times\delta(b_2-a_1^{-1}a_2b_1)\,\delta(b_3-a_1^{-1}a_3b_1)\end{matrix} & 0 & 0 \\ \frac{1}{p}\xi_p^{-u'_2a_2-u'_3a_3} & 0 & \begin{matrix}\xi_p^{-ua_2^{-1}b_2-u'b_2^{-1}a_2} \\ \times\delta(b_3-a_2^{-1}a_3b_2)\end{matrix} & 0 \\ \frac{1}{p}\xi_p^{-u'_3a_3} & 0 & 0 & \xi_p^{-ua_3^{-1}b_3-u'b_3^{-1}a_3} \end{pmatrix}$$

where the blocks correspond to the subsets $\{a = e\}$, $\{a_1 \neq 0\}$, $\{a_1 = 0,\ a_2 \neq 0\}$, and $\{a_1 = a_2 = 0,\ a_3 \neq 0\}$.

The entries $T^\alpha_{(a,\chi)(a,\chi)}$ are 1 for $a = e = (0,0,0)$ and $\xi_p^{u-a_1a_2a_3/2}$ in any other case.

In the next section, we observe that the modular data of the extraspecial p-group gives the same fusion coefficients with this twisted case. Hence we can conclude that these two fusion algebras are the same.

3.4 Modular data for extraspecial p-group

When discussing what he called electric/magnetic duality in [35], Propitius observed that the modular data for $(\mathbb{Z}/2)^3$ with a particular twist equals that of the untwisted dihedral group D_8 (for an appropriate identification of primary fields). In the more recent work of [20] it is claimed that there are many more such examples; when calculating the modular data for an appropriately twisted $(\mathbb{Z}/p)^3$, it is noted that the quantum dimensions and number of primaries suggest that this twisted data would yield the modular data for the extraspecial p-group H of order p^3, which is the central extension of the cyclic subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$. This group may be represented as

$$\langle A, B, C | A^p = B^p = C^p = [A, C] = [B, C] = 1, [A, B] = C \rangle.$$

In this section we prove that this is indeed the case.

It is convenient to use an isomorphic realization of H via $(r, s, t) \in (\mathbb{Z}/p)^3$ with the following defining relations.

$$(r, s, t)^{-1} = (-r, -s, -t - rs),$$

$$(r, s, t)(r', s', t') = (r + r', s + s', t + t' - sr').$$

The irreducible characters are calculated in [29]. For fixed x, y, z in $\{1, 2, \ldots, p-1\}$, we define the character as

$$\chi(r, s, t) = \begin{cases} 0, & \text{if } \frac{p}{d} \nmid s \vee (\frac{p}{d} \nmid r \wedge z \neq 0); \\ p\xi_p^{tz+rx+sy}, & \text{otherwise}; \end{cases}$$

where $d = gcd(z, p)$. It may be rewritten in a more compact form using the

Kronecker delta function:

$$\chi(r,s,t) = \begin{cases} \xi_p^{rx+sy}, & \text{if z=0;} \\ \delta(p|r)\delta(p|s)p\xi_p^{rx+sy+tz}, & \text{if } z \neq 0; \end{cases}$$

where $\delta(p|r) = 1$ if $p = r$ and 0 otherwise. Using this triple realization, the conjugacy class of the element (a,b,c) can be given as $C_{(a,b,c)} = \{(a,b,c+bx-ay) \mid x,y \in \mathbb{Z}/p\}$. Thus two elements, (a,b,c) and (a',b',c'), belong to the same conjugacy class if and only if $a = a'$, $b = b'$ and there exist integers x, y and z such that $c = c' - ay + bx + pz$. This equation is solvable if and only if $c \equiv c' \pmod{gcd(a,b,p)}$. Therefore every conjugacy class contains exactly one element of the set

$$L = \{(a,b,c) \in \{0.\dots,p-1\}^3 | c < gcd(a,b,p)\}.$$

If neither a nor b is equal to 0 then $gcd(a,b,p) = 0$. Hence c should be equal to 0. If $a = b = 0$ then c can be any number in $\{0,\dots,p-1\}$ as $gcd(0,0,p) = p$ in this case. So the representative of the conjugacy classes are either in the form of $(a,b,0) \neq (0,0,0)$ or $(0,0,c)$. We now calculate the centralizer for each element. The elements (r,s,t) of $C_G(a,b,0)$ should satisfy

$$(r,s,t)(a,b,0) = (a,b,0)(r,s,t),$$

$$(a+r,s+b,t-as) = (a+r,s+b,t-rb).$$

Therefore $as = rb$ and $(r,s,t) = (a,asr^{-1},c)$. $C_G(a,b,0)$ is of order p^2. From the basic results of finite group theory there are two groups of order p^2 up to isomorphism. $C_G(a,b,0)$ cannot be the cyclic one as it has at least two elements of order p, namely $(a,0,0)$ and $(0,0,c)$. Thus, $C_G(a,b,0)$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$.

The elements (r, s, t) of $C_G(0, 0, c)$ should satisfy

$$(r, s, t)(0, 0, c) = (0, 0, c)(r, s, t),$$

$$(r, s, c + t) = (r, s, c + t).$$

Hence $C_G(0, 0, c) = G$.

