Essays on Dynamic Matching Markets

by

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Abstract

This dissertation studies dynamic matching and bargaining games with two-sided private information bargaining. There is a market in which a large number of heterogeneous buyers and sellers search for trading partners to trade with. Traders in the market are randomly matched pairwise. Once a buyer and a seller meet, they bargain following the random-proposer protocol: either the buyer or the seller (randomly chosen) makes a take-it-or-leave-it offer to the other party. The traders leave once they successfully trade, and the market is continuously replenished with new-born buyers and sellers who voluntarily choose to enter. We study the steady state with positive entry. There are (except the asymmetric information) two kinds of frictions: time discounting and explicit search costs. Chapter 2 addresses existence and uniqueness of equilibrium. It provides a simple necessary and sufficient condition for the existence of a nontrivial steady-state equilibrium. The equilibrium is unique if the discount rate is small relative to the search costs. This chapter also analyzes how the composition of frictions affects the patterns of equilibria. It shows that if the discount rate is small relative to the search costs, in equilibrium every meeting results in trade. If the discount rate is relatively large, some meetings do not result in trade. Chapter 3 shows that private information typically deters entry. Because of search externalities, this entry-deterring effect may be socially desirable or undesirable. We provide and interpret a simple condition under which private information improves welfare. Chapter 4 studies the convergence properties of equilibria as frictions vanish. It not only shows that, as frictions vanish, the equilibrium price range collapses to the Walrasian price and the equilibrium welfare converges to the Walrasian welfare level, but also provides the rate of convergence. Under random-proposer bargaining, welfare converges at the fastest possible rate among all bargaining mechanisms. If we assume double auction instead of random-proposer bargaining, equilibria might converge at a slower rate or even not converge at all. These results also hold under full information bargaining. It suggests that private information does not affect asymptotic efficiency, but bargaining protocol might.

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Chapter 1

Introduction

1.1 Dynamic matching and bargaining games

This dissertation contributes to the literature on dynamic matching and bargaining games (DMBG). This strand of literature stands in between two rather extreme paradigms in economic theory: the Walrasian theory and the bargaining theory.

On one extreme, the Walrasian theory assumes that trading happens in a centralized market where every agent has no market power at all (i.e. every agent takes the market price as given). The concept of Walrasian equilibrium is usually justified by telling a story with a large number of buyers and sellers, perfect information, and/or a Walrasian auctioneer enforcing the trading process.

On the other extreme, the bargaining theory assumes that a small number of agents (say a pair of buyer and seller) strategically bargain over the economic outcome (say the quantity transacted and the trading price), possibly with private information. The methodology and equilibrium concepts in the arena of bargaining theory are game-theoretic.

The literature on DMBG, which is in the middle, assumes that the market has a large number of buyers and sellers, but they cannot gather together to come up with trades. Instead, the quantities transacted and the trading prices are determined by a lot of small bargaining games, each of which is among a small coalition of agents (usually a pair of buyer and seller). The trading process is dynamic rather than a one-shot game. Also, all the agents in the market are connected through the way that the formation of bargaining coalitions varies over time, such that each agent is able to meet many other agents at different points of time. But this connection is only imperfect, because meeting other agents takes time, and it is costly due to either impatience, or a probability of death, or a fixed search cost, depending on the specific modeling. These "costs of delay" are called *frictions* of the market. And we call such a market with frictions a dynamic matching and bargaining market, or simply a *dynamic matching market*. The labor and the housing markets are often cited examples of this kind of markets.

The nature of DMBG is suitable for economic theorists to build a game-theoretic foundation for the Walrasian theory. Indeed, a main focus of this strand of literature has been on the following question: as the frictions vanish, do the (game-theoretic) equilibrium outcomes of dynamic matching and bargaining games converge to the perfectly competitive outcome predicted by Walrasian equilibrium?

Until very recently, most papers in the literature assume that the bargaining games are bilateral and under full information, i.e. a buyer and a seller bargain knowing each other's willingness-to-pay and cost. They include: Mortensen (1982), Rubinstein and Wolinsky (1985, 1990), Gale (1986a,b, 1987) and Mortensen and Wright (2002), among others.[1] Satterthwaite and Shneyerov (2007) have recently introduced two-sided private information in a dynamic matching market where sellers use auctions, and have shown that the presence of private information does not prevent convergence to perfect competition.[2]

[1] A notable exception is the unpublished manuscript Butters (1979). Other papers that have incorporated private information in some form include Wolinsky (1988), De Fraja and Sakovics (2001) and Serrano (2002).

[2] Several recent papers have explored convergence under private information in more detail: Satterthwaite and Shneyerov (2008) show convergence in the model that is a replica of Satterthwaite and Shneyerov (2007) except that it has exogenous exit rate. Lauermann (2008) shows convergence even if one side of the market has all the bargaining power, and Lauermann (2006b) shows that in that case, the welfare under private information may be higher than under full information. Atakan (2008) provides a generalization to multiple units. Lauermann (2006a) derives a set of general conditions for convergence. In addition, Hurkens and Vulkan (2006, 2007) study the role of privately observed deadlines in a matching and bargaining market.

1.2 Baseline model

This dissertation studies a dynamic matching market, modeled as a DMBG. Our baseline model is a replica of the one in Mortensen and Wright (2002), modified with two-sided private information bargaining. It is roughly described as follows.[3] There is a market in which a large number (more precisely, continuum) of risk-neutral buyers and sellers search for trading partners to trade with. Each buyer has a unit demand for a homogeneous and indivisible good; and each seller has a unit supply of the same good. The buyers and sellers are heterogeneous: different buyers have different valuations (or willingness-to-pay) and different sellers have different costs. Traders in the market are randomly matched pairwise. The mass of total matched pairs per unit time is determined by some unspecified Pissarides (2000) style matching function.[4] Once a buyer and a seller meet, they bargain following the so-called *random-proposer* protocol: either the buyer or the seller (randomly chosen) makes a take-it-or-leave-it offer to the other party. The traders leave once they successfully trade, and the market is continuously replenished with new-born buyers and sellers who voluntarily choose to enter. We assume the market is in steady state and with positive mass of traders in it. From each trader's point of view, searching for a trading partner takes time, and it is costly both because traders are impatient (parameterized by a discount rate) and they have to spend other resources like money or effort to search (parameterized by explicit search costs). Thus, the "costs of delay", or frictions, are multi-dimensional in

[3] The modeling methods for DMBG can be divided into two classes: non-steady-state models and steady-state models. In a non-steady-state model (e.g. Moreno and Wooders (2002)), the market starts with a fixed number of agents and no more agent comes in later on. As time collapses, the number of agents left in the market decreases. On the other hand, in a steady-state model, new agents keep coming in, and attention is restricted to the steady-state equilibrium. Gale (1987) shows convergence for both versions of his model. We will use the steady-state approach throughout this dissertation. As a matter of fact, our model (or Mortensen-Wright model) is also a version of search models.

[4] A Pissarides-style matching function assigns a mass of total matched pairs for each combination of buyers' and sellers' masses currently participating in the market. With such a matching function, the precise matching process need not be specified, pretty much like a production function in macro models assigns a level of output for each combination of inputs, without specifying the precise production process.

our model.

1.3 Summary

Here we briefly summarize what will be seen in the following three chapters. More detailed summaries will be presented in the introduction sections of those chapters.

Chapter 2 proves the existence of equilibrium for our baseline model, characterizes the equilibrium patterns, and develops a bunch of results that are useful for the subsequent two chapters. Chapter 3 analyzes the impacts of the private information in bargaining. Chapter 4 derives convergence properties of the equilibrium outcome as frictions vanish. Roughly speaking, Chapter 2 and Chapter 3 are concerned with markets with significant frictions; while Chapter 4 is concerned with markets with small frictions. More precisely, in Chapter 2 and Chapter 3 the level and composition of frictions are considered to be fixed; while frictions are considered to be vanishing in Chapter 4.[5]

A main interest throughout this dissertation is how the private information in bargaining games shapes the equilibrium outcome of a dynamic matching market. A recent paper by Satterthwaite and Shneyerov (2007) shows in a similar but different model that equilibrium outcome converges towards perfect competition as frictions vanish, even when traders hold private information. However their model might have no natural full information counterpart.[6] Furthermore, even when private information in bargaining does not prevent convergence, would it make the convergence any slower? This question has not been addressed in the literature. On the other hand, if the private information has no impact

[5]This dichotomy is not strict. The core results of Chapter 4 apply to any level of frictions.

[6]The main differences between the model of Satterthwaite and Shneyerov (2007) and ours are that: in their model, time is discrete; every buyer is randomly matched with one seller in each period, so that a seller might be matched with several buyers, one buyer, or no buyer; and sellers sell their goods through first-price auctions without committed reserve price. Thus their model assumes a specific multi-lateral matching and bargaining process. In contrast, our model assumes bilateral matching and bargaining (which is common in the literature of DMBG and search models), with a general matching function and a general distribution of bargaining power.

in the limit, one might wonder what impacts it has in the "out-of-the-limit" case. Is the private information always bad in the social point of view? The analyses in this dissertation shed some light on all these issues.

1.4 Other related literature

The market games we analyze in this dissertation can also be counted as a search model. Indeed, our modeling choices include random matching, a Pissarides-style matching function, and steady state. All these are common features of classic labor search models (although recent developments allow directed search, on-the-job search, etc.).

The literature on search models of labor market, surveyed for example in Mortensen and Pissarides (1999) and more recently Rogerson, Shimer, and Wright (2005), is large. While the literature on DMBG is micro-oriented, the literature on labor search models is macro-oriented. The latter studies topics like equilibrium unemployment, wage dispersions, and the constrained efficiency when the market is plagued by search frictions.[7]

Most if not all of the labor search literature neglects the private information in bargaining.[8] Besides, many labor search models simply assume homogeneous workers and homogeneous firms, so that the information structure at the bargaining stages is irrelevant. This dissertation, in contrast, emphasizes the role of information structure at the bargaining stages. In this regard, this dissertation contributes to the search theory, by allowing private information.

Another strand of literature related to this dissertation is the literature on static double auction.

As we mentioned before, DMBG stands in between the Walrasian theory and bargaining theory, and hence theorists build the foundation of Walrasian equilibrium on it. The

[7] Search theory has also been applied to monetary models and marriage models. But the models in these areas are not as relevant to this dissertation as the labor search models.

[8] In the labor search literature, when matching is random (rather than directed), typically the (generalized) Nash bargaining solution is assumed.

literature on static double auction is very similar to DMBG in this aspect. A number of papers on static double auction ask whether the equilibrium outcomes converge to the Walrasian outcome as the number of traders n gets large. More importantly, this literature also looks at the rate of convergence. In particular, Rustichini, Satterthwaite, and Williams (1994) show robust convergence of double-auction equilibria in the symmetric class at the rate $O(1/n)$ for the bid/ask strategies and the rate $O(1/n^2)$ for the ex-ante traders' welfare.[9] Moreover, this literature also has asymptotic efficiency results: the double auction converges at the rate that is fastest among all incentive-compatible and individually rational mechanisms (Satterthwaite and Williams (2002); Tatur (2005)). Cripps and Swinkels (2005) substantially enrich the model by allowing correlation among bidders' valuations, and show convergence at the rate $O(1/n^{2-\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small.[10]

In contrast, as far as I know, the rate of convergence has not been addressed in the literature of DMBG. Comparing with the rather sophisticated literature on static double auction, there is a gap in the DMBG literature.[11] Our rate of convergence analysis in Chapter 4 takes a step to fill this gap.

[9] Other related papers include Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Satterthwaite (1989), and Williams (1991).

[10] Reny and Perry (2006) allow interdependent values and show that it is almost efficient and almost fully aggregates information as $n \to \infty$, although the rate of convergence is not addressed.

[11] Of course, the natures of convergence in these two strands of literature are different. For the static double auction, we let the number of traders get large. For DMBG, we let the level of frictions get small. However, they are analogous to each other, because both of them reflect increasing intensity of competition, as we will discuss in Chapter 4. After all, both of them lead to the convergence towards the perfect competition outcome.

Chapter 2

Dynamic Matching and Two-sided Private Information Bargaining

2.1 Introduction

This chapter starts our formal analysis of dynamic matching and bargaining markets.[12] We will study a replica of Mortensen and Wright (2002) model, modified with two-sided private information bargaining. There is a market in which continua of risk-neutral buyers and sellers search for trading partners to trade with. Each buyer has a unit demand for a homogeneous and indivisible good; and each seller has a unit supply of the same good. The buyers and sellers are heterogeneous: different buyers have different valuations $v \in [0,1]$ and different sellers have different costs $c \in [0,1]$.

Our model has features of a steady-state search-theoretic model, where the matching between buyers and sellers is pairwise, random, and described by a Pissarides-style matching function $M(B,S)$ that gives the matching rate as a function of the masses of buyers B and sellers S currently participating in the market. In steady state and from the standpoint of a particular trader, matchings come up according to a Poisson process. The Poisson arrival rate of being matched is $\alpha_B \equiv M(B,S)/B$ for a buyer, or $\alpha_S \equiv M(B,S)/S$ for a seller. Since we assume that M exhibits constant returns to scale, the arrival rates α_B and α_S only depend on the buyer-seller ratio $\zeta \equiv B/S$.

Our model also has the feature of two-sided asymmetric information bilateral bargaining.

[12] The chapter significantly includes the materials in my manuscript "Bilateral Matching and Bargaining with Private Information", which is joint with my dissertation co-supervisor Artyom Shneyerov.

The bargaining game between a pair of buyer and seller follows what we call the random-proposer protocol: with probability $\beta_B \in (0,1)$ the buyer makes a take-it-or-leave-it price offer to the seller; and with probability $\beta_S \equiv 1-\beta_B$ the seller makes a take-it-or-leave-it price offer to the buyer. The traders leave once they successfully trade, and the market is continuously replenished with new-born buyers and sellers who voluntarily choose to enter.

Another important feature of our model is that our notion of frictions is multi-dimensional. There are two kinds of search frictions: searching for a trading partner takes time, parameterized by an instantaneous discount rate $r>0$, and also takes other resources (e.g. money, effort), parameterized by explicit search costs $\kappa_B>0$ for buyers and $\kappa_S>0$ for sellers, per unit time.

The main purpose of this chapter is to prove the existence of equilibrium, and to understand the equilibrium patterns and properties, under different combinations of frictions. This chapter will also be the foundation of the analyses of the next two chapters.

Our fundamental result (Theorem 3) is: at least one nontrivial (i.e. with positive mass of traders participating) steady-state equilibrium exists if and only if

$$\frac{\kappa_B}{\alpha_B(\zeta_0)}+\frac{\kappa_S}{\alpha_S(\zeta_0)}<1, \tag{2.1}$$

where $\zeta_0 \equiv \beta_B\kappa_S/\beta_S\kappa_B$, and $\alpha_B(\zeta_0)$ (resp. $\alpha_S(\zeta_0)$) is a buyer's (resp. seller's) Poisson arrival rate of being matched when the (steady-state) buyer-seller ratio is ζ_0.

An uninteresting trivial equilibrium, in which nobody participates, always exists. Indeed, if searching for trading partners is very costly, only the trivial equilibrium can exist. Roughly speaking, our fundamental existence result says that some nontrivial steady-state equilibrium also exists if the search costs κ_B and κ_S are moderate. To get more sense of the above necessary and sufficient condition (2.1) for the existence of some nontrivial steady-state equilibrium, let us restrict attention to the no-discounting case, i.e. $r\to 0$. In this case, equilibrium analysis becomes very tractable and it is easy to show that the equilibrium buyer-seller ratio must be ζ_0, which simply reflects the ratio of buyers' and sellers' bargaining powers, and the ratio of buyers' and sellers' per-unit-time search costs. Then for an unmatched buyer to bring himself matched, the expected total search costs is

$\kappa_B/\alpha_B(\zeta_0)$. Similarly, for an unmatched seller the expected total search costs is $\kappa_S/\alpha_S(\zeta_0)$. On the other hand, since we normalize the supports of buyers' valuations and sellers' costs to be $[0,1]$, the maximum gain of trade that a pair of buyer and seller can realize is 1. Therefore, condition (2.1) simply says that the maximum gain a buyer-seller pair can realize is greater than the expected total search costs they incur to get matched. While this existence condition is rather natural in the no-discounting case, our result shows that the existence condition does not change at all in the general case.[13]

We distinguish two kinds of nontrivial steady-state equilibrium: "full-trade equilibrium" (i.e. in which every meeting results in trade) and "non-full-trade equilibrium". Given that some nontrivial steady-state equilibrium exists, we make predictions on the equilibrium pattern (full-trade vs non-full-trade): there are two critical levels of discount rate r^* and $\underline{r}$, with $0 < \underline{r} < r^*$, such that a full-trade equilibrium exists if and only if $r \leq r^*$ (Theorem 1); and only a full-trade equilibrium, but no non-full-trade one, exists if $r \leq \underline{r}$ (Theorem 2).[14] The formulas for r^* and $\underline{r}$ are explicitly derived, in terms of parameters including (κ_B, κ_S). In particular, both r^* and $\underline{r}$ are increasing in (κ_B, κ_S); and both r^* and $\underline{r}$ tend to 0 as $(\kappa_B, \kappa_S) \to \mathbf{0}$. These results suggest that: in (nontrivial steady-state) equilibrium whether every meeting results in a trade mainly depends not on the level of frictions, but the relativity of the two kinds of frictions. More concretely, if r is small relative to (κ_B, κ_S), then in equilibrium every meeting results in a trade; if on the other hand r is large relative to (κ_B, κ_S), in equilibrium some meetings lead to bargaining breakdowns.

Satterthwaite and Shneyerov (2007) also provide an existence theorem for a dynamic matching market with two-sided asymmetric information. Their existence theorem (SS existence theorem), in our language, is: there exists a full-trade equilibrium if κ_B, κ_S and $r/\min\{\kappa_B, \kappa_S\}$ are sufficiently small. Their model involves one-to-many matchings

[13] If $r > 0$, then both the left-hand side and the right-hand side of (2.1) are subject to discounting. Since the two effects cancel out, the generality of (2.1) is possible.

[14] Since there can be at most one full-trade equilibrium, $r \leq \underline{r}$ is also a sufficient condition for the uniqueness of nontrivial steady-state equilibrium.

and auctions, unlike ours.[15] By switching to a bilateral matching and bargaining model like Mortensen and Wright (2002), we are able to prove much sharper results than theirs. Compared with our results, SS existence theorem has several limitations. First, how small κ_B, κ_S and $r/\min\{\kappa_B, \kappa_S\}$ have to be is unknown. Second, when r is large relative to (κ_B, κ_S), it is unknown whether some nontrivial steady-state equilibrium exists. Third, SS existence theorem makes no prediction on the equilibrium pattern for any friction profile: small r relative to (κ_B, κ_S) does not imply full trade; and large r relative to (κ_B, κ_S) does not imply non-full trade. Our results in this chapter do not have these limitations.

The aforementioned second limitation of SS existence theorem also brings about a limitation of their convergence theorem. Their convergence theorem (SS convergence theorem), again in our language, is: along any sequence of nontrivial steady-state equilibria associated with a sequence of (r, κ_B, κ_S) that tends to $\mathbf{0}$ proportionally, the set of transaction prices and the welfare of every agent must converge to their counterparts under perfect competition. The limitation of SS convergence theorem is: it does not preclude the possibility that, even when we let (r, κ_B, κ_S) tend to $\mathbf{0}$ proportionally, nontrivial steady-state equilibrium keeps absent. In contrast, our convergence results presented in Chapter 4, with the foundation of our existence results, do not have this limitation.

The rest of this chapter proceeds as follows. Section 2.2 introduces the model. Section 2.3 defines the equilibrium concept. Section 2.4 analyzes the basic equilibrium properties. Section 2.5 studies full-trade equilibria and the condition under which this kind of equilibria exist. Section 2.6 proves that the equilibrium is unique when the discount rate is small. Section 2.7 presents and proves the "general existence theorem". Section 2.8 concludes. Additional details for the general existence proof is in Appendix A.

2.2 The model

The agents in our model are potential buyers and sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller has a unit supply. All traders

[15] For more details of Satterthwaite and Shneyerov (2007) model, see footnote 6.

are risk neutral. Potential buyers are heterogeneous in their valuations (or types) v of the good. Potential sellers are also heterogeneous in their costs (or types) c of providing the good. For simplicity, we assume $v, c \in [0, 1]$. Time is continuous and infinite horizon. The instantaneous discount rate is $r > 0$. The details of the model are described as follows:

- Entry: Potential buyers and sellers are continuously born at rates b and s respectively. We normalize the aggregate born rate to be 1, i.e. $b + s = 1$. The type of a new-born buyer is drawn i.i.d. from the c.d.f. $F(v)$ and the type of a new-born seller is drawn i.i.d. from the c.d.f. $G(c)$. Each trader's type will not change once it is drawn. Entry (or participation, or being active) is voluntary. Each potential trader decides whether to enter the market once he is born. Those who do not enter will get zero payoff. Those who enter must incur the search cost continuously at the rate κ_B for buyers and κ_S for sellers, until they leave the market.

- Matching: Active buyers and sellers are randomly and continuously matched pairwise at a flow rate given by a Pissarides (2000) style matching function $M(B, S)$, where B and S are the masses of active buyers and active sellers currently in the market.

- Bargaining: Once a pair of buyer and seller are matched, they bargain without observing each other's type. The bargaining protocol is what we call *random-proposer bargaining*: with probability $\beta_B \in (0, 1)$, the buyer makes a take-it-or-leave-it price offer to the seller, then the seller chooses either to accept or reject. And with probability $\beta_S \equiv 1 - \beta_B$ the seller proposes and the buyer responds. (The "bargaining weights" β_B and β_S can be interpreted as the buyer's and seller's relative bargaining powers.) We also assume the market is anonymous, so that bargainers do not know their partners' market history, e.g. how long they have been in the market, what they proposed previously, and what offers they rejected previously.

- If a type v buyer and a type c seller trade at a price p, then they leave the market with (current value) payoff $v - p$ and $p - c$ respectively. If the bargaining between the matched pair breaks down, both traders can either stay in the market waiting for another match (and incur the search costs) as if they were never matched, or simply exit and never come back.

We make the following assumptions on the primitives of our model.

Assumption 1 (distributions of inflow types) *The cumulative distributions $F(v)$ and $G(c)$ of inflow types have densities $f(v)$ and $g(c)$ on $(0,1)$, bounded away from 0 and ∞:*

$$0 < \underline{f} \leq f(v) \leq \bar{f} < \infty,$$

$$0 < \underline{g} \leq g(c) \leq \bar{g} < \infty.$$

Assumption 2 (matching function) *The matching function M is continuous on $\mathbb{R}_+^2$, nondecreasing in each argument, exhibits constant returns to scale (i.e. homogeneous of degree one), and satisfies*

$$M(B,S) = 0 \text{ if } B = 0 \text{ or } S = 0.$$

Given the current mass of buyers $B > 0$ and the mass of sellers $S > 0$, trading opportunities for a buyer come at the Poisson arrival rate $M(B,S)/B$.[16] Similarly, trading opportunities for a seller come at the Poisson arrival rate $M(B,S)/S$.

It is more convenient to work with a normalized matching function. Let

$$\zeta \equiv B/S$$

be the ratio of buyers to sellers (or market tightness), and define

$$m(\zeta) \equiv M(\zeta, 1).$$

Since the matching technology is assumed to exhibit constant returns to scale, it is easy to see that $m(\zeta)$ is also equal to $M(B,S)/S$, which is a seller's Poisson arrival rate of being

[16] That is, $M(B,S)/B$ is the probability that a buyer is matched over a short time period of length dt divided by the length dt.

matched. Similarly, $m(\zeta)/\zeta$ is equal to $M(B,S)/B$, a buyer's Poisson arrival rate of being matched. We denote these two arrival rates as $\alpha_B(\zeta)$ and $\alpha_S(\zeta)$:

$$\alpha_B(\zeta) \equiv \frac{m(\zeta)}{\zeta}, \quad \alpha_S(\zeta) \equiv m(\zeta).$$

Assumption 2 implies that

1. $\alpha_B(\zeta)$ is continuous and nonincreasing;

2. $\alpha_S(\zeta)$ is continuous and nondecreasing; and

3. $\alpha_B(\infty) = \alpha_S(0) = 0$.

2.3 Nontrivial steady-state equilibria

Throughout this dissertation we will restrict attention to *steady-state* equilibria, i.e. ones in which the market distribution of active traders and the agents' strategies are time-invariant.

Like other DMBG in the literature, our model always has an uninteresting perfect Bayesian equilibrium, in which no potential trader enters. Indeed, if no potential trader enters, not to enter is optimal to every potential trader. Throughout this dissertation we will only consider *nontrivial* equilibria, i.e. ones in which positive entry occurs (or equivalently, positive trade occurs, or the steady-state market mass of active traders is positive). We thus call our equilibrium notion *nontrivial steady-state equilibrium*, which will be formally defined in a moment.

Let $W_B, W_S : [0,1] \to \mathbb{R}_+$ be the value functions for buyers and sellers: $W_B(v)$ is the continuation payoff of a type v buyer whenever he has not traded and is unmatched; and $W_S(c)$ is the continuation payoff of a type c seller whenever she has not traded and is unmatched. Let $N_B, N_S : [0,1] \to \mathbb{R}_+$ be the (stock) market distribution functions: $N_B(v)$ is the mass of buyers in the market with valuations less than or equal to v; and $N_S(c)$ is the mass of sellers with costs less than or equal to c. (In this notation, $B = N_B(1)$ and $S = N_S(1)$.) Let $\chi_B, \chi_S : [0,1] \to \{0,1\}$ be the entry strategies: buyers with valuation v enter if and only if $\chi_B(v) = 1$; sellers with cost c enter if and only if $\chi_S(c) = 1$. Also let $p_B, p_S : [0,1] \to [0,1]$ be the proposing strategies: buyers with valuation v propose the take-it-or-leave-it price offer $p_B(v)$; sellers with cost c propose $p_S(c)$.

Sequential optimality requires that the value functions in steady state satisfy the following Bellman equations. For a type v buyer,

$$\begin{aligned} rW_B(v) &= \max_{\chi\in\{0,1\}} \chi \cdot \{\alpha_B(\zeta)[\beta_B \pi_B(v) \\ &\quad + \beta_S \int_{\{c: v-p_S(c)\geq W_B(v)\}} (v - p_S(c) - W_B(v)) \frac{dN_S(c)}{S}] - \kappa_B\} \end{aligned} \tag{2.2}$$

where $\pi_B(v)$ is the buyer's capital gain when he becomes a proposer:

$$\pi_B(v) \equiv \max_{p\in[0,1]} \left\{ \int_{\{c: p-c\geq W_S(c)\}} (v - p - W_B(v)) \frac{dN_S(c)}{S} \right\}. \tag{2.3}$$

The buyer's equilibrium entry strategy $\chi_B(v)$ must be an optimal value of χ in (2.2), and his equilibrium proposing strategy $p_B(v)$ must be an optimal value of p in (2.3). The intuition is that, contingent on entry, a buyer's flow value of search $rW_B(v)$ is equal to the expected capital gain due to matching a partner, net of the flow search cost. Specifically, the buyer's proposed price $p_B(v)$ is accepted by the seller if her trade surplus is weakly greater than the value of search, i.e. if $p_B(v) - c \geq W_S(c)$. The seller's proposed price $p_S(c)$ is accepted by the buyer if his trade surplus is weakly greater than his value of search, i.e. if $v - p_S(c) \geq W_B(v)$. When the buyer trades, the capital gain is his trade surplus minus the value of search.

The Bellman equation for the sellers has a similar form: for a type c seller,

$$\begin{aligned} rW_S(c) &= \max_{\chi\in\{0,1\}} \chi \cdot \{\alpha_S(\zeta)[\beta_S \pi_S(c) + \\ &\quad \beta_B \int_{\{v: p_B(v)-c\geq W_S(c)\}} (p_B(v) - c - W_S(c)) \frac{dN_B(v)}{B}] - \kappa_S\} \end{aligned} \tag{2.4}$$

where

$$\pi_S(c) \equiv \max_{p\in[0,1]} \left\{ \int_{\{v: v-p\geq W_B(v)\}} (p - c - W_S(c)) \frac{dN_B(v)}{B} \right\}. \tag{2.5}$$

The seller's equilibrium entry strategy $\chi_S(c)$ must be an optimal value of χ in (2.4), and her equilibrium proposing strategy $p_S(c)$ must be an optimal value of p in (2.5).

It is convenient to define the trading probabilities in a given meeting, $q_B(v)$ for buyers and $q_S(c)$ for sellers:

$$\begin{aligned} q_B(v) &\equiv \beta_B \int\limits_{\{c: p_B(v)-c\geq W_S(c)\}} \frac{dN_S(c)}{S} + \beta_S \int\limits_{\{c: v-p_S(c)\geq W_B(v)\}} \frac{dN_S(c)}{S}, \\ q_S(c) &\equiv \beta_S \int\limits_{\{v: v-p_S(c)\geq W_B(v)\}} \frac{dN_B(v)}{B} + \beta_B \int\limits_{\{v: p_B(v)-c\geq W_S(c)\}} \frac{dN_B(v)}{B}. \end{aligned}$$

In steady state, the rate of inflow of the traders of each type is equal to the rate of the outflow due to trading:[17]

$$b\chi_B(v)\,dF(v) = \alpha_B(\zeta) q_B(v)\,dN_B(v), \tag{2.6}$$

$$s\chi_S(c)\,dG(c) = \alpha_S(\zeta) q_S(c)\,dN_S(c). \tag{2.7}$$

We now formally define nontrivial steady-state equilibrium.[18]

Definition 1 *A tuple* $(W_B, W_S, \chi_B, \chi_S, p_B, p_S, N_B, N_S)$ *is a nontrivial steady-state equilibrium if* $B \equiv N_B(1) > 0$, $S \equiv N_S(1) > 0$, *equations* (2.2), (2.4), (2.6) *and* (2.7) *hold, and* χ_B, p_B, χ_S, p_S *solve the optimization problems in* (2.2), (2.3), (2.4) *and* (2.5) *respectively.*

2.4 Basic equilibrium properties

Our characterization of equilibrium patterns begins with showing that the slopes of equilibrium value functions $W_B(v)$ and $W_S(c)$ are the corresponding "ultimate probabilities of trade", which can be defined as the present value of one dollar to be received at the time of next successful trade. Since every active trader must recover their search costs, these

[17] Exiting without trade never occurs in steady-state equilibrium.

[18] We implicitly assume that traders use symmetric pure strategies. But this is essentially without loss of generality and merely for simplicity of exposition. We will come back to this point in the conclusion chapter.

ultimate probabilities of trade must be strictly positive on the active regions, i.e. the supports of N_B and N_S. Therefore the active regions must be intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ for some $\underline{v}$ and $\bar{c}$. Furthermore, we show that W_B and W_S are convex, which implies that trading probabilities q_B and q_S are monotonic.[19]

Lemma 1 *In any nontrivial steady-state equilibrium, there are marginal entering types* $\underline{v}, \bar{c} \in (0,1)$ *such that the supports of* N_B *and* N_S *are* $[\underline{v}, 1]$ *and* $[0, \bar{c}]$ *respectively. Marginal entrants (i.e. type* $\underline{v}$ *buyers and type* $\bar{c}$ *sellers) are indifferent between entering or not, while the entry preferences of all others are strict.* $\{v : \chi_B(v) = 1\}$ *is either* $[\underline{v}, 1]$ *or* $(\underline{v}, 1]$. $\{c : \chi_S(c) = 1\}$ *is either* $[0, \bar{c}]$ *or* $[0, \bar{c})$. W_B *is absolutely continuous, convex, nondecreasing on* $[0,1]$*, strictly increasing on* $[\underline{v}, 1]$*, with* $W_B(\underline{v}) = 0$*; whenever differentiable,*

$$W_B'(v) = \chi_B(v) \frac{\alpha_B(\zeta) q_B(v)}{r + \alpha_B(\zeta) q_B(v)}. \tag{2.8}$$

W_S *is absolutely continuous, convex, nonincreasing on* $[0,1]$*, strictly decreasing on* $[0, \bar{c}]$*, with* $W_S(\bar{c}) = 0$*; whenever differentiable,*

$$W_S'(c) = -\chi_S(c) \frac{\alpha_S(\zeta) q_S(c)}{r + \alpha_S(\zeta) q_S(c)}. \tag{2.9}$$

The trading probability q_B *is strictly positive and nondecreasing on* $[\underline{v}, 1]$*, while* q_S *is strictly positive and nonincreasing on* $[0, \bar{c}]$.

Proof. We prove the results for buyers only. We use an argument from mechanism design. For any $v \in [0,1]$, define

$$\begin{aligned} t_B(v) \equiv \; & \beta_B \int_{\{c: p_B(v) - c \geq W_S(c)\}} p_B(v) \frac{dN_S(c)}{S} \\ & + \beta_S \int_{\{c: v - p_S(c) \geq W_B(v)\}} p_S(c) \frac{dN_S(c)}{S}. \end{aligned}$$

The buyers' Bellman equation (2.2) implies for any $v, \hat{v} \in [0,1]$ and any $\chi \in \{0,1\}$,

$$rW_B(v) \geq \chi \cdot \{\alpha_B [q_B(\hat{v}) v - t_B(\hat{v}) - q_B(\hat{v}) W_B(v)] - \kappa_B\}$$

[19] Lemma 1 is generally true for any bargaining protocol, as long as the bargainers' types are private information. We therefore provide a proof that can easily be generalized.

or equivalently

$$W_B(v) \geq \chi \cdot u_B(v, \hat{v})$$

where

$$u_B(v, \hat{v}) \equiv \frac{\alpha_B [q_B(\hat{v}) v - t_B(\hat{v})] - \kappa_B}{r + \alpha_B q_B(\hat{v})}.$$

And the inequality becomes equality if $\hat{v} = v$ and $\chi = \chi_B(v)$. Let $U_B(v) \equiv \max_{\hat{v} \in [0,1]} u_B(v, \hat{v})$. We then have $W_B(v) = \chi_B(v) u_B(v, v) = \chi_B(v) U_B(v) = \max\{U_B(v), 0\}$. For any $\hat{v}$, $u_B(v, \hat{v})$ is affine and nondecreasing in v. Milgrom and Segal (2002) Envelope Theorem implies $U_B(v)$ is absolutely continuous, convex, nondecreasing, and with slope $\alpha_B q_B(v) / (r + \alpha_B q_B(v))$ whenever differentiable. The same properties are inherited by $W_B(v)$, except that its slope becomes $\chi_B(v) \alpha_B q_B(v) / (r + \alpha_B q_B(v))$.

Obviously $U_B(0) < 0$. Let $\underline{v} \equiv \sup\{v \in [0,1] : U_B(v) < 0\}$. By continuity of U_B, we have $\underline{v} > 0$ and $U_B(\underline{v}) \leq 0$. But $U_B(\underline{v}) < 0$ is impossible in nontrivial equilibrium because it implies $\chi_B(v) = 0 \ \forall v \in [0,1]$ and hence $B = 0$. Thus $U_B(\underline{v}) = W_B(\underline{v}) = 0$. By monotonicity of U_B, for all $v < \underline{v}$, we have $U_B(v) < 0$ and hence $\chi_B(v) = W_B(v) = 0$. Moreover, $q_B(v) > 0$ for all $v \geq \underline{v}$. It is because for all $v \geq \underline{v}$, the fact $U_B(v) \geq 0$ implies $\alpha_B q_B(v) \geq \kappa_B > 0$. It furthermore implies $U_B'(\underline{v}+) \geq \alpha_B q_B(\underline{v}+) / (r + \alpha_B q_B(\underline{v}+)) > 0$. Thus for all $v > \underline{v}$, we have $U_B(v) > 0$ and hence $\chi_B(v) = 1$ and $W_B(v) = U_B(v)$. From steady-state equation (2.6), $[\underline{v}, 1]$ is the support of N_B. Since the inflow distribution F does not have atom point, neither does N_B. Hence $B > 0$ implies $\underline{v} < 1$. Finally, the convexity of U_B implies that q_B is nondecreasing on $[\underline{v}, 1]$. ■

We call the $\underline{v}$ and $\bar{c}$ in Lemma 1 the buyers' and sellers' *marginal entering types*, and call the traders with such types *marginal entrants*. Since the flow and stock masses of marginal entrants (who are indifferent between entering or not) are zero anyway, we will without loss of generality assume throughout they enter, i.e. $\chi_B(\underline{v}) = \chi_S(\bar{c}) = 1$.

Before providing further equilibrium properties, let us make a note on the traders' bargaining strategies (i.e. proposing and responding strategies) by introducing a pair of important notions. Define

$$\rho_B(v) \equiv v - W_B(v), \tag{2.10}$$

$$\rho_S(c) \equiv c + W_S(c). \tag{2.11}$$

We call $\rho_B(v)$ type v buyers' *dynamic valuation*, and $\rho_S(c)$ type c sellers' *dynamic cost*. Both of them are called *dynamic types*. The reason is that, as far as we are concerned with bargaining between a buyer and a seller in our dynamic model, $\rho_B(v)$ and $\rho_S(c)$ play the same roles as v and c do in a static bargaining game. Indeed, as captured in our equilibrium definition, a type v buyer is willing to accept a price offer p if and only if $p \leq \rho_B(v)$; and a type c seller is willing to accept a price offer p if and only if $p \geq \rho_S(c)$. Thus the dynamic types fully characterize the responding strategies played in equilibrium. Furthermore, the proposing problems in (2.3) and (2.5) are nothing more than static take-it-or-leave-it problems with types replaced by dynamic types.

Lemma 1 implies that ρ_B and ρ_S are absolutely continuous and increasing:

$$\rho_B'(v) = \frac{r}{r + \alpha_B(\zeta)\, q_B(v)} > 0 \quad \text{a.e. } v \in [\underline{v}, 1] \tag{2.12}$$

$$\rho_S'(c) = \frac{r}{r + \alpha_S(\zeta)\, q_S(c)} > 0 \quad \text{a.e. } c \in [0, \bar{c}]. \tag{2.13}$$

Since $W_B(\underline{v}) = W_S(\bar{c}) = 0$, the marginal entering types are equal to the corresponding dynamic types:

$$\begin{aligned} \rho_S(\bar{c}) &= \bar{c}, \\ \rho_B(\underline{v}) &= \underline{v}. \end{aligned}$$

Thus the buyers' lowest and highest reservation prices are $\underline{v}$ and $\rho_B(1)$. The sellers' lowest and highest reservation prices are $\rho_S(0)$ and $\bar{c}$. We will see in the next lemma that the proposing strategies p_B and p_S are also monotonic on the active intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$. Thus the lowest and highest price offers by buyers are $p_B(\underline{v})$ and $p_B(1)$. The lowest and highest price offers by sellers are $p_S(0)$ and $p_S(\bar{c})$.

