Observables and Dynamics in Quantum Gravity

by

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A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

OCTOBER 2009

Abstract

The treatment of quantum constraint theories in physics is considered. These systems exist in many different branches of physics. One noticeable case is the quantum theory of gravity (which is a *completely* constraint system).

We contrast two modern methods of solving such quantum constraint systems, namely the method used in the String Theory - *BRST Quantization*, and that used in Loop Quantum Gravity - the *Loop Quantization*.

We investigate these methods and the quantum solution space they produce - along with proper observables - for the case of a *discrete closed string*. This system has the advantage that it lends itself to the investigation of these two methods of quantization easily - thereby highlighting some of their principle features - yet it is simple enough so to side-step the many complications that arise in more general systems.

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Acknowledgments

I would like to acknowledge the help and supervision of my supervisor, Prof. William G. Unruh. This work would not have been possible without your infinite patience and good guidance. I thank you for the privilege of working with you.

I would also like to thank the members of the gravity group for helpful discussions and comments. Thank you.

Dedication

I would like to dedicate this work to my family,

My Father, My Mother and My Sister,

whose help and support as always has and will be invaluable.

Thank you ....

Chapter 1

Introduction

In physics symmetries play a very central role. In the most general case, a physical system is described by an *action* in terms of a *Lagrangian* or a *Hamiltonian.* The only restriction that is (reasonably) applied on the action is that it should obey the symmetries of the system[1]. From such an action then, two sets of equations can be derived; one set are the *equations of motion* (EOM), and the other are the *constraint equations* (CE). This second set ensures that the solutions of the EOM obey the symmetries of the said system.

One of the important physical systems where the question of constraints is raised, is the *quantum theory of gravity.* This theory is a Generally Covariant system. A Generally Covariant (GC) theory, is a theory whose action is invariant under a general transformation of the spacetime coordinate system. The Hamiltonian of a GC turns out to be a total constraint[2]. That is to say, *the Hamiltonian of a GC is just a Gauge Transformation.* Notice that the Hamiltonian also defines the dynamics of the system. However if the Hamiltonian itself is a constraint, then the dynamics it defines can only be trivial[3]. These facts therefore, have significant implications for the *observables* of the theory ([1] and [2]-[7]).

The mathematical formulation of these difficulties, is the contexts of *Dirac's Canonical Quantization.* We will say more about this method, shortly.

The main goal of this report is to consider the exact role of the constraint equations and the various methods of analyzing constraint systems in generally covariant theories. In particular we will be looking at the effect the constraints have on the *physical observables* of the theory, and the *dynamics* associated with these observables.

[1] It is also usually required that the action be *minimal*, i.e. that it be the simplest action which can describe the system.

[2] *If* the "position", X and "momenta", Π transform as scalars with respect to the transformation parameter.

[3] This makes sense because the symmetry of the system basically implies that translation in the time is only a coordinate transformation.

Now as we mentioned a quantum theory of gravity is also a GC. However, in quantum gravity understanding the question of constraints is particularly crucial for the consistent construction and interpretation of the quantization process of gravity. As such it is reasonable to ask how do the candidates of a quantum theory of gravity propose to solve the question of observables and their dynamics in GCs.

Therefore in this report we will consider two of the major candidates of quantum gravity, namely *String Theory* and *Loop Quantum Gravity*. There are two questions which we will be specifically interested in:

1. How do the *mathematical* concepts of these theories allow for alternate formulation of quantum observables, compared to the Dirac's quantization?

2. What are the *physical* interpretations of these formulations, and how do they differ from Dirac's quantization?

However, the context in which each of these two theories - String Theory and Loop Quantum Gravity - is defined in, is much larger than the above questions require - String theory is not merely a quantum theory of gravity; it is supposed to be a *theory of everything*! And Loop Quantum Gravity (LQG) has to consider many more questions associated with the *4D* nature of spacetime which are not relevant to the questions which we wish to address.

Therefore we will attempt to cast these theories in a model. This model is the *Discrete String*. This model is useful for the following reasons:

- From the perspective of String Theory, *the model is a discrete version of the full string*. This modification - we hope - will allow for the simplification of the full string solution space. This simplification will then allow for a more focused analysis of the questions of observables and their dynamics, associated with the solution space.

- From the LQG's perspective, *the model represents D scalars on a 2D spacetime*. However in LQG, it is argued that the discreteness of spacetime is "natural" and not *assumed*. Therefore the discrete model represents the "natural" solution set for the Loop quantized string.

This then outlines the theses of this report.

In the rest of this introduction we will further discuss the questions of observability, and dynamics. Then we will review Dirac's quantization formulation which should then, *mathematically*, identify the problems associated with this quantization procedure in a GC. Then we will shortly discuss the solutions that the String quantization and the Loop quantization propose for these problems. At the end we will discuss the model - which we have proposed - in more detail, thereby motivating the remainder of this thesis.

1.1 The Question of an Observable

To start then let us first address two key questions:

- What is an *observable*; what should be defined as a *physically observable quantity*?
- What can be defined as *dynamics* for these observables and what limitations are placed on the dynamics by the action of the constraints in a GC?

First let us briefly consider the question of observables in physics and physical systems.

A physical observable is defined as some function, $f(X, \Pi)$ of the phase space variables of the system, X and Π, *which is invariant under the symmetry transformations of the system*[4].

The implications of this definition and the class of observables which can therefore be defined however, is drastically different in classical mechanics and quantum mechanics. And although this difference is not the focus of this report and we will be concerned with quantum observables only, it will help if we briefly reviewed these differences so to see (at least phenomenologically) *how the quantization of a system can greatly demand a different prescription for what may be defined as an observable and its dynamics.*

In the classical theory an observable must, again, be symmetry-invariant. However, with respect to the question of observables, there exists one crucial difference between classical and quantum theories.

This difference is due to the distinction between *measurables* and *observables.*
In the classical case, we can measure quantities that change value under a symmetry transformation, which nonetheless have physical meaning. One simple example is the momentum of a particle moving at a given speed in a given frame (of reference). The value of this quantity is certainly observer(frame)-dependent. Nonetheless, its value in any given frame has a physical meaning and significance. Therefore in classical theories, in addition to observables it is possible to use the rest of the measurable quantities to define physically "useful" quantities.

In contrast, in the quantum theory, the act of measurement is not at all straight-froward; it can be argued that a measurement "alters" a quantum system. Hence measurable quantities may not all be warranted to be physically "useful" quantities. Only a subset of these measurables *which can be predicted without measurement* are considered "useful" in the quantum theory.
However notice that quantities that are measurable but cannot be predicted *without* measurement, must be coordinate or gauge-dependent[5]. Hence we see that in the quantum theory only gauge-independent quantities are given significance and by the above definition these are the observables. Said another way, in the quantum theory only observables are physically significant and these are symmetry-invariant.

[4]Because these transformations do not represent any real physical transformations.

[5]Since $observables \subset measurables \quad \Rightarrow \quad \frac{measurables}{observables} \sim gauge - dependent.$

Therefore we see that the crucial difference between classical and quantum theories - as regards the question of observables and dynamics - is in *what may be considered as a physically useful quantity* in these theories. As we discussed, the answer to this depends to a large extend on the *physical interpretation of the mathematical structure of these theories.* This is precisely why we would like to see how the different mathematical apparatus of the two theories of quantum gravity propose to solve this problem.

However despite the above mentioned difference which allows us to construct a much larger class of "useful" quantities for classical theories, there exist specific problems associated with generally covariant theories. let us therefore say a few words on the observables of GCs.

Observables of GCs

As we mentioned in a GC the Hamiltonian vanishes; the Hamiltonian is a total constraint. The problem of vanishing Hamiltonian can be traced back to a very specific feature of a GC: not having a preferred/chosen time-parameter. In the classical theory, in most cases this problem is dealt with using boundary conditions. A boundary can be used in different ways. In one respect, the boundary terms of the action can be used as a *preferred frame of reference from which to look at the system.* In this way a time parameter can be derived and the dynamics can be defined in terms of it.

In a quantum theory however, it is not as easy to consider boundaries. In particular *a boundary usually defines some sort of classical gauge-fixing.* The consequences of such a gauge-fixing combined with quantization can be quite non-trivial. In fact in the quantization of the string via String's method, (*chap.*(2)), we will see that such a gauge fixing is done. There we will ask the validity of such a method and its possible effects on the solution space of the resulting quantum theory.

However notice that a given system does not have to *naturally* have a boundary. In the absence of a boundary, in a GC, even in the classical theory the bag of observables which can be constructed is very limited. Therefore it may be argued that the problem of producing observables in a GC is a problem for physics in general and not a specific artifact of quantization.

Hence we can easily perceive that coupling the question of "useful" physical quantities in quantum theories, to the problem of constraints in a GC, can be quite challenging to analyze and understand in many respects.

This motivates the focus of this report.

Next therefore let us briefly discuss the Dirac Quantization procedure, which is the first prescription for handling quantum constraint equations. Here we will see the major problems which arise as a result of this prescription when applied to a GC.

1.2 The Dirac Quantization

The original method of dealing with constraints in the Hamiltonian formulation of quantum mechanics (also called *The Canonical Quantization Method (CQ)*), is due to Dirac *et.al.* [8] (see also [9] and [10]). In this method, first the *the solution space* is found by solving the EOM, and then the constraints are “imposed” on the individual quantum states of the solution space (hence “ensuring” that they are invariant under the transformations due to the constraints) as follows:
Let:

$$\begin{aligned} \hat{\mathcal{C}} \quad &\sim \quad \textit{Constraint} \\ |\Psi\rangle \quad &\sim \quad \textit{The state in the solution space}\,, \end{aligned} \tag{1.1}$$

then Dirac’s *Physicality Condition* is that:

$$\hat{\mathcal{C}}|\Psi_{physical}\rangle = 0 \quad \textit{The Physicality Condition} \tag{1.2}$$

States which obey this condition are called *physical states*. By going through the entire solution space in the same way, the *physical solution space* can be isolated from the solution space itself[6].
However, in a GC, since the Hamiltonian is a total constraint, this requirement implies that:

$$|\dot{\Psi}\rangle \propto \hat{\mathcal{H}}|\Psi\rangle = 0 \tag{1.3}$$

In other words all states, representing the physical system are static!

In terms of the operators, the Dirac method is as follows. For an operator $\hat{\mathcal{O}}$, corresponding to some observable of the system, $\hat{\mathcal{O}}$ must obey the condition:

$$[\hat{\mathcal{O}}, \hat{\mathcal{C}}] \approx 0 \tag{1.4}$$

where the $\approx$ is named *weakly equal to*, and it means that the resulting operator from the commutation must be a function of the constraints. However, in a GC, with $\hat{\mathcal{H}} \approx 0$, we need therefore have:

$$\dot{\hat{\mathcal{O}}} \propto [\hat{\mathcal{O}}, \hat{\mathcal{H}}] \approx 0 \tag{1.5}$$

Hence, all observables of the theory are non-dynamical. This conclusion is equivalent to the one above concerning the states, $|\Psi\rangle$.

This is a brief description of the problems of quantizing a GC via Dirac’s method.

Now let us condensly discuss the solutions which are set forth to solve these problems, via the *new* quantization methods of two candidates for a quantum theory of gravity, namely String Theory and Loop Quantum Gravity.

[6] The original solution space would, in general, contain null-normed states and also negative-normed states. In *appendix*(E) we argue that the later problem is related to the general covariance of the theory.

1.3 The String Quantization

The canonical method of quantizing the string via the method of String theory is called the Old Covariant Quantization (OCQ)[7]. OCQ can be described as follows.
In this method, to determine the physicality of a given state, $|\Psi\rangle$, one modifies Dirac's prescription by "splitting" the constraint, and applying only part of it onto the states.
The rational behind this method is that in the quantum theory, one is interested in *expectation values*.
Hence, it is argued the constraint placed on the states:

$$\hat{\mathcal{C}}|\Psi\rangle = 0$$

should instead be replaced by the weaker condition:

$$\langle\Psi|\hat{\mathcal{C}}|\Psi\rangle = 0 \tag{1.6}$$

and notice that if:

$$\begin{aligned}
&Let \quad \hat{\mathcal{C}} \equiv \hat{\mathcal{C}}^+ + \hat{\mathcal{C}}^- \quad where \quad (\hat{\mathcal{C}}^-)^\dagger = \hat{\mathcal{C}}^+ \\
&If \quad \hat{\mathcal{C}}^-|\Psi\rangle = 0 \quad or \quad \hat{\mathcal{C}}^+|\Psi\rangle = 0 \\
&Then \quad \langle\Psi|\hat{\mathcal{C}}|\Psi\rangle = 0
\end{aligned} \tag{1.7}$$

which confirms that this condition is much weaker; we only need demand,

$$\hat{\mathcal{C}}^-|\Psi\rangle = 0 \quad or \quad \hat{\mathcal{C}}^+|\Psi\rangle = 0 \tag{1.8}$$

for *eqn*(1.6) to be satisfied - which is only "half" the constraints.

This is the main element of the quantization of constraint systems in String theory:

A new definition for a "physically useful quantity".

We will expand upon it in the part on the String quantization.

However we may ask if this method can be justified. Moreover, what is the resulting physical solution space and many more questions arise from this *new* definition for a *quantum physical solution space.*

This and the other questions which were mentioned are some of the questions which we will address in this report.

Next let us briefly describe the Loop quantization.

[7] The most general method of considering the constraints in String Theory is via the gauge-fixing method of BRST. In *appendix*(C) we argue the equivalence of this Lagrangian method to the Old-Covariant-Quantization (OCQ) which is a Hamiltonian formulation.

1.4 The LQG Quantization

The analysis of the methods of LQG are more involved, and a full analysis will be required to grasp the model of observables and dynamics which is prescribed by LQG. However, very briefly we may describe this method of quantization as follows.

Loop Quantum Gravity (LQG) is a background independent quantization of gravity. In LQG although the theory starts with a continuum (manifold), a discrete structure (lattice) emerges as the space of solutions. As a result the constraint equations effectively act on a lattice. The basic principle and method of Loop quantization however, differs significantly from the previous two method.

Here the requirement that the theory be gauge-invariant and the requirement that the state be a solution of the system (minus the constraints) are split into two requirements and applied on *two different objects*. The combination of which defines the physical useful quantity; the Dirac's *bracket*:

- The *ket* of Dirac's *bracket* belongs to a subset of the Hilbert space, called the *cylindrical solutions*:

$$|\Psi\rangle \in Cyl \tag{1.9}$$

  These provide a "dense" solution space for the quantum theory.

- The *bra* is the object, outside the Hilbert space which is gauge-invariant; it belongs to the dual space of the Cyl, i.e. Cyl^*:

$$(\Psi| \in Cyl^* \tag{1.10}$$

Then the combination, $(\Psi|\Psi\rangle$, is *posited* to carry the proper (gauge-invariant) physical observable information.
This method, also known as the *Gelfand-triple*:

$$Cyl \subseteq \mathcal{H}_{space} \subseteq Cyl^* \,, \tag{1.11}$$

is based on the idea that in a constrained system, the question of having a proper measure and that of obeying the constraints are in principle contradictory *if they were to be applied on the same object*, and therefore they should be considered separately.
As such the measure is defined for the Cyl solutions, $|\Psi\rangle$, whereas the dual of these, $(\Psi|$ are gauge-invariant. However no measure can be defined on the dual space Cyl^*. Therefore to extract the physical information we need look at the inner product between the elements of Cyl and those of Cyl^*, i.e. the $(\Psi|\Psi\rangle$, carry the physical information.

So we see that the inner-product is again highlighted in the Loop quantization as the main physical object (however due to very different physical requirements as that in the String quantization).

The quantization methods of LQG can be said to be on a firm mathematical footing. However the physical questions concerning dynamics, observability, the time evolution and so on are not clear. This is the second method which we will look at, with the focus to see where within the mathematical structure we can extract the answers to the above questions.

Now as we mentioned we will be looking to cast the above two quantization methods in a model, where we could then effectively investigate the questions regarding observability and dynamics of GCs without further convolution due to other physical problems.

This brings us to our model.

1.5 The Model: *Discrete String* vs. *2D Spacetime + Scalars*

As we mentioned the model which we will consider is the Discrete String. Different features of this model can be used in the two quantization methods which we described above. We mentioned some of these features:

- **Discreteness - Finite H space, Less Ambiguity**
  In the case of the String theory, we will assume, *a-priori* that the string is discrete (i.e that it is composed of a discrete set of points). By doing this, we hope that with a discrete theory - which would immediately mean less degrees of freedom and also avoidance of the problems associated with the Hilbert space of a continuous theory - we should be able to side-step the many issues which are not directly relevant with the problems of constraints in generally covariant theories. However the theory should still be general enough so that these questions (even though in a simplified context) can be asked and addressed.
- **Spacetime - Natural Solution Space of LQG**
  As we should see the quantization procedure of LQG asks for a certain "sampling" of the string . Hence in the LQG case the discrete structure is "emergent" and not "assumed". Therefore the discrete string emerges as the proper solution for the loop quantized string.

However notice that the treatments of the constraints in these two theories are drastically different from each other. Therefore this discretization - "natural" or assumed - will enable us to compare these formulations more transparently with each other.

Hence we see that this model has one very important feature; it can be viewed in two very different ways; This model is a *Discrete String* but it also represents a *2D Spacetime + Scalars*. This feature therefore enables us to consider both of the quantization methods which we discussed above, in the same simple context. And, in each of these, different features of the model will enable us to greatly simplify the system, so to be able to directly concentrate on the questions regarding observability and dynamics.

This concludes our introduction. The outline of the report is as follows:

1.6 Outline

In *chap.*(2) we will look at the Continuous String Theory. This chapter will mainly serve as a review. We use this chapter to extract the main features which contribute to the analysis of observables and dynamics of the string. The main focus of the chapter will therefore be to:

- Study the method of quantization used in string theory, and the specific method used to solve the constraints of the theory. But more importantly
- To analyze the *assumptions that go into the gauge-fixing of the theory* and *the possible effects of this method on the quantization of the string.*

These results will be crucial in the analysis of the following chapter. In addition we will carry out the full calculation of the solution space of the theory. This will highlight the difficulties that exist for discussing observables in the context of the full string theory, which motivates the model of the discrete string.

In *chap.*(3) we will use the findings and results of the previous chapter, to analyze the discrete string via String quantization. Within the context of this model we will try to look at the questions concerning the quantization of a GC. However, as we shall see, we will face an unanticipated problem; *the gauge-algebra of the discrete theory does not close!* This indicates that the solution space may be too restricted.

To see if it is possible to get around this problem, we therefore carry out a full analysis of the source of the non-closure of the algebra. Although the answer will turn out to be in the negative, through this analysis a number of very interesting features of the discrete theory are found. These include:

- Firstly that the "proper" discrete string theory, is *non-local.*
- Secondly, that the non-locality is not a mere artifact of the discretization which is used; i.e. other discretization of the string will also be non-local.
- And thirdly that in the quantum theory, this non-locality cannot be trivially removed.

The *non-locality* indicates very strong conditions on the theory. As a result it may be expected that the final solution space will be too restricted. One way of getting around this problem, maybe to assume that some of the symmetries are broken; *not all the constraints need be satisfied.* This conjecture is akin to the types of conjectures which are made in some of the most recent analysis done in LQG, and other theories of quantized GCs.

Interestingly enough we will see that these features have a very general origin, and therefore they in fact may be present in the discretization of any generally covariant theory. Therefore, though as a side-note, through this analysis we find that a strong statement may be made with regards to any GC which possesses a discrete structure.

At the end of the chapter, for completion we will carry out the full analysis of the solution space of the discrete string. Here we will explicitly confirm that the resulting space is too restricted and therefore only contains the trivial solution.

This fact, along with other questions regarding the gauge-fixing method of the string quantization, then motivates the analysis of the last chapter.

In *chap.*(4) we consider the Loop Quantization method. To do so, first we will consider the full Loop Quantum Gravity. The focus will be to 1). Briefly discuss the method of this quantization procedure and 2). To discuss the questions which arise regarding observability in this method and the dynamics associated with them.

In the next section of the same chapter, then we apply what we have learned to the string. We will carry out the full loop quantization of the discrete string. First we will lay out all the similarities which exist between our model's and the full LQG's solution spaces. We will see that the states of the resulting solution space - what we will call *charge-networks* - are in complete analogy to the *spin-networks* of LQG.
These similarities will utilize a comparative analysis of the effects of the constraint operator algebras of the two theories. Within this model, however we perform a *full analysis of the action of the constraint operators on the solutions.* We will fully discuss the action of the *Diff* constraint - something which is rarely done for LQG. This will clarify many of the formal arguments which exist in the Loop quantization of systems. In particular, it will precisely identify the role and the action of the *Scalar* constraint, which is essential in describing the dynamics of the system. These findings therefor enable us to conduct a full analysis of the possible solution space, the notion of an observable, and dynamics, in the context of a fully Loop quantized theory.

As we mentioned, many of the features which exist in the full LQG exist in this model - which is however far simpler to analyze - therefore the results which are found through the analysis of the discrete string, can be generalized to the full theory of LQG, and for that matter to any GC which is to be quantized via the method of LQG.

Through this analysis therefore we manage to clearly analyze and clarify some of the fundamental notions that exist in LQG. And in particular we will be able to clearly discuss the questions of observables and dynamics in this theory.

In *chap.*(5) we will summarize the findings of the different chapters. We will end with a few concluding remarks as to:

- The implications of these modern methods of quantization of systems, and
- The fate of the question of observability in generally covariant theories.

Chapter 2

The Continuous String Case

In this chapter we examine and discuss the *Closed Continuous Bosonic String Theory*. The main purpose of this is to outline the different features of the theory as concerned with the gauge-choices and the application of the constraints on the solution space which results in the physical solution space, via the method of the string theory ([12]-[15]).
As such we will first outline the main theory, and the EOM and Constraints. Then we look at its Gauge freedom and contrast it against the Coordinate-choice- freedom of the theory, thereby identifying the exact assumptions and conditions that go into the solution space which is used to describe bosonic strings. And finally we look at the Virasoro constraint algebra to investigate the possible physical solution space of the theory.
In *chap.*(3) we use what we analyze in this chapter on the discrete string model to investigate the corresponding physical solution space that can be constructed, were we to quantize the theory using the methods of this chapter.

2.1 Initial Setup

The action of the Continuous String is a generalization of the action of the relativistic free particle; the *Area* that the string spans as it evolves through time is required to be minimal, just as the *Length* of the path a particle travels through time is minimized. The following action, known as the Polyakov action (which is a "generalization" of the Nambu-Goto action) meets this requirement:

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^2\lambda\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu \\ g_{ab}(\sigma,t) &= \gamma^2\begin{pmatrix} -N^2+M^2 & M \\ M & 1 \end{pmatrix} \end{aligned} \tag{2.1}$$

Where σ and t are parameterization on the string, and the N and M are the *lapse* and the *shift* functions associated with this parameterization, *fig.*(2.1).

Notice that this theory is diffeomorphism invariant (i.e. invariant under a general coordinate transformation). This produces two of the three symmetries of the theory. The

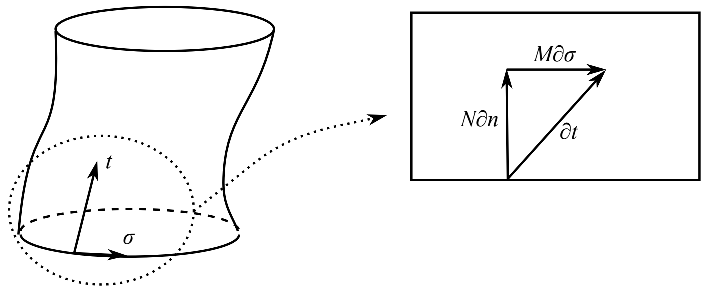

Figure 2.1: The string is parametrized by σ in the *spacial* direction and by t in the *temporal* direction. n is the normal to σ. The N and M are the *lapse* and *shift* functions, respectively.

other symmetry comes about because the theory is $2 - Dimensional$, and as is known in this dimension all metrics are conformally flat. In other words we should be able to recast the general metric g_{ab} above, into the form $\gamma^2\eta_{ab}$, where the γ^2 is the conformal factor. Therefore the choice of this factor produces the 3rd symmetry of the system.
So we already know that we should anticipate a constrained theory. Let us see how string theory handles such theories. First let us look for the EOM and the Constraints which we should find from this theory.

2.1.1 The EOM and The Constraints

First following the method of Dirac, let us identify all the constraints (primary and secondary[1]) of this theory, whereby then we can define the Hamiltonian and the correct EOM, and also to see that it is a fully constraint system (i.e. $\mathcal{H} \approx 0$).

Looking at the momenta (this is where we anticipate primary constraints to be found):

$$\Pi_\mu(\sigma, t) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{1}{N}\left(\dot{X}_\mu - M X'_\mu\right)(\sigma, t) \tag{2.2}$$

$$\left.\begin{aligned} P_N &= \frac{\partial \mathcal{L}}{\partial \dot{N}} \cong 0 \\ P_M &= \frac{\partial \mathcal{L}}{\partial \dot{M}} \cong 0 \end{aligned}\right\} \qquad The\, Primary\, Constraints \tag{2.3}$$

From this we can rewrite the Lagrangian and define the full Hamiltonian:

[1]Fortunately, as we shall see this theory only has First Class constraints which simplifies things greatly.

$$\begin{aligned}
\mathcal{L}(t) &= \frac{1}{2}\int \mathrm{d}\sigma N\big(\Pi^2 - X'^2\big)(\sigma,t) \qquad (2.4)\\
\mathcal{H}(t) &= \frac{1}{2}\int \mathrm{d}\sigma \left(N\big(\Pi^2 + X'^2\big) + 2M\Pi^\mu X'_\mu\right)(\sigma,t)\\
\mathcal{H}'(t) &= \mathcal{H} + \int \mathrm{d}\sigma \left(\mathcal{U}^1 P_N + \mathcal{U}^2 P_M\right)(\sigma,t) \qquad (2.5)
\end{aligned}$$

Next we look for secondary constraints by assuring that the primary constraints are time-independent, we find the secondary (and the main) constraints of the system:

$$\begin{array}{rclcl}
\dot{P}_N = \{P_N, \mathcal{H}\} &=& -\frac{1}{2}\left(\Pi^2 + X'^2\right)(\sigma,t) &\equiv& C^0(\sigma,t) \approx 0\\
\dot{P}_M = \{P_M, \mathcal{H}\} &=& -\Pi^\mu X'_\mu(\sigma,t) &\equiv& C^1(\sigma,t) \approx 0
\end{array} \qquad (2.6)$$

So we immediately see that the Hamiltonian is a total constraint:

$$\mathcal{H}' = \int \mathrm{d}\sigma \left(NC^0 + MC^1 + \mathcal{U}^1 P_N + \mathcal{U}^2 P_M\right) \approx 0 \qquad (2.7)$$

Now to check whether there exist any more constraints we need consider the Poisson bracket of the new constraints. However since these are distributions we should actually look at the corresponding *smeared* Gauges[2].
These constraints therefore define the following two gauge transformations:

$$\begin{aligned}
G^0[\xi] &\equiv \frac{1}{2}\int \mathrm{d}\sigma\, \xi\big(\Pi^2 + X'^2\big) \qquad (2.8)\\
G^1[\eta] &\equiv \int \mathrm{d}\sigma\, \eta \Pi^\mu X'_\mu \qquad (2.9)
\end{aligned}$$

These form the following gauge algebra:

$$\begin{aligned}
\{G^0[\xi], G^0[\eta]\} &= G^1[\eta'\xi - \xi'\eta] \equiv G^1[\mathcal{L}_\eta \xi]\\
\{G^0[\xi], G^1[\eta]\} &= G^0[\eta'\xi - \xi'\eta] \equiv G^0[\mathcal{L}_\eta \xi]\\
\{G^1[\xi], G^1[\eta]\} &= G^1[\eta'\xi - \xi'\eta] \equiv G^1[\mathcal{L}_\eta \xi]
\end{aligned} \qquad (2.10)$$

This indicates that the algebra of the constraints (or the gauge algebra) is closed, which confirms that there does not exist anymore constraints in this theory.[3] Hence *eqn.*(2.7) is the most general Hamiltonian for this system. From which we can derive the equations of motion for the system:

$$\begin{aligned}
\dot{X}^\mu &= \{X^\mu, \mathcal{H}'\} = N\Pi^\mu + MX'^\mu\\
\dot{\Pi}^\mu &= \{\Pi^\mu, \mathcal{H}'\} = NX^\mu + M\Pi'^\mu
\end{aligned} \qquad (2.11)$$

[2] See *appendix*(B) for a full discussion

[3] This is the very point to which we will come back in *sec.*(3.2) of *chap.*(3), where we will see that it is this very assumption that, "one should deal with the smeared version of the constraints rather than the distributions themselves", which provides the continuous string theory with a closed constraint algebra.

Now to solve these, or their Lagrangian-variants, in the usual method of string theory, one first makes a gauge choice, where $g_{ab} \rightarrow \eta_{ab}$. The justification for this is that (as mentioned) gauges correspond to coordinate (and conformal factor)-choice degrees of freedom of the theory, and as such they can be fixed to simplify the EOM and the corresponding Constraint equations.
Two questions arise from this choice however:

- First: Is this true; do the gauge and the coordinate choices correspond exactly?
- Is this a proper procedure? In other words, is this gauge-fixing an appropriate step, in lieu of the future quantization of the system?

We will address the first of these in the next section.

However, the second question is related to the quantization-anomalies.[4] This method of quantization can be contrasted to other methods which we review briefly in *appendix*(C). However one particular example where no gauge-fixing takes place is exactly in the Loop quantization which we will look at in *chap.*(4). There then we may compare the results found here to those suggested by the loop quantization.

2.2 The Transformations of the Action

Recall that the entity that should remain invariant under the gauge and coordinate transformations is the action. So let us look at these transformations and try to find the relation between them. The ultimate goal is to realize that the gauge and the coordinate transformations are "equivalent" so that gauge fixing conditions which are used in the string quantization procedure, for finding the solution space of the string can be justified from (at least) this perspective.

2.2.1 Gauge Transformation of the Action[5]

We now consider the transformation of the action under a general gauge transformation:

$$G[\xi,\eta] = G^0[\xi] + G^1[\eta] \tag{2.12}$$

The variation can be given by:

$$\delta_G S_{cov} = \int \mathrm{d}^2\lambda \delta_G \left(\Pi\dot{X} - \mathcal{H}\right) = \int \mathrm{d}^2\lambda \left\{\Pi\dot{X} - \mathcal{H}, G\right\} \tag{2.13}$$

[4]It is found that in gauge theories, often after quantization, having first gauge fixed the theory, the final solutions are not necessarily gauge-invariant, these are called quantization anomalies. For example the requirement that the Bosonic string theory be embedded in a 26-Dim spacetime is the requirement so that the theory recovers form one such anomaly.

[5] The variation of a variable, *delta(Variable)* always has a subscript corresponding to a coordinate transformation, $\delta_C X$ or a gauge transformation, $\delta_G X$. This is not to be confused with the infinitesimal *amount* of spatial transformation δ which has no subscript.

Now we have:

$$\begin{aligned}\mathcal{H} &= G^0[N] + G^1[M] &(2.14)\\ \Rightarrow \delta_G\mathcal{H} &= \big\{G^0[N] + G^1[M], G[\xi,\eta]\big\} \\ Using \quad eqn.\,(2.10) & \\ &= G^0[\mathcal{L}_\xi M + \mathcal{L}_\eta N] + G^1[\mathcal{L}_\xi N + \mathcal{L}_\eta M] &(2.15)\\ where: \quad \mathcal{L}_\xi\eta &\equiv \xi'\eta - \eta'\xi &(2.16)\end{aligned}$$

Also (for a full calculation of this see *appendix*(A)):

$$\int \mathrm{d}t \int \mathrm{d}\sigma \big\{\Pi\dot{X}, G^0[\xi]\big\} = G^0[\dot{\xi}] \tag{2.17}$$

$$\int \mathrm{d}t \int \mathrm{d}\sigma \big\{\Pi\dot{X}, G^1[\eta]\big\} = G^1[\dot{\eta}] \tag{2.18}$$

$$\Rightarrow \int \mathrm{d}t \int \mathrm{d}\sigma \delta_G\big(\Pi\dot{X}\big) = \int \mathrm{d}t \int \mathrm{d}\sigma \big\{\Pi\dot{X}, G\big\} = \int \mathrm{d}t\big(G^0[\dot{\xi}] + G^1[\dot{\eta}]\big) \tag{2.19}$$

Therefore the total effect of the gauge transformation on the action becomes:

$$\begin{aligned}\delta_G S_{cov} &= \big\{S_{cov}, G^0[\xi] + G^1[\eta]\big\} &(2.20)\\ Gives &: \\ \mathcal{H} &= G^0[N] + G^1[M] \rightarrow G^0[N + \mathcal{L}_\xi M + \mathcal{L}_\eta N - \dot{\xi}] \\ &\qquad + G^1[N + \mathcal{L}_\xi N + \mathcal{L}_\eta M - \dot{\eta}] &(2.21)\end{aligned}$$

We can conclude two results from this calculation:

- That the transformation is indeed a gauge whose effect on the action is to redefine the Lagrange multipliers N and M.
- The Gauge transformation (seems) to be identical to coordinate transformations (which effectively change the metric, and hence redefine N and M.)

This second result is the one which we will investigate next: i.e. we would like to know if the gauge transformations are identical to general coordinate transformations[6].
We therefore next look at the coordinate transformations of the action.

[6]Note that we indeed know that the *source* of the gauge trns. are the coordinate choice freedom of the theory, however in order for a set of (X^μ, Π^μ) to be gauge invariant a certain condition is necessary; *they need to obey the equation of motion.* This may seem obvious; i.e. we are stating the well-known classical statement that *Two solutions of the eom which are obtained by different choice of coordinate systems, will be related by a gauge transformation.* Or said another way: *If two solution of the eom are related by a gauge transformation, they simply correspond to two choices of coordinate system in which the eom where solved. This in turn corresponds to two different observers.* But now note that the obviousness of the above statement relies on the requirement that (X^μ, Π^μ) be solutions of the *classical* eom. This corresponds to saying (X^μ, Π^μ) are *on-shell.* When however we wish to quantize the theory the situation changes entirely. One way of regarding the quantization of a given system is to allow (X^μ, Π^μ) to *go off-shell*; this is the Feynman description of quantization: It is not just the classical solutions that are allowed; all solutions are allowed and they are weighted-summed. Hence, here with this perspective if one where to choose a

2.2.2 The Coordinate Transformation of the Action

Under a general coordinate transformation:

$$\vec{\lambda} = (t, \sigma) \to \tilde{\lambda} = \vec{\lambda} + \vec{\zeta} \qquad where \qquad \vec{\zeta} = (\zeta^0, \zeta^1) \equiv (\Delta, \delta)\,, \tag{2.22}$$

the metric transforms as:

$$\begin{aligned} \delta_\zeta g_{ab} &= \mathcal{L}_\zeta g_{ab} \\ &= g_{ac}\partial_b \zeta^c + g_{cb}\partial_a \zeta^c + \zeta^c \partial_c g_{ab}\,. \end{aligned} \tag{2.23}$$

From this we find:

$$\begin{aligned} \delta_C g_{11} &= \delta\left(\gamma^2\right) = -\left[2\gamma^2\left(M\Delta' + \delta'\right) + \left(\dot{\overline{(\gamma^2)}}\Delta + \overline{(\gamma^2)}'\delta\right)\right] \\ \delta_C g_{01} &= \delta\left(\gamma^2 M\right) = -\left[\gamma^2\left(\left(M\dot{\Delta} + \dot{\delta}\right) + \left(\left(-N^2 + M^2\right)\Delta' + M\delta'\right)\right)\right. \\ &\qquad \left. + \left(\dot{\overline{(M\gamma^2)}}\Delta + \overline{(M\gamma^2)}'\delta\right)\right] \\ \delta_C g_{00} &= \delta\left(\gamma^2\left(-N^2 + M^2\right)\right) = -\left[2\gamma^2\left(\left(-N^2 + M^2\right)\dot{\Delta} + M\dot{\delta}\right)\right. \\ &\qquad \left. - \left(\dot{\overline{(\gamma^2\left(-N^2 + M^2\right))}}\Delta + \overline{(\gamma^2\left(-N^2 + M^2\right))}'\delta\right)\right] \end{aligned} \tag{2.24}$$

Rewriting this we can calculate the induced change on N and M:

$$\begin{aligned} \delta_C N &= -N\big(\dot{\Delta} - \delta' - 2M\Delta'\big) - \big(\Delta\dot{N} + \delta N'\big) \\ \delta_C M &= -\big[(M\dot{\Delta} + \dot{\delta}) - (N^2 + M^2)\Delta' - M\delta'\big] - \big(\Delta\dot{M} + \delta M'\big) \end{aligned} \tag{2.25}$$

We are now in the position to compare the Gauge transformations to the Coordinate transformations.

2.2.3 Gauge vs. Coordinate Transformations

If we consider the Coordinate Transformation:

$$\zeta = (\Delta, \delta) \Rightarrow \begin{cases} \delta(t) = \Delta \\ \delta(\sigma) = \delta \end{cases} \tag{2.26}$$

we see that these correspond to one gauge transformation along the t direction, and one along σ direction.
The later transformation, has a direct correspondence with the gauge transformations, since the σ transformations correspond to spatial gauge transformations, and these are generated

certain coordinate system in which to solve the eom and then quantize the system, the solutions obtained may very well not be gauge-related to another set of solutions obtained by another choice of the coordinate system. This is precisely what we wish to examine: i.e. can one relax the requirement that (X^μ, Π^μ) necessary be solutions of the eom. For this we need to draw up exactly how the C-Transformations and the G-transformations correspond.

by $G^1[\eta]$ hence for the equality we simply need: $\eta = \delta$. The second one is slightly more complicated because the t direction is defined by:

$$\partial_t \equiv \tilde{N}\partial_{\hat{n}} + \tilde{M}\partial_\sigma, \tag{2.27}$$

and since the gauge transformations along the t direction are defined by $\mathcal{H}[\tilde{N}, \tilde{M}]$, we see that we need $\tilde{N} = \Delta N$ and $\tilde{M} = \Delta M$ for the coordinate transformations, and using *eqn.*(2.14), this corresponds to a gauge transformation of:

$$\mathcal{H}[\Delta N, \Delta M] = G^0[\Delta N] + G^1[\Delta M] .$$

Together then we need:

$$G[N, M, \Delta, \delta] = G^0[\Delta N] + G^1[\Delta M + \delta] . \tag{2.28}$$

It can then easily be verified that using the above correspondence, we have:

$$\begin{cases} \delta_C N = \delta_G N \, , \\ \delta_C M = \delta_G M \, . \end{cases}$$

So we have shown that the gauge transformations and the coordinate transformations can be related; a given gauge choice (for example the conformal flat gauge, which we will use in *sec.*(2.3.2) to find the solution space) corresponds to a coordinate choice[7].

We are ready to discuss the solution space of the string; i.e. we have provided a "plausible argument" for gauge-fixing the theory. This is nevertheless not a proof and only the technique used in the String quantization. As we mentioned we will look at the Loop quantization in the last chapter where this assumption is not made.

However before we proceed let us summarize the Gauge-Coordinate relation which we found. In addition let consider the individual action of these transformations on the scalar field variables, X^μ and Π^μ. This will further highlight the concern one may have for gauge-fixing any theory *prior to quantization*.

[7] However we must point out that while a gauge choice fixes the *unit of length* and *relative angles*, does not fix the *origin of the coordinate system*. Hence a gauge-choice does not fully fix the coordinate system; there is a slight mismatch. This however is a mere technicality, which we mention for completeness, and it does not affect or relate to our calculations. For a quick look at this you may wish to consult [11].

2.2.4 The G vs. C Transformation of X and Π

Another question which we should ask is whether the $GT \to CT$ correspondence is sufficient to also match the $\delta_C(X \text{ and } \Pi)$ and $\delta_G(X \text{ and } \Pi)$. This is the question that we will next consider.

So far we can summarize the way the two transformations compare as follows, *table*(2.1):

Gauge Tr. (GT)	Coordinate Tr. (CT)
$G[\Delta,\delta] = G^0[\Delta N] + G^1[\Delta M + \delta]$	$\begin{cases} t & \to & \tilde{t} = t + \Delta \\ \sigma & \to & \tilde{\sigma} = \sigma + \delta \end{cases}$
$S \to S$ $\mathcal{H}[N,M] \to \mathcal{H}[\tilde{N},\tilde{M}]$	$S \to S$ $g_{ab}[N,M] \to g_{ab}[\tilde{N},\tilde{M}]$
$\begin{cases} \delta_G N & = & \mathcal{L}_{(\Delta N)}M + \mathcal{L}_{(\Delta M+\delta)}N - \dot{\overline{(\Delta N)}} \\ \delta_G M & = & \mathcal{L}_{(\Delta N)}N + \mathcal{L}_{(\Delta M+\delta)}M - \dot{\overline{(\Delta M + \delta)}} \end{cases}$	$\begin{cases} \delta_C N & = & -N(\dot{\Delta} - \delta' - 2M\Delta') - (\Delta\dot{N} + \delta N') \\ \delta_C M & = & -[(M\dot{\Delta} + \dot{\delta}) - (N^2 + M^2)\Delta' - M\delta'] \\ & & -[\Delta\dot{M} + \delta M'] \end{cases}$

Table 2.1: The gauge to coordinate transformations correspondence; with respect to the spacetime variables degrees of freedom, N and M there exists an exact correspondence between these two transformations.

Now the gauge, G and coordinate C, transformation of a scalar field, $\mathcal{F}$ can be given by:

$$\delta_G \mathcal{F} = \{\mathcal{F}, G\} \qquad\qquad \delta_C \mathcal{F} = \dot{\mathcal{F}}\Delta + \mathcal{F}'\delta$$

and since X^μ is a scalar, we have:

$$\delta_G X^\mu = \{X^\mu, G\} = \Delta(N\Pi^\mu + MX'^\mu) + \delta X'^\mu \qquad \delta_C X^\mu = \Delta\dot{X}^\mu + \delta X'^\mu \tag{2.29}$$

Notice first that we used used *eqn.*(2.28). However in order to equate these two expressions we, in addition will require to use one of the EOM, *eqn.*(2.11):

$$\dot{X}^\mu = \{X^\mu, \mathcal{H}[N,M]\} = N\Pi^\mu + MX'^\mu \tag{2.30}$$

Therefore at least one of the EOM are required to have $\delta_C X = \delta_G X$.
Now let us consider the variation of Π^μ. Notice that Π^μ is not a scalar[8]. Hence even to evaluate the two variations of it we need resort to the same EOM as was used for the case of X^μ. In effect we are comparing the following two expressions:

$$\delta_G \Pi^\mu = \{\Pi^\mu, G\} = \overline{(\Delta\,(NX'^\mu + M\Pi^\mu))}' + \overline{(\delta\Pi^\mu)}' \qquad \delta_C \Pi^\mu = \delta_C\left(\frac{1}{N}(\dot{X}^\mu - MX'^\mu)\right) \tag{2.31}$$

[8]Due to the $\frac{1}{N}$ factor in the definition for $\Pi = \frac{1}{N}(\dot{X} - MX')$.

Carrying out the calculation and writing everything in terms of X^μ we find (we omit the μ index for simplicity):

$$\delta_G\Pi = \dot{X}\overline{\left(\frac{\Delta M+\delta}{N}\right)}' + X'\overline{\left(\Delta N - M\left(\frac{\Delta M+\delta}{N}\right)\right)}' \\ + \dot{X}'\left(\frac{\Delta M+\delta}{N}\right) + X''\left(\Delta N - \frac{M}{N}(\Delta M+\delta)\right) \tag{2.32}$$

$$\begin{aligned}\delta_C\Pi =& \dot{X}\left(\frac{1}{N^2}\left(\dot{\overline{\Delta N}} + \delta N' - N\delta' - 2M\Delta'\right)\right) \\ &+ X'\left(-\frac{M}{N^2}\left(\dot{\overline{\Delta N}} + \delta M' - M\delta'\right) + \frac{1}{N}\left(\dot{\overline{\Delta M+\delta}} + \delta M' - M\delta' - \left(N^2+M^2\right)\Delta'\right)\right) \\ &+ \dot{X}'\frac{(\delta - M\Delta)}{N} - X''\frac{M\delta}{N} + \ddot{X}\frac{\Delta}{N}\end{aligned} \tag{2.33}$$

Comparing the terms in the two expressions above we see that the $\ddot{X}^\mu$ term only appears in $\delta_C X^\mu$[9]. And it cannot be written in terms of the others *without the use of the second EOM*:

$$\left.\begin{aligned}\dot{\Pi} &= \overline{(NX' + M\Pi)}' \\ \dot{X} &= N\Pi + MX'\end{aligned}\right\} \quad \Rightarrow \quad \dot{\overline{\left(\frac{\dot{X} - MX'}{N}\right)}} = \overline{\left(NX' + \frac{M}{N}(\dot{X} - MX')\right)}' \tag{2.34}$$

This then leads to the relation:

$$\ddot{X}\left(\frac{1}{N}\right) + \dot{X}'\left(\frac{-2M}{N}\right) + X''\left(\frac{M^2-N^2}{N}\right) \\ - \dot{X}\left(-\dot{\overline{\left(\frac{1}{N}\right)}} + \overline{\left(\frac{M}{N}\right)}'\right) - X'\left(\dot{\overline{\left(\frac{M}{N}\right)}} + \overline{\left(\frac{N^2-M^2}{N}\right)}'\right) = 0 \tag{2.35}$$

With these, the expressions for the gauge and coordinate transformations of Π^μ become:

$$\begin{aligned}\delta_G\Pi^\mu &= (\Delta\dot{\Pi} + \delta\Pi') + \Pi(\Delta' M + \delta') + X'(\Delta' N) &(2.36)\\ \delta_C\Pi^\mu &= (\Delta\dot{\Pi} + \delta\Pi') + \Pi(\dot{\Delta} - \delta' - 2M\Delta' + 2\Delta\frac{\dot{N}}{N} + 2\delta\frac{N'}{N}) \\ &\quad + X'\frac{1}{N}\left(M\dot{\Delta} + \dot{\delta} - (N^2+M^2)\Delta' - M\delta' + 2\dot{M} + 2\delta M'\right) &(2.37)\end{aligned}$$

Therefore we can conclude, that the only way we can hope to equate the two variations of the Π^μ is to resort to *both* equations of motion. i.e. this verifies the classical requirement that only the solutions of the EOM are gauge invariant.

[9] In fact the only reason we have $\dot{\overline{Variable}}$ in $\delta_G X^\mu$ is that we have used the EOM for X^μ, otherwise the gauge transformation is *alien* to the ∂_t; it only knows of the Poisson Bracket $\{X^\mu, G\}$ which (in the case of our Hamiltonian) can only lead to ∂_σ.

This is the point of concern; the GT of field variables can be considered a CT only if Π and X are the solutions of the classical EOM. Said differently: it is only for *on-shell* values of the X and Π that the GT of these correspond to a different choice of the coordinate systems. But in quantization, for example in the path integration this is exactly the point which we depart from: X and Π do not just *follow the classical solution orbits; they follow* *all orbits*. So whether we can actually gauge-fix the theory in lieu of quantization is no longer transparent. Indeed due care must be given, and more often than not one does run into anomalies.

2.3 The Solution Space

With these remarks and having emphasized that the gauge-fixing conditions imply very strong assumptions for the theory, we proceed to the solution space making the said gauge choices. To do so, however let us note an interesting relation which exists between the constraints and the energy momentum tensor. This will help us in the next section where the constraints are written in this language (rather than the Dirac's canonical language).

2.3.1 T_{ab} vs. $C^\alpha(\sigma,\sigma^{'})$

The relation that we seek comes from the two alternate definitions of the action:

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^2\lambda\sqrt{-g}g^{cd}\partial_c X^\mu \partial_d X_\nu \\ S_{cov} &= \int \mathrm{d}t \int \mathrm{d}\sigma \Big(\Pi\dot{X} - \mathcal{H}[N,M]\Big) \end{aligned} \tag{2.38}$$

From these we can define T_{ab} and $C(\sigma,\sigma^{'})$:

$$T_{ab} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = \frac{-2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g^{ab}} = \partial_a X^\mu \partial_b X_\nu - \frac{1}{2}g_{ab}g^{cd}\partial_c X^\mu \partial_d X_\mu \tag{2.39}$$

$$\begin{aligned} C^0(\sigma,\sigma^{'}) &= -\big\{P_N,\mathcal{H}\big\} = \frac{\partial\mathcal{H}}{\partial N(\sigma,\sigma')} = \frac{1}{2}\left(\Pi^2 + X'^2\right) \\ C^1(\sigma,\sigma^{'}) &= -\big\{P_M,\mathcal{H}\big\} = \frac{\partial\mathcal{H}}{\partial M(\sigma,\sigma')} = \Pi^\mu X'_\mu \end{aligned} \tag{2.40}$$

where, again P_N and P_M are the primary constraints:

$$\begin{cases} P_N \equiv \dfrac{\partial\mathcal{L}}{\dot{N}} \\ P_M \equiv \dfrac{\partial\mathcal{L}}{\dot{M}} \end{cases} \tag{2.41}$$

Now let us relate the constraints and the energy-momentum tensor. Notice that since N and M exist only in g^{ab}:

$$g_{ab} = \gamma^2 \begin{pmatrix} -N^2+M^2 & M \\ M & 1 \end{pmatrix} \quad \rightarrow \quad g^{ab} = \frac{1}{N^2\gamma^2}\begin{pmatrix} -1 & M \\ M & N^2-M^2 \end{pmatrix} \tag{2.42}$$

we can write:

$$\frac{\partial}{\partial N} = \frac{\partial g^{ab}}{\partial N}\frac{\partial}{\partial g^{ab}} \equiv \mathcal{A}^{ab}\frac{\partial}{\partial g^{ab}} \tag{2.43}$$

$$\frac{\partial}{\partial M} = \frac{\partial g^{ab}}{\partial M}\frac{\partial}{\partial g^{ab}} \equiv \mathcal{B}^{ab}\frac{\partial}{\partial g^{ab}} \tag{2.44}$$

Where:

$$\mathcal{A}^{ab} \equiv \frac{\partial g^{ab}}{\partial N} = \frac{1}{N^3\gamma^2}\begin{pmatrix} 1 & -M \\ -M & 2N^2 \end{pmatrix} \tag{2.45}$$

$$\mathcal{B}^{ab} \equiv \frac{\partial g^{ab}}{\partial M} = \frac{1}{N^2\gamma^2}\begin{pmatrix} 0 & 1 \\ 1 & -2M \end{pmatrix} \tag{2.46}$$

Now if we use the obvious equality:

$$S = S_{cov} = \int \mathrm{d}t \int \mathrm{d}\sigma \Big(\Pi\dot{X} - \mathcal{H}\Big)$$

we can write:

$$T_{ab} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{cov}}{\delta g^{ab}} = \frac{-2}{\sqrt{-g}}\frac{\partial}{\partial g^{ab}}(\Pi\dot{X} - \mathcal{H}) \tag{2.47}$$

Therefore we have:

$$C^{\alpha} = \frac{\partial \mathcal{H}}{\partial N^{\alpha}} = \alpha^{ab}\frac{\partial \mathcal{H}}{\partial g^{ab}} = \alpha^{ab}\Big(\frac{\partial}{\partial g^{ab}}\Pi\dot{X} + \frac{1}{2}\sqrt{-g}T_{ab}\Big) \tag{2.48}$$

Where:

$$\alpha = \begin{cases} 0 \\ 1 \end{cases} \qquad N^{\alpha} = \begin{cases} N \\ M \end{cases} \qquad \alpha^{ab} = \begin{cases} \mathcal{A}^{ab} \\ \mathcal{B}^{ab} \end{cases}$$

This establishes the relation between the energy momentum tensor and the constraints. Writing the form of T_{ab} explicitly we find:

$$\begin{cases} T_{00} = (N^2 + M^2)\frac{1}{2}(\Pi^2 + X'^2) + 2NM(\Pi X') \\ T_{01} = M\frac{1}{2}(\Pi^2 + X'^2) + N(\Pi X') \\ T_{11} = \frac{1}{2}(\Pi^2 + X'^2) \end{cases} \tag{2.49}$$

which gives:

$$T_{ab} = \begin{pmatrix} (N^2 + M^2)C^0 + 2NMC^1 & MC^0 + NC^1 \\ MC^0 + NC^1 & C^0 \end{pmatrix} \tag{2.50}$$

Therefore we see that the energy momentum tensor is also just a combination of the constraints. In fact, it can be considered a certain representation (in complex conformal coordinates) of the constraints. This representation is the one which is used in string theory.