Let us denote the irreducible characters by $\Gamma^i|_{i=1}^{p^2}$ and $\chi^i|_{i=1}^{p^2+p+1}$ for $\mathbb{Z}/p \times \mathbb{Z}/p$ and G, respectively. The primary fields of the modular data may be written as $\big((0, 0, c), \chi^i\big)$ for all $c \in G$ and $\big((a, b, 0), \Gamma^i\big)$ where $(a, b, 0) \neq (0, 0, 0)$. We find that the number of the primary fields is $p(p^2 + p - 1) + (p^2 - 1)p^2 = p^4 + p^3 - p$ which is equal to the number of primary fields of twisted modular data of $(\mathbb{Z}/p)^3$.

We next calculate the entries of the S matrix. We first consider the primary fields (a, χ^i) and (b, χ^j) where $a = (0, 0, r)$ and $b = (0, 0, t)$. We use the formula given in the second section:

$S_{(a,\chi^i)(b,\chi^j)} = \frac{1}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \chi^i(gbg^{-1})^* \chi^j(g^{-1}ag)^*$.

We know $|C_G(a)||C_G(b)| = p^6$, $G(a, b) = \{g \in G | agbg^{-1} = gbg^{-1}a\} = \{g \in G | (0, 0, r + t) = (0, 0, r + t)\} = G$ as well as $\chi^i(gbg^{-1}) = \chi^i(b)$ for $g \in G$. Hence

$$S_{(a,\chi^i)(b,\chi^j)} = \frac{1}{p^3} \chi^i(b)^* \chi^j(a)^*.$$

By the above formula for χ, if z in the formula is equal to 0 then there are p^2 characters such that $\chi(0, 0, t) = 0$. We denote these characters by $\chi^{i,j}$ where i and j are in $\{0, \ldots, p - 1\}$. If $z \neq 0$ then there are $p - 1$ characters of the form as $z \in \{1, \ldots, p - 1\}$. We denote these characters by χ^{z_i} after indexing the $(p - 1)$ z's. Hence, by the formula for the S matrix, we have

$$S_{((0,0,r),\chi^{i,j})((0,0,t),\chi^{i',j'})} = 1/p^3,$$

$$S_{((0,0,r),\chi^{z_i})((0,0,t),\chi^{z_j})} = 1/p\xi_p^{-rz_j-tz_i},$$

$$S_{((0,0,r),\chi^{i,j})((0,0,t),\chi^{z})} = 1/p^2\xi_p^{-rz}.$$

One may compare these results with the S^α matrix of the twisted modular data for $(\mathbb{Z}/p)^3$. In order to get the same entries as S^α, we change $((0,0,t),\chi^{z_i})$ to $((0,0,z_i^{-1}t),\chi^{z_i})$, and notice that z_i corresponds to a_3 in the notation of the twisted matrix. We basically proved that the blocks at the corners of each matrix are the same. It is trivial to see that the dimension of these blocks is also the same.

We now need to check the other blocks. For the other blocks we need to consider the primary fields $((r,s,0),\Gamma^i)$ where s or t are not equal to 0. We have the following formula

$$S_{((r,s,0),\Gamma^i)((0,0,t),\chi^z)} = \frac{1}{|C_G(a)||C_G(b)|}\sum_{g\in G(a,b)} \Gamma^i(gbg^{-1})^*\chi^z(g^{-1}ag)^*$$

where $a = (0,0,t)$ and $b = (r,s,0)$. By the Kronecker delta functions $\delta(p|r)$ and $\delta(p|s)$ in the definition of χ, the character $\chi^z(g^{-1}ag)$ should be equal to zero as at least one of r or s is nonzero. These zeros corresponds to the $(4,2)$, $(4,3)$, $(2,4)$ and $(3,4)$ block entries of the S^α matrix of the twisted case if we index the blocks starting from the left upper corner which is the $(1,1)$ block.

Another kind of entry that we need to consider is

$$S_{((0,0,t),\chi^{i,j})((r,s,0),\Gamma^i)} = \frac{1}{|C_G(a)||C_G(b)|}\sum_{g\in G(a,b)} \chi^{i,j}(gbg^{-1})^*\Gamma^i(g^{-1}ag)^*$$

where $a = (0, 0, t)$ and $b = (r, s, 0)$. Here, we know $|C_G(a)| = p^3$, $|C_G(b)| = p^2$ and

$$G(a, b) = \{g \in G | agbg^{-1} = gbg^{-1}a\}$$

$$= \{g \in G | (r, s, t - as + rb) = (r, s, t - as + rb)\} = G.$$

Moreover, we have $g^{-1}ag = a$ for $g \in G$ and

$$gbg^{-1} = (g_1, g_2, g_3)(r, s, 0)(-g_1, -g_2, -g_3 - g_1g_2) = (r, s, g_2r - g_1s)$$

