Couplings Between 11-Dimensional Supergravity Fields and The ABJM Multiple M2-Brane Theory

by

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Abstract

In this work we study the proposed model for multi M2-Branes by ABJM(Aharony, Bergman, Jafferis, Maldacena). We take an specific limit in moduli space of vacua of this model, and we will show explicitly that at this limit ABJM model becomes equivalent to maximally supersymmetric (2+1)-dimensional Yang-Mills model. Then we find the currents that are coupled to background fields of ABJM model and will try to show that these currents are some of the currents of the relevant Yang-Mills model at that limit. We also discuss a field theory approach to get these results.

Table of Contents

Appendices

List of Figures

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Dedication

To my parents, my brothers and my sister

Introduction

String theory, since its inception, has changed our understanding of fundamental physics. It began by unifying all different particles as distinct vibrational excitations of a fundamental one-dimensional object, named string. It is a quantum theory that has in itself gravity. And it is a promising candidate to our final goal, the unification of all the forces.

By discovery of different dualities in the 1980's and 1990's, and pioneering work of Witten, string theory opened new doors and steps forward to reach that goal, in the context of M-theory. In the late 1990's by breakthrough work of Maldacena[5], the AdS/CFT or gauge/gravity duality, people could use string theory to obtain some phenomenological models of particle physics such as Sakai-Sugimoto model. And also for the first time since the discovery of string theory, it could be used to predict some results that can be compared with experiment in quark-gluon plasmas. Recently people have used the AdS/CFT correspondence to get remarkable results in different areas of physics, specially Condensed Matter physics. This lead us to a better understanding of Strongly Correlated systems and High T_c superconductors.

In this section I will briefly introduce basic ideas of string theory, dualities, D-brane and their dynamics and finally introduce M-theory and its relation to string theory.

1.1 Perturbative String Theory

1.1.1 Free Strings

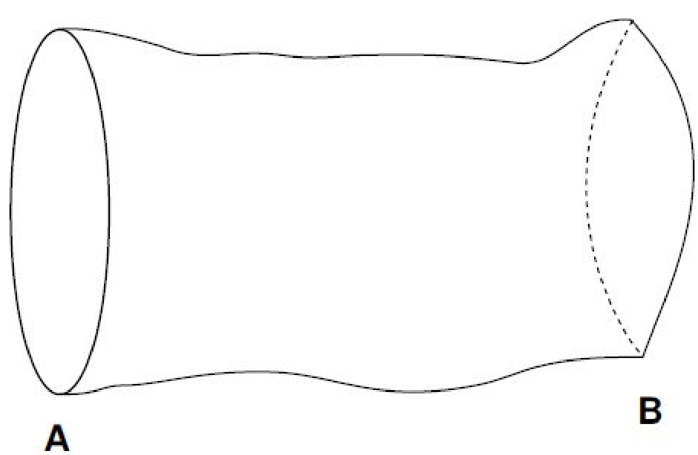
To study the behaviour of string we need the action that describes the dynamics of string . The general action for interacting strings is a big goal and no one knows how to get that ; only some low energy effective actions around particular vacua have been found. The starting point to study strings and

their spectrum is to find the action for free strings[2],[3].
The action of free string is a simple generalization of free relativistic point like particle. We know that the action of free particle is,

$$S = -m \int d\tau$$

The geometrical description of this action is straightforward and generalizable. This action gives the length of the world-line of the particle from a starting point to some final point. So by equation of motion, the stationary path would be an straight line.

Figure 1.1: Propagation of Closed String



Now suppose that string is propagating from point A to B,Fig[1.1] . As string is one-dimensional, it sweeps out a surface, this surface is called the world-sheet of the string. By analogy of point particle action we can simply find the action for a propagating string.
The resulting action is called Nambu-Goto(N-G) action, and that is the area of swept surface,

$$S_{NG} = -\frac{T}{2} \int d\sigma d\tau \sqrt{det(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu})} \tag{1.1}$$

In above action (σ, τ) are world-sheet parameters, X^μ's are space time coordinates and $G_{\mu\nu}$ is the metric, T is the tension of the string.

As it can be easily seen this action is highly non-linear and it is very difficult to quantize, even for flat space time. There is an equivalent way to get this action by introducing an auxiliary world-sheet metric. The resulting action is the Polyakov action ,

$$S_{Polyakov} = -\frac{T}{2}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \tag{1.2}$$

And $h = det(h_{\alpha\beta})$. The Polyakov action and Nambu-Goto action are equivalent at least at the classical level.
By finding the classical equation of motion of the world-sheet metric $h_{\alpha\beta}$, we find that

$$T_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}h_{\alpha\beta}h^{\alpha'\beta'}\partial_{\alpha'} X^\mu \partial_{\beta'} X^\nu G_{\mu\nu} = 0 \tag{1.3}$$

By taking a determinant of right hand side of above equation it can be verified that

$$\sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} = \sqrt{det(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu})}$$

This will return us back to N-G action. Both Polyakov and N-G actions have re-parametrization invariance. It means that by choosing different parametrization of world-sheet the action does not change. But Polyakov action is easier to dealt with. First, it is quadratic. And second, it manifestly has world-sheet Weyl scaling symmetry, such that under transformation $h_{\alpha\beta} \to e^\phi h_{\alpha\beta}$, the action is invariant.
By using re-parametrization invariance we can fix two of three degrees of freedom of world-sheet metric. By help of Weyl invariance we can fix the last degree of freedom of metric, and choose the so called *Conformal Gauge* for the metric

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

Then for a flat space time metric we will have

$$S = -\frac{T}{2}\int d\tau d\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

With this action the equation of motion simply is the wave equation,

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu = 0 \tag{1.5}$$

But to fully satisfy the invariance of the action, we need to also impose some boundary conditions. There are two type of boundary conditions,

$$X^{\mu}(\sigma, \tau) = X^{\mu}(\sigma + \pi, \tau)$$

or

$$\partial_{\tau} X^{\mu}(0, \tau) = \partial_{\tau} X^{\mu}(\pi, \tau) = 0 \tag{1.6}$$

These boundary conditions leads to two different type of strings, Closed and Open string, respectively.

1.1.2 Spectrum of Closed String

It is very straightforward to quantize string by the regular canonical quantization method. Having done that it becomes evident that vibrational excitations of string are carried out by operation of some creation operators, $\alpha^{\dagger}_{\mu}$ over the vacuum. By repeated operation of these operators we can build tower of all possible states known as Fock space.

The mass spectrum of closed string is very interesting. First, the vacuum is Tachyonic, it means it has negative mass squared. Second, in mass-less sector , there are three kinds of excitations:
1- Scalar field, ϕ. It is called Dilaton and its expectation value determines the string interaction coupling, $g_s = e^{<\phi>}$.
2-Mass-less spin=2 excitation. This is the graviton, the field responsible for gravity. So string theory manifestly has gravity in itself and it lives consistently in a quantum theory.
3-Anti-Symmetric rank-2 field, The Kalb-Ramond field.

The quantization of string action has another remarkable consequence: And it becomes evident that to have self-consistent bosonic string theory space time dimension has to be 26!. Bosonic string theory has some problems such as

- It only has bosons, but for ordinary matter we need fermions
- D=26; space time dimension is 26, but we live in four dimensional world.

- Its vacuum is Tachyonic!

Trying to solve these problems leads people to study string theories with fermionic degrees of freedom, or superstring theory that can address some of above questions.

1.1.3 Superstring Theories

We know that the action for free fermion is the Dirac action

$$\mathcal{L}_{Dirac} = -i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

We can simply generalize the bosonic string action to include fermionic degrees of freedom[7]. We add two dimensional world-sheet fermions to the string action, to get the free superstring action. In conformal gauge we have

$$S_{superstring} = -\frac{T}{2}\int d\tau d\sigma(\partial_{\alpha}X^{\mu}\partial^{\alpha}X_{\mu} - \bar{\psi}^{\mu}\rho^{\alpha}\partial_{\alpha}\psi_{\mu}) \tag{1.7}$$

Here ρ^{α} are 2×2 Dirac matrices, and $\psi_{\mu} = \begin{pmatrix} \psi_{\mu}^{+} \\ \psi_{\mu}^{-} \end{pmatrix}$ is a two component Majorana spinor. The Majorana condition simply means that ψ_{μ}^{+} and ψ_{μ}^{-} are real in a suitable representation of the Dirac algebra.
The above action is also globally supersymmetric, And it is important to mention that in contrast to bosonic string theory, superstring theory lives in space-time with dimension equal to 10.

By the mid 1980's it became evident that there are five distinct superstring theories,

- $\mathcal{N} = 2$, Non-Chiral type IIA superstring theory
- $\mathcal{N} = 2$, Chiral type IIB superstring theory
- $\mathcal{N} = 1$, $E_8 \times E_8$ heterotic superstring theory
- $\mathcal{N} = 1$, $SO(32)$ heterotic superstring theory
- $\mathcal{N} = 1$, Type I superstring theory

Having found these five superstring theories, some relations between them were found soon after their discovery. But a clear picture emerged only in the mid 1990's by help of different dualities. This led to M-theory, such that different superstring theories are different limits of M-theory on different vacua. We will back to these relations between M-theory and string theory later.

The last problem with string theory that has to be solved is 6 extra dimensions in superstring theory. The solution to this question is *Compactification*. It means that extra dimensions are compactified on a small manifold, $\mathcal{M}$, so snmall that the present day experiments cannot resolve that distance. Of course manifold $\mathcal{M}$, cannot be any arbitrary manifold, it has to satisfy the symmetry and equation of motions of string theory. Despite the fact that these manifold are so small that they cannot be seen directly by today experiments, their topological properties have significant effect on the types of field theories and phenomenological models on remaining 4-Dimensional space time.

1.2 Dualities and D-Branes

1.2.1 T and S-Duality

Even if we have not found a unique connection of different types of string theory to our 4-dimensional world, string theory greatly contributed to our understanding of physics by use of different dualities. Dualities are transformations that transforms the states and vacua of one theory to states and vacua of a distinct theory. There are many kinds of dualities in string theory, but in this section I will briefly introduce two of most important of them[6, 8, 9].