With this, we are now ready to consider the solution space of the theory.

2.3.2 Solving the EOM

So now we come to describing the solution space which is prescribed by the gauge-fixing method of string theory. The Gauge fixed equations of motion are very simple[10]:

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b X^\mu) \quad \xrightarrow{\underline{G.\,Fixing}} \quad \eta^{ab}\partial_a\partial_b X^\mu \tag{2.51}$$

$$\Rightarrow \quad \ddot{X}_n - X_n'' = 0 \quad is\,a\,wave\,equation. \tag{2.52}$$

This equation has the wave equation solution:

$$\begin{aligned} X^\mu(\sigma,t) &= x_0^\mu + \frac{p_0^\mu}{L}t + \left(\frac{\hbar}{2L}\right)^{\frac{1}{2}} \sum_{m\neq 0} \frac{1}{\omega_m} e^{i\omega_m t}\left(\alpha_m^\mu e^{i\kappa_m \sigma} + \beta_m^\mu e^{-i\kappa_m\sigma}\right) \\ \Pi^\mu(\sigma,t) &= \frac{p_0^\mu}{L} + i\left(\frac{\hbar}{2L}\right)^{\frac{1}{2}} \sum_{m\neq 0} e^{i\omega_m t}\left(\alpha_m^\mu e^{i\kappa_m \sigma} + \beta_m^\mu e^{-i\kappa_m\sigma}\right) \end{aligned} \tag{2.53}$$

Where L is the length of the sting, and:

$$\omega_m^2 - \kappa_m^2 = 0 \quad and \quad \kappa_m = \frac{2m\pi}{L}\,.$$

These must satisfy the proper commutation relations:

$$\begin{aligned} [X^\mu(\sigma,t),\Pi^\nu(\sigma',t)] &= \frac{1}{L}\left[x_0^\mu, p_0^\nu\right] \\ &\quad + i\left(\frac{\hbar}{2L}\right)\sum_{lk\neq 0}\frac{e^{i(\omega_l+\omega_k)t}}{\omega_l}\left[\alpha_l^\mu e^{i\kappa_l\sigma} + \beta_l^\mu e^{-i\kappa_l\sigma}\,, \alpha_k^\mu e^{i\kappa_k\sigma'} + \beta_k^\mu e^{-i\kappa_k\sigma'}\right] \end{aligned} \tag{2.54}$$

with:

$$\begin{aligned} &[\alpha_l^\mu, \alpha_k^\nu] = \omega_l\delta_{l,-k}\delta^{\mu\nu} \\ &[\beta_l^\mu, \beta_k^\nu] = \omega_l\delta_{l,-k}\delta^{\mu\nu} \qquad l,k\neq 0 \\ &[\alpha_l^\mu, \beta_k^\nu] = 0 \end{aligned} \tag{2.55}$$

This commutation relation is satisfied.
We are now in the position to look at the Fock-Space solution and the corresponding Virasoro constraints.

[10]We go through the derivation here quickly, where the results are well-known. However in the discrete case we show and derive every step carefully. Not so surprisingly the steps are very similar and the basic physics is in fact identical.

2.3.3 The Fock Space

We can represent the most general Fock solution vector, $|\Psi_{A,B}(K^\mu)\rangle$ as:

$$|\Psi_{A,B}(K^\mu)\rangle = \prod_{a,b=1}^{\infty} |\Psi_{A_a,B_b};K\rangle \tag{2.56}$$

where :

$$|\Psi_{A_a,B_b}\rangle = \prod_{\mu=1}^{D} |A_a^\mu, B_b^\mu\rangle = \prod_{\mu=1}^{D} \frac{\left(\alpha_{-a}^\mu\right)^{A_a^\mu} \otimes \left(\beta_{-b}^\mu\right)^{B_b^\mu}}{\sqrt{\omega_a^{A_a^\mu}\omega_b^{B_b^\mu} A_a^\mu! B_b^\mu!}} |0,0\rangle \tag{2.57}$$

In other words, the general solution is the product of all modes of oscillation of both the left and the right moving waves, hence the $\prod_{a,b}$ and the product of the oscillation in each individual dimension, hence the $\prod_\mu$.

We also use the following convention:

$$\alpha_a^\mu |A_a^\mu, B_b^\mu\rangle = \begin{cases} \sqrt{\omega_a}\sqrt{A_a^\mu + 1}\,|A_a^\mu + 1, B_b^\mu\rangle & if \quad a < 0 \\ \sqrt{\omega_a}\sqrt{A_a^\mu}\,|A_a^\mu - 1, B_b^\mu\rangle & if \quad a > 0 \end{cases} \tag{2.58}$$

$$\beta_b^\mu |A_a^\mu, B_b^\mu\rangle = \begin{cases} \sqrt{\omega_b}\sqrt{B_b^\mu + 1}\,|A_a^\mu, B_b^\mu + 1\rangle & if \quad b < 0 \\ \sqrt{\omega_b}\sqrt{B_b^\mu}\,|A_a^\mu, B_b^\mu - 1\rangle & if \quad b > 0 \end{cases} \tag{2.59}$$

2.3.4 The Virasoro Generators

Let us now look at the corresponding Virasoro constraints. First note that these come form gauge fixing of the elements of the energy momentum tensor *eqn.*(2.49), T_{ab} and there are only two of these:

$$\begin{cases} T_{00}(\sigma,t) = C^0(\sigma,t) = \dfrac{1}{2}\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) \\ T_{01}(\sigma,t) = C^1(\sigma,t) = \dfrac{1}{2}\left(\dot{X}^\mu X'_\mu + \dot{X}^\mu X'_\mu\right) \end{cases} \tag{2.60}$$

These then become:

$$\begin{aligned} 2T_{00}(\sigma;t) = \frac{p_0^\mu p_{0\mu}}{L^2} - 2\aleph^2 \sum_{l,k\neq 0} e^{i(\omega_l+\omega_k)t}\left(\alpha_l^\mu \alpha_{k\mu} e^{i(\kappa_n+\kappa_m)\sigma} + \beta_l^\mu \beta_{k\mu} e^{-i(\kappa_n+\kappa_m)\sigma}\right) \\ + 2i\aleph\frac{p_0^\mu}{L}\sum_{l\neq 0} e^{i\omega_l t}\left(\alpha_{l\mu} e^{i\kappa_l\sigma} + \beta_{l\mu} e^{-i\kappa_l\sigma}\right) \\ 2T_{01}(\sigma;t) = -2\aleph^2 \sum_{l,k\neq 0} e^{i(\omega_l+\omega_k)t}\left(\alpha_l^\mu \alpha_{k\mu} e^{i(\kappa_n+\kappa_m)\sigma} - \beta_l^\mu \beta_{k\mu} e^{-i(\kappa_n+\kappa_m)\sigma}\right) \\ + 2i\aleph\frac{p_0^\mu}{L}\sum_{l\neq 0} e^{i\omega_l t}\left(\alpha_{l\mu} e^{i\kappa_l\sigma} - \beta_{l\mu} e^{-i\kappa_l\sigma}\right) \end{aligned} \tag{2.61}$$

Where we have defined:

$$\aleph \equiv \left(\frac{\hbar}{2L}\right)^{\frac{1}{2}}$$

Now we have:

$$\begin{cases} T_{00}(n,m;t) & \approx & 0 \\ T_{01}(n,m;t) & \approx & 0 \end{cases} \qquad \forall \quad t\,, \tag{2.62}$$

Making this more symmetric:

$$\begin{aligned} 2\left(T_{00}+T_{01}\right) &= \frac{p_0^2}{L^2} - 4\aleph^2 \sum_{k,l\neq 0} e^{i(\omega_k+\omega_l)t}\left(\alpha_k^\mu \alpha_{l\mu} e^{i(\kappa_n+\kappa_m)\sigma}\right) + 4i\aleph\frac{p_0^\mu}{L}\sum_{l\neq 0} e^{i\omega_l t}\alpha_{l\mu}e^{i\kappa_l\sigma} \\ 2\left(T_{00}-T_{01}\right) &= \frac{p_0^2}{L^2} - 4\aleph^2 \sum_{k,l\neq 0} e^{i(\omega_k+\omega_l)t}\left(\beta_k^\mu \beta_{l\mu} e^{-i(\kappa_n+\kappa_m)\sigma}\right) + 4i\aleph\frac{p_0^\mu}{L}\sum_{l\neq 0} e^{i\omega_l t}\beta_{l\mu}e^{-i\kappa_l\sigma} \end{aligned} \tag{2.63}$$

If we rewrite this in the following form, we notice something interesting:

$$\begin{aligned} \Gamma^{0\mu}\Gamma^{0}{}_{\mu} \equiv 2(T_{00}+T_{01}) &= \left(\frac{p_0}{L} + 2i\aleph\sum_{k\neq 0} e^{i\omega_k t}\alpha_k e^{i\kappa_k\sigma}\right)^{\mu}\left(\frac{p_0}{L} + 2i\aleph\sum_{l\neq 0} e^{i\omega_l t}\alpha_l e^{i\kappa_l\sigma}\right)_{\mu} \approx 0 \\ \Gamma^{1\mu}\Gamma^{1}{}_{\mu} \equiv 2(T_{00}-T_{01}) &= \left(\frac{p_0}{L} + 2i\aleph\sum_{k\neq 0} e^{i\omega_k t}\beta_k e^{i\kappa_k\sigma}\right)^{\mu}\left(\frac{p_0}{L} + 2i\aleph\sum_{l\neq 0} e^{i\omega_l t}\beta l e^{i\kappa_l\sigma}\right)_{\mu} \approx 0 \end{aligned} \tag{2.64}$$

In other words, $\Gamma^{0\mu}$ and $\Gamma^{1\mu}$ are two null vectors.

Here, based on what we know about p_0^μ, that it represents the momentum of the centre of mass, we can equate its value to be:

$$\frac{p_0^2}{L^2} \approx M^2\,, \tag{2.65}$$

where M is the mass-density of the string.

In addition we can posit that the direction of motion of the centre of mass, is orthogonal to that of the modes of oscillations:

$$\begin{cases} p_0^\mu \alpha_{k\mu} \approx 0 \\ p_0^\mu \beta_{k\mu} \approx 0\,, \end{cases} \qquad \forall k \tag{2.66}$$

i.e. the oscillations are transverse.[11]

[11] This posit can be justified in a variety of different ways. A physical explanation is that the longitudinal modes will have *negative norm*. We show this in *appendix*(E).

With this then the conditions on the $\Gamma^{\alpha\mu}$ $\alpha = 0, 1$ can be translated into statements about the modes of oscillations. Let:

$$\begin{aligned}\mathcal{K}^0(\sigma) &\equiv \sum_{k,l\neq 0}\left(\alpha_k^\mu \alpha_{l\mu} e^{i(\kappa_l+\kappa_k)\sigma}\right) &= \left(\frac{M}{2\aleph}\right)^2 \\ \mathcal{K}^1(\sigma) &\equiv \sum_{k,l\neq 0}\left(\beta_k^\mu \beta_{l\mu} e^{-i(\kappa_l+\kappa_k)\sigma}\right) &= \left(\frac{M}{2\aleph}\right)^2\end{aligned} \tag{2.67}$$

With this we arrive at the following:

$$\begin{cases} \dfrac{p_0^2}{L^2} - M^2 \approx 0 \\ \dfrac{p_0^\mu}{L}\displaystyle\sum_{l\neq 0}\alpha_{l\mu}e^{i\kappa_l\sigma} \approx 0 \\ \dfrac{p_0^\mu}{L}\displaystyle\sum_{l\neq 0}\beta_{l\mu}e^{i\kappa_l\sigma} \approx 0 \\ \mathcal{K}^0(\sigma) - \left(\dfrac{p_0}{2L}\right)^2 \equiv \displaystyle\sum_{k,l\neq 0}\left(\alpha_k^\mu\alpha_{l\mu}e^{i(\kappa_l+\kappa_k)\sigma}\right) - \left(\dfrac{M}{2\aleph}\right)^2 \approx 0 \\ \mathcal{K}^1(\sigma) - \left(\dfrac{p_0}{2L}\right)^2 \equiv \displaystyle\sum_{k,l\neq 0}\left(\beta_k^\mu\beta_{l\mu}e^{-i(\kappa_l+\kappa_k)\sigma}\right) - \left(\dfrac{M}{2\aleph}\right)^2 \approx 0 \end{cases} \tag{2.68}$$

as the set of the constraints of the theory[12].
By performing the Fourier transform[13]of these we obtain the Virasoro Generators:

$$\begin{cases} \mathcal{Z} \equiv \dfrac{p_0^\mu}{L}\displaystyle\int \mathrm{d}\sigma \sum_{l\neq 0}\alpha_{l\mu}e^{i\kappa_l} = \dfrac{p_0^\mu}{L}\alpha_{l\mu} \\ \widetilde{\mathcal{Z}} \equiv \dfrac{p_0^\mu}{L}\displaystyle\int \mathrm{d}\sigma \sum_{l\neq 0}\beta_{l\mu}e^{-i\kappa_l} = \dfrac{p_0^\mu}{L}\beta_{l\mu} \\ \mathcal{L}_m \equiv \dfrac{1}{L}\displaystyle\int \mathrm{d}\sigma e^{-i\kappa_m\sigma}\left(\mathcal{K}^0(\sigma) - \left(\dfrac{p_0}{2L}\right)^2\right) = \sum_{k\neq 0}\alpha_{m-k}^\mu\alpha_{k\mu} - \left(\dfrac{M}{2\aleph}\right)^2\delta_{m,0} \\ \widetilde{\mathcal{L}}_m \equiv \dfrac{1}{L}\displaystyle\int \mathrm{d}\sigma e^{i\kappa_m\sigma}\left(\mathcal{K}^1(\sigma) - \left(\dfrac{p_0}{2L}\right)^2\right) = \sum_{k\neq 0}\beta_{m-k}^\mu\beta_{k\mu} - \left(\dfrac{M}{2\aleph}\right)^2\delta_{m,0} \end{cases} \tag{2.70}$$

[12] Also note the following relations:

$$\begin{cases} \mathcal{K}^0(\sigma) - \left(\dfrac{p_0}{2L}\right)^2 = \dfrac{1}{4\aleph^2}\Gamma^{0\mu}\Gamma^0_\mu = \dfrac{1}{2\aleph^2}(T_{00}+T_{01}) = \dfrac{1}{2\aleph^2}(C^0+C^1) = \dfrac{1}{2\aleph^2}\left((\partial_0+\partial_1)X\right)^2 \approx 0 \\ \mathcal{K}^1(\sigma) - \left(\dfrac{p_0}{2L}\right)^2 = \dfrac{1}{4\aleph^2}\Gamma^{1\mu}\Gamma^1_\mu = \dfrac{1}{2\aleph^2}(T_{00}-T_{01}) = \dfrac{1}{2\aleph^2}(C^0-C^1) = \dfrac{1}{2\aleph^2}\left((\partial_0-\partial_1)X\right)^2 \approx 0 \end{cases} \tag{2.69}$$

[13]Note that the transformation of the constraints had to be performed because the solutions are the momentum space solutions (in its representation) therefore the corresponding generators should also be in the same space.

These are the constraints which validity we should test by applying them on the solution space *eqn.*(2.56).
However before doing so let us make a remark on the $\mathcal{L}_0$ and $\widetilde{\mathcal{L}}_0$. Note that they are the only ones in which the ordering of the two operators in the sum does matter. This ordering however has only the effect of leading to an infinite term (due to the residue of the commutation). As such we require that the constraints be *operator-ordered*. This then effectively redefines the last three constraints as follows:

$$\begin{cases} \mathcal{L}_0 = 2\sum_{k=1}^{\infty} \alpha^{\mu}_{-k}\alpha_{k\mu} - \left(\frac{M}{2\aleph}\right)^2 \approx 0 \\ \widetilde{\mathcal{L}}_0 = 2\sum_{k=1}^{\infty} \beta^{\mu}_{-k}\beta_{k\mu} - \left(\frac{M}{2\aleph}\right)^2 \approx 0 \\ \mathcal{L}_m = \sum_{k\neq 0} \alpha^{\mu}_{m-k}\alpha_{k\mu} \approx 0 & m \neq 0 \\ \widetilde{\mathcal{L}}_m = \sum_{k\neq 0} \beta^{\mu}_{m-k}\beta_{k\mu} \approx 0 & m \neq 0 \end{cases} \tag{2.71}$$

Let us now consider these to isolate the physical solution space.

2.3.5 The Physical Space

First notice that the first and the second constraints $\mathcal{Z}, \widetilde{\mathcal{Z}}$, are restrictions on μ whereas the last two $\mathcal{L}_m, \widetilde{\mathcal{L}}_m$ are restrictions on the occupation numbers of states.
Therefore we can satisfy the first two conditions by a convenient choice of the *direction of oscillation* and that of the *direction of propagation*. Let:

$$\begin{aligned} &\mu = \{1; i\} \qquad s.t. \quad i = 2, \cdots, D \\ &\qquad\qquad\qquad p_0^{\mu} = \{NM; \overrightarrow{0}\} \\ &and \qquad\qquad \alpha^{\mu}_a = \{0; \alpha^i_a\} \\ &\qquad\qquad\qquad \beta^{\mu}_b = \{0; \beta^i_b\} \end{aligned} \tag{2.72}$$

Therefore we should rewrite the Fock solutions of the theory as:

$$|\Psi_{A,B}(K^0)\rangle = \prod_{a,b=1}^{\infty} |\Psi_{A_a,B_b}; M\rangle \tag{2.73}$$

where :

$$|\Psi_{A_a,B_b}\rangle = \prod_{\mu=2}^{D} |A^{\mu}_a, B^{\mu}_b\rangle = \prod_{\mu=2}^{D} \frac{\left(\alpha^{\mu}_{-a}\right)^{A^{\mu}_a} \otimes \left(\beta^{\mu}_{-b}\right)^{B^{\mu}_b}}{\sqrt{\omega_a^{A^{\mu}_a}\omega_b^{B^{\mu}_b} A^{\mu}_a! B^{\mu}_b!}} |0,0\rangle \tag{2.74}$$

Now the last two constraints serve the purpose of isolating the physical sector.

This is the place where String Theory's definition of physicality comes into picture. Note that all the previous assumptions as to the Mass-density of the string and the transversality of the oscillations are respectively, assumptions about the basic theory and a convenient coordinate system. However in order to satisfy the last two constraints, we must make a certain choice as to the definition of physicality. In the canonical quantization of Dirac, as discussed in *sec.*(1.2), we demand that the "whole" of the constraints be satisfied, as in:

$$\begin{cases} \mathcal{L}_m|\Psi_{phys}\rangle = 0 \\ \widetilde{\mathcal{L}}_m|\Psi_{phys}\rangle = 0 \end{cases} \tag{2.75}$$

This of course has the effect of rendering the solution space trivial (as is expected of any completely constrained theory), as we will show at the end of this section.

Here, in string theory however, a physical state $|\Psi_{phys}\rangle$, is defined as one which obeys the following condition:

$$\begin{cases} \langle\Psi_{phys}|\mathcal{L}_m|\Psi_{phys}\rangle = 0 \\ \langle\Psi_{phys}|\widetilde{\mathcal{L}}_m|\Psi_{phys}\rangle = 0 \end{cases} \tag{2.76}$$

In addition we get rid of the null states:

$$|\Psi_{null}\rangle \equiv \left\{ |\Psi_{phys}\rangle\ s.t.\ |\Psi_{null}\rangle = \sum_k A_k|\Psi_{phys}\rangle_k \right\}$$
$$\Rightarrow \mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}} = \frac{\mathcal{H}_{Phys}}{\mathcal{H}_{null}}$$

Let us now show how this results in a non-trivial solution space. We shall focus on the requirement *eqn.*(2.76) for now. To do so first we note an identity:

$$\begin{cases} (\mathcal{L}_m)^\dagger = \mathcal{L}_{-m} \\ (\widetilde{\mathcal{L}}_m)^\dagger = \widetilde{\mathcal{L}}_{-m} \end{cases}$$

Therefore in order to satisfy *eqn.*(2.76), it is sufficient if we had:

$$\begin{cases} \mathcal{L}_m|\Psi_{phs}\rangle = 0 \\ \widetilde{\mathcal{L}}_m|\Psi_{phs}\rangle = 0 \end{cases} \quad for \qquad m \geq 0 \tag{2.77}$$

i.e. as compared to *eqn.*(2.75), we are applying only "half" of the constraints. This is the very reason why the physical solution space will now be more viable.
Because of the symmetry of the situation we will consider the $\mathcal{L}_m$ only.
For the $m = 0$ case consider:

$$\mathcal{L}_0|A^\mu_a\rangle = \left(2\sum_{n=1}^{\infty}\alpha^\nu_{-n}\alpha_{n\nu} - \left(\frac{M}{2\aleph}\right)^2\right)|A^\mu_a\rangle \tag{2.78}$$

Notice that since $a > 0$ only the $n = a$ affects this expression,

$$\mathcal{L}_0|A_a^\mu\rangle \to \left(2\alpha_{-a}^\nu\alpha_{a\nu} - \left(\frac{M}{2\aleph}\right)^2\right)|A_a^\mu\rangle \tag{2.79}$$

Now using:

$$\alpha_{-a}^\nu\alpha_{a\nu}\left(\alpha_{-a}^\mu\right)^{A_a^\mu} = \alpha_{-a}^\nu\left(\left(\alpha_{-a}^\mu\right)^{A^m{}_{u_a}}\alpha_{a\nu} + A_a^\mu\delta_\nu^\mu\left(\alpha_{-a}^\mu\right)^{(A_a^\mu-1)}\right) \quad \xrightarrow{|0\rangle} \quad A_a^\mu\left(\alpha_{-a}^\mu\right)^{A_a^\mu} \tag{2.80}$$

we get:

$$\mathcal{L}_0|A_a^\mu\rangle \to \left(2(A_a^\mu) - \left(\frac{M}{2\aleph}\right)^2\right)|A_a^\mu\rangle \tag{2.81}$$

With similar results for the $\widetilde{\mathcal{L}}_0$. In other words, the $\mathcal{L}_0$ and $\widetilde{\mathcal{L}}_0$ (less the mass-density constant) give the occupation number of their corresponding states.

Now let us consider the $m > 0$ case. Let us first rewrite the Virasoro constraints:

$$\begin{aligned}
\mathcal{L}_m\prod_{a,b=1}^{\infty}|A_a,B_b\rangle &= \sum_{n\neq 0}^{\infty}\alpha_{m-n}^\mu\alpha_{n\mu}\prod_{a,b=1}^{\infty}|A_a,B_b\rangle \approx 0 \qquad\qquad m>0\\
&= \sum_{n=1}^{\infty}\left(\alpha_{m+n}^\mu\alpha_{-n\mu} + \alpha_{m-n}^\mu\alpha_{n\mu}\right)\prod_{a,b=1}^{\infty}|A_a,B_b\rangle\\
&= \left[\sum_{n=m+1}^{\infty}\left(\alpha_n^\mu\alpha_{m-n\mu} + \alpha_{m-n}^\mu\alpha_{n\mu}\right) + \sum_{n=1}^{m}\alpha_{m-n}^\mu\alpha_{n\mu}\right]\prod_{a,b=1}^{\infty}|A_a,B_b\rangle\\
&= \left[2\sum_{n=m+1}^{\infty}\alpha_n^\mu\alpha_{m-n\mu} + \sum_{n=1}^{m-1}\alpha_{m-n}^\mu\alpha_{n\mu} + \alpha_0^\mu\alpha_{m\mu}\right]\prod_{a,b=1}^{\infty}|A_a,B_b\rangle
\end{aligned} \tag{2.82}$$

We now throw out the n=m case which gives an $\alpha_0^\mu\alpha_{m\mu} \sim p_0^\mu\alpha_{m\mu} \approx 0$.
This now can be written in the form:

$$\begin{aligned}
\mathcal{L}_m &\equiv \sum_{\substack{n=1\\ n\neq m}}^{\infty}\alpha_n{}^\mu\alpha_{m-n}{}_\mu\left(2\Theta_{n-m} + \Theta_{m-n}\right) \qquad m>0\\
&\mathit{where:}\quad \Theta_x \equiv \begin{cases}1 & x>0\\ 0 & x<0\end{cases}
\end{aligned} \tag{2.83}$$

Applying this on the states *eqn.*(2.73), and taking into account that each term of the Θ_{m-n} actually picks up 2 terms in the domain $1 \leq n \leq m-1$, we have:

$$\begin{aligned}
\mathcal{L}_m\prod_{\mu=2}^{D}\prod_{a=1}^{\infty}|A_a^\mu\rangle = &\sum_{\substack{n=1\\ n\neq m}}^{\infty}\prod_{\mu=2}^{D}\prod_{\substack{a\neq n\\ a\neq|m-n|}}^{\infty}|A_a^\mu\rangle\sqrt{\left(w_{|n|}w_{|m-n|}A_{|n|}^\mu\right)}|A_{|n|}^\mu - 1\rangle\\
&\times 2\left(\sqrt{\left(A_{|m-n|}^\mu + 1\right)}|A_{|m-n|}^\mu + 1\rangle\Theta_{n-m} + \sqrt{\left(A_{|m-n|}^\mu\right)}|A_{|m-n|} - 1\rangle\Theta_{m-n}\right)
\end{aligned}$$

(2.84)

This is a complicated linear combination of the states. Note that under this construction of the states, two states of the same mode, say $|C_n^\mu\rangle$ and $|D_n^\mu\rangle$, however of differing occupation numbers: $C \neq D$ are orthogonal (at least in the original construction). Hence if a linear combination of these is zero, it means that either the coefficients must be zero or otherwise the original states themselves are not all independent.
However, looking at the coefficients for a given n in the sum:

$$\begin{cases} n < m & \rightarrow \sqrt{\left(w_{|n|}w_{|m-n|}A^{\mu}_{|n|}\right)}\sqrt{\left(A^{\mu}_{|m-n|}\right)} & \qquad n = 1, 2, \cdots, m-1 \\ n > m & \rightarrow \sqrt{\left(w_{|n|}w_{|m-n|}A^{\mu}_{|n|}\right)}\sqrt{\left(A^{\mu}_{|m-n|}+1\right)} & \qquad n = m+1, m+2, \cdots \end{cases} \tag{2.85}$$

we see that these are very tight restrictions on the occupation numbers of physical states.

Therefore we may conjecture that the complicated expression of the constraints on the states, has the effect of rendering some of the states dependent on the rest. Yet the restrictions may not be so strong as to make the physical solution space - as defined via this string theory method - trivial.
The problem of demonstrating and attempting to prove this conjecture was initiated in the works of *Dual multi-particle theory* [16] (the precursor to string which was initially used to describe the strong force in nuclear physics.) These lead to the works of [17] which culminated in the now-known *Goddard-Thorn Theorems* (also popularly known as the *No-Ghost Theorems*.) We do not present the proof here, and will only state it at the end of *appendix*(C)[14]. Instead the main features which we would like to take from these discussions are the following:

- The complicatedness of the expressions of the constraints for the string, *eqn.*(2.84), is quite puzzling; it is very unusual that from such a set of constraints, a non-trivial solution space may survive. Now obviously the above-mentioned works do suggest (and perhaps prove) this conjecture. However we may be able to consider another way of looking at the problem which may reveal something interesting.

  Notice that, abstractly, we know that Dirac's quantization method is too restrictive, and will therefore only produce a non-dynamical solution space (in contrast to the String's method, where - as we show in *appendix*(F) - it can at least abstractly be argued that the solution space may still contain dynamics due to the weaker constraint conditions.) However observe that *Dirac's constraints are String's constraints, augmented with the $m < 0$ constraints*. Therefore, by comparing the $m > 0$ case and the $m < 0$ case, we may be able to find a simple method of relating the two quantization methods with each other.

  This is therefore what we will consider next.

[14]For a proof based on the BRST formalism also see [18].

2.3.6 Comparison with Dirac's Quantization

Going back, as we argued, Dirac's quantization can be satisfied, if in addition to *eqn.*(2.77), we demand the following conditions:

$$\begin{cases} \mathcal{L}_{-m}|\Psi_{phys}\rangle = 0 \\ \widetilde{\mathcal{L}}_{-m}|\Psi_{phys}\rangle = 0 \end{cases} \qquad m > 0 \tag{2.86}$$

Then repeating the same steps of the calculation for $\mathcal{L}_m$ of the String, we will have:

$$\begin{aligned} \mathcal{L}_m \prod_{a,b=1}^{\infty} |A_a, B_b\rangle &= \sum_{n\neq 0}^{\infty} \alpha^{\mu}_{m-n}\alpha_{n\mu} \prod_{a,b=1}^{\infty} |A_a, B_b\rangle \approx 0 \qquad\qquad m<0 \\ &= \sum_{n=-\infty}^{-1} \left(\alpha^{\mu}_{m+n}\alpha_{-n\mu} + \alpha^{\mu}_{m-n}\alpha_{n\mu}\right) \prod_{a,b=1}^{\infty} |A_a, B_b\rangle \\ &= \left[\sum_{n=-\infty}^{m-1} \left(\alpha^{\mu}_{n}\alpha_{m-n\mu} + \alpha^{\mu}_{m-n}\alpha_{n\mu}\right) + \sum_{n=m}^{-1} \alpha^{\mu}_{m-n}\alpha_{n\mu}\right] \prod_{a,b=1}^{\infty} |A_a, B_b\rangle \\ &= \left[2\sum_{n=-\infty}^{m-1} \alpha^{\mu}_{n}\alpha_{m-n\mu} + \sum_{n=m+1}^{-1} \alpha^{\mu}_{m-n}\alpha_{n\mu} + \alpha^{\mu}_{0}\alpha_{m\mu}\right] \prod_{a,b=1}^{\infty} |A_a, B_b\rangle \end{aligned} \tag{2.87}$$

We again throw out the n=-m case which gives an $\alpha^{\mu}_0\alpha_{m\mu} \sim p^{\mu}_0\alpha_{m\mu} \approx 0$.

Again we write:

$$\mathcal{L}_m = \sum_{\substack{n=-\infty \\ n\neq m}}^{-1} {\alpha_n}^{\mu}{\alpha_{m-n}}_{\mu}\left(2\Theta_{m-n} + \Theta_{n-m}\right) \qquad m<0 \tag{2.88}$$

With the corresponding action on the states (with the same note as to the Θ_{n-m} terms as for the Θ_{m-n} in the $m>0$ case):

$$\begin{aligned} \mathcal{L}_m \prod_{\mu=2}^{D}\prod_{a=1}^{\infty} |A^{\mu}_a\rangle &= \sum_{\substack{n=-\infty \\ n\neq m}}^{-1} \prod_{\mu=2}^{D} \prod_{\substack{a\neq n \\ a\neq |m-n|}}^{\infty} |A^{\mu}_a\rangle \sqrt{w_{|n|}w_{|m-n|}\left(A^{\mu}_{|n|}+1\right)}|A^{\mu}_n + 1\rangle \\ &\times 2\left(\sqrt{\left(A^{\mu}_{|m-n|}\right)}|A^{\mu}_{|m-n|} - 1\rangle\Theta_{m-n} + \sqrt{\left(A^{\mu}_{|m-n|}+1\right)}|A_{|m-n|}+1\rangle\Theta_{n-m}\right) \end{aligned} \tag{2.89}$$

This can now be compared to *eqn.*(2.84), of the String.

The striking feature is the symmetry that exists between these two expressions; i.e. they seem very similar. It is very difficult to see how a solution should exist that will obey only one of these sets of constraints, and not the other. In fact it would seem quite improbable that this should be the case. It may therefore be more reasonable to assume that a given solution of one of these constraints will obey the other constraint, than not.

Hence, this comparison may suggest that, despite the results of Goddard-Thorn Theorems, in the specific case of the string the solution space may still be quite trivial - in the sense that the solution space is as restricted by String's constraints as it is by Dirac's constraints.

Of course these are suggestions, and do not form a proof. The proof, as we discussed, is complicated due to the complicated nature of the solution space of the *continuous* string. This is precisely what motivates the analysis of the *discrete* string; we hope that the solution space there will be easier to analyze and discuss.

This finishes the analysis of this chapter, let us end with a discussion.

2.4 Discussion

In the last section we carried out the full analysis of the solution space and the constraint algebra thereof string theory. As mentioned, through the works of [16], [17] and others, it has been shown that the continuous string theory is *ghost-free* (i.e. the physical space is positive-definite normed). These results however do not indicate whether the resulting solution space is trivial or not.
In the last section, we considered the possibility that the solution space may in fact be trivial.
Now let us assume the alternative possibility. Let us assume:

- That there exists a non-trivial solution to these constraints, with a complexly inter-related set of physical states, restricted by the constraints *eqn.*(2.84).

Let us discuss the implications of this possibility.

Again based on the symmetry of the $m > 0$ and $m < 0$ cases of the Virasoro constraints - *eqn.*(2.84) and *eqn.*(2.89) - we may argue that, if we assume that there exists a nontrivial solution space, it perhaps obeys both sets of constraints. However if this is the case, given the complexity of the constraint relations, this realization would strongly suggest that the restrictions are obeyed *a-priori*; all constraints seem to be already satisfied!

If this is true, we may ask what construct in the analysis may have warranted it?
The answer must lie in the only step we have taken to get to this result, i.e. the gauge-fixing.
In other words, due to the gauge fixing which we have performed, it would seem that we have removed all degrees of freedom which contribute to the constraints.
If this conclusion is correct, then we may be able to propose the following conjecture for any gauge theory that is to be quantized:

Either *gauge-fix then quantize* or *quantize then apply constraints*

Said another way, we need only do one of these; gauge fix (completely) before quantization, and then satisfy the quantum EOM, or Quantize and then apply the constraints. Hence in the analysis of the string of this chapter, the checking of the states against $\mathcal{L}_m$ is just that, only a check.

Of course this does not mean that these two methods - gauge-fixing before vs. applying constraints after - would lead to the same description of the theory. In other words the question whether the gauge fixing method is acceptable, or whether it is consistent with the non-gauge-fixing method of Dirac, and also whether the results are still covariant are questions which remain unanswered.
This provide further motivation for the analysis of *chap.*(4), were we consider the *covariant* quantization of the string via the Loop Quantum Gravity method. In other words, by considering and comparing the results which may follow from the Loop quantization, which is a fully covariant theory, it may be possible to shed some light on the above-mentioned questions.

On another note, the question of dynamics still remains; in what sense (in light of the weaker constraint conditions) are the solutions dynamical? Notice that via the gauge fixing, we have fixed a time parameter, and have therefore conceived of a *preferred time*, and a *true Hamiltonian*, which can then provide nontrivial dynamics. But in order to discuss this dynamic explicitly we will need to have access to the physical solution space.

Therefore the investigation of this question is also compounded by the complexity of the interrelations of the solution space of the continuum string.

As such the possible usefulness of a simplified version of the string, which may then have a simpler solution space to analyze, is again emphasized. This again, provides the motivation to consider the Discrete String, in the next *chap.*(3). We hope that this system will be simpler to discuss and therefore more accessible for analysis, so that within its context we may try to re-ask the questions regarding the observables and dynamics of the string.

Chapter 3

The Discrete String

In this chapter we analyze a discrete version of the closed (Bosonic) string. The main motivation for this is, as mentioned, to utilize a simple model where some of the questions regarding observables and their dynamics, in generally covariant theories could be answered more clearly than in the more complex situation of the continuous string.

3.1 Initial Setup - First Picture

To start we would like to discretize the original continuous string's action:

$$S = -\frac{1}{2}\int \mathrm{d}^2\lambda\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu$$

There, of course exist a wide class of discretizations which can be tried. We will focus on the following:

$$\begin{aligned}
\int \mathrm{d}\sigma \quad &\to \quad \sum_{i=1}^{n}\epsilon_i\,, \\
X^\mu(\sigma,t) \quad &\to \quad X^\mu_{(i)}(t)\,, \\
\partial_\sigma X^\mu(\sigma,t) \quad &\to \quad \frac{1}{\epsilon_i}\left(X^\mu_{(i+1)} - X^\mu_{(i-1)}\right)(t)\,, \\
g_{ab}(\sigma,t) \quad &\to \quad \gamma^2\begin{pmatrix} -N^2_{(i)} + M^2_{(i)} & M_{(i)} \\ M_{(i)} & 1 \end{pmatrix}(t)\,,
\end{aligned} \tag{3.1}$$

however, as we shall show in *sec*(3.2.4), this discretization represents the class of all discretization which are related by a *linear, time-independent transformation.*

The ϵ_i is the unit length on the string, which is allowed to be position-dependent (via the index i).

The resulting action from this discretization is:

$$
\begin{aligned}
S \rightarrow -\frac{1}{2}\int \mathrm{d}t \sum_{i=1}^{n} \frac{\epsilon_i}{N_{(i)}} \Bigg[& -\dot{X}^{\mu}_{(i)}\dot{X}_{(i)\mu} + \frac{2}{\epsilon_i} M_{(i)} \dot{X}^{\mu}_{(i)} (X_{(i+1)} - X_{(i-1)})_{\mu} \\
& + \frac{1}{\epsilon_i^2}(N_{(i)}^2 - M_{(i)}^2)(X_{(i+1)} - X_{(i-1)})^2 \Bigg]
\end{aligned} \tag{3.2}
$$

However notice that the N_i and the M_i are arbitrary. And since ϵ_i is finite (none-zero), it can be absorbed into the definition of these:

$$
\begin{cases}
N_i & \rightarrow \quad \frac{1}{\epsilon_i} N_i \equiv \tilde{N}_i \\
M_i & \rightarrow \quad \frac{1}{\epsilon_i} M_i \equiv \tilde{M}_i
\end{cases} \tag{3.3}
$$

which has the effect of redefining the metric:

$$
g_{ab}(\sigma, t) \rightarrow \quad \frac{\gamma^2}{\epsilon_i^2} \begin{pmatrix} -N_{(i)}^2 + M_{(i)}^2 & \epsilon_i M_{(i)} \\ \epsilon_i M_{(i)} & \epsilon_i^2 \end{pmatrix} (t) \quad \equiv \quad \gamma^2 \begin{pmatrix} -\tilde{N}_{(i)}^2 + \tilde{M}_{(i)}^2 & \tilde{M}_{(i)} \\ \tilde{M}_{(i)} & 1 \end{pmatrix} (t) \tag{3.4}
$$

So we can effectively assume that the discretization which is used is in fact the following:

$$
\begin{aligned}
X^{\mu}(\sigma, t) \quad & \rightarrow \quad X^{\mu}_{(i)}(t)\,, \\
\partial_{\sigma} X^{\mu}(\sigma, t) \quad & \rightarrow \quad \left(X^{\mu}_{(i+1)} - X^{\mu}_{(i-1)} \right)(t)\,, \\
g_{ab}(\sigma, t) \quad & \rightarrow \quad \gamma^2 \begin{pmatrix} -N_{(i)}^2 + M_{(i)}^2 & M_{(i)} \\ M_{(i)} & 1 \end{pmatrix} (t)\,.
\end{aligned} \tag{3.5}
$$

i.e. the ϵ_i does not contribute to the analysis. We will use this convention in the remainder of this chapter.

With this the discrete action becomes:

$$
\begin{aligned}
S \rightarrow -\frac{1}{2}\int \mathrm{d}t \sum_{i=1}^{n} \frac{1}{N_{(i)}} \Big[& -\dot{X}^{\mu}_{(i)}\dot{X}_{(i)\mu} + 2 M_{(i)} \dot{X}^{\mu}_{(i)} (X_{(i+1)} - X_{(i-1)})_{\mu} \\
& + (N_{(i)}^2 - M_{(i)}^2)(X_{(i+1)} - X_{(i-1)})^2 \Big]
\end{aligned} \tag{3.6}
$$

With this let us start the analysis of the discrete string.

To start, let us first work out a number of preliminary calculations, which will be useful in the remainder of the analysis of this chapter.

3.1.1 Lagrangian Equations of Motion

Varying the action with respect to all its variables we have: (δ_v stands for variation, see *footnote 5* in sec. 2.2.1).

$$\begin{aligned}
\delta_v S \quad &= \quad -\frac{1}{2}\int \mathrm{d}t \sum_{i=1}^{n} \delta_v \Bigg\{ \frac{1}{N_{(i)}}\Big[-\dot{X}^{\mu}_{(i)}\dot{X}_{(i)\mu} + 2M_{(i)}\dot{X}^{\mu}_{(i)}(X_{(i+1)} - X_{(i-1)})_{\mu} \\
&\qquad + (N^2_{(i)} - M^2_{(i)})(X_{(i+1)} - X_{(i-1)})^2\Big]\Bigg\} \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
\delta_v X^{\mu}_{(i)} \quad &\rightarrow \quad \Bigg[\partial_t\Big[\frac{1}{N_{(i)}}\Big(\dot{X}_{(i)} - M_{(i)}(X_{(i+1)} - X_{(i-1)})\Big)\Big] \\
&- \Big((\frac{M_{(i+1)}}{N_{(i-1)}}\dot{X}_{(i+1)} - \frac{M_{(i-1)}}{N_{(i-1)}}\dot{X}_{(i-1)}\Big) \qquad (3.8) \\
&- \Big(\frac{N^2_{(i+1)} - M^2_{(i+1)}}{N_{(i+1)}}(X_{(i+2)} - X_{(i)}) - \frac{N^2_{(i-1)} - M^2_{(i-1)}}{N_{(i-1)}}(X_{(i)} - X_{(i-2)})\Big)\Bigg]^{\mu} \\
\delta_v N_{(i)} \quad &\rightarrow \quad \frac{1}{N^2_{(i)}}\Big(\dot{X}_{(i)} - M_{(i)}(X_{(i+1)} - X_{(i-1)})\Big)^2 + \Big(X_{(i+1)} - X_{(i-1)}\Big)^2 \qquad (3.9) \\
\delta_v M_{(i)} \quad &\rightarrow \quad \frac{1}{N_{(i)}}\Big(\dot{X}_{(i)} - M_{(i)}(X_{(i+1)} - X_{(i-1)})\Big)^{\mu}\Big(X_{(i+1)} - X_{(i-1)}\Big)_{\mu} \qquad (3.10)
\end{aligned}$$

Using the definition of $\Pi^{\mu}_{(i)}$ *eqn.*(3.15), we have:

$$\begin{aligned}
\delta_v X^{\mu}_{(i)} \quad &\rightarrow \quad \Bigg[\dot{\Pi}_{(i)} - \Big((\frac{M_{(i+1)}}{N_{(i+1)}}\dot{X}_{(i+1)} - \frac{M_{(i-1)}}{N_{(i-1)}}\dot{X}_{(i-1)}\Big) \qquad (3.11) \\
&- \Big(\frac{N^2_{(i+1)} - M^2_{(i+1)}}{N_{(i+1)}}(X_{(i+2)} - X_{(i)}) - \frac{N^2_{(i-1)} - M^2_{(i-1)}}{N_{(i-1)}}(X_{(i)} - X_{(i-2)})\Big)\Bigg]^{\mu} \\
&= \quad \Big[\dot{\Pi}^{\mu}_{(i)} - \big(M_{(i+1)}\Pi^{\mu}_{(i+1)} - M_{(i-1)}\Pi^{\mu}_{(i-1)}\big) \qquad (3.12) \\
&- \big(N_{(i+1)}(X^{\mu}_{(i+2)} - X^{(i)}) - N_{(i-1)}(X^{\mu}_{(i)} - X^{\mu}_{(i-2)})\big)\Big] \\
\delta_v N_{(i)} \quad &\rightarrow \quad \Pi^2_{(i)} + \Big(X_{(i+1)} - X_{(i-1)}\Big)^2 \qquad (3.13) \\
\delta_v M_{(i)} \quad &\rightarrow \quad \Pi^{\mu}_{(i)}\Big(X_{(i+1)} - X_{(i-1)}\Big)_{\mu} \qquad (3.14)
\end{aligned}$$

From these expressions, we can deduce the resulting Lagrangian EOM, for the field variables, X^{μ}, *eqn.*(3.12), and the resulting constraint equations on these variables from the variation of the Lagrange multipliers N and M, *eqn.*(3.13, 3.14). We will repeatedly come back to see these expressions in the remainder of this chapter.
Now let us proceed with the canonical analysis of the discrete string, starting from its constraint algebra.

3.1.2 The Constraints

First let us determine the complete Hamiltonian of the system. To do so we will follow Dirac's procedure for the integration of the constraints into the Hamiltonian. This method - for a system without any Second-Class Constraints - consists of the following steps:[1]

- Identifying the *Primary Constraints*; these are the vanishing momenta.
- Adding the Primary Constraints to the Hamiltonian
- Finding *Secondary Constraints*; these are expressions that one finds from the Poisson Bracket of the Primary Constraints with the Hamiltonian.
- Now we need to test the Secondary Constraints against all the other constraints and themselves; their Poisson Bracket with each other should all vanish.
- This process continues until all constraints are identified. The full Hamiltonian is the original Hamiltonian plus all the constraints weighted be an arbitrary function (multiplier).

Here then we will follow these steps for this particular discretization of the string, to see what sort of dynamics is predicted by the Hamiltonian.

The momenta are given by:

$$\Pi_{(i)\mu} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}_{(i)}} = \frac{1}{N_{(i)}} \left[\dot{X}_{(i)\mu} - M_{(i)} (X_{(i+1)} - X_{(i-1)})_{\mu} \right] \tag{3.15}$$

$$\left. \begin{aligned} P_{N_{(i)}} &= \frac{\partial \mathcal{L}}{\partial \dot{N}_{(i)}} \approx 0 \\ P_{M_{(i)}} &= \frac{\partial \mathcal{L}}{\partial \dot{M}_{(i)}} \approx 0 \end{aligned} \right\} \qquad The\, Primary\, Constraints \tag{3.16}$$

The last two expressions identify the primary constraints of the theory. Based on the definition of the momenta we can rewrite the Lagrangian and the Hamiltonian - which

[1] In the presence of second class constraints there are different identical approaches which can be taken. They effectively lead to the redefinition of the Poisson Bracket to a Dirac Bracket. To understand this, notice that a Second Class Constraint ultimately refers to a condition: $\{X^{\mu}, \Pi^{\mu}\} \approx 0$. This means that there exist certain degrees of freedom in the theory which:

- Are redundant in the description of the system
- While existing in the formulation, they cannot be allowed to evolve arbitrarily.

Effectively, therefore, with a proper redefinition of the variables, they can be removed from the formulation. As far as the Poisson Bracket is concerned, this means that the evolution of the real degrees of freedom should not be affected by these sets of (X^{μ}, Π^{μ}). Hence we need to remove these sets of variables from the definition of the PB. This is the essence in which a $\{\ \ ,\ \ \}_{PB} \rightarrow \{\ \ ,\ \ \}_{DB}$.

Also notice that first class constraints can be changed to second class constraints, and hence removed from the theory via gauge-fixing the conjugate variable associated with the first class constraint variable, see [38].

would be useful in the upcoming calculations - as:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\left[N_{(i)}\left(\Pi_{(i)}^{2} - \left(X_{(i+1)} - X_{(i-1)}\right)^{2}\right)\right] \tag{3.17}$$

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n}\left[N_{(i)}\left(\Pi_{(i)}^{2} + (X_{(i+1)} - X_{(i-1)})^{2}\right) + 2M_{(i)}\Pi_{(i)}^{\mu}\left(X_{(i+1)} - X_{(i-1)}\right)_{\mu}\right] \tag{3.18}$$

$$\mathcal{H}' = \mathcal{H} + \sum_{i=0}^{n}\mathcal{U}_{(i)}^{1}P_{N_{(i)}} + \mathcal{U}_{(i)}^{2}P_{M_{(i)}} \tag{3.19}$$

Using these expressions and those of the primary constraints, we now look for any secondary constraints:

$$\begin{aligned} \dot{P}_{N_{(i)}} &= \left\{P_{N_{(i)}}, \mathcal{H}\right\} = -\frac{1}{2}\left(\Pi_{(i)}^{2} + \left(X_{(i+1)} - X_{(i-1)}\right)^{2}\right) \equiv -\mathcal{C}_{(i)}^{0} \approx 0 \\ \dot{P}_{M_{(i)}} &= \left\{P_{M_{(i)}}, \mathcal{H}\right\} = -\Pi_{(i)}^{\mu}\left(X_{(i+1)} - X_{(i-1)}\right)_{\mu} \equiv -\mathcal{C}_{(i)}^{1} \approx 0 \end{aligned} \tag{3.20}$$

These are the secondary constraints of the theory, which however as we should see, are more important than the primary constraints. These expressions also imply that the $N_{(i)}$ and $M_{(i)}$ are to be treated as arbitrary Lagrange multipliers.