by the formula for χ, $\chi^{i,j}(gbg^{-1}) = \xi_p^{rx+sy}$ where x, $y \in G$ are fixed for each character $\chi^{i,j}$. We can change the notation x and y to u_1 to u_2. On the other hand, $\Gamma^i(0, 0, t) = \xi_p^{a_3t}$ where we can define $\Gamma^i(r, s, t) = \xi_p^{vs}\xi_p^{a_3t}$ where v and a_3 are fixed for each character Γ^i. We recall that, for $(x, y, z) \in C_G(r, s, 0)$, x and y are the same up to a constant while z is independent than the other coordinates. For the character Γ^i of $\mathbb{Z}/p \times \mathbb{Z}/p$ we thus consider the second and third coordinates without loss of generality. We use this fact again in the next calculation. As a result, we obtain

$$S_{((0,0,t),\chi^{i,j})((r,s,0),\Gamma^i)} = 1/p^2\xi_p^{-ru_1-su_2-ta_3}.$$

Hence if $r \neq 0$ this block coincides with $(1, 2)$ and $(2, 1)$ blocks of the S^α matrix. If $r = 0$ it coincides with the $(1, 3)$ and $(3, 1)$ blocks if we change r, s and t to a_1, a_2 and u_3, respectively.

The remaining blocks that we need to check are the four blocks at the center. Let us first check $S_{((0,s,0),\Gamma^i)((r',s',0),\Gamma^j)}$ where $r' \neq 0$. In this case we see that $G(a, b) = \{g \in G | agbg^{-1} = gbg^{-1}a\} = \{g \in G | (r', s + s', g_2r' - g_1s') = (r', s + s', g_2r' - g_1s' - sr') = \emptyset$ because $sr' \neq 0$. Hence

$S_{((0,s,0),\Gamma^i)((r',s',0),\Gamma^j)} = 0$ corresponds to the zero blocks ((2,3) and (3,2) blocks) at the center.

We continue with the block corresponding to $S_{((0,s,0),\Gamma^i)((0,s',0),\Gamma^j)}$. Here we have $G(a,b) = \{g \in G | agbg^{-1} = gbg^{-1}a\} = \{g \in G | (0, s+s', -g_1 s') = (0, s+s', -g_1 s')\} = G$, $gbg^{-1} = (0, s', -s'g_1)$ and $g^{-1}ag = (0, s, sg_1)$ where $g = (g_1, g_2, g_3)$, and recall that we defined Γ^i and Γ^j as $\Gamma^i(r,s,t) = \xi_p^{vs}\xi_p^{a_3 t}$ and $\Gamma^j(r,s,t) = \xi_p^{v's}\xi_p^{b_3 t}$. If $a_3 s' = b_3 s$ then

$$\Gamma^i(gbg^{-1})^*\Gamma^j(g^{-1}ag)^* = \xi_p^{-vs'}\xi_p^{a_3 s' g_1}\xi_p^{-v's}\xi_p^{-b_3 s g_1}$$

$$= \xi_p^{-vs'-v's}\xi_p^{g_1(s'a_3 - b_3 s)} = \xi_p^{-vs'-v's}.$$

If $a_3 s' \neq b_3 s$ which means $\delta(a_3 s' - b_3 s) = 0$, then

$$\Gamma^i(gbg^{-1})^*\Gamma^j(g^{-1}ag)^* = \xi_p^{-vs'-v's}\xi_p^{g_1(s'a_3 - b_3 s)}.$$

We now consider the S-matrix formula

$$S_{((0,s,0),\Gamma^i)((0,s',0),\Gamma^j)} = 1/p^4 \sum_{g\in G} \xi_p^{-vs'-v's}\xi_p^{g_1(s'a_3 - b_3 s)}.$$

If $a_3 s' \neq b_3 s$, then $\xi_p^{g_1(s'a_3 - b_3 s)}$ is summed while g_1 is covering all $\mathbb{Z}/p$ p^2 times. As the sum of all the roots of unity is 0, the entry $S_{((0,s,0),\Gamma^i)((0,s',0),\Gamma^j)}$ must be zero. Otherwise $S_{((0,s,0),\Gamma^i)((0,s',0),\Gamma^j)} = 1/p\xi_p^{-vs'-v's}$. Any non-zero power of ξ_p is also a pth root unity. Hence we can choose $v = ua_2^{-1}$ and $v' = u'b_2^{-1}$. If we change the notation $s = a_2$ and $s' = b_2$, we obtain the $(3,3)$ block of the twisted S matrix of $(\mathbb{Z}/p)^3$. Note that this assignment does not change the order of the entries in the first column or in the first row as they are determined by second factor $\xi_p^{a_3 t}$ of the character Γ^i.

The $(2,2)$ block is the only block left in the twisted S^α of $(\mathbb{Z}/p)^3$. We need to calculate $S_{((r,s,0),\Gamma^i)((r',s',0),\Gamma^j)}$. Here $G(a,b) = \{g \in G | agag^{-1} = gbg^{-1}a\} = \{g \in G | (r+r', s+s', g_2 r - g_1 s' - r's) = (r+r', s+s', -g_1 s' - s'r)$ is G if and only if $r's = s'r$. Otherwise $G(a,b)$ is an empty set, a case which is covered by examining the Kronecker delta function$\delta(s' - r^{-1}sr')$, as these terms are in the abelian group $\mathbb{Z}/p$.