T-Duality

T-duality was discovered in the late of 1980's. It relates one string theory with a circular compactified dimension of radius R to a different string theory with compactified dimension of radius $\frac{1}{R}$. To see how it could happen , let us consider a closed string theory with one compactified dimension. There are two kinds of excitations for string. First, we have Kaluza-Klein(K-K) excitations. These excitations are general for any quantum field theory with a compactified dimension. For any such quantum theory to have well defined and single-valued wave functions for free particles we need, $\psi(x) = \psi(x + 2\pi R)$, with x is coordinate along compactified direction. So to satisfy this condition we need that momentum along that direction to be a multiple of $\frac{1}{R}$

$$p = \frac{n}{R}$$

so K-K excitations have contribution of $\left(\frac{n}{R}\right)^2$ to energy squared.
The second kind of excitations are peculiar to the character of closed strings as one-dimensional objects. For a closed string with tension T, that wind m-times around the compactifed direction there is another contribution of $(2\pi RmT)^2$ in energy squared. So in units of $2\pi T = 1$ we have,

$$E^2 = \left(\frac{n}{R}\right)^2 + (mR)^2 + ... \tag{1.8}$$

In the above equation ... means contributions from excitations of harmonic modes. From above equation it is obvious that if we do do the following transformations the energy does not change.

$$m \leftrightarrow n \qquad\qquad R \leftrightarrow \frac{1}{R}$$

It means that K-K excitations of one string theory on compactified dimension R, are winding modes of the other string theory with compactified dimension $\frac{1}{R}$, and vice versa.
The interesting point here is that from the viewpoint of closed strings there is no difference between a theory with compact dimension of R of $\frac{1}{R}$?! What about open strings?
For open strings there is no winding modes so it seems there is a paradox, that the dimension of world looks different in the limit of $R \to \infty$, for closed and open strings. This paradox can be easily resolved by introducing D-branes, higher dimensional extended objects to which the ends of open strings attach.

S-Duality

Another exciting duality that discovered in the mid 1990's is S-duality. S-duality maps the states and vacua of one string theory with coupling constant g, to other states and vacua of the same or different theory with coupling $\frac{1}{g}$. It is very interesting that the strong coupling regime of one theory is dual to the weak coupling regime of other theory. So by use of S-duality on can study a theory in the weak coupling regime by perturbative methods to learn about the strong coupling regime of the other theory.

Web of Dualities

Till now we have mentioned that there are five different consistent superstring theories in ten dimension. As we will see these theories are not exactly distinct, and are related to each other by different dualities. Here are few examples of different dual pairs in $D = 10$[9],

- Type IIA string theory is T-Dual to type IIB string theory
- $SO(32)$ and $E_8 \times E_8$ heterotic string theories are also T-dual to each other.
- Type I and $SO(32)$ heterotic are S-dual to each other.
- Type IIB string theory is S-dual to itself.

1.2.2 Dirichlet Branes

In string theory the strings are fundamental objects. There are also a variety of other objects like D-branes. As we saw, to minimize the action of strings we needed that strings have to satisfy some boundary conditions. There are two types of boundary conditions for open strings. One is that the ends have to be free. This is called the Neumann boundary condition. The other is that the ends of string have to be stationary. This is called the Dirichlet boundary condition. For Dirichlet boundary conditions it is needed some extended objects to restrict the ends of open string on them. These objects are called Dirichlet-branes or simply D-branes.
D-branes can have any dimension between 0 to 9. D0-branes are like point particles and D9-branes are called worldvolume branes. Generally Dp-branes are D-branes that their spatial dimensions are p. Actually it is possible to extract D-branes from interacting string action, and D-branes are solitonic excitations of string field equations. Their tension is proportional to $\frac{1}{g_s}$, and exactly because of this they do not appear in perturbative study of string theory, as the perturbation is around $g_s = 0$, so they are hidden in perturbation.

Gauge Theories and D-Branes

At the early stages of string theory, there was a way to introduce gauge symmetries to string theory. by use of non-dynamical Chan-Paton factors[6]. It is consistent with all symmetries of string theory, to add non-dynamical degrees of freedom to the ends of open string. By non-dynamical we mean that an end of open string prepared in one state will remain in that state or the hamiltonian of those degrees of freedom is zero. So in addition to the usual Fock space label there are other labels, i and j for each end of open string, so state of string is,

$$|p,a\rangle = \sum_{i,j=1}^{N} |p,ij\rangle \lambda_{ij}^{a} \tag{1.9}$$

λ_{ij}^{a} is an $N \times N$ matrix and is called Chan-Paton(C-P) factor. The non-dynamical structure of C-P factors, forces them to appear as a trace in interactions. So any amplitude of string interaction is proportional to,

$$\lambda_{ij}^{1}\lambda_{jk}^{2}...\lambda_{ki}^{n} = Tr(\lambda^{1}\lambda^{2}...\lambda^{n}) \tag{1.10}$$

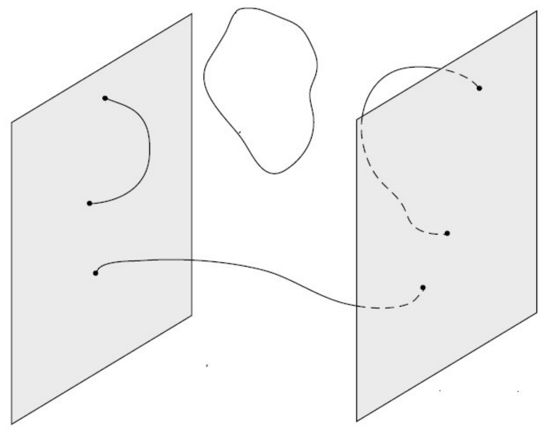
As it is evident, such an amplitude is invariant under $U(N)$ transformation as,

$$\lambda \leftrightarrow U\lambda U^{-1}$$

An important thing about D-branes is that, they provide us a remarkable, geometrical picture of introducing non-Abelian gauge symmetries in string theory and understand the nature of Chan-Paton factors.

For now suppose that we have two parallel D-branes, 1 and 2, and are separated by distance d,Fig-1.2. There are two kind of open strings,
1- Strings that both ends are on the same brane: the lowest excitations of these strings along D-brane are mass-less and represents $U(1)$ symmetry. Then we have $U(1) \times U(1)$ gauge symmetry.
2- Strings that the ends are on different branes, the lowest excitations of these strings have mass proportional to $d * T$.
Now if we take the limit $d \rightarrow 0$, an interesting thing happens: the lowest excitations of the second kind of strings will also become massless and with the first kind of string will represents the $U(2)$ gauge symmetry. It is easy to generalize this picture to an arbitrary number of D-branes. So by use of D-branes, the meaning of Chan-Paton factors is that each number, i, will be

Figure 1.2: Open Strings End on D-Branes



the label of each brane. And also the nice geometrical picture of symmetry breaking will emerge: symmetry breaking is nothing but making D-branes separate.

D-Brane Dynamics

As we saw open strings are attached to D-branes, so D-branes interact with strings. Also as they are massive they can propagate closed strings and interact with other D-branes. It is very important to understand the dynamics of D-branes as they are reason for gauge symmetries and building blocks of constructing phenomenological models. For a single Dp-brane, the low energy effective action is described by Dirac-Born-Infeld (DBI) action

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \tag{1.11}$$

In the above action ξ_a are world volume coordinate of D-brane and $G_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ and $B_{ab} = \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}$, are pull-back of space-time fields, metric and Kalb-Ramond field. In the above action the $det(G_{ab})$, is giving the world-volume of the Dp-brane and is similar to string Nambu-Goto action.

For N coincident D-branes the fields become $N \times N$ matrices, so the simple generalization of the DBI action is

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} Tr \sqrt{det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + O([X^a, X^b]^2) \quad (1.12)$$

The X^a fields are describing the transverse coordinates, or position, of D-branes and for a single D-brane the second part is obviously zero.

1.3 M-Theory

Since the early stages of string theory, parallel to string theory, people studied various supergravity and supersymmetric theories[13]. One of them was the eleven dimensional supergravity(SUGRA) that was constructed at the late 1970's. Based on supersymmetric arguments the largest possible spacetime dimension for a supersymmetric theory of gravity($spin < 2$) is eleven. It has three kinds of fields, the graviton, gravitino(supersymmetric counterpart of graviton) and three-index anti-symmetric tensor gauge field $C_{\mu\nu\rho}$.
This $D = 11$ supergravity was thought to have major problems. First, it is non-renormalizable so it cannot be a fundamental theory. Second, it was impossible to do chiral compactification on it[8, 13]. Based on dimensional reduction and some other techniques people constructed other supergravity theories like

i) Non-Chiral $\mathcal{N} = 2$ supergravity (IIA)
ii) Chiral $\mathcal{N} = 2$ supergravity (IIB)
iii) $\mathcal{N} = 1$ supergravity/YM with $E_8 \times E_8$ gauge group
iii) $\mathcal{N} = 1$ supergravity/YM with $SO(32)$ gauge group

All of these SUGRAs are effective field theory of similar superstring theories in infinite tension limit, except D=11 supergravity that is not the effective field theory of any string theory.

As we have seen, different string theories are related to each other by dualities, and effective field theories of string theories are supergravity theories. All this evidence lead people to conjecture that all of these theories are different limits of a unique, fundamental underlying theory that people called it M-Theory. And D=11 supergravity is low energy effective field theory of it, and also strong coupling limit of IIA string theory is M-theory.
In IIA string theory there are non-perturbative point-like excitations, D0-branes. The mass of D0-branes is $(l_s g_s)^{-1}$. The claim is that D0-branes can

be interpreted as Kaluza-Klein excitations of M-theory on one circular compactified dimension.
The mass of graviton in M-theory is zero so

$$M_{11}^2 = -P_M P^M = 0 \qquad\qquad M = 0, ..., 10 \tag{1.13}$$

then in 10-dimension

$$-P_\mu P^\mu = P_{11}^2 \qquad\qquad \mu = 0, ..., 9 \tag{1.14}$$

as the momentum along the compactified dimension has to be multiple of $\frac{1}{R_{11}}$ then

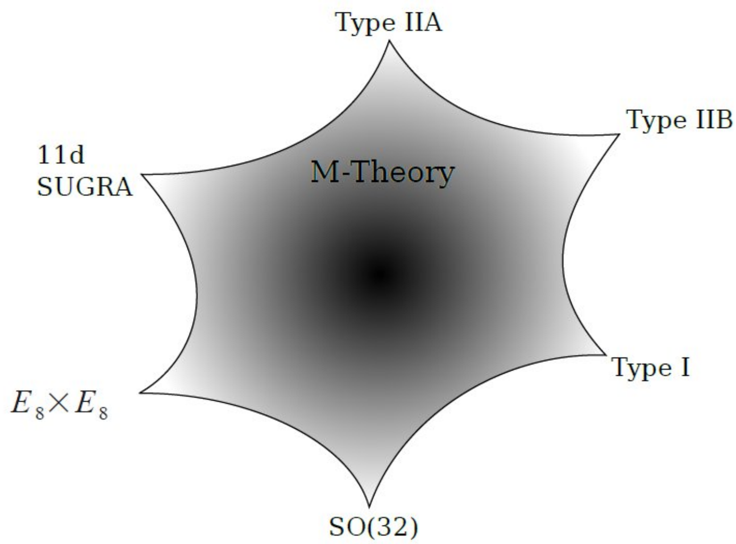
$$M^2 = \left(\frac{n}{R_{11}}\right) \tag{1.15}$$

so if the conjecture is true then $R_{11} = l_s g_s$. This is the relation between the string coupling and the radius of the compactified direction. It can be verified, if D=11 SUGRA as low energy limit of M-theory and compactified D=11 SUGRA on a circle be effective field theory of IIA string theory. And it is true because dimensional reduction of D=11 SUGRA is IIA supergravity Fig.1.4.
From the relation $R_{11} = l_s g_s$, something very interesting happens if we take the limit of $g_s \to \infty$. In this limit IIA string theory becomes eleven dimensional, How can it happen?
The point is that we studied string theory by perturbation around $g_s = 0$, so actually we studied string theory around $R_{11} = 0$ so by perturbation theory we cannot resolve the extra dimension. Now by all this we can complete the web that relates all the theories. In Fig.1.3 and Fig.1.4, the complete picture of dualities are depicted.