Lemma 2 *In any nontrivial steady-state equilibrium,*

(a) *for all* $v \in [\underline{v}, 1]$*,* $\rho_B(v) > p_B(v) \in [\rho_S(0), \bar{c}]$*; for all* $c \in [0, \bar{c}]$*,* $\rho_S(c) < p_S(c) \in [\underline{v}, \rho_B(1)]$*;*

(b) *the proposing strategies* $p_B(v)$ *and* $p_S(c)$ *are nondecreasing on* $[\underline{v}, 1]$ *and* $[0, \bar{c}]$ *respectively;*

(c) $\alpha_B(\zeta)\beta_B\pi_B(\underline{v}) = \kappa_B$ *and* $\alpha_S(\zeta)\beta_S\pi_S(\bar{c}) = \kappa_S$*.*

Proof. *Step 1:* Suppose, by way of contradiction, $p_B(v) > \bar{c}$ for some $v \in [\underline{v}, 1]$. Then $p_B(v)$ is accepted by any active seller (because $\rho_S(c)$ is increasing in c). A type v buyer can lower his offer without losing acceptance probability. But then $p_B(v)$ does not solve the proposing problem in (2.3). Therefore $p_B(v) \leq \bar{c}$ for all $v \in [\underline{v}, 1]$. Similarly $p_S(c) \geq \underline{v}$ for all $c \in [0, \bar{c}]$.

Step 2: The buyers with type $\underline{v}$ cannot get positive bargaining surplus when he is a responder, i.e. the second term inside the square bracket of (2.2), evaluated at $v = \underline{v}$, is 0. It is because, from step 1, $\underline{v} - W_B(\underline{v}) = \underline{v}$ is no higher than $p_S(c)$ proposed by any active seller. Then, since $W_B(\underline{v}) = 0$ from Lemma 1, the Bellman equation (2.2) evaluated at $v = \underline{v}$ implies $\alpha_B(\zeta)\beta_B\pi_B(\underline{v}) - \kappa_B = 0$. It follows that $\pi_B(\underline{v}) > 0$ and hence $\pi_B(v) > 0$ for all $v \in [\underline{v}, 1]$ (because any buyer can choose $p = p_B(\underline{v})$ in his proposing problem in (2.3)). Similarly, we can prove $\alpha_S(\zeta)\beta_S\pi_S(\bar{c}) - \kappa_S = 0$ and $\pi_S(c) > 0$ for all $c \in [0, \bar{c}]$.

Step 3: Fix any $v \in [\underline{v}, 1]$. From $\pi_B(v) > 0$ given by step 2, we have $v - p_B(v) > W_B(v)$ (or equivalently $\rho_B(v) > p_B(v)$) and $p_B(v) - c \geq W_S(c)$ for some c. The last result is equivalent to $p_B(v) \geq \rho_S(0)$ because $\rho_S(c)$ is increasing in c. Similarly we can prove for all $c \in [0, \bar{c}]$, $\rho_S(c) < p_S(c) \leq \rho_B(1)$.

Step 4: Let $\Gamma_S(p) \equiv \int_{\{c:p-c\geq W_S(c)\}} \frac{dN_S(c)}{S}$. Obviously Γ_S is nondecreasing. Then the buyers' proposing problem in (2.3) can be written as $\pi_B(v) = \max_{p\in[0,1]}[v - W_B(v) - p]\Gamma_S(p)$. Pick any $v_1, v_2 \in [\underline{v}, 1]$. Let $p_1 \equiv p_B(v_1)$ and $p_2 \equiv p_B(v_2)$. Revealed preference implies

$$[v_1 - W_B(v_1) - p_1]\Gamma_S(p_1) \geq [v_1 - W_B(v_1) - p_2]\Gamma_S(p_2) \tag{2.14}$$

and

$$[v_2 - W_B(v_2) - p_2]\Gamma_S(p_2) \geq [v_2 - W_B(v_2) - p_1]\Gamma_S(p_1).$$

Sum these two inequalities and then simplify. We obtain

$$[(v_2 - W_B(v_2)) - (v_1 - W_B(v_1))] \cdot [\Gamma_S(p_2) - \Gamma_S(p_1)] \geq 0.$$

Suppose, by way of contradiction, $v_2 > v_1$ and $p_2 < p_1$. Then the above inequality implies $\Gamma_S(p_2) \geq \Gamma_S(p_1)$ and the monotonicity of Γ_S implies $\Gamma_S(p_2) \leq \Gamma_S(p_1)$. We thus have $\Gamma_S(p_2) = \Gamma_S(p_1) > 0$, where the last inequality is from step 2. Substitute back into (2.14), we have $p_2 \geq p_1$, a contradiction. ■

The intuition is, in equilibrium, the marginal entrants do not get bargaining surplus in responding stages (the worst types in the market do not have information rent) so that these marginal entrants must earn positive surpluses in proposing stages, otherwise they cannot recover their search costs. Since even the marginal entrants earn positive proposing surplus, all entrants do as well. Then any buyer's offer must be lower than his dynamic valuation and within the support of sellers' reservation prices. Of course, a symmetric argument can be made by switching the roles of buyers and sellers. We thus have part (a) of Lemma 2. Part (b), the monotonicity of proposing strategies, is due to standard revealed-preference argument. Part (c), the marginal type equations, simply says that, for the marginal entrants to be indifferent between entering or not, their expected gain from searching in the market, net of search cost, must be zero. (Recall that marginal entrants makes positive bargaining surplus only when they are proposer.)

In equilibrium, it could be the case that $\underline{v} \leq \bar{c}$, or $\underline{v} > \bar{c}$. If the former one is the case, we say it is an *equilibrium with overlapping supports*. If the latter one is the case, we say it is an *equilibrium with separated supports*. Figure 2.1 and Figure 2.2 visualize the pattern of proposing and responding strategies of a possible equilibrium of each kind.

Before closing this section, we compare the equilibrium price range with the Walrasian price. Define Walrasian price p^* as the price that clears the flow demand and flow supply:[20]

$$b[1 - F(p^*)] = sG(p^*).$$

[20] This is the appropriate concept of market-clearing price in the steady-state context, as first pointed out by Gale (1987).

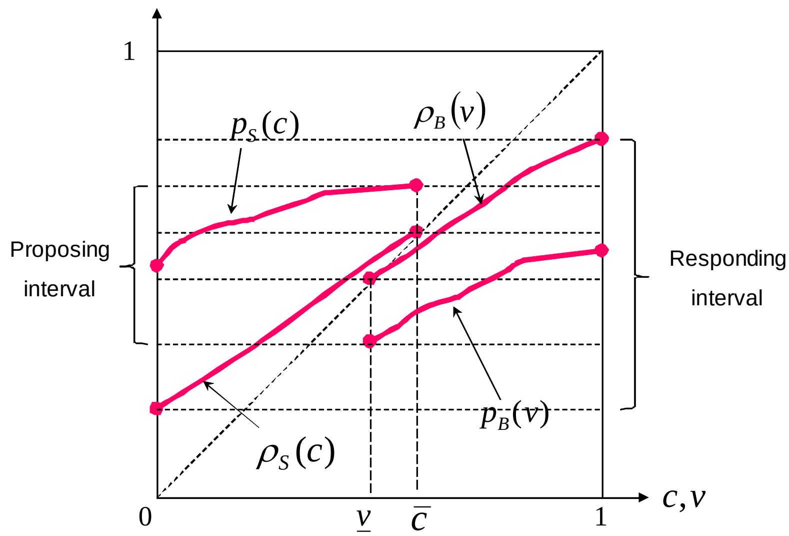

Figure 2.1: Proposing and responding strategies in an equilibrium with overlapping supports (which must be non-full-trade)

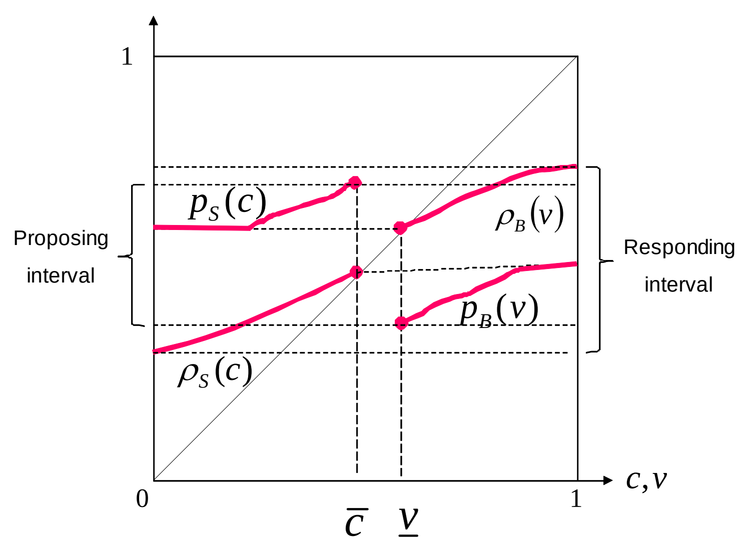

Figure 2.2: Proposing and responding strategies in a non-full-trade equilibrium with separated supports

Also define the *responding interval* as $[\rho_S(0), \rho_B(1)]$, and the *proposing interval* as $[p_B(\underline{v}), p_S(\bar{c})]$. We see from Lemma 2(a) that $[p_B(\underline{v}), p_S(\bar{c})] \subset [\rho_S(0), \rho_B(1)]$. In words, the proposing interval is contained in the responding interval.

Since buyers and sellers always leave the market in pairs, the entry flows must also be balanced in steady state, i.e. $b[1-F(\underline{v})] = sG(\bar{c})$.[21] Then it is clear that the marginal entering types $\underline{v}$ and $\bar{c}$ must be on different sides of the Walrasian price p^*. Then from Lemma 2(a), it is not hard to prove that Walrasian price p^* must fall within the proposing interval $[p_B(\underline{v}), p_S(\bar{c})]$.

Lemma 3 *In any nontrivial steady-state equilibrium,* $p^* \in [p_B(\underline{v}), p_S(\bar{c})] \subset [\rho_S(0), \rho_B(1)]$.

Proof. The second inclusion is straight implication of Lemma 2(a). To see the first inclusion, simply notice that

$$p_B(\underline{v}) \leq \min\{\bar{c}, \underline{v}\} \leq p^* \leq \max\{\bar{c}, \underline{v}\} \leq p_S(\bar{c}).$$

The first and last inequalities are from Lemma 2(a). The other two inequalities in the middle are due to the facts that $b[1-F(\underline{v})] = sG(\bar{c})$ and $b[1-F(p^*)] = sG(p^*)$. ■

2.5 Full-trade equilibria

Although the previous section provides a series of results that characterize equilibrium patterns, in general there is no analytic solution for a nontrivial steady-state equilibrium. However, we have more to say about the qualitative properties of equilibria.

There are two qualitatively different possibilities that could happen in equilibrium. First, it may happen that in an equilibrium every meeting results in a trade. In contrast, it can be that not every meeting results in a trade. We call these two types of equilibria *full-trade equilibria* and *non-full-trade equilibria* respectively.

[21] It can be formally derived from (2.6) and (2.7).

Definition 2 *A nontrivial steady-state equilibrium is called a full-trade equilibrium if in this equilibrium every meeting results in a trade. A nontrivial steady-state equilibrium is called a non-full-trade equilibrium if it is not a full-trade equilibrium.*

The above definition is sensible for any bargaining mechanism, and for either private information or full information bargaining. In the current context, Lemma 2(a) implies that full-trade equilibria must have the following properties: (i) the supports for active buyers' types and active sellers' types are separate, i.e. $\underline{v} > \bar{c}$; (ii) the lowest buyers' offer $p_B(\underline{v})$ is exactly at the level acceptable to all active sellers, i.e. $p_B(\underline{v}) = \bar{c}$; and (iii) the highest sellers' offer $p_S(\bar{c})$ is exactly at the level acceptable to all active buyers, i.e. $p_S(\bar{c}) = \underline{v}$. It is easy to see that the converse is also true. (Clearly, a full-trade equilibrium must be with separated supports; or equivalently an equilibrium with overlapping supports must be non-full-trade.) Thus we could alternatively define a full-trade equilibrium to be a nontrivial steady-state equilibrium with $p_B(\underline{v}) = \bar{c}$ and $p_S(\bar{c}) = \underline{v}$.

Non-full-trade equilibria are illustrated in Figure 2.1 and Figure 2.2 in the previous section. Figure 2.3 illustrates the qualitative features of strategies played in a full-trade equilibrium. In particular, the proposing strategies must be flat and the dynamic type functions must be linear.[22]

We are interested in full-trade equilibria for several reasons. First, our uniqueness and existence results are closely related to full-trade equilibria. Second, our discussions on bargaining efficiency and the effect of information structure on entry in the next chapter will also be intimately related to full-trade equilibria. Third, full-trade equilibria admit a very simple characterization, which we present now.[23]

In full-trade equilibria (if any), the marginal type equations in Lemma 2(c) take the

[22] If a full-trade equilibrium and a non-full-trade equilibrium coexist (whether they can coexist is an open question), our results do not imply the full-trade equilibrium Pareto dominates the non-full-trade one. Indeed, the non-full-trade equilibrium could have more entry (i.e. lower $\underline{v}$ and higher $\bar{c}$) than the full-trade one, so that the marginal entrants strictly prefer the non-full-trade equilibrium.

[23] As a matter of fact, the analysis of Mortensen and Wright (2002) is based only on full-trade equilibria (although they do not use this term).

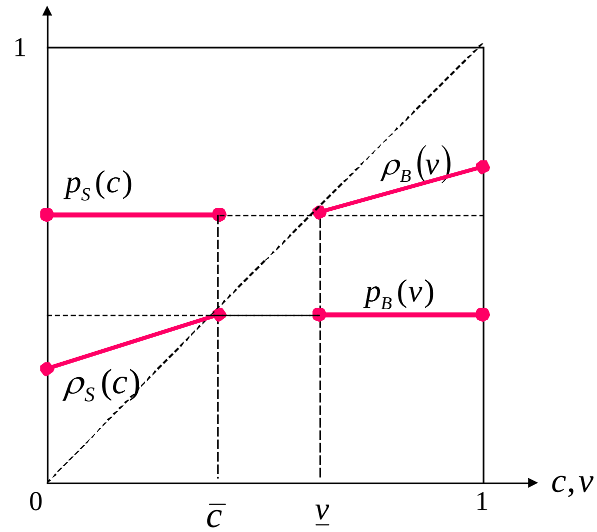

Figure 2.3: Proposing and responding strategies in a full-trade equilibrium

form

$$\alpha_B(\zeta)\beta_B(\underline{v}-\bar{c}) = \kappa_B, \tag{2.15}$$

$$\alpha_S(\zeta)\beta_S(\underline{v}-\bar{c}) = \kappa_S. \tag{2.16}$$

Noticing that $\alpha_S(\zeta)/\alpha_B(\zeta) = \zeta$, (2.15) and (2.16) can be easily solved for ζ and $\underline{v}-\bar{c}$:

$$\zeta = \frac{\beta_B}{\beta_S}\frac{\kappa_S}{\kappa_B} \equiv \zeta_0, \tag{2.17}$$

$$\underline{v}-\bar{c} = K(\zeta_0), \tag{2.18}$$

where

$$K(\zeta) \equiv \frac{\kappa_B}{\alpha_B(\zeta)} + \frac{\kappa_S}{\alpha_S(\zeta)} \quad \forall\zeta. \tag{2.19}$$

In steady state, the inflow of active buyers must equal the inflow of active sellers:

$$b[1-F(\underline{v})] = sG(\bar{c}). \tag{2.20}$$

Since $\underline{v}-\bar{c}$ is determined from (2.18), $\underline{v}$ and $\bar{c}$ are uniquely pinned down by (2.20). It is clear that equations (2.18) and (2.20) have a solution for $\underline{v} < 1$ and $\bar{c} > 0$ if and only if

$K(\zeta_0) < 1$. Let us suppose $K(\zeta_0) < 1$ and denote such a solution for $(\underline{v}, \bar{c})$ by $(\underline{v}_0, \bar{c}_0)$. That is to say, a full-trade equilibrium, if exists, must have its buyer-seller ratio and marginal entering types given by $(\zeta_0, \underline{v}_0, \bar{c}_0)$. Other equilibrium objects are also easily obtained. In particular,

$$\chi_B(v) = \begin{cases} 1 & \text{if } v \geq \underline{v}_0 \\ 0 & \text{otherwise} \end{cases}, \quad \chi_S(c) = \begin{cases} 1 & \text{if } c \leq \bar{c}_0 \\ 0 & \text{otherwise} \end{cases},$$

$$W_B(v) = \chi_B(v) \frac{\alpha_B(\zeta_0)}{r + \alpha_B(\zeta_0)} (v - \underline{v}_0)$$

$$W_S(c) = \chi_S(c) \frac{\alpha_S(\zeta_0)}{r + \alpha_S(\zeta_0)} (\bar{c}_0 - c)$$

$$N_B(v) = \chi_B(v) \frac{b\left[F(v) - F(\underline{v}_0)\right]}{\alpha_B(\zeta_0)}$$

$$N_S(c) = [1 - \chi_S(c)] \frac{sG(\bar{c}_0)}{\alpha_S(\zeta_0)} + \chi_S(c) \frac{sG(c)}{\alpha_S(\zeta_0)}$$

$$p_B(v) = \bar{c}_0 \quad \forall v \geq \underline{v}_0$$

$$p_S(c) = \underline{v}_0 \quad \forall c \leq \bar{c}_0.$$

Throughout this dissertation, we identify an equilibrium with another one if they differs only in the proposing strategies of non-entrants and entry strategies of marginal entrants. Under this convention, there is at most one full-trade equilibrium. We call the above possible full-trade equilibrium the *full-trade equilibrium candidate*.

We have seen that a unique full-trade equilibrium candidate exists if and only if $K(\zeta_0) < 1$.[24] The function $K(\zeta)$, especially the value $K(\zeta_0)$, will play an important role in our analysis. It can be interpreted as the expected search costs incurred by a pair of buyer and seller when the buyer-seller ratio is ζ and there is no discounting. In the full-trade equilibrium, this expected search cost, $K(\zeta_0)$, is equal to the *entry gap* $\underline{v}_0 - \bar{c}_0$, as shown in (2.18). This value $K(\zeta_0)$ has yet an alternative interpretation. The following simple lemma, which will be used many times in our proofs, shows that $K(\zeta_0)$ can be interpreted

[24] Recall that for expositional simplicity we have assumed that the types are distributed on $[0, 1]$. If the support is $[a_1, a_2]$, then the condition would read $K(\zeta_0) < a_2 - a_1$.

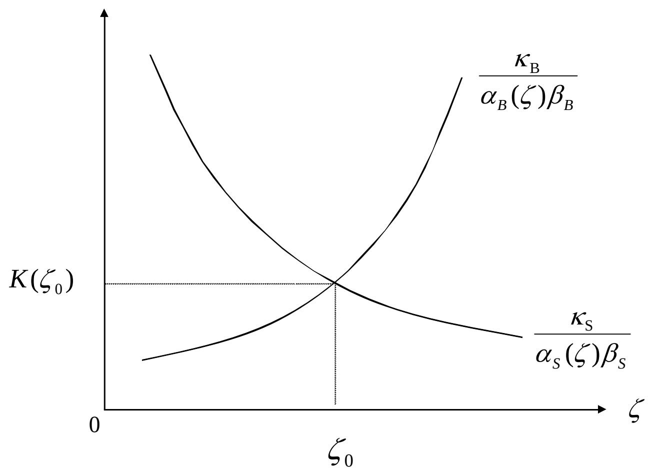

Figure 2.4: Interpretation of ζ_0 and $K(\zeta_0)$

either as a maximin or a minimax value of adjusted accumulated search costs until the next meeting.

Lemma 4 *For any matching function satisfying Assumption 2, we have*

$$\begin{aligned} K(\zeta_0) &= \max_{\zeta>0}\min\left\{\frac{\kappa_B}{\alpha_B(\zeta)\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}\right\} \\ &= \min_{\zeta>0}\max\left\{\frac{\kappa_B}{\alpha_B(\zeta)\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}\right\} \\ &= \frac{\kappa_B}{\alpha_B(\zeta_0)\beta_B} = \frac{\kappa_S}{\alpha_S(\zeta_0)\beta_S}. \end{aligned}$$

Proof. Consult Figure 2.4. Note that $\alpha_B(\zeta)$ is a nonincreasing function, while $\alpha_S(\zeta)$ is a nondecreasing function. The maximin and minimax values are realized at the intersection of the curves

$$\frac{\kappa_B}{\alpha_B(\zeta)\beta_B} = \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}$$

which occurs if and only if $\zeta = \zeta_0$. ■

Corollary 1 *For any matching function satisfying Assumption 2, the following statements are equivalent. (i)* $K(\zeta_0) < 1$*. (ii) For some* $\zeta > 0$*, we have* $\alpha_B(\zeta)\beta_B > \kappa_B$ *and*

$\alpha_S(\zeta)\beta_S > \kappa_S$. *(iii) For all* $\zeta > 0$, *we have* $\alpha_B(\zeta)\beta_B > \kappa_B$ *or* $\alpha_S(\zeta)\beta_S > \kappa_S$. *(iv)* $\alpha_B(\zeta_0)\beta_B > \kappa_B$. *(v)* $\alpha_S(\zeta_0)\beta_S > \kappa_S$.[25]

Even if $K(\zeta_0) < 1$, so that a full-trade equilibrium candidate exists, this candidate may not constitute an equilibrium, since buyers may have an incentive to bid lower than $\bar{c}_0$, and similarly sellers may have an incentive to bid above $\underline{v}_0$. Nevertheless, Theorem 1, which is the main result of this section, provides a necessary and sufficient condition under which such deviations are unprofitable and hence a full-trade equilibrium exists. Before stating this main result, we need to introduce the so-called *virtual types* of buyers and sellers. The buyers' and sellers' virtual type functions are respectively defined as:

$$\begin{aligned} J_B(v) &\equiv v - \frac{1-F(v)}{f(v)}, \\ J_S(c) &\equiv c + \frac{G(c)}{g(c)}. \end{aligned}$$

It is well-known that the virtual type functions are nondecreasing for most usual probability distributions. We therefore take the monotonicity of J_B and J_S as a regularity condition. And this condition guarantees that only the first-order conditions for the proposers' problems in (2.3) and (2.5) are sufficient for optimal proposing.

Now we are ready to state the main theorem of this section.

Theorem 1 (Existence of full-trade equilibrium) *Assume the regularity condition that the virtual type functions* J_B *and* J_S *are nondecreasing. Then a (unique) full-trade equilibrium exists if and only if*

(i) $K(\zeta_0) < 1$ *where* $\zeta_0 \equiv \beta_B\kappa_S/\beta_S\kappa_B$, *and*

(ii) $r \leq r^*$ *where* r^* *is given by:*

$$r^* \equiv \min\left\{\frac{\kappa_B/\beta_B}{\max\{\bar{c}_0 - J_B(\underline{v}_0), 0\}}, \frac{\kappa_S/\beta_S}{\max\{J_S(\bar{c}_0) - \underline{v}_0, 0\}}\right\}. \tag{2.21}$$

(If both denominators are 0, *there is no upper bound so a full-trade equilibrium exists for all* r. *In this case we define* $r^* = \infty$.*)*

[25] We will see in Section 2.7 that all these statements are equivalent to the existence of some nontrivial steady-state equilibrium.

Proof. We have already seen that a unique full-trade equilibrium candidate described before exists if and only if $K(\zeta_0) < 1$. For proving existence of full-trade equilibrium, it suffices to verify that this candidate is really an equilibrium. Almost all equilibrium conditions are satisfied by construction, except that we need to verify $p_B(v) = \bar{c}_0\ \forall v \geq \underline{v}_0$ and $p_S(c) = \underline{v}_0\ \forall c \leq \bar{c}_0$ are buyers' and sellers' optimal proposing strategies. For notational simplicity, we omit the subscript "0". We focus on sellers' proposing problem in (2.5), which, according to our construction of the equilibrium candidate, can be rewritten as $\max_{p\in[0,1]} \hat{\pi}_S(c,p)$, where

$$\hat{\pi}_S(c,p) = \left(p - \frac{rc + \alpha_S \bar{c}}{r + \alpha_S}\right) \int_{\underline{v}}^{1} I\left[p \leq \frac{rv + \alpha_B \underline{v}}{r + \alpha_B}\right] \frac{dF(v)}{1 - F(\underline{v})}$$

where $I[\cdot]$ is 1 if the condition inside the bracket holds, and is 0 otherwise.

Notice that $\partial\hat{\pi}_S(c,p)/\partial p = 1$ if $p < \underline{v}$, so that any $p < \underline{v}$ is not optimal; moreover any $p \geq \frac{r + \alpha_B \underline{v}}{r + \alpha_B}$ is also not optimal because it implies $\hat{\pi}_S(c,p) = 0$. The partial derivative of $\hat{\pi}_S$ w.r.t. p for any $p \in \left[\underline{v}, \frac{r+\alpha_B \underline{v}}{r+\alpha_B}\right]$ (it is right-hand derivative at the left boundary; and left-hand derivative at the right boundary) is:

$$\frac{\partial\hat{\pi}_S(c,p)}{\partial p} = -\frac{f\left(\frac{(r+\alpha_B)p - \alpha_B \underline{v}}{r}\right)}{1 - F(\underline{v})} \frac{r + \alpha_B}{r} \left\{ \frac{rJ_B\left(\frac{(r+\alpha_B)p - \alpha_B \underline{v}}{r}\right) + \alpha_B \underline{v}}{r + \alpha_B} - \frac{rc + \alpha_S \bar{c}}{r + \alpha_S} \right\}.$$

For $p = \underline{v}$ being optimal for all $c \leq \bar{c}$, a necessary condition is that $\partial\hat{\pi}_S(\bar{c},p)/\partial p \leq 0$ at $p = \underline{v}$, because otherwise a type $\bar{c}$ seller would deviate upward. This is also a sufficient condition because (i) $\partial\hat{\pi}_S(c,p)/\partial p$ is increasing in c, so that $\partial\hat{\pi}_S(\bar{c},p)/\partial p \leq 0$ implies $\partial\hat{\pi}_S(c,p)/\partial p \leq 0\ \forall c \leq \bar{c}$; and (ii) due to the monotonicity of J_B, $\partial\hat{\pi}_S(c,p)/\partial p \leq 0$ at $p = \underline{v}$ implies $\partial\hat{\pi}_S(c,p)/\partial p \leq 0$ at any $p \in \left[\underline{v}, \frac{r+\alpha_B \underline{v}}{r+\alpha_B}\right]$. That is, we only need to verify

$$\frac{rJ_B(\underline{v}) + \alpha_B \underline{v}}{r + \alpha_B} - \bar{c} \geq 0.$$

Similarly considering the buyers' proposing problem, we would see that $p_B(v) = \bar{c}$ $\forall v \geq \underline{v}$ is optimal if and only if

$$\underline{v} - \frac{rJ_S(\bar{c}) + \alpha_S \bar{c}}{r + \alpha_S} \geq 0.$$

Thus full-trade equilibrium exists if and only if both of these two inequalities hold, or equivalently,

$$r \leq \min\left\{\frac{\alpha_B(\zeta)(\underline{v}-\bar{c})}{\max\{\bar{c}-J_B(\underline{v}),0\}}, \frac{\alpha_S(\zeta)(\underline{v}-\bar{c})}{\max\{J_S(\bar{c})-\underline{v},0\}}\right\}.$$

Finally, applying the full-trade equilibrium marginal type equations (2.15) and (2.16), we obtain the upper bound r^* in (2.21). ■

Corollary 2 *Suppose the virtual type functions J_B and J_S are nondecreasing. Then,*

(a) *In the region where $r^* < \infty$, if κ_B and κ_S increase, then r^* increases, and vice versa.*

(b) *Given any $r > 0$, there is a $\bar{\kappa} > 0$ such that full-trade equilibrium does not exist whenever $\kappa_B, \kappa_S < \bar{\kappa}$.*

(c) *Given any $r > 0$, a full-trade equilibrium exists when (κ_B, κ_S) is such that $K(\zeta_0)$ is less than but sufficiently close to 1.*

(d) *Given any (κ_B, κ_S) such that $K(\zeta_0) < 1$, a full-trade equilibrium exists when r is sufficiently close to 0.*

Proof. Consult Figure 2.4. The curve $\frac{\kappa_B}{\alpha_B(\zeta)\beta_B}$ shifts up when κ_B goes up. The curve $\frac{\kappa_S}{\alpha_S(\zeta)\beta_S}$ shifts up when κ_S goes up. Both of the two curves pointwise converge to 0 on $\{\zeta : \zeta > 0\}$ as $(\kappa_B, \kappa_S) \to \mathbf{0}$. Obviously, $K(\zeta_0)$, as the height of the intersection, increases as κ_B and κ_S increase, and vice versa. An increase in $K(\zeta_0)$ in turn implies that $\underline{v}_0$ rises and $\bar{c}_0$ drops, and and vice versa. Also, as $(\kappa_B, \kappa_S) \to \mathbf{0}$, we have $K(\zeta_0) \to 0$, $\underline{v}_0 \to p^*$ and $\bar{c}_0 \to p^*$. As $K(\zeta_0) \to 1$ from below, we have $\underline{v}_0 \to 1$ and $\bar{c}_0 \to 0$.

From monotonicity of J_B and J_S, $\bar{c}_0 - J_B(\underline{v}_0)$ and $J_S(\bar{c}_0) - \underline{v}_0$ drop as κ_B and κ_S increase. Then (a) follows.

To prove (b), it suffices to prove $r^* \to 0$ as $(\kappa_B, \kappa_S) \to \mathbf{0}$. Notice that $\bar{c}_0 - J_B(\underline{v}_0) \geq \bar{c}_0 - \underline{v}_0 + (1-\underline{v}_0)\underline{f}/\bar{f}$ and $J_S(\bar{c}_0) - \underline{v}_0 \geq \bar{c}_0 + \bar{c}_0\underline{g}/\bar{g} - \underline{v}_0$. Therefore, as $\underline{v}_0 \to p^*$ and $\bar{c}_0 \to p^*$, $\liminf[\bar{c}_0 - J_B(\underline{v}_0)] \geq (1-p^*)\underline{f}/\bar{f} > 0$ and $\liminf[J_S(\bar{c}_0) - \underline{v}_0] \geq p^*\underline{g}/\bar{g} > 0$. As a result, $r^* \to 0$ as $(\kappa_B, \kappa_S) \to \mathbf{0}$, and (b) follows.

To prove (c), notice that $\bar{c}_0 - J_B(\underline{v}_0) \leq \bar{c}_0 - \underline{v}_0 + (1-\underline{v}_0)\bar{f}/\underline{f}$ and $J_S(\bar{c}_0) - \underline{v}_0 \leq \bar{c}_0 + \bar{c}_0\bar{g}/\underline{g} - \underline{v}_0$. Thus both of them are negative when $\underline{v}_0$ and $\bar{c}_0$ are sufficiently close to 1 and 0 respectively. But $\underline{v}_0$ and $\bar{c}_0$ can be made arbitrarily close to 1 and 0 respectively by letting $K(\zeta_0)$ be less than but close enough to 1. Hence $r^* = \infty$ if $K(\zeta_0)$ is less than but close to 1. Then (c) follows.

(d) is simply from $r^* > 0$ for any $\kappa_B, \kappa_S > 0$ such that $K(\zeta_0) < 1$. ■

Remark 1 *We need the monotonicities of virtual type functions J_B and J_S only in the proof of Theorem 1 and the proof of Corollary 2. Moreover, even if we do not assume these monotonicities, $r \leq r^*$ is still a necessary condition for the existence of full-trade equilibrium.*

2.6 Uniqueness of equilibrium

In this section we will show that a full-trade equilibrium is a unique equilibrium for small r. That is to say, there cannot be a non-full-trade equilibrium when r is small. The proof of this will utilize the following lemma.

Lemma 5 *In any nontrivial steady-state equilibrium, we have*

$$1 > \rho_B(1) - \rho_S(0) > K(\zeta_0), \tag{2.22}$$

$$\underline{v} - \bar{c} \leq K(\zeta_0). \tag{2.23}$$

Proof. Pick any nontrivial steady-state equilibrium. Lemma 1 implies $W_B(1) > 0$ and $W_S(0) > 0$. The first inequality in (2.22), which is equivalent to $W_B(1) + W_S(0) > 0$, follows. From the definition of π_B and Lemma 2(a), we have $\pi_B(\underline{v}) \leq \underline{v} - p_B(\underline{v}) < \rho_B(1) - \rho_S(0)$. Then Lemma 2(c) implies

$$\alpha_B(\zeta)\,\beta_B(\rho_B(1) - \rho_S(0)) > \kappa_B.$$

We can similarly prove

$$\alpha_S(\zeta)\,\beta_S(\rho_B(1) - \rho_S(0)) > \kappa_S.$$

It follows that

$$\rho_B(1) - \rho_S(0) > \max\left\{\frac{\kappa_B}{\alpha_B(\zeta)\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}\right\} \geq K(\zeta_0).$$

The last inequality is from Lemma 4. It proves the second inequality in (2.22).

To prove (2.23), notice that $\pi_B(\underline{v}) \geq \underline{v} - \bar{c}$ because a type $\underline{v}$ buyer can always propose $p = \bar{c}$ in his proposing problem (2.3) and this offer would be accepted with probability 1. Thus Lemma 2(c) implies

$$\alpha_B(\zeta)\beta_B(\underline{v} - \bar{c}) \leq \kappa_B.$$

Similarly, we have $\pi_S(\bar{c}) \geq \underline{v} - \bar{c}$, so that

$$\alpha_S(\zeta)\beta_S(\underline{v} - \bar{c}) \leq \kappa_S.$$

It follows that

$$\underline{v} - \bar{c} \leq \min\left\{\frac{\kappa_B}{\alpha_B(\zeta)\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}\right\} \leq K(\zeta_0). \tag{2.24}$$

The last inequality is again from Lemma 4. ■

Lemma 5 is of interest on its own. Firstly, (2.22) implies an (nontrivial steady-state) equilibrium could exist only if $K(\zeta_0) < 1$. Moreover, (2.22) can be written as $W_B(1) + W_S(0) \in (0, 1 - K(\zeta_0))$. It means in equilibrium the joint lifetime payoff of the best buyer-seller pair (i.e. type 1 buyer and type 0 seller) must be positive but smaller than their gains from trade, net of the expected accumulated search costs evaluated at $\zeta = \zeta_0$. Roughly speaking, (2.23) says that in equilibrium the entry gap $\underline{v} - \bar{c}$ cannot be too large relative to the search costs, otherwise extramarginal traders would have strict incentives to enter.

In order to prove non-full-trade equilibria cannot exist when r is close to 0, recall that a non-full-trade equilibrium could be either with overlapping supports (i.e. $\underline{v} \leq \bar{c}$), or with separated supports (i.e. $\underline{v} > \bar{c}$). Now we shall claim neither exists for small r. The following lemma implies that an equilibrium with overlapping supports cannot exist whenever r is lower than the search costs κ_B and κ_S.

Lemma 6 *In any nontrivial steady-state equilibrium,*

$$\frac{\underline{v} - \bar{c}}{\rho_B(1) - \rho_S(0)} > \frac{\kappa - r}{r + \kappa}$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Proof. Pick any nontrivial steady-state equilibrium.

Step 1. Since q_B is nondecreasing (from Lemma 1),

$$q_B(v) \geq q_B(\underline{v}) \geq \beta_B \int_{\{c: p_B(\underline{v}) - c \geq W_S(c)\}} \frac{dN_S(c)}{S}$$

for any $v \geq \underline{v}$.

Step 2. From Lemma 2(a) we have $\underline{v} > p_B(\underline{v}) \geq \rho_S(0)$.

Step 3. Combining the previous two steps and Lemma 2(c), we obtain $\alpha_B q_B(v)(\underline{v} - \rho_S(0)) \geq \kappa_B\ \forall v \geq \underline{v}$. From Lemma 1, we have

$$\rho_B'(v) = 1 - W_B'(v) = \frac{r}{r + \alpha_B q_B(v)} \leq \frac{r}{r + \kappa_B/(\underline{v} - \rho_S(0))}.$$

Hence

$$\rho_B(1) - \underline{v} = \int_{\underline{v}}^{1} \rho_B'(v)dv \leq \frac{r}{r + \kappa_B/(\underline{v} - \rho_S(0))} < \frac{r}{\kappa_B/(\underline{v} - \rho_S(0))},$$

$$\frac{\rho_B(1) - \underline{v}}{\underline{v} - \rho_S(0)} < \frac{r}{\kappa_B},$$

$$\begin{aligned} \frac{\rho_B(1) - \underline{v}}{\rho_B(1) - \rho_S(0)} &= \frac{(\rho_B(1) - \underline{v})/(\underline{v} - \rho_S(0))}{1 + (\rho_B(1) - \underline{v})/(\underline{v} - \rho_S(0))} \\ &< \frac{r/\kappa_B}{1 + (r/\kappa_B)} = \frac{r}{r + \kappa_B} \leq \frac{r}{r + \kappa}, \end{aligned}$$

where $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Step 4. Repeat the previous three steps with the roles of buyers and sellers interchanged, we can also get

$$\frac{\bar{c} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} < \frac{r}{r + \kappa}.$$

Sum these two inequalities up and rearrange terms. Then we get the desired inequality. ■

Corollary 3 *If $r \leq \kappa \equiv \min\{\kappa_B, \kappa_S\}$, then a non-full-trade equilibrium with overlapping supports cannot exist (i.e. any nontrivial steady-state equilibrium has $\underline{v} > \bar{c}$).*

Now we turn to the proof that a non-full-trade equilibrium with separated supports cannot exist. It is based on the following idea. As $r \to 0$, the dynamic types ρ_B and ρ_S, as functions of v and c, get flat, so that the support of dynamic types narrows down to a singleton. Consequently, a marginal entering trader who makes an interior offer in the support of his partner's dynamic types gains little relative to proposing at the boundary of the support (i.e. seller proposing $\underline{v}$ and buyer proposing $\bar{c}$), but risks a substantially reduced probability of trading. We are able to show that bidding the endpoint of the support is the best response, so for small r it must be that $p_B(\underline{v}) = \bar{c}$ and $p_S(\bar{c}) = \underline{v}$. This leads to the following uniqueness result.