We now assume that $r's = s'r$. We also have $gbg^{-1} = (r', s', r'g_2 - s'g_1)$ and $g^{-1}ag = (r, s, sg_1 - rg_1)$ where $g = (g_1, g_2, g_3)$. By the formulae $\Gamma^i(r,s,t) = \xi_p^{vs}\xi_p^{a_3 t}$ and $\Gamma^j(r,s,t) = \xi_p^{v's}\xi_p^{b_3 t}$ we get the multiplication

$$\Gamma^i(gbg^{-1})^*\Gamma^j(g^{-1}ag)^* = \xi_p^{-vs'-v's}\xi_p^{g_1(s'a_3 - b_3 s) + g_2(r'a_3 - rb_3)}.$$

We note that $s'a_3 - b_3 s = 0$ if and only if $r'a_3 - rb_3$ as $r's = s'r$. If these exponents are non-zero, the entry in our S is zero because g_i is covering the set G by resolving the power sum of the ξ_p which is equal to zero. Hence another Kronecker delta function appears in our expression which is $\delta(r'a_3 - rb_3)$. If $\delta(r'a_3 - rb_3) = 1$ then $S_{((r,s,0),\Gamma^i)((r',s',0),\Gamma^j)} = 1/p\xi_p^{-vs'-v's}$. We now make the conventional choice of v and v' as $ua_2^{-1} - a_1 a_3/2$ and $u'b^{2-1} - b_1 b_3/2$. Hence

$$S_{((r,s,0),\Gamma^i)((r',s',0),\Gamma^j)} = 1/p\xi_p^{-(ua_2^{-1} - a_1 a_3/2)s' - (u'b^{2-1} - b_1 b_3/2)s}.$$

We also change the notation $r = a_1$, $s = a_2$, $r' = b_1$ and $s' = b_2$ to obtain

$$\xi_p^{-ua_2^{-1}b_2 - u'b_2^{-1}a_2 + (a_1 a_3 b_2 + b_1 b_3 b_2)/2} = \xi_p^{-ua_1^{-1}b_1 - u'b_1^{-1}a_1 + (a_2 a_3 b_1 + b_2 b_3 a_1)/2},$$

where the equality follows from $b_1 a_2 = a_1 b_2$ and $b_2 a_3 - b_3 a_2 = 0$. The choice of v and v' is the same as the last part we have calculated because $r = a_1 = 0$

in that case. We note that the change in variables does not change the order of the entries in first column or row as they are determined by second factor $\xi_p^{a_3 t}$ of the character Γ^i. This completes our calculation of the S matrix of H which as a result is the same as the twisted S^α matrix of the group $(\mathbb{Z}/p)^3$

We can write the change of variables as a map from the primary field $((r,s,0),\Gamma^i)$ to $((a_1,a_2,a_3),\chi_u)$ in the twisted fusion algebra where r and s are mapped to a_1 and a_2, respectively. The two factors in the character Γ^i are mapped to a_3 and u.

We next calculate the T matrix. We again start with the primary field of type $((0,0,t),\chi^{i,j})$ for which

$$T_{((0,0,t),\chi^{i,j})((0,0,t),\chi^{i,j})} = \chi^{i,j}(0,0,t)/\chi^{i,j}(0,0,0) = 1.$$

If the field is $((0,0,t),\chi^{z_i})$ then

$$T_{((0,0,t),\chi^{z_i})((0,0,t),\chi^{z_i})} = \chi^{z_i}(0,0,t)/\chi^{z_i}(0,0,0) = \xi_p^{tz_i}.$$

Remember that we replace $(0,0,t)$ by $(0,0,tz_i^{-1})$. Hence $T_{((0,0,t),\chi^{z_i})((0,0,t),\chi^{z_i})}$ is ξ_p^t.

If the field is of type $((0,s,0),\Gamma^i)$ then

$$T_{((0,s,0),\Gamma^i)((0,s,0),\Gamma^i)} = \Gamma^i(0,s,0)/\Gamma^i(0,0,0) = \xi_p^{sv}.$$

By our previous change of v and s to ua_2^{-1} and a_2, respectively, we obtainξ_p^u for this primary field.

The last primary field to check is the field of type $((r,s,0),\Gamma^i)$ for which

$$T_{((r,s,0),\Gamma^i)((r,s,0),\Gamma^i)} = \Gamma^i(r,s,0)/\Gamma^i(0,0,0) = \xi_p^{sv},$$

where we replace s and v by a_2 and $ua_2^{-1} - a_1a_3/2$, respectively. Therefore we obtain the term $\xi_p^{u-a_1a_2a_3/2}$. We conclude that the matrix T^α of the group $(\mathbb{Z}/p)^3$ for a particular twist α and the T matrix of the extraspecial p-group H of order p^3 with exponent p coincide.

As a result, we have proved the following theorem.

Theorem 3.4.1. *The twisted modular data of the group $(\mathbb{Z}/p)^3$ are the same as the modular data of the extraspecial p-group of order p^3 with exponent p.*

We also have the following corollary.