We should now check for closure of the constraint algebra; i.e. are there any more constraints?

For $\left\{\mathcal{C}_{(i)}^{0}, \mathcal{C}_{(j)}^{0}\right\}$ We find:

$$\begin{aligned} \left\{\mathcal{C}_{(i)}^{0}, \mathcal{C}_{(i+1)}^{0}\right\} &= \Pi_{(i+1)}\left(X_{(i+1)} - X_{(i-1)}\right) + \Pi_{(i)}\left(X_{(i+2)} - X_{(i)}\right) \\ \left\{\mathcal{C}_{(i)}^{0}, \mathcal{C}_{(i+1)}^{1}\right\} &= \Pi_{(i)}\Pi_{(i+1)} + \left(X_{(i+1)} - X_{(i-1)}\right)\left(X_{(i+2)} - X_{(i)}\right) \\ \left\{\mathcal{C}_{(i)}^{1}, \mathcal{C}_{(i+1)}^{1}\right\} &= \Pi_{(i)}\left(X_{(i+2)} - X_{(i)}\right) + \Pi_{(i+1)}\left(X_{(i+1)} - X_{(i-1)}\right) \end{aligned} \tag{3.21}$$

As is evident we see that there indeed are more constraints. These have the form:

$$\begin{aligned} \mathcal{D}_{(i)}^{a} &\equiv \frac{1}{2}\left[\Pi_{(i)}\Pi_{(i+1)} + \left(X_{(i+1)} - X_{(i-1)}\right)\left(X_{(i+2)} - X_{(i)}\right)\right] \\ \mathcal{D}_{(i)}^{b} &\equiv \frac{1}{2}\left[\Pi_{(i)}\left(X_{(i+2)} - X_{(i)}\right) + \Pi_{(i+1)}\left(X_{(i+1)} - X_{(i-1)}\right)\right] \end{aligned} \tag{3.22}$$

We are already departing form the continuum case; in the continuum string the constraint algebra of the secondary constraints was closed, and no new constraints existed. Here we are forming new constraints due to the discretization. This can be quite bad since too many constraints may make the final solution space, trivial. As a result we may start to anticipate that this model will not be too useful for the questions we set out to consider.

Nonetheless let us continue to find the ultimate implications of these findings, and the reasons behind them. To do so we need to find all the constraints. Therefore let us now consider the PB of the new Constraints. Another set of calculations will show that:

For $\left\{\mathcal{C}^0_{(i)}, \mathcal{D}^b_{(j)}\right\}$ We find:

$$\begin{aligned}
\left\{\mathcal{C}^0_{(i)}, \mathcal{D}^b_{(i-2)}\right\} &= -\frac{1}{2}\left[\Pi_{(i)}\Pi_{(i-2)} + (X_{(i+1)} - X_{(i-1)})(X_{(i-1)} - X_{(i-3)})\right] \\
\left\{\mathcal{C}^0_{(i)}, \mathcal{D}^b_{(i-1)}\right\} &= -\frac{1}{2}\left[{\Pi_{(i)}}^2 + (X_{(i+1)} - X_{(i-1)})^2\right] \quad = \quad -\mathcal{C}^0_{(i)} \\
\left\{\mathcal{C}^0_{(i)}, \mathcal{D}^b_{(i)}\right\} &= +\frac{1}{2}\left[{\Pi_{(i)}}^2 + (X_{(i+1)} - X_{(i-1)})^2\right] \quad = \quad +\mathcal{C}^0_{(i)} \\
\left\{\mathcal{C}^0_{(i)}, \mathcal{D}^b_{(i+1)}\right\} &= +\frac{1}{2}\left[\Pi_{(i)}\Pi_{(i+2)} + (X_{(i+1)} - X_{(i-1)})(X_{(i+3)} - X_{(i+1)})\right]
\end{aligned} \tag{3.23}$$

For $\left\{\mathcal{C}^1_{(i)}, \mathcal{D}^b_{(j)}\right\}$ We find:

$$\begin{aligned}
\left\{\mathcal{C}^1_{(i)}, \mathcal{D}^b_{(i-2)}\right\} &= -\frac{1}{2}\left[\Pi_{(i)}(X_{(i-1)} - X_{(i-3)}) + \Pi_{(i-2)}(X_{(i+1)} - X_{(i-1)})\right] \\
\left\{\mathcal{C}^1_{(i)}, \mathcal{D}^b_{(i-1)}\right\} &= -\mathcal{C}^0_{(i)} \\
\left\{\mathcal{C}^1_{(i)}, \mathcal{D}^b_{(i)}\right\} &= +\mathcal{C}^0_{(i)} \\
\left\{\mathcal{C}^1_{(i)}, \mathcal{D}^b_{(i+1)}\right\} &= +\frac{1}{2}\left[\Pi_{(i)}(X_{(i+3)} - X_{(i+1)}) + \Pi_{(i+2)}(X_{(i+1)} - X_{(i-1)})\right]
\end{aligned} \tag{3.24}$$

At this stage we get a hint that all the constraints are of the form:

$$\begin{cases}
\mathcal{C}^0_{(ij)} \equiv & \frac{1}{2}\left[\Pi_{(i)}\Pi_{(j)} + (X_{(i+1)} - X_{(i-1)})(X_{(j+1)} - X_{(j-1)})\right] \\
\mathcal{C}^1_{(ij)} \equiv & \frac{1}{2}\left[\Pi_{(i)}(X_{(j+1)} - X_{(j-1)}) + \Pi_{(j)}(X_{(i+1)} - X_{(i-1)})\right]
\end{cases} \tag{3.25}$$

If we calculate the only remaining cases, namely $\left\{\mathcal{C}^0_{(i)}, D^a_{(j)}\right\}$ and $\left\{\mathcal{C}^1_{(i)}, D^b_{(j)}\right\}$ we see that this is indeed the case. So this shows that the algebra of these new constraints, *eqn.*(3.25), is in fact closed:

$$\begin{aligned}
\left\{\mathcal{C}^0_{(ij)}, \mathcal{C}^0_{(kl)}\right\} = \frac{1}{2}\Big[&\big(\mathcal{C}^1_{ik}\delta_{j,l-1} + \mathcal{C}^1_{il}\delta_{j,k-1} + \mathcal{C}^1_{jk}\delta_{i,l-1} + \mathcal{C}^1_{jl}\delta_{i,k-1}\big) \\
&- \big(\mathcal{C}^1_{(ik}\delta_{j,l+1} + \mathcal{C}^1_{il}\delta_{j,k+1} + \mathcal{C}^1_{jk}\delta_{i,l+1} + \mathcal{C}^1_{jl}\delta_{i,k+1}\big)\Big]
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\left\{\mathcal{C}^0_{(ij)}, \mathcal{C}^1_{kl)}\right\} = \frac{1}{2}\Big[&\big(\mathcal{C}^0_{ik}\delta_{j,l-1} + \mathcal{C}^0_{il}\delta_{j,k-1} + \mathcal{C}^0_{jk}\delta_{i,l-1} + \mathcal{C}^0_{jl}\delta_{i,k-1}\big) \\
&- \big(\mathcal{C}^0_{ik}\delta_{j,l+1} + \mathcal{C}^0_{il}\delta_{j,k+1} + \mathcal{C}^0_{jk}\delta_{i,l+1} + \mathcal{C}^0_{jl}\delta_{i,k+1}\big)\Big]
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\left\{\mathcal{C}^1_{(ij)}, \mathcal{C}^1_{(kl)}\right\} = \frac{1}{2}\Big[&\big(\mathcal{C}^1_{ik}\delta_{j,l-1} + \mathcal{C}^1_{il}\delta_{j,k-1} + \mathcal{C}^1_{jk}\delta_{i,l-1} + \mathcal{C}^1_{jl}\delta_{i,k-1}\big) \\
&- \big(\mathcal{C}^1_{ik}\delta_{j,l+1} + \mathcal{C}^1_{il}\delta_{j,k+1} + \mathcal{C}^1_{jk}\delta_{i,l+1} + \mathcal{C}^1_{jl}\delta_{i,k+1}\big)\Big]
\end{aligned} \tag{3.28}$$

These constraints therefore define the complete set of the constraints of the system. The interesting feature of these is that they are *non-local*. The alarming feature of them is that *there are too many of them* relative to the (local) continuum case ($\infty \times \infty \quad vs. \quad \infty$.) We will investigate these feature in the upcoming sections.

We can also rewrite the PB of the new (non-local) constraints in a more suggestive way, the reason for which will become apparent in the next section:

$$\left\{\mathcal{C}^0_{(ij)}, \mathcal{C}^0_{(kl)}\right\} = \frac{1}{2}\Big[\big(\mathcal{C}^1_{ik}(\delta_{j+1,l} - \delta_{j-1,l}) + \mathcal{C}^1_{il}(\delta_{j+1,k} - \delta_{j-1,k}) + \mathcal{C}^1_{jk}(\delta_{i+1,l} - \delta_{i-1,l}) + \mathcal{C}^1_{jl}(\delta_{i+1,k} - \delta_{i-1,k})\Big] \tag{3.29}$$

$$\left\{\mathcal{C}^0_{(ij)}, \mathcal{C}^1_{kl)}\right\} = \frac{1}{2}\Big[\big(\mathcal{C}^0_{ik}(\delta_{j+1,l} - \delta_{j-1,l}) + \mathcal{C}^0_{il}(\delta_{j+1,k} - \delta_{j-1,k}) + \mathcal{C}^0_{jk}(\delta_{i+1,l} - \delta_{i-1,l}) + \mathcal{C}^0_{jl}(\delta_{i+1,k} - \delta_{i-1,k})\Big] \tag{3.30}$$

$$\left\{\mathcal{C}^1_{(ij)}, \mathcal{C}^1_{(kl)}\right\} = \frac{1}{2}\Big[\big(\mathcal{C}^1_{ik}(\delta_{j+1,l} - \delta_{j-1,l}) + \mathcal{C}^1_{il}(\delta_{j+1,k} - \delta_{j-1,k}) + \mathcal{C}^1_{jk}(\delta_{i+1,l} - \delta_{i-1,l}) + \mathcal{C}^1_{jl}(\delta_{i+1,k} - \delta_{i-1,k})\Big] \tag{3.31}$$

At this point let us make a few remarks:

- As was mentioned the above algebra is a generalization of the algebra of the constraints, *eqn.*2.10 of the continuum case. This fact can have significant implications which we will try to analyze fully in sections to come. For now however for completeness, let us just mention what the resulting Hamiltonian and the EOM thereof would be as a result of these calculations so far. The complete Hamiltonian, using the complete set of non-local constraints will have the form:

$$\mathcal{H}_E = \frac{1}{2}\sum_{ij}\left[\lambda^0_{(ij)}\mathcal{C}^0_{(ij)} + \gamma_{(ij)}\mathcal{C}^1_{(ij)}\right] + \sum_k\left[\mathcal{U}^1_{(k)}P_{N_{(k)}} + \mathcal{U}^2_{(k)}P_{M_{(k)}}\right] \tag{3.32}$$

  where the $\lambda^\alpha_{(ij)}$ are completely symmetric, arbitrary yet nonlocal "discrete smearing" functions. This smearing feature will turn out to be very important when we analyze the reason for the enlargement of the constraint group in *sec.*(3.2).

- In order to recover the original $N_{(i)}$ and $M_{(i)}$ we need:

$$\left\{\begin{array}{lcl} \lambda^\alpha_{(i\,j)} & \propto & \delta_{i\,j} \\ N_{(i)} & = & \lambda_{(i\,i)} \\ M_{(i)} & = & \gamma_{(i\,i)} \end{array}\right. \tag{3.33}$$

  These conditions are extremely nontrivial; there is no good reason why they should be obeyed. This is an indication that there may not exist any a good connection which would then relate the emerging non-local theory to the original local theory. We shall investigate this question as well.

- Note that this theory is consisted only of First-Class constraints. This is very good news for the quantization of the system because the elimination procedure ($\{\ \ ,\ \ \}_{PB} \to \{\ \ ,\ \ \}_{DB}$) which was briefly explained in *footnote*1 of sec.3.1.2 usually leads to nonlinearity in the constraints which then leads to certain quantum anomalies/ ambiguities.

So now lets us briefly look at the EOM of the Hamiltonian before delving into the problem of the "runaway" non-local constraints.

3.1.3 Hamiltonian Equations of Motion

Using the Hamiltonian which was found:

$$\mathcal{H}_E = \frac{1}{2}\sum_{ij}\left[\lambda^0_{(ij)}\mathcal{C}^0_{(ij)} + \gamma_{(ij)}\mathcal{C}^1_{(ij)}\right] + \sum_k\left[\mathcal{U}^1_{(k)}P_{N_{(k)}} + \mathcal{U}^2_{(k)}P_{M_{(k)}}\right]$$

and the relations in *table*(3.1), (which for completeness we have also shown their correspondence to the continuum analogues),

<u>*Discrete Case*</u>

$$\begin{aligned}
\{X_{(k)}, \mathcal{C}^0_{(ij)}\} &= \tfrac{1}{2}\left[\Pi_{(i)}\delta_{kj} + \Pi_{(j)}\delta_{ik}\right]\\
\{X_{(k)}, \mathcal{C}^1_{(ij)}\} &= \tfrac{1}{2}\left[(X_{(i+1)} - X_{(i-1)})\delta_{kj} + (X_{(j+1)} - X_{(j-1)})\delta_{ki}\right]\\
\{\Pi_{(k)}, \mathcal{C}^0_{(ij)}\} &= \tfrac{1}{2}\left[(X_{(i+1)} - X_{(i-1)})(\delta_{k,j-1} - \delta_{k,j+1})\right.\\
&\qquad \left. + (X_{(j+1)} - X_{(j-1)})(\delta_{k,i-1} - \delta_{k,i-1})\right]\\
\{\Pi_{(k)}, \mathcal{C}^1_{(ij)}\} &= \tfrac{1}{2}\left[\Pi_{(i)}(\delta_{k,j-1} - \delta_{k,j+1}) + \Pi_{(j)}(\delta_{k,i-1} - \delta_{k,i+1})\right]
\end{aligned}$$

<u>*Continuum Case*</u>

$$\begin{aligned}
\{X^\mu, G^0[\xi]\} &= \xi\Pi^\mu\\
\{X^\mu, G^1[\eta]\} &= \eta\acute{X}^\mu\\
\{\Pi^\mu, G^0[\xi]\} &= \overline{\xi\acute{X}}\\
\{\Pi^\mu, G^1[\eta]\} &= \overline{\eta\Pi}
\end{aligned}$$

Table 3.1: The transformations of the field variables via the constraint/ gauges, in the discrete and the continuum theories.

we can calculate the nonlocal Hamiltonian EOM:

$$\dot{X}^\mu_{(i)} = \left\{X^\mu_{(i)}, \mathcal{H}_E\right\} = \sum_{i=1}^{n}\left[\lambda_{(ik)}\Pi^\mu_{(i)} + \gamma_{(ik)}(X^\mu_{(i+1)} - X^\mu_{i-1)})\right] \tag{3.34}$$

$$\dot{\Pi}^\mu_{(i)} = \left\{\Pi^\mu_{(i)}, \mathcal{H}_E\right\} = \sum_{i=1}^{n}\left[\Pi^\mu_{(i)}(\gamma_{i,k+1} - \gamma_{i,k-1}) + (X^\mu_{(i+1)} - X^\mu_{(i-1)})(\lambda_{i,k+1} - \lambda_{i,k-1})\right] \tag{3.35}$$

Notice that using *eqns.*(3.33), we will get:

$$\dot{X}^\mu_{(i)} = N_{(i)}\Pi^\mu_{(i)} + M_{(i)}\left(X^\mu_{(i+1)} - X^\mu_{(i-1)}\right) \tag{3.36}$$

$$\begin{aligned}
\dot{\Pi}^\mu_{(i)} = &\left(M_{(i+1)}\Pi^\mu_{(i+1)} - M_{(i-1)}\Pi^\mu_{(i-1)}\right)\\
&+ \left(N_{(i+1)}(X^\mu_{(i+2)} - X^\mu_{(i)}) - N_{(i-1)}(X^\mu_{(i)} - X^\mu_{(i-2)})\right)
\end{aligned} \tag{3.37}$$

which corresponds correctly to *eqn.*(3.12), however again there is no physical reason for this reduction, of the nonlocal set of equations to the local ones.

3.2 Constraints Revisited

Let us now turn our attention to the problem of the constraints. The main question we are interested in is whether the (now nonlocal) theory can be recovered. To do so we need to analyze the main (physical and mathematical) reason for the enlargement of the constraint group, and then to see if this problem can somehow be remedied. These are the objectives of this section.

To do these we will find it helpful to repeatedly draw up the exact relation between the continuum case and the discrete case.

Let us first consider the *similarities* that exist between the two cases. Notice that in the continuum case we actually use the smeared versions of the constraints, or the gauges:

$$G^0[\xi] = \int \mathrm{d}\sigma C^0(\sigma)\xi(\sigma) \quad \textit{where} \quad C^0(\sigma) = \frac{1}{2}(\Pi^2 + X^2)(\sigma) \tag{3.38}$$

$$G^1[\eta] = \int \mathrm{d}\sigma C^1(\sigma)\eta(\sigma) \quad \textit{where} \quad C^1(\sigma) = \frac{1}{2}(\Pi^\mu X_\mu + X^\mu \Pi_\mu)(\sigma) \tag{3.39}$$

Yet if we compute the commutation relations of the constraints, $C^\alpha(\sigma)$ themselves:

$$\big\{C^0(\sigma), C^0(\acute{\sigma})\big\} = \Pi(\sigma)\acute{X}(\acute{\sigma})\partial_{\acute{\sigma}}\delta(\acute{\sigma}-\sigma) + \Pi(\acute{\sigma})\acute{X}(\sigma)\partial_\sigma\delta(\sigma-\acute{\sigma}) \tag{3.40}$$

i.e. the same situation arises as the discrete case. And if we were to follow the steps of the Dirac quantization (*sec.*(3.1.2)), and take the PB of these constraints, we would find:

$$\begin{aligned}\Big\{C^0(\sigma,\sigma^{'}), C^0(\sigma^{''},\sigma^{'''})\Big\} = \frac{1}{2}\Big[&C^1(\sigma,\sigma^{''})\partial_{\sigma^{'}}\delta(\sigma^{'}-\sigma^{'''}) + C^1(\sigma,\sigma^{'''})\partial_{\sigma^{'}}\delta(\sigma^{'}-\sigma^{''}) \\ &+ C^1(\sigma^{'},\sigma^{''})\partial_\sigma\delta(\sigma-\sigma^{'''}) + C^1(\sigma^{'},\sigma^{'''})\partial_\sigma\delta(\sigma-\sigma^{''})\Big]\end{aligned} \tag{3.41}$$

which is completely comparable to the case of nonlocal constraint algebra of the discrete case, *eqn.*(3.26). In other words if we consider the constraints rather than the gauges, we might be forced to consider a similar enlargement of the continuum constraints, as follows:

$$\begin{cases} C^0(\sigma,\acute{\sigma}) \equiv & \frac{1}{2}\big[\Pi(\sigma)\Pi(\acute{\sigma}) + \acute{X}(\sigma)\acute{X}(\acute{\sigma})\big] \\ C^1(\sigma,\acute{\sigma}) \equiv & \frac{1}{2}\big[\Pi(\sigma)\acute{X}(\acute{\sigma}) + \Pi(\acute{\sigma})\acute{X}(\sigma)\big] \end{cases} \tag{3.42}$$

Hence we realize that the continuum and the discrete theories are not that unlike. But more importantly through the above analysis we realize the importance of the smearing procedure in the continuum case; without this procedure the continuum case would be as over-constrained as the discrete case.

The main reason for using the smearing functions in the continuum case, and hence the gauges:

$$G^\alpha[\xi] = \int \mathrm{d}\sigma\xi(\sigma)C^\alpha(\sigma)\,.$$

rather than the constraints, $C^{\alpha}(\sigma)$, are the Dirac delta-functions $\delta(\sigma-\sigma')$ which appear in *eqn.*(3.41). The delta-function is properly defined within an integration only; otherwise the delta function is not a *proper function*. Hence it may be argued that the integration of the constraints are somewhat necessitated due to this mathematical feature.

We might therefore consider a similar procedure for the discrete case. However notice that the delta-functions will be replaced with kronecker deltas in the discrete case, and as such the main motivation for the "discrete smearing" does not exist. Regardless let us see if this intuition can be somehow fruitful in revealing something deeper about the non-locality of the discrete theory.

Therefore let us define the "discrete-smeared" constraints of the discrete string as follows:

$$\mathcal{G}^0[\lambda] \equiv \sum_{ij} \lambda_{ij}\mathcal{C}^0_{ij} = \sum_{ij} \lambda_{ij}\frac{1}{2}\left[\Pi_{(i)}\Pi_{(j)} + (X_{(i+1)} - X_{(i-1)})(X_{(j+1)} - X_{(j-1)})\right] \tag{3.43}$$

$$\mathcal{G}^1[\gamma] \equiv \sum_{ij} \gamma_{ij}\mathcal{C}^1_{ij} = \sum_{ij} \gamma_{ij}\frac{1}{2}\left[\Pi_{(i)}(X_{(j+1)} - X_{(j-1)}) + \Pi_{(j)}(X_{(i+1)} - X_{(i-1)})\right] \tag{3.44}$$

where the λ_{ij} and γ_{ij} are "discrete smearing" functions.

Now if we carrying out the calculation of the commutation relations, we have:[2]

$$\begin{aligned}
\left\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\right\} &= \sum_{ijkl} \lambda_{ij}\gamma_{kl}\left\{\mathcal{C}^0_{ij}, \mathcal{C}^0_{kl}\right\} &(3.45)\\
&= \sum_{ijkl} \lambda_{ij}\gamma_{kl}\frac{1}{2}\Big[\big(\mathcal{C}^1_{ik}\delta_{j,l-1} + \mathcal{C}^1_{il}\delta_{j,k-1} + \mathcal{C}^1_{jk}\delta_{i,l-1} + \mathcal{C}^1_{jl}\delta_{i,k-1}\big)\\
&\qquad - \big(\mathcal{C}^1_{ik}\delta_{j,l+1} + \mathcal{C}^1_{il}\delta_{j,k+1} + \mathcal{C}^1_{jk}\delta_{i,l+1} + \mathcal{C}^1_{jl}\delta_{i,k+1}\big)\Big] &(3.46)\\
&\equiv \sum_{ij} \mathcal{C}^1_{ij}\big(\mathcal{D}_\gamma\lambda\big)_{ij} &(3.47)\\
\Rightarrow \left\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\right\} &= \mathcal{G}^1[\mathcal{D}_\gamma\lambda] &(3.48)
\end{aligned}$$

where we have defined:

$$\begin{aligned}
\big(\mathcal{D}_\gamma\lambda\big)_{ij} \equiv \frac{1}{2}\sum_k \Big[&\lambda_{ik}(\gamma_{k+1,j} - \gamma_{k-1,j}) - \gamma_{ik}(\lambda_{k+1,j} - \lambda_{k-1,j})\\
&+ \lambda_{kj}(\gamma_{i,k+1} - \gamma_{i,k-1}) - \gamma_{kj}(\lambda_{i,k+1} - \lambda_{i,k-1})\Big].
\end{aligned} \tag{3.49}$$

And the complete "discrete gauge" algebra is:

$$\begin{aligned}
\left\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\right\} &= \mathcal{G}^1[\mathcal{D}_\gamma\lambda]\\
\left\{\mathcal{G}^0[\lambda], \mathcal{G}^1[\gamma]\right\} &= \mathcal{G}^0[\mathcal{D}_\gamma\lambda]\\
\left\{\mathcal{G}^1[\lambda], \mathcal{G}^1[\gamma]\right\} &= \mathcal{G}^1[\mathcal{D}_\gamma\lambda]
\end{aligned} \tag{3.50}$$

[2] We include this detail calculation here because it reflects on the relationship between the discrete and the continuous case

i.e. the algebra of the "discrete-smeared" constraints is closed. In addition these are in complete analogy to the continuum expressions:

$$
\begin{aligned}
\left\{G^0[\xi], G^0[\eta]\right\} &= G^1[\mathcal{L}_\eta \xi] \\
\left\{G^0[\xi], G^1[\eta]\right\} &= G^0[\mathcal{L}_\eta \xi] \\
\left\{G^1[\xi], G^1[\eta]\right\} &= G^1[\mathcal{L}_\eta \xi] \\
\textit{where}: & \\
\mathcal{L}_\eta \xi &\equiv \eta' \xi - \xi' \eta
\end{aligned}
$$

In particular we see that the $\mathcal{D}$ is "discrete" analogue of the Lie-derivative:

$$\mathcal{L}_\eta \xi \equiv \eta' \xi - \xi' \eta$$

which appears in the expression for the PB of the continuum gauges.

These results then confirm that the smearing of the constraints is a crucial procedure when we wish to analyze the constraint algebra of any theory.

Nevertheless, one important difference still remains; for the constraint algebra of the discrete case to close, the constraints, $\mathcal{C}^\alpha$ need still be nonlocal, whereas the continuum counterpart's constraint algebra can close with the original local constraints. We will investigate this in the next *sec.*(3.2.1).

In other words despite the similarity which we discovered above, the discreteness does produce features which are not present in the continuum case, and these features can upset the usefulness of the discrete theory. This investigation will be the content of *sec.*(3.2.2).

Let us therefore address these concerns.

3.2.1 Summary Comparison

In this section let us show that indeed the smeared version of the local constraints, does not produce a closed algebra in the discrete case.
This can be done with a simple example. If we define the local "discrete gauges" as:

$$\mathcal{G}^0[\lambda] \equiv \sum_i \lambda_i \mathcal{C}^0_i = \frac{1}{2}(\Pi_i^2 + (X_{i+1} - X_{i-1})^2) \tag{3.51}$$

$$\mathcal{G}^1[\gamma] \equiv \sum_i \gamma_i \mathcal{C}^1_i =\equiv (\Pi_i (X_{i+1} - X_{i-1})) \tag{3.52}$$

and consider the algebra that follows, we for example have:

$$
\begin{aligned}
\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\} &= \sum_{ij} \lambda_i \gamma_j \{\mathcal{C}_i^0, \mathcal{C}_j^0\} \\
&\vdots \\
&= \sum_i (X_{i+1} - X_{i-1}) \big[\Pi_{i+1}(\gamma_i \lambda_{i+1} - \gamma_{i+1}\lambda_i) - \Pi_{i-1}(\gamma_i \lambda_{i-1} - \gamma_{i-1}\lambda_i)\big]
\end{aligned}
\tag{3.53}
$$

First notice that the form of the combination of the X_i and Π_j, suggests that the results are of the form $\sim \mathcal{C}^1$. However, the local version of the $\mathcal{C}_k^1 \sim \Pi_k(X_{k+1} - X_{k-1})$. But as we can see the terms that are formed in the above calculation - i.e. the combination of the Π_i and $(X_{j+1} - X_{j-1})$ - are not in this format. Hence the final expression of the *eqn.*(3.53) cannot be hoped to be in the group of the local gauges.

This result confirms that smearing alone is not sufficient to make the algebra closed; we need to make the discrete constraints nonlocal if we wish to close the algebra. Hence the non-locality of the discrete theory persists.
This realization has two important implications:

1. The nonlocal discrete theory, is not related to the original continuous theory in the sense that the final solution space of these two theories will be under severely different restrictions.

2. As a result, the discrete theory is in fact perhaps related to, or has a different continuum limit.

While in the next two sections we will still continue to dig deeper into the mathematical feature which forces the discrete theory to be nonlocal, it is important to see what this alternate continuum limit is.

To answer this we take another look at the continuous case.
If we allow for the non-locality of the theory this can be done at the level of the constraints. Hence let us assume we have nonlocal continuous constraints:

$$G^0[\xi] = \int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \xi(\sigma, \sigma^{'}) C^0(\sigma, \sigma^{'}) \tag{3.54}$$

$$G^1[\eta] = \int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \eta(\sigma, \sigma^{'}) C^1(\sigma, \sigma^{'}) \tag{3.55}$$

where

$$C^0(\sigma, \sigma^{'}) \equiv \frac{1}{2}\Big(\Pi(\sigma)\Pi(\sigma^{'}) + \acute{X}(\sigma)\acute{X}(\sigma^{'})\Big) \tag{3.56}$$

$$C^1(\sigma, \sigma^{'}) \equiv \frac{1}{2}\Big(\Pi(\sigma)\acute{X}(\sigma^{'}) + \Pi(\sigma^{'})\acute{X}(\sigma)\Big) \tag{3.57}$$

Of course the algebra that follows is identical to the discrete case:

$$\begin{aligned}\left\{G^{\alpha}[\xi],G^{\beta}[\eta]\right\} &= \int \mathrm{d}\sigma\mathrm{d}\sigma^{''}C^{\{\alpha,\beta\}}(\sigma,\sigma^{''})\int \mathrm{d}\sigma^{'}\frac{1}{2}\big[(\partial_{\sigma^{'}}\eta(\sigma,\sigma^{'})\xi(\sigma^{'},\sigma^{''})-\partial_{\sigma^{'}}\xi(\sigma,\sigma^{'})\eta(\sigma^{'},\sigma^{''})\\ &\qquad +\partial_{\sigma^{'}}\eta(\sigma^{''},\sigma^{'})\xi(\sigma^{'},\sigma)-\partial_{\sigma^{'}}\xi(\sigma^{''},\sigma^{'})\eta(\sigma^{'},\sigma)\big]\\ &\equiv \int \mathrm{d}\sigma\mathrm{d}\sigma^{''}C^{\{\alpha,\beta\}}(\sigma,\sigma^{''})(\mathcal{L}_{\eta}\xi)(\sigma,\sigma^{''}) \qquad (3.58)\\ &= G^{\{\alpha,\beta\}}[\mathcal{L}_{\eta}\xi] \qquad (3.59)\end{aligned}$$

where

$$\{\alpha,\beta\}\equiv\begin{cases}1 & if \quad \alpha=\beta\\ 0 & if \quad \alpha\neq\beta\end{cases} \qquad (3.60)$$

and $(\mathcal{L}_{\eta}\xi)(\sigma,\sigma^{''})$ *is the nonlocal lie derivative.*

Now in order for the above nonlocal gauges, $G^{\alpha}[\xi]$, to represent gauges of the theory, they need to leave the action invariant:

$$\{S,G[\xi]\}=\delta^{\alpha}_{\xi}S=\delta^{\alpha}_{\xi}\int \mathrm{d}t\int \mathrm{d}\sigma\left(\Pi\dot{X}-\mathcal{H}\right)\approx 0 \qquad (3.61)$$

We saw in the beginning of this chapter that gauge transformations operate on the action by redefining the N and M functions, in the definition of the Hamiltonian, (*sec.*(2.2.1)):

$$H[N,M]=\int\mathcal{H} \qquad (3.62)$$

However, if we consider the transformations of the field variables due to the nonlocal gauges:

$$\delta^{\alpha}_{\xi}F(\sigma)\equiv\left\{F(\sigma),G^{\alpha}[\xi]\right\} \qquad (3.63)$$

which gives :

$$\begin{cases}\delta^{0}_{\xi}X(\sigma)=\int \mathrm{d}\sigma^{'}\xi(\sigma,\sigma^{'})\Pi(\sigma^{'})\\ \delta^{1}_{\eta}X(\sigma)=\int \mathrm{d}\sigma^{'}\eta(\sigma,\sigma^{'})\acute{X}(\sigma^{'})\\ \delta^{0}_{\xi}\Pi(\sigma)=\int \mathrm{d}\sigma^{'}\partial_{\sigma}\xi(\sigma,\sigma^{'})\acute{X}(\sigma^{'})\\ \delta^{1}_{\eta}X(\sigma)=\int \mathrm{d}\sigma^{'}\partial_{\sigma}\eta(\sigma,\sigma^{'})\Pi(\sigma^{'})\end{cases} \qquad (3.64)$$

we can see that in order for the action to remain invariant under the nonlocal gauge transformations, the N and M functions need be nonlocal as well:

$$H[N,M]=\int\mathcal{H}=G^{0}[N]+G^{1}[M]=\int \mathrm{d}\sigma\mathrm{d}\sigma^{'}\left[N(\sigma,\sigma^{'})C^{0}(\sigma,\sigma^{'})+M(\sigma,\sigma^{'})C^{1}(\sigma,\sigma^{'})\right] \qquad (3.65)$$

In fact a quick calculation will show that the transformation of S_{cov} by a general nonlocal gauge $G = G^0[\xi] + G^1[\eta]$, is as follows:

$$\delta_G S_{cov} = \left\{S_{cov}, G^0[\xi] + G^1[\eta]\right\} \tag{3.66}$$

Gives :

$$\begin{aligned} H[N,M] = G^0[N] + G^1[M] \rightarrow G^0&[N + \mathcal{L}_\xi M + \mathcal{L}_\eta N - \dot{\xi}] \\ &+ G^1[M + \mathcal{L}_\xi N + \mathcal{L}_\eta M - \dot{\eta}] \end{aligned} \tag{3.67}$$

where, N and M are nonlocal functions, and $\mathcal{L}_\xi \eta$ is the generalized nonlocal lie derivative which we defined above.
This of course means that the resulting action is nonlocal[3] :

$$S = \int \mathrm{d}t \int \mathrm{d}\sigma \mathrm{d}\sigma' \left[\Pi \dot{X} \delta(\sigma - \sigma') - N(\sigma, \sigma^{'}) C^0(\sigma, \sigma^{'}) - M(\sigma, \sigma^{'}) C^1(\sigma, \sigma^{'})\right] \tag{3.68}$$

Hence the proper continuum limit of the nonlocal discrete case is also the nonlocal string, defined via the above action. This is the action which we will in effect use in *sec.*(3.3) to analyze the solution space.

To make these results concrete, let us demonstrate that indeed the same results are obtained for the discrete case, and that the proper discrete action which we should use needs to be nonlocal.

The transformations of the field variables are again given by:

$$\delta^a_\lambda F_k = \left\{F_k, \mathcal{G}^a[\lambda]\right\} = \sum_{ij} \lambda_{ij} \left\{F_k, \mathcal{C}^a_{ij}\right\} \tag{3.69}$$

which gives :

$$\begin{cases} \delta^0_\lambda X_k = \sum_i \lambda_{ik} \Pi_i \\ \delta^1_\gamma X_k = \sum_i \gamma_{ik} (X_{i+1} - X_{i-1}) \\ \delta^0_\lambda \Pi_k = \sum_i (X_{i+1} - X_{i-1})(\lambda_{i,k+1} - \lambda_{i,k-1}) \\ \delta^1_\gamma \Pi_k = \sum_i \Pi_i (\gamma_{i,k+1} - \gamma_{i,k-1}) \end{cases} \tag{3.70}$$

And consequently a general nonlocal gauge $\mathcal{G} = \mathcal{G}^0[\lambda] + \mathcal{G}^1[\gamma]$ transforms the action, S_{cov}:

$$S_{cov} = \frac{1}{2} \int \mathrm{d}t \sum_i \Pi_i \dot{X}_i - \mathcal{H}_E \tag{3.71}$$

where

$$\mathcal{H}_E = \mathcal{G}^0[\mathcal{N}] + \mathcal{G}^1[\mathcal{M}] = \sum_{ij} \left(\mathcal{N}_{ij} \mathcal{C}^0_{ij} + \mathcal{M}_{ij} \mathcal{C}^1_{ij}\right) \tag{3.72}$$

[3]The reason for the $\delta\sigma - \sigma'$ in the first term is that we only have reason to make the Hamiltonian nonlocal.

as follows:

$$\delta_{\mathcal{G}} S_{cov} = \big\{ S_{cov}, \mathcal{G}^0[\lambda] + \mathcal{G}^1[\gamma] \big\} \tag{3.73}$$

Gives :

$$\begin{aligned} \mathcal{H} = \mathcal{G}^0[\mathcal{N}] + \mathcal{G}^1[\mathcal{M}] \rightarrow \mathcal{G}^0[&\mathcal{N} + \mathcal{D}_\lambda \mathcal{M} + \mathcal{D}_\gamma \mathcal{N} - \dot{\lambda}] \\ &+ \mathcal{G}^1[\mathcal{M} + \mathcal{D}_\lambda \mathcal{N} + \mathcal{D}_\gamma \mathcal{M} - \dot{\gamma}] \end{aligned} \tag{3.74}$$

where, again:

$$\begin{aligned} (\mathcal{D}_\gamma \lambda)_{ij} = \frac{1}{2} \sum_k \big[&\lambda_{ik}(\gamma_{k,j+1} - \gamma_{k,j-1}) - \gamma_{ik}(\lambda_{k,j+1} - \lambda_{k,j-1}) \\ &+ \lambda_{jk}(\gamma_{k,i+1} - \gamma_{k,i-1}) - \gamma_{jk}(\lambda_{k,i+1} - \lambda_{k,i-1}) \big] \end{aligned}$$

This reconfirms the result that wherever the $X_i^\mu - \Pi_i^\mu - combination$ and $\mathcal{N}_i, \mathcal{M}_i$ appear in the action, they must be replaced by their more general, non-local substitutes.

Now let us return to analyzing the exact reason that causes the discrete case to be ultimately nonlocal, when its continuum counterpart can be made to be local. In other words let us see what mathematical construct does not translate completely in the transition of $Continuum \rightarrow Discrete$.

3.2.2 The Link

In this section let us identify the exact cause of the discrepancy between the local discrete theory and the local continuum theory. To do this we will again run a parallel calculation between these two cases.

To start, let us notice that the results of the previous sections do show that the smearing is an important procedure; *it is the smearing, in the continuum case, which allows the nonlocal algebra of the constraints to define a local algebra of gauges.* Hence we want to ask what is it in the *integration* that is not present in the *summation*, which then does not produce the same results for the discrete case.

At first it may seem that it is indeed some property of the *integration*, in the continuum case, which is not present for the *sum*, in the discrete case which leads to this difference. However as we shall see, this difference is in fact related to much more fundamental concepts in the theory of continuum than these operations.

Therefore let us by considering the algebra of the constraints. We had:

$$\Big\{ C^0(\sigma), C^0(\sigma^{'}) \Big\} = \Big(\Pi(\sigma) \acute{X}(\sigma^{'}) + \Pi(\sigma^{'}) \acute{X}(\sigma) \Big) \partial_{\sigma'} \delta(\sigma^{'} - \sigma) = 2C^1(\sigma, \sigma^{'}) \partial_\sigma \delta(\sigma - \sigma^{'}) \tag{3.75}$$

$$\Big\{ \mathcal{C}_i^0, \mathcal{C}_j^0 \Big\} = \Big(\Pi_i (X_{j+1} - X_{j-1}) + \Pi_j (X_{i+1} - X_{i-1}) \Big) (\delta_{i,j-1} - \delta_{i,j+1}) = 2C_{ij}^1 (\delta_{i+1,j} - \delta_{i-1,j}) \tag{3.76}$$

As mentioned both these, the continuum and the discrete, produce nonlocal constraint algebras. However when we smear these we get different outcomes. Let us therefore carry out the steps of the smearing to locate the exact place where this difference comes from.

In the continuum case the step by step calculation of the smearing is as follows:

$$\begin{aligned}\int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \xi(\sigma)\eta(\sigma^{'})\{C^0(\sigma), C^0(\sigma^{'})\} &= \int\int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \xi(\sigma)\eta(\sigma^{'})\Big[\Pi(\sigma)\acute{X}(\sigma^{'} + \Pi(\sigma^{'})\acute{X}(\sigma))\Big]\partial_\sigma \delta(\sigma - \sigma^{'})) \\ &= -\int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \eta(\sigma^{'})\big[\acute{X}(\sigma^{'})\partial_\sigma(\xi(\sigma)\Pi(\sigma)) + \Pi(\sigma^{'})\partial_\sigma(\xi(\sigma)\acute{X}(\sigma)\big]\delta(\sigma - \sigma^{'}) \\ &= -\int \mathrm{d}\sigma \mathrm{d}\sigma^{'} \Big\{\eta(\sigma^{'}\acute{\xi}(\sigma)\big[\Pi(\sigma)\acute{X}(\sigma^{'}) + \Pi(\sigma^{'})\acute{X}(\sigma) \\ &\qquad + \eta(\sigma^{'})\xi\sigma\big[\acute{X}(\sigma^{'})\partial_\sigma \Pi(\sigma) + \Pi(\sigma^{'})\partial_\sigma \acute{X}(\sigma)\big]\Big\}\delta(\sigma - \sigma^{'}) \\ &= -\int \mathrm{d}\sigma \eta(\sigma)\Big\{2\partial_\sigma \xi(\sigma)\acute{X}(\sigma)\Pi(\sigma) + \xi(\sigma)\partial_\sigma(\acute{X}(\sigma)\Pi(\sigma)\Big\} \\ &= \int \mathrm{d}\sigma\big(\partial_\sigma(\eta\xi) - 2\eta\acute{\xi}\big)\Pi\acute{X} \\ &= \int \mathrm{d}\sigma(\eta^{'}\xi - \eta\xi^{'})\Pi\acute{X} \\ &= G^1[\mathcal{L}_\eta \xi]\end{aligned} \tag{3.77}$$

The corresponding step by step "discrete smearing" procedure is as follows:

$$\begin{aligned}\sum_{ij} \lambda_i \gamma_j \{\mathcal{C}_i^0, \mathcal{C}_j^0\} &= \sum_{ij} \lambda_i \gamma_j \big(\Pi_i(X_{j+1} - X_{j-1}) + \Pi_j(X_{i+1} - X_{i-1})\big)(\delta_{i+1,j} - \delta_{i-1,j}) \\ &= \sum_{ij} \Big[\gamma_j(X_{j+1} - X_{j-1})\big(\lambda_{i-1}\Pi_{i-1} - \lambda_{i+1}\Pi_{i+1}\big)\gamma_j \Pi_j\big((X_i - X_{i-2}) - (X_{i+2} - X_i)\big)\Big]\delta_{i,j}\end{aligned} \tag{3.78}$$

As mentioned, by comparing the above two results, it can easily be seen that the sum does indeed mirror the integration; it is in fact the *differentiation* that fails in the following sense.

Note that when we cary the first integration by part we get:

$$\xi(x)F(x)\partial_x \delta(x - x^{'}) \to -\partial_x(\xi(x)F(x))\delta(x - x^{'})\,, \tag{3.79}$$

Whereas in the discrete case:

$$\lambda_m \mathcal{F}_m(\delta_{m,n+1} - \delta_{m,n-1}) \to (\lambda\mathcal{F})_{n+1} - (\lambda\mathcal{F})_{n-1}\,. \tag{3.80}$$

In other words *the discrete case seems not to be mirroring the differentiation chain-rule!* Let us confirm this. In other words let us ask if there actually exists some identity that relates the above two expressions - in which case then these are not different - or that these two expressions cannot *generally* be set to be equal.

For this let us consider the discrete *differentiation*. We find the following identity:

$$\begin{aligned}(\lambda\mathcal{F})_{n+1} - (\lambda\mathcal{F})_{n-1} &= (\lambda_n + \Delta)(\mathcal{F}_n + \Delta^{'}) - (\lambda_n - \Delta'')(\mathcal{F}_n - \Delta^{'''}) \\ &= \big(\lambda_n\mathcal{F}_{n+1} + \lambda_{n+1}\mathcal{F}_n - \lambda_n\mathcal{F}_n + \Delta\Delta^{'}\big) \\ &\qquad - \big(\lambda_n\mathcal{F}_{n-1} + \lambda_{n-1}\mathcal{F}_n - \lambda_n\mathcal{F}_n + \Delta^{''}\Delta^{'''}\big) \\ &= \lambda_n\big(\mathcal{F}_{n+1} - \mathcal{F}_{n-1}\big) + \big(\lambda_{n+1} - \lambda_{n-1}\big)\mathcal{F}_n + (\Delta\Delta^{'} - \Delta^{''}\Delta^{'''})_n\end{aligned}$$

which we define as:

$$\Rightarrow (\lambda\mathcal{F})_{n+1} - (\lambda\mathcal{F})_{n-1} \equiv \lambda_n Dif(\mathcal{F}_n) + Dif(\lambda_n)\mathcal{F}_n + f(\Delta\Delta^{'} - \Delta^{''}\Delta^{'''})_n \qquad (3.81)$$

Notice that in the limit that we take $\Delta \to 0$, the above identity *eqn.*(3.81) does produce the equivalence - between the *continuum chain rule*, *eqn.*(3.79), and the *"discrete chain rule"*, *eqn.*(3.80) - which we seek.
Therefore the main question we need to ask is, under what conditions will this limit be warranted, whether these conditions exit in our theory, and if not what are the consequences?

However before we address these questions, for completion let us check that if we set the $f(\Delta\Delta^{'} - \Delta^{''}\Delta^{'''}) = 0$, the "discrete smeared" gauge algebra does close.

Let us therefore conduct an example calculation. Using the identity, *eqn.*(3.81), we can then see:

$$\begin{aligned}\sum_{ij}\lambda_i\gamma_j\big\{\mathcal{C}_i^0,\mathcal{C}_j^0\big\} &= \sum_{ij}\lambda_i\gamma_j\big(\Pi_i(X_{j+1} - X_{j-1}) + \Pi_j(X_{i+1} - X_{i-1})\big)(\delta_{i+1,j} - \delta_{i-1,j}) \\ &= \frac{1}{2}\sum_i\Big\{\lambda_i\big[\Pi_i(\gamma_{i+1}(X_{i+2} - X_i) - \gamma_{i-1}(X_i - X_{i-2})) + (X_{i+1} - X_{i-1})(\gamma_{i+1}\Pi_{i+1} - \gamma_{i-1}\Pi_{i-1})\big] \\ &\qquad - \gamma_i\big[(X_{i+1} - X_{i-1})(\Pi_{i+1}\lambda_{i+1} - \Pi_{i-1}\lambda_{i-1}) + \Pi_i((X_{i+2} - X_i)\lambda_{i+1} - (X_i - X_{i-2}\lambda_{i-1})\big]\Big\}\end{aligned}$$

now if we assume $f = 0$, and therefore using:

$$\lambda_{n+1}\mathcal{F}_{n+1} - \lambda_{n-1}\mathcal{F}_{n-1} = (\lambda_{i+1} - \lambda_{i-1})\mathcal{F}_i + \lambda_i(\mathcal{F}_{i+1} - \mathcal{F}_{i-1}) \qquad (3.82)$$

the local "discrete smeared" gauge algebra becomes:

$$\begin{aligned}\Rightarrow \big\{\mathcal{G}^0[\lambda],\mathcal{G}^0[\gamma]\big\} &= \frac{1}{2}\sum_i\Big\{\lambda_i\big[\Pi_i\big((\gamma_{i+1} - \gamma_{i-1})(X_{i+1} - X_{i-1}) + \gamma_i(X_{i+2} - 2X_i - X_{i-2})\big) \\ &\qquad + (X_{i+1} - X_{i-1})\big((\gamma_{i+1} - \gamma_{i-1})\Pi_i + \gamma_i(\Pi_{i+1} - \Pi_{i-1})\big)\big] \\ &\qquad - \gamma_i\big[(X_{i+1} - X_{i-1})\big((\Pi_{i+1} - \Pi_{i-1})\lambda_i + (\lambda_{i+1} - \lambda_{i-1})\Pi_i\big) \\ &\qquad + \Pi_i\big((X_{i+2} - 2X_i - X_{i-2})\lambda_i + (X_{i+1} - X_{i-1})(\lambda_{i+1} - \lambda_{i-1})\big)\big]\Big\} \\ &= \sum_i\big[\lambda_i(\gamma_{i+1} - \gamma_{i-1}) - \gamma_i(\lambda_{i+1} - \lambda_{i-1})\big]\Pi_i(X_{i+1} - X_{i-1})\end{aligned}$$

Therefore for this example we have:

$$\left\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\right\} = \mathcal{G}^1[\mathcal{D}_\gamma \lambda]$$

where

$$(\mathcal{D}_\gamma \lambda)_i = \lambda_i(\gamma_{i+1} - \gamma_{i-1}) - \gamma_i(\lambda_{i+1} - \lambda_{i-1}) \tag{3.83}$$

is the local version of the $\mathcal{D}$ in the previous sections, and is analogous to the local lie derivative of the continuum.

This calculation can be done for the other sets of constraints and the same result is found. Hence we see that if we could ignore the $f(\Delta\Delta^{'} - \Delta^{''}\Delta^{'''})$ the local discrete case would produce closed algebras, if we smeared the constraints - in complete analogy to the local continuum case.

Let us now consider the question of the Deltas, Δ.

3.2.3 The Delta's, Δ's

In the first place let us realize that in the discrete case there does not exist any good reason to *a-priori* set the $\Delta \to 0$. In fact these deltas are related to the the discrete structure. Hence unless at some point we are going to take the limit $Discrete \to Continuum$, we cannot hope to arbitrary set the $\Delta = 0$. However even if we did wish to take such a limit, notice that a problem arises when we consider that the theory is to be quantized. Let us discuss this at the end of this section.

Instead let us first consider the *consequences* of the nonzero Δ. Recall that we are dealing with two different notions:

1. We found that the discrete theory has a nonlocal constraint algebra
2. We realized that the reason that the local constraint algebra does not close are because of the Δ's, or the discreteness of the theory

A natural question is then, what is the *relation* between these Δ's and the nonlocal parts of the constraint operators.

To see this let us therefore consider the nonlocal constraint algebra, and contrast it against the extra pieces which are produced in the calculations of the local constraint algebra, were we to use the full identity, *eqn.*(3.81), (i.e. were we to consider nonzero Δ.)