Theorem 2 (Uniqueness of equilibrium) *There is at most one nontrivial steady-state equilibrium, which is full-trade, if* $r \leq \underline{r}$ *where* $\underline{r}$ *is given by:*

$$\underline{r} \equiv \kappa \cdot \frac{K(\zeta_0)\,\underline{\phi}}{1 + K(\zeta_0)\,\underline{\phi}}, \tag{2.25}$$

where

$$\kappa \equiv \min\{\kappa_B, \kappa_S\}, \qquad \underline{\phi} \equiv \frac{\min\{b\underline{f}, s\underline{g}\}}{M(\bar{B}, \bar{S})},$$

$$\bar{B} \equiv \frac{b}{\kappa_B}, \qquad \bar{S} \equiv \frac{s}{\kappa_S}.$$

Proof. We have seen in the text that there cannot be more than one full-trade equilibrium. It suffices to prove that, if r is small, then in any (non-trivial steady-state) equilibrium, $p_B(\underline{v}) = \bar{c}$ and $p_S(\bar{c}) = \underline{v}$. We will only consider $r < \kappa \equiv \min\{\kappa_B, \kappa_S\}$, which through Lemma 3 implies $\underline{v} > \bar{c}$ in equilibrium. Now pick any equilibrium and focus on sellers. To prove $p_S(\bar{c}) = \underline{v}$, it suffices to prove that $p = \underline{v}$ is the only maximizer of $\max_{p\in[0,1]} \hat{\pi}_S(\bar{c}, p)$, where

$$\hat{\pi}_S(\bar{c}, p) = (p - \bar{c}) \int_{\underline{v}}^{1} I[p \leq \rho_B(v)] \frac{dN_B(v)}{B}$$

where $I[\cdot]$ is 1 if the condition inside the bracket holds, and is 0 otherwise.

Since $\hat{\pi}_S(\bar{c}, p)$ is absolutely continuous in p, it is differentiable in p almost everywhere. Notice that $\partial\hat{\pi}_S(\bar{c}, p)/\partial p$ is 1 if $p < \underline{v}$, so that any $p < \underline{v}$ is never optimal. Proposing

$p > \rho_B(1)$, which implies $\hat{\pi}_S(\bar{c}, p) = 0$, is also never optimal. If $\underline{v} < p < \rho_B(1)$, whenever differentiable, we have

$$\frac{\partial \pi_S(\bar{c}, p)}{\partial p} = 1 - \Gamma_B(p) - (p - \bar{c})\gamma_B(p), \tag{2.26}$$

where $\Gamma_B(p) \equiv \int_{\underline{v}}^1 I[p \le \rho_B(v)] \frac{dN_B(v)}{B}$ and $\gamma_B(p) \equiv \Gamma'_B(p)$. Define $\phi_B(x) \equiv N'_B(x)/B$. The function ρ_B is strictly increasing (from Lemma 1 and $r > 0$), so that its inverse function ρ_B^{-1} is well-defined on the range of ρ_B, and is also strictly increasing. Then

$$\gamma_B(p) = \frac{\phi_B\left(\rho_B^{-1}(p)\right)}{\rho'_B\left(\rho_B^{-1}(p)\right)} \;\forall p \in [\underline{v}, \rho_B(1)].$$

We want to show that, when $\underline{v} < p < \rho_B(1)$ the r.h.s. of (2.26) must be negative for all sufficiently small $r > 0$. Firstly, from $r < \kappa$, Lemma 6 and 5, we obtain

$$p - \bar{c} > \underline{v} - \bar{c} \ge K(\zeta_0)\left(\frac{\kappa - r}{r + \kappa}\right) > 0. \tag{2.27}$$

Moreover, for all $v \ge \underline{v}$, we have $\rho'_B(v) = \frac{r}{r + \alpha_B q_B(v)}$ (from Lemma 1) and $\alpha_B q_B(v) \ge \kappa_B$ (from Lemma 2(c)). Thus $\rho'_B(v) \le r/(r+\kappa)$, and hence

$$\gamma_B(p) \ge \left(1 + \frac{\kappa}{r}\right)\phi_B\left(\rho_B^{-1}(p)\right). \tag{2.28}$$

We now derive a lower bound on the market probability density of buyers' types ϕ_B. From the steady-state equation (2.6), we can deduce

$$\phi_B(v) = \frac{bf(v)}{M(B,S)\,q_B(v)} \ge \frac{b\underline{f}}{M(B,S)} \quad \forall v \ge \underline{v} \tag{2.29}$$

and

$$B = \int_{\underline{v}}^1 \frac{bf(v)\,dv}{\alpha_B q_B(v)} < \frac{b}{\kappa_B} \equiv \bar{B}.$$

Similarly (2.7) implies

$$S < \frac{s}{\kappa_S} \equiv \bar{S}.$$

Since $M(B,S)$ is nondecreasing in each of its arguments, $M(B,S) \le M\left(\bar{B}, \bar{S}\right)$. Substituting this bound into (2.29) we obtain

$$\phi_B(v) \ge \frac{b\underline{f}}{M\left(\bar{B}, \bar{S}\right)} \equiv \underline{\phi}_B \quad \forall v \ge \underline{v}. \tag{2.30}$$

Then apply (2.27), (2.28) and (2.30) to (2.26), and simplify, we find that for almost all $p \in [\underline{v}, \rho_B(1)]$,

$$\frac{\partial \pi_S(\bar{c}, p)}{\partial p} < 1 - K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\phi}_B.$$

Similarly, we can consider a type $\underline{v}$ buyer's proposing problem and find that $p_B(\underline{v}) \in [\rho_S(0), \bar{c}]$, and for almost all $p \in [\rho_S(0), \bar{c}]$, we have

$$\frac{\partial \pi_B(\underline{v}, p)}{\partial p} > -1 + K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\phi}_S$$

where

$$\begin{aligned} \pi_B(\underline{v}, p) &\equiv (\underline{v} - p)\int_0^{\bar{c}} I[p \geq \rho_S(c)]\frac{dN_S(c)}{S}, \\ \underline{\phi}_S &\equiv \frac{s\underline{g}}{M(\bar{B}, \bar{S})}. \end{aligned}$$

Therefore, if $1 - K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\phi}_B \leq 0$ and $-1 + K(\zeta_0)\left(\frac{\kappa}{r} - 1\right)\underline{\phi}_S \geq 0$, or equivalently $r \leq \underline{r}$, then we have $r < \kappa$, $\frac{\partial \pi_S(\bar{c},p)}{\partial p} < 0$ for almost every $p \in (\underline{v}, \rho_B(1))$ and $\frac{\partial \pi_B(\underline{v},p)}{\partial p} > 0$ for almost every $p \in (\rho_S(0), \bar{c})$. Hence $p_S(\bar{c}) = \underline{v}$ and $p_B(\underline{v}) = \bar{c}$. ■

The following corollary provides the main properties of our uniqueness bound $\underline{r}$ and relates it to the other bounds, r^* and $\min\{\kappa_B, \kappa_S\}$, in Theorem 1 and Corollary 3.

Corollary 4 *We have*

- ***(a)*** *If κ_B and κ_S increase, then $\underline{r}$ increases, and vice versa;*
- ***(b)*** $0 < \underline{r} < \min\{\kappa_B, \kappa_S\}$*;*
- ***(c)*** *$\underline{r}$ goes to 0 as κ_B and κ_S go to 0; and*
- ***(d)*** *if $K(\zeta_0) < 1$ then $\underline{r} < r^*$.*

Proof. (a)-(c) are obvious. Our derivation of $\underline{r}$ (in the proof of Theorem 2) shows that $r \leq \underline{r}$ is equivalent to $K(\zeta_0)\left(\frac{\kappa - r}{r + \kappa}\right) - \frac{1}{\underline{\phi}_B}\frac{r}{r+\kappa} \geq 0$ and $K(\zeta_0)\left(\frac{\kappa - r}{r + \kappa}\right) - \frac{1}{\underline{\phi}_S}\frac{r}{r+\kappa} \geq 0$, where $\underline{\phi}_B$ and $\underline{\phi}_S$ (defined in Theorem 2) are lower bounds of the market probability densities of buyers' and sellers' types in any equilibrium. On the other hand, $r < r^*$ is equivalent to $K(\zeta_0) - \frac{1}{\phi_{B0}(\underline{v}_0)}\frac{r}{r+\alpha_B(\zeta_0)} > 0$ and $K(\zeta_0) - \frac{1}{\phi_{S0}(\bar{c}_0)}\frac{r}{r+\alpha_S(\zeta_0)} > 0$ where ϕ_{B0} and ϕ_{S0} are the market probability densities of buyers' and sellers' types in the full-trade equilibrium. Then it is easy to verify that, given $K(\zeta_0) < 1$ (so that $\alpha_B(\zeta_0) > \kappa$ and $\alpha_S(\zeta_0) > \kappa$), $r \leq \underline{r}$ implies $r < r^*$. In other words, $\underline{r} < r^*$ if $K(\zeta_0) < 1$. ■

Before closing this section, we give a simple example that visualizes the main results of this and the previous sections. In particular, whether in equilibrium every meeting results in a trade does not hinge on the level of frictions, but rather on the composition of different kinds of frictions (discount rate r and explicit costs κ_B, κ_S). More precisely, in the friction space of (r, κ_B, κ_S), any neighborhood of $\mathbf{0}$, no matter how small, must contain a region (where r is small relative to κ_B, κ_S) in which only full-trade equilibria exist, and also contain another region (where r is large relative to κ_B, κ_S) in which only non-full-trade equilibria exist.

Example 1 *Buyers and sellers are born at the same rate, i.e.* $b = s = 1/2$. *The distributions of buyers' valuations and sellers' costs are both uniform* $[0,1]$, *i.e.* $F(v) = v$, $G(c) = c$. *(It is easy to check that the monotonicity of the virtual type functions* J_B *and* J_S *is satisfied.) The bargaining power is evenly distributed, i.e.* $\beta_B = \beta_S = 1/2$. *The matching function is given by* $M(B,S) = BS/(B+S)$.[26] *One can check that the entry gap and marginal types in a full-trade equilibrium are given by*

$$\underline{v}_0 - \bar{c}_0 = K(\zeta_0) = 2(\kappa_B + \kappa_S),$$

$$\underline{v}_0 = \frac{1}{2} + \kappa_B + \kappa_S$$

$$\bar{c}_0 = \frac{1}{2} - \kappa_B - \kappa_S.$$

Also, r^* *can be calculated as*

$$r^* = \frac{4\min\{\kappa_B, \kappa_S\}}{\max\{1 - 6(\kappa_B + \kappa_S), 0\}}.$$

[26] Gale (1987) assumes this matching function, although the matching function is not explicitly stated there. In his model each trader in the market is randomly matched for each period with another trader, either a buyer or a seller. So, for a buyer, the probability per period of being matched with a seller is $S/(B+S)$. Similarly a seller is matched with a buyer with probability $B/(B+S)$. With continua of buyers and sellers in the market, the total mass of matches made per period is $BS/(B+S)$.

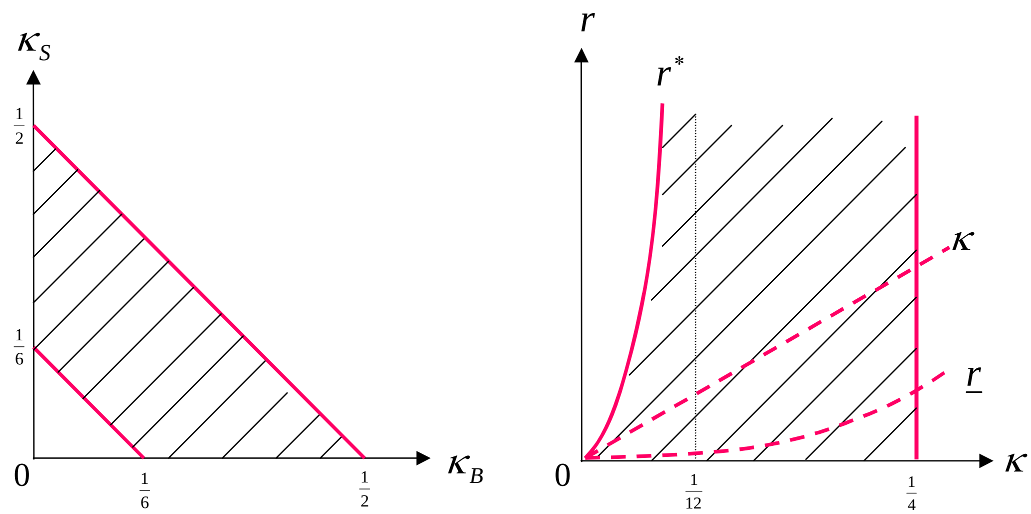

Figure 2.5: Different patterns of equilibria in different regions of friction space

Therefore, a full-trade equilibrium exists for all discount rate r *if* $\frac{1}{6} \leq \kappa_B + \kappa_S < \frac{1}{2}$*, shown in the left panel of Figure 2.5. If* $\kappa_B + \kappa_S < \frac{1}{6}$*, full-trade equilibrium may or may not exist, depending on whether* r *is sufficiently small. Now let us assume* $\kappa_B = \kappa_S = \kappa$*, then we have*

$$r^* = \frac{4\kappa}{\max\{1 - 12\kappa, 0\}},$$

$$\underline{r} \equiv \frac{8\kappa^3}{1 + 8\kappa^2}.$$

The shaded area in the right panel of Figure 2.5 shows the values of r *and* κ *for which a full-trade equilibrium exists. Under the dashed ray* κ*, a non-full-trade equilibrium with overlapping supports cannot exist. Under the dashed curve* $\underline{r}$*, a unique equilibrium, which is full-trade, exists.*

2.7 Necessary and sufficient condition for existence

In this section we prove that the condition $K(\zeta_0) < 1$ alone is a necessary and sufficient for the existence of a (full-trade or non-full-trade) nontrivial steady-state equilibrium.

Theorem 3 (General existence) *At least one nontrivial steady-state equilibrium exists if and only if* $K(\zeta_0) < 1$.

Taken together with Corollary 2(b), Theorem 3 implies that a non-full-trade equilibrium exists if the search costs are sufficiently small relative to the discount rate.

Corollary 5 (Existence of a non-full-trade equilibrium) *Given any* $r > 0$, *there is some* $\bar{\kappa} > 0$ *such that a non-full-trade equilibrium exists whenever* $\kappa_B, \kappa_S < \bar{\kappa}$.

It is relatively easy to see that the condition $K(\zeta_0) < 1$, a necessary condition for the existence of a full-trade equilibrium, is also necessary for the existence of any nontrivial equilibrium of our model. Indeed, it is already proved by (2.22) in Lemma 5.

Perhaps surprisingly, the condition $K(\zeta_0) < 1$ is also sufficient for the existence of a nontrivial equilibrium of our model.

It might be natural to guess that a nontrivial equilibrium exists if and only if the expected search cost incurred by a buyer-seller pair (i.e. $K(\zeta)$) is smaller than the maximum gains from trade, which is 1. However, this alone does not give us a meaningful condition for existence. It is because the buyer-seller ratio ζ in equilibrium (if any) is endogenous, and the set $\{K(\zeta) : \zeta > 0\}$ is unbounded since $\lim_{\zeta \to 0} K(\zeta) = \lim_{\zeta \to \infty} K(\zeta) = \infty$.

However, Theorem 3 tells us that in order to know whether a friction profile is compatible with a nontrivial market, it suffices to check only the expected search costs in the full-trade equilibrium candidate, although the true equilibrium (if any) might be non-full-trade.

This result can be informally understood as follows. The market might have to break down because the expected search cost $K(\zeta)$ is too high that it does not pay for traders to enter. So a case where $K(\zeta)$ is very small is an "inframarginal situation". What matters to the existence condition is the "marginal situation" where $K(\zeta)$ is close to 1. If we insert the full-trade equilibrium buyer-seller ratio ζ_0 into $K(\zeta)$ and then consider the marginal situation, Corollary 2(c) asserts that a full-trade equilibrium does exist, which in turn validates ζ_0 in the first place.

The rest of this section is devoted to the main elements of the formal proof of Theorem 3. Additional details are provided in the Appendix. As usual, we want to construct a mapping T such that its fixed point characterizes an (nontrivial steady-state) equilibrium, and prove that T has a fixed point. The mapping T is informally described as follows. Start with a pair of value functions (W_B, W_S) and a pair of distribution functions (N_B, N_S), we construct best-response proposing strategies (p_B, p_S) and entry strategies (χ_B, χ_S). Then from those strategies and the original functions (W_B, W_S, N_B, N_S), we define a new pair of value functions (W_B^*, W_S^*) through the Bellman equations, and a new pair of distribution functions (N_B^*, N_S^*) through the steady-state equations. Thus a fixed point of T (i.e. $(W_B, W_S, N_B, N_S) = (W_B^*, W_S^*, N_B^*, N_S^*)$) characterizes an equilibrium.

We will apply the *Schauder fixed point theorem*: if D is a nonempty compact convex subset of a Banach space and T is a continuous function from D to D, then T has a fixed point. In order to make this theorem applicable, certain difficulties need to be overcome. The main difficulty is that as we apply the mapping T, we need to preserve positive entry. To deal with this difficulty, we first prove existence of what we call an ε-*equilibrium*, which is an actual equilibrium of the ε-*model* described below.

The ε-model differs from our original model in three ways. First, we add a subsidy that ensures that all buyers with type $v \geq 1-\varepsilon$ and all sellers with type $c \leq \varepsilon$ enter. Every newborn trader is qualified to receive a flow of subsidy for her market participation, provided that (i) her type satisfies $v \geq 1-\varepsilon$ or $c \leq \varepsilon$, and (ii) she would choose not to participate if no subsidy were available. Further, the flow rate of the subsidy for a qualified trader is the least amount sufficient to make this trader participate, i.e. the flow subsidies are infimum subsidies to attain $W_B(v) \geq 0$ and $W_S(c) \geq 0$ for $v \in [1-\varepsilon, 1]$ and $c \in [0, \varepsilon]$. Because any subsidized traders are simply indifferent between entering or staying out, the Bellman equations for (W_B, W_S) and optimality conditions for (p_B, p_S) do not need to be changed.

Although we now have a positive lower bound for the inflows of traders, we may not have a positive lower bound for the mass of traders in the market because the outflow rate (i.e. $\alpha_B(\zeta)q_B(v)$ or $\alpha_S(\zeta)q_S(c)$) could be potentially very large. To overcome this difficulty, we impose the second modification, which ensures that the arrival rates $\alpha_B(\zeta)$ and $\alpha_S(\zeta)$ are bounded from above by some $\bar{\alpha}$. We modify the matching function $M(B,S)$ as $\min\{M(B,S), B\bar{\alpha}, S\bar{\alpha}\}$. Notice that this modified one inherits all the properties of a matching function. But under the modified matching function we make sure that $\alpha_B(\zeta), \alpha_S(\zeta) \leq \bar{\alpha}$.

While the first two modifications are made to make the mass of traders bounded from below, we also want it to be bounded from above, because our domain D needs to be compact. It suffices to have a lower bound for the outflow rate ($\alpha_B(\zeta)q_B(v)$ or $\alpha_S(\zeta)q_S(c)$). For a type who chooses to enter without subsidy, there is naturally an upper bound for its mass because her expected trading surplus must be larger than her search cost. More precisely, for a participating v-buyer who is not subsidized, $\alpha_B(\zeta)q_B(v) \geq \kappa_B$. However, a subsidized buyer could have $\alpha_B(\zeta)q_B(v) < \kappa_B$. Our third modification is to disqualify subsidized traders in a way that ensures the outflow rates of subsidized types are at least κ_B or κ_S. The disqualification process is a Poisson process, with the rate equal to the minimum that makes the outflow rate at least κ_B or κ_S. For example, a currently qualified v-buyer with $\alpha_B(\zeta)q_B(v) < \kappa_B$ will be disqualified and exit immediately at a Poisson rate $\kappa_B - \alpha_B(\zeta)q_B(v)$; while a currently qualified v-buyer with $\alpha_B(\zeta)q_B(v) \geq \kappa_B$ will not be disqualified. Notice that for any v-buyer, either subsidized or not, the gross outflow rate must be $\max\{\alpha_B(\zeta)q_B(v), \kappa_B\}$. Therefore, in the steady-state equations (2.6) and (2.7) that define N_B^* and N_S^* we now use $\max\{\alpha_B(\zeta)q_B(x), \kappa_B\}$ and $\max\{\alpha_S(\zeta)q_S(x), \kappa_S\}$ instead of $\alpha_B(\zeta)q_B(x)$ and $\alpha_S(\zeta)q_S(x)$. It completes the descriptions of our ε-model.

We will show that our ε-model has at least one equilibrium, which we shall call an ε-equilibrium (Proposition 1). Next, we will prove that if $\varepsilon > 0$ is sufficiently small and $\bar{\alpha}$ sufficiently large, then an ε-equilibrium is an equilibrium of our original model (Proposition 1). The main ideas of the proof are illustrated graphically in Figure 2.6.

First, as in Lemma 5, we show that in any ε-equilibrium, we must have $\underline{v} - \bar{c} \leq K(\zeta_0)$. Second, we show that the trading flows are almost balanced, the discrepancy bounded in absolute value by (a multiple of) ε. Imposing these constraints on the set of values

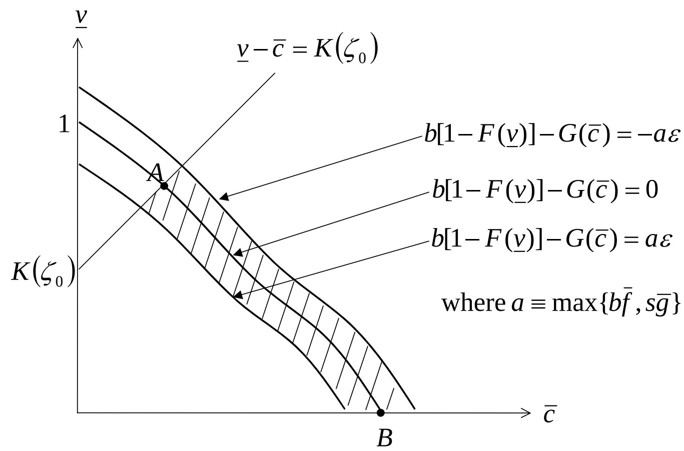

Figure 2.6: Illustration of the idea behind the existence proof

$(\bar{c}, \underline{v})$, we obtain the set of feasible values given by the shaded area in Figure 2.6. As the graph makes clear, the shaded area collapses to the curvilinear segment AB. Consequently, as ε gets arbitrarily small, the minimum $\bar{c}$ in the shaded area is arbitrarily close to the horizontal coordinate of point A, and the maximum feasible $\underline{v}$ is arbitrarily close to the vertical coordinate of A. It follows that for small enough $\varepsilon > 0$, the constraints $\bar{c} \geq \varepsilon$ and $\underline{v} \leq 1 - \varepsilon$ become non-binding. In other words, our subsidy policy does not have a bite because no entrant is subsidized. It further implies the marginal entrants must be able to recover their search costs, and hence ζ is bounded away from 0 and ∞. Thus as long as $\bar{\alpha}$ are chosen to be large enough, our modification of the matching function does not have a bite. It follows that if $\varepsilon > 0$ is small and $\bar{\alpha}$ large, then an ε-equilibrium is an equilibrium of our original model.

The following is our formal treatments. We first define an appropriate domain D_ε, and then a mapping $T_\varepsilon : D_\varepsilon \to D_\varepsilon$.

Definition 3 *Fix* $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ *and* $\varepsilon \in (0, \bar{\varepsilon}]$, *where*

$$\bar{\varepsilon} \equiv \min\left\{1, \frac{\bar{f}\bar{\alpha}}{\kappa_B \underline{f}}, \frac{\bar{g}\bar{\alpha}}{\kappa_S \underline{g}}\right\}.$$

Let $C[0,1]$ *be the Banach space of real continuous bounded functions defined on* $[0,1]$, *en-*

dowed with the supremum norm. Then we define $D_\varepsilon \subset (C[0,1])^4$ *as the set of all tuples* (W_B, W_S, N_B, N_S) *such that*

(i) W_B, N_B *and* N_S *are nondecreasing, while* W_S *is nonincreasing,*

(ii) W_B, W_S, N_B *and* N_S *have Lipschitz constants no greater than* $\bar{\alpha}/(r+\bar{\alpha})$, $\bar{\alpha}/(r+\bar{\alpha})$, $b\bar{f}/\kappa_B$ *and* $s\bar{g}/\kappa_S$ *respectively, and*

(iii) $W_B(0) = W_S(1) = N_B(0) = N_S(0) = 0$ *and* $N_B(1) \geq \varepsilon b\underline{f}/\bar{\alpha}$, $N_S(1) \geq \varepsilon s\underline{g}/\bar{\alpha}$.

Lemma 7 D_ε *is nonempty, convex and compact for any* $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ *and any* $\varepsilon \in (0, \bar{\varepsilon}]$.

Proof. In Appendix A. ■

Definition 4 *Fix* $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ *and* $\varepsilon \in (0, \bar{\varepsilon}]$. *Define a mapping* $T_\varepsilon : D_\varepsilon \to D_\varepsilon$ *as follows. For any* $(W_B, W_S, N_B, N_S) \in D_\varepsilon$, *define* $B \equiv N_B(1)$, $S \equiv N_S(1)$, $\alpha_B \equiv \min\{M(B,S), B\bar{\alpha}, S\bar{\alpha}\}/B$ *and* $\alpha_S \equiv \min\{M(B,S), B\bar{\alpha}, S\bar{\alpha}\}/S$. *Then construct* p_B, p_S, W_B^*, W_S^*, χ_B, χ_S, N_B^*, N_S^* *by*

$$p_B(v) \equiv \max\left\{\arg\max_{p\in[0,1]} \int_{\{c:p-c\geq W_S(c)\}} (v - p - W_B(v)) \frac{dN_S(c)}{S}\right\} \tag{2.31}$$

$$p_S(c) \equiv \min\left\{\arg\max_{p\in[0,1]} \int_{\{v:v-p\geq W_B(v)\}} (p - c - W_S(c)) \frac{dN_B(v)}{B}\right\}. \tag{2.32}$$

$$\begin{aligned} W_B^*(v) \equiv & \max_{\chi\in\{0,1\}} \chi \cdot \{\frac{\alpha_B}{r+\alpha_B}[\beta_B \pi_B(v) \\ & +\beta_S \int_{\{c:v-p_S(c)\geq W_B(v)\}} (v - p_S(c) - W_B(v)) \frac{dN_S(c)}{S}] \\ & -\frac{\kappa_B}{r+\alpha_B}\} + \frac{\alpha_B}{r+\alpha_B} W_B(v) \end{aligned} \tag{2.33}$$

$$\begin{aligned} W_S^*(c) \equiv & \max_{\chi\in\{0,1\}} \chi \cdot \{\frac{\alpha_S}{r+\alpha_S}[\beta_S \pi_S(c) \\ & +\beta_B \int_{\{v:p_B(v)-c\geq W_S(c)\}} (p_B(v) - c - W_S(c)) \frac{dN_B(v)}{B}] \\ & -\frac{\kappa_S}{r+\alpha_S}\} + \frac{\alpha_S}{r+\alpha_S} W_S(c). \end{aligned} \tag{2.34}$$

$\chi_B(v)$ *and* $\chi_S(c)$ *are defined as the maximizers in* (2.33) *and* (2.34) *respectively; wherever multiple maximizers exist, we pick* 1.

$$N_B^*(v) \equiv \int_0^v \frac{\chi_B^*(x)\, b}{\max\{\alpha_B q_B(x), \kappa_B\}} dF(x) \tag{2.35}$$

where $\chi_B^*(v)$ *is* 1 *if* $\chi_B(v) = 1$ *or* $v \geq 1 - \varepsilon$*, and is* 0 *otherwise.*

$$N_S^*(c) \equiv \int_0^c \frac{\chi_S^*(x)\, s}{\max\{\alpha_S q_S(x), \kappa_S\}} dG(x) \tag{2.36}$$

where $\chi_S^*(c)$ *is* 1 *if* $\chi_S(c) = 1$ *or* $c \leq \varepsilon$*, and is* 0 *otherwise.*

Now $T_\varepsilon(W_B, W_S, N_B, N_S)$ *is defined by the constructed* $(W_B^*, W_S^*, N_B^*, N_S^*)$.

In Appendix A we show that our definition of T_ε is legitimate, i.e. it is well-defined and $T_\varepsilon(D_\varepsilon) \subset D_\varepsilon$.

Lemma 8 *The mapping* $T_\varepsilon : D_\varepsilon \to D_\varepsilon$ *is continuous for any* $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ *and any* $\varepsilon \in (0, \bar{\varepsilon}]$.

Proof. In Appendix A. ■

Lemma 9 *Fix any* $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ *and any* $\varepsilon \in (0, \bar{\varepsilon}]$*. There exists some* $E \in D_\varepsilon$ *such that* $T_\varepsilon(E) = E$*. (That is, there exists an* ε*-equilibrium).*

Proof. As claimed before, D_ε is a nonempty, convex and compact set in a Banach space $(C[0,1])^4$ and the mapping T_ε is continuous. Then we obtain our result by applying the Schauder Fixed Point Theorem (which is stated before). ■

Proposition 1 *Suppose* $K(\zeta_0) < 1$*. Then if* $\varepsilon > 0$ *is small enough and* $\bar{\alpha}$ *large enough, any fixed point of* T_ε *characterizes a nontrivial steady-state equilibrium. (That is, if* $\varepsilon > 0$ *is small and* $\bar{\alpha}$ *large, any* ε*-equilibrium is in fact an equilibrium of our original model.)*

Proof. Suppose $E = (W_B, W_S, N_B, N_S) \in D_\varepsilon$ is a fixed point of T_ε. Then E, together with the constructed objects through the transformation from E to $T_\varepsilon(E)$, constitutes what we call an ε-equilibrium. Moreover, an ε-equilibrium satisfies all the equilibrium conditions in Definition 1 if one can verify that

(i) $\underline{v}^* \equiv \inf\{v : \chi_B^*(v) = 1\} < 1 - \varepsilon$, and $\bar{c}^* \equiv \sup\{c : \chi_S^*(c) = 1\} > \varepsilon$;

(ii) $\alpha_B q_B(v) \geq \kappa_B$ if $\chi_B^*(v) = 1$, and $\alpha_S q_S(c) \geq \kappa_S$ if $\chi_S^*(c) = 1$; and

(iii) $\alpha_B, \alpha_S < \bar{\alpha}$.

The following steps 1-6 will show that, for any $(r, \bar{\alpha}) \gg (0, \max\{\kappa_B, \kappa_S\})$, any $\varepsilon \in (0, \bar{\varepsilon}]$, and any associated fixed point of T_ε, a bunch of equilibrium properties hold. Then steps 7 and 8 will show that (i)-(iii) also hold if $\varepsilon > 0$ is small enough and $\bar{\alpha}$ large enough.

Step 1. $E \in D_\varepsilon$ implies $v - W_B(v)$ and $c + W_S(c)$ are strictly increasing. Thus, from (2.31) and (2.32), we have $p_B(v) \leq \bar{c}^* + W_S(\bar{c}^*)$ and $p_S(c) \geq \underline{v}^* - W_B(\underline{v}^*)$.

Step 2. The expression inside the curly bracket in (2.33) can be written as

$$\frac{\alpha_B}{r + \alpha_B}\left[\beta_B \pi_B(v) + \beta_S \int \max\{v - p_S(c) - W_B(v), 0\} \frac{dN_S(c)}{S} - \frac{\kappa_B}{\alpha_B}\right], \tag{2.37}$$

which is continuous in v. Then by definition of $\underline{v}^*$, $\chi = 0$ is a maximizer in (2.31) when $v = \underline{v}^*$. In other words, (2.37) is non-positive when $v = \underline{v}^*$. Now evaluate (2.33) at $v = \underline{v}^*$. From the above result and that $W_B^* = W_B$, we have $W_B(\underline{v}^*) = \frac{\alpha_B}{r+\alpha_B} W_B(\underline{v}^*)$, or $W_B(\underline{v}^*) = 0$.

Step 3. The fact that (2.37) is non-positive when $v = \underline{v}^*$ can be simplified as $\beta_B \pi_B(\underline{v}^*) \leq \frac{\kappa_B}{\alpha_B}$ because $\underline{v}^* - W_B(\underline{v}^*) = \underline{v}^*$ is no greater than $p_S(c)$, due to step 1. The logic in this and the previous step can be applied to the sellers' side. Thus we also have $W_S(\bar{c}^*) = 0$ and $\beta_S \pi_S(\bar{c}^*) \leq \frac{\kappa_S}{\alpha_S}$.

Step 4. Notice that $\pi_B(\underline{v}^*) \geq \underline{v}^* - \bar{c}^*$ since the choice variable p in the definition (2.3) of π_B can be taken as $\bar{c}^*$. Similarly $\pi_S(\bar{c}^*) \geq \underline{v}^* - \bar{c}^*$. Then step 3 implies

$$\underline{v}^* - \bar{c}^* \leq \min\left\{\frac{\kappa_B}{\alpha_B \beta_B}, \frac{\kappa_S}{\alpha_S \beta_S}\right\} \leq K(\zeta_0). \tag{2.38}$$

Step 5. The expression inside the curly bracket in (2.33), which can be written as (2.37), is increasing in v. Hence χ_B and χ_B^* are increasing. Therefore, if $v \geq \underline{v} \equiv \inf\{v : \chi_B(v) = 1\}$, then (2.37) is non-negative, which implies $\alpha_B q_B(v) \geq \kappa_B$. Similarly, χ_S and χ_S^* are decreasing, and for any $c \leq \bar{c} \equiv \sup\{c : \chi_S(c) = 1\}$, we have $\alpha_S q_S(c) \geq \kappa_S$.

Step 6. Equation (2.35), $N_B^* = N_B$, and step 5 imply

$$b\left[1-F\left(\underline{v}^*\right)\right]-\int_{\underline{v}^*}^{1}\alpha_B q_B\left(v\right)dN_B\left(v\right)=\int_{\underline{v}^*}^{1}\max\left\{0,\kappa_B-\alpha_B q_B\left(v\right)\right\}dN_B\left(v\right). \tag{2.39}$$

The r.h.s. of (2.39) is clearly non-negative. Moreover, it is also no greater than $b\bar{f}\varepsilon$. To see this, consider two (exhaustive) cases: $\underline{v}^* = \underline{v}$ and $\underline{v}^* < \underline{v}$. First consider the case that $\underline{v}^* = \underline{v}$. From step 5 the r.h.s. of (2.39) is 0. Then consider the case that $\underline{v}^* < \underline{v}$. Due to the definition of $\underline{v}^*$ and $\underline{v}$, we have $\underline{v}^* = 1-\varepsilon$. The r.h.s. of (2.39) is no greater than $b\bar{f}\varepsilon$ because $dN_B\left(v\right) \le \frac{b\bar{f}}{\kappa_B}$. Similar logic can be applied to the sellers' side. Therefore we obtain

$$0 \le b\left[1-F\left(\underline{v}^*\right)\right]-\int_{\underline{v}^*}^{1}\alpha_B q_B\left(v\right)dN_B\left(v\right) \le b\bar{f}\varepsilon$$

$$0 \le sG\left(\bar{c}^*\right)-\int_{0}^{\bar{c}^*}\alpha_S q_S\left(c\right)dN_S\left(c\right) \le s\bar{g}\varepsilon.$$

On the other hand, by definition of $\alpha_B, q_B, \alpha_S, q_S$, we have

$$\int_{\underline{v}^*}^{1}\alpha_B q_B(v)dN_B(v)=\int_{0}^{\bar{c}^*}\alpha_S q_S(c)dN_S(c).$$

Therefore,

$$\left|b[1-F(\underline{v}^*)]-sG(\bar{c}^*)\right| \le \max\left\{b\bar{f}, s\bar{g}\right\}\cdot\varepsilon. \tag{2.40}$$

Step 7. The previous six steps work with a particular fixed point of T_ε given $(\varepsilon, \bar{\alpha})$. In this and the next step, we let $(\varepsilon, \bar{\alpha}) \to (0, \infty)$ and consider an associated sequence of fixed points. Along any subsequence, $\bar{c}^*$ cannot approach to 0 because otherwise (2.40) implies $\underline{v}^* \to 1$ and hence $\underline{v}^* - \bar{c}^* \to 1$, violating (2.38) and $K(\zeta_0) < 1$. Similarly, $\underline{v}^*$ cannot approach to 1 along any subsequence. Therefore, in the tail of the sequence, we have $\bar{c}^* > \varepsilon$ and $\underline{v}^* < 1-\varepsilon$, i.e. (i) holds. Notice that (i) implies $\underline{v}^* = \underline{v}$ and $\bar{c}^* = \bar{c}$. Thus step 5 implies (ii) also holds in the tail.

Step 8. From steps 5 and 7, we have $\alpha_B\left(\zeta\right) \ge \kappa_B$ and $\alpha_S\left(\zeta\right) \ge \kappa_S$ in the tail as $(\varepsilon, \bar{\alpha}) \to (0, \infty)$. Thus $\zeta \equiv B/S$ is bounded away from 0 and ∞. It follows that, in the tail, $\alpha_B < \bar{\alpha}$ and $\alpha_S < \bar{\alpha}$, i.e. (iii) holds. ■

Proof of Theorem 3. The necessity of the condition $K\left(\zeta_0\right) < 1$ has been proved by (2.22) in Lemma 5. The sufficiency is implied by Lemma 9 and Proposition 1. ■

2.8 Concluding remarks

We have analyzed a steady-state search-theoretic model with two-sided private information bargaining. Although the model is not complicated, analyzing the equilibrium is a highly nontrivial job. It is because traders' best-response bargaining behaviors in general depend on the buyers' and sellers' distributions in a nontrivial manner. And these distributions in turn depend on the traders' bargaining behaviors through steady-state equations. Moreover the existence of equilibrium is also elusive. It is because a trivial no-entry equilibrium always exists, so that we cannot solely apply a fixed-point argument to prove existence of some nontrivial equilibrium. In spite of these difficulties, we are able to provide quite a few results.