Corollary 3.4.2. *The twisted fusion algebra of $(\mathbb{Z}/p)^3$, which is isomorphic to the twisted orbifold K-theory ${}^{\alpha}K_{orb}(\wedge[*/G])$ in the sense of [8], is isomorphic to the fusion algebra of the extraspecial p-group of order p^3 with exponent p, which is isomorphic to the orbifold K-theory $K_{orb}(\wedge[*/H])$.*

Chapter 4

Cohomology of Toroidal Orbifolds

4.1 Introduction

Let G be a finite group and $\varphi : G \to GL_n(\mathbb{Z})$ an integral representation of G. In this way G acts linearly on $\mathbb{R}^n$ preserving the integral lattice $\mathbb{Z}^n$, thus inducing a G-action on the torus $X_\varphi = X := \mathbb{R}^n/\mathbb{Z}^n$. The quotient $X \to X/G$ naturally has the structure of an orbifold as a global quotient, and these kind of orbifolds are usually referred to as toroidal orbifolds. The goal of this chapter is to compute the cohomology groups $H^*(X/G; \mathbb{Z})$ for the particular case where $G = \mathbb{Z}/p$ for a prime number p.

The quotients of the form X/G appear naturally in different contexts. For example, given a topological space Y, the m-th cyclic product of Y is defined to be the quotient

$$CP^m(Y) := Y^m/\mathbb{Z}/m,$$

where $\mathbb{Z}/m$ acts by cyclically permuting the product Y^m. In the particular case where the representation $\varphi : G \to GL_n(\mathbb{Z})$ induces the $\mathbb{Z}G$-module $(\mathbb{Z}G)^n$, the associated torus X is $(S^1)^p)^n$, where G acts cyclically on each

$(S^1)^p$ and diagonally on the product $((S^1)^p)^n$. In this case

$$X/G = (((S^1)^p)^n)/\mathbb{Z}/p \cong ((S^1)^n)^p/\mathbb{Z}/p$$

where now $\mathbb{Z}/p$ acts cyclically on the $((S^1)^n)^p$. Therefore

$$X/G \cong CP^p((S^1)^n).$$

The homology groups of quotient spaces of the form X^m/K, where $K \subset \Sigma_m$, have long been studied. In particular, in [38] Swan formulated a method for the computation of homology of cyclic products of topological spaces.

4.2 Preliminaries

Let G be a finite group and $\varphi : G \to GL_n(\mathbb{Z})$ an integral representation of G. Consider $X = X_\varphi$ the standard torus with the action of G induced from φ. Then there is a fibration sequence

$$X \to X \times_G EG \to BG. \tag{4.1}$$

Using the long exact sequence in homotopy groups associated to this fibration, it follows that $X \times_G EG$ is an Eilenberg-Maclane space of type $K(\Gamma, 1)$, where $\Gamma := \pi_1(X \times_G EG)$. The G-action on X makes $L := \pi_1(X) \cong H_1(X;\mathbb{Z}) \cong \mathbb{Z}^n$ into a $\mathbb{Z}G$-module. Moreover, since $[0] \in \mathbb{R}^n/\mathbb{Z}^n$ is always a fixed point for the action of G on X, it follows that the fibration (4.1) has a section. The existence of such a section implies that the short exact sequence

$$1 \to \pi_1(X) \to \pi_1(X \times_G EG) \to \pi_1(BG) \to 1$$

has a section and therefore $\Gamma \cong L \rtimes G$is a semi-direct product. For example, when the representation φ is injective, the group Γ is a crystallographic group.

The cohomology groups of the groups of the form $\Gamma \cong L \rtimes G$, when $G = \mathbb{Z}/p$ for a prime number p, were computed in [4, Theorem 1.1] where it is shown that a certain special free resolution $\epsilon : F \to \mathbb{Z}$ of $\mathbb{Z}$ as a $\mathbb{Z}[L]$-module admits an action of G compatible with φ. This implies that the Lyndon-Hochshild-Serre spectral sequence associated to the short exact sequence

$$1 \to L \to \Gamma \to G \to 1$$

collapses on the E_2-term without extension problems, thus for any $k \geq 0$

$$H^k(\Gamma; \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G; \bigwedge^j (L^*)),$$

where L^* denotes the dual module $Hom(L, \mathbb{Z})$.

One application of this, by [4, Theorem 1.2] is when $G = \mathbb{Z}/p$ acts on X via a representation $\varphi : G \to GL_n(\mathbb{Z})$, then

$$H^k(X \times_G EG; \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G; H^j(X; \mathbb{Z})).$$

for each $k \geq 0$. The strategy we will use to compute the cohomology groups $H^*(X/G; \mathbb{Z})$ is as follows. Let F denote the subspace of fixed points under the G-action. In general, F will be a disjoint union of product of circles. Then the long exact sequence in cohomology associated to the pair $(X/G, F)$ can be used to compute the cohomology groups $H^*(X/G; \mathbb{Z})$. To make this work, the relative cohomology groups $H^*(X/G, F; \mathbb{Z})$ need to be determined. For this, we consider the equivariant projection $\pi_1 : X \times EG \to X$. On the level of orbit spaces π_1 induces a map

$$\phi : X \times_G EG \to X/G.$$

By [17, Proposition V 1.1], ϕ induces the isomorphism

$$\phi^* : H^*(X/G, F; \mathbb{Z}) \to H^*_G(X, F; \mathbb{Z}).$$

This reduces the problem to one of determining the cohomology groups $H^*_G(X, F; \mathbb{Z})$. These groups will be computed using representation theory and the fact that the Lyndon-Hochshild-Serre spectral sequence associated to the fibration sequence

$$X \to X \times_G EG \to BG$$

collapses on the E_2-term without extension problems.