Recall that the full identity was of the form:

$$\lambda_{i+1}\mathcal{F}_{i+1} - \lambda_{i-1}\mathcal{F}_{i-1} = (\lambda_{i+1} - \lambda_{i-1})\mathcal{F}_i + \lambda_i(\mathcal{F}_{i+1} - \mathcal{F}_{i-1}) + (\Delta\Delta^{'} - \Delta^{''}\Delta^{'''})_i \tag{3.84}$$

Where we define, (*fig.*(3.1)):

$$\Delta_i = \Delta^+_{\lambda_i} \qquad \Delta^{''}_i = \Delta^-_{\lambda_i} \qquad \Delta^{'}_i = \Delta^+_{\mathcal{F}_i} \qquad \Delta^{'''}_i = \Delta^-_{\mathcal{F}_i}$$

$$and:$$

$$f(\Delta\Delta^{'} - \Delta^{''}\Delta^{'''})_i = (\Delta^+_\lambda\Delta^+_\mathcal{F} - \Delta^-_\lambda\Delta^-_\mathcal{F})_i \equiv \Delta_{[\lambda\mathcal{F}]_i}$$

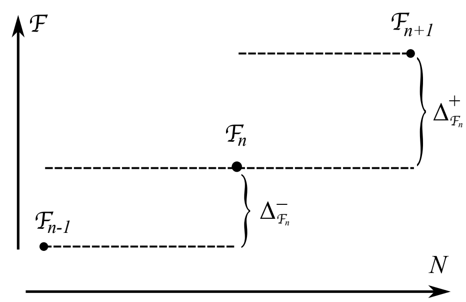

Figure 3.1: Discrete Functions on the string, and the definition of the Deltas.

Then the full local gauge algebra becomes:

$$\begin{aligned}\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\} = \frac{1}{2}\sum_i \Big\{\lambda_i\big[\Pi_i\big((\gamma_{i+1}-\gamma_{i-1})(X_{i+1}-X_{i-1}) + \gamma_i(X_{i+2}-2X_i-X_{i-2}) + \Delta_{[\gamma\acute{X}]_i}\big) \\ + (X_{i+1}-X_{i-1})\big((\gamma_{i+1}-\gamma_{i-1})\Pi_i + \gamma_i(\Pi_{i+1}-\Pi_{i-1}) + \Delta_{[\gamma\Pi]_i}\big)\big] \\ - \gamma_i\big[(X_{i+1}-X_{i-1})\big((\Pi_{i+1}-\Pi_{i-1})\lambda_i + (\lambda_{i+1}-\lambda_{i-1})\Pi_i + \Delta_{[\lambda\Pi]_i}\big) \\ + \Pi_i\big((X_{i+2}-2X_i-X_{i-2})\lambda_i + (X_{i+1}-X_{i-1})(\lambda_{i+1}-\lambda_{i-1})\Delta_{[\lambda\acute{X}]_i}\big)\big]\Big\}\end{aligned}$$

$$\begin{aligned}= \sum_i \big[\lambda_i(\gamma_{i+1}-\gamma_{i-1}) - \gamma_i(\lambda_{i+1}-\lambda_{i-1})\big]\Pi_i(X_{i+1}-X_{i-1}) \\ + \frac{1}{2}\sum_i\Big\{\lambda_i\Big(\Pi_i\Delta_{[\gamma\acute{X}]_i} + (X_{i+1}-X_{i-1})\Delta_{[\gamma\Pi]_i}\Big) - \gamma_i\Big(\Pi_i\Delta_{[\lambda\acute{X}]_i} + (X_{i+1}-X_{i-1})\Delta_{[\lambda\Pi]_i}\Big)\Big\}\end{aligned}$$

$$\begin{aligned}\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\} = \mathcal{G}^1[\mathcal{D}_\gamma\lambda] + \frac{1}{2}\sum_i\Big\{\lambda_i\Big(\Pi_i\Delta_{[\gamma\acute{X}]_i} + (X_{i+1}-X_{i-1})\Delta_{[\gamma\Pi]_i}\Big) \\ - \gamma_i\Big(\Pi_i\Delta_{[\lambda\acute{X}]_i} + (X_{i+1}-X_{i-1})\Delta_{[\lambda\Pi]_i}\Big)\Big\}\end{aligned} \tag{3.85}$$

where:

$$(\mathcal{D}_\gamma\lambda)_i = \lambda_i(\gamma_{i+1}-\gamma_{i-1}) - \gamma_i(\lambda_{i+1}-\lambda_{i-1})$$

Comparing this to the nonlocal constraint algebra:

$$
\begin{aligned}
\{\mathcal{G}^0[\lambda], \mathcal{G}^0[\gamma]\} &= \sum_{ijkl} \lambda_{ij}\gamma_{kl}\{\mathcal{C}^0_{ij}, \mathcal{C}^0_{kl}\} \\
&= \sum_{ij} \mathcal{C}^1_{ij}(\mathcal{D}_\gamma \lambda)_{ij} \qquad (3.86) \\
&= \sum_{ij} \frac{1}{2}\big[\Pi_i\,(X_{j+1} - X_{j-1}) + \Pi_j\,(X_{i+1} - X_{i-1})\,\big] \\
&\qquad \times \frac{1}{2}\sum_k \big[\lambda_{ik}\,(\gamma_{k,j+1} - \gamma_{k,j-1}) - \gamma_{ik}\,(\lambda_{k,j+1} - \lambda_{k,j-1}) \\
&\qquad\qquad + \lambda_{jk}\,(\gamma_{k,i+1} - \gamma_{k,i-1}) - \gamma_{jk}\,(\lambda_{k,i+1} - \lambda_{k,i-1})\,\big] \\
&= \sum_i \mathcal{C}^1_{ii}(\mathcal{D}_\gamma \lambda)_{ii} + \sum_{i\neq j} \mathcal{C}^1_{ij}(\mathcal{D}_\gamma \lambda)_{ij} \qquad (3.87)
\end{aligned}
$$

we see that all nonlocal terms, i.e. the off-diagonal terms of $\mathcal{C}^\alpha_{ij}$ are indeed proportional to terms of order $\Delta_{[\lambda\mathcal{F}]_i} = (\Delta^+_\lambda \Delta^+_\mathcal{F} - \Delta^-_\lambda \Delta^-_\mathcal{F})_i$. This is the nontrivial result of this section; *the off-diagonal terms of the nonlocal discrete theory are related to the* Δ*-terms of the local discrete theory.*

Using this result we can now answer the other questions, which were posed at the beginning of this section. Notice that the off-diagonal terms therefore vanish in the classical continuum case limit, because - based on the above finding - this is the same limit where we take $\Delta(function) \rightarrow 0$. Therefore the non-locality, in the discrete case, would vanish *classically*, when we take the continuum limit.

We may ask if this is true for the quantum theory as well. Notice that in the quantum theory, we wish to compare the quantum solution spaces of the discrete and the continuum case. Hence we will be quantizing the system before taking the continuum limit. In this case then Δ terms become operators and cannot be trivially set to zero. Hence in the quantum theory, the off-diagonal terms of the nonlocal constraints can be anticipated to contribute to the final analysis of the solution space and cannot therefore *a-priori* be ignored.

We therefore see clearly why the discrete and the continuum formulations lead to different theories, and can clearly identify the source of this difference.

Let us now ask the only question that remains; namely is it possible to get rid of the extra constraints using a different discretization method. This is the question we will address in the next section. The answer will be in the negative. We will see that the mathematical impossibility of the simultaneous diagonalization of the two constraint matrices indeed has physical consequences.

3.2.4 Non-Diagonalizability

The question we wish to ask is if the nonlocal constraint matrices can be diagonalized via a new choice of the discretization. Namely, if we were to choose a new set of X_i^μ and Π_j^μ, would the constraints:

$$
\begin{aligned}
\mathcal{C}^0_{ij} &= \frac{1}{2}\Big(\Pi_i\Pi_j + (X_{i+1} - X_{i-1})(X_{j+1} - X_{j-1})\Big) \\
\mathcal{C}^1_{ij} &= \frac{1}{2}\Big(\Pi_i(X_{j+1} - X_{j-1}) + \Pi_j(X_{i+1} - X_{i-1})\Big)\,,
\end{aligned}
$$

become diagonalized? Of course there may in fact be a large class of transformations that can be performed to redefine the discretization. And it may not be possible to analyze all of these. However let us motivate two physical requirements, which will then greatly reduce the class of possible transformations. Let us denote a transformation which redefines the discretization by **T**. Let us posit that these should have the following properties:

- A given transformation **T**, must be a constant (independent of σ and t). It can be argued that more general transformations will simply relocate a portion of the dynamics of the field variables to **T**.
- **T** must also only be a linear mixing of the space nodes; a nonlinear mixing would require **T** to be a function of the field variables X^μ and Π^μ which is again not useful for the same reason as above.

Based on these requirements, the only "physically interesting" transformations we will consider are *Constant Linear Transformations.*

Now let us analyze this class of transformations. First let us define the *field-vector*:

$$
\overrightarrow{X^\mu} \equiv \begin{pmatrix} X_1^\mu \\ X_2^\mu \\ \vdots \\ X_N^\mu \end{pmatrix} \tag{3.88}
$$

A constant linear transformation of $\overrightarrow{X^\mu}$ produces a linear combinations of the elements of this vector. The **T** can therefore be presented as linear matrix transformations, on the space of the field-vectors:

$$
\mathbf{T} : \overrightarrow{X^\mu} \to \widetilde{X^\mu} = \mathbf{T}\overrightarrow{X^\mu} \tag{3.89}
$$

Similarly we can define the following field vectors:

$$
\vec{\Pi} \equiv \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_N \end{pmatrix} \qquad \overrightarrow{X'} \equiv \begin{pmatrix} X'_1 \\ X'_2 \\ \vdots \\ X'_N \end{pmatrix} \qquad \textit{where} \quad X'_i \equiv X_{i+1} - X_{i-1} \tag{3.90}
$$

Since $\mathbf{T}$ are independent of t and σ, we can see that:

$$\mathbf{T}:\begin{cases}\overrightarrow{X'^{\mu}} \rightarrow \widetilde{X'^{\mu}} \equiv \mathbf{T}\overrightarrow{X'^{\mu}}\\ \overrightarrow{\dot{X}^{\mu}} \rightarrow \widetilde{\dot{X}^{\mu}} \equiv \mathbf{T}\overrightarrow{\dot{X}^{\mu}}\end{cases} \tag{3.91}$$

However a simple comparison will show that:

$$\Pi = \frac{1}{N}(\dot{X} - MX') \quad \Rightarrow \quad \mathbf{T}:\overrightarrow{\Pi^{\mu}} \rightarrow \widetilde{\Pi^{\mu}} \equiv \mathbf{T}\overrightarrow{\Pi^{\mu}} \tag{3.92}$$

however since :

$$\widetilde{\Pi} = \frac{\delta\mathcal{L}}{\delta\dot{\widetilde{X}}} = \frac{\delta X}{\delta\widetilde{X}}\frac{\delta\mathcal{L}}{\delta\dot{X}} = (\mathbf{T}^{-1})^t\Pi \tag{3.93}$$

comparing the above two equations we get:

$$\mathbf{T}^{-1} = \mathbf{T}^t \tag{3.94}$$

i.e. the $\mathbf{T}$ are real-unitary matrices. This fact will be useful later.

We can also denote the field-vectors as kets: $\overrightarrow{Y} \equiv |Y\rangle$. We will use this notation in what follows. Now in order to consider the constraints, let us define the following matrices:

$$\mathbf{\Pi^2} \quad \equiv |\Pi\rangle\langle\Pi| \quad = \begin{pmatrix}\Pi_1^2 & \Pi_1\Pi_2 & \cdots & \Pi_1\Pi_N\\ \Pi_2\Pi_1 & \Pi_2^2 & \cdots & \Pi_2\Pi_N\\ \vdots & \vdots & \ddots & \vdots\\ \Pi_N\Pi_1 & \Pi_N\Pi_2 & \cdots & \Pi_N^2\end{pmatrix} \tag{3.95}$$

$$\mathbf{X'^2} \quad \equiv |X'\rangle\langle X'| \quad = \begin{pmatrix}X'^2_1 & X'_1X'_2 & \cdots & X'_1X'_N\\ X'_2X'_1 & X'^2_2 & \cdots & X'_2X'_N\\ \vdots & \vdots & \ddots & \vdots\\ X'_NX'_1 & X'_NX'_2 & \cdots & X'^2_N\end{pmatrix} \tag{3.96}$$

$$\mathbf{\Pi X'} \quad \equiv |\Pi\rangle\langle X'| \quad = \begin{pmatrix}\Pi_1X'_1 & \Pi_1X'_2 & \cdots & \Pi_1X'_N\\ \Pi_2X'_1 & \Pi_2X'_2 & \cdots & \Pi_2X'_N\\ \vdots & \vdots & \ddots & \vdots\\ \Pi_NX'_1 & \Pi_NX'_2 & \cdots & \Pi_NX'_N\end{pmatrix} \tag{3.97}$$

with a similar expression for:

$$\mathbf{X'\Pi} \equiv |X'\rangle\langle\Pi| \tag{3.98}$$

Using these we can now rewrite the constraint matrices as:

$$\begin{cases}\mathbf{C^0} = \dfrac{1}{2}(\mathbf{\Pi^2} + \mathbf{X'^2})\\ \mathbf{C^1} = \dfrac{1}{2}(\mathbf{\Pi X'} + \mathbf{X'\Pi})\end{cases} \tag{3.99}$$

This provides us with the expressions which we can now manipulate to analyze the possible effects of the linear transformations on the constraints. This analysis then will enable us to answer whether a re-discretization (of the form we have argued) can make the nonlocal constraints diagonal, and therefore local.

Notice that a given linear transformation, transforms the constraints in the following way:

$$\begin{cases} \mathbf{C}^0 & \rightarrow \quad \tilde{\mathbf{C}^0} = \mathbf{T}\mathbf{C}^0\mathbf{T}^t \\ \mathbf{C}^1 & \rightarrow \quad \tilde{\mathbf{C}^1} = \mathbf{T}\mathbf{C}^1\mathbf{T}^t \,. \end{cases} \tag{3.100}$$

Hence in order to diagonalize these simultaneously, (via a linear transformation on the basic phase-space variables) requires that $\mathbf{C^0}$ and $\mathbf{C^1}$ to commute. However from the constraint algebra we know that the constraints do not commute. Hence these constraints cannot be diagonalized simultaneously by a transformation of the kind we have considered. This means that the nonlocal constraints cannot be made local by a re-descretization (subject to the physical conditions defined at the beginning of this section.)

This concludes our analysis of the non-locality of the constraints of the discrete theory. We investigated all aspect of the implications of this feature of the discrete theory:

- We found that the non-locality has significant implications for the action of the system - *the action has to be nonlocal for constraints to represent gauge transformations.*
- We analyzed the fundamental reason behind the non-locality, and in the process found interesting consequences of discretization when the system is quantized - *the non-locality cannot be trivially removed in the quantum discrete theory, when we take the continuum limit.*
- Furthermore we investigated alternatives of the discretization which would result in a local theory, however the results of this section shows that such a discretization (under certain restrictions) does not exist. *Hence non-locality is an unremovable feature of the discrete theory.*

We conclude that a discrete string theory will therefore have a nonlocal constraint algebra and a nonlocal action. We already anticipate that the gauge-group is too large to allow for non-trivial physics in the solution space of this theory. However there exist several reasons which still motivate this investigation:

- Firstly we should confirm this triviality.
- Secondly, as was discussed, there are interesting relations between the non-locality and the quantization process.

Let us therefore next look at the solution space that the discrete string theory describes.

3.3 The Solution Space

We have seen that the discrete string's constraint algebra closes if the constraints and therefore the action are nonlocal. We therefore start with the nonlocal discrete action of *sec.*(3.2.1). We will first show that this action does produce the proper equations of motion and the corresponding closed constraint algebra.
Then we proceed to develop a solution space, using the methods of the continuous string of *chapter*(2).
Then we ask to see what, if any, nontrivial physics can be extracted from the solution space, after it has be treated with the nonlocal constraints via the methods of Virasoro algebra of String theory[4].

3.3.1 The Action, The EOM and The Constraints

The action we start with is therefore the following (the i and j run over N nodes):

$$S = -\frac{1}{2}\int \mathrm{d}t \sum_{ij=1}^{N} (\sqrt{-g}g^{ab})(i,j;t)(\partial_a X_i^\mu \partial_b X_j^\mu)(t)$$

with the *nonlocal metric*:

$$\begin{aligned} g_{ab}(i,j;t) &= \gamma_{ij}^2 \begin{pmatrix} -N_{ij}^2 + M_{ij}^2 & M_{ij} \\ M_{ij} & 1 \end{pmatrix}(t) \\ g^{ab}(i,j;t) &= N^{ij}{\gamma^{ij}}^2 \begin{pmatrix} -1 & M_{ij} \\ M_{ij} & N_{ij}^2 - M_{ij}^2 \end{pmatrix}(t) \\ &\quad where \\ &\qquad N^{ij} = (N_{ij})^{-1} \end{aligned} \tag{3.101}$$

Therefore:

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}t \sum_{ij} N^{ij}[-\dot{X}_i\dot{X}_j + 2M_{ij}\dot{X}_i X_j' + (N_{ij}^2 - M_{ij}^2)X_i' X_j'] \\ &\quad where \\ X_i' &\equiv X_{i+1} - X_{i-1} \end{aligned} \tag{3.102}$$

(or with respect to the findings of *sec.*(3.2.4), any other linear variation of this.)
The momenta can be found as:

$$\Pi^k = \frac{\delta S}{\delta \dot{X}_k} = \sum_i N^{ki}(\dot{X}_i - M_{ki}X_i') \tag{3.103}$$

[4] In this section we make the following notational restrictions:

- There is no sum over repeated indices without the presence of a $\sum$,
- We suppress the background field $D-$*dimensional* superscript, μ unless its presence is nontrivial.

However at this step we will need to impose the following restriction on the N and M matrices:

$$\sum_i N^{ij} M_{ik} \propto \delta^i_k \quad \rightarrow \quad \sum_i N^{ij} M_{ik} X'_k = N^{ij} M_{ij} X'_j \tag{3.104}$$

Using this we can show that:

$$\dot{X}_k = \sum_k (N_{ik}\Pi^k + M_{ik} X'_k) \tag{3.105}$$

Notice that this restriction is necessary so that the theory would resemble the continuous theory as closely as possible. However the resulting N and M matrices are still nonlocal under the above restrictions; this restriction relates these two matrices in certain ways and they still remain nonlocal.

With this we now have :

$$\begin{aligned}\mathcal{L} &= \sum_{ij}[(\dot{X}_i - M_{ij}X'_i)(\dot{X}_j - M_{ij}X'_j) - N^2_{ij}X'_iX'_j] \\ &= \frac{1}{2}\sum_{ij} N_{ij}(\Pi^i\Pi^j - X'_iX'_j)\end{aligned} \tag{3.106}$$

The Hamiltonian can then be verified to be of the form:

$$\begin{aligned}\mathcal{H} &= \sum_i \Pi_i \dot{X}_i - \mathcal{L} \\ &= \frac{1}{2}\sum_{ij}[N_{ij}(\Pi^i\Pi^j + X'_iX'_j) + M_{ij}(\Pi^i X'_j + \Pi^j X'_i)] \\ &= \sum_{ij} N_{ij}\mathcal{C}^0_{ij} + M_{ij}\mathcal{C}^1_{ij}\end{aligned} \tag{3.107}$$

in agreement with the findings of *sec.*(3.2.1). However to get to this point we can summarize that we have made the following changes to the original local action:

- We have used nonlocal combinations of the field-variables in the action *eqn.*(3.102), *yet the field variables themselves are still local functions of spacetime.*
- A nonlocal metric: $g_{ab}(i,j;t)$ *eqn.*(3.101); i.e. the N and M are nonlocal functions

The first of these two changes - and in particular the fact the field variables are still local - will have significant consequences in regards to the physical space of this theory, which we shall discuss at the end of this chapter.

At this stage however $\mathcal{C}^0_{ij}$ and $\mathcal{C}^1_{ij}$ are not yet identified as constraints. Let us confirm this next.

3.3.2 The Constraint/ Virasoro Algebra

Starting with the action *eqn.*(3.102), we find the following first class primary constraints:

$$\mathcal{P}_{N_{ij}} \approx 0 \approx \mathcal{P}_{M_{ij}} \tag{3.108}$$

Checking for secondary constraints we find:

$$\{P_{N_{ij}}, \mathcal{H}\} = C^0_{ij} \approx 0 \tag{3.109}$$

$$\{P_{M_{ij}}, \mathcal{H}\} = C^1_{ij} \approx 0\,, \tag{3.110}$$

and with these we can verify the (smeared) constraint algebra:

$$\{\mathcal{G}^\alpha[\lambda], \mathcal{G}^\beta[\gamma]\} \equiv \sum_{ijkl} \lambda_{ij}\gamma_{kl}\{C^\alpha_{ij}, C^\beta_{kl}\} = \sum_{ij} (\mathcal{D}_\lambda \gamma)_{ij} C^{\{\alpha,\beta\}}_{ij}$$

$$\alpha, \beta = 0\, or\, 1 \quad and \quad \{\alpha,\beta\} = \begin{cases} 0 & if\, \alpha \neq \beta \\ 1 & if\, \alpha = \beta \end{cases}\,,$$

$$where$$

$$(\mathcal{D}_\gamma \lambda)_{ij} = \frac{1}{2}\sum_k \big[\lambda_{ik}(\gamma_{k,j+1} - \gamma_{k,j-1}) - \gamma_{ik}(\lambda_{k,j+1} - \lambda_{k,j-1}) \\ + \lambda_{jk}(\gamma_{k,i+1} - \gamma_{k,i-1}) - \gamma_{jk}(\lambda_{k,i+1} - \lambda_{k,i-1})\big]$$

in agreement with the findings of *sec.*(3.2).
Recall the following relation - from the results of *sec.*(2.3.1) - between the constraints and the energy momentum tensor:

$$T_{ab} \sim \frac{\delta S}{\delta g^{ab}} \sim (\frac{\delta S}{\delta N}, \frac{\delta S}{\delta M})\,,$$

which gives:

$$T_{ab} = \begin{pmatrix} (N^2 + M^2)C^0 + 2NMC^1 & MC^0 + NC^1 \\ MC^0 + NC^1 & C^0 \end{pmatrix}.$$

Now based on the arguments of *sec*(3.2.1) - which lead to the above mentioned "non-localization" of the action - we can perform a gauge fixing of the metric components[5]:

$$\begin{cases} N_{ij} \to \delta_{ij} \\ M_{ij} \to 0\,, \end{cases} \tag{3.111}$$

as a result of which, the components of the energy-momentum tensor become:

$$T_{ab}(ij) = \begin{pmatrix} C^0_{ij} & C^1_{ij} \\ C^1_{ij} & C^0_{ij} \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} \Pi_i\Pi_j + X'_i X'_j & \Pi_i X'_j + \Pi_j X'_i \\ \dots & \dots \end{pmatrix} \tag{3.112}$$

These are the constraints of the nonlocal theory. We will use these to isolate the physical sector of the solution space. So now let us formulate the solution space of this theory.

[5]The questions regarding general covariance of this gauge fixing are the same ones as in the continuum string case, see end-remarks of the last chapter for this discussion.

3.3.3 The Solution Space

The Gauge fixed equations of motion are very simple:

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^a b\partial_b X^\mu) \quad \xrightarrow{G.\,Fixing} \quad \eta^{ab}\partial_a\partial_b X^\mu \tag{3.113}$$

$$\Rightarrow \quad \ddot{X}_n - X_n'' = 0 \quad is\,a\,wave\,equation. \tag{3.114}$$

The only difference between this "wave equation" and a continuous wave equation is that:

$$X_n'' \equiv X_{n+2} - 2X_n + X_{n-2}\,,$$

nonetheless, we shall see that this merely changes the relation between the wave number and the wave frequency. Hence let us posit a solution of the form:

$$\begin{aligned} X_n^\mu = \sum_m X_n^\mu(m) &= \aleph \sum_m \frac{1}{\omega_m}\delta(\sigma - \frac{nL}{N})e^{i\omega_m t}\left(\alpha_m^\mu e^{i\kappa_m \sigma} + \beta_m^\mu e^{-i\kappa_m \sigma}\right) \\ &= \aleph \sum_m \frac{e^{i\omega_m t}}{\omega_m}\left(\alpha_m^\mu e^{i\kappa_m \frac{nL}{N}} + \beta_m^\mu e^{-i\kappa_m \frac{nL}{N}}\right), \end{aligned} \tag{3.115}$$

Note:

- *L is the length of the string and N the number of "nodes" of the closed string. By definition then: $\Delta L = \frac{L}{N}$.*
- *The factor $\frac{1}{\omega_m}$ is a "convention" which we choose in this report; there exists a certain freedom in placing this factor which we analyze fully in* appendix*(D), where we motive our choice.*
- *The factor $\aleph$ is a normalization factor to be determined in the quantization.*

In other words the only difference is the $\delta(\sigma - \frac{nL}{N})$ which enforces the solution to vanish on points which are off the "nodes" of the discrete string. However we now need to apply the boundary conditions (BC) and the EOM to find out the restriction and the relation between ω_m and κ_m; we find that:

$$\begin{aligned} EOM: \quad \ddot{X}_n - X_n'' = 0 \quad &\rightarrow \quad \omega_m^2 - \left(2sin\left(\kappa_m \frac{L}{N}\right)\right)^2 = 0 \\ BC: \quad X_{n+N} = X_n \quad &\rightarrow \quad \kappa_m = \frac{2m\pi}{L}\,, \end{aligned}$$

which we may compare with the continuum case:

$$\omega_m^2 - \kappa_m^2 = 0 \quad and \quad \kappa_m = \frac{2m\pi}{L}\,.$$

i.e. in the $\lim_{\Delta L \to 0}$, *these are equivalent.*

This is a solution with somewhat strange requirements, yet note:

- We cannot expect the symmetry between ω_m and κ_m because of the broken symmetry between the t and σ in the discrete case.
- It has what we need in terms of satisfying both the EOM and the BC.

However we can already foresee a problem for the $m = 0$ case, corresponding to α_0^μ and β_0^μ. In other words since these will eventually be identified as creation ($m < 0$) and annihilation ($m > 0$) operators, we need to clarify what the $m = 0$ cases stand for from here.
Note that the $m = 0 \to \omega_m = 0$ corresponds to a non-wave motion. This is indeed the *centre of mass mode* and needs to be separated from the sum. Hence let us posit a solution of the form:

$$X_n^\mu = \sum_m X_n^\mu(m) = x_0^\mu + Ap_0^\mu t + \aleph \sum_{m\neq 0} \frac{e^{i\omega_m t}}{\omega_m}\Big(\alpha_m^\mu e^{i\kappa_m \frac{nL}{N}} + \beta_m^\mu e^{-i\kappa_m \frac{nL}{N}}\Big), \tag{3.116}$$

Where A is a constant which is to be determined so to satisfy the required quantization relations (commutation relations), i.e. this is the only restriction on A since the EOM and the BC do not impose anything extra.
So, now let us proceed to quantization; imposing commutation relations:
First note that:

$$\Pi_n^\mu \quad \xrightarrow{\underline{G.F.}} \quad \dot{X}_n^\mu = Ap_0 + i\aleph \sum_{m\neq 0} e^{i\omega_m t}\Big(\alpha_m^\mu e^{i\kappa_m \frac{nL}{N}} + \beta_m^\mu e^{-i\kappa_m \frac{nL}{N}}\Big), \tag{3.117}$$

which gives:

$$\begin{aligned}[X_n^\mu(t), \Pi_m^\nu(t)] &= A\,[x_0^\mu, p_0^\nu] \\ &+ i\aleph^2 \sum_{lk\neq 0} \frac{e^{i(\omega_l+\omega_k)t}}{\omega_l}\Big[\alpha_l^\mu e^{i\kappa_l \frac{nL}{N}} + \beta_l^\mu e^{-i\kappa_l \frac{nL}{N}}\,, \alpha_k^\mu e^{i\kappa_k \frac{nL}{N}} + \beta_k^\mu e^{-i\kappa_k \frac{nL}{N}}\Big]\end{aligned} \tag{3.118}$$

we notice that in order for this to be equal to a constant, and therefore be t-independent, we must have:

$$\omega_l + \omega_k = 0 \quad \to \quad sin(\frac{l\pi}{N}) = -sin(\frac{k\pi}{N}) \quad \to \quad l = -k + 2aN \quad a \in \mathbb{Z} \tag{3.119}$$

However the $2aN$ does not affect any of the results and we can generally consider: $l = -k$. This requirement then indicates that we must have:

$$\begin{aligned}&[\alpha_l^\mu, \alpha_k^\nu] \quad and \quad [\beta_l^\mu, \beta_k^\nu] \quad \propto \delta_{l,-k} \\ &[\alpha_l^\mu, \beta_k^\nu] = 0\,,\end{aligned} \tag{3.120}$$

where the last condition was added because we also wish commutativity between the left-moving and the right-moving waves. We also need a term $\propto \omega_l$ to cancel the $\frac{1}{\omega_l}$ present in the commutation. The requirement $\delta^{\mu\nu}$ is also obvious. Hence let us posit the following form for the commutations:

$$\begin{aligned}&[\alpha_l^\mu, \alpha_k^\nu] = K\omega_l \delta_{l,-k}\delta^{\mu\nu} \\ &[\beta_l^\mu, \beta_k^\nu] = \widetilde{K}\omega_l \delta_{l,-k}\delta^{\mu\nu} \\ &[\alpha_l^\mu, \beta_k^\nu] = 0\end{aligned} \tag{3.121}$$

With this we have:

$$[X_n^\mu(t), \Pi_m^\nu(t)] = i\hbar A\delta^{\mu\nu} + i\aleph^2\delta^{\mu\nu}\sum_{l\neq 0}\left[Ke^{2i\frac{l\pi}{N}(n-m)} + \widetilde{K}e^{-2i\frac{l\pi}{N}(n-m)}\right] \tag{3.122}$$

we recognize that the prop. constants must equal $K = \widetilde{K}$.

Therefore we have:

$$\begin{aligned}
[X_n^\mu(t), \Pi_m^\nu(t)] &= i\hbar A\delta^{\mu\nu} + i2\aleph^2K\delta^{\mu\nu}\sum_{l\neq 0} cos\left(\frac{2l\pi}{N}(n-m)\right) \\
&= i\hbar A\delta^{\mu\nu} + i4\pi\aleph^2K\delta^{\mu\nu}\left(\delta\left(\frac{2\pi}{N}(n-m)\right) - \frac{1}{2\pi}\right) \\
&= i\hbar A\delta^{\mu\nu} + i2\aleph^2NK\delta^{\mu\nu}\left(\delta_{n,m} - \frac{1}{N}\right) \\
&= i2\aleph^2NK\delta^{\mu\nu}\delta_{n,m} + i(\hbar A - 2K) \\
must\, equal &= i\hbar\delta^{\mu\nu}\delta_{n,m}
\end{aligned} \tag{3.123}$$

We wish to have $K = \widetilde{K} = 1$ so that the theory resembles a simple harmonic oscillator as much as possible. This gives:

$$\Rightarrow K = \widetilde{K} = 1 \quad \aleph = \left(\frac{\hbar}{2N}\right)^{\frac{1}{2}} \qquad and \qquad A = \frac{1}{N} \tag{3.124}$$

The solution space can then be given by:

$$\begin{aligned}
X_n^\mu(t) &= x_0^\mu + \frac{p_0^\mu}{N}t + \left(\frac{\hbar}{2N}\right)^{\frac{1}{2}}\sum_{m\neq 0}\frac{1}{\omega_m}e^{i\omega_m t}\Big(\alpha_m^\mu e^{i\kappa_m\frac{nL}{N}} + \beta_m^\mu e^{-i\kappa_m\frac{nL}{N}}\Big) \\
\Pi_n^\mu(t) &= \frac{p_0^\mu}{N} + i\left(\frac{\hbar}{2N}\right)^{\frac{1}{2}}\sum_{m\neq 0}e^{i\omega_m t}\Big(\alpha_m^\mu e^{i\kappa_m\frac{nL}{N}} + \beta_m^\mu e^{-i\kappa_m\frac{nL}{N}}\Big)
\end{aligned} \tag{3.125}$$

$$with:$$

$$[\alpha_l^\mu, \alpha_k^\nu] = [\beta_l^\mu, \beta_k^\nu] = \omega_l\delta^{\mu\nu}\delta_{l,-k} \qquad [\alpha_l^\mu, \beta_k^\nu] = 0 \tag{3.126}$$

$$where:$$

$$\begin{aligned}
\omega_k &= 2sin\left(\frac{2k\pi}{N}\right) = 2sin\left(\frac{\kappa_k L}{N}\right) \\
\kappa_k &= \frac{2k\pi}{L}
\end{aligned} \tag{3.127}$$

Also notice the association of *the zeroth mode oscillations* $\alpha_0^\mu, \beta_0^\mu$ and p_0^μ:

$$p_0^\mu p_{0\mu} \equiv (R_0^\mu\ R_{0\mu} + L_0^\mu L_{0\mu}) = (2N\aleph)^2\,(\alpha_0^\mu\alpha_{0\mu} + \beta_0^\mu\beta_{0\mu}) \tag{3.128}$$

where, R_0^μ and L_0^μ are, respectively the momenta of the *right* and the *left* moving waves.

We are now in the position to look at the Fock-Space solution and the corresponding Virasoro constraints.

3.3.4 The Physical Fock-Space

We can represent the most general solution vector, $|\Psi_{A,B}(K^\mu)\rangle$ as:

$$|\Psi_{A,B}(K^\mu)\rangle = \prod_{a,b=1}^{\infty} |\Psi_{A_a,B_b};K\rangle \tag{3.129}$$

where :

$$|\Psi_{A_a,B_b}\rangle = \prod_{\mu=1}^{D} |A_a^\mu, B_b^\mu\rangle = \prod_{\mu=1}^{D} \frac{\left(\alpha_{-a}^\mu\right)^{A_a^\mu} \otimes \left(\beta_{-b}^\mu\right)^{B_b^\mu}}{\sqrt{\omega_a^{A_a^\mu}\omega_b^{B_b^\mu} A_a^\mu! B_b^\mu!}} |0,0\rangle \tag{3.130}$$

In other words, the general solution is the product of all modes of oscillation of both the left and the right moving waves, hence the $\prod_{a,b}$ and the product of the oscillation in each individual dimension, hence the $\prod_\mu$.

We also use the following convention:

$$\alpha_a^\mu |A_a^\mu, B_b^\mu\rangle = \begin{cases} \sqrt{\omega_a}\sqrt{A_a^\mu+1}\,|A_a^\mu+1, B_b^\mu\rangle & if \quad a<0 \\ \sqrt{\omega_a}\sqrt{A_a^\mu}\,|A_a^\mu-1, B_b^\mu\rangle & if \quad a>0 \end{cases} \tag{3.131}$$

$$\beta_b^\mu |A_a^\mu, B_b^\mu\rangle = \begin{cases} \sqrt{\omega_b}\sqrt{B_b^\mu+1}\,|A_a^\mu, B_b^\mu+1\rangle & if \quad b<0 \\ \sqrt{\omega_b}\sqrt{B_b^\mu}\,|A_a^\mu, B_b^\mu-1\rangle & if \quad b>0 \end{cases} \tag{3.132}$$

Let us now look at the corresponding Virasoro constraints. First note that these come form the elements of the energy momentum tensor, T_{ab} and there is only two of these:

$$\begin{cases} 2T_{00}(n,m) & = \dot{X}_n\dot{X}_m + X'_nX'_m \quad \sim \dot{X}_n\dot{X}_m + (X_{n+1}-X_{n-1})(X_{m+1}-X_{m-1}) \\ 2T_{01}(n,m) & = \dot{X}_nX'_m + \dot{X}_mX'_n \quad \sim \dot{X}_n(X_{m+1}-X_{m-1}) + \dot{X}_m(X_{n+1}-X_{n-1}) \end{cases} \tag{3.133}$$

using *eqn.*(3.125), these become:

$$\begin{aligned} 2T_{00}(n,m;t) = {} & \frac{p_0^\mu p_{0\mu}}{N^2} - 2\aleph^2 \sum_{l,k\neq 0} e^{i(\omega_l+\omega_k)t}\left(\alpha_l^\mu \alpha_{k\mu} e^{i\frac{2\pi}{N}(ln+km)} + \beta_l^\mu \beta_{k\mu} e^{-i\frac{2\pi}{N}(ln+km)}\right) \\ & + i\aleph\frac{p_0^\mu}{N}\sum_{l\neq 0} e^{i\omega_l t}\left(\alpha_{l\mu}\left(e^{i\frac{2\pi}{N}ln} + e^{i\frac{2\pi}{N}lm}\right) + \beta_{l\mu}\left(e^{-i\frac{2\pi}{N}ln} + e^{-i\frac{2\pi}{N}lm}\right)\right) \\ 2T_{01}(n,m;t) = {} & -2\aleph^2 \sum_{l,k\neq 0} e^{i(\omega_l+\omega_k)t}\left(\alpha_l^\mu \alpha_{k\mu} e^{i\frac{2\pi}{N}(ln+km)} - \beta_l^\mu \beta_{k\mu} e^{-i\frac{2\pi}{N}(ln+km)}\right) \\ & + i\aleph\frac{p_0^\mu}{N}\sum_{l\neq 0} e^{i\omega_l t}\left(\alpha_{l\mu}\left(e^{i\frac{2\pi}{N}ln} + e^{i\frac{2\pi}{N}lm}\right) - \beta_{l\mu}\left(e^{-i\frac{2\pi}{N}ln} + e^{-i\frac{2\pi}{N}lm}\right)\right) \end{aligned}$$

Now we have:

$$\begin{cases} T_{00}(n,m;t) & \approx \quad 0 \\ T_{01}(n,m;t) & \approx \quad 0 \end{cases} \qquad \forall \quad t\,, \tag{3.134}$$

Making this more symmetric:

$$
\begin{aligned}
2(T_{00}+T_{01}) &= \frac{p_0^2}{N^2} - 4\aleph^2 \sum_{k,l\neq 0} e^{i(\omega_k+\omega_l)t}\left(\alpha_k^\mu \alpha_{l\mu} e^{i\frac{2\pi}{N}(kn+lm)}\right) + 2i\aleph\frac{p_o^\mu}{N}\sum_{l\neq 0} e^{i\omega_l t}\alpha_{l\mu}\left(e^{i\frac{2\pi}{N}n}+e^{i\frac{2\pi}{N}m}\right)\\
2(T_{00}-T_{01}) &= \frac{p_0^2}{N^2} - 4\aleph^2 \sum_{k,l\neq 0} e^{-i(\omega_k+\omega_l)t}\left(\beta_k^\mu \beta_{l\mu} e^{-i\frac{2\pi}{N}(kn+lm)}\right) + 2i\aleph\frac{p_o^\mu}{N}\sum_{l\neq 0} e^{i\omega_l t}\beta_{l\mu}\left(e^{-i\frac{2\pi}{N}n}+e^{-i\frac{2\pi}{N}m}\right)
\end{aligned}
\tag{3.135}
$$

Let us now try to interpret these. Notice that by picking p_0^μ to be the centre of mass momenta, we can have:

$$
\begin{cases}
\dfrac{p_0^\mu p_{0\mu}}{N^2} \approx M^2 \\
p_0^\mu \alpha_{k\mu} \approx 0 \qquad \mathit{and} \qquad p_0^\mu \beta_{k\mu} \approx 0 \qquad \forall \quad k \neq 0
\end{cases}
\tag{3.136}
$$

The first equation makes the association between the momenta of the centre of mass and the mass-density of the string. The second set posits that the direction of motion of the centre of mass is perpendicular to the oscillatory motion of the string. So we can separate the constraints to five (5) different sets:

$$
\begin{cases}
\dfrac{p_0^2}{N^2} - M^2 \approx 0 \\
\dfrac{p_0^\mu}{N}\Lambda^0_{n\mu} \equiv \dfrac{p_0^\mu}{N}\displaystyle\sum_{l\neq 0}\alpha_{l\mu}e^{i\kappa_l n} \approx 0 \\
\dfrac{p_0^\mu}{N}\Lambda^1_{n\mu} \equiv \dfrac{p_0^\mu}{N}\displaystyle\sum_{l\neq 0}\beta_{l\mu}e^{i\kappa_l n} \approx 0 \\
\mathcal{K}^0_{nm} - \left(\dfrac{p_0^\mu}{2N\aleph}\right)^2 \equiv \displaystyle\sum_{k,l\neq 0}\alpha_k^\mu\alpha_{l\mu}e^{i(\kappa_k n + k_l m)} - \left(\dfrac{M}{2\aleph}\right)^2 \approx 0 \\
\mathcal{K}^1_{nm} - \left(\dfrac{p_0^\mu}{2N\aleph}\right)^2 \equiv \displaystyle\sum_{k,l\neq 0}\beta_k^\mu\beta_{l\mu}e^{-i(\kappa_k n + k_l m)} - \left(\dfrac{M}{2\aleph}\right)^2 \approx 0
\end{cases}
\tag{3.137}
$$

Notice that, in order to obtain the second and the third equations here (i.e. the posit that the oscillations are transverse), we actually made another assumption:

$$
\frac{p_0^\mu}{N}\sum_{l\neq 0}\alpha_{l\mu}\left(e^{i\frac{2\pi}{N}ln}+e^{i\frac{2\pi}{N}lm}\right) \to \frac{p_0^\mu}{N}\sum_{l\neq 0}\alpha_{l\mu}\left(e^{i\frac{2\pi}{N}ln}\right) = \frac{p_0^\mu}{N}\Lambda^0_n
\tag{3.138}
$$

and similarly for Λ^1_n.

This transformation or posit corresponds to the following assumption:

$$
\begin{aligned}
& \dot{X}_n X'_m + \dot{X}_m X'_n \approx 0 \quad \to \quad \dot{X}_n X'_m \approx 0 \approx \dot{X}_m X'_n \\
\Rightarrow \quad & \dot{X}_n \dot{X}_m + \dot{X}_m X'_n \approx 0 \quad \to \quad \dot{X}_n \dot{X}_m \approx 0
\end{aligned}
$$

for the centre of mass degree of freedom.

First note that this may seem like a more restrictive assumption than is a-priori granted. Yet as we argue from a different perspective in *appendix*(E), the non-transverse oscillations correspond to the negative-norm solutions of the theory, which are therefore unphysical. Hence their exclusion at this stage is justified in that respect. This also helps provide correspondence between the discrete case and the results of *sec.*(2.3.4) of the continuum case.

Now, let us look at the Virasoro generators.

We note, (as we shall make this point more clear shortly) that the fact that n and m need not be equal is the embodiment of the presence of the extra (nonlocal) constraints.
Next, let us perform the Fourier-transform of these equations to obtain the Virasoro Generators (for the same reason as outlined in the continuum case):
We define:

$$\begin{cases} \mathcal{Z}_{00} \equiv \dfrac{p_0^\mu p_{\mu 0}}{N^2} - M^2 \\ \mathcal{Z}_{0c}^\diamond \equiv \dfrac{1}{N}\displaystyle\sum_n \Lambda_n^\diamond e^{-i\frac{2\pi}{N}(nc)} & \diamond = 0\,or\,1 \\ \mathcal{L}_{cd}^\diamond \equiv \dfrac{1}{N}\displaystyle\sum_{nm} e^{\mp i\frac{2\pi}{N}(nc+md)}\left(\mathcal{K}_{nm}^\diamond - \left(\dfrac{M}{2\aleph}\right)^2\right) \end{cases} \tag{3.139}$$

which gives:

$$\begin{cases} \mathcal{Z}_{00} = \dfrac{p_0^\mu p_{\mu 0}}{N^2} - M^2 \\ \mathcal{Z}_{0c}^0 = \dfrac{p_0^\mu}{N}\alpha_{c\mu} \\ \mathcal{Z}_{0c}^1 = \dfrac{p_0^\mu}{N}\beta_{c\mu} \\ \mathcal{L}_{cd}^0 = {\alpha_c}^\mu \alpha_{d\mu} - \left(\dfrac{M}{2\aleph}\right)^2 \delta_{c,0}\delta_{d,0} \\ \mathcal{L}_{cd}^1 = {\beta_c}^\mu \beta_{d\mu} - \left(\dfrac{M}{2\aleph}\right)^2 \delta_{c,0}\delta_{d,0} \end{cases} \tag{3.140}$$

Let us make a few remarks:

- Note we may need factors of the form: $(\omega_a\omega_b)^{-(r+1)}$ and $(\omega_a)^{-(r+1)}$ in the definition of these expressions, where the value of r depends on the choice of the factor in the definition of $X_n^\mu \sim (\omega_l)^r$. (More on this in *appendix*(D).) [6]

[6] Incidentally these could have been removed:

i in the definition of $X_n^\mu(t)$ and $\Pi_n^\mu(t)$ or

ii in the definition of α_m^μ and β_m^μ.

- However, now notice using *eqn*(3.128), $\left(\frac{p_0^\mu}{N}\right)^2 = 4\aleph^2\left(\alpha_0^2+\beta_0^2\right)$ the $c,d=0$ cases of $\mathcal{L}^\diamond_{cd}$ reduces to to the first equation, and the $d=0$ reduce to the second and the third equation. As a result we can rewrite these equations as:

$$\begin{cases} \mathcal{Z}_{00} = \dfrac{p_0^\mu p_{\mu 0}}{N^2} - M^2 \qquad \mathcal{Z}^0_{0c} = \dfrac{p_0^\mu}{N}\alpha_{c\mu} \qquad \mathcal{Z}^1_{0c} = \dfrac{p_0^\mu}{N}\beta_{c\mu} \\ \begin{cases} \mathcal{L}^0_{cd} = \alpha_c{}^\mu \alpha_{d\mu} \\ \mathcal{L}^1_{cd} = \beta_c{}^\mu \beta_{d\mu} \end{cases} \qquad\qquad c,d \neq 0 \end{cases} \tag{3.141}$$

- This reduction was absent in the continuum case precisely because of the intertwined-ness of the constraint equations. In the discrete case the assumption of non-locality makes it possible to separate these degrees of freedom completely, hence obtaining the above set of independent equations.

So we may now continue with this new set of constraints to isolate the physical solution space. First let us consider the first three constraints: $\mathcal{Z}_{00}, \mathcal{Z}^0_{01}$ and $\mathcal{Z}^1_{01}$.
$\mathcal{Z}_{00}$ is simply a restriction of the value of the four-momenta of the centre of mass. Whereas $\mathcal{Z}^0_{01}$ and $\mathcal{Z}^1_{01}$ are restrictions on μ.
Therefore the first three conditions can be satisfied by a convenient choice of the *direction of oscillation* and that of the *direction of propagation*. Let:

$$\begin{aligned} &\mu = \{1; i\} \qquad s.t. \quad i = 2, \cdots, D \\ &\qquad\qquad\qquad p_0^\mu = \{M; \overrightarrow{0}\} \\ &and \qquad\qquad\quad \alpha_a^\mu = \{0; \alpha_a^i\} \\ &\qquad\qquad\qquad \beta_b^\mu = \{0; \beta_b^i\} \end{aligned} \tag{3.142}$$

Therefore we should rewrite the Fock solutions of the theory as:

$$|\Psi_{A,B}(K^0)\rangle = \prod_{a,b=1}^{\infty} |\Psi_{A_a,B_b}; M\rangle \tag{3.143}$$

$$where:$$

$$|\Psi_{A_a,B_b}\rangle = \prod_{\mu=2}^{D} |A_a^\mu, B_b^\mu\rangle = \prod_{\mu=2}^{D} \frac{\left(\alpha^\mu_{-a}\right)^{A_a^\mu} \otimes \left(\beta^\mu_{-b}\right)^{B_b^\mu}}{\sqrt{\omega_a^{A_a^\mu}\omega_b^{B_b^\mu} A_a^\mu! B_b^\mu!}} |0,0\rangle \tag{3.144}$$

Now as in the case of the continuous string the last two constraints serve the purpose of isolating the physical sector of the solution space, where a physical solution $|\Psi_{phys}\rangle$ must obey the following condition (per the prescription of string theory):

$$\langle\Psi_{phys}|\mathcal{L}^\diamond_{cd}|\Psi_{phys}\rangle = 0 \qquad \forall \quad c,d \neq 0\,. \tag{3.145}$$

We also get rid of the null states:

$$|\Psi_{null}\rangle \equiv \left\{ |\Psi_{phys}\rangle\; s.t.\, |\Psi_{null}\rangle = \sum_k A_k |\Psi_{phys}\rangle_k \right\}$$

$$\Rightarrow \mathcal{H}_{BRST} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}} = \frac{\mathcal{H}_{Phys}}{\mathcal{H}_{null}}$$

We shall focus on the requirement *eqn.*(3.145) for now. To do so first we note an identity:

$$\begin{aligned}
&(X^{\mu}{}_{a})^{\dagger} = X^{\mu}{}_{a} \quad and \quad \omega_{-l} = -\omega_{l} \\
&\quad \Rightarrow (\alpha^{\mu}{}_{l})^{\dagger} = \alpha^{\mu}_{-l} \quad and \quad (\beta^{\mu}{}_{l})^{\dagger} = \beta^{\mu}_{-l} \\
&\quad \Rightarrow (\mathcal{L}^{\diamond}_{cd})^{\dagger} = \mathcal{L}^{\diamond}_{-c-d}
\end{aligned} \tag{3.146}$$

Therefore in order to satisfy *eqn.*(3.145), it is sufficient if we had:

$$\mathcal{L}^{\diamond}_{cd}|\Psi_{phys}\rangle = 0 \quad for \quad c > 0 \quad and \quad \forall d \neq 0\,, \tag{3.147}$$

as we can see from *fig.*(3.2).

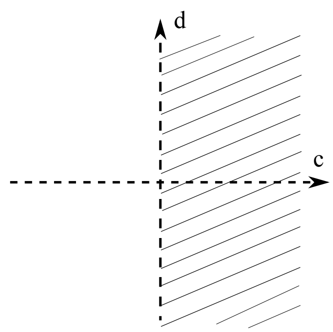

Figure 3.2: Here we can see how the Generalized Virasoro Space can be split into two parts so that the expectation value is kept invariant.