1.3.1 M2-Branes vs D2-Branes

M-theory in not a string theory, because we know that a consistent superstring theory can live only in 10 dimensions but M-theory is 11-dimensional. So what are the fundamental objects in M-theory?
To answer this question we need to look at the field content of M-theory. In low energy limit it becomes D=11 SUGRA and we know that it has a three-index gauge field $C_{\mu\nu\rho}$, in addition to graviton.

Figure 1.3: Web of Dualities and M-Theory



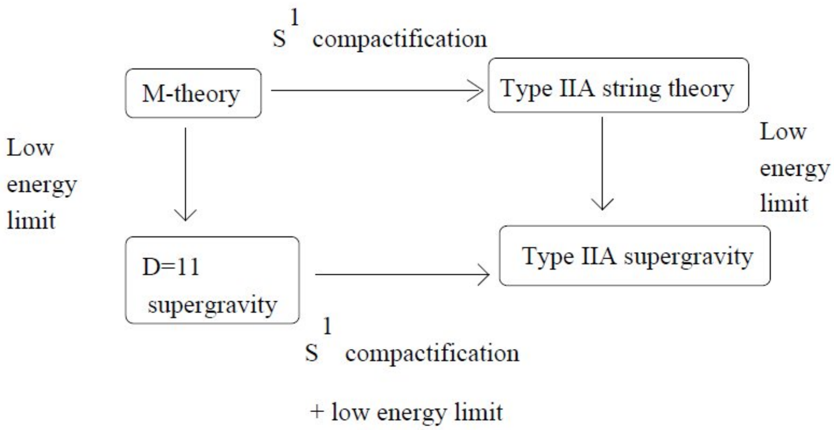
For a point particle, the one-form field is coupled to its world-line, and for the string, a two-form field $B_{\mu\nu}$ is coupled to string world-sheet. So we expect that this three-form gauge field has to be coupled to 3-dimensional world-volume of a 2-dimensional object, people called this 2-dimensional object, membrane or M2-brane. So the fundamental object of M-theory is M2-brane. What is the relation between strings in 10-dimensional string theories and M2-branes?

The answer is very simple as we said IIA string theory is low energy limit of compactified M-theory, so if we wrap an M2-brane around compactified dimension in the other 10-dimensions it looks like an string. And if M2-brane is not wrapped then it becomes ordinary D2-brane in D=10, string theory.

1.3.2 Dynamics of M2-Brane

Having known the fundamental object in M-theory, It is important to find the action that describes the dynamics of it. For a single M2-brane the action

Figure 1.4: Relation Between M-theory and Various Supergravity, Superstring Theories



is a generalized version of Nambu-Goto action (only for bosonic part),

$$S_{M2} = -T \int d^3\sigma (\sqrt{-det(h_{\alpha\beta})} + \epsilon^{\alpha\beta\gamma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho C_{\mu\nu\rho}) \tag{1.16}$$

In above action T is the tension of M2-brane, $C_{\mu\nu\rho}$ is the three-form background field and $h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$, is the pull-back of metric to the world volume. With an auxiliary world-volume metric $h_{\alpha\beta}$, this action can be written as,

$$S_{M2} = -\frac{T}{2} \int d^3\sigma [\sqrt{-h}(h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}) + \epsilon^{\alpha\beta\gamma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho C_{\mu\nu\rho})] \tag{1.17}$$

Finding the action that describes the world-volume of multiple M2-branes was a challenging work, it could not be simply generalized as we did for D2-branes. The reason is that the number of low energy degrees of freedom for N D2-branes is N^2, this is reasonable as we are dealing with $U(N)$ gauge symmetry that have N^2 generators.
But for stacks of N M2-branes, with many different methods, such as entropy calculations and supergravity methods, people found that the number of degrees of freedom is proportional to $N^{\frac{3}{2}}$. So it was very difficult to find an

action that gives these results, until the recent pioneering work of Bagger and Lambert[11] and Gustavson. A few month later, ABJM(Aharony, Bergman, Jafferis and Maldacena)[14] conjectured that an $U(N)\times U(N)$, 3-dimensional superconformal Chern-Simons matter theory can describe the multiple M2-brane action.

The rest of this work is as follows. In the next section we will describe, the ABJM proposal and we will show that in some particular limit it becomes the maximally supersymmetric $2+1$-dimensional Yang-Mills theory. Then we will try to find coupling of supergravity fields, metric and three-form, to sources and compare them to results for the D2-brane theory.

Chapter 2

ABJM Proposal

As we described in the introduction, from the results of supergravity, the number of degrees of freedom for N stacks of M2-branes, is proportional to $N^{\frac{3}{2}}$[12]. This scaling, suggests that the underlying microscopic theory that describes the dynamics of M2-branes is not trivial generalization of action of single M2-brane.

It was believed, from SUGRA, that world-volume theory of multi M2-branes is dual to M-theory on $AdS_4 \times S^7$, space time[15]. Since the late of 1990's, people have tried to find the microscopic theory of multi M2-branes with different number of supersymmetry, but till recently that by pioneering work of Bagger and Lambert has resulted in the construction of three dimensional conformal field theory that describes low energy effective field theory of multi M2-branes, by ABJM, in different M-theory backgrounds.
In this section we first describe the ABJM proposal, and then by taking certain limit, will derive the multi D2-brane action from ABJM action in specific limit.

2.1 ABJM Lagrangian

The ABJM proposal is believed to describe dynamics of multi M2-branes in C^4/Z_k orbifold[14–16]. It is a $\mathcal{N} = 6$ superconformal gauge theory with gauge group $G = U(N) \times U(N)$, with gauge fields A_μ and $\tilde{A}_\mu$ that have Chern-Simons kinetic term of level $(k, -k)$. Its matter contents are composed of four complex scalars Z_A(A=1,2,3,4), this scalar fields are describing complexified eight coordinates transverse to M2-branes world-volume , and four 3-dimensional spinors Ψ^A, and their adjoints $\bar{Z}^A$ and $\bar{\Psi}_A$. Which both of them transforms as $(N, \bar{N})$ for Z_A, Ψ^A, and $(\bar{N}, N)$ for $\bar{Z}^A, \bar{\Psi}_A$, under the gauge group G.
In addition to gauge symmetry, it also has global $SU(4)$ R-symmetry, which fields(scalar and spinors) with lower index , labels the 4 representation of

R-symmetry, and fields with upper index labels the complex-conjugate $\bar{4}$ representation.

The Lagrangian of ABJM model is

$$\mathcal{L} = \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm potential} \tag{2.1}$$

Where

$$\begin{aligned}
\mathcal{L}_{\rm CS} + \mathcal{L}_{\rm kin} &= \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}{\rm tr}\big(A_\mu\partial_\nu A_\rho - i\frac{2}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu\partial_\nu\tilde{A}_\rho + i\frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\big) \\
&\quad -{\rm tr}\left(D_\mu\bar{Z}^A D^\mu Z_A - i\bar{\Psi}_A\gamma^\mu D^\mu\Psi^A\right) \ , \\
\mathcal{L}_{\rm Yukawa} &= -\frac{2\pi i}{k}{\rm tr}\left(\bar{Z}^A Z_A\bar{\Psi}_B\Psi^B - Z_A\bar{Z}^A\Psi^B\bar{\Psi}_B\right) \\
&- \frac{2\pi i}{k}{\rm tr}\left(2\bar{Z}^A\Psi^B\bar{\Psi}_A Z_B - 2Z_A\bar{\Psi}_B\Psi^A\bar{Z}^B\right) \\
&- \frac{2\pi i}{k}\epsilon^{ABCD}{\rm tr}\left(Z_A\bar{\Psi}_B Z_C\bar{\Psi}_D\right) + \frac{2\pi i}{k}\epsilon_{ABCD}{\rm tr}\left(\bar{Z}^A\Psi^B\bar{Z}^C\Psi^D\right)
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\mathcal{L}_{\rm potential} &= +\frac{4\pi^2}{3k^2}{\rm tr}\Big(Z_A\bar{Z}^A Z_B\bar{Z}^B Z_C\bar{Z}^C + \bar{Z}^A Z_A\bar{Z}^B Z_B\bar{Z}^C Z_C \\
&\qquad +4Z_A\bar{Z}^C Z_B\bar{Z}^A Z_C\bar{Z}^B - 6Z_A\bar{Z}^A Z_B\bar{Z}^C Z_C\bar{Z}^B\Big)
\end{aligned} \tag{2.3}$$

We are basically using the convention of [15], and the covariant derivatives are

$$\begin{aligned}
D_\mu Z_A &= \partial_\mu Z_A - iA_\mu Z_A + iZ_A\tilde{A}_\mu \\
D_\mu\bar{Z}^A &= \partial_\mu Z^A - i\tilde{A}_\mu\bar{Z}^A + i\bar{Z}^A A_\mu
\end{aligned} \tag{2.4}$$

As we said the ABJM model has $\mathcal{N} = 6$ supersymmetry, whose transformaton rules for gauge fields are

$$\begin{aligned}
\delta A_\mu &= \frac{2\pi i}{k}\left(\eta^{AB}\gamma_\mu Z_A\bar{\Psi}_B + \eta_{AB}\gamma_\mu\Psi^B\bar{Z}^A\right) \\
\delta\tilde{A}_\mu &= \frac{2\pi i}{k}\left(\eta^{AB}\gamma_\mu\bar{\Psi}_B Z_A + \eta_{AB}\gamma_\mu\bar{Z}^A\Psi^B\right)
\end{aligned} \tag{2.5}$$

And for the matter fields

$$\begin{aligned}\delta Z_A &= -i\eta_{AB}\Psi^B \\ \delta\Psi^A &= \left[\gamma^\mu D_\mu Z_B - \frac{4\pi}{3k}\left(Z_{[C}\bar{Z}^C Z_{B]}\right)\right]\eta^{BA} \\ &+ \frac{8\pi}{3k}\left(Z_B\bar{Z}^A Z_C\right)\eta^{BC} - \frac{4\pi}{3k}\epsilon^{ABCD}\left(Z_B\bar{Z}^E Z_C\right)\eta_{DE}\end{aligned} \tag{2.6}$$

In above equation, γ_μ is two dimensional Dirac matrices and η^{AB} is supersymmetry transformation parameter that has the following peoperties,

$$\eta^{AB} = -\eta^{BA} \qquad \eta_{AB} = \left(\eta^{AB}\right)^* = \frac{1}{2}\epsilon_{ABCD}\eta^{CD} \tag{2.7}$$

And finally the infinitesimal gauge transformations, by Λ and $\tilde{\Lambda}$, are given by

$$\begin{aligned}\delta A_\mu &= D_\mu\Lambda = \partial_\mu\Lambda + i[A_\mu, \Lambda] \\ \delta\tilde{A}_\mu &= D_\mu\tilde{\Lambda} = \partial_\mu\tilde{\Lambda} + i[\tilde{A}_\mu, \tilde{\Lambda}] \\ \delta Z_A &= -i\Lambda Z_A + iZ_A\tilde{\Lambda}\end{aligned} \tag{2.8}$$

There are some interesting points about ABJM model. First, The Chern-Simons level k is quantized integer, and because of its quantized nature the conformal symmetry persists at the quantum level. In the large N limit, it is easier to introduce a 't Hooft like coupling $\lambda = \frac{N}{k}$, so it is evident that for small k the model is strongly coupled and very hard to dealt with, but for large k it is weakly coupled. We will return to discuss this situation more.
Finally it is constructive to study the vacuum moduli space of the model at classical level $V(\phi) = 0$. It is known that the potential part of Lagrangian can be written in quadratic form as

$$V = \frac{2\pi^2}{3k^2}Tr(W^C_{AB}\bar{W}^{AB}_C) \tag{2.9}$$

with

$$\begin{aligned}W^C_{AB} &= \left(2Z_A\bar{Z}^C Z_B - \delta^C_B Z_A\bar{Z}^D Z_D - \delta^C_A Z_D\bar{Z}^D Z_B\right) - (A \leftrightarrow B) \\ \bar{W}^{AB}_C &= \left(2\bar{Z}^A Z_C\bar{Z}^B - \delta^B_C\bar{Z}^A Z_D\bar{Z}^D - \delta^A_C\bar{Z}^D Z_D\bar{Z}^B\right) - (A \leftrightarrow B)\end{aligned} \tag{2.10}$$

to have $V = 0$, it leads to the equations of motion

$$Z_A \bar{Z}^C Z_B = Z_B \bar{Z}^C Z_A \qquad \bar{Z}^A Z_C \bar{Z}^B = \bar{Z}^B Z_C \bar{Z}^A \tag{2.11}$$

This implies that the hermitian matrices, $Z_A \bar{Z}^C$, has to commute with each other.
So up to some gauge, Z_A can be written in diagonal form,

$$Z_A = diag(z_A^1, z_A^2, ..., z_A^N) \tag{2.12}$$

It means in that point of vacuum moduli space the gauge group $U(N)\times U(N)$ is broken to $U(N)^N$, by keeping also the diagonal elements of the gauge fields A_μ and $\tilde{A}_\mu$,

$$A_\mu = diag(a_\mu^1, a_\mu^2, ..., a_\mu^n) \qquad \tilde{A}_\mu = diag(\tilde{a}_\mu^1, \tilde{a}_\mu^2, ..., \tilde{a}_\mu^n)$$

The classical low energy, bosonic part of the Lagrangian becomes,

$$\mathcal{L} = -\sum_i |D_\mu z_A^i|^2 + \frac{k}{4\pi} \sum_i \epsilon^{\mu\nu\rho}(a_\mu^i - \tilde{a}_\mu^i) f_{\mu\nu}^i \tag{2.13}$$

Where $D_\mu z_A^i = \partial_\mu z_A^i - i(a_\mu^i - \tilde{a}_\mu^i)$, and $f^i = d(a^i + \tilde{a}^i)/2$. Now we add a total derivative term to above Lagrangian, that term is

$$\frac{1}{4}\epsilon^{\mu\nu\rho} \sum \partial_\mu \theta^i f_{\nu\rho}^i$$

and θ^i is normalized to have 2π period. Why this term is total derivative is because

$$\partial_\mu \sum \epsilon^{\mu\nu\rho} \theta^i f_{\nu\rho}^i = \epsilon^{\mu\nu\rho} \sum (\partial_\mu \theta^i f_{\nu\rho}^i + \theta^i \partial_\mu f_{\nu\rho}^i) \tag{2.14}$$

By use of Bianchi identity the last expression in right hand side of above equation is zero, so the added term to Lagrangian is total derivative. Now by integrating out the field $f_{\mu\nu}^i$ we will have

$$k(a_\mu^i - \tilde{a}_\mu^i) = \partial_\mu \theta^i \tag{2.15}$$

Then the action simplifies to

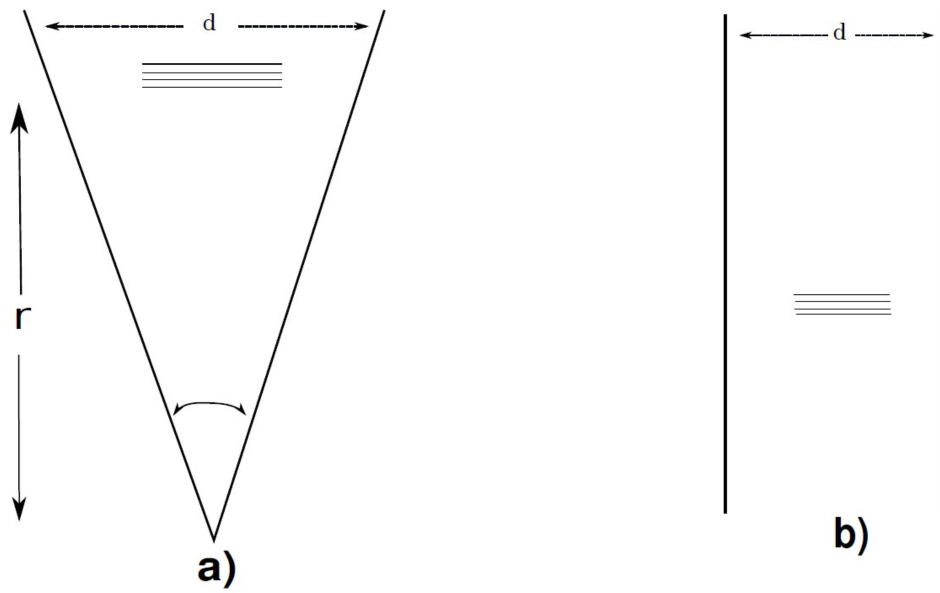
$$\mathcal{L} = -\sum_i |\partial_\mu \tilde{z}_A^i| \qquad \tilde{z}_A^i = e^{-i\theta^i/k} z_A^i \tag{2.16}$$

So by use of 2π periodicity of θ^i, we will have the vacuum moduli space is

$$\left(C^4/Z_k\right)^N / S_N \tag{2.17}$$

in here, $(C^4/Z_k)^N$ is because z_A^i(A=1,2,3,4) are complex fields and are important up to a phase,$e^{\frac{2\pi i}{k}}$, as θ^i have periodicity of 2π, and the power of N is because i is going from 1 to N.

Figure 2.1: From M2-Brane Theory to D2-Brane Theory



a) is multi M2-branes at distance r of orbifold origin with cone angle $\frac{2\pi}{k}$ b) by taking the limit of $(r, k) \to \infty$ and $\frac{r}{k}$ constant, the orbifold will become like a cylinder

2.2 Multi D2-Branes Action From ABJM Action

In this section we want to derive the action of mltiple D2-branes from multi M2-brane action of ABJM model. For doing that we put N stacks of M2-branes at distance r, of the orbifold origin. This means we are probing the action, at specific point of vacuum moduli space, such that $< Z_A >= r\delta^4_A$. By doing this we explicitly are breaking the $U(N) \times U(N)$ gauge symmetry to $U(N)$.

We expect that by taking the limit of $(k, r) \to \infty$, we can get the low energy effective action of multi D2-branes of IIA string theory, and will show it explicitly in this section. But for now let us get some physics of behind the calculations. In Fig(2.1) we depicted the orbifold background at different limits, Fig.2.1(a) is N stacks of M2-branes at distance r of orbifold origin, $< Z_A >= r\delta^4_A$, with orbifold cone angle of $\frac{2\pi}{k}$, and $d = \frac{2\pi r}{k}$. In the limit of

$(k, r) \to \infty$ by keeping $\frac{r}{k}$ as constant and finite, d will remain finite and we reach to Fig.2.1(b), and for M2-branes the geometry looks like one direction is compactified on circle. We said that M-theory compactified on circle is IIA superstring, so in this limit we expect to get multi D2-brane action with g_{YM} coupling as

$$g_{YM} = \frac{4\pi r}{k\sqrt{2N}} \tag{2.18}$$

To clarify the procedure, we choose $Z_A = r\delta_A^4 + Z_A$, such that Z_A are fluctuations around vacuum. Then we will see that some fields will become massive with mass proportional to r, then by integrating out those fields we will find a low energy effective Lagrangian that is equivalent to, $\mathcal{N} = 4$ Super Yang-Mills matrix model.

$$\begin{aligned}\mathcal{L}_{SYM} &= \frac{-1}{2g_{YM}^2} Tr(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu x_i D^\mu x_i - i\Psi\Gamma^\mu D_\mu \Psi \\ &+ \sum_{i,j} [x_i, x_j]^2 + i\bar{\Psi}\Gamma^i [x_i, \Psi])\end{aligned} \tag{2.19}$$

First to see which fields become massive, let us assume that $Z_A = r\delta_A^4$, then from Eq.(2.4) we see that

$$D_\mu Z_A = -(A_\mu - \tilde{A}_\mu) r\delta_A^4$$

$$D_\mu \bar{Z}_A = +(A_\mu - \tilde{A}_\mu) r\delta_A^4$$

Then by using above equation we have

$$\mathcal{L}_{ABJM} = \mathcal{L}_{CS} + r^2 Tr([A_\mu - \tilde{A}_\mu]^2) \tag{2.20}$$

So we see that $A_\mu - \tilde{A}_\mu$ becomes massive. By use of this, we introduce new combinations of gauge fields as

$$A_\mu^+ = A_\mu + \tilde{A}_\mu \qquad\qquad A_\mu^- = A_\mu - \tilde{A}_\mu \tag{2.21}$$

as we have broken the $U(N) \times U(N)$ gauge symmetry to $U(N)$, so in Eq.2.8 we have $\Lambda = \tilde{\Lambda}$, then it is evident that A_μ^+ and A_μ^- are transforming as a gauge field under broken symmetry.
By using this combinations, the kinetic part of gauge fields become

$$\mathcal{L}_{\text{gauge}} = \frac{k}{\pi} \epsilon^{\mu\nu\rho} Tr(A_\mu^- (\partial_\nu A_\rho^+ - i A_\nu^+ A_\rho^+) - \frac{i}{3} A_\mu^- A_\nu^- A_\rho^-) \tag{2.22}$$

for the rest of this section, I will take $Z_A = r_A + Z_A$, but at the end will replace r_A by $r\delta^4_A$.
For sake of simplicity, we will absorb a factor of k into the A^-_μ. As we will integrate out this field, at the end, this does not affect anything. So by changing $kA^-_\mu \to A^-_\mu$ the Eq.2.22 becomes

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}Tr(A^-_\mu F_{\nu\rho} - \frac{2i}{3k^2}A^-_\mu A^-_\nu A^-_\rho) \\ F_{\mu\nu} = \partial_\mu A^+_\nu - \partial_\nu A^+_\mu - i[A^+_\mu, A^+_\nu] \end{aligned} \tag{2.23}$$

now by defining, $e_A = \frac{r_A}{k}$, the kinetic part of scaling fields become

$$\begin{aligned} D_\mu \bar{Z}^A D^\mu Z_A &= \left(\tilde{D}_\mu \bar{Z}^A + 2i\bar{e}_A A^-_\mu + \frac{i}{k}(A^-_\mu \bar{Z}^A + \bar{Z}^A A^-_\mu)\right) \\ &* \left(\tilde{D}^\mu Z_A - 2i\bar{e}_A A^-_\mu - \frac{i}{k}(A^-_\mu Z_A + Z_A A^-_\mu)\right) \\ \tilde{D}_\mu Z_A &:= \partial_\mu Z_A - iA^+_\mu Z_A + iZ_A \tilde{A}^+_\mu \end{aligned} \tag{2.24}$$

Now by taking the limit of $(k, r) \to \infty$ and keeping e_A as finite, we see that terms proportional to $\frac{1}{k}$ and $\frac{1}{k^2}$ in Eqs 2.23 and 2.24, will disappear and Chern-Simons + kinetic part of scalar fields become,

$$\mathcal{L} = Tr\left(\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A^-_\mu F_{\nu\rho} - \left(\tilde{D}_\mu \bar{Z}^A + 2i\bar{e}^A A^-_\mu\right)\left(\tilde{D}^\mu Z_A - 2ie_A A^-_\mu\right)\right) \tag{2.25}$$

As we said before, from above action it can be easily seen that A^-_μ is massive and action is quadratic in terms of it. So by easily integrating it out we have

$$\begin{aligned} \mathcal{L} &= -\tilde{D}_\mu \bar{Z}^A \tilde{D}^\mu Z_A - \frac{1}{16e^2}(2i\tilde{D}^\mu(\bar{Z} - Z) + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho})^2 \\ Z &= \bar{e}^A Z_A \end{aligned} \tag{2.26}$$

As we said before Z_A are complexified, eight coordinates transverse to world-volume of M2-branes and actually

$$Z_A = x_A + ix_{A+4} \qquad\qquad \bar{Z}^A = x_A - ix_{A+4} \tag{2.27}$$

and x_i(i=1,...,8) are transverse coordinates of world-volume. Now by taking $\bar{e}^A = e_A = e\delta^4_A$, with $e = \frac{r}{k}$ we have

$$\mathcal{L} = -Tr\left[(\tilde{D}_\mu \bar{Z}^A \tilde{D}^\mu Z_A) + (\tilde{D}^\mu x_8)^2 - \frac{1}{32\pi^2 e^2}F^2 - \frac{1}{4e\pi}\epsilon^{\mu\nu\rho}\tilde{D}_\mu x_8 F_{\nu\rho}\right] \tag{2.28}$$

It is obvious that in the first and second term, the kinetic part of x_8 will cancel out. It only remains to show that the last term is zero or is a total derivative to get rid of x_8. We need to only have seven scalar field, because D2-brane of IIA string theory has 7 coordinate transverse to itself, as superstring is 10-dimensional.
The last term of Eq.2.28 is actually a total derivative, to see this let us take the following equation

$$\tilde{D}_\mu(\epsilon^{\mu\nu\rho}x_8 F_{\nu\rho}) = \epsilon^{\mu\nu\rho}\tilde{D}_\mu x_8 F_{\nu\rho} + x_8\epsilon^{\mu\nu\rho}\tilde{D}_\mu F_{\nu\rho}$$

The rhs of above equation is total derivative. By use of Bianchi identity we have

$$Tr(\epsilon^{\mu\nu\rho}\tilde{D}_\mu F_{\nu\rho})$$

finally by use of this and using x_i in terms of Z_A we reach to

$$\mathcal{L}_{CS+kinetic} = -\sum_{i=1}^{7}(\tilde{D}_\mu x_i \tilde{D}^\mu x_i) - \frac{1}{32\pi^2 e^2}F_{\mu\nu}F^{\mu\nu} \tag{2.29}$$

Now we want to find the above limit for the other terms of ABJM action. First, for interaction part Eq.A.3. As the same as before by replacing $Z_A \to r_A + Z_A$ and doing tedious algebra we find that

$$\begin{aligned} \mathcal{L}_{potential} \;\propto\; & [Z_A, Z_4 + \bar{Z}^4][\bar{Z}^A, Z_4 + \bar{Z}^4] - [Z_4, Z_4 + \bar{Z}^4][\bar{Z}^4, Z_4 + \bar{Z}^4] \\ + \; & [\bar{Z}^{A'}, \bar{Z}^{B'}][Z_{A'}, Z_{B'}] + [\bar{Z}^{A'}, Z_{B'}][Z_{A'}, \bar{Z}^{B'}] \end{aligned} \tag{2.30}$$

In above relation $A = 1, 2, 3, 4$ and $A' = 1, 2, 3$. It is easy to see that there is no term with x_8 in above equation. Because if we take $A = 4$ then the first and second term will cancel each other. So finally by using x_i as the same as before the above equation will simplifies a lot and

$$\mathcal{L}_{potential} = -8e^2\pi^2\sum_{i=1}^{7}[x_i, x_j][x_i, x_j] \tag{2.31}$$

It only remains to find kinetic part of spinors and Yukawa part of the action. For kinetic part we have

$$\begin{aligned} D_\mu\Psi_A &= \tilde{D}_\mu\Psi_A - \frac{i}{k}(A^-_\mu\Psi_A + \Psi_A A^-_\mu) \\ \tilde{D}_\mu\Psi_A &= \partial_\mu\Psi_A - iA^+_\mu\Psi_A + i\Psi_A A^+_\mu \end{aligned}$$

so in the limit of $k \to \infty$ we have $D_\mu \Psi_A = \tilde{D}_\mu \Psi_A$. The only remaining part is Yukawa part by doing exactly the same as before, we get

$$\begin{aligned} \mathcal{L}_{Yukawa} &= 4\pi e[x_8(\bar{\psi}_A \psi^A - \psi^A \bar{\psi}_A) + \bar{\psi}_4(Z_A \psi^A - \psi^A Z_A) \\ &+ \psi^4(\bar{\psi}_A \bar{Z}^A - \bar{Z}^A \bar{\psi}_A) + \epsilon^{A'B'C'}(\bar{\psi}_{A'} Z_{B'} \bar{\psi}_{C'} - \psi^{A'} \bar{Z}^{B'} \psi^{C'})] \end{aligned} \tag{2.32}$$

As the same as Z_A we can write Ψ_A's in term of $SO(8)$ Majorana spinors ψ_i such that

$$\begin{aligned} \Psi_A &= i\psi_A - \psi_{A+4} \\ \bar{\Psi}_A &= i\psi_A + \psi_{A+4} \end{aligned} \tag{2.33}$$

by using Majorana conditions on ψ_i it can be easily shown that there is no term that contain x_8 in Eq.2.32. It is also possible to write down the Yukawa term in a compact form [18, 19] as

$$\mathcal{L}_{Yukawa} = -\frac{g_{YM}}{2} Tr(\bar{\Psi}\Gamma^i[x_i, \Psi]) \tag{2.34}$$

In above equation g_{YM} is defined in Eq.2.18 and Ψ is $SO(2,1) \times SO(7)$ spinor

$$\Psi = \left(\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \psi^7, \psi^8\right)^T$$

which Γ^i are defined in [18]. So finally we got all the terms of $\mathcal{N} = 4$ SYM, and we explicitly showed that in the limit of weak coupling of ABJM model we will get the multi D2-branes action of IIA superstring theory as expected before.

Chapter 3

Supergravity Currents from ABJM Model

As we described in previous chapter, ABJM proposal is believed to describe the low-energy physics of world-volume of multi M2-branes on R^8/Z_k. By complexifying the coordinates, we introduced z^I such that there is an equivalence relation on z^I

$$z^I \equiv e^{2\pi i/k} z^I$$

We also found that by probing the multi ABJM model at specific point in moduli space, we will get the action that describes the dynamics of multi D2-branes of IIA string theory.
In this chapter we want to derive the coupling between M2-brane degrees of freedom (currents) with the bulk of eleven dimensional supergravity fields (sources). Then by taking the limit that we introduced in previous chapter, we expect that in that limit those currents of ABJM should reduce to the known the currents of D2-branes in IIA string Theory.

3.1 Setup

As we have known the degrees of freedom, field content, in ABJM model are: h_{AB} the metric that is the field of graviton, Ψ_A the gravitino field and A_{ABC} the anti-symmetric gauge field. There is also a magnetic dual of gauge field which is five-index anti-symmetric tensor, and is supposed to couple to world-volume of some M5-brane, $A_{A_1 \ldots A_5}$.
In general the coupling between above fields and background fields is

$$S = \int d^3x d^4z d^4\bar{z} \left[\frac{1}{2} h_{AB} T^{AB} + \frac{1}{3!} A_{ABC} J^{ABC} + \frac{1}{5!} A_{A_1 \ldots A_5} J^{A_1 \ldots A_5} + i\psi_A S^A \right] \tag{3.1}$$

It is convenient to write above equation in terms of fields that only depends on coordinates along M2-brane. To do that we have to integrate out

the transverse coordinates of world-volume of M2-brane.
For a field ϕ on R^8/Z_k, we can expand it as a Taylor expansion along coordinates transverse to M2-brane,

$$\phi(z) = \sum_{k|(m-n)}^{\infty} \frac{1}{m!}\frac{1}{n!}\phi_{I_1\cdots I_m}{}^{J_1\cdots J_n} z^{I_1}\cdots z^{I_m}\bar{z}_{J_1}\cdots\bar{z}_{J_n} \tag{3.2}$$

In above equation we have to restrict the only values of k, such that $k|(m-n)$, because we assumed that our fields are invariant under transformation $z \to \omega_k z$,such that ω_k is k^{th}-root of unity. In equation (3.2) we have

$$\phi_{I_1\cdots I_n}{}^{J_1\cdots J_n} \equiv \partial_{I_1}\cdots\partial_{I_m}\bar{\partial}^{J_1}\cdots\bar{\partial}^{J_n}\phi(0) \tag{3.3}$$

by pluging above equation into Eq(3.1) we have

$$\begin{aligned} S \;=\; & \int d^3x \sum_{k|(m-n)}^{\infty} \frac{1}{2}\frac{1}{m!}\frac{1}{n!}\partial_{I_1}\cdots\partial_{I_m}\bar{\partial}^{J_1}\cdots\bar{\partial}^{J_n} h_{AB}(x) * \\ & T^{AB;I_1\cdots I_m}{}_{J_1\cdots J_n}(x) \\ + & \int d^3x \sum_{k|(m-n)}^{\infty} \frac{1}{3!}\frac{1}{m!}\frac{1}{n!}\partial_{I_1}\cdots\partial_{I_m}\bar{\partial}^{J_1}\cdots\bar{\partial}^{J_n} A_{ABC}(x) * \\ & J^{ABC;I_1\cdots I_m}{}_{J_1\cdots J_n}(x) \\ + & \int d^3x \sum_{k|(m-n)}^{\infty} \frac{1}{6!}\frac{1}{m!}\frac{1}{n!}\partial_{I_1}\cdots\partial_{I_m}\bar{\partial}^{J_1}\cdots\bar{\partial}^{J_n} A^{D}_{ABCDEF}(x) * \\ & M^{ABCDEF;I_1\cdots I_m}{}_{J_1\cdots J_n}(x) \\ + & \int d^3x \sum_{k|(m-n)}^{\infty} i\frac{1}{m!}\frac{1}{n!}\partial_{I_1}\cdots\partial_{I_m}\bar{\partial}^{J_1}\cdots\bar{\partial}^{J_n} \psi_{A}(x) * \\ & S^{A;I_1\cdots I_m}{}_{J_1\cdots J_n}(x) \end{aligned} \tag{3.4}$$

In above equation the multipole moments are defined as

$$T^{AB;I_1\cdots I_m}{}_{J_1\cdots J_n}(x^\mu) = \int d^4z d^4\bar{z} z^{I_1}\cdots z^{I_m}\bar{z}_{J_1}\cdots\bar{z}_{J_n} T^{AB}(x,z)$$
$$J^{ABC;I_1\cdots I_m}{}_{J_1\cdots J_n}(x^\mu) = \int d^4z d^4\bar{z} z^{I_1}\cdots z^{I_m}\bar{z}_{J_1}\cdots\bar{z}_{J_n} J^{ABC}(x,z)$$

$$M^{ABCDEF;I_1\cdots I_m}{}_{J_1\cdots J_n}(x^\mu) = \int d^4zd^4\bar{z}z^{I_1}\cdots z^{I_m}\bar{z}_{J_1}\cdots\bar{z}_{J_n}M^{ABCDEF}(x,z)$$
$$S^{A;I_1\cdots I_m}{}_{J_1\cdots J_n}(x^\mu) = \int d^4zd^4\bar{z}z^{I_1}\cdots z^{I_m}\bar{z}_{J_1}\cdots\bar{z}_{J_n}S^{A}(x,z) \tag{3.5}$$

In these equations the indices $A, B, ...$, can be on M2-brane world-volume $(\mu, \nu, ...)$ and also transverse coordinates $(I, J, ...)$. Then it is convenient to write down above currents in $SO(2,1)\times SU(4)$ representations. For example for T^{AB} we have,$T^{\mu\nu}, T^{\mu I}, T^{\mu}{}_J, T^{IJ}, T^I{}_J, T_{IJ}$ in $SO(2,1)\times SU(4)$ representations. These are the fields that we are trying to find in this chapter.
Before going to find the expressions for currents, It is useful to say general things about the currents. As these currents are responsible for some symmetries, these currents are conserved so we have

$$\partial_A J^{\alpha A} = 0 \tag{3.6}$$

in here α could be any index. Then

$$\partial_\mu J^{\alpha\mu} + \partial_I J^{\alpha I} + \bar{\partial}^I J^\alpha{}_I = 0$$

by using the definition of multipole moments of currents, we have

$$\int \left(\partial_\mu J^{\alpha\mu} + \partial_I J^{\alpha I} + \bar{\partial}^I J^\alpha{}_I\right) z^{I_1}...z^{I_m}\bar{z}_{J_1}...\bar{z}_{J_n}d^4zd^4\bar{z} = 0$$

so by partial integration we

$$\partial_\mu J^{\alpha\mu;I_1...I_m}{}_{J_1...J_n} = \sum_{i=1}^{m} J^{\alpha I_i;I_1..\hat{I}_i..I_m}{}_{J_1...J_n} + \sum_{i=1}^{n} J^{\alpha}{}_{J_i}{}^{;I_1...I_n}{}_{J_1..\hat{J}_i..J_m} \tag{3.7}$$

In above equation the hatted indices means that those are omitted in multipole moments. This equation is very useful, as it gives a relation between higher multipole moments and lower ones. In this work we will use these equations to find currents.

Bosonic Currents for $U(1)\times U(1)$

For $N = 1$, ABJM model describes the dynamics of single M2-brane. We know the action for one M2-brane is described by Eq(1.16)

$$S_{M2} = -T\int d^3\sigma(\sqrt{-det(h_{ab})} + \epsilon^{abc}\partial_a X^\mu\partial_b X^\nu\partial_c X^\rho C_{abc}) \tag{3.8}$$

In this equation $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$. Now by assuming the background fields are weak, we can expand metric around flat spacetime so we have $g_{\mu\nu} = \eta_{\mu\nu} + l_{\mu\nu}$, then for the pull back of metric we have

$$h_{ab} = \eta_{ab} + l_{ab} + \partial_a X^I \partial_b X_I + 2\partial_a X^I l_{bI} + \partial_a X^I \partial_b X^J l_{IJ}$$

In above equation $a, b, ...$ represents coordinates along M2-brane. To find above equation we have used the physical gauge along M2-branes such that $X^a = \xi^a$, such that ξ's are coordinates along M2-branes. Now by plug above equation into Eq(1.16) and comparing it with Eq(3.1), it is easy to see that

$$\begin{aligned} J^{\mu\nu\rho;i_1...i_n} &= \epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\rho X^{i_1}...X^{i_n} \\ T^{ab;i_1...i_n} &= \eta^{ab} X^{i_1}...X^{i_n} \\ T^{ai;i_1...i_n} &= \partial^a X^i X^{i_1}...X^{i_n} \\ T^{ij;i_1...i_n} &= \partial_a X^i \partial^a X^j X^{i_1}...X^{i_n} \end{aligned} \tag{3.9}$$

in above equation a,b are along D2-brane coordinates, μ, ν, ρ are all space time coordinates and finally $i_1, ..., i_n$ are coordinates transverse to D2-brane.

Generalization to $U(N) \times U(N)$

The generalization of above currents to $U(N) \times U(N)$ case is not straight forward. The first subtlety is that, now we are dealing with matrices rather than functions. The second and more important one is that, The scalars Z and $\bar{Z}$ are transforming differently under $U(N) \times U(N)$ gauge symmetry,

$$Z \to UZV^{-1} \qquad\qquad \bar{Z} \to V\bar{Z}U^{-1}$$

and, as we want to our currents to be invariant under gauge symmetry so we have to put every Z adjacent to $\bar{Z}$, or if we want to put two Z or $\bar{Z}$ adjacent to each other we have to use an operator called 't Hooft disorder operator in between to respect the gauge symmetry, like

$$(...ZZ...) \to (...ZTZ...) \qquad\qquad (...\bar{Z}\bar{Z}...) \to (...\bar{Z}\bar{T}\bar{Z}...)$$

such that T is 't Hooft disorder operator and $\bar{T}$ is conjugate of it. In this work we restrict ourself to the case that in multipole operator the number of Z and $\bar{Z}$ are equal. In this case we do not need to use 't Hooft operator to satisfy the gauge invariance. The problem with 't Hooft operator is that

does not have explicit local form so it is very difficult to use it.
The last subtlety in our approach to find the curernts is that, the currents that we are trying to find them are not unique in the sense that these are defined up to terms that are coming from equations of motion.

Now let us show with an example, to how find the currents. We are only interested to the case of $m = n$, because of two reason. First, in this case the number of Z and $\bar{Z}$ are equal and for generalization we do not need to use the t' Hooft operator, Second, as we saw in Taylor expansion we have to restrict m, n such that $k|(m-n)$, so we see that the terms with $m = n$ are the only terms that satisfies this condition for any value of k, so these terms are present for any k. From single M2-brane currents Eq(3.9) we have

$$J^{\mu\nu\rho;i_1...i_n}{}_{j_1...j_n} = \epsilon^{\mu\nu\rho} z^{i_1}...z^{i_n}\bar{z}_{j_1}...\bar{z}_{j_n}$$

a point about above current is that it is symmetric in i's and j's, as it coupled to a term like $\partial_{i_1}...\partial_{i_n}\phi(0)$. So the generalized current that is gauge invariant and is symmetric in higher moments is,

$$J^{\mu\nu\rho;I_1...I_n}{}_{J_1...J_n} = \epsilon^{\mu\nu\rho} STr(Z^{i_1}\bar{Z}_{j_1}...Z^{i_n}\bar{Z}_{j_n}) \tag{3.10}$$

Now let us to use Eq(3.7) to find J along transverse directions. To do that we first have to take derivative of above equation so,

$$\begin{aligned}\partial_\rho J^{\mu\nu\rho;I_1...I_n}{}_{J_1...J_n} &= \epsilon^{\mu\nu\rho}\sum_i Tr[Z^{(I_1}\bar{Z}_{(J_1}..\partial_\rho Z^{I_i}\bar{Z}_{J_i}..Z^{I_n)}\bar{Z}_{J_n)}] \\ &+ \epsilon^{\mu\nu\rho}\sum_i Tr[Z^{(I_1}\bar{Z}_{(J_1}..Z^{I_i}\partial_\rho\bar{Z}_{J_i}..Z^{I_n)}\bar{Z}_{J_n)}]\end{aligned} \tag{3.11}$$

by use of Eq(3.7) we also have

$$\partial_\rho J^{\mu\nu\rho;I_1...I_n}{}_{J_1...J_n} = \sum_i J^{\mu\nu I_i;I_1..\hat{I}_i..I_n}{}_{J_1...J_n} + \sum_i J^{\mu\nu}{}_{J_i}{}^{;I_1...I_n}{}_{J_1..\hat{J}_i..J_n} \tag{3.12}$$

It is easy to see that above current also satisfies Eq(3.11) By use of above equations and comparing it with Eq(3.11) the naive guess says that the curent could be

$$J^{\mu\nu I;I_1...I_{n-1}}{}_{J_1...J_n} = Tr[\partial_\rho Z^{(I}\bar{Z}_{J_1}Z^{I_1}\cdots Z^{I_{n-1})}\bar{Z}_{J_n)}]$$

but simple observation says that above current is not invariant under gauge transformation, unless we replace partial derivative by covariant derivative. So

$$J^{\mu\nu I;I_1...I_{n-1}}{}_{J_1...J_n} = Tr[D_\rho Z^{(I}\bar{Z}_{J_1}Z^{I_1}\cdots Z^{I_{n-1})}\bar{Z}_{J_n)}] \tag{3.13}$$

By similar procedure we can find that

$$
\begin{aligned}
J^{\alpha\beta\gamma;I_1\cdots I_n}{}_{J_1\cdots J_n} &= \epsilon^{\alpha\beta\gamma}Tr[Z^{(I_1}\bar{Z}_{(J_1}\cdots Z^{I_n)}Z_{J_n)}] \\
J^{\alpha\beta I;I_1\cdots I_{n-1}}{}_{J_1\cdots J_n} &= \epsilon^{\alpha\beta\gamma}Tr[D_\rho Z^{(I}\bar{Z}_{J_1}Z^{I_1}\cdots Z^{I_{n-1})}\bar{Z}_{J_n)}] \\
J^{\alpha\beta}{}_J{}^{;I_1\cdots I_n}{}_{J_1\cdots J_{n-1}} &= \epsilon^{\alpha\beta\gamma}Tr[Z^{(I_1}D_\rho\bar{Z}_J Z^{I_2}\cdots Z^{I_n)}\bar{Z}_{J_{n-1})}] \\
J^{\alpha II';I_1\cdots I_{n-2}}{}_{J_1\cdots J_n} &= \epsilon^{\alpha\beta\gamma}Tr[D_\gamma Z^{(I}\bar{Z}_{J_1}D_\beta Z^{I'}\bar{Z}_{J_2}Z^{I_1}\cdots Z^{I_{n-2}}\bar{Z}_{J_n}] \\
J^{\alpha}{}_J{}^{I;I_1\cdots I_{n-1}}{}_{J_1\cdots J_{n-1}} &= \epsilon^{\alpha\beta\gamma}Tr[D_\gamma Z^{(I}D_\beta\bar{Z}_{(J}Z^{I_1}\bar{Z}_{J_1}\cdots Z^{I_{n-1})}\bar{Z}_{J_{n-1})} \\
&+ \frac{1}{2n}F_{\beta\gamma}Z^I\bar{Z}_J Z^{(I_1}\bar{Z}_{(J_1}\cdots Z^{I_{n-1})}\bar{Z}_{J_{n-1})} \\
&- \frac{1}{2n}\bar{Z}_J Z^I\tilde{F}_{\beta\gamma}\bar{Z}_{(J_1}Z^{(I_1}\cdots\bar{Z}_{J_{n-1})}Z^{I_{n-1})}
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
J^{II'I'';I_1\cdots I_{n-3}}{}_{J_1\cdots J_n} &= \epsilon^{\alpha\beta\gamma}Tr[D_\alpha Z^{(I}\bar{Z}_{J_1}D_\beta Z^{I'}\bar{Z}_{J_2}D_\gamma Z^{I''}\bar{Z}_{J_3}Z^{I_1}\cdots Z^{I_{n-3})}\bar{Z}_{n)}] \\
J_J{}^{II';I_1\cdots I_{n-2}}{}_{J_1\cdots J_{n-1}} &= \epsilon^{\alpha\beta\gamma}Tr[D_\alpha Z^{(I}D_\beta\bar{Z}_{(J}D_\gamma Z^{I'}\bar{Z}_{J_1}Z^{I_1}\cdots Z^{I_{n-2})}\bar{Z}_{J_{n-1})} \\
&+\frac{1}{2n}F_{\alpha\beta}Z^{I'}\bar{Z}_J D_\gamma Z^{(I}\bar{Z}_{(J_1}Z^{I_1}\cdots Z^{I_{n-2})}\bar{Z}_{J_{n-1})} \\
&-\frac{1}{2n}F_{\alpha\beta}Z^{I}\bar{Z}_J D_\gamma Z^{(I'}\bar{Z}_{(J_1}Z^{I_1}\cdots Z^{I_{n-2})}\bar{Z}_{J_{n-1})} \\
&-\frac{1}{2n}\bar{Z}_J Z^{I'}\tilde{F}_{\alpha\beta}\bar{Z}_{(J_1}D_\gamma Z^{(I}\bar{Z}_{J_2}\cdots\bar{Z}_{J_{n-1})}Z^{I_{n-2})} \\
&+\frac{1}{2n}\bar{Z}_J Z^{I}\tilde{F}_{\alpha\beta}\bar{Z}_{(J_1}D_\gamma Z^{(I'}\bar{Z}_{J_2}\cdots\bar{Z}_{J_{n-1})}Z^{I_{n-2})}
\end{aligned}
$$

All of above equations are correct up to corrections from equations of motion, and also some commutator of fields that are definitely hidden in $U(1)\times U(1)$ case and are higher order corrections.

3.2 Relation of ABJM Currents to D2-Brane Currents

Till now we have found the form of bosonic currents of ABJM model and also in chapter two we showed that in some specific limit in moduli space we will get the action of multi D2-branes. In this section we want to check that weather the ABJM current should also reduce to D2-brane currents in the limit that described before.

General Setup

Suppose we have turned on some weak background SUGRA fields ϕ_α on R^8/Z_k, we assume that these fields does not depend on Z^4, as it will simplifies the calculations. In language of D2-brane, this means that fields does not depend on coordinate along compactified dimension. So we have

$$S = S_{ABJM} + \int d^3x \frac{1}{m!n!}\partial_{a_1}...\partial_{a_n}\bar{\partial}^{b_1}....\bar{\partial}^{b_n}\phi_\alpha(x,0)J^{\alpha;a_1...a_n}_{b_1...b_n} \tag{3.15}$$

here a,b are taking only values of 1,2,3. As we showed by taking the limit as described before and integrating out $B_\mu = A_\mu - \tilde{A}_\mu$, it leads to $B_\mu \to \epsilon_{\mu\nu\rho}F^{\nu\rho}$, we reach to multi D2-brane action.
On the other hand in IIA string theory for multi D2-branes in weak background field we have

$$S = S_{D2} + \int d^3x \frac{1}{m!n!}\partial_{a_1}...\partial_{a_n}\bar{\partial}^{b_1}....\bar{\partial}^{b_n}\phi^{IIA}_\alpha(x,0)J^{\alpha;a_1...a_n}_{D2,b_1...b_n} \tag{3.16}$$

by comparing Eq(3.15) and Eq(3.16), we will be able to determine the expressions for M2-brane currents in terms of D2-brane currents, or test that the determined M2-brane currents to reduce to D2-brane ones by taking the above limit. So the final expression for currents is

$$\left\{J^{\alpha;a_1\cdots a_m}{}_{b_1\cdots b_n}(x;Z^4=\delta^I_4 r, B\sim\epsilon F)\right\}_{k\to\infty\ r/k\ \text{fixed}} = J^{\alpha;a_1\cdots a_m}_{D2}{}_{b_1\cdots b_n}(x;X^7=0) \tag{3.17}$$

by using above equation, we expect that we could check that Eq(3.14) to reduce to D2-brane currents. But it does not happen. The reason that why it did not happen, is not clear to us. Some possibilities are, the currents that we have found are simple generalizations from $U(1)\times U(1)$ case, and are defined up to some terms from equations of motions, may be these extra terms could change the results.

To find currents explicitly there are other approaches to find them. One way is to do field theory calculations. To do that we need to turn on some weak background fields in ABJM action. Then by expanding the action in second order in terms of perturbations around the background, and using perturbative methods to find the effective action. We also tried to find the effective action, but we found that in $U(N)\times U(N)$ case, the propagators and vertices become very complicated and very hard to deal with them, and also to find the effective action by simple analysis we need to do perturbation to fifth order.

Till now we only found the bosonic currents, What about fermionic currents? The easiest way to find the fermionic currents is to use the invariance of the action under supersymmetric transformations. As we know SUSY transformations relates the variation of bosonic fields to fermionic ones and vice versa. So by restricting the action to be invariant under these transformation we will get some relations between variation of bosonic currents and fermionic currents So by knowing the bosonic currents and using SUSY transformations we could find fermionic currents up to some terms from equations of motion.

3.3 Summary

This work was mainly a review of ABJM model and relation of this candidate of multi M2-branes to type IIA superstring theory and multi D2-brane action. We also tried to find the multi M2-brane currents and check the relation between these currents and currents of D2-branes, but the results are in contrast with the results that we expected to find. Some possible work for future could be, to find the currents from field theory approach and check them with the currents that we have found and also check the limit that we tried to do in this case, to find out that, what was the problem with our approach. Definitely this approach is very cumbersome as we tried to do. With field theory approach it is also possible to find the fermionic currents directly but it will complicate the problem more.

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Appendix A

Field Theory Calculations

In this appendix we want to derive the vertex coefficient that are required to do perturbative calculations in field theory approach to find the currents of ABJM model. We only assume that a weak scalar background is turned on and we do the perturbation around that background. The ABJM Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm potential} \tag{A.1}$$

Where

$$\begin{aligned}
\mathcal{L}_{\rm CS} + \mathcal{L}_{\rm kin} &= \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}{\rm tr}\big(A_\mu\partial_\nu A_\rho - i\frac{2}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu\partial_\nu\tilde{A}_\rho + i\frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\big) \\
&\quad -{\rm tr}\left(D_\mu\bar{Z}^A D^\mu Z_A - i\bar{\Psi}_A\gamma^\mu D^\mu\Psi^A\right) \ , \\
\mathcal{L}_{\rm Yukawa} &= -\frac{2\pi i}{k}{\rm tr}\left(\bar{Z}^A Z_A\bar{\Psi}_B\Psi^B - Z_A\bar{Z}^A\Psi^B\bar{\Psi}_B\right) \\
&\quad - \frac{2\pi i}{k}{\rm tr}\left(2\bar{Z}^A\Psi^B\bar{\Psi}_A Z_B - 2Z_A\bar{\Psi}_B\Psi^A\bar{Z}^B\right) \\
&\quad - \frac{2\pi i}{k}\epsilon^{ABCD}{\rm tr}\left(Z_A\bar{\Psi}_B Z_C\bar{\Psi}_D\right) + \frac{2\pi i}{k}\epsilon_{ABCD}{\rm tr}\left(\bar{Z}^A\Psi^B\bar{Z}^C\Psi^D\right)
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
\mathcal{L}_{\rm potential} &= +\frac{4\pi^2}{3k^2}{\rm tr}\Big(Z_A\bar{Z}^A Z_B\bar{Z}^B Z_C\bar{Z}^C + \bar{Z}^A Z_A\bar{Z}^B Z_B\bar{Z}^C Z_C \\
&\qquad +4Z_A\bar{Z}^C Z_B\bar{Z}^A Z_C\bar{Z}^B - 6Z_A\bar{Z}^A Z_B\bar{Z}^C Z_C\bar{Z}^B\Big)
\end{aligned} \tag{A.3}$$

The covariant derivative is defined

$$D_\mu Z_A = \partial_\mu Z_A - iA_\mu Z_A + iZ_A\tilde{A}_\mu \ , \tag{A.4}$$

As I said we only turn on a scalar background field so the form of our fields become,

$$Z_A = \begin{pmatrix} X_A & Y_A \\ \hat{Y}_A & \hat{X}_A \end{pmatrix} \qquad\qquad \bar{Z}^A = \begin{pmatrix} \bar{X}_A & \bar{\hat{Y}}_A \\ \bar{Y}_A & \bar{\hat{X}}_A \end{pmatrix}$$

$$\Psi^A = \begin{pmatrix} 0 & \psi^A \\ \hat{\psi}_A & 0 \end{pmatrix} \qquad \bar{\Psi}_A = \begin{pmatrix} 0 & \bar{\hat{\psi}}_A \\ \bar{\psi}_A & 0 \end{pmatrix}$$

$$A_\mu = \begin{pmatrix} 0 & a_\mu \\ \bar{a}_\mu & 0 \end{pmatrix} \qquad \tilde{A}_\mu = \begin{pmatrix} 0 & \tilde{a}_\mu \\ \bar{\tilde{a}}_\mu & 0 \end{pmatrix}$$

In above relations as you can see we only consider the fluctuations that are off-diagonal part of our field matrices , X and $\hat{X}$ are background fields and the other are fluctuations. The reason for doing that is very simple, we only want to keep low energy fluctuations and integrate out the high-energy ones, from ABJM Lagrangian it is evident that all the diagonal fluctuation of the fields are massless and only the off-diagonal parts are massive with mass proportional to $\frac{r}{k}$, despite the diagonal part of $A-\tilde{A}$ that we have to integrate it out later. Now by plug in above relations into the Lagrangian and expanding it to quadratic order in terms of fluctuations we have:

First for potential term

$$\delta\mathcal{L}_{\text{potential}} = \frac{4\pi^2}{k^2} \begin{pmatrix} \bar{\hat{Y}}_A & Y_A \end{pmatrix} \mathcal{M}^b_{AB} \begin{pmatrix} \hat{Y}_B \\ \bar{Y}_B \end{pmatrix}$$

the elements of matrix $\mathcal{M}^b_{AB}$ are

$$\begin{aligned}
\mathcal{M}^{b(11)}_{AB} &= \delta_{AB}(K_C\bar{K}_C K_D\bar{K}_D + \hat{K}_C\bar{\hat{K}}_C\hat{K}_D\bar{\hat{K}}_D - 2\hat{K}_C\bar{\hat{K}}_C K_D\bar{K}_D) \\
&- 2\delta_{AB}(K_C\bar{K}_D K_D\bar{K}_C + \hat{K}_C\bar{\hat{K}}_D\hat{K}_D\bar{\hat{K}}_C - 2\hat{K}_C\bar{\hat{K}}_D K_D\bar{K}_C) \\
&+ K_A\bar{K}_C K_C\bar{K}_B + \hat{K}_A\bar{\hat{K}}_C\hat{K}_C\bar{\hat{K}}_B + K_C\bar{K}_C\hat{K}_A\bar{\hat{K}}_B \\
&+ \hat{K}_C\bar{\hat{K}}_C K_A\bar{K}_B + 4K_C\bar{K}_B K_A\bar{K}_C + 4\hat{K}_C\bar{\hat{K}}_B\hat{K}_A\bar{\hat{K}}_C \\
&- 2K_A\bar{K}_B K_C\bar{K}_C - 2K_C\bar{K}_C K_A\bar{K}_B - 2K_A\bar{K}_C\hat{K}_C\bar{\hat{K}}_B \\
&- 2\hat{K}_A\bar{\hat{K}}_C K_C\bar{K}_B - 2\hat{K}_A\bar{\hat{K}}_B\hat{K}_C\bar{\hat{K}}_C - 2\hat{K}_C\bar{\hat{K}}_C\hat{K}_A\bar{\hat{K}}_B
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\mathcal{M}^{b(12)}_{AB} &= K_A\bar{K}_C K_C\hat{K}_B + \hat{K}_A\bar{\hat{K}}_C\hat{K}_C K_B + K_C\bar{K}_C K_B\hat{K}_A \\
&+ \hat{K}_C\bar{\hat{K}}_C\hat{K}_B K_A + 4K_B\bar{K}_C K_A\hat{K}_C + 4\hat{K}_B\bar{\hat{K}}_C\hat{K}_A K_C \\
&- 2K_A\bar{K}_C K_B\hat{K}_C - 2\hat{K}_A\bar{\hat{K}}_C\hat{K}_B K_C - 2K_C\bar{K}_C K_A\hat{K}_B \\
&- 2\hat{K}_C\bar{\hat{K}}_C\hat{K}_A K_B - 2K_B\bar{K}_C K_C\hat{K}_A - 2\hat{K}_B\bar{\hat{K}}_C\hat{K}_C K_A
\end{aligned} \tag{A.6}$$

In above and the rest of work the definition for K and $\hat{K}$ is

$$K := \mathbb{1} \otimes X \qquad\qquad \hat{K} := \hat{X} \otimes \mathbb{1}$$

for $\mathcal{M}^{b(22)}_{AB}$ and $\mathcal{M}^{b(21)}_{AB}$ we only need to do following replacements

$$K \longleftrightarrow \bar{K} \qquad\qquad \hat{K} \longleftrightarrow \bar{\hat{K}}$$

There is also contribution to scalar part from gauge fixing function, this contribution has to be added to $\mathcal{M}$,

$$\mathcal{N}^b_{AB} = \begin{pmatrix} \xi_+ K_A \bar{K}_B + \xi_- \hat{K}_A \bar{\hat{K}}_B & -(\xi_+ K_A \hat{K}_B + \xi_- \hat{K}_A K_B) \\ -(\xi_+ \bar{\hat{K}}_A \bar{K}_B + \xi_- \bar{\hat{K}}_B \bar{K}_A) & \xi_+ \bar{\hat{K}}_A \hat{K}_B + \xi_- \bar{K}_A K_B \end{pmatrix} \tag{A.7}$$

The gauge fixing function that I have used is:

$$f = \frac{1}{\sqrt{\xi_+}} \left[\partial_\mu A^\mu - i\xi_+(\mathcal{X}\bar{\mathcal{Y}} - \mathcal{Y}\bar{\mathcal{X}})\right]$$

$$\tilde{f} = \frac{1}{\sqrt{\xi_-}} \left[\partial_\mu \tilde{A}^\mu - i\xi_-(\bar{\mathcal{X}}\mathcal{Y} - \bar{\mathcal{Y}}\mathcal{X})\right]$$

In above relation $\mathcal{X}$ and $\mathcal{Y}$ are diagonal and off-diagonal parts of Z, respectively. Actually $\mathcal{X}$ in above relation contains both X and $\hat{X}$, the same is true for $\mathcal{Y}$.

Second, for Yukawa potential term

$$\delta\mathcal{L}_{\text{Yukawa}} = \frac{-2\pi i}{k} \begin{pmatrix} \bar{\hat{\psi}}_A & \psi^A \end{pmatrix} \mathcal{M}^f_{AB} \begin{pmatrix} \hat{\psi}_B \\ \bar{\psi}_B \end{pmatrix}$$

The elements of matrix M^f_{AB} are

$$\begin{aligned} M^{f(11)}_{AB} &= \delta_{AB}(K_C \bar{K}_C - \hat{K}_C \bar{\hat{K}}_C) + 2(\hat{K}_B \bar{\hat{K}}_A - K_B \bar{K}_A) \\ M^{f(12)}_{AB} &= -2\epsilon^{ABCD} \hat{K}_C K_D \\ M^{f(21)}_{AB} &= 2\epsilon^{ABCD} \bar{\hat{K}}_C \bar{K}_D \\ M^{f(22)}_{AB} &= -\delta_{AB}(\bar{K}_C K_C - \bar{\hat{K}}_C \hat{K}_C) - 2(\bar{\hat{K}}_B \hat{K}_A - \bar{K}_B K_A) \end{aligned} \tag{A.8}$$

Then for gauge field part we have

$$\delta\mathcal{L}_{\text{gauge}} = \begin{pmatrix} \tilde{a}_\mu & a_\mu \end{pmatrix} \mathcal{M}^g_{\mu\nu} \begin{pmatrix} \bar{\tilde{a}}_\nu \\ \bar{a}_\nu \end{pmatrix}$$

And elements of $\mathcal{M}^g$ are

$$\begin{aligned} M^{g(11)}_{\mu\nu} &= \frac{\partial^\mu\partial^\nu}{\xi_+} + \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho - \eta^{\mu\nu}(\bar{\hat{K}}_C\hat{K}_C + K_C\bar{K}_C) \\ M^{g(12)}_{\mu\nu} &= 2\eta^{\mu\nu}\bar{\hat{K}}_C K_C \\ M^{g(21)}_{\mu\nu} &= 2\eta^{\mu\nu}\hat{K}_C\bar{K}_C \\ M^{g(22)}_{\mu\nu} &= \frac{\partial^\mu\partial^\nu}{\xi_-} - \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho - \eta^{\mu\nu}(\hat{K}_C\bar{\hat{K}}_C + \bar{K}_C K_C) \end{aligned} \tag{A.9}$$

there is also a gauge-scalar vertex that it comes as follow,

$$-2i \begin{pmatrix} \tilde{a}_\mu & a_\mu \end{pmatrix} \mathcal{Q}^{\mu,A}_1 \begin{pmatrix} \hat{Y}_A \\ \bar{Y}_A \end{pmatrix} + 2i \begin{pmatrix} \bar{\hat{Y}}_A & Y_A \end{pmatrix} \mathcal{Q}^{\mu,A}_2 \begin{pmatrix} \bar{\tilde{a}}_\mu \\ a_\mu \end{pmatrix}$$

And,

$$\mathcal{Q}^{\mu,A}_1 = \begin{pmatrix} \partial^\mu\bar{\hat{K}}_A & -\partial^\mu K_A \\ -\partial^\mu\bar{K}_A & \partial^\mu\hat{K}_A \end{pmatrix} \qquad \mathcal{Q}^{\mu,A}_2 = \begin{pmatrix} \partial^\mu\hat{K}_A & -\partial^\mu K_A \\ -\partial^\mu\bar{K}_A & \partial^\mu\bar{\hat{K}}_A \end{pmatrix}$$

And Finally the Ghost part, I only write the the vertex of it

$$G = \delta^{AB} \begin{pmatrix} \partial^2 - \xi_+(\hat{K}_C\bar{\hat{K}}_C + K_C\bar{K}_C) & 0 \\ 0 & \partial^2 - \xi_-(\bar{\hat{K}}_C\hat{K}_C + \bar{K}_C K_C) \end{pmatrix}$$

So if we have some stack of M2-branes at the origin ,say X, and some of them at distance r ,say $\hat{X}$. Then in All of above relations we have to replace $\hat{X}_A$ by $r_A + \hat{X}_A$.

As you can see the form of the vertex coefficients are very complicated and very hard to deal with, only in the case of $U(1) \times U(1)$ case the matrices becomes two by two matrices and the vertices can be simplified ,[19]