Let $L := H^1(X; \mathbb{Z}) \cong \pi_1(X)$. As explained, L has the structure of a $\mathbb{Z}G$-lattice; this structure determines the cohomology groups of X/G. Let $R = \mathbb{Z}_{(p)}$ be the ring of integers localized at the prime p. Then (see [21]) there are only three distinct isomorphism classes of RG-lattices, namely the trivial module R, the augmentation ideal IG and the group ring RG. Moreover, if L is any finitely generated $\mathbb{Z}G$-lattice, then there is a $\mathbb{Z}G$-lattice $L' \cong \mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$ and $\mathbb{Z}G$ homomorphism $f : L' \to L$ such that f is an isomorphism after tensoring with R. In general, a $\mathbb{Z}G$-module of the form $\mathbb{Z}^r \oplus \mathbb{Z}G^s \oplus IG^t$ is called a module of type (r, s, t).

A fundamental tool in the computation of the cohomology groups of the toroidal orbifolds is the next lemma.

Lemma 4.2.1. *Suppose that p is a prime number. Let $G = \mathbb{Z}/p$ act on a finite dimensional space X with fixed point set F. If there is an integer N such that $H^k(X, F; \mathbb{Z}) = 0$ for $k > N$, then $H^k_G(X, F; \mathbb{Z}) = 0$ for $k > N$.*

Proof. This follows by applying [17, Exercise III.9] and [17, Proposition VII 1.1]. □

From now on G denotes the group $\mathbb{Z}/p$ for a prime number p. Given a representation $\varphi : G \to GL_n(\mathbb{Z})$, X denotes the G-space $\mathbb{R}^n/\mathbb{Z}^n$ with the G-action induced by φ and F denotes the fixed point set under this action. The representation φ makes $L : H^1(X; \mathbb{Z})$ into a $\mathbb{Z}G$-module whose structure completely determines the cohomology groups $H^*(X/G; \mathbb{Z})$ as it will be shown in the next section in the case of $L = IG^m$.

4.3 The case $L = IG^m$

In this section the particular case where the $\mathbb{Z}G$-lattice L equals IG^m is considered. Let $\rho_m := \oplus_m \rho : G \to GL_N(\mathbb{Z})$ be the integral representation inducing the $\mathbb{Z}G$-module L, where $N = m(p-1)$. The fixed point set of such an action is easily identified by the following straightforward lemma.

Lemma 4.3.1. *If $X = X_{\oplus_m \rho}$ is the G-space induced by the representation $\oplus_m \rho$ then the fixed point set F under this G-action is a discrete set with p^m points.*

Proof. Consider the short exact sequence of G-modules defining the G-space X

$$0 \to L \to L \rtimes \mathbb{R} \to (L \rtimes \mathbb{R})/L = X \to 0.$$

This short exact sequence induces a long exact sequence on the level of group cohomology

$$0 \to H^0(G,L) \to H^0(G,L\rtimes\mathbb{R}) \to H^0(G,X) \to H^1(G,L) \to H^1(G,L\rtimes\mathbb{R}) \to \dots$$

Note that $H^1(G,L\rtimes\mathbb{R}) = 0$ and $H^0(G,L) = H^0(G,L\rtimes\mathbb{R}) = 0$, thus there is an isomorphism $F = H^0(G,X) \cong H^1(G,L) \cong (\mathbb{Z}/p)^m$. □

We are interested in computing the cohomology groups $H^*_G(X,F)$. These can be computed using the Serre spectral sequence for the pair (X,F)

$$E_2^{i,j} = H^i(G, H^j(X,F)) \Longrightarrow H^{i+j}_G(X,F).$$

Therefore the structure of $H^j(X,F)$ as a G-module needs to be studied. To this end, we consider the long exact sequence in cohomology associated to the pair (X,F). Since $H^j(F) = 0$ when $j \geq 1$, it follows that $H^j(X,F) \cong H^j(X)$ for $j \geq 2$ and there is a short exact sequence

$$0 \to \mathbb{Z}^{p^m-1} \to H^1(X,F) \to (IG)^m \to 0. \tag{4.2}$$

By the classification theorem for $\mathbb{Z}G$-modules in [21], there are integers a_j, b_j and $c_j \geq 0$, ideals $A_1, ..., A_{a_j}$ of $\mathbb{Z}G$ rank $p-1$ and projective indecomposable modules $P_1, ..., P_{b_j}$ such that

$$H^j(X,F) \cong \left(\bigoplus_1^{a_j} A_i\right) \oplus \left(\bigoplus_1^{b_j} P_j\right) \oplus \left(\bigoplus_1^{c_j} \mathbb{Z}\right).$$