Now let us look at the physical solution space that these restrictions prescribe.
Here because of the symmetry of the situation we will assume $\diamond = 0$, i.e. consider the $\mathcal{L}^{0}_{cd}$ only (and μ runs from $2, \cdots, D$) :

$$\mathcal{L}^{0}_{cd}\prod_{a,b}^{\infty}|A_a, B_b\rangle = \alpha^{\mu}_{c}\alpha_{d\mu} \times \prod_{a,b}^{\infty}|A_a, B_b\rangle \qquad c > 0 \quad d \neq 0 \tag{3.148}$$

$$= \begin{cases} \alpha^{\mu}_{c}\alpha_{d\mu} \\ \alpha^{\mu}_{c}\alpha_{-d\mu} \end{cases} \times \prod_{a,b}^{\infty}|A_a, B_b\rangle \quad c, d > 0 \tag{3.149}$$

$$= \prod_{a,b}^{\infty}\prod_{\nu=2}^{D} \begin{cases} \alpha^{\mu}_{c}\alpha_{d\mu} \\ \alpha^{\mu}_{c}\alpha_{-d\mu} \end{cases} \times \frac{\left(\alpha^{\nu}_{-a}\right)^{A^{\nu}_{a}}\left(\beta^{\nu}_{-b}\right)^{B^{\nu}_{b}}}{\sqrt{\cdots}}|0;0\rangle \tag{3.150}$$

- In the first case either c or d or both do not commute with a
- In the second case d commutes and increases the occupation number of the $|A^{\nu}_{d}\rangle$ state by 1, c may not commute with a

$$\rightarrow = \prod_{a}^{\infty}\prod_{\nu=2}^{D}\begin{cases}\alpha_c^{\mu}\left(\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}}\alpha_{d\mu} + A_a^{\nu}\omega_d\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}-1}\delta_{d,a}\delta_{\mu}{}^{\nu}\right)\\ \alpha_{-d\mu}\left(\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}}\alpha_d^{\mu} + A_a^{\nu}\omega_c\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}-1}\delta_{c,a}\delta^{\mu\nu}\right)\end{cases} \times |0;B\rangle \tag{3.151}$$

The first term of each expression vanished since $c, d > 0$, and we are left with:

$$\rightarrow = \prod_{a}^{\infty}\prod_{\nu=2}^{D}\begin{cases}A_a^{\nu}(A_a^{\nu}-1)\omega_d\omega_c\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}-2}\delta_{c,a}\delta_{d,a}\delta^{\mu\nu}\delta_{\mu}{}^{\nu}\\ \alpha_{-d\mu}A_a^{\nu}\omega_c\left(\alpha_{-a}^{\nu}\right)^{A_a^{\nu}-1}\delta_{c,a}\delta^{\mu\nu}\end{cases} \times |0;B\rangle \tag{3.152}$$

- The first condition indicates that if $c = d > 0$ then only states with $occupation\,number\ = 1\,or\,0$ survive (because of the $A_a^{\nu}(A_a^{\nu}-1)$ factor).
  If however, $c \neq d$ we must have $occupation\,number\ = 0$, but this is easily seen from:
- The second condition, which forces the occupation number to be 0, because of the A_a^{ν}.

Hence the only remaining state is the trivial or the ground state $\sim A_a^{\nu} = 0\ \forall a > 0$ and ν.

Now let us see why this is; i.e. why is it that this theory does not lead to a reasonable solution space whereas it continuous counterpart, namely the continuous string does succeed in doing so (as we saw in *sec.*(3.3).)
In other words let us see why the restrictions:

$$\begin{aligned}&\mathcal{L}_{cd}^{\diamond}\prod_{a,b=1}^{\infty}|A_a,B_b\rangle = \overset{\diamond}{O}{}_c^{\mu}\overset{\diamond}{O}_{d\mu}\prod_{a,b=1}^{\infty}|A_a,B_b\rangle \qquad c>0 \quad d\neq 0\\ &\quad where\quad \overset{\diamond}{O}{}_c^{\mu}\quad stands\,for\,either \qquad \alpha_c^{\mu}\quad or\quad \beta_c^{\mu}\end{aligned} \tag{3.153}$$

are too-strong a conditions. This problem can be traced back to the definition of the $\mathcal{L}_{cd}^{\diamond}$ in the discrete case as compared to its continuum counterpart. Notice that here we do not have the $\sum\overset{\diamond}{O}_c\overset{\diamond}{O}_d$, as we did in the continuum case *eqn.*(2.70), relating the c to d. This problem then traces back to when we transformed $T_{ab}(n,m) \rightarrow \mathcal{L}_{cd}$ via:

$$\sum_{n,m} e^{-i(k_c n + k_d m)}\mathcal{K}_{nm}^{\diamond}$$

where we had to use *two separate sums* over the n and the m, $\sum_n\sum_m$ because of the form of the $\mathcal{K}_{nm}^{\diamond}$:

$$\mathcal{K}_{nm}^{\diamond} = \sum_{k,l\neq 0}\overset{\diamond}{O}{}_n^{\mu}\overset{\diamond}{O}_{m\mu}e^{i(k_k n + k_l m)}$$

Or more accurately because of the $e^{i(k_k n + k_l m)}$ where $n \neq m$ necessarily, which was precisely the posit that the theory is nonlocal.

Hence we can clearly see why the solution space is “too restricted”: as a direct consequence of the non-locality of the theory brought upon by the nonlocal constraints, this theory’s physical content is trivial.

This result was anticipated from the first time we encountered the nonlocal constraints. However notice that having had considered a nonlocal action, we were hoping that the solution space may have been enlarged enough to be able to produce non-trivial results. We can now see that this hope was unfounded for the following reason:

As we mentioned in sec.*(3.3.1), although the action was generalized by allowing nonlocal spacetime degrees of freedom, the field variables,* X^μ *and* Π^μ *are still defined locally;* $X^\mu = X^\mu(\sigma)$ *and* $\Pi^\mu = \Pi^\mu(\sigma)$.

So we see, that since the degrees of freedom of the constraints, $C^\alpha(\sigma, \sigma')$*, are reflected on the degrees of freedom of the field variables and not the spacetime coordinates, the solution space defined by the* local *variables is too small to survive the enlarged* nonlocal *constraint group.*

Had we considered a nonlocal versions of the field variables, perhaps nontrivial results would have been expected, but as it were there did not exist any physical reason to consider this change.

Therefore as we had anticipated we have shown that the solution space of the discrete string, *per the prescription of the quantization of the continuous string theory* fails to produce a physically useful solution space.

As a result - for reasons which were nonetheless not transparent initially - we are unable to use the discrete string to analyze the method of the string quantization, and to consider the questions regarding observables and their dynamics. In this respect the model has certainly failed.

However along the way we discovered many a interesting features of this model, which can be true of any discrete or lattice theory in general. For an example the non-locality of the resulting constraint algebra is perhaps present in more general cases. As such the questions regarding the implications and meaning of this can be quite important and interesting. The same can be said of other "side-results" which we discussed in this chapter.

Nevertheless, as regards the question of observables and dynamics in generally covariant theories, and in particular the question of dynamics in string theory, this model has proven unsuccessful. This provides another motivation for considering a completely different approach to these problems.

In the next chapter therefore let us consider the method of quantizing the string (and generally covariant theories in general), used in Loop Quantum Gravity, and to re-ask the questions regarding dynamics and observability in the context of this method.

Chapter 4

LQG and Loop Quantization of the String

Loop quantum gravity (LQG) is an elegant and novel approach to the quantization of the gravitational field. This theory has already managed to produce a background independent description of gravity, avoiding therefore all singularity and non-renormalizability issues which concern background-dependent field theory quantizations. In addition to having a full description of the solution space of gravity and its coupling to matter, LQG has many startling findings about the quantum structure of the spacetime, such as that spatial area and volume are granule, and perhaps so is time (in spin-foam formulations). Nonetheless despite these great advances there exist many open issues and problems which concern quantization of generally-covariant theories (GC)[1] in general which LQG does not provide an answer to yet. These include questions regarding the *Hamiltonian*[2], and the *True Hamiltonian*[3] of such theories, *the problem of time*, and the related issue of *observables* and their *dynamics*.

These issues are conceptually quite complicated and their description in LQG is further compounded due to the difficulties inherent with the mathematical structure of *4D* spacetime gravity. This is the fact that motivates the use of simpler models which can nonetheless capture some of the features of the Loop quantization.

In this chapter therefore, we first review the basic constructs that enter into the full LQG, whereby highlighting its *method* and then we consider the above mentioned *problems* by

[1]Note that gravity is a very special form of a generally covariant theory, in that in addition to being *passively* covariant, it is also *actively* covariant. See [39] for a full discussion. How this special character may affect different aspects of the analysis is highlighted in *sec.*(4.8), where we consider the *Scalar* constraint.

[2]In LQG the *Scalar* constraint's algebra with the *Diff* constraint does not *close* (*sec.*(4.2.2)) in the BRST sense. As such via Thiemann's *Master Constraint Programme*, [40] which is perhaps the best known method to handle this problem, one is lead to consider variations of this constraint, i.e. consider *Hamiltonian-type functions* which can act as the Hamiltonian yet form closed algebras. The problem that arises then is that there may not exist a unique choice of such a Hamiltonian.

[3]Since a GC's Hamiltonian is a constraint, in some models, to extract dynamics one needs to construct (by gauge-fixing or some other mean) a so called *True Hamiltonian*.

loop quantizing The String model. This model, as we shall see has the interesting advantage of possessing many of the interesting features of a constrained system, yet it is simple enough so that the methods of loop quantization could be considered and analyzed in an effectively very simple framework.

4.1 Introduction

Here we will present a brief introduction to LQG, however since our main goal is to extract and use the methods and ideas which are applied in this theory, we should mention that this introduction is by no means self-contained. In addition we will gloss-over many theorems which although fundamental to the whole construction of LQG, are not relevant to our considerations. We provide a somewhat detailed investigation of the full mathematical framework which reproduces LQG in *appendix*(G). More details as well as better reads on the topic may be found in [27] [28] [29].

The interesting feature of LQG is that it is a *quantum theory of constraints*. In other words, since General Relativity's equations are constraint equations, in essence *LQG is a consistent method of quantizing constraint systems*.
This is the feature that enables us to use the methods of the Loop quantization to analyze the constraints that are present in the string model.

4.2 Overview

The quantization of theory of gravity via the methods of Loop Quantization may be briefly summarized in the following steps (we will later try to explain each step):

1. **The New variables** The canonical variables of the theory are redefined
2. **The Constraints** The equations of motion are the constraints. In addition to the *diffeomorphism* and the *scalar* constraints we now also have an additional constraint brought about by the introduction of the new variables. These are the *internal* degrees of freedom or the $SU(2)$ gauges.
   - **The $SU(2)$ Solns.** The solutions to the $SU(2)$ constraints are the *spin-networks*
   - **The Diff. Inv. Solns.** The spin-networks, in a compact space obey the *Diff.* constraints, and in more general spaces can be made to do so by utilizing the concept of *Group Averaging method*[4]
   - **The Scalar Solns.** This is where the problem of Loop quantization lies, and where this method may be considered incomplete. However in a specific sense, the spin-networks "can be made to be" the solution to the *Scalar* constraint. In addition due to the nonlinearity of this constraint many different methods for its quantization exist (such as Thiemann's procedure, the consistent discretization

[4]This method is also called the *Gelfand-Naimark-Segal (GNS) construction*, which are the generalization of the so called *Rigged Hilbert Spaces*.

procedure, and the spin foam quantization). These methods may be said to be well-defined, however they all face many unknown questions.

3. **Observables** Depending on the method of quantization of the Scalar constraint, different observables may be constructed. However, 1) Many of these do not have a physical interpretation and 2)They are *Ultralocal* meaning that it is not known how global features of the classical and/or semiclassical gravity may be extracted from them.

The last two of these steps - namely the fate of the Scalar constraint and that of the observables - are precisely the focus of the quantization of the string model via the Loop quantization method, which we will address in *sec.*(4.4) through *sec.*(4.9).

We will now explain each of the above steps and provide a quick formulation of the *4D* LQG theory.

4.2.1 The Action and the New Variables

The action of the classical general relativity can be written as:

$$S[g_{\mu\nu}] = \frac{1}{2\kappa}\int_M \mathrm{d}^4x\sqrt{-g}R \tag{4.1}$$

where $g_{\mu\nu}$ is the 4-metric on the manifold M.

The *3+1 D* foliation of the *4D* manifold, $M \to \Sigma + t$, then gives the action:

$$S[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{2\kappa}\int \mathrm{d}t\int_\Sigma \mathrm{d}^3x\left[\pi^{ab}\dot{q}_{ab} - \underbrace{\left(N_bV^b(q_{ab},\pi^{ab}) + NS(q_{ab},\pi^{ab})\right)}_{\mathcal{H}[N_b,N]}\right] \tag{4.2}$$

Where:

$$V^b(q_{ab},\pi^{ab}) = -2\,\nabla^{(3)}_a\left(\frac{1}{\sqrt{q}}\pi^{ab}\right) \qquad \textit{Diff. Constraint} \tag{4.3}$$

$$S(q_{ab},\pi^{ab}) = -\frac{1}{\sqrt{q}}\left[R^{(3)} + \frac{1}{q}\left(\frac{1}{2}\left(Tr(\pi)\right)^2 - Tr(\pi^2)\right)\right] \qquad \textit{Scalar Constraint} \tag{4.4}$$

The q_{ab} is the 3-metric on the spacial slice Σ of M, and π^{ab} is its conjugate momenta. The N_b and N are the *shift* and the *lapse* functions respectively.

The Hamiltonian (density), $\mathcal{H}[N_b, N]$, is a total constraint, as expected. The last two equations are the *Diffeomorphism* and *Scalar* Constraints respectively. At this stage the conjugate variables of the theory are q_{ab} and π^{ab}:

$$\left\{q_{ab}(x),\pi^{cd}(y)\right\} = 2\kappa\delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x-y), \qquad \left\{q_{ab}(x),q_{cd}(y)\right\} = 0 = \left\{\pi^{ab}(x),\pi^{cd}(y)\right\} \tag{4.5}$$

We now introduce the set of the new variables, which will enable us to write the constraints in a more suitable form. The main motivation for this change of variables is that the solution space of the theory has a surprisingly simple formulation in terms of these variables (see *sec.*(G.1) for more details.)

New Variables

First define the *densitized triad*:

$$E^a_i \equiv \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c, \tag{4.6}$$

$Where:$

$$e^i_a \quad \textit{is the einbein} \quad \Rightarrow \quad q_{ab} = e^i_a e^j_b \delta_{ij} \tag{4.7}$$

The conjugate variable to the densitized triad is the tensor

$$K^i_a \equiv \frac{1}{\sqrt{E}} K_{ab} E^b_j \delta^{ij} \tag{4.8}$$

$Where:$

$$K_{ab} = \frac{1}{\sqrt{q}}\big(\pi^{ab} - \frac{1}{2}\pi q^{ab}\big) \quad \textit{is the extrinsic curvature.} \tag{4.9}$$

There are two points that we have to pay attention to at this point:

- First note that the introduction of the new variables has introduced 3 additional degrees of freedom into the theory. These correspond to the $so(3)$ rotation of the orthonormal coordinate. These represent a new $su(2)$[5] gauge freedom. Hence we will have to augment our constraint equations with the equation:

  $$G_i(E^a_i, K^j_b) \equiv \epsilon_{ijk} E^{aj} K^k_a = 0 \tag{4.10}$$

- Secondly, because of this new degree of freedom there exists a more "natural" conjugate variable for the E^a_i namely the connection (for a more complete explanation, see *sec.*(G.1)):

  $$A^i_a \equiv \Gamma^i_a + \gamma K^i_a \tag{4.11}$$

  Where Γ^i_a is called the *Spin Connection*, and the γ is the Immirzi parameter which identifies the ambiguity in choosing the connection.

In terms of these variables we have:

$$S[E^a_j, A^i_a, N_a, N, M_a] = \frac{1}{\kappa}\int \mathrm{d}t \int_\Sigma \mathrm{d}x^3 \big[E^a_i \dot{A}^a_i - N^b V_b(E^a_j, A^j_a) - N S(E^a_j, A^j_a) - M^i G_i(E^a_j, A^j_a)\big] \tag{4.12}$$

Where:

$$V_b(E^a_j, A^j_a) = E^a_j F_{ab} - (1+\gamma^2) K^i_b G_i \qquad Diff.\,Constraint \tag{4.13}$$

$$S(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{E}}\Big(\epsilon^{ij}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}\Big) \qquad Scalar\,Constraint \tag{4.14}$$

$$G_i(E^a_j, A^j_a) = D_a E^a_i \qquad SU(2)\,Constraint \tag{4.15}$$

[5] $SO(3)$ and $SU(2)$ have the same Lie algebra. However this is a generalization; see *appendix*(G) for a discussion on this.

Where:

$$F^i_{ab} \equiv \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b \qquad \textit{is the curvature of the connection} A^i_a \tag{4.16}$$

$$D_a E^a_i \equiv \partial_a E^a_i + \epsilon^k_{ij} A^j_a E^a_k \qquad \textit{is the covariant divergence of} E^a_i \tag{4.17}$$

With the commutation relations:

$$\left\{E^a_j(s), A^i_b(y)\right\} = \kappa\gamma\delta^a_b\delta^i_j\delta(x-y), \qquad \left\{E^a_j(x), E^b_i(y)\right\} = 0 = \left\{A^j_a(x, A^i_b(y)\right\} \tag{4.18}$$

4.2.2 The Constraint Algebra

Defining the smeared versions of the constraints:

$$G(\alpha) = \int_\Sigma \mathrm{d}x^3 \alpha^i G_i(A^i_a, E^a_i) \tag{4.19}$$

$$V(N^a) = \int_\Sigma \mathrm{d}x^3 N^a V_a(A^i_a, E^a_i) \tag{4.20}$$

$$S(N) = \int_\Sigma \mathrm{d}x^3 N S(A^i_a, E^a_i) \tag{4.21}$$

the resulting constraint algebra of the new constraints (both in terms of the new variables and the new set of the $SU(2)$ constraints) become:

$$\left\{G(\alpha), G(\beta)\right\} = G([\alpha,\beta]), \tag{4.22}$$

$$\left\{G(\alpha), V(N^a)\right\} = -G(\mathcal{L}_N\alpha), \tag{4.23}$$

$$\left\{G(\alpha), S(N)\right\} = 0, \tag{4.24}$$

$$\left\{V(N^a), V(M^a)\right\} = V(N^b\partial_b M^a - M^b\partial_b N^a), \tag{4.25}$$

$$\left\{S(N), V(N^a)\right\} = -S(\mathcal{L}_N N), \tag{4.26}$$

$$\left\{S(N), S(M)\right\} = V(S^a) + (1+\gamma^2)G\left(\frac{[E^a_i\partial_a N, E^{bi}\partial_b M]}{|q|}\right), \tag{4.27}$$

Where:

$$S^a = \frac{E^a_i E^b_j \delta^{ij}}{|E|}(N\partial_b M - M\partial_b N). \tag{4.28}$$

Notice that in the BRS sense, because of the form of this last equation(its dependance on field variables rather than structure constants) the algebra is *not closed*! As was mentioned in the introduction, this is a significant source of ambiguity in the definition of the Scalar constraints and therefore the Hamiltonian. We will discuss this issue further in *sec.*(4.9.3)[6].

Having rewritten the canonical structure of GR in terms of the new variable, we are now in the position to ask what the solution space to these constraints may look like.

[6] You may also see section 6 of [36] for further discussion.

4.3 The Solution Spaces

The solution space, as mentioned will be the solution to all the constraints of the theory. So let us go step by step and see what the solutions to each set of the constraints ($SU(2)$, *Diff* and then *Scalar*) will be.
This is the second place where LQG introduces new methods for the analysis of the constraint equations of GR.

4.3.1 The Solutions to $G_i(E^a_j, A^j_a)$; the Spin Networks

The solutions to the $SU(2)$ constraint are the so called *Cylindrical Functions*, Cyl, of all graphs $\gamma \subset \Sigma \sim 3D$ *spacial slice*:

$$Cyl = \bigcup_\gamma Cyl_\gamma \qquad \forall \quad graphs \quad \gamma \subset \Sigma \tag{4.29}$$

$$Cyl_\gamma \equiv \left\{\psi_{\gamma,f} = f(h_{e_1}[A], h_{e_2}[A], ..., h_{e_N}[A]) \middle| f : SU(2) \to \mathbb{C}\right\} \tag{4.30}$$

where N is the number of the nodes of γ, and

$$h_e[A] = Pexp(-\int_e A) \tag{4.31}$$

is the holonomy[7] *of the connection A along the edge e.*

These are the elementary blocks of the construction of the *spin-networks.* To see that the spin-networks form the solutions to the $SU(2)$ gauge condition, notice that the $SU(2)$ gauge:

$$\begin{aligned} G(\alpha) &= \int_\Sigma \mathrm{d}x^3 \alpha^i G_i(A^i_\alpha, E^a_i) = \int_\Sigma \mathrm{d}x^3 \alpha^i D_a E^a_i, & (4.32)\\ g &\sim exp(G), & (4.33)\end{aligned}$$

transforms the holonomy of the connection as:

$$A \to g^{-1}Ag + g^{-1}\mathrm{d}g \quad \Rightarrow \quad h'_e[A] = g(x(0))h_e[A]g^{-1}(x(1)). \tag{4.34}$$

Therefore certain functions of the holonomy, such as the $tr(h_e[A])$ (given e is a *loop*), are invariant under this transformation.

[7] The holonomy is defined as the solution to the equation

$$\frac{d}{ds}h_e[A, s] + \dot{x}^\mu(s)A_\mu h_e[A, s] = 0$$

which is comparable to $U(t)$ in Shrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}U(t) + H(t)U(t) = 0$$

i.e. the holonomy induces parallel transportation along the edge e.

Another function of the holonomy which is also $SU(2)$ invariant is

$$W_e^j[A] \equiv tr[\Pi^j(h_e[A])] \equiv \frac{1}{\sqrt{2j+1}} Tr[\Phi^j(h_e[A])] \tag{4.35}$$

where $\Pi^j_{mm'}$ is the $SU(2)$ unitary irreducible representation matrix of spin j, and

$$\Phi^j_{mn} \equiv \sqrt{2j+1}\Pi^j_{mn}, \tag{4.36}$$

is the "normalized" (with respect to the measure) $SU(2)$ representation of spin j.

Now, expanding on this idea, for a graph γ, with N nodes (such that the entire graph has no "loose" edges; it forms *a mesh of loops*), the function of the holonomy:

$$\Psi_{\gamma,f}[A] = \sum_{j_1 \cdots j_N} f^{m_1 \cdots m_N, n_1 \cdot n_N}_{j_1 \cdots j_N} \Phi^{j_1}_{m_1 n_1}(h_{e1}[A]) \cdots \Phi^{j_N}_{m_N n_N}(h_{e_N}[A]), \tag{4.37}$$

where $\Psi_{\gamma,f}[A] \in Cyl_\gamma$ is $SU(2)$ invariant. Such a function is called a ***Spin Network*** of the graph γ.

The collection of all Cyl_γ, for all graphs $\gamma \subset \Sigma$, forms Cyl. These are all solutions of the $SU(2)$ gauge, and it can then be shown (*sec.*(G.3)), [29], that Cyl (given an appropriate measure) forms a (possibly *over*-) complete solution space for the $SU(2)$ gauge equation.

4.3.2 The Treatment of $U_D[\phi]$; the Diffeomorphism

In this and the next section, we will be looking to see how the Cyl functions could "be made" to *obey* the diffeomorphism and the scalar constraints of the theory - the main constraints of GR.
Immediately there exist formal problems with the diffeomorphism gauge, which are independent of the LQG formulation. To illustrate these we will first discuss a simple example. This will help us to motivate and introduce the concepts of the *group averaging techniques* (or the so called *Gelfand-Naimark-Segal, (GNS)* construction or that of the *Rigged Hilbert Space.*) You may consult [27] and [37] and other literature therein for further reading on these topics.

Gelfand-Naimark-Segal Construction:

The basis of the GNS construction is that the Hilbert space, H (consisted of square integrable functions) is *not* sufficient to consider more general class of physical systems with certain symmetries (such as the general Diff. transformation). Instead it is posited that the following triple:

$$Cyl \subseteq H \subseteq Cyl^* \tag{4.38}$$

known as the *Gelfand triple* should be used to extract the physical information of the system. Let us see how this is done in the following example.

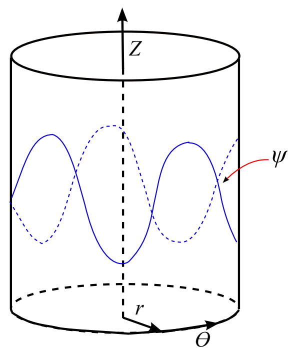

Figure 4.1: The Display of Compact vs. NonCompact Dimensions, and the resulting issue of finite-integrability or finite-norm (normalization) of the state.

The Example we will use is that of a particle confined on a cylinder, *Fig.*(4.1):
Let us denote the solutions by, $\psi(z, \theta)$, which we posit must obey the constraints:

$$\hat{p}_\theta \psi = 0 \qquad and \qquad \hat{p}_z \psi = 0 \tag{4.39}$$

where, $\hat{p}_\alpha$ is the gauge translation along the direction α.
A solution of the Schrödinger's equation which obeys the θ boundary condition, $\psi(\theta + 2\pi) = \psi(\theta)$, yet is neither normalized nor gauge-invariant, is:

$$\psi_{n,k}(z, \theta) \sim e^{in\theta} e^{ikz} \qquad\qquad n \in \mathbb{Z} \quad and \quad k \in \mathbb{R} \tag{4.40}$$

However, recall that the solutions of a *Hilbert Space* $H \subset \mathcal{L}^2(M)$ - where M is the manifold which the solution is defined on - i.e. are *square integrable* functions. Therefore $\psi_{n,k}(z, \theta) \notin H$, in fact $\psi_{n,k}(z, \theta) \in Cyl_{n,k}$.

In addition, the square integrability requirement introduces other problems for constrained systems. In fact it demands that any solution, $\psi \in H$ *to have special properties along the directional dimensions of* M:

- Along compact dimensions, x_c, ψ is finite-integrable
- Along non-compact dimensions, x_{nc}, ψ must die "fast-enough" as we go to $\pm\infty$

We therefore see, that it is *along compact dimensions, that* ψ *can be made to be translationally invariant,* $\hat{p}_{x_c}\psi = 0$.

This is the limitation of a Hilbert space; non-compact dimensions pose formal problems in defining invariant states.

This is exactly the situation we have for $\psi(z, \theta)$ along the z direction.

To remedy this we use the GNS construction or the concept of *Group Averaging*. Notice that a solution to the two constraints, *eqn.*(4.39), on ψ is c, a constant:

$$c \sim Average\left(e^{in\theta}e^{ikz}\right) \sim \int \mathrm{d}\theta \mathrm{d}z e^{in\theta}e^{ikz} \tag{4.41}$$

i.e. if we average over all the resulting expressions of the transformations of ψ by the constraints, we find a gauge-invariant solution, c.
However notice that c is neither $\mathcal{L}^2$ - and therefore not part of H - nor is it part of Cyl. Instead c is in the *algebraic dual space*, Cyl^* of Cyl.
The elements of Cyl^* are however only accessible via inner products with the elements of Cyl:

$$(c|\psi(z',\theta')\rangle \sim \int \mathrm{d}\theta \mathrm{d}z e^{-in\left(\theta-\theta'\right)}e^{-ik\left(z-z'\right)} \tag{4.42}$$

and these inner products are well-defined; only a finite number of terms in the average for c will survive the inner product. This suggests that instead of considering only Cyl or H, or the Cyl^*, we need to look at:

$$Cyl \subseteq H \subseteq Cyl^* \tag{4.43}$$

the *Gelfand triple*, and use the *Gelfand-Naimark-Segal (GNS)* construction - which uses the inner product of the Cyl and the dual Cyl^* - to extract the physical information of the system.

This is the programme of GNS, or the *Group-Averaging* technique used in Loop Quantum Gravity to treat the quantization of gravity.

As we have seen in LQG the holonomies present themselves as the natural variables of the theory, and the spin-networks as the solution space of it. Here, then the group-averaging method is applied on the spin-networks to produce diffeomorphism-invariant states. We anticipate these will belong to Cyl^*. The physical content of the theory is then extracted via the inner product of these states with elements of Cyl, the spin-networks.

4.3.3 The Scalar Constraint

The "croaks" of LQG is the dynamics ([36],[34],[35]). Here the ideas of group-averaging cannot be applied because finite transformations of the scalar constraint even in the classical gravity is not well-understood. Hence the only remaining method of trying to solve the scalar constraint is to seek its kernel.

There exist consistent methods which attempt, and perhaps it can be said that they achieve this goal. En-route to this goal however a number of non-trivial choices are made. These produce ambiguities in the method. Hence the results are not unique.

Nonetheless the very fact that this feat can be achieved has been one of the great successes of LQG. Yet, despite the large possibility of the space of the solutions there exist a number of very significant problems which remain unsolved.

One such problem is the question of observables.

- In the first place it is not clear what these mean. Are they properties associated with the states in Cyl or in Cyl^* or neither, or perhaps both?
- Secondly how can we assume these to be dynamical? Is there a sense in which this can be achieved, hence providing a canonical method of quantizing generally covariant theories with dynamics?
- Thirdly, if so, can these be related to semiclassical or classical observables of the theory?
- $\vdots$

We can see that these are fundamental questions which a full quantum theory of spacetime, such as LQG, is expected to provide answers to.

At the moment however the answer to these questions, in LQG are at best unclear. This is the main motivation behind trying to pursue the answer to these questions in a simpler model, such as this model of a closed string; D scalar fields on a *2D* spacetime.

4.3.4 The Closed String

To do so, let us therefore first restate the main principles of loop quantization.

The main idea that can be derived from using the spin-networks (and their variants) as the solutions in LQG, is that the physical content of the infinite theory (infinite points on the manifold), can be extracted by using an infinite set of finite samplings of the full theory; the spin-networks are "samplings" of the the $3D$ spatial manifold.

This theory then, works on the basis of finite-"sampling" the infinite theory, infinitely many times.

The interesting feature of the string model is that despite its simplicity, it embodies most of the main features of the full LQG. In particular most of the ambiguities which are inherent in the treatment of the scalar constraint of LQG are present in this model as well (cast in a much simpler format). This model therefore provides a very simple structure where we can hope to explore some of the problems which are very difficult to answer in the full LQG.

However, let us make a comment as to the discreteness which manifests itself throughout this method of quantization.

The Discreteness

The discreteness comes about when we consider samples of the fields in question (here these are the spacetime). From the perspective of the continuum theory, the sampling may be seen as a coarse-graining, and the description provided as a kind of approximation of the full theory.

In LQG the discreteness of the spacetime comes about because the spin-networks present themselves as the "natural" solutions of the $SU(2)$ gauge[8], and these have a discrete structure.
However in the String case there does not exist an equivalent to the $SU(2)$ gauge. Hence such a discrete structure may not be warranted a-priori.

However, the fundamental posit of LQG is that *the discreteness of spacetime is fundamental*. Therefore it is reasonable to think that perhaps, in extending this method of quantization to other theories, *the sampling is also assumed as a fundamental feature* of the theory rather than an approximation to it.

This is the view point which we will take in the Loop quantization of the string, i.e. we will assume a discrete string as the background for the scalar fields on it.

In the remaining sections of this chapter, *sec.*(4.4) through *sec.*(4.9), therefore, we will explore the method of the quantization of LQG, as well as questions regarding observability and dynamics in generally covariant theories by attempting a full loop quantization of the *2D* spacetime + scalar fields, i.e. the closed bosonic string.

[8]Technically however this discreteness is not fully automatic in LQG; the fundamental group, i.e. the $SU(2)$ gauge, which is used is in fact a mathematically convenient choice. Other possible choices exist which however do not have fully developed mathematical structures. More on this in *appendix*(G).

4.4 Loop Quantization of the String

From here to the end of this chapter we apply the *full* method of loop quantization to the D-dimensional scaler field theory on a *2D* spacetime, where the field theory is treated in analogy to the connection variable in LQG.

As mentioned this model has two interesting features; On the one hand it is a simple arena where many of the questions regarding the observables of a GC and many of the ambiguities concerning the loop quantization may be investigated.

On the other hand it represents the *String in string theory.* As such, although we do not address the issues of the background (the background is flat here), the results of the loop quantization may then be compared to the results of the quantization of string via the usual string theory methods in *chap.*(2) and *chap.(3)*.

4.5 Basic Construction

As in LQG the choice of the operators are interrelated to the possible solution spaces. However in contrast to LQG where the spin-networks arise from the requirement that the solution space be a solution to the $SU(2)$ gauge, here we have no such restriction. Hence we shall start from one step further; from the idea of sampling/probing. We will assume that the continuum string is to be modeled by considering *all* possible discrete samplings of it.

Let us therefore define what a sample is. This also defines the *Kinematic Hilbert Space* of the theory.

4.5.1 *Kinematic* Hilbert Space

A sample is defined as a *fixed*[9] set of probes which are points $n \in [0, L)$ where L is the length of the string. A sample is simply then a set of nodes on the string.
A sample is also called a *graph*. A graph composed of N *ordered*[10] nodes, is denoted by α_N, *fig.*(4.2):

$$\alpha_N \equiv \{e_1, e_2, \cdots, e_N\} \quad e_n \in [0, L) \tag{4.44}$$

[9] Two samples of equal number of nodes which are however differently positioned on the string continuum are considered different samples. Hence a sample in reality is a fixed lattice. We shall see that this is the very reason why the *kinematic* Hilbert space, $\mathcal{H}_{kin}$ (defined as the Cauchy completion of Cyl) is not separable; two physically equal states with different graphs are not necessarily orthogonal, however the *Diff.* solution space (defined in *sec.*(4.7) as the quotient of the $\mathcal{H}$-space with the *Diff* transformations) is composed of states which are orthogonal if the states are physically different. Hence $\mathcal{H}_{Diff}$ is separable whereas $\mathcal{H}_{kin}$ is not.

[10] The ordering does not pose any restriction in this theory; it further facilitates analysis. It will later appear in the consideration of *Diff* constraints, where we will see that the ordering represents a diffeomorphism transformation, and hence is quite irrelevant.

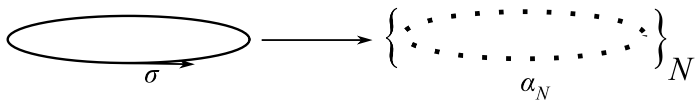

Figure 4.2: A sample or graph from the continuum string

Now a solution $\Psi^{\mu}_{\alpha_N}(f)$ is defined as follows[11]:

$$\begin{aligned} \Psi^{\mu}_{\alpha_N}[f] &= \sum_{j_1^{\mu}, j_2^{\mu}, \cdots, j_N^{\mu}} f_{j_1^{\mu}, j_2^{\mu}, \cdots, j_N^{\mu}} e^{i j_1^{\mu} X_1^{\mu}} \otimes e^{i j_2^{\mu} X_2^{\mu}} \otimes \cdots \otimes e^{i j_N^{\mu} X_N^{\mu}} \\ &= \sum_{\overrightarrow{j^{\mu}}} f_{\overrightarrow{j^{\mu}}} \bigotimes_{n=1}^{N} \phi^{\mu}_{j_n^{\mu}}[X^{\mu}] \end{aligned} \tag{4.45}$$

$$where \quad \phi^{\mu}_{j_n^{\mu}}[X^{\mu}] \equiv e^{i j_n^{\mu} X_n^{\mu}} \tag{4.46}$$

where j_n^{μ} are *charges* at the node n (in analogy to *spins* on an edge in LQG), X_n^{μ} is the value of the field X^{μ} at the node n, and $f_{\overrightarrow{j^{\mu}}}$ are *proper smearing functions*[12].
A general solution, $\Psi_{\alpha_N}(f)$, may then be defined as:

$$\Psi_{\alpha_N}[f] = (\prod_{\mu=1}^{D} \Psi^{\mu}_{\alpha_N})[f] \tag{4.47}$$

In analogy to LQG we call these the ***Charge-Networks***. The class of all functions $\Psi_{\alpha_N}(f)$ on the graph α_N forms Cyl_{α_N}, *the cylindrical functions of the graph* α_N. The set of all cylindrical spaces, for all possible graphs α_N forms the cylindrical solution space of the system (let us drop the N index as it is redundant and understood):

$$Cyl = \bigcup_{\alpha} Cyl_{\alpha} \tag{4.48}$$

The Cauchy-completion of Cyl, forms the *Kinematical Hilbert space* of the system, $\mathcal{H}_{kin}$. This Hilbert space therefore has the following structure:

$$\mathcal{H}_{kin} \to \bigoplus_{\alpha} \mathcal{H}_{\alpha} \to \bigoplus_{\alpha, [\overrightarrow{j}]} \mathcal{H}_{\alpha, [\overrightarrow{j}]}, \tag{4.49}$$

where the $[\cdots]$ indicates the $\mu = 1, \cdots D$ sets of charges, $\overrightarrow{j}^{\mu} = j_1^{\mu}, j_2^{\mu}, \cdots, j_N^{\mu}$.

[11] Note that at this stage we consider j_n^{μ} to take on discrete values. However note that in contrast to LQG where these need to be eigenvalues of the discrete $SU(2)$ group, and hence are discrete, here they are not subject to any such restriction and may take on a continuum of values. In later sections when we analyze the states we shall make these continuous.

[12] We define the proper functions as those whose norm, with respect to the measure of the theory $d\mu_N$, which we discuss in the next section, is finite.

4.5.2 The Measure

Here we will present the semantics for the definition of the measure in this theory.
The measure for this theory should have two very important properties:

- It should be graph-independent. In other words if Ψ is a solution on α_N and α_M is just an extension of the α_N by adding nodes for which Ψ has null values, then the inner product must stay invariant.
- Second we wish to define the measure $\mathrm{d}\mu_N$ such that $\langle \phi^{\mu}_{j^{\mu}_n} | \phi^{\mu}_{j^{\mu}_m} \rangle \propto \delta_{n,m}$.

The measure on this solution space is therefore defined as follows:

$$\langle \Psi_{\alpha_N}(f) | \Psi_{\alpha_N}(g) \rangle = \int \mathrm{d}\mu_N \bar{f} g \tag{4.50}$$

For an example we could allow:

$$\mathrm{d}\mu_N \equiv \left(\frac{1}{2\pi} \right)^N \mathrm{d}X_1 \mathrm{d}X_2 \cdots \mathrm{d}X_N \tag{4.51}$$

whence we have:

$$\langle \Psi_{\alpha}(f) | \Psi_{\alpha}(g) \rangle = \int \mathrm{d}\mu_N \sum_{[\vec{j}][\vec{K}]} \bar{f}_{[\vec{j}]} g_{[\vec{k}]} \bigotimes_{n=1}^{N} e^{-i(j_n - k_n)X_n} \tag{4.52}$$

$$= \sum_{[\vec{j}]} \bar{f}_{[\vec{j}]} g_{[\vec{j}]} \tag{4.53}$$

Hence we see that through this construction $\bigotimes_{\mu=1}^{D} \bigotimes_{n=1}^{N} \phi^{\mu}_{j^{\mu}_n}$ forms a complete basis for a given graph α_N. The questions regarding consistency of the measure and so on, are mathematical questions regarding the measures for discrete theories, known as Lebesgue-measure theory which are well-established. We provide a semi-rigorous development of the main ideas behind these in the *appendix*(G.3.1).

4.5.3 The Operators

Now we can address the question of the operators of the theory. We define these as follows:

The "j^{μ}th representation of the holonomy" of the field

$$\hat{h}^{\mu}_{e,j^{\mu}}[X] \equiv e^{ij^{\mu}\hat{X}^{\mu}_e} \tag{4.54}$$

The smeared conjugate momenta to the field

$$\hat{P}^{\mu}_{f,l} \equiv \int_l \mathrm{d}\sigma f(\sigma) \hat{\Pi}^{\mu}(\sigma) \tag{4.55}$$

the X-representation:

$$\hat{P}^{\mu}_{f,l} \to -i\hbar \int_l \mathrm{d}\sigma f(\sigma) \frac{\delta}{\delta X^{\mu}} \tag{4.56}$$

where $l \in [0, L)$, and $f(\sigma)$ is the smearing function.

The holonomy operator is point-wise, i.e. it is defined in the same manner as the sampling is used to define the states. This feature is in fact the reason for this definition; the action of this operator on a given graph α_N is rather trivial: it adds a charge j^μ at the node e of the string. If the node already exists, it increases the value of the charge at that node by the amount j^μ, and if the node does not exist, then it creates a new node with charge j^μ hence changing the graph; $\alpha_N(\overrightarrow{j}) \to \alpha_{N+1}(\overrightarrow{j'})$.

The action of the momenta is a little more involved and we will shortly describe it in full detail, however intuitively it can be seen that $\hat{P}^\nu_{f,l}\phi^\mu_{j^\mu_n}$ will give the charge of the node n *if 1)* $\mu = \nu$ but more importantly *if 2)* $n \cap l \neq \emptyset$ *i.e. if the node and the interval intersect.* This suggest that the action of this operator on the basis $\phi^\mu_{j^\mu_n}$ is case-wise. This is therefore also be true of the operator algebra. So let us consider the operator algebra which then would also detail the action of this operator on the states.

4.6 Operator Algebra and Action on Solution Spaces

We wish to evaluate:

$$\left\{\hat{h}^\mu_{e,j}[X], \hat{P}^\nu_{f,l}\right\} = \sum_k \frac{i^k}{k!}\int_l \mathrm{d}\sigma f(\sigma)\left\{(X^\mu_e)^k, \Pi^\nu(\sigma)\right\} \tag{4.57}$$

We see that the Poisson bracket (P.B.) will depend on whether the interval l and the node e intersect. There exist in general three possible cases *fig.*(4.3):

- l and e do not intersect, in which case the P.B. is null
- e is within the interval l in which case we wish the P.B. to return a value of $\delta^{\mu\nu}$
- e falls on one of the end-points of the interval l. To handle this case we need to know whether the string is oriented or not.

To handle this case we invoke two consistency requirements: the interval should be "breakable" into "fundamental" pieces. These pieces should be able to incorporate the orientation of the string should the theory requires it. So let us explain this in a bit more detail. There exist two cases:

4.6.1 Unoriented String

Note that if the string is not oriented, then we need not bother with two cases of the left and right intersection, we do as follows:
Break the interval l into fundamental pieces which may be defined as follows: *a fundamental piece of l, l_f is a closed-open interval (closed on "left" end and open on "right" end*[13]*, and no nodes exist in the interval l_f.* Of course as in the second case in

[13] The "left" and the "right" can be identified via the direction of the $d\sigma$ on the string.

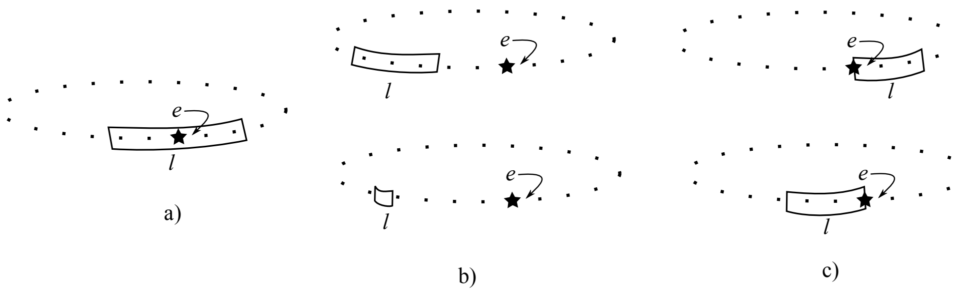

Figure 4.3: The Operator Algebra, a) The node intersects the interval b) The node does not intersect the interval c) The node intersect the interval at the endpoint

fig.(C.1 b), we see that it may be that l cannot be decomposed into fundamental nodes. If so, then the P.B. is immediately null, (since that l is a closed interval.) Hence a fundamental interval is graph-dependent and intersects one node of the graph α_N at one of its end-points. Each of these then contribute a factor of $\delta_{l_e,n}$ to the P.B.. With this then expression for the P.B. can be written as:

$$\left\{\hat{h}^{\mu}_{e,j^{\mu}}[X], \hat{P}^{\nu}_{f,l}\right\} = \sum_{l_f} \left\{\hat{h}^{\mu}_{e,j^{\mu}}[X], \hat{P}^{\nu}_{f,l_f}\right\} \tag{4.58}$$

and each of these is easily defined.
In the case of the un-oriented string we avoid another complication which exists in describing the interaction of the holonomy and the flux operator in LQG. The analogue of this problem is when we consider the oriented string. Let us then consider this next. The expression which we shall find there is a general expression which also includes the unoriented string.

4.6.2 Oriented String

Consider *fig.*(4.4). We define the P.B. based on the requirement that for case *iii)* of *fig.*(4.4):

$$\left[\hat{h}^{\mu}_{e,j}[X], \hat{P}^{\nu}_{l_{iii},f}\right] = \hbar j f_e \delta^{\mu\nu} \hat{h}^{\mu}_{e,j}[X] \tag{4.59}$$

i.e. *it gives the value of the charge at that node*, and the requirement that *if the length, $l_{iii} \sim l_i \oplus l_{ii}$, then:*

$$\left[\hat{h}^{\mu}_{e,j}[X], \hat{P}_{iii,f}\right] = \left[\hat{h}^{\mu}_{e,j}[X], \hat{P}_{i,f} + \hat{P}_{ii,f}\right] \tag{4.60}$$

each with an equal contribution. Now we can see that due to the dσ if the string is oriented we can pick up a sign. To get around this we make two definitions so to achieve the above goals:

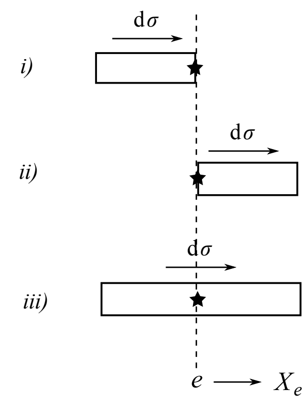

Figure 4.4: The effect of the orientation of the string, and the principle which is used to define the operator algebra.

1. *Source* vs. *Target*
   if l_f is the source or target of node e, which reflects the orientation of the interval
2. Whether the node is on the *Left* or on the *Right* of l_f:

$$\Rightarrow if \quad \begin{cases} Source & \sim + \\ Target & \sim - \end{cases} \qquad if \quad \begin{cases} Left & \sim + \\ Right & \sim - \end{cases} \tag{4.61}$$

Let us denote these by the variable $k_{l_f} = \pm$, which is a multiple of the two factors above. For example if e is on the left of the interval l_f and l_f is its source, $k_{l_f} = + \times + = +$. Therefore to evaluate the commutator first we need cut the interval l into "elementary length", such that each l_f contains one node at one of its end points. Hence we have:

$$\left[\hat{h}_{e,j}[X], \hat{P}_{l,f}\right] = \sum_{l_f} k_{l_f} \left[\hat{h}_{e,j}[X], \hat{P}_{l_f,f}\right] \tag{4.62}$$

As explained this expression includes the unoriented string which has $k_{l_e} = +$ for all its fundamental intervals.
Hence we have defined the operator algebras. We are now in the position to discuss the first of the two constraints of the system.

4.7 The *Diff* Constraint

We will treat the *Diff* constraints in two stages. In the first we set up the construct for *group averaging* a state, which constructionally renders a state Diffeomorphism-invariant. In the second step we actually take a look at the action of the Diff. operator on basis states.

However since the second step is closely related to the treatment of the Scalar constraint, we will postpone it to the next section where we attempt to consider this constraint.
Now to *define* a diffeomorphism-invariant state first we need to recognize the different families of diffeomorphisms in the theory. These can be divided into three groups:

- $Diff_\alpha$: Subgroup of $Diff$ which map $\alpha \rightarrow \alpha$ (itself)
  This subgroup can in turn be divided into two groups: $TDiff_\alpha$ and GS_α.
  - ▷ $TDiff_\alpha$: Subgroup of $Diff_\alpha$ which has trivial action on α (i.e. preserves every node and its charge.) These can be assumed to be obeyed trivially.
  - ▷ $GS_\alpha \equiv \frac{Diff_\alpha}{TDiff_\alpha}$ are the *graph symmetries* of α. To satisfy this set of Diff. we group-average over the states as follows:
    Let a graph-$Diff$-invariant state be denoted by:

$$\hat{GS}_\alpha \Psi_\alpha \equiv \frac{1}{N_\alpha} \sum_{\varphi \in GS_\alpha} \varphi * \Psi_\alpha \qquad group - averaged\, state\,. \tag{4.63}$$

    This state is invariant under *graph-Diff.* These states are however still in the Cyl_α; this is not the group-averaging which is needed for the next set of Diff. constraints:

- $\frac{Diff}{Diff_\alpha}$ which *move/ change* the graph α on the continuum string. This is a large group, its action takes elements of Cyl_α to Cyl^*_α, (the *Algebraic Dual* of Cyl). Hence a solution to the $Diff = \frac{Diff}{Diff_\alpha} \oplus GS_\alpha \oplus TDiff_\alpha$ constraint, belongs to the algebraic dual of Cyl. And since the elements of this space are understood only via their action on the elements of Cyl, we therefore need to consider the inner product for the definition of this operation.
  We therefore define a $\frac{Diff}{Diff_\alpha}$-transformed state, by its action on the elements of $Cyl = \bigcup_\alpha Cyl_\alpha$ as follows:
  let: $\quad \varphi \in \frac{Diff}{Diff_\alpha}$
  Then we define the $\frac{Diff}{Diff_\alpha} - averaged - state: \quad (\eta(\Psi_\alpha)| \in Cyl^*$ by:

$$(\eta(\Psi_\alpha)|\Phi_\beta\rangle = \sum_{\varphi \in \frac{Diff}{Diff_\alpha}} \langle \varphi * \hat{GS}_\alpha \Psi_\alpha, \Phi_\beta \rangle \tag{4.64}$$

  $(\eta(\Psi_\alpha)|$ is well-defined because for any given β only a finite number of the transformations of Ψ_α are not orthonormal to Φ_β, (although $\varphi \in \frac{Diff}{Diff_\alpha}$ is an infinite group.)

Let us make a note on the graph-symmetries. In contrast to LQG where in *3D* space, 2 edges may be related via homotopies (via rotation in the third dimension, i.e. no crossing is allowed), in our case since two non-adjacent nodes *cannot* be swapped, the graph-symmetries exist *only* if two adjacent nodes have the same charge (and orientation if the string is oriented.)

This is then the group-averaging through which we make a given state, Ψ_α on the graph α, diffeomorphism-invariant. Notice that there is an immense emphasis on the role of the *Dual*

space Cyl^* in this definition. We will come to see that this role *can* have great significance in the definition of observables and their dynamics in LQG.
This role will be explored the in next section where we shall see in some detail the action of the diffeomorphism operator on the states and analyze its operation.

4.8 The *Scalar* Constraint

Now in the expression of the Hamiltonian (constraint):

$$H[N, M] = \int \mathrm{d}\sigma \frac{N}{2} \left(\Pi^2 + X'^2\right) + \frac{M}{2} \left(\Pi^\mu X'_\mu + X'^\mu \Pi_\mu\right) \tag{4.65}$$

we see that the second expression is the *Diff* constraint, $C^1[M]$. Hence we identify the *Scalar* constraint as the first term:

$$C^0[N] = \int \mathrm{d}\sigma \frac{N}{2} \left(\Pi^2 + X'^2\right) \tag{4.66}$$

Notice that the method of group-averaging for the scalar constraint is not suitable in LQG because "finite scalar transformations are not well-understood even at the classical level" [36]. However the scalar transformations here are not due to the spacetime, they are due to the scalar fields, so the group-averaging may still apply. However to draw as close a comparison to LQG as possible let us consider LQG's method.