We have provided a necessary and sufficient condition $K(\zeta_0) < 1$ under which some nontrivial steady-state equilibrium exists. Not surprisingly, in equilibrium the market must breakdown (i.e. nobody enters) if search costs are too large. Besides, the qualitative pattern of equilibrium (whether every meeting results in a trade) mainly depends not on the level of frictions, but the relativity of the two kinds of frictions (time discounting and explicit search costs). This result can only be obtained in a model with the coexistence of two kinds of frictions.

Before closing this chapter, we make a few remarks on the existence condition $K(\zeta_0) < 1$.

First, although the trivial no-entry equilibrium always exists because of the coordination problem, it is appealing to assume that the trivial equilibrium will not be selected as long as a nontrivial one exists. After all, every nontrivial equilibrium Pareto dominates the trivial one. With this assumption, we predict that the market will open if and only if $K(\zeta_0) < 1$.

Second, the discount rate r does not enter into the condition $K(\zeta_0) < 1$. One way to understand it is that in steady state, when a trader is going to decide whether to enter or not, he just need to compare the expected gain from search for a period of very short length dt and the corresponding search costs incurred for the same period. Since this dt can be arbitrarily small, the discount rate has to be irrelevant for this entry decision.

Third, the condition $K(\zeta_0) < 1$ depends on the distribution of bargaining power between buyers and sellers. In particular, if the relative bargaining power of sellers β_S is close to 0 or 1, the market must breakdown. The intuition is that if β_S is close to zero, sellers do not have enough incentive to participate. If β_S is close to one, buyers do not have enough incentive to participate. The openness of the market requires some balance between the interests of the two sides.

Chapter 3

Role of Information Structure in Dynamic Matching Markets

3.1 Introduction

This chapter studies how the information structure at the bargaining stages affects the equilibrium outcome of a dynamic matching market.[27] We analyze and compare two search-theoretic dynamic matching and bargaining games. They are called the private information model and the full information model. The former is the one we have seen in Chapter 2; the latter is the one in Mortensen and Wright (2002).

In both models, searching for a trading partner takes time and other resources (e.g. money, effort). Thus there are two kinds of search frictions, one parameterized by a discount rate and one parameterized by explicit search costs. The two models are identical except for only one aspect: in the private information model, when a buyer and a seller meet each other they bargain without knowing each other's characteristics; while in the full information model they observe each other's characteristics once they meet.

We show that the private information and full information models have some similarities. They have the same necessary and sufficient condition for the existence of a nontrivial steady-state equilibrium (Theorem 4); or putting it another way, information structure has no impact on whether the market would open or breakdown. Moreover, in both models, whether there exists a full-trade equilibrium (i.e. in which every meeting results in a trade)

[27] The chapter includes materials in my manuscript "Bilateral Matching and Bargaining with Private Information", which is joint with my thesis co-supervisor Artyom Shneyerov.

mainly depends on the relative magnitudes of the two kinds of frictions (Corollary 2 and Corollary 6). Furthermore, the two models become completely identical if the discount rate is zero (Proposition 2); or in other words, information structure has no impact if agents are perfectly patient.

Information structure also makes qualitative differences. Due to private information the bargaining between a buyer and a seller might breakdown even when the gain from trade is larger than the total value of their outside options. We show that whether this bargaining inefficiency occurs in equilibrium again depends on the relative magnitudes of the two kinds of frictions (Proposition 3).

Besides, private information has an entry-deterring effect, so that typically less potential traders enter in the private information model (Proposition 4). Why this is so can be understood through the following logic. Suppose we start with a nontrivial steady-state equilibrium under full information. In this equilibrium there are marginal entrants: the lowest-value active buyers and the highest-cost active sellers. They are indifferent between entering or not. Now let us think about how the entry incentives of the marginal entrants would change if we make the bargaining to be under private information. Recall that the bargaining protocol in our market is the so-called random-proposer protocol. Notice the followings: First, when being a responder, a buyer with the lowest value in the market (or a seller with the highest cost in the market) would never receive an offer that makes him better off on top of his outside option. This is true no matter information is full or private. Second, when being a proposer, not knowing the responder's type would make a marginal entrant lose some information rent. Summing up these two concerns, the marginal entrants would expect less participating gains on average, if information is switched to be private. They, originally indifferent between entering or not, would become non-participants. Hence less potential traders enter in the private information model.

Because entry decisions have externalities through the matching process, this entry-deterring effect could either improve or deteriorate the aggregate social welfare. We also provide and interpret sufficient conditions under which this entry effect improves or deteriorates social welfare (Theorem 6).

The rest of this chapter is organized as follows. Section 3.2 reviews the private information model. Section 3.3 presents the full information model, and the associated results. Section 3.4 solves the no-discounting case for both models. Section 3.5 analyzes the concept of bargaining efficiency and its relation with full-trade equilibria. Section 3.6 studies how the information structure affects social welfare through its impact on entry decisions. Section 3.7 concludes. Appendix B contains the calculations for Section 3.6.

3.2 Private information model

Our model of dynamic matching market with private information bargaining, or *private information model* for short, is the one we use in Chapter 2. To make this chapter somehow self-contained and at the same time avoid too much repetition, let us for now only briefly review the model, recall the notations, write down the equations that define our equilibrium concept, and present our central results in Chapter 2.

This is a continuous time, steady state model of a decentralized market with continua of risk-neutral traders (buyers and sellers). Different buyers (with unit demand) have different valuations $v \in [0, 1]$ for an indivisible good, and different sellers (with unit supply) have different costs $c \in [0, 1]$ for the good. Traders in the market are randomly matched pairwise at the aggregate flow rate $M(B, S)$, which depends on the mass of buyers B and the mass of sellers S currently in the market. Once a buyer and a seller meet, they bargain following the *random-proposer* protocol: with probability $\beta_B \in (0, 1)$ the buyer makes a take-it-or-leave-it offer to the seller, and with probability $\beta_S \equiv 1 - \beta_B$ the seller makes a take-it-or-leave-it offer. The traders leave once they successfully trade. New potential buyers are born at the rate b and sellers at the rate s. We normalize the aggregate born rate to be 1, i.e. $b + s = 1$. Once a potential buyer (seller) is born, his valuation (cost) is drawn i.i.d. from the c.d.f. $F(v)$ ($G(c)$). The market is continuously replenished with new-born buyers and sellers who voluntarily choose to enter. We study the steady-state perfect Bayesian equilibria with positive entry, so called *nontrivial steady-state equilibria*. There are (except the asymmetric information) two kinds of frictions: time discounting at rate $r > 0$ and explicit search costs at rates $\kappa_B > 0$ for buyers and $\kappa_S > 0$ for sellers. The matching function M exhibits constant returns to scale. (For other assumptions we make on the functions F, G and M, see Assumptions 1 and 2 in Section 2.2.)

Definition 5 *Under the private information model, a nontrivial steady-state equilibrium is a pair of value functions* $W_B, W_S : [0,1] \to \mathbb{R}_+$, *a pair of entry strategies* $\chi_B, \chi_S : [0,1] \to \{0,1\}$, *a pair of proposing strategies* $p_B, p_S : [0,1] \to [0,1]$, *and a pair of distribution functions* $N_B, N_S : [0,1] \to \mathbb{R}_+$ *such that* $B \equiv N_B(1) > 0$, $S \equiv N_S(1) > 0$,

$$rW_B(v) = \max_{\chi\in\{0,1\}} \chi \cdot \{\alpha_B(\zeta)[\beta_B \pi_B(v) + \beta_S \int_{\{c: v - p_S(c) \geq W_B(v)\}} (v - p_S(c) - W_B(v)) \frac{dN_S(c)}{S}] - \kappa_B\} \tag{3.1}$$

$$rW_S(c) = \max_{\chi\in\{0,1\}} \chi \cdot \{\alpha_S(\zeta)[\beta_S \pi_S(c) + \beta_B \int_{\{v: p_B(v) - c \geq W_S(c)\}} (p_B(v) - c - W_S(c)) \frac{dN_B(v)}{B}] - \kappa_S\} \tag{3.2}$$

$$b\chi_B(v)\, dF(v) = \alpha_B(\zeta) q_B(v)\, dN_B(v) \tag{3.3}$$

$$s\chi_S(c)\, dG(c) = \alpha_S(\zeta) q_S(c)\, dN_S(c) \tag{3.4}$$

where

$$\zeta \equiv B/S, \quad \alpha_B(\zeta) \equiv M(1, 1/\zeta), \quad \alpha_S(\zeta) \equiv M(\zeta, 1),$$

$$\pi_B(v) \equiv \max_{p\in[0,1]} \left\{ \int_{\{c: p - c \geq W_S(c)\}} (v - p - W_B(v)) \frac{dN_S(c)}{S} \right\} \tag{3.5}$$

$$\pi_S(c) \equiv \max_{p\in[0,1]} \left\{ \int_{\{v: v - p \geq W_B(v)\}} (p - c - W_S(c)) \frac{dN_B(v)}{B} \right\} \tag{3.6}$$

$$q_B(v) \equiv \beta_B \int_{\{c: p_B(v) - c \geq W_S(c)\}} \frac{dN_S(c)}{S} + \beta_S \int_{\{c: v - p_S(c) \geq W_B(v)\}} \frac{dN_S(c)}{S}$$

$$q_S(c) \equiv \beta_S \int_{\{v: v - p_S(c) \geq W_B(v)\}} \frac{dN_B(v)}{B} + \beta_B \int_{\{v: p_B(v) - c \geq W_S(c)\}} \frac{dN_B(v)}{B},$$

and χ_B, χ_S, p_B, p_S *solve the optimization problems in* (3.1), (3.2) (3.5), *and* (3.6) *respectively.*

The equilibrium objects have the following interpretations:

- $W_B(v), W_S(c)$: buyers' and sellers' continuation payoffs when unmatched,
- $\chi_B(v), \chi_S(c)$: buyers' and sellers' entry strategies (1 represents "enter" and 0 represents "not enter"),
- $p_B(v), p_S(c)$: buyers' and sellers' proposing strategies, i.e. what trading prices they propose,
- $N_B(v), N_S(c)$: buyers' and sellers' steady-state distributions of types in the market,
- B, S: buyers' and sellers' steady-state masses in the market,
- ζ: steady-state buyer-seller ratio (or market tightness),
- $\alpha_B(\zeta), \alpha_S(\zeta)$: buyers' and sellers' Poisson arrival rates of being matched,
- $\pi_B(v), \pi_S(c)$: buyer's and sellers' capital gains when they become a proposer, and
- $q_B(v), q_S(c)$: buyer's and sellers' trading probabilities in a given meeting.

Equations (3.1) and (3.2) are buyers' and sellers' Bellman equations. Equations (3.3) and (3.4) are the steady-state equations: the inflow rate of the traders of each type is equal to the outflow rate due to trading. The buyer's and sellers' responding strategies are also captured in the above equilibrium definition: A type v buyer accepts a price offer p if and only if $p \leq v - W_B(v)$; a type c seller accepts a price offer p if and only if $p \geq c + W_S(c)$. Thus buyers' and sellers' reservation prices, also called dynamic types, are given by

$$\rho_B(v) \equiv v - W_B(v), \tag{3.7}$$

$$\rho_S(c) \equiv c + W_S(c). \tag{3.8}$$

In general there is no analytic solution for the system of equations (3.1) through (3.4). However, we know from Theorem 3 (in Chapter 2) that our private information model has at least one nontrivial steady-state equilibrium if and only if $K(\zeta_0) < 1$ where

$$\zeta_0 \equiv \frac{\beta_B}{\beta_S}\frac{\kappa_S}{\kappa_B}, \tag{3.9}$$

$$K(\zeta) \equiv \frac{\kappa_B}{\alpha_B(\zeta)} + \frac{\kappa_S}{\alpha_S(\zeta)} \quad \forall \zeta. \tag{3.10}$$

By Theorem 2 and Corollary 4, if the discount rate r is small relative to the search costs κ_B and κ_S, then the (nontrivial steady-state) equilibrium is unique and has the property that every meeting results in a trade. We call this kind of equilibria *full-trade equilibria*. By Theorem 1 and Corollary 2 (see also Remark 1), if the discount rate is large relative to the search costs, then in equilibrium some meetings do not result in a trade. We call this kind of equilibria *non-full-trade equilibria*.

In particular, whether there exists a nontrivial equilibrium depends on the search costs (κ_B, κ_S) (but not on the discount rate r) and the distribution of bargaining power (β_B, β_S). Whether in equilibrium every meeting results in a trade depends on the relative magnitudes of r and (κ_B, κ_S).

3.3 Full information (Mortensen-Wright) model

3.3.1 Model

Our model of dynamic matching market with full information bargaining, or *full information model* for short, is the one in Mortensen and Wright (2002). Mortensen and Wright (2002) consider a model that differs from our private information model only in one respect: they assume full information bargaining, i.e. bargainers know each other's type once they meet. Consequently, proposers hold their partners to their reservation values (i.e., to their dynamic types), and the proposing strategies depend on both the proposer's and the responder's type. In other words, for a meeting between a type v buyer and a type c seller, if the buyer proposes, he will propose the offer $p_B(v,c) = \rho_S(c)$ if $v - \rho_S(c) \geq W_B(v)$, while the offer can be defined as any price less than $\rho_S(c)$ if $v - \rho_S(c) < W_B(v)$ (such a price will be rejected by the seller). Similarly, if the seller proposes, she will propose the offer $p_S(v,c) = \rho_B(v)$ if $\rho_B(v) - c \geq W_S(c)$.

In the context of full information bargaining, the random-proposer protocol is equivalent to the generalized Nash bargaining solution. To see this, notice that under random-proposer protocol, a meeting between a type v buyer and a type c seller results in a trade if and only if $\rho_S(c) \leq \rho_B(v)$. And conditional on trade, the expected trading price $p(v,c)$ is the weighted average of the seller's offer $\rho_B(v)$ and the buyer's offer $\rho_S(c)$:

$$p(v,c) = \beta_S \rho_B(v) + \beta_B \rho_S(c). \tag{3.11}$$

Now consider the generalized Nash bargaining with the buyer's relative bargaining power being $\beta_B \in (0,1)$, and the seller's relative bargaining power being $\beta_S \equiv 1 - \beta_B$. The joint matching surplus to be shared is $v - c - W_B(v) - W_S(c)$ and the threat points of the buyer and the seller is $W_B(v)$ and $W_S(c)$ respectively. Therefore, a trade occurs if and only if

$$v - c - W_B(v) - W_S(c) \geq 0$$

or equivalently $\rho_S(c) \leq \rho_B(v)$. Conditional on that, the trading price $p(v,c)$ is determined by

$$p(v,c) \in \arg\max_p \left[v - p - W_B(v)\right]^{\beta_B} \left[p - c - W_S(c)\right]^{\beta_S},$$

for which the solution is exactly (3.11).

Thus, no matter we use random-proposer protocol or generalized Nash bargaining, the buyer's and seller's capital gains from the meeting are given respectively by

$$v - p(v,c) - W_B(v) = \beta_B \cdot (\rho_B(v) - \rho_S(c)),$$

$$p(v,c) - c - W_S(c) = \beta_S \cdot (\rho_B(v) - \rho_S(c)).$$

In this regard, the random-proposer bargaining is an extension of Nash bargaining into the environment of private information.

Here we define nontrivial steady-state equilibria for the full information model in a way parallel to Definition 5.

Definition 6 *Under the full information model, a nontrivial steady-state equilibrium is a pair of value functions* $W_B, W_S : [0,1] \to \mathbb{R}_+$*, a pair of entry strategies* $\chi_B, \chi_S : [0,1] \to \{0,1\}$*, and a pair of distribution functions* $N_B, N_S : [0,1] \to \mathbb{R}_+$ *such that* $B \equiv N_B(1) > 0$*,* $S \equiv N_S(1) > 0$*,*

$$rW_B(v) = \max_{\chi \in \{0,1\}} \chi \cdot \{\alpha_B(\zeta)\beta_B \int_{\{c:\rho_B(v) \geq \rho_S(c)\}} (\rho_B(v) - \rho_S(c)) \frac{dN_S(c)}{S} - \kappa_B\} \tag{3.12}$$

$$rW_S(c) = \max_{\chi \in \{0,1\}} \chi \cdot \{\alpha_S(\zeta)\beta_S \int_{\{v:\rho_B(v) \geq \rho_S(c)\}} (\rho_B(v) - \rho_S(c)) \frac{dN_B(v)}{B} - \kappa_S\} \tag{3.13}$$

$$b\chi_B(v)\, dF(v) = \alpha_B(\zeta) q_B(v)\, dN_B(v)$$

$$s\chi_S(c)\, dG(c) = \alpha_S(\zeta) q_S(c)\, dN_S(c)$$

where

$$\zeta \equiv B/S, \quad \alpha_B(\zeta) \equiv M(1, 1/\zeta), \quad \alpha_S(\zeta) \equiv M(\zeta, 1),$$

$$\rho_B(v) \equiv v - W_B(v)$$

$$\rho_S(c) \equiv c + W_S(c)$$

$$q_B(v) \equiv \int_{\{c:\rho_B(v) \geq \rho_S(c)\}} \frac{dN_S(c)}{S} \tag{3.14}$$

$$q_S(c) \equiv \int_{\{v:\rho_B(v) \geq \rho_S(c)\}} \frac{dN_B(v)}{B} \tag{3.15}$$

and χ_B, χ_S *solve the optimization problems in* (3.12) *and* (3.13) *respectively.*

The interpretations for the equilibrium conditions and equilibrium objects in Definition 6 are the same as in the previous section.

3.3.2 Basic equilibrium properties

The analysis of Mortensen and Wright (2002) is based only on full-trade equilibria (although they do not use this term). That would not be enough for the purposes of this and the next Chapter. Now let us provide some lemmas for nontrivial steady-state equilibria in general. Our methodology here is similar to the one in Section 2.4.

Lemma 10 *Under full information, in any nontrivial steady-state equilibrium, there are marginal entering types* $\underline{v}, \bar{c} \in (0,1)$ *such that the supports of* N_B *and* N_S *are* $[\underline{v}, 1]$ *and* $[0, \bar{c}]$ *respectively. Marginal entrants (i.e. type* $\underline{v}$ *buyers and type* $\bar{c}$ *sellers) are indifferent between entering or not, while the entry preferences of all others are strict.* $\{v : \chi_B(v) = 1\}$ *is either* $[\underline{v}, 1]$ *or* $(\underline{v}, 1]$*.* $\{c : \chi_S(c) = 1\}$ *is either* $[0, \bar{c}]$ *or* $[0, \bar{c})$*.* W_B *is absolutely continuous, convex, nondecreasing on* $[0,1]$*, strictly increasing on* $[\underline{v}, 1]$*, with* $W_B(\underline{v}) = 0$*; whenever differentiable,*

$$W_B'(v) = \chi_B(v) \frac{\alpha_B(\zeta)\beta_B q_B(v)}{r + \alpha_B(\zeta)\beta_B q_B(v)}. \tag{3.16}$$

W_S *is absolutely continuous, convex, nonincreasing on* $[0,1]$*, strictly decreasing on* $[0, \bar{c}]$*, with* $W_S(\bar{c}) = 0$*; whenever differentiable,*

$$W_S'(c) = -\chi_S(c) \frac{\alpha_S(\zeta)\beta_S q_S(c)}{r + \alpha_S(\zeta)\beta_S q_S(c)}. \tag{3.17}$$

The trading probability q_B *is strictly positive and nondecreasing on* $[\underline{v}, 1]$*, while* q_S *is strictly positive and nonincreasing on* $[0, \bar{c}]$*.*

Proof. We prove the results for buyers only. We use an argument parallel to that for Lemma 1. For any $v, \hat{v} \in [0,1]$, define

$$\Pi_B(v, \hat{v}) \equiv \int_{\{c:\rho_B(\hat{v}) \geq \rho_S(c)\}} (v - \rho_S(c)) \frac{dN_S(c)}{S}.$$

The buyers' Bellman equation (3.12) implies for any $v, \hat{v} \in [0,1]$ and any $\chi \in \{0,1\}$,

$$rW_B(v) \geq \chi \cdot \{\alpha_B\beta_B [\Pi_B(v, \hat{v}) - q_B(\hat{v}) W_B(v)] - \kappa_B\}$$

or equivalently

$$W_B(v) \geq \chi \cdot u_B(v, \hat{v})$$

where

$$u_B(v, \hat{v}) \equiv \frac{\alpha_B \beta_B \Pi_B(v, \hat{v}) - \kappa_B}{r + \alpha_B \beta_B q_B(\hat{v})}.$$

And the inequality becomes equality if $\hat{v} = v$ and $\chi = \chi_B(v)$. Let $U_B(v) \equiv \max_{\hat{v} \in [0,1]} u_B(v, \hat{v})$. We then have $W_B(v) = \chi_B(v) u_B(v, v) = \chi_B(v) U_B(v) = \max\{U_B(v), 0\}$. For any $\hat{v}$, $u_B(v, \hat{v})$ is affine and nondecreasing in v. Milgrom and Segal (2002) Envelope Theorem implies $U_B(v)$ is absolutely continuous, convex, nondecreasing, and with slope $\alpha_B \beta_B q_B(v) / (r + \alpha_B \beta_B q_B(v))$ whenever differentiable. The same properties are inherited by $W_B(v)$, except that its slope becomes $\chi_B(v) \alpha_B \beta_B q_B(v) / (r + \alpha_B \beta_B q_B(v))$.

Obviously $U_B(0) < 0$. Let $\underline{v} \equiv \sup\{v \in [0,1] : U_B(v) < 0\}$. By continuity of U_B, we have $\underline{v} > 0$ and $U_B(\underline{v}) \leq 0$. But $U_B(\underline{v}) < 0$ is impossible in nontrivial equilibrium because it implies $\chi_B(v) = 0 \ \forall v \in [0,1]$ and hence $B = 0$. Thus $U_B(\underline{v}) = W_B(\underline{v}) = 0$. By monotonicity of U_B, for all $v < \underline{v}$, we have $U_B(v) < 0$ and hence $\chi_B(v) = W_B(v) = 0$. Moreover, $q_B(v) > 0$ for all $v \geq \underline{v}$. It is because $q_B(v) \geq \Pi_B(v, v)$, and for all $v \geq \underline{v}$, the fact $U_B(v) \geq 0$ implies $\alpha_B \beta_B \Pi_B(v, v) \geq \kappa_B > 0$. It furthermore implies $U'_B(\underline{v}+) \geq \alpha_B \beta_B q_B(\underline{v}+) / (r + \alpha_B \beta_B q_B(\underline{v}+)) > 0$. Thus for all $v > \underline{v}$, we have $U_B(v) > 0$ and hence $\chi_B(v) = 1$ and $W_B(v) = U_B(v)$. From the buyers' steady-state equation, $[\underline{v}, 1]$ is the support of N_B. Since the inflow distribution F does not have atom point, neither does N_B. Hence $B > 0$ implies $\underline{v} < 1$. Finally, the convexity of U_B implies that q_B is nondecreasing on $[\underline{v}, 1]$. ■

The thresholds $\underline{v}$ and $\bar{c}$ in Lemma 10 are called *marginal entering types*. Those buyers with type $\underline{v}$ and those sellers with type $\bar{c}$ are called *marginal entrants*. Since the flow and stock masses of marginal entrants (who are indifferent between entering or not) is zero anyway, we will without loss of generality assume throughout they enter, i.e. $\chi_B(\underline{v}) = \chi_S(\bar{c}) = 1$.

Comparing Lemma 10 above with Lemma 1 in Chapter 2, we see that they looks almost the same, except that the trading probabilities q_B and q_S in Lemma 1 are replaced by $\beta_B q_B$

and $\beta_S q_S$ respectively in Lemma 10. To see the intuition, recall that under full information bargaining a buyer can gain from a meeting (on top of his outside option) only when he proposes, the probability of which is β_B. In other words, he will be indifferent between accepting or rejecting an offer whenever he is a responder. Therefore, keeping α_B and q_B unchanged, we can evaluate the buyer's lifetime payoff W_B as if he will reject any offer. If so, his counterfactual trading probability becomes $\beta_B q_B$ instead of q_B. A similar logic applies to sellers.

As a direct implication, keeping α_B and q_B unchanged, private information bargaining makes the slopes of lifetime payoffs $W_B(v)$ and $W_S(c)$ steeper. It should not be surprising because it is well-known that information rents are monotone in types. As another direct implication, again keeping α_B and q_B unchanged, the slopes of dynamic types $\rho_B(v)$ and $\rho_S(c)$ become flatter:

$$\rho_B'(v) = \frac{r}{r + \alpha_B(\zeta)\beta_B q_B(v)} > 0 \quad \text{a.e. } v \in [\underline{v}, 1] \tag{3.18}$$

$$\rho_S'(c) = \frac{r}{r + \alpha_S(\zeta)\beta_S q_S(v)} > 0 \quad \text{a.e. } c \in [0, \bar{c}]\,. \tag{3.19}$$

The following lemma provides the indifference conditions for the marginal entrants.

Lemma 11 *Under full information, in any nontrivial steady-state equilibrium,* $\rho_B(\underline{v}) = \underline{v}$ *and* $\rho_S(\bar{c}) = \bar{c}$. *Moreover,*

$$\alpha_B(\zeta)\beta_B \int \max\{\underline{v} - \rho_S(c), 0\} \frac{dN_S(c)}{S} = \kappa_B \tag{3.20}$$

$$\alpha_S(\zeta)\beta_S \int \max\{\rho_B(v) - \bar{c}, 0\} \frac{dN_B(v)}{B} = \kappa_S. \tag{3.21}$$

Proof. From Lemma 10 we have $W_B(\underline{v}) = W_S(\bar{c}) = 0$, hence $\rho_B(\underline{v}) = \underline{v}$ and $\rho_S(\bar{c}) = \bar{c}$. Evaluate (3.12) and (3.13) at $v = \underline{v}$ and $c = \bar{c}$, we get the results. ■

Since the buyers' and sellers' reservation prices ρ_B and ρ_S (also called dynamic types) are increasing, the buyers' lowest and highest reservation prices are $\underline{v}$ and $\rho_B(1)$. The sellers' lowest and highest reservation prices are $\rho_S(0)$ and $\bar{c}$.

The following lemma shows that $\rho_S(0) < \underline{v}$, otherwise type $\underline{v}$ buyers prefer not to enter as they cannot recover the search costs. Similarly $\bar{c} < \rho_B(1)$, otherwise type $\bar{c}$ sellers prefer not to enter as they cannot recover the search costs.

Lemma 12 *Under full information, in any nontrivial steady-state equilibrium,* $\rho_S(0) < \underline{v}$ *and* $\bar{c} < \rho_B(1)$.

Proof. We prove $\rho_S(0) < \underline{v}$ first. Suppose $\underline{v} \leq \rho_S(0)$. Then the left-hand side of (3.20) is 0, while the right-hand side is strictly positive, a contradiction. To prove $\bar{c} < \rho_B(1)$, simply apply (3.21) instead of (3.20). ■

As in Chapter 2, define the Walrasian price p^* as the price that clears the flow demand and flow supply:

$$b[1 - F(p^*)] = sG(p^*).$$

Since buyers and sellers always leave the market in pairs, the entry flows of buyers and sellers must be balanced in steady state, i.e. $b[1 - F(\underline{v})] = sG(\bar{c})$.[28] Therefore the marginal entering types $\underline{v}$ and $\bar{c}$ must lie on different sides of the Walrasian price p^*. Although both $\underline{v} \leq p^* \leq \bar{c}$ and $\bar{c} < p^* < \underline{v}$ are possible, the comparisons between $\rho_B(1)$, $\rho_S(0)$ and p^* are, as under private information, unambiguous.

Lemma 13 *Under full information, in any nontrivial steady-state equilibrium,* $\rho_S(0) < p^* < \rho_B(1)$.

Proof. We prove $\rho_S(0) < p^*$ only. The other part is completely parallel. Suppose $\rho_S(0) \geq p^*$. Then clearly $\bar{c} \geq p^*$. Moreover, from Lemma 12 we have $\underline{v} > \rho_S(0) \geq p^*$. But then

$$b[1 - F(\underline{v})] < b[1 - F(p^*)] = sG(p^*) \leq sG(\bar{c}),$$

a contradiction. ■

The following lemma is parallel to Lemma 5 in Chapter 2.

[28] It can be formally derived from steady-state equations (3.3) and (3.4).

Lemma 14 *Under full information, in any nontrivial steady-state equilibrium, we have*

$$1 > \rho_B(1) - \rho_S(0) > K(\zeta_0), \tag{3.22}$$

$$\underline{v} - \bar{c} < K(\zeta_0). \tag{3.23}$$

Proof. Pick any nontrivial steady-state equilibrium. Lemma 10 implies $W_B(1) > 0$ and $W_S(0) > 0$. The first inequality in (3.22), which is equivalent to $W_B(1) + W_S(0) > 0$, follows. Condition (3.20) implies

$$\alpha_B(\zeta)\,\beta_B(\rho_B(1) - \rho_S(0)) > \kappa_B,$$

because (i) $\rho_B(1) > \underline{v}$ and (ii) $\rho_S(c) > \rho_S(0)$ for any c on $[0, \bar{c}]$ (which is the support of N_S).

Similarly (3.21) implies

$$\alpha_S(\zeta)\,\beta_S(\rho_B(1) - \rho_S(0)) > \kappa_S,$$

so that

$$\rho_B(1) - \rho_S(0) > \max\left\{\frac{\kappa_B}{\alpha_B(\zeta)\,\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\,\beta_S}\right\} \geq K(\zeta_0).$$

The last inequality is from Lemma 4 in Chapter 2. This proves (3.22).

We turn to prove (3.23). Notice that (3.20) implies

$$\alpha_B(\zeta)\,\beta_B \max\{\underline{v} - \bar{c}, 0\} < \kappa_B,$$

because $\rho_S(c) < \bar{c}$ for any c on $[0, \bar{c})$. Similarly, (3.21) implies

$$\alpha_S(\zeta)\,\beta_S \max\{\underline{v} - \bar{c}, 0\} < \kappa_S,$$

from which it follows that

$$\max\{\underline{v} - \bar{c}, 0\} < \min\left\{\frac{\kappa_B}{\alpha_B(\zeta)\,\beta_B}, \frac{\kappa_S}{\alpha_S(\zeta)\,\beta_S}\right\} \leq K(\zeta_0). \tag{3.24}$$

The last inequality is again from Lemma 4 in Chapter 2. This proves (3.23). ■

3.3.3 Necessary and sufficient condition for existence

Mortensen and Wright (2002) do not provide a necessary and sufficient condition under which a nontrivial steady-state equilibrium exists. We can fill this gap by applying the technique we developed in Chapter 2. Indeed, our general existence proof for private information model (see Section 2.7 and Appendix A) adapts to full information model with minor changes. In particular the necessary and sufficient condition for the existence is the same as before, which is $K(\zeta_0) < 1$.[29]

Having developed the results in the previous subsection, it is now easy to see the necessity of $K(\zeta_0) < 1$. Indeed, if there exists some nontrivial steady-state equilibrium, then Lemma 14 implies the condition $K(\zeta_0) < 1$.

For the sufficiency part, the proof for the full information model is strictly easier than that for the private information model (which is provided in Section 2.7 and Appendix A) because we do not have to consider proposing strategies in our construction mapping T, whose fixed point characterizes an equilibrium. The essential changes involved are to modify Definition 4 of T_ε by (i) deleting the proposers' problems (2.31) and (2.32), (ii) replacing the expressions inside the square brackets in (2.33) and (2.34) by

$$\beta_B \int_{\{c:\rho_B(v)\geq\rho_S(c)\}} (\rho_B(v) - \rho_S(c)) \frac{dN_S(c)}{S}$$

and

$$\beta_S \int_{\{v:\rho_B(v)\geq\rho_S(c)\}} (\rho_B(v) - \rho_S(c)) \frac{dN_B(v)}{B}$$

respectively, and (iii) redefining q_B and q_S according to (3.14) and (3.15).

Theorem 4 *Given the parameters $(b, s, F, G, M, \beta_B, \beta_S, r, \kappa_B, \kappa_S)$, a nontrivial steady-state equilibrium exists in the full information model if and only if a nontrivial steady-state equilibrium exists in the private information model. More precisely, for either the private in-*

[29] The value $\zeta_0 \equiv \frac{\beta_B\kappa_S}{\beta_S\kappa_B}$ in the full information model should not be interpreted as the buyer-seller ratio in full-trade equilibrium. Nevertheless, it can be, like in the private information model, interpreted as the equilibrium buyer-seller ratio when $r = 0$.

formation or the full information model, a necessary and sufficient condition for existence of a nontrivial steady-state equilibrium is

$$K(\zeta_0) < 1,$$

where ζ_0 *and the function* K *are defined by* (3.9) *and* (3.10).

For the intuition of the existence condition $K(\zeta_0) < 1$, see Section 2.7. Here let us discuss the intuition of the invariance of this condition across different information structures. It suffices to consider the "marginal situation" where the search costs are such that $K(\zeta_0)$ is smaller than but very close to 1. Then only those potential buyers with valuations very close to 1 and those potential sellers with costs very close to 0 would enter. That is to say, all buyers (sellers) in the market are virtually homogeneous in their valuations (costs). It is no wonder that the information structure at the bargaining stages does not alter the existence condition in this situation.

3.3.4 Full-trade equilibrium

In the context of full information bargaining, a nontrivial steady-state equilibrium is full-trade (i.e. every meeting results in a trade) if and only if $\underline{v} \geq \bar{c}$. (That is, the dichotomy of full-trade/non-full-trade and the dichotomy of separated/overlapping supports we introduce in Chapter 2 are the same thing in the context of full information.) We will characterize the full-trade equilibria under full information later (see Section 3.6). For now, let us present the results on the existence and uniqueness of full-trade equilibrium, which are due to Mortensen and Wright (2002). We need some definitions in order to state the following theorem. Let $\psi_B : (0,\infty) \to (0,\infty]$ and $\psi_S : (0,\infty) \to (0,\infty]$ be

$$\psi_B(\zeta) \equiv \frac{\beta_S \kappa_B \zeta}{\beta_B \max\left\{\int_0^{p^*} (p^* - c)\frac{dG(c)}{G(p^*)} - \frac{\kappa_B}{\alpha_B(\zeta)\beta_B}, 0\right\}},$$

$$\psi_S(\zeta) \equiv \frac{\beta_B \kappa_S}{\beta_S \zeta \max\left\{\int_{p^*}^1 (v - p^*)\frac{dF(v)}{1-F(p^*)} - \frac{\kappa_S}{\alpha_S(\zeta)\beta_S}, 0\right\}}.$$

(These functions take the value ∞ whenever their defining expressions have a denominator 0.) Now let $\hat{r} \in (0, \infty]$ be the unique value such that $\hat{r} = \psi_B(\hat{\zeta}) = \psi_S(\hat{\zeta})$ for some $\hat{\zeta} > 0$.[30]

Theorem 5 (Mortensen and Wright, 2002) *Under full information, a (unique) full-trade equilibrium exists if and only if $K(\zeta_0) < 1$ and $r \leq \hat{r}$. Moreover, if r is sufficiently close to 0, then non-full-trade equilibrium does not exist, implying uniqueness of equilibrium.*

From the above definition of $\hat{r}$ we can prove the following results (for full information model), which is parallel to Corollary 2 (for private information model).

Corollary 6 ***(a)*** *In the region where $\hat{r} < \infty$, if κ_B and κ_S increase, then $\hat{r}$ increases, and vice versa.*

(b) *Given any $r > 0$, there is a $\bar{\kappa} > 0$ such that full-trade equilibrium in the full information model does not exist whenever $\kappa_B, \kappa_S < \bar{\kappa}$.*

(c) *Given any $r > 0$, a full-trade equilibrium in the full information model exists when (κ_B, κ_S) is such that $K(\zeta_0)$ is less than but sufficiently close to 1.*

(d) *Given any (κ_B, κ_S) such that $K(\zeta_0) < 1$, a full-trade equilibrium exists when r is sufficiently close to 0.*

Proof. For any $\zeta > 0$ such that $\psi_B(\zeta)$ is finite, $\psi_B(\zeta)$ is strictly increasing in κ_B. Similarly, for any $\zeta > 0$ such that $\psi_S(\zeta)$ is finite, $\psi_S(\zeta)$ is strictly increasing in κ_S. Hence (a) holds.

For any $\zeta > 0$, we have $\psi_B(\zeta) \to 0$ as $\kappa_B \to 0$; and $\psi_S(\zeta) \to 0$ as $\kappa_S \to 0$. Therefore $\hat{r} \to 0$ as $(\kappa_B, \kappa_S) \to \mathbf{0}$, and (b) follows.

It follows from Lemma 4 that, for any $\zeta > 0$, we must have either $\frac{\kappa_B}{\alpha_B(\zeta)\beta_B} \geq K(\zeta_0)$ or $\frac{\kappa_S}{\alpha_S(\zeta)\beta_S} \geq K(\zeta_0)$. Also notice that both $\int_0^{p^*}(p^* - c)\frac{dG(c)}{G(p^*)}$ and $\int_{p^*}^1 (v - p^*)\frac{dF(v)}{1-F(p^*)}$ are constants strictly smaller 1. Then according to the definitions of $\psi_B(\cdot)$ and $\psi_S(\cdot)$, it is impossible to

[30] It is easy to see that ψ_B and ψ_S are continuous. Moreover, ψ_B is nondecreasing and ψ_S is nonincreasing, and

$$\lim_{\zeta \to 0} \psi_B(\zeta) = \lim_{\zeta \to \infty} \psi_S(\zeta) = 0.$$

Therefore $\hat{r}$ is well-defined.

keep both $\psi_B(\zeta)$ and $\psi_S(\zeta)$ finite if we let $K(\zeta_0)$ go to 1 from below. Therefore $\hat{r} = \infty$ when $K(\zeta_0)$ is less than but sufficiently close to 1. Hence (c) follows.

(d) is simply from $\hat{r} > 0$ for any $\kappa_B, \kappa_S > 0$ such that $K(\zeta_0) < 1$. ■

Usually a full-trade equilibrium is easier to exist under full information bargaining.