In particular,

$$H^i(G, H^j(X,F)) \cong \begin{cases} \mathbb{Z}^{b_j+c_j} & \text{if} \quad i = 0, \\ (\mathbb{Z}/p)^{a_j} & \text{if} \quad i \text{ is odd and } i > 0, \\ (\mathbb{Z}/p)^{c_j} & \text{if} \quad i \text{ is even and } i > 0. \end{cases}$$

On the other hand, for $j \geq 2$ we obtain

$$H^j(X,F) \cong H^j(X) \cong \bigwedge^j (IG)^m.$$

Hence the rank of $\bigwedge^j (IG)^m$ is equal to the rank of $H^j(X,F) \cong (\bigoplus_1^{a_j} A_i) \oplus (\bigoplus_1^{b_j} P_i) \oplus (\bigoplus_1^{c_j} \mathbb{Z})$. This implies the following equation when $j \geq 2$:

$$\binom{m(p-1)}{j} = a_j(p-1) + b_j p + c_j. \tag{4.3}$$

For each $m, j \geq 0$, we define $p_m(j)$ to be the number of all possible sequences of integers $l_1, ..., l_m$ such that $0 \leq l_r \leq p-1$ and $l_1 + \cdots l_m = j$. This forces $p_m(j) = 0$ for $j > N = (p-1)m$, $p_m(0) = 1$, $p_m(1) = m$ and by induction it is easy to see that

$$\sum_{j=0}^{m(p-1)} p_m(j) = p^m.$$

When $j \geq 2$, it follows from [3, Proposition 1.10] that

$$H^i(G, H^j(X,F)) \cong \begin{cases} (\mathbb{Z}/p)^{p_m(j)} & \text{if} \quad i+j \text{ is even and } i > 0, \\ 0 & \text{if} \quad i+j \text{ is odd and } i > 0. \end{cases}$$

This result implies

$$a_j = \begin{cases} p_m(j) & \text{if } j \text{ is odd and} \quad j \geq 2, \\ 0 & \text{if } j \text{ is even and} \quad j \geq 2, \end{cases}$$

and

$$c_j = \begin{cases} 0 & \text{if } j \text{ is odd and} \quad j \geq 2, \\ p_m(j) & \text{if } j \text{ is even and} \quad j \geq 2 \end{cases}$$

We also need to find $H^0(G, H^j(X, F)) \cong \mathbb{Z}^{b_j + c_j}$. When $j \geq 2$ by equation (4.3) we obtain

$$b_j = \begin{cases} \frac{1}{p}\left[\binom{m(p-1)}{j} - (p-1)p_m(j)\right] & \text{if} \quad j \text{ is odd,} \\ \frac{1}{p}\left[\binom{m(p-1)}{j} - p_m(j)\right] & \text{if} \quad j \text{ is even.} \end{cases}$$

The $E_2^{i,j}$ of the this spectral sequence is described below

$$E_2^{i,j} = \begin{cases} 0 & \text{if } j = 0, \\ \mathbb{Z}^{b_j + c_j} & \text{if} \quad i = 0 \text{ and } j > 0, \\ (\mathbb{Z}/p)^{p_m(j)} & \text{if} \quad i + j \text{ is even and } i > 0, \quad j \geq 2, \\ 0 & \text{if} \quad i + j \text{ is odd and } i > 0, \quad j \geq 2, \\ (\mathbb{Z}/p)^{a_1} & \text{if} \quad i \text{ is odd and} \quad j = 1, \\ (\mathbb{Z}/p)^{c_1} & \text{if} \quad i \text{ is even and} \quad j = 1. \end{cases}$$

Consider the Serre spectral sequence

$$\tilde{E}_2^{i,j} = H^i(G, H^j(X)) \Longrightarrow H_G^{i+j}(X)$$

associated to the fibration sequence $X \to X \times_G EG \to BG$. As it was pointed out before this sequence collapses on the E_2-term. The inclusion $X \to (X, F)$ defines a map of spectral sequences $f_r^{i,j} : E_r^{i,j} \to \tilde{E}_r^{i,j}$. Notice that $f_2^{i,j}$ is an isomorphism when $j \geq 2$. This implies that the only possible nontrivial differentials in the spectral sequence $\{E_r^{i,j}\}$ must land in $E_r^{i,1}$. On the other hand, by Lemma 4.2.1 it follows that $H_G^k(X, F) = 0$ for $k > N$; this means that there are no permanent cocycles of total degree k with $k > N$. We notice that all the differentials that end at $E_2^{2k+1,1} \cong (\mathbb{Z}/p)^{a_1}$ start at trivial groups. This implies that if $2k + 1 > N$ then $E_2^{2k+1,1}$ must be the trivial group and thus $a_1 = 0$. On the other hand, when $2 \leq j \leq N$ and $i+j$ even, the only possible nonzero differential starting at $E_2^{i,j} \cong (\mathbb{Z}/p)^{p_m(j)}$ is