Therefore to proceed with loop quantization, we seek to find the *kernel* of the scalar operator. The first step in this direction is defining the Scalar constraint operator in terms of well-known operators of the theory, namely the "holonomy" $\hat{h}^\mu_{e,j^\mu}[X]$ and the "momenta flux", $\hat{P}^\nu_{f,l}$.

The Π^μ in the scalar constraint's expression is easily read; in the X^μ representation it can be $\sim -i\hbar\frac{\delta}{\delta X^\mu}$. The difficulty lies in the X'^μ. To define it we need discuss regularization, and this is precisely where we make contact with similar issues in defining the *Scalar* constraint, and therefore the Hamiltonian, in the LQG.

To define this operator, let us consider a discretization of the continuum string into regular ϵ-sized pieces, *which may be independent of the graph being considered.* At this point, ϵ is constant throughout the string and not related to the discrete structure of any one of the graphs on the string. Later we will come to find a relationship between these. Now using:

$$X'^\mu_e = \frac{1}{2\epsilon} \left(X^\mu_{e+\epsilon} - X^\mu_{e-\epsilon}\right) + O(\epsilon^3) \quad e \in [0, L) \tag{4.67}$$

we can see (dropping the μ index for simplicity):

$$X'_{e,j} \sim e^{ijX_{e+\epsilon}} - e^{ijX_{e-\epsilon}} = e^{ijX_e} \left(e^{ij\epsilon X'_e} - e^{-ij\epsilon X'_e}\right) = e^{ijX_e} \left(2ij\epsilon X'_e + O(\epsilon^3)\right) \tag{4.68}$$

Notice that whereas j cannot be generally considered to be small (i.e. we cannot take $\lim_{j\to 0}$), ϵ can. Furthermore this definition does make physical sense. Nonetheless there does exist a certain ambiguity in the definition of this operator. This ambiguity is likened (though not on the same footing) to the ambiguity in the definition of the *volume* operator in the regularized expression of LQG's scalar constraint. The effects of this ambiguity can be plainly seen at the end of the calculations of this section.

As such we then define (after self-conjugating):

$$\hat{X'}^{\mu}_{j^\mu} \equiv \frac{1}{2ij^\mu\epsilon}\left[\hat{h}^{\mu\dagger}_{e,j^\mu}[X]\left(\hat{h}^{\mu}_{e+\epsilon,j^\mu}[X]-\hat{h}^{\mu}_{e-\epsilon,j^\mu}[X]\right)-\hat{h}^{\mu}_{e,j^\mu}[X]\left(\hat{h}^{\mu\dagger}_{e+\epsilon,j^\mu}[X]-\hat{h}^{\mu\dagger}_{e-\epsilon,j^\mu}[X]\right)\right] \tag{4.69}$$

We call this the *the j^μth representation of the operator* $\hat{X'}^{\mu}$.

Now we are in the position to try to analyze the details of the actions of the constraints on the solution space. We shall do this in two distinctly different ways. In the first we will apply an intuitive reasoning to the problem. In the second we will consider detailed calculation. We do so to illustrate and highlight an interesting intuitive-discrepancy in LQG which one should be aware of. This will also help in the discussion which we will have as to the possible observables of LQG, and their dynamics.

4.8.1 Intuitive Picture

At this point we are equipped with the required elements to define the j^μth representation of the constraints, C^0_{ϵ,j^μ} and C^1_{ϵ,j^μ}, using a regularization ϵ. Using these we can also consider the effects of the Diff. constraint. We will in fact *use* the effect of this constraint to later interpret the effects of the Scalar constraint. To do so, let us consider the action of these operators on a give graph, α_N.

What we may ask is *What is the definition of a Diff. transformation of a state* $\Psi_{\alpha_N}[f]$ *defined on this graph?*

Intuitively, Spatial Diffeomorphism should *shift* the graph on the spacial slice, i.e. along the length of the string. As such for a given graph the action of the operator defining this transformation should not change or alter the discrete topology, the only feature of which is the number of nodes of the graph, and their relative positions, (recall based on the discussion at the end of the last section that nodes cannot be swapped unless they were adjacent and possessed the same charge.) *Hence theDiff. operator corresponding to a graph must be tailored not to alter its topology.* In other words the diffeomorphism operator must be defined graph-dependently.

Now notice that the only mathematical constructs that are introduced in the definition of the *Diff.* constraint are the e and the ϵ. However notice that given a certain ϵ, we therefore must have that $\frac{L}{\epsilon} \in \mathbb{N}$, i.e. the ϵ must slice the string into an integer-number of intervals. This defines a (chargeless) graph. Hence we find that there exists a one-one relation between a given regularization, given by ϵ, and a given graph, denoted by α_N. This then suggests the following:

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha \quad \rightarrow \quad \mathcal{O} = \bigoplus_\alpha \mathcal{O}_\alpha \quad \mathcal{H} \sim Hilbert\, Space \quad \forall\, operators\, \mathcal{O} \tag{4.70}$$

i.e. just as the Hilbert space can be split into graph-spaces (the Hilbert space itself is *not* however separable[14]), so can (and should) any operator $\mathcal{O}$.

Now that we have made this connection, we may suggest how the *Diff.* constraint or any other operator, should act on the states:
Since a given state Ψ_α is group-averaged, we can pick the one graph in the average which "matches" the operator's regularization exactly. The effect of the Diff. operator on this state is to shift it to another one of the states of the same group-average state. And as such the state remains invariant.

Hence we *intuitively* reach the conclusion that the *Scalar* constraint should also be "matched" to the graph. If so however since the only part of these operators which can alter a graph is X', despite the internal difference of the two constraints (i.e. the detail of their action on the state), the *Scalar* constraint should also leave the topological structure of the state unchanged. Hence nodes cannot be created or destroyed even due to the scalar constraint. As such we see that there is no change of the state via the Hamiltonian; *the state, in this sense, is static.*

Through this intuitive analysis therefore we find that if we were to associate dynamics to the topological changes of a given graph on the string, then due to the restrictions which are imposed on operators due to the (intuitive) definition for a Diff., we anticipate all charge-networks to be static and non-dynamical. In the next section however, we shall see that this intuition may not be well-justified.

4.8.2 Detailed Structure

Now let us analyze this question rigorously. We will in fact see that the *Diff.* constraint does not simply "move" the graph; it can also change its topology. This indicates that we cannot think of these operators intuitively as we did in the previous section. As an immediate consequence we therefore realize that we need not impose the requirement that the state and the operator be *regularization-matched.*

In this analysis we will try to make contact with the same considerations in LQG. We will however consider an additional important case; we will perform a detail analysis of both the *Scalar* <u>and</u> the *Diff.* constraint, not just the former.

To do so let us reconsider the regularized constraint operators. In the first step we have:

$$\int \mathrm{d}\sigma \quad \rightarrow \quad \sum_{m=1}^{N} \epsilon \tag{4.71}$$

[14] See for example [28].

where the ϵ comes about from the ΔL. Let us still demand that $\epsilon = \frac{L}{N}$[15]. Putting this in the expression for the constraints we have:

$$C^1_{\epsilon,[\overrightarrow{j}]}[M] = \sum_{\mu=1}^{D}\sum_{k=1}^{N} \epsilon M_{e_k} C^1_{e_k,\epsilon,j^\mu} \tag{4.72}$$

$$C^0_{e,\epsilon,[\overrightarrow{j}]}[N] = \sum_{\mu=1}^{D}\sum_{k=1}^{N} \epsilon N_{e_k} C^0_{e_k,\epsilon,j^\mu} \tag{4.73}$$

where:

$$C^1_{e_k,\epsilon,j^\mu} = \frac{-\hbar}{2j\epsilon}\frac{\delta}{\delta X^\mu_{e_k}}\left[{h^\mu}^\dagger{}_{e_k,j^\mu}\left(h^\mu_{e_k+\epsilon,j^\mu} - h^\mu_{e_k-\epsilon,j^\mu}\right) - h^\mu{}_{m,j^\mu}\left({h^\mu}^\dagger{}_{e_k+\epsilon,j^\mu} - {h^\mu}^\dagger{}_{e_k-\epsilon,j^\mu}\right)\right] \tag{4.74}$$

$$C^0_{e_k,\epsilon,j^\mu} = -\left[\hbar^2\frac{\delta}{\delta X^\mu}\frac{\delta}{\delta X^\mu} + \frac{1}{(2j^\mu\epsilon)^2}\left[{h^\mu}^\dagger{}_{e_k,j^\mu}\left(h^\mu_{e_k+\epsilon,j^\mu} - h^\mu_{e_k-\epsilon,j^\mu}\right) - h^\mu{}_{e_k,j^\mu}\left({h^\mu}^\dagger{}_{e_k+\epsilon,j^\mu} - {h^\mu}^\dagger{}_{e_k-\epsilon,j^\mu}\right)\right]^2\right] \tag{4.75}$$

Note that the regulator ϵ vanishes in the first expression (just as it does in LQG's *Scalar* constraint regularization), however it is present in the *Scalar* constraint's expression. However a look at LQG's *Diff.* constraints will show that, based on the same regularization as the *Scalar* constraint, the regulator does *not* vanish[16]. Given the parallel nature of these constraints, and in light of what we will later further discuss, we may say that our regularization is just as reliable as that of LQG's. (We will have other notes to add to the discussion of the regulator, which we will differ to *sec.*(4.8.2).)

Now let us analyze the action of these operators.
In LQG, the action of the *Scalar* constraint is tailored to the nodes[17]. However the *Diff.* constraint is not subject to this restriction[18]. Interestingly enough we have the opposite situation in our toy model. In our case the *Diff.* can only act exactly on the nodes of the graph, because of the $\frac{\delta}{\delta X_e}$; it annihilates the state if no node exists where it is applied[19]. However the *Scalar* is not subjected to this restriction. Two different cases in fact exist for this constraint:

- if on the node, both terms of the constraint contribute
- if not on a node of the graph, only the ${X'}^2$ contributes.

Hence in the case of the Diff. constraint we have $e_k \to k \in N$, coincide with the nodes of the graph, α_N (it is irrelevant which because of the $\sum_{e_m}$), whereas this restriction is not applied automatically on the action of the Scalar constraint. Of course this situation is no different than for the constraints of LQG (with the $Diff \leftrightarrow Scalar$.)

[15]This is because σ is compact and the string is closed; no equivalent requirement to this exists in LQG.
[16]In this model$C^1 \sim \frac{\epsilon}{\epsilon}$ and $C^0 \sim \frac{\epsilon}{1} + \frac{\epsilon}{\epsilon^2}$. In LQG $C_{Scalar} \sim \frac{\epsilon^3}{\epsilon^3}$ whereas, $C_{Diff} \sim \frac{\epsilon^3}{\epsilon^2} + \frac{\epsilon^3}{\epsilon}$.
[17]This is because in that expression there exists a *volume* operator. This operator only acts on the nodes.
[18]It is only a function of the holonomy.
[19]Pending certain ordering ambiguities which we will discuss in *sec.*(4.8.2).

At this point ϵ is still arbitrary. Yet it too must be tailored. Following LQG we demand that $\epsilon < \epsilon_n$ i.e. *less than the spatial separations of adjacent nodes of the node n.* The (LQG) reasoning is that the *new* node is the result of the evolution of the original node, hence it should be created *locally.*

Now let us look at the action of the operators as defined above on a given node of the graph. We have:

$$X'^{\mu}_{m,j^\mu} = \frac{1}{2ij\epsilon}\left[h^{\mu\dagger}_{m,j^\mu}\left(h^\mu_{m+\epsilon,j^\mu} - h^\mu_{m-\epsilon,j^\mu}\right) - c.c.\right] \tag{4.76}$$

$$\Pi^\mu_m = -i\hbar\frac{\delta}{\delta X^\mu_m} \tag{4.77}$$

on the states. A general state is again give by:

$$\Phi_{[\overrightarrow{j}],\alpha} = \prod_{\mu=1}^{D}\Phi^\mu_{\overrightarrow{j}^\mu,\alpha} \tag{4.78}$$

$$where \quad \Phi^\mu_{\overrightarrow{j}^\mu,\alpha} = \bigotimes_{n=1}^{N}\phi^\mu_{j^\mu_n} = \bigotimes_{n=1}^{N} e^{ij^\mu_n X^\mu_n} \tag{4.79}$$

$$\rightarrow \quad \Psi_\alpha[f] = \int[\mathrm{d}\overrightarrow{j}]f([\overrightarrow{j}])\Phi_{[\overrightarrow{j}],\alpha} \tag{4.80}$$

Again, the *arrow*, $\rightarrow$ indicates the $n = 1\cdots N$ nodes, whereas the square brackets, $[\cdots]$ indicate the $\mu = 1\cdots D$ scalar fields.

The action of the momenta Π^μ_m is rather trivial:

$$\Pi^\mu_m \rightarrow \Phi^\mu_{\overrightarrow{j}^\mu,\alpha} = \hbar j^\mu_m \Phi^\mu_{\overrightarrow{j}^\mu,\alpha} \tag{4.81}$$

The action of the $X'^{\mu}_{m,j}$ may seem more complicated at first. It is however also more instructive if we therefore looked at it in pieces. We have:

$$\begin{cases} h^\dagger_{m,j^\mu}h_{m+\epsilon,j^\mu} \rightarrow \Phi^\mu_{j^\mu,\alpha} \rightarrow \bigotimes\limits_{n\neq m} e^{ij^\mu_n X^\mu_n}\bigotimes e^{i(j^\mu_m - j^\mu)X^\mu_m}\bigotimes e^{ij^\mu X^\mu_{m+\epsilon}} \\ h^\dagger_{m,j^\mu}h_{m-\epsilon,j^\mu} \rightarrow \Phi^\mu_{j^\mu,\alpha} \rightarrow \bigotimes\limits_{n\neq m} e^{ij^\mu_n X^\mu_n}\bigotimes e^{i(j^\mu_m - j^\mu)X^\mu_m}\bigotimes e^{ij^\mu X^\mu_{m-\epsilon}} \end{cases} \tag{4.82}$$

Then subtracting the second from the first we have:

$$\begin{aligned} &\rightarrow \bigotimes_{n\neq m} e^{ij^\mu_n X^\mu_n}\bigotimes\left[e^{i(j^\mu_m - j^\mu)X^\mu_m}\otimes e^{ij^\mu X^\mu_{m+\epsilon}} - e^{ij_\mu X^\mu_{m-\epsilon}}\otimes e^{i(j^\mu_m - j^\mu)X^\mu_m}\right] \\ &= \bigotimes_{n\neq m} e^{ij^\mu_n X^\mu_n}\otimes e^{i(j^\mu_m - j^\mu)X^\mu_m}\otimes\left(e^{ij^\mu X^\mu_{m+\epsilon}} - e^{ij^\mu X^\mu_{m-\epsilon}}\right) \\ &= e^{-ij^\mu X^\mu_m}\left(e^{ij^\mu X^\mu_{m+\epsilon}} - e^{ij^\mu X^\mu_{m-\epsilon}}\right)\bigotimes\Phi^\mu_{\overrightarrow{j}^\mu,\alpha} \end{aligned} \tag{4.83}$$

In other words we see the action of the X'^{μ}_{m} is in fact rather trivial on the state. With its complex conjugate portion then we have:

$$
\begin{aligned}
X'^{\mu}_{m,j^\mu}\Phi^{\mu}_{j^\mu,\alpha} =& \frac{1}{2ij^\mu\epsilon}\left[e^{-ij^\mu X^\mu_m}\left(e^{ij^\mu X^\mu_{m+\epsilon}} - e^{ij^\mu X^\mu_{m-\epsilon}}\right) - e^{ij^\mu X^\mu_m}\left(e^{-ij^\mu X^\mu_{m+\epsilon}} - e^{-ij^\mu X^\mu_{m-\epsilon}}\right)\right]\Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \\
=& \frac{1}{j^\mu\epsilon}\left[sin\left(j^\mu(X^\mu_{m+\epsilon} - X^\mu_m)\right) + sin\left(j^\mu(X^\mu_m - X^\mu_{m-\epsilon})\right)\right]\Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \qquad (4.84)
\end{aligned}
$$

Hence the action of the $X'^{\mu}_{j^\mu}$ on a charge-network, gives a superposition of four (4) charge-networks *fig.*(4.5).

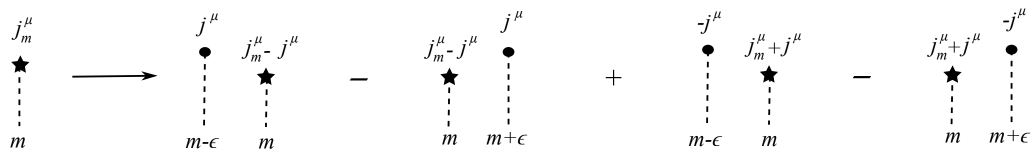

Figure 4.5: The action of the X'_m on a node m of the graph α_N.

Each of these is a simple modification of the original charge-network by the addition of another node of charge $\pm j^\mu$ an $\pm\epsilon$-away from every node, and the reduction/ addition of the same charge from the original node.

These modifications are exact analogues to those in LQG. Note that what this means is that *the graph does actually change, by addition of nodes* via the action of X'. This operator is present in both the *Diff.* and the *Scalar* constraints. As such we see that the *action of the* Diff. *is* <u>*not*</u> *a simple diffeomorphism of the graph; it may indeed change the topology of the graph.*

This finding is also in correspondence with LQG, although this is not the usual treatment. In LQG if we look at the *Diff.* constraint we see that it includes the curvature term; the same term in the *Scalar* constraint which is responsible for the addition of loops at nodes. The only difference is that this operator is not bound to act only at the nodes (because there is no *volume* operator present.) Hence in addition to adding loops to nodes, it can add an edge to another edge hence making it into a node, *fig.*(4.6).

Nonetheless the greater lesson here is that the *Diff* actually changes the graph, *in much the same way that the Scalar constraint changes the graph.* This has two consequences.

1) The scalar constraint's changes on the graph cannot be trivially considered as dynamical; these are identical to diffeomorphism transformations.

2) It can be shown that the space of *Diff.-Invariant* states, also obeys the *Scalar* constraint. We will see this in the next section, *sec.*(4.8.2).

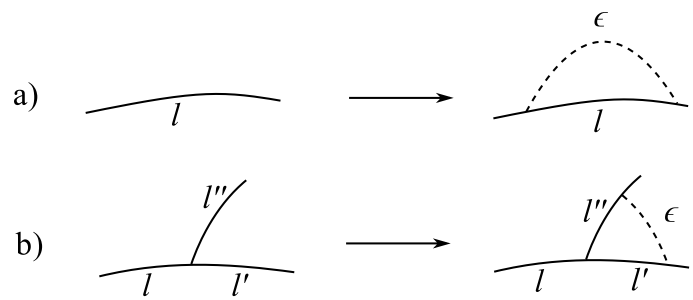

Figure 4.6: The *Diff.* in LQG can act in two ways on a graph. a) On an edge b) At a node

Hence one may ask if the dynamics is once again lost here. In the next section, *sec.*(4.9), we will see that this may not necessarily be the case. However let us finalize the action of the constraints in our model before proceeding.

Now putting together *eqn.*(4.81) and *eqn.*(4.84), and recalling that in the expression of the full constraints, we have two sums: $\sum_{k=1}^{N}$ and $\sum_{\mu=1}^{D}$, and therefore to consider for example:

$$\Pi^{\mu}\Pi_{\mu} + X'^{\mu}X'_{\mu} \rightarrow \sum_{\mu=1}^{D} \left(X'^{\mu}\right)^2 + (\Pi^{\mu})^2 \tag{4.85}$$

we need consider the general solution, $\Phi_{[\vec{j}],\alpha} = \prod_{\mu} \Phi^{\mu}_{\vec{j}^{\mu},\alpha}$, we find the following results:

$$\begin{aligned} C^{0}_{[\vec{j}],\epsilon}[N]\Phi_{[\vec{j'}],\alpha} &\rightarrow \sum_{m=1}^{N} \epsilon N_m \sum_{\mu=1}^{D} \left\{ \frac{1}{(j^{\mu}\epsilon)^2} \left[sin\left(j^{\mu}(X^{\mu}_{m+\epsilon} - X^{\mu}_{m})\right) + sin\left(j^{\mu}(X^{\mu}_{m} - X^{\mu}_{m-\epsilon})\right)\right]^2 + (\hbar j^{\mu}_{m})^2 \right\} \Phi_{[\vec{j'}],\alpha} \\ C^{1}_{[\vec{j}],\epsilon}[M]\Phi_{[\vec{j'}],\alpha} &\rightarrow \sum_{m=1}^{N} \epsilon M_m \sum_{\mu=1}^{D} \left(\frac{\hbar}{2j^{\mu}\epsilon}\right) j^{\mu}_{m} \left[sin\left(j^{\mu}(X^{\mu}_{m+\epsilon} - X^{\mu}_{m})\right) + sin\left(j^{\mu}(X^{\mu}_{m} - X^{\mu}_{m-\epsilon})\right)\right] \Phi_{[\vec{j'}],\alpha} \end{aligned} \tag{4.86}$$

The latter of these is satisfied via the group-averaging. Therefore in the next section let us look to see what the solution space of the first constraint can be.

The $Kernel$ of C_{Scalar}

First note that the solution space is such that the different scalar fields, (i.e. the μ index) are truly decoupled from each other. Hence we may focus on just one of these. So let us ignore the $\sum_{\mu}$. The constraint equations then become:

$$C^0_{\epsilon,\vec{j}^{\,\mu}}[N]\Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \to \sum_{n=1}^{N} \epsilon N_n \left\{ (\hbar j^{\mu}_n)^2 + \left(\frac{1}{2i\epsilon j^{\mu}_n}\right)^2 \left[e^{-ij^{\mu}X^{\mu}_n}\left(e^{ij^{\mu}X^{\mu}_{n+\epsilon}} - e^{ij^{\mu}X^{\mu}_{n-\epsilon}}\right) - c.c.\right]^2 \right\} \Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \tag{4.87}$$

$$C^1_{\epsilon,\vec{j}^{\,\mu}}[M]\Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \to \sum_{n=1}^{N} \epsilon M_n \frac{\hbar}{2ij^{\mu}\epsilon} j^{\mu}_n \left[e^{-ij^{\mu}X^{\mu}_n}\left(e^{ij^{\mu}X^{\mu}_{n+\epsilon}} - e^{ij^{\mu}X^{\mu}_{n-\epsilon}}\right) - c.c.\right] \Phi^{\mu}_{\vec{j}^{\,\mu},\alpha} \tag{4.88}$$

Again the action of the diffeomorphism constraint are as displayed in *fig(4.5)*. A *Diff. Inv.* state is then averaged over all these changes. By definition then it belongs to the dual space Cyl^*. However then such an averaged-state also obeys the *Scalar* constraint, i.e. is in its *kernel*. We can see this as follows:

$$C^0_{\epsilon} \sim \Pi^2 + X'^2 \quad \Rightarrow \quad C^0_{\epsilon}\Phi \sim j^2\Phi + X'(X'\Phi) \tag{4.89}$$

However, the $X'\Phi$ term is indeed the term in *Diff.* which requires the group-averaging of the state. So we have:

$$C^0_{\epsilon}\Phi \to j^2\Phi + X'(X'\Phi) \to j^2\Phi + X'(\Phi_{Diff}) \to j^2\Phi + \Phi_{Diff} \sim \Phi_{Diff} \tag{4.90}$$

Hence, the *Scalar* constraint does not change the states of the *Diff* constraint; they share the same solution space. We may also see this by writing:

$$C^0_{\epsilon,\vec{j}^{\,m}u_e}\Phi^{\mu} \sim \frac{1}{\beta}\left[C^1_{\epsilon,\vec{j}^{\,\mu},e}\left(C^1_{\epsilon,\vec{j}^{\,\mu},e}\right) - K\right]\Phi^{\mu} \tag{4.91}$$

where, β and K are (graph-dependent) constants. Hence *under the same regularization*,

$$if \quad \Phi \in \{\Phi_{Diff}\} \to \left(C^1\right)^n \Phi \sim \Phi \quad \forall n \in \mathbb{N} \tag{4.92}$$

and therefore the solutions to the C^1_{ϵ} constraint are also in $kernel(C^0_{\epsilon})$.

However this also means that the C^0 does not change the state. As such the *averaged* eigenvalues of the observables ($\sim$ *area*, *volume*) should be unchanging, or static hence non-dynamical.

However the perspective (as we will discuss in the sections on *Dynamics sec.*(4.9.2)) on the observables in the loop quantization is that these eigenvalues, should only be considered in the case of the un-averaged states, i.e. on the elements of Cyl not Cyl^*. These do change via the action of the Hamiltonian (a combination of the *Scalar* and *Diff* constraints.) However they are not solutions to the constraints.

As such the observables are associated with the states in Cyl and these are dynamical, however it is the elements of the Cyl^* which obey the constraints. Therefore the resulting inner product between the elements of Cyl^* and the elements of Cyl will carry the dynamical yet invariant physical information of the theory.

If we are to take this perspective, we see precisely how LQG separates the requirements of *constraint-invariance* and those o f the *dynamics*.

The ϵ Limit

We said a little bit about the ϵ and the $\lim_{\epsilon\to 0}$. Let us make a few more remarks here. First notice that, as in the *Diff* case, the solutions to the *Scalar* constraint belong to Cyl^*, and can be understood via their inner product on the elements of Cyl. Hence in LQG it is posited that the limit is to be understood in the following sense:

$$(C^0\Psi)(\Phi) = (\Psi)(\lim_{\epsilon\to 0} C^0_\epsilon \Phi) \equiv \lim_{\epsilon\to 0}(\Psi)(C^0_\epsilon \Phi) \tag{4.93}$$

where the last step is a *definition* of this limit, which is used in LQG. Now notice, that if Ψ is *Diff. Inv.* this limit exists and is proper.

Ordering Ambiguity

Notice that the $\Pi \sim \frac{\delta}{\delta X}$ gives the charge of the node. Hence if no node exists at that point, it will annihilate the state. In our case the *Diff.* is a combination of Π and X'. If Π is put to the "right" of X', then it acts on the state first. Hence the operator must be "matched" to the nodes of the graph. However if we put Π to the "left" of X', since X' creates nodes at the exact position where Π will act, then the action of this operator will never be null, but it will be completely arbitrary. Therefore there exits an *ordering ambiguity.* The equivalent of this ambiguity is also present in the *Scalar* constraint of LQG[20]. These similarities are extremely interesting specially when we realize that the
$Diff_{our} \sim Scalar_{LQG}$ and vice versa.

Hence we have analyzed the detailed structure of the constraints in this section. The conclusion is that the consideration of dynamics and dynamical observables in LQG may be quite non-trivial and unintuitive. Therefore let us spend the next and final section of this chapter to finalize the possible solution of Loop quantization for the question of observability and dynamics in generally covariant theories.

4.9 Observables and Dynamics

Let us first discuss the observable space of the model in question, and then ask how, if at all, may these be assumed to be dynamical.

4.9.1 Observables

First, let us make a superficial comparison of our observables to those of LQG's.
Local observables in LQG are defined as the functions of the flux operator. These become the *area* and *volume* operators of the theory. The analogues of these in this toy model are functions of the momentum-flux operator, $P^\mu_{l,f}$:

$$\hat{\mathcal{O}} \sim f(\hat{P}^\mu_l) \tag{4.94}$$

[20]In the expression of the Scalar constraint in LQG, there exists a volume operator which annihilates trivalent nodes. The curvature term in the same expression is capable of adding edges to a graph. Hence the ordering of these may determine if a state will be annihilated or not.

However in contrast to LQG, (where the *area* and *volume* define the geometric observables of the theory), these operators will not be geometrical; they are related to the field variables not the spacetime variables. Hence they cannot be given such interesting interpretations as their LQG counterparts.

In particular, a very important feature will be missing if we were to analyze these observables in the same manner; the eigenvalues of these are the j_n^μ:

$$\hat{P}^\mu_{l,f}\Phi^\mu_{\vec{j}^\mu,\alpha} \rightarrow \left(\hbar \sum_{n\in(l\cap\alpha)} j_n^\mu\right)\Phi^\mu_{\vec{j}^\mu,\alpha} \tag{4.95}$$

where the $l \cap \alpha$ was defined fully in sec.(4.6)

However these are not constrained to be discrete; they take on a continuum of values. Hence the interesting discreteness of spacetime, described in the form of the discrete eigenvalues of the geometrical observables (*area* and *volume*) does not appear here. Hence some of the very interesting findings of LQG, such as that the geometric structure of the spacetime is granule cannot be confirmed within this toy model.

However this has not been the goal. We had set out to cast the loop quantization method in an effective and simple framework, where the questions regarding observables and their dynamics may be addressed. We have produced an accessible solution and observable space in the form of $\Phi^\mu_{\vec{j}^\mu,\alpha}$, and $F(\hat{P}^\mu_{l,f})$, functions of the momenta. Now we may consider the dynamics.

4.9.2 Dynamics

To address the question of dynamics and dynamical observables, let us restate a crucial construct which enters the treatment of the constraints in the loop quantization. This construct is the utilization of the Gelfand triple:

$$Cyl \subseteq \mathcal{H} \subseteq Cyl^*$$

In other words as we saw, we started of by defining the states of the system as being the functions $\Psi \in Cyl$. These were the charge-networks. The Cauchy-completion of Cyl is then $\mathcal{H}$, the Hilbert space.
However in the next step we saw that in order to make sense of the constraints we need resort to the dual space of these functions, Cyl^*. For example we realized that the *Diff. Inv.* states belong to Cyl^* (We saw in detail that the *Diff* constraint changes the graph. The solution, invariant under such changes cannot belong to Cyl.)

In the case of the *Scalar* constraint, which is more relevant to the question of dynamics, we found two crucial results:

- The changes induced by the two constraints are very similar
- Solutions to the *Diff* constraint are also solutions of the *Scalar* constraint

Put together, the solutions invariant under the Scalar constraint will again belong to Cyl^*.

Therefore while the states in Cyl change under the scalar transformations, the elements of Cyl^* are invariant under it. So we may consider the elements of Cyl to be dynamical but they are not invariant. The Cyl^* are invariant but they are static. The question is, can these two, somehow be combined to represent the observables or the physical information of the system which is dynamical while obeying some sort of invariance under the constraints.

LQG claims that the answer to this comes from the mathematical structure. Namely that, mathematically the elements of Cyl^* are to be understood *only* via their action (inner-product) on the elements of Cyl. In other words the physical content of the theory can be extracted only via this inner product. But notice that while one portion of this inner product is invariant the other part is dynamical. Hence it may be possible to define dynamical physical quantities which are in some special sense invariant via a certain interpretation of this inner product.

In effect although, we may define *physical* states, as those which are $Diff \oplus Scalar$ invariant, *it is the inner product*:

$$(C_{Diff}\Psi)(\Phi) = (\Psi)(C_{Diff}\Phi) = (\Psi)(\Phi) \tag{4.96}$$

which produces the physical content of the theory.

This is a very important distinction from Dirac's treatment of constraints; it is certainly a much weaker condition. As such (*appendix*(F)) therefore observables, $\hat{\mathcal{O}}$, need *not* commute with the constraints. Hence they need not commute with the Hamiltonian constraints. And since this is the (Dirac's) criteria which would otherwise render observables static, we see that through loop quantization it is still possible to maintain dynamics and dynamical observables.

Now let us see what these could be; let us give a more intuitive picture of these observables, and their possible dynamics.

As we discussed in *sec.*(4.9.1), the observables of the theory, are functions, $F(P_{f,l})$ of the momenta-flux. These are in fact any polynomial combination of the momenta-flux - just as the area and the volume operators of LQG are polynomials of the momenta-flux (of the connection.) The eigenvalues of the $P_{f,l}$, as we showed in *sec.*(4.6), are the charges at the nodes of the charge-network state.

Now as we saw, the scalar constraint can actually change the graph of a given charge-network state, by adding/subtracting nodes and also changing the charge of the already existing nodes.
Hence we see that evolution can be described in terms of the changes in the values of the eigenvalues of the momenta-flux operator functions.

This is picture of dynamics which the Loop quantization provides, and the arguments for it seem viable. Nonetheless we may question their validity. Let us finish by making a comment as to this.

Validity

Notice that from a physical point of view the above construction may not seem that straight-forward; there is a large degree of freedom which enters the construction there, and therefore the last assertion of the loop quantization regarding the dynamics may not be fully automatic; from a specific perspective it is an artifact of the mathematical structure. In other words it may be said that *no new physical intuition* seems to have been used to particularly answer the problem of dynamics.

Of course it may be argued that the $Cyl \leftrightarrow Cyl^*$ - which lies at the heart of the possible dynamics of the theory - is a direct artifact of the "sampling" method which is used in the loop quantization. And if we recall this sampling is a physical statement regarding the fundamental structure of spacetime; i.e. the sampling implies that spacetime is discrete and not continuous. Hence it may be argued that the possibility of dynamics in this formulation of generally covariant theories is a direct consequence of the physical granularity of the spacetime.

If so, this argument is at best quite tenuous. Because, physically it is extremely hard to see why these two seemingly disparate concepts - namely the granule structure of spacetime and dynamics in a GC - should be related. In fact there exist perhaps as much counter-intuitive physical considerations in the loop quantization to render any such argument unacceptable. Furthermore, it is a simple realization that LQG simply does neither anticipate nor provide a physical construct which should, eventually allow for the possibility of dynamics within the framework of GC theories.

From this perspective then the above mentioned $Cyl \leftrightarrow Cyl^*$ construct, becomes merely a mathematical tool; we do not know any other way to extract useful information from the theory.

Therefore we conclude that despite its elegant method of providing a full solution space with interesting outcomes as to the geometrical structure of spacetime, LQG may not be physically constructed in such a way to be able to solve the issues related to time evolution and dynamics in generally covariant theories. In other words there does not exist any new a intuition as regards this problem within the framework of LQG. Hence in fact we cannot hope that its, nonetheless elegant description for a quantum theory of spacetime can resolve such issues.

Let us conclude with a few more remarks as to the problems associated with the Hamiltonian of the loop-quantized theories, and a note on the perspective of *Partial vs. Complete* observables, which may also play a role in describing the dynamics of a loop quantized system.

4.9.3 A Final Note on the Hamiltonian

So far we have discussed a number of problems in defining the Hamiltonian constraint expression. These included the *Proper action of the constraints on the states*, ϵ-regularization, and the *ordering ambiguity*. We tried to explore these to some extent and explain their origin and their effects in the description of the theory.
There are however other questions and problems which present themselves in the formulation of the Hamiltonian of a system which is to be quantized via loop quantization.

The Constraint Closure ambiguity

This is another ambiguity associated with the Hamiltonian in LQG, due to constraint algebra. The algebra of the *Scalar* constraint with the *Diff* constraint does not close in the BRST sense[21]. (As was mentioned at the beginning of the Loop quantization.)
In one of the major attempts to address this problem one seeks *Hamiltonian-like functions* which satisfy two properties:

1) They form a closed algebra

2) They can be used to isolate the physical space of the solution space.

These requirements however do not uniquely identify such a function. As such one has a class of possible functions that can act as the *"Hamiltonian"*. Each with its own possible prescription as to *LQG-type* evolution.
This however does not happen in our toy model, because the algebra closes. Hence the original Hamiltonian of the system is the unique candidate for isolating the physical space *and* prescribing dynamics. This can be viewed as both good an bad; on the one hand it is good because we need not worry about this complication in our analysis, however on the other hand the model fails in trying to consider such problems associated with the full LQG.

The Operator Ambiguity

Another ambiguity is due to the definition of the fundamental operators that go into the definition of the Hamiltonian constraint. For example in LQG one changes the scalar constraint to a series of Poisson brackets of the *Holonomy* (around test loops) and the *Volume* operator. Of these two , the volume operator can be defined in at least two consistent methods (the external vs. the internal definition; the former assumes the operator knows about the differential properties of the nodes of the graph, whereas the latter does not). And different physical descriptions can be expected of each of these.

This problem can be related to the ambiguity that exists in our theory, in the definition for X' in terms of the holonomy, *sec.*(4.8). Of course the method used and the expression which we chose is not unique. This choice has a significant effect on the final solution space.

[21] This is because the original lagrangian which is used in LQG is a modified version of the original Einstein-Hilbert action (for various reasons). The action is in fact Palatini's action, based on the connection variable A_a^i, augmented by a canonical term which introduces the Barbero-Immirizi parameter, γ. In the expression for the commutator of two scalar constraints, one gets terms which are related to this parameter, which cause the algebra not to close.

If we recall, the expression *eqn.*(4.87), that needs to be satisfied for a state to be physical, crucially depends on the choice of this operator. Hence we see that this issue of identifying the Hamiltonian, can in fact be the source of great ambiguities in describing the physical space and the definition of dynamics.

The Representation Ambiguity

Yet another source of the ambiguity is the jth representation. In principle there is an infinity of this representation (continuous in our model, and discrete in LQG) for the constraints. The question is, which is to be used, if we must choose. A criteria may exist in the discrete LQG-type theories, where we can for example choose the $spin = \frac{1}{2}$ over the other representations. However such a criteria may not exist in continuous theories like ours. Another solution may be to average over all representations, but this cannot be hoped to be convergent.
However a more reasonable solution may be to assume all representations, just as we assume all representations for all graphs. In this sense we relate the extension of the solution space via the *charges* to the extension that happens in the definition of the corresponding operators that act on these states. However there is certainly an ambiguity related as to whether this is a reasonable-enough criteria.

Let us now summarize the results of the Loop quantization of the String, within the content of the new concepts of observables - namely *Partial* vs. *Complete* observables - which are commonly used in some interpretations of LQG.

4.9.4 *Partial* vs. *Complete* observables

There is also another discussion that enters the question of observables in LQG. This is the distinction between so-called *Partial* and *Complete* observables. For a full discussion of the ideas here see [28] and [32].
In short, Partial observables can be measured, however their value (probabilistic or otherwise) cannot be predicted by the theory.
Partial observables are not gauge invariant. Hence in the Dirac sense they are not an observable. Nonetheless they can be measured. As such, for example they need not commute with the Hamiltonian in a GC. Hence their values can change (although these cannot be predicted without being measured), and are therefore dynamical.

The rational behind the consideration of Partial observables is that in a theory of gravity, due to the principle of relativity one cannot hope to have an independent time variable. Hence "$\cdots$ notions of time evolution, instantaneous states and observables at a fixed time" are not relevant, [28]. It is posited that only relations between quantities exist, and these can change though none can be predicted by the theory until they are measured.

Here again the idea that LQG attempts to answer the question of dynamics in a GC by motivating new notions which can be considered as physically useful quantities - not just the observables - is highlighted; Partial vs. Complete observables are akin to the concepts of $Cyl \leftrightarrow Cyl^*$, where *new* physical quantities are suggested as *useful physical quantities*. Via such redefinitions, new considerations are developed which then may allow for certain notions of dynamics in GCs.

We faced the same situation in the String quantization case, *chapters*(2) and (3). There, physically useful quantities, were defined by a *very similar inner product* as that in LQG. As a result these quantities were again required to satisfy much weaker constraints. These new requirements then - as we discussed in *appendix*(F) - *does* allow for a consistent description of the theory with possible dynamical physically useful quantities.

That these two theories should agree on this conclusion is extremely striking. For one, this conclusion is reached based on completely different physical and mathematical constructs. Hence we would not, trivially anticipate this agreement on the basis of the construction.

However notice that the physical requirement that leads to this result is the same; we want to devise dynamical quantities. The full set of constraints, and their application ala Dirac, are simply too strong to allow for this. So it is posited that the conditions should be relaxed. Of course many a different ways may exist to exercise such a relaxation. Hence we may ask why is it that the inner product should be the preferred solution.

Recall that as was discussed in the very beginning of this report, that the question of observability and dynamics in a GC is largely an artifact of quantization. What the above theories are suggesting is that, the inner product is the quantity that all other quantum quantities are based on. Hence it is natural that the question of what is a useful quantum physical quantity should be related to the inner product.

Of course there is a great deal of hand-waving and interpretation which enters into this suggestion.

However notice that between the two prescriptions - the LQG and the String - it *may* be argued that the appearance of the notion of the dual spaces as a solution to the constraints, and therefor the need to consider both the Cyl and Cyl^*, "inescapably" motivates the importance of the inner product, whereas in the String case, the inner product's application is quite ad-hoc; it has no other motivation than *" if we apply the full constraints the theory is static."*

Of course this is a matter of physical principle. Operationally however the end results are strikingly similar; while the solution spaces are described based on completely different notions and constructs, they convey a very similar picture. This picture is that the inner product, and not simply the states, should be considered for defining observability and dynamics.

This concludes our discussion of the Loop quantization. Let us conclude this investigation on the observables and the dynamics of generally covariant theories by summarizing our findings in the final chapter of this report.

Chapter 5

Summary

Throughout this report we have analyzed the question of the constraints and the resulting effect on the observables, in Generally covariant (GC) systems; systems whose Hamiltonian is a complete constraint.

The question of what this means, as a physical principle, and as a mathematical restriction on physical theories has posed many a challenges to theories which poses general covariance. These include what a physical observable may be and what could be regarded as dynamics for it.

In many of these theories - through different methods - it is possible to assume a limited version of, or an approximation to the original system and try to analyze that system within that framework.

However in a fundamental theory, such as Quantum Gravity (QG), this sort of approximation is neither appropriate physically, nor does it seem feasible mathematically. Therefore the fate of the effect of the constraints on the dynamics of the observables of the system becomes difficult to describe.

As such, a theory which wishes to claim a successful quantization of gravity is expected to provide a satisfactory resolution for this problem. Hence, in any such theory the answer to questions of observables and dynamics become a key feature for checking the validity of the construction and the method of that theory. Therefore it is important to at least clarify the exact definitions for observables and dynamics, that are made in such theories of Quantum Gravity.

In this report - by taking different perspectives on the specific case of a String - we attempted to cast some of these questions in a more manageable, yet relevant framework. In particular we fully analyzed the methods of the Loop Quantization of Loop Quantum Gravity (as a Ddim. field on *2D* spacetime) and also the Virasoro methods of the String Theory (as a *2D* Discrete field on a flat background.)

In each of these we highlighted the above mentioned questions and tried to see how the particular method of the theory, attempts to develop an answer to these questions.

The striking feature which was found throughout our analysis, was that fundamentally in each of these theories, the concepts of observables and dynamics *are defined in a new way.*

In other words, despite the new methods and mathematical framework of either one of these theories, ultimately the same kinds of restrictions will need to be applied on observables. The remedy that these new theories suggest is to relax the constraint conditions by redefining what the *physically useful quantities* of the theory are. In this way then it is possible to allow for a larger solution space and the possibility of dynamics *for these quantities.*

The physical intuition behind most of these new definitions come from quantum mechanics and the mathematical construct which is exploited to exercise this intuition, is the ambiguities that exist in interpreting *a quantum theory's physical content.* This is why throughout our analysis we repeatedly came back to visit the inner product of the quantum operators. In both LQG and String Theory, it was repeatedly argued that the inner product is the element which carries the physically useful content of the theory and therefore the constraints should be relaxed to be obeyed within the structure of the inner product.

Of course - although it may be argued that no other method is known to rescue the theory from becoming trivial - there exists a large degree of *ad-hocness* to any physical theory where *only* interpretation is used as the guiding principle. This is the concern which greatly discredits the results of both the Loop quantization and the String quantization.

However comparatively it may be said that there exists less of this ad-hocness in the Loop quantization as compared to the String quantization; the mathematical structure of LQG is constructed such that it (more or less) suggests the group-averaging techniques as the only alternative.

Hence it may be concluded that the two major candidates of a quantum theory of gravity, namely LQG and String Theory, attempt to resolve the question of observables and dynamics in quantum gravity (and therefore other generally covariant theories) by redefining what should be considered as a physically useful quantity. Nonetheless there exists inherent ambiguities in what this new definition should be.

It may be argued that it will ultimately be future experiments which may prove one or other of these theories to be the better candidate, and as a result the new definitions within that theory to be the proper physical definitions for physically useful quantities. However it should be clear that even the concept of experimentation requires a clear definition of an observable to be able to make any such distinction.

Therefore one may say that these questions remain open whose full description, despite their significant and fundamental nature are still lacking.

The answer may perhaps lay in one of the already existing theories; however if so what is certain is that some form of new physical principle is perhaps missing - which would then explain the above-mentioned ad-hocness (or ambiguities) - which may then adequately address the questions of observability and dynamics in quantum gravity and therefore generally covariant theories in general.

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Appendix A

The Proper Treatment of $\delta_G(\Pi\dot{X}(\sigma,t))$

In this appendix we carry out the proper calculation of $\delta_G(\Pi\dot{X}(\sigma,t))$ - the gauge transformation of $\Pi\dot{X}(\sigma,t)$ - which we encountered in *chap.*(2), for a general gauge:

$$\begin{aligned} G &\equiv G^0[\xi(\sigma,\sigma^{'},t)] + G^1[\eta(\sigma,\sigma^{'},t)] && \text{(A.1)}\\ &= \int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\frac{1}{2}\Big[\xi(\sigma,\sigma^{'},t)\Big(\Pi(\sigma,t)\Pi(\sigma^{'},t) + X'(\sigma,t)X'(\sigma^{'},t)\Big)\\ &\qquad + \eta(\sigma,\sigma',t)\Big(\Pi(\sigma,t)X'(\sigma^{'},t) + \Pi(\sigma^{'},t)X'(\sigma,t)\Big)\Big]. && \text{(A.2)}\end{aligned}$$

Notice that we are considering the *nonlocal* gauges. We use these because the results will then be more general. However the analysis which follows, and the results we should find, apply also to the case of the local gauges.

In particular we wish to see if - similar to using different σ parameters for the variables of the gauge function and the $\Pi\dot{X}(\sigma,t)$ - we need to use different temporal, t parameters as well, to calculate the above mentioned variation.
To do so, let us start with this as a posit, and show that it is well-motivated; proper results will be achieved only if we use two temporal parameters. Therefore let us consider:

$$\begin{aligned}\delta_G(\Pi\dot{X}(\sigma,t)) &= \int \mathrm{d}t\mathrm{d}\sigma\Big\{\Pi\dot{X}(\sigma,t), G[\xi,\eta(\sigma^{'},\sigma^{''},t^{'})]\Big\} && \text{(A.3)}\\ &= \int \mathrm{d}t\mathrm{d}\sigma\Big\{\Pi\dot{X}(\sigma,t), G^0[\xi] + G^1[\eta](\sigma^{'},\sigma^{''},t^{'})\Big\} && \text{(A.4)}\end{aligned}$$

where the G^α are gauges we defined in *chap.*(2). We use the convention that, a (σ,t) applies to all terms prior to it.

Now we will carry out a careful calculation of the above transformation, with a focus on the role of the temporal parameters in the calculations.

Let us first calculate the first term of the general gauge transformation *eqn.*(A.4), i.e. the G^0 term. So we have:

$$\delta^0_\xi(\Pi\dot{X}) \equiv \int \mathrm{d}\sigma \mathrm{d}t \left\{\Pi\dot{X}(\sigma,t), G^0[\xi(\sigma^{'},\sigma^{''},t^{'})]\right\} \tag{A.5}$$

$$\begin{aligned} &= \int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''} \left\{\Pi\dot{X}(\sigma,t), \frac{1}{2}\xi(\sigma^{'},\sigma^{''},t^{'})\big(\Pi(\sigma^{'},t^{'})\Pi(\sigma^{''},t^{'}) + X'(\sigma^{'},t^{'})X'(\sigma^{''},t^{'})\big)\right\} \\ &= \frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},t^{'}) \Big[\Pi(\sigma,t)\left\{\dot{X}(\sigma,t), \Pi(\sigma^{'},t^{'})\Pi(\sigma^{''},t^{'})\right\} \\ &\qquad + \dot{X}(\sigma,t)\left\{\Pi(\sigma,t), X'(\sigma^{'},t^{'})X'(\sigma^{''},t^{'})\right\}\Big] \end{aligned} \tag{A.6}$$

There are therefore two PB's to consider. We will analyze these separately.

First however notice that the functions which appear in these calculations and necessitate the use of the smearing - and therefore the integrations - are the partial derivatives of the delta functions, $\partial_x\delta(x-x^{'})$; to deal with these terms, we need to perform an integration-by-parts, on the variable which is being differentiated.

Now the ∂_x comes from the terms in a PB, such as $\{\partial_x f, g\}$. The crucial observation is that, *to analyze this PB, the ∂_x should be able to be taken out of the PB. The only way that is possible is if the $x-$dependence of the two terms of PB are different; <u>they depend on different x parameters</u>.* For example:

$$f = f(x) \quad g = g(x') \quad \Rightarrow \quad \{\partial_x f(x), g(x')\} = \partial_x \{f(x), g(x')\}\,. \tag{A.7}$$

In the above variation therefore we get a $\partial_{\sigma'}\delta(\sigma-\sigma^{'})$ term from the second Poisson Bracket, which is dealt with by the $\int \mathrm{d}\sigma^{'}$. However for the first Poisson Bracket we get a $\partial_t\delta(t-t^{'})$ term, which then requires a $\int \mathrm{d}t$ to integrate by part. Note that *had we considered the same time parameter $t^{'}\to t$, we would not be able to pull the ∂_t outside the Poisson Bracket.*

This is the result that motivates the posit; *we need to use different time parameters.* Whether, however this result is a mathematical artifact or a physical requirement is not clear. Nonetheless from this result it is evident that *a proper treatment is only possible if two different time parameters are considered.*

Now let us return to the calculation of the gauge transformation.