Example 2 *Take the parameters as in Example 1, i.e.* $b = s = 1/2$, $F(v) = v$, $G(c) = c$, $\beta_B = \beta_S = 1/2$, *and* $M(B, S) = BS/(B + S)$. *Also take* $\kappa_B = \kappa_S = \kappa$. *The* $\hat{r}$ *in Theorem 5 (the threshold of* r *below which a full-trade equilibrium exists under full information) is*

$$\hat{r} = \frac{4\kappa}{\max\{1 - 16\kappa, 0\}},$$

and the r^* *in Theorem 1 (the threshold of* r *below which a full-trade equilibrium exists under private information) is*

$$r^* = \frac{4\kappa}{\max\{1 - 12\kappa, 0\}}.$$

Obviously $\hat{r} \geq r^*$ *and it is strict unless* $\hat{r} = r^* = \infty$. *In other words, a full-trade equilibrium is strictly easier to exist in full information model than in private information model.*

3.4 No-discounting case

The previous two sections describe the private information model and the full information model respectively. The equilibrium properties of the two models exhibit some similarities. They have the same necessary and sufficient condition for the existence of some (nontrivial steady-state) equilibrium. In either model, keeping other parameters unchanged, any equilibrium must be full-trade if the discount rate r is sufficiently close to 0; and any equilibrium must be non-full-trade if r is sufficiently large.

In this section, we completely solve the equilibria for both of the two models for the case where there is no time discounting. Formally, we extend our private information model to allow $r = 0$, and define an associated nontrivial steady-state equilibrium as a tuple $(W_B, W_S, \chi_B, \chi_S, p_B, p_S, N_B, N_S)$ such that (i) it satisfies the conditions in Definition 5 evaluated at $r = 0$, and (ii) it is the limit of some private information equilibrium sequence as $r \to 0$ from above. Similarly, we extend our full information model to allow $r = 0$, and define an associated nontrivial steady-state equilibrium as a tuple $(W_B, W_S, \chi_B, \chi_S, N_B, N_S)$ such that (i) it satisfies the conditions in Definition 6 evaluated at $r = 0$, and (ii) it is the limit of some full information equilibrium sequence as $r \to 0$ from above.[31]

Both models are greatly simplified in the no-discounting case. Furthermore, the no-discounting case provides a benchmark in which the information structure at the bargaining stages plays no role. Indeed, if $r = 0$, the two models are equivalent, in the sense that any equilibrium of the full information model must be an equilibrium of the private information model, and conversely any equilibrium of the private information model must be an equilibrium of the full information model.

To see this, consider the full information model with $r > 0$ and let $r \to 0$. From (3.18) and (3.19), in the limit we have $\underline{v} = \rho_B(1)$ and $\rho_S(0) = \bar{c}$. In words, all buyers (sellers) in the market have the same dynamic valuation (dynamic cost). Participating traders are homogeneous in their dynamic types although they are heterogeneous in their original types. It follows that a trader within a meeting does not really need to observe his partner's types, because all that matter for bargaining are the dynamic types rather than the original types. Therefore, any equilibrium under no discounting would still be an equilibrium when we switch the information structure into the private one.

Conversely, consider the private information model with $r > 0$ and let $r \to 0$. From (2.12) and (2.13), in the limit we have $\underline{v} = \rho_B(1)$ and $\rho_S(0) = \bar{c}$. Clearly a seller (buyer) whenever being a proposer would have no choice but propose the trading price $\underline{v}$ $(\bar{c})$, and this offer would be accepted. Letting the proposer know the responder's type does not have a bite, because the proposer already knows the responder's dynamic type, which is all he needs to make the decision of proposing. Thus, this equilibrium would still be an equilibrium when we switch into the full information model.

We now solve the no-discounting case analytically.[32] We have already claimed that $\underline{v} = \rho_B(1)$ and $\rho_S(0) = \bar{c}$ in any of the two models. Then both Lemma 2(c) (for private

[31] We will see (ii) actually implies (i) in either models.

[32] Mortensen and Wright (2002) have already solved it for their full information model.

information) and Lemma 11 (for full information) are (in the limit) reduced to:

$$\alpha_B(\zeta)\beta_B \max\{\underline{v}-\bar{c},0\} = \kappa_B, \tag{3.25}$$

$$\alpha_S(\zeta)\beta_S \max\{\underline{v}-\bar{c},0\} = \kappa_S. \tag{3.26}$$

Equation (3.25) simply means that the type $\underline{v}$ buyers are indifferent between entering or not: the left-hand side is the expected gain from participating in the market per unit time, and the right-hand side is the corresponding search cost. Similarly equation (3.26) is the indifference condition for marginal sellers.

Noticing that $\alpha_S(\zeta)/\alpha_B(\zeta) = \zeta$, equations (3.25) and (3.26) uniquely pin down the buyer-seller ratio ζ and entry gap $\underline{v}-\bar{c}$:

$$\zeta = \zeta_0, \tag{3.27}$$

$$\underline{v}-\bar{c} = K(\zeta_0) > 0, \tag{3.28}$$

where ζ_0 and $K(\cdot)$ are defined by (3.9) and (3.10).

In steady state, the incoming flow of active buyers must equal the incoming flow of active sellers. Thus in either model we have the following inflow balance equation:

$$b[1-F(\underline{v})] = sG(\bar{c}). \tag{3.29}$$

Given that $K(\zeta_0) < 1$, equations (3.28) and (3.29) have a unique solution for $(\underline{v},\bar{c})$, which is denoted as $(\underline{v}_0,\bar{c}_0)$ (see Figure 3.1). Hence, when $r \to 0$ and $K(\zeta_0) < 1$, the equilibrium buyer-seller ratio and marginal types are unique and given by $(\zeta_0,\underline{v}_0,\bar{c}_0)$.

Other endogenous variables are easily obtained. In particular, the equilibrium is full-trade, so that $q_B(v) = q_S(c) = 1$ for any $v \in [\underline{v}_0, 1]$ and $c \in [0,\bar{c}_0]$. As a result, the aggregate inflow-outflow balance equations become

$$b\,[1-F(\underline{v}_0)] = B\alpha_B(\zeta_0),$$

$$sG(\bar{c}_0) = S\alpha_S(\zeta_0),$$

which pin down the steady-state masses of buyers and sellers in the market:

$$B = \frac{b\,[1-F(\underline{v}_0)]}{\alpha_B(\zeta_0)}, \quad S = \frac{sG(\bar{c}_0)}{\alpha_S(\zeta_0)}.$$

Furthermore, the market distributions of types must be proportional to the corresponding distributions of inflow types:

$$N_B(v) = B \cdot \frac{F(v) - F(\underline{v}_0)}{1 - F(\underline{v}_0)},$$

$$N_S(c) = S \cdot \frac{G(c)}{G(\bar{c}_0)}.$$

From either Lemma 1 (for private information) or Lemma 10 (for full information), we have

$$W_B'(v) = -W_S'(c) = 1$$

for any $v \in [\underline{v}_0, 1]$ and $c \in [0, \bar{c}_0]$. Therefore the equilibrium lifetime payoffs are

$$W_B(v) = \max\{v - \underline{v}_0, 0\},$$

$$W_S(c) = \max\{\bar{c}_0 - c, 0\}.$$

It is easy to verify that these equilibrium objects indeed satisfy Definition 5 and Definition 6. The above analysis yields the following proposition.

Proposition 2 *If $r = 0$, the private information model and the full information model are equivalent, in the sense that the two models have the same set of equilibria.*[33] *In fact, this set is either empty (if $K(\zeta_0) \geq 1$) or a singleton ($K(\zeta_0) < 1$).*[34]

As in Mortensen and Wright (2002), we generally define the welfare measure W as the aggregate lifetime payoffs of a cohort:

$$W \equiv bW_B^{ea} + sW_S^{ea} \tag{3.30}$$

[33] Formally, this statement is not completely rigorous, because by Definition 5 an equilibrium of the private information model is a tuple $(W_B, W_S, \chi_B, \chi_S, p_B, p_S, N_B, N_S)$, while by Definition 6 an equilibrium of the full information model is a collection $(W_B, W_S, \chi_B, \chi_S, N_B, N_S)$; and the proposing strategies (p_B, p_S) in the private information model are functions with one argument, while in the full information model they are functions with two arguments (own type and partner's type). Evidently these can be taken care of, but it would not be interesting at all and we do not bother to do so.

[34] This is under the convention that we identify an equilibrium with another one if they differs only in the proposing strategies of non-entrants and entry strategies of marginal entrants.

where W_B^{ea} (W_S^{ea}) is a buyer's (seller's) ex-ante utility, i.e.

$$W_B^{ea} \equiv \int W_B(v)dF(v), \tag{3.31}$$

$$W_S^{ea} \equiv \int W_S(c)dG(c). \tag{3.32}$$

The welfare measure W is also interpreted as the ex-ante utility of an agent before knowing whether he is a buyer or a seller and what his valuation/cost is. (Recall the normalization that $b + s = 1$ so that the measure of a cohort is 1.)

For the no-discounting case, the welfare measure W can be written as

$$\begin{aligned} W &= b\int_{\underline{v}_0}^{1} (v - \underline{v}_0)\, dF(v) + s\int_0^{\bar{c}_0} (\bar{c}_0 - c)\, dG(c) \\ &= \int_{\underline{v}_0}^{1} b\,[1 - F(v)]\, dv + \int_0^{\bar{c}_0} sG(c)dc. \end{aligned}$$

Figure 3.1 illustrates the equilibrium for the no-discounting case. The black area represents the welfare W. The Walrasian price p^* must be bracketed by the marginal types $\underline{v} = \underline{v}_0$ and $\bar{c} = \bar{c}_0$. Intuitively, it is analogous to the standard demand-supply analysis, with a transaction cost $K(\zeta_0)$ that must be incurred for each transaction.

3.5 Full-trade equilibria and bargaining efficiency

The previous section shows that private information in bargaining has no effect in the special case where there is no time discounting. However, when the discount rate is strictly positive, private information will have impacts. This section turns to the question of whether the private information affects the efficiency with respect to bargaining. Let us start with a definition.

Definition 7 *In either the full information or the private information model, a nontrivial steady-state equilibrium is said to be bargaining-efficient if in this equilibrium the bargaining outcome of every meeting is always ex-post efficient, in the sense that every buyer-seller meeting (on the equilibrium path) results in a trade if and only if the matching surplus is*

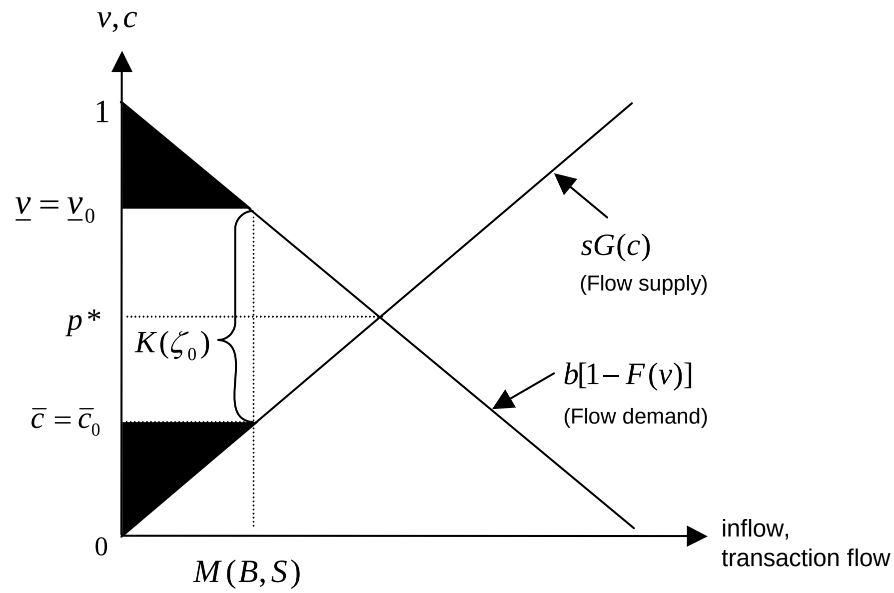

Figure 3.1: Equilibrium when discount rate is zero

non-negative (i.e. $v - c \geq W_B(v) + W_S(c)$), or equivalently the buyer's dynamic value is at least as high as the seller's dynamic cost (i.e. $\rho_B(v) \geq \rho_S(c)$).

A nontrivial steady-state equilibrium is said to be bargaining-inefficient if it is not bargaining-efficient.

Clearly, in the full information model, any nontrivial steady-state equilibrium is bargaining-efficient.

In the private information model, it is not hard to see that any nontrivial steady-state equilibrium is bargaining-efficient if and only if it is full-trade. Suppose an equilibrium is full-trade, then the entry gap $\underline{v} - \bar{c}$ must be strictly positive, so that every meeting (on the equilibrium path) must have positive matching surplus. Thus the equilibrium is also bargaining-efficient. Now suppose an equilibrium is non-full-trade, then either (i) the marginal buyer's offer $p_B(\underline{v})$ will not be accepted with probability 1, or (ii) the marginal seller's offer $p_S(\bar{c})$ will not be accepted with probability 1. To be concrete, let us say (i) is the case. Then there must be sellers with $\rho_S(c) \in (p_B(\underline{v}), \underline{v})$. But then the marginal buyers, when they propose, would not trade with those sellers, although the matching surplus is positive. Therefore the equilibrium is also bargaining-inefficient.

Proposition 3 *Under full information, any nontrivial steady-state equilibrium is bargaining-efficient. Under private information, any nontrivial steady-state equilibrium is bargaining-efficient if and only if it is full-trade.*

We therefore, in the context of private information, only need to recall our results in Chapter 2 on the full-trade equilibrium. By Theorem 2 and Corollary 4, if the discount rate r is small relative to the search costs κ_B and κ_S, then in equilibrium bargaining efficiency is attained. In contrast, by Theorem 1 and Corollary 2 (see also Remark 1), if the discount rate is large relative to the search costs, then in equilibrium some meetings do not result in a trade although there is a positive matching surplus.

Before closing this section, we want to point out that even when private information does not result in bargaining inefficiency in equilibrium, the equilibrium welfare level would still be altered. It is because the private information affects the way bargainers split the matching surplus. This redistribution effect would in turn have impacts on potential traders' entry decisions and hence aggregate welfare. We will discuss it in the next section.

3.6 Entry effect of private information

If the frictions (r, κ_B, κ_S) are zero and the agents behave as if in the Walrasian equilibrium (i.e. type v buyers enter if and only $v \geq p^*$, type c sellers enter if and only if $c \leq p^*$, and every participating trader proposes p^* whenever she is a proposer, and takes p^* as his reservation price whenever she is a responder), then the welfare measure of our models (either full information or private information) would be at the Walrasian level, denoted as

W^*:

$$\begin{aligned} W^* &\equiv b\int_{p^*}^{1}(v-p^*)dF(v)+s\int_0^{p^*}(p^*-c)dG(c) \\ &= b\int_{p^*}^{1}vdF(v)+s\int_0^{p^*}cdG(c) \\ &= \int_{p^*}^{1}bF(v)dv+\int_0^{p^*}sG(c)dc. \end{aligned}$$

Our search models (either full information or private information) necessarily have lower welfare than the Walrasian level. There are three sources of welfare loss. The first source of welfare loss is the direct effect of frictions (r, κ_B, κ_S): search takes time (and traders discount), and search costs have to be paid. The second one is the entry effect: the entry (or search) of a trader induces positive externality to the opposite side of the market (so called "thick market effect") and negative externality to the same side (so called "congestion effect"). The entry of buyers and sellers could be either too much or too little relative to the constrained optimal level. The third effect is bargaining inefficiency, which occurs only in non-full-trade equilibria of the private information model.

In the next Chapter, we will see "convergence results" that imply as the frictions vanish, the last two "behavioral effects" also vanish.[35] However, in this Chapter we are interested in a market with positive frictions, which is why search theory was developed in the first place. We have seen in the previous section that bargaining inefficiency may or may not exist under private information, by and large depending on the composition of frictions.

As for entry effect, it is well understood since Diamond (1981) and Mortensen (1982) that, even under full information, search equilibria are generally not constrained efficient due to entry externalities (or search externalities). However, because this strand of literature typically assumes full information bargaining, it does not tell us the interaction between the entry effect and the information structure. In this section, we shall see private information in bargaining affects welfare even when bargaining inefficiency does not arise. Basically, even when the private information does not reduce the pie of matching surplus, it redistributes

[35] Various versions of this claim for other comparable models can be found in Gale (1987), Mortensen and Wright (2002), Satterthwaite and Shneyerov (2007), Atakan (2008) and Lauermann (2008), etc.

the surplus and hence affects the incentives of entry. As a result, private information alters the level of welfare through the channel of entry externalities.

To elaborate the above point, let us compare the full-trade equilibria of the two models. To ensure both models have some full-trade equilibrium, we assume $K(\zeta_0) < 1$ and $0 \leq r \leq \min\{r^*, \hat{r}\}$, where r^* is given by Theorem 1 and $\hat{r}$ is given by Theorem 5. Also, we shall use subscript "p" to denote private information (e.g. ζ_p) and use subscript "f" to denote full information (e.g. ζ_f).

Section 2.5 shows that, in the context of private information, the unique full-trade equilibrium can be characterized by the following three simple equations with three unknowns $(\zeta_p, \underline{v}_p, \bar{c}_p)$:[36]

$$\alpha_B(\zeta_p)\beta_B(\underline{v}_p - \bar{c}_p) = \kappa_B, \tag{3.33}$$

$$\alpha_S(\zeta_p)\beta_S(\underline{v}_p - \bar{c}_p) = \kappa_S, \tag{3.34}$$

$$b[1 - F(\underline{v}_p)] = sG(\bar{c}_p). \tag{3.35}$$

Equations (3.33) and (3.34) are marginal entering buyers' and sellers' indifference conditions between entering or not. Equation (3.35) is inflow balance equation, which must hold in steady state.

We now turn to the full information full-trade equilibrium.[37] Since the equilibrium is full-trade, we have $\underline{v}_f \geq \bar{c}_f$, and

$$\frac{N_{Bf}(v)}{B_f} = \frac{F(v) - F(\underline{v}_f)}{1 - F(\underline{v}_f)}, \quad \frac{N_{Sf}(c)}{S_f} = \frac{G(c)}{G(\bar{c}_f)}.$$

Also, (3.18) and (3.19) gives the dynamic types:

$$\rho_{Bf}(v) = \frac{rv + \alpha_B(\zeta_f)\beta_B\underline{v}_f}{r + \alpha_B(\zeta_f)\beta_B}, \quad \rho_{Sf}(c) = \frac{rc + \alpha_S(\zeta_f)\beta_S\bar{c}_f}{r + \alpha_S(\zeta_f)\beta_S}.$$

[36] Equations (3.33) through (3.35) are the same as equations (3.25), (3.26) and (3.29) for the no discounting case. It is simply because in the private information model, the full-trade equilibrium $(\zeta_p, \underline{v}_p, \bar{c}_p)$ does not vary with r.

[37] This is already characterized by Mortensen and Wright (2002).

Substituting these into the marginal type equations (3.20) and (3.21) in Lemma 11, we obtain

$$\alpha_B\left(\zeta_f\right)\beta_B\int_0^{\bar{c}_f}\left[\underline{v}_f-\rho_{Sf}(c)\right]\frac{dG\left(c\right)}{G\left(\bar{c}_f\right)}=\kappa_B, \tag{3.36}$$

$$\alpha_S\left(\zeta_f\right)\beta_S\int_{\underline{v}_f}^{1}\left[\rho_{Bf}(v)-\bar{c}_f\right]\frac{dF\left(v\right)}{1-F\left(\underline{v}_f\right)}=\kappa_S. \tag{3.37}$$

The inflow balance equation still holds here:

$$b\left[1-F(\underline{v}_f)\right]=sG(\bar{c}_f). \tag{3.38}$$

Equations (3.36) – (3.38) uniquely pin down $\left(\zeta_f,\underline{v}_f,\bar{c}_f\right)$ for all $r\leq\hat{r}$.

We are now ready to claim that private information in bargaining deters entry, at least within the full-trade class of equilibria.

Proposition 4 *Fix the parameters $(b,s,F,G,M,\beta_B,\beta_S,r,\kappa_B,\kappa_S)$ such that $r>0$ and both the full information model and the private information model have a full-trade equilibrium (i.e. $K\left(\zeta_0\right)<1$ and $0<r\leq\min\left\{r^*,\hat{r}\right\}$). Comparing the two full-trade equilibria, we must have $\underline{v}_p>\underline{v}_f$ and $\bar{c}_p<\bar{c}_f$.*

Proof. From the inflow balance equations (3.35) and (3.38), the two inequalities $\underline{v}_p>\underline{v}_f$ and $\bar{c}_p<\bar{c}_f$ are equivalent. Therefore it suffices to prove $\underline{v}_p-\bar{c}_p>\underline{v}_f-\bar{c}_f$. We will consider two cases: $\zeta_p\geq\zeta_f$ and $\zeta_p<\zeta_f$.

Suppose $\zeta_p\geq\zeta_f$ first. Then $\alpha_B(\zeta_p)\leq\alpha_B(\zeta_f)$ since α_B is nonincreasing. Now the buyers' marginal equations (3.33) and (3.36) imply

$$\alpha_B(\zeta_p)(\underline{v}_p-\bar{c}_p)=\alpha_B(\zeta_f)\left[\underline{v}_f-\int_0^{\bar{c}_f}\rho_{Sf}(c)\frac{dG(c)}{G(\bar{c}_f)}\right]$$

$$\underline{v}_p-\bar{c}_p\geq\underline{v}_f-\int_0^{\bar{c}_f}\rho_{Sf}(c)\frac{dG(c)}{G(\bar{c}_f)}.$$

Moreover, $r>0$ implies $\rho_{Sf}\left(0\right)<\bar{c}_f$ and hence

$$\int_0^{\bar{c}_f}\rho_{Sf}(c)\frac{dG(c)}{G(\bar{c}_f)}<\bar{c}_f.$$

Combining the above results, we have $\underline{v}_p-\bar{c}_p>\underline{v}_f-\bar{c}_f$, as desired.

Now suppose $\zeta_p < \zeta_f$. Applying a symmetric logic to the sellers' marginal equations (3.34) and (3.37), we can easily obtain $\underline{v}_p - \bar{c}_p > \underline{v}_f - \bar{c}_f$ again. ∎

To see the intuition of Proposition 4, we need to compare the entry incentives of the two models. In the full information model, a bargainer extracts the full surplus of matching if he proposes, but gets none if he responds. In contrast, if types are private information, then typically some information rent is redistributed from the proposer of a meeting to the responder. Now recall that the proposer of a meeting is randomly chosen and notice that the responder's expected information rent is higher if his type is better (i.e. higher value or lower cost). As a result, the redistribution of rent from proposers to responders would be translated into a redistribution from inefficient (i.e. low value or high cost) traders to efficient (i.e. high value or low cost) traders in the market.

Therefore, if the information structure is switched from full information into private information, inframarginal entrants have higher incentive to enter while marginal entrants have lower incentive to enter. Now it is clear that the private information model tends to induce less entry, because what matters to the equilibrium amount of entry is the entry incentive of marginal entrants. Proposition 4 simply says that this is unambiguously true when $r > 0$ and we are comparing two full-trade equilibria.

Put it more concretely. Suppose, as a thought experiment, that the marginal types $\underline{v}$ and $\bar{c}$ are the same across the two models. Notice that a marginal entering buyer as a responder gets zero rent from bargaining anyway. But as a proposer, a marginal entering buyer under full information extracts the full rent from the seller he meets; while he is (in a full-trade equilibrium) only able to extract the rent of the most inefficient type $\bar{c}$ of sellers under private information. If $r = 0$, it makes no difference because the distribution of sellers' dynamic types collapses to a single point. But if r is positive, that marginal entering buyer tends to be worse off under private information than under full information. This tendency might be reversed only when the buyer-seller ratio is lower (so that the buyers' arrival rate of being matched is higher) under private information, i.e. $\zeta_p < \zeta_f$. Similarly, if $r > 0$, a marginal seller is worse off under private information than under full information, unless $\zeta_p > \zeta_f$. Since $\zeta_p < \zeta_f$ and $\zeta_p > \zeta_f$ cannot hold together, either more buyers or more sellers must be attracted to enter. Finally, the entry must be more for both the buyers' side and the sellers' side, because the inflows of the two sides have to balance in steady state.

Then why is the result in Proposition 4 only for full-trade equilibria? It is because in non-full-trade equilibria of the private information model the marginal entrants' proposing gains also depend on the steady-state distributions of dynamic types in the market. These distributions and the traders' bargaining behaviors affect each other in a highly non-trivial way. Thus it is conceivable that, from a marginal buyer's (seller's) standpoint, the distribution of sellers' (buyers') dynamic types in the private information model is much more favorable than the full information counterpart. And this effect might dominate the aforementioned information rent effect.

Our next goal is to evaluate the impact of private information on the equilibrium buyer-seller ratio ζ and level of welfare W. To proceed, in the rest of this section we shall focus on cases where r is positive but sufficiently close to 0. In such cases, both models have a unique equilibrium, which is full-trade. Doing this has several advantages, both methodologically and technically. First, we can annihilate bargaining inefficiency so that the entry effect is isolated out. Second, by virtue of uniqueness we do not need to worry about the selection of equilibria. Third, the relatively simple structures of full-trade equilibria in both models make it feasible to compare the levels of welfare under private and full information. Fourth, by virtue of equivalence between the two models in the no-discounting case, studying sufficiently small discounting case only amounts to working out the "first-order effects" of r around the $r = 0$ case. Yet the main insight gained from our analysis should also enlighten our understanding of the main driving forces in the general case.

From now on, we shall think of the equilibrium objects as functions of r. For example, we shall write $\zeta_p(r)$, $\zeta_f(r)$, $W_{Bp}(v; r)$, $W_{Sf}(c; r)$ etc., although the dependency on r might be suppressed for notational simplicity.

The welfare measure (defined by (3.30) in general) in the full information model is:

$$
\begin{aligned}
W_f(r) &= b\int W_{Bf}(v;r)dF(v) + s\int W_{Sf}(c;r)dG(c) \qquad (3.39)\\
&= \frac{\alpha_B(\zeta_f)\beta_B}{r+\alpha_B(\zeta_f)\beta_B} b \int_{\underline{v}_f}^{1} (v-\underline{v}_f)\, dF(v) \\
&\quad + \frac{\alpha_S(\zeta_f)\beta_S}{r+\alpha_S(\zeta_f)\beta_S} s \int_0^{\bar{c}_f} (\bar{c}_f - c)\, dG(c).
\end{aligned}
$$

And, the welfare measure in the private information model is:

$$
\begin{aligned}
W_p(r) &= b\int W_{Bp}(v;r)dF(v) + s\int W_{Sp}(c;r)dG(c) \qquad (3.40)\\
&= \frac{\alpha_B(\zeta_0)}{r+\alpha_B(\zeta_0)} bW_{B0}^{ea} + \frac{\alpha_S(\zeta_0)}{r+\alpha_S(\zeta_0)} sW_{S0}^{ea},
\end{aligned}
$$

where W_{B0}^{ea} (W_{S0}^{ea}) is a buyer's (seller's) ex-ante utility in the no-discounting case, i.e.

$$
W_{B0}^{ea} \equiv \int_{\underline{v}_0}^{1} (v-\underline{v}_0)\, dF(v), \quad W_{S0}^{ea} \equiv \int_0^{\bar{c}_0} (\bar{c}_0 - c)\, dG(c). \qquad (3.41)
$$

It is clear from (3.33) – (3.35) that, under private information and small r, the equilibrium buyer-seller ratio ζ_p and the marginal entering types $\underline{v}_p$ and $\bar{c}_p$ do not change when r varies. Thus ζ_p, $\underline{v}_p$ and $\bar{c}_p$ are simply at the levels of the no-discounting case. Mathematically, $\zeta_p(r) = \zeta_0$, $\underline{v}_p(r) = \underline{v}_0$ and $\bar{c}_p(r) = \bar{c}_0$ for all r sufficiently close to 0. Furthermore, as we have claimed in Section 3.4, when $r = 0$, the two models are equivalent. Indeed, it is easy to verify that $\zeta_f(0) = \zeta_0$, $\underline{v}_f(0) = \underline{v}_0$, $\bar{c}_f(0) = \bar{c}_0$ and $W_p(0) = W_f(0)$. By virtue of these, the comparison between the two models for sufficiently small $r > 0$ amounts only to working out the derivatives $\zeta_f'(0)$, $W_p'(0)$ and $W_f'(0)$.

Proposition 5 *For all sufficiently small $r > 0$, the private information model has higher (resp. lower) buyer-seller ratio compared to the full information model if*

$$
\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}
$$

is negative (resp. positive).

Proof. It is shown in Appendix B that the sign of $\zeta_f'(0)$ is the same as that of the expression $\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}$. This, together with $\zeta_f(0) = \zeta_p(r)$ for sufficiently small r, implies the result. ∎

Since $\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}$ could be positive or negative, Proposition 5 implies that the private information model may have higher or lower buyer-seller ratio compared to the full information model.

Under private information, the slope of the welfare $W_p'(r)$ evaluated at $r = 0$ is

$$W_p'(0) = -\frac{bW_{B0}^{ea}}{\alpha_B(\zeta_0)} - \frac{sW_{S0}^{ea}}{\alpha_S(\zeta_0)}. \tag{3.42}$$

This is simply the direct effect of discounting. In particular, the effect of discounting on buyers' (resp. sellers') welfare is proportional to their expected searching time $1/\alpha_B$ (resp. $1/\alpha_S$).

Under full information, in contrast, the slope of the welfare $W_f'(r)$ evaluated at $r = 0$, as shown in Appendix B, is

$$W_f'(0) = -\frac{bW_{B0}^{ea}}{\alpha_B(\zeta_0)} - \frac{sW_{S0}^{ea}}{\alpha_S(\zeta_0)} - sG(\bar{c}_0)K'(\zeta_0)\zeta_f'(0). \tag{3.43}$$

Other than the direct effect, the increase in r away from 0, by inducing additional entry, could increase or decrease the buyer-seller ratio ζ_f, which in turn affects the expected searching time $1/\alpha_B$ and $1/\alpha_S$. Thus the indirect effect on the total accumulated search costs incurred by a cohort is the last term in (3.43).

In Appendix B, we also show that the difference of the two slopes can be written as

$$\begin{aligned} W_p'(0) - W_f'(0) &= sG(\bar{c}_0)K'(\zeta_0)\zeta_f'(0) \\ &= K(\zeta_0)[\sigma_S(\zeta_0) - \beta_S]\left(\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}\right) \end{aligned} \tag{3.44}$$

where $\sigma_S(\zeta) \equiv 1 - \zeta m'(\zeta)/m(\zeta)$ is the elasticity of the matching function with respect to the mass of sellers (i.e. $\sigma_S(\zeta) = SM_2(B,S)/M(B,S)$). We thus have the following theorem.

Theorem 6 *For all sufficiently small $r > 0$, the private information welfare $W_p(r)$ is higher (resp. lower) than the full information welfare $W_f(r)$, if*

$$[\sigma_S(\zeta_0) - \beta_S]\left(\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}\right)$$

is positive (resp. negative).

It is easy to see that the difference $W_p'(0) - W_f'(0)$ may be either positive or negative, depending on the elasticity of the matching function, the search costs, the new-born rates and the new-born distributions. For example, if the new-born rates are equal (i.e. $b = s$), the new-born distributions F and G are flips of each other (i.e. $1 - F(x) = G(1 - x)$ for all $x \in [0, 1]$, so that $W_{S0}^{ea} = W_{B0}^{ea}$) and the matching function is Cobb-Douglas $\sqrt{BS}$ (so that $\sigma_S = 1/2$), then the sign of $W_p'(0) - W_f'(0)$ is the same as $\left(\frac{1}{2} - \beta_S\right)(\kappa_B - \kappa_S)$. In other words, when the discount rate is positive but small, the private information welfare is higher (resp. lower) than the full information welfare if the side with greater bargaining power incurs higher (resp. lower) search costs.

The intuition behind Theorem 6 is the following. Basically, the first factor $\sigma_S(\zeta_0) - \beta_S$ summarizes entry externalities, while the second factor $\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}$ represents how information structure affects the equilibrium buyer-seller ratio. The product of the two hence represents the interaction between entry externalities and information structure.

To get more insight, recall that traders' entry imposes positive externality to the opposite side of the market and negative externality to the same side. In case of constant returns to scale matching technology and zero discount rate, Mortensen and Wright (2002) show that the positive and negative externalities completely cancel out only when the Hosios (1990) condition holds, i.e. $\sigma_S = \beta_S$. If, for example, the elasticity of matching function with respect to the mass of sellers, σ_S, is larger than sellers' bargaining weight β_S, then the equilibrium buyer-seller ratio is higher than the constrained optimal level, hence decreasing ζ would be welfare enhancing. On the other hand, Proposition 5 implies that for small positive r, if $\frac{sW_{S0}^{ea}}{\kappa_S} > \frac{bW_{B0}^{ea}}{\kappa_B}$, then the private information model has smaller ζ. Therefore, the private information model could have better welfare performance if both $\sigma_S > \beta_S$ and $\frac{sW_{S0}^{ea}}{\kappa_S} > \frac{bW_{B0}^{ea}}{\kappa_B}$ hold.

3.7 Concluding remarks

Until recently, the literature of search models and dynamic matching and bargaining games usually adopts (generalized) Nash bargaining solution, which inevitably requires that the bargainers know each other's type during the bargaining. This might not be an appealing assumption for many applications.

In order to understand the impact of releasing this common assumption, we have analyzed and compared two models of dynamic matching markets: the private information model and the full information model. The two models differ in only one aspect: whether the bargainers observe each other's type during the bargaining.

There are two kinds of frictions: discount rate and search costs. If the discount rate is zero, private information bargaining has no impact at all. More generally, the smaller the discount rate relative to the search costs, the more alike the two models are. The bargaining efficiency, an equilibrium property of the full information model, is maintained under private information bargaining as long as the discount rate is small enough relative to the search costs. Furthermore, private information bargaining does not affect when the market would breakdown and when open.

The private information model induces less potential traders to enter, at least when the discount rate is small. Intimately relating to Hosios (1990) condition, the impact of private information on social welfare could be either positive or negative.

Before closing this chapter, we note that the dynamic structure of our models is important to obtain our results. An easy way to see this is to notice that the discount rate plays a crucial role in our analyses and results; and the discount rate can play a role only in dynamic models. Indeed, the uniqueness of full-trade equilibria hinges on small discounting. Our results on entry effect hinges on the uniqueness and simple characterization of full-trade equilibria.

Chapter 4

Rate of Convergence towards Perfect Competition

4.1 Introduction

This chapter continues our study of decentralized dynamic matching markets.[38] The previous two chapters analyze the markets with non-vanishing frictions. This chapter, in contrast, studies the convergence properties of dynamic matching markets, as search frictions vanish. Our baseline model is the one in Chapter 2. In particular, the buyers and sellers participating in the market are matched pairwise; and every pair of buyer and seller bargains over the trading outcome under the so-called random-proposer bargaining protocol, and under two-sided private information.

In order to reduce repetition, this chapter is not prepared to be self-contained. Readers should have read either Chapter 2 or Chapter 3 before reading this chapter.

Our basic results are as follows. As frictions vanish, we not only show that the equilibrium price range collapses to the Walrasian (or market-clearing) price, but also show that the rate of convergence is linear, i.e. of the same order as frictions (Theorem 7 and Corollary 7). Furthermore, under random-proposer bargaining, equilibrium welfare also converges to the first best Walrasian level at the linear rate, which is shown to be the fastest possible rate among all bargaining mechanisms (Theorem 9, Theorem 10 and Corollary 11).

[38] The chapter significantly includes the materials in my manuscript "The Rate of Convergence to Perfect Competition of a Simple Matching and Bargaining Mechanism", which is joint with my thesis co-supervisor Artyom Shneyerov.

We also provide two robustness checks for our basic results. The first one is to assume full information bargaining, as in Mortensen and Wright (2002), rather than private information bargaining. We show that our basic results are robust to this switch of information structure (Theorem 8 and Corollary 10). The second robustness check is to assume bilateral double auction bargaining, as first introduced by Chatterjee and Samuelson (1983), instead of random-proposer take-it-or-leave-it bargaining. We show that our basic results are not robust to this switch of bargaining protocol. More precisely, along some sequences of nontrivial steady-state equilibria under double auction bargaining, the equilibrium price range does not collapse to the Walrasian price, and the equilibrium welfare level does not converge to the Walrasian welfare level. These results suggest that information structure at the bargaining stages does not affect asymptotic efficiency, but bargaining protocol might.

To understand why the random-proposer bargaining has robust convergence property but the double auction bargaining does not, first notice the well known fact that double auction bargaining generates plethora of equilibria. As it turns out, although there are sequences that are convergent to perfect competition, we are also able to select sequences of progressively inefficient equilibria that keep far from it, no matter how small the frictions are.

This inefficiency along our non-convergent sequences is due to a positive entry gap $\underline{v} - \bar{c}$ that is bounded away from zero even when frictions vanish. (Recall that $\underline{v}$ denotes the lowest valuation of those buyers who enter; and $\bar{c}$ denotes the highest cost of sellers who enter.) It is because under double auction, the bargaining power of any bargainer is not guaranteed. One can construct a double auction full-trade equilibrium with large entry gap by giving one side of the market, say sellers, large bargaining power. It makes the buyers reluctant to enter. But it does not mean the sellers have strong incentive to enter. It is because in steady state, more entry of sellers makes the stock of sellers accumulate, so that the steady-state buyer-seller ratio is so low, cancelling the incentive of entry brought by the high bargaining power. This is the intuition behind Theorem 11.

In short, under double auction, the potentially unbalanced distribution of bargaining power between buyers and sellers can seriously deter entry of both of the two sides, leading to large inefficiency for arbitrarily small frictions. Random-proposer bargaining, on the other hand, distributes the bargaining power between buyers and sellers rather evenly, which guarantees that both sides have the right entry incentives in the limit.

We also discuss another class of progressively inefficient equilibria under double auction, which are so-called *two-step equilibria*. In such equilibria, there is again positive entry gap that is bounded away from zero, but there is no unbalancedness between buyers and sellers. Instead, we play with some kind of unbalancedness between good traders (i.e. high valuation buyers and low cost sellers) and bad traders (Theorem 13).

The structure of this chapter is as follows. Section 4.2 borrows the framework in Chapter 2, which assumes private information random-proposer bargaining, as our baseline model. Section 4.3 derives for our baseline model the rate of convergence of equilibrium price range to the Walrasian price as frictions are removed. This section also shows that the rate of convergence remains unchanged if we assume full information bargaining instead. Section 4.4 gives the rate of convergence of equilibrium welfare to the Walrasian welfare level. This section also proves that this rate is the fastest possible rate among all bargaining mechanisms, either under private or full information. Section 4.5 presents and proves our results for the double auction bargaining. Section 4.6 concludes.

4.2 The baseline model

We take the dynamic matching and bargaining game we study in Chapter 2 as the baseline model of this Chapter. In particular, the market we study is decentralized; searching for a trading partner is costly; the trading decisions and trading prices are determined by bilateral bargaining under two-sided incomplete information. Our equilibrium concept is the one we call nontrivial steady-state equilibrium. Readers can consult Chapter 2 for the details and Section 3.2 for a brief review.

Recall that the flow rate of pairwise matching generated in the market is given by a matching function $M(B, S)$, where B and S are the masses of active buyers and active sellers currently in the market. Since we want to study the convergence properties of our model as search frictions vanish, let us embed a shifter τ in the matching function, and write $M(B,S;\tau)$ instead of $M(B,S)$. This shifter τ is chosen to be inversely proportional to the rate of matching. That is,

$$M(B,S;\tau) \equiv \frac{\tilde{M}(B,S)}{\tau}$$

for some function $\tilde{M} : \mathbb{R}_+^2 \to \mathbb{R}_+$ that satisfies Assumption 2 (see p.12). Therefore τ is analogous to the time length between matches in discrete time models, e.g. Satterthwaite and Shneyerov (2007).

The parameter τ plays a crucial role throughout this chapter, because we are interested in the asymptotic properties of our model as $\tau \to 0$.[39] Let us say a bit more about τ.

Note that τ is proportional to an active trader's expected waiting time until his next meeting. To see this, notice that, given steady-state active trader masses B and S, trading opportunities for a buyer arrive at the Poisson rate $M(B,S;\tau)/B$ or equivalently $\tilde{M}(B,S)/\tau B$. Therefore the expected waiting time is $\tau \cdot B/\tilde{M}(B,S)$. Similarly, the expected waiting time for the seller is $\tau \cdot S/\tilde{M}(B,S)$.

The inverse of τ can also be interpreted as the level of competition intensity that is analogous to the number of traders in the centralized double auction literature. To see why, recall that in a centralized market, traders are competing intratemporally with all other traders on the same side. In contrast, in the dynamic matching environment here, traders, owing to the matching frictions, are not directly competing with all other traders on the same side whenever they bargain with their partners. But they do intertemporally compete with others in the sense that their partners have the option to search another to trade with. Since $1/\tau$ is proportional to arrival rates, it measures the *local market size* that reflects the intensity of this intertemporal competition.

Since the buyers' and sellers' arrival rates of being matched, denoted as α_B and α_S in Chapter 2, directly depend on τ, let us write $\alpha_B(\zeta,\tau)$ and $\alpha_S(\zeta,\tau)$ instead of $\alpha_B(\zeta)$ and

[39] All of our results hold equally well if we fix τ and let the discount rate r, search costs κ_B and κ_S tend to 0 proportionally, instead of letting $\tau \to 0$.

$\alpha_S(\zeta)$. (Recall that $\zeta \equiv B/S$.) More precisely,

$$\alpha_B(\zeta,\tau) \equiv \frac{\tilde{M}(\zeta,1)}{\tau\zeta},$$

$$\alpha_S(\zeta,\tau) \equiv \frac{\tilde{M}(\zeta,1)}{\tau}.$$

The function K (see p.24) also directly depends on τ, so we write $K(\zeta,\tau)$ rather than $K(\zeta)$. Note that, given any ζ, $K(\zeta,\tau)$ is proportional to τ. Specifically,

$$\begin{aligned} K(\zeta,\tau) &\equiv \frac{\kappa_B}{\alpha_B(\zeta,\tau)} + \frac{\kappa_S}{\alpha_S(\zeta,\tau)} \\ &= \tau \cdot K(\zeta,1). \end{aligned}$$

All other notations are left unchanged.

4.3 Rate of convergence of trading prices

In (nontrivial steady-state) equilibrium, trading prices are different across transactions, simply because buyers and sellers in the market are heterogeneous, and the matching and the selection of proposer are random. Recall that, in our terminology and notations in Chapter 2, the price offer proposed by the proposer (either buyer or seller) of a meeting must fall within what we call the *proposing interval* $[p_B(\underline{v}), p_S(\bar{c})]$; while the reservation price (or dynamic type) of the responder must fall within what we call the *responding interval* $[\rho_S(0), \rho_B(1)]$. (Of course, these intervals implicitly depend on τ, and on which equilibrium is prevailing.) Besides, the Walrasian price (or market-clearing price) p^* is the price that clears the flow demand $b[1-F(\cdot)]$ and flow supply $sG(\cdot)$:

$$b[1-F(p^*)] = sG(p^*).$$

The purpose of this section is to prove that both the proposing interval $[p_B(\underline{v}), p_S(\bar{c})]$ and the responding interval $[\rho_S(0), \rho_B(1)]$ collapse to Walrasian price p^* as $\tau \to 0$, and furthermore to show the speed of it.

At this point it is helpful to recall Lemma 3, which asserts that in any nontrivial steady-state equilibrium,

$$p^* \in [p_B(\underline{v}), p_S(\bar{c})] \subset [\rho_S(0), \rho_B(1)].$$

Hence it suffices to consider convergence of the length $\rho_B(1) - \rho_S(0)$. Indeed, we will show that $\rho_B(1) - \rho_S(0)$ is $O(\tau)$. In other words, as $\tau \to 0$, the length $\rho_B(1) - \rho_S(0)$ converges to 0 at the linear rate.

4.3.1 Convergence of full-trade equilibria

Before proving our general rate of convergence theorem, we show how the linear rate is obtained when we restrict our attention to full-trade equilibria. This can be done in a simple manner because a full-trade equilibrium admits a simple characterization.

Recall that, in a full-trade equilibrium (if it exists), the buyer-seller ratio ζ, and the marginal entering types $\underline{v}$ and $\bar{c}$ are uniquely determined by the following three equations:

$$\zeta = \frac{\beta_B}{\beta_S}\frac{\kappa_S}{\kappa_B} \equiv \zeta_0,$$

$$\underline{v} - \bar{c} = K(\zeta_0, \tau),$$

$$b[1 - F(\underline{v})] = sG(\bar{c}).$$

It follows that the entry gap $\underline{v} - \bar{c}$ converges to 0 at the linear rate in τ. Recall Lemma 1 (see p.16). Since $q_B(v) = q_S(c) = 1$ and $\zeta = \zeta_0$ in the full-trade equilibrium, the slopes of responding strategies also converge to 0 linearly in τ. Consequently, $\rho_B(1) - \rho_S(0)$ converges at that rate as well.

4.3.2 General convergence theorem

Proving that *all* equilibria (i.e. also non-full-trade) converge at the linear rate in τ is much harder. However, our result is neat.

Theorem 7 (Rate of convergence for trading prices) *Fix $\tau > 0$. In any nontrivial steady-state equilibrium, we have*

$$K(\zeta_0, \tau) \leq p_S(\bar{c}) - p_B(\underline{v}) \leq \rho_B(1) - \rho_S(0) \leq K(\zeta_0, \tau)\left(1 + \frac{2r}{\kappa}\right)^3,$$

where $\zeta_0 \equiv \beta_B \kappa_S / \beta_S \kappa_B$ and $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

We will prove Theorem 7 in the next subsection. Notice that both the upper and lower bounds in the theorem are proportional to τ. We thus conclude that the proposing interval and responding interval collapse at the linear rate as $\tau \to 0$.

Moreover, the upper bound provided in Theorem 7 converges to the lower bound as r gets small relative to $\kappa \equiv \min\{\kappa_B, \kappa_S\}$. It indicates that our bounds are tight at least when the discount rate is small relative to the search costs.

As a corollary of Theorem 7, traders' proposing and responding strategies must converge to the Walrasian price at no-slower-than-linear convergence rate.

Corollary 7 *Fix $(r, \kappa_B, \kappa_S) \gg \mathbf{0}$. For any sequence of nontrivial steady-state equilibria parametrized by τ such that $\tau \to 0$, the proposing interval $[p_{B\tau}(\underline{v}), p_{S\tau}(\bar{c})]$ and responding interval $[\rho_{S\tau}(0), \rho_{B\tau}(1)]$ collapse to the Walrasian price $\{p^*\}$ at no-slower-than-linear convergence rate. More precisely,*

$$\begin{aligned} & \max\{|p_{B\tau}(\underline{v}) - p^*|, |p_{S\tau}(\bar{c}) - p^*|, |\rho_{S\tau}(0) - p^*|, |\rho_{B\tau}(1) - p^*|\} \\ \leq\ & K(\zeta_0, \tau)\left(1 + \frac{2r}{\kappa}\right)^3. \end{aligned}$$

Before turning to the proof of Theorem 7, we make two remarks.

Remark 2 *In the previous two chapters and Mortensen and Wright (2002), frictions are represented by the discount rate and search costs. Our result can equally well be interpreted that way: fix the matching function and let the discount rate and search costs be $\tau \cdot (r, \kappa_B, \kappa_S)$, then the equilibrium responding interval and proposing interval would collapse at linear rate as $\tau \to 0$. Indeed, the upper and lower bounds in Theorem 7 do not change if we replace τ by 1 and then (r, κ_B, κ_S) by $\tau \cdot (r, \kappa_B, \kappa_S)$ (note that $K(\zeta_0, 1)$ will also be replaced by $\tau \cdot K(\zeta_0, 1)$).*

Remark 3 *We interpret Theorem 7 as a rate of convergence result because our interest of this chapter is the convergence of decentralized market towards perfect competition. But it is clear that Theorem 7 is much more than merely an asymptotic result. More precisely, Theorem 7 provides upper and lower bounds of the lengths* $p_S(\bar{c}) - p_B(\underline{v})$ *and* $\rho_B(1) - \rho_S(0)$ *(and hence the deviation of trading prices from the Walrasian price) for any parameter profile and any nontrivial steady-state equilibrium. In other words, it is also a result for the world of non-vanishing frictions. In this regard, it is complementary to our results in the previous two chapters.*

The above two remarks can be made for all the results throughout this and the next sections.

4.3.3 Proof of Theorem 7

We are now ready to prove Theorem 7. The following formal proof will be followed by some intuition behind it.

Proof of Theorem 7. *Step 1*: We claim that

$$\text{(a):}\quad \frac{\underline{v} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} \geq \frac{\kappa_B}{r + \kappa_B}$$

$$\text{(b):}\quad \frac{\rho_B(1) - \bar{c}}{\rho_B(1) - \rho_S(0)} \geq \frac{\kappa_S}{r + \kappa_S}.$$

We provide the proof for part (a) only. The proof for part (b) is the flip of that for part (a). The buyers' marginal type equation in Lemma 2(c) (see p.18) can be written as $\alpha_B(\zeta)\beta_B\Gamma_S(p_B(\underline{v}))[\underline{v} - p_B(\underline{v})] = \kappa_B$ where $\Gamma_S(p) \equiv \int_{\{c:p-c\geq W_S(c)\}} \frac{dN_S(c)}{S}$. Notice that $q_B(v) \geq \beta_B\Gamma_S(p_B(\underline{v})) > 0$ whenever $v \in [\underline{v}, 1]$, and that $\underline{v} - \rho_S(0) \geq \underline{v} - p_B(\underline{v}) > 0$, we have $\alpha_B q_B(v)(\underline{v} - \rho_S(0)) \geq \kappa_B$ whenever $v \in [\underline{v}, 1]$. Then for almost all $v \in [\underline{v}, 1]$,

$$\rho_B'(v) = \frac{r}{r + \alpha_B q_B(v)} \leq \frac{r}{\kappa_B/(\underline{v} - \rho_S(0))}.$$

Hence

$$\rho_B(1) - \underline{v} = \int_{\underline{v}}^{1} \rho_B'(v)dv \leq \frac{r}{\kappa_B/(\underline{v} - \rho_S(0))},$$

$$\frac{\rho_B(1) - \underline{v}}{\underline{v} - \rho_S(0)} \le \frac{r}{\kappa_B},$$

$$\frac{\underline{v} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} = \frac{1}{1 + (\rho_B(1) - \underline{v})/(\underline{v} - \rho_S(0))} \ge \frac{1}{1 + \frac{r}{\kappa_B}} = \frac{\kappa_B}{r + \kappa_B}.$$

Step 2: We claim that

$$\text{(a)}: \quad \min\{\underline{v}, \bar{c}\} - \rho_S(0) \le \frac{4r(r + \kappa_B)}{\alpha_S \beta_S \kappa_B}$$

$$\text{(b)}: \quad \rho_B(1) - \max\{\underline{v}, \bar{c}\} \le \frac{4r(r + \kappa_S)}{\alpha_B \beta_B \kappa_S}.$$

Again by symmetry, we only provide a proof for (a). Recall that $p_S(c)$ solves the sellers' proposing problem in (2.5). In other words, $p_S(c) \in \arg\max_{p\in[0,1]} [1 - \Gamma_B(p)] [p - \rho_S(c)]$ where $\Gamma_B(p) \equiv \int_{\{v: v-p \ge W_B(v)\}} \frac{dN_B(v)}{B}$.

Let $y \equiv \min\{\underline{v}, \bar{c}\} - \rho_S(0)$. Consider a type c seller with $\rho_S(c) \le \rho_S(0) + y/2$, then

$$\begin{aligned}
[1 - \Gamma_B(p_S(c))] [p_S(c) - \rho_S(c)] &\ge [1 - \Gamma_B(\underline{v})] [\underline{v} - \rho_S(c)] \\
&\ge \underline{v} - \left(\rho_S(0) + \frac{y}{2}\right) \ge \frac{\underline{v} - \rho_S(0)}{2}.
\end{aligned}$$

Consequently, such a seller's probability of trade in a given meeting, $q_S(c)$, is bounded from below by

$$\begin{aligned}
q_S(c) &\ge \beta_S[1 - \Gamma_B(p_S(c))] \ge \frac{\beta_S}{p_S(c) - \rho_S(c)} \frac{\underline{v} - \rho_S(0)}{2} \\
&\ge \frac{\beta_S}{2} \frac{\underline{v} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} \ge \frac{\beta_S \kappa_B}{2(r + \kappa_B)}.
\end{aligned}$$

The last inequality is from step 1(a).

Then from (2.13) in Lemma 1 (see p.16),

$$\rho_S'(c) = \frac{r}{r + \alpha_S q_S(c)} \le \frac{r}{\alpha_S \beta_S \kappa_B / 2r(r + \kappa_B)} = \frac{2r(r + \kappa_B)}{\alpha_S \beta_S \kappa_B}.$$

Now we can see that

$$\frac{y}{2} = \int_{\{c: \rho_S(c) \in [\rho_S(0), \rho_S(0) + \frac{y}{2}]\}} \rho_S'(c)\, dc \le \frac{2r(r + \kappa_B)}{\alpha_S \beta_S \kappa_B},$$

which is the same as (a).

Step 3: Let κ be $\min\{\kappa_B, \kappa_S\}$. We claim that

$$\rho_B(1) - \rho_S(0) \leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} \cdot \left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2.$$

To prove it, first notice that from step 2(a) and inequality (2.24) (see p.31), we have

$$\underline{v} - \rho_S(0) = \min\{\underline{v}, \bar{c}\} - \rho_S(0) + \max\{\underline{v} - \bar{c}, 0\} \leq \frac{4r(r + \kappa_B)}{\alpha_S\beta_S\kappa_B} + \frac{\kappa_S}{\alpha_S\beta_S}.$$

Then from step 1(a),

$$\begin{aligned} \rho_B(1) - \rho_S(0) &\leq \frac{r + \kappa_B}{\kappa_B}(\underline{v} - \rho_S(0)) \leq \frac{r + \kappa_B}{\alpha_S\beta_S\kappa_B}\left[\frac{4r(r + \kappa_B)}{\kappa_B} + \kappa_S\right] \\ &= \frac{\kappa_S}{\alpha_S\beta_S}\left(1 + \frac{r}{\kappa_B}\right)\left[1 + \frac{4r}{\kappa_S}\left(1 + \frac{r}{\kappa_B}\right)\right] \\ &\leq \frac{\kappa_S}{\alpha_S\beta_S}\left(1 + \frac{r}{\kappa}\right)\left[1 + \frac{4r}{\kappa}\left(1 + \frac{r}{\kappa}\right)\right] \\ &= \frac{\kappa_S}{\alpha_S\beta_S}\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2. \end{aligned}$$

Similarly, from step 2(b), inequality (2.24) and step 1(b),

$$\rho_B(1) - \rho_S(0) \leq \frac{\kappa_B}{\alpha_B\beta_B}\left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2.$$

We get our claim by combining the above two upper bounds of $\rho_B(1) - \rho_S(0)$.

Step 4: We claim that

$$\rho_B(1) - \rho_S(0) \geq p_S(\bar{c}) - p_B(\underline{v}) \geq \max\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}.$$

To prove it, simply observe that Lemma 2(a,c) (see p.18) implies

$$\kappa_B \leq \alpha_B\beta_B(\underline{v} - p_B(\underline{v})) \leq \alpha_B\beta_B(p_S(\bar{c}) - p_B(\underline{v})),$$

$$\kappa_S \leq \alpha_S\beta_S(p_S(\bar{c}) - \bar{c}) \leq \alpha_S\beta_S(p_S(\bar{c}) - p_B(\underline{v})),$$

and $p_S(\bar{c}) - p_B(\underline{v}) \leq \rho_B(1) - \rho_S(0)$.

Step 5: Combine steps 3 and 4, we get

$$\begin{aligned} \max\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} &\leq p_S(\bar{c}) - p_B(\underline{v}) \leq \rho_B(1) - \rho_S(0) \\ &\leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} \cdot \left(1 + \frac{r}{\kappa}\right)\left(1 + \frac{2r}{\kappa}\right)^2 \\ &\leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} \cdot \left(1 + \frac{2r}{\kappa}\right)^3. \end{aligned} \tag{4.1}$$

From Lemma 4 (see p.26) we have

$$\min\left\{\frac{\kappa_S}{\alpha_S\beta_S},\frac{\kappa_B}{\alpha_B\beta_B}\right\}\leq K\left(\zeta_0,\tau\right)\leq\max\left\{\frac{\kappa_S}{\alpha_S\beta_S},\frac{\kappa_B}{\alpha_B\beta_B}\right\}.$$

Combine the above two results, we obtain the theorem. ■

As a by-product of the above proof, we also obtain upper and lower bounds for the equilibrium buyer-seller ratio ζ. These bounds do not depend on τ, which implies that ζ is $O(1)$ as $\tau\to 0$.

Corollary 8 *In any nontrivial steady-state equilibrium, we have*

$$\zeta_0\cdot\left(1+\frac{2r}{\kappa}\right)^{-3}\leq\zeta\leq\zeta_0\cdot\left(1+\frac{2r}{\kappa}\right)^{3}$$

where $\zeta_0\equiv\beta_B\kappa_S/\beta_S\kappa_B$ *and* $\kappa\equiv\min\{\kappa_B,\kappa_S\}$.

Proof. From (4.1) we have

$$\frac{\kappa_S}{\alpha_S\beta_S}\leq\frac{\kappa_B}{\alpha_B\beta_B}\left(1+\frac{2r}{\kappa}\right)^{3}$$

and

$$\frac{\kappa_B}{\alpha_B\beta_B}\leq\frac{\kappa_S}{\alpha_S\beta_S}\left(1+\frac{2r}{\kappa}\right)^{3}.$$

Recall that $\alpha_S/\alpha_B=\zeta$. Then we get the result by simple rearranging of terms. ■

We now turn to the intuition behind the proof of Theorem 7. The main parts of the above proof are step 2 through step 4. Steps 2 and 3 derive an upper bound (proportional to τ) for the length of responding interval $\rho_B(1)-\rho_S(0)$, while step 4 derives a lower bound for the length of proposing interval $p_S(\bar{c})-p_B(\underline{v})$. The lower bound part is relatively easy. We have seen in Lemma 2(a) (on p.18) that the marginal entering types $\underline{v}$ and $\bar{c}$ must fall within the proposing interval $[p_B(\underline{v}),p_S(\bar{c})]$ in equilibrium. If the length $p_S(\bar{c})-p_B(\underline{v})$ is too small, the marginal entrants would not be able to recover the search costs they incur. (The accumulated search cost is $O(\tau)$.) Therefore $p_S(\bar{c})-p_B(\underline{v})$ is bounded below by τ multiplied by some constant.

The upper bound part is subtler. Following the logic we use to show the linear rate of convergence for full-trade equilibria, we want to show the slopes of responding strategies

$\rho_B(v)$ and $\rho_S(c)$ are $O(\tau)$. Look at sellers for example, from (2.13) (on p.18), $\rho'_S(c)$ is indeed $O(\tau)$ for those c such that the probability of trade $q_S(c)$ is bounded away from 0. Such a boundedness of $q_S(c)$ in turn can be obtained for low cost sellers ($\rho_S(c) \leq \rho_S(0) + y/2$ in step 2) since those sellers, with substantial profitability, would never prefer to make an offer that is accepted with a too low probability. Therefore, $\rho'_S(c)$ is $O(\tau)$ for a subset of active sellers. Moreover, our choice of the subset allows us to extend the result to bound $\min\{\underline{v}, \bar{c}\} - \rho_S(0)$; and then the statement claimed in step 1 further extends the boundedness to the whole length of responding interval $\rho_B(1) - \rho_S(0)$.

4.3.4 Full information model

This subsection deviates from our baseline model by considering full-information bargaining as introduced in Section 3.3. We will see that the rate of convergence remains unchanged in this full information model. Furthermore, the proof of the rate of convergence for the full information model is similar to (actually a bit easier than) that for our baseline (private information) model, although it seems not convenient to unify the two proofs.

Recall that in the full information model we study in Chapter 3, we assume that, as in our baseline model, traders bargain using the random-proposer protocol with buyers' bargaining weight $\beta_B \in (0,1)$ and sellers' bargaining weight $\beta_S \equiv 1 - \beta_B$. All trading prices must fall within the interval $[\rho_S(0), \rho_B(1)]$, where $\rho_S(0)$ is the lowest dynamic type of active sellers and $\rho_B(1)$ is the highest dynamic type of active buyers. In this full information context, one could also equivalently assume that the bargaining outcome of a matched pair is given by the generalized Nash bargaining solution with buyer's and seller's relative bargaining powers being β_B and β_S.

Then the following theorem shows that, as $\tau \to 0$, the length $\rho_B(1) - \rho_S(0)$ converges to 0 at the linear rate in τ.[40]

[40] Like in Theorem 7, the upper bound provided in Theorem 8 converges to the lower bound as r gets small relative to $\kappa \equiv \min\{\kappa_B, \kappa_S\}$. It indicates that our bounds are tight at least when the discount rate is small relative to the search costs.

Theorem 8 *Under full information bargaining, in any nontrivial steady-state equilibrium, we have*

$$K(\zeta_0, \tau) \leq \rho_B(1) - \rho_S(0) \leq K(\zeta_0, \tau)\left(1 + \frac{r}{\kappa}\right)^2,$$

where $\zeta_0 \equiv \beta_B \kappa_S / \beta_S \kappa_B$ *and* $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Proof. *Step 1*: We claim that

$$\text{(a):} \quad \frac{\underline{v} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} \geq \frac{\kappa_B}{r + \kappa_B} \quad \text{and} \quad \text{(b):} \quad \frac{\rho_B(1) - \bar{c}}{\rho_B(1) - \rho_S(0)} \geq \frac{\kappa_S}{r + \kappa_S}.$$

We provide the proof for part (a) only. The proof for part (b) is the flip of that for part (a). Applying (3.20) (which is on p.58), we have

$$\frac{\kappa_B}{\alpha_B \beta_B} = \int_{\underline{v} \geq \rho_S(c)} [\underline{v} - \rho_S(c)] \frac{dN_S(c)}{S} \leq \int_{\underline{v} \geq \rho_S(c)} [\underline{v} - \rho_S(0)] \frac{dN_S(c)}{S} = q_B(\underline{v})(\underline{v} - \rho_S(0)).$$

Thus for any $v \geq \underline{v}$, we have $\alpha_B \beta_B q_B(v) \geq \kappa_B / (\underline{v} - \rho_S(0))$. Then for almost all $v \in [\underline{v}, 1]$,

$$\rho_B'(v) = \frac{r}{r + \alpha_B \beta_B q_B(v)} \leq \frac{r}{\kappa_B / (\underline{v} - \rho_S(0))}.$$

Hence

$$\rho_B(1) - \underline{v} = \int_{\underline{v}}^1 \rho_B'(v) dv \leq \frac{r}{\kappa_B / (\underline{v} - \rho_S(0))},$$

$$\frac{\rho_B(1) - \underline{v}}{\underline{v} - \rho_S(0)} \leq \frac{r}{\kappa_B},$$

$$\frac{\underline{v} - \rho_S(0)}{\rho_B(1) - \rho_S(0)} = \frac{1}{1 + (\rho_B(1) - \underline{v})/(\underline{v} - \rho_S(0))} \geq \frac{1}{1 + \frac{r}{\kappa_B}} = \frac{\kappa_B}{r + \kappa_B}.$$

Step 2: We claim that

$$\text{(a):} \quad \min\{\underline{v}, \bar{c}\} - \rho_S(0) \leq \frac{r}{\alpha_S \beta_S} \quad \text{and} \quad \text{(b):} \quad \rho_B(1) - \max\{\underline{v}, \bar{c}\} \leq \frac{r}{\alpha_B \beta_B}.$$

Again by symmetry, we only provide a proof for (a). It is clear that $q_S(c) = 1$ if $\rho_S(c) \leq \min\{\underline{v}, \bar{c}\}$. Thus,

$$\min\{\underline{v}, \bar{c}\} - \rho_S(0) = \int_{\rho_S(c) \leq \min\{\underline{v}, \bar{c}\}} \rho_S'(c) dc \leq \frac{r}{r + \alpha_S \beta_S} \leq \frac{r}{\alpha_S \beta_S}.$$

Step 3: We claim that

$$\rho_B(1) - \rho_S(0) \leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

To prove it, first notice that from step 2(a) and inequality (3.24) (see p.60), we have

$$\begin{aligned} \underline{v} - \rho_S(0) &= \min\{\underline{v}, \bar{c}\} - \rho_S(0) + \max\{\underline{v} - \bar{c}, 0\} \\ &\leq \frac{r}{\alpha_S\beta_S} + \frac{\kappa_S}{\alpha_S\beta_S} = \frac{\kappa_S}{\alpha_S\beta_S}\left(1 + \frac{r}{\kappa_S}\right). \end{aligned}$$

Then from step 1(a),

$$\rho_B(1) - \rho_S(0) \leq \frac{r + \kappa_B}{\kappa_B}\left(\underline{v} - \rho_S(0)\right) \leq \frac{\kappa_S}{\alpha_S\beta_S}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

Similarly, from step 2(b), inequality (3.24), and step 1(b), we have

$$\rho_B(1) - \rho_S(0) \leq \frac{\kappa_B}{\alpha_B\beta_B}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right).$$

Step 4: We claim that

$$\rho_B(1) - \rho_S(0) \geq \max\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}.$$

To prove it, observe that Lemma 11 (on p.58), together with Lemma 12 (p.59), implies

$$\kappa_B \leq \alpha_B\beta_B(\rho_B(1) - \rho_S(0))$$

$$\kappa_S \leq \alpha_S\beta_S(\rho_B(1) - \rho_S(0)).$$

Step 5: Combine steps 3 and 4, we get

$$\begin{aligned} \max\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} &\leq \rho_B(1) - \rho_S(0) \\ &\leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}\left(1 + \frac{r}{\kappa_B}\right)\left(1 + \frac{r}{\kappa_S}\right) \\ &\leq \min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}\left(1 + \frac{r}{\kappa}\right)^2. \end{aligned} \tag{4.2}$$

From Lemma 4 (see p.26) we have

$$\min\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\} \leq K(\zeta_0, \tau) \leq \max\left\{\frac{\kappa_S}{\alpha_S\beta_S}, \frac{\kappa_B}{\alpha_B\beta_B}\right\}.$$

Combine the above two results, we obtain the theorem. ■

As a by-product of the above proof, we also obtain upper and lower bounds for the equilibrium buyer-seller ratio ζ. These bounds do not depend on τ, which implies that ζ is $O(1)$ as $\tau \to 0$.

Corollary 9 *Under full information bargaining, in any nontrivial steady-state equilibrium, we have*

$$\zeta_0 \cdot \left(1 + \frac{r}{\kappa}\right)^{-2} \leq \zeta \leq \zeta_0 \cdot \left(1 + \frac{r}{\kappa}\right)^2$$

where $\zeta_0 \equiv \beta_B \kappa_S / \beta_S \kappa_B$ *and* $\kappa \equiv \min\{\kappa_B, \kappa_S\}$.

Proof. From (4.2) we have

$$\frac{\kappa_S}{\alpha_S \beta_S} \leq \frac{\kappa_B}{\alpha_B \beta_B} \left(1 + \frac{r}{\kappa}\right)^2$$

and

$$\frac{\kappa_B}{\alpha_B \beta_B} \leq \frac{\kappa_S}{\alpha_S \beta_S} \left(1 + \frac{r}{\kappa}\right)^2.$$

Recall that $\alpha_S / \alpha_B = \zeta$. Then we get the result by simple rearranging of terms. ■

Corollary 10 *Under full information bargaining, for any sequence of nontrivial steady-state equilibria parametrized by* τ *such that* $\tau \to 0$, *the proposing interval* $[\rho_{S\tau}(0), \rho_{B\tau}(1)]$ *collapses to the Walrasian price* $\{p^*\}$ *at no-slower-than-linear convergence rate. More precisely,*

$$\max\{|\rho_{S\tau}(0) - p^*|, |\rho_{B\tau}(1) - p^*|\} < K(\zeta_0, \tau) \left(1 + \frac{r}{\kappa}\right)^2.$$

Proof. Recall from Lemma 13 (on p.59) that $\rho_S(0) < p^* < \rho_B(1)$. Then the result is a straight implication of Theorem 8. ■

4.4 Rate of convergence of welfare

In this section, we turn to the rate of convergence of welfare. We will consider both the private information model and the full information model in a unified way.

Recall that the lifetime payoff of a particular new-born type v buyer (type c seller) is denoted as $W_B(v)$ ($W_S(c)$). To provide a benchmark for our results, we define their Walrasian counterparts in the usual manner, as

$$W_B^*(v) \equiv \max\{v - p^*, 0\}, \qquad W_S^*(c) \equiv \max\{p^* - c, 0\}.$$

Recall that we generally define (on p.67) the measure of aggregate welfare W as the aggregate lifetime payoffs of a cohort:

$$W \equiv bW_B^{ea} + sW_S^{ea} \tag{4.3}$$

where W_B^{ea} (W_S^{ea}) is a buyer's (seller's) ex-ante utility, i.e.

$$W_B^{ea} \equiv \int W_B(v)dF(v),$$

$$W_S^{ea} \equiv \int W_S(c)dG(c).$$

The Walrasian counterpart of W is:

$$W^* \equiv b\int_{p^*}^{1} (v - p^*)\, dF(v) + s\int_0^{p^*} (p^* - c)\, dG(c).$$

The following lemma shows that, in either the private or full information model, every trader's interim lifetime utility converges no slower than the length of responding interval $\rho_B(1) - \rho_S(0)$.

Lemma 15 *In either the private or full information model, and in any nontrivial steady-state equilibrium, we have* $|W_B^*(v) - W_B(v)| \leq \rho_B(1) - \rho_S(0)$ *and* $|W_S^*(c) - W_S(c)| \leq \rho_B(1) - \rho_S(0)$*, for any* $v, c \in [0, 1]$*.*

Proof. We will only prove the result for buyers. That for sellers can be proved by a symmetric argument. Recall that if $v \geq \underline{v}$ then $W_B(v) = v - \rho_B(v)$; and if $v < \underline{v}$ then

$W_B(v) = 0$. Consequently,

$$\begin{aligned} W_B^*(v) - W_B(v) &= \max\{v - p^*, 0\} - W_B(v) \\ &= \begin{cases} \rho_B(v) - p^* & \text{if} \quad v \geq p^* \text{ and } v \geq \underline{v} \\ \rho_B(v) - v & \text{if} \quad v < p^* \text{ and } v \geq \underline{v} \\ v - p^* & \text{if} \quad v \geq p^* \text{ and } v < \underline{v} \\ 0 & \text{if} \quad v < p^* \text{ and } v < \underline{v} \end{cases}. \end{aligned}$$

In any of the four cases, we must have $|W_B^*(v) - W_B(v)| < \rho_B(1) - \rho_S(0)$. It is obvious for the fourth case. For the other three cases, recall Lemma 2 and Lemma 3 in Section 2.4, Lemma 12 and 13 in Subsection 3.3.2, and the monotonicity of ρ_B in both models. We can see that (i) $p^* \in [\rho_S(0), \rho_B(1)]$, (ii) $\rho_B(v) \in [\rho_S(0), \rho_B(1)]$ under the conditions of the first and second cases, and (iii) $v \in [\rho_S(0), \rho_B(1)]$ under the conditions of the second or third cases. ■

Combine Lemma 15, Theorem 7 and Theorem 8, we obtain the following rate of convergence theorem for interim lifetime utilities.

Theorem 9 (Rate of convergence for interim lifetime utilities) *Fix* $(r, \kappa_B, \kappa_S) \gg 0$. *Then the interim lifetime utilities* $W_{B\tau}(v)$, $W_{S\tau}(c)$ *converge to their Walrasian counterparts* $W_B^*(v)$ *and* $W_S^*(c)$ *at least as fast as linear rate, as* $\tau \to 0$. *More precisely, for all* $v, c \in [0, 1]$, *we have*

$$\max\{|W_B^*(v) - W_{B\tau}(v)|, |W_S^*(c) - W_{S\tau}(c)|\} \leq K(\zeta_0, \tau)\left(1 + \frac{2r}{\kappa}\right)^3$$

for both the private information model and the full information model.

Remark 4 *In Theorem 9, absolute values for both* $W_B^*(v) - W_{B\tau}(v)$ *and* $W_S^*(c) - W_{S\tau}(c)$ *are needed because they are not guaranteed to be positive. Indeed, if* $\underline{v}_\tau < p^*$, *then buyers with type* $v \in (\underline{v}_\tau, p^*]$ *would have strictly positive utilities in equilibrium but have* 0 *Walrasian utilities. Furthermore, we have not precluded the possibility that some interim utility converges at a faster-than-linear rate. It is because we do not have a positive lower bound for* $\frac{1}{\tau}|W_B^*(v) - W_{B\tau}(v)|$ *and* $\frac{1}{\tau}|W_S^*(c) - W_{S\tau}(c)|$. *Indeed, for some types* v, *we could have* $W_B^*(v) = W_{B\tau}(v) = 0$.

Our baseline model assumes a random-proposer take-it-or-leave-it bargaining game. But the treatment can be straightforwardly extended to any bargaining protocol as long as traders' types are private information and attention is still restricted to steady state. In particular Lemma 1 (on p.16) holds for the double auction bargaining protocol as well, although, as shown later, our convergence results fail for arbitrary protocol.

We now show that no bargaining mechanism can generate the (steady-state) aggregate welfare W converging at a faster than linear rate in τ, regardless of whether information is full or private. Any bargaining game played in each meeting results in a trading probability $q(v,c) \in [0,1]$ and expected payment $t(v,c)$ from the buyer to the seller, as functions of traders' types. In steady-state equilibrium, the bargaining outcomes q and t are unchanged over time. Then, *contingent on entry*, buyers' and sellers' lifetime payoff W_B and W_S are given by the following Bellman equations:

$$rW_B(v) = \alpha_B(\zeta)[q_B(v)v - t_B(v) - q_B(v)W_B(v)] - \kappa_B$$

$$rW_S(c) = \alpha_S(\zeta)[t_S(c) - q_S(c)c - q_S(c)W_S(c)] - \kappa_S$$

where

$$q_B(v) \equiv \int q(v,c)\frac{dN_S(c)}{S}, \quad q_S(c) \equiv \int q(v,c)\frac{dN_B(v)}{B},$$

$$t_B(v) \equiv \int t(v,c)\frac{dN_S(c)}{S}, \quad t_S(c) \equiv \int t(v,c)\frac{dN_B(v)}{B}.$$

The functions $q_B(v)$ and $q_B(c)$ are, as before, the trading probabilities in a given meeting conditional only on traders' own types; $t_B(v)$, $t_S(c)$ are the expected payments conditional only on own types.

Given the bargaining mechanism, the entry of traders is voluntary. We assume the entry of every trader is a one-time decision: once being born, every trader can choose either to stay away from the market forever (in which case his payoff is 0), or to stay in the market until he trades successfully. In steady state this restriction is not binding.

Let $\chi_B(v)$ and $\chi_S(c)$ be the buyers' and sellers' entry probabilities respectively. From the above Bellman equations, we can write

$$W_B(v) = \chi_B(v) \cdot \frac{\alpha_B(\zeta)[q_B(v)v - t_B(v)] - \kappa_B}{r + \alpha_B(\zeta)q_B(v)} \tag{4.4}$$

$$W_S(c) = \chi_S(c) \cdot \frac{\alpha_S(\zeta)[t_S(c) - q_S(c)\, c] - \kappa_S}{r + \alpha_S(\zeta) q_S(c)}. \tag{4.5}$$

Individual rationality requires that $W_B(v) \geq 0$ and $W_S(c) \geq 0$ for all $v, c \in [0,1]$. Equivalently, individual rationality requires

$$\begin{aligned} \alpha_B[q_B(v)v - t_B(v)] \geq \kappa_B \quad &\text{if} \quad \chi_B(v) > 0, \\ \alpha_S[t_S(c) - q_S(c)c] \geq \kappa_S \quad &\text{if} \quad \chi_S(c) > 0. \end{aligned} \tag{4.6}$$

The steady-state equations for market distributions N_B and N_S are maintained as before.

We now prove that no individually rational bargaining mechanism can have a faster-than-linear rate of convergence for the steady-state welfare W, by establishing an explicit lower bound on $W^* - W$.

Theorem 10 *For any individually rational bargaining protocol, in steady-state equilibrium we have*

$$W^* - W \geq \mu \cdot \min_{\zeta > 0} K(\zeta, \tau), \tag{4.7}$$

where μ is the equilibrium mass of buyers (or sellers) who enter the market per unit time.

Proof. Rewrite the Walrasian welfare level W^*:

$$\begin{aligned} W^* &= b\int_{p^*}^{1} (v - p^*) dF(v) + s\int_0^{p^*} (p^* - c) dG(c) \\ &= \max_{\chi_B, \chi_S} \left\{ \begin{array}{c} b\int \chi_B(v) v dF(v) - s\int \chi_S(c) c dG(c) \\ \text{s.t. } b\int \chi_B(v) dF(v) = s\int \chi_S(c) dG(c), \\ 0 \leq \chi_B(v) \leq 1, \qquad 0 \leq \chi_S(c) \leq 1 \end{array} \right\}. \end{aligned} \tag{4.8}$$

On the other hand, the equilibrium welfare level W for any individually rational bargaining mechanism can be bounded as follows. For any active buyer type v (i.e. $\chi_B(v) \neq 0$), individual rationality requires $\alpha_B[q_B(v)v - t_B(v)] \geq \kappa_B$. Hence, from (4.4) we have

$$\begin{aligned} W_B(v) &\leq \chi_B(v) \cdot \frac{\alpha_B[q_B(v)v - t_B(v)] - \kappa_B}{\alpha_B q_B(v)} \\ &= \chi_B(v) \cdot \left[v - \frac{\kappa_B}{\alpha_B q_B(v)} - \frac{t_B(v)}{q_B(v)} \right]. \end{aligned}$$

Similarly for sellers:

$$W_S(c) \leq \chi_S(c) \cdot \left[-c - \frac{\kappa_S}{\alpha_S q_S(c)} + \frac{t_S(c)}{q_S(c)} \right].$$

Substituting these bounds into the definition (4.3),

$$\begin{aligned} W \quad \leq \quad & b \int \chi_B(v) v dF(v) - s \int \chi_S(c) c dG(c) \\ & -b \int \chi_B(v) \frac{\kappa_B}{\alpha_B} dF(v) - s \int \chi_S(c) \frac{\kappa_S}{\alpha_S} dG(c) \\ & -b \int \chi_B(v) \frac{t_B(v)}{q_B(v)} dF(v) + s \int \chi_S(c) \frac{t_S(c)}{q_S(c)} dG(c). \end{aligned} \tag{4.9}$$

(In the second line, we have used $q_B(v) \leq 1$ and $q_S(c) \leq 1$.) In view of (4.8), the terms in the first line of the right hand side do not exceed the Walrasian surplus W^*. Also, since the steady-state condition implies that $b\chi_B(v)dF(v) = \alpha_B q_B(v) dN_B(v)$ for buyers and $s\chi_S(c)dG(c) = \alpha_S q_S(c) dN_S(c)$ for sellers, and the transfers are balanced,

$$\int t_B(v) \frac{dN_B(v)}{B} = \int t_S(c) \frac{dN_S(c)}{S},$$

the last line in (4.9) is 0.

Taking all these into account, we have

$$W \leq W^* - b\frac{\kappa_B}{\alpha_B} \int \chi_B(v) dF(v) - s\frac{\kappa_S}{\alpha_S} \int \chi_S(c) dG(c)$$

and therefore

$$W^* - W \geq \left(\frac{\kappa_B}{\alpha_B} + \frac{\kappa_S}{\alpha_S} \right) \mu,$$

where

$$\mu \equiv b \int \chi_B(v) dF(v) = s \int \chi_S(c) dG(c)$$

is the equilibrium mass of buyers (or sellers) who enter the market per unit time. Furthermore,

$$\frac{\kappa_B}{\alpha_B(\zeta,\tau)} + \frac{\kappa_S}{\alpha_S(\zeta,\tau)} = K(\zeta,\tau) \geq \min_{\zeta>0} K(\zeta,\tau).$$

The inequality (4.7) follows. ∎

As $\tau \to 0$, we must have $\mu_\tau \to sG(p^*)$ whenever $W_\tau \to W^*$. We therefore have the following corollary.

Corollary 11 *No individually rational bargaining mechanism can attain a faster-than-linear convergence rate for the traders' aggregate welfare level* W_τ *as* $\tau \to 0$.

Remark 5 *Since Theorem 10 does not require incentive compatibility, it in particular implies that even with full information, as in Mortensen and Wright (2002), convergence cannot be faster than linear.*

Remark 6 *Theorem 9 and Theorem 10 together imply that the traders' aggregate welfare level* W_τ*, in either the private or full information model, converges to* W^* *at exactly linear rate as* τ *tends to* 0.

It follows that the random-proposer bargaining mechanism attains the fastest possible convergence towards the first best (i.e. Walrasian welfare level), independent of the information structure of bargaining. The intuition for why no other bargaining mechanism can attain a faster rate for welfare is that matching delays will still be present regardless of the efficiency of bargaining. Even if only the buyers with $v \geq p^*$ and sellers with $c \leq p^*$ enter and always trade to full efficiency, there still will be welfare loss at rate τ because of costly search (and discounting), since the expected time between matches is proportional to τ.

This might motivate one to separate the welfare loss into the loss directly due to delay and search costs, and the loss indirectly due to equilibrium behaviors. More precisely, let us explicitly think of buyers' lifetime utility as functions of $\sigma_B \equiv (\chi_B, q_B, t_B)$ and (r, κ_B):

$$\hat{W}_B(v; \sigma_B; r, \kappa_B) \equiv \chi_B(v) \cdot \frac{\alpha_B [q_B(v)v - t_B(v)] - \kappa_B}{r + \alpha_B q_B(v)}.$$

And similarly for sellers,

$$\hat{W}_S(c; \sigma_S; r, \kappa_S) \equiv \chi_S(c) \cdot \frac{\alpha_S [t_S(c) - q_S(c)c] - \kappa_S}{r + \alpha_S q_S(c)}.$$

Obviously $\hat{W}_B(\cdot; \sigma_B; r, \kappa_B)$ and $\hat{W}_S(\cdot; \sigma_S; r, \kappa_S)$ become the Walrasian counterparts W_B^* and W_S^* when (i) $(r, \kappa_B, \kappa_S) = \mathbf{0}$, and (ii) σ_B and σ_S are at their Walrasian values, i.e.

$$\chi_B(v) = I[v \geq p^*], \quad \chi_S(c) = I[c \leq p^*]$$

where $I[\cdot]$ is 1 if the condition inside the bracket holds, and is 0 otherwise; and for all $v \geq p^*$ and all $c \leq p^*$,

$$q_B(v) = q_S(c) = 1, \quad t_B(v) = t_S(c) = p^*.$$

Then we can define the welfare loss indirectly due to equilibrium behaviors as

$$b \int \left[W_B^*(v) - \hat{W}_B(v; \sigma_B; 0, 0) \right] dF(v) + s \int \left[W_S^*(c) - \hat{W}_S(c; \sigma_S; 0, 0) \right] dG(c),$$

and define the welfare loss directly due to delay and search costs as

$$b \int \left[\hat{W}_B(v; \sigma_B; 0, 0) - \hat{W}_B(v; \sigma_B; r, \kappa_B) \right] dF(v)$$
$$+ s \int \left[\hat{W}_S(c; \sigma_S; 0, 0) - \hat{W}_S(c; \sigma_S; r, \kappa_S) \right] dG(c).$$

The driving force of Theorem 10 is that the direct part is $O(\tau)$, for any bargaining protocol. However the indirect part could vanish at a faster-than-linear rate. To see this, notice that this indirect part can be simplified as

$$b \int \left[I(v \geq p^*) - \chi_B(v) \right] v dF(v) - s \int \left[I(c \leq p^*) - \chi_S(c) \right] c dG(c).$$

Under random-proposer bargaining (or any other protocol with private information), the entry strategies must be cutoff strategies, i.e. $\chi_B(v) = I[v \geq \underline{v}]$ and $\chi_S(c) = I[c \leq \bar{c}]$. Hence the indirect loss is simply the familiar deadweight loss triangle:

$$b \int_{p^*}^{\underline{v}} v dF(v) - s \int_{\bar{c}}^{p^*} c dG(c).$$

It is easy to see that in our baseline model (i.e. random-proposer bargaining with private information), along a sequence of full-trade equilibria with $\tau \to 0$, this indirect welfare loss vanishes at quadratic rate in τ, because the entry gap $\underline{v} - \bar{c}$ vanishes at linear rate.[41]

4.5 Results for k-double auction

We have seen that the rate of convergence results for our baseline model are robust to other information structure of bargaining. In this section, we are interested in whether our

[41] It is not hard to verify that this is also true for our full information model.

results are robust to other bargaining protocol. Although we have not been able to prove a general theorem in this direction, we have a theorem showing that another well-studied trading mechanism, the double auction, does not have robust convergence properties. In other words, some sequences of equilibria do not converge to perfect competition.

Recall the rules of the bilateral k-double auction introduced by Chatterjee and Samuelson (1983): once a meeting occurs, the buyer and the seller simultaneously and independently submit a bid price p_B and an ask price p_S respectively, and then trade occurs if and only if the buyer's bid is at least as high as the seller's ask, at the weighted average price $(1-k)p_S + kp_B$, where $k \in (0,1)$. As in the baseline model, we assume that the buyer and the seller do not observe each other's type during the bargaining.

We maintain the notation as before up to a bit reinterpretations. The functions $p_B(v)$ and $p_S(c)$ are now the strategies of submitting bids and asks respectively. There is no responding strategy under double auction, but $\rho_B(v)$ and $\rho_S(c)$ are still buyers' and sellers' reservation prices, and also called dynamic types.

The definition for nontrivial steady-state equilibria can be obtained as a straightforward revision from the baseline (random-proposer) case. Furthermore, Lemma 1 (on p.16) still holds here. The proof goes through almost word-by-word, with the trading probability function replaced with

$$q_B(v) \equiv \int_{p_S(c) \le p_B(v)} \frac{dN_S(c)}{S}$$

and the expected payment function replaced with

$$t_B(v) \equiv \int_{p_S(c) \le p_B(v)} \left[kp_B\left(v\right) + (1-k)p_S\left(c\right)\right] \frac{dN_S(c)}{S}.$$

In this k-double auction model, as in the baseline model, a nontrivial steady-state equilibrium could be either full-trade or non-full-trade. We claim that the full-trade class of double auction equilibria includes equilibria that are very inefficient, even with arbitrarily small frictions. (But at the same time, this class also includes equilibria that converge to perfect competition.)

The set of full-trade equilibria is even easier to characterize for the double auction model. In particular, from Lemma 1, the sets of active buyers' and sellers' types are still intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ for some marginal types $\underline{v}$ and $\bar{c}$; and we also have $\rho_B(v) < v$ and $\rho_S(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Since all active traders' trading probabilities are strictly positive, they must in equilibrium submit serious bids/asks, and therefore, we must have $p_B(v) \leq \rho_B(v) < v$ and $p_S(c) \geq \rho_S(c) > c$ for all $v > \underline{v}$ and all $c < \bar{c}$. Now it is clear that for an equilibrium to be full-trade, we must have $\bar{c} \leq \underline{v}$, and all traders submit a common bid/ask p and hence every matched pair trades at the price p. Furthermore, the marginal entrants (i.e. type $\underline{v}$ buyers and type $\bar{c}$ sellers) have to recover their search costs, thus in any full-trade equilibrium we have $\bar{c} < p < \underline{v}$ for some $p \in (0, 1)$.

Any full-trade equilibrium for the double auction model must satisfy the following marginal type equations and inflow balance equation:

$$\alpha_B(\zeta, \tau)(\underline{v} - p) = \kappa_B, \tag{4.10}$$

$$\alpha_S(\zeta, \tau)(p - \bar{c}) = \kappa_S, \tag{4.11}$$

$$b[1 - F(\underline{v})] = sG(\bar{c}). \tag{4.12}$$

Unlike in the baseline model, it is easy to see that the converse is also true, i.e. any quadruple $\{p, \zeta, \underline{v}, \bar{c}\}$ satisfying (4.10), (4.11), (4.12) and $K(\zeta, \tau) < 1$ must characterize a full-trade equilibrium. In particular, any trader's best-response bid/ask strategy is p, given that all other active traders submit p.[42]

From equations (4.10) and (4.11), it follows that the entry gap is

$$\underline{v} - \bar{c} = K(\zeta, \tau). \tag{4.13}$$

The next proposition shows that $\underline{v} - \bar{c}$ can be arbitrarily close to 1 for all τ (such that an equilibrium exists), so that the equilibrium outcomes can be arbitrarily far from efficiency even with small frictions. The set of equilibrium entry gaps converges to the full-range $(0, 1)$ as frictions disappear, so the set of full-trade equilibria ranges from the perfectly competitive one to the almost perfectly inefficient ones. Moreover, the set of equilibrium prices also converges to the full-range $(0, 1)$ as frictions disappear. Thus indeterminacy

[42] Clearly, equations (4.10)-(4.12) still characterize a full-trade equilibrium even if we assume *full information* double auction bargaining.

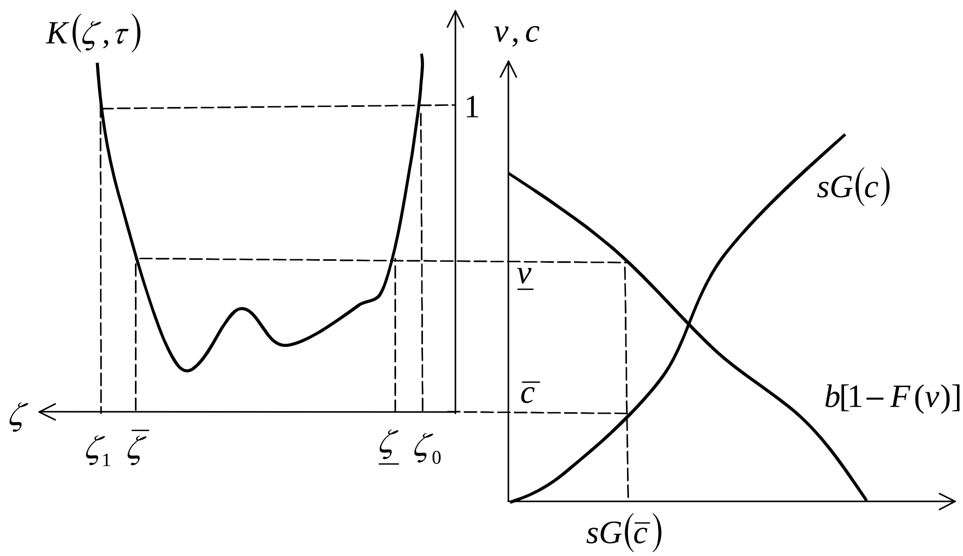

Figure 4.1: Construction of a double auction full-trade equilibrium

grows rather than vanishes with competition, contrary to the results in the static double auction literature.

Theorem 11 *Under double auction, a full-trade equilibrium exists if and only if*

$$\min_{\zeta>0} K(\zeta,\tau) < 1. \tag{4.14}$$

The set of equilibrium values of $\underline{v}-\bar{c}$ *in full-trade equilibria is an interval* $[\min_{\zeta>0} K(\zeta,\tau), 1)$*. As* $\tau \to 0$*, this set and the set of equilibrium prices converge to* $(0,1)$*. In particular, there exist sequences of full-trade equilibria that converges to perfect competition, and also sequences that do not converge.*

Proof. The proof follows the graphical argument shown in Figure 4.1. Given τ, the right panel shows the marginal types $\underline{v}$ and $\bar{c}$ in a steady-state equilibrium. The left panel shows the supportable values of buyer-seller ratio $\underline{\zeta}$ and $\bar{\zeta}$ that correspond to the given entry gap $\underline{v}-\bar{c}<1$. (In general, there can be one, two or more such values.)

Fix any $\tau>0$. Our assumption $M(0,S;\tau)=M(B,0;\tau)=0$ implies $\alpha_B(\infty,\tau)=$

$\alpha_S(0,\tau) = 0$. It in turn implies

$$\lim_{\zeta\to 0} K(\zeta,\tau) = \lim_{\zeta\to\infty} K(\zeta,\tau) = \infty, \tag{4.15}$$

as depicted in the left panel.

Given that (4.15) holds, a solution ζ to the equation $K(\zeta,\tau) = \underline{v} - \bar{c}$ exists if and only if $\underline{v} - \bar{c} \in [\min_{\zeta>0} K(\zeta,\tau), 1)$. Since $\lim_{\tau\to 0} K(\zeta,\tau) = 0$ for any $\zeta > 0$, we also must have $\min_{\zeta>0} K(\zeta,\tau) \to 0$ as $\tau \to 0$. It proves that the set of supportable values of entry gap $\underline{v} - \bar{c}$ converges to the interval $(0,1)$.

Now fix any τ such that $\min_{\zeta>0} K(\zeta,\tau) < 1$. Consider the longest interval $[\zeta_0, \zeta_1]$ such that $K(\zeta_0,\tau) = K(\zeta_1,\tau) = 1$ and $K(\zeta,\tau) < 1$ for $\zeta \in (\zeta_0,\zeta_1)$. For any $\zeta \in (\zeta_0,\zeta_1)$, $\underline{v}$ and $\bar{c}$ can be found uniquely from (4.13) and (4.12) (graphically shown in Figure 4.1). Denote $\underline{v}_\tau(\zeta)$ and $\bar{c}_\tau(\zeta)$ as the results. The equilibrium price p can also be found uniquely from equation (4.10) or equation (4.11):

$$p_\tau(\zeta) \equiv \bar{c}_\tau(\zeta) + \frac{\kappa_S}{\alpha_S(\zeta,\tau)} \tag{4.16}$$

$$\left(= \underline{v}_\tau(\zeta) - \frac{\kappa_B}{\alpha_B(\zeta,\tau)}\right). \tag{4.17}$$

This formally defines a continuous mapping $p_\tau(\cdot)$ of $[\zeta_0,\zeta_1]$ into $\mathbb{R}_+$. Consequently, its image is a closed interval that contains the points $p(\zeta_0)$ and $p(\zeta_1)$; and the set of supportable equilibrium price contains this interval. The definitions of ζ_0 and ζ_1 imply that $\zeta_0 \to 0$ and $\zeta_1 \to \infty$ as $\tau \to 0$. Now $\bar{c}_\tau(\zeta_1) = 0$ for all τ and $\alpha_S(\zeta_1,\tau) \to \infty$ as $\tau \to 0$, therefore (4.16) implies that $\lim_{\tau\to 0} p_\tau(\zeta_1) = 0$. Similarly, $\underline{v}_\tau(\zeta_0) = 1$ for all τ and $\alpha_B(\zeta_0,\tau) \to \infty$ as $\tau \to 0$, so that (4.17) implies that $\lim_{\tau\to 0} p_\tau(\zeta_0) = 1$. It proves that the set of supportable equilibrium price converges to $(0,1)$. ■

It is not hard to see that the condition $\min_{\zeta>0} K(\zeta,\tau) < 1$ is also necessary for any nontrivial steady-state equilibrium to exist. We thus have the following theorem.

Theorem 12 *Under double auction, there exists a nontrivial steady-state equilibrium (either full-trade or non-full-trade) if and only if*

$$\min_{\zeta>0} K(\zeta,\tau) < 1.$$

Proof. Having Theorem 11, it now suffices to claim the necessity of $\kappa_B/\alpha_B(\zeta,\tau) + \kappa_S/\alpha_S(\zeta,\tau) < 1$ for a nontrivial equilibrium to exist. Recall the notation for a general bargaining game introduced in Section 4.4. Individual rationality (4.6) implies

$$\frac{\kappa_B}{\alpha_B(\zeta,\tau)} \leq \int\int [q(v,c)v - t(v,c)] \frac{dN_B(v)}{B}\frac{dN_S(c)}{S},$$

$$\frac{\kappa_S}{\alpha_S(\zeta,\tau)} \leq \int\int [t(v,c) - q(v,c)c] \frac{dN_B(v)}{B}\frac{dN_S(c)}{S},$$

and hence

$$\frac{\kappa_B}{\alpha_B(\zeta,\tau)} + \frac{\kappa_S}{\alpha_S(\zeta,\tau)} \leq \int\int (v-c) \frac{dN_B(v)}{B}\frac{dN_S(c)}{S} < 1.$$

■

Remark 7 *Compare the necessary and sufficient conditions for equilibrium existence under double auction (given by Theorem 12) and under random-proposer bargaining (given by Theorem 3 on p.37). The condition under double auction is weaker than the one under the baseline (random-proposer bargaining) model, which is* $K(\zeta_0,\tau) < 1$*. In this sense, the market is easier to open under double auction.*

Theorem 11 shows that the set of double-auction equilibria, even if we restrict attention to the full-trade ones, is very large. For more intuition, rewrite the first two marginal type equations of the double-auction full-trade equilibrium in a parallel way to the baseline model:

$$\alpha_B(\zeta,\tau)\, \beta_B^{DA}(\underline{v} - \bar{c}) = \kappa_B,$$

$$\alpha_S(\zeta,\tau)\, \beta_S^{DA}(\underline{v} - \bar{c}) = \kappa_S,$$

where

$$\beta_B^{DA} \equiv 1 - \beta_S^{DA}, \quad \beta_S^{DA} \equiv \frac{p - \bar{c}}{\underline{v} - \bar{c}}.$$

We may call β_B^{DA} and β_S^{DA} the buyers' and sellers' relative bargaining powers under double-auction full-trade equilibrium. These equations are the same as the marginal type equations (2.15) and (2.16) (on p.24) that characterize a full-trade equilibrium in our baseline model, with the only difference that the exogenous bargaining power β_S is now replaced with the endogenous bargaining power β_S^{DA}. (The remaining inflow balance equation is the same in both models.) If $\beta_S^{DA} = \beta_S$, the equilibria in both models have the same marginal types $\underline{v}$ and $\bar{c}$, and once these are solved for, the price p is uniquely determined from the equation $\beta_S^{DA} = \beta_S$, or equivalently $p = \bar{c} + (\underline{v} - \bar{c})\,\beta_S$. In other words, to any $\beta_S \in (0,1)$ there corresponds a double-auction full-trade equilibrium with $\beta_S^{DA} = \beta_S$ and the same marginal types $\underline{v}$ and $\bar{c}$ as in the random-proposer full-trade equilibrium candidate.

The above discussion has the following two implications. First, since β_S^{DA} can be arbitrary, in the double auction model, there is a great multiplicity of equilibria.[43] Second, since we know that full-trade equilibria of the baseline model converge in terms of welfare level at the linear rate, it follows immediately that there is a sequence of double-auction equilibria that also converges at the linear rate to perfect competition. We state this finding as a corollary.

Corollary 12 *As $\tau \to 0$, there are double-auction full-trade equilibria that converge, in terms of welfare level, at the linear rate in τ.*

Remark 8 *The logic of Theorem 12 and Corollary 12 has nothing to do with the assumption of private information bargaining. They hold equally well if every pair of buyer and seller bids and asks knowing each other's type, because all they need to know is only the equilibrium price p.*

The above discussion explains why double auction full-trade equilibria can have non-Walrasian limit while it cannot be the case in our baseline model. But Figure 4.1 and the logic in the proof of Theorem 11 also make it clear that for the double auction full-trade equilibria to be non-convergent to the Walrasian outcome, we have to let the bargaining power of one side vanish (i.e. either $\underline{v}-p \to 0$ or $p-\bar{c} \to 0$ as $\tau \to 0$) and also let the market become extremely unbalanced (i.e. either $\zeta \to 0$ or $\zeta \to \infty$ as $\tau \to 0$). One might wonder if all equilibria (e.g. non-full-trade) will converge to the Walrasian outcome if we preclude

[43] The nature of indeterminacy here is analogous to that in the Nash demand game. As is well-known, the outcome of double auction is highly indeterminate even when information is complete.

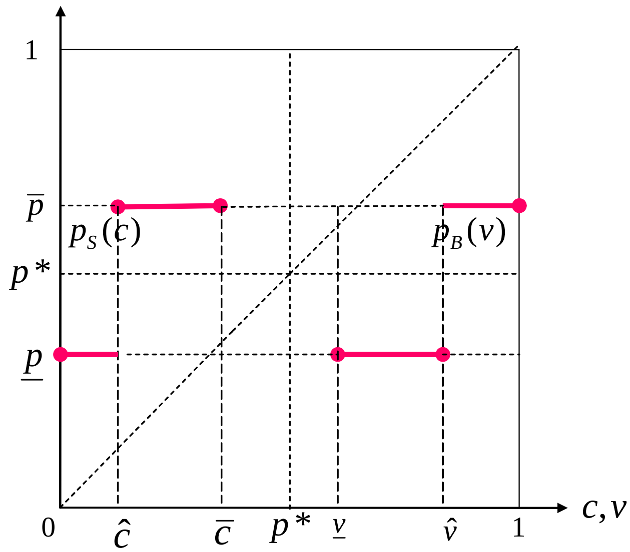

Figure 4.2: A two-step equilibrium under double auction

that class of equilibria (which is perhaps a natural restriction on equilibrium selection). It turns out that this is not so, as we show next.

We construct a non-full-trade equilibrium of the following nature (see Figure 4.2). There are two seller cutoff types $\hat{c} \in (0,1)$ and $\bar{c} \in (0,1)$ with $\hat{c} < \bar{c}$, and two buyer cutoff types $\hat{v} \in (0,1)$ and $\underline{v} \in (0,1)$ with $\hat{v} > \underline{v}$. The sellers with $c \in [0, \hat{c})$ enter and submit $p_S(c) = \underline{p}$, where $\underline{p}$ is some constant strictly below the Walrasian price p^*. The sellers with $c \in [\hat{c}, \bar{c}]$ enter and submit $p_S(c) = \bar{p}$, where $\bar{p} > p^*$. The sellers with $c \in (\bar{c}, 1]$ do not enter. Similarly, the buyers with $v \in (\hat{v}, 1]$ enter and submit $\bar{p}$, the buyers with $v \in [\underline{v}, \hat{v}]$ enter and submit $\underline{p}$, and the buyers with $v \in [0, \underline{v})$ do not enter. We call the equilibria of this kind *two-step equilibria.*

The following theorem gives our non-convergence result for the two-step (non-full-trade) equilibria.[44]

Theorem 13 *For any constant $a \in (0,1)$, there exist $r_0 > 0$, $\tau_0 > 0$ and $\bar{W} < W^*$ such that for all $r \in (0, r_0)$ and $\tau \in (0, \tau_0)$, there exists a two-step equilibrium in which the price*

[44] As a by-product, we also prove the existence of non-full-trade equilibrium for small τ and r.

spread is larger than a, i.e. $\bar{p} - \underline{p} > a$*, and the welfare level is lower than* $\bar{W}$*, i.e.* $W < \bar{W}$*.*

Proof. We derive a system of equations characterizing the set of two-step equilibria. But before doing so, it is convenient to introduce some notations. In a two-price equilibrium, the buyers with $v > \hat{v}$ who submit the high bid price $\bar{p}$, trade with any seller they meet. Buyers with $v \in [\underline{v}, \hat{v}]$, who submit the low bid price $\underline{p}$, trade only with those sellers with $c < \hat{c}$, who submit $\underline{p}$; their probability of trading is equal to $N_S(\hat{c})/S$. Similarly sellers with $c < \hat{c}$ trade with any buyer they meet, and sellers with $c \in [\hat{c}, \bar{c}]$ trade only with those buyers with $v > \hat{v}$; their probability of trading is equal to $1 - N_B(\hat{v})/B$.

In our constructed equilibria $N_S(\hat{c})/S$ and $1 - N_B(\hat{v})/B$ will converge to 0 as τ goes to 0, so it is convenient to divide them by τ:

$$\lambda_B \equiv \frac{1}{\tau}\left[1 - \frac{N_B(\hat{v})}{B}\right], \qquad \lambda_S \equiv \frac{1}{\tau}\frac{N_S(\hat{c})}{S}.$$

Since type $\underline{v}$ buyers and type $\bar{c}$ sellers are indifferent between entering or not, we have

$$\alpha_B \tau \lambda_S(\underline{v} - \underline{p}) = \kappa_B \tag{4.18}$$

$$\alpha_S \tau \lambda_B(\bar{p} - \bar{c}) = \kappa_S. \tag{4.19}$$

Since type $\hat{v}$ buyers are indifferent between biding $\underline{p}$ or $\bar{p}$, and type $\hat{c}$ sellers are indifferent between asking $\underline{p}$ or $\bar{p}$, we have

$$\tau\lambda_S[\rho_B(\hat{v}) - \underline{p}] = \tau\lambda_S\left\{\rho_B(\hat{v}) - [(1-k)\underline{p} + k\bar{p}]\right\} + (1 - \tau\lambda_S)\left[\rho_B(\hat{v}) - \bar{p}\right] \tag{4.20}$$

$$\tau\lambda_B[\bar{p} - \rho_S(\hat{c})] = \tau\lambda_B\left\{[(1-k)\underline{p} + k\bar{p}] - \rho_S(\hat{c})\right\} + (1 - \tau\lambda_B)\left[\underline{p} - \rho_S(\hat{c})\right]. \tag{4.21}$$

Since Lemma 1 still hold here, we have

$$\hat{W}_B = (\hat{v} - \underline{v})\frac{\tilde{m}\left(\zeta\right)\lambda_S}{\zeta r + \tilde{m}\left(\zeta\right)\lambda_S} \tag{4.22}$$

$$\hat{W}_S = (\bar{c} - \hat{c})\frac{\tilde{m}\left(\zeta\right)\lambda_B}{r + \tilde{m}\left(\zeta\right)\lambda_B}. \tag{4.23}$$

where we denoted $\tilde{m}(\zeta) \equiv \tilde{M}(\zeta, 1)$, $\hat{W}_B \equiv W_B\left(\hat{v}\right)$ and $\hat{W}_S \equiv W_S\left(\hat{c}\right)$.

To complete the description of the two-step equilibrium, the indifference conditions are supplemented with steady-state inflow balance conditions for each interval of types. Here, it suffices to require that the total inflows into the intervals $[\underline{v}, 1]$ and $[0, \bar{c}]$ are balanced with outflows,

$$b\left[1 - F\left(\underline{v}\right)\right] = S\tilde{m}(\zeta)\left[\lambda_S + \lambda_B\left(1 - \tau\lambda_S\right)\right], \tag{4.24}$$

$$sG(\bar{c}) = S\tilde{m}(\zeta)\left[\lambda_B + \lambda_S(1 - \tau\lambda_B)\right] \tag{4.25}$$

and that the inflows into the intervals $v \in [\hat{v}, 1]$ and $[0, \hat{c}]$ are also balanced with outflows,

$$b[1 - F(\hat{v})] = S\tilde{m}(\zeta)\lambda_B, \tag{4.26}$$

$$sG(\hat{c}) = S\tilde{m}(\zeta)\lambda_S. \tag{4.27}$$

(Observe that the matching rate is $S\tilde{m}\left(\zeta\right)/\tau$ for both buyers and sellers, and that τ cancels out.) We also define the price spread,

$$a_0 \equiv \bar{p} - \underline{p}.$$

Then equations (4.18) through (4.27) form a 10-equation system with 12 endogenous variables $\{\underline{p}, a_0, \zeta, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \lambda_B, \lambda_S, S, \hat{W}_B, \hat{W}_S\}$. This system does characterize an equilibrium. Indeed, one can easily see that buyers with $v \in (\hat{v}, 1]$ strictly prefer to bid $\bar{p}$, buyers with $v \in (\underline{v}, \hat{v})$ strictly prefer to bid $\underline{p}$, and buyers with $v \in [0, \underline{v})$ strictly prefer not to enter. Similar remark applies for sellers.

Since we have two degrees of freedom, we can fix some $\zeta > 0$ and $a_0 \in (a, 1)$ and then let equations (4.18) - (4.27) determine $\{\underline{p}, \underline{v}, \bar{c}, \hat{v}, \hat{c}, \lambda_B, \lambda_S, S, \hat{W}_B, \hat{W}_S\}$. We claim that solution exists for small enough τ and r. To see this, one can check that when $\tau = r = 0$, we have a (unique) solution with $\underline{p}$ implicitly determined by $b[1 - F(\underline{p} + a_0)] = sG(\underline{p})$, and all other variables given by

$$\bar{c} = \underline{p}, \quad \underline{v} = \bar{p} = \underline{p} + a_0, \quad \lambda_B = \frac{\kappa_S}{\tilde{m}(\zeta)a_0}, \quad \lambda_S = \frac{\kappa_B\zeta}{\tilde{m}(\zeta)a_0},$$

$$S = \frac{sG(\underline{p})a_0}{\kappa_B\zeta + \kappa_S}, \quad 1 - F(\hat{v}) = \frac{[1 - F(\bar{p})]\kappa_S}{\kappa_B\zeta + \kappa_S}, \quad G(\hat{c}) = \frac{G(\underline{p})\kappa_B\zeta}{\kappa_B\zeta + \kappa_S},$$

$$\hat{W}_B = \hat{v} - \bar{p}, \;\; \hat{W}_S = \underline{p} - \hat{c}.$$

One can also check that the Jacobian evaluated at $\tau = r = 0$ is not zero.[45] Therefore the Implicit Function Theorem applies. Because $\bar{p} - \underline{p} \equiv a_0 > a$, there exists a two-step equilibrium with $\bar{p} - \underline{p} > a$ when τ and r are small enough. Moreover, since $\underline{v} \to \bar{p}$ and $\bar{c} \to \underline{p}$ as $(\tau, r) \to (0,0)$, the spread $\underline{v} - \bar{c}$ is also bounded below by a. It follows that the associated welfare W is bounded away from the Walrasian welfare W^*. ■

Unlike Theorem 11, the construction in the proof of Theorem 13 treats buyers and sellers symmetrically. In particular, ζ could be fixed at any value. Then why does the double auction mechanism has non-Walrasian limit equilibria while the random-proposer mechanism does not?[46] One can verify that the dynamic types do collapse to singletons even in the two-step non-convergent equilibria. Thus to fix the idea, let us simply suppose the discount rate r is 0 so that the ultimate trading probabilities are 1 and therefore the dynamic types are constant and equal to $\rho_S = \bar{c} \to \underline{p}$ and $\rho_B = \underline{v} \to \bar{p}$. Also suppose τ is very small. Then all buyers have dynamic types approximately $\bar{p}$ and all sellers have dynamic types approximately $\underline{p}$. Unlike under random-proposer bargaining, the dynamic types are no longer the acceptance levels. Effectively the bids/asks also play this role. A seller submitting an ask lower than the dynamic types of all buyers does not guarantee herself a successful trade. To guarantee a trade, she has to ask lower than all buyers' bids. Consider a seller with $c < \hat{c}$. This seller's equilibrium ask price is $\underline{p}$. She realizes fully that the buyer's dynamic willingness-to-pay is always $\bar{p}$ approximately, and would like to demand that much if acceptance is guaranteed, as it would be under the random-proposer bargaining. However, demanding that much under the double auction protocol runs into the risk of being countered with the buyer's bid of $\underline{p}$, resulting in no trade. In our equilibrium with τ small, most of the active buyers bid $\underline{p}$. Weighing these trade-offs carefully, the seller

[45] The Mathematica® notebook that contains the evaluation of the Jacobian is available at http://grad.econ.ubc.ca/adamwong.

[46] Note that the non-convergent sequence constructed in the proof of Theorem 13 does not converge to the trivial no-trade equilibrium. Indeed, as revealed in the proof, when r is small and τ tends to 0, both the entry levels and steady-state stocks of buyers and sellers do not go to 0.

decides to submit $\underline{p}$ rather than $\bar{p}$. Similar logic applies to the buyers.

Now consider a seller with $c = \underline{p} + \varepsilon$ where $\varepsilon > 0$ is small. Although her type (or dynamic type) is significantly lower than buyers' dynamic types, which is $\bar{p}$ approximately, she chooses not to enter even when the expected search costs incurred to obtain a meeting is very small as τ becomes very small. It is again because most of the active buyers bid $\underline{p}$, making her prospect of trade meager. Similar logic applies to the buyers.

Finally, to complete our logic, we explain why the fraction of active buyers bidding $\underline{p}$ is very high relative to the fraction bidding $\bar{p}$. It is because, in our equilibrium, buyers bidding $\underline{p}$ can only trade with those sellers asking $\underline{p}$, which makes their outflow rate tiny. On the other hand, buyers bidding $\bar{p}$ trade in any meeting. Thus in steady state, the buyers who bid $\underline{p}$ accumulate and dominate the buyers' side of the market. Similar logic applies to the sellers. These arguments together explain why marginal traders do not enter to quest the significant size of the unexploited surplus $\underline{v} - \bar{c}$, keeping a positive gap between $\underline{v}$ and $\bar{c}$.

The rules of the double auction do not provide a tight connection between the dynamic types and actual acceptance levels as would be the case under the random-proposer bargaining. Here, a bid/ask is both an offer and an acceptance level. On the other hand, under random-proposer bargaining, proposing strategies and responding strategies are separate decisions because traders are clear about who is proposer and who is responder. Ex-post, the bargaining power is given to one party, and thus well-defined. Therefore the responder is always held to her acceptance level, which creates strong incentive to enter. Ex-ante, both parties could have the full bargaining power. Therefore the incentives to enter are evenly distributed over both sides of the market, driving the marginal entering types close to each other and to the Walrasian price, and leading to rapid convergence.

4.6 Concluding remarks

This chapter studies the equilibrium convergence properties of a decentralized dynamic matching and bargaining market, as search frictions vanish. The literature on dynamic matching and bargaining games has concentrated on whether the game-theoretic equilibrium outcome converges to the perfect competition. Although other papers (as reviewed in Chapter 1) have shown convergence in the contexts of similar models (for the sake of providing foundation of Walrasian equilibrium), this chapter has fundamental contributions on top of the literature.

First, we not only prove the convergence, but also derive the *rate of convergence*, for our baseline model in which the decentralized bargaining is under two-sided private information and the random-proposer take-it-or-leave-it protocol. Second, we show that the market with such a simple bargaining protocol has the property that equilibrium welfare converges to the Walrasian (first best) welfare at the fastest possible rate among all bargaining protocols. Although we have not been able to characterize the most efficient bargaining mechanism for our decentralized market, our result can be interpreted to be an asymptotic efficiency result. Third, we show that the information structure of bargaining does not alter the convergence and its speed, but the convergence might fail if we assume another bargaining protocol, double auction. It suggests that information structure is not essential to the asymptotic efficiency of a dynamic matching and bargaining market, but the bargaining protocol might.

Before closing this chapter, we make two remarks. Our first remark is a caveat on our non-convergence results for the double auction model. Under double auction, there is a great deal of multiplicity of equilibria, and some sequences of equilibria do converge to perfect competition. Also, the non-convergent equilibria we have constructed might be rather special. Our possible approach to address this is to impose additional assumptions on equilibrium selection (e.g. continuity of strategies and boundedness of the ratio of buyers to sellers) with the purpose of proving convergence.

Secondly, as we point out in Remark 2, we can think of frictions as the "cost of delay", i.e. the discount rate r and the search costs κ_B and κ_S, as we did in the previous two chapters. Then Theorem 7 (for private information bargaining) and Theorem 8 (for full information bargaining) imply that market equilibria converge to perfect competition as the friction profile (r, κ_B, κ_S) tends to zero proportionally. But what if (r, κ_B, κ_S) tends to zero non-proportionally?

It might be natural that the search costs (κ_B, κ_S) would vanish slower than the discount rate r. Let us discuss in the language of a discrete time model, so that the matching among the market participants occurs once per period; and the discount rate and search costs are measured per period. Then as the period length is shortened (in other words matches are made more frequently), the discount rate per period would decrease at the same rate as the period length. But the search costs per period might decrease at a slower rate, reflecting that making matches more frequently is costly.

It is easy to see from our theorems that, as friction profile (r, κ_B, κ_S) tends to $\mathbf{0}$, convergence (for the baseline model) holds as long as the search costs (κ_B, κ_S) vanish not faster than the discount rate r. As a matter of fact, convergence holds even when the vanishing of (κ_B, κ_S) is mildly faster. To be more concrete, let us say $\kappa_B = \kappa_S = r^\theta$ for some $\theta > 0$. Then Theorem 7 implies that, for private information bargaining, convergence holds if $\theta < \frac{3}{2}$; while Theorem 8 implies that, for full information bargaining, convergence holds if $\theta < 2$. Finally, what if the vanishing of (κ_B, κ_S) is much faster? Is there a "uniform convergence" result? This is still an open question.

Chapter 5

Conclusion

5.1 Summary

This dissertation studies a decentralized market with frictions (e.g. labor market, housing market). In the market, which we call a dynamic matching market, there are a large number of traders and the trading decisions and prices are determined by countless bilateral negotiations. More precisely, we model our market as a steady-state dynamic matching and bargaining game. The bargaining games are always bilateral, i.e. between a buyer and a seller; and each bargainer holds private information about his own willingness-to-pay or cost of providing the good.

The main purpose of Chapter 2 is to prove the existence of equilibrium for our baseline model, and to understand the equilibrium patterns and properties, under different combinations of frictions. While the results in this chapter are interesting on their own right, they are also the foundation of the analyses of Chapter 3 and Chapter 4.

Chapter 3 studies the role of private information bargaining in our baseline model. Our approach is to compare the equilibrium predictions of our baseline model (in which bargaining is under private information) with those of the full information bargaining version of the same model (i.e. Mortensen-Wright model).[47] We find both qualitative similarities and differences between them. In particular, the two models have completely the same predictions if agents are perfectly patient. Besides, if agents are impatient, private information bargaining has an entry-deterring effect. In other words, typically less potential

[47] Part of this chapter's contribution is that we have derived new results (most importantly the general condition for equilibrium existence) for Mortensen-Wright model. It is done by applying the techniques we developed in Chapter 2.

traders enter in the private information model. We also show when the private information bargaining would generate a higher level of social welfare.

Unlike most works in the literature on DMBG, Chapter 2 and Chapter 3 focus on "out-of-the-limit" results (i.e. the frictions are fixed rather than vanishing). They are particularly of interest when we are concerned with those markets with significant frictions (e.g. labor market, housing market), rather than concerned with providing a foundation for the Walrasian equilibrium.

The concern of Chapter 4 is convergence. However it is different than the literature in that this chapter does not merely provide a foundation of Walrasian equilibrium based on the convergence of a DMBG, but also shows how fast the equilibrium outcome converges to the Walrasian first best outcome. In other words, this chapter studies the "asymptotic efficiency", in terms of the rate of convergence, of dynamic matching and bargaining markets. Our results suggest that whether there is private information in bargaining does not affect the asymptotic efficiency, but the choice of bargaining protocol could have a significant effect.

5.2 Discussions

Here let us discuss which underlying assumptions we have made are crucial, and which are not. Some of the following discussions are based on conjectures.

5.2.1 Continuous time, continuous types

First of all, our assumption that time is continuous does not matter. All of our results hold under the discrete time version of our model, with only minor modifications. Our assumption that types are continuous (together with strictly positive densities) should not matter in any significant way. However, if types are discrete, we have to allow mixed (or asymmetric) strategies in order to have nontrivial equilibrium. For example, the marginal entrants must be allowed to enter probabilistically (or asymmetrically). The proposing strategies would probably have to be mixed (with nondegenerate support) as well.[48]

5.2.2 Symmetric pure strategies

Although we implicitly assume that traders use symmetric pure strategies, this is merely for simplicity of exposition. At a cost in notation we could define trader-specific and mixed strategies and then prove that they must be (essentially) symmetric and pure. To see this intuitively, recall that the matching in the market is anonymous and random. Even if different traders follow distinct strategies, every buyer with the same type v would still face the same market environment. (This is strictly true because we assume a continuum of traders.) Therefore, for a given value v, every buyer will have the identical continuation payoff, implying essentially identical responding and entry strategies. Moreover, every buyer has identical best-response correspondence for proposing strategy. Lemma 2(b) still holds so that every selection from this correspondence is nondecreasing. Consequently, the best-response is single-valued apart from a measure zero set of values where jumps could occur. But because the set is of measure zero, the selection/mixing over that set has no consequence for the maximization problems of the other traders. The same logic applies to sellers.

5.2.3 Random-proposer bargaining

If the bargaining games proceed under full information, then assuming our random-proposer bargaining protocol is equivalent to assuming the generalized Nash bargaining solution (see Subsection 3.3.1 for more details). While the Nash bargaining solution is so standard in the context of full information bargaining, there is no standard modeling method for a bilateral bargaining with two-sided private information.

The tractability of our model relies on the assumption of random-proposer bargaining even under private information. Under this bargaining protocol, the signaling issue is assumed away, because the proposers directly make take-it-or-leave price offers so that responders do not need to know their proposers' types. Also, this bargaining protocol ensures

[48] Gale (1987) proves convergence in a model with discrete type setting.

the bargaining games are one-shot.

We justify our assumption of random-proposer bargaining under private information as follows. First, it is a natural generalization of the Nash bargaining solution to a private information setting. Second, it is used in some of the recent labor search literature, e.g. Kennan (2007). In addition, it is actually much less restrictive than it looks. We can allow the proposers to propose a general mechanism (which is an informed principal mechanism design problem), and shows that in equilibrium the proposers would still make take-it-or-leave-it price offers, as in Atakan (2008).[49]

5.2.4 Choice of friction space

Recall that our notion of frictions includes two things: time discount rate r and search costs (κ_B, κ_S).[50] For our analyses to be interesting, we have to include both of them. If search costs are positive and there is no time discounting, as we have seen in Section 3.4, the private information in bargaining plays no role at all. Equilibrium existence and convergence can all be proved in a very simple manner.

On the other hand, if search costs are zero, it is impossible to have a nontrivial steady-state equilibrium, given that entry is endogenous. The reason is that, if search costs are zero and there exists a nontrivial steady-state equilibrium, the marginal entrants (who are indifferent between entering or not) must have zero probability of trade. But then these marginal entrants would accumulate and eventually clog the matching process.[51,52]

Assuming an exogenous death rate (or exit rate) δ as in Satterthwaite and Shneyerov (2008) can restore the nontrivial steady-state equilibrium. What if we take (r, δ) or $(r, \delta, \kappa_B, \kappa_S)$ as our notion of frictions? This is an open question.

[49] Atakan (2008) does that by extending the results of Riley and Zeckhauser (1983) and Yilankaya (1999). His logic can be applied here as well.

[50] The parameter τ in Chapter 4 can be interpreted as a common multiplier of the discount rate and the search costs.

[51] The argument here is rather loose, but it can be made rigorous.

[52] In Gale (1987), this problem is resolved by adding an entry fee.

5.2.5 Constant-returns-to-scale matching function

We assume that the matching function exhibits constant returns to scale. I conjecture that our main results would not be changed qualitatively if the matching function exhibits decreasing returns instead. What if the matching function exhibits increasing returns? Then things could be different. It is well-known that it is easy to have multiplicity of equilibria under increasing returns. Hence at least the uniqueness of full-trade equilibrium would not hold any more. Our convergence results should also have to be modified. I conjecture that as frictions vanish, some sequence of nontrivial steady-state equilibria still converges to perfect competition, but some other sequence converges to the trivial (i.e. no-entry) equilibrium, since now the trivial equilibrium becomes "stable". Besides, our proof of equilibrium existence does rely on constant returns. It is not clear how the necessary and sufficient condition for the existence of a nontrivial steady-state equilibrium would change if we release the assumption of constant returns.

5.2.6 Continuum of traders

We assume the market has continua of buyers and sellers. It is a common assumption in the literature, and it is technically crucial to our analysis. If the number of traders in the market is finite (of course, it is endogenous, so we need to assume the number of traders born within any finite length of time being finite), then the number and distribution of traders in the market cannot stay at some steady-state value. They have to follow some stochastic process because the matching is random and the law of large number does not apply.

The equilibrium analysis would become much less tractable, but I conjecture that the equilibrium (defined appropriately) of such a "finite market" converges to the equilibrium of the "corresponding continuum market", at least in some sense.

5.3 Further research

The previous section has pointed out some unanswered questions that are left for future research. This section suggests several more.

First, our ε-equilibrium technique (see Section 2.7) is seemingly applicable to prove existence of nontrivial equilibrium for other dynamic matching and bargaining games with heterogeneous types and free entry. For example, Satterthwaite and Shneyerov (2008) have been unable to prove existence of equilibrium, unless a distribution of new-born types is assumed to be concave. But, as they points out, "concavity is not an economically plausible assumption to impose on type distributions". Besides, the existence theorem in Satterthwaite and Shneyerov (2007) requires sufficiently small discount rate relative to the search costs (together with sufficiently small search costs); and it is only for full-trade equilibria. As another example, the existence theorem in Atakan (2008) requires what he calls Free First Draw for Low Cost Sellers, which is an artificial assumption. To sum up, all these papers have gaps in the equilibrium existence, and I expect our ε-equilibrium technique is useful to fill those gaps.

Another line of related research could be introducing competitive search (or directed search), like in Moen (1997). In particular, we could ask: would competitive search make the convergence faster? If the discount rate is zero, the competitive search version of our model (which is analyzed in Mortensen and Wright (2002)) is equivalent to the random-proposer model with a specific bargaining weight. If the discount rate is positive, different buyers and different sellers would choose to enter different submarkets. One might conjecture that even when the discount rate is positive, our rate of convergence results for the random-proposer model maintain in the competitive search model.

Another further research could be on general bargaining mechanism. We have touched on that in Section 4.4, but there are still unanswered questions. In particular, we have not solved the socially optimal bargaining mechanism when both the discount rate and the search costs are positive.[53] Moreover, what kind of mechanisms ensure convergence, and at

[53] Mortensen and Wright (2002) solve it for the no-discounting case.

what rate? This is also interesting to explore.

Bibliography

Apostol, T. (1974): *Mathematical analysis*. Addison Wesley.

Atakan, A. (2008): "Competitive Equilibria in Decentralized Matching with Incomplete Information," Working paper, Northwestern University.

Butters, G. (1979): "Equilibrium price distributions in a random meetings market," Working paper.

Chatterjee, K., and W. Samuelson (1983): "Bargaining under Incomplete Information," *Operations Research*, 31(5), 835–851.

Cripps, M., and J. Swinkels (2005): "Efficiency of Large Double Auctions," *Econometrica*, 74(5), 47–92.

De Fraja, G., and J. Sakovics (2001): "Walras Retrouve: Decentralized Trading Mechanisms and the Competitive Price," *Journal of Political Economy*, 109(4), 842–863.

Diamond, P. (1981): "Mobility Costs, Frictional Unemployment, and Efficiency," *The Journal of Political Economy*, 89(4), 798–812.

Gale, D. (1986a): "Bargaining and Competition Part I: Characterization," *Econometrica*, 54(4), 785–806.

——— (1986b): "Bargaining and Competition Part II: Existence," *Econometrica*, 54(4), 807–818.

——— (1987): "Limit Theorems for Markets with Sequential Bargaining," *Journal of Economic Theory*, 43(1), 20–54.

GRESIK, T., AND M. SATTERTHWAITE (1989): "The Rate at which a Simple Market Becomes Efficient as the Number of Traders Increases: an Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

HOSIOS, A. (1990): "On the Efficiency of Matching and Related Models of Search and Unemployment," *The Review of Economic Studies*, 57(2), 279–298.

HURKENS, S., AND N. VULKAN (2006): "Dynamic Matching and Bargaining: The Role of Private Deadlines," Working Paper, IAE and Said Business School.

——— (2007): "Dynamic Matching and Bargaining: The Role of Deadlines," Working Paper.

KENNAN, J. (2007): "Private Information, Wage Bargaining and Employment Fluctuations," Working Paper, University of Wisconsin.

LAUERMANN, S. (2006a): "Dynamic Matching and Bargaining Games: A General Approach," Working paper, University of Bonn.

——— (2006b): "When Less Information is Good for Efficiency: Private Information in Bilateral Trade and in Markets," Working Paper, University of Bonn.

——— (2008): "Price Setting in a Decentralized Market and the Competitive Outcome," Working paper.

MILGROM, P., AND I. SEGAL (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70(2), 583–601.

MOEN, E. (1997): "Competitive Search Equilibrium," *Journal of Political Economy*, 105(2), 385–411.

MORENO, D., AND J. WOODERS (2002): "Prices, Delay, and the Dynamics of Trade," *Journal of Economic Theory*, 104(2), 304–339.

Mortensen, D. (1982): "The Matching Process as a Noncooperative Bargaining Game," in *The Economics of Information and Uncertainty*, ed. by J. McCall, pp. 233–58. University of Chicago Press, Chicago.

Mortensen, D., and C. Pissarides (1999): *New Developments in Models of Search in the Labour Market*. Centre for Economic Policy Research.

Mortensen, D., and R. Wright (2002): "Competitive Pricing and Efficiency In Search Equilibrium," *International Economic Review*, 43(1), 1–20.

Pissarides, C. (2000): *Equilibrium Unemployment Theory*. Mit Press.

Reny, P., and M. Perry (2006): "Toward a Strategic Foundation for Rational Expectations Equilibrium," *Econometrica*, 74(5), 1231–1269.

Riley, J., and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *The Quarterly Journal of Economics*, 98(2), 267–289.

Rogerson, R., R. Shimer, and R. Wright (2005): "Search-Theoretic Models of the Labor Market: A Survey," *Journal of Economic Literature*, 43(4), 959–988.

Royden, H. (1988): *Real Analysis*. Prentice Hall, NJ, 3 edn.

Rubinstein, A., and A. Wolinsky (1985): "Equilibrium in a Market with Sequential Bargaining," *Econometrica*, 53(5), 1133–1150.

——— (1990): "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome," *The Review of Economic Studies*, 57(1), 63–78.

Rustichini, A., M. Satterthwaite, and S. Williams (1994): "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica*, 62(5), 1041–1063.

Satterthwaite, M. (1989): "Bilateral trade with the sealed bid k-double auction: Existence and efficiency.," *Journal of Economic Theory*, 48(1), 107–133.

Satterthwaite, M., and A. Shneyerov (2007): "Dynamic Matching, Two-sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, 75(1), 155–200.

Satterthwaite, M., and A. Shneyerov (2008): "Convergence to Perfect Competition of a Dynamic Matching and Bargaining Market with Two-sided Incomplete Information and Exogenous Exit Rate," *Games and Economic Behavior*, 63(1), 435–467.

Satterthwaite, M., and S. Williams (1989): "The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large," *The Review of Economic Studies*, 56(4), 477–498.

——— (2002): "The Optimality of a Simple Market Mechanism," *Econometrica*, 70(5), 1841–1863.

Serrano, R. (2002): "Decentralized Information and the Walrasian Outcome: A Pairwise Meetings Market with Private Values," *Journal of Mathematical Economics*, 38(1), 65–89.

Shneyerov, A., and A. Wong (2007): "The Rate of Convergence to Perfect Competition of a Simple Matching and Bargaining Mechanism," Working Paper, UBC.

——— (2009): "Bilateral Matching and Bargaining with Private Information," Working Paper, UBC.

Tatur, T. (2005): "On the Trade off Between Deficit and Inefficiency and the Double Auction with a Fixed Transaction Fee," *Econometrica*, 73(2), 517–570.

Williams, S. (1991): "Existence and Convergence of Equilibria in the Buyer's Bid Double Auction," *The Review of Economic Studies*, 58(2), 351–374.

Wolinsky, A. (1988): "Dynamic Markets with Competitive Bidding," *The Review of Economic Studies*, 55(1), 71–84.

Yilankaya, O. (1999): "A Note on the Seller's Optimal Mechanism in Bilateral Trade with Two-Sided Incomplete Information," *Journal of Economic Theory*, 87(1), 267–271.

Appendix A

Additional Details for Existence of Nontrivial Steady-state Equilibrium

This appendix provides the additional details for the proof of Theorem 3, which asserts that: In the private information model, at least one nontrivial steady-state equilibrium exists if and only if $K(\zeta_0) < 1$. (For the adaptations needed for the full information model, see subsection 3.3.3.) We first claim that our definition of mapping T_ε is legitimate.

Definition 4 of T_ε is legitimate. Fix $\bar{\alpha} > \max\{\kappa_B, \kappa_S\}$ and $\varepsilon \in (0, \bar{\varepsilon}]$. We need to claim that T_ε is well-defined and its range, as stated in the definition, is contained in its domain D_ε. The restrictions we impose on D_ε are important to claim that. Pick any $E \equiv (W_B, W_S, N_B, N_S) \in D_\varepsilon$. Firstly, by construction $B > 0$ and $S > 0$, so that α_B and α_S are well-defined. Second, $N_B(v)$ and $\rho_B(v) \equiv v - W_B(v)$ are continuous in v; $N_S(c)$ and $\rho_S(c) \equiv c + W_S(c)$ are continuous in c. Third, ρ_B and ρ_S are strictly increasing (since $E \in D_\varepsilon$ and $r > 0$). It follows that the objective functions in (2.31) and (2.32) are continuous in p. Therefore the $\arg\max$ correspondences in (2.31) and (2.32) are nonempty-valued and compact-valued. Thus p_B and p_S are well-defined. Now it is obvious that all other constructed objects, in particular $W_B^*, W_S^*, N_B^*, N_S^*$, are well-defined.

It remains to verify that $(W_B^*, W_S^*, N_B^*, N_S^*) \in D_\varepsilon$. First, by our construction W_B^*, W_S^*, N_B^*, N_S^* are absolutely continuous; and whenever differentiable,

$$W_B^{*\prime}(v) = \chi_B(v)\frac{\alpha_B}{r+\alpha_B}q_B(v)\left[1 - W_B'(v)\right] + \frac{\alpha_B}{r+\alpha_B}W_B'(v),$$

$$W_S^{*\prime}(c) = -\chi_S(c)\frac{\alpha_S}{r+\alpha_S}q_S(c)\left[1+W_S'(c)\right] + \frac{\alpha_S}{r+\alpha_S}W_S'(c),$$

$$N_B^{*\prime}(v) \equiv \frac{\chi_B^*(v)\,bf(v)}{\max\{\alpha_B q_B(v), \kappa_B\}}, \quad N_S^{*\prime}(c) \equiv \frac{\chi_S^*(c)\,sg(c)}{\max\{\alpha_S q_S(c), \kappa_S\}}.$$

From these derivatives we see $(W_B^*, W_S^*, N_B^*, N_S^*)$ satisfies the conditions (i) and (ii) in Definition 4. Second, it is easy to verify that $(W_B^*, W_S^*, N_B^*, N_S^*)$ also satisfies the condition (iii) in Definition 4. Therefore $(W_B^*, W_S^*, N_B^*, N_S^*) \in D_\varepsilon$. We conclude that Definition 4 of T_ε is legitimate. ■

Next, we prove D_ε is nonempty, convex and compact (i.e. Lemma 7).

Proof of Lemma 7. Obviously, D_ε is convex and closed. To see D_ε is nonempty, let $W_B(v) = W_S(c) = 0$ for all v, c, and $N_B(v) = b\bar{f}v/\kappa_B$, $N_S = s\bar{g}c/\kappa_S$. Since $\varepsilon \le \bar{\varepsilon}$, we have $N_B(1) \ge \varepsilon b\underline{f}/\bar{\alpha}$ and $N_S(1) \ge \varepsilon s\underline{g}/\bar{\alpha}$. All other restrictions of D_ε are obviously satisfied, thus D_ε is nonempty. To see the compactness, notice that D_ε is a uniformly bounded family of functions on a compact set $[0,1]$, and is also an equicontinuous family of functions because the Lipschitz constant for every function in D_ε is at most $\max\left\{1, b\bar{f}/\kappa_B, s\bar{g}/\kappa_S\right\}$. By Ascoli-Arzela Theorem (see e.g. Royden (1988) p.169), D_ε is compact. ■

It remains to prove the continuity of T_ε (i.e. Lemma 8). It requires the following lemma.

Lemma 16 *Let $\{\Phi_n\}$ be a sequence of continuous c.d.f.'s with supports contained in $[0,1]$ and $\{\psi_n\}$ a sequence of real functions on $[0,1]$. Suppose*

(i) $\{\Phi_n\}$ is uniformly convergent to some c.d.f. Φ;

(ii) $\{\psi_n\}$ is convergent to some real function ψ almost everywhere on $[0,1]$; and

(iii) the absolute values and total variations of $\{\psi_n\}$ and ψ are bounded by some constant C.

Then ψ_n is Riemann integrable with respect to Φ_n for each n; and ψ is Riemann integrable with respect to Φ. Moreover,

$$\lim_{n\to\infty}\int_0^1 \psi_n(x)\,d\Phi_n(x) = \int_0^1 \psi(x)\,d\Phi(x).$$

Proof. For each n, since ψ_n is of bounded variation and Φ_n is continuous, hence ψ_n is Riemann integrable with respect to Φ_n (see e.g. Apostol (1974) p.159 Theorem 7.27 and p.144 Theorem 7.6). Similarly, ψ is of bounded variation and Φ (as the uniform limit of a sequence of continuous functions) is continuous, hence ψ is Riemann integrable with respect to Φ. Moreover,

$$\left|\int_0^1 \psi_n d\Phi_n - \int_0^1 \psi d\Phi\right| \leq \left|\int_0^1 \psi_n d\Phi_n - \int_0^1 \psi_n d\Phi\right| + \left|\int_0^1 \psi_n d\Phi - \int_0^1 \psi d\Phi\right|.$$

The first part of the right-hand side can be written, through integration by parts for Riemann-Stieltjes integrals (see e.g. Apostol (1974) p.144 Theorem 7.6), as $\left|\int [\Phi - \Phi_n] \, d\psi_n\right|$ and hence is bounded by $C \cdot \sup_{x \in [0,1]} |\Phi(x) - \Phi_n(x)|$, which converges to 0 as $n \to \infty$, due to the uniform convergence of $\{\Phi_n\}$. The second part also converges to 0 as $n \to \infty$, due to Lebesgue's dominated convergence theorem (see e.g. Apostol (1974) p.270 Theorem 10.27).

■

Proof of Lemma 8. Fix $(r, \bar{\alpha}) \gg (0, \max\{\kappa_B, \kappa_S\})$ and $\varepsilon \in (0, \bar{\varepsilon}]$. We write the constructed objects in Definition 4 as functions of $E \equiv (W_B, W_S, N_B, N_S)$ explicitly, e.g. $B(E)$, $\alpha_B(E)$, $p_B(v, E)$, $W_B(v, E)$, $N_B(v, E)$ etc. We need to show that: for any sequence $\{E_n\}$ on D_ε, $E_n \to E$ implies $T_\varepsilon(E_n) \to T_\varepsilon(E)$. (Recall that we use the uniform metric on D_ε.)

Step 1. Obviously $B(E)$, $S(E)$, $\alpha_B(E)$ and $\alpha_S(E)$ are continuous in E.

Step 2. It is easy to see that: $I[p \geq c + W_S(c)]$ (where $I[\cdot]$ is 1 if the condition inside the bracket holds, and 0 otherwise), as a function of (c, p, E), is continuous on $\{(c, p, E) : p \neq c + W_S(c)\}$. Similarly, $I[p \leq v - W_B(v)]$, as a function of (v, p, E), is continuous on $\{(v, p, E) : p \neq v - W_B(v)\}$.

Step 3. $\hat{\pi}_B(v, p, E) \equiv [v - p - W_B(v)] \int_0^1 I[p \geq c + W_S(c)] \frac{dN_S(c)}{S(E)}$ is continuous in (v, p, E). To see this, let $(v_n, p_n, E_n) \to (v, p, E)$. Then firstly $v_n - p_n - W_{Bn}(v_n) \to v - p - W_B(v)$ (note that the convergence $W_{Bn} \to W_B$ is uniform); secondly from step 2, $I[p_n \geq c + W_{Sn}(c)] \to I[p \geq c + W_S(c)]$ except at the c such that $p = c + W_S(c)$ (note that there is at most one such c since $r > 0$ and $E \in D_\varepsilon$ imply $c + W_S(c)$ is strictly increasing). Applying Lemma 16, we obtain $\hat{\pi}_B(v_n, p_n, E_n) \to \hat{\pi}_B(v, p, E)$. Thus $\hat{\pi}_B(v, p, E)$ is continuous. Similarly, $\hat{\pi}_S(c, p, E) \equiv [p - c - W_S(c)] \int_0^1 I[p \leq v - W_B(v)] \frac{dN_B(v)}{B(E)}$ is continuous in (c, p, E).

Step 4. From step 3 and Berge's maximum theorem, $\pi_B(v, E)$ (which is equal to

$\max_{p\in[0,1]} \hat{\pi}_B(v,p,E))$ is continuous in (v,E), and $P_B(v,E) \equiv \arg\max_{p\in[0,1]} \hat{\pi}_B(v,p,E)$ is nonempty-valued, compact-valued, and upper-hemicontinuous in (v,E).

Analogous results can be proved for $\pi_S(c,E)$ and $P_S(c,E) \equiv \arg\max_{p\in[0,1]} \hat{\pi}_S(c,p,E)$.

Step 5. $p_B(v,E)$ is continuous on $\{(v,E) : P_B(v,E)$ is a singleton$\}$. To see this, let $(v_n,E_n) \to (v,E)$ and let $p_B(v_n,E_n) \to p$. Then from step 4, $p \in P_B(v,E)$. Thus, if $p \neq p_B(v,E)$ then $P_B(v,E)$ is not a singleton. Moreover, $p_B(v,E)$ is continuous on $\{(v,E) : v - W_B(v) > W_S(0)\}$. Analogous result can be proved for p_S.

Step 6. Let $E \in D_\varepsilon$ and $E_n \to E$. Then $p_B(v,E_n) \to p_B(v,E)$ a.e. $v \in [0,1]$. To see this, firstly consider those v with $v - W_B(v) < W_S(0)$. Then it is easy to see that $\pi_B(v,E_n) = 0 = \pi_B(v,E)$ and $P_B(v,E_n) = [0, W_{Sn}(0)] = P_B(v,E)$. Thus $p_B(v,E_n) \to W_S(0) = p_B(v,E)$. Now consider those v with $v - W_B(v) > W_S(0)$. By a standard revealed preference argument, any selection of $P_B(\cdot,E)|_{\{v:v-W_B(v)>W_S(0)\}}$ is nondecreasing. It follows that, for all but countably many v's in $\{v : v - W_B(v) > W_S(0)\}$, $P_B(v,E)$ is a singleton. Then $p_B(v,E_n) \to p_B(v,E)$ a.e. from step 5. Analogous result can be proved for p_S.

Step 7. Let $E \in D_\varepsilon$ and $E_n \to E$. Then, from steps 1, 2, 4, 6, and Lemma 16, $W_B^*(v,E_n) \to W_B^*(v,E)\ \forall v$ and $W_S^*(c,E_n) \to W_S^*(c,E)\ \forall c$.

Step 8. It is easy to see that $\chi_B(v,E)$ is continuous on $\{(v,E) : \alpha_B(E)\Pi_B(v,E) \neq \kappa_B\}$, where $\Pi_B(v,E)$ is the expression inside the square bracket in (2.33). Furthermore, given E, there is at most one v such that $\alpha_B(E)\Pi_B(v,E) = \kappa_B$. To see this, notice that $\alpha_B(E)\Pi_B(v,E)$ is nondecreasing in v, and if $\alpha_B(E)\Pi_B(v,E) = \kappa_B$ then $\alpha_B(E)q_B(v,E) \geq \kappa_B$ and hence $\frac{\partial}{\partial v}[\alpha_B(E)\Pi_B(v,E)] = \alpha_B(E)q_B(v,E)[1 - W_B'(v)] \geq \frac{\kappa_B r}{r+\alpha_B(E)} > 0$.

As a result, given any $E \in D_\varepsilon$, if $E_n \to E$ then $\chi_B(v,E_n) \to \chi_B(v,E)$ a.e. $v \in [0,1]$. Obviously χ_B^* has the same property, and analogous results can be proved for χ_S and χ_S^*.

Step 9. Let $E \in D_\varepsilon$ and $E_n \to E$. Then, from steps 1, 2, 6, and Lemma 16, $q_B(v,E_n) \to q_B(v,E)$ a.e. $v \in [0,1]$, and $q_S(c,E_n) \to q_S(c,E)$ a.e. $c \in [0,1]$. This together with step 8 implies that $N_B^*(v,E_n) \to N_B^*(v,E)\ \forall v$ and $N_S^*(c,E_n) \to N_S^*(c,E)\ \forall c$, again due to Lemma 16.

Step 10. Let $E \in D_\varepsilon$ and $E_n \to E$. From steps 7 and 9, $W_B^*(\cdot, E_n)$, $W_S^*(\cdot, E_n)$, $N_B^*(\cdot, E_n)$ and $N_S^*(\cdot, E_n)$ converge pointwise to $W_B^*(\cdot, E)$, $W_S^*(\cdot, E)$, $N_B^*(\cdot, E)$ and $N_S^*(\cdot, E)$ respectively. Moreover, the pointwise convergence is equivalent to uniform convergence, because each of those function sequences form an equicontinuous family of functions with a compact domain $[0, 1]$ (see e.g. Royden (1988) p.168). We therefore conclude that $T_\varepsilon(E_n) \to T_\varepsilon(E)$. ■

Appendix B

Calculations for Section 3.6

The goal of this Appendix is to derive the slopes $\zeta_f'(0)$ and $W_f'(0)$ in Section 3.6. As a by-product, we also show that $\underline{v}_f'(0) < 0$ and $\bar{c}_f'(0) > 0$.

First of all, divide the buyers' marginal type equation (3.36) through by $\alpha_B\left(\zeta_f\right)$, apply integration by parts to the integral in left-hand side, differentiate through at $r = 0$, and rearrange:

$$\frac{d}{dr}\left[\beta_B \int_0^{\bar{c}_f}\left[\underline{v}_f - \frac{rc + \alpha_S\left(\zeta_f\right)\beta_S\bar{c}_f}{r + \alpha_S\left(\zeta_f\right)\beta_S}\right]\frac{dG\left(c\right)}{G\left(\bar{c}_f\right)}\right]_{r=0} = \frac{d}{dr}\left[\frac{\kappa_B}{\alpha_B\left(\zeta_f\right)}\right]_{r=0}$$

$$\beta_B \cdot \frac{d}{dr}\left[\underline{v}_f - \bar{c}_f + \frac{r}{r + \alpha_S\left(\zeta_f\right)\beta_S}\int_0^{\bar{c}_f}\frac{G\left(c\right)}{G\left(\bar{c}_f\right)}dc\right]_{r=0} = \kappa_B\eta_B(\zeta_0)\zeta_f'\left(0\right)$$

$$\beta_B\left[\underline{v}_f'(0) - \bar{c}_f'(0) + \frac{W_{S0}^{ea}}{\alpha_S\left(\zeta_0\right)\beta_S G\left(\bar{c}_0\right)}\right] = \kappa_B\eta_B(\zeta_0)\zeta_f'\left(0\right) \tag{B.1}$$

where

$$\eta_B(\zeta_0) \equiv \frac{d}{d\zeta}\left[\frac{1}{\alpha_B\left(\zeta\right)}\right]_{\zeta=\zeta_0} = -\frac{\alpha_B'\left(\zeta_0\right)}{\left[\alpha_B\left(\zeta_0\right)\right]^2} > 0.$$

Work with the sellers' marginal type equation (3.37) in the same fashion, we have

$$\beta_S\left[\underline{v}_f'\left(0\right) - \bar{c}_f'\left(0\right) + \frac{W_{B0}^{ea}}{\alpha_B\left(\zeta_0\right)\beta_B\left[1 - F\left(\underline{v}_0\right)\right]}\right] = -\kappa_S\eta_S(\zeta_0)\zeta_f'\left(0\right) \tag{B.2}$$

where

$$\eta_S(\zeta_0) \equiv -\frac{d}{d\zeta}\left[\frac{1}{\alpha_S\left(\zeta\right)}\right]_{\zeta=\zeta_0} = \frac{\alpha_S'\left(\zeta_0\right)}{\alpha_S\left(\zeta_0\right)^2} > 0.$$

Equations $(B.1)$ and $(B.2)$ can be solved for $\bar{c}_f'\left(0\right) - \underline{v}_f'\left(0\right)$ and $\zeta_f'\left(0\right)$. After some rewriting from the characterizing equations of $(\zeta_0, \underline{v}_0, \bar{c}_0)$, we get

$$\zeta_f'\left(0\right) = \frac{K\left(\zeta_0\right)}{sG\left(\bar{c}_0\right)}\left[\frac{\kappa_S\eta_S(\zeta_0)}{\beta_S} + \frac{\kappa_B\eta_B(\zeta_0)}{\beta_B}\right]^{-1}\left(\frac{sW_{S0}^{ea}}{\kappa_S} - \frac{bW_{B0}^{ea}}{\kappa_B}\right), \tag{B.3}$$

$$\bar{c}_f'(0) - \underline{v}_f'(0) = \frac{K(\zeta_0)}{sG(\bar{c}_0)} \left[\frac{\kappa_S \eta_S(\zeta_0)}{\beta_S} + \frac{\kappa_B \eta_B(\zeta_0)}{\beta_B}\right]^{-1} \cdot \left[\frac{\kappa_S \eta_S(\zeta_0)}{\beta_S} \frac{sW_{S0}^{ea}}{\kappa_S} + \frac{\kappa_B \eta_B(\zeta_0)}{\beta_B} \frac{bW_{B0}^{ea}}{\kappa_B}\right].$$

Notice that

$$\frac{\kappa_B \eta_B(\zeta_0)}{\beta_B} \left[\frac{\kappa_S \eta_S(\zeta_0)}{\beta_S} + \frac{\kappa_B \eta_B(\zeta_0)}{\beta_B}\right]^{-1} = 1 - \frac{\zeta_0 m'(\zeta_0)}{m(\zeta_0)} \equiv \sigma_S(\zeta_0) > 0$$

and

$$\frac{\kappa_S \eta_S(\zeta_0)}{\beta_S} \left[\frac{\kappa_S \eta_S(\zeta_0)}{\beta_S} + \frac{\kappa_B \eta_B(\zeta_0)}{\beta_B}\right]^{-1} = \frac{\zeta_0 m'(\zeta_0)}{m(\zeta_0)} \equiv \sigma_B(\zeta_0) > 0.$$

Then $\bar{c}_f'(0) - \underline{v}_f'(0)$ can be further simplified:

$$\bar{c}_f'(0) - \underline{v}_f'(0) = \frac{K(\zeta_0)}{sG(\bar{c}_0)} \left[\sigma_B(\zeta_0) \frac{sW_{S0}^{ea}}{\kappa_S} + \sigma_S(\zeta_0) \frac{bW_{B0}^{ea}}{\kappa_B}\right] > 0. \tag{B.4}$$

Now $(B.3)$ gives the result for $\zeta_f'(0)$, while $(B.4)$ and the flow balance equation (3.38) imply that $\underline{v}_f'(0) < 0$ and $\bar{c}_f'(0) > 0$.

Next, by direct calculation, the private information slope of welfare $W_p'(0)$ is what we state in (3.42). The full information slope of welfare $W_f'(0)$ is

$$W_f'(0) = -\frac{1}{\beta_B} \frac{bW_{B0}^{ea}}{\alpha_B(\zeta_0)} - \frac{1}{\beta_S} \frac{sW_{S0}^{ea}}{\alpha_S(\zeta_0)} + sG(\bar{c}_0)\left[\bar{c}_f'(0) - \underline{v}_f'(0)\right]. \tag{B.5}$$

Sum $(B.1)$ and $(B.2)$, and insert the resulting $\bar{c}_f'(0) - \underline{v}_f'(0)$ into $(B.5)$, and cancel terms, we obtain:

$$\begin{aligned} W_f'(0) &= -\frac{bW_{B0}^{ea}}{\alpha_B(\zeta_0)} - \frac{sW_{S0}^{ea}}{\alpha_S(\zeta_0)} - sG(\bar{c}_0)\left[\kappa_B \eta_B(\zeta_0) - \kappa_S \eta_S(\zeta_0)\right] \zeta_f'(0) \\ &= W_p'(0) - sG(\bar{c}_0) K'(\zeta_0) \zeta_f'(0) \end{aligned}$$

which gives (3.43).

To obtain (3.44), simply substitute $(B.4)$ into $(B.5)$ and rewrite.