$$d_j : E_j^{i,j} \to E_j^{i+j,1}.$$

If $i + j > N$, there are no permanent cocycles in $E_j^{i,j}$ and thus d_j must be injective. We note that the spectral sequence $E_r^{i,j}$ is a spectral sequence of $H^*(BG)$-modules and we can find $t \in H^2(BG)$ such that multiplication by t^k is an isomorphism $t^k : E_j^{i,j} \to E_j^{i+2k,j}$. Since d_j is a map of $H^*(BG)$ modules, we are able to choose k big enough such that $i + j + 2k > N$; it follows that $d_j : E_j^{i,j} \to E_j^{i+j,1}$ is injective for all $i + j > 0$ and its image has rank $c_j = p_m(j)$ when $i = 0$ and j is even. Moreover $\mathbb{Z}^{b_j} \subset (H^j(X, F))^G$ lies inside the image of the norm map.

We next consider the transfer map associated with the trivial subgroup $\{1\} \to G$. This map preserves the filtrations that induce the Serre spectral sequence and thus it induces a map of the corresponding spectral sequences

$$\tau_1^G : H^i(\{1\}, H^j(X, F)) \to H^i(G, H^j(X, F)).$$

Since the image of the transfer map

$$\tau_1^G : H^0(\{1\}, H^j(X,F)) \to H^0(G, H^j(X,F))$$

consists of elements in the image of the norm map, it follows that all the differentials in the Serre spectral sequence vanish on the elements that are in the image of the norm map; in particular, d_j is trivial on the summand $\mathbb{Z}^{b_j}$.

We conclude that when $2k > N$, $E_2^{2k,1} \cong (\mathbb{Z}/p)^{c_1}$, $E_3^{2k,1} \cong (\mathbb{Z}/p)^{c_1 - p_m(2)}$, ... ,$E_{N+1}^{2k,1} \cong (\mathbb{Z}/p)^{c_1 - (\sum_{j=2}^{N} p_m(j))}$. Also, if $2k+1 > N$, the group $E_{N+1}^{2k,1}$ must vanish because its elements are not permanent cocycles and there are no nontrivial differentials with target $E_{N+1}^{2k,1}$ as $E_{N+1}^{i,j} = 0$ for $j \geq 2$. This shows that $c_1 = (\sum_{j=2}^{N} p_m(j)) = p^m - 1 - m$. By the short exact sequence (4.2), we obtain $b_1 = m$.

Therefore $E_\infty^{0,1} \cong \mathbb{Z}^{(\sum_{j=1}^{N} p_m(j))} = \mathbb{Z}^{p^m - 1}$, $E_\infty^{k,1} \cong (\mathbb{Z}/p)^{(\sum_{j=k+1}^{N} p_m(j))}$ for $0 < k \leq N$ even, $E_\infty^{0,k} \cong \mathbb{Z}^{b_k + c_k}$ for $1 < k \leq N$ and the other groups are trivial.

By studying the Serre spectral sequence for the pair (X, F) with coefficients in $\mathbb{Z}/q$ with q a prime number different from p and also with rational coefficients, we can conclude that there are no extension problems and the following theorem is obtained.

Theorem 4.3.2. *Suppose that $G = \mathbb{Z}/p$ acts on $X \cong ((S^1)^{p-1})^m$ via the* *representation* $\oplus_m \rho$*. If* F *denotes the fixed point set under this action, then*

$$H^k_G(X,F) \cong \begin{cases} 0 & \text{if} \quad k=0, \\ \mathbb{Z}^{p^m-1} & \text{if} \quad k=1, \\ \mathbb{Z}^{\gamma_k} & \text{if} \quad k \text{ is even and } k>0, \\ \mathbb{Z}^{\gamma_k} \oplus (\mathbb{Z}/p)^{(\sum_{j=k}^{m(p-1)} p_m(j))} & \text{if} \quad k \text{ is odd and } k>0. \end{cases}$$

where $\gamma_k = b_k + c_k = 1/p[\binom{m(p-1)}{k} + (-1)^k(p-1)p_m(k)$*.*

Consider now the long exact sequence in cohomology associated to the pair $(X/G, F)$. Since F is a discrete set in this case, it follows at once that $H^k(X/G,F) \cong H^k(X/G)$ whenever $k \geq 2$. On the other hand, for $k=1$ it is easy to see directly that $H^1(X/G)=0$. This fact, together with [17, Proposition VII 1.1] proves the following corollary.

Corollary 4.3.3. *Suppose that* $G=\mathbb{Z}/p$ *acts on* $X \cong ((S^1)^{p-1})^m$ *via the representation* ρ_m*, then*

$$H^k(X/G;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad k=0, \\ 0 & \text{if} \quad k=1, \\ \mathbb{Z}^{\gamma_k} & \text{if} \quad k \text{ is even and } k>0, \\ \mathbb{Z}^{\gamma_k} \oplus (\mathbb{Z}/p)^{(\sum_{j=k}^{m(p-1)} p_m(j))} & \text{if} \quad k \text{ is odd and } k>0. \end{cases}$$

where $\gamma_k = b_k + c_k = \frac{1}{p}\left[\binom{m(p-1)}{k} + (-1)^k(p-1)p_m(k)\right]$*.*

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