Notice that in the second PB we immediately obtain a $\delta(t-t^{'})$ term, which reduces the $\int \mathrm{d}t^{'}$ integral. Therefore the posit is not needed for this term, however the results of the calculation of this term serves to show that calculation of the PBs is still consistent if we consider two temporal parameters:

$$
\begin{aligned}
&\frac{1}{2}\int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t^{'})\dot{X}(\sigma,t)\Big\{\Pi(\sigma,t),X'(\sigma^{'},t^{'})X'(\sigma^{''},t^{'})\Big\}\\
&\to -\frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t)\dot{X}(\sigma,t)\Big[\partial_{\sigma'}\delta(\sigma-\sigma^{'})X'(\sigma^{''},t)+\partial_{\sigma''}\delta(\sigma-\sigma^{''})X'(\sigma^{'},t)\Big]\\
&\to \frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t)\dot{X}(\sigma,t)\Big[\partial_{\sigma}\delta(\sigma-\sigma^{'})X'(\sigma^{''},t)+\partial_{\sigma}\delta(\sigma-\sigma^{''})X'(\sigma^{'},t)\Big]\\
&\to -\int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\xi(\sigma,\sigma^{'},t)\dot{X}^{'}(\sigma,t)X'(\sigma^{'},t)\\
&= -\frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\xi(\sigma,\sigma^{'},t)\partial_t\Big[X'(\sigma,t)X'(\sigma^{'},t)\Big] \qquad \text{(A.8)}
\end{aligned}
$$

This shows that the posit does not upset the otherwise well-understood results of the PB.

However, as we mentioned, in the first term in order to pull the ∂_t out of the PB, we need different temporal parameters:

$$
\begin{aligned}
&\frac{1}{2}\int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t^{'})\Pi(\sigma,t)\{\dot{X}(\sigma,t),\Pi(\sigma^{'},t^{'})\Pi(\sigma^{''},t^{'})\}\\
&= \frac{1}{2}\int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t^{'})\Pi(\sigma,t)\Big[\partial_t\delta(t-t^{'})\Big(\Pi(\sigma^{'},t^{'})\delta(\sigma-\sigma^{''})+\Pi(\sigma^{''},t^{'})\delta(\sigma-\sigma^{'})\Big)\Big]\\
&\to \frac{1}{2}\int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t^{'})\partial_t\delta(t-t^{'})\Big[\Pi(\sigma^{''},t)\Pi(\sigma^{'},t^{'})+\Pi(\sigma^{''},t^{'})\Pi(\sigma^{'},t)\Big]\\
&\to -\frac{1}{2}\int \mathrm{d}t\mathrm{d}t^{'}\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\delta(t-t^{'})\xi(\sigma^{'},\sigma^{''},t^{'})\Big[\Pi(\sigma^{'},t^{'})\partial_t\Pi(\sigma^{''},t)+\Pi(\sigma^{''},t^{'})\partial_t\Pi(\sigma^{'},t)\Big]\\
&= -\frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t)\Big[\Pi(\sigma^{'},t)\dot{\Pi}(\sigma^{''},t)+\Pi(\sigma^{''},t)\dot{\Pi}(\sigma^{''},t)\Big]\\
&= -\frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma^{'}\mathrm{d}\sigma^{''}\xi(\sigma^{'},\sigma^{''},t)\partial_t\Big(\Pi(\sigma^{'},t)\Pi(\sigma^{''},t)\Big) \qquad \text{(A.9)}
\end{aligned}
$$

In other words without the posit, this PB could not be properly treated. Hence this calculation confirms the necessity of the posit.

Putting the two PB's together, we then get:

$$
\begin{aligned}
\delta^0_\xi(\Pi\dot{X}) &= -\frac{1}{2}\int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\xi(\sigma,\sigma^{'},t)\partial_t\Big[\Pi(\sigma,t)\Pi(\sigma^{'},t)+X'(\sigma,t)X'(\sigma^{'},t)\Big]\\
&\to \int \mathrm{d}t\mathrm{d}\sigma\mathrm{d}\sigma^{'}\dot{\xi}(\sigma,\sigma^{'},t)C^0(\sigma,\sigma^{'},t)\\
&= \int \mathrm{d}tG^0[\dot{\xi}(\sigma,\sigma^{'},t)] \qquad \text{(A.10)}
\end{aligned}
$$

which is the expected result. Therefore we conclude that the posit of using different time parameters (in addition to different spacial parameters), is both *consistent* with previous results (where the posit is not necessary), and *necessary* to find the correct results (for all other PBs).

Now to complete the calculation of the full gauge transformation of G, we need to calculate the second term of this gauge transformation, G^1 as well. A similar calculation to the one above -under the same posit for the temporal parameter - for $G^1[\eta(\sigma, t)]$ gives:

$$\delta_\eta^1(\Pi\dot{X}) = \int \mathrm{d}t G^1[\dot{\eta}] \tag{A.11}$$

Which again confirms our results.

Therefore the general gauge transformation of the $\Pi\dot{X}$ under a general gauge $G[\xi, \eta]$, assuming temporal smearing as well as spacial smearing gives:

$$\delta_G(\Pi\dot{X}) = \delta_\xi^0(\Pi\dot{X}) + \delta_\eta^1(\Pi\dot{X}) = \int \mathrm{d}t \Big(G^0[\dot{\xi}] + G^1[\dot{\eta}] \Big) \tag{A.12}$$

Appendix B

On Poisson Bracket of Distributions

In this appendix we consider the Poisson bracket (PB) of Distributions. The results which we will try to motivate are the following:

$$\Big\{X^{\mu}(\sigma,t),\Pi^{\nu}(\sigma^{'},t^{'})\Big\}=\delta^{\mu\nu}\delta(\sigma-\sigma^{'})\delta(t-t^{'}) \tag{B.1}$$

$$\Big\{\Pi^{\mu}(\sigma,t),X^{\nu}(\sigma^{'},t^{'})\Big\}=-\delta^{\mu\nu}\delta(\sigma^{'}-\sigma)\delta(t^{'}-t) \tag{B.2}$$

These equations derive from the concept of "induced change" via the conjugate variable, in the following sense.
Π induce an (incremental, infinitesimal) change on the X, (since it is X's momentum, suggesting motion of the variable), and vice versa; i.e. X also induces a change on Π.

This is the origin of the Poisson Bracket equation for functions:

$$\{x^{i},p_{j}\}=\sum_{k}\frac{\partial x^{i}}{\partial x^{k}}\frac{\partial p_{j}}{\partial p_{k}}-\frac{\partial x^{i}}{\partial p_{k}}\frac{\partial p_{j}}{\partial x^{k}}=\sum_{k}\delta^{i}{}_{k}\delta_{j}{}^{k}=\delta^{i}{}_{j} \tag{B.3}$$

The same idea extends to Distribution functions. A given smeared functions:

$$\widetilde{O}(t)\equiv\int \mathrm{d}s f(s)O(s,t)\qquad \textit{where } f(s) \textit{ is the smearing function} \tag{B.4}$$

must obey the same basic principle which defines the Poisson bracket. However $O(s,t)$ is a distribution.The problem that arises is that the PB is defined for functions. As such technically a PB can only be properly defined for the *smeared versions of the distribution*, or *functional* of $O(s,t)$ - i.e. for $\widetilde{O}(t)$.

This generalization leads to the following complication. A functional of the distribution will always involve an arbitrary function. As a result the calculation of the PB for a distribution can involve some arbitrariness. In fact the results of the PB of two functionals are always

only valid up to boundary terms of the smearing functions - which are due to the freedom in the integration by parts .
For example consider the following conjugate distributions: $X(\sigma, t)$ and $\Pi(\sigma, t)$, with the smeared versions: $\widetilde{X}_f(t)$ and $\widetilde{P}_g(t)$, for smearing functions $f(\sigma)$ and $g(\sigma)$. Now if wish to calculate the following PB:

$$\left\{\widetilde{X}'_f(t), \widetilde{P}_g(t)\right\} = \int \mathrm{d}\sigma\mathrm{d}\sigma' f(\sigma)g(\sigma')\left\{\partial_\sigma X(\sigma, t), \Pi(\sigma', t)\right\} \tag{B.5}$$

$$= \int \mathrm{d}\sigma\mathrm{d}\sigma' f(\sigma)g(\sigma')\partial_\sigma\delta(\sigma - \sigma')$$

$$= -\int \mathrm{d}\sigma\mathrm{d}\sigma'\partial_\sigma f(\sigma)g(\sigma')\delta(\sigma - \sigma')$$

$$= -\int \mathrm{d}\sigma(f'g)(\sigma) \tag{B.6}$$

$$= \int \mathrm{d}\sigma(fg')(\sigma) \tag{B.7}$$

$$= \int \mathrm{d}\sigma\frac{1}{2}\left(fg' - f'g\right)(\sigma) \tag{B.8}$$

The last three integrals, *eqns.*(B.6, B.7, B.8), represent the arbitrariness in the calculation of PB of functionals, as we mentioned. The resolution to this problem comes from physics, and physical systems.

Notice that in any physical situation, a PB for a distribution will involve several terms. For example when we take the PB of the scalar field, X with the Hamiltonian H, there are usually several terms in the Hamiltonian which will contribute to the PB. To obtain the final result of any such PB we need to "put all the resulting terms together".

The method that we put these terms together defines the proper method of handling the PB of functionals.

Now to put the different terms of any such PB together, we notice that it is the field variables which are the physical quantities of the system, and not the smearing functions. Therefore *we put the resulting terms of the PB together in such a way that they all have the same multiplicative smearing functions.*

This is because the resulting expression, minus the smearing functions, is the result of the PB of the distributions. Hence to remove the smearing functions all the terms of the PB must have the same form of smearing multiplicative factor.

The result which is then obtained in this way - for the PB - can be shown to be unique.

For example, in the example above, if we were calculating the following PB instead:

$$\left\{ \widetilde{X}'_f(t), \widetilde{P^2}_g(t) \right\} = \int \mathrm{d}\sigma \mathrm{d}\sigma' f(\sigma) g(\sigma') \left\{ \partial_\sigma X(\sigma, t), \Pi^2(\sigma', t) \right\} \tag{B.9}$$

$$= -2 \int \mathrm{d}\sigma (f' g)(\sigma) \Pi(\sigma, t) \tag{B.10}$$

Notice that if we perform an integration by part on the ∂_σ of f', we would get two terms in the integral with different smearing multiplicative functions:

$$= 2 \int \mathrm{d}\sigma \left[(f g')(\sigma) \Pi(\sigma, t) + (f g)(\sigma) \Pi'(\sigma, t) \right] \tag{B.11}$$

As a result, we can say that this solution is not useful, and we would use the original solution instead.

This is then the proper method to calculate the PB of the distribution functionals.

Following this method, we can see that the equations given at the beginning of this appendix, do define the proper distributional Poisson brackets for the fields X and Π of the string case.

Appendix C

Methods of Quantization of Constraint Systems

In this appendix we will describe the OCQ of String theory completely. We will then briefly review the problems that can arise due to the gauge fixings which enter this formulation. These motivate the BRST formulation of String theory which is a non-gauge-fixing approach.
At the end we will briefly represent the result of the BRST formulation regarding the physical space of the string.

C.1 String's OCQ

Let us first summarize the method of quantization due to the Old Covariant Quantization (OCQ). This method has two very important features:

1. This method has a Lagrangian-formulation origin. Therefore it places emphasis on the importance of the expectation value in quantum theories.
2. As we should see this is in fact the way this method is related to the BRST method, which is also a Lagrangian formulation.

This method therefore starts from the Path integral formulation of quantum theory. Then it argues that by varying the Lagrangian with respect to Lagrange multipliers, N^k, we have:

$$\delta_{N^k}\langle i|f\rangle = \int \mathrm{d}X(\cdots\langle\hat{C}_k\rangle\delta N^k) = 0\,, \tag{C.1}$$

Notice that the expectation value is naturally highlighted in this way, and it is posited that *the constraints are satisfied if this variation vanishes.* Which means:

$$\langle\hat{C}_k\rangle = 0 \tag{C.2}$$

for all constraints $\hat{C}_k$ and physical states.

Now recall that starting from the energy-momentum tensor, we have:

$$T_{ab} \propto \frac{\delta S}{\delta g_{ab}} = 0 \sim \{C^0, C^1\} \tag{C.3}$$

In other words, the constraints of the string system are the restriction $T_{ab} \approx 0$. As we mentioned in the text the constraints are associated with the $Diff \times Weyl$ invariance of the theory. We also carried out the calculation where the above expressions were transformed into the Virasoro generators, L_n:

$$T_{ab} \approx 0 \quad \Rightarrow \quad L_n \approx 0 \tag{C.4}$$

Putting these together therefore the method of OCQ can be summarized as follows:

$$\begin{aligned} & (L_n + A\delta_{n,0})|\Psi\rangle = 0 \quad for \quad n \geq 0 \\ \Rightarrow \quad & using: \quad L_n^\dagger = L_{-n} \quad for \quad n < 0 \quad \rightarrow \quad \langle\Psi|L_n|\Psi'\rangle = \langle L_{-n}\Psi|\Psi'\rangle = 0 \end{aligned}$$

which satisfies $\langle\hat{C}_k\rangle = 0$.
In addition it is required that:

$$\begin{aligned} & if \quad |\chi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle \\ & \Rightarrow \quad \langle\chi_{phys}|\chi\rangle = \sum_{n=1}^{\infty}\langle L_n\chi_{phys}|\chi\rangle = 0 \quad is\, spurious \\ and\, if \quad & |\chi\rangle \in \{|\Psi_{phys}\rangle\} \quad \Rightarrow \quad \langle\chi|\chi\rangle = 0 \quad is\, null \end{aligned}$$

However:

$$\begin{aligned} & if \quad |\Psi\rangle \in Phys \quad \& \quad |\chi\rangle \in null\,(= Phys \oplus Spurious) \\ & \Rightarrow \quad |\tilde{\Psi}\rangle = |\Psi\rangle + |\chi\rangle \quad is\, Phys \\ & \& \quad \langle\tilde{\Psi}|\tilde{\Psi}\rangle = \langle\Psi|\Psi\rangle \\ & \Rightarrow \quad |\tilde{\Psi}\rangle\, \&\, |\Psi\rangle \quad are\, physically\, indistinguishable \end{aligned}$$

$\therefore$

$$\mathcal{H}_{OCQ} \equiv \frac{\mathcal{H}_{Phys}}{\mathcal{H}_{null}} \quad (Set\, of\, equivalence\, classes) \tag{C.5}$$

This is the process of identifying the physical sector of the Hilbert space, $\mathcal{H}_{OCQ}$, via the method of OCQ.

Notice that in this report, in order to get to the solution space we also applied a gauge fixing (which we tried to justify to some extent.) However a number of problems arise whenever we gauge-fix a theory. These problems, and also the requirement that the OCQ be generalized for more general theories, motivates the formulation of the BRST quantization. Let us therefore briefly consider the problems of gauge fixing a theory, and then describe the BRST method.

C.2 Gauge Fixing of Theories

Let us first recall where the gauge fixing comes from.

- Recall that classically, in a constraint system, we obtain two sets of equations:

  1. E.O.M — Denote by: $E = 0$
  2. Constraint equations — Denote by: $C = 0$

  Classically therefore, using (2.) we could get rid of a number of the variables in (1.)

- However in quantization the above equations become operator equations:

$$\begin{cases} E & \rightarrow & \hat{E} \\ C & \rightarrow & \hat{C} \end{cases} \tag{C.6}$$

  and the corresponding variables become quantum states:

$$\{q\} \quad \rightarrow \quad |\Psi_q\rangle \,. \tag{C.7}$$

  Therefore one can view the quantization process as:

$$\textbf{quantization} : \{E, C \oplus^{cl} q_i\} \quad \rightarrow \quad \{\hat{E}, \hat{C} \oplus^{qm} |\Psi_i\rangle\} \tag{C.8}$$

  with one very important difference:

$$\oplus^{cl} \quad \not\leftrightarrow \quad \oplus^{qm} \tag{C.9}$$

  i.e. the relation between the classical equations and the solutions to these and the relation between the quantum states and the corresponding operators is not on the same footing.

As such it is not at all clear if the removal of the classically "redundant" variables *before* or *after* quantization will lead to equivalent formulations. This "choice" leads to two major methods of quantization: the gauge-fixing vs. non-gauge-fixing.
We extensively discussed the latter method in this report. Now let us briefly discuss the gauge-fixed approaches:

Gauge-Fixing (OCQ) $\rightarrow$ BRST

As mentioned in this method of quantization, the "extra" degrees of freedom are removed *prior to quantization*, i.e. we reduce the variable space. Two major problems arise as a consequence to this:

1. We need to show that the variables do reduce properly (i.e. gauge freedoms correspond properly to d.o.f.-reduction.) This is because gauges are local transformations, with changing global behavior. However the reduction of the configuration/phase space is a global restriction . Hence the equivalence of these two needs be considered carefully, [11][1].

[1] We see this explicitly in String Theory where the $Diff \times Weyl$ (3 gauges) gauge of the string does not correspond exactly to the complete removal of the 3 terms (variables) of the metric, g_{ab}.

2. Due to the reduction of the variable space, non-linearities arise. These are the source of quantization ambiguities. In general, however these ambiguities do not have a unique solution and the resulting solution space may be degenerate.

These problems are some of the main motivations behind the BRST formulation.

However before we proceed to see how this is done, let us make a note on the relation between this method of quantization and the Dirac's non-gauge-fixing (which was discussed in the report).

As we discussed, the Lagrangian non-gauge-fixing method of OCQ is related to the Hamiltonian method of Dirac's procedure. However we saw that these prescribe different procedures for the constraints, *based on different physical requirements* - the latter applies the "full" set of constraints whereas the former applies only "half" of these. As a result the physical pictures which they describe are very different.
Therefore we expect that the BRST - as a generalization of the OCQ - will also be different from the Dirac quantization.

C.3 BRST Quantization (of the Path Integral)

The main ideas of this method of quantization come from the generalization of gauge-fixing and the OCQ methods. In BRST quantization the phase-space is extended - instead of getting reduced (as it is done in gauge-fixing methods) - via the the so-called *ghost variables*[2].

Briefly we can describe this method as follows:

This method is a path integral formulation of quantization. The above mentioned extension is done in such a way that the gauges invariances are removed - by "tagging" the gauge degrees of freedom using the ghost variables - *before quantization*, and then constraints are applied *within* the path integral - as variations on the ghost variables - to ensure the path integration is gauge-invariant.

Let us now consider the details of this formulation. Mathematically, the following steps are taken:

I. Augmenting the path integral action using Faddeev-Popov determinant Δ_{FP} and as a result including the gauge conditions, $F_A = 0$ in the path integral.

II. It is checked that the path integral is invariant with respect to the variation of the gauge-condition ($F_A = 0$). This leads to the the BRST transformation and its conserved quantity, the BRST charge, Q_B.

Let us explain these one by one.

[2] And therefore BRST avoids the quantization ambiguities which are inherent to gauge-fixed theories.

I. **Faddeev-Popov Determinant:**

The main motivation behind this Determinant is to determine the factor: $V_{[Diff \times Weyl]}$ in the path integral:

$$\int \frac{\mathrm{d}X \mathrm{d}g}{V_{[Diff \times Weyl]}} e^{-S_\alpha} \tag{C.10}$$

This factor represents the *over-counting* of states related by a gauge degree of freedom. To define this determinant let us consider the following example.

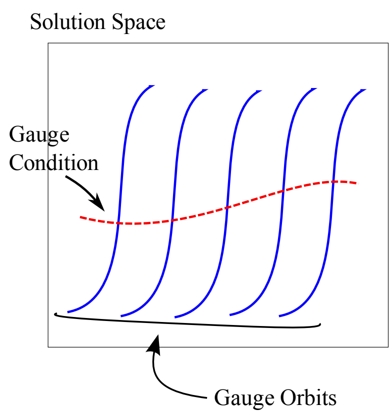

Figure C.1: Gauge Orbits and Gauge Fixing

States which are related by a choice of gauge form *gauge orbits*, *fig.*(C.1). However, recall that the states in a given orbit describe the same system and are therefore redundant. To remove the extra states we use gauge conditions:

$$\begin{aligned} F^A[\phi] = 0 \quad & A \sim No.\, of\, conditions\, (gauges) \\ & \phi \sim the\, field\, in\, the\, system\, being\, considered. \end{aligned} \tag{C.11}$$

Now notice that the gauge conditions can be realized from the following integral:

$$\int \mathrm{d}B_A e^{iB_A F^A} \quad \xrightarrow{\delta B_A} \quad F^A = 0\,, \tag{C.12}$$

Hence by adding this integral to the path integral, we can "fix" a specific slice of the gauge orbits:

$$\int \frac{\mathrm{d}\phi}{V_{[D\times W]}} e^{-S_\phi} \quad \rightarrow \quad \int \mathrm{d}\phi \mathrm{d}B_A e^{-S_\phi - S_g} \tag{C.13}$$

The B_A are called "ghost variables", and:

$$S_g = iB_A F^A \tag{C.14}$$

the "ghost action". And the integral which acts as a "measure", or a Jacobian determinant is called the *Faddeev-Popov determinant*:

$$1 = \Delta_{FP} \int \mathrm{d}B^A F_A \qquad (notice\, the\, inversion\, of\, the\, indices.) \tag{C.15}$$

Let us see this for the example of the string:
For Example:
For the String:

$$Z[\hat{g}] = \int \frac{\mathrm{d}X\mathrm{d}g}{V_{[D\times W]}} e^{-S_X[g,X]} \tag{C.16}$$

The $Diff \times Weyl$ transformations of the metric are:

$$\delta g_{ab} = 2\delta w \delta g_{ab} - (\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a) \tag{C.17}$$
$$\equiv (2\delta w - \nabla_c \delta\sigma^c) g_{ab} - (P^1 \nabla\delta\sigma)_{ab} \tag{C.18}$$

The $F.P.$ is defined as:

$$1 = \Delta_{FP}(\hat{g}) \int \mathrm{d}\xi \delta(\hat{g} - g^{\xi}) \tag{C.19}$$

and goes into $Z[\hat{g}]$ as:

$$Z[\hat{g}] = \int \frac{\mathrm{d}X\mathrm{d}\xi\mathrm{d}g}{V_{[D\times W]}} \Delta_{FP}(\hat{g})\delta(\hat{g} - g^{\xi}) e^{-S_x[g,X]} \tag{C.20}$$
$$= \int \frac{\mathrm{d}X^{\xi}\mathrm{d}\xi}{V_{[D\times W]}} \Delta_{FP}(g^{\xi}) e^{-S_X[g^{\xi},X^{\xi}]} \tag{C.21}$$

However:

$$\Delta_{FP}^{-1}(g^{\xi}) = \int \mathrm{d}\xi' \delta(g^{\xi} - g^{\xi'}) = \int \mathrm{d}\xi' \delta(\hat{g} - g^{\xi^{-1}\xi'}) = \int \mathrm{d}\xi'' \delta(\hat{g} - g^{\xi''}) = \Delta_{FP}^{-1}(\hat{g}) \tag{C.22}$$

and by construction X, g and therefore S are invariant under ξ:

$$\Rightarrow Z = \int \frac{\mathrm{d}X\mathrm{d}\xi}{V_{[D\times W]}} \Delta_{FP}(\hat{g}) e^{-S[\hat{g},X]} \tag{C.23}$$

$$Now\, using: \int \frac{\mathrm{d}\xi}{V_{[D\times W]}} \equiv 1 \tag{C.24}$$

where in this step we are setting *Local G Trans.* equal to the *Local C Trans.* of the metric, we get:

$$\boxed{Z[\hat{g}] = \int \mathrm{d}X \Delta_{FP}(\hat{g}) e^{-S[\hat{g},X]}} \tag{C.25}$$

This is then the role of the F.P. determinant. Now let us consider the second step in the BRST quantization.

II. **Variation of the Gauge**

Now we wish to add another term to the gauge-fixed path integral to make sure under arbitrary variations of *the choice of the gauge* it remains invariant.
Hence we need also include a term of the form:

$$\int \mathrm{d}b_A \mathrm{d}c^\alpha e^{-b_A c^\alpha \delta_\alpha F^A} \qquad b_A \, and \, c^\alpha \; more\; ghost\; variables \tag{C.26}$$

This is the BRST requirement which is intended to make sure the gauge fixing is effective, i.e. the $\langle i|f\rangle$ is truly independent of the choice of the gauge.

This variation is called the *BRST transformation*. Now if we consider:

$$\delta\langle i|f\rangle \sim \frac{\delta}{\delta(\delta F^A)}\langle i|f\rangle \tag{C.27}$$

$$\rightarrow \langle i|\delta_B(b_A \delta F^A)|f\rangle = 0 \tag{C.28}$$

this introduces the BRST (conserved) charge, Q_B:

$$\delta_B(b_A\delta F^A) \equiv \{Q_B, b_A\delta F^A\}_+ \qquad (anticommutator;\; ghost\, variables\, are\, Grassmannian) \tag{C.29}$$

However this must be true for all δF^A (arbitrary). Therefore putting everything together we have:

$$\Rightarrow \quad \langle\Psi|\{Q_B, b_A\delta F^A\}_+|\Psi'\rangle = 0 \tag{C.30}$$

$$\Rightarrow \quad Q_B|\Psi\rangle_{Phys} = 0 \tag{C.31}$$

Side remark on Q_B:

Q_B is a conserved charge $\quad \Rightarrow \quad [Q_B, H] = 0$

In particular addition of $\delta_B(b_A\delta F^A)$ to H (as is done in BRST) should not change this condition:
i.e. let: $\quad H' = H + b_A\delta F^A$

$$\Rightarrow \quad 0 = [Q_B, \{Q_B, b_A\delta F^A\}_+] = [Q_B^2, b_A\delta F^A] \quad \forall \quad \delta F^A$$

$$\Rightarrow \quad Q_B^2 = 0$$

i.e. Q_B is nilpotent.

This is the physicality condition which then BRST prescribes, and we can of course immediately notice its resembles to OCQ.

In fact in addition to the above condition, we will again need to remove the null (Physical $\oplus$ Spurious) states - (since null states do satisfy $Q_B|\,null\,\rangle = 0$ - by imposing the following conditions):

$$if \quad |\Psi\rangle = Q_B|\chi\rangle \qquad for\, some \quad |\chi\rangle \tag{C.32}$$

$$\Rightarrow \quad |\tilde{\Psi}\rangle = |\Psi'\rangle + |\Psi\rangle \approx |\Psi'\rangle \tag{C.33}$$

because the norm of $|\Psi\rangle$ is $0 \Rightarrow |\tilde{\Psi}\rangle\&|\Psi'\rangle$ are physically indistinguishable.
We need therefore consider the *Cohomology* of Q_B:

$$\boxed{\mathcal{H}_{BRST} \equiv \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}}} \qquad \textit{which can be compared to eqn.(C.5) of OCQ.} \tag{C.34}$$

where:

$$\mathcal{H}_{closed} \equiv \Big\{|\Psi\rangle\Big|Q_B|\Psi\rangle = 0\Big\} \tag{C.35}$$

$$\mathcal{H}_{exact} \equiv \Big\{\Psi\rangle\Big||\Psi\rangle = Q_B|\chi\rangle \quad for\, some \quad |\chi\rangle\Big\} \tag{C.36}$$

This summarizes the BRST quantization.

Remarks:

- As compared to the gauge-fixing, BRST could be considered as a gauge-fixing quantization *with a "check" for gauge invariance*. However since, in effect we impose the gauge-fixing without reducing the variable space (indeed we expand it via the ghost variables), this procedure side-steps many of the issues associated with the ambiguities which can otherwise occur.

- As compared to OCQ, the Q_B of BRST exacts the action of $\hat{C}^-$ in the decomposition of the constraints (Gauge) condition: $\hat{C} = \hat{C}^+ + \hat{C}^-$. i.e. just as Q_B is the conserved charge (wrst BRST, i.e. changes in the gauge-choice condition), $\hat{C}^-$ is the *part of the conserved charge which is affected by the choice of the gauge.*

Let us conclude by *stating* the *Goddard-Thorn Theorem*, which is the culmination of applying the BRST quantization to the string theory, see [11] for details.

C.4 Goddard-Thorn Theorem

Suppose that V is a vector space with a non-singular bilinear form $(\cdot,\cdot)$.
Further suppose that

a. V is acted on by the Virasoro algebra in such a way that the adjoint of the operator L_m is L_{-m},

b. The central element of the Virasoro algebra acts as multiplication by 24,

c. Any vector of V is the sum of eigenvectors of L_0 with non-negative integral eigenvalues and

d. All eigenspaces of L_0 are finite-dimensional.

Now:

- Let V_m be the subspace of V on which L_0 has eigenvalue m.
- Assume that V is acted on by a group G which preserves all of its structure. Now let $V_{II_{1,1}}$ be the vertex algebra of the double cover $\hat{I}I_{1,1}$ of the two-dimensional even unimodular Lorentzian lattice $II_{1,1}$ (so that $V_{II_{1,1}}$ is $II_{1,1} - graded$, has a bilinear form $(\cdot,\cdot)$ and is acted on by the Virasoro algebra).
- Furthermore, let P^1 be the subspace of the vertex algebra $V \otimes V_{II_{1,1}}$ of vectors v with $L_0(v) = v\,, L_m(v) = 0\, for\, m > 0$ and let P^1_r be the subspace of P^1 of degree $r \in II_{1,1}$. (All these spaces inherit an action of G from the action of G on V and the trivial action of G on $V_{II_{1,1}}$ and $\mathbb{R}^2$).

Then, the quotient of P^1_r by the nullspace of its bilinear form is naturally isomorphic (as a G module with an invariant bilinear form) to $V^{1-(r,r)/2}\, if\, r \neq 0$, and to $V^1 \oplus \mathbb{R}^2\, if\, r = 0$.

The name "no-ghost theorem" stems from the fact that in the original statement of the theorem by Goddard and Thorn, V was part of the underlying vector space of the vertex algebra of a positive definite lattice so that the inner product on V_i was positive definite; thus, had no "ghosts" (vectors of negative norm) for $r \neq 0$. The name "no-ghost theorem" is also a word play on the phrase no-go theorem.

Appendix D

The Choice of The Solution Space

As we mentioned - both in the continuous and the discrete cases - there exists a choice as to where the oscillation frequency of the modes of the string, ω_m can be placed, in the definition of X^μ and Π^μ:

$$X^{\mu}{}_{n} = \sum_m X^{\mu}{}_{n}(m) = \aleph \sum_m (\omega_m)^r e^{i\omega_m t} \left(\alpha^\mu_m e^{i\kappa_m \frac{nL}{N}} + \beta^\mu_m e^{-i\kappa_m \frac{nL}{N}} \right) \tag{D.1}$$

$$\Pi^{\mu}{}_{n} = \dot{X}^\mu_n = \sum_m \Pi^{\mu}{}_{n}(m) = i\aleph \sum_m (\omega_m)^{r+1} e^{i\omega_m t} \left(\alpha^\mu_m e^{i\kappa_m \frac{nL}{N}} + \beta^\mu_m e^{-i\kappa_m \frac{nL}{N}} \right) \tag{D.2}$$

where, r represents the degree of freedom of this choice; the exponent of the frequency factor is undetermined at this stag.

In this appendix we will investigate this choice, thereby motivating the solution set which was used in the main text of this report.
To do so, first recall that:

$$\omega_m \sim \begin{cases} m & \textit{continuous case} \\ sin(m) & \textit{discrete case} \end{cases} \tag{D.3}$$

Therefore as we mentioned, the ω_0 case is anomalous; in fact ω_0 corresponds to the centre of mass d.o.f., i.e. $frequency = 0$. However we wish to provide a consistent formulation, where we do not need to treat this mode separately. This can be done if wherever ω_0 appears in the definition of the *relevant* operators of the theory, it is in the numerator and not in the denominator. This is the relation between the choice of r and this anomalous case; the "proper" choice of r will determine whether *the operator algebra on the resulting Hilbert space of solutions* is well-defined or not.

The *relevant* operators of this theory, are the operators which define the final solution space. These are the constraint operators, $T_{ab} \rightarrow \mathcal{K}^\diamond \rightarrow \mathcal{L}^\diamond$. Now as we saw these operators are functions of the creation/ annihilation operators α^μ_m, and β^μ_m.

Therefore the first step to take is to properly define the algebra of the α^μ_m and β^μ_m, $\forall\, m$, including $m = 0$; i.e. we need to define their commutation relations.

To do this, we need to look at the commutation relation between X^μ_n and Π^ν_m. If we use the *naive* expressions:

$$[\alpha^\mu_k, \alpha^\nu_l] = (\omega_k)^{-2r-1}\eta^{\mu\nu}\delta_{k,-l} = [\beta^\mu_k, \beta^\nu_l] \tag{D.4}$$

we have:

$$[X^\mu{}_n, \Pi^\nu{}_m] = \sum_{kl}[X^\mu{}_n(k), \Pi^\nu{}_m(l)] = i\hbar\eta^{\mu\nu}\delta_{n,m} \tag{D.5}$$

$$where\, we\, used: \qquad \aleph = \left(\frac{\hbar}{2N}\right)^{\frac{1}{2}}$$

Notice however, that the expression for the commutation of α^μ_k and β^μ_k, is well-defined, $\forall\, l, k$, including $k = 0$, if and only if:

$$2r + 1 < 0 \quad \Rightarrow \quad r < -\frac{1}{2} \tag{D.6}$$

This highlights the difference between the $l, k = 0 \sim \omega_0$ mode and the other modes. With this in mind let us now calculate the commutation relation more carefully by separating the anomalous cases:

$$[X^\mu{}_n, \Pi^\nu{}_m] = [X_n(0), \Pi_m(0)] + \sum_{k,l\neq 0}[X_n(k), \Pi_m(l)] + \sum_{k\neq 0}[X_n(k), \Pi_m(0)] + \sum_{l\neq 0}[X_n(0), \Pi_m(l)] \tag{D.7}$$

If we dismiss the last two terms of this expression by imposing orthogonality, *appendix*(E), we have:

$$\begin{aligned}
[X^\mu{}_n, \Pi^\nu{}_m] &= [X_n(0), \Pi_m(0)] + \frac{i\hbar}{2N}\delta^{\mu\nu}\sum_{k\neq 0} 2cos\left(k\frac{2\pi}{N}(n-m)\right) \\
&= [X_n(0), \Pi_m(0)] + \frac{i\hbar}{N}\delta^{\mu\nu}2\sum_{k=1}^{\infty} cos\left(k\frac{2\pi}{N}(n-m)\right) \\
&= [X_n(0), \Pi_m(0)] + \frac{2i\hbar}{N}\delta^{\mu\nu}\left[\pi\delta\left(\frac{2\pi}{N}(n-m)\right) - \frac{1}{2}\right] \\
&= [X_n(0), \Pi_m(0)] + \frac{2i\hbar}{N}\delta^{\mu\nu}\left[\frac{N}{2}\delta\left((n-m)\right) - \frac{1}{2}\right] \\
&= [X_n(0), \Pi_m(0)] + i\hbar\delta^{\mu\nu}\left[\delta_{n,m} - \frac{1}{N}\right]
\end{aligned} \tag{D.8}$$

$$must\, equal \quad i\hbar\delta^{\mu\nu}\delta_{n,m}$$

$$\therefore \quad [X_n(0), \Pi_m(0)] = \frac{i\hbar}{N}\delta^{\mu\nu} \tag{D.9}$$

We can now use this to *define* the commutation relation for α_0^μ and β_0^μ :

$$
\begin{aligned}
[X_n(0), \Pi_m(0)] &\sim i(\omega_0)^{2r+1} \left(\frac{\hbar}{2N}\right) ([\alpha_0^\mu, \alpha_0^\nu] + [\beta_0^\mu, \beta_0^\nu]) \\
&= \frac{i\hbar}{N} \delta^{\mu\nu} \qquad i.e.\, take \quad \omega_0 \neq 0
\end{aligned}
$$

Therefore we conclude the following expressions for the α_k^μ and β_k^μ:

$$
\Rightarrow \begin{cases} [\alpha_0^\mu, \alpha_0^\nu] = {\omega_0}^{-2r-1} \eta^{\mu\nu} & = [\beta_0^\mu, \beta_0^\nu] \\ [\alpha_k^\mu, \alpha_l^\nu] = {\omega_k}^{-2r-1} \eta^{\mu\nu} \delta_{k,-l} & = [\beta_k^\mu, \beta_l^\nu] \end{cases} \tag{D.10}
$$

where, $\omega_k \sim k \quad \forall k \neq 0$, and $\omega_0 \sim arbitrary$.

With these definitions then the commutation relations will be well-defined. The only condition on r so far is $r < -\frac{1}{2}$. Now let us consider the operators which are defined from these to see if other restrictions exist.
As we mentioned the operators which *define* the solution space are the constraints.

As, we saw the constraints originated from the energy momentum tensor, $T_{ab}(n,m)$ which was then broken up to the $\mathcal{K}_{nm}^\diamond$ and then transformed to Virasoro generators, $\mathcal{L}_{cd}^\diamond$:

$$
T_{ab}(n,m) \sim \mathcal{K}_{nm}^{\diamond_{ab}} \sim (\dot{X}\dot{X}, \dot{X}X', X'X') \sim \sum_{kl} (\omega_k \omega_l)^{r+1} e^{\mp i(kn+lm)} \times \cdots \tag{D.11}
$$

$$
\Rightarrow \quad \text{to have} \quad \mathcal{L}_{cd}^\diamond \sim f(\omega_c \omega_d) \sum_{nm} e^{\pm i(nc+md)} \mathcal{K}_{nm}^\diamond \quad \rightarrow \quad \alpha_c^\mu \alpha_{d_\mu}
$$

$$
\text{need} \quad f(\omega_c, \omega_d) = (\omega_c \omega_d)^{(-r-1)} \tag{D.12}
$$

Now notice that - based on their initial definition - depending on the choice of the value of r, $\mathcal{K}^\diamond$ and $\mathcal{L}^\diamond$ will have anomalous cases;

$$
i.e. \quad \begin{cases} \mathcal{K} & \text{needs to consider the} \quad k\, and/or\, l = 0 \quad \text{and} \\ \mathcal{L} & \text{the} \quad c\, and/or\, d = 0 \end{cases} \qquad \text{specially.} \tag{D.13}
$$

There however exists one unique choice of r which will make both these cases vanish:

$$
\left.\begin{aligned} r + 1 \geq 0 \\ -r - 1 \geq 0 \end{aligned}\right\} \quad \Rightarrow r = -1 \tag{D.14}
$$

This choice is in agreement with the previous condition on r, *eqn.*(D.6).
The choice $r = -1$ then stands out as being the choice that defines the proper operator algebra for the solution space of the field variables, X and Π. This verifies the correctness of the convention used in the main text.

Appendix E

Negative Norm $\sim$ Longitudinal Oscillations

In this appendix, we briefly discuss the origin of the negative norm states, hence providing an argument for removing these from the theory via the method in the main text.

The main function of the constraint equations is to remove the unphysical states of the solution space. In the quantum theory there exist one important argument for un-physicality of a given state is that it should have negative or null norm.

Now in the case of the string, we sought the solutions to the operators X^μ and Π^μ. However notice that the superscript $\mu = 1, 2, \cdots D$ represents the *spacetime* coordinates of the background - *and not just the space.* As such the commutation relation between the field operators, are of the form:

$$[X^\mu, \Pi^\nu] = i\hbar\eta^{\mu\nu} \tag{E.1}$$

where, $\eta_{\mu,\nu}$ is the flat (background metric. And the following convention is used:

$$\eta_{\mu\nu} = (-1; \underbrace{+1, +1, \cdots, +1}_{D-1}) \tag{E.2}$$

For this to be true, based on the calculations of the main text, *chap.*(3), we need to have:

$$[\alpha_n^\mu, \alpha_m^\nu] = \omega_n \delta_{n,-m} \eta^{\mu\nu} \tag{E.3}$$

A simple calculation using this commutation relation will then give the following result:

$$\left[(\alpha_a^\mu)^{A_a^\mu}, \left(\alpha_{-a}^\nu\right)^{A_a^\nu}\right] = \eta^{\mu\nu} A_a^\mu! (\omega_a)^{A_a^\mu} \tag{E.4}$$

However, the norm of a given state in the solution space of this problem:

$$|A_a^\mu\rangle = \frac{\left(\alpha_{-a}^\mu\right)^{A_a^\mu}}{\sqrt{\omega_a^{A_a^\mu} A_a^\mu!}} |0\rangle \tag{E.5}$$

can be calculated to give:

$$\langle A^{\mu}_{a} | A^{\nu}_{a} \rangle = \frac{1}{\sqrt{\cdots}} \langle 0 | \left(\alpha^{\mu}_{a}\right)^{A^{\mu}_{a}} \left(\alpha^{\nu}_{-a}\right)^{A^{\nu}_{a}} |0\rangle = \eta^{\mu\nu} \tag{E.6}$$

i.e. the norm of the $D = 1$ dimension can be negative. This confirms that there can exist negative normed states.

However notice that since the negative-normed states are related to the $\eta_{11} \sim D = 1$, these are associated with longitudinal oscillations of the string.

From this result then it is argued that the longitudinal oscillations are unphysical and must be removed from the solution space. This is the feature which allows for the separation of the transformed constraints, in the calculation of the solution space of the *chap.*(3).

Appendix F

Consistency and Dynamics via String Quantization

In this appendix we consider the *possibility* and *consistency* of the dynamics which may exist if the theory is quantized due to the weaker constraint conditions of string theory.

Recall that in Dirac's method we check the constraints:

$$C_\alpha \approx 0 \tag{F.1}$$

are obeyed, by checking the states, as follows:

$$C_\alpha |\Psi\rangle = 0 \tag{F.2}$$

So we can think of this as the realization of the constraint conditions. Hence, in the string case, where the condition is now redefined to:

$$\langle\Psi|C_\alpha|\Psi\rangle = 0 \quad \Rightarrow \quad C^+_\alpha|\Psi\rangle = 0 \tag{F.3}$$

instead of considering that:

$$[O, C] \approx 0 \quad \Rightarrow \quad [O, C]|\Psi\rangle = 0 \tag{F.4}$$

we need consider:

$$\langle\Psi|[O, C]|\Psi\rangle = 0 \tag{F.5}$$

In this respect, we can then see that in a theory where $H = \sum C^\alpha$ (is a total constraint) such as the string, whereas Dirac's theory prescribes:

$$[O, H]|\Psi\rangle = \dot{O}|\Psi\rangle = 0 \tag{F.6}$$

in string's case we have:

$$\langle\Psi|[O, H]|\Psi\rangle = 0 \tag{F.7}$$

which has the potential of leaving a few of the $\dot{O}$ eigenvalues:

$$\dot{\hat{O}}|\Psi\rangle = \dot{o}|\Psi\rangle \tag{F.8}$$

non-trivial. Let us examine this.

$$If \quad \langle\Psi|[O,H]|\Psi\rangle = 0 \quad \Rightarrow \langle\Psi|\dot{O}|\Psi\rangle = 0 \tag{F.9}$$

which is satisfied if via:

$$\dot{O} = \dot{O}^{+} + \dot{O}^{-} \tag{F.10}$$

we require that:

$$\dot{O}^{+}|\Psi\rangle = 0 \tag{F.11}$$

Hence the nontrivial eigenvalues of $\dot{O}$ come from $\dot{O}^{-}$:

$$\dot{O}|\Psi\rangle = \dot{O}^{-}|\Psi\rangle \quad not\,necessarily\ = 0 \tag{F.12}$$

Therefore we have the possibility of having dynamics. Let us check the consistency of this method as applied to the variations of the observables and their dynamics.

F.1 Dynamics and Gauge Transformations

If we analyze the expression for the expectation value:

$$\begin{aligned}
\langle\Psi|\dot{O}|\Psi\rangle &= \langle\Psi|O(H^{+}+H^{-}) - (H^{+}+H^{-})O|\Psi\rangle \\
&= \langle\Psi|OH^{-} - H^{+}O|\Psi\rangle \\
&\equiv i\langle\Psi|F^{-} + F^{+}|\Psi\rangle \\
&= 0 \\
&= \langle\Psi|\dot{O}^{-} + \dot{O}^{+}|\Psi\rangle
\end{aligned} \tag{F.13}$$

where, for example we can therefore define:

$$\dot{O}^{-} \equiv iOH^{-} \qquad and \qquad \dot{O}^{+} \equiv -iH^{+}O \tag{F.14}$$

We could have of-course defined these in the other order, however note that this, convention is the one which will lead to a consistent formulation without need of resorting to any more restrictions on the observables. For notice that:

$$\dot{O}^{+}|\Psi\rangle = 0 \rightarrow \begin{cases} iH^{+}O|\Psi\rangle & = 0 \quad or \\ iOH^{-}|\Psi\rangle & = 0 \end{cases} \tag{F.15}$$

Note that we previously had:

$$H^{+}|\Psi\rangle = 0$$

Without loss of generality let us assume now that $|\Psi\rangle$ is an eigenstate of O. Then:

$$\begin{cases} If & \dot{O}^{+} \equiv H^{+}O \Rightarrow O^{+}|\Psi\rangle = H^{+}O|\Psi\rangle = oH^{+}|\Psi\rangle = 0 \quad \forall\, O \\ whereas\, if & \dot{O}^{+} \equiv OH^{-} \Rightarrow O^{+}|\Psi\rangle = OH^{-}|\Psi\rangle \quad need\, not\, necessarily\, = 0 \end{cases} \tag{F.16}$$

So we find that the constraint conditions can be satisfied without further restrictions, Similar results can be established for a general gauge transformation induced by any constraint; the eigenvalues of the variation of an observable O via a given constraint C^{α} can be consistently defined as:

$$\begin{cases} \delta_\alpha O & = (\delta_\alpha O^{-} + \delta_\alpha O^{+}) \approx \delta_\alpha O^{-} = OC_\alpha^{-} \\ \delta_\alpha O|\Psi\rangle & = (\delta_\alpha O^{-} + \delta_\alpha O^{+})\,|\Psi\rangle = \delta_\alpha O^{-}|\Psi\rangle = OC_\alpha^{-}|\Psi\rangle \end{cases} \tag{F.17}$$

So we can see that these are not trivial and to stress:

$$\dot{O}|\Psi\rangle = OH^{-}|\Psi\rangle = \dot{o}|\Psi\rangle \tag{F.18}$$

Hence observables can be seen to consistently poses nontrivial dynamics, without further restrictions on the theory.

F.2 The String C-Algebra

Now let us address the question of the closure of the algebra. In other words, we would like to see what the combination of this new method with the requirement of the closure of the algebra will lead to. Notice that in Dirac's quantization we'd require that:

$$if \quad [C_\alpha, C_\beta] = K_{\alpha\beta}^{\lambda} C_\lambda \tag{F.19}$$

Whereas in the string's case, since we need only have:

$$\langle\Psi|[C_\alpha, C_\beta]|\Psi\rangle \approx \langle\Psi|C_\alpha^{+}C_\beta^{-} - C_\beta^{+}C_\alpha^{-}|\Psi\rangle = \langle\Psi|K_{\alpha\beta}^{\lambda}C_\lambda^{+} - K_{\beta\alpha}^{\lambda}C_\lambda^{-}|\Psi\rangle = 0 \tag{F.20}$$

where we have used $K_{\alpha\beta}^{\lambda} = -K_{\beta\alpha}^{\lambda}$.
Therefore we set:

$$\begin{cases} K_{\alpha\beta}^{\lambda}C_\lambda^{+} & \equiv C_\alpha^{+}C_\beta^{-} \\ K_{\beta\alpha}^{\lambda}C_\lambda^{-} & \equiv C_\beta^{+}C_\alpha^{-} \end{cases} \tag{F.21}$$

which is also consistent with the consistent convention needed for other observables O, we see that the $\langle\Psi|[C_\alpha, C_\beta]|\Psi\rangle = 0$ is satisfied. So not only the algebra is closed, its closure can be met with much weaker conditions which are nonetheless consistent with the rest of the formulation.

Appendix G

The Foundations of LQG

In this appendix we will try to lay out the foundations of the formulation of LQG. Our emphasis will be to consider all the assumptions which enter into its formulation, and the consequences of these assumptions for the final results of the theory - in particular on its "manifest" discreteness.

To do so let us consider the full mathematical structure of LQG.

G.1 The Origin of the Action

The action of LQG is derived from GR's original action. However a number of modifications are made to this action, which have significant ramifications for the eventual theory that emerges - LQG. Let us consider these modifications.
Recall that, GR's Einstein-Hilbert action in a general coordinate system, $x^\mu = (x^0, x^1, x^2, x^3)$ can be written in the form:

$$S_{EH} = \frac{1}{2K} \int \mathrm{d}^4 x \sqrt{-g} R \tag{G.1}$$

The first modification of the LQG, is that it considers the *Palatini's* action, which is based on the connection rather than the metric variable. The second modification is to write this action in an *orthonormal basis*. Why we move to an orthonormal basis, and the consequences of this modification are questions which we consider in *sec.*(G.2). First let us see how this modification is done.

The index notations which we will use are as follows:

$$\begin{cases} \textit{Greek indices,} & \alpha, \beta, \mu, \nu \cdots & \rightarrow \textit{4-D general coordinates} & \rightarrow & 0,1,2,3 \\ \textit{Lower case Roman indices} & a,\ b,\ c, \cdots & \rightarrow \textit{3-D spatial-general coordinates} & \rightarrow & 1,2,3 \quad t=0 \\ \textit{Upper case Roman indices} & I\ J,\ K \cdots & \rightarrow \textit{4-D orthonormal coordinates} & \rightarrow & 0,1,2,3 \\ \textit{Lower case Roman indices} & i,\ j,\ k, \cdots & \rightarrow \textit{3-D spatial-orthonormal coordinates} & \rightarrow & 1,2,3 \quad t=0 \end{cases}$$

Now to rewrite the action we need to make a few definitions, and rewrite the geometrical functions in the new coordinate system. The *vierbein* (4-leg "thing") is defined as follows:

$$\mathfrak{e}^I_\alpha : x^\alpha \to \mathfrak{e}^I = \mathfrak{e}^I_\alpha x^\alpha \tag{G.2}$$

which defines the elementary transformation between the two coordinate systems.
The general *connection* $S^\alpha{}_{\mu\nu}$ is defined as follows:

$$\nabla_{\mathfrak{e}_\nu}\mathfrak{e}_\mu = \mathfrak{e}_\nu(\mathfrak{e}_\mu) + \Gamma^\alpha{}_{\mu\nu}\mathfrak{e}_\alpha \equiv S^\alpha{}_{\mu\nu}\mathfrak{e}_\alpha \tag{G.3}$$

Then using the general definition of the Riemann Curvature Tensor (next section), we have:

$$R^\alpha{}_{\beta\mu\nu} \equiv [\nabla_{\mathfrak{e}_\mu}, \nabla_{\mathfrak{e}_\nu}]\mathfrak{e}_\beta = S^\alpha{}_{\beta\nu,\mu} - S^\alpha{}_{\beta\mu,\nu} + S^\alpha{}_{\lambda\mu}S^\lambda{}_{\beta\nu} - S^\alpha{}_{\lambda\nu}S^\lambda{}_{\beta\mu} \tag{G.4}$$

Notice however that $R^\alpha{}_{\beta\mu\nu}$ is asymmetric over the its last two indices. This property is manipulated in the Palatini's formulation of GR to simplify calculations, as follows:

$$\begin{aligned}
R^\alpha{}_{\beta\mu\nu}(\mathfrak{e}^\mu \otimes \mathfrak{e}^\nu) &= \frac{1}{2}R^\alpha{}_{\beta\mu\nu}(\mathfrak{e}^\mu \otimes \mathfrak{e}^\nu - \mathfrak{e}^\nu \otimes \mathfrak{e}^\mu) \\
&= R^\alpha{}_{\beta\mu\nu}\mathfrak{e}^\mu \wedge \mathfrak{e}^\nu \\
\textit{using eqn.(G.4)} &= 2\left(S^\alpha{}_{\beta\nu,\mu} + S^\alpha{}_{\lambda\mu}S^\lambda{}_{\beta\nu}\right)\mathfrak{e}^\mu \wedge \mathfrak{e}^\nu \\
&= 2\left(\mathrm{d}S^\alpha{}_\beta + S^\alpha{}_\lambda \wedge S^\lambda{}_\beta\right)
\end{aligned} \tag{G.5}$$

Let us define this as:

$$\frac{1}{2}R^\alpha{}_{\beta\mu\nu}\mathfrak{e}^\mu \wedge \mathfrak{e}^\nu \equiv \Omega^\alpha{}_\beta = \mathrm{d}S^\alpha{}_\beta + S^\alpha{}_\lambda \wedge S^\lambda{}_\beta \tag{G.6}$$

Rewriting this then gives[1]:

$$\begin{aligned}
R^\alpha{}_{\beta\mu\nu} &= R^\alpha{}_{\beta\gamma\sigma}\left(\mathfrak{e}^\gamma \otimes \mathfrak{e}^\sigma\right)\cdot\left(\mathfrak{e}_\mu \otimes \mathfrak{e}_\nu\right) \\
&= \frac{1}{2}R^\alpha{}_{\beta\gamma\sigma}\left(\mathfrak{e}^\gamma \otimes \mathfrak{e}^\sigma - \mathfrak{e}^\sigma \otimes \mathfrak{e}^\gamma\right)\cdot\left(\mathfrak{e}_\mu \otimes \mathfrak{e}_\nu\right) \\
&= R^\alpha{}_{\beta\gamma\sigma}\left(\mathfrak{e}^\gamma \wedge \mathfrak{e}^\sigma\right)\cdot\left(\mathfrak{e}_\mu \otimes \mathfrak{e}_\nu\right) \\
&= 2\Omega^\alpha{}_\beta \cdot \frac{1}{2}\left(\mathfrak{e}_\mu \otimes \mathfrak{e}_\nu - \mathfrak{e}_\nu \otimes \mathfrak{e}_\mu\right) \\
&= 2\Omega^\alpha{}_\beta \cdot \mathfrak{e}_\mu \wedge \mathfrak{e}_\nu \\
&= 2\Omega^\alpha{}_{\beta\,IJ}\left(\mathfrak{e}^I \wedge \mathfrak{e}^J\right)\cdot\left(\mathfrak{e}^K_\mu\mathfrak{e}_K \wedge \mathfrak{e}^L_\nu\mathfrak{e}_L\right) \\
&= 2\Omega^\alpha{}_{\beta\,IJ}\mathfrak{e}^K_\mu\mathfrak{e}^L_\nu\left(\mathfrak{e}^I \wedge \mathfrak{e}^J\right)\cdot\left(\mathfrak{e}_K \wedge \mathfrak{e}_L\right) \\
&= 2\Omega^\alpha{}_{\beta\,IJ}\mathfrak{e}^K_\mu\mathfrak{e}^L_\nu\frac{1}{2}\left(\delta^I{}_K\delta^J{}_L - \delta^I{}_L\delta^J{}_K\right) \\
&= \Omega^\alpha{}_{\beta\,IJ}\left(\mathfrak{e}^I_\mu\mathfrak{e}^J_\nu - \mathfrak{e}^J_\mu\mathfrak{e}^I_\nu\right) \\
&= \Omega^\alpha{}_{\beta\,IJ}\mathfrak{e}^I_\mu\mathfrak{e}^J_\nu \qquad \textit{is asymmetric over } I \leftrightarrow J \textit{ and therefore over } \mu \leftrightarrow \nu
\end{aligned} \tag{G.7}$$

[1] We go through every single step of this procedure for two reasons.

1 It would help clarify many initial steps which are taken in the formalism which are nonetheless not stated clearly in any literature.

2 It will help highlight the exact relations between the general and the vierbein coordinates, hidden in the notational indices, which are however oversimplified by the seeming simplicity of the relations.

Now, if we calculate the Ricci scalar, we have:

$$R = R^{\alpha\beta}{}_{\alpha\beta} = \Omega^{\alpha\beta}{}_{IJ}\mathfrak{e}^I_\alpha \mathfrak{e}^J_\beta$$

and using the identities:

$$\begin{cases} \mathfrak{e}_I{}^\alpha \mathfrak{e}_\beta{}^I = \delta^\alpha{}_\beta \\ \epsilon_{IJKL} = \mathfrak{e}_I{}^\alpha \mathfrak{e}_J{}^\beta \mathfrak{e}_K{}^\lambda \mathfrak{e}_L{}^\gamma \epsilon_{\alpha\beta\lambda\gamma} \end{cases} \tag{G.8}$$

we find:

$$\Omega^{\alpha\beta}{}_{IJ}\mathfrak{e}^I_\alpha \mathfrak{e}^J_\beta = \frac{1}{2}\epsilon_{IJKL}\mathfrak{e}^I \wedge \mathfrak{e}^J \wedge \Omega^{KL} \tag{G.9}$$

So, we find:

$$\Rightarrow \boxed{R = \frac{1}{2}\epsilon_{IJKL}\mathfrak{e}^I \wedge \mathfrak{e}^J \wedge \Omega^{KL}} \tag{G.10}$$

This is the main result of this section; we have derived the Palatini action for GR in the orthonormal coordinates. Also note that:

$$g_{\alpha\beta} = \mathfrak{e}^I_\alpha \mathfrak{e}^J_\beta \eta_{IJ} \quad \rightarrow \quad \mathbf{g} = \mathfrak{e}\eta\mathfrak{e}^{\mathfrak{t}} \quad \Rightarrow \quad \boxed{det(\mathbf{g}) = -\left(det(\mathfrak{e})\right)^2} \tag{G.11}$$

whence the action becomes

$$S_P = \frac{1}{4k}\int_M det(\mathfrak{e})\epsilon_{IJKL}\mathfrak{e}^I \wedge \mathfrak{e}^J \wedge \Omega^{KL} = \frac{1}{2k}\int_M det(\mathfrak{e})\Omega^{\alpha\beta}{}_{IJ}\mathfrak{e}^I{}_\alpha \mathfrak{e}^J{}_\beta \tag{G.12}$$

The important mathematical transformation which has taken place so far is the formulation of the theory in terms of the connection. As we saw, calculation of the terms of the curvature tensor, the Ricci tencsor and the scalar are greatly simplified in the formulation - as a direct consequence of the manipulation of the connections. However in this format - Palatini's original formulation - " $\cdots$ in the Hamiltonian formulation of this theory $\cdots$ all references to the connection-dynamics will be lost", [36]. Therefore in LQG the following term is added to the Palatini action, to maintain the useful properties of the connection-formulation.

$$I = -\frac{1}{2k\gamma}\int_M det(\mathfrak{e})\Omega_{IJ} \wedge \mathfrak{e}^I \wedge \mathfrak{e}^J = -\frac{1}{4k\gamma}\int_M det(\mathfrak{e})\epsilon_{IJKL}\mathfrak{e}^I{}_\alpha \mathfrak{e}^J{}_\beta \Omega^{KL\alpha\beta} \tag{G.13}$$

where γ is the famous *Barbero-Immirizi parameter*, whose choice has significant mathematical and physical consequences which we shall discuss in the next section. This integral is posited to be a canonical transformation [41]. The combination of this "canonical" term, with the Palatini action, leads to the Holst's action for GR:

$$\begin{aligned} S_H &= -\frac{1}{2k\gamma}\int_M det(\mathfrak{e})\mathfrak{e}^I \wedge \mathfrak{e}^J \wedge \left(\Omega_{IJ} - \frac{\gamma}{2}\epsilon_{IJKL}\Omega^{KL}\right) && \textit{Ashtekar} \\ &= \frac{1}{2k}\int_M det(\mathfrak{e})\mathfrak{e}^I{}_\alpha \mathfrak{e}^J{}_\beta \left(\Omega^{\alpha\beta}{}_{IJ} - \frac{1}{2\gamma}\epsilon_{IJKL}\Omega^{KL\alpha\beta}\right) && \textit{Hölst} \end{aligned} \tag{G.14}$$

In Asktekar's formulation the $det(\mathfrak{e})$ is absorbed into the expression. This action defines the basic action of LQG.

The next step is to adopt an *ADM-type* formulation from this action. To do this the following steps are taken:

- Make a gauge fixing to define a *3D* spatial slice (but we consider all slicings so the gauge fixing does not ruin covariance.)
  Define, n^I as the normal to this surface, with $n^I n_I = signature(\eta_{IJ}) \equiv \sigma = -1$.
  The metric on the *3D* surface is $\eta_{ij} = q_i^I q_j^J \eta_{IJ}$, where the q_i^I are the projection matrices.

- The connection (1-form) S_α^{IJ} defines:

$$A_a^i = n_J q_I^i q_a^\alpha A_\alpha^{IJ} \equiv n_J q_I^i q_a^\alpha \left(\frac{1}{2} \epsilon^{IJ}{}_{KL} S_\alpha^{KL} + \gamma S_\alpha^{IJ} \right) \equiv \Gamma^i{}_a + \gamma K^i{}_a \,, \tag{G.15}$$

  the *connection 1-form* on the 3-D spatial M. This connection along with its conjugate:

$$P_i^a \equiv \frac{1}{2k\gamma} e_b^i e_c^k \epsilon^{abc} \epsilon_{ijk} = \frac{1}{k\gamma} E_i^a \tag{G.16}$$

  form the basic conjugate pair of the LQG theory:

$$\left\{ A_a^i(x), P_j^b(y) \right\} = \delta_j^i \delta_a^b \delta(x,y) = \frac{1}{k\gamma} \left\{ A_a^i(x), E_i^a \right\} \tag{G.17}$$

Through this calculation we also see how the *new* definition of the action leads to the definition of the *new variables* of LQG.

The rest of the formulation then follows as discussed in the chapter LQG.

G.1.1 On the Spin-Connection[2]

In this subsection we will formulate the definition and the mathematical concept of the connection, which then leads to the spin-connections. As we have seen the spin-connections are the main elements which are manipulated in the Palatini's formulation of GR.

The basic concept here is that of covariant derivative (CD). The basic idea behind CD is to enable us to properly compare properties of tensors at different points of the manifold. CD is needed because *in a general coordinate system $\{e_\alpha\}$, when we go from one point on the manifold to another the coordinate system can change.*

As such we define the CD as:

$$\nabla_{e_\nu} e_\mu = e_\nu(e_\mu) + \Gamma^\alpha{}_{\mu\nu} e_\alpha \equiv S^\alpha{}_{\mu\nu} e_\alpha \tag{G.18}$$

[2]We include this "basic" subsection mainly because despite the simplicity of its context, in the literature there rarely *exists* a well-composed explanation of these basics.

- The first term of this definition, is a simple differential (in a general coordinate system different coordinates may not be independent of each other; a coordinate system whose different coordinates are independent is called a coordinate basis.)
- The second term, takes into account the possible change in the coordinate system.
- The last term, $S^{\alpha}{}_{\mu\nu}$ is the total connection.

From the CD we may now calculate the following fundamental forms of differential geometry:

$$\begin{aligned}
w^{\alpha}{}_{\mu\nu}e_{\alpha} &= e_{\mu}(e_{\nu}) - e_{\nu}(e_{\mu}) && \text{is the } \textit{Spin-connection} \\
&\equiv [e_{\mu}, e_{\nu}]e_{\alpha} && \text{(G.19)} \\
T^{\alpha}{}_{\mu\nu}e_{\alpha} &\equiv \nabla_{e_{\mu}}e_{\nu} - \nabla_{e_{\nu}}e_{\mu} - [e_{\mu}, e_{\nu}] && \text{is the } \textit{Torsion} \\
&= \left(S^{\alpha}{}_{\nu\mu} - S^{\alpha}{}_{\mu\nu} - w^{\alpha}{}_{\mu\nu}\right) e_{\alpha} \\
&= \left(\Gamma^{\alpha}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\mu\nu}\right) e_{\alpha} && \text{(G.20)} \\
R^{\alpha}{}_{\beta\mu\nu}e_{\alpha} &= \left([\nabla_{e_{\mu}}, \nabla_{e_{\nu}}] - \nabla_{[e_{\mu}, e_{\nu}]}\right) e_{\beta} && \text{is the } \textit{Riemann Curvature} \\
&= \left[S^{\alpha}{}_{\beta\nu|\mu} - S^{\alpha}{}_{\beta\mu|\nu} + S^{\alpha}{}_{\lambda\mu}S^{\lambda}{}_{\beta\nu} - S^{\alpha}{}_{\lambda\nu}S^{\lambda}{}_{\beta\mu} + S^{\alpha}{}_{\beta\lambda}\left(S^{\lambda}{}_{\nu\mu} - S^{\lambda}{}_{\mu\nu} - w^{\lambda}{}_{\mu\nu}\right)\right] e_{\alpha} \\
&= \left[S^{\alpha}{}_{\beta\nu|\mu} - S^{\alpha}{}_{\beta\mu|\nu} + S^{\alpha}{}_{\lambda\mu}S^{\lambda}{}_{\beta\nu} - S^{\alpha}{}_{\lambda\nu}S^{\lambda}{}_{\beta\mu} + S^{\alpha}{}_{\beta\lambda}T^{\lambda}{}_{\mu\nu}\right] e_{\alpha} && \text{(G.21)}
\end{aligned}$$

If we assume the torsion is zero - as it is done in GR- we will have:

$$\begin{cases} T^{\alpha}{}_{\mu\nu} = 0 \\ w^{\alpha}{}_{\mu\nu} = -\left(S^{\alpha}{}_{\mu\nu} - S^{\alpha}{}_{\nu\mu}\right) \\ R^{\alpha}{}_{\beta\mu\nu} = S^{\alpha}{}_{\beta\nu|\mu} - S^{\alpha}{}_{\beta\mu|\nu} + S^{\alpha}{}_{\lambda\mu}S^{\lambda}{}_{\beta\nu} - S^{\alpha}{}_{\lambda\nu}S^{\lambda}{}_{\beta\mu} \end{cases} \tag{G.22}$$

Now we can work out the general form of $S^{\alpha}{}_{\mu\nu}$ in a general metric $g_{\alpha\beta}$:

$$S_{\alpha\mu\nu} = \frac{1}{2}\left(g_{\alpha\nu,\mu} - g_{\nu\mu,\alpha} + g_{\mu\alpha,\nu} + w_{\alpha\mu\nu} - w_{\mu\nu\alpha} + w_{\nu\alpha\mu}\right) \tag{G.23}$$

which then leads to the familiar results:

$$\begin{cases} S_{\alpha\mu\nu} = \dfrac{1}{2}\left(g_{\alpha\nu,\mu} - g_{\nu\mu,\alpha} + g_{\mu\alpha,\nu}\right) & \textit{in coordinate basis} \\ S_{\alpha\mu\nu} = \dfrac{1}{2}\left(w_{\alpha\mu\nu} - w_{\mu\nu\alpha} + w_{\nu\alpha\mu}\right) & \textit{in orthonormal basis} \end{cases} \tag{G.24}$$

So we see that in the orthonormal basis, which is used in LQG, the connection $S^{\alpha}{}_{\mu\nu}$ is completely dependent on the *spin-connection* $w^{\alpha}{}_{\mu\nu}$.

G.2 The Internal Gauge Group

As we mentioned there are certain mathematical advantages which motivates the transformation to the orthonormal basis. Let us now discuss some of these.

Notice that in the conversion from general coordinates to orthonormal ones, we picked up an internal index. This new index represents a new degree of freedom. This d.o.f. is related to the choice of the orientation of the orthonormal basis, and is therefore just a gauge. The physical ramifications of *the choice of the gauge group associated with this gauge freedom*, are the single motivation for the transformation to the orthonormal basis in LQG.

Let us discuss these ramifications.

As we mentioned the gauge freedom is related to the freedom to the *rotation of the vierbein*, therefore it is related to an $SO(3,1)$ gauge. However, although $SO(3,1)$ is globally different, this gauge is locally isomorphic to $SU(2)$ and therefore shares the same Lie-Algebra with $SU(2)$. In LQG therefore - for reasons which we will shortly see - the internal gauge group is *assumed* to be $SU(2)$ instead. This change is definitely a generalization.

This generalization facilitates LQG with two features:

1. The internal gauge group is now that of the (simplest) Yang-Mills theory. Hence the fully developed machinery of Yang-Mills quantization can now be used in LQG
2. $SU(2)$ gives us the possibility of including half-integer "objects" into the theory therefore utilizing the possibility of adding Fermions to the theory.

Regardless though, *this is a choice!* As a result alternatives to LQG exist which make other choices. For example the $SO(3,1)$ itself, or $SL(2,\mathbb{C})$ gauge groups are sometimes used, [43].

However *mathematically*, these alternatives are problematic. The most important of these problems is that most of these alternatives have a *non-compact* gauge group, whereas $SU(2)$ is compact. The important feature of compact groups is that a *Haar measure* can be defined on them. This measure has the property that it is easily normalized, and therefore can be given a probability interpretation, which is precisely what is needed in quantization. None of these features are present in non-compact groups.

Therefore we can say that $SU(2)$ is *mathematically* motivated as the preferred gauge group to represent the internal degrees of freed of LQG.

We can demonstrate above-mentioned property of compact groups with a simple example.

Compact Group Example

The example we use is a particle on a circle. This example represents the compact $U(1)$ group, *fig.*(G.1).

A solution on the circle, parametrized by θ, is of the form:

$$\psi \sim e^{iw\theta} \quad \xrightarrow{BC} \quad \psi_n(\theta) = ce^{in\theta} \quad c \quad normalization\,constant \tag{G.25}$$

A measure on this circle is simply $\mathrm{d}\theta$.

However notice that *due to the compactness*, the measure can be normalized:

$$d\theta \to \frac{1}{2\pi} d\theta \,,$$

As a result the solutions have well-defined norms, and the inner product:

$$\langle \psi_n(\theta) | \psi_m(\theta) \rangle$$

can be given proper quantum-probabilistic interpretation.

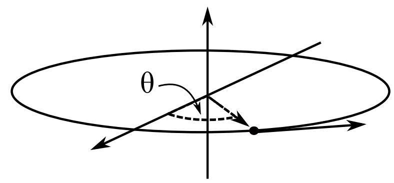

Figure G.1: Demonstration of normalizability and discreteness of a compact group; particle on a circle.

The generalization of this property is precisely what distinguishes $SU(2)$ as the preferred gauge-group for LQG.

Now let us consider the physical and mathematical consequences of this choice. As a direct result of this choice of gauge group - via a generalization of the above example - it can be shown that the structure of compact groups is discrete. This discreteness is precisely the discreteness which is manifest in the spin-networks - the formal solutions of (compact) LQG.

However notice that the discreteness of the spin-networks, is the reason that the eigenvalue spectrum of the geometrical operators of LQG - namely the *area* and *volume* operators - are discrete. However, recall that the discrete spectrum of these operators is usually interpreted as one of the *physical* features of the loop quantization of spacetime; i.e. it is claimed that spacetime's discreteness is a manifest feature of the loop quantization.

However as we have shown, the *physical* discreteness results are a direct consequence of this initial *mathematical* "choice" of the internal gauge group[3].

To conclude let us say a few words on the γ parameter and its relation to the the internal gauge.

The γ Parameter

Mathematically, the choice of the internal gauge-group is reflected in the freedom of choosing the *Barbero-Immirizi parameter*, γ. Let us see how.

Recall that the γ entered the formulation in the process $S_{Palatini} \to S_{H\ddot{o}lst}$, when the new integral, *eqn.*(G.13), was added to define the latter action.

Notice that, although classically the addition of this integral to the action is *posited* to be a canonical transformation - and therefore the choice of the γ is not physically significant -

[3]Some of the results on the non-compact representations of LQG, confirm the validity of these conclusions. Interestingly in one of these alternative formulations, (see for example [43]) discreteness is only found along the time-direction.

quantum mechanically the different choices of the γ parameter are not unitarily-equivalent [36].

The γ later appears in the definition of the internal gauge transformation. From that expression it can be seen that the choice of the value of the γ (whether it is real or not) will then determine if the internal gauge group is compact or not. This is the way that these two constructs - the internal gauge group and the Barbero-Immirizi Parameter - then get related in LQG.

From this relation however an interesting result is found with respect to the *Scalar constraint* of LQG.

On the Scalar Constraint

In the compact formulation, or Ashtekar's formulation γ needs to be real-valued, which then makes the internal group compact.

It is interesting to note that initially however, γ was taken to be imaginary, because *this choice leads to great simplifications of the scalar constraint of LQG*. Amazingly enough this "coincidence", was actually the motivation that lead to the development of LQG! (Because recall that the scalar constraint is nonlinear and therefore extremely difficult to quantize.)

But later - having discovered that the non-real choice of γ would lead to a non-compact structure - research was redirected to look at compact groups. However in compact formulations the "nice" simplifications of the scalar constraint did not exist. This to a great extend halted further development of LQG.
In fact it was not until Thiemann's discovery, [42] of a unique method of quantization of the nonlinear constraints, that the way for further developments was found.

In this method the constraints can be recast as the Poisson Brackets of the *volume* and the *Holonomy* operators. These have well-defined behavior in LQG.[4] Therefore one can devise *a method* to quantize the (still nonlinear but now compact) group theory.

This concludes the basic construction of the LQG.

In the final section of this appendix we will discuss the concept of measure which is also very crucial to the discussion of the solution space of this theory. First we will discuss the general features of this concept, and then we will develop the measure which is particularly used in the main part of this report for the string.

[4]We should mention that another ambiguity enters at this point; the *volume* operator, in contrast to the *holonomy* and the *area* operator can either be quantized ***internally*** (Ashtekar and Thiemann) or ***externally*** (Smolin and Rovelli). The physical difference is that in the former the resulting operator will know about the differential structure at the vertices of graphs whereas in the latter it is topological[36].

G.3 The Measure

Let us now briefly take a look at the issue of defining the measure in LQG.

The most important question which one faces in dealing with the measure of LQG, is whether it is consistent.

To discuss the concept of measure in LQG we will need the following definitions:

- **Sigma-Algebra**
  In mathematics, a σ**-algebra** over a set X is a nonempty collection S of subsets of X (including X itself) that is *closed* under complementation and countable unions of its members.
  The pair (X, S) is called a σ**-field** or a **measurable space**.

- **Borel Measure**
  In mathematics, the Borel algebra is the smallest σ-algebra on X, a locally compact Hausdroff space.
  The **Borel measure** is any measure μ on this σ-algebra.
  Note: Every Borel measurable set is also Lebesgue measurable.

- **Lebesgue Measure**
  Let D be a domain of a set S. Now Let $S = \{e_i\} \quad i = 1, 2, 3, \cdots \quad e_i$ are the measurable subsets of elements of S.
  i.e. we can define a "size-function" (measure) for them, $\mu(e_i)$, such that:

$$\begin{aligned} & \mu(\emptyset) = 0 \\ & if \quad e_i \cap e_j = 0 \quad \Rightarrow \quad \mu(e_i \cup e_j) = \mu(e_i) + \mu(e_j) \\ For\, disjoint \quad & e_i \ : \quad \mu(e_i \cup e_j \cup e_k \cup \cdots) = \mu(e_i) + \mu(e_j) + \mu(e_k) + \cdots \end{aligned}$$

  Next we need a notion of a *measurable function*, $f : S \to R$ where R is a topological space.
  i.e. f sends members of S to a topological space.
  Then for every open subset $A : A \in R$, if f^{-1} is a measurable subset $\in S$, then f is said to be measurable.
  Now we are in a position to construct the Lebesgue Integral.
  Notice that for a measurable subset $\{e_i\}$ of $S\, f(e_i)$ is well-defined.
  For each of these, a well-defined range exists: $t_{i-1} \leq f(e_i) \leq t_i$.
  Assume that these e_i are disjoint, and denote their measure as $m(e_i)$
  The sum

$$\sum_{i=1}^{n} m(e_i)(t_i - t_{i-1}) \,,$$

  can be defined as the Lebesgue sum (*fig.*(G.2)).

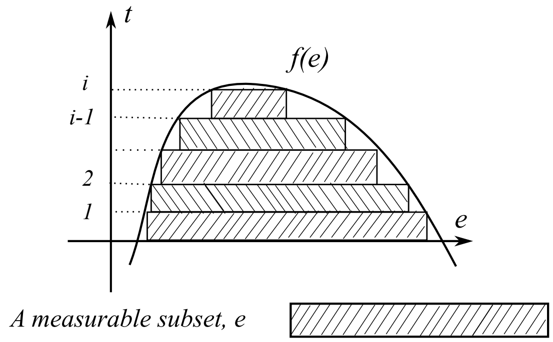

Figure G.2: The Lebesgue measure, and Lebesgue Integral.

- **Haar Measure**
  Lets now consider having a *locally compact* topological group G on S.
  The **Haar measure** is the invariant (under-group transformations G on S) measure defined on this Borel-algebra. i.e. it is the (can be shown to be unique) G-invariant Borel-measure.
  And (as mentioned before) due to this compact-ness, the Haar measure is normalizable and therefore suitably adaptable for quantization.

G.3.1 The Measure of the String

Now let us develop the above mentioned concept of the *measure on a set*, and use it to define the measure for the String.

For this let us assume the space we start with is the real line $\mathbb{R}$. A *set* on $\mathbb{R}$ is defined as N points on $\mathbb{R}$. N may be infinite (yet it must be countable). Call this set of points a *graph* α_N.
Denote by $\mathbb{S}$ the set of *all* such graphs on $\mathbb{R}$. The graphs may vary in two different ways:

1. α_N vs. β_N
   The same number but different selection

2. α_N vs. α_M
   Different number of points, yet with some overlap, in particular ex. $\alpha_N \subset \alpha_M$

Notice that these span the methods in which a graph may be different from another. Put together, we in general have:
α_N vs. $\beta_M \rightarrow$ Different selections with different numbers (of nodes) with possible overlap (in the selection.)

We call two graphs which differ in any one (or more) of these ways, *different graphs*. Hence α_N is *different* from $\alpha_M \quad N \neq M$, is *different* from β_N, and is certainly *different* from β_M.

Now the main goals of the sampling are the following:

1. The set $\mathbb{S} = \{\alpha_N\}$, of all α_N *covers* $\mathbb{R}$ (in a special sense which we shall elaborate on.)
2. To each sample or graph one may associate a *measure* or *size-function* $m(\alpha_N)$ (in a consistent manner which we will also describe shortly.)

The *measure* of α_N is the "size" of α_N. Therefore *measure* is a relative concept, that is why this method requires the set to be a Borel (or Lebesgue) set, i.e. a set with the following properties:

- If $\alpha_N \in \mathbb{S}$, so is its complement, $C(\alpha_N)$
- $\mathbb{S}$ is closed $\quad\rightarrow\quad$ if $\alpha_N \in \mathbb{S}$, and $\beta_M \in S \quad\Rightarrow\quad \alpha_N \oplus \beta_M \in \mathbb{S}$

The second requirement allows for a definition of the *measure of* S. The former realizes that if say $m_T = \oplus_N m_N$, the sum of all *elementary subsets/ graphs* (we will define these later), then if $m(\alpha_N) = m_N \quad \Rightarrow m(C(\alpha_N)) = m_T \ominus m_N$. In this sense the measure theory is intuitively consistent.
Notice however that $\{\alpha_N\}$ forms an *over-complete* basis for $\mathbb{S}$ due to the closure requirement. Therefore let us consider a subset of these, $\mathbb{A} \subset \mathbb{S}$; which we call the *fundamental* or *elementary* set of $\mathbb{R}$:

$$\mathbb{A} = \{\alpha_N\}_{fundamental} \qquad \alpha_N \cap \alpha_M = \emptyset \quad N \neq M \tag{G.26}$$

where we define $\mathbb{A}$ to be the set of *all non-overlapping sets of* $\mathbb{R}$. Notice that a certain degree of freedom exists in choosing this set, yet it can be done, and since the concept of the measure is a relative one (not absolute), the measure will be *set-dependent*[5].

Now let us proceed to see how this may be done. Define a function on $\mathbb{A}$ as:

$$\Psi_{\mathbb{A}}[f] \sim \sum_N f_N \alpha_N \tag{G.27}$$

The inner-product of two such functions:

$$\langle \Psi_{\mathbb{A}}[f] | \Phi_{\mathbb{A}}[g] \rangle \,, \tag{G.28}$$

may be defined as:

$$\rightarrow \langle \Psi_{\mathbb{A}}[f] | \Phi_{\mathbb{A}}[g] \rangle \sim \sum_{N,M} \langle f_N \alpha_N | g_M \alpha_M \rangle \tag{G.29}$$

and by construction we have:

$$\begin{cases} \langle \alpha_N | \alpha_N \rangle = m(\alpha_N) & \textit{The size of}\, \alpha_N \\ \langle \alpha_N | \alpha_M \rangle = m(\alpha_N \cap \alpha_M) & \textit{The size of the overlap of}\, \alpha_N \,\textit{and}\, \alpha_M \end{cases} \tag{G.30}$$

[5]In physical terms since the set will be related to the set of all possible physical states, we can see how the measure begin relative is actually rather appropriate.

which gives:

$$\langle\Psi_{\mathbb{R}}[f]|\Phi_{\mathbb{R}}[g]\rangle \equiv \langle\Psi_{\mathbb{A}}[f]|\Phi_{\mathbb{A}}[g]\rangle = \sum_N \bar{f}_N g_N m(\alpha_N) - \sum_{N,M} \bar{f}_N g_M m(\alpha_N \cap \alpha_M) \tag{G.31}$$

However the second term vanishes for a fundamental set $\mathbb{A}$, and we have:

$$\langle\Psi_{\mathbb{R}}[f]|\Phi_{\mathbb{R}}[g]\rangle = \sum_N \bar{f}_N g_N m(\alpha_N) \tag{G.32}$$

This sets the basic construction of the sampling theory, its representation, the choice of the basis, the definition of the measure and its relation to the inner product of the space of the functions.

We can summarize these as follows:

- The concept is to consider all possible α_N non-overlapping
- Assign $m(\alpha_N)$ to each
- We have an inner product on the state-space defined as above

A state is initially defined on: $\mathbb{R} \to \Psi_{\mathbb{R}}[f]$.
We break $\mathbb{R}$ into all it possible infinite, countable subsets α_N, the collection of which is $\mathbb{S}$. From this choose a subset $\mathbb{A}$ whose elements are non-overlapping; these then are *all possible un-overlapping graphs of* $\mathbb{R}$. $\mathbb{S}$ is an *overcomplete* representation of $\mathbb{R}$, whereas $\mathbb{A}$ is the *minimal* representation of it. To see the $\mathbb{A}$ actually forms a representation notice the following.

Consider the contrary situation; an overlapping set. The description in the overlap sector is simply redundant; it does not describe $\mathbb{R}$ any better. Hence the non-overlapping basis is not only the most convenient, it is also the minimal set (using which the information of $\mathbb{R}$ may be represented completely, so far as this representation is concerned.)
The basis set of $\mathbb{A}$ forms a *complete/ minimal* basis for $\mathbb{R}$ which in the sense of the measure, $m(\alpha_N \cap \alpha_M) = 0, \quad if \quad N \neq M$, is said to be "orthogonal" as well.
A function-state on this set is then defined as:

$$\Psi_{\mathbb{S}}[f] \sim \sum_N f_N \alpha_N$$

The main result is that any function on $\mathbb{R}$ can be represented in this way. Hence we can consider $\{\alpha_N\}$ as an (over-complete) basis of the space $\mathbb{R}$. And the $\{\alpha_N\}_{fundamental}$ is the minimal basis for it.

The *Cauchy completion* of $\mathbb{S}$ or $\mathbb{A}$ is $\mathbb{R}$.

The connection with the quantum theory is that *The Hilbert space, $\mathcal{H}$ can also be defined as the Cauchy completion* of Cyl.

To construct a quantum theory then we start from the $\mathbb{A}$, instead of $\mathbb{R}$. We assume α_N as the basis for the quantum states. In analogy we can have many different kinds of sets $\mathbb{A}$ just as in quantum mechanics we can have different basis. We *choose* an orthonormal basis because in many cases it is convenient.
So long as the requirements for a basis-to-the-space is met there does not exist any inconsistency. The requirement here is that $\{\alpha_N\}_f$ *represent* $\mathbb{R}$, (this is the definition of the *cover* which we used earlier).
A state function of $\mathbb{R}$ can be represented in $\mathbb{A}$ as:

$$\Psi_A[f] = \sum_N f_N \alpha_N \equiv \sum_N \psi_{\alpha_N}[f] \tag{G.33}$$

Let us now look at the fine structure of $\mathbb{A}$. The last equation implies that we could in fact write:

$$\psi_{\alpha_N}[f] = \sum_{k\in N} f^k{}_N e_{N,k} \tag{G.34}$$

where $e_{N,k}$ are the nodes, or the *basic elements of the sampling.*
In principle we may even associate

$$\begin{cases} m(e_{N,k}) = \langle e_{N,k}|e_{N,k}\rangle \\ m(e_{N,k}\cap e_{N,l}) = \langle e_{N,k}|e_{N,l}\rangle \propto \delta_{k,l} \end{cases} \tag{G.35}$$

to these (and the "overlapping vs. non-overlapping basis" discussion disappears completely.) Furthermore it would make sense to let $m(e_{N,k})$ be equal for all (N,k), since these are constructionally-equivalent. Hence the proportionality constant becomes unity.

This concludes the discussion of the basic structure which leads to the definition of the measure in σ-algebras.

Now notice that the expression of *eqn.*(G.35), is often satisfied by the variables of the theory. In other words the measure is defined using these variables. In our construction, these are the X^μ. So we wish to make the relation:

$$\mathrm{d}\mu_N \sim \prod_{\mu=1}^{D} \mathrm{d}X_1^\mu \mathrm{d}X_2^\mu \cdots \mathrm{d}X_N^\mu \tag{G.36}$$

However we would like to do see in a way so that a number of criteria regarding the charge-states are satisfied. Hence let us first recall these. In this model then the step by step construction of the state is again:

$$\phi^\mu_{n,j_n^\mu}[X] = e^{ij_n^\mu X_n^\mu} \qquad\qquad n\in\alpha \tag{G.37}$$

$$\Phi^\mu_{\vec{j}^{\,\mu},\alpha}[X] = \bigotimes_{n\in\alpha} \phi^\mu_{n,j_n^\mu}[X] = \bigotimes_{n=1}^{N} e^{ij_n^\mu X_n^\mu} \tag{G.38}$$

$$\Phi_{[\vec{j}\,],\alpha}[X] = \bigotimes_{\mu=1}^{D} \Phi^\mu_{\vec{j}^{\,\mu},\alpha} = \bigotimes_{\mu=1}^{D}\bigotimes_{n=1}^{N} e^{ij_n^\mu X_n^\mu} \tag{G.39}$$

with the smeared functions:

$$\begin{aligned}\Psi^{\mu}_{\alpha}[f] &= \int \mathrm{d}j^{\mu}_1 \mathrm{d}j^{\mu}_2 \cdots \mathrm{d}j^{\mu}_N f_{j^{\mu}_1 \cdots j^{\mu}_N} e^{i j^{\mu}_1 X^{\mu}_1} \otimes \cdots \otimes e^{i j^{\mu}_n X^{\mu}_n} \\ &= \int f_{j^{\mu}_1 \cdots j^{\mu}_N} \prod_{n=1}^{N} \mathrm{d}j^{\mu}_n \phi^{\mu}_{n, j^{\mu}_n, \alpha}[X] \end{aligned} \tag{G.40}$$

$$\Psi_{\alpha}[f] = \bigotimes_{\mu=1}^{D} \Psi^{\mu}_{\alpha}[f] \tag{G.41}$$

We wish to define a measure which would therefore realize the separability of the :

$$\mathcal{H}_{kin} = \bigoplus_{\alpha} \mathcal{H}_{\alpha} = \bigoplus_{\alpha, [\overrightarrow{j}]} \mathcal{H}_{\alpha, [\overrightarrow{j}]} = \bigoplus_{\alpha, [\overrightarrow{j}], \mu} \mathcal{H}_{\alpha, \overrightarrow{j}^{\mu}} \tag{G.42}$$

The $\Phi_{[\overrightarrow{j}], \alpha}[X]$ forms a basis for the $\mathcal{H}_{\alpha, [\overrightarrow{j}]}$, and the $\Phi^{\mu}_{\overrightarrow{j}^{\mu}, \alpha}$ forms the basis for $\mathcal{H}_{\alpha, \overrightarrow{j}^{\mu}}$. This can be verified via the following relations:

$$\begin{cases} \langle \Psi_{\alpha}[f] | \Psi_{\beta}[g] \rangle \propto \delta_{\alpha, \beta} \\ \langle \Psi^{\mu}_{\alpha}[f] | \Psi^{\nu}_{\alpha}[g] \rangle \propto \delta^{\mu, \nu} \end{cases} \tag{G.43}$$

We see that the measure $\mathrm{d}\mu_N$ defined above, *eqn.*(G.36) with proper normalization factors, will give exactly these results.

Appendix H

On “Background Independence” of Field Theories. Gravity vs. Other Fields

As we have argued in this report, we know that GR is a GC, and so it does not have a true Hamiltonian and this leads to the mentioned problem with respect to its observables. However the naive question which arises is, why is this problem so specific to GR; after all shouldn't all theories of physical systems be generally covariant? Why is it that this problem seems to surface for gravity only?
Is there a principle physical concept that differentiates GR from the other field theories of physics?

In this appendix we wish to consider questions like this, from a phenomenological perspective. We will try to produce a phenomenological comparison of the theory of gravity (*gravity* for short) and the other field theories of physics, using the electromagnetic field as a particular example.

However, let us state the result which we will try to argue for, right away . This will help clarify the direction which we are headed for:

We will argue that in a field theory, if the theory is formulated relationally with respect to its own degrees of freedom - i.e. with respect to its own “background” - *it will produce only static observables.*

Therefore It is not surprising that Gravity - which in GR, is formulated based on the internal degrees of freedom of the spacetime field - fails to produce non-static observables. Alternatively, we will suggest that:

Only if a field theory is measured against another field theory - i.e. one field is used as the “background” *to the other* - dynamical observables - where the dynamics is measured against the background field - *can be produced*[1].

In this sense gravity should not be singled out as *an odd-behaving theory*, nor can it be concluded that the methods used to treat field theories cannot be used to analyze it.

Now to start our argument, let us first consider the simplest question: is it consistent to consider gravity as a *field theory*? To answer this question then let us start by assuming that gravity is a field theory and see if there are any technical problems associated with this assumption.

In the analysis of any field theory there exist two mathematical constructs which decide whether the quantization of that field theory is consistent or not. These are the *Renormalizability* and the *Proper Gauge-invariance* of the quantized field theory. Let us therefore analyze what these constructs describe for gravity.

H.1 Renormalizability

As we mentioned we will assume gravity is a field theory. One method of implementing this is by assuming the spacetime metric is a field on the spacetime. In other words:

- $g_{\alpha\beta}(x^\mu)$ is just a function of spacetime, as $A^\alpha(x^\mu)$ in E&M is a field on spacetime.

Now recall that renormalization is an artifact of perturbative field theory.
Briefly then the perturbative field quantization of gravity is comprised of the following two steps:

1. The metric, $g_{\alpha\beta}$ is expanded around a flat spacetime, $\eta_{\alpha\beta}$
2. Quantum corrections are added around this expansion

There exist many different physical descriptions which attempt to justify renormalization. Technically however, renormalization is supposed to take care of the divergences that occur in calculating the quantum corrections (step 2). Naively speaking this is done by "summing the infinite divergences in the corrections and adding them to constants of the theory which do not contribute to the analysis", [30]. However for the analysis to work, the number of corrections which are needed should be finite.

In the case of gravity however, this method fails because an infinite number of corrections are required[2]. By definition therefore, gravity is *not* renormalizable.

[1] Note that this kind of construction is fraught with conceptual (and perhaps technical) problems (see *appendix*(I)). One such complication is that it can be argued that a theory which is formulated in this fashion is *incomplete*, or only an *approximation* to the full theory. This is because, when a field is fixed as the *background*, that field is being given special properties. For example it may be assumed that the degrees of freedom of the background field are fixed and non-dynamical. In this sense, it may be said that the description is *ignoring* the background. The resulting theory, under this assumption can therefore be considered an approximation.

[2] In the calculation of the propagator; the (energy) dimensions of the coupling constants is such that, the propagator diverges infinitely, and to normalize the theory, an infinite number of terms is required.

However notice that ultimately, there is a great deal of mathematics involved in the renormalization of a given field theory. In fact it can be said that *the process of renormalization is merely the way that we know how to "fix" perturbative quantization!* The adjusting of the running-constants via renormalization does not have a good physical interpretation. Consequently, if renormalization fails, it can be argued that it is the mathematical method that fails; non-renormalizability does not imply physicality conditions.

Hence it can be said that renormalizability does not provide a good physical argument for dismissing gravity as a field theory, nor does it *physically* distinguish it from the other field theories. As such it can be said that *physically* gravity can still be treated as any other field theory of physics, even though it is not renormalizable.

Next let us consider Gauge transformability of gravity.

H.2 Gauges

We have seen that the results of the renormalization argument are that:
1). it is in fact the method of quantization that fails, and
2). that this does not produce a physical difference between gravity and other physical fields.

Now let us consider the question of gauge invariance and to ask if this feature can physically distinguish between gravity and the other field theories.
In literature sometimes the following distinction is emphasized with regards to gauge invariance of gravity vs. other fields.

It is stated, [28] [31], that there exist two different types of diffeomorphism transformations:

1. **Passive Diffeomorphism** i.e. invariance under general coordinate transformation. Any physical theory can be made to be invariant under Passive Diffeomorphism[3].
2. **Active Diffeomorphism** It is stressed that gravity is the only theory which is invariant under active diffeomorphisms. Let us therefore analyze the meaning of this gauge transformation and consider its consequences.

To do so, consider a solution of the gravity field, $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$. We can think of this solution, $g_{\mu\nu}$, in different ways:

1. A quantum correction of $\epsilon_{\mu\nu}$ to a <u>vacuum</u> $\eta_{\mu\nu}$.

   or,

[3]The idea is a generalization of the *Point Particle* case where we can include the time-parameter as a coordinate and include it as part of the Hamiltonian.

2. A classical solution of gravity where *another* field - for example a given matter-field - contributes to the nontrivial portion: $\epsilon_{\mu\nu} \sim \frac{\partial \mathcal{L}_{field}}{\partial g^{\mu\nu}}$

   or,

3. via the Active Diff. of GR, $\eta_{\mu\nu} + \epsilon_{\mu\nu}$, could be interpreted as $\eta_{\mu\nu}$ viewed by another observer - i.e. in a different coordinate system.

The first of these was related to quantization which we discussed in the renormalization section. Therefore let us consider the latter two viewpoints.

Notice that the first of these interpretations, (2.) could be realized in any field theory, and is not specific to gravity. However, the last viewpoint *seems* to be specific to gravity, because a *nontrivial* solution is not usually *equivalent* to the *trivial* solution of the theory. However in gravity this can be true in the following sense:

This viewpoint is related to the concept that *an observer cannot tell the difference between a gravitational field, or that she is being accelerated (in vacuum)* - This is the *Equivalence Principle of GR* - Now, since a G-field is related to a non-zero energy-momentum tensor, $T_{\mu\nu}$, and acceleration (in vacuum) to a special coordinate (curved) system with $T_{\mu\nu} = 0$, we see that:

Equivalence Principle:

$$T_{\mu\nu} \oplus \text{"special" coordinate system} \simeq T_{\mu\nu} \neq 0 \oplus \text{"not special" coordinate system}$$

This is the essence of active diffeomorphism, which is assumed true only for GR; *active diffeomorphism is the mathematical embodiment of the* Equivalence Principle *in GR.*

This is the argument that claims gravity is different from other field theories, based on its gauge transformation properties.

Now let us analyze this difference completely. The results of this analysis are the conjectures which were stated at the beginning of this appendix.

Under the equivalence principle we can see that a massive body (the observer) can interpret the mass of another body, either as a gravitational field (in a flat coordinate system), or as acceleration - free fall - (in a curved coordinate system). In other words, the coordinate system can be transformed to *reinterpret* the presence of matter, or the source of gravitational field.

Let us consider the analogy in electromagnetism. Lets assume there exists a charge-source, which creates an electromagnetic field, A_μ.
However notice that there exist gauge transformations which take $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$, for arbitrary scalars Λ.
However A_μ and A'_μ represent the same physical system; just as $g_{\mu\nu}$ and $\eta_{\mu\nu}$ represented the same system.

As such a charge (in analogy to the massive body above) can interpret the charge of another body either as an electromagnetic field, or as acceleration (in a special gauge). Hence the gauge transformation can be used to *reinterpret* the presence of charge, or the source of the EM field.

The only objection which we can raise to this interpretation is if we claim that coordinate transformations cannot be considered as gauge transformations.

As we can see therefore, if we "generalize" coordinate systems to include gauge choices of other fields no such difference exists. Indeed this difference is instilled because we insist on differentiating the spacetime from other fields. In other words, if we allowed for coordinate transformations to be on the same footing as other gauge transformations, then we could conclude that *with respect to themselves - i.e. with respect to their own gauge transformations - all fields are actively diffeomorphic!*

Hence if the field - any field - is considered *against its own background*, it will be actively diffeomorphic. In other words different solutions will be related via the gauges of the theory. However notice that this would mean that all solutions will be constants of motion in the sense that they simply represent the gauge transformed versions of the same physical system. As such the famous problem of time - regarded as a problem specific to gravity - exists for any field theory, *if that field theory's dynamics is described with respect to itself*; i.e. if the field theory's own internal degrees of freedom are used in relation to each other to describe dynamics, the evolution that is described will be static.

However an obvious question remains:

Why is it then that this isn't what is seen in the formulation of other field theories - at least not trivially?

The answer is that in other field theories, we do not formulate the theory based on its own background; In these theories we *assume a spacetime background <u>on which</u> the field theory exists. And by assuming that the background has special properties - it is static, or semi-classical etc. - we could then extract some meaning for dynamics of the field.*

Based on our discussion however, gravity (i.e. spacetime) is no different from other field theories. As such if, for instance we were to formulate gravity *in relation* to a given E&M field - where the E&M field could be assumed the "background" in the above sense - then we could extract a dynamical theory for gravity.

This is the result which we summarized at the beginning of the appendix. It motivates a *fully* relational description of physical systems. Such a formulation can be quite difficult to consider or define, and many concepts relating to it are not understood.

Let us therefore say a few words on the concepts and difficulties that enter into the relational formulation of physics in the next *appendix*(I).

Appendix I

On the Observables of GR

The question of the observables in GC's has been the main topic of the investigation in this report. However in this appendix we would like to consider the question of observables in the context of relational formulations which we described in the previous *appendix*(H).

As we discussed, the relational way of describing physics may in fact be the proper method of treating GR's observables. This method - while being properly facilitated classically - may have fundamental concerns quantum mechanically.

The problem can be summarized in trying to define a proper time (what it means, how it should be defined etc.) in the quantum framework.

We can explain this problem as follows:

- In a classical theory one can propose a relational method of defining a local time variable - but not globally - based on the arguments of the previous appendix.
- However in any such consideration, the "related-to" or the "background" field, must act classically for the time parameter to operate properly - for any reasonable length of the given time.
- Therefore it would seem that some part of the full theory will always be classical. This then means that the theory can never be fully realized as a quantum theory.

This, however, fundamental problem, has an interesting facet: *it arises from insisting on describing physics as a theory of evolution in a time parameter.* If one were to consider a shift from this stance, an interesting resolution emerges:

Let us recapitulate; if we consider theories to be *relational* **and** demand a *time-evolution*, we cannot make the theory fully quantum mechanical. However let us notice something interesting. The set of relational quantities, can provide dynamics, without some sort of external time; the change in the relations between these quantities defines the dynamics of the system.

In addition we find something even more interesting. Notice that if we demand that the theory be *fully* relational, *then there will always exist a degree of* uncertainty *in the theory.* Regularly we would regard this as a sign of approximating some degree of freedom of the theory. However we propose to consider the following possibility:

Instead of assuming the uncertainties to be a shortcoming of the relational formulation, let us alleviate its consequences to the status of a physical principle, in the following way.

If we define quantization in the following broad way:

Quantization is the principle which puts a limit on the determinism of physical processes,

we can then identify the inherent uncertainty in the relational formulation of theories, as the mechanism for quantization. Namely, requiring that physical theories must be relational, will demand that quantum mechanics emerge as the fundamental mathematical apparatus suitable for analyzing physical processes.

Hence we conclude:

- Physics can only be consistently formulated in a relational way.
- As an immediate consequence, there always exists a degree of uncertainty in formulating physical systems.
- This uncertainty is akin to the uncertainty so-postulated in Quantization.
- We identify these two uncertainties.

Postulate

"Quantum mechanics is the very reason that - i.e. is necessitated (emerges), from the basic requirement that fundamentally and intrinsically - physics must be formulated as relations between things. The classical limit is where the intrinsic relational description of physics could be ignored - where then one would ignore the intrinsic uncertainty (principle) or indeterminism - at which point one could adopt a time-parametrized dynamical picture."

Of course there exist many questions which need to be addressed, some of which are:

- Is this a coincidence? I.e. is there any degree of *ad-hocness* in this realization, or is it a self-contained, consistent construction?
- How does the classical limit emerge where we see two phenomena: a) *size-dependence*; what is the origin of size-dependent? b) How and why does a time-parametrized dynamical picture emerge in the classical limit?

The accumulative answer to these questions - and others like them - will determine whether the relational proposal is both consistent and fruitful, in providing an alternate and more fundamental picture of physical processes.