Asymptotics for Fermi curves of electric and magnetic periodic fields

by

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Abstract

This work is concerned with some geometrical properties of (complex) Fermi curves of electric and magnetic periodic fields. These are analytic curves in \( \mathbb{C}^2 \) that arise from the study of the eigenvalue problem for periodic Schrödinger operators. More specifically, we characterize a certain class of these curves in the region of \( \mathbb{C}^2 \) where at least one of the coordinates has “large” imaginary part. The new results obtained in this thesis extend previous results in the absence of magnetic field to the case of “small” magnetic field. Our theorems can be used to show that generically these Fermi curves belong to a class of Riemann surfaces of infinite genus.
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Chapter 1

Introduction and summary

This work is concerned with some geometrical properties of (complex) Fermi curves. These are analytic curves in $\mathbb{C}^2$ that arise from the study of the eigenvalue problem for periodic Schrödinger operators. More specifically, we characterize a certain class of these curves in the region of $\mathbb{C}^2$ where at least one of the coordinates has “large” imaginary part.

In order to describe the contents of this thesis we need to introduce the definition of Fermi curve. We stress that our purpose here is only to grasp the main ideas. We will return to the definitions (and results) later in much more detail.

Let $A_1, A_2$ and $V$ be functions on $\mathbb{R}^2$ that are periodic with respect to $\mathbb{Z}^2$ and consider the operator $H = (i\nabla + A)^2 + V$ acting on $L^2(\mathbb{R}^2)$, where $\nabla$ is the gradient operator in $\mathbb{R}^2$ and $A = (A_1, A_2)$. For $k \in \mathbb{C}^2$ let $E_n(k)$ be the $n$-th eigenvalue of the boundary value problem

$$H\psi = \lambda\psi,$$

$$\psi(x + \gamma) = e^{ik\cdot\gamma}\psi(x)$$

for all $x \in \mathbb{R}^2$ and all $\gamma \in \mathbb{Z}^2$. By definition, the $n$-th band $B_n \subset \mathbb{C}$ is the range of the function $k \mapsto E_n(k)$. It is known that the spectrum of $H$ is the union of the bands $B_n$ (restricted to real $k$) for $n \geq 1$. The (complex) Fermi curve of $A$ and $V$ with energy $\lambda \in \mathbb{C}$ is defined as

$$\mathcal{F}_{\lambda, A, V} = \{k \in \mathbb{C}^2 \mid \text{the above boundary value problem has a nonzero solution } \psi\}.$$  

Similarly one can define Fermi “surfaces” in any dimension greater than two.
The above operator $H$ (and its three-dimensional counterpart) is important in solid state physics. It is the Hamiltonian of a single electron under the influence of the magnetic field with vector potential $A$, and the electric field with scalar potential $V$, in the independent electron model of a two-dimensional solid [12]. The classical framework for studying the spectrum of a differential operator with periodic coefficients is the Floquet (or Bloch) theory [12, 8]. Roughly speaking, the main idea of this theory is to “decompose” the original eigenvalue problem, which usually has continuous spectrum, into a family of boundary value problems, each one having discrete spectrum. In our context this leads to decomposing the problem $H \psi = \lambda \psi$ (without boundary conditions) into the above $k$-family of boundary value problems.

The Fermi surface—and in particular the Fermi curve—has the following remarkable property. There exists a function $F(k, \lambda, A, V)$ analytic on $C^d \times C \times A \times V$, where $A$ and $V$ are suitable spaces of functions, such that $\mathcal{F}_{\lambda,A,V} = \{ k \in C^d | F(k, \lambda, A, V) = 0 \}$. In other words, the Fermi surface is a complex analytic variety. In Chapter 2 we provide a detailed proof of this (well-known) theorem following the proof in [3].

It is believed that for “generic” sufficiently regular potentials $A$ and $V$ the union of the surfaces $z = E_n(k)$ for $n \geq 1$ is a complex analytic manifold. In fact, this statement was made precise in [4] for Fermi curves without magnetic potential ($A = 0$), and for heat curves. The later are the spectral curves (defined similarly as the Fermi curves) associated to the “heat” equation $\psi_{x_1} - \psi_{x_2 x_2} + V \psi = 0$ with $V$ periodic, while the former are a particular case of the curves studied in this thesis (where $A \neq 0$). This picture is believed to hold for other differential operators with sufficiently regular periodic coefficients as well. In Chapters 3 and 4 we prove a theorem, that holds only for “small” $A$, which is the main step for showing that the above picture is also true for Fermi curves with “small” magnetic potential. This is the main contribution of this thesis. We have followed the same strategy that Feldman, Knörrer and Trubowitz implemented in [4, §16] for proving a similar result for $A = 0$. Below we provide more details about our results.

As we have already mentioned in the last paragraph, there is a relationship between Riemann surfaces and differential operators with periodic coefficients. We briefly mention two examples here. In one dimension, the solution of the KdV equation $u_t = 3uu_x - \frac{1}{2}u_{xxx}$ with initial data $u_0$ is related to the Schrödinger curve $\mathcal{S}(V)$ by $\mathcal{S}(u(\cdot, t)) = \mathcal{S}(u_0)$, where
$S(V)$ is the spectral curve associated to the one-dimensional analogue of the operator $H$ with $A = 0$ [10]. In two dimensions, there is a relationship between Riemann surfaces of finite genus and solutions of the KP equation $u_{x_1x_1} + \frac{2}{3}(u_t + 3uu_{x_2} + \frac{1}{2}u_{x_2x_2x_2})_{x_2} = 0$ [8]. If the initial data for this problem is a function on $\mathbb{R}^2$ that is periodic with respect to a certain lattice, this relationship is even more explicit [4]. In this case the solution of the KP equation is related to the heat curve mentioned above. This turns out to be, for “generic” potentials $V$, a Riemann surface of infinite genus according to the theory proposed in [4]. The class of surfaces introduced in that work yields an extension of the classical theory of finite genus that has analogues of many theorems of finite genus theory.

When $A$ and $V$ are zero the (free) Fermi curve can be found explicitly. It consists of two copies of $\mathbb{C}$ with the points $-b_2 + ib_1$ (in the first copy) and $b_2 + ib_1$ (in the second copy) identified for all $(b_1, b_2) \in \pi \mathbb{Z}^2$ with $b_2 \neq 0$. The purpose of this thesis is to show that in the region of $\mathbb{C}^2$ where $k \in \mathbb{C}^2$ has “large” imaginary part the Fermi curve (for nonzero $A$ and $V$) is “close” to the free Fermi curve. When $A$ is zero this was proved by Feldman, Knörrer and Trubowitz in [4, §16]. Very briefly, the main result of this thesis is essentially the following.

"Theorem”. Suppose that $A$ and $V$ have some regularity and assume that $A$ is sufficiently small in a suitable norm. Write $k$ in $\mathbb{C}^2$ as $k = u + iv$ with $u$ and $v$ in $\mathbb{R}^2$ and suppose that $|v|$ is sufficiently large. (Recall that the free Fermi curve is two copies of $\mathbb{C}$ with certain points in one copy identified with points in the other one.) Then, in this region of $\mathbb{C}^2$, the Fermi curve of $A$ and $V$ is very close to the free Fermi curve, except that instead of two planes we may have two deformed planes, and identifications between points can open up to handles that look like $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1z_2 = \text{constant}\}$ in suitable local coordinates.

To prove this theorem we follow the same strategy as [4]. The proof has basically three steps. We first derive very detailed information about the free Fermi curve (which is explicitly known). This is done in Chapter 3. Then, to compute the interacting Fermi curve we have to find the kernel of $H - \lambda I$ in $L^2(\mathbb{R}^2)$ with the above boundary condition. In the second step of the proof we derive a number of estimates for showing that this kernel has finite dimension for small $A$ and $k \in \mathbb{C}^2$ with large imaginary part. Furthermore, in the complement of this kernel in $L^2(\mathbb{R}^2)$, after a suitable (invertible) change of variables in $L^2(\mathbb{R}^2)$, the operator $H - \lambda I$ multiplied by the inverse of the operator that implements this
change of variables is a (compact) perturbation of the identity. This reduces the problem
of finding the kernel to finite dimension and thus we can write (local) defining equations
for the Fermi curve. In the third step of the proof we use these equations to study the
Fermi curve. A few more estimates and the implicit function theorem gives us the deformed
planes. The handles are obtained using a quantitative Morse lemma available in [4] that we
prove at end of Chapter 4. Steps two and three are contained in Chapter 4, which is the
core of this thesis.

We have not yet mentioned an important property of $H$—namely, gauge invariance.
Briefly, gauge invariance implies that the spectrum of $(i \nabla + A)^2 + V$ is the same as the
spectrum of $(i \nabla + A + \nabla \Psi)^2 + V$, where $\Psi$ is function on $\mathbb{R}^2$ (under suitable hypotheses),
and $\nabla \Psi$ is periodic with respect to $\mathbb{Z}^2$. Consequently, the Fermi curve of $A$ and $V$ is equal
to the Fermi curve of $A + \nabla \Psi$ and $V$. In Chapter 5, by choosing a convenient gauge $\Psi$,
we are able to indicate how to simplify the proof of the above theorem and improve some
constants in it. After performing this gauge transformation “some terms vanish” and the
analysis becomes simpler.

The critical part of the proof is the second step. The main difficulty arises due to
the presence of the term $A \cdot i \nabla$ in $H$. When $A$ is large, taking the imaginary part of
$k \in \mathbb{C}^2$ arbitrarily large is not enough to control this term—it is not enough to make its
contribution small and hence have the interacting Fermi curve as a perturbation of the free
Fermi curve. (The term $V$ in $H$ is easily controlled by this method.) However, the proof
can be implemented by assuming that $A$ is small.

We finally make some remarks about the interdependence of chapters. Since we have
attempted to write this thesis in order to minimize referencing, the reader may notice
some minor repetitions along the text. Chapter 2 is completely independent of the others;
Chapter 3 is independent and only needs Chapter 4 to be fulfilled; Chapter 4 depends on
Chapter 3, and Chapter 5 depends on Chapters 3 and 4, and can be seen as a bonus chapter.
The most demanding (but still elementary) part to read in this thesis is the proof of the
lemmas and propositions in Chapter 4. These proofs may be (safely) skipped in a first
reading.
Chapter 2

Periodic Schrödinger operators

2.1 Bloch theory

Below we briefly describe the main ideas of Bloch (or Floquet) theory without providing proofs. Our discussion is based on [2, 12].

Let \( \Gamma \) be a lattice of static ions in \( \mathbb{R}^d \) and suppose that the ions generate an electric potential \( V(x) \) and a magnetic potential \( A(x) = (A_1(x), \ldots, A_d(x)) \) that are periodic with respect to \( \Gamma \). Then the Hamiltonian of a single electron moving in \( \mathbb{R}^d \) under the influence of this lattice is

\[
H = (i\nabla + A)^2 + V,
\]

where \( \nabla \) is the gradient operator in \( \mathbb{R}^d \). This operator acts on \( L^2(\mathbb{R}^d) \). For simplicity we suppress the physical constants in the Hamiltonian.

For \( \gamma \in \Gamma \) consider the translation operator \( T_\gamma \) acting on \( L^2(\mathbb{R}^d) \) as

\[
T_\gamma : \varphi(x) \mapsto \varphi(x + \gamma).
\]

This operator is unitary on \( L^2(\mathbb{R}^d) \) and \( T_\gamma T_{\gamma'} = T_{\gamma + \gamma'} \) for all \( \gamma, \gamma' \in \Gamma \). Furthermore, it is not difficult to verify that the Hamiltonian \( H \) (formally) commutes with all the translation operators,

\[
HT_\gamma = T_\gamma H
\]

for all \( \gamma \in \Gamma \). Under suitable hypotheses on the potentials \( A \) and \( V \) one can in fact prove that this property holds on an appropriate domain. This translational symmetry is the main ingredient of Bloch theory.
Since our goal here is only describing the main ideas of Bloch theory without providing proofs, for the rest of this section we pretend that $H$ and $T_\gamma$ are matrices. A rigorous version of the procedure below can be implemented for the actual operators.

The matrices $\{H \text{ and } T_\gamma \text{ for all } \gamma \in \Gamma\}$ form a family of commuting normal matrices. Thus, there exists an orthonormal basis of simultaneous eigenvectors $\{\varphi_\alpha\}$ for this family. For all $\gamma \in \Gamma$ these eigenvectors obey

$$H \varphi_\alpha = E_\alpha \varphi_\alpha,$$
$$T_\gamma \varphi_\alpha = \lambda_{\alpha,\gamma} \varphi_\alpha,$$

where $E_\alpha$ and $\lambda_{\alpha,\gamma}$ are numbers.

Let $\{v_1, \ldots, v_d\}$ be a basis for $\Gamma$. Then for any $\gamma \in \Gamma$ there are integers $n_1, \ldots, n_d$ such that $\gamma = n_1 v_1 + \cdots + n_d v_d$. If we set $\gamma_i = n_i v_i$ for $1 \leq i \leq d$, then we can write any element $\gamma \in \Gamma$ as $\gamma = \gamma_1 + \cdots + \gamma_d$.

As the operator $T_\gamma$ is unitary, all its eigenvalues are complex numbers of modulus one. Hence, there exist real numbers $\beta_{\alpha,\gamma}$ such that

$$\lambda_{\alpha,\gamma} = e^{i \beta_{\alpha,\gamma}}.$$

Using the properties of $T_\gamma$ we find that

$$e^{i \beta_{\alpha,\gamma} + \gamma'} \varphi_\alpha = T_{\gamma + \gamma'} \varphi_\alpha = T_\gamma T_{\gamma'} \varphi_\alpha = T_\gamma e^{i \beta_{\alpha,\gamma}} e^{i \beta_{\alpha,\gamma'}} \varphi_\alpha = e^{i (\beta_{\alpha,\gamma} + \beta_{\alpha,\gamma'})} \varphi_\alpha,$$

so that

$$\beta_{\alpha,\gamma} + \gamma' = \beta_{\alpha,\gamma} + \beta_{\alpha,\gamma'} \mod 2\pi$$

for all $\gamma, \gamma' \in \Gamma$. Consequently,

$$\beta_{\alpha,\gamma} = \beta_{\alpha,\gamma_1 + \cdots + \gamma_d} = \beta_{\alpha,\gamma_1} + \cdots + \beta_{\alpha,\gamma_d} \mod 2\pi$$

for all $\gamma \in \Gamma$. That is, the number $\beta_{\alpha,\gamma}$ is determined (mod $2\pi$) by $\beta_{\alpha,\gamma_1}, \ldots, \beta_{\alpha,\gamma_d}$.

Now observe that, given any $d$ numbers $\beta_1, \ldots, \beta_d$, the system of linear equations

$$\gamma_i \cdot k = \beta_i \quad \text{for} \quad 1 \leq i \leq d$$

for the unknowns $k_1, \ldots, k_d$ has a unique solution. This follows from the fact that $\gamma_1, \ldots, \gamma_d$ are linear independent. Hence, for each $\alpha$ there exists a $k_\alpha \in \mathbb{R}^d$ such that $\gamma_i \cdot k_\alpha = \beta_{\alpha,\gamma_i}$ for $1 \leq i \leq d$. Therefore,

$$\beta_{\alpha,\gamma} = \gamma \cdot k_\alpha \mod 2\pi$$
for all $\gamma \in \Gamma$. Note that, for each $\alpha$ the vector $k_\alpha$ is not uniquely determined. Indeed,

$$\beta_{\alpha,\gamma} = \gamma \cdot k_\alpha \mod 2\pi \quad \text{and} \quad \beta_{\alpha,\gamma} = \gamma \cdot k'_\alpha \mod 2\pi$$

for all $\gamma \in \Gamma$, if and only if

$$(k_\alpha - k'_\alpha) \cdot \gamma \in 2\pi\mathbb{Z}$$

for all $\gamma \in \Gamma$. If we define the dual lattice of $\Gamma$ as

$$\Gamma^\# = \{ b \in \mathbb{R}^d \mid b \cdot \gamma \in 2\pi\mathbb{Z} \quad \text{for all} \quad \gamma \in \Gamma \},$$

where $b \cdot \gamma$ is the usual scalar product on $\mathbb{R}^d$, the last expression can be rewritten as

$$k_\alpha - k'_\alpha \in \Gamma^\#.$$

Summarizing, the numbers $\{\beta_{\alpha,\gamma} \mid \gamma \in \Gamma\}$ determine $k_\alpha$ up to a vector in $\Gamma^\#$. Hence, the vector $k_\alpha$ is unique in $\mathbb{R}^d/\Gamma^\#$. This establishes a correspondence between $\alpha$ and $k_\alpha$.

Now, relabel the eigenvalues and eigenvectors by replacing the index $\alpha$ by the corresponding vector $k \in \mathbb{R}^d/\Gamma^\#$ and another index $n$. The index $n$ is needed because many $k_\alpha$ with different values of $\alpha$ can be equal. Under the new labelling the eigenvalue-eigenvector equations become

$$H \varphi_{n,k} = E_n(k) \varphi_{n,k},$$

$$T_\gamma \varphi_{n,k} = e^{ik \cdot \gamma} \varphi_{n,k}$$

for all $\gamma \in \Gamma$. The eigenvalue of $H$ is denoted $E_n(k)$ rather than $E_{n,k}$ because, while $k$ runs over the continuous set $\mathbb{R}^d/\Gamma^\#$, it turns out that $n$ runs over a countable set. Observing the definition of $T_\gamma$, for each $k \in \mathbb{R}^d$ the above equations can be rewritten as

$$H \varphi_{n,k} = E_n(k) \varphi_{n,k},$$

$$\varphi_{n,k}(x + \gamma) = e^{ik \cdot \gamma} \varphi_{n,k}(x)$$

for all $x \in \mathbb{R}^d$ and all $\gamma \in \Gamma$. Under suitable hypotheses on the potentials $A$ and $V$ this boundary value problem is self-adjoint. As we have already mentioned, its spectrum is discrete, it consists of a sequence of real eigenvalues

$$E_1(k) \leq E_2(k) \leq \cdots \leq E_n(k) \leq \cdots$$
For each integer \( n \geq 1 \) the eigenvalue \( E_n(k) \) defines a continuous function of \( k \) that is periodic with respect to the dual lattice \( \Gamma^\# \). It is customary to refer to \( k \) as the crystal momentum and to \( E_n(k) \) as the \( n \)-th band function. The corresponding normalized eigenfunctions \( \varphi_{n,k} \) are called Bloch eigenfunctions.

Let \( U_k \) be the unitary transformation on \( L^2(\mathbb{R}^d) \) that acts as

\[
U_k : \varphi(x) \mapsto e^{ik \cdot x} \varphi(x).
\]

By applying this transformation we can rewrite the above problem and put the boundary conditions into the operator. Indeed, if we define

\[
H_k = U_k^{-1} H U_k \quad \text{and} \quad \psi_{n,k} = U_k^{-1} \varphi_{n,k},
\]

then the above problem is unitary equivalent to

\[
H_k \psi_{n,k} = E_n(k) \psi_{n,k},
\]

\[
\psi_{n,k}(x + \gamma) = \psi_{n,k}(x)
\]

for all \( x \in \mathbb{R}^d \) and all \( \gamma \in \Gamma \), or, using a more compact notation,

\[
H_k \psi_{n,k} = E_n(k) \psi_{n,k} \quad \text{for} \quad \psi_{n,k} \in L^2(\mathbb{R}^d/\Gamma).
\]

To see that these problems are equivalent we proceed (formally) as follows. On the one hand, from the original problem and using the above transformation we find that

\[
0 = U_k^{-1} 0 = U_k^{-1} (H - E_n(k)) \varphi_{n,k} = U_k^{-1} (H - E_n(k)) U_k U_k^{-1} \varphi_{n,k} = (U_k^{-1} H U_k - E_n(k)) U_k^{-1} \varphi_{n,k} = (H - E_n(k)) \psi_{n,k}
\]

and

\[
\psi_{n,k}(x + \gamma) = (U_k^{-1} \varphi_{n,k})(x + \gamma) = e^{-ik \cdot (x + \gamma)} \varphi_{n,k}(x + \gamma) = e^{-ik \cdot (x + \gamma)} e^{ik \cdot \gamma} \varphi_{n,k}(x) = e^{-ik \cdot x} \varphi_{n,k}(x) = (U_k^{-1} \varphi_{n,k})(x) = \psi_{n,k}(x).
\]

On the other hand, by a similar computation (in the reverse order), using the last two equalities and the above transformation we derive the original problem. This (formally) implies unitary equivalence. Furthermore, a simple (formal) calculation shows that

\[
H_k = (i \nabla + A - k)^2 + V.
\]
In fact,

$$H_k\psi = U_k^{-1}HU_k\psi$$

$$= e^{-ik \cdot x} [(i\nabla + A)^2 + V] e^{ik \cdot x} \psi$$

$$= e^{-ik \cdot x} [(i\nabla + A) \cdot e^{ik \cdot x} (-k\psi + i\nabla \psi + A\psi)] + V\psi$$

$$= (-k + A) \cdot (-k + i\nabla + A)\psi + i\nabla \cdot (-k + i\nabla + A)\psi + V\psi$$

$$= [(i\nabla + A - k)^2 + V] \psi.$$  

Of course, the unitary transformation $U_k$ preserves self-adjointness and does not change the spectrum $\{E_n(k)\}_{n=1}^{\infty}$.

Finally, denote by $N_k$ the set of values of $n$ that appear in the pairs $\alpha = (k,n)$ and define

$$\mathcal{H}_k = \text{span}\{\varphi_{n,k} \mid n \in N_k\}.$$

Then, formally, and in particular ignoring that $k$ runs over an uncountable set,

$$L^2(\mathbb{R}^d) = \text{span}\{\varphi_{n,k} \mid k \in \mathbb{R}^d/\Gamma^# \text{ and } n \in N_k\}$$

$$= \bigoplus_{k \in \mathbb{R}^d/\Gamma^#} \mathcal{H}_k.$$

Set

$$\widetilde{\mathcal{H}}_k = \text{span}\{\psi_{n,k} \mid n \in N_k\}.$$ 

Observe that, as the operator $U_k^{-1}$ is unitary on $L^2(\mathbb{R}^d)$, the space $\mathcal{H}_k$ is unitary equivalent to $\widetilde{\mathcal{H}}_k$, and $L^2(\mathbb{R}^d)$ is unitary equivalent to $\bigoplus_{k \in \mathbb{R}^d/\Gamma^#} \widetilde{\mathcal{H}}_k$. The restriction of $U_k^{-1}HU_k$ to $\widetilde{\mathcal{H}}_k$ is $H_k$ applied to functions that are periodic with respect to $\Gamma$.

Therefore, at least formally, to find the spectrum of $H$ on $L^2(\mathbb{R}^d)$ it suffices to find the spectrum of $H_k$ on $L^2(\mathbb{R}^d/\Gamma)$ for all $k \in \mathbb{R}^d/\Gamma^#$. Unlike $H$, the operator $H_k$ has compact resolvent. Thus, the spectrum of $H_k$ is discrete, unlike the spectrum of $H$ that is continuous and is given by

$$\sigma(H) = \{E_n(k) \mid n \in \mathbb{N} \text{ and } k \in \mathbb{R}^d/\Gamma^#\}.$$ 

(See Figure 2.1 below.) All these statements can be made precise and rigorously proved. We prove a few of them in §2.4; the proof of some others can be found in [2, 12].
In this section we define the Fermi surfaces in $\mathbb{R}^d$. We first state more precisely some definitions already given above.

Let $\Gamma$ be a lattice of maximal rank in $\mathbb{R}^d$ with $d \geq 2$. Consider a real number $r > d$ and set

$$A := \{ (A_1, \ldots, A_d) \mid A_j \in L^r_{\mathbb{R}}(\mathbb{R}^d/\Gamma) \text{ for } 1 \leq j \leq d \} \quad \text{and} \quad \mathcal{V} := L^{r/2}_{\mathbb{R}}(\mathbb{R}^d/\Gamma),$$

where $L^r_{\mathbb{R}}(\mathbb{R}^d/\Gamma)$ is the space of real-valued functions $f$ such that $|f|^r$ is integrable on the torus $\mathbb{R}^d/\Gamma$. For real-valued potentials $(A, V) \in \mathcal{A} \times \mathcal{V}$ and for $k \in \mathbb{R}^d$ define the operator

$$H_k(A, V) := (i \nabla + A - k)^2 + V$$

acting on $L^2(\mathbb{R}^d/\Gamma)$, where $\nabla$ is the gradient operator in $\mathbb{R}^d/\Gamma$. Recall the discussion in the last section and consider the eigenvalue-eigenvector problem

$$H_k(A, V)\psi_{n,k} = E_n(k, A, V)\psi_{n,k} \quad \text{for} \quad \psi_{n,k} \in L^2(\mathbb{R}^d/\Gamma).$$

The real “lifted” Fermi surface of $(A, V)$ with energy $\lambda \in \mathbb{R}$ is defined as

$$\mathcal{F}_{\lambda, \mathbb{R}}(A, V) := \{ k \in \mathbb{R}^d \mid E_n(k, A, V) = \lambda \text{ for some } n \geq 1 \}.$$
Equivalently,
\[
\tilde{F}_{\lambda,R}(A,V) = \{ k \in \mathbb{R}^d \mid (H_k(A,V) - \lambda)\psi = 0 \text{ for some } \psi \in \mathcal{D}_{H_k(A,V)} \setminus \{0\} \},
\]
where \(\mathcal{D}_{H_k(A,V)} \subset L^2(\mathbb{R}^d/\Gamma)\) denotes the (dense) domain of \(H_k(A,V)\). The adjective “lifted” indicates that \(\tilde{F}_{\lambda,R}(A,V)\) is a subset of \(\mathbb{R}^d\) rather than \(\mathbb{R}^d/\Gamma^\#\). When \(d = 2\) the Fermi surfaces are called Fermi curves, in particular. This is the main case of interest in this work.

We next describe the physical context in which the concept of Fermi surface arises.

### 2.3 Electrons in a crystal

The purpose of this section is to motivate the definition of Fermi surfaces (for \(d \leq 3\)) and briefly describe its physical meaning. For simplicity we consider the case \(d = 3\). Our discussion is based on [9, 12].

A full rigorous theoretical description of crystalline solids is unavailable. In fact, no one has given an explanation from first principles of why crystals form. That is, no one has proven that a large number of heavy nuclei with enough electrons to produce neutrality, interacting via Coulomb potential, have a ground state that is approximately a crystal. Nevertheless, we may consider a simplified model for solids in attempt to describe some of the observed phenomena. This is what we discuss next.

It is observed experimentally that the nuclei in a solid lie more or less in a regular array. For example, the common crystalline phase of iron has crystal structure given approximately by \(\alpha e_1 \mathbb{Z} \oplus \alpha e_2 \mathbb{Z} \oplus \frac{\alpha}{2}(e_1 + e_2)\mathbb{Z} \subset \mathbb{R}^3\), where \(\alpha \approx 2.87\text{Å}\) and \(\{e_j\}_{j=1}^3\) is the canonical base of \(\mathbb{R}^3\). Thus, we postulate in our model that at each site of a lattice \(\Gamma\) in \(\mathbb{R}^3\) there is a fixed nucleus with a number of core electrons. Furthermore, we assume that the solid is filling all of \(\mathbb{R}^3\). Hence, if we ignore electron-electron interactions, and suppose that the fixed nuclei generate an electric potential \(V(x)\) and a magnetic potential \(A(x) = (A_1(x), \ldots, A_3(x))\) that are periodic with respect to \(\Gamma\), we have a cloud of valence electrons moving in \(\mathbb{R}^3\) subjected only to the influence of this lattice. Each one of these electrons has Hamiltonian
\[
H = \frac{1}{2m_e} (ih\nabla + eA)^2 + V,
\]
where \(m_e\) and \(e\) are the mass and the charge of the electron, respectively, and \(h\) is the Planck constant. This model is known as the independent electron model of solids. We shall outline how to use this model to describe the notion of density of states and give a qualitative explanation of the difference between metal and insulators.
Let $B$ be a fundamental cell in $\mathbb{R}^3$ for the dual lattice $\Gamma^\#$ and let $\{E_n(k)\}_{n=1}^\infty$ be the eigenvalues of $H_k$. Given a set $X \subset \mathbb{R}^3$, denote by $|X|$ its Lebesgue measure. The (integrated) density of states measure $\rho$ is the measure on $\mathbb{R}$ defined by

$$\rho(-\infty, E) := \frac{2}{|B|} \sum_{n=1}^\infty \left| \{k \in B \mid E_n(k) \leq E \} \right|.$$ 

Since $E_n(k) \to \infty$ uniformly in $k$ as $n \to \infty$, the number $\rho(-\infty, E)$ is finite. Furthermore, one can show that $\rho$ is absolutely continuous with respect to $dE$, the Lebesgue measure on $\mathbb{R}$. The Radon-Nikodym derivative $d\rho/dE$ is usually called the density of states. The (integrated) density of states is a concept of fundamental importance in condensed matter physics. It measures the “number of quantum states per unit volume” below a given energy. In fact, to understand the definition of $\rho(-\infty, E]$ given above, suppose for simplicity that for a given energy $E$ there are exactly $N$ bands $E_n(k)$ such that $E_n(k) < E$ for all $k \in B$. Then we have

$$\rho(-\infty, E) = \frac{2}{|B|} \sum_{n=1}^\infty \left| \{k \in B \mid E_n(k) \leq E \} \right| = \sum_{n=1}^N \frac{2|B|}{|B|} = 2N.$$ 

That is, the “number of quantum states per volume $|B|$” below the energy $E$ is $2N$. Here, the factor of 2 comes from the Pauli principle. Since we are considering electrons (fermions), this principle asserts that each quantum state corresponding to the energy level $E_n(k)$ can have at most two electrons. Thus, in the definition of $\rho(-\infty, E]$ above, we include a factor of 2 in the numerator. (We shall explain below how to implement the Pauli principle in our model).

To explain the importance of $\rho$ for the study of solids we introduce one more concept. Let $D$ be a fundamental cell in $\mathbb{R}^3$ for the lattice $\Gamma$ and, given an integer $m$, let $D^{(m)}$ be the set of volume $m^3|D|$ obtained by gluing together an $m \times m \times m$ set of $D$’s (see figure 2.2). Consider the operator

$$H_m = \frac{1}{2m_e} (i\hbar \nabla_p + eA)^2 + V$$

on $L^2(D^{(m)})$, where $\nabla_p$ is the gradient operator with periodic boundary conditions. Let $\Omega$ be a subset of $\mathbb{R}$ and let $P_m(\Omega)$ be the spectral projection for $H_m$. Define

$$\rho_m(-\infty, E] := \frac{2}{m^3} \dim P_m(-\infty, E].$$

It is possible to show that $\rho_m \to \rho$ as $m \to \infty$ in the sense that $\rho_m(-\infty, E] \to \rho(-\infty, E]$ for all $E \in \mathbb{R}$ [12]. We can now return to our model of solids.
Suppose that each nucleus in free space is surrounded by \( l \) electrons. Then in our model we wish to have \( l \) valence electrons per unit cell. While we ignore the interaction between electrons, we cannot ignore the Pauli principle. Again, this principle asserts that the quantum state corresponding to each eigenvalue of \( H \) can have at most two electrons. How do we take this into account when \( H \) does not have eigenfunctions and when there are infinitely many electrons (in our infinite crystal lattice)? We claim that a reasonable way of taking the Pauli principle into account is to say that in the ground state the electrons fill up the continuum eigenstates up to that energy \( E \) where \( \rho(-\infty, E] = l \). If we have a large but finite \( m \times m \times m \) crystal with periodic boundary conditions, there are \( m^3l \) electrons, and in the ground state these fill up the eigenstates of \( H_m \) up to an energy \( E_m \) determined by \( \rho_m(-\infty, E_m] = l \). The smallest number \( E \) with \( \rho(-\infty, E] = l \) is called the Fermi energy \( E_F \). The set of \( k \in B \) with \( E_n(k) = E_F \) for some \( n \) is called the Fermi surface. Observe that this agrees with the definition of \( \hat{\mathcal{F}}_{\lambda,R}(A,V) \) given above. This picture is similar to the elementary discussion of the periodic table based on the hydrogen atom but with the complication of continuum states.

We are now in a position to explain why electron conduction is hard in some solids (insulators) and easy in others (metals). In the ground state one can use complex conjugation symmetry to prove that there is no net movement of electrons—the expected value of the total momentum is zero. To get flow of electrons one must excite some of the electrons. Usually, a periodic Schrödinger operator has gaps in its spectrum. There is a qualitative difference if \( E_F \) occurs at the bottom of a gap or not. If \( E_F \) is at the bottom of a gap, then \( H \) has no spectrum in \( (E_F, E_F + \varepsilon) \), and there is a discrete amount of energy needed to set up a current. In this case one has an insulator. If \( E_F \) is not at the bottom of a gap, one
has a metal. Of course, if $E_F$ is at the bottom of a small gap ($\varepsilon$ small), or if $E_F$ is not at the bottom of a gap but is fairly close to the bottom of a gap, then one has an intermediate case where the metal/insulator distinction is not sharp (semiconductors). We should also note that in dealing with real solids one must take into account the fact that the solid is not in the ground state but rather in a finite temperature state determined by statistical mechanics. Notice that the gaps in the spectrum are crucial for this theory of insulators versus metals. The Fermi surface plays a fundamental role in this context. We refer the reader to [9] for more details.

![Diagram of energy bands in conductors and insulators.](TypicalInsulator.png)

Figure 2.3: Energy bands in conductors and insulators.

### 2.4 Basic properties of $H_k(A, V)$

Recall that $r > d \geq 2$ and let $\mathcal{A}_C$ and $\mathcal{V}_C$ be the “complexifications” of $\mathcal{A}$ and $\mathcal{V}$, respectively. That is,

$$
\mathcal{A}_C := \{(A_1, \ldots, A_d) \mid A_j \in L^r(\mathbb{R}^d/\Gamma) \text{ for } 1 \leq j \leq d\} \quad \text{and} \quad \mathcal{V}_C := L^{r/2}(\mathbb{R}^d/\Gamma).
$$

In this section we prove the following properties of the operator $H_k(A, V)$.

**Theorem 2.4.1** (Properties of $H_k(A, V)$).

(a) Let $k \in \mathbb{C}^d$ and let $(A, V) \in \mathcal{A}_C \times \mathcal{V}_C$. Then if $\lambda$ is not in the spectrum of $H_k(A, V)$ the resolvent $(H_k(A, V) - \lambda)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d/\Gamma)$.

(b) For $k \in \mathbb{R}^d$ and $(A, V) \in \mathcal{A} \times \mathcal{V}$, the operator $H_k(A, V)$ is self-adjoint in a dense domain $\mathcal{D}_{H_k(A,V)} \subset L^2(\mathbb{R}^d/\Gamma)$. 

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In order to prove this theorem we first introduce some notation and prove a number of propositions.

Write \( H := L^2(\mathbb{R}^d/\Gamma) \) and denote by \( L(\mathcal{H}) \) the set of all bounded linear operators on \( \mathcal{H} \) with respect to the operator norm \( \| \cdot \| \) on \( \mathcal{H} \). Recall that \( \mathcal{H} \) is a separable Hilbert space and that the trace class \( \mathcal{S}_1 \) is the set of linear operators \( A \) on \( \mathcal{H} \) such that \( \text{tr}(|A|) < \infty \). Here, the operator \( |A| \) is the positive square root of \( A^*A \). The trace ideals \( \mathcal{S}_r \) and its associated norms are defined as

\[
\mathcal{S}_r := \{ A \in L(\mathcal{H}) \mid \text{tr}(|A|^r) < \infty \} \quad \text{and} \quad \| A \|_r := (\text{tr}(|A|^r))^{1/r} \quad \text{for } 1 \leq r < \infty,
\]

\[
\mathcal{S}_\infty := \{ A \in L(\mathcal{H}) \mid A \text{ is compact} \} \quad \text{and} \quad \| A \|_\infty := \| A \| \quad \text{for } r = \infty.
\]

Let \( \mu(A) = \{ \mu_n(A) \}_{n=1}^\infty \) be the singular values of \( A \), that is, the eigenvalues of \( |A| \). One can show that \( \mathcal{S}_r \) is the set of compact operators on \( \mathcal{H} \) whose singular values are in \( l^r(\mathbb{N}) \) and \( \| A \|_r = \| \mu(A) \|_{l^r(\mathbb{N})} \). This is true for any \( 1 \leq r \leq \infty \) [15]. The properties of \( \| \cdot \|_r \) that we need are collected below.

**Proposition 2.4.2** (Properties of \( \| \cdot \|_r \) [15]). Let \( 1 \leq r < s \leq \infty \) and suppose that \( A \in L(\mathcal{H}) \) and \( B \in \mathcal{S}_r \). Then:

(a) \( AB \in \mathcal{S}_r \) and \( \| AB \|_r \leq \| A \| \| B \|_r \);

(b) \( \mathcal{S}_1 \subset \mathcal{S}_r \subset \mathcal{S}_s \subset \mathcal{S}_\infty \) with \( \| \cdot \| \leq \| \cdot \|_s \leq \| \cdot \|_r \leq \| \cdot \|_1 \);

(c) \( B^* \in \mathcal{S}_r \) and \( \| B^* \|_r = \| B \|_r \).

For any \( \varphi \in L^2(\mathbb{R}^d/\Gamma) \) define \( \hat{\varphi} : \Gamma^\# \to \mathbb{C} \) as

\[
\hat{\varphi}(b) := (\mathcal{F}\varphi)(b) := \frac{1}{|\Gamma|} \int_{\mathbb{R}^d/\Gamma} \varphi(x) e^{-ib \cdot x} \, dx,
\]

where \( |\Gamma| := \int_{\mathbb{R}^d/\Gamma} dx \). Then,

\[
\varphi(x) = (\mathcal{F}^{-1}\hat{\varphi})(x) = \sum_{b \in \Gamma^\#} \hat{\varphi}(b) e^{ib \cdot x}
\]

and

\[
\| \varphi \|_{L^2(\mathbb{R}^d/\Gamma)} = |\Gamma|^{1/2} \| \hat{\varphi} \|_{l^2(\Gamma^\#)}.
\]

To simplify the notation we sometimes write \( L^p \) and \( l^p \) in place of \( L^p(\mathbb{R}^d/\Gamma) \) and \( l^p(\Gamma^\#) \), respectively.
For \( f \in L^r \) and \( g \in L^r \) define the operators \( g(-i\nabla) \) and \( f \) on a suitable domain of \( L^2 \) as
\[
g(-i\nabla): \varphi \mapsto \sum_{b \in \Gamma^\#} g(b) \hat{\varphi}(b) e^{ib \cdot x} \quad \text{and} \quad f: \psi \mapsto f\psi.
\]
It turns out that \( f(x)g(-i\nabla) \) and \( g(-i\nabla)f(x) \) are in the trace ideal \( \mathcal{I}_r \). More precisely, as in [15, Theorem 4.1], using complex interpolation we prove the following result.

**Proposition 2.4.3.** Let \( 2 \leq r \leq \infty \) and suppose that \( f \in L^r(\mathbb{R}^d/\Gamma) \) and \( g \in L^r(\Gamma^\#). \) Then \( f(x)g(-i\nabla) \in \mathcal{I}_r \) and \( g(-i\nabla)f(x) \in \mathcal{I}_r \). Furthermore,
\[
\|f(x)g(-i\nabla)\|_r \leq |\Gamma|^{-1/r} \|f\|_{L^r} \|g\|_{L^r} \tag{2.4.1}
\]
and
\[
\|g(-i\nabla)f(x)\|_r \leq |\Gamma|^{-1/r} \|f\|_{L^r} \|g\|_{L^r}. \tag{2.4.2}
\]

**Proof.** We first prove (2.4.1) and (2.4.2) for \( r = 2 \) and \( r = \infty \); then we interpolate using [15, Theorem 2.9]. For \( r = 2 \),
\[
\|f(x)g(-i\nabla)\|_2^2 = \text{tr} \left((f(x)g(-i\nabla))^*(f(x)g(-i\nabla))\right) = \sum_{b \in \Gamma^\#} \left\|f(x)g(-i\nabla) |\Gamma|^{-1/2} e^{ib \cdot x}\right\|_{L^2}^2
\]
\[
= \sum_{b \in \Gamma^\#} \left\|f(x) |\Gamma|^{-1/2} g(b)e^{ib \cdot x}\right\|_{L^2}^2 = |\Gamma|^{-1} \sum_{b \in \Gamma^\#} |g(b)|^2 \|f\|_{L^2}^2
\]
\[
= |\Gamma|^{-1} \|f\|_{L^2}^2 \|g\|_{L^2}^2 \tag{2.4.3}
\]
and
\[
\|g(-i\nabla)f(x)\|_2^2 = \text{tr} \left((g(-i\nabla)f(x))^*(g(-i\nabla)f(x))\right) = \sum_{b \in \Gamma^\#} \left\|g(-i\nabla)f(x) |\Gamma|^{-1/2} e^{ib \cdot x}\right\|_{L^2}^2
\]
\[
= \sum_{b \in \Gamma^\#} \left\|g(-i\nabla) \sum_{c \in \Gamma^\#} \hat{f}(c) |\Gamma|^{-1/2} e^{i(b+c) \cdot x}\right\|_{L^2}^2 = \sum_{b \in \Gamma^\#} \left\|\sum_{c \in \Gamma^\#} g(b + c) \hat{f}(c) |\Gamma|^{-1/2} e^{i(b+c) \cdot x}\right\|_{L^2}^2
\]
\[
= \sum_{b \in \Gamma^\#} \sum_{c \in \Gamma^\#} |g(b + c)|^2 |\hat{f}(c)|^2 = \|\hat{f}\|_{L^2}^2 \|g\|_{L^2}^2 = |\Gamma|^{-1} \|f\|_{L^2}^2 \|g\|_{L^2}^2, \tag{2.4.4}
\]
while for \( r = \infty \),
\[
\|f(x)g(-i\nabla)\|_{L^2}^2 \leq \|f\|_{L^\infty}^2 \|g(-i\nabla)\|_{L^2}^2 = |\Gamma| \|f\|_{L^\infty}^2 \|g(-i\nabla)\|_{L^2}^2
\]
\[
= |\Gamma| \|f\|_{L^\infty}^2 \|g\|_{L^\infty}^2 \|\varphi\|_{L^2}^2
\]
\[
= \|f\|_{L^\infty}^2 \|g\|_{L^\infty}^2 \|\varphi\|_{L^2}^2
\]
\[
\|g(-i\nabla)f(x)\varphi\|_{L^2}^2 = |\Gamma| \| (g(-i\nabla)f\varphi)^\wedge \|_{L^2}^2 = |\Gamma| \| g(f\varphi)^\wedge \|_{L^2}^2
\]
\[
\leq |\Gamma| \| g\|_\infty^2 \| (f\varphi)^\wedge \|_{L^2}^2 = \| g\|_\infty^2 \| f\varphi\|_{L^2}^2
\]
\[
\leq \| g\|_\infty^2 \| f\|_{L^\infty} \| \varphi\|_{L^2}^2.
\]

Hence,
\[
\|f(x)g(-i\nabla)\| \leq \|f\|_{L^\infty} \|g\|_{L^\infty}
\]  \hspace{1cm} (2.4.5)
and
\[
\|g(-i\nabla)f(x)\| \leq \|f\|_{L^\infty} \|g\|_{L^\infty}
\]  \hspace{1cm} (2.4.6)

The general \(2 \leq r < \infty\) case now follows by interpolation. Consider the family of norms
\[
\Phi_t(A) := \begin{cases} 
(\text{tr}(|A|^{1/t}))^{t} & \text{for } 0 < t \leq \frac{1}{2} \\
\|A\| & \text{for } t = 0.
\end{cases}
\]
One can show that \(\lim_{t \downarrow 0} \Phi_t(A) = \Phi_0(A)\) by applying that \(\|\cdot\|_{L^\infty} = \lim_{p \to \infty} \|\cdot\|_{L^p}\) to the singular values of \(A\). Thus, we have a continuous family of norms. Let
\[
z \mapsto F_1(z) := |\Gamma|^z e^{i \arg f(x)} |f|^{2\tau} e^{i \arg g(-i\nabla)} |g|^{2\tau} (-i\nabla)
\]
and
\[
z \mapsto F_2(z) := |\Gamma|^z e^{i \arg g(-i\nabla)} |g|^{2\tau} (-i\nabla) e^{i \arg f(x)} |f|^{2\tau}(x)
\]
be maps from the strip \(S := \{z \in \mathbb{C} \mid 0 \leq \text{Re } z \leq 1/2\}\) to the set of linear operators on \(L^2\).
Suppose for the moment that \(f\) and \(g\) are simple functions. We may assume, without loss of generality, that \(\|f\|_{L^r} = \|g\|_{L^r} = 1\). We shall shortly prove the following property.

\textbf{Proposition 2.4.4.} For any \(\varphi, \psi \in L^2\), and for \(j \in \{1, 2\}\), the function
\[
z \mapsto \langle \varphi, F_j(z)\psi \rangle_{L^2}
\]
is continuous in \(S\), analytic in its interior, and bounded.

Furthermore, in view of (2.4.3) to (2.4.6), for \(j \in \{1, 2\}\) we have
\[
\Phi_0(F_j(iy)) \leq \| \Gamma \|_{t^p} e^{i \arg f} \| f \|_{L^\infty} \| e^{i \arg g} \|_{t^p} = 1
\]
and

$$\Phi_{1/2}(F_j(\frac{1}{2} + iy)) \leq |\Gamma|^{-\frac{1}{2}} \|\Phi_{\frac{1}{2}+iy} e^{i\arg f} |f|^{\frac{1}{2}+iy} \|_{L^2} \|e^{i\arg g} |g|^{\frac{1}{2}+iy}\|_{L^2}$$

$$= \|f\|_{L^2} \|g\|_{L^2} = 1.$$  

Hence, \(F_j(iy) \in \mathcal{I}_\infty\) and \(F_j(\frac{1}{2} + iy) \in \mathcal{I}_2\) with \(\|F_j(iy)\| \leq 1\) and \(\|F_j(\frac{1}{2} + iy)\|_2 \leq 1\) for all \(y \in \mathbb{R}\) and \(j \in \{1, 2\}\). By complex interpolation [15, Theorem 2.9], it follows that \(F_j(z) \in \mathcal{I}_{(Re\ z)}\) with \(\Phi(Re\ z)(F_j(z)) \leq 1\) for all \(z \in S\) and \(j \in \{1, 2\}\). In particular, for \(z = 1/r\),

$$|\Gamma|^{1/r} \|f(x)g(-i\nabla)\|_r = \Phi_{1/r}(F_1(1/r)) \leq 1$$

and

$$|\Gamma|^{1/r} \|g(-i\nabla)f(x)\|_r = \Phi_{1/r}(F_2(1/r)) \leq 1.$$  

This proves (2.4.1) and (2.4.2) for \(f\) and \(g\) as above. Since the set of simple functions is dense in \(L^r(\mathbb{R}^d/\Gamma)\) and \(l^r(\Gamma^\#)\) for \(2 \leq r \leq \infty\), an approximation argument (using the triangle inequality) completes the proof.  

We now prove Proposition 2.4.4.

**Proof of Proposition 2.4.4.** Recall that \(f\) and \(g\) are simple functions and suppose for the moment that \(\psi\) is a \(C^\infty\)-function on \(\mathbb{R}^d/\Gamma\) with compact support. We first prove that, for all \(x \in \mathbb{R}^d/\Gamma\), the function

$$z \mapsto (e^{i\arg g(-i\nabla)} |g|^{2r}(-i\nabla)\psi)(x)$$

is continuous in \(S\), analytic in its interior and bounded. Indeed, let

$$h_{x,b}(z) := e^{i\arg g(b)} |g(b)|^{2r} \hat{\psi}(b)e^{ib\cdot x}.$$  

Observe that \(z \mapsto h_{x,b}(z)\) has the desired properties. Then, since \(|g(b)|^r \leq \|g\|_r^r \leq 1\) for all \(b \in \Gamma^\#\), it follows that \(\sum_{b \in \Gamma^\#} |h_{x,b}(z)| \leq \sum_{b \in \Gamma^\#} |\hat{\psi}(b)| < \infty\) (because \(\hat{\psi} \in l_1\) since we have assume that \(\psi\) is a \(C^\infty\)-function). Thus, the sum

$$(e^{i\arg g(-i\nabla)} |g|^{2r}(-i\nabla)\psi)(x) = \sum_{b \in \Gamma^\#} h_{x,b}(z)$$

converges uniformly in \(S\) (in the uniform norm) by the Weierstrass M-test. Hence it has the same properties as \(h_{x,b}(z)\) and the claim follows.
Now observe that, for all \( x \in \mathbb{R}^d / \Gamma \), the maps
\[
z \mapsto F_1(z) \psi(x) = |\Gamma|^z e^{i \arg f(x)} |f(x)|^{zr} (e^{i \arg g(-i \nabla)} |g|^{zr} (-i \nabla) \psi)(x)
\]
and
\[
z \mapsto F_2(z) \psi(x) = |\Gamma|^z e^{i \arg g(-i \nabla)} |g|^{zr} (e^{i \arg f(x)} |f(x)|^{zr} \psi)(x)
\]
have the same properties as the above functions. (Note that, in the second map, we can think of \( e^{i \arg f(x)} |f(x)|^{zr} \psi(x) \) as a new \( \tilde{\psi}(x) \) with the same properties of \( \psi(x) \).) Thus, for \( j \in \{1, 2\} \), the function \( z \mapsto \langle \phi, F_j(z) \psi \rangle_{L^2} \) is continuous and bounded in \( S \). This remains true for any \( \psi \in L^2 \) because the set of \( C^\infty \)-functions with compact support is dense in \( L^2 \). Furthermore, for \( j \in \{1, 2\} \) the function \( z \mapsto \langle \phi, F_j(z) \psi \rangle_{L^2} \) is also analytic in the interior of \( S \) because, by Cauchy’s Formula and Fubini’s Theorem,
\[
\langle \phi, F_j(z) \psi \rangle_{L^2} = \int_{\mathbb{R}^d / \Gamma} \overline{\phi(x)} \left( \frac{1}{2\pi i} \int_{\gamma} F_j(\xi) \psi(x) \frac{d\xi}{\xi - z} \right) dx = \frac{1}{2\pi i} \int_{\gamma} \langle \phi, F_j(\xi) \psi \rangle_{L^2} \frac{d\xi}{\xi - z},
\]
where \( \gamma \subset S \) is a closed path around \( z \). This completes the proof of the proposition. \( \square \)

The following notation will be used whenever we consider vector-valued quantities. Let \( X \) be a Banach space and let \( A = (A_1, \ldots, A_d) \in X^d \) and \( B = (B_1, \ldots, B_d) \in X^d \). Then,
\[
\|A\|_{X} := (\|A_1\|^2_X + \cdots + \|A_d\|^2_X)^{1/2} \quad \text{and} \quad A \cdot B := A_1 B_1 + \cdots + A_d B_d.
\]

The next three propositions are simple straightforward calculations.

**Proposition 2.4.5.** Let \( A \in (\mathcal{L}(\mathcal{H}))^d \) and \( B \in \mathcal{S}^d_r \). Then \( A \cdot B \in \mathcal{S}^d_r \) and \( B \cdot A \in \mathcal{S}^d_r \) with
\[
\|A \cdot B\|_r \leq \|A\| \cdot \|B\|_r \quad \text{and} \quad \|B \cdot A\|_r \leq \|A\| \cdot \|B\|_r.
\]

**Proof.** Using the properties of \( \| \cdot \|_r \) (see Proposition 2.4.2) and the Cauchy-Schwarz inequality we have
\[
\|A \cdot B\|_r = \|A_1 B_1 + \cdots + A_d B_d\|_r \leq \|A_1 B_1\|_r + \cdots + \|A_d B_d\|_r \\
\leq \|A_1\| \cdot \|B_1\|_r + \cdots + \|A_d\| \cdot \|B_d\|_r \leq \|A\| \cdot \|B\|_r,
\]
and
\[
\|B \cdot A\|_r = \|B_1 A_1 + \cdots + B_d A_d\|_r \leq \|B_1 A_1\|_r + \cdots + \|B_d A_d\|_r \\
= \|A_1 B_1\|_r + \cdots + \|A_d B_d\|_r \leq \|A_1\| \cdot \|B_1\|_r + \cdots + \|A_d\| \cdot \|B_d\|_r \\
\leq \|A\| \cdot \|B\|_r,
\]
as desired. \( \square \)
Proposition 2.4.6. The following inequalities hold:
\[ \left\| \frac{1}{\sqrt{I - \Delta}} \nabla \right\| \leq d^{1/2} \quad \text{and} \quad \left\| \frac{1}{\sqrt{I - \Delta}} k \right\| \leq |k|. \]

Proof. Let \( \varphi \in L^2 \). Then, for \( 1 \leq j \leq d \),
\[ \left\| \frac{1}{\sqrt{I - \Delta}} \frac{\partial}{\partial x^j} \varphi \right\|_{L^2}^2 = \left\| \frac{j}{\sqrt{1 + b^2}} \varphi(b) \right\|_{l^2}^2 \leq |\Gamma| \left\| \varphi \right\|_{l^2}^2 = \left\| \varphi \right\|_{L^2}^2 \]
and
\[ \left\| \frac{1}{\sqrt{I - \Delta}} k_j \varphi \right\|_{L^2}^2 = |\Gamma| \left\| \frac{k_j}{\sqrt{1 + b^2}} \varphi(b) \right\|_{l^2}^2 \leq |\Gamma| |kJ| \left\| \varphi \right\|_{l^2}^2 = |kJ| \left\| \varphi \right\|_{L^2}^2. \]

Now observe that, if \( T \) is a bounded linear operator and \( \|T\varphi\|_{L^2} \leq C \|\varphi\|_{L^2} \) for all \( \varphi \in L^2 \) then \( \|T\| \leq C \). Hence,
\[ \left\| \frac{1}{\sqrt{I - \Delta}} \nabla \right\| = \left( \sum_{j=1}^{d} \left\| \frac{1}{\sqrt{I - \Delta}} \frac{\partial}{\partial x^j} \varphi \right\|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{d} 1 \right)^{1/2} = d^{1/2} \]
and
\[ \left\| \frac{1}{\sqrt{I - \Delta}} k \right\| = \left( \sum_{j=1}^{d} \left\| \frac{1}{\sqrt{I - \Delta}} k_j \varphi \right\|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{d} |kJ|^2 \right)^{1/2} = |kJ|, \]
as was to be shown. \( \Box \)

Proposition 2.4.7. The function \( g : \Gamma^# \rightarrow \mathbb{R} \) given by \( g(b) = (1 + b^2)^{-1/2} \) is in \( l^r \) for \( r > d \).

Proof. There are constants \( C_{\Gamma^#} \) and \( C_{\Gamma^#,d} \) such that
\[ \|g\|_{l^r} = \sum_{b \in \Gamma^#} |g(b)|^r \leq C_{\Gamma^#} \int_{\mathbb{R}^d} (1 + b^2)^{-r/2} db = C_{\Gamma^#} \int_0^\infty \left( \int_{\partial B(0, \rho)} dS \right) (1 + \rho^2)^{-r/2} d\rho \]
\[ = C_{\Gamma^#} \int_0^\infty \rho^{d-1}(1 + \rho^2)^{-r/2} d\rho < \infty, \]
since the above integral converges for \( d - 1 - r < -1 \). \( \Box \)

Now write
\[ H_k(A, V) - \lambda = I - \Delta + u(k, \lambda) + w(k, A, V), \]
where \( \Delta \) is the Laplace operator in \( \mathbb{R}^d \),
\[ u(k, \lambda) := -2ik \cdot \nabla + k^2 - \lambda - I, \]
and
\[ w(k, A, V) := i\nabla \cdot A + iA \cdot \nabla - 2k \cdot A + A^2 + V. \]

Applying Propositions 2.4.2 to 2.4.7 we prove the following estimates.
Lemma 2.4.8. Let \( r > d \). There is a constant \( C = C_{\Gamma, r, d} \) such that

\[
\left\| \frac{1}{\sqrt{1-\Delta}} w(k, A, V) \frac{1}{\sqrt{1-\Delta}} \right\|_{L^r} \leq C \left( (1 + |k|)\|A\|_{L^r} + \|A\|^2_{L^r} + \|V\|_{L^{r/2}} \right)
\]

and

\[
\left\| \frac{1}{\sqrt{1-\Delta}} u(k, \lambda) \frac{1}{\sqrt{1-\Delta}} \right\|_{L^r} \leq C (1 + |k|^2 + |\lambda|).
\]

Furthermore, let \( 0 \leq \varepsilon \leq \frac{r-d}{2r} \). There is a constant \( C = C_{\Gamma, r, d, k, \lambda, A, V} \) such that

\[
|\langle (u(k, \lambda) + w(k, A, V)) \varphi, \psi \rangle_{L^2}|
\leq C \left( \|(I - \Delta)^{(1-\varepsilon)/2} \varphi\|_{L^2} \|(I - \Delta)^{1/2} \psi\|_{L^2} + \|(I - \Delta)^{(1-\varepsilon)/2} \varphi\|_{L^2} \|(I - \Delta)^{1/2} \psi\|_{L^2} \right)
\]

\[
\leq 2C \left( \|(I - \Delta)^{1/2} \varphi\|_{L^2} \|(I - \Delta)^{1/2} \psi\|_{L^2} \right)
\]

for all \( \varphi, \psi \in L^2(\mathbb{R}^d/\Gamma) \).

Proof. (a) Write

\[
g = \frac{1}{\sqrt{1-b^2}} g(-i\nabla) \quad \text{with} \quad g(b) = \frac{1}{\sqrt{1-b^2}}.
\]

Then, using the properties of \( \| \cdot \|_r \), in particular that

\[
\|gA\|_r = \|gA\|^* r = \|A^* g\|_r = \|A^* g\|_r,
\]

and Proposition 2.4.6, it follows that

\[
\|gwg\|_r = \|g(i\nabla \cdot A + iA \cdot \nabla - 2k \cdot A + A^2 + |V|^2 e^{i\arg V})g\|_r
\]

(by the triangle inequality)

\[
\leq \|g(\nabla \cdot A)g\|_r + 2\|gA \cdot \nabla g\|_r + 2\|gk \cdot Ag\|_r + \|gA \cdot Ag\|_r + \|gV|^{1/2} e^{i\arg V} |V|^{1/2} g\|_r
\]

(by Proposition 2.4.5)

\[
\leq \|g\nabla \|_r \|Ag\|_r + \|\nabla g\|_r \|gAg\|_r + 2\|gk \cdot Ag\|_r + \|gA \cdot Ag\|_r + \|gV|^{1/2} \|V|^{1/2} g\|_r
\]

\[
= \|g\nabla \|_r \|Ag\|_r + \|\nabla g\|_r \|A^* g\|_r + 2\|gk \cdot Ag\|_r + \|A^* g\|_r \|Ag\|_r + \|V|^{1/2} \|V|^{1/2} g\|_r
\]

\[
\leq (d^{1/2} + 2|k|)\|Ag\|_r + d^{1/2} \|A^* g\|_r + \|A^* g\|_r \|Ag\|_r + \|V|^{1/2} \|V|^{1/2} g\|_r^2.
\]

Now, since \( g \in L^r \) for \( r > d \) by Proposition 2.4.7, applying Proposition 2.4.3 we obtain

\[
\|gwg\|_r \leq 2\left( d^{1/2} + |k| \right) \|Ag\|_{L^r} \|g\|_{L^r} + \|\nabla g\|_{L^r}^2 \|g\|_{L^r}^2 + \|Ag\|_{\Gamma^{-1/r}}^2 \|A\|_{\Gamma^{-2/r}}^2 \|Ag\|_{L^r}^2 \|g\|_{L^r}^2 + \|\nabla g\|_{L^r}^2 \|V|^{1/2} \|V|^{1/2} \|g\|_{L^r}^2
\]

\[
\leq C_{\Gamma, r, d} \left( (1 + |k|) \|A\|_{L^r} + \|A\|^2_{L^r} + \|V\|_{L^{r/2}} \right).
\]

This proves part (a).
(b) The spectrum of $g(-i\nabla)u(k, \lambda)g(-i\nabla)$ is

$$\left\{ \frac{2k \cdot b + k^2 - \lambda - 1}{1 + b^2} \mid b \in \Gamma^\# \right\}.$$ 

Hence,

$$\|gug\|_r = \left\| \frac{2k \cdot b + k^2 - \lambda - 1}{1 + b^2} \right\|_{L^r} \leq \left\| \frac{|b|}{1 + b^2} \right\|_{L^r} 2|k| + \left\| \frac{1}{1 + b^2} \right\|_{L^r} (1 + |k|^2 + |\lambda|) \leq C_{r, \Gamma} (1 + |k|^2 + |\lambda|),$$

which proves part (b).

(c) Write $D = \sqrt{I - \Delta}$. The condition on $\varepsilon$ implies that $r(1 - \varepsilon) > \frac{r+d}{2} > d$ so that, by Proposition 2.4.7,

$$(1 + b^2)^{-r(1-\varepsilon)/2} \in l^1(\Gamma^#),$$

and hence $D^{-1-\varepsilon} \in \mathcal{S}_r$. Thus, as in part (a),

$$\|D^{-1}\nabla \cdot AD^{-(1-\varepsilon)}\|_r \leq C_{r, \Gamma, d} A_{L^r},$$

$$\|k \cdot AD^{-(1-\varepsilon)}\|_r \leq C_{r, \Gamma, d} |k| A_{L^r},$$

$$\|D^{-(1-\varepsilon)}A \cdot AD^{-(1-\varepsilon)}\|_r \leq C_{r, \Gamma, d} |A|_{L^r}^2,$$

$$\|D^{-(1-\varepsilon)}VD^{-(1-\varepsilon)}\|_r \leq C_{r, \Gamma, d} \|V\|_{L^{r/2}}.$$ 

Furthermore,

$$\|D^{-1}u\| \leq \sup_{b \in \Gamma^#} \left| \frac{2k \cdot b + k^2 - \lambda - 1}{\sqrt{1 + b^2}} \right| \leq 2 (1 + |k|^2 + |\lambda|).$$

Consequently,

$$\langle i\nabla \cdot A\varphi, \psi \rangle = \langle D^{-1}i\nabla \cdot AD^{-(1-\varepsilon)}D^{1-\varepsilon}\varphi, D\psi \rangle$$

$$\leq \|D^{-1}\nabla \cdot AD^{-(1-\varepsilon)}D^{1-\varepsilon}\varphi\|_{L^2} \|D\psi\|_{L^2}$$

$$\leq \|D^{-1}\nabla \cdot AD^{-(1-\varepsilon)}\|_{L^r} \|D^{1-\varepsilon}\varphi\|_{L^2} \|D\psi\|_{L^2}$$

$$\leq \|D^{-1}\nabla \cdot AD^{-(1-\varepsilon)}\|_{L^r} \|D^{1-\varepsilon}\varphi\|_{L^2} \|D\psi\|_{L^2}$$

$$\leq C_{\Gamma, r, d} A_{L^r} \|D^{1-\varepsilon}\varphi\|_{L^2} \|D\psi\|_{L^2}. $$

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Similarly,
\[
|\langle iA \cdot \nabla \varphi, \psi \rangle| \leq C_{d^r,d} \|A\|_{L^r} \|D\varphi\|_{L^2} \|D^{1-\varepsilon}\psi\|_{L^2},
\]
\[
|\langle 2k \cdot A \varphi, \psi \rangle| \leq C_{d^r,d} \|k\| \|A\|_{L^r} \|D^{1-\varepsilon}\varphi\|_{L^2} \|\psi\|_{L^2},
\]
\[
|\langle A \cdot A \varphi, \psi \rangle| \leq C_{d^r,d} \|A\|^2_{L^r} \|D^{1-\varepsilon}\varphi\|_{L^2} \|D^{1-\varepsilon}\psi\|_{L^2},
\]
\[
|\langle V \varphi, \psi \rangle| \leq C_{d^r,d} \|V\|_{L^{r/2}} \|D^{1-\varepsilon}\varphi\|_{L^2} \|D^{1-\varepsilon}\psi\|_{L^2},
\]
\[
|\langle u \varphi, \psi \rangle| \leq C_{d^r,d} (1 + |k|^2 + |A|) \|\varphi\|_{L^2} \|D\psi\|_{L^2}.
\]

Finally, since \(\|D^{q}\varphi\|_{L^2} \leq \|D^{s}\varphi\|_{L^2}\) for all \(0 \leq q \leq s\) and all \(\varphi \in L^2\), the above estimates imply inequality (c). The proof of the lemma is complete.

Lemma 2.4.8 says that the operators given in (a) and (b) belong to \(I_r\) and that the quadratic form in (c) is well-defined on the domain \(H^1(R^d/\Gamma) \times H^1(R^d/\Gamma)\). Here, the space \(H^1(R^d/\Gamma)\) is the usual Sobolev space with norm \(\|\cdot\|_{H^1(R^d/\Gamma)} = (I - \Delta)^{1/2} \cdot \|\cdot\|_{L^2(R^d/\Gamma)}\). We now prove the following general property of Sobolev norms.

**Proposition 2.4.9.** Let \(0 < r < s < t\). Then, given \(\delta > 0\) there is a positive constant \(C = C_{\delta,r,s,t}\) such that
\[
\|\varphi\|_{H^s(R^d/\Gamma)} \leq \delta \|\varphi\|_{H^t(R^d/\Gamma)} + C \|\varphi\|_{H^r(R^d/\Gamma)}
\]
for all \(\varphi \in H^t(R^d/\Gamma)\).

**Proof.** We first find a constant \(C > 0\) such that
\[
(1 + b^2)^s \leq \delta^2 (1 + b^2)^t + C^2 (1 + b^2)^r
\]
for all \(b \in \Gamma^\#\) or, equivalently, setting \(y := 1 + b^2\), such that
\[
\delta^2 y^{t-s} + C^2 \frac{1}{y^{s-r}} \geq 1
\]
for all \(y \geq 1\). Let \(\phi(y) := \delta^2 y^{t-s} + C^2 \frac{1}{y^{s-r}}\). Then,
\[
\phi'(y) = \frac{1}{y^{s+1}} (\delta^2 (t-s) y^t + C^2 (r-s) y^r),
\]
so that \(\phi(y)\) has only one critical point for \(y > 0\), namely,
\[
y^* := \left[ \frac{C^2 (s-r)}{\delta^2 (t-s)} \right]^{\frac{1}{t-r}}.
\]

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Since \( \lim_{y \to 0} \phi(y) = \lim_{y \to \infty} \phi(y) = \infty \), the point \( y^* \) is a point of minimum and, for all \( y \geq 1 \),
\[
\phi(y) \geq \phi(y^*) = C \frac{2(t-s)}{\pi} \left[ \frac{s-r}{\delta^2(t-s)} \right]^{\frac{r-s}{t-s}} + \left[ \frac{s-r}{\delta^2(t-s)} \right]^{\frac{r-s}{2(t-s)}} = 1
\]
if we choose
\[
C = \left[ \delta^2 \left( \frac{s-r}{\delta^2(t-s)} \right)^{\frac{r-s}{t-s}} + \left( \frac{s-r}{\delta^2(t-s)} \right)^{\frac{r-s}{2(t-s)}} \right].
\]
Consequently,
\[
\|
\phi \|^2_{H^s} = \|(I - \Delta)^{s/2} \phi \|^2_{L^2} = |\Gamma| \|((I - \Delta)^{s/2} \phi)^{\wedge} \|^2_{L^2} = |\Gamma| \sum_{b \in \Gamma^\#} (1 + b^2)^{s/2} |\phi(b)|^2
\]
\[
\leq \delta^2 |\Gamma| \sum_{b \in \Gamma^\#} (1 + b^2)^{s/2} |\phi(b)|^2 + C^2 |\Gamma| \sum_{b \in \Gamma^\#} (1 + b^2)^{r/2} |\phi(b)|^2
\]
\[
= \delta^2 \|
\phi \|^2_{H^s} + C^2 \|
\phi \|^2_{H^r} \leq (\delta \|
\phi \|^2_{H^s} + C \|
\phi \|^2_{H^r})^2.
\]
Taking the square root of both sides of the last expression we obtain the desired inequality.

We now show that \( H_k(A, V) \) is a self-adjoint semibounded operator (on a suitable domain) when \( k, A \) and \( V \) are real. To prove this property we consider the quadratic form associated to this operator (see [11, §VIII.6]).

**Proof of Theorem 2.4.1(b).** Let \( D = H^1(\mathbb{R}^d/\Gamma) \) be the domain of \( \sqrt{I - \Delta} \). By Lemma 2.4.8, the operator
\[
(i\nabla + A - k)^2 + V - \lambda = I - \Delta + u(k, \lambda) + w(k, A, V)
\]
gives a well-defined quadratic form
\[
q : D \times D \rightarrow \mathbb{C},
\]
\[
(\varphi, \psi) \mapsto \langle (I - \Delta + u(k, \lambda) + w(k, A, V)) \varphi, \psi \rangle_{L^2}.
\]
Furthermore, for \( \varphi \in D \), when \( k, A \) and \( V \) are real,
\[
q(\varphi, \varphi) = \|\sqrt{I - \Delta} \varphi\|^2 + \langle (u(k, \lambda) + w(k, A, V)) \varphi, \varphi \rangle_{L^2}
\]
\[
\geq \|\sqrt{I - \Delta} \varphi\|^2 - C \|I - \Delta\|^{(1-\varepsilon)/2} \|\sqrt{I - \Delta} \varphi\|,
\]
where \( C > 0 \) is a constant and \( \| \cdot \| = \| \cdot \|_{L^2} \) (to simplify the notation). By Proposition 2.4.9, for any \( \delta > 0 \) there is a constant \( C_\delta > 0 \) such that
\[
\|(I - \Delta)^{1-\varepsilon/2} \varphi\| \leq \delta \|\sqrt{I - \Delta} \varphi\| + C_\delta \|\varphi\|
\]

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for all \( \varphi \in \mathcal{D} \). Choosing \( \delta = (2C)^{-1} \) we obtain

\[
q(\varphi, \varphi) \geq (1 - C\delta)\sqrt{T - \Delta} \varphi^2 - C C_\delta \|\varphi\| \|\sqrt{T - \Delta} \varphi\| = \frac{1}{4} \|\sqrt{T - \Delta} \varphi\|^2 - C C_\delta \|\varphi\| \|\sqrt{T - \Delta} \varphi\|
\]

\[
= \frac{1}{4} \|\sqrt{T - \Delta} \varphi\|^2 - (C C_\delta)^2 \|\varphi\|^2 + C \|\varphi\| \left( C_\delta \sqrt{C} \|\varphi\| - \frac{\|\sqrt{T - \Delta} \varphi\|}{2\sqrt{C} \|\varphi\|}\right)^2
\]

\[
\geq \frac{1}{4} \|\sqrt{T - \Delta} \varphi\|^2 - (C C_\delta)^2 \|\varphi\|^2 \geq -(C C_\delta)^2 \|\varphi\|^2.
\]

Hence, the form \( q \) is semibounded and

\[
\frac{1}{4} \|\sqrt{T - \Delta} \varphi\|^2 - (C C_\delta)^2 \|\varphi\|^2 \leq q(\varphi, \varphi) \leq C \|\sqrt{T - \Delta} \varphi\|^2. \tag{2.4.7}
\]

Since \( \mathcal{D} = H^1(\mathbb{R}^d/\Gamma) \) is complete and \( \|\varphi\|_{H^1} = \|\sqrt{T - \Delta} \varphi\| \), whenever \( \{\varphi_n\} \subset \mathcal{D} \) with \( \varphi_n \to \varphi \) in \( L^2 \) and \( q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0 \) as \( n, m \to \infty \), it follows from (2.4.7) that \( \|\varphi_n - \varphi_m\|_{H^1} \to 0 \) as \( n, m \to \infty \) and consequently that \( \varphi \in \mathcal{D} \) with \( q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0 \).

This implies that the form \( q \) is closed. By [11, Theorem VIII.15], there is a unique self-adjoint semibounded operator \( H_k(A, V) \) densely defined on \( \mathcal{D} \subset L^2(\mathbb{R}^d/\Gamma) \) associated to \( q \).

\[
\square
\]

Finally, we outline the proof of Theorem 2.4.1(a).

**Outline of the proof of Theorem 2.4.1(a).** To simplify the notation write

\[H_k = H_k(A, V),\]

and denote by \( \sigma(H_k) \) the spectrum of \( H_k \). If \( \lambda \notin \sigma(H_k) \), the resolvent \( (H_k - \lambda)^{-1} \) exists and is bounded. This is just the definition of spectrum. We first claim that, if \( (H_k - \lambda)^{-1} \) is compact for some \( \lambda \notin \sigma(H_k) \), then it is compact for all \( \lambda \notin \sigma(H_k) \). In fact, suppose that \( \lambda, \lambda' \notin \sigma(H_k) \). Then, by the resolvent identity we have

\[
(H_k - \lambda)^{-1} = (H_k - \lambda)^{-1} + (H_k - \lambda)^{-1}(\lambda - \lambda')(H_k - \lambda')^{-1}
\]

\[
= (H_k - \lambda)^{-1}[I + (\lambda - \lambda')(H_k - \lambda')^{-1}].
\]

The factor \( [I + (\lambda - \lambda')(H_k - \lambda')^{-1}] \) is a bounded operator. Thus, if \( (H_k - \lambda)^{-1} \) is compact for some \( \lambda \notin \sigma(H_k) \), the above product is compact and consequently \( (H_k - \lambda')^{-1} \) is compact for all \( \lambda' \notin \sigma(H_k) \). This shows that is enough to prove part (a) of theorem for a single \( \lambda \notin \sigma(H_k) \). This is what we do next.
Write

\[ H_k = -\Delta + Q \]

with

\[ Q := i\nabla \cdot (A - k) + (A - k) \cdot i\nabla + (A - k)^2 + V. \]

Then, if \( \lambda \notin \sigma(H_k) \), and if \(|\lambda|\) is sufficiently large, the operator

\[ \left( I + \frac{1}{\sqrt{-\Delta - \lambda}} Q \frac{1}{\sqrt{-\Delta - \lambda}} \right)^{-1} \]

exists and is bounded. Furthermore, we can choose the \( \lambda \) so that \( \frac{1}{\sqrt{-\Delta - \lambda}} \) exists, and similarly as in the proof of Lemma 2.4.8(c), we can prove that this operator is compact. Thus we can write

\[ (H_k - \lambda)^{-1} = \frac{1}{\sqrt{-\Delta - \lambda}} I + \frac{1}{\sqrt{-\Delta - \lambda}} Q \frac{1}{\sqrt{-\Delta - \lambda}} \]

and similarly as above conclude that \((H_k - \lambda)^{-1}\) is compact for a suitable \( \lambda \notin \sigma(H_k) \). In view of the above remark, this implies the statement for all \( \lambda \notin \sigma(H_k) \) and completes the outline of the proof. \( \square \)

2.5 The complex analytic structure of the spectrum

For complex-valued potentials \((A, V) \in \mathcal{A}_C \times \mathcal{V}_C\) and for \( k \in \mathbb{C}^2 \) the problem

\[ H \varphi_{n,k} = E_n(k) \varphi_{n,k}, \]

\[ \varphi_{n,k}(x + \gamma) = e^{ik \cdot \gamma} \varphi_{n,k}(x) \]

for all \( x \in \mathbb{R}^d \) and all \( \gamma \in \Gamma \) is no longer self-adjoint. Its spectrum, however, remains discrete. It is a sequence of eigenvalues in the complex plane. From the above boundary condition it is easy to see that for each \( n \geq 1 \) the function \( k \mapsto E_n(k) \) remains periodic with respect to \( \Gamma^\# \). Furthermore, the transformation \( U_k \) is no longer unitary but it is still bounded and invertible and it still preserves the spectrum. That is, we can still rewrite this problem in the form

\[ H_k \psi_{n,k} = E_n(k) \psi_{n,k} \quad \text{for} \quad \psi_{n,k} \in L^2(\mathbb{R}^d/\Gamma) \]

without modifying the eigenvalues. Thus, as above we define the \((\text{complex}) \ \text{“lifted” Fermi surface}\) of \((A, V)\) with energy \( \lambda \in \mathbb{C} \) to be

\[ \tilde{\mathcal{F}}(A, V) := \{ k \in \mathbb{C}^d \mid (H_k(A, V) - \lambda) \psi = 0 \ \text{for some} \ \psi \in \mathcal{D}_{H_k(A, V)} \setminus \{0\} \}. \]
We shall prove below that this surface has the following (well-known) property.

**Theorem 2.5.1** ([3]). There exists an analytic function $F$ on $\mathbb{C}^d \times \mathbb{C} \times \mathcal{A}_\mathbb{C} \times \mathcal{V}_\mathbb{C}$ such that

$$\tilde{\mathcal{F}}(A, V) = \{ k \in \mathbb{C}^d \mid F(k, \lambda, A, V) = 0 \}.$$

In particular, for $k$, $A$ and $V$ real,

$$\lambda \in \text{Spec}(H_k(A, V)) \quad \text{if and only if} \quad F(k, \lambda, A, V) = 0.$$

Furthermore, such an analytic function $F$ is given by (2.5.1).

To prove this theorem we follow [3]. Another proof of a similar statement can be found in [8, Theorem 4.4.2]. We shall use the definition and some properties of the regularized determinant on $\mathcal{S}_r$ (see [15, Chapter 9]).

**Proof.** Since $L^p(\mathbb{R}^d/\Gamma) \supset L^q(\mathbb{R}^d/\Gamma)$ for all $1 \leq p < q$, we may assume, without loss of generality, that $r \leq d + 1$. Then, since $\| \cdot \|_{d+1} \leq \| \cdot \|_r$, Lemma 2.4.8 implies that

$$F(k, \lambda, A, V) := \det_{d+1} \left( I + \frac{1}{\sqrt{1-\Delta}} u(k, \lambda) \frac{1}{\sqrt{1-\Delta}} + \frac{1}{\sqrt{1-\Delta}} w(k, A, V) \frac{1}{\sqrt{1-\Delta}} \right) \quad (2.5.1)$$

is a well-defined function on $\mathbb{C}^d \times \mathbb{C} \times \mathcal{A}_\mathbb{C} \times \mathcal{V}_\mathbb{C}$. Here $\det_{d+1}(I + B)$ is the regularized determinant for $B \in \mathcal{S}_{d+1}$. This function is Fréchet differentiable [15] and hence analytic on its domain [1, Theorem 14.13]. Analyticity may also be proved as in [6] by approximating $B$ by a sequence of finite rank operators.

Now, let $\mu$ be a complex number that is not in the spectrum of $H_k(A, V)$. Then, by Theorem 2.4.1(a), the resolvent $(H_k(A, V) - \mu)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d/\Gamma)$. Thus, the spectrum of $(H_k(A, V) - \mu)^{-1}$ is discrete, and so is the spectrum of $H_k(A, V)$. Therefore, $\lambda \in \text{Spec}(H_k(A, V))$ if and only if there exists $\psi \in \mathcal{D}_{H_k(A, V)} \subset \mathcal{D}$ such that

$$(H_k(A, V) - \lambda) \frac{1}{\sqrt{1-\Delta}} \sqrt{1-\Delta} \psi = 0.$$

This is the case if and only if $\frac{1}{\sqrt{1-\Delta}} (H_k(A, V) - \lambda) \frac{1}{\sqrt{1-\Delta}}$ is not invertible. Since this operator is the sum of the identity and a compact operator, it fails to be invertible if and only if it has a nontrivial kernel. By [15, Theorem 9.2], this is the case if and only if $F(k, \lambda, A, V) = 0$. \qed
2.6 Gauge invariance

In this section we outline the proof of the following (well-known) properties.

**Theorem 2.6.1** (Gauge invariance). For $1 \leq j \leq d$, let $A_j \in C^1(\mathbb{R}^d/\Gamma)$, $V \in C^0(\mathbb{R}^d/\Gamma)$, and $\Psi \in C^2(\mathbb{R}^d/\Gamma)$. Then

$$D_{H_k(A,V)} = D_{-\Delta + I} \quad \text{and} \quad e^{i\Psi} : D_{H_k(A,V)} \rightarrow D_{H_k(A,V)}.$$

Furthermore:

(a) $\text{Ker}(H_k(A,V) - \lambda) \neq \{0\}$ if and only if $\text{Ker}(e^{-i\Psi}H_k(A,V)e^{i\Psi} - \lambda) \neq \{0\}$;

(b) $e^{-i\Psi}H_k(A,V)e^{i\Psi} = H_k(A - \nabla \Psi, V)$;

(c) $\hat{F}_\lambda(A,V) = \hat{F}_\lambda(A - \nabla \Psi, V)$.

**Outline of the Proof.** We first prove parts (a) to (c). (a) Consider the linear transformation $e^{i\Psi} : \varphi(x) \mapsto e^{i\Psi(x)}\varphi(x)$ acting on $D_{H_k(A,V)}$. Since the function $\Psi$ is bounded, it is clear that $e^{i\Psi}$ is a bounded operator with bounded inverse on $L^2(\mathbb{R}^d/\Gamma)$ given by $e^{-i\Psi}$. Define

$$H_k^\Psi(A,V) := e^{-i\Psi}H_k(A,V)e^{i\Psi} \quad \text{and} \quad D_{H_k^\Psi(A,V)} := e^{-i\Psi}D_{H_k(A,V)}.$$

Observe that, by hypothesis, $D_{H_k^\Psi(A,V)} = D_{H_k(A,V)}$. Furthermore, we claim that

$$H_k(A,V)\psi = \lambda\psi$$

for $\psi \in D_{H_k(A,V)}$ with $\psi \neq 0$, if and only if

$$H_k^\Psi(A,V)\varphi = \lambda\varphi,$$

where $\varphi = e^{-i\Psi}\psi$ with $\varphi \in D_{H_k(A,V)}$ and $\varphi \neq 0$. Indeed, observe that the former equation implies the last one,

$$0 = e^{-i\Psi}0 = e^{-i\Psi}(H_k(A,V) - \lambda)\psi = e^{-i\Psi}(H_k(A,V) - \lambda)e^{i\Psi}e^{-i\Psi}\psi$$

$$= (e^{-i\Psi}H_k(A,V)e^{i\Psi} - \lambda)e^{-i\Psi}\psi = (H_k^\Psi(A,V) - \lambda)\varphi,$$

where $\varphi \neq 0$ since $\psi \neq 0$, and that a similar calculation (in the reverse order) implies the converse statement. This proves part (a).
(b) To simplify the notation write

\[ A' = A - \nabla \Psi. \]

Observe that, formally (without worry about domains and weak derivatives),

\[
e^{-i\Psi}H_k(A, V)e^{i\Psi}\varphi
= e^{-i\Psi}((i\nabla + A - k)^2 + V)e^{i\Psi}\varphi
= e^{-i\Psi}(i\nabla + A - k)\cdot e^{i\Psi}(-(\nabla \Psi) + i\nabla + A - k))\varphi + \nabla \varphi
= -(\nabla \Psi) + A - k \cdot (-i\nabla + i\nabla + A - k)\varphi + i\nabla \cdot (-i\nabla + i\nabla + A - k)\varphi + V\varphi
= (k^2 - 2ik \cdot \nabla - \Delta + A' \cdot (i\nabla - k) + (i\nabla - k) \cdot A' + (A')^2 + V)\varphi
= ((i\nabla - k)^2 + (i\nabla - k) \cdot A' + A' \cdot (i\nabla - k) + (A')^2 + V)\varphi
= ((i\nabla + A' - k)^2 + V)\varphi.
\]

However, since \( e^{-i\Psi}D_{H_k(A,V)} = D_{H_k(A,V)} \), we claim that this calculation can be rigorously justified indeed. Assuming this we have

\[
e^{-i\Psi}H_k(A, V)e^{i\Psi} = H_k(A - \nabla \Psi, V),
\]

which proves part (b).

(c) As a consequence of parts (a) and (b) we obtain

\[
\hat{\mathcal{F}}_\lambda(A, V) = \{ k \in \mathbb{C}^d \mid (H_k(A, V) - \lambda)\psi = 0 \text{ for some } \psi \in D_{H_k(A,V)} \setminus \{0\} \}
= \{ k \in \mathbb{C}^d \mid (e^{-i\Psi}H_k(A, V)e^{i\Psi} - \lambda)\varphi = 0 \text{ for some } \varphi \in D_{H_k(A,V)} \setminus \{0\} \}
= \{ k \in \mathbb{C}^d \mid (H_k(A - \nabla \Psi, V) - \lambda)\varphi = 0 \text{ for some } \varphi \in D_{H_k(A - \nabla \Psi, V)} \setminus \{0\} \}
= \hat{\mathcal{F}}_\lambda(A - \nabla \Psi, V),
\]

as desired.

We now outline the proof of the first part of the theorem. First observe that, since \( D_{-\Delta+I} = H^2(\mathbb{R}^d/\Gamma) \), and \( H^2(\mathbb{R}^d/\Gamma) \) is dense in \( L^2(\mathbb{R}^d/\Gamma) \), the set \( D_{-\Delta+I} \) is dense in \( D_{H_k(A,V)} \). Furthermore, it is easy to verify that \( D_{-\Delta+I} \subset D_{H_k(A,V)} \). Thus, if we can prove that \( H_k(A, V) \) with domain \( D_{-\Delta+I} \) is closed, then it follows that \( D_{H_k(A,V)} = D_{-\Delta+I} \). That is, we can choose \( D_{-\Delta+I} \) as a domain for \( H_k(A, V) \). Furthermore, since \( \Psi \in C^2(\mathbb{R}^d/\Gamma) \), it is clear then that \( e^{i\Psi} \) maps \( D_{H_k(A,V)} \) to \( D_{H_k(A,V)} \). Therefore, to conclude the proof of the theorem it suffices to show that \( H_k(A, V) \) is closed in \( D_{-\Delta+I} \).
The operator $H_k(A, V)$ is closed in $D_{-\Delta+I}$ if its graph is closed in $D_{-\Delta+I} \times D_{-\Delta+I}$. Equivalently, its graph is closed if for any $\{\varphi_n\} \subset D_{-\Delta+I}$ such that $\lim_{n \to \infty} \varphi_n =: \varphi$ exists and $\lim_{n \to \infty} H_k(A, V) \varphi_n =: \psi$ exists, then $\varphi \in D_{-\Delta+I}$ and $H_k(A, V)\varphi = \psi$ with $\psi \in D_{-\Delta+I}$.

To prove that $H_k(A, V)$ is closed write

$$H_k(A, V) = -\Delta + Q$$

with

$$Q := i\nabla \cdot (A - k) + (A - k) \cdot i\nabla + (A - k)^2 + V,$$

and assume that $\lim_{n \to \infty} \varphi_n =: \varphi$ and $\lim_{n \to \infty} H_k(A, V)\varphi_n =: \psi$ exist. Then, for a suitable constant $\lambda$,

$$(H_k(A, V) + \lambda)\varphi_n = (-\Delta + \lambda + Q)\varphi_n = [I + Q(-\Delta + \lambda)^{-1}] (-\Delta + \lambda)\varphi_n,$$

where the operator $Q(-\Delta + \lambda)^{-1}$ is bounded. Furthermore, we can choose $\lambda$ sufficiently large so that the operator norm of $Q(-\Delta + \lambda)^{-1}$ is strictly less than 1. Consequently, the operator $[I + Q(-\Delta + \lambda)^{-1}]$ is bounded and has a bounded inverse. Since by hypothesis $\{(H_k(A, V) + \lambda)\varphi_n\}$ converges, we conclude that $\lim_{n \to \infty} (-\Delta + \lambda)\varphi_n$ exists. It follows then that $\varphi \in D_{-\Delta+I}$ and $(-\Delta + \lambda)\varphi \in D_{-\Delta+I}$ because $-\Delta + \lambda$ is closed in $D_{-\Delta+I}$ (we are using this fact without proof). Finally, since $[I + Q(-\Delta + \lambda)^{-1}]$ is bounded and has a bounded inverse, it follows that $\lim_{n \to \infty} (H_k(A, V) + \lambda)\varphi_n$ is in $D_{-\Delta+I}$. All this together shows that $H_k(A, V)$ is closed in $D_{-\Delta+I}$ and completes the proof of the theorem.  

\[\square\]
Chapter 3

Asymptotics for Fermi curves

3.1 Fermi curves

Below we define the Fermi curves and briefly describe some of its properties.

Let $\Gamma$ be a lattice in $\mathbb{R}^2$ and let $A_1$, $A_2$ and $V$ be real-valued functions in $L^2(\mathbb{R}^2/\Gamma)$. Set $A := (A_1, A_2)$ and define the operator

$$H(A, V) := (i\nabla + A)^2 + V$$

acting on $L^2(\mathbb{R}^2)$, where $\nabla$ is the gradient operator in $\mathbb{R}^2$. For $k \in \mathbb{R}^2$ consider the following self-adjoint eigenvalue-eigenvector problem with boundary conditions,

$$H(A, V)\varphi = \lambda \varphi,$$

$$\varphi(x + \gamma) = e^{ik \cdot \gamma} \varphi(x)$$

for all $x \in \mathbb{R}^2$ and all $\gamma \in \Gamma$. The spectrum of this problem is discrete. It consists of a sequence of real eigenvalues

$$E_1(k, A, V) \leq E_2(k, A, V) \leq \cdots \leq E_n(k, A, V) \leq \cdots$$

For each integer $n \geq 1$ the eigenvalue $E_n(k, A, V)$ defines a continuous function of $k$. From the above boundary condition it is easy to see that this function is periodic with respect to the dual lattice

$$\Gamma^\# := \{b \in \mathbb{R}^2 \mid b \cdot \gamma \in 2\pi \mathbb{Z} \text{ for all } \gamma \in \Gamma\},$$
where \( b \cdot \gamma \) is the usual scalar product on \( \mathbb{R}^2 \). It is customary to refer to \( k \) as the crystal momentum and to \( E_n(k, A, V) \) as the \( n \)-th band function. The corresponding normalized eigenfunctions \( \varphi_{n,k} \) are called Bloch eigenfunctions.

Let \( U_k \) be the unitary transformation on \( L^2(\mathbb{R}^2) \) that acts as

\[
U_k : \varphi(x) \mapsto e^{ik \cdot x} \varphi(x).
\]

By applying this transformation we can rewrite the above problem and put the boundary conditions into the operator. Indeed, if we define

\[
H_k(A, V) := U_k^{-1} H(A, V) U_k \quad \text{and} \quad \psi := U_k^{-1} \varphi,
\]

then the above problem is unitary equivalent to

\[
H_k(A, V) \psi = \lambda \psi,
\]

\[
\psi(x + \gamma) = \psi(x)
\]

for all \( x \in \mathbb{R}^2 \) and all \( \gamma \in \Gamma \), or, using a more compact notation,

\[
H_k(A, V) \psi = \lambda \psi \quad \text{for} \quad \psi \in L^2(\mathbb{R}^2 / \Gamma).
\]

To see that these problems are equivalent we proceed (formally) as follows. On the one hand, from the original problem and using the above transformation we find that

\[
0 = U_k^{-1} 0 = U_k^{-1} (H(A, V) - \lambda) \varphi = U_k^{-1} (H(A, V) - \lambda) U_k U_k^{-1} \varphi
\]

\[
= (U_k^{-1} H(A, V) U_k - \lambda) U_k^{-1} \varphi = (H_k(A, V) - \lambda) \psi
\]

and

\[
\psi(x + \gamma) = (U_k^{-1} \varphi)(x + \gamma) = e^{-ik \cdot (x + \gamma)} \varphi(x + \gamma)
\]

\[
= e^{-ik \cdot (x + \gamma)} e^{ik \cdot \gamma} \varphi(x) = e^{-ik \cdot x} \varphi(x) = (U_k^{-1} \varphi)(x) = \psi(x).
\]

On the other hand, by a similar computation (in the reverse order), using the last two equalities and the above transformation we derive the original problem. This (formally) implies unitary equivalence. Furthermore, a simple (formal) calculation shows that

\[
H_k(A, V) = (i \nabla + A - k)^2 + V.
\]
In fact,
\[ H_k(A,V)\psi = U_k^{-1}H(A,V)U_k\psi \]
\[ = e^{-ik \cdot x}[(i\nabla + A)^2 + V]e^{ik \cdot x}\psi \]
\[ = e^{-ik \cdot x}[(i\nabla + A) \cdot e^{ik \cdot x}(-k\psi + i\nabla \psi + A\psi)] + V\psi \]
\[ = (-k + A) \cdot (-k + i\nabla + A)\psi + i\nabla \cdot (-k + i\nabla + A)\psi + V\psi \]
\[ = [(i\nabla + A - k)^2 + V]\psi. \]

Of course, the unitary transformation \( U_k \) preserves self-adjointness and does not change the spectrum \( \{E_n(k,A,V)\}_{n=1}^{\infty} \).

The real “lifted” Fermi curve of \((A,V)\) with energy \( \lambda \in \mathbb{R} \) is defined as
\[ \hat{F}_{\lambda,R}(A,V) := \{k \in \mathbb{R}^2 \mid E_n(k,A,V) = \lambda \text{ for some } n \geq 1\}. \]

Equivalently,
\[ \hat{F}_{\lambda,R}(A,V) = \{k \in \mathbb{R}^2 \mid (H_k(A,V) - \lambda)\varphi = 0 \text{ for some } \varphi \in \mathcal{D}_{H_k(A,V)} \setminus \{0\}\}, \]
where \( \mathcal{D}_{H_k(A,V)} \subset L^2(\mathbb{R}^2/\Gamma) \) denotes the (dense) domain of \( H_k(A,V) \). The adjective “lifted” indicates that \( \hat{F}_{\lambda,R}(A,V) \) is a subset of \( \mathbb{R}^2 \) rather than \( \mathbb{R}^2/\Gamma^\# \). As we may replace \( V \) by \( V - \lambda \), we only discuss the case \( \lambda = 0 \) and write \( \hat{F}_R(A,V) \) in place of \( \hat{F}_{0,R}(A,V) \) to simplify the notation. Furthermore, since \( H_k(A,V) = H_{k - \hat{A}(0)}(A - \hat{A}(0),V) \), if we perform the change of coordinates \( k \rightarrow k + \hat{A}(0) \) and redefine \( A - \hat{A}(0) \rightarrow A \) we may assume, without loss of generality, that
\[ \hat{A}(0) = \frac{1}{|\Gamma|} \int_{\mathbb{R}^2/\Gamma} A(x) \, dx = 0. \]

The dual lattice \( \Gamma^\# \) acts on \( \mathbb{R}^2 \) by translating \( k \mapsto k + b \) for \( b \in \Gamma^\# \). This action maps \( \hat{F}_R(A,V) \) to itself because for each \( n \geq 1 \) the function \( k \mapsto E_n(k,A,V) \) is periodic with respect to \( \Gamma^\# \). In other words, the real lifted Fermi curve “is periodic” with respect to \( \Gamma^\# \).

Define
\[ F_R(A,V) := \hat{F}_R(A,V)/\Gamma^#. \]

We call \( F_R(A,V) \) the real Fermi curve of \((A,V)\). It is a curve in the torus \( \mathbb{R}^2/\Gamma^\# \).

The above definitions and the real Fermi curve have physical meaning. It is useful and interesting, however, to study the “complexification” of these curves. Knowledge about the complexified curves may provide information about the real counterparts.
For complex-valued functions $A_1$, $A_2$ and $V$ in $L^2(\mathbb{R}^2/\Gamma)$ and for $k \in \mathbb{C}^2$ the above problem is no longer self-adjoint. Its spectrum, however, remains discrete. It is a sequence of eigenvalues in the complex plane. From the boundary condition in the original problem it is easy to see that the family of functions $k \mapsto E_n(k, A, V)$ remains periodic with respect to $\Gamma^\#$. Furthermore, the transformation $U_k$ is no longer unitary but it is still bounded and invertible and it still preserves the spectrum, that is, we can still rewrite the original problem in the form

$$H_k(A, V)\psi = \lambda \psi \quad \text{for} \quad \psi \in L^2(\mathbb{R}^2/\Gamma)$$

without modifying the eigenvalues. Thus, it makes sense to define

$$\hat{\mathcal{F}}(A, V) := \{ k \in \mathbb{C}^2 \mid H_k(A, V)\varphi = 0 \text{ for some } \varphi \in \mathcal{D}_{H_k(A, V)} \setminus \{0\} \}$$

and

$$\mathcal{F}(A, V) := \hat{\mathcal{F}}(A, V)/\Gamma^\#.$$

We call $\hat{\mathcal{F}}(A, V)$ and $\mathcal{F}(A, V)$ the (complex) “lifted” Fermi curve and the (complex) Fermi curve, respectively. When there is no risk of confusion we shall refer to either simply as Fermi curve.

Let $\{\gamma_1, \gamma_2\}$ be a basis of $\Gamma$ and set $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Define the exponential map $E$ as

$$E : \mathbb{C}^2 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*,$$

$$k \longmapsto (e^{ik\cdot\gamma_1}, e^{ik\cdot\gamma_2}).$$

This map is holomorphic and the pair $(\mathbb{C}^2, E)$ is a covering space of $\mathbb{C}^* \times \mathbb{C}^*$. In fact, every point of $\mathbb{C}^* \times \mathbb{C}^*$ has an open neighbourhood $W \subset \mathbb{C}^* \times \mathbb{C}^*$ such that the inverse image of $W$ under $E$ is a disjoint union of open sets $U_j \subset \mathbb{C}^2$, with the map $E$ sending each $U_j$ homeomorphically onto $W$. If we recall that $\hat{\mathcal{F}}(A, V)$ is invariant under the action of $\Gamma^\#$ and observe that $b \cdot \gamma_1 \in 2\pi\mathbb{Z}$ and $b \cdot \gamma_2 \in 2\pi\mathbb{Z}$ for all $b \in \Gamma^\#$, it is not difficult to see that

$$E(\hat{\mathcal{F}}(A, V)) \cong \mathcal{F}(A, V).$$

That is, up to the isomorphism

$$J : E(\hat{\mathcal{F}}(A, V)) \longrightarrow \mathcal{F}(A, V),$$

$$(e^{ik\cdot\gamma_1}, e^{ik\cdot\gamma_2}) \longmapsto [k],$$

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where \([k]\) denotes a point (or equivalence class) in \(\mathbb{R}^2/\Gamma\), the curve \(\mathcal{F}(A, V)\) is the image of \(\hat{\mathcal{F}}(A, V)\) under the exponential map. We sometimes assume that the isomorphism \(J\) is understood and simply write
\[
E(\hat{\mathcal{F}}(A, V)) = \mathcal{F}(A, V).
\]
Alternatively, we could have used this expression to define \(\mathcal{F}(A, V)\) (and avoid talking about the isomorphism \(J\)).

### 3.2 The free Fermi curve

When the potentials \(A\) and \(V\) are zero the curve \(\hat{\mathcal{F}}(A, V)\) can be found explicitly. In this section we collect some properties of this curve.

For \(\nu \in \{1, 2\}\) and \(b \in \Gamma\) set
\[
N_{b, \nu}(k) := (k_1 + b_1) + i(-1)^\nu(k_2 + b_2),
\]
\[
\mathcal{N}_{\nu}(b) := \{ k \in \mathbb{C}^2 \mid N_{b, \nu}(k) = 0 \},
\]
\[
\mathcal{N}_b(k) := N_{b,1}(k)N_{b,2}(k),
\]
\[
\mathcal{N}_b := \mathcal{N}_1(b) \cup \mathcal{N}_2(b),
\]
\[
\theta_{\nu}(b) := \frac{1}{2}((-1)^\nu b_2 + ib_1).
\]
Observe that \(\mathcal{N}_{\nu}(b)\) is a line in \(\mathbb{C}^2\). The free lifted Fermi curve is an union of these lines. Here is the precise statement.

**Theorem 3.2.1** (The free Fermi curve). The curve \(\hat{\mathcal{F}}(0, 0)\) is the locally finite union
\[
\bigcup_{b \in \Gamma\#} \mathcal{N}_{\nu}(b).
\]
In particular, the curve \(\mathcal{F}(0, 0)\) is a complex analytic curve in \(\mathbb{C}^2/\Gamma\#\) and \(\mathcal{F}(0, 0) \cong E(N_0)\).

Before we prove this theorem we shall prove some simple properties of the lines \(\mathcal{N}_{\nu}(b)\).

**Proposition 3.2.2** (Properties of \(\mathcal{N}_{\nu}(b)\)). Let \(\nu \in \{1, 2\}\) and let \(b, c, d \in \Gamma\#\). Then:

(a) \(\mathcal{N}_{\nu}(b) \cap \mathcal{N}_{\nu}(c) = \emptyset\) if \(b \neq c\);

(b) \(\text{dist}(\mathcal{N}_{\nu}(b), \mathcal{N}_{\nu}(c)) = \frac{1}{\sqrt{2}}|b - c|\);
(c) \( \mathcal{N}_1(b) \cap \mathcal{N}_2(c) = \{(i\theta_1(c) + i\theta_2(b), \theta_1(c) - \theta_2(b))\}; \)

(d) the map \( k \mapsto k + d \) maps \( \mathcal{N}_\nu(b) \) to \( \mathcal{N}_\nu(b - d); \)

(e) the map \( k \mapsto k + d \) maps \( \mathcal{N}_1(b) \cap \mathcal{N}_2(c) \) to \( \mathcal{N}_1(b - d) \cap \mathcal{N}_2(c - d). \)

**Proof.** (a) By contradiction. Suppose that \( \mathcal{N}_\nu(b) \cap \mathcal{N}_\nu(c) \) is not empty. Then there is at least one \( k \in \mathbb{C}^2 \) such that

\[
k_1 + b_1 + i(-1)^\nu(k_2 + b_2) = 0, \\
k_1 + c_1 + i(-1)^\nu(k_2 + c_2) = 0.
\]

This is true if and only if

\[
b_1 - c_1 + i(-1)^\nu(b_2 - c_2) = 0.
\]

But this is impossible because \( b \neq c \) and \( b_1, b_2, c_1 \) and \( c_2 \) are real numbers. Thus, the intersection \( \mathcal{N}_\nu(b) \cap \mathcal{N}_\nu(c) \) is empty. This proves part (a).

(b) Let \( k \in \mathcal{N}_\nu(b) \) and \( k' \in \mathcal{N}_\nu(c). \) Then,

\[
k_1 + b_1 + i(-1)^\nu(k_2 + b_2) = 0, \\
k'_1 + c_1 + i(-1)^\nu(k'_2 + c_2) = 0.
\]

Write \( d := c - b. \) Hence, using the above equations we obtain

\[
\text{dist}(\mathcal{N}_\nu(b), \mathcal{N}_\nu(c)) = \inf\{|k - k'| \mid k \in \mathcal{N}_\nu(b) \text{ and } k' \in \mathcal{N}_\nu(c)\}
\]

\[
= \inf\{|(k_1 - k'_1, k_2 - k'_2)| \mid k \in \mathcal{N}_\nu(b) \text{ and } k' \in \mathcal{N}_\nu(c)\}
\]

\[
= \inf\{|(c_1 - b_1 + i(-1)^\nu(c_2 - b_2 + k'_2 - k_2), k_2 - k'_2)| \mid k_2, k'_2 \in \mathbb{C}\}
\]

\[
= \inf\{|(d_1 + i(-1)^\nu(d_2 + z), -z)| \mid z \in \mathbb{C}\}
\]

\[
= \inf\{|(d^2 - 2 \text{ Re}(i(-1)^\nu d_1 \bar{z} - d_2 \bar{z}) + 2|z|^2)^{1/2} \mid z \in \mathbb{C}\}
\]

\[
= \inf\{|(d^2 + 2(d_2 x - (-1)^\nu d_1 y) + 2(x^2 + y^2))^{1/2} \mid x, y \in \mathbb{R}\}
\]

\[
= \inf\{|(\frac{1}{2}d^2 + \frac{1}{\sqrt{2}}(d_2, (-1)^\nu d_1) + \sqrt{2}(x, y))^2)^{1/2} \mid (x, y) \in \mathbb{R}^2\}
\]

\[
= \frac{1}{\sqrt{2}}|d| = \frac{1}{\sqrt{2}}|b - c|,
\]

as claimed.

(c) Let \( k \in \mathcal{N}_1(b) \cap \mathcal{N}_2(c). \) Then,

\[
k_1 + b_1 - i(k_2 + b_2) = 0, \\
k_1 + c_1 + i(k_2 + c_2) = 0,
\]

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which hold if and only if

\[ k_1 = \frac{1}{2} (-b_1 - c_1 + i(b_2 - c_2)) = i\theta_1(c) + i\theta_2(b), \]
\[ k_2 = \frac{1}{2} (-b_2 - c_2 + i(c_1 - b_1)) = \theta_1(c) - \theta_2(b). \]

This proves part (c).

(d) Observe that

\[ \mathcal{N}_\nu(b) + d = \{ k + d \in \mathbb{C}^2 \mid k_1 + b_1 + i(-1)^\nu(k_2 + b_2) = 0 \} \]
\[ = \{ k' \in \mathbb{C}^2 \mid k'_1 + b_1 - d_1 + i(-1)^\nu(k'_2 + b_2 - d_2) = 0 \} \]
\[ = \{ k' \in \mathbb{C}^2 \mid N_{b-d,\nu}(k') = 0 \} = \mathcal{N}_\nu(b - d). \]

This shows that the map \( k \mapsto k + d \) maps \( \mathcal{N}_\nu(b) \) to \( \mathcal{N}_\nu(b - d) \), as desired.

(e) Similarly as in part (d), the statement of part (e) follows from the equality

\[ \mathcal{N}_1(b) \cap \mathcal{N}_2(c) + d = \{ (\frac{1}{2} (-b_1 - c_1 + i(b_2 - c_2)) + d_1, \frac{1}{2} (-b_2 - c_2 + i(c_1 - b_1)) + d_2) \} \]
\[ = \{ (\frac{1}{2} (-b_1 - d_1) - (c_1 - d_1) + i((b_2 - d_2) - (c_2 - d_2))), \]
\[ \frac{1}{2} (-b_2 - d_2) - (c_2 - d_2) + i((c_1 - d_1) - (b_1 - d_1)) \} \]
\[ = \mathcal{N}_1(b - d) \cap \mathcal{N}_2(c - d). \]

The proof of the proposition is complete. \( \square \)

We now prove Theorem 3.2.1.

**Proof of Theorem 3.2.1.** For all \( k \in \mathbb{C}^2 \) the functions \( \{ e^{ib \cdot x} \mid b \in \Gamma^\# \} \) form a complete set of eigenfunctions for \( H_k(0, 0) \) in \( L^2(\mathbb{R}^2/\Gamma) \) satisfying

\[ H_k(0, 0) e^{ib \cdot x} = (i \nabla - k)^2 e^{ib \cdot x} = (b + k)^2 e^{ib \cdot x} = N_b(k) e^{ib \cdot x}. \]

Hence,

\[ \hat{\mathcal{F}}(0, 0) = \{ k \in \mathbb{C}^2 \mid N_b(k) = 0 \text{ for some } b \in \Gamma^\# \} = \bigcup_{b \in \Gamma^\#} \mathcal{N}_b = \bigcup_{b \in \Gamma^\#} \bigcup_{\nu \in \{1,2\}} \mathcal{N}_\nu(b). \]

This is the desired expression for \( \hat{\mathcal{F}}(0, 0) \). We next prove that this union is locally finite.

First observe that, since the lattice \( \Gamma^\# \) is discrete, there is a constant \( C > 0 \) such that \( |b - c| \geq C \) for all distinct elements \( b, c \in \Gamma^\# \). Thus, by Proposition 3.2.2(b), the distance
between any two distinct lines $\mathcal{N}_\nu(b)$ and $\mathcal{N}_\nu(c)$ is bounded below by

$$\text{dist}(\mathcal{N}_\nu(b), \mathcal{N}_\nu(c)) = \frac{1}{\sqrt{2}} |b - c| \geq \frac{C}{\sqrt{2}} > 0.$$ 

Now, let $U$ be an open bounded subset of $\mathbb{C}^2$. We claim that only a finite number of lines $\mathcal{N}_\nu(b)$ can intersect $U$. Indeed, suppose this is not the case. Then for at least one $\nu \in \{1, 2\}$ there is an infinite number of lines $\mathcal{N}_\nu(b)$ crossing $U$. In particular, there is at least one point of each of these lines inside $U$. By the above inequality, all these points are apart from each other by at least $C/\sqrt{2}$. Since we have an infinite number of such points inside $U$ this implies that $U$ is unbounded. But this is a contradiction. Therefore, only a finite number of lines $\mathcal{N}_\nu(b)$ can intersect $U$. Consequently, given $U$ there exits a finite set $B \subset \Gamma^\#$ such that

$$\hat{\mathcal{F}}(0, 0) \cap U = U \cap \bigcup_{b \in \Gamma^\#} \mathcal{N}_b = \bigcup_{b \in \Gamma^\#} \mathcal{N}_b \cap U = \bigcup_{b \in B} \mathcal{N}_b \cap U = \{ k \in U \mid N_0(k) = 0 \text{ for } b \in B \}.$$ 

That is, the union $\hat{\mathcal{F}}(0, 0)$ is locally finite. Furthermore, the curve $\hat{\mathcal{F}}(0, 0)$ is locally the zero set of a finite number of polynomials $N_b(k)$ and hence it is a complex analytic curve in $\mathbb{C}^2$. Clearly the same conclusion holds for $\mathcal{F}(0, 0)$ in $\mathbb{C}^2/\Gamma^\#$. Indeed, consider $\mathcal{F}(0, 0) \cap W$ for some open bounded subset $W$ in $\mathbb{C}^2/\Gamma^\#$. Then if we embed this set in $\hat{\mathcal{F}}(0, 0) \subset \mathbb{C}^2$ “around some $b \in \Gamma^\# \setminus \{0\}”, we obtain a finite number of defining equations for $\mathcal{F}(0, 0) \cap W$. This can be properly done by exploiting the covering property $\mathcal{F}(0, 0) \cong E(\mathcal{N}_0)$. To prove this relation we proceed as follows. Let $\mathcal{N}_0 - b$ be the translation of $\mathcal{N}_0$ by $-b \in \Gamma^\#$. Then, by Proposition 3.2.2(d) we have $\mathcal{N}_b = \mathcal{N}_0 - b$ for all $b \in \Gamma^\#$. Hence,

$$\mathcal{F}(0, 0) \cong E(\hat{\mathcal{F}}(0, 0)) = E\left( \bigcup_{b \in \Gamma^\#} \mathcal{N}_b \right) = \bigcup_{b \in \Gamma^\#} E(\mathcal{N}_b) = \bigcup_{b \in \Gamma^\#} E(\mathcal{N}_0 - d) = E(\mathcal{N}_0) = E(\mathcal{N}_1(0) \cup \mathcal{N}_2(0)) = E(\mathcal{N}_1(0) \cup E(\mathcal{N}_2(0)))$$

$$= E(\{(ik_2, k_2) \mid k_2 \in \mathbb{C}\}) \cup E(\{(-ik_2, k_2) \mid k_2 \in \mathbb{C}\})$$

$$= \{ (e^{ik_2(1,1)}, e^{ik_2(1,1)}) \mid k_2 \in \mathbb{C}\} \cup \{ (e^{ik_2(1,-1)}, e^{ik_2(-1,1)}) \mid k_2 \in \mathbb{C}\}.$$ 

In particular this shows that $\mathcal{F}(0, 0) \cong E(\mathcal{N}_0)$ and completes the proof of the theorem. \qed

Let us briefly describe what the free Fermi curve looks like. In the Figure 3.1 there is a sketch of the set of $(k_1, k_2) \in \hat{\mathcal{F}}(0, 0)$ for which both $ik_1$ and $k_2$ are real, for the case where
the lattice $\Gamma^#$ has points over the coordinate axes, that is, it has points of the form $(b_1,0)$ and $(0,b_2)$. Observe that, in particular, Proposition 3.2.2 yields

$$\mathcal{N}_1(0) \cap \mathcal{N}_2(b) = \{(i\theta_1(b),\theta_1(b))\},$$  
$$\mathcal{N}_1(-b) \cap \mathcal{N}_2(0) = \{(i\theta_2(-b),\theta_2(b))\},$$

the map $k \mapsto k + b$ maps $\mathcal{N}_1(0) \cap \mathcal{N}_2(b)$ to $\mathcal{N}_1(-b) \cap \mathcal{N}_2(0)$.

Recall that points in $\hat{\mathcal{F}}(0,0)$ that differ by elements of $\Gamma^#$ correspond to the same point in $\mathcal{F}(0,0)$. Thus, in the sketch on the left, we should identify the lines $k_2 = -b_2/2$ and $k_2 = b_2/2$ for all $b \in \Gamma^#$ with $b_2 \neq 0$, to get a pair of helices climbing up the outside of a cylinder, as illustrated by the figure on the right. The helices intersect each other twice on each cycle of the cylinder—once on the front half of the cylinder and once on the back half. Hence, viewed as a “manifold” (with singularities), the pair of helices are just two copies of $\mathbb{R}$ with points that corresponds to intersections identified. We can use $k_2$ as a coordinate in each copy of $\mathbb{R}$ and then the pairs of identified points are $k_2 = b_2/2$ and $k_2 = -b_2/2$ for all $b \in \Gamma^#$ with $b_2 \neq 0$ (see Figure 3.2).

So far we have only considered $k_2$ real. The full $\hat{\mathcal{F}}(0,0)$ is just two copies of $\mathbb{C}$ with $k_2$ as a coordinate in each copy, provided we identify the points $\theta_1(b) = \frac{1}{2}(-b_2 + ib_1)$ (in the first copy) and $\theta_2(b) = \frac{1}{2}(b_2 + ib_1)$ (in the second copy) for all $b \in \Gamma^#$ with $b_2 \neq 0$ (see Figure 3.3).

Figure 3.1: Sketch of $\hat{\mathcal{F}}(0,0)$ and $\mathcal{F}(0,0)$ when both $ik_1$ and $k_2$ are real.
To conclude this section we give a (global) defining equation for \( \tilde{F}(0, 0) \). For \( b \in \Gamma^\# \) set

\[
R_b(k) := \exp \left[ -\frac{2k \cdot b + k^2 - 1}{1 + b^2} + \frac{1}{2} \left[ \frac{2k \cdot b + k^2 - 1}{1 + b^2} \right]^2 \right].
\]

We have the following proposition.

**Proposition 3.2.3** (Global defining equation for \( \tilde{F}(0, 0) \)). The following equality holds:

\[
\tilde{F}(0, 0) = \left\{ k \in \mathbb{C}^2 \left| \prod_{b \in \Gamma^\#} \frac{(k + b)^2}{1 + b^2} R_b(k) = 0 \right. \right\},
\]

where the infinite product inside brackets converges to an entire function on \( \mathbb{C}^2 \).

Before we prove this proposition we remark that the above product is analogous to the canonical product associated to a sequence of complex numbers \( \{z_n\} \). Such product defines an entire function on \( \mathbb{C} \) which has a zero at each point \( z_n \) (see [13, p 302]).
Proof. We give only a sketch of the proof. By Theorem 2.5.1 we have
\[ \hat{F}(0,0) = \{ k \in \mathbb{C}^2 \mid F(k,0,0,0) = 0 \} \]
with
\[ F(k,0,0,0) = \det_3 \left( I + \frac{1}{\sqrt{I - \Delta}} u(k,0) \frac{1}{\sqrt{I - \Delta}} \right). \]
Write
\[ T := \frac{1}{\sqrt{I - \Delta}} u(k,0) \frac{1}{\sqrt{I - \Delta}} = \frac{1}{\sqrt{I - \Delta}} (-2ik \cdot \nabla + k^2 - I) \frac{1}{\sqrt{I - \Delta}} \]
and set
\[ \mathcal{R}_3(T) := \left( I + T \right) \exp \left[ \sum_{j=1}^{2} \frac{(-1)^j}{j} T^j \right] - I. \]
The eigenvalues of \( \mathcal{R}_3(T) \), which we denote by \( \{ \lambda_b(\mathcal{R}_3(T)) \}_{b \in \Gamma^*} \), are given by
\[ \lambda_b(\mathcal{R}_3(T)) = \frac{(k + b)^2}{1 + b^2} R_b(k) - 1. \]
Hence (see [15, Chapter 9] for details),
\[ F(k,0,0,0) = \det_3(I + T) = \det(I + \mathcal{R}_3(T)) = \prod_{b \in \Gamma^*} \left( 1 + \lambda_b(\mathcal{R}_3(T)) \right) = \prod_{b \in \Gamma^*} \frac{(k + b)^2}{1 + b^2} R_b(k). \]
Furthermore, according to Theorem 2.5.1 the function \( F(k,0,0,0) \) is analytic on \( \mathbb{C}^2 \). This proves the proposition.

3.3 The \( \varepsilon \)-tubes about the free Fermi curve

We now introduce real and imaginary coordinates in \( \mathbb{C}^2 \) and define \( \varepsilon \)-tubes about the free Fermi curve. We derive some properties of the \( \varepsilon \)-tubes as well.

For \( k \in \mathbb{C}^2 \) write
\[ k_1 = u_1 + iv_1 \quad \text{and} \quad k_2 = u_2 + iv_2, \]
where \( u_1, u_2, v_1 \) and \( v_2 \) are real numbers. Then,
\[ N_{b,\nu}(k) = (k_1 + b_1) + i(-1)^{\nu}(k_2 + b_2) \]
\[ = i(v_1 + (-1)^{\nu}(u_2 + b_2)) - (-1)^{\nu}(v_2 - (-1)^{\nu}(u_1 + b_1)), \]
so that

\[ |N_{b,\nu}(k)| = |v + (-1)^\nu(u + b)^\perp|, \]

where

\[ (y_1, y_2)^\perp := (y_2, -y_1). \]

Observe that \((y^\perp)^\perp = -y\) and \(|y^\perp| = |y|\). Furthermore, for any real number \(\lambda\) we have \((\lambda y)^\perp = \lambda y^\perp\).

Since \(N_b(k) = N_{b,1}(k)N_{b,2}(k)\), it follows that \(N_b(k) = 0\) if and only if

\[ v - (u + b)^\perp = 0 \quad \text{or} \quad v + (u + b)^\perp = 0. \]

Let \(2\Lambda\) be the length of the shortest nonzero "vector" in \(\Gamma^\#\). Then there is at most one \(b \in \Gamma^\#\) with \(|v + (u + b)^\perp| < \Lambda\) and at most one \(b \in \Gamma^\#\) with \(|v - (u + b)^\perp| < \Lambda\). Indeed, suppose there is another \(b' \neq b\) such that \(|v + (u + b')^\perp| < \Lambda\) or \(|v - (u + b')^\perp| < \Lambda\). Then,

\[ |b - b'| = |(b - b')^\perp| = |v \pm (u + b)^\perp - (v \pm (u + b')^\perp)| < \Lambda + \Lambda = 2\Lambda, \]

which contradicts the definition of \(\Lambda\). Thus, there is no such \(b'\) (see Figure 3.4).

![Figure 3.4: On the left: definition of \(\Lambda\). On the right: possible configuration.](image)

Let \(\varepsilon\) be a constant satisfying

\[ 0 < \varepsilon < \frac{\Lambda}{6}. \]

For \(\nu \in \{1, 2\}\) and \(b \in \Gamma^\#\) define the \(\varepsilon\)-tube about \(N_{\nu}(b)\) as

\[ T_{\nu}(b) := \{k \in \mathbb{C}^2 \mid |N_{b,\nu}(k)| = |v + (-1)^\nu(u + b)^\perp| < \varepsilon\}, \]
and the $\varepsilon$-tube about $N_b = N_1(b) \cup N_2(b)$ as

$$T_b := T_1(b) \cup T_2(b).$$

Since $(v + (u + b)^\perp) + (v - (u + b)^\perp) = 2v$, at least one of the factors $|v + (u + b)^\perp|$ or $|v - (u + b)^\perp|$ in $|N_b(k)|$ must always be greater or equal to $|v|$. If $k \notin T_b$ both factors are also greater or equal to $\varepsilon$. If $k \in T_b$ one factor is bounded by $\varepsilon$ and the other must lie within $\varepsilon$ of $|2v|$. Thus,

$$k \notin T_b \implies |N_b(k)| \geq \varepsilon|v|,$$

$$k \in T_b \implies |N_b(k)| \leq \varepsilon(2|v| + \varepsilon). \quad (3.3.1)$$

The pairwise intersection $T_b \cap T_{b'}$ is compact whenever $b \neq b'$. Here $T_b$ denotes the closure of $T_b$. Indeed, to prove this first observe that, the intersection $T_{b'}(b) \cap T_{b'}(b')$ is empty if $b \neq b'$ because, if it were not, we would have

$$|v + (-1)^\varepsilon(u + b)^\perp - v - (-1)^\varepsilon(u + b')^\perp| \leq 2\varepsilon < \frac{\Lambda}{3},$$

which contradicts

$$|v + (-1)^\varepsilon(u + b)^\perp - v - (-1)^\varepsilon(u + b')^\perp| = |b - b'| \geq 2\Lambda,$$

which is certainly true according to the definition of $\Lambda$. Furthermore, if $k \in T_1(b) \cap T_2(b')$ then

$$|u + \frac{1}{2}(b + b')| = \frac{1}{2}|v - (u + b)^\perp - v - (u + b')^\perp| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$|v - \frac{1}{2}(b - b')^\perp| = \frac{1}{2}|v - (u + b)^\perp + v + (u + b')^\perp| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which defines a compact set. Thus, the intersection

$$T_b \cap T_{b'} = (T_1(b) \cap T_2(b')) \cup (T_2(b) \cap T_1(b')) \quad (3.3.3)$$

is compact. Finally, it is easy to see that $T_b \cap T_{b'} \cap T_{b''}$ is empty for all distinct elements $b, b', b'' \in \Gamma^\#$. In fact, in view of (3.3.3),

$$T_b \cap T_{b'} \cap T_{b''} = \left[ T_1(b) \cap T_2(b') \cap T_1(b'') \right] \cup \left[ T_1(b) \cap T_2(b') \cap T_2(b'') \right]$$

$$\cup \left[ T_2(b) \cap T_1(b') \cap T_1(b'') \right] \cup \left[ T_2(b) \cap T_1(b') \cap T_2(b'') \right]$$

$$= \emptyset \cup \emptyset \cup \emptyset \cup \emptyset = \emptyset.$$
Figure 3.5: The ε-tubes about the free “lifted” Fermi curve.

If a point \( k \) belongs to the free Fermi curve the function \( N_b(k) \) vanishes for some \( b \in \Gamma^\# \).

To conclude this section we give a lower bound for this function when \((b, k)\) is away from the zero set.

**Proposition 3.3.1** (Lower bound for \(|N_b(k)|\)).

(a) If \(|b + u + v^\perp| \geq \Lambda \) and \(|b + u - v^\perp| \geq \Lambda \), then \(|N_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|)\).

(b) If \(|v| > 2\Lambda \) and \(k \in T_0\), then \(|N_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|)\) for all \(b \neq 0\) but at most one \(b \neq 0\). This exceptional \(\tilde{b}\) obeys \(|\tilde{b}| > |v|\) and \(||u + \tilde{b}^{}| - |v|| < \Lambda\).

(c) If \(|v| > 2\Lambda \) and \(k \in T_0 \cap T_d\) with \(d \neq 0\), then \(|N_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|)\) for all \(b \notin \{0, d\}\). Furthermore we have \(|d| > |v|\) and \(| |u + d| - |v|| < \Lambda\).

**Proof.** (a) By hypothesis, both factors in

\[ |N_b(k)| = |v + (u + b)^\perp| |v - (u + b)^\perp| \]

are greater or equal to \(\Lambda\). We now prove that at least one of the factors must also be greater or equal to \(\frac{\Lambda}{2}(|v| + |u + b|)\). Suppose that \(|v| \geq |u + b|\). Then, since

\[ (v + (u + b)^\perp) + (v - (u + b)^\perp) = 2v, \]

we have

\[ |N_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|) \]
at least one of the factors must also be greater or equal to
\[ |v| = \frac{1}{2}(|v| + |v|) \geq \frac{1}{2}(|v| + |u + b|). \]

Now suppose that \(|v| < |u + b|\). Then, since
\[ (v + (u + b)^{\perp}) - (v - (u + b)^{\perp}) = 2(u + b)^{\perp}, \]
at least one of the factors must also be greater or equal to
\[ |u + b| = \frac{1}{2}(|u + b| + |u + b|) > \frac{1}{2}(|v| + |u + b|). \]

All this together implies that
\[ |\mathcal{N}_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|), \]
which proves part (a).

(b) By hypothesis \(\varepsilon < \Lambda/6 < 2\Lambda < |v|\). Let \(k \in T_0\). Then, by (3.3.2),
\[ |\mathcal{N}_0(k)| \leq \varepsilon(2|v| + \varepsilon) < 3\varepsilon|v| < \frac{\Lambda}{2}|v|. \] (3.3.4)
Thus we have either \(|u + v^{\perp}| < \Lambda\) or \(|u - v^{\perp}| < \Lambda\) (otherwise apply part (a) to get a contradiction). Suppose that \(|u + v^{\perp}| < \Lambda\). Then there is no \(b \in \Gamma^{\#}\setminus\{0\}\) with \(|b + u + v^{\perp}| < \Lambda\) and there is at most one \(\tilde{b} \in \Gamma^{\#}\setminus\{0\}\) satisfying \(|\tilde{b} + u - v^{\perp}| < \Lambda\). This inequality implies \(|u + \tilde{b} - |v|| < \Lambda\). Furthermore, for this \(\tilde{b}\),
\[ |\tilde{b}| = |2v^{\perp} - (u + v^{\perp}) + (\tilde{b} + u - v^{\perp})| > 2|v| - 2\Lambda > |v|, \]
since \(-2\Lambda > -|v|\). Now suppose that \(|u - v^{\perp}| < \Lambda\). Then there is no \(b \in \Gamma^{\#}\setminus\{0\}\) obeying \(|b + u - v^{\perp}| < \Lambda\) and there is at most one \(\tilde{b} \in \Gamma^{\#}\setminus\{0\}\) satisfying \(|\tilde{b} + u + v^{\perp}| < \Lambda\). This inequality implies \(|u + \tilde{b} - |v|| < \Lambda\). Consequently,
\[ |\tilde{b}| = |2v^{\perp} - (v^{\perp} - u) - (v^{\perp} + \tilde{b} + u)| > 2|v| - 2\Lambda > |v|. \]
Finally observe that, if \(b \not\in \{0, \tilde{b}\}\) then \(|b + u + v^{\perp}| \geq \Lambda\) and \(|b + u - v^{\perp}| \geq \Lambda\). Hence, applying part (a) it follows that \(|\mathcal{N}_b(k)| \geq \frac{\Lambda}{2}(|v| + |u + b|)|. This proves part (b).

(c) As in the proof of part (b), if \(k \in T_0 \cap T_d\) then in addition to (3.3.4) we have
\[ |\mathcal{N}_d(k)| \leq \varepsilon(2|v| + \varepsilon) < 3\varepsilon|v| < \frac{\Lambda}{2}|v|. \]
Thus, applying part (b) we conclude that \(d\) must be the exceptional \(\tilde{b}\) of part (b). The statement of part (c) follows then from part (b). This completes the proof of the proposition. \(\square\)
3.4 Motivation and main results

Below we state our main results. The proofs come later divided in many steps.

In [4], the authors introduced a class of Riemann surfaces of infinite genus that are “asymptotic to” a finite number of complex lines joined by infinite many handles. These surfaces are constructed by pasting together a compact submanifold of finite genus, plane domains, and handles. More precisely, these surfaces can be decomposed into

\[ X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}, \]

where \( X^{\text{com}} \) is a compact submanifold with smooth boundary and finite genus, \( X^{\text{reg}} \) is a finite union of open “regular pieces”, and \( X^{\text{han}} \) is an infinite union of closed “handles”. All these components satisfy a number of geometric/analytic hypotheses stated in [4] that specify the asymptotic holomorphic structure of the surface. The class of surfaces obtained in this way yields an extension of the classical theory of compact Riemann surfaces that has analogues of many theorems of the classical theory.

The choice of geometric/analytic hypotheses was guided by two requirements. First, that the classical theory of compact Riemann surfaces could be developed in the new context. Secondly, that a number of interesting examples satisfy the hypotheses. In fact, it was proven in [4] that this new class of surfaces includes quite general hyperelliptic surfaces, heat curves (which are related to the Kadomcev-Petviashvili equation), and Fermi curves with zero magnetic potential \( \mathcal{F}(A = 0, V) \). In order to verify the geometric/analytic hypotheses for \( \mathcal{F}(0, V) \) the authors proved two “asymptotic” theorems similar to the ones we prove below. This is the main step needed to verify the geometric/analytic hypotheses. In this thesis we extend their results to Fermi curves \( \mathcal{F}(A, V) \) with “small” magnetic potential \( A \).

We have followed their strategy of analysis. The main idea is to consider the eigenvalue-eigenvector problem for \( H_k(A, V) \) for \( k \in \mathbb{C}^2 \) with large imaginary part as a perturbation of the problem for \( H_k(0, 0) \). When \( A \) is zero, they are able to prove asymptotic theorems for \( \mathcal{F}(0, V) \) for arbitrary large \( V \). When \( A \) is not zero, however, new difficulties arise due to the presence of the term \( A \cdot (i \nabla - k) \) in the Hamiltonian \( H_k \). When \( A \) is large, taking the imaginary part of \( k \in \mathbb{C}^2 \) arbitrarily large is not enough to control this term—it is not enough to make its contribution small and hence have the interacting Fermi curve as a perturbation of the free Fermi curve. (The term \( V \) in \( H_k \) is easily controlled by this
method.) However, as we shall see below, the proof can be implemented by assuming that $A$ is small.

Before we proceed we need to introduce some notation. For any $\varphi \in L^2(\mathbb{R}^2/\Gamma)$ define $\hat{\varphi} : \Gamma^\# \to \mathbb{C}$ as

$$\hat{\varphi}(b) := (\mathcal{F}\varphi)(b) := \frac{1}{|\Gamma|} \int_{\mathbb{R}^2/\Gamma} \varphi(x) e^{-ib \cdot x} \, dx,$$

where $|\Gamma| := \int_{\mathbb{R}^2/\Gamma} dx$. Then,

$$\varphi(x) = (\mathcal{F}^{-1}\hat{\varphi})(x) = \sum_{b \in \Gamma^\#} \hat{\varphi}(b) e^{ib \cdot x} \quad \text{and} \quad \|\varphi\|_{L^2(\mathbb{R}^2/\Gamma)} = |\Gamma|^{1/2} \|\hat{\varphi}\|_{L^2(\Gamma^\#)}.$$

Let $\rho$ be a positive constant and set

$$\mathcal{K}_\rho := \{ k \in \mathbb{C}^2 \mid |v| \leq \rho \}.$$

Consider the projection

$$pr : \mathbb{C}^2 \longrightarrow \mathbb{C},$$

$$(k_1, k_2) \longmapsto k_2,$$

and define

$$q := (i\nabla \cdot A) + A^2 + V.$$

Finally, recall that

$$T_\nu(b) = \{ k \in \mathbb{C}^2 \mid |N_{b,\nu}(k)| = |v + (-1)^\nu(u + b)^\perp| < \varepsilon \} \quad \text{and} \quad T_b := T_1(b) \cup T_2(b).$$

Clearly, the set $\mathcal{K}_\rho$ is invariant under the action of $\Gamma^\#$ and $\mathcal{K}_\rho / \Gamma^\#$ is compact. Hence, the image of $\hat{\mathcal{F}}(A, V) \cap \mathcal{K}_\rho$ under the exponential map $E : \hat{\mathcal{F}}(A, V) \to \mathcal{F}(A, V)$ is compact in $\mathcal{F}(A, V)$. It will essentially play the role of $X^{\text{com}}$ in the decomposition of $\mathcal{F}(A, V)$. Our first theorem characterizes the regular piece $X^{\text{reg}}$.

**Theorem 3.4.1** (The regular piece). Let $0 < \varepsilon < \Lambda/6$ and suppose that $A_1$, $A_2$ and $V$ are functions in $L^2(\mathbb{R}^2/\Gamma)$ obeying $\|b^2q(b)\|_{L^1(\Gamma^\#)} < \infty$ and $\|(1 + b^2)\hat{A}(b)\|_{L^1(\Gamma^\# \setminus \{0\})} < 2\varepsilon/63$. Then there is a constant $\rho = \rho_{\Lambda, \varepsilon, q, A}$ such that, for $\nu \in \{1, 2\}$, the projection $pr$ induces a biholomorphic map between

$$\left( \hat{\mathcal{F}}(A, V) \cap T_\nu(0) \right) \setminus \left( \mathcal{K}_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right)$$
and its image in \( C \). This image component contains
\[
\left\{ z \in C \mid 8|z| > \rho \text{ and } |z + (-1)^\nu \theta(b)| > \varepsilon \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\}
\]
and is contained in
\[
\left\{ z \in C \mid |z + (-1)^\nu \theta(b)| > \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right) \text{ for all } b \in \Gamma^\# \setminus \{0\} \right\},
\]
where \( \theta(b) = \frac{1}{2}((-1)^\nu b_2 + ib_1) \). Furthermore,
\[
pr^{-1} : \text{Image}(pr) \longrightarrow T_\nu(0),
\]
y \mapsto (-\beta^{(1,0)}_2 - i(-1)^\nu y - r(y), y),
\]
where \( \beta^{(1,0)}_2 \) is a constant given by (4.6.8) that depends only on \( \rho \) and \( A \),
\[
|\beta^{(1,0)}_2| < \frac{\varepsilon^2}{100\Lambda} \quad \text{and} \quad |r(y)| \leq \frac{\varepsilon^3}{50\Lambda^2} + \frac{C}{\rho},
\]
where \( C = C_{\Lambda,\varepsilon,q,A} \) is a constant.

Since \( T_b + c = T_{b+c} \) for all \( b, c \in \Gamma^\# \), the complement of \( E(\mathcal{F}(A, V) \cap \mathcal{K}_\rho) \) in \( \mathcal{F}(A, V) \) is the disjoint union of
\[
E\left((\mathcal{F}(A, V) \cap T_0) \setminus \left( \mathcal{K}_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right) \right)
\]
and
\[
\bigcup_{b \in \Gamma^\# \setminus \{0\}} E(\mathcal{F}(A, V) \cap T_0 \cap T_b).
\]
Basically, the first of the two sets will be the regular piece of \( \mathcal{F}(A, V) \), while the second set will be the handles. The map \( \Phi \) parametrizing the regular part will be the composition of the exponential map \( E \) with the inverse of the map discussed in the above theorem. The detailed information about the handles comes from our second main theorem.

**Theorem 3.4.2 (The handles).** Let \( 0 < \varepsilon < \Lambda/6 \) and suppose that \( A_1, A_2 \) and \( V \) are functions in \( L^2(\mathbb{R}^2/\Gamma) \) with \( \|b^2 \hat{g}(b)\|_{H^1(\Gamma^\#)} < \infty \) and \( \|(1+b^2)\hat{A}(b)\|_{H^1(\Gamma^\# \setminus \{0\})} < 2\varepsilon/63 \). Then, for every sufficiently large constant \( \rho \) and for every \( d \in \Gamma^\# \setminus \{0\} \) with \( 2|d| > \rho \), there are maps
\[
\phi_{d,1} : \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2}\} \longrightarrow T_1(0) \cap T_2(d),
\]
\[
\phi_{d,2} : \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \varepsilon\} \longrightarrow T_1(-d) \cap T_2(0),
\]
and a complex number \( t_d \) with \( |t_d| \leq \frac{C}{|d|^2} \) such that:
For $\nu \in \{1, 2\}$ the domain of the map $\phi_{d,\nu}$ is biholomorphic to its image, and the image contains
\[
\left\{ k \in \mathbb{C}^2 \mid |k_1 + i(-1)^\nu k_2| \leq \frac{\varepsilon}{8} \text{ and } |k_1 + (-1)^{\nu+1} d_1 - i(-1)^\nu (k_2 + (-1)^{\nu+1} d_2)| \leq \frac{\varepsilon}{8} \right\}.
\]
Furthermore,
\[
D\hat{\phi}_{d,\nu} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i(-1)^\nu & i(-1)^\nu \end{pmatrix} \left( I + O\left(\frac{1}{|d|^2}\right) \right)
\]
and
\[
\phi_{d,\nu}(0) = (i\theta_{\nu}(d), (-1)^{\nu+1}\theta_{\nu}(d)) + O\left(\frac{\varepsilon}{900}\right) + O\left(\frac{1}{\rho}\right).
\]

(ii) $\phi_{d,1}(T_1(0) \cap T_2(d) \cap \hat{F}(A,V)) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_d, |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2} \right\}$,
\[
\phi_{d,2}^{-1}(T_1(-d) \cap T_2(0) \cap \hat{F}(A,V)) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t_d, |z_1| \leq \frac{\varepsilon}{2} \text{ and } |z_2| \leq \frac{\varepsilon}{2} \right\}.
\]

(iii)
\[
\phi_{d,1}(z_1, z_2) = \phi_{d,2}(z_2, z_1) - d.
\]

These are the main new results of this thesis. Using these theorems one should be able to verify that $\hat{F}(A,V)$ satisfies the geometric/analytic hypotheses as was done in [4] for $\hat{F}(0,V)$. In the next section we outline the strategy for proving these results. The proofs are given in the next chapter divided in many steps.

We finally mention a small simplification (or modification) that we were able to make in the Theorem 3.4.1 (the regular piece). We shall not go into details here. We refer the reader to Chapter 5. First, recall an important property of $\hat{F}(A,V)$—namely, gauge invariance. Briefly, gauge invariance implies that $\hat{F}(A,V) = \hat{F}(A + \nabla \Psi, V)$, where $\Psi$ is function on $\mathbb{R}^2$ (under suitable hypotheses), and $\nabla \Psi$ is periodic with respect to $\Gamma$. In Chapter 5, by choosing a convenient gauge $\Psi$, we are able to indicate how to simplify the proof of Theorem 3.4.1 and improve some constants in it. After performing this gauge transformation “some terms vanish” and the analysis becomes simpler. We do not provide this “new” proof because it is essentially the same as the proof given below, up to some minor modifications. In fact, we believe that the simplifications introduced by the gauge will become clear after the reader gets familiar with the proof of the above results.
3.5 Strategy outline

Below we briefly describe the general strategy of analysis used to prove our results. It was applied by Feldman, Knörrer and Trubowitz in [4, §16] for studying the case where $A = 0$. We implement it in detail in the subsequent sections for $A \neq 0$.

Let us first introduce some notation and definitions. Observe that

$$H_k(A,V)\varphi = ((i\nabla + A - k)^2 + V)\varphi$$

$$= ((i\nabla - k)^2 + A \cdot (i\nabla - k) + (i\nabla - k) \cdot A + A^2 + V)\varphi$$

$$= ((i\nabla - k)^2 + A \cdot (i\nabla - 2k) + (i\nabla \cdot A) + A \cdot i\nabla + A^2 + V)\varphi$$

$$= ((i\nabla - k)^2 + 2A \cdot (i\nabla - k) + (i\nabla \cdot A) + A^2 + V)\varphi,$$

and write

$$H_k(A,V) = \Delta_k + h(k,A) + q(A,V)$$

with

$$\Delta_k := (i\nabla - k)^2, \quad h(k,A) := 2A \cdot (i\nabla - k) \quad \text{and} \quad q(A,V) := (i\nabla \cdot A) + A^2 + V.$$

For each finite subset $G$ of $\Gamma^\#$ set

$$G' := \Gamma^\# \setminus G \quad \text{and} \quad C^2_G := C^2 \setminus \bigcup_{b \in G'} N_b,$$

$$L^2_G := \text{span}\{e^{ib \cdot x} \mid b \in G\} \quad \text{and} \quad L^2_{G'} := \text{span}\{e^{ib \cdot x} \mid b \in G'\}.$$

To simplify the notation write $L^2$ in place of $L^2(\mathbb{R}^2/\Gamma)$, let $I$ be the identity operator on $L^2$, and let $\pi_G$ and $\pi_{G'}$ be the orthogonal projections from $L^2$ onto $L^2_G$ and $L^2_{G'}$, respectively. Then,

$$L^2 = L^2_G \oplus L^2_{G'} \quad \text{and} \quad I = \pi_G + \pi_{G'}.$$ 

For $k \in C^2_G$ define the partial inverse $(\Delta_k)^{-1}_G$ on $L^2$ as

$$(\Delta_k)^{-1}_G := \pi_G + \Delta_k^{-1}\pi_{G'}.$$ 

Its matrix elements are

$$(\Delta_k)^{-1}_G)_{b,c} := \left< e^{ib \cdot x}, (\Delta_k)^{-1}_G e^{ic \cdot x} \right>_{L^2} = \begin{cases} 
\delta_{b,c} & \text{if } c \in G, \\
\delta_{b,c} N_c(k) & \text{if } c \notin G,
\end{cases}$$

where $b,c \in \Gamma^\#$. 

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Here is the main idea: By definition, a point $k$ is in $\hat{\mathcal{F}}(A, V)$ if $H_k(A, V)$ has a nontrivial kernel in $L^2$. Hence, to study the part of the curve in the intersection of $\bigcup_{d' \in G} T_{d'}$ with $C^2 \setminus \bigcup_{b \in G'} T_b$ for some finite subset $G$ of $\Gamma^\#$ (see Figure 3.6), it is natural to look for a nontrivial solution of

$$\left(\Delta_k + h + q\right)(\psi_G + \psi_{G'}) = 0,$$

where $\psi_G \in L^2_G$ and $\psi_{G'} \in L^2_{G'}$. Equivalently, if we make the following (invertible) change of variables in $L^2$,

$$\left(\psi_G + \psi_{G'}\right) = (\Delta_k)^{-1}_G (\varphi_G + \varphi_{G'}),$$

where $\varphi_G \in L^2_G$ and $\varphi_{G'} \in L^2_{G'}$, we may consider the equation

$$\left(\Delta_k + h + q\right)\varphi_G + \left(I + (h + q)\Delta^{-1}_k\right)\varphi_{G'} = 0. \quad (3.5.1)$$

The projections of this equation onto $L^2_{G'}$ and $L^2_G$ are, respectively,

$$\pi_{G'}(h + q)\varphi_G + \pi_{G'}(I + (h + q)\Delta^{-1}_k)\varphi_{G'} = 0, \quad (3.5.2)$$

$$\pi_G(\Delta_k + h + q)\varphi_G + \pi_G(h + q)\Delta^{-1}_k\varphi_{G'} = 0. \quad (3.5.3)$$

Now define $R_{G'G'}$ on $L^2$ as

$$R_{G'G'} := \pi_{G'}(I + (h + q)\Delta^{-1}_k)\pi_{G'}.$$

Observe that $R_{G'G'}$ is the zero operator on $L^2_G$. Then, if $R_{G'G'}$ has a bounded inverse on $L^2_{G'}$, the equation (3.5.2) is equivalent to

$$\varphi_{G'} = -R_{G'G'}^{-1}\pi_{G'}(h + q)\varphi_G.$$

Substituting this into (3.5.3) yields

$$\pi_G(\Delta_k + h + q - (h + q)\Delta^{-1}_k R_{G'G'}^{-1}\pi_{G'}(h + q))\varphi_G = 0.$$

This equation has a nontrivial solution if and only if the (finite) $|G| \times |G|$ determinant

$$\det \left[ \pi_G(\Delta_k + h + q - (h + q)\Delta^{-1}_k R_{G'G'}^{-1}\pi_{G'}(h + q))\pi_G \right] = 0$$

or, equivalently, expressing all operators as matrices in the basis $\{ |\Gamma|^{-1/2} e^{ib \cdot x} | b \in \Gamma^\# \}$,

$$\det \left[ N_{d''}(k) \delta_{d',d''} + w_{d',d''} - \sum_{b,c \in G'} \frac{w_{d'',b}}{N_{b}(k)} (R_{G'G'}^{-1})_{b,c} w_{c,d''} \right]_{d',d'' \in G} = 0, \quad (3.5.4)$$

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where
\[ w_{b,c} := h_{b,c} + \hat{q}(b - c) = -2(c + k) \cdot \hat{A}(b - c) + \hat{q}(b - c). \]

Therefore, if \( R_{G'G'} \) has a bounded inverse on \( L^2_{G'} \)—which is in fact the case under suitable conditions—in the region under consideration we can study the Fermi curve in detail using the (local) defining equation (3.5.4). In order to implement this strategy we shall first derive a number of analytic estimates.

Figure 3.6: Sketch of \((\cup_{d' \in G} T_{d'}) \cap (C^2 \setminus \cup_{b \in \Gamma^* \setminus \{0\}} T_b)\) for \( G = \{0\} \) and \( G = \{0, d\} \).
3.6 Notation and remarks

We summarize here some notation and remarks that will be used henceforth.

(i) For $\nu \in \{1, 2\}$ define the (complementary) index $\nu'$ as

$$
\nu' := \nu - (-1)^\nu.
$$

Observe that

$$
\nu' = \begin{cases} 2 & \text{if } \nu = 1 \\ 1 & \text{if } \nu = 2 \end{cases}
$$

and $(-1)^{\nu'} = -(1)^{\nu'}$. 

(ii) Let $f(x)$ and $g(x)$ be multivariable functions and let $p$ be a real number. The notation

$$
f(x) = O(|x|^p)
$$

means that there is a constant $C > 0$ such that

$$
|f(x)| \leq C|x|^p
$$

for all $x$ in a suitable domain. Similarly, the statement

$$
O(f(x)) = O(g(x))
$$

is equivalent to say that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$
C_1|f(x)| \leq |g(x)| \leq C_2|f(x)|.
$$

(iii) Let $z$ and $w$ be vectors in $C^2$. We denote the length of $z$ by

$$
|z| := (z_1 \bar{z}_1 + z_2 \bar{z}_2)^{1/2} = (|z_1|^2 + |z_2|^2)^{1/2}.
$$

Note that

$$
z \cdot w := z_1 w_1 + z_2 w_2
$$

is not an inner product on $C^2$. However, we still have the property

$$
|z \cdot w| = |z_1 w_1 + z_2 w_2| \leq |z_1| |w_1| + |z_2| |w_2| = (|z_1|, |z_2|) \cdot (|w_1|, |w_2|) \leq |z| |w|.
$$

Here we have used the Schwarz inequality (for the inner product on $R^2$).
(iv) Let $T$ be a linear operator from $L^2_B$ to $L^2_C$ with $B, C \subset \Gamma^\#$. We denote its matrix elements in the basis \{\(|\Gamma|^{-1/2}e^{ib \cdot x} \mid b \in \Gamma^\#\}\} by

$$T_{b,c} := \langle e^{ib \cdot x} | \Gamma | \frac{1}{2}, T e^{ic \cdot x} | \Gamma | \frac{1}{2} \rangle_{L^2},$$

where $b \in B$ and $c \in C$. The operator $T$ represented as a matrix $[T_{b,c}]$ acts on $l^2(\Gamma^\#)$.

(v) Consider a linear operator $T : X \to Y$. The operator norm of $T$ is defined as

$$\|T\| := \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X}.$$

(vi) In general, we denote by

$$C_{X_1, \ldots, X_n}$$

a constant that depends only on $X_1, \ldots, X_n$. We may use the same symbol $C_{X_1, \ldots, X_n}$ to denote different constants that change from line to line in our calculations without further notice.

(vii) The following notation will be used whenever we consider vector-valued quantities. Let $\mathcal{X}$ be a Banach space and let $A = (A_1, \ldots, A_d) \in \mathcal{X}^d$ and $B = (B_1, \ldots, B_d) \in \mathcal{X}^d$. Then,

$$\|A\|_{\mathcal{X}} := (\|A_1\|_\mathcal{X}^2 + \cdots + \|A_d\|_\mathcal{X}^2)^{1/2} \quad \text{and} \quad A \cdot B := A_1 B_1 + \cdots + A_d B_d.$$

(viii) Finally, recall the Neumann series for bounded linear operators,

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j,$$

and the geometric series for complex numbers,

$$\frac{1}{1 - z} = \sum_{j=0}^{\infty} z^j,$$

which are convergent if $\|T\| < 1$ and $|z| < 1$. 

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Chapter 4

Weak magnetic potential

4.1 Invertibility of $R_{G'G'}$

A simple strategy for inverting $R_{G'G'}$ is the following. Since this operator has the form $I + T$, it is easily invertible if $\|T\| < 1$. In this section we manage to get this bound and prove that for a suitable choice of $G \subset \Gamma^\#$, large $|v|$, and weak magnetic potential, the operator $R_{G'G'}$ is invertible on $L_2^G$.

In general, for any $B, C \subset \Gamma^\#$ ($C$ such that $\Delta_k^{-1}\pi_C$ exists) define the operator $R_{BC}$ as

$$R_{BC} := \pi_B (I + (h + q)\Delta_k^{-1})\pi_C$$

$$= \pi_B \pi_C + \pi_B q \Delta_k^{-1}\pi_C + \pi_B (2A \cdot i\nabla)\Delta_k^{-1}\pi_C - \pi_B (2k \cdot A)\Delta_k^{-1}\pi_C. \quad (4.1.1)$$

Its matrix elements are

$$(R_{BC})_{b,c} = \delta_{b,c} + \frac{\hat{q}(b - c)}{N_c(k)} - \frac{2c \cdot \hat{A}(b - c)}{N_c(k)} - \frac{2k \cdot \hat{A}(b - c)}{N_c(k)}, \quad (4.1.2)$$

where $b \in B$ and $c \in C$. We shall first estimate the norm of the last three terms on the right hand side of (4.1.1). We begin with the following proposition.

**Proposition 4.1.1.** Let $k \in \mathbb{C}^2$ and let $B, C \subset \Gamma^\#$ with $C \subset \{b \in \Gamma^\# \mid N_b(k) \neq 0\}$. Then,

$$\|\pi_B q \Delta_k^{-1}\pi_C\| \leq \|\hat{q}\|_{\mu} \sup_{c \in C} \frac{1}{|N_c(k)|},$$

$$\|\pi_B (A \cdot i\nabla)\Delta_k^{-1}\pi_C\| \leq \|\hat{A}\|_{\mu} \sup_{c \in C} \frac{|c|}{|N_c(k)|},$$

$$\|\pi_B (k \cdot A)\Delta_k^{-1}\pi_C\| \leq \|\hat{A}\|_{\mu} |k| \sup_{c \in C} \frac{1}{|N_c(k)|}.$$
To prove this proposition we apply the following inequality, which we prove later in §4.9.

**Proposition 4.1.2.** Consider a linear operator $T : L^2_C \to L^2_B$ with matrix elements $T_{b,c}$. Then,

$$\|T\| \leq \max \left\{ \sup_{c \in C} \sum_{b \in B} |T_{b,c}|, \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \right\}.$$ 

**Proof of Proposition 4.1.1.** Write $T_1 := \pi_B q \Delta_k^{-1} \pi_C$. Then, in view of (4.1.1) and (4.1.2),

$$\sup_{c \in C} \sum_{b \in B} |(T_1)_{b,c}| \leq \sup_{c \in C} \sum_{b \in B} \left| \hat{q}(b - c) \right| \leq \sup_{c \in C} \frac{1}{|N_c(k)|} \left\| \hat{q} \right\|_1,$$

$$\sup_{b \in B} \sum_{c \in C} |(T_1)_{b,c}| \leq \sup_{b \in B} \sum_{c \in C} \left| \hat{q}(b - c) \right| \leq \sup_{c \in C} \frac{1}{|N_c(k)|} \left\| \hat{q} \right\|_1.$$ 

By Proposition 4.1.2, these estimates imply the first inequality.

Now, let $T_2 := \pi_B (A \cdot i \nabla) \Delta_k^{-1} \pi_C$ and $T_3 := \pi_B (k \cdot A) \Delta_k^{-1} \pi_C$. Similarly as above, the second and third inequalities follow from the estimates

$$\sup_{c \in C} \sum_{b \in B} |(T_2)_{b,c}| \leq \sup_{c \in C} \sum_{b \in B} \left| c \right| \left| \hat{A}(b - c) \right| \leq \sup_{c \in C} \frac{|c|}{|N_c(k)|} \left\| \hat{A} \right\|_1,$$

$$\sup_{b \in B} \sum_{c \in C} |(T_2)_{b,c}| \leq \sup_{b \in B} \sum_{c \in C} \left| c \right| \left| \hat{A}(b - c) \right| \leq \sup_{c \in C} \frac{|c|}{|N_c(k)|} \left\| \hat{A} \right\|_1,$$

and

$$\sup_{c \in C} \sum_{b \in B} |(T_3)_{b,c}| \leq \sup_{c \in C} \sum_{b \in B} \left| k \right| \left| \hat{A}(b - c) \right| \leq \sup_{c \in C} \frac{1}{|N_c(k)|} \left\| \hat{A} \right\|_1,$$

$$\sup_{b \in B} \sum_{c \in C} |(T_3)_{b,c}| \leq \sup_{b \in B} \sum_{c \in C} \left| k \right| \left| \hat{A}(b - c) \right| \leq \sup_{c \in C} \frac{1}{|N_c(k)|} \left\| \hat{A} \right\|_1.$$ 

This proves the proposition. \( \square \)

The key estimate for the existence of $R^{-1}_{G^G'}$ is given below.

**Proposition 4.1.3** (Estimate of $\|R_{SS} - \pi_S\|$). Let $k \in \mathbb{C}^2$ with $|u| \leq 2|v|$ and $|v| > 2\Lambda$. Suppose that $S \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon |v| \}$. Then,

$$\|R_{SS} - \pi_S\| \leq \|\hat{q}\|_1 \frac{1}{\varepsilon |v|} + \frac{14}{\varepsilon} \|\hat{A}\|_1.$$ 

(4.1.3)

If $A = 0$ the right hand side of (4.1.3) can be made arbitrarily small for any $q(0,V) = V$ by taking $|v|$ sufficiently large. If $A \neq 0$, however, we need to take $\|\hat{A}\|_1$ small to make that quantity less than 1. The term $\frac{14}{\varepsilon} \|\hat{A}\|_1$ in (4.1.3) comes from the estimate we have for $\|\pi_{G'} h \Delta_k^{-1} \pi_{G'}\|$. 

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Proof. By hypothesis, for all \( b \in S \),

\[
\frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon |v|}.
\]  (4.1.4)

We now show that, for all \( b \in S \),

\[
\frac{|b|}{|N_b(k)|} \leq \frac{4}{\varepsilon}.
\]  (4.1.5)

First suppose that \( |b| \leq 4|v| \). Then,

\[
\frac{|b|}{|N_b(k)|} \leq \frac{4|v|}{\varepsilon |v|} = \frac{4}{\varepsilon}.
\]

Now suppose that \( |b| \geq 4|v| \). Again, by hypothesis we have

\[
|u| \leq 2|v| \quad \text{and} \quad |v| > 2\Lambda > \Lambda/6 > \varepsilon.
\]

Hence,

\[
|v \pm (u + b)\perp| \geq |b| - |u| - |v| \geq |b| - 3|v| \geq |b| - \frac{3}{4}|b| = \frac{|b|}{4}.
\]

Consequently,

\[
\frac{|b|}{|N_b(k)|} = \frac{|b|}{|v + (u + b)\perp|} \leq \frac{|b|}{|b|} \frac{4}{|b|} = \frac{4}{|b|} \leq \frac{4}{\varepsilon}.
\]

This proves (4.1.5).

The expression for \( R_{SS} - \pi_S \) is given by (4.1.1). Observe that

\[
|k| = |u + iv| \leq |u| + |v| \leq 3|v|.
\]

Then, applying Proposition 4.1.1 and using (4.1.4) and (4.1.5) we obtain

\[
\|R_{SS} - \pi_S\| \leq (6|v|\| \hat{A}\|_{\ell^1} + \|\hat{q}\|_{\ell^1}) \sup_{b \in S} \frac{1}{|N_c(k)|} + 2\|\hat{A}\|_{\ell^1} \sup_{b \in S} \frac{|c|}{|N_c(k)|} \\
\leq (6|v|\| \hat{A}\|_{\ell^1} + \|\hat{q}\|_{\ell^1}) \frac{1}{\varepsilon |v|} + \frac{8}{\varepsilon} \|\hat{A}\|_{\ell^1} = \|\hat{q}\|_{\ell^1} \frac{1}{\varepsilon |v|} + \frac{14}{\varepsilon} \|\hat{A}\|_{\ell^1}.
\]

This is the desired inequality.

From the last proposition it follows easily that \( R_{SS} \) has a bounded inverse for large \( |v| \) and weak magnetic potential.

Lemma 4.1.4 (Invertibility of \( R_{SS} \)). Let \( k \in C^2 \),

\[
|u| \leq 2|v|, \quad |v| > \max \left\{ 2\Lambda, \frac{2}{\varepsilon} \right\}, \quad \|\hat{q}\|_{\ell^1} < \infty \quad \text{and} \quad \|\hat{A}\|_{\ell^1} < \frac{2}{63}\varepsilon.
\]

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Suppose that $S \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v|\}$. Then the operator $RSS$ has a bounded inverse with

$$\|RSS - \pi_S\| < \|\hat{q}\| \frac{1}{|v|} + \|\hat{A}\| \frac{14}{\varepsilon} < \frac{17}{18}$$

and

$$\|R^{-1}_{SS} - \pi_S\| < 18\|RSS - \pi_S\|.$$  

**Proof.** Write $RSS = \pi_S + T$ with $T = RSS - \pi_S$. Then, by Proposition 4.1.3,

$$\|T\| = \|RSS - \pi_S\| \leq \|\hat{q}\| \frac{1}{|v|} + \|\hat{A}\| \frac{14}{\varepsilon} < \frac{1}{2} + \frac{4}{9} = \frac{17}{18} < 1.$$  

Hence, the Neumann series for $R_{SS}^{-1} = (\pi_S + T)^{-1}$ converges (and is a bounded operator). Furthermore,

$$\|R_{SS}^{-1} - \pi_S\| = \|(\pi_S + T)^{-1} - \pi_S\| = \|(\pi_S + T)^{-1} - (\pi_S + T)^{-1}(\pi_S + T)\| = \|(\pi_S + T)^{-1}T\| \leq (1 - \|T\|)^{-1}\|T\| < 18\|RSS - \pi_S\|,$$

as was to be shown. \qed

Lemma 4.1.4 says that if $G$ is such that $G' \subset \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v|\}$, the operator $R_{G'G'}$ has a bounded inverse on $L^2_G$, for $|u| \leq 2|v|$, large $|v|$, and weak magnetic potential. We are now able to write (local) defining equations for $\tilde{F}(A, V)$ under such conditions.

### 4.2 Local defining equations

In this section we derive (local) defining equations for the Fermi curve. We begin with a simple proposition.

**Proposition 4.2.1.** Suppose either (i) or (ii) or (iii) where:

(i) $G = \{0\}$ and $k \in T_0 \setminus \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b$;

(ii) $G = \{0, d\}$ and $k \in T_0 \cap T_d$;

(iii) $G = \emptyset$ and $k \in C^2 \setminus \bigcup_{b \in \Gamma^\#} T_b$.

Then $G' = \Gamma^\# \setminus G = \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon|v|\}$. 

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Proof. The proposition follows easily if we observe that \( G' = \Gamma^# \setminus G \) and recall from (3.3.1) that
\[
k \notin T_b \implies |N_b(k)| \geq \varepsilon |v|.
\]

We now need some notation. Let \( B \) be a fundamental cell for \( \Gamma^# \subset \mathbb{R}^2 \) (see [12, p 310]). Then any vector \( u \in \mathbb{R}^2 \) can be written as
\[
u = \xi + u,
\]
for some \( \xi \in \Gamma^# \) and \( u \in B \). Define
\[
\alpha := \sup\{|u| \mid u \in B\}, \quad R := \max\left\{\alpha, 2\Lambda, \|\hat{q}\|_{L^1} \frac{2}{\varepsilon}\right\}, \quad K_R := \{k \in \mathbb{C}^2 \mid |v| \leq R\}.
\]

We first show that in \( \mathbb{C}^2 \setminus K_R \) the Fermi curve is contained in the union of \( \varepsilon \)-tubes about the free Fermi curve.

Proposition 4.2.2 (\( \hat{F}(A, V) \setminus K_R \) is contained in the union of \( \varepsilon \)-tubes).

\[
\hat{F}(A, V) \setminus K_R \subset \bigcup_{b \in \Gamma^#} T_b.
\]

Proof. Recall that \( \hat{F}(A, V) \) is invariant under the action of \( \Gamma^# \), that is, that \( k \in \hat{F}(A, V) \) if and only if \( k + \xi \in \hat{F}(A, V) \) for all \( \xi \in \Gamma^# \). Hence, given any \( k \in \hat{F}(A, V) \), we can always perform a change of coordinates \( k \to k - \xi \) for some suitable \( \xi \in \Gamma^# \) so that \( k - \xi \in \hat{F}(A, V) \) and \( \text{Re}(k - \xi) \in B \). Here \( \text{Re}(k) \in \mathbb{R}^2 \) denotes the real part of \( k \in \mathbb{C}^2 \). Thus, without loss of generality we may assume that \( \text{Re}(k) \in B \).

We now prove that any point outside the region \( K_R \) and outside the union of \( \varepsilon \)-tubes does not belong to \( \hat{F}(A, V) \). Suppose that \( k \in \mathbb{C}^2 \setminus (K_R \cup \bigcup_{b \in \Gamma^#} T_b) \) and recall that \( k \) is in \( \hat{F}(A, V) \) if and only if (3.5.1) has a nontrivial solution. If we choose \( G = \emptyset \) then \( G' = \Gamma^# \) and this equation reads
\[
R_{G'G'} \varphi_{G'} = 0.
\]

By Proposition 4.2.1(iii) we have \( G' = \Gamma^# = \{b \in \Gamma^# \mid |N_b(k)| \geq \varepsilon |v|\} \). Furthermore, since \( u \in B \) and \( |v| > R \geq \alpha \), it follows that \( |u| \leq \alpha < |v| < 2|v| \). Consequently, the operator \( R_{G'G'} \) has a bounded inverse by Lemma 4.1.4. Thus, the only solution of the above equation is \( \varphi_{G'} = 0 \). That is, there is no nontrivial solution of this equation and therefore \( k \notin \hat{F}(A, V) \). \( \square \)
We are left to study the Fermi curve inside the $\varepsilon$-tubes. There are two types of regions to consider: intersections and non-intersections of tubes. To study non-intersections we choose $G = \{0\}$ and consider the region $(T_0 \setminus \cup_{b \in \Gamma^\# \setminus \{0\}} T_b) \setminus \mathcal{K}_R$. For intersections we take $G = \{0, d\}$ for some $d \in \Gamma^\# \setminus \{0\}$ and consider $(T_0 \cap T_d) \setminus \mathcal{K}_R$ (see Figure 4.1). Observe that, since the tubes $T_b$ have the following translational property, $T_b + c = T_{b+c}$ for all $b, c \in \Gamma^\#$, and the curve $\hat{\mathcal{F}}(A, V)$ is invariant under the action of $\Gamma^\#$, there is no loss of generality in considering only the two regions above. Any other part of the curve can be reached by translation.

Recall that $G' = \Gamma^\# \setminus G$, and for $d', d'' \in G$ and $i, j \in \{1, 2\}$ set

$$
B_{ij}^{d'd''}(k; G) := -4 \sum_{b,c \in G'} \frac{\hat{A}_i(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b,c} \hat{A}_j(c - d''),
$$

$$
C_i^{d'd''}(k; G) := -2 \hat{A}_i(d' - d'') + 2 \sum_{b,c \in G'} \frac{\hat{q}(d' - b) - 2b \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b,c} \hat{A}_i(c - d'')
+ 2 \sum_{b,c \in G'} \frac{\hat{A}_i(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b,c} (\hat{q}(c - d'') - 2d'' \cdot \hat{A}(c - d'')),
$$

$$
C_0^{d'd''}(k; G) := \hat{q}(d' - d'') - 2d'' \cdot \hat{A}(d' - d'')
- \sum_{b,c \in G'} \frac{\hat{q}(d' - b) - 2b \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'G'}^{-1})_{b,c} (\hat{q}(c - d'') - 2d'' \cdot \hat{A}(c - d'')),
$$

(4.2.1)
Then,
\[ D_{d',d''}(k; G) := w_{d',d''} - \sum_{b,c \in G'} \frac{w_{d',b}}{N_0(k)} (R_{G'G'}^{-1})_{b,c} w_{c,d''} \]
\[ = \hat{q}(d' - d'') - 2(d'' + k) \cdot \hat{A}(d' - d'') \]
\[ - \sum_{b,c \in G'} (\hat{q}(d' - b) - 2(b + k) \cdot \hat{A}(d' - b)) \frac{(R_{G'G'}^{-1})_{b,c}}{N_0(k)} (\hat{q}(c - d'') - 2(d'' + k) \cdot \hat{A}(c - d'')) \]
\[ = B_{11}^{d',d''} k_1^2 + B_{22}^{d',d''} k_2^2 + (B_{12}^{d',d''} + B_{21}^{d',d''}) k_1 k_2 + C_0^{d',d''} k_1 + C_1^{d',d''} k_2 + C_2^{d',d''}. \]

Furthermore, we shall shortly prove the following property.

**Proposition 4.2.3.** For \( d', d'' \in G \) and \( i, j \in \{1, 2\} \), the functions \( B_{ij}^{d',d''}, C_i^{d',d''}, C_0^{d',d''} \) (and consequently \( D_{d',d''} \)) are analytic on \((T_0 \setminus \cup_{b \in \Gamma^* \setminus \{0\}} T_b) \setminus K_R \) and \((T_0 \cap T_d) \setminus K_R \) for \( G = \{0\} \) and \( G = \{0, d\} \), respectively.

Using the above functions we can write (local) defining equations for the Fermi curve.

**Lemma 4.2.4 (Local defining equations for \( \tilde{F}(A, V) \)).**

(i) Let \( G = \{0\} \) and \( k \in (T_0 \setminus \cup_{b \in \Gamma^* \setminus \{0\}} T_b) \setminus K_R \). Then \( k \in \tilde{F}(A, V) \) if and only if
\[ N_0(k) + D_{0,0}(k) = 0. \]

(ii) Let \( G = \{0, d\} \) and \( k \in (T_0 \cap T_d) \setminus K_R \). Then \( k \in \tilde{F}(A, V) \) if and only if
\[ (N_0(k) + D_{0,0}(k))(N_d(k) + D_{d,d}(k)) - D_{0,d}(k)D_{d,0}(k) = 0. \]

We now prove this lemma and then Proposition 4.2.3. The proof of this lemma is easy once we have that \( R_{G'G'} \) is invertible.

**Proof.** (i) First, by Proposition 4.2.1(i) we have \( G' = \Gamma^* \setminus \{0\} = \{b \in \Gamma^* \mid |N_b(k)| \geq \varepsilon|v|\} \). Furthermore, since \( k \in T_0 \), we have either \(|v - u^\perp| < \varepsilon \text{ or } |v + u^\perp| < \varepsilon\). In either case this implies
\[ |u| < \varepsilon + |v| < 2\Delta + |v| < 2|v|. \quad (4.2.2) \]

Hence, the operator \( R_{G'G'} \) has a bounded inverse by Lemma 4.1.4. Thus, in the region under consideration \( \tilde{F}(A, V) \) is given by (3.5.4):
\[ 0 = N_0(k) + w_{0,0} - \sum_{b,c \in G'} \frac{w_{0,b}}{N_0(k)} (R_{G'G'}^{-1})_{b,c} w_{c,0} = N_0(k) + D_{0,0}(k). \]

This proves part (i).
Similarly, by Proposition 4.2.1(ii), $G' = \Gamma^\# \setminus \{0, d\} = \{ b \in \Gamma^\# | |N_b(k)| \geq \varepsilon|v|\}$. Furthermore, since $k \in T_0 \cap T_d \subset T_0$, similarly as above we obtain (4.2.2). Thus, by Lemma 4.1.4 the operator $R_{G'G'}$ has a bounded inverse. Hence, in the region under consideration $\hat{F}(A,V)$ is given by (3.5.4):

$$(N_0(k) + D_{0,0}(k))(N_d(k) + D_{d,d}(k)) - D_{0,d}(k)D_{d,0}(k) = 0.$$ 

This proves part (ii) and completes the proof of the lemma.

As promised, here is the (sketchy) proof of Proposition 4.2.3.

**Proof of Proposition 4.2.3.** It suffices to show that $B_{ij}^{d''}, C_i^{d''}$ and $C_0^{d''}$ are analytic functions. This property follows from the fact that all the series involved in the definition of these functions are uniformly convergent sums of analytic functions (see [16, Theorem 4.1]). The argument is similar for all cases. We give only a sketch of the proof.

First observe that, in view of Lemma 4.1.4, for each $b, c \in G'$,

$$|(R_{G'G'}^{-1})_{b,c}| = \left|\left(\left(\pi_{G'} - (\pi_{G'} - R_{G'G'})^{-1}\right)\right)_{b,c}\right|$$

$$= \sum_{j=0}^{\infty} \left|\left((\pi_{G'} - R_{G'G'})^j\right)_{b,c}\right|$$

$$\leq \sum_{j=0}^{\infty} \left|\pi_{G'} - R_{G'G'}\right|^j < \sum_{j=0}^{\infty} \left(\frac{17}{18}\right)^j = 18.$$ 

Hence, the above sum converges uniformly by the Weierstrass M-test. Since $(\pi_{G'} - R_{G'G'})_{b,c}$ is an analytic function of $k$, so is $(R_{G'G'}^{-1})_{b,c}$. 

Now observe that $B_{ij}^{d''}, C_i^{d''}$ and $C_0^{d''}$ are given by sums of the form

$$\text{const} + \sum_{b,c \in G'} \frac{f(b, d')}{N_b(k)} (R_{G'G'}^{-1})_{b,c} g(c, d''),$$

where $f$ and $g$ are known functions. Furthermore, all the terms in these series are analytic functions, and the sum converges uniformly because of the uniform bounds

$$\frac{|b|}{|N_b(k)|} \leq \frac{4}{\varepsilon}, \quad \frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon|v|} \leq \frac{1}{\varepsilon R} \quad \text{and} \quad \left|(R_{G'G'}^{-1})_{b,c}\right| \leq \left\|R_{G'G'}^{-1}\right\| \leq 18$$

for all $b, c \in G'$, and because $f(\cdot, d')$ and $g(\cdot, d'')$ are in $l^1(\Gamma^\#)$ in all cases by hypothesis. Consequently, all the limits $B_{ij}^{d''}, C_i^{d''}$ and $C_0^{d''}$ are analytic functions. 

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To study in detail the defining equations above we shall estimate the asymptotic behaviour of the functions $B_{ij}^{d,d''}, C_i^{d,d''}, C_0^{d,d''}$ and $D_{d',d''}$ for large $|v|$. (We sometimes refer to these functions as coefficients.) Since all these functions have a similar form it is convenient to prove these estimates in a general setting and specialize them later. This is the contents of §4.4 and §4.5. We next introduce a change of variables in $C^2$ that will be useful for proving these bounds.

### 4.3 Change of coordinates

The following change of coordinates in $C^2$ will be useful for our analysis.

For $\nu \in \{1, 2\}$ and $d', d'' \in G$ define the functions $w_{\nu,d'}, z_{\nu,d'} : C^2 \to \mathbb{C}$ as

\[
\begin{align*}
w_{\nu,d'}(k) &:= k_1 + d_1' + i(-1)^\nu(k_2 + d_2'), \\
&= \nu,d' \in C^2 \to \mathbb{C}
\end{align*}
\]

(4.3.1)

Observe that, the transformation $(k_1, k_2) \mapsto (w_{\nu,d'}, z_{\nu,d'})$ is just a translation composed with a rotation. Furthermore, if $k \in T_\nu(d') \setminus \mathcal{K}_R$ then $|w_{\nu,d'}(k)|$ is “small” and $|z_{\nu,d'}(k)|$ is “large”.

Indeed,

\[
|w_{\nu,d'}(k)| = |N_{d',\nu}(k)| < \varepsilon \quad \text{and} \quad |z_{\nu,d'}(k)| = |N_{d',\nu}(k)| \geq |v| > R.
\]

Define also

\[
\begin{align*}
J_{\nu,d''}^d &:= \frac{1}{4}(B_{11}^{d,d''} - B_{22}^{d,d''} + i(-1)^\nu(B_{12}^{d,d''} + B_{21}^{d,d''})) , \\
K_{d,d''}^d &:= \frac{1}{2}(B_{11}^{d,d''} + B_{22}^{d,d''}) , \\
L_{\nu,d''}^d &:= -d_1' B_{11}^{d,d''} - i(-1)^\nu d_2' B_{22}^{d,d''} - \frac{1}{2}(d_2' - i(-1)^\nu d_1')(B_{12}^{d,d''} + B_{21}^{d,d''}) + \frac{1}{2}(C_1^{d,d''} + i(-1)^\nu C_2^{d,d''}), \\
M_{d,d''}^d &:= d_1' B_{11}^{d,d''} + d_2' B_{22}^{d,d''} + d_1 d_2' (B_{12}^{d,d''} + B_{21}^{d,d''}) - d_1 C_1^{d,d''} - d_2 C_2^{d,d''} + C_0^{d,d''},
\end{align*}
\]

where $J_{\nu,d''}^d, K_{d,d''}^d, L_{\nu,d''}^d$ and $M_{d,d''}^d$ are functions of $k \in C^2$ that also depend on the choice of $G \subset \Gamma^\#$. Using these functions we can express $N_{d'}(k) + D_{d',d''}(k)$ and $D_{d',d''}(k)$ as follows.

**Proposition 4.3.1.** Let $\nu \in \{1, 2\}$ and let $d', d'' \in G$. Then,

\[
N_{d'} + D_{d',d''} = J_{\nu,d''}^d w_{\nu,d'}^2 + J_{\nu,d''}^d z_{\nu,d'}^2 + (1 + K_{d,d''}^d) w_{\nu,d'} z_{\nu,d'} + L_{\nu,d''}^d w_{\nu,d'} + L_{\nu,d''}^d z_{\nu,d'} + M_{d,d''}^d
\]

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and
\[ D_{\nu'}^{d'\nu''} = J_{\nu'}^{d'\nu''} \nu, \nu' + J_{\nu'}^{d'\nu''} \nu, \nu' + K^{d'\nu''} \nu, \nu' + L_{\nu'}^{d'\nu''} \nu, \nu' + M^{d'\nu''}. \]

Furthermore,
\[ J_{\nu'}^{d'\nu''}(k) = -\sum_{b,c \in G'} \frac{(1, -i(-1)^\nu) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'}^{-1})_{b,c} (1, -i(-1)^\nu) \cdot \hat{A}(c - d''), \]
\[ K^{d'\nu''}(k) = -2 \sum_{b,c \in G'} \frac{\hat{A}(d' - b) \cdot \hat{A}(c - d'')} {N_b(k)} (R_{G'}^{-1})_{b,c} (1, -i(-1)^\nu) \cdot \hat{A}(c - d'') \]
\[ + \sum_{b,c \in G'} \frac{(1, i(-1)^\nu) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'}^{-1})_{b,c} (\hat{q}(c - d'') + 2(d' - d'') \cdot \hat{A}(c - d'')) \]
\[ - (1, i(-1)^\nu) \cdot \hat{A}(d' - d''), \]

and
\[ M^{d'\nu''}(k) = -\sum_{b,c \in G'} \frac{\hat{q}(d' - b) + 2(d' - b) \cdot \hat{A}(d' - b)}{N_b(k)} (R_{G'}^{-1})_{b,c} \hat{q}(c - d'') \]
\[ + \hat{q}(d' - d'') + 2(d' - d'') \cdot \hat{A}(d' - d''). \]

\textbf{Proof.} To simplify the notation write
\[ w = w_{\nu', \nu'}, \quad z = z_{\nu', \nu'}, \quad B_{ij} = B_{ij}^{d'\nu''} \quad \text{and} \quad C_i = C_i^{d'\nu''}. \]

First observe that, in view of (4.3.1),
\[ N_{d''} = (k_1 + d'_1 + i(-1)^\nu(k_2 + d'_2))(k_1 + d'_1 - i(-1)^\nu(k_2 + d'_2)) = wz. \]

Furthermore,
\[ k_1 = \frac{1}{2}(w + z) - d'_1, \]
\[ k_2 = \frac{(-1)^\nu}{2i}(w - z) - d'_2, \]
\[ k_1^2 = \frac{1}{4}(w^2 + z^2) + \frac{1}{2}wz - d'_1(w + z) + d'_1^2, \]
\[ k_2^2 = -\frac{1}{4}(w^2 + z^2) + \frac{1}{2}wz + i(-1)^\nu d'_2(w - z) + d'_2^2, \]
\[ k_1k_2 = \frac{i(-1)^\nu}{4}(z^2 - w^2) - \frac{1}{2}(d'_2 - i(-1)^\nu d'_1)w - \frac{1}{2}(d'_2 + i(-1)^\nu d'_1) + d'_1d'_2. \]
Hence,

\[ D_{d',d''} = B_{11}k_1^2 + B_{22}k_2^2 + (B_{12} + B_{21})k_1k_2 + C_1k_1 + C_2k_2 + C_0 \]

\[ = \frac{1}{4}(B_{11} - B_{22} - i(-1)^\nu(B_{12} + B_{21}))w^2 + \frac{1}{4}(B_{11} - B_{22} + i(-1)^\nu(B_{12} + B_{21}))z^2 \]

\[ + \left( -d_1'B_{11} + i(-1)^\nu d_2'B_{22} - \frac{1}{2}(d_2' - i(-1)^\nu d_1')(B_{12} + B_{21}) + \frac{1}{2}(C_1 - i(-1)^\nu C_2) \right)w \]

\[ + \left( -d_1'B_{11} + i(-1)^\nu d_2'B_{22} - \frac{1}{2}(d_2' + i(-1)^\nu d_1')(B_{12} + B_{21}) + \frac{1}{2}(C_1 + i(-1)^\nu C_2) \right)z \]

\[ + d_1^2B_{11} + d_2^2B_{22} + d_1'd_2'(B_{12} + B_{21}) - d_1'C_1 - d_2'C_2 + C_0 + \frac{1}{2}(B_{11} + B_{22})wz \]

\[ = J_{\nu}^{d,d'} w^2 + J_{\nu}^{d,d'} z^2 + K^{d,d'} wz + L_{\nu}^{d,d'} w + L_{\nu}^{d,d'} z + M^{d,d'} . \]

This proves the first claim. Consequently,

\[ N_{d'} + D_{d',d''} = J_{\nu}^{d,d'} w^2 + J_{\nu}^{d,d'} z^2 + (1 + K^{d,d'}) wz + L_{\nu}^{d,d'} w + L_{\nu}^{d,d'} z + M^{d,d'} , \]

which proves the second claim.

Now, again to simplify the notation write

\[ fg = \sum_{b,c \in G'} \hat{f}(b,d') \hat{g}(c,d'') \]

that is, to represent sums of this form suppress the summation and the other factors. Note that \( fg \neq gf \) according to this notation. Then, substituting (4.2.1) into the definitions of \( J_{\nu}^{d,d''} , K^{d,d''}, L_{\nu}^{d,d''} \) and \( M^{d,d''} \) we obtain

\[ J_{\nu}^{d,d''} = \frac{1}{4}(B_{11} - B_{22} + i(-1)^\nu(B_{12} + B_{21})) \]

\[ = -A_1A_1 + A_2A_2 - i(-1)^\nu(A_1A_2 + A_2A_1) \]

\[ = (A_1 - i(-1)^\nu A_2)(-A_1 + i(-1)^\nu A_2) \]

\[ = -((1, -i(-1)^\nu) \cdot A) ((1, -i(-1)^\nu) \cdot A) \]

\[ = -\sum_{b,c \in G'} \frac{(1, -i(-1)^\nu) \cdot \hat{A}(d' - b) - (R_{G'G'}^{-1})_{b,c} (1, -i(-1)^\nu) \cdot \hat{A}(c - d'')}{N_b(k)} \]

and

\[ K^{d,d''} = \frac{1}{2}(B_{11} + B_{22}) = -2(A_1A_1 + A_2A_2) \]

\[ = -2\sum_{b,c \in G'} \frac{\hat{A}(d' - b) \cdot \hat{A}(c - d'')}{N_b(k)} (R_{G'G'}^{-1})_{b,c} \]

and

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\[ L^{d'',d'}_\nu = -d'_1 B_{11} - i(-1)^\nu d'_2 B_{22} - \frac{1}{2}(d'_2 + i(-1)^\nu d'_1)(B_{12} + B_{21}) + \frac{1}{2}(C_1 + i(-1)^\nu C_2) \]

\[ = 4d'_1 A_1 A_1 + 4i(-1)^\nu d'_2 A_2 A_2 + 2d'_2 (A_1 A_2 + A_2 A_1) + 2i(-1)^\nu d'_1 (A_1 A_2 + A_2 A_1) \]

\[ + (q - 2b \cdot A)A_1 + (A_1 A_2 + A_2 A_1) + i(-1)^\nu (q - 2b \cdot A) A_2 + i(-1)^\nu (q - 2d' \cdot A) \]

\[ - (\hat{A}_1 (d' - d'') + i(-1)^\nu \hat{A}_2 (d' - d'')) \]

\[ = 2(2d'_1 - d''_1) A_1 A_1 + 2i(-1)^\nu (2d'_2 - d''_2) A_2 A_2 + 2(d'_2 - d''_2 + i(-1)^\nu d'_1) A_1 A_2 \]

\[ + 2(d'_2 + i(-1)^\nu d'_1 - i(-1)^\nu d'_1) A_2 A_1 + (q - 2b \cdot A) A_1 + A_1 q \]

\[ + i(-1)^\nu (q - 2b \cdot A) A_2 + i(-1)^\nu A_2 q - (1, i(-1)^\nu) \cdot \hat{A}(d' - d'') \]

\[ = 2(d'_1 A_1 + d'_2 A_2)(A_1 + i(-1)^\nu A_2) + 2(1, i(-1)^\nu) \cdot A((d' - d'') \cdot A) \]

\[ + (q - 2b \cdot A)(A_1 + i(-1)^\nu A_2) + (A_1 + i(-1)^\nu A_2) q - (1, i(-1)^\nu) \cdot \hat{A}(d' - d'') \]

\[ = (q + 2(d - b) \cdot A)(1, i(-1)^\nu) \cdot A + (1, i(-1)^\nu) \cdot A(q + 2(d - d') \cdot A) \]

\[ - (1, i(-1)^\nu) \cdot \hat{A}(d' - d'') \]

so that

\[ L^{d',d''}_\nu = \sum_{b,c \in G'} \frac{\hat{q}(d' - b) + 2(d' - b) \cdot \hat{A}(d' - b)}{N_b(k)} (R^{-1}_{G'G'})_{b,c}(1, i(-1)^\nu) \cdot \hat{A}(c - d'') \]

\[ + \sum_{b,c \in G'} \frac{(1, i(-1)^\nu) \cdot \hat{A}(d' - b)}{N_b(k)} (R^{-1}_{G'G'})_{b,c}(\hat{q}(c - d'') + 2(d' - d'') \cdot \hat{A}(c - d'')) \]

\[ - (1, i(-1)^\nu) \cdot \hat{A}(d' - d'') \]

\[ M^{d',d''} = d''_1 B_{11} + d''_2 B_{22} + d'_1 d'_2 (B_{12} + B_{21}) - d'_1 C_1 - d''_2 C_2 + C_0 \]

\[ = -4(d''_1 A_1 A_1 + d''_2 A_2 A_2 + d'_1 d'_2 A_1 A_2 + d'_1 d'_2 A_2 A_1) + 2d' \cdot \hat{A}(d' - d'') \]

\[ - 2d'_1 (q - 2b \cdot A) A_1 - 2d'_1 (q - 2d' \cdot A) - 2d'_2 (q - 2b \cdot A) A_2 - 2d'_2 A_2 (q - 2d' \cdot A) \]

\[ - (q - 2b \cdot A)(q - 2d' \cdot A) + \hat{q}(d' - d'') + 2d' \cdot \hat{A}(d' - d'') \]

\[ = -4(d'' \cdot A)(d' \cdot A) - 2(q - 2b \cdot A)(d' \cdot A) - 2(d' \cdot A)(q - 2d' \cdot A) \]

\[ - (q - 2b \cdot A)(q - 2d' \cdot A) + \hat{q}(d' - d'') + 2(d' - d'') \cdot \hat{A}(d' - d'') \]

\[ = -qq - 2(d - b) \cdot A q + \hat{q}(d' - d'') + 2(d' - d'') \cdot \hat{A}(d' - d'') \]

\[ = - \sum_{b,c \in G'} \frac{\hat{q}(d' - b) + 2(d' - b) \cdot \hat{A}(d' - b)}{N_b(k)} (R^{-1}_{G'G'})_{b,c} \hat{q}(c - d'') \]

\[ + \hat{q}(d' - d'') + 2(d' - d'') \cdot \hat{A}(d' - d''). \]

We next use this change of variables for deriving asymptotics for certain functions.
4.4 Asymptotics for the coefficients

Let \( f \) and \( g \) be functions on \( \Gamma^# \) and for \( k \in \mathbb{C}^2 \) and \( d', d'' \in G \) set
\[
\Phi_{d',d''}(k; G) := \sum_{b,c \in G'} \frac{f(d' - b)}{N_b(k)} R_{G'}^{-1} b \cdot g(c - d').
\] (4.4.1)

In this section we study the asymptotic behaviour of the function \( \Phi_{d',d''}(k) \) for \( k \) in the union of \( \varepsilon \)-tubes with large \(|v|\). We first give all the statements and then the proofs.

Reset the constant \( R \) as
\[
R := \max \left\{ 1, \alpha, 2\Lambda, 140 \| \hat{A} \|_{l^1}, \| (1 + b^2) \hat{q}(b) \|_{l^1} \frac{4}{\varepsilon} \right\},
\] (4.4.2)
and make the following hypothesis.

**Hypothesis 4.4.1.**
\[
\| b^2 \hat{q}(b) \|_{l^1} < \infty \quad \text{and} \quad \| (1 + b^2) \hat{A}(b) \|_{l^1} < \frac{2}{63} \varepsilon.
\]

Our first lemma provides and expansion for \( \Phi_{d',d''}(k) \) “in powers of \( 1/|z_{\mu,d'}(k)| \).”

**Lemma 4.4.1 (Asymptotics for \( \Phi_{d',d''}(k) \)).** Under Hypothesis 4.4.1, let \( \nu \in \{1, 2\} \) and let \( f \) and \( g \) be functions on \( \Gamma^# \) with \( \| b^2 f(b) \|_{l^1} < \infty \) and \( \| b^2 g(b) \|_{l^1} < \infty \). Suppose either (i) or (ii) where:

(i) \( G = \{0\} \) and \( k \in (T_{\nu}(0) \setminus \cup_{b \in G'} T_b) \setminus \mathcal{K}_R \);

(ii) \( G = \{0, d\} \) and \( k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus \mathcal{K}_R \).

Then, for \( (\mu, d') = (\nu, 0) \) if (i) or \( (\mu, d') \in \{ (\nu, 0), (\nu', d) \} \) if (ii),
\[
\Phi_{d',d''}(k) = \alpha^{(1)}_{\mu,d'}(k) + \alpha^{(2)}_{\mu,d'}(k) + \alpha^{(3)}_{\mu,d'}(k),
\]
where for \( 1 \leq j \leq 2 \),
\[
|\alpha^{(j)}_{\mu,d'}(k)| \leq \frac{C_j}{(2|z_{\mu,d'}(k)| - R)^j} \quad \text{and} \quad |\alpha^{(3)}_{\mu,d'}(k)| \leq \frac{C_3}{|z_{\mu,d'}(k)| R^2},
\]
where \( C_j = C_{j;\Lambda,A,q,f,g} \) and \( C_3 = C_{3;\Lambda,A,q,f,g} \) are constants. Furthermore, the functions \( \alpha^{(j)}_{\mu,d'}(k) \) are given by (4.4.25) and (4.4.28) and are analytic in the region under consideration.
Below we have more information about the function $\alpha^{(1)}_{\mu,d'}(k)$.

**Lemma 4.4.2** (Asymptotics for $\alpha^{(1)}_{\mu,d'}(k)$). Consider the same hypotheses of Lemma 4.4.1. Then, for $(\mu, d') = (\nu, 0)$ if (i) or $(\mu, d') \in \{(\nu, 0), (\nu', d)\}$ if (ii),

$$z_{\mu,d'}(k) \alpha^{(1)}_{\mu,d'}(k) = \alpha^{(1,0)}_{\mu,d'} + \alpha^{(1,1)}_{\mu,d'}(w(k)) + \alpha^{(1,2)}_{\mu,d'}(k) + \alpha^{(1,3)}_{\mu,d'}(k),$$

where $\alpha^{(1,0)}_{\mu,d'}$ is a constant given by (4.4.38), and the remaining functions $\alpha^{(1,j)}_{\mu,d'}$ are given by (4.4.39). Furthermore, for $0 \leq j \leq 2$,

$$|\alpha^{(1,j)}_{\mu,d'}| \leq C_j$$

and

$$|\alpha^{(1,3)}_{\mu,d'}| \leq \frac{C_3}{2|z_{\mu,d'}(k)| - R},$$

where $C_j = C_{j; A, f, g}$ and $C_3 = C_{3; A, f, g}$ are constants given by (4.4.40).

The next lemma estimates the decay of $\Phi_{d', d''}(k)$ with respect to $z_{\nu', d}(k)$ for $d' \neq d''$.

**Lemma 4.4.3** (Decay of $\Phi_{d', d''}(k)$ for $d' \neq d''$). Under Hypothesis 4.4.1, let $\nu \in \{1, 2\}$ and let $f$ and $g$ be functions on $\Gamma^\#$ with $\|b^2f(b)\|_{l_1} < \infty$ and $\|b^2g(b)\|_{l_1} < \infty$. Suppose further that $G = \{0, d\}$ and $k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus K_R$. Then, for $d', d'' \in G$ with $d' \neq d''$,

$$|\Phi_{d', d''}(k)| \leq \frac{C_{\Gamma^\#, \nu; f, g}}{|z_{\nu', d}(k)|^{3-10^{-1}}},$$

where $C_{\Gamma^\#, \nu; f, g}$ is a constant.

The above lemmas are the main statements of this section. Before we move to the proofs we give one more proposition that provides relations between the quantities $|v|$, $|k_2|$, $|z_{\nu,0}(k)|$ and $|d|$ for $k$ in the $\varepsilon$-tubes with large $|v|$.

**Proposition 4.4.4.** For $\nu \in \{1, 2\}$ we have:

(i) Let $k \in T_{\nu}(0) \setminus K_R$. Then,

$$\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|}$$

and

$$\frac{1}{|z_{\nu, d}(k)|} \leq \frac{1}{|d|} \leq \frac{8}{|v|}.$$

(ii) Let $k \in (T_{\nu}(0) \cap T_{\nu'}(d)) \setminus K_R$. Then,

$$\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|},$$

$$\frac{1}{|z_{\nu, d}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu', d}(k)|}$$

and

$$\frac{1}{2|z_{\nu', d}(k)|} \leq \frac{1}{|d|} \leq \frac{2}{|z_{\nu', d}(k)|}.$$

This proposition will be used several times henceforth. We next prove all the above statements.
Proof of Proposition 4.4.4

Proof of Proposition 4.4.4. We first derive a more general inequality and then we prove parts (i) and (ii). First observe that, if \( k \in T_{\mu}(d') \setminus K_R \) then

\[
|v + (-1)^\mu(u + d')| = |N_{\mu,d}(k)| < \varepsilon < |v|.
\]

Hence,

\[
|v| \leq |2v - (v + (-1)^\mu(u + d'))| \leq 3|v|.
\]

But

\[
|2v - (v + (-1)^\mu(u + d'))| = |v - (-1)^\mu(u + d')|
\]

\[
= |k_1 + d'_1 - i(-1)^\mu(k_2 + d'_2)| = |z_{\mu,d'}(k)|.
\]

Thus,

\[
|v| \leq |z_{\mu,d'}(k)| \leq 3|v|.
\]

Therefore,

\[
\frac{1}{|z_{\mu,d'}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\mu,d'}(k)|}. \tag{4.4.3}
\]

(i) The first inequality of part (i) follows from the above estimate setting \((\mu, d') = (\nu, 0)\).

To prove the second inequality observe that, since \(|v| > R \geq 2\Lambda > 12\varepsilon\) by hypothesis and \(|v| \leq |z_{\nu,0}(k)|\) by (4.4.3), on the one hand we have

\[
\frac{1}{4}|v| \leq \frac{11}{42}|v| = |v| - \frac{11}{12}|v| \leq |v| - \frac{1}{6}\varepsilon \leq |v| - |z_{\nu,0}(k)| - |k_1 + i(-1)^\nu k_2|
\]

\[
\leq |z_{\nu,0}(k) - k_1 - i(-1)^\nu k_2| = 2|k_2|.
\]

On the other hand, since \(|z_{\nu,0}(k)| < 3|v|\) by (4.4.3),

\[
|k_2| = |2i(-1)^\nu k_2| = |k_1 + i(-1)^\nu k_2 - (k_1 - i(-1)^\nu k_2)|
\]

\[
= |k_1 + i(-1)^\nu k_2 - z_{\nu,0}(k)| \leq \varepsilon + 3|v| \leq 4|v|.
\]

Combining these estimates we obtain the second inequality of part (i).

(ii) Similarly, in view of (4.4.3), if \( k \in T_{\mu}(d') \setminus K_R \) for \((\mu, d') \in \{(\nu, 0), (\nu', d)\}\) then

\[
\frac{1}{|z_{\nu,0}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu,0}(k)|} \quad \text{and} \quad \frac{1}{|z_{\nu',d}(k)|} \leq \frac{1}{|v|} \leq \frac{3}{|z_{\nu',d}(k)|}. \tag{4.4.4}
\]
These are the first two inequalities of part (ii). Now, since
\[ z_{\nu',d}(k) = k_1 - i(-1)^\nu k_2 + d_1 - i(-1)^\nu d_2 \]
\[ = z_{\nu',0}(k) + d_1 - i(-1)^\nu d_2 = w_{\nu',0}(k) + d_1 - i(-1)^\nu d_2, \]
\[ |w_{\nu',0}(k)| < \varepsilon, \text{ and } |d_1 - i(-1)^\nu d_2| = |d|, \]
it follows that
\[ |z_{\nu',d}(k)| - \varepsilon \leq |d| \leq |z_{\nu',d}(k)| + \varepsilon. \]
Furthermore, by (4.4.4),
\[ \varepsilon < \frac{\Lambda}{6} \leq \frac{|v|}{12} \leq \frac{|z_{\nu',d}(k)|}{12}. \]
Thus,\[ \frac{1}{2}|z_{\nu',d}(k)| \leq |d| \leq 2|z_{\nu',d}(k)|. \]
This yields the third inequality of part (ii) and completes the proof.

\textbf{Proof of Lemma 4.4.1}

\textit{Proof of Lemma 4.4.1.} We consider all cases at the same time. Therefore, we have either hypothesis (i) with \((\mu, d') = (\nu, 0)\) or hypothesis (ii) with \((\mu, d') \in \{ (\nu, 0), (\nu', d) \}\). Note that either \((\nu, \nu') = (1, 2)\) or \((\nu, \nu') = (2, 1)\).

[Step 1] Recall the change of variables
\[ w_{\mu,d'}(k) = k_1 + d_1' + i(-1)^\mu(k_2 + d_2'), \]
\[ z_{\mu,d'}(k) = k_1 + d_1' - i(-1)^\mu(k_2 + d_2'), \]
and set
\[ G'_1 := \{ b \in G' \mid |b - d'| < \frac{1}{4} R \}, \]
\[ G'_2 := \{ b \in G' \mid |b - d'| \geq \frac{1}{4} R \}. \]
Observe that
\[ G' = G'_1 \cup G'_2. \]
By Proposition 4.2.1,
\[ G'_1, G'_2 \subset G' = \Gamma^\# \backslash G = \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon |v| \}. \]
Furthermore, by Proposition 4.4.4, for \((\mu, d') = (\nu, 0)\) if (i) or \((\mu, d') \in \{(\nu, 0), (\nu', d)\}\) if (ii),
\[
\frac{1}{|\nu|} \leq \frac{3}{|\mu, d'|}.
\]
Thus, using the above remarks and observing the definition of \(G'_2\) we have
\[
|R_1(k)| := \left| \sum_{b \in G'_1} \sum_{c \in G'_2} \frac{f(d' - b)}{\mu_b(k)(R_{G'G'})_{b,c}} g(c - d') \right| \leq \frac{1}{|\nu|} \left\| R_{G'G'}^{-1} \right\| \sum_{b \in G'_1} \left| f(d' - b) \right| \sum_{c \in G'_2} \left| \frac{c - d'}{c - d'} \right|^2 \left| g(c - d') \right| \leq \frac{1}{|\nu|} \left\| R_{G'G'}^{-1} \right\| \left\| f \right\|_1 \frac{16}{R^2} \left\| g(c) \right\|_1 \leq \frac{C_{\epsilon,f,g}}{|\nu, d'| R^2} \quad \text{(4.4.5)}
\]
and
\[
|R_2(k)| := \left| \sum_{b \in G'_2} \sum_{c \in G'} \frac{f(d' - b)}{\mu_b(k)(R_{G'G'})_{b,c}} g(c - d') \right| \leq \frac{1}{|\nu|} \left\| R_{G'G'}^{-1} \right\| \sum_{b \in G'_2} \left| \frac{d' - b}{d' - b} \right|^2 \left| f(d' - b) \right| \sum_{c \in G'} \left| g(c - d') \right| \leq \frac{1}{|\nu|} \left\| R_{G'G'}^{-1} \right\| \left\| b \right\|^2 \left\| f(b) \right\|_1 \frac{16}{R^2} \left\| g(b) \right\|_1 \leq \frac{C_{\epsilon,f,g}}{|\nu, d'| R^2} \quad \text{(4.4.6)}
\]
Hence,
\[
\Phi_{d',d'}(k) = \left[ \sum_{b,c \in G'_1} + \sum_{b \in G'_1, c \in G'_2} + \sum_{c \in G'_2} \right] \frac{f(d' - b)}{\mu_b(k)(R_{G'G'})_{b,c}} g(c - d') \leq \frac{C_{\epsilon,f,g}}{|\nu, d'| R^2} \quad \text{(4.4.7)}
\]
with
\[
|R_1(k) + R_2(k)| \leq \frac{C_{\epsilon,f,g}}{|\nu, d'| R^2} \quad \text{(4.4.8)}
\]
Now, if we set
\[
T_{G'G'} := \pi_{G'} - R_{G'G'}
\]
and recall the convergent series expansion
\[
R_{G'G'}^{-1} = (\pi_{G'} - T_{G'G'})^{-1} = \sum_{j=0}^{\infty} T_{G'G'}^j,
\]
we can write
\[
\sum_{b,c \in G'_1} \frac{f(d' - b)}{\mu_b(k)} (R_{G'G'})_{b,c} g(c - d') = \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} \frac{f(d' - b)}{\mu_b(k)} (T_{G'G'}^j)_{b,c} g(c - d') \quad \text{(4.4.9)}
\]
Note, the above equality is fine because $G'_1$ is finite set. Let
\[ G'_3 := \{ b \in G' \mid |b - d'| < \frac{1}{2} R \}, \]
\[ G'_4 := \{ b \in G' \mid |b - d'| \geq \frac{1}{2} R \}. \]

Again, observe that
\[ G' = G'_3 \cup G'_4. \]

Thus, we can break $T_{G'G'}$ into
\[ T_{G'G'} = \pi_{G'} T_{\pi_{G'}} = (\pi_{G'_3} + \pi_{G'_4}) T (\pi_{G'_3} + \pi_{G'_4}) = T_{33} + T_{43} + T_{34} + T_{44}, \]
where
\[ T_{ij} := \pi_{G_i} T_{\pi_{G_j}} \]
for $i, j \in \{3, 4\}$. Using this decomposition we are able to prove the following.

**Proposition 4.4.5.** Under the hypotheses of Lemma 4.4.1 we have
\[ \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} f(d' - b) \frac{1}{N_b(k)} (T_{G'G'})_{b,c} g(c - d') = \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} f(d' - b) \frac{1}{N_b(k)} (T_{33})_{b,c} g(c - d') + R_3(k) \]
with $R_3(k)$ given by (4.4.34) and
\[ |R_3(k)| \leq \frac{C_{\Lambda,f,g}}{|z_{\mu,d'}| R^2}. \]

This proposition will be proved below. Combining this with (4.4.7) and (4.4.9) we obtain
\[ \Phi_{d',d'}(k) = \sum_{j=0}^{\infty} \sum_{b,c \in G'_1} f(d' - b) \frac{1}{N_b(k)} (T_{33})_{b,c} g(c - d') + \sum_{j=1}^{3} R_j(k). \]

**Step 2** We now look in detail to the operator $T_{33}$ and its powers $T_{33}^j$. Recall that
\[ \theta_{\mu}(b) = \frac{1}{2}((1)^{\mu} b_2 + i b_1) \]
and set $\mu' := \mu - (1)^{\mu}$ so that $(1)^{\mu} = (1)^{\mu'}$. Then,
\[ N_b(k) = N_{b,\mu}(k) N_{b,\mu'}(k) = (k_1 + b_1 + i(1)^{\mu}(k_2 + b_2))(k_1 + b_1 + i(1)^{\mu}(k_2 + b_2)) \]
\[ = (k_1 + d'_1 + i(1)^{\mu}(k_2 + d'_2) + b_1 - d'_1 + i(1)^{\mu}(b_2 - d'_2)) \]
\[ \times (k_1 + d'_1 + i(1)^{\mu}(k_2 + d'_2) + b_1 - d'_1 + i(1)^{\mu}(b_2 - d'_2)) \]
\[ = (k_1 + d'_1 + i(1)^{\mu}(k_2 + d'_2) + b_1 - d'_1 - i(1)^{\mu}(b_2 - d'_2)) \]
\[ \times (k_1 + d'_1 - i(1)^{\mu}(k_2 + d'_2) + b_1 - d'_1 - i(1)^{\mu}(b_2 - d'_2)) \]
\[ = (w_{\mu,d'} - 2i\theta_{\mu'}(b - d')(z_{\mu,d'} - 2i\theta_{\mu}(b - d'))) \]
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We next prove the following estimates, Consequently, Thus, for \( b \in \Gamma' \), we have the same length as the complex number

\[
2i\theta_\mu(b) = |(b_1, b_2)| = |b_1 + i(-1)^\mu b_2| = |2i\theta_\mu(b)|. \tag{4.4.16}
\]

Thus, for \( b \in G'_3 \),

\[
\frac{|2i\theta_\mu(b - d')|}{R} = \frac{|b - d'|}{R} < \frac{1}{2}.
\]

Consequently,

\[
\left| \frac{1}{z_{\mu,d'} - 2i\theta_\mu(b - d')} \right| \leq \frac{1}{|z_{\mu,d'}| - |2i\theta_\mu(b - d')|} < \frac{1}{|z_{\mu,d'}| - \frac{1}{2}R} = \frac{2}{|z_{\mu,d'}|}. \tag{4.4.17}
\]
Furthermore, for $b \in G'$,
\[
\frac{1}{|w_{\mu,d'} - 2i\theta \mu_d(b - d')|} \leq \frac{1}{|b - d'| - |w_{\mu,d'}|} \leq \frac{1}{|b - d'| - \varepsilon} \leq \frac{1}{2\Lambda - \Lambda} = \frac{1}{\Lambda}.
\] (4.4.18)

Here we have used that $|w_{\mu,d'}| < \varepsilon < \Lambda$ and $|b - d'| \geq 2\Lambda$ for all $b \in G'$. Using again that $\varepsilon < \Lambda \leq |c - d'|/2$ for all $c \in G'$ we have
\[
\frac{|c - d'|}{|c - d'| - \varepsilon} < 2.
\] (4.4.20)

Finally recall that
\[
\frac{\varepsilon}{\Lambda} < \frac{1}{6} \quad \text{and} \quad \frac{1}{|z_{\mu,d'}|} \leq \frac{1}{|v|} < \frac{1}{R},
\] (4.4.21)
where the last inequality follows from Proposition 4.4.4 since $|v| > R$ by hypothesis. Then, using the above inequalities and Proposition 4.1.2, the bounds (4.4.15) for $\|X_{33}\|$ and $\|Y_{33}\|$ follow from the estimates
\[
\left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] [X_{b,c}] \leq \left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + |2i\theta \mu_d(\hat{A}(b - c))| |w_{\mu,d'}|}{|w_{\mu,d'} - 2i\theta \mu_d(c - d')| |z_{\mu,d'} - 2i\theta \mu_d(c - d')|} \leq \frac{2}{|z_{\mu,d'}| R} \left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + \varepsilon \sqrt{2} |\hat{A}(b - c)|}{|w_{\mu,d'} - 2i\theta \mu_d(c - d')|} \leq \frac{2}{|z_{\mu,d'}| R} \left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + \varepsilon \sqrt{2} |\hat{A}(b - c)|}{|c - d'| - \varepsilon} \leq \frac{2}{|z_{\mu,d'}| R} \left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + \varepsilon \sqrt{2} |\hat{A}(b - c)|}{|c - d'| - \varepsilon} \leq \frac{2}{|z_{\mu,d'}| R} \left[ \sup_{c \in G_3} \sum_{b \in G_3} \sum_{c \in G_3} \right] \frac{2|c - d'| |\hat{A}(b - c)| + |\hat{q}(b - c)| + \varepsilon \sqrt{2} |\hat{A}(b - c)|}{\Lambda} \leq \left[ 20 \|\hat{A}\|_{l^1} + \frac{4 \|\hat{q}\|_{l^1}}{\Lambda} \right] \frac{1}{|z_{\mu,d'}| R} \leq \left[ 20 \|\hat{A}\|_{l^1} + \frac{4 \|\hat{q}\|_{l^1}}{\Lambda} \right] \frac{1}{R} < \frac{1}{7} + \frac{1}{4} = \frac{1}{3}.
\]
and
\[
\left[ \sup_{c \in G_3'} \sum_{b \in G_3'} + \sup_{b \in G_3'} \sum_{c \in G_3'} \right] |Y_{b,c}|
\leq \left[ \sup_{c \in G_3'} \sum_{b \in G_3'} + \sup_{b \in G_3'} \sum_{c \in G_3'} \right] \frac{|2i\theta_{\mu'}(\hat{A}(b - c))| |z_{\mu,d'}|}{|w_{\mu,d'} - 2i\theta_{\mu'}(c - d')| |z_{\mu,d'} - 2i\theta_{\mu}(c - d')|}
\]

(now apply (4.4.16), (4.4.17) and (4.4.19))
\[
\leq \frac{2}{\Lambda} \left[ \sup_{c \in G_3'} \sum_{b \in G_3'} + \sup_{b \in G_3'} \sum_{c \in G_3'} \right] \frac{|2\theta_{\mu'}(\hat{A}(b - c))| |z_{\mu,d'}|}{|z_{\mu,d'}| R}
\leq \frac{8}{\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1} \leq \frac{8\sqrt{2}}{\Lambda} \|\hat{A}\|_{l^1} \leq \frac{16\sqrt{2} \varepsilon}{63}
< \frac{16\sqrt{2}}{63} \frac{1}{6} < \frac{1}{14}.
\]

**Step 3** We now look in detail to $T_{33}^j$. For each integer $j \geq 1$ write
\[
T_{33}^j = (X_{33} + Y_{33})^j = Z_j + W_j + Y_{33}^j,
\]
where $W_j$ is the sum of the $j$ terms containing only one factor $X_{33}$ and $j - 1$ factors $Y_{33}$,
\[
W_j := X_{33} Y_{33} \cdots Y_{33} + Y_{33} X_{33} Y_{33} \cdots Y_{33} + \cdots + Y_{33} \cdots Y_{33} X_{33}
= \sum_{m=1}^{j} (Y_{33})^{m-1} X_{33} (Y_{33})^{j-m},
\]
and
\[
Z_j := (X_{33} + Y_{33})^j - W_j - Y_{33}^j.
\]
In view of (4.4.15) we have
\[
\|Y_{33}\|^j \leq \left( \frac{1}{14} \right)^j,
\]
\[
\|W_j\| \leq j \|X_{33}\| \|Y_{33}\|^{j-1} \leq \frac{C_{\Lambda,A,q}}{|z_{\mu,d'}| R} j \left( \frac{1}{14} \right)^{j-1},
\]
\[
\|Z_j\| \leq (2^j - j - 1) \|X_{33}\|^2 \left( \frac{1}{3} \right)^{j-2} \leq \frac{C_{\Lambda,A,q}}{|z_{\mu,d'}|^2 R} \left( \frac{2}{3} \right)^j.
\]
Hence, the series
\[
S := \sum_{j=0}^{\infty} Y_{33}^j = (I - Y_{33})^{-1}, \quad W := \sum_{j=1}^{\infty} W_j \quad \text{and} \quad Z := \sum_{j=2}^{\infty} Z_j
\]
(4.4.23)
converge, and the operator norm of $W$ and $Z$ decay with respect to $|z_{\mu,d'}|$. Indeed,

\[
\|S\| \leq \sum_{j=0}^{\infty} \|Y_{33}^j\|^j \leq \sum_{j=0}^{\infty} \left( \frac{1}{14} \right)^j < C,
\]

\[
\|W\| \leq \sum_{j=1}^{\infty} \|W_j\| \leq \frac{C'_{\Lambda,A,q}}{|z_{\mu,d'}| - R} \sum_{j=1}^{\infty} j \left( \frac{1}{14} \right)^{j-1} < \frac{C_{\Lambda,A,q}}{|z_{\mu,d'}| R},
\]

\[
\|Z\| \leq \sum_{j=2}^{\infty} \|Z_j\| \leq \frac{C'_{\Lambda,A,q}}{|z_{\mu,d'}| R^2} \sum_{j=2}^{\infty} \left( \frac{2}{3} \right)^j < \frac{C_{\Lambda,A,q}}{|z_{\mu,d'}| R^2}.
\]

Thus, we have the expansion

\[
\sum_{j=0}^{\infty} T_{33}^j = S + W + Z.
\]

**Step 4** Consequently,

\[
\sum_{j=0}^{\infty} \sum_{b,c \in G_1} f(d' - b) \frac{(T_{33}^j)_{b,c} g(c - d')}{N_0(k)} = \sum_{b,c \in G_1} \frac{f(d' - b) (S + W + Z)_{b,c} g(c - d')}{(w_{\mu,d'} - 2i\mu(b - d'))(z_{\mu,d'} - 2i\mu(b - d'))}
\]

\[
= \alpha^{(1)}_{\mu,d'} + \alpha^{(2)}_{\mu,d'} + \mathcal{R}_4,
\]

(4.4.24)

where

\[
\alpha^{(1)}_{\mu,d'}(k) := \sum_{b,c \in G_1} \frac{f(d' - b) S_{b,c}(k) g(c - d')}{(w_{\mu,d'}(k) - 2i\mu(b - d'))(z_{\mu,d'}(k) - 2i\mu(b - d'))},
\]

(4.4.25)

\[
\alpha^{(2)}_{\mu,d'}(k) := \sum_{b,c \in G_1} \frac{f(d' - b) W_{b,c}(k) g(c - d')}{(w_{\mu,d'}(k) - 2i\mu(b - d'))(z_{\mu,d'}(k) - 2i\mu(b - d'))},
\]

(4.4.26)

and

\[
\mathcal{R}_4(k) := \sum_{b,c \in G_1} \frac{f(d' - b) Z_{b,c}(k) g(c - d')}{(w_{\mu,d'}(k) - 2i\mu(b - d'))(z_{\mu,d'}(k) - 2i\mu(b - d'))}.
\]

By a short calculation as in (4.4.33), using (4.4.17) and (4.4.19) we find that

\[
|\alpha^{(1)}_{\mu,d'}(k)| \leq \frac{1}{\Lambda} \frac{2}{|z_{\mu,d'}| - R} \|f\|_1 \|g\|_1 \|S\| \leq \frac{C_{\Lambda,f,g}}{|z_{\mu,d'}| R},
\]

(4.4.27)

\[
|\alpha^{(2)}_{\mu,d'}(k)| \leq \frac{1}{\Lambda} \frac{2}{|z_{\mu,d'}| - R} \|f\|_1 \|g\|_1 \|W\| \leq \frac{C_{\Lambda,A,q,f,g}}{|z_{\mu,d'}| R^2},
\]

\[
|\mathcal{R}_4(k)| \leq \frac{1}{\Lambda} \frac{2}{|z_{\mu,d'}| - R} \|f\|_1 \|g\|_1 \|Z\| \leq \frac{C_{\Lambda,A,q,f,g}}{|z_{\mu,d'}| R^3}.
\]

Hence, recalling (4.4.11) we conclude that

\[
\Phi_{d',d'} = \alpha^{(1)}_{\mu,d'} + \alpha^{(2)}_{\mu,d'} + \mathcal{R}_4,
\]

(4.4.28)
where
\[ \alpha_{\mu,d}^{(3)}(k) := \sum_{j=1}^{4} R_j(k). \] (4.4.28)

Furthermore, in view of (4.4.8), (4.4.10) and (4.4.27), since
\[
\frac{1}{|z_{\mu,d}'(k)|^3} = \frac{1}{(2|z_{\mu,d}'| - R)^3} < \frac{1}{|z_{\mu,d}'| R^2},
\]
for \(1 \leq j \leq 2\) we have
\[
|\alpha_{\mu,d}^{(j)}(k)| \leq \frac{C_j}{|z_{\mu,d}'(k)|^3} \quad \text{and} \quad |\alpha_{\mu,d}^{(3)}(k)| \leq \frac{C_3}{|z_{\mu,d}'(k)| R^2},
\]
where \(C_j = C_{j;\Lambda,A,q,f,g}\) and \(C_3 = C_{3;\varepsilon,\Lambda,A,q,f,g}\) are constants. This proves the main statement of the lemma. Finally observe that, since \(G_3'\) is a finite set, the matrices \(X_{33}\) and \(Y_{33}\) are analytic in \(k\) because their matrix elements are analytic functions of \(k\). (Note, the functions \(w_{\mu,d}(k)\) and \(z_{\mu,d}'(k)\) are analytic.) Consequently, the matrices \(W_j\) and \(Z_j\) are also analytic and so are \(S_{b,c}, W_{b,c}\) and \(Z_{b,c}\) because the series (4.4.23) converge uniformly with respect to \(k\). Thus, all the functions \(\alpha_{\mu,d}^{(j)}(k)\) are analytic in the region under consideration. This completes the proof of the lemma.

We now prove Proposition 4.4.5, which was used in the above proof.

**Proof of Proposition 4.4.5.** [Step 1] Recall that
\[
T_{G'G'} = T_{33} + T_{34} + T_{43} + T_{44} \quad \text{with} \quad T_{ij} = \pi_{G'_i} T \pi_{G'_j},
\]
and set
\[
X_{33}^{(0)} := 0, \quad Y_{34}^{(0)} := T_{34}, \quad W_{43}^{(0)} := T_{43} \quad \text{and} \quad Z_{44}^{(0)} := T_{44}.
\]
First observe that
\[
T_{G'G'}^2 = T_{33}^2 + X_{33}^{(1)} + Y_{34}^{(1)} + W_{43}^{(1)} + Z_{44}^{(1)},
\]
where
\[
X_{33}^{(1)} := T_{33} X_{33}^{(0)} + T_{34} W_{43}^{(0)} \quad : \quad L_{G'_3}^2 \to L_{G'_3}^2,
\]
\[
Y_{34}^{(1)} := T_{33} Y_{34}^{(0)} + T_{34} Z_{44}^{(0)} \quad : \quad L_{G'_3}^2 \to L_{G'_4}^2,
\]
\[
W_{43}^{(1)} := T_{33} T_{33} + T_{33} X_{33}^{(0)} + T_{44} W_{43}^{(0)} \quad : \quad L_{G'_4}^2 \to L_{G'_4}^2,
\]
\[
Z_{44}^{(1)} := T_{34} Y_{34}^{(0)} + T_{44} Z_{44}^{(0)} \quad : \quad L_{G'_4}^2 \to L_{G'_4}^2.
\]
Now suppose that

\[ T^{j}_{G'G'} = T^{j}_{33} + X^{(j-1)}_{33} + Y^{(j-1)}_{34} + W^{(j-1)}_{43} + Z^{(j-1)}_{44} \]

with

\[ X^{(j-1)}_{33} : L^{2}_{G'3} \rightarrow L^{2}_{G'3}, \]
\[ Y^{(j-1)}_{34} : L^{2}_{G'4} \rightarrow L^{2}_{G'4}, \]
\[ W^{(j-1)}_{43} : L^{2}_{G'4} \rightarrow L^{2}_{G'3}, \]
\[ Z^{(j-1)}_{44} : L^{2}_{G'4} \rightarrow L^{2}_{G'4}. \]

It is straightforward to verify that

\[ T^{j+1}_{G'G'} = T^{j+1}_{33} + X^{(j)}_{33} + Y^{(j)}_{34} + W^{(j)}_{43} + Z^{(j)}_{44}, \] (4.4.29)

where

\[ X^{(j)}_{33} := T^{j}_{33} X^{(j-1)}_{33} + T^{j}_{34} W^{(j-1)}_{43} : L^{2}_{G'3} \rightarrow L^{2}_{G'3}, \]
\[ Y^{(j)}_{34} := T^{j}_{33} Y^{(j-1)}_{34} + T^{j}_{34} Z^{(j-1)}_{44} : L^{2}_{G'4} \rightarrow L^{2}_{G'4}, \]
\[ W^{(j)}_{43} := T^{j}_{43} T^{j}_{33} + T^{j}_{43} X^{(j-1)}_{33} + T^{j}_{44} W^{(j-1)}_{43} : L^{2}_{G'4} \rightarrow L^{2}_{G'3}, \]
\[ Z^{(j)}_{44} := T^{j}_{43} Y^{(j-1)}_{34} + T^{j}_{44} Z^{(j-1)}_{44} : L^{2}_{G'4} \rightarrow L^{2}_{G'4}. \] (4.4.30)

Thus, it follows by induction that (4.4.29) and (4.4.30) hold for any \( j \geq 0 \).

**Step 2** Since \( \pi_{G'3} \pi_{G'4} = \pi_{G'4} \pi_{G'3} = 0 \) and \( \pi_{G'3} \pi_{G'4} = \pi_{G'4} \pi_{G'3} = \pi_{G'3} \), substituting (4.4.29) into the sum below for the terms where \( j \geq 1 \) we have, recalling that \( X_{33}^{(0)} = 0 \),

\[
\sum_{j=0}^{\infty} \sum_{b,c \in G'3} \frac{f(d'-b)}{N_b(k)} (T^{j}_{G'G'})_{b,c} g(c-d') = \left[ \sum_{j=0}^{\infty} + \sum_{j=1}^{\infty} \right] (\cdots) \\
= \sum_{j=0}^{\infty} \sum_{b,c \in G'3} \frac{f(d'-b)}{N_b(k)} (T^{j}_{33})_{b,c} g(c-d') + \sum_{j=1}^{\infty} \sum_{b,c \in G'4} \frac{f(d'-b)}{N_b(k)} (X^{(j-1)}_{33})_{b,c} g(c-d'). \] (4.4.31)

Now recall from (4.4.17) and (4.4.19) that, for all \( b \in G'3 \),

\[
\frac{1}{|N_b(k)|} \leq \frac{2}{\Lambda} \frac{1}{|\mu_{d'}|_R}, \] (4.4.32)

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and observe that $G_1' \subset G_3'$. Let $\mathcal{M}$ be either $T_{G'G'}$ or $T_{33}$. Then, the estimate

$$
\left| \sum_{b,c \in G_1'} \frac{f(d' - b)}{N(b)} (\mathcal{M}^j)_{b,c} g(c - d') \right|
$$

implies that the left hand side and the first term on the right hand side of (4.4.31) converge because $\|\mathcal{M}\| < 17/18$. Thus, the last term in (4.4.31) also converges. Hence, we are left to show that

$$
\mathcal{R}_3(k) := \sum_{j=1}^{\infty} \sum_{b,c \in G_1'} \frac{f(d' - b)}{N(b)} (X^{(j)}_{33})_{b,c} g(c - d')
$$

obeys

$$
|\mathcal{R}_3(k)| \leq \frac{C_{A,f,g}}{|z_{\mu,d'}| R^2}.
$$

In order to do this we need the following inequality, which we prove later.

**Proposition 4.4.6.** Consider a constant $\beta \geq 0$ and suppose that $\|(1 + |b|^{\beta}) \hat{q}(b)\|_{11} < \infty$ and $\|(1 + |b|^{\beta}) \hat{A}(b)\|_{11} < 2\varepsilon/63$. Suppose further that $|v| > \frac{2}{\varepsilon} \|(1 + |b|^{\beta}) \hat{A}(b)\|_{11}$. Then, for any $B,C \subset G'$ and $m \geq 1$,

$$
\|\pi_B T_{G'G'}^m \pi_C\| \leq (1 + (2\Lambda)^{\beta - [\beta]} [\beta] m^{[\beta] - 1}) \left(\frac{17}{18}\right)^m \sup_{b \in B} \sup_{c \in C} \frac{1}{1 + |b - c|^{[\beta]}},
$$

where $[\beta]$ is the smallest integer greater or equal than $\beta$.

**Step 3** Now observe that, if $b \in G_1'$ and $c \in G_4'$ then

$$
|b - c| = |b - d' - (c - d')| \geq |c - d'| - |b - d'| \geq \frac{R}{2} - \frac{R}{4} = \frac{R}{4}.
$$

Thus, applying the last proposition with $\beta = 2$ and recalling that $G_3' \subset G'$, for $m \geq 0$ we have

$$
\|\pi_{G_1'} T_{33}^{m+1} T_{34}\| \leq \|\pi_{G_1'} T_{G'G'}^m T_{G'G'}\| = \|\pi_{G_1'} T_{G'G'}^m \pi_{G_4'}\| \leq 3(m + 1) \left(\frac{17}{18}\right)^{m+1}.
$$

Furthermore, since $\pi_{G_4'} \pi_{G_3'} = \pi_{G_4'} \pi_{G_1'} = 0$ and $\pi_{G_3'} \pi_{G_1'} = \pi_{G_1'}$, from (4.4.29) we obtain

$$
W^{(j)}_{43, G_1'} = \pi_{G_4'} T_{G'G'}^{j+1} \pi_{G_4'} \pi_{G_1'} = \pi_{G_4'} T_{G'G'}^{j+1} \pi_{G_1'}.
$$

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Hence,
\[ \|W_{43}(j)\pi G_1\| = \|\pi G_1^T G G \pi G_1\| \leq \|T G G\|^{j+1} < \left( \frac{17}{18} \right)^{j+1}. \]

Therefore, for \(0 \leq m < j\),
\[ \|\pi G_1^T m T_{34} W^{(j-m-1)} \pi G_1\| \leq \|\pi G_1^T m T_{34} \| \|W_{43}(j-m-1) \pi G_1\| \leq 3(m+1) \left( \frac{17}{18} \right)^{j+1}. \]

Iterating the first expression in (4.4.30) we find that
\[
X_{33}^{(j)} = T_{34} W_{43}^{(j-1)} + T_{33} X_{33}^{(j-1)} \\
= T_{34} W_{43}^{(j-1)} + T_{33} T_{34} W_{43}^{(j-2)} + T_{33} X_{33}^{(j-2)} \\
\vdots \\
= T_{34} W_{43}^{(j-1)} + T_{33} T_{34} W_{43}^{(j-2)} + \cdots + T_{33} T_{34} W_{43}^{(0)} + T_{33} T_{34} W_{43}^{(0)} \\
= \sum_{m=0}^{j-1} T_{33} m T_{34} W_{43}^{(j-m-1)}.
\]

Thus, using the above inequality,
\[
\|\pi G_1^T X_{33}^{(j)} \pi G_1\| = \left\| \sum_{m=0}^{j-1} \pi G_1^T m T_{33} T_{34} W_{43}^{(j-m-1)} \pi G_1 \right\| \leq \sum_{m=0}^{j-1} \|\pi G_1^T m T_{33} T_{34} W_{43}^{(j-m-1)} \pi G_1\| \\
\leq \frac{3}{1 + 16 R^2} \left( \frac{17}{18} \right)^{j+1} \sum_{m=0}^{j-1} (m+1) = \frac{3}{2} + \frac{1}{8} R^2 (j^2 + j) \left( \frac{17}{18} \right)^{j+1}. \]

Consequently,
\[
\left\| \pi G_1^T \sum_{j=1}^{\infty} X_{33}^{(j)} \pi G_1 \right\| \leq \sum_{j=1}^{\infty} \|\pi G_1^T X_{33}^{(j)} \pi G_1\| \\
\leq \frac{3}{2} + \frac{1}{8} R^2 \sum_{j=1}^{\infty} (j^2 + j) \left( \frac{17}{18} \right)^{j+1} \leq \frac{C}{R^2},
\]

where \(C\) is an universal constant. Finally, using this and (4.4.32), since \(|z_{\mu,d'}| \leq 3|v|\) we have
\[
|R_3(k)| = \left| \sum_{b,c \in G_1^T} \frac{f(d'-b)}{N_b(k)} \left[ \sum_{j=1}^{\infty} X_{33}^{(j)} \right]_{b,c} g(c-d') \right| \\
\leq \frac{2}{|v|} \left\| f \right\|_{L^1} \left\| \pi G_1^T \sum_{j=1}^{\infty} X_{33}^{(j)} \pi G_1 \right\| \left\| g \right\|_{L^1} \leq \frac{6C}{\Lambda} \left\| f \right\|_{L^1} \left\| g \right\|_{L^1} \frac{1}{|z_{\mu,d'}| R^2},
\]

In view of (4.4.31) and (4.4.34) this completes the proof. \(\square\)
We now prove Proposition 4.4.6, which was used above and left behind without proof. This is the last step we need to finish the proof of Lemma 4.4.1 indeed.

**Proof of Proposition 4.4.6.** For any \( b, c \in \Gamma^\# \) set

\[ Q_{b,c} := (1 + |b - c|^\beta)T_{b,c}. \]

We first claim that, for any \( B, C \subset G' \),

\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}| < \frac{17}{18} \quad \text{and} \quad \sup_{c \in C} \sum_{b \in B} |Q_{b,c}| < \frac{17}{18}. \tag{4.4.36}
\]

In fact, using the bounds (4.1.4), (4.1.5) and (4.2.2), namely, for all \( b \in G' \),

\[
\frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon|v|}, \quad \frac{|b|}{|N_b(k)|} \leq \frac{4}{\varepsilon} \quad \text{and} \quad |k| \leq |u| + |v| \leq 3|v|,
\]

it follows that

\[
\sup_{b \in B} \sum_{c \in C} |Q_{b,c}| = \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^\beta) \left| \frac{\hat{q}(b - c)}{N_c(k)} - \frac{2c \cdot \hat{A}(b - c)}{N_c(k)} - \frac{2k \cdot \hat{A}(b - c)}{N_c(k)} \right| \leq \|(1 + |b|^\beta)\hat{q}(b)\|_1 \frac{1}{\varepsilon|v|} + \frac{14}{\varepsilon} \|(1 + |b|^\beta)\hat{A}(b)\|_1 < \frac{1}{2} + \frac{4}{9} = \frac{17}{18}
\]

and

\[
\sup_{c \in C} \sum_{b \in B} |Q_{b,c}| = \sup_{c \in C} \sum_{b \in B} (1 + |b - c|^\beta) \left| \frac{\hat{q}(b - c)}{N_c(k)} - \frac{2c \cdot \hat{A}(b - c)}{N_c(k)} - \frac{2k \cdot \hat{A}(b - c)}{N_c(k)} \right| \leq \|(1 + |b|^\beta)\hat{q}(b)\|_1 \frac{1}{\varepsilon|v|} + \frac{14}{\varepsilon} \|(1 + |b|^\beta)\hat{A}(b)\|_1 < \frac{1}{2} + \frac{4}{9} = \frac{17}{18}.
\]

Furthermore, since \( |T_{b,c}| \leq |Q_{b,c}| \) for all \( b, c \in \Gamma^\# \), for any integer \( m \geq 1 \) we have

\[
\sup_{b \in B} \sum_{c \in C} |(T_{BC}^m)_{b,c}| < \left( \frac{17}{18} \right)^m \quad \text{and} \quad \sup_{c \in C} \sum_{b \in B} |(T_{BC}^m)_{b,c}| < \left( \frac{17}{18} \right)^m.
\]

Now, let \( p \) be the smallest integer greater or equal than \( \beta \), and for any integer \( m \geq 1 \) and any \( \xi_0, \xi_1, \ldots, \xi_m \in \Gamma^\# \), let \( b = \xi_0 \) and \( c = \xi_m \). Then,

\[
|b - c|^\beta = (2\Lambda)^\beta \left[ \frac{|b - c|}{2\Lambda} \right]^\beta \leq (2\Lambda)^\beta \left[ \frac{|b - c|}{2\Lambda} \right]^p = \frac{(2\Lambda)^\beta}{(2\Lambda)^p} \sum_{i_1, \ldots, i_p = 1}^m |\xi_{i_1} - \xi_i| \cdots |\xi_{i_p} - \xi_{i_p}|\]

\[
\leq (2\Lambda)^{\beta - p} \sum_{i_1, \ldots, i_p = 1}^m \max \left\{ |\xi_{i_1} - \xi_i|^p, \ldots, |\xi_{i_p} - \xi_{i_p}|^p \right\}
\]

\[
\leq (2\Lambda)^{\beta - p} \sum_{i_1, \ldots, i_p = 1}^m (|\xi_{i_1} - \xi_i|^p + \cdots + |\xi_{i_p} - \xi_{i_p}|^p)
\]

\[
= (2\Lambda)^{\beta - p} p m^{p-1} \sum_{i=1}^m |\xi_{i-1} - \xi_i|^p \leq (2\Lambda)^{\beta - p} p m^{p-1} \prod_{i=1}^m (1 + |\xi_{i-1} - \xi_i|^p).
\]

\[\tag{4.4.37}\]
To simplify the notation write

$$s := \sup_{b \in B} \frac{1}{1 + \beta |b - c|}.$$

Hence,

$$\sup_{b \in B} \sum_{c \in C} |(T_{G'}^{m}G')_{b,c}| \leq \sup_{b \in B} \frac{1}{1 + \beta |b - c|} \sup_{c \in C} \sum_{b \in B} (1 + \beta |b - c|) |(T_{G'}^{m})_{b,c}|$$

$$\leq s \left[ \sup_{b \in B} \sum_{c \in C} |(T_{G'}^{m})_{b,c}| + (2\Lambda)^{\beta - p} m^{p-1} \sup_{b \in B \xi_1 \in G'} (1 + \beta |b - \xi_1|) |T_{b}\xi_1| \right.$$  

$$\times \sum_{\xi_2 \in G'} (1 + \beta |\xi_2|) |T_{\xi_1}\xi_2| \cdots \sum_{c \in C} (1 + \beta |c|) |T_{\xi_{m-1},c}|$$

$$\leq s \left[ \left( \frac{17}{18} \right)^{m} + (2\Lambda)^{\beta - p} m^{p-1} \sup_{b \in B \xi_1 \in G'} (1 + \beta |b - \xi_1|) |T_{b}\xi_1| \right.$$  

$$\times \sup_{\xi_1 \in G'} \sum_{\xi_2 \in G'} (1 + \beta |\xi_2|) |T_{\xi_1}\xi_2| \cdots \sup_{\xi_{m-1} \in G'} \sum_{c \in C} (1 + \beta |c|) |T_{\xi_{m-1},c}|$$

$$= s \left[ \left( \frac{17}{18} \right)^{m} + (2\Lambda)^{\beta - p} m^{p-1} \sup_{b \in B \xi_1 \in G'} \sum_{c \in C} |Q_{b}\xi_1| \cdots \sup_{\xi_{m-1} \in G'} \sum_{c \in C} |Q_{\xi_{m-1},c}| \right.$$  

$$\leq s (1 + (2\Lambda)^{\beta - p} m^{p-1}) \left( \frac{17}{18} \right)^{m}.$$

Similarly,

$$\sup_{c \in C} \sum_{b \in B} |(T_{G'}^{m}G')_{b,c}| \leq s \left[ \sup_{c \in C} \sum_{b \in B} |(T_{G'}^{m})_{b,c}| + (2\Lambda)^{\beta - p} m^{p-1} \sup_{c \in C} \sum_{\xi_{m-1} \in G'} (1 + \beta |\xi_{m-1}|) |T_{\xi_{m-1},c}| \right.$$  

$$\times \sum_{\xi_{m-2} \in G'} (1 + \beta |\xi_{m-2} - \xi_{m-1}|) |T_{\xi_{m-2},\xi_{m-1}}| \cdots \sum_{b \in B} (1 + \beta |b - \xi_1|) |T_{b}\xi_1|$$

$$\leq s \left[ \left( \frac{17}{18} \right)^{m} + (2\Lambda)^{\beta - p} m^{p-1} \sup_{c \in C} \sum_{\xi_{m-1} \in G'} (1 + \beta |\xi_{m-1}|) |T_{\xi_{m-1},c}| \right.$$  

$$\times \sup_{\xi_{m-1} \in G'} \sum_{\xi_{m-2} \in G'} (1 + \beta |\xi_{m-2} - \xi_{m-1}|) |T_{\xi_{m-2},\xi_{m-1}}| \cdots \sup_{\xi_1 \in G'} \sum_{b \in B} (1 + \beta |b - \xi_1|) |T_{b}\xi_1|$$

$$\leq s (1 + (2\Lambda)^{\beta - p} m^{p-1}) \left( \frac{17}{18} \right)^{m}.$$
Therefore, by Proposition 4.1.2,
\[
\| \pi_B T_{G'^G}^m \pi_C \| \leq (1 + (2\Lambda)^{\beta - \lceil \beta \rceil} \lceil \beta \rceil m^{\lceil \beta \rceil - 1}) \left( \frac{17}{18} \right)^m \sup_{b \in B} \sup_{c \in C} \frac{1}{1 + |b - c|^{\beta}},
\]
where \( \lceil \beta \rceil \) is the smallest integer greater or equal than \( \beta \). This is the desired inequality. \( \square \)

**Proof of Lemma 4.4.2**

**Proof of Lemma 4.4.2.** To simplify the notation write
\[
w = w_{\mu,d'}, \quad z = z_{\mu,d'} \quad \text{and} \quad |z|_R = 2|z| - R.
\]

First observe that
\[
\frac{1}{w - 2i\theta_{\mu'}(c - d')} = \frac{-1}{2i\theta_{\mu'}(c - d')} + \frac{w}{2i\theta_{\mu'}(c - d')(w - 2i\theta_{\mu'}(c - d'))},
\]
so that
\[
N_c(k) = \frac{z}{(w - 2i\theta_{\mu'}(c - d'))(z - 2i\theta_{\mu}(c - d'))} = \frac{1}{w - 2i\theta_{\mu'}(c - d')} \left( 1 + \frac{2i\theta_{\mu}(c - d')}{z - 2i\theta_{\mu}(c - d')} \right)
\]
\[
= \frac{1}{w - 2i\theta_{\mu'}(c - d')} + \frac{2i\theta_{\mu}(c - d')}{w - 2i\theta_{\mu'}(c - d')} \frac{1}{z - 2i\theta_{\mu}(c - d')}
\]
\[
= \frac{-1}{2i\theta_{\mu'}(c - d')} + \frac{2i\theta_{\mu}(c - d')}{w - 2i\theta_{\mu'}(c - d')} \frac{1}{z - 2i\theta_{\mu}(c - d')}
\]
\[
=: \eta_c^{(0)} + \eta_c^{(w)} + \eta_c^{(z)},
\]
where, in view of (4.4.17) to (4.4.20), since \( |w| < \epsilon \),
\[
|\eta_c^{(0)}| \leq \frac{1}{2\Lambda}, \quad |\eta_c^{(w)}| \leq \frac{\epsilon}{2\Lambda^2} \quad \text{and} \quad |\eta_c^{(z)}| \leq \frac{4}{|z|_R}.
\]

Hence,
\[
Y_{b,c} = \frac{-2i\theta_{\mu'}(\hat{A}(b - c))z}{N_c(k)}
\]
\[
= -2i\theta_{\mu'}(\hat{A}(b - c))\eta_c^{(0)} - 2i\theta_{\mu'}(\hat{A}(b - c))\eta_c^{(w)} - 2i\theta_{\mu'}(\hat{A}(b - c))\eta_c^{(z)}
\]
\[
=: Y_{b,c}^{(0)} + Y_{b,c}^{(w)} + Y_{b,c}^{(z)}.
\]

Let \( Y^{(\cdot)} \) be the operator whose matrix elements are \( Y_{b,c}^{(\cdot)} \) and set \( Y_{33}^{(\cdot)} := \pi_{G'^G} Y^{(\cdot)} \pi_{G'^G} \).

Then, similarly as we estimated \( \|Y_{33}\| \), using (4.4.17) to (4.4.20) and Proposition 4.1.2, it
follows easily that
\[
\|Y_{33}^{(0)}\| \leq \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1}, \quad \|Y_{33}^{(w)}\| \leq \frac{\varepsilon}{2\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_{l^1}, \quad \|Y_{33}^{(z)}\| \leq \frac{4}{|z|_R} \|\theta_{\mu'}(\hat{A})\|_{l^1}.
\]
Furthermore,
\[
S = (I - Y_{33})^{-1} = 1 + (1 - Y_{33})^{-1}Y_{33} = 1 + SY_{33}
\]
\[
= 1 + (1 + SY_{33})Y_{33} = 1 + Y_{33} + SY_{33}^2
\]
\[
= 1 + Y_{33}^{(0)} + Y_{33}^{(w)} + Y_{33}^{(z)} + SY_{33}^2,
\]
where, recalling (4.4.15),
\[
\|SY_{33}^2\| \leq \|(1 - Y_{33})^{-1}\| \|Y_{33}\|^2 \leq \frac{\|Y_{33}\|^2}{1 - \|Y_{33}\|} < \frac{14}{13} \left(\frac{8}{\Lambda}\right)^2 \|\theta_{\mu'}(\hat{A})\|_{l^1}^2.
\]
Combining all this we have
\[
z \frac{S_{b,c}}{N_b(k)} = (\eta_b^{(0)} + \eta_b^{(w)} + \eta_b^{(z)})S_{b,c}
\]
\[
= (\eta_b^{(0)})^2 (\delta_{b,c} + Y_{b,c}^{(0)} + Y_{b,c}^{(w)} + Y_{b,c}^{(z)} + (SY_{33}^2)_{b,c}) + \eta_b^{(z)}S_{b,c}
\]
\[
= \left[\eta_b^{(0)}(\delta_{b,c} + Y_{b,c}^{(0)})\right] + \left[\eta_b^{(0)}Y_{b,c}^{(w)} + \eta_b^{(w)}(\delta_{b,c} + Y_{b,c}^{(0)} + Y_{b,c}^{(w)})\right]
\]
\[
+ \left[\eta_b^{(0)} + \eta_b^{(w)}(SY_{33}^2)_{b,c}\right] + \left[(\eta_b^{(0)} + \eta_b^{(w)})Y_{b,c}^{(z)} + \eta_b^{(z)}S_{b,c}\right]
\]
\[
= K_{b,c}^{(0)} + K_{b,c}^{(1)} + K_{b,c}^{(2)} + K_{b,c}^{(3)}
\]
with
\[
|K_{b,c}^{(0)}| \leq \frac{1}{2\Lambda} \left(1 + \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1}\right),
\]
\[
|K_{b,c}^{(1)}| \leq \frac{\varepsilon}{4\Lambda^3} \|\theta_{\mu'}(\hat{A})\|_{l^1} + \frac{\varepsilon}{2\Lambda^2} \left(1 + \frac{1}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1} + \frac{\varepsilon}{\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_{l^1}\right)
\]
\[
< \frac{\varepsilon}{2\Lambda^2} \left(1 + \frac{7}{6\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1}\right),
\]
\[
|K_{b,c}^{(2)}| \leq \frac{1}{\Lambda} \left(\frac{8}{\Lambda}\right)^2 \|\theta_{\mu'}(\hat{A})\|_{l^1}^2 < \frac{64}{\Lambda^2} \|\theta_{\mu'}(\hat{A})\|_{l^1}^2,
\]
\[
|K_{b,c}^{(3)}| \leq \frac{3}{2\Lambda} \|\theta_{\mu'}(\hat{A})\|_{l^1} \frac{4}{|z|_R} + \frac{14}{13} \frac{4}{|z|_R} < \frac{C_{A,A}}{|z|_R}
\]
for all \( b, c \in G_3' \). Here, to estimate \(|K_{b,c}^{(1)}|\) we have used that \( \varepsilon < \Lambda/6 \).
Finally, recalling (4.4.25) and using the above estimates we find that

\[ z_{\mu,d}(k) \alpha_{\mu,d}(1)(k) = \sum_{b,c \in G_1} f(d' - b) \sum_{j=0}^{3} K_{b,c}^{(j)} g(c - d') \]

where, in particular,

\[ \alpha_{\mu,d}(1,0) = - \sum_{b,c \in G_1} f(d' - b) \left[ \delta_{b,c} + \frac{\theta_{\mu'}(\hat{A}(b - c))}{\theta_{\mu'}(c - d')} \right] g(c - d'). \]

Furthermore, it follows easily from (4.4.38) that, for \( 0 \leq j \leq 2 \),

\[ |\alpha_{\mu,d}(1,j)| \leq C_j \]

with

\[ C_0 := \frac{1}{2\Lambda} \left( 1 + \frac{1}{2\Lambda} ||\theta_{\mu'}(\hat{A})||_{l^1} \right) ||f||_{l^1} ||g||_{l^1}, \]

\[ C_1 := \frac{\varepsilon}{2\Lambda^2} \left( 1 + \frac{\theta_{\mu'}(\hat{A})}{6\Lambda} ||f||_{l^1} ||g||_{l^1} \right), \]

\[ C_2 := \frac{64}{\Lambda^3} ||\theta_{\mu'}(\hat{A})||_{l^1}^2 ||f||_{l^1} ||g||_{l^1}, \]

while for \( j = 3 \),

\[ |\alpha_{\mu,d}(1,3)| \leq C_{\Lambda,A,f,g} \frac{1}{|z|_R}. \]

This completes the proof of the lemma.

\[ \square \]

**Proof of Lemma 4.4.3**

We first derive the following inequality.

**Proposition 4.4.7.** Let \( \alpha \) and \( \delta \) be constants with \( 1 < \alpha \leq 2 \) and \( 1 < \delta \leq 2 \). Suppose that \( f \) is a function on \( \Gamma^\# \) obeying \( ||b|^{\alpha} f(b)||_{l^1} < \infty \). Then, for any \( \xi_1, \xi_2 \in \Gamma^\# \) with \( \xi_1 \neq \xi_2 \),

\[ \sum_{b \in \Gamma^\# \setminus \{\xi_1, \xi_2\}} \frac{|f(b - \xi_1)|}{|b - \xi_2|^\delta} \leq \frac{C}{|\xi_1 - \xi_2|^{\alpha + \delta - 2}} \times \begin{cases} 1 & \text{if } \alpha, \delta < 2, \\ \ln |\xi_1 - \xi_2| & \text{if } \alpha = 2 \text{ or } \delta = 2, \end{cases} \]

where \( C = C_{\Gamma^\#, \alpha, \delta, f} \) is a constant.

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Proof. We can obtain the desired estimate as follows:

\[
\sum_{b \in \Gamma \setminus \{\xi_1, \xi_2\}} \frac{|f(b - \xi_1)|}{|b - \xi_2|^\delta} \leq \sup_{b \in \Gamma} \left( |b - \xi_1|^\alpha |f(b - \xi_1)| \right) \sum_{b \in \Gamma \setminus \{\xi_1, \xi_2\}} \frac{1}{|b - \xi_2|^\delta |b - \xi_1|^\alpha} \\
\leq \|b\|^\alpha \|f(b)\|_{L^1} \sum_{b \in \Gamma \setminus \{\xi_1, \xi_2\}} \frac{1}{|b - \xi_2|^\delta |b - \xi_1|^\alpha} \\
= \|b\|^\alpha \|f(b)\|_{L^1} \sum_{b \in \Gamma \setminus \{0, \xi_1 - \xi_2\}} \frac{1}{|b|^\delta |b - (\xi_1 - \xi_2)|^\alpha} \\
\leq \|b\|^\alpha \|f(b)\|_{L^1} C_{\Gamma^\#} \int_{|x| \geq \Lambda} \frac{d^2x}{|x - (\xi_1 - \xi_2)|^\alpha} \\
(\text{by a rotation in } \mathbb{R}^2 \text{ such that } \xi_1 - \xi_2 \to |\xi_1 - \xi_2|(1, 0)) \\
= \|b\|^\alpha \|f(b)\|_{L^1} C_{\Gamma^\#} \int_{|y| \geq \Lambda} \frac{d^2y}{|y - \xi_1 - \xi_2(1, 0)|^\alpha} \\
= \|b\|^\alpha \|f(b)\|_{L^1} C_{\Gamma^\#} \int_{|z| \geq \Lambda/|\xi_1 - \xi_2|} \frac{d^2z}{|z|^\delta |z - (1, 0)|^\alpha} \\
\leq \frac{C_{\Gamma^\#, \alpha, \delta, f}}{|\xi_1 - \xi_2|^\delta + \alpha - 2} \times \begin{cases} 
1 & \text{if } \alpha, \delta < 2, \\
\ln |\xi_1 - \xi_2| & \text{if } \alpha = 2 \text{ or } \delta = 2.
\end{cases}
\]

\[
\square
\]

We now apply this proposition for proving Lemma 4.4.3.

Proof of Lemma 4.4.3. First observe that

\[
\|\pi(b) T_{G'G'}^m \pi(c) \| = \sup_{\varphi \in L^2} \|\pi(b) T_{G'G'}^m \pi(c) \varphi\|_{L^2} = \|\pi(b) T_{G'G'}^m e^{i c x} \|_{L^2} \\
= \|\langle e^{i b x} \| \Gamma^{1/2} \| T_{G'G'}^m e^{i c x} \| \Gamma^{1/2} \rangle_{L^2} = \langle e^{i b x} \| \Gamma^{1/2} \| T_{G'G'}^m e^{i c x} \| \Gamma^{1/2} \rangle_{L^2} \\
= \| (T_{G'G'}^m)_{b,c} \|.
\]

Hence, by Proposition 4.4.6 with \( \beta = 2 \), for all \( b, c \in G' \) and \( m \geq 1 \),

\[
|(T_{G'G'}^m)_{b,c}| = \|\pi(b) T_{G'G'}^m \pi(c)\| \leq (1 + 2m) \left( \frac{17}{18} \right)^m \frac{1}{1 + |b - c|^2}.
\]

This inequality is also valid for \( m = 0 \) because

\[
|(T_{G'G'}^0)_{b,c}| = |\delta_{b,c}| = \begin{cases} 
1 & \text{if } b = c, \\
0 & \text{if } b \neq c,
\end{cases} \leq \frac{1}{1 + |b - c|^2}.
\]

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Thus,

\[
|\Phi_{d',d''}(k)| = \sum_{m=0}^{\infty} \sum_{b,c \in G'} \frac{f(d' - b)}{N_b(k)} (T_{m_{G'}})_{b,c} g(c - d'')
\]

\[
\leq \frac{1}{\varepsilon |v|} \left[ \sum_{m=0}^{\infty} (1 + 2m) \left( \frac{17}{18} \right)^m \right] \sum_{b \in G'} |f(d' - b)| \sum_{c \in G'} |g(c - d'')| \sum_{c \in G' \setminus \{b\}} \frac{|g(c - d'')|}{|b - c|^2} \qquad (4.4.41)
\]

\[
\leq \frac{C}{\varepsilon |v|} \sum_{b \in G'} |f(d' - b)| \left[ |g(b - d'')| + \sum_{c \in G' \setminus \{b\}} \frac{|g(c - d'')|}{|b - c|^2} \right],
\]

where \( C \) is an universal constant.

Now, by the triangle inequality, Hölder’s inequality, and since \( || \cdot ||_{l_2} \leq || \cdot ||_{l_1} \),

\[
\sum_{b \in G'} |f(d' - b)| |g(b - d'')| = \sum_{b \in G'} \left| \frac{|d' - d''|^2}{|d' - d''|^2} |f(d' - b)| |g(b - d'')| \right|
\]

\[
\leq \frac{4}{|d' - d''|^2} \sum_{b \in G'} \left( |d' - b|^2 + |b - d''|^2 \right) |f(d' - b)| |g(b - d'')| \leq \frac{4}{|d' - d''|^2} \left( ||b^2 f(b)||_{l_2} ||g||_{l_2} + ||f||_{l_2} ||b^2 g(b)||_{l_2} \right)
\]

\[
\leq \frac{4}{|d' - d''|^2} \left( ||b^2 f(b)||_{l_1} ||g||_{l_1} + ||f||_{l_1} ||b^2 g(b)||_{l_1} \right) \leq \frac{C_{f,g}}{|d' - d''|^2}.
\]

(4.4.42)

Furthermore, by Proposition 4.4.7 with \( \alpha = \delta = 2 \), for any \( 0 < \epsilon_1 < 2 \),

\[
\sum_{c \in G' \setminus \{b\}} \frac{|g(c - d'')|}{|b - c|^2} \leq C_{r^*,g} \frac{\ln |b - d''|}{|b - d''|^2} \leq \frac{C_{r^*,g,\epsilon_1}}{|b - d''|^{2-\epsilon_1}}.
\]

Applying this inequality and (4.4.42) to (4.4.41) we obtain

\[
|\Phi_{d',d''}(k)| \leq \frac{C}{\varepsilon |v|} \left[ \frac{C_{f,g}}{|d' - d''|^2} + C_{r^*,g,\epsilon_1} \sum_{b \in G'} \frac{|f(d' - b)|}{|b - d''|^2} \right].
\]

Applying again Proposition 4.4.7 with \( \alpha = 2 \) and \( \delta = 2 - \epsilon_1 \) we conclude that, for any \( 0 < \epsilon_2 < 2 - \epsilon_1 \),

\[
|\Phi_{d',d''}(k)| \leq \frac{C}{\varepsilon |v|} \left[ \frac{C_{f,g}}{|d' - d''|^2} + C_{r^*,f,g,\epsilon_1,\epsilon_2} \frac{\ln |d' - d''|}{|d' - d''|^{2-\epsilon_1}} \right] \leq \frac{C_{r^*,f,g,\epsilon_1,\epsilon_2}}{|v|} \frac{C_{r^*,f,g,\epsilon_1,\epsilon_2}}{|d' - d''|^{2-\epsilon_1-\epsilon_2}}.
\]

Finally, recall from Proposition 4.4.4(ii) that \( |z_{v',d}| < 3|d| \) and \( |z_{v',d}| < 3|v| \), observe that \( |d' - d''| = |d| \), and set \( \epsilon = \epsilon_1 + \epsilon_2 \). Then, for any \( 0 < \epsilon < 2 \),

\[
|\Phi_{d',d''}(k)| \leq \frac{C_{r^*,f,g,\epsilon_1,\epsilon_2}}{|d|} \frac{C_{r^*,f,g,\epsilon_1,\epsilon_2}}{|d|^2-\epsilon_1-\epsilon_2} \leq \frac{C_{r^*,f,g,\epsilon_1,\epsilon_2}}{|z_{v',d}|^3-\epsilon}.
\]

Choosing \( \epsilon = 10^{-1} \) we obtain the desired inequality. \( \square \)
4.5 Bounds on the derivatives

In the last section we expressed \( \Phi_{d',d''}(k) \) as a sum of certain functions \( \alpha_{\mu,d'}^{(j)}(k) \) for \( k \) in the \( \varepsilon \)-tubes with large \(|v|\). In this section we provide bounds for the derivatives of all these functions. We first give all the statements and then the proofs.

Our first lemma concerns the derivatives of \( \Phi_{d',d''}(k) \).

Lemma 4.5.1 (Derivatives of \( \Phi_{d',d''}(k) \)). Under Hypothesis 4.4.1, let \( f \) and \( g \) be functions in \( l^1(\Gamma^\#) \) and suppose either (i) or (ii) where:

(i) \( G = \{0\} \) and \( k \in (T_0 \setminus \cup_{b \in G} T_b) \setminus \mathcal{K}_R \);

(ii) \( G = \{0, d\} \) and \( k \in (T_0 \cap T_d) \setminus \mathcal{K}_R \).

Then, for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for any \( d', d'' \in G \),

\[
\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \right| \leq \frac{C}{|v|},
\]

where \( C \) is a constant with \( C = C_{\varepsilon,A,f,g,m,n} \) if (i) or \( C = C_{A,f,g,m,n} \) if (ii).

We now improve the estimate of Lemma 4.5.1(ii) for \( d' \neq d'' \).

Lemma 4.5.2 (Derivatives of \( \Phi_{d',d''}(k) \) for \( d' \neq d'' \)). Consider a constant \( \beta \geq 2 \) and suppose that \( \|b|^{\beta} \hat{q}(b)\|_{11} < \infty \) and \( \|(1 + |b|^\beta)\hat{A}(b)\|_{11} < 2\varepsilon/63 \). Let \( \nu \in \{1, 2\} \) and let \( f \) and \( g \) be functions on \( \Gamma^\# \) obeying \( \|b|^{\beta} f(b)\|_{11} < \infty \) and \( \|b|^{\beta} g(b)\|_{11} < \infty \). Suppose further that \( G = \{0, d\} \) and \( k \in T_0 \cap T_d \) with \( |v| > \frac{2}{\varepsilon} \|b|^{\beta} \hat{q}(b)\|_{11} \). Then, for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for any \( d', d'' \in G \) with \( d' \neq d'' \),

\[
\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Phi_{d',d''}(k) \right| \leq \frac{C}{|d|^{1+\beta}},
\]

where \( C = C_{\varepsilon,A,f,g,m,n} \) is a constant.

Observe that, in particular, this Lemma with \( m = n = 0 \) generalizes Lemma 4.4.3.

We next have bounds for the derivatives of \( \alpha_{\mu,d'}^{(j)}(k) \).

Lemma 4.5.3 (Derivatives of \( \alpha_{\mu,d'}^{(j)}(k) \)). Under Hypothesis 4.4.1, let \( \nu \in \{1, 2\} \) and let \( f \) and \( g \) be functions in \( l^1(\Gamma^\#) \). Suppose either (i) or (ii) where:

(i) \( G = \{0\} \) and \( k \in (T_\nu(0) \setminus \cup_{b \in G'} T_b) \setminus \mathcal{K}_R \);

(ii) \( G = \{0, d\} \) and \( k \in (T_\nu(0) \cap T_\nu(d)) \setminus \mathcal{K}_R \).
Then, there is a constant \( \rho = \rho_{\epsilon, \Lambda, q, m, n} \) with \( \rho \geq R \) such that, for \( |v| \geq \rho \) and for \( (\mu, d') = (\nu, 0) \) if (i) or \( (\mu, d') \in \{(\nu, 0), (\nu', d)\} \) if (ii), for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for \( 1 \leq j \leq 2 \),

\[
\left| \frac{\partial^{n+m}}{\partial k^n \partial k^m} \alpha_{\mu,d'}^{(j)}(k) \right| \leq \frac{C_j}{(2|z_{\mu,d'}(k)| - R)^j} \quad \text{and} \quad \left| \frac{\partial^{n+m}}{\partial k^n \partial k^m} \alpha_{\mu,d'}^{(3)}(k) \right| \leq \frac{C_3}{|z_{\mu,d'}(k)|R^2},
\]

where \( C_l = C_{l; f, g, \Lambda, q, m, n, m} \) for \( 1 \leq l \leq 3 \) are constants. Furthermore,

\[
C_{1; f, g, \Lambda, 1, 0}, C_{1; f, g, \Lambda, 0, 1} \leq 13\Lambda^{-2} \|f\|_1 \|g\|_1 \quad \text{and} \quad C_{1; f, g, \Lambda, 1, 1} \leq 65\Lambda^{-3} \|f\|_1 \|g\|_1.
\]

**Proof of Lemma 4.5.1**

**Proof of Lemma 4.5.1.** [Step 0] When there is no risk of confusion we shall use the same notation to denote an operator or its matrix. Define

\[
\mathcal{F}_{BC} := [f(b - c)]_{b \in B, c \in C}, \quad \mathcal{G}_{BC} := [g(b - c)]_{b \in B, c \in C}, \quad \Phi_G(k) := \left[ \Phi_{b,d'}(k; G) \right]_{d',d'' \in G}.
\]

Here \( \mathcal{F}_{BC} \) and \( \mathcal{G}_{BC} \) are \( |B| \times |C| \) matrices and \( \Phi_G(k) \) is a \( |G| \times |G| \) matrix. First observe that

\[
\Phi_G(k) = \left[ \sum_{b',c \in G} f(d' - b) \frac{(R_{G'}^{-1})_{b',c}}{N_b(k)} g(c - d'') \right]_{d',d'' \in G}
\]

can be written as the product of matrices

\[
\mathcal{F}_{GG'} \Delta_k^{-1} R_{G'}^{-1} \mathcal{G}(G').
\]

Furthermore, since on \( L^2_G \), we have \( \Delta_k^{-1} R_{G'}^{-1} = (R_{G'} \Delta_k)^{-1} = H_k^{-1} \), we can write \( \Phi_G(k) \) as

\[
\mathcal{F}_{GG'} H_k^{-1} \mathcal{G}(G').
\]

Hence,

\[
\frac{\partial^{n+m}}{\partial k^n \partial k^m} \Phi_G(k) = \mathcal{F}_{GG'} \frac{\partial^{n+m}}{\partial k^n \partial k^m} H_k^{-1} \mathcal{G}(G'). \tag{4.5.1}
\]

This is the quantity we want to estimate.

**Step 1** Let \( T = T(k) \) be an invertible matrix. Then applying \( \frac{\partial^m}{\partial k^m} \) to the identity \( TT^{-1} = I \) and using the Leibniz rule for \( \frac{\partial^m}{\partial k^m}(TT^{-1}) \) we find that

\[
\frac{\partial^m}{\partial k^m} T^{-1} = -T^{-1} \sum_{m=0}^{m-1} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \frac{\partial^{m_0-m_1} T}{\partial k^{m_0-m_1}} \frac{\partial^{m_1} T^{-1}}{\partial k^{m_1}}.
\]

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Iterating this formula \( m_0 - 1 \) times we obtain

\[
\frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \left( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \right) = \left[ \prod_{j=1}^m \left( m_j \right)^{-1} \cdot \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \right] H^{-1}_{k_2},
\]

where \( \sum_{j=1}^m n_j = m \). Thus, when we compute \( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \), the derivative \( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \) acts either on \( H^{-1}_{k_2} \) or \( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \). However, since \( \frac{\partial H_{k_2}^{(b,c)}}{\partial x^2_{k_2}} = 2(k_2 + b_2)\delta_{b,c} - 2 \hat{A}_2(b, c) \), we have \( \frac{\partial^m H_{k_2}^{(b,c)}}{\partial k^{-1}} = 0 \) if \( n_j \geq 1 \), and \( \frac{\partial^m H_{k_2}^{(b,c)}}{\partial k^{-1}} \) if \( n_j = 0 \). Similarly, using again (4.5.2) one can see that \( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \) is given by a finite linear combination of terms of the form (4.5.3), with \( m \) and \( k_2 \) replaced by \( n \) and \( k_1 \), respectively, and \( \sum_{j=1}^n n_j = n \). Therefore, combining all this we conclude that \( \frac{\partial^m H^{-1}_{k_2}}{\partial k^{-1}} \) is given by a finite linear combination of terms of the form

\[
\frac{\partial^m H_{k_2}^{(b,c)}}{\partial x^2_{k_2}} \Delta^{-1}_{k_2} R^{-1}_{G'G'},
\]

where \( \sum_{j=1}^{n+m} n_j \delta_{2,i_j} = m \) and \( \sum_{j=1}^{n+m} n_j \delta_{1,i_j} = n \), that is, where the sum of \( n_j \) for which \( i_j = 2 \) is equal to \( m \), and the sum of \( n_j \) for which \( i_j = 1 \) is equal to \( n \).
Furthermore, by Proposition 4.2.1,

\[ \frac{1}{|N_b(k)|} \leq \frac{1}{\varepsilon |v|} \]

for all \( b \in G' \), while by Proposition 3.3.1 we have

\[ \frac{1}{|N_b(k)|} \leq \frac{2}{\Lambda |v|} \]  

(4.5.5)

and

\[ |k_i + b_i| \leq |u_i + b_i| + |v_i| \]
\[ \leq |v| + |u + b| \]
\[ \leq \frac{2}{\Lambda} |N_b(k)| \]

for all \( b \in G' \) if \( G = \{0, d\} \), and for all \( b \in G' \setminus \{\tilde{b}\} \) if \( G = \{0\} \). Furthermore,

\[ |	ilde{b}| \leq \Lambda + |u| + |v| \]
\[ < \Lambda + 3|v|, \]

(4.5.6)

since \(|u| < 2|v| \) because \( k \in T_0 \) (see (4.2.2)). Now, let \( 1_B(x) \) be the characteristic function of the set \( B \). Then, using the above estimates,

\[
\sup_{c \in G'} \sum_{b \in G'} \left| \left( \frac{\partial^\mu H_k}{\partial k_{ij}^\nu} \Delta_k^{-1} \pi_{G'} \right)_{b,c} \right|
\leq \sup_{c \in G'} \sum_{b \in G'} \left[ \frac{2|k_{ij} + b_{ij}| |\delta_{n_{ij},1} + 2\delta_{n_{ij,2}} + 2|\hat{A}_{ij}(b - c)|}{|N_b(k)|} |\delta_{n_{ij,1}}| \right]
\leq \sup_{c \in G'} \left[ \frac{2|k_{ij} + \tilde{b}_{ij}| + 2|\hat{A}_{ij}(\tilde{b} - c)|}{|N_b(k)|} |\delta_{b,c}| \right] 1_{G'}(\tilde{b})
\]

\[
+ \sup_{c \in G'} \sum_{b \in G' \setminus \{\tilde{b}\}} \left[ \frac{2|k_{ij} + b_{ij}| + 2|\hat{A}_{ij}(b - c)|}{|N_b(k)|} |\delta_{b,c}| \right]
\leq \frac{2|k_{ij} + \tilde{b}_{ij}| + 2 + 2||\hat{A}||\mu}{\varepsilon |v|} 1_{G'}(\tilde{b}) + \sup_{c \in G'} \sum_{b \in G' \setminus \{\tilde{b}\}} \left[ \frac{4}{\Lambda} + \frac{2}{|N_b(k)|} \right] |\delta_{b,c}| + \frac{2}{|N_b(k)|} |\hat{A}_{ij}(b - c)|
\leq \frac{2}{\varepsilon |v|} \left( 2(\varepsilon |v| + |\tilde{b}|) + 2 + 2||\hat{A}||\mu \right) 1_{G'}(\tilde{b}) + \frac{4}{\Lambda} + \frac{4}{\Lambda |v|} + \frac{4}{\Lambda |v|} ||\hat{A}||l_i
\leq \frac{2}{\varepsilon |v|} \left( 12|v| + 2\Lambda + 2 + 2||\hat{A}||\mu \right) 1_{G'}(\tilde{b}) + \frac{4}{\Lambda} + \frac{4}{\Lambda |v|} + \frac{4}{\Lambda |v|} ||\hat{A}||l_i
\]

(recall that \(|v| > 1\))

\[ \leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{\Lambda,A} + C_{\Lambda,A}. \]
Similarly, 

\[
\sup_{b \in G'} \sum_{c \in G'} \left| \left( \frac{\partial^{n_j} H_k}{\partial k^{n_j}_{ij}} \Delta_k^{-1} \pi_{G'} \right)_{b,c} \right| \leq \sup_{b \in G'} \sum_{c \in G'} \left[ \frac{2|k_{ij} + b_{ij}| |\delta_{n_j,1}|}{|N_b(k)|} + \frac{2|\hat{A}_{ij}(b - \tilde{b})|}{|N_b(k)|} \right] 1_{G'}(\tilde{b}) \\
+ \sup_{b \in G'} \sum_{c \in G' \setminus \{b\}} \left[ \frac{2|k_{ij} + b_{ij}| + 2|\hat{A}_{ij}(b - \tilde{b})|}{|N_b(k)|} \right] 1_{G'}(\tilde{b})
\]

\[
\leq \frac{2|k_{ij} + b_{ij}| + 2 + 2\|\hat{A}\|_\nu}{\varepsilon|v|} 1_{G'}(\tilde{b}) + \sup_{b \in G'} \sum_{c \in G' \setminus \{b\}} \left[ \frac{4}{\Lambda} + \frac{2}{|N_b(k)|} \right] \delta_{b,c} + \frac{2|\hat{A}_{ij}(b - c)|}{|N_b(k)|}
\]

\[
\leq \frac{2}{\varepsilon|v|} (2(|u| + |v| + |\tilde{b}|) + 2 + 2\|\hat{A}\|_\nu) 1_{G'}(\tilde{b}) + \frac{4}{\Lambda} + \frac{4}{\Lambda|v|} \|\hat{A}\|_\nu \\
\leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{A,A} + C_{A,A}.
\]

Hence, by Proposition 4.1.2,

\[
\left\| \frac{\partial^{n_j} H_k}{\partial k^{n_j}_{ij}} \Delta_k^{-1} \pi_{G'} \right\| \leq 1_{G'}(\tilde{b}) \varepsilon^{-1} C_{A,A} + C_{A,A}.
\]

[Step 4] By a similar (and much simpler) calculation (using Proposition 4.1.2) we get

\[
\| F_{G|G'} \| \leq \| f \|_\nu, \\
\| G_{G|G'} \| \leq \| g \|_\nu, \\
\| \Delta_k^{-1} \pi_{G'} \| \leq 1_{G'}(\tilde{b}) \frac{1}{\varepsilon|v|} + (1 - 1_{G'}(\tilde{b})) \frac{2}{\Lambda|v|}.
\]

From Lemma 4.1.1 we have

\[
\| R^{-1}_{G|G'} \| \leq 18.
\]

Thus, the operator norm of (4.5.4) is bounded by

\[
\left\| \prod_{j=1}^{n+m} \Delta_k^{-1} R_{G|G'}^{-1} \frac{\partial^{n_j} H_k}{\partial k^{n_j}_{ij}} \Delta_k^{-1} R_{G|G'}^{-1} \right\| \leq \| \Delta_k^{-1} \| \| R_{G|G'}^{-1} \| \left\| \prod_{j=1}^{n+m} \frac{\partial^{n_j} H_k}{\partial k^{n_j}_{ij}} \Delta_k^{-1} \pi_{G'} \right\| \| R_{G|G'}^{-1} \|,
\]

which is bounded either by

\[
\frac{1}{\varepsilon|v|} 18 \prod_{j=1}^{n+m} (\varepsilon^{-1} C_{A,A} + C_{A,A}) 18 \leq \varepsilon^{-(n+m+1)} C_{A,A,n,m} \frac{1}{|v|}
\]

if \( G = \{0\} \), or by

\[
\frac{1}{\Lambda|v|} 18 \prod_{j=1}^{n+m} C_{A,A} 18 \| g \|_\nu \leq C_{A,A,n,m} \frac{1}{|v|}
\]

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if \( G = \{0,d\} \). Therefore,
\[
\left\| \frac{\partial^{n+m} H^{-1}_{k_1 \partial k_2}}{\partial k_1^n \partial k_2^m} \Phi_G(k) \right\| \leq \sum_{\text{finite sum where \# of terms depend on } n \text{ and } m} \frac{C'}{|v|} \leq C_n,m \frac{C'}{|v|} \leq \frac{C'}{|v|},
\]
(4.5.8)

with \( C = C_{\varepsilon,\Lambda,A,n,m} \) if \( G = \{0\} \) or \( C = C_{\Lambda,A,n,m} \) if \( G = \{0,d\} \). Finally, recalling (4.5.1) and (4.5.7) we have
\[
\left\| \frac{\partial^{n+m} H^{-1}_{k_1 \partial k_2}}{\partial k_1^n \partial k_2^m} \right\| \leq \|F\| \|G\| \leq \|f\| \|g\| \leq \frac{C}{|v|},
\]
where \( C = C_{\varepsilon,\Lambda,A,n,m,f,g} \) if \( G = \{0\} \) or \( C = C_{\Lambda,A,n,m,f,g} \) if \( G = \{0,d\} \). This is the desired inequality. The proof of the lemma is complete.  

**Proof of Lemma 4.5.2**

Let \( \mathbb{R}^+ \) be the set of non-negative real numbers and let \( \sigma \) be a real-valued function on \( \mathbb{R}^+ \) such that:

(i) \( \sigma(t) \geq 1 \) for all \( t \in \mathbb{R}^+ \) with \( \sigma(0) = 1 \);

(ii) \( \sigma(s) \sigma(t) \geq \sigma(s+t) \) for all \( s, t \in \mathbb{R}^+ \);

(iii) \( \sigma \) increases monotonically.

For example, for any \( \beta \geq 0 \) the functions \( t \mapsto e^{\beta t} \) and \( t \mapsto (1+t)^\beta \) satisfy these properties.

Now, let \( T \) be a linear operator from \( L^2_C \) to \( L^2_B \) with \( B, C \subset \Gamma^# \), (or a matrix \( T = [T_{b,c}] \) with \( b \in B \) and \( c \in C \)), and consider the \( \sigma \)-norm
\[
\|T\|_{\sigma} := \max \left\{ \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \sigma(|b-c|), \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \sigma(|b-c|) \right\}.
\]

In §4.9 we prove that this norm has the following properties.

**Proposition 4.5.4** (Properties of \( \| \cdot \|_\sigma \)). Let \( S \) and \( T \) be linear operators from \( L^2_C \) to \( L^2_B \) with \( B, C \subset \Gamma^# \). Then:

(a) \( \|T\| \leq \|T\|_{\sigma=1} \leq \|T\|_{\sigma} \);

(b) If \( B = C \), then \( \|ST\|_{\sigma} \leq \|S\|_{\sigma} \|T\|_{\sigma} \);
(c) If $B = C$, then $\| (I + T)^{-1} \|_\sigma \leq (1 - \| T \|_\sigma)^{-1}$ if $\| T \|_\sigma < 1$;

(d) $|T_{b,c}| \leq \frac{1}{\sigma(b-c)} \| T \|_\sigma$ for all $b \in B$ and all $c \in C$.

Using these properties we prove Lemma 4.5.2.

**Proof of Lemma 4.5.2.** We follow the same notation as above. First observe that, similarly as in the last proof we can write

$$
\Phi_{d',d''}(k) = \mathcal{F}_{d'} \mathcal{G}_{d'} \Delta^{-1}_k R^{-1}_{G'} \mathcal{G}_{G'}(d'') = \mathcal{F}_{d'} \mathcal{G}_H^{-1} \mathcal{G}_{G'}(d'').
$$

Now, let $\sigma(|b|) = (1 + |b|)^{\beta}$, and observe that there is a positive constant $C_{\beta}$ such that $\sigma(|b|) \leq C_{\beta}(1 + |b|^{\beta})$ for all $b \in \Gamma^\#$. Then, it is easy to see that

$$
\| \mathcal{F}_{d'} \mathcal{G}_\sigma \| = \| f \|_\sigma \leq C_{\beta}(1 + |b|^{\beta}) f(b) \|_V
$$

and

$$
\| \mathcal{G}_{G'} \|_\sigma \| = \| g \|_\sigma \leq C_{\beta}(1 + |b|^{\beta}) g(b) \|_V.
$$

Furthermore, by (4.4.36) and Proposition 4.1.2,

$$
\| R^{-1}_{G'} \|_\sigma = \| (I + T_{G'})^{-1} \|_\sigma \leq \sum_{j=0}^{\infty} \| T_{G'} \|_\sigma^j < 18,
$$

and since for diagonal operators the $\sigma$-norm and the operator norm agree, from (4.5.7) we have

$$
\| \Delta^{-1}_k \|_\sigma \leq \frac{2}{\Lambda |v|}.
$$

Hence, in view of Proposition 4.5.4(b) and Proposition 4.4.4(ii),

$$
|\Phi_{d',d''}(k)| \leq \| \mathcal{F}_{d'} \mathcal{G}_\sigma \| \| (I + T_{G'})^{-1} \|_\sigma \leq C_{\beta,f,g,\Lambda,A,m,n} \frac{1}{|d|^1},
$$

and by repeating the proof of Lemma 4.5.1 with the operator norm replaced by the $\sigma$-norm we obtain

$$
\left\| \frac{\partial^n + m}{\partial k_1^n \partial k_2^n} \Phi_{d',d''}(k) \right\|_\sigma \leq C_{\beta,f,g,\Lambda,A,m,n} \frac{1}{|d|^1}.
$$

Therefore, by Proposition 4.5.4(d), for any integers $n$ and $m$ with $n + m \geq 0$,

$$
\left\| \frac{\partial^n + m}{\partial k_1^n \partial k_2^n} \Phi_{d',d''}(k) \right\|_\sigma \leq C_{\beta,f,g,\Lambda,A,m,n} \frac{1}{|d|^{1+\beta}}.
$$

This is the desired inequality. \qed
Proof of Lemma 4.5.3

Define the operator \( M^{(j)} : L^2_{G_4} \to L^2_{G_3} \) as

\[
M^{(j)} := \begin{cases} 
    S & \text{if } j = 1, \\
    W & \text{if } j = 2, \\
    Z & \text{if } j = 3,
\end{cases}
\]

where \( S, W \) and \( Z \) are given by (4.4.23). In order to prove Lemma 4.5.3 we first prove the following proposition.

Proposition 4.5.5. Assume the same hypotheses of Lemma 4.5.3. Then, for any integers \( n \) and \( m \) with \( n + m \geq 1 \) and for \( 1 \leq j \leq 3 \),

\[
\left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Delta_k^{-1} M^{(j)} \right\| \leq \frac{C_j}{(2|z_{\mu,d'}(k)| - R)^j},
\]

where \( C_1 = C_1;A,n,m \) and \( C_j = C_j;A,n,m \) for \( 2 \leq j \leq 3 \) are constants. Furthermore,

\[
C_1;A,0,0 \leq \frac{13}{2}, \quad C_1;A,0,1 \leq \frac{13}{2}, \quad \text{and} \quad C_1;A,1,1 \leq \frac{65}{2}.
\]

Proof. \[\text{Step 0}\] To simplify the notation write

\[
w = w_{\mu,d'}, \quad z = z_{\mu,d'} \quad \text{and} \quad |z|_R = 2|z| - R.
\]

First observe that, for any analytic function of the form \( h(k) = \tilde{h}(w(k), z(k)) \), we have

\[
\frac{\partial}{\partial k_1} h = \left( \frac{\partial w}{\partial k_1} \frac{\partial}{\partial w} + \frac{\partial z}{\partial k_1} \frac{\partial}{\partial z} \right) h = \left( \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \right) h, \\
\frac{\partial}{\partial k_2} h = \left( \frac{\partial w}{\partial k_2} \frac{\partial}{\partial w} + \frac{\partial z}{\partial k_2} \frac{\partial}{\partial z} \right) h = i(-1)^\nu \left( \frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right) h.
\]

Thus,

\[
\left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \Delta_k^{-1} M^{(j)} \right\| = \left\| (i(-1)^\nu)^m \sum_{p=0}^n \sum_{r=0}^n \binom{m}{p} \binom{n}{r} (-1)^{m-p} \frac{\partial^{n-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta_k^{-1} M^{(j)} \right\|
\]

\[
\leq 2^{n+m} \sup_{p \leq r \leq n} \left\| \frac{\partial^{n-r+m-p}}{\partial z^{n-r+m-p}} \frac{\partial^{r+p}}{\partial w^{r+p}} \Delta_k^{-1} M^{(j)} \right\|.
\]

Now, by the Leibniz rule,

\[
\left\| \frac{\partial^n}{\partial z^n} \frac{\partial^m}{\partial w^m} \Delta_k^{-1} M^{(j)} \right\| = \left\| \sum_{p=0}^m \sum_{r=0}^n \binom{m}{p} \binom{n}{r} \frac{\partial^{n-r+m-p}}{\partial z^{n-r}} \frac{\partial^p}{\partial w^p} \Delta_k^{-1} \frac{\partial^{r+p}}{\partial z^r \partial w^p} M^{(j)} \right\|
\]

\[
\leq 2^{n+m} \sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{n-r+m-p}}{\partial z^{n-r}} \frac{\partial^p}{\partial w^p} \Delta_k^{-1} \frac{\partial^{r+p}}{\partial z^r \partial w^p} \right\|.
\]

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Furthermore, we shall prove below that
\[
\sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{n-r+m-p} \Delta_{k}^{-1}}{\partial z^{n-r} \partial w^{m-p}} \right\| \sup_{p \leq m} \left\| \frac{\partial^{r+p} M(j)}{\partial z^{r} \partial w^{p}} \right\| \leq \frac{C_{j,n,m}}{|z|^{n+J}},
\]
(4.5.10)
with constants \(C_{1,n,m} = C_{1,0,n,m;\Lambda,\Lambda,0}\) and \(C_{j,n,m} = C_{j,0,n,m;\Lambda,\Lambda,0}\) for \(2 \leq j \leq 3\). Hence,
\[
\left\| \frac{\partial^{n} \partial^{m} \Delta_{k}^{-1} M(j)}{\partial z^{n} \partial w^{m}} \right\| \leq 2^{n+m} \frac{C_{j,n,m}}{|z|^{n+J}}.
\]
Therefore, being careful with the indices,
\[
\left\| \frac{\partial^{n+m} k^{1} \partial k^{2}}{\partial k^{1} \partial k^{2}} \Delta_{k}^{-1} M(j) \right\| \leq 2^{n+m} \sup_{p \leq m} \sup_{r \leq n} 2^{n-r+m-p+r+p} \frac{C_{j,n,m} \Delta_{k}^{-1} M(j)}{\partial z^{n-r} \partial w^{m-p}} \leq \frac{C_{j}}{|z|^{n+J}},
\]
where \(C_{1} = C_{1;\Lambda,A,0,n,m}\) and \(C_{j} = C_{j;\Lambda,A,0,n,m}\) for \(2 \leq j \leq 3\). This is the desired inequality.

We are left to prove (4.5.10) and estimate the constants \(C_{1;\Lambda,A,i,j}\) for \(i,j \in \{0,1\}\) to finish the proof of the proposition.

**Step 1** The first step for obtaining (4.5.10) is to estimate
\[
\left\| \frac{\partial^{r+p} \Delta_{k}^{-1}}{\partial z^{r} \partial w^{p}} \right\|.
\]
Observe that
\[
\left( \frac{\partial^{r+p} \Delta_{k}^{-1}}{\partial z^{r} \partial w^{p}} \right)_{b,c} = \left( \frac{\partial^{r+p} \Delta_{k}^{-1}}{\partial z^{r} \partial w^{p}} \right)_{b,c} = \left| \frac{\partial^{p} 1}{\partial z^{p}} \frac{1}{w - 2i \theta_{\mu}(b - d')} \frac{\partial^{r} \delta_{b,c}}{\partial z^{r} \partial w^{p}} \right| = \left| \frac{(-1)^{p} p!}{w - 2i \theta_{\mu}(b - d')}^{p+1} \frac{\partial^{r} \delta_{b,c}}{\partial z^{r} \partial w^{p}} \right| \leq \frac{p! r! \delta_{b,c}}{|w - 2i \theta_{\mu}(b - d')|^{p+1}} \leq \frac{2}{|z - 2i \theta_{\mu}(b - d')|},
\]
and recall from (4.4.17) and (4.4.18) that, for all \(b \in G_{3}',\)
\[
\frac{1}{|z - 2i \theta_{\mu}(b - d')|} \leq \frac{1}{|z|} \quad \text{and} \quad \frac{1}{|w - 2i \theta_{\mu}(b - d')|} \leq \frac{1}{|z|}.
\]
(4.5.11)
Then,
\[
\left( \frac{\partial^{r+p} \Delta_{k}^{-1}}{\partial z^{r} \partial w^{p}} \right)_{b,c} \leq \frac{p! r! 2^{r+1} \delta_{b,c}}{\Lambda^{r+1} |z|^{r+1}},
\]
and consequently,
\[
\left[ \sup_{b \in G_{3}'} \sum_{c \in G_{3}'} + \sup_{c \in G_{3}'} \sum_{b \in G_{3}'} \right] \left( \frac{\partial^{r+p} \Delta_{k}^{-1}}{\partial z^{r} \partial w^{p}} \right)_{b,c} \leq \frac{p! r! 2^{r+1} \delta_{b,c}}{\Lambda^{r+1} |z|^{r+1}},
\]
and
\[
\left[ \sup_{b \in G_{3}'} \sum_{c \in G_{3}'} + \sup_{c \in G_{3}'} \sum_{b \in G_{3}'} \right] \delta_{b,c} \leq \frac{p! r! 2^{r+2} \delta_{b,c}}{\Lambda^{r+2} |z|^{r+2}}.
\]
We now estimate the second factor in (4.5.10). Let us first consider the case $j = 1$, that is, $M^{(1)} = S$. Since $S = (I - Y_{33})^{-1}$, the operator $S$ is clearly invertible. Thus, by applying (4.5.2) with $T = S^{-1}$, one can see that $\frac{\partial r S}{\partial w^p}$ is given by a finite linear combination of terms of the form

$$\left[ \prod_{j=1}^{p} S \frac{\partial^{n_j} S^{-1}}{\partial w^{n_j}} \right] S,$$

(4.5.13)

where $\sum_{j=1}^{p} n_j = p$. Hence, when we compute $\frac{\partial r}{\partial x} \frac{\partial r S}{\partial w^p}$, the derivative $\frac{\partial r}{\partial x}$ acts either on $S$ or $\frac{\partial^{n_j} S^{-1}}{\partial w^{n_j}}$. Similarly, using again (4.5.2) with $T = S^{-1}$, one can see that $\frac{\partial^r S}{\partial z^r}$ is given by a finite linear combination of terms of the form (4.5.13), with $p$ and $w$ replaced by $r$ and $z$, respectively, and $\sum_{j=1}^{r} m_j = r$. Thus, we conclude that $\frac{\partial^{r+p} S}{\partial z^r \partial w^p}$ is given by a finite linear combination of terms of the form

$$\left[ \prod_{j=1}^{r+p} S \frac{\partial^{m_j+n_j} S^{-1}}{\partial z^{m_j} \partial w^{n_j}} \right] S,$$

(4.5.14)

where $\sum_{j=1}^{r+p} m_j = r$ and $\sum_{j=1}^{r+p} n_j = p$. Indeed, observe that the general form of the terms (4.5.14) follows directly from (4.5.2) because that identity is also valid for mixed derivatives.

Since $S = (I - Y_{33})^{-1}$ with $\|Y_{33}\| < 1/14$ and

$$Y_{b,c} = \frac{-2i \theta_{\mu'}(\hat{A}(b - c)) z}{(w - 2i \theta_{\mu'}(c - d'))(z - 2i \theta_{\mu'}(c - d'))},$$

(4.5.15)

we have

$$\|S\| = \|(I - Y_{33})^{-1}\| \leq \frac{1}{1 - \|Y_{33}\|} \leq \frac{14}{13}$$

(4.5.16)

and

$$\left| \frac{\partial^{j+l} S}{\partial z^j \partial w^l} \right|_{b,c} = \left| \frac{\partial^{j+l} (I - Y_{33})}{\partial z^j \partial w^l} \right|_{b,c} = \left| \frac{\partial^j}{\partial z^j} \frac{\partial^{l} Y_{b,c}}{\partial w^l} \right| = \frac{\partial^j}{\partial z^j} \frac{-2i \theta_{\mu'}(\hat{A}(b - c)) z}{z - 2i \theta_{\mu'}(c - d')} \frac{\partial^{l} \frac{1}{w - 2i \theta_{\mu'}(c - d')}}{w - 2i \theta_{\mu'}(c - d')}. $$

Furthermore,

$$\frac{\partial^j}{\partial z^j} \frac{-2i \theta_{\mu'}(\hat{A}(b - c)) z}{z - 2i \theta_{\mu'}(c - d')} = \frac{(-1)^{j-1} j! 2i \theta_{\mu'}(\hat{A}(b - c)) 2i \theta_{\mu'}(c - d')}{(z - 2i \theta_{\mu'}(c - d'))^{j+1}}$$

for $j \geq 1$, and

$$\frac{\partial^l}{\partial w^l} \frac{1}{w - 2i \theta_{\mu'}(c - d')} = \frac{(-1)^l !}{(w - 2i \theta_{\mu'}(c - d'))^{l+1}}$$

for $l \geq 0$. 

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Recall from (4.4.18) and (4.4.20) that, for all \( c \in G' \),

\[
\frac{|c - d'|}{|w - 2i\theta(c - d')|} \leq \frac{|c - d'|}{|c - d'| - \varepsilon} \leq 2. \tag{4.5.17}
\]

Then, using this and (4.5.11), for \( j \geq 1 \) and \( l \geq 0 \),

\[
\left| \left( \frac{\partial^{j+l}}{\partial z^j \partial w^l} S^{-1} \right)_{b,c} \right| \leq \frac{j! l! |\hat{A}(b - c)|}{|z - 2i\theta(c - d')|^{j+1} |w - 2i\theta(c - d')|^{l+1}} \left| \frac{c - d'}{\Lambda^l |z|^{j+1}_R} \right|, \tag{4.5.18}
\]

while for \( j = 0 \) and \( l \geq 0 \),

\[
\left| \left( \frac{\partial^{j+l}}{\partial z^j \partial w^l} S^{-1} \right)_{b,c} \right| \leq \frac{l! |\hat{A}(b - c)| |z|}{|z - 2i\theta(c - d')|^{j+1} |w - 2i\theta(c - d')|^{l+1}} \leq \frac{2l! |\hat{A}(b - c)|}{\Lambda^{l+1}}. \tag{4.5.19}
\]

Consequently,

\[
\left( \sup_{b \in G_3} \sum_{c \in G_3} + \sup_{c \in G_3} \sum_{b \in G_3} \right) \left| \left( \frac{\partial^{j+l}}{\partial z^j \partial w^l} S^{-1} \right)_{b,c} \right| \leq \left( 1 - \delta_{0,j} + \frac{|z|_R}{2\Lambda} \delta_{0,j} \right) \frac{2^{j+l+3}j! l!}{\Lambda^l |z|^{j+1}_R} \left| \hat{A} \right|_1. \tag{4.5.20}
\]

Therefore, by Proposition (4.1.2),

\[
\left| \frac{\partial^{j+l}}{\partial z^j \partial w^l} S^{-1} \right| \leq \left( 1 - \delta_{0,j} + \frac{|z|_R}{2\Lambda} \delta_{0,j} \right) \frac{2^{j+l+3}j! l!}{\Lambda^l |z|^{j+1}_R} \left| \hat{A} \right|_1. \tag{4.5.20}
\]

Thus, for \( r \geq 1 \), in view of (4.5.14) where \( \sum_{j=1}^{r+p} m_j = r \),

\[
\left| \frac{\partial^{r+p}}{\partial z^r \partial w^p} S \right| \leq C_{r,p} \left[ \prod_{j=1}^{r+p} \|S\| \left| \frac{\partial^{m_j+n_j}}{\partial z^{m_j} \partial w^{n_j}} S^{-1} \right| \right] \|S\| \leq C_{r,p} \left[ \prod_{j=1}^{r+p} C_{\Lambda,A} \frac{2^{m_j+n_j} \delta_{0,m_j}}{\Lambda^{m_j}} \left| \hat{A} \right|_1 \right] C_{\Lambda,A} \left( 1 - \delta_{0,m_j} + \frac{|z|_R}{2\Lambda} \delta_{0,m_j} \right) \frac{1}{|z|^{m_j+1}_R},
\]

since \( m_j \geq 1 \) for at least one \( 1 \leq j \leq r + p \). Similarly, if \( r = 0 \) then

\[
\left| \frac{\partial^{r+p}}{\partial z^r \partial w^p} S \right| \leq C_{\Lambda,A,r,p}.
\]
Hence, in view of (4.5.12),

\[
\sup_{p \leq m} \sup_{r \leq n} \left| \frac{\partial^{n-r+m-p} \Delta_k^{-1}}{\partial z^{n-r} \partial w^{m-p}} \right| \leq \sup_{p \leq m} \sup_{r \leq n} \frac{(m-p)! (n-r)! 2^{n-r+r}}{\Lambda^{m-p+1} |z|^{n-r+1}} C_{\Lambda,A,r,p} \| \hat{A} \|_1^1 \left( 1 - \delta_{0,r} + \frac{|z|_R \delta_{0,r}}{2 \Lambda |z|_R^4} \right) \frac{1}{|z|_R^{r+1}}
\]

\[
\leq C_{\Lambda,A,m,n} \frac{1}{|z|_R^{n+1}}.
\]

This proves (4.5.10) for \( j = 1 \).

**Step 3** We now estimate the constant \( C_{1: \Lambda,A,i,j} \) for \( i, j \in \{0,1\} \). First observe that

\[
\left| \frac{\partial w}{\partial k_j} \right| = |\delta_{1,j} + i(-1)^\nu \delta_{2,j}| = 1 \quad \text{and} \quad \left| \frac{\partial z}{\partial k_j} \right| = |\delta_{1,j} - i(-1)^\nu \delta_{2,j}| = 1.
\]

Thus, in view of (4.5.16) and (4.5.20), since \( |z| \geq |v| > R \geq 2\Lambda \),

\[
\left| \frac{\partial S}{\partial k_j} \right| = \left| -S \frac{\partial S^{-1}}{\partial k_j} S \right| = \left| -S \left( \frac{\partial w \partial S^{-1}}{\partial k_j} \frac{\partial w}{\partial w} + \frac{\partial z \partial S^{-1}}{\partial k_j} \frac{\partial z}{\partial z} \right) S \right|
\]

\[
\leq \|S\|^2 \left( \left| \frac{\partial S^{-1}}{\partial w} \right| + \left| \frac{\partial S^{-1}}{\partial z} \right| \right) \leq \left( \frac{3}{2} \right)^2 \left( \frac{2^4 \|\hat{A}\|_1^1}{\Lambda |z|_R^2} + \frac{2^2 \|\hat{A}\|_1^1}{\Lambda^2} \right)
\]

\[
\leq \left( \frac{3}{2} \right)^2 \frac{8\|\hat{A}\|_1^1}{\Lambda^2} = \frac{18\|\hat{A}\|_1^1}{\Lambda^2}.
\]

Similarly,

\[
\frac{\partial^2 S}{\partial k_i \partial k_j} = \frac{\partial S}{\partial k_i} \left( \frac{\partial w \partial S^{-1}}{\partial k_j} \frac{\partial w}{\partial w} + \frac{\partial z \partial S^{-1}}{\partial k_j} \frac{\partial z}{\partial z} \right) S - S \left( \frac{\partial w \partial S^{-1}}{\partial k_j} \frac{\partial w}{\partial k_i} + \frac{\partial z \partial S^{-1}}{\partial k_j} \frac{\partial z}{\partial k_i} \right) \frac{\partial S}{\partial k_i}
\]

\[
- S \left( \frac{\partial w \partial S^{-1}}{\partial k_i} \frac{\partial w}{\partial k_j} + \frac{\partial z \partial S^{-1}}{\partial k_i} \frac{\partial z}{\partial k_j} \right) \frac{\partial S}{\partial k_j} S
\]

so that, using the above inequality as well,

\[
\left| \frac{\partial^2 S}{\partial k_i \partial k_j} \right| \leq 2 \|S\| \left| \frac{\partial S}{\partial k_i} \right| \left( \left| \frac{\partial S^{-1}}{\partial w} \right| + \left| \frac{\partial S^{-1}}{\partial z} \right| \right)
\]

\[
+ \|S\|^2 \left( \left| \frac{\partial^2 S^{-1}}{\partial w^2} \right| + 2 \left| \frac{\partial^2 S^{-1}}{\partial w \partial z} \right| + \left| \frac{\partial^2 S^{-1}}{\partial z^2} \right| \right)
\]

\[
\leq 2 \left( \frac{3}{2} \right)^2 \frac{18\|\hat{A}\|_1^1}{\Lambda^2} \frac{8\|\hat{A}\|_1^1}{\Lambda^2} + \left( \frac{3}{2} \right)^2 \left( \frac{2^3 \|\hat{A}\|_1^1}{\Lambda^3} + \frac{2^5 \|\hat{A}\|_1^1}{\Lambda^3} + \frac{2^6 \|\hat{A}\|_1^1}{\Lambda^3} \right)
\]

\[
\leq \frac{432}{\Lambda^4} \|\hat{A}\|_1^1 + \frac{54}{\Lambda^3} \|\hat{A}\|_1^1 \leq \frac{55}{\Lambda^3} \left( \frac{8\|\hat{A}\|_1^1}{\Lambda} + 1 \right).
\]

Furthermore, by (4.5.12),

\[
\left| \frac{\partial \Delta_k^{-1}}{\partial k_j} \right| \leq \left| \frac{\partial \Delta_k^{-1}}{\partial w} \right| + \left| \frac{\partial \Delta_k^{-1}}{\partial z} \right| \leq \frac{2^2}{\Lambda^2 |z|_R} + \frac{2^3}{\Lambda^2 |z|_R^2} \leq \frac{8}{\Lambda^2 |z|_R}
\]

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and
\[
\left\| \frac{\partial^2 \Delta^{-1}}{\partial k_i \partial k_j} \right\| \leq \left\| \frac{\partial^2 \Delta^{-1}}{\partial w^2} \right\| + 2 \left\| \frac{\partial^2 \Delta^{-1}}{\partial z \partial w} \right\| + \left\| \frac{\partial^2 \Delta^{-1}}{\partial z^2} \right\|
\]
\[
\leq \frac{2^3}{\Lambda^3 |z|_R} + \frac{2^4}{\Lambda^2 |z|^2} + \frac{2^6}{\Lambda |z|^3} < \frac{5 \cdot 2^3}{\Lambda^3} \frac{1}{|z|_R}.
\]
Hence, since \( \| \hat{A} \|_{r_1} < 2\varepsilon / 63 \) and \( \varepsilon < \Lambda / 6 \),
\[
\left\| \frac{\partial}{\partial k_j} \Delta^{-1} S \right\| \leq \left\| \frac{\partial \Delta^{-1}}{\partial k_j} \right\| \| S \| + \| \Delta^{-1} \| \left\| \frac{\partial S}{\partial k_j} \right\| \leq \frac{8}{\Lambda^2} \frac{3}{2} + \frac{2}{\Lambda |z|_R} \frac{18 \| \hat{A} \|_{\varepsilon}}{\Lambda^2} \leq \frac{13}{\Lambda^2} \frac{1}{|z|_R}
\]
and
\[
\left\| \frac{\partial^2}{\partial k_i \partial k_j} \Delta^{-1} S \right\| \leq \left\| \frac{\partial^2 \Delta^{-1}}{\partial k_i \partial k_j} \right\| \| S \| + \left\| \frac{\partial \Delta^{-1}}{\partial k_i} \right\| \left\| \frac{\partial S}{\partial k_i} \right\| + \left\| \frac{\partial \Delta^{-1}}{\partial k_j} \right\| \left\| \frac{\partial S}{\partial k_j} \right\| + \left\| \Delta^{-1} \right\| \left\| \frac{\partial^2 S}{\partial k_i \partial k_j} \right\|
\]
\[
\leq \frac{1}{|z|_R} \left( \frac{5 \cdot 2^3}{\Lambda^3} \frac{3}{2} + \frac{8}{\Lambda^2} \frac{18 \| \hat{A} \|_{\varepsilon}}{\Lambda^2} + \frac{2}{\Lambda} \frac{55 \| \hat{A} \|_{\varepsilon}}{\Lambda^3} \left( \frac{8 \| \hat{A} \|_{\varepsilon}}{\Lambda} + 1 \right) \right) < \frac{65}{\Lambda^3} \frac{1}{|z|_R}.
\]
Therefore,
\[
C_{1;\Lambda,A,1,0} \leq \frac{13}{\Lambda^2}, \quad C_{1;\Lambda,A,0,1} \leq \frac{13}{\Lambda^2} \quad \text{and} \quad C_{1;\Lambda,A,1,1} \leq \frac{65}{\Lambda^3},
\]
as was to be shown.

**Step 4** To prove (4.5.10) for \( j = 2 \) we need to bound
\[
\left\| \frac{\partial^{r+p} M^{(2)}}{\partial z^r \partial w^p} \right\| = \left\| \frac{\partial^{r+p} W}{\partial z^r \partial w^p} \right\|.
\]
Recall from (4.4.23) that
\[
W = \sum_{j=1}^{\infty} W_j = \sum_{j=1}^{\infty} \sum_{m=1}^{j} (Y_{33})^{m-1} X_{33}(Y_{33})^{j-m},
\]
where \( Y_{b,c} \) is given above by (4.5.15) and \( \| X_{33} \| \leq C / |z| < 1 / 3 \) with
\[
X_{b,c} = \frac{(c - d') \cdot \hat{A}(b - c) - \hat{q}(b - c) - 2i\theta_{\mu}(\hat{A}(b - c)) w}{(w - 2i\theta_{\mu}(c - d'))(z - 2i\theta_{\mu}(c - d'))}.
\]
First observe that
\[
\frac{\partial^{r+p}}{\partial z^r w^p} (Y_{33})^{m-1} X_{33}(Y_{33})^{j-m}
\]
is given by a sum of \( j^{r+p} \) terms of the form
\[
\frac{\partial^{r+n_1} Y_{33}}{\partial z_{l_1} \partial w_{n_1}} \ldots \frac{\partial^{m-1+n_m-1} Y_{33}}{\partial z_{l_{m-1}} \partial w_{n_{m-1}}} \frac{\partial^{m+n_m} X_{33}}{\partial z_{l_m} \partial w_{n_m}} \frac{\partial^{m+1+n_{m+1}} Y_{33}}{\partial z_{l_{m+1}} \partial w_{n_{m+1}}} \ldots \frac{\partial^{r+j+n_j} Y_{33}}{\partial z_{l_j} \partial w_{n_j}},
\]

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where there are \( j \) factors ordered as in the product \((Y_{33})^{m-1}X_{33}(Y_{33})^{j-m}\). Furthermore, for each term in the sum we have \( \sum_{i=1}^{j} l_i = r \) and \( \sum_{i=1}^{j} n_i = p \). Thus,

\[
\left\| \frac{\partial^{r+p}}{\partial z^r w^p} W \right\| = \left\| \sum_{j=1}^{\infty} \frac{\partial^{r+p}}{\partial z^r w^p} W_j \right\| = \left\| \sum_{j=1}^{\infty} \sum_{m=1}^{j} \frac{\partial^{r+p}}{\partial z^r w^p} (Y_{33})^{m-1}X_{33}(Y_{33})^{j-m} \right\| \quad (4.5.21)
\]

\[
\leq \sum_{j=1}^{\infty} \sum_{m=1}^{j} \left\| \frac{\partial^{r+p}}{\partial z^r w^p} (Y_{33})^{m-1}X_{33}(Y_{33})^{j-m} \right\|
\]

\[
\leq \sum_{j=1}^{\infty} \sum_{m=1}^{j} \sup_{\mathcal{I}} \left\| \frac{\partial^{l+1+n_1}Y_{33}}{\partial z^l \partial w^{n_1}} \cdots \frac{\partial^{l+m+n_m}X_{33}}{\partial z^l \partial w^{n_m}} \cdots \frac{\partial^{l+j+n_j}Y_{33}}{\partial z^l \partial w^{n_j}} \right\|, \quad (4.5.22)
\]

where

\[
\mathcal{I} := \left\{ (l_i, n_i) \mid l_i \leq r \text{ and } n_i \leq p \text{ for } 1 \leq i \leq j \text{ with } \sum_{i=1}^{j} l_i = r \text{ and } \sum_{i=1}^{j} n_i = p \right\}.
\]

Note, we can differentiate the series \((4.5.21)\) term-by-term because the sum \( \sum_{j=1}^{\infty} W_j \) converges uniformly and the sum \( \sum_{j=1}^{\infty} W_j \) is finite. We next estimate the factors in \((4.5.22)\).

Combining \((4.5.18)\) and \((4.5.19)\) we have

\[
\left\| \frac{\partial^{l+1+n_1}Y_{b,c}}{\partial z^l \partial w^{n_1}} \right\| \leq \left( 1 - \delta_0 l_i + \frac{|z|_R \delta_0 l_i}{2\Lambda} \right) \frac{2^{l_i+1} l_i! n_i!}{\Lambda^{n_i} |z|_{R}^{l_i+1}} |\hat{A}(b - c)|. \quad (4.5.24)
\]

Furthermore, using \((4.5.11)\) and \((4.17)\),

\[
\left\| \frac{\partial^{l+1+n_1}X_{b,c}}{\partial z^l \partial w^{n_1}} \right\| \leq \frac{2^{l_i+1} l_i! n_i! |\hat{A}(b - c)|}{\Lambda^{n_i} |z|_{R}^{l_i+1}} \frac{1}{\Lambda} \left( 4|\hat{A}(b - c)| + \frac{1}{\Lambda} |\hat{q}(b - c)| \right). \quad (4.5.25)
\]
Thus, by Proposition (4.1.2), since $|z| \geq |v| > R \geq 2\Lambda$,
\[
\left\| \frac{\partial^{l_1+n_i}}{\partial z_1^l \partial w_1^{n_i}} Y_{b,c} \right\| \leq \left( 1 - \delta_{0,l_1} + \frac{|z|_R \delta_{0,l_1}}{2\Lambda} \right) \frac{2^{l_1+3} l_1! n_i!}{\Lambda^n_i |z|_R^{l_1+1}} \| \hat{A} \|_{l_1} \\
\leq \left( \frac{1}{|z|_R} + \frac{1}{2\Lambda} \right) \frac{2^{l_1+3} l_1! n_i!}{\Lambda^n_i |z|_R^{l_1+1}} \| \hat{A} \|_{l_1} \leq \frac{2^{l_1+3} l_1! n_i!}{\Lambda^n_i |z|_R^{l_1+1}} \| \hat{A} \|_{l_1}
\]

and
\[
\left\| \frac{\partial^{l_1+n_i}}{\partial z_1^l \partial w_1^{n_i}} X_{b,c} \right\| \leq \frac{2^{l_1+2} l_1! n_i!}{\Lambda^n_i |z|_R^{l_1+1}} \left( 4 \| \hat{A} \|_{l_1} + \frac{\| \hat{q} \|_{l_1}}{\Lambda} \right) = \left( 2\Lambda + \frac{\| \hat{q} \|_{l_1}}{2\| \hat{A} \|_{l_1}} \right) \frac{1}{|z|_R} \frac{2^{l_1+3} l_1! n_i!}{\Lambda^n_i+1 |z|_R^{l_1+1}} \| \hat{A} \|_{l_1}.
\]

Applying these estimates to (4.5.22) and recalling that $\sum_{i=1}^j l_i = r$ and $\sum_{i=1}^j n_i = p$ we have
\[
\left\| \frac{\partial^{r+p}}{\partial z_1^r w_1^p} W \right\| \leq \sum_{j=1}^{\infty} j^{r+p} \sum_{m=1}^{j} \left\| \frac{\partial^{l_1+n_i}}{\partial z_1^l \partial w_1^{n_i}} Y_{b,c} \right\| \cdots \left\| \frac{\partial^{l_{m+n_m}}}{\partial z_1^{l_m} \partial w_1^{n_m}} X_{b,c} \right\| \\
\leq \sum_{j=1}^{\infty} j^{r+p} \sum_{m=1}^{j} \sup_{\hat{A}} \left\{ \left( 2\Lambda + \frac{\| \hat{q} \|_{l_1}}{2\| \hat{A} \|_{l_1}} \right) \frac{1}{|z|_R} \frac{2^{l_1+3} l_1! n_i!}{\Lambda^n_i+1 |z|_R^{l_1+1}} \| \hat{A} \|_{l_1} \right\} \\
= \left( 2\Lambda + \frac{\| \hat{q} \|_{l_1}}{2\| \hat{A} \|_{l_1}} \right) \frac{1}{|z|_R} \frac{2^r}{\Lambda^p |z|_R^{r+1}} \sum_{j=1}^{\infty} j^{r+p} \left( 8\| \hat{A} \|_{l_1} \right) \frac{1}{\Lambda} \sup_{\hat{A}} \left\{ \prod_{i=1}^{j} l_i \prod_{m=1}^{j} n_m! \right\}^{j} \leq C_{A,A,q,r,p} \frac{1}{|z|_R^{r+1}}.
\]
This is the inequality we needed to prove (4.5.10) for \( j = 2 \). In fact, using (4.5.12) we obtain

\[
\sup_{p \leq m} \sup_{r \leq n} \left\| \frac{\partial^{n-r+m-p} \Delta_k^{\lambda-p}}{\partial z^{n-r} \partial w^{m-p}} \right\| \left\| \frac{\partial^{r+p} M^{(2)}}{\partial z^r \partial w^p} \right\| \leq \sup_{p \leq m} \sup_{r \leq n} \frac{(m-p)!(n-r)!2^{n-r+2}}{\Lambda^{m-p+1}|z|^{n-r+1}} \frac{C'_{\Lambda,q,r,p}}{|z|^{r+p+1}}.
\]

\[
\leq C_{\Lambda,A,q,m,n} 1 \text{ if } |z|^{n-r+2}.
\]

**Step 5** To prove (4.5.10) for \( j = 3 \) we need to estimate

\[
\left\| \frac{\partial^{r+p} M^{(3)}}{\partial z^r \partial w^p} \right\| = \left\| \frac{\partial^{r+p} Z}{\partial z^r \partial w^p} \right\|,
\]

where

\[
Z = \sum_{j=2}^{\infty} Z_j = \sum_{j=2}^{\infty} (X_{33} + Y_{33})^j - W_j - Y_{33}^j.
\]

First observe that

\[
\frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j = \frac{\partial^{r+p}}{\partial z^r \partial w^p} ((X_{33} + Y_{33})^j - W_j - Y_{33}^j)
\]

is given by a sum of \((2^j - j - 1) \cdot j^{r+p}\) terms of the form

\[
\frac{\partial^{l_1+n_1} Y_{33}}{\partial z^{l_1} \partial w^{n_1}} \cdots \frac{\partial^{l_m+n_m} X_{33}}{\partial z^{l_m} \partial w^{n_m}} \cdots \frac{\partial^{l_j+n_j} Y_{33}}{\partial z^{l_j} \partial w^{n_j}},
\]

where there are \( j - 2 \) factors involving \( X_{33} \) or \( Y_{33} \) and two factors containing \( X_{33} \). Furthermore, for each term in the sum we have \( \sum_{i=1}^{j} l_i = r \) and \( \sum_{i=1}^{j} n_i = p \). Thus,

\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\| \leq (2^j - j - 1) j^{r+p} \sup_{I} \left\| \frac{\partial^{l_1+n_1} Y_{33}}{\partial z^{l_1} \partial w^{n_1}} \right\| \cdots \left\| \frac{\partial^{l_m+n_m} X_{33}}{\partial z^{l_m} \partial w^{n_m}} \right\| \cdots \left\| \frac{\partial^{l_j+n_j} Y_{33}}{\partial z^{l_j} \partial w^{n_j}} \right\|,
\]

where the set \( I \) is given above by (4.5.23). Now observe that, the estimate for the derivatives of \( X_{33} \) in (4.5.27) is better than the estimate for the derivatives of \( Y_{33} \) in (4.5.26) because the former has an extra factor \( C_{\Lambda,A,q}/|z|_R < 1 \). Since the product (4.5.28) has at least two factors containing \( X_{33} \), we can estimate any of these products by considering the worst case. This happens when there are exactly two factors involving \( X_{33} \). Hence, by proceeding in this way, for each \( j \geq 2 \) we have

\[
\left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\| \leq (2^j - j - 1) j^{r+p} \sup_{I} \left\{ 2\Lambda + \frac{\|\hat{q}\|_1}{2\|A\|_1} \right\} \frac{1}{|z|_R} \prod_{i=1}^{j} \frac{2^{i+3q_i!n_i!}}{\Lambda^{n_i+1}|z|_R} \frac{\|\hat{A}\|_1}{|A|_t} \bigg( \frac{\Lambda^t}{2} \bigg)^j \leq C'_{\Lambda,A,q,r,p} j^{r+p} \left( \frac{2}{21} \right)^j \frac{1}{|z|^{r+p+2}}.
\]
since $\|A\|_{l_1} \leq 2\varepsilon/63$ and $\varepsilon < \Lambda/6$. Thus,
\[ \left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z \right\| \leq \sum_{j=2}^{\infty} \left\| \frac{\partial^{r+p}}{\partial z^r \partial w^p} Z_j \right\| \leq C_{\Lambda,A,q,r,p} \sum_{j=2}^{\infty} j^{r+p} \left( \frac{2}{21} \right)^j \leq C_{\Lambda,A,q,r,p} \frac{1}{|z|^{r+2}}. \]

Therefore, recalling (4.5.12),
\[ \sup_{r \leq m} \sup_{r \leq n} \left| \frac{\partial^{n-r+m-p}}{\partial z^{n-r} \partial w^{m-p}} \right| \left\| \frac{\partial^{r+p} M^{(3)}}{\partial z^r \partial w^p} \right\| \leq \sup_{r \leq m} \sup_{r \leq n} \frac{(m-p)! (n-r)! 2^{n-r+2}}{\Lambda^{m-p+1} |z|^{n-r+1}} C_{\Lambda,A,q,r,p} \frac{1}{|z|^{r+2}} \leq C_{\Lambda,A,q,m,n} \frac{1}{|z|^{n+3}}. \]

This is the desired inequality for $j = 3$. The proof of the proposition is complete.

We can now prove Lemma 4.5.3. We first prove it for $1 \leq j \leq 2$ and then for $j = 3$ separately.

**Proof of Lemma 4.5.3 for $1 \leq j \leq 2$.** Define the $|B| \times |C|$ matrices
\[ F_{BC} := [f(b - c)]_{b \in B, c \in C} \quad \text{and} \quad G_{BC} := [g(b - c)]_{b \in b, c \in C}, \]
and write
\[ w = w_{\mu,d'}, \quad z = z_{\mu,d'} \quad \text{and} \quad |z|_R = 2|z| - R. \]

First observe that, for $1 \leq j \leq 2$, the functions
\[ \left[ \alpha_{\mu,d'}^{(j)}(k) \right]_{d' \in G} = \sum_{b,c \in G_1} \frac{f(d' - b) M_{b,c}^{(j)} g(c - d')}{(w - 2i\theta_{\mu'}(b - d'))(z - 2i\theta_{\mu'}(b - d'))} \]
are the diagonal entries of the matrix
\[ F_{G_1'} \Delta_k^{-1} M^{(j)} G_{G_1'} \]
Thus, similarly as in the proof of Lemma 4.5.1, by Proposition 4.5.5, for $1 \leq j \leq 2$,
\[ \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha_{\mu,d'}^{(j)}(k) \right\| \| F_{G_1'} \| \left\| \frac{\partial^{n}}{\partial k_1^n} \frac{\partial^{m}}{\partial k_2^m} \Delta_k^{-1} M^{(j)} \right\| \| G_{G_1'} \| \leq C_j \| f \| \| g \| \| \frac{\partial^{n}}{\partial k_1^n} \frac{\partial^{m}}{\partial k_2^m} \Delta_k^{-1} M^{(j)} \| \| \frac{\partial^{n}}{\partial k_1^n} \frac{\partial^{m}}{\partial k_2^m} \Delta_k^{-1} M^{(j)} \| \leq C_1 \| z \|_R \]

where $C_1 = C_{1;\Lambda,A,m,n,f,g}$ and $C_2 = C_{2;\Lambda,A,m,n,f,g}$ are constants. Furthermore,
\[ C_{1;\Lambda,A,1,0,f,g} \leq C_{1;\Lambda,A,1,0,f,g} \leq \frac{13}{\Lambda^2} \| f \|_l \| g \|_l, \quad C_{1;\Lambda,A,0,1,f,g} \leq C_{1;\Lambda,A,0,1,f,g} \leq \frac{13}{\Lambda^2} \| f \|_l \| g \|_l \]

and
\[ C_{1;\Lambda,A,1,1,f,g} \leq C_{1;\Lambda,A,1,1,f,g} \leq \frac{65}{\Lambda^3} \| f \|_l \| g \|_l. \]

This proves the lemma for $1 \leq j \leq 2$. 

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Proof of Lemma 4.5.3 for j = 3. We need to estimate
\[ \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha^{(3)}_{A, \delta}(k) = \sum_{j=1}^{4} \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_j(k), \]
where \( R_1, \ldots, R_4 \) are given by (4.4.5), (4.4.6), (4.4.34) and (4.4.26), respectively.

Step 1 We begin with the terms involving \( R_1 \) and \( R_2 \), which are easier. We follow the same notation as above. First observe that, similarly as in the proof of Lemma 4.5.1, since \( \Delta_k R^{-1}_{G'} = H^{-1}_k \) on \( L^2_{G'} \), we have
\[ \left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{R}_1(k) \right| = \left\| \mathcal{F}_{(d') G_1} \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{G}_{G_2}(d') \right\| \leq \left\| \mathcal{F}_{(d') G_1} \right\| \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} H^{-1}_k \mathcal{G}_{G_2}(d') \right\|, \]
\[ \left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{R}_2(k) \right| = \left\| \mathcal{F}_{(d') G_2} \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \mathcal{G}_{G'}(d') \right\| \leq \left\| \mathcal{F}_{(d') G_2} \right\| \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} H^{-1}_k \mathcal{G}_{G'}(d') \right\|. \]
Furthermore, we have already proved that (see (4.5.7) and (4.5.8))
\[ \left\| \mathcal{F}_{(d') G_1} \right\| \leq \| f \|_{l^1}, \quad \left\| \mathcal{G}_{G'}(d') \right\| \leq \| g \|_{l^1} \]
and, since \( |z| \leq 3|v| \) by Proposition 4.4.4,
\[ \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} H^{-1}_k \right\| \leq \varepsilon^{-(n+m+1)} C_{A,A,n,m} \frac{1}{|z|}. \]

Now recall that \( G_2' = \{ b \in G' \mid |b - d'| > \frac{1}{4} R \} \). Then,
\[ \sup_{b \in \{d'\}} \sum_{c \in G_2} \frac{|d' - c|^2}{|d' - c|^2} |f(b) - c| \leq \sup_{c \in G_2} \frac{1}{|d' - c|^2} \leq \frac{16}{R^2} \| b^2 f(b) \|_{l^1}, \]
\[ \sup_{c \in G_2} \sum_{b \in \{d'\}} |f(b) - c| \leq \sup_{c \in G_2} \frac{1}{|d' - c|^2} \| f(b) \|_{l^1} \leq \frac{16}{R^2} \| b^2 f(b) \|_{l^1}. \]

Hence, by Proposition (4.1.2),
\[ \| \mathcal{F}_{(d') G_2} \| \leq 16 \| b^2 f(b) \|_{l^1} \frac{1}{R^2}. \]

Similarly,
\[ \| \mathcal{G}_{G_2'}(d') \| \leq 16 \| b^2 f(b) \|_{l^1} \frac{1}{R^2}. \]

Therefore, combining all this, for \( 1 \leq j \leq 2 \) we obtain
\[ \left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_j(k) \right| \leq \varepsilon^{-(n+m+1)} C_{A,A,n,m,f,g} \frac{1}{|z| R^2}. \]
Step 2 Recall from (4.4.26) the expression for $R_4$. Then, similarly as above, by applying Proposition 4.5.5 for $j = 3$ we find that
\[
\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_4(k) \right| \leq \| F(d') \| G_1 \| \left( \| \partial^{n+m} G_1 \| + \| \partial^{n} \Delta_k^1 Z \| \right) \| \partial^{n} \Delta_k^1 Z \| \| f \|_p \| g \|_{tA}^{\Delta, A, q, n, m, 1} \frac{1}{|z|^3}.
\]

Step 3 To bound the derivatives of $R_3$ (which is given by (4.4.34)) we need a few more estimates. Recall from (4.4.29) that $W^{(j)} = \pi G_1^j T_{G'G'} \pi G_3$. First observe that
\[
\left| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1 \Delta_k^{-1} T_{33} T_{34} W^{(j-m-1)} \right| = \left| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1 \Delta_k^{-1} T_{33} T_{34} W^{(j-m-1)} \right|
\]
is given by a sum of $(j + 2)^{r+p}$ terms of the form
\[
\frac{\partial^{1+n_1} \Delta_k^{-1}}{\partial k_1^{1+n_1} \partial k_2^{n_1}} \pi G_1 \frac{\partial^{1+n_2} \Delta_k^{-1}}{\partial k_1^{1+n_2} \partial k_2^{n_2}} \frac{\partial^{m_1} \Delta_k^{-1} \Delta_k^{m_2} T_{33}}{\partial k_1^{m_1} \partial k_2^{m_2}} \frac{\partial^{m_3} \Delta_k^{-1} \Delta_k^{m_4} T_{34}}{\partial k_1^{m_3} \partial k_2^{m_4}} \frac{\partial^{m_5} \Delta_k^{-1} \Delta_k^{m_6} T_{G'G'}}{\partial k_1^{m_5} \partial k_2^{m_6}} \frac{\partial^{m_7} \Delta_k^{-1} \Delta_k^{m_8} T_{G'G'}}{\partial k_1^{m_7} \partial k_2^{m_8}} \frac{\partial^{m_9} \Delta_k^{-1} \Delta_k^{m_{10}} T_{G'G'}}{\partial k_1^{m_9} \partial k_2^{m_{10}}} \pi G_3.
\]
Moreover, for each term in the sum we have $\sum_{i=1}^{j+2} t_i = r$ and $\sum_{i=1}^{j+2} n_i = p$. Thus, \[
\left| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1 \Delta_k^{-1} T_{33} T_{34} W^{(j-m-1)} \right| \leq (j + 2)^{r+p} \sup_{T'} \left( \prod_{i=1}^{j+2} \frac{\partial^{t_i+n_i}}{\partial k_1^{t_i} \partial k_2^{n_i}} \right) \pi G_3 \right), \quad (4.5.29)
\]
where the set $T'$ is given by (4.5.23) with $j$ replaced by $j + 2$ and
\[
T_{(i)} := \begin{cases} 
\Delta_k^{-1} \pi G_1 & \text{for } i = 1, \\
T_{33} & \text{for } 2 \leq i \leq m + 1, \\
T_{34} & \text{for } i = m + 2, \\
T_{G'G'} & \text{for } m + 3 \leq i \leq j + 2.
\end{cases} \quad (4.5.30)
\]

Step 3a The first step in bounding (4.5.29) is to estimate
\[
\left| \frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^r \partial k_2^p} \pi G_1 \right|.
\]
We follow the same argument that we have used in the proof of Lemma 4.5.1 to bound
\[
\left| \frac{\partial^{n+m} H_k^{-1}}{\partial k_1^n \partial k_2^m} \right|.
\]
In fact, in view of (4.5.2) one can see that
\[
\frac{\partial^{p} \Delta_k^{-1}}{\partial k_2^p} = \sum_{\text{finite sum}} \left( \prod_{j=1}^{p} \frac{\partial^{n_j} \Delta_k^{n_j}}{\partial k_2^{n_j}} \right) \Delta_k^{-1}, \quad (4.5.31)
\]
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where $\sum_{j=1}^{p} n_j = p$. Hence, when we compute $\frac{\partial^r \varphi \Delta^{-1}}{\partial k_1 \partial k_2}$, the derivative $\frac{\partial^r}{\partial k_1}$ acts either on $\Delta_k^{-1}$ or $\frac{\partial^p \varphi}{\partial k_2}$. However, since $\left( \frac{\partial \Delta_k}{\partial k_2} \right)_{b,c} = 2(k_2 + c_2)\delta_{b,c}$, we have $\frac{\partial^r}{\partial k_1} \frac{\partial^p \varphi}{\partial k_2} \Delta_k = 0$ if $n_j \geq 1$ and $\frac{\partial^r}{\partial k_1} \frac{\partial^p \varphi}{\partial k_2} \Delta_k = \frac{\partial^r}{\partial k_1} \Delta_k$ if $n_j = 0$. Similarly, using again (4.5.2) one can see that $\frac{\partial^r \Delta_k^{-1}}{\partial k_1}$ is given by a finite sum as in (4.5.31), with $p$ and $k_2$ replaced by $r$ and $k_1$, respectively, and $\sum_{j=1}^{r} n_j = r$. Thus, combining all this we conclude that

$$\frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^r \partial k_2^p} = \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \left[ \prod_{j=1}^{r+p} \Delta_k^{-1} \frac{\partial^{n_j} \Delta_k}{\partial k_{ij}^{n_j}} \right] \Delta_k^{-1},$$

(4.5.32)

where $\sum_{j=1}^{r+p} n_j \delta_{2,ij} = p$ and $\sum_{j=1}^{r+p} n_j \delta_{1,ij} = r$. If we observe that

$$\left( \frac{\partial^{n_j} \Delta_k}{\partial k_{ij}^{n_j}} \right)_{b,c} = \begin{cases} 2(k_{ij} + c_{ij})\delta_{b,c} & \text{if } n_j = 1, \\ 2\delta_{b,c} & \text{if } n_j = 2, \\ 0 & \text{if } n_j \geq 3, \end{cases}$$

and extract the “leading term” from the summation in (4.5.32), in a sense that will be clear below, we can rewrite (4.5.32) in terms of matrix elements as

$$\frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^r \partial k_2^p} \frac{1}{N_c(k)} = \frac{(-1)^r (r+p)!}{N_c(k)} \left[ \frac{2(k_1 + c_1)}{N_c(k)} \right]^r \left[ \frac{2(k_2 + c_2)}{N_c(k)} \right]^p + \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \frac{(2(k_1 + c_1))^{\alpha_j} (2(k_2 + c_2))^{\beta_j}}{N_c(k)^{r+p+1}},$$

(4.5.33)

where $\alpha_j + \beta_j < r + p$ for every $j$ in the summation. Recall from (4.5.5) and (4.5.6) that, for all $c \in G' \setminus \{\tilde{c}\}$,

$$\frac{|k_1 + c_1|}{|N_c(k)|} \leq \frac{2}{3} < \frac{1}{\varepsilon} < \frac{7}{2\varepsilon} \quad \text{and} \quad \frac{|k_1 + \tilde{c}_i|}{|N_{\tilde{c}}(k)|} \leq \frac{\Lambda + 3|v|}{\varepsilon|v|} \leq \frac{7}{2\varepsilon}. \quad (4.5.33)$$

Hence,

$$\left| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \frac{1}{N_c(k)} \right| \leq \frac{(r+p)!}{|N_c(k)|} \left( \frac{7}{\varepsilon} \right)^{r+p} + \sum_{\text{finite sum where } \# \text{ of terms depend on } r \text{ and } p} \left( \frac{7}{\varepsilon} \right)^{\alpha_j + \beta_j} \frac{1}{|N_c(k)|^2}$$

(4.5.34)

$$\leq \frac{(r+p)!}{|N_c(k)|} \left( \frac{7}{\varepsilon} \right)^{r+p} + C_{\varepsilon,r,p} \frac{1}{|N_c(k)|^2}.$$ 

Thus, by Proposition 4.1.2, since $|N_c(k)| \geq \varepsilon|v| \geq \varepsilon|z|/3$ for all $c \in G'$, we have

$$\left| \frac{\partial^{r+p} \Delta_k^{-1}}{\partial k_1^r \partial k_2^p \pi G'_1} \right| \leq \frac{7^{r+p} (r+p)!}{\varepsilon^{r+p+1}} \frac{3}{|z|} + C_{\varepsilon,r,p} \frac{1}{|z|^2}. \quad (4.5.35)$$

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Now, let \( \rho_1 = \rho_{1; \varepsilon, r, p} \) be the constant
\[
\rho_{1; \varepsilon, r, p} := \max_{\substack{l_1 \leq r \\ n_1 \leq p}} \frac{\varepsilon^{l_1 + n_1 + 1} C_{\varepsilon, l_1, n_1}}{4(l_1 + n_1)! 7^{l_1 + n_1}},
\]
where \( C_{\varepsilon, l_1, n_1} \) is the constant in (4.5.35). Then, for \(|z| > \rho_1\) and for any \( l_1 \leq r \) and any \( n_1 \leq p \),
\[
\left\| \frac{\partial^{l_1 + n_1} \Delta_k^{-1}}{\partial k_1^{l_1} \partial k_2^{n_1}} \right\| \leq \frac{7^{l_1 + n_1} (l_1 + n_1)!}{\varepsilon^{l_1 + n_1 + 1} |z|} + \frac{7^{l_1 + n_1} (l_1 + n_1)!}{\varepsilon^{l_1 + n_1 + 1} |z|} - 3 \frac{1}{|z|}.
\]

This is the first inequality we need to bound (4.5.29). We next estimate the other factors in that expression.

**Step 3b** Recall from (4.4.12) that
\[
T_{b,c} = \frac{1}{N_c(k)} (2(c + k) \cdot \hat{A}(b - c) - \hat{q}(b - c)).
\]

By direct calculation we have
\[
\frac{\partial^{r+p} T_{b,c}}{\partial k_1^r \partial k_2^p} = \left( \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \frac{1}{N_c(k)} \right) (2(c + k) \cdot \hat{A}(b - c) - \hat{q}(b - c)) + p \left( \frac{\partial^{r+p-1}}{\partial k_1^r \partial k_2^{p-1}} \frac{1}{N_c(k)} \right) 2 \hat{A}_j(b - c).
\]

Hence, using (4.5.33) and (4.5.34), since \(|N_c(k)| \geq \varepsilon|v| \geq \varepsilon|z|/3\) for all \( c \in G' \) and \(|v| > 1\),
\[
\left| \frac{\partial^{r+p} T_{b,c}}{\partial k_1^r \partial k_2^p} \right| \leq (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p} + C_{\varepsilon, r, p} \left( \frac{7}{\varepsilon} \right)^{r+p} \hat{A}_j(b - c) + \frac{|\hat{q}(b - c)|}{|v|} + \frac{C_{\varepsilon, r, p}}{|v|} \hat{A}_j(b - c)\right) - \frac{1}{|z|}.
\]

Therefore, by Proposition 4.1.2,
\[
\left\| \frac{\partial^{r+p} T_{G', G'}}{\partial k_1^r \partial k_2^p} \right\| \leq \Theta_{r,p},
\]

where
\[
\Theta_{r,p} := (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \| \hat{A} \|_{H^1} + C_{\varepsilon, A, q, r, p} \frac{1}{|z|}.
\]

This is the second estimate we need to bound (4.5.29). We next derive one more inequality.
Step 3c: Set

\[ Q_{b,c}^{r,p} := (1 + |b - c|^2) \frac{\partial^{r+p} T_{b,c}}{\partial k_1^r \partial k_2^p}. \]

We first prove that, for any \( B, C \subset G' \),

\[ \sup_{b \in B} \sum_{c \in C} |Q_{b,c}^{r,p}| \leq \Omega_{r,p} \]

and

\[ \sup_{c \in C} \sum_{b \in B} |Q_{b,c}^{r,p}| \leq \Omega_{r,p}, \]

where

\[ \Omega_{r,p} := (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \|(1 + b^2) \hat{A}(b)\|_{l^1} + C_{\varepsilon,A,q,r,p} \frac{1}{|z|}. \]  \hspace{1cm} (4.5.40)

In fact, in view of (4.5.37) we have

\[ \sup_{b \in B} \sum_{c \in C} |Q_{b,c}^{r,p}| = \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^2) \left| \frac{\partial^{r+p} T_{b,c}}{\partial k_1^r \partial k_2^p} \right| \]

\[ \leq \sup_{b \in B} \sum_{c \in C} (1 + |b - c|^2) \]

\[ \times \left( r + p! \left( \frac{7}{\varepsilon} \right)^{r+p+1} |\hat{A}(b - c)| + C_{\varepsilon,r,p} \frac{1}{|z|} \right) \]

\[ \leq (r + p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \|(1 + b^2) \hat{A}(b)\|_{l^1} + C_{\varepsilon,A,q,r,p} \frac{1}{|z|}, \]

and similarly we estimate \( \sup_{c \in C} \sum_{b \in B} |Q_{b,c}^{r,p}| \). Now observe that, as in (4.4.37), for any integer \( m \geq 0 \) and for any \( \xi_0, \xi_1, \ldots, \xi_{m+2} \in \Gamma' \), let \( b = \xi_0 \) and \( c = \xi_{m+2} \). Then,

\[ |b - c|^2 \leq 2(m + 2) \sum_{i=1}^{m+2} |\xi_i - \xi_{i-1}|^2. \]

To simplify the notation write

\[ \partial^{l_i,n_i} = \frac{\partial^{l_i+n_i}}{\partial k_1^{l_i} \partial k_2^{n_i}}, \]

and recall from (4.5.30) and (4.5.40) the definition of \( T_{(i)} \) and \( \Omega_{r,p} \). Hence, similarly as in the proof of Proposition 4.4.6, since \( |b - c| \geq R/4 \) for all \( b \in G'_1 \) and \( c \in G'_4 \),
\[
\sup_{b \in G'_1} \left| \sum_{c \in G'_4} \left( \prod_{i=2}^{m+2} \partial^{l_{i,n_i}} T_{(i)} \right) \right|_{b,c} \\
\leq \sup_{b \in G'_1} \frac{1}{1 + |b-c|^2} \sup_{c \in G'_4} \left| \sum_{c \in G'_4} (1 + |b-c|^2) \left( \prod_{i=2}^{m+2} \partial^{l_{i,n_i}} T_{(i)} \right) \right|_{b,c} \\
\leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{b \in G'_1} \sum_{c \in G'_4} (1 + |b - \xi|)^2 \left| \partial^{l_{2,n_2}} T_{b,\xi_1} \right| \\
\times \sum_{\xi_2 \in G'_3} (1 + |\xi_1 - \xi_2|^2) \left| \partial^{l_{3,n_3}} T_{\xi_1,\xi_2} \right| \cdots \sum_{\xi_{m+1} \in G'_3} (1 + |\xi_{m+1} - c|^2) \left| \partial^{l_{m+2,n_{m+2}}} T_{\xi_{m+1},c} \right| \\
\leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{b \in G'_1} \sum_{\xi_1 \in G'_3} (1 + |b - \xi_1|^2) \left| \partial^{l_{2,n_2}} T_{b,\xi_1} \right| \sup_{\xi_1 \in G'_3} \sum_{\xi_2 \in G'_3} (1 + |\xi_1 - \xi_2|^2) \left| \partial^{l_{3,n_3}} T_{\xi_1,\xi_2} \right| \\
\times \sup_{\xi_{m+1} \in G'_3} \sum_{c \in G'_4} (1 + |\xi_{m+1} - c|^2) \left| \partial^{l_{m+2,n_{m+2}}} T_{\xi_{m+1},c} \right| \\
= \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{b \in G'_1} \left| \sum_{\xi_1 \in G'_3} Q_{b,\xi_1}^{l_{2,n_2}} \cdots \sum_{\xi_{m+1} \in G'_3} Q_{\xi_{m+1},c}^{l_{m+2,n_{m+2}}} \right| \leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{i,n_i} \cdot \]

and

\[
\sup_{c \in G'_4} \left| \sum_{b \in G'_1} \left( \prod_{i=2}^{m+2} \partial^{l_{i,n_i}} T_{(i)} \right) \right|_{b,c} \\
\leq \sup_{b \in G'_1} \frac{1}{1 + |b-c|^2} \sup_{c \in G'_4} \left| \sum_{c \in G'_4} (1 + |b-c|^2) \left( \prod_{i=2}^{m+2} \partial^{l_{i,n_i}} T_{(i)} \right) \right|_{b,c} \\
\leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{b \in G'_1} \sum_{c \in G'_4} (1 + |b - \xi|^2) \left| \partial^{l_{m+2,n_{m+2}}} T_{\xi_{m+1},c} \right| \\
\times \sum_{\xi_{m+1} \in G'_3} (1 + |\xi_{m+1}|^2) \left| \partial^{l_{m+1,n_{m+1}}} T_{\xi_{m+1},\xi_{m+1}} \right| \cdots \sum_{b \in G'_1} (1 + |b - \xi|^2) \left| \partial^{l_{2,n_2}} T_{b,\xi_1} \right| \\
\leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{b \in G'_1} \sum_{\xi_{m+1} \in G'_3} (1 + |\xi_{m+1}|^2) \left| \partial^{l_{m+2,n_{m+2}}} T_{\xi_{m+1},c} \right| \\
\times \sup_{\xi_{m+1} \in G'_3} \sum_{c \in G'_4} (1 + |\xi_{m+1} - c|^2) \left| \partial^{l_{m+1,n_{m+1}}} T_{\xi_{m+1},\xi_{m+1}} \right| \\
\times \sup_{\xi_1 \in G'_3} \sum_{b \in G'_1} (1 + |b - \xi|^2) \left| \partial^{l_{2,n_2}} T_{b,\xi_1} \right| \\
= \frac{2(m+2)}{1 + \frac{1}{16} R^2} \sup_{c \in G'_4} \sum_{\xi_{m+1} \in G'_3} Q_{\xi_{m+1},c}^{l_{m+2,n_{m+2}}} \cdots \sum_{\xi_1 \in G'_3} Q_{\xi_1,b}^{l_{2,n_2}} \leq \frac{2(m+2)}{1 + \frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{i,n_i} 
\]
Therefore, by Proposition 4.1.2,

$$
\left\| \pi_{G_i} \prod_{i=2}^{m+2} \frac{\partial^{l_i+n_t} T_{i(i)}}{\partial k_{1i}^{l_i} \partial k_{2i}^{n_t}} \right\| \leq \frac{2(m+2)}{1+\frac{1}{16} R^2} \prod_{i=2}^{m+2} \Omega_{l_i, n_i}.
$$

We have all we need to bound (4.5.29).

**Step 3d** From (4.5.38) and (4.5.36) it follows that

$$
\left\| \prod_{i=m+3}^{j+2} \frac{\partial^{l_i+n_t} T_{i(i)}}{\partial k_{1i}^{l_i} \partial k_{2i}^{n_t}} \right\| \leq \prod_{i=m+3}^{j+2} \Theta_{l_i, n_i}
$$

and

$$
\left\| \frac{\partial^{l_i+n_t} T_{i(i)}}{\partial k_{1i}^{l_i} \partial k_{2i}^{n_t}} \right\| \leq (r+p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} \frac{1}{|z|}.
$$

Thus, recalling (4.5.29) we get

$$
\left\| \frac{\partial^{r+p}}{\partial k_1^{l_1} \partial k_2^{n_t}} \Delta_k^{-1} \pi_{G_3} T_{33} T_{34} W_{43}^{(j-m-1)} \right\| \leq (j+2)^{r+p} \sup_{T'} \left\{ \frac{1}{|z|} \frac{2(m+2)(r+p)!}{1+\frac{1}{16} R^2} \left( \frac{7}{\varepsilon} \right)^{r+p+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
$$

$$
\leq (j+2)^{r+p}(m+2) C \frac{C}{|z| R^2} \sup_{T'} \left\{ (l_1+n_1)! \left( \frac{7}{\varepsilon} \right)^{l_1+n_1+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
$$

where $C$ is an universal constant. Now, recall the definition of $\Theta_{r,p}$ and $\Omega_{r,p}$ in (4.5.39) and (4.5.40), observe that $\| \hat{A} \|_{l_1} < \|(1+b^2)\hat{A}\|_{l_1}$, and let $\rho_2 = \rho_{2;\varepsilon,A,q,r,p}$ be a sufficiently large constant such that, for $|z| > \rho_2$ and for any $l_i \leq r$ and any $n_i \leq p$,

$$
\Theta_{l_i, n_i}, \Omega_{l_i, n_i} \leq 2(l_i + n_i)! \left( \frac{7}{\varepsilon} \right)^{l_i+n_i+1} \| (1+b^2)\hat{A}(b) \|_{l_1}.
$$

Then,

$$
\left\| \frac{\partial^{r+p}}{\partial k_1^{l_1} \partial k_2^{n_t}} \Delta_k^{-1} \pi_{G_3} T_{33} T_{34} W_{43}^{(j-m-1)} \right\|
$$

$$
\leq (j+2)^{r+p}(m+2) C \frac{C}{|z| R^2} \sup_{T'} \left\{ (l_1+n_1)! \left( \frac{7}{\varepsilon} \right)^{l_1+n_1+1} \prod_{i=2}^{m+2} \Omega_{l_i, n_i} \right\}
$$

$$
\leq (j+2)^{r+p}(m+2) C \frac{C}{|z| R^2} \left( 2\| (1+b^2)\hat{A}(b) \|_{l_1} \right)^{j+1} \left( \frac{7}{\varepsilon} \right)^{j+2} \sup_{T'} \left\{ \left( \frac{7}{\varepsilon} \right)^{l_i+n_i+1} \prod_{i=1}^{j+2} (l_i + n_i)! \right\}
$$

(since $\sum_{i=1}^{j+2} l_i = r$, $\sum_{i=1}^{j+2} n_i = p$ and $\prod_{i=1}^{j+2} (l_i + n_i)! < (r+p)!$)

$$
\leq C(r+p)! \left( \frac{7}{\varepsilon} \right)^{r+p+1} (m+2)(j+2)^{r+p} \left( \frac{14}{\varepsilon} \right)^{j+1} \frac{1}{|z| R^2}
$$

$$
\leq C_{\varepsilon, r,p} (m+2)(j+2)^{r+p} \left( \frac{4}{9} \right)^{j+1},
$$
We now apply the last inequality for deriving an estimate for the derivatives of $R_3$ and complete the proof of the lemma for $j = 3$. Recall from (4.4.35) that

$$X_{33}^{(j)} = \sum_{m=0}^{j-1} T_{33}^{m} T_{34} W_{43}^{(j-m-1)}.$$  

Then,

$$\left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1 \Delta_k^{-1} X_{33}^{(j)} \right\| \leq \sum_{m=0}^{j-1} \left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \Delta_k^{-1} \pi G_1 T_{33}^{m} T_{34} W_{43}^{(j-m-1)} \right\| \leq \sum_{m=0}^{j-1} \frac{C_{\varepsilon,r,p}}{|z| R^2} (m+2)(j+2)^{r+p} \frac{4}{9} \leq \frac{C_{\varepsilon,r,p}}{|z| R^2} (j+2)^{r+p} \left( \frac{4}{9} \right)^{j+1} \sum_{m=0}^{j-1} (m+2).$$

Thus, since $G_1' \subseteq G_3'$,

$$\left\| \pi G_1' \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \right\| \leq \sum_{j=1}^{\infty} \left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \pi G_1' \Delta_k^{-1} X_{33}^{(j)} \right\| \leq C_{\varepsilon,r,p} \frac{1}{|z| R^2}.$$

where $C$ is an universal constant. Therefore,

$$\left\| \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} R_3 (k) \right\| = \left\| \mathcal{F} (d') G_1' \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \mathcal{G} G_1' (d') \right\| \leq \left\| \mathcal{F} (d') G_1' \right\| \left\| \pi G_1' \frac{\partial^{r+p}}{\partial k_1^r \partial k_2^p} \left[ \Delta_k^{-1} \sum_{j=1}^{\infty} X_{33}^{(j)} \right] \pi G_1' \right\| \leq C_{\varepsilon,r,p} \frac{1}{|z| R^2}.$$

Finally, combining all the estimates we have

$$\left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} \alpha_{\mu,d}^{(3)} (k) \right\| \leq \sum_{j=1}^{4} \left\| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} R_j (k) \right\| \leq 3 \frac{C}{|z| R^2} + \frac{C}{|z|^3 R} \leq 4C \frac{1}{|z| R^2},$$

where $C = C_{\varepsilon,A,q,f,g,m,n}$ is a constant. Set $\rho_{\varepsilon,A,q,m,n} := \max \{ \rho_{1;\varepsilon,m,n}, \rho_{2;\varepsilon,A,q,m,n} \}$. The proof of the lemma for $j = 3$ is complete.
4.6 The regular piece

Proof of Theorem 3.4.1. Step 1 (defining equation) We first derive a defining equation for the Fermi curve. Without loss of generality we can assume that \( \hat{A}(0) = 0 \) (see the discussion in §3.1). Let \( G = \{0\} \), recall that \( G' = \Gamma^\# \setminus \{0\} \), and consider the region \( (T_{\nu}(0) \cup b \in G \cup T_b) \setminus K_\rho \), where \( \rho \) is a constant to be chosen sufficiently large obeying \( \rho \geq R \). By Proposition 4.2.1(i) we have \( G' = \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon |v| \} \). To simplify the notation write

\[
M_\nu := \left( \hat{F}(A, V) \cap T_{\nu}(0) \right) \setminus \left( K_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right).
\]

By Lemma 4.2.4(i), a point \( k \) is in \( M_\nu \) if and only if

\[
N_0(k) + D_{0,0}(k) = 0.
\]

By Proposition 4.3.1, if we set

\[
w(k) := w_{\nu,0}(k) = k_1 + i(-1)^\nu k_2
\]

and

\[
z(k) := z_{\nu,0}(k) = k_1 - i(-1)^\nu k_2,
\]

this equation becomes

\[
\beta_1 w^2 + \beta_2 z^2 + (1 + \beta_3)wz + \beta_4 w + \beta_5 z + \beta_6 + \hat{q}(0) = 0,
\]

(4.6.1)

where

\[
\beta_1 := J^{00}_\nu, \quad \beta_2 := J^{00}_\nu, \quad \beta_3 := K^{00},
\]

\[
\beta_4 := L^{00}_\nu, \quad \beta_5 := L^{00}_\nu, \quad \beta_6 := M^{00} - \hat{q}(0),
\]

with \( J^{00}_\nu, K^{00}, L^{00}_\nu \) and \( M^{00} \) given by Proposition 4.3.1. Observe that all the coefficients \( \beta_1, \ldots, \beta_6 \) have exactly the same form as the function \( \Phi_0,0(k) \) of Lemma 4.4.1(i) (see (4.4.1)). Thus, by this lemma, for \( 1 \leq i \leq 6 \) we have

\[
\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)},
\]

(4.6.2)

where the functions \( \beta_i^{(j)} \) are analytic in the region under consideration with

\[
|\beta_i^{(j)}(k)| \leq \frac{C}{(2|z(k)| - \rho)^j} \leq \frac{C}{|z(k)|^j} \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and} \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z(k)| \rho^2},
\]
where \( C = C_{\varepsilon, \Lambda, q, A} \) is a constant. The exact expressions for \( \beta^{(j)} \) can be easily obtained from the definitions and from Lemma 4.4.1(i). Substituting (4.6.2) into (4.6.1) and dividing both sides of the equation by \( z \) yields

\[
w + \beta^{(1)}_2 z + g = 0,
\]

where

\[
g := \frac{\beta_1 w^2}{z} + (\beta^{(2)}_2 + \beta^{(3)}_2)z + \beta_3 w + \frac{\beta_4 w}{z} + \beta_5 + \frac{\beta_6}{z} + \hat{q}(0)
\]

obeys

\[
|g(k)| \leq \frac{C}{\rho}
\]

with a constant \( C = C_{\varepsilon, \Lambda, q, A} \). Therefore, a point \( k \) is in \( M_\nu \) if and only if

\[
F(k) = 0,
\]

where

\[
F(k) := w(k) + \beta^{(1)}_2(k) z(k) + g(k)
\]

is an analytic function (in the region under consideration).

**Step 2 (candidates for a solution)** Let us now identify which points are candidates to solve the equation \( F(k) = 0 \). First observe that, by Proposition 3.2.2(c) we have

\[
\mathcal{N}_1(0) \cap \mathcal{N}_2(d) = \{(i\theta_1(d), \theta_1(d))\}
\]

and

\[
\mathcal{N}_1(d) \cap \mathcal{N}_2(0) = \{(i\theta_2(d), -\theta_2(d))\}.
\]

Thus, the lines \( \mathcal{N}_\nu(0) \) and \( \mathcal{N}_\nu(d) \) intersect at

\[
\mathcal{N}_\nu(0) \cap \mathcal{N}_\nu'(d) = \{(i\theta_\nu(d), (-1)^{\nu'} \theta_\nu(d))\}.
\]

Hence, the second coordinate of this point and the second coordinate of a point \( k \) differ by

\[
pr(k) - pr(\mathcal{N}_\nu(0) \cap \mathcal{N}_\nu'(d)) = k_2 - (-1)^{\nu'} \theta_\nu(d) = k_2 + (-1)^{\nu} \theta_\nu(d).
\]

Now observe that, if \( k \in T_\nu(0) \cap T_\nu'(d) \) then

\[
|k_1 + i(-1)^{\nu} k_2| < \varepsilon
\]

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and
\[
|k_2 + (-1)\nu \theta_\nu(d)| = |i(-1)\nu k_2 + i\theta_\nu(d)| = |i(-1)\nu k_2 + \frac{1}{2}(i(-1)\nu d_2 - d_1)|
\]
\[
= \left| \frac{1}{2}(k_1 + i(-1)\nu k_2) - \frac{1}{2}(k_1 + d_1 - i(-1)\nu (k_2 + d_2)) \right|
\]
\[
\leq \frac{1}{2}|N_{0,\nu}(k) - N_{d,\nu}(k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

That is, the second coordinate of \( k \) and the second coordinate of \( N_{\nu}(0) \cap N_{\nu}(d) \) must be apart from each other by at most \( \varepsilon \). This gives a necessary condition on the second coordinate of a point \( k \) for being in \( M_\nu \).

Conversely, if a point \( k \) is in the \((\varepsilon/4)\)-tube inside \( T_\nu(0) \), that is,
\[
|k_1 + i(-1)\nu k_2| < \frac{\varepsilon}{4},
\]
and its second coordinate differ from the second coordinate of \( N_{\nu}(0) \cap N_{\nu}(d) \) by at most \( \varepsilon/4 \), that is,
\[
|k_2 + (-1)\nu \theta_\nu(d)| < \frac{\varepsilon}{4},
\]
then
\[
|N_{d,\nu}(k)| = |N_{0,\nu}(k) - 2(k_2 + (-1)\nu \theta_\nu(d))| \leq \frac{\varepsilon}{4} + 2 \frac{\varepsilon}{4} < \varepsilon,
\]
that is, the point \( k \) is also in \( T_\nu(d) \) and hence lie in the intersection \( T_\nu(0) \cap T_\nu(d) \). This gives a sufficient condition on the first and second coordinates of a point \( k \) for being in \( T_\nu(0) \cap T_\nu(d) \).

For \( y \in \mathbb{C} \) define the set of candidates for a solution of \( F(k) = 0 \) as
\[
M_\nu(y) := pr^{-1}(y) \cap \left( T_\nu(0) \setminus \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right) = pr^{-1}(y) \cap \left( T_\nu(0) \setminus \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_\nu(b) \right).
\]

Observe that, if \(|y + (-1)\nu \theta_\nu(b)| \geq \varepsilon \) for all \( b \in \Gamma^\# \setminus \{0\} \) then
\[
M_\nu(y) = pr^{-1}(y) \cap T_\nu(0) = \{(k_1, y) \in \mathbb{C}^2 \mid |k_1 + i(-1)\nu y| < \varepsilon\}. \tag{4.6.6}
\]
On the other hand, if \(|y + (-1)\nu \theta_\nu(d)| < \varepsilon \) for some \( d \in \Gamma^\# \setminus \{0\} \), then there is at most one such \( d \) and consequently
\[
M_\nu(y) = pr^{-1}(y) \cap (T_\nu(0) \setminus T_\nu(d)) = \{(k_1, y) \in \mathbb{C}^2 \mid |k_1 + i(-1)\nu y| < \varepsilon \text{ and } |k_1 + d_1 + i(-1)\nu(y + d_2)| \geq \varepsilon\}. \tag{4.6.7}
\]
Indeed, suppose there is another $d' \neq 0$ such that $|y + (-1)^\nu \theta_\nu(d')| < \varepsilon$. Then,

$$|d - d'| = |2(-1)^\nu \theta_\nu(d - d')| = |y + (-1)^\nu \theta_\nu(d) - (y + (-1)^\nu \theta_\nu(d'))| \leq 2\varepsilon < 2\Lambda,$$

which contradicts the definition of $\Lambda$. Thus, there is no such $d' \neq 0$.

**Step 3 (uniqueness)** We now prove that, given $k_2$, if there exists a solution $k_1(k_2)$ of $F(k_1, k_2) = 0$, then this solution is unique and it depends analytically on $k_2$. This follows easily using the implicit function theorem and the estimates below, which we prove later.

**Proposition 4.6.1.** Under the hypotheses of Theorem 3.4.1 we have

$$|F(k) - w(k)| \leq \frac{\varepsilon}{900} + \frac{C_1}{\rho},$$

and

$$\left|\frac{\partial F}{\partial k_1}(k) - 1\right| \leq \frac{1}{7 \cdot 3^4} + \frac{C_2}{\rho},$$

where the constants $C_1$ and $C_2$ depend only on $\varepsilon$, $\Lambda$, $q$ and $A$.

Now, suppose that $(k_1, y) \in M_\nu(y)$. Then,

$$\left|\frac{\partial F}{\partial k_1}(k_1, y) - 1\right| \leq \frac{1}{7 \cdot 3^4} + \frac{C_2}{\rho}.$$

Hence, by the implicit function theorem [16, Theorem 3.7.1], by choosing the constant $\rho \geq R$ sufficiently large, if $F(k_1^*, y) = 0$ for some $(k_1^*, y) \in M_\nu(y)$, then there is a neighbourhood $U \times V \subset \mathbb{C}^2$ which contains $(k_1^*, y)$, and an analytic function $\eta : V \to U$ such that $F(k_1, k_2) = 0$ for all $(k_1, k_2) \in U \times V$ if and only if $k_1 = \eta(k_2)$. In particular, this implies that the equation $F(k_1, k_2) = 0$ has at most one solution $(\eta(y), y)$ in $M_\nu(y)$ for each $y \in \mathbb{C}$.

We next look for conditions on $y$ to have a solution or have no solution in $M_\nu(y)$.

**Step 4 (existence)** We first state an improved version of Proposition 4.6.1(a).

**Proposition 4.6.2.** Under the hypotheses of Theorem 3.4.1 we have

$$F(k) - w(k) = \beta_2^{(1,0)} + \beta_2^{(1,1)}(w(k)) + \beta_2^{(1,2)}(k) + h(k),$$

where

$$\beta_2^{(1,0)} = -2i \sum_{b,c \in \mathbb{G}'} \frac{\theta_\nu'(\hat{A}(b))}{\theta_\nu'(b)} \left[ \delta_{b,c} + \frac{\theta_\nu'(\hat{A}(b - c))}{\theta_\nu'(c)} \right] \theta_\nu(\hat{A}(c))$$

is a constant that depends only on $\rho$ and $A$ and

$$h := \beta_2^{(1,3)} + g.$$
Furthermore,
\[ |\beta_2^{(1,0)}| < \frac{1}{100\Lambda} \varepsilon^2, \quad |\beta_2^{(1,1)}(k)| < \frac{1}{40\Lambda^2} \varepsilon^3, \]
\[ |\beta_2^{(1,2)}(k)| < \frac{1}{74\Lambda^3} \varepsilon^4, \quad |h(k)| \leq C_{\varepsilon,\Lambda,q,\Lambda} \frac{1}{\rho}. \]

We now derive conditions for the existence of solutions. Suppose that \( F(\eta(y), y) = 0 \). Then, since \( \eta(y) + i(-1)^\nu y = w(\eta(y), y) \) and \( \varepsilon < \Lambda/6 \), using the above proposition we obtain

\[ |\eta(y) + i(-1)^\nu y| = |w(\eta(y), y)| = |F(\eta(y), y) - w(\eta(y), y)| \leq \frac{\varepsilon^2}{100\Lambda} + \frac{\varepsilon^3}{40\Lambda^2} + \frac{\varepsilon^4}{74\Lambda^3} + \frac{C}{\rho} \leq \frac{\varepsilon^2}{50\Lambda} + \frac{C}{\rho}. \]

Hence, by choosing the constant \( \rho \) sufficiently large we find that

\[ |\eta(y) + i(-1)^\nu y| < \frac{\varepsilon^2}{40\Lambda}. \]

In view of (4.6.7), there is no solution in \( M_\nu(y) \) if for some \( d \in \Gamma^\# \setminus \{0\} \) we have

\[ |y + (-1)^\nu \theta_\nu(d)| < \varepsilon \quad \text{and} \quad |\eta(y) + d_1 + i(-1)^\nu (y + d_2)| < \varepsilon. \]

This happens if

\[ |y + (-1)^\nu \theta_\nu(d)| \leq \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right) \]

because in this case

\[ |\eta(y) + d_1 + i(-1)^\nu (y + d_2)| = |\eta(y) + i(-1)^\nu y - 2i(-1)^\nu y + d_1 - i(-1)^\nu d_2| \leq |\eta(y) + (-1)^\nu y| + 2|y + (-1)^\nu \theta_\nu(d)| < \varepsilon. \]

Therefore, the image set of \( pr \) is contained in

\[ \Omega_1 := \left\{ z \in \mathbb{C} \mid |z + (-1)^\nu \theta_\nu(b)| > \frac{1}{2} \left( \varepsilon - \frac{\varepsilon^2}{40\Lambda} \right) \quad \text{for all } b \in \Gamma^\# \setminus \{0\} \right\}. \]

On the other hand, in view of (4.6.6), there is a solution in \( M_\nu(y) \) if \( |y + (-1)^\nu \theta_\nu(b)| > \varepsilon \) for all \( b \in \Gamma^\# \setminus \{0\} \). Recall from Proposition 4.4.4(a) that \( \rho < |v| < 8|k_2| \). Thus, the image set of \( pr \) contains the set

\[ \Omega_2 := \left\{ z \in \mathbb{C} \mid 8|z| > \rho \quad \text{and} \quad |z + (-1)^\nu \theta_\nu(b)| > \varepsilon \quad \text{for all } b \in \Gamma^\# \setminus \{0\} \right\}. \]

**Step 5 (conclusion)** Summarizing, we have the following biholomorphic correspondence:

\[ \mathcal{M}_\nu \ni k \xrightarrow{pr} k_2 \in \Omega, \]

\[ \mathcal{M}_\nu \ni (\eta(y), y) \xleftarrow{pr^{-1}} y \in \Omega, \]
where
\[ \Omega_2 \subset \Omega \subset \Omega_1 \]
and
\[ \eta(y) = -\beta_2^{(1,0)} - i(-1)\nu y - r(y), \]
with the constant \( \beta_2^{(1,0)} \) given by (4.6.8),
\[ |\beta_2^{(1,0)}| < \frac{\varepsilon^2}{100\Lambda} \quad \text{and} \quad |r(y)| \leq \frac{\varepsilon^3}{50\Lambda^2} + \frac{C}{\rho}. \]
This completes the proof of the theorem. \( \square \)

**Proof of Propositions 4.6.1 and 4.6.2**

We follow the same notation as above.

**Proof of Proposition 4.6.1.** (a) Recall that \( \beta_2 = J_\nu^{(0)} \). First observe that, by Proposition 4.3.1, Lemma 4.4.1, and (4.4.25), we have
\[ \beta_2^{(1)}(k) = (J_\nu^{(0)})^{(1)}(k) = \sum_{b,c \in G_1} \frac{1}{N_b(k)} \cdot 2 \cdot (1, i(-1)\nu \cdot \hat{A}(b)) \cdot S_{b,c} \cdot (1, i(-1)\nu \cdot \hat{A}(c)). \] (4.6.9)
Thus, by (4.5.11) and (4.5.16),
\[ |\beta_2^{(1)}(k)| \leq \frac{1}{\Lambda} \|\hat{A}\|_{l_1} \leq \frac{1}{\Lambda} \|\hat{A}\|_{l_1} \leq \frac{4}{45} \frac{\Lambda(z(k))}{\Lambda(z(k))} \leq \frac{4}{45} \frac{1}{\Lambda(z(k))} \leq \frac{1}{900} |z(k)|. \] (4.6.10)

Now recall that
\[ |g(k)| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}. \]
Hence,
\[ |F(k) - w(k)| = |\beta_2^{(1)}(k)z(k) + g(k)| \leq \frac{\varepsilon}{900} + C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}. \]
This proves part (a).

(b) We first compute
\[ \frac{\partial g}{\partial k_1} = \frac{\partial}{\partial k_1} \left( \frac{\beta_1 w^2}{z} + (\beta_2^{(2)} + \beta_3^{(3)})z + \beta_3 w + \frac{\beta_4 w}{z} + \beta_5 + \frac{\beta_6 z}{z} + \hat{q}(0) \right) \]
\[ = \frac{\partial\beta_1}{\partial k_1} \frac{w^2}{z} + \frac{\beta_1}{z} \left( 2w z - w^2 \right) + \left( \frac{\partial\beta_2^{(2)}}{\partial k_1} + \frac{\partial\beta_3^{(3)}}{\partial k_1} \right) z + \beta_2^{(2)} + \beta_3^{(3)} + \frac{\partial\beta_3}{\partial k_1} z + \beta_3 \quad (4.6.11) \]
\[ + \frac{\partial\beta_4}{\partial k_1} \frac{w}{z} + \beta_4 \frac{z - w}{z^2} + \frac{\partial\beta_5}{\partial k_1} + \frac{\partial\beta_6}{\partial k_1} \frac{1}{z} - \frac{\beta_6}{z^2} - \hat{q}(0). \]
Furthermore, by Lemmas 4.4.1(i), 4.5.1(i) and 4.5.3(i), for $1 \leq i \leq 6$ and $1 \leq j \leq 2$,\n\n$$|\beta_i(k)| \leq \frac{C}{|z(k)|}, \quad |\beta_i^{(j)}(k)| \leq \frac{C}{|z(k)|^j}, \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z(k)|\rho^2},$$\n\n$$\frac{\partial \beta_i(k)}{\partial k_1} \leq \frac{C}{|z(k)|}, \quad \frac{\partial \beta_i^{(j)}(k)}{\partial k_1} \leq \frac{C}{|z(k)|^j}, \quad \frac{\partial \beta_i^{(3)}(k)}{\partial k_1} \leq \frac{C}{|z(k)|\rho^2},$$\n\n(4.6.12)\n\nwhere $C = C_{\varepsilon,\Lambda,q,A}$ in all cases. Hence,\n\n$$\left| \frac{\partial g(k)}{\partial k_1} \right| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}. \quad (4.6.13)$$\n\nBy Lemma 4.5.3(i) with $f = g = (1, -i(-1)^{\nu}) \cdot \hat{A}$, we obtain\n\n$$\left| z(k) \frac{\partial \beta_2^{(1)}(k)}{\partial k_1} \right| \leq |z(k)| \frac{13}{\Lambda^2 |z(k)|} \|(1, -i(-1)^{\nu}) \cdot \hat{A}\|_1^2$$\n\n$$\leq \frac{26}{\Lambda^2} \|\hat{A}\|_1^2 \leq \frac{26 \cdot 4 \varepsilon^2}{(63)^2} \leq \frac{26 \cdot 4}{63^2} \frac{1}{\varepsilon^2} < \frac{1}{7 \cdot 3^4}. \quad (4.6.14)$$\n\nTherefore,\n\n$$\frac{\partial F}{\partial k_1}(k) - 1 = \frac{\partial}{\partial k_1} (F(k) - w(k)) = \left| \frac{\partial}{\partial k_1} (\beta_2^{(1)}(k)z(k) + g(k)) \right|$$\n\n$$= \left| \frac{\partial \beta_2^{(1)}(k)}{\partial k_1} z(k) + \beta_2^{(1)}(k) + \frac{\partial g}{\partial k_1}(k) \right| \leq \frac{1}{7 \cdot 3^4} + C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}. \quad (4.6.9)$$\n\nThis proves part (b) and completes the proof of the proposition. \hfill \Box

Here is the proof of Proposition 4.6.2.

**Proof of Proposition 4.6.2.** First observe that\n\n$$(1, i(-1)^{\nu}) \cdot A = A_1 + i(-1)^{\nu} A_2 = A_1 - i(-1)^{\nu} A_2 = -2i \frac{1}{2}(iA_1 + (-1)^{\nu} A_2) = -2i \theta_\nu(A).$$\n\nThus, recalling (4.6.9),\n\n$$\beta_2^{(1)}(k) = (J^{(0)}_\nu)_2^{(1)}(k) = \sum_{b,c \in G'_1} \frac{2i \theta_\nu(\hat{A}(b))}{N_b(k)} S_{b,c} 2i \theta_\nu(\hat{A}(c)).$$\n\nNow, by Lemma 4.4.2 we have\n\n$$z(k) \beta_2^{(1)}(k) = \beta_2^{(1,0)} + \beta_2^{(1,1)}(w(k)) + \beta_2^{(1,2)}(k) + \beta_2^{(1,3)}(k),$$\n\nwhere\n\n$$\beta_2^{(1,0)} = -2i \sum_{b,c \in G'_1} \frac{\theta_\nu(\hat{A}(b))}{\theta_\nu(b)} \left[ \delta_{b,c} + \frac{\theta_\nu(\hat{A}(b-c))}{\theta_\nu(c)} \right] \theta_\nu(\hat{A}(c))$$

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and

\[ |\beta^{(1,3)}_3(k)| \leq C_{\Lambda,A} \frac{1}{|z(k)|} < C_{\Lambda,A} \frac{1}{\rho}. \]

Hence,

\[ F(k) - w(k) = z(k)\beta^{(1)}_2(k) + g(k) = \beta^{(1,0)}_2 + \beta^{(1,1)}_2(w(k)) + \beta^{(1,2)}_2(k) + h(k) \]

with \( h := \beta^{(1,3)}_3 + g \). Furthermore, in view of (4.6.5),

\[ |h(k)| \leq |\beta^{(1,3)}_3(k)| + |g(k)| < C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}. \]

This proves the first part of the proposition. Finally, by (4.4.40), since \( \|\hat{A}\|_{l^1} < 2\varepsilon/63 \) and \( \varepsilon < \Lambda/6 \), we find that

\[ |\beta^{(1,0)}_2| \leq \frac{1}{2\Lambda} \left( 1 + \frac{1}{2\Lambda} \|\theta_{\nu}(\hat{A})\|_{l^1} \right) \|2i\theta_{\nu}(\hat{A})\|_{l^1} \|2i\theta_{\nu}(\hat{A})\|_{l^1} \leq \frac{1}{2\Lambda} \left( 1 + \frac{1}{2\Lambda} \frac{2\varepsilon}{63} \right) 4\|\hat{A}\|^2_{l^1} \leq \frac{4}{\Lambda} \|\hat{A}\|^2_{l^1} < \frac{1}{100\Lambda} \varepsilon^2 \]

and

\[ |\beta^{(1,1)}_2| \leq \frac{\varepsilon}{\Lambda^2} \left( 1 + \frac{7}{6\Lambda} \|\theta_{\nu}(\hat{A})\|_{l^1} \right) \|2i\theta_{\nu}(\hat{A})\|_{l^1} \|2i\theta_{\nu}(\hat{A})\|_{l^1} \leq \frac{\varepsilon}{\Lambda^2} \left( 1 + \frac{7}{6\Lambda} \frac{2\varepsilon}{63} \right) 4\|\hat{A}\|^2_{l^1} \leq \frac{8}{\Lambda^2} \varepsilon \|\hat{A}\|^2_{l^1} < \frac{1}{40\Lambda^2} \varepsilon^3 \]

and

\[ |\beta^{(1,2)}_2| \leq \frac{64}{\Lambda^4} \|\theta_{\nu}(\hat{A})\|^2_{l^1} \|2i\theta_{\nu}(\hat{A})\|_{l^1} \|2i\theta_{\nu}(\hat{A})\|_{l^1} \leq \frac{256}{\Lambda^4} \|\hat{A}\|^3_{l^1} < \frac{1}{74\Lambda^3} \varepsilon^4. \]

This completes the proof. \( \square \)

### 4.7 The handles

**Proof of Theorem 3.4.2.** We first derive a defining equation for the Fermi curve. Without loss of generality we can assume that \( \hat{A}(0) = 0 \). Let \( G = \{0,d\} \), recall that \( G' = \Gamma^\# \setminus \{0,d\} \), and consider the region \( (T_{\nu}(0) \cap T_{\nu}(d)) \setminus K_\rho \), where \( \rho \) is a constant to be chosen sufficiently large obeying \( \rho \geq R \). Observe that, this requires \( d \) being sufficiently large for \( (T_{\nu}(0) \cap T_{\nu}(d)) \setminus K_\rho \) being not empty. In fact, by Proposition 4.4.4(ii), for \( k \) in this region we have

\[ \rho < |v| \leq 2|d|. \]
Now, recall from Proposition 4.2.1(ii) that \( G' = \{ b \in \Gamma^\# \mid |N_b(k)| \geq \varepsilon |v| \} \), and to simplify the notation write
\[
H_\nu := \hat{\mathcal{F}}(A, V) \cap (T_\nu(0) \cap T_{\nu'}(d)) \setminus \mathcal{K}_\rho.
\]
By Lemma 4.2.4(ii), a point \( k \) is in \( H_\nu \) if and only if
\[
(N_0(k) + D_{0,0}(k))(N_d(k) + D_{d,d}(k)) - D_{0,d}(k)D_{d,0}(k) = 0. \tag{4.7.1}
\]
Define
\[
w_1(k) := w_{\nu,0} = k_1 + i(-1)^\nu k_2,
\]
\[
z_1(k) := z_{\nu,0} = k_1 - i(-1)^\nu k_2,
\]
\[
w_2(k) := w_{\nu',d} = k_1 + d_1 + i(-1)^{\nu'}(k_2 + d_2),
\]
\[
z_2(k) := z_{\nu',d} = k_1 + d_1 - i(-1)^{\nu'}(k_2 + d_2).
\]
Note that, by Proposition 4.4.4(ii),
\[
|v| \leq |z_1| \leq 3|v|, \quad |v| \leq |z_2| \leq 3|v| \quad \text{and} \quad |d| \leq |z_2| \leq 2|d|.
\]
By Proposition 4.3.1,
\[
N_0 + D_{0,0} = \beta_1 w_1^2 + \beta_2 z_1^2 + (1 + \beta_3)w_1 z_1 + \beta_4 w_1 + \beta_5 z_1 + \beta_6 + \hat{q}(0),
\]
\[
N_d + D_{d,d} = \eta_1 w_2^2 + \eta_2 z_2^2 + (1 + \eta_3)w_2 z_2 + \eta_4 w_2 + \eta_5 z_2 + \eta_6 + \hat{q}(0), \tag{4.7.2}
\]
where
\[
\beta_1 := J^00_\nu, \quad \beta_2 := J^00_{\nu'}, \quad \beta_3 := K^00, \quad \beta_4 := L^00_{\nu'}, \quad \beta_5 := L^00_{\nu'}, \quad \beta_6 := M^00 - \hat{q}(0),
\]
and
\[
\eta_1 := J^{dd}_\nu, \quad \eta_2 := J^{dd}_{\nu'}, \quad \eta_3 := K^{dd}, \quad \eta_4 := L^{dd}_\nu, \quad \eta_5 := L^{dd}_{\nu'}, \quad \eta_6 := M^{dd} - \hat{q}(0),
\]
with \( J^{dd}_\nu, K^{dd}, L^{dd}_\nu \) and \( M^{dd} \) given by Proposition 4.3.1. Observe that all the coefficients \( \beta_1, \ldots, \beta_6 \) and \( \eta_1, \ldots, \eta_6 \) have exactly the same form as the function \( \Phi_{d',d'}(k) \) of Lemma 4.4.1(ii) (see (4.4.1)). Thus, by this lemma, for \( 1 \leq i \leq 6 \) we have
\[
\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)},
\]
\[
\eta_i = \eta_i^{(1)} + \eta_i^{(2)} + \eta_i^{(3)}, \tag{4.7.4}
\]
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where the functions $\beta_i^{(j)}$ and $\eta_i^{(j)}$ are analytic in the region under consideration with

$$|\beta_i^{(j)}(k)| \leq \frac{C}{(2 |z_1(k)| - \rho)^2} \leq \frac{C}{|z_1(k)|^2} \quad \text{for } 1 \leq j \leq 2 \quad \text{and} \quad |\beta_i^{(3)}(k)| \leq \frac{C}{|z_1(k)| \rho^2},$$

$$|\eta_i^{(j)}(k)| \leq \frac{C}{(2 |z_2(k)| - \rho)^2} \leq \frac{C}{|z_2(k)|^2} \quad \text{for } 1 \leq j \leq 2 \quad \text{and} \quad |\eta_i^{(3)}(k)| \leq \frac{C}{|z_2(k)| \rho^2},$$

where $C = C_{\epsilon, \Lambda, q, A}$ is a constant. The exact expressions for $\beta_i^{(j)}$ and $\eta_i^{(j)}$ can be easily obtained from the definitions and from Lemma 4.4.1(ii). Substituting (4.7.4) into (4.7.3) yields

\[
\left\{
\begin{array}{l}
\frac{1}{z_1} (N_0 + D_{0,0}) = w_1 + \beta_2^{(1)} z_1 + g_1, \\
\frac{1}{z_2} (N_d + D_{d,d}) = w_2 + \eta_2^{(1)} z_2 + g_2,
\end{array}
\right.
\]  

(4.7.5)

where

\[
g_1 := \frac{\beta_1 w_1}{z_1} + (\beta_2^{(2)} + \beta_2^{(3)}) z_1 + \beta_3 w_1 + \frac{\beta_4 w_1}{z_1} + \beta_5 + \frac{\beta_6}{z_1} + \frac{\hat{q}(0)}{z_1},
\]

\[
g_2 := \frac{\eta_1 w_2}{z_2} + (\eta_2^{(2)} + \eta_2^{(3)}) z_2 + \eta_3 w_2 + \frac{\eta_4 w_2}{z_2} + \eta_5 + \frac{\eta_6}{z_2} + \frac{\hat{q}(0)}{z_2}
\]

(4.7.6)

obey

\[
|g_1(k)| \leq \frac{C}{\rho} \quad \text{and} \quad |g_2(k)| \leq \frac{C}{\rho}
\]  

(4.7.7)

with a constant $C = C_{\epsilon, \Lambda, q, A}$. This gives us more information about the first term in (4.7.1).

We next consider the second term in that equation.

Write

\[
D_{0,d} = c_1(d) + p_1 \quad \text{and} \quad D_{d,0} = c_2(d) + p_2
\]

(4.7.8)

with

\[
c_1(d) := \hat{q}(-d) - 2 d \cdot \hat{A}(-d), \quad p_1 := D_{0,d} - \hat{q}(-d) + 2 d \cdot \hat{A}(-d),
\]

\[
c_2(d) := \hat{q}(d) + 2 d \cdot \hat{A}(d), \quad p_2 := D_{d,0} - \hat{q}(d) - 2 d \cdot \hat{A}(d).
\]

We shall shortly prove the following estimates.

**Proposition 4.7.1.** Under the hypotheses of Theorem 3.4.2 we have, for any integers $n$ and $m$ with $n + m \geq 0$ and for $1 \leq j \leq 2$,

\[
\left| \frac{\partial^{n+m}}{\partial k_1^n \partial k_2^m} p_j(k) \right| \leq \frac{C_1}{|d|} \quad \text{and} \quad |c_j(d)| \leq \frac{C_2}{|d|},
\]

where the constants $C_1$ and $C_2$ depend only on $\epsilon$, $\Lambda$, $q$ and $A$.  

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Thus, by dividing both sides of (4.7.1) by $z_1z_2$ and substituting (4.7.5) and (4.7.8) we find that
\[
0 = \frac{1}{z_1z_2} \left[ (N_0 + D_{0,0})(N_d + D_{d,d}) - D_{0,d}D_{d,0} \right] = (w_1 + \beta_2^{(1)} z_1 + g_1)(w_2 + \eta_2^{(1)} z_2 + g_2) - \frac{1}{z_1z_2}(c_1(d) + p_1)(c_2(d) + p_2). \tag{4.7.9}
\]

We now introduce a (nonlinear) change of variables in $C^2$. Set
\[
x_1(k) := w_1(k) + \beta_2^{(1)}(k) z_1(k) + g_1(k),
\]
\[
x_2(k) := w_2(k) + \eta_2^{(1)}(k) z_2(k) + g_2(k). \tag{4.7.10}
\]

We shall prove below that this transformation satisfies the following estimates.

**Proposition 4.7.2.** Under the hypotheses of Theorem 3.4.2 we have:

(i) For $1 \leq j \leq 2$ and for $\rho$ sufficiently large,
\[
|x_j(k) - w_j(k)| \leq \frac{\varepsilon}{900} + \frac{C}{\rho} < \frac{\varepsilon}{8}.
\]

(ii)
\[
\begin{pmatrix}
\frac{\partial x_1}{\partial k_1} & \frac{\partial x_1}{\partial k_2} \\
\frac{\partial x_2}{\partial k_1} & \frac{\partial x_2}{\partial k_2}
\end{pmatrix} = \begin{pmatrix}
1 & i(-1)^\nu \\
1 & i(-1)^{\nu'}
\end{pmatrix} (I + M)
\]

and
\[
\begin{pmatrix}
\frac{\partial k_1}{\partial x_1} & \frac{\partial k_1}{\partial x_2} \\
\frac{\partial k_2}{\partial x_1} & \frac{\partial k_2}{\partial x_2}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
i(-1)^{\nu'} & i(-1)^\nu
\end{pmatrix} (I + N)
\]

with
\[
\|M\| \leq \frac{4}{7 \cdot 3^4} + \frac{C}{\rho} < \frac{1}{2} \quad \text{and} \quad \|N\| \leq 4\|M\|.
\]

Furthermore, for all $m, i, j \in \{1, 2\}$,
\[
\left| \frac{\partial^2 k_m}{\partial x_i \partial x_j} \right| \leq \frac{3}{A^3} \varepsilon^2 + \frac{C}{\rho}.
\]

Here, all the constants $C$ depend only on $\varepsilon$, $A$, $q$ and $A$.

By the inverse function theorem, these estimates imply that the above transformation is invertible. Therefore, by rewriting the equation (4.7.9) in terms of these new variables, we conclude that a point $k$ is in $H_{\nu}$ if and only if $x_1(k)$ and $x_2(k)$ satisfy the equation
\[
x_1x_2 + v(x_1, x_2) = 0, \tag{4.7.11}
\]

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where
\[ r(x_1, x_2) := -\frac{1}{z_1 z_2} (c_1(d) + p_1)(c_2(d) + p_2). \]

This is the defining equation we shall study in detail below. In order to do this we need some estimates.

**Step 2 (estimates)** Using the above inequalities we have, for \( i, j, l \in \{1, 2\} \),
\[
\left| \frac{\partial}{\partial x_i} p_j(k(x)) \right| \leq \sum_{m=1}^{2} \left| \frac{\partial p_j}{\partial k_m} \frac{\partial k_m}{\partial x_i} \right| \leq \frac{C}{|d|},
\]
and
\[
\left| \frac{\partial^2}{\partial x_i \partial x_l} p_j(k(x)) \right| \leq \sum_{m,n=1}^{2} \left| \frac{\partial^2 p_j}{\partial k_m \partial k_n} \frac{\partial k_m}{\partial x_i} \frac{\partial k_n}{\partial x_l} \right| + \sum_{m=1}^{2} \left| \frac{\partial p_j}{\partial k_m} \frac{\partial^2 k_m}{\partial x_i \partial x_l} \right| \leq \frac{C}{|d|^4},
\]
so that
\[
|r(x)| \leq C \frac{1}{|d|^2} \frac{1}{|d|} \frac{1}{|d|} \leq \frac{C}{|d|^4},
\]
\[
\left| \frac{\partial}{\partial x_i} r(x) \right| \leq C \frac{1}{|d|^3} \frac{1}{|d|} \frac{1}{|d|} + C \frac{1}{|d|^2} \frac{1}{|d|} \frac{1}{|d|} \leq \frac{C}{|d|^4}
\]
and
\[
\left| \frac{\partial^2}{\partial x_i \partial x_j} r(x) \right| \leq \frac{C}{|d|^3}.
\]

Here, all the constants depend only on \( \varepsilon, \Lambda, q \) and \( A \).

**Step 3 (Morse lemma)** We now apply the quantitative Morse lemma in §4.8 for studying the equation (4.7.11). We consider this lemma with
\[
a = b = \frac{C}{|d|^4}, \quad \delta = \varepsilon,
\]
and \( d \) sufficiently large so that \( b \leq \max\{\frac{2}{3}, \frac{1}{55}, \frac{\varepsilon}{3}\} \). Observe that, under this condition we have
\[
(\delta - a)(1 - 19b) > \frac{\varepsilon}{2} \quad \text{and} \quad (\delta - a)(1 - 55b) > \frac{\varepsilon}{4}.
\]

According to this lemma, there is a biholomorphism \( \Phi_\nu \) defined on
\[
\Omega_1 := \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < \frac{\varepsilon}{2} \text{ and } |z_2| < \frac{\varepsilon}{2} \}
\]
with range containing
\[
\{ (x_1, x_2) \in \mathbb{C}^2 \mid |x_1| < \frac{\varepsilon}{4} \text{ and } |x_2| < \frac{\varepsilon}{4} \} \quad (4.7.12)
\]

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such that
\[ \|D\Phi_\nu - I\| \leq \frac{C}{|d|^2}, \]
\[ ((x_1x_2 + r) \circ \Phi_\nu)(z_1, z_2) = z_1z_2 + t_d, \]
\[ |t_d| \leq \frac{C}{|d|^4}, \]
\[ |\Phi_\nu(0)| \leq \frac{C}{|d|^4}, \]
where \( D\Phi_\nu \) is the derivative of \( \Phi_\nu \) and \( t_d \) is a constant that depends on \( d \). Hence, if for \( \nu = 1 \) we define
\[ \phi_{d,1} : \Omega_1 \rightarrow T_1(0) \cap T_2(d) \]
as
\[ \phi_{d,1}(z_1, z_2) := (k_1(\Phi_1(z_1, z_2)), k_2(\Phi_1(z_1, z_2))), \]
where \( k(x) \) is the inverse of the transformation (4.7.10), we obtain the desired map. Note that the conclusion (ii) of the theorem is immediate. We next prove (i) and (iii).

Step 4 (proof of (i)) By Proposition 4.7.2(i), for \( 1 \leq j \leq 2 \),
\[ |x_j(k) - w_j(k)| \leq \frac{\varepsilon}{8}. \]
Now, recall from (4.7.2) the definition of \( w_1(k) \) and \( w_2(k) \). Then, since
\[ |x_j(k)| \leq |x_j(k) - w_j(k)| + |w_j(k)| < \frac{\varepsilon}{8} + |w_j(k)|, \]
the set
\[ \{(k_1, k_2) \in C^2 \mid |w_1(k)| < \frac{\varepsilon}{8} \text{ and } |w_2(k)| < \frac{\varepsilon}{8}\} \]
is contained in the set (4.7.12). This proves the first part of (i). To prove the second part we use Proposition 4.7.2 and (4.7.13). First observe that
\[ D\phi_{d,1} = \frac{\partial k}{\partial x} D\Phi_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} (I + N)(I + D\Phi_1 - I) \]
\[ = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} (I + N + R), \]
where
\[ \|N\| \leq \frac{1}{3^3} + \frac{C}{\rho} \quad \text{and} \quad \|R\| \leq \frac{C}{|d|^2}. \]
Furthermore, from (4.7.2) and (4.7.10) we have

\[ k_1 = i\theta_\nu(d) + \frac{1}{2}(w_1 + w_2) = i\theta_\nu(d) + \frac{1}{2}(x_1 + x_2 + \beta_2^{(1)} z_1 + \eta_2^{(1)} z_2 + g_1 + g_2), \]

\[ k_2 = -(-1)^{\nu} \theta_\nu(d) + \frac{(-1)^{\nu}}{2i}(w_1 - w_2) = -(-1)^{\nu} \theta_\nu(d) + \frac{(-1)^{\nu}}{2i}(x_1 - x_2 - \beta_2^{(1)} z_1 + \eta_2^{(1)} z_2 - g_1 + g_2), \]

so that

\[ \phi_{d,1}(0) = k(\Phi_1(0)) = k \left( O \left( \frac{1}{|d|^4} \right) \right) = (i\theta_\nu(d), -(-1)^{\nu} \theta_\nu(d)) + O \left( \frac{\varepsilon}{900} \right) + O \left( \frac{1}{\rho} \right). \]

**Step 5 (proof of (iii))** To prove part (iii) it suffices to note that \( T_1(0) \cap T_2(d) \cap  \hat{\mathcal{F}}(A, V) \) is mapped to \( T_1(-d) \cap T_2(0) \cap  \hat{\mathcal{F}}(A, V) \) by translation by \( d \) and define \( \phi_{d,2} \) by

\[ \phi_{d,2}(z_1, z_2) := \phi_{d,1}(z_2, z_1) + d. \]

This completes the proof of the theorem.

**Proof of Propositions 4.7.1 and 4.7.2**

We follow the same notation as above.

**Proof of Proposition 4.7.1.** It suffices to estimate

\[ c_{d',d''} := \hat{q}(d' - d'') - 2(d' - d'') \cdot \hat{A}(d' - d'') \quad \text{and} \quad p_{d',d''} := D_{d',d''} - c_{d',d''} \]

for \( d', d'' \in \{0,d\} \) with \( d' \neq d'' \). Define

\[ t_{d',d''} := (1, i(-1)^{\nu}) \cdot \hat{A}(d' - d''). \]

Observe that, since

\[ |\hat{q}(d' - d'')| = \frac{1}{|d' - d''|^2} |d' - d''| |\hat{q}(d' - d'')| \leq \frac{1}{|d' - d''|^2} \sum_{b \in \Gamma} |b|^2 |\hat{q}(b)| < \|b^2 \hat{q}(b)\|_1 \frac{1}{|d|^2}, \]

and similarly

\[ |\hat{A}(d' - d'')| \leq \|b^2 \hat{A}(b)\|_{l_1} \frac{1}{|d|^2}, \]

it follows that

\[ |c_{d',d''}| \leq C_{A,q} \frac{1}{|d|} \quad \text{and} \quad |t_{d',d''}| \leq C_A \frac{1}{|d|^2} \]

This gives the desired bounds for \( c_1 \) and \( c_2 \).
Now, by Proposition 4.3.1 we have

\[ p = J_{\nu}^{d',d''} w_{\nu,d'}^2 + J_{\nu}^{d',d''} z_{\nu,d'}^2 + K_{\nu}^{d',d''} w_{\nu,d} z_{\nu,d'} + (\tilde{L}_{\nu}^{d',d''} - l_{\nu}^{d',d''}) w_{\nu,d'} + (\tilde{M}_{\nu}^{d',d''} - l_{\nu}^{d',d''}) z_{\nu,d'} + \tilde{M}^{d,d'} \]

with

\[ \tilde{L}_{\nu}^{d',d''} := L_{\nu}^{d',d''} + l_{\nu}^{d',d''} \quad \text{and} \quad \tilde{M}_{\nu}^{d',d''} := M_{\nu}^{d',d''} - c. \]

Observe that all the coefficients \( J_{\nu}^{d',d''}, K_{\nu}^{d',d''}, \tilde{L}_{\nu}^{d',d''} \) and \( \tilde{M}_{\nu}^{d',d''} \) have exactly the same form as the function \( \Phi_{d',d''}(k) \) of Lemma 4.5.2 (see Proposition 4.3.1 and (4.4.1)). Thus, by this lemma with \( \beta = 2 \), for any integers \( n \) and \( m \) with \( n + m \geq 0 \), the absolute value of the \( \partial^{n+m}_{k_1 k_2} \)-derivative of each of these functions is bounded above by

\[ C_{\varepsilon,\Lambda,A,q,m,n} \frac{1}{|d|^2}. \]

Hence, if we recall from Proposition 4.4.4(ii) that

\[ |z_1(k)| \leq 6|d| \quad \text{and} \quad |z_2(k)| \leq 2|d|, \]

and apply the Leibniz rule we find that

\[ \left| \frac{\partial^{n+m}_{k_1 k_2}}{\partial k_1^{n} \partial k_2^{m}} p_{d',d''}(k) \right| \leq C_{m,n} \frac{C}{|d|}. \]

This yields the desired bounds for \( p_1 \) and \( p_2 \) and completes the proof.

We now prove Proposition 4.7.2.

**Proof of Proposition 4.7.2.** (i) Similarly as in (4.6.10) we have

\[ |\beta_2^{(1)}(k)| \leq \frac{\varepsilon}{900} \frac{1}{|z_1(k)|} \quad \text{and} \quad |\eta_2^{(1)}(k)| \leq \frac{\varepsilon}{900} \frac{1}{|z_2(k)|}. \]

Thus, in view of (4.7.7), and by choosing \( \rho \) sufficiently large,

\[ |x_1(k) - w_1(k)| \leq |\beta_2^{(1)}(k) z_1(k) + g_1(k)| \leq \frac{\varepsilon}{900} \frac{C}{\rho} < \frac{\varepsilon}{8}, \]

\[ |x_2(k) - w_2(k)| \leq |\eta_2^{(1)}(k) z_2(k) + g_2(k)| \leq \frac{\varepsilon}{900} \frac{C}{\rho} < \frac{\varepsilon}{8}. \]

This proves part (i).

(ii) Recall (4.7.2) and (4.7.10). Then, for \( 1 \leq j \leq 2 \),

\[ \frac{\partial x_1}{\partial k_j} = \frac{\partial}{\partial k_j} (w_1 + z_1 \beta_2^{(1)} + g_1) = \frac{\partial w_1}{\partial k_j} + z_1 \frac{\partial \beta_2^{(1)}}{\partial k_j} + \frac{\partial z_1}{\partial k_j} \beta_2^{(1)} + \frac{\partial g_1}{\partial k_j}, \]

\[ \frac{\partial x_2}{\partial k_j} = \frac{\partial}{\partial k_j} (w_2 + z_2 \eta_2^{(1)} + g_2) = \frac{\partial w_2}{\partial k_j} + z_2 \frac{\partial \eta_2^{(1)}}{\partial k_j} + \frac{\partial z_2}{\partial k_j} \eta_2^{(1)} + \frac{\partial g_2}{\partial k_j}. \]
To estimate the terms above on the right hand side we use a calculation that we have already done in the proof of Proposition 4.6.1. First observe that the functions \( g_1 \) and \( g_2 \) are similar to the function \( g \) (see (4.7.6) and (4.6.4)). Thus, it is easy to see that \( \frac{\partial g_1}{\partial k_j} \) and \( \frac{\partial g_2}{\partial k_j} \) are given by expressions similar to (4.6.11). Since \( k \in T_{\nu}(0) \cap T_{\nu}(d) \) we have \( |w_1(k)| < \varepsilon \) and \( |w_2(k)| < \varepsilon \). Recall also the inequalities in Proposition 4.4.4(ii). Hence, by Lemmas 4.4.1(ii), 4.5.1(ii) and 4.5.3(ii), we obtain (4.6.12) with \( k \) and \( 2 \) replaced by \( k_j \) and \( 1 \), respectively, and for \( k_1, z(k) \) and \( \beta \) replaced by \( k_j, z_2(k) \) and \( \eta \), respectively. Consequently, similarly as in (4.6.13) and using again Lemma 4.4.1(ii), for \( 1 \leq j \leq 2 \) we have

\[
\left| \frac{\partial z^1}{\partial k_j^1} \beta_2^{(1)} + \frac{\partial g_1}{\partial k_j} \right| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho} \quad \text{and} \quad \left| \frac{\partial z^2}{\partial k_j^2} \eta_2^{(1)} + \frac{\partial g_2}{\partial k_j} \right| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}.
\]

Now recall that \( \beta_2 = J_{\nu}^{00} \) and \( \eta_2 = J_{\nu}^{dd} \). Then, by Proposition 4.3.1, Lemma 4.4.1(ii), and (4.4.25), it follows that

\[
\beta_2^{(1)}(k) = (J_{\nu}^{00})^{(1)}(k) = \sum_{b,c \in G_1'} \frac{1}{N_b(k)} (1, i(-1)\nu' \cdot \hat{\Delta}(b - d) - b, 1, -i(-1)\nu' \cdot \hat{\Delta}(c),
\]

\[
\eta_2^{(1)}(k) = (J_{\nu}^{dd})^{(1)}(k) = \sum_{b,c \in G_1'} \frac{1}{N_b(k)} (1, i(-1)\nu' \cdot \hat{\Delta}(d - b) - b, 1, -i(-1)\nu' \cdot \hat{\Delta}(c - d).
\]

Hence, by Lemma 4.5.3(ii), similarly as in (4.6.14), for \( 1 \leq j \leq 2 \),

\[
\left| z_1(k) \frac{\partial \beta_2^{(1)}(k)}{\partial k_j} \right| \leq \frac{13}{\Lambda^2} \left\| (1, i(-1)\nu') \cdot \hat{\Delta} \right\|_1^2 < \frac{1}{7 \cdot 3^4},
\]

\[
\left| z_2(k) \frac{\partial \eta_2^{(1)}(k)}{\partial k_j} \right| \leq \frac{13}{\Lambda^2} \left\| (1, i(-1)\nu') \cdot \hat{\Delta} \right\|_1^2 < \frac{1}{7 \cdot 3^4}.
\]

Therefore,

\[
\begin{pmatrix}
\frac{\partial z_1}{\partial k_1} & \frac{\partial z_1}{\partial k_2} \\
\frac{\partial z_2}{\partial k_1} & \frac{\partial z_2}{\partial k_2}
\end{pmatrix} = \begin{pmatrix}
1 & i(-1)\nu' \\
1 & i(-1)\nu'
\end{pmatrix} + \begin{pmatrix}
z_1(k) \frac{\partial \beta_2^{(1)}(k)}{\partial k_1} & z_1(k) \frac{\partial \beta_2^{(1)}(k)}{\partial k_2} \\
z_2(k) \frac{\partial \eta_2^{(1)}(k)}{\partial k_1} & z_2(k) \frac{\partial \eta_2^{(1)}(k)}{\partial k_2}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\beta_2^{(1)} & -i(-1)\nu' \beta_2^{(1)} \\
\eta_2^{(1)} & -i(-1)\nu' \eta_2^{(1)}
\end{pmatrix} + \begin{pmatrix}
\frac{\partial g_1}{\partial k_1} & \frac{\partial g_1}{\partial k_2} \\
\frac{\partial g_2}{\partial k_1} & \frac{\partial g_2}{\partial k_2}
\end{pmatrix}.
\]

\[
= \begin{pmatrix}
1 & i(-1)\nu' \\
1 & i(-1)\nu'
\end{pmatrix} (I + M_1 + M_2 + M_3),
\]

where

\[
\left\| M_1 \right\| \leq \frac{2}{7 \cdot 3^4} \quad \text{and} \quad \left\| M_2 + M_3 \right\| \leq C_{\varepsilon,\Lambda,q,A} \frac{1}{\rho}.
\]

Set \( M := M_1 + M_2 + M_3 \). This proves the first claim.
Now, by choosing \( \rho \) sufficiently large we can make
\[
\|M\| < \frac{1}{2}.
\]
Write
\[
P := \begin{pmatrix}
1 & i(-1)^\nu \\
1 & i(-1)^\nu
\end{pmatrix}.
\]
Then, by the inverse function theorem and using the Neumann series,
\[
\begin{pmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}
\end{pmatrix}^{-1} = (I + M)^{-1}P^{-1} = (I + \tilde{M})P^{-1} =: P^{-1}(I + \tilde{M}P^{-1}) = \frac{1}{2}\begin{pmatrix}
1 & 1 \\
i(-1)^\nu & i(-1)^\nu
\end{pmatrix}(I + \tilde{M}P^{-1}),
\]
with
\[
\|P\tilde{M}P^{-1}\| \leq 2\|\tilde{M}\|1 \leq \frac{2\|M\|}{1 - \|M\|} \leq 4\|M\|.
\]
Set \( N := P\tilde{M}P^{-1} \). This proves the second claim.

Differentiating the matrix identity \( TT^{-1} = I \) and applying the chain rule we find that
\[
\frac{\partial^2 k_m}{\partial x_i \partial x_j} = -\sum_{l,p=1}^2 \frac{\partial k_m}{\partial x_l} \frac{\partial}{\partial x_i} \left( \frac{\partial x_l}{\partial k_p} \right) \frac{\partial k_p}{\partial x_j} = -\sum_{l,p=1}^2 \frac{\partial k_m}{\partial x_l} \frac{\partial^2 x_l}{\partial k_r \partial x_p} \frac{\partial k_r}{\partial x_i} \frac{\partial k_p}{\partial x_j}.
\]
Furthermore, in view of the above calculations we have
\[
\left| \frac{\partial k_i}{\partial x_j} \right| \leq \frac{1}{2}(1 + \|N\|) \leq \frac{1}{2}(1 + 4\|M\|)
\]
\[
\leq \frac{1}{2} \left( 1 + 4 \frac{1}{2} \right) < \frac{3}{2}.
\]
Thus,
\[
\left| \frac{\partial^2 k_m}{\partial x_i \partial x_j} \right| \leq 4 \left( \frac{3}{2} \right)^3 \sup_{l,r,p} \left| \frac{\partial^2 x_l}{\partial k_r \partial x_p} \right|.
\]

We now estimate
\[
\frac{\partial^2 x_1}{\partial k_i \partial k_j} = \frac{\partial z_1}{\partial k_i} \frac{\partial \beta_2^{(1)}}{\partial k_j} + z_1 \frac{\partial^2 \beta_2^{(1)}}{\partial k_i \partial k_j} + \frac{\partial z_1}{\partial k_j} \frac{\partial \beta_2^{(1)}}{\partial k_i} + \frac{\partial^2 g_1}{\partial k_i \partial k_j}
\]
and
\[
\frac{\partial^2 x_2}{\partial k_i \partial k_j} = \frac{\partial z_2}{\partial k_i} \frac{\partial \eta_2^{(1)}}{\partial k_j} + z_2 \frac{\partial^2 \eta_2^{(1)}}{\partial k_i \partial k_j} + \frac{\partial z_2}{\partial k_j} \frac{\partial \eta_2^{(1)}}{\partial k_i} + \frac{\partial^2 g_2}{\partial k_i \partial k_j}.
\]
From (4.6.11) with $g, w$ and $z$ replaced by $g_1, w_1$ and $z_1$, respectively, we obtain

\[
\frac{\partial^2 g_1}{\partial k_1^2} = \frac{\partial^2 \beta_1 w_1^2}{\partial k_1^2} + 2 \frac{\partial \beta_1 2w_1 z_1 - w_1^2}{\partial k_1} + \frac{\beta_1 (2z_1^2 - 6w_1 z_1 + 4w_1^2)}{z_1^3} + \left( \frac{\partial^2 \beta_2 (2)}{\partial k_1^2} + \frac{\partial^2 \beta_3 (3)}{\partial k_1^2} \right) z_1
\]

\[+ 2 \left( \frac{\partial \beta_2 (2)}{\partial k_1} + \frac{\partial \beta_3 (3)}{\partial k_1} \right) w_1 + 2 \frac{\partial \beta_3 \beta_4}{\partial k_1} + \frac{\partial^2 \beta_4 w_1}{\partial k_1} + 2 \frac{\partial \beta_4}{\partial k_1} \frac{z_1 - w_1}{z_1^2}
\]

\[+ \frac{\beta_4 (2w_1 - z_1)}{z_1^4} + \frac{\partial^2 \beta_5}{\partial k_1^3} \frac{1}{z_1} - \frac{2 \beta_6}{\partial k_1} \frac{1}{z_1} + \frac{2 \beta_6}{\partial k_1} + \frac{2 \dot{q}(0)}{z_1^3}.
\]

Hence, by Lemmas 4.4.1(ii), 4.5.1(ii) and 4.5.3(ii),

\[\left| \frac{\partial^2 g_1}{\partial k_1^2} \right| \leq C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}.
\]

Similarly we prove that

\[\left| \frac{\partial^2 g_l}{\partial k_i \partial k_j} \right| \leq C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}
\]

for all $l, i, j \in \{1, 2\}$ because all the derivatives acting on $g_l$ are essentially the same (up to constant factors). Indeed,

\[\frac{\partial}{\partial k_i} g_l = \left( \frac{\partial w_l}{\partial k_i} \frac{\partial}{\partial w_l} + \frac{\partial z_l}{\partial k_i} \frac{\partial}{\partial z_l} \right) g_l
\]

with

\[\left( \frac{\partial w_1}{\partial k_1} \frac{\partial w_1}{\partial k_2} \right) = \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu \end{pmatrix} \text{ and } \left( \frac{\partial z_1}{\partial k_1} \frac{\partial z_1}{\partial k_2} \right) = \begin{pmatrix} 1 & i(-1)^\nu \\ 1 & i(-1)^\nu \end{pmatrix}.
\]

In particular this implies

\[\left| \frac{\partial z_i}{\partial k_j} \right| = 1.
\]

Furthermore, again by Lemma 4.5.3(ii),

\[\left| \frac{\partial \beta_2 (1)}{\partial k_j} \right| \leq C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}, \quad \left| \frac{\partial \eta_2 (1)}{\partial k_j} \right| \leq C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho},
\]

and

\[\left| z_1(k) \frac{\partial^2 \beta_2 (1)}{\partial k_j \partial k_j} \right| \leq \frac{65}{\Lambda^3} \|(1, -i(-1)^\nu) \cdot \hat{A} \|_{H_1}^2 < \frac{1}{5 \Lambda^3} \varepsilon^2,
\]

\[\left| z_2(k) \frac{\partial^2 \eta_2 (1)}{\partial k_j \partial k_j} \right| \leq \frac{65}{\Lambda^3} \|(1, -i(-1)^\nu) \cdot \hat{A} \|_{H_1}^2 < \frac{1}{5 \Lambda^3} \varepsilon^2.
\]

Hence,

\[\left| \frac{\partial^2 x_l}{\partial k_i \partial k_j} \right| \leq \frac{1}{5 \Lambda^3} \varepsilon^2 + C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}.
\]

Therefore,

\[\left| \frac{\partial^2 k_m}{\partial x_i \partial x_j} \right| \leq 4 \left( \frac{3}{2} \right)^3 \sup_{i, r, p} \left| \frac{\partial^2 x_l}{\partial k_i \partial x_p} \right| \left| \frac{\partial^2 x_l}{\partial k_i \partial x_p} \right| \leq \frac{3}{\Lambda^3} \varepsilon^2 + C_{\varepsilon, \Lambda, q, A} \frac{1}{\rho}.
\]

This completes the proof of the proposition. \qed
4.8 Quantitative Morse lemma

In this section we prove a quantitative Morse lemma from [4] that is used above for proving Theorem 3.4.2.

**Lemma 4.8.1** (Quantitative Morse lemma [4]). Let δ be a constant with 0 < δ < 1 and assume that
\[ f(x_1, x_2) = x_1x_2 + r(x_1, x_2) \]
is an holomorphic function on
\[ D_δ = \{ (x_1, x_2) ∈ \mathbb{C}^2 \mid |x_1| ≤ δ \text{ and } |x_2| ≤ δ \}. \]
Suppose further that, for all \( x ∈ D_δ \) and \( 1 ≤ i ≤ 2 \) the function \( r \) satisfies
\[
\left| \frac{∂r}{∂x_i}(x) \right| ≤ a < δ \quad \text{and} \quad \left\| \left[ \frac{∂^2 r}{∂x_i∂x_j}(x) \right]_{i,j∈\{1,2\}} \right\| ≤ b < \frac{1}{55},
\]
where \( a \) and \( b \) are constants. Then \( f \) has a unique critical point \( ξ = (ξ_1, ξ_2) ∈ D_δ \) with
\[
|ξ_1| ≤ a \quad \text{and} \quad |ξ_2| ≤ a.
\]
Furthermore, let \( s = \max\{|ξ_1|, |ξ_2|\} \). Then there is a biholomorphic map \( Φ \) from the domain
\[ D(δ−s)(1−19b) \] to a neighbourhood of \( ξ ∈ D_δ \) that contains
\[ \{ (z_1, z_2) ∈ \mathbb{C}^2 \mid |z_i − ξ_i| < (δ − s)(1 − 55b) \text{ for } 1 ≤ i ≤ 2 \} \]
such that
\[ (f ◦ Φ)(z_1, z_2) = z_1z_2 + c, \]
where \( c ∈ \mathbb{C} \) is a constant fulfilling \( |c − r(0,0)| ≤ a^2 \). The differential \( DΦ \) obeys
\[ \|DΦ − I\| ≤ 18b. \]
If \( \frac{∂r}{∂x_1}(0,0) = 0 \) and \( \frac{∂r}{∂x_2}(0,0) = 0 \), then \( ξ = 0 \) and \( s = 0 \).

**Proof.** Step 1 For \( 1 ≤ i ≤ 2 \) set
\[
C_i := \left\{ (x_1, x_2) ∈ D_δ \mid \frac{∂f}{∂x_i}(x) = 0 \right\}.
\]
To prove the first claim we show that \( C_1 \) and \( C_2 \) have a unique point of intersection. For each \( x_1 \) with \( |x_1| \leq \delta \) consider the functions
\[
 u(x_2) := x_2 \quad \text{and} \quad v(x_2) := \frac{\partial r}{\partial x_1}(x_1, x_2)
\]
on the domain \( \Omega := \{x_2 \in \mathbb{C} \mid |x_2| \leq \delta \} \) with boundary \( \partial \Omega := \{x_2 \in \mathbb{C} \mid |x_2| = \delta \} \). By hypothesis, for all \( x_2 \in \partial \Omega \) we have
\[
 |u(x_2)| = \delta \quad \text{and} \quad |v(x_2)| \leq a < \delta,
\]
so that
\[
 |u(x_2)| > |v(x_2)| \quad \text{and} \quad u(x_2) \neq 0.
\]
Thus, by Rouche’s theorem [13, Theorem 10.43], the functions \( u(x_2) \) and \( u(x_2) + v(x_2) \) have the same number of zeros in \( \Omega \). Since \( u(x_2) \) has only one zero, the equation
\[
 0 = u(x_2) + v(x_2) = x_2 + \frac{\partial r}{\partial x_1}(x_1, x_2) = \frac{\partial f}{\partial x_1}(x_1, x_2)
\]
has a unique solution \( \tilde{x}_2(x_1) \). By the implicit function theorem, this solution is an holomorphic function of \( x_1 \). Furthermore, differentiating with respect to \( x_1 \) the identity
\[
 \tilde{x}_2(x_1) = -\frac{\partial r}{\partial x_1}(x_1, \tilde{x}_2(x_1))
\]
we find that
\[
 \frac{\partial \tilde{x}_2(x_1)}{\partial x_1} = -\frac{\partial^2 r}{\partial x_1^2}(x_1, \tilde{x}_2(x_1)) \left(1 + \frac{\partial^2 r}{\partial x_2 \partial x_1}(x_1, \tilde{x}_2(x_1)) \right)^{-1}.
\]
Hence, using the hypotheses,
\[
 |\tilde{x}_2(x_1)| \leq a \quad \text{and} \quad \left| \frac{\partial \tilde{x}_2(x_1)}{\partial x_1} \right| \leq \frac{b}{1 - b}.
\]
Similarly we can parametrize the curve \( C_2 \) by a map \( x_2 \mapsto (\tilde{x}_1(x_2), x_2) \) satisfying
\[
 |\tilde{x}_1(x_2)| \leq a \quad \text{and} \quad \left| \frac{\partial \tilde{x}_1(x_2)}{\partial x_2} \right| \leq \frac{b}{1 - b}.
\]
Therefore, the curves \( C_1 \) and \( C_2 \) intersect in a unique point \((\xi_1, \xi_2)\) with \( \tilde{x}_1(\xi_2) = \xi_1 \) and \( \tilde{x}_2(\xi_1) = \xi_2 \). Thus we have \( |\xi_1| \leq a \) and \( |\xi_2| \leq a \). This proves the first claim.

**Step 2** Without loss of generality we may assume that \( r(0, 0) = 0 \). (If \( r(0, 0) \neq 0 \) then instead of \( f \) consider \( f_0(x_1, x_2) = x_1 x_2 + r_0(x_1, x_2) \) with \( f_0 := f - r(0, 0) \) and \( r_0 := r - r(0, 0) \).)

Define
\[
 \tilde{f}(x_1, x_2) := f(x_1 + \xi_1, x_2 + \xi_2)
\]
and
\[
\tilde{r}(x_1, x_2) := x_1\xi_2 + \xi_1 x_2 + \xi_1 \xi_2 + r(x_1 + \xi_1, x_2 + \xi_2).
\]
Then,
\[
\tilde{f}(x_1, x_2) = x_1 x_2 + \tilde{r}(x_1, x_2),
\]
where \(\tilde{r}\) is an holomorphic function on \(D_{\delta - s}\) with \(s := \max\{|\xi_1|, |\xi_2|\}\). Since \((\xi_1, \xi_2)\) is a critical point of \(f\), we have
\[
\tilde{r}(0, 0) = \xi_1 \xi_2 + r(\xi_1, \xi_2),
\]
\[
\frac{\partial \tilde{r}}{\partial x_1}(0, 0) = \xi_2 + \frac{\partial r}{\partial x_1}(\xi_1, \xi_2) = 0,
\]
\[
\frac{\partial \tilde{r}}{\partial x_2}(0, 0) = \xi_1 + \frac{\partial r}{\partial x_2}(\xi_1, \xi_2) = 0.
\]
Thus, if we translate the system of coordinates in \(\mathbb{C}^2\) so that \((\xi_1, \xi_2)\) is mapped to \((0, 0)\), we obtain
\[
\tilde{r}(0, 0) = r(0, 0) = 0, \quad \frac{\partial \tilde{r}}{\partial x_1}(0, 0) = 0 \quad \text{and} \quad \frac{\partial \tilde{r}}{\partial x_2}(0, 0) = 0.
\]
Furthermore, since
\[
\frac{\partial^2 \tilde{r}}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j},
\]
we still have the bound
\[
\left\| \begin{bmatrix} \frac{\partial^2 \tilde{r}}{\partial x_i \partial x_j}(x) \end{bmatrix}_{i,j\in\{1,2\}} \right\| \leq b.
\]
This shows that it suffices to prove the lemma in the special case that
\[
r(0, 0) = \frac{\partial r}{\partial x_1}(0, 0) = \frac{\partial r}{\partial x_2}(0, 0) = 0
\]
and then replace \(\delta\) by \(\delta - s\).

**Step 3** For \(0 \leq t \leq 1\) set
\[
f_t(x_1, x_2) := x_1 x_2 + t r(x_1, x_2).
\]
Below we construct a \(t\)-dependent vector field \(X^t(x)\) that is holomorphic on \(D_{\delta(1-4b)}\) and satisfies
\[
r(x) + (\nabla f_t)(x) \cdot X^t(x) = 0, \quad (4.8.2)
\]
\[
\|X^t(x)\| \leq 5b(|x_1| + |x_2|) \leq 10b\delta, \quad (4.8.3)
\]
\[
\left\| \frac{\partial X^t}{\partial x_i}(x) \right\| \leq 8b \quad \text{for} \ 1 \leq i \leq 2. \quad (4.8.4)
\]
Now, for $0 \leq \tau \leq 1$ consider the map

$$\Phi_{\tau} : D_{\delta(1+10\delta)} \to \mathbb{C}^2,$$

where $\Phi_{\tau}(x)$ is the solution of the initial value problem

$$\frac{d}{d\tau} \Phi_{\tau}(x) = X^\tau(\Phi_{\tau}(x)) \quad \text{for} \quad 0 \leq \tau \leq 1,$$

$$\Phi_0(x) = x.$$

This solution exists because $X^\tau(x)$ is holomorphic. Consequently, in view of (4.8.2),

$$\Phi_0(x) = x \quad \text{and} \quad \frac{d}{d\tau} f_\tau(\Phi_{\tau}(x)) = r(\Phi_{\tau}(x)) + (\nabla f_\tau)(\Phi_{\tau}(x)) \cdot \frac{d}{d\tau}\Phi_{\tau}(x) = 0.$$

Furthermore, by (4.8.3) we have

$$\frac{d}{d\tau} \| \Phi_{\tau}(x) \| = \frac{\Phi_{\tau}(x)}{\| \Phi_{\tau}(x) \|} \cdot \frac{d}{d\tau} \Phi_{\tau}(x) = \frac{\Phi_{\tau}(x)}{\| \Phi_{\tau}(x) \|} \cdot X^\tau(\Phi_{\tau}(x)) \leq \| X^\tau(\Phi_{\tau}(x)) \| \leq 10b\| \Phi_{\tau}(x) \|,$$

so that

$$\| \Phi_{\tau}(x) \| \leq e^{10b\tau} \| \Phi_0(x) \| = e^{10b\tau} \| x \|.$$

This implies

$$\left\| \frac{d}{d\tau} \Phi_{\tau}(x) \right\| = \| X^\tau(\Phi_{\tau}(x)) \| \leq 10b\| \Phi_{\tau}(x) \| \leq 10be^{10b\tau} \| x \|.$$

Hence, after integrating with respect to $\tau$,

$$\| \Phi_{\tau}(x) - x \| \leq 10be^{10b\tau} \| x \| \leq 15b\delta.$$

This shows that $\Phi_{\tau}(x)$ remains in the domain of $X^\tau(x)$ for all $0 \leq \tau \leq 1$ and $x \in D_{\delta(1-19\delta)}$.

Now observe that

$$\frac{d}{d\tau} (D_x \Phi_{\tau})(x) = D_x (X^\tau(\Phi_{\tau}(x))) = (D_x X^\tau)(\Phi_{\tau}(x)) (D_x \Phi_{\tau})(x),$$

so that, by (4.8.4),

$$\frac{d}{d\tau} \| (D_x \Phi_{\tau})(x) \| \leq \left\| \frac{d}{d\tau} (D_x \Phi_{\tau})(x) \right\|$$

$$\leq \| (D_x X^\tau)(\Phi_{\tau}(x)) \| \| (D_x \Phi_{\tau})(x) \| \leq \sqrt{2} 8b \| (D_x \Phi_{\tau})(x) \|.$$

Consequently,

$$\| (D_x \Phi_{\tau})(x) \| \leq e^{12b} \| (D_x \Phi_0)(x) \| = e^{12b}$$

$$\left\| \frac{d}{d\tau} (D_x \Phi_{\tau})(x) \right\| \leq 18b.$$
and
\[ \|D_x \Phi_t - I\| \leq \int_0^t \left\| \frac{d}{d\tau}(D_x \Phi_\tau)(x) \right\| d\tau \leq 18b < 1. \]

Therefore, by the inverse function theorem, the map \( \Phi_t \) is biholomorphic into its image. If we set \( \Phi := \Phi_1 \), then \( \Phi \) has the desired properties because
\[ f_1 \circ \Phi_1 = f_1 \circ \Phi_1 - f_0 + f_0 = \int_0^1 \frac{d}{d\tau}f_\tau(\Phi_\tau) d\tau + f_0 = f_0 \]
and, by the contraction mapping theorem, the image of \( D_δ(1-19b) \) under \( \Phi \) contains \( B_δ' \) with
\[ \delta' = \frac{1 - 18b}{1 + 18b} (1 - 19b)\delta \geq (1 - 55b)\delta. \]

**Step 4** To construct \( X^t(x) \) observe that the equation (4.8.2) is
\[ r(x_1, x_2) + \begin{pmatrix} x_2 + t \frac{\partial r}{\partial x_1}(x) \\ x_1 + t \frac{\partial r}{\partial x_2}(x) \end{pmatrix} = 0. \quad (4.8.5) \]

By the assumptions on \( r \), for \( 1 \leq i \leq 2 \) we have
\[ \left| \frac{\partial r}{\partial x_i}(x) \right| \leq b(|x_1| + |x_2|) \leq 2b\delta. \quad (4.8.6) \]

Hence,
\[ |r(x)| \leq b(|x_1| + |x_2|)^2 < 4b\delta^2 < 4b\delta. \]

Thus, since \( b < 1/55 \), by the inverse function theorem, for \( 0 \leq t \leq 1 \) the map
\[ P_t : D_\delta \longrightarrow \mathbb{C}^2 \]
\[ (x_1, x_2) \mapsto \begin{pmatrix} x_2 + t \frac{\partial r}{\partial x_1}(x) \\ x_1 + t \frac{\partial r}{\partial x_2}(x) \end{pmatrix} \]
is biholomorphic into its image, and the image contains \( D_\delta(1-2b) \). Furthermore,
\[ \left\| D_x P_t - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| = \left\| t \frac{\partial^2 r}{\partial x_i \partial x_j} \right\| \leq tb \quad (4.8.7) \]
and
\[ \left\| D_x P_t^{-1} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \leq \frac{tb}{1 - tb}. \quad (4.8.8) \]
The last inequality follows by inverse function theorem and a estimate similar to (4.8.1).
Set $g(y) := -(r \circ P_t^{-1})(y)$. To solve (4.8.5) we first solve the equation

$$g(y_1, y_2) = \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} Y_1^t(y) \\ Y_2^t(y) \end{pmatrix} \right\rangle$$

on $D_{b(1-2b)}$. This is done by the functions

$$Y_1^t(y) = \frac{1}{y_1} g(y_1, 0) \quad \text{and} \quad Y_2^t(y) = \frac{1}{y_2} (g(y_1, y_2) - g(y_1, 0)).$$

We now show that these functions are holomorphic and derive some estimates for them. Observe that, in view of (4.8.6), the change of variables

$$y = P_t(x) = \left( x_2 + t \frac{\partial r}{\partial x_1}(x), x_1 + t \frac{\partial r}{\partial x_2}(x) \right)$$

obeys

$$(1 - 2b)(|x_1| + |x_2|) \leq |y_1| + |y_2| \leq (1 + 2b)(|x_1| + |x_2|). \quad (4.8.9)$$

By the chain rule, (4.8.6), (4.8.8), and the last inequality, for $1 \leq i \leq 2$,

$$\left| \frac{\partial g}{\partial y_i}(y) \right| \leq b(|x_1| + |x_2|) \left( 1 + \frac{\sqrt{2b}}{1-b} \right) \leq b \frac{1 + 2b}{1 - 2b} (|y_1| + |y_2|),$$

since $\sqrt{2b}/(1-b) < 2b$ because $b < 1/55$. Therefore,

$$|g(y_1, y_2)| \leq \frac{b}{2} \frac{1 + 2b}{1 - 2b} (|y_1| + |y_2|)^2.$$

This shows that $Y_1^t$ and $Y_2^t$ are not singular at $y = 0$ and thus are holomorphic. Furthermore, it is easy to see that

$$|Y_1^t(y)| \leq \frac{b}{2} \frac{1 + 2b}{1 - 2b} |y_1|, \quad \left| \frac{\partial Y_1^t}{\partial y_1}(y) \right| \leq \frac{3b}{2} \frac{1 + 2b}{1 - 2b} \quad \text{and} \quad \frac{\partial Y_1^t}{\partial y_1}(y) = 0.$$ 

These are the estimates we need for $Y_1^t$. We next consider $Y_2^t$.

To derive bounds for $Y_2^t$ we consider the regions $|y_2| \geq |y_1|$ and $|y_2| \leq |y_1|$ separately. First, if $|y_2| \geq |y_1|$ then

$$|Y_2^t(y)| \leq \frac{1}{|y_2|} (|g(y_1, y_2)| + |g(y_1, 0)|) \leq \frac{b}{1 - 2b} \frac{1 + 2b}{|y_2|} (|y_1| + |y_2|)^2 \leq 2b \frac{1 + 2b}{1 - 2b} (|y_1| + |y_2|),$$

and similarly

$$\left| \frac{\partial Y_2^t}{\partial y_1}(y) \right| \leq \frac{1}{|y_2|} \left( \left| \frac{\partial g}{\partial y_1}(y_1, y_2) \right| + \left| \frac{\partial g}{\partial y_1}(y_1, 0) \right| \right) \leq 4b \frac{1 + 2b}{1 - 2b},$$

$$\left| \frac{\partial Y_2^t}{\partial y_2}(y) \right| \leq \frac{1}{|y_2|} |Y_2^t(y)| + \frac{1}{|y_2|} \left| \frac{\partial g}{\partial y_2}(y_1, y_2) \right| \leq 6b \frac{1 + 2b}{1 - 2b}.$$
Observe that, in particular, these estimates hold for \(|y_1| = |y_2|\). Now, for fixed \(y_1\) we can apply the maximum modulus principle to derive bounds for the functions \(Y_2^t(y_1,\cdot)\) and \(\frac{\partial}{\partial y_2}Y_2(y_1,\cdot)\) in the disk \(|z| \leq |y_1|\). (Note, this is the case \(|y_2| \leq |y_1|\).) By this principle, the modulus of these functions inside the disk is bounded by the maximum modulus at the boundary \(|z| = |y_1|\). Thus, using the above estimates, which are valid for \(|y_1| = |y_2|\), we obtain

\[
|Y_2^t(y_1, z)| \leq \max_{|z|=|y_1|} |Y_2^t(y_1, z)| \leq \max_{|z|=|y_1|} 2b \frac{1 + 2b}{1 - 2b} (|y_1| + |z|)
= 4b \frac{1 + 2b}{1 - 2b} |y_1| \leq 4b \frac{1 + 2b}{1 - 2b} (|y_1| + |z|).
\]

Similarly,

\[
\left| \frac{\partial Y_2^t}{\partial y_1}(y_1, z) \right| \leq 4b \frac{1 + 2b}{1 - 2b} \quad \text{and} \quad \left| \frac{\partial Y_2^t}{\partial y_2}(y_1, z) \right| \leq 6b \frac{1 + 2b}{1 - 2b}.
\]

We have all the bounds we need for \(Y_2^t\).

Let \(Y^t(y) = (Y_1^t(y), Y_2^t(y))\). Then, combining all the above estimates we find that, for \(1 \leq i \leq 2\) and for all \(y \in D_\delta(1-2b)\),

\[
\|Y^t(y)\| \leq \sqrt{16 \frac{1 + 2b}{1 - 2b} (|y_1| + |y_2|)},
\]

\[
\left\| \frac{\partial Y^t}{\partial y_i}(y) \right\| \leq 6b \frac{1 + 2b}{1 - 2b}.
\]

Furthermore, by construction the vector field \(Y^t(y)\) satisfies the equation

\[
\langle y, Y^t(y) \rangle = -h(P_t^{-1}(y)).
\]

If we recall the change of variables \(y = P_t(x)\), this equation is equivalent to

\[
\langle P_t(x), Y^t(P_t(x)) \rangle = -r(x).
\]

Finally, if we set \(X^t \coloneqq Y_t \circ P_t\), then \(X^t(x)\) satisfies the desired equation

\[
\langle P_t(x), X^t(x) \rangle = -r(x)
\]
on \(P_t^{-1}(D_\delta(1-2b))\). Note, this is in fact equation (4.8.5). That is, we have proved (4.8.2). Furthermore, in view of (4.8.6), the region \(P_t^{-1}(D_\delta(1-2b))\) contains \(D_\delta(1-4b)\). From (4.8.10), (4.8.11), (4.8.7) and (4.8.9), we obtain the estimates (4.8.3) and (4.8.4), namely,

\[
\|X^t(x)\| \leq \sqrt{16 \frac{1 + 2b}{1 - 2b} (|x_1| + |x_2|)} < 5b(|x_1| + |x_2|) < 10b\delta,
\]

\[
\left\| \frac{\partial X^t}{\partial x_i}(x) \right\| \leq 6b \frac{1 + 2b}{1 - 2b} < 8b.
\]

This completes the proof of the theorem. \(\square\)
4.9 Appendix

We now prove Propositions 4.1.2 and 4.5.4. Proposition 4.1.2 follows from part (a) of Proposition 4.5.4, which we reproduce below.

Let $T$ be a linear operator from $L^2_C$ to $L^2_B$ with $B, C \subset \Gamma^\#$, and recall the definition

$$
\|T\|_\sigma := \max \left\{ \sup_{b \in B} \sum_{c \in C} |T_{b,c}| \sigma(|b-c|), \sup_{c \in C} \sum_{b \in B} |T_{b,c}| \sigma(|b-c|) \right\},
$$

where $\sigma$ satisfies the hypotheses stated in p. 93. We next prove that this norm has the following properties.

**Proposition 4.5.4** (Properties of $\| \cdot \|_\sigma$). Let $S$ and $T$ be linear operators from $L^2_C$ to $L^2_B$ with $B, C \subset \Gamma^\#$. Then:

(a) $\|T\| \leq \|T\|_{\sigma \equiv 1} \leq \|T\|_\sigma$;

(b) If $B = C$, then $\|ST\|_\sigma \leq \|S\|_\sigma \|T\|_\sigma$;

(c) If $B = C$, then $\|(I + T)^{-1}\|_\sigma \leq (1 - \|T\|_\sigma)^{-1}$ if $\|T\|_\sigma < 1$;

(d) $|T_{b,c}| \leq \frac{1}{\sigma(|b-c|)} \|T\|_\sigma$ for all $b \in B$ and all $c \in C$.

**Proof.** (a) By the Cauchy-Schwarz inequality we have

$$
|\langle T \hat{\varphi} \rangle(b) | \leq \left[ \sum_{c \in C} |T_{b,c} \hat{\varphi}(c)| \right]^{1/2} \left[ \sum_{c \in C} |\hat{\varphi}(c)|^2 \right]^{1/2} 
$$

Hence,

$$
\|\langle T \hat{\varphi} \rangle \|_{l^2}^2 = \sum_{b \in B} |\langle T \hat{\varphi} \rangle(b) |^2 \leq \left[ \sup_{b \in B} \sum_{c' \in C} |T_{b,c'}| \right] \left[ \sum_{c \in C} \sum_{b \in B} |T_{b,c}| \right] \left[ \sum_{c \in C} |\hat{\varphi}(c)|^2 \right]^{1/2} 
$$

$$
\leq \left[ \sup_{b \in B} \sum_{c' \in C} |T_{b,c'}| \right] \left[ \sup_{c \in C} \sum_{b \in B} |T_{b,c}| \right] \left[ \sum_{c \in C} |\hat{\varphi}(c)|^2 \right] \leq \max \left\{ \sup_{b \in B} \sum_{c' \in C} |T_{b,c'}|, \sup_{c \in C} \sum_{b \in B} |T_{b,c}| \right\} \|\hat{\varphi}\|_{l^2}.
$$

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Thus, 
\[
\frac{\|T \varphi\|_{L^2_B}}{\|\varphi\|_{L^2_C}} = \frac{\|(T \varphi)^{\land}\|_{L^2}}{\|\varphi\|_{L^2}} \leq \max \left\{ \sup_{b \in B} \sum_{c', c'' \in C} |T_{b, c'}|, \sup_{c \in C} \sum_{b \in B} |T_{b, c}| \right\}
\]
for all \( \varphi \in L^2_C \). This implies that
\[
\|T\| \leq \max \left\{ \sup_{b \in B} \sum_{c', c'' \in C} |T_{b, c'}|, \sup_{c \in C} \sum_{b \in B} |T_{b, c}| \right\}.
\]
That is, 
\[
\|T\| \leq \|T\|_{\sigma \equiv 1}.
\]
This is the first inequality of part (a). The second inequality, namely,
\[
\|T\|_{\sigma \equiv 1} \leq \|T\|_{\sigma},
\]
follows immediately if we observe that \( \sigma(t) \geq 1 \) for all \( t \geq 0 \) by hypothesis (see p. 93). This proves part (a).

(b) By hypothesis \( B = C \). First observe that, since \( \sigma \) increases monotonically, and \( \sigma(s + t) \leq \sigma(s)\sigma(t) \) for all \( s, t \in \mathbb{R}^+ \), we have
\[
\sigma(|b - c|) = \sigma(|b - d + d - c|) \leq \sigma(|b - d| + |d - c|) \leq \sigma(|b - d|)\sigma(|d - c|)
\]
for all \( b, c, d \in B \). Hence,
\[
\sup_{b \in B} \sum_{c \in B} |(ST)_{b, c}| \sigma(|b - c|) = \sup_{b \in B} \sum_{c \in B} \left| \sum_{d \in B} S_{b, d} T_{d, c} \right| \sigma(|b - c|) \\
\leq \sup_{b \in B} \sum_{d \in B} |S_{b, d}| \sigma(|b - d|) \sum_{c \in B} |T_{d, c}| \sigma(|d - c|) \\
\leq \left[ \sup_{d \in B} \sum_{c \in B} |T_{d, c}| \sigma(|d - c|) \right] \left[ \sup_{b \in B} \sum_{d \in B} |S_{b, d}| \sigma(|b - d|) \right],
\]
and similarly we prove that
\[
\sup_{c \in B} \sum_{b \in B} |(ST)_{b, c}| \sigma(|b - c|) \leq \left[ \sup_{c \in B} \sum_{d \in B} |T_{d, c}| \sigma(|d - c|) \right] \left[ \sup_{b \in B} \sum_{d \in B} |S_{b, d}| \sigma(|b - d|) \right].
\]
Thus,
\[
\|ST\|_{\sigma} \leq \|S\|_{\sigma} \|T\|_{\sigma},
\]
as claimed.
(c) By hypothesis $B = C$. Since $\|T\|_\sigma < 1$, the Neumann series for $(I + T)^{-1}$ converges (and its $\sigma$-norm is bounded). Hence, using part (b),
\[
\|(I + T)^{-1}\|_\sigma = \left\| \sum_{j=0}^{\infty} T^j \right\|_\sigma \leq \sum_{j=0}^{\infty} \|T\|^j_{\sigma} \leq (1 - \|T\|_{\sigma})^{-1},
\]
as was to be shown.

(d) Since $\sigma(t) \geq 1$ for all $t \geq 0$, observing the definition of $\| \cdot \|_\sigma$ it is easy to see that
\[
|T_{b,c}| = \frac{1}{\sigma(|b - c|)} \cdot T_{b,c} \sigma(|b - c|) \leq \|T\|_\sigma
\]
for all $b \in B$ and all $c \in C$. This completes the proof.

We finally prove Proposition 4.1.2.

**Proof of Proposition 4.1.2.** By applying Proposition 4.5.4(a) we find that
\[
\|T\| \leq \|T\|_{\sigma \equiv 1} = \max \left\{ \sup_{b \in B} \sum_{c \in C} |T_{b,c}|, \sup_{c \in C} \sum_{b \in B} |T_{b,c}| \right\}.
\]
This is the desired inequality.
Chapter 5

Exploiting gauge invariance

5.1 A gauge transformation

In this section we introduce a gauge transformation \( A \to A^{(\nu)} \) that simplify (or modify) the main estimates and results in Chapter 4.

For \( \nu \in \{1, 2\} \) define \( \hat{\Psi}_\nu : \Gamma^\# \to \mathbb{C} \) as

\[
\hat{\Psi}_\nu(b) := \begin{cases} 
(1, i(-1)^\nu) \cdot \hat{A}(b) / i(1, i(-1)^\nu) \cdot b & \text{if } b \neq 0, \\
0 & \text{if } b = 0.
\end{cases}
\]

Observe that, for all \( b \in \Gamma^\# \setminus \{0\} \) and \( \nu \in \{1, 2\} \),

\[
|(1, i(-1)^\nu) \cdot b| = |b_1 + i(-1)^\nu b_2| = |b| \geq 2\Lambda
\]

and

\[
|\hat{\Psi}_\nu(b)| = \left| \frac{(1, i(-1)^\nu) \cdot \hat{A}(b)}{|b_1 + i(-1)^\nu b_2|} \right| = \left| \frac{(1, i(-1)^\nu) \cdot \hat{A}(b)}{|b|} \right| \leq \sqrt{2} \frac{|\hat{A}(b)|}{|b|}.
\]

Thus, the function \( \hat{\Psi}_\nu \) is well-defined and for any \( \beta \geq 0 \) we have

\[
\|b|^{1+\beta} \hat{\Psi}_\nu(b)\|_{L^1} \leq \sqrt{2} \|b|^\beta \hat{A}(b)\|_{L^1}.
\]

Now set

\[
A^{(\nu)} := A - \nabla \hat{\Psi}_\nu
\]

where

\[
\hat{\Psi}_\nu := (\hat{\Psi}_\nu)^\dagger.
\]
and to simplify the notation write

\[ \zeta := (1, i(-1)^\nu). \]

Since without loss of generality we have assumed that \( \hat{A}(0) = 0 \), it follows that \( \hat{A}^{(\nu)}(0) = 0 \).

Furthermore, for all \( b \in \Gamma^\# \setminus \{0\} \),

\[
\begin{align*}
\hat{A}^{(\nu)}(b) &= \hat{A}(b) - ib\Psi_{\nu}(b) = \hat{A}(b) - b \frac{\zeta \cdot \hat{A}(b)}{\zeta \cdot b} = \frac{(\zeta \cdot b)\hat{A}(b) - (\zeta \cdot \hat{A}(b))b}{\zeta \cdot b} \\
&= \frac{1}{\zeta \cdot b} \left( (\zeta_1 b_1 + \zeta_2 b_2)\hat{A}_1(b) - (\zeta_1 \hat{A}_1(b) + \zeta_2 \hat{A}_2(b))b_1, \\
&\quad (\zeta_1 b_1 + \zeta_2 b_2)\hat{A}_2(b) - (\zeta_1 \hat{A}_1(b) + \zeta_2 \hat{A}_2(b))b_2 \right) \\
&= \frac{1}{\zeta \cdot b} \left( (b_2 \hat{A}_1(b) - b_1 \hat{A}_2(b))\zeta_2, (b_2 \hat{A}_1(b) - b_1 \hat{A}_2(b))(-\zeta_1) \right) \\
&= \frac{1}{\zeta \cdot b} (b_2, -b_1) \cdot (\hat{A}(b), \hat{A}(b)) (\zeta_2, -\zeta_1) = \frac{b^\perp \cdot \hat{A}(b)}{\zeta \cdot b} \zeta^\perp \\
&= \frac{b^\perp \cdot \hat{A}(b)}{(1, i(-1)^\nu) \cdot \hat{A}(b)} (1, i(-1)^\nu)^\perp.
\end{align*}
\]

Hence, for all \( b \in \Gamma^\# \),

\[ |\hat{A}^{(\nu)}(b)| \leq \sqrt{2} |\hat{A}(b)|, \]

and for any \( \beta \geq 0 \),

\[ \|b|^{\beta} \hat{A}^{(\nu)}(b)\|_{l^1} \leq \sqrt{2} \|b|^{\beta} \hat{A}(b)\|_{l^1}. \]

The transformation \( A \to A^{(\nu)} \) is useful because of the following proposition, which is a particular case of Theorem 2.6.1 (gauge invariance).

**Proposition 5.1.1** (Gauge invariance). Let \( \nu \in \{1, 2\} \) and assume that \( A_1, A_2 \in C^1(\mathbb{R}^2/\Gamma) \) and \( V \in C^0(\mathbb{R}^2/\Gamma) \) so that \( \Psi_\nu \in C^2(\mathbb{R}^2/\Gamma) \). Then,

\[ \text{Ker}(H_k(A, V)) \neq \{0\} \quad \text{if and only if} \quad \text{Ker}(H_k(A^{(\nu)}, V)) \neq \{0\}, \]

and consequently

\[ \tilde{F}(A, V) = \tilde{F}(A^{(\nu)}, V). \]

Therefore, to study the Fermi curve of \((A, V)\) we may replace \( A \) by \( A^{(1)} \) or \( A^{(2)} \). We shall exploit this property below.
5.2 The regular piece revisited

From now on we shall make the following hypothesis.

Hypothesis 5.2.1.

\[ A_1, A_2 \in C^1(\mathbb{R}^2/\Gamma) \quad \text{and} \quad V \in C^0(\mathbb{R}^2/\Gamma). \]

We now describe the main simplifications (or modifications) introduced by the gauge transformation \( A \rightarrow A^{(\nu)} \). Our first observation is that the expressions for \( N_{d'} + D_{d',d'} \) and \( D_{d',d'} \) (in Proposition 4.3.1) become simpler.

**Proposition 5.2.1.** Under hypothesis 5.2.1, let \( \nu \in \{1, 2\} \) and replace \( A \) by \( A^{(\nu)} \). Then \( J_{d'}^{d''} = 0 \) and \( L_{d'}^{d''} = 0 \) for any \( d', d'' \in G \) and consequently

\[ N_{d'} + D_{d',d'} = J_{d'}^{d''} w_{d',d''} + (1 + K_{d'}^{d''}) w_{d',d'} z_{d',d''} + L_{d'}^{d''} z_{d',d''} + M_{d'}^{d''} \]

and

\[ D_{d',d'} = J_{d'}^{d''} w_{d',d''} + K_{d'}^{d''} w_{d',d'} z_{d',d''} + L_{d'}^{d''} z_{d',d''} + M_{d'}^{d''}. \]

**Proof.** First recall from (5.1.1) that

\[ A^{(\nu)}(b) = \frac{b^\perp \cdot \hat{A}(b)}{(1, i(-1)^\nu \cdot b(i(-1)^\nu), -1) \cdot b(i(-1)^\nu), -1} \]

for all \( b \in \Gamma^\# \setminus \{0\} \), and that \( A^{(\nu)}(0) = 0 \). Now, substitute this expression into the definitions of \( J_{d'}^{d''} \) and \( L_{d'}^{d''} \) in Proposition 4.3.1. Then, if we observe that

\[ (-1)^\nu = -(1)^\nu, \]

and compute

\[ (1, -i(-1)^\nu) \cdot (i(-1)^\nu, -1) = i((-1)^\nu + (1)^\nu) = 0, \]

it follows easily that \( J_{d'}^{d''} = 0 \) and \( L_{d'}^{d''} = 0 \). Thus, by Proposition 4.3.1,

\[ N_{d'} + D_{d',d'} = J_{d'}^{d''} w_{d',d''} + (1 + K_{d'}^{d''}) w_{d',d'} z_{d',d''} + L_{d'}^{d''} z_{d',d''} + M_{d'}^{d''} \]

and

\[ D_{d',d'} = J_{d'}^{d''} w_{d',d''} + K_{d'}^{d''} w_{d',d'} z_{d',d''} + L_{d'}^{d''} z_{d',d''} + M_{d'}^{d''}, \]

as was to be shown. \( \square \)
We now inspect the proof of Theorem 3.4.1 (the regular piece) to see what we have gained by performing this transformation. The last proposition immediately implies that the defining equation (4.6.1), namely,
\[
\beta_1 w^2 + \beta_2 z^2 + (1 + \beta_3) wz + \beta_4 w + \beta_5 z + \beta_6 + \hat{q}(0) = 0,
\]
is reduced to
\[
\beta_1 w^2 + (1 + \beta_3) wz + \beta_5 z + \beta_6 + \hat{q}(0) = 0,
\]
because \(\beta_2 = J^{00}_\nu = 0\) and \(\beta_4 = L^{00}_\nu = 0\). Thus, instead of equation (4.6.3), namely,
\[
w + \beta_2^{(1)} z + g = 0,
\]
we have
\[
w + g = 0,
\]
where
\[|g(k)| \leq \frac{C}{\rho}.
\]
In the above equation, observe the absence of term \(\beta_2^{(1)} z\) which does not decay with respect to \(\rho\) (it is only \(O(1)\)). This yields better bounds and makes the analysis simpler. In fact, Proposition 4.6.1 becomes

**Proposition 5.2.2.** Under the hypotheses of Theorem 3.4.1 and Hypothesis 5.2.1, after the gauge transformation \(A \rightarrow A^{(\nu')}\) we have
\[
|F(k) - w(k)| \leq \frac{C_1}{\rho}
\]
(a) and
\[
\left| \frac{\partial F}{\partial k_1}(k) - 1 \right| \leq \frac{C_2}{\rho},
\]
(b)
where the constants \(C_1\) and \(C_2\) depend only on \(\epsilon, \Lambda, q\) and \(A\).

Consequently, after replacing Proposition 4.6.1 by the above proposition, we no longer need Proposition 4.6.2, which is an improvement of Proposition 4.6.1 necessary to take care of the term \(\beta_2^{(1)} z\) (which vanishes after the gauge transformation). Thus, Lemma 4.4.2 that was used to prove Proposition 4.6.2 is not necessary anymore. This simplifies the analysis. Then, using the new bounds, the proof of Theorem 3.4.1 can be carried out in exactly the same way yielding the following improved version of Theorem 3.4.1.
Theorem 5.2.3 (The regular piece after the transformation $A \to A^{(\nu')}$. Let $0 < \varepsilon < \Lambda/6$ and assume that $A_1$, $A_2$ and $V$ satisfy Hypothesis 5.2.1 with $\| (1+b^2) \hat{A}(b) \|_{l^1(\Gamma^\# \setminus \{0\})} < 2\varepsilon/63$ and $\| b^2 \hat{q}(b) \|_{l^1(\Gamma^\#)} < \infty$. Then, after performing the transformation $A \to A^{(\nu')}$, there is a constant $\rho = \rho_{\Lambda,\varepsilon,q,A}$ such that, for $\nu \in \{1,2\}$, the projection $pr$ induces a biholomorphic map between

$$\left( \hat{F}(A,V) \cap T_\nu(0) \right) \setminus \left( K_\rho \cup \bigcup_{b \in \Gamma^\# \setminus \{0\}} T_b \right)$$

and its image in $\mathbb{C}$. This image component contains

$$\left\{ z \in \mathbb{C} \mid 8|z| > \rho \quad \text{and} \quad |z + (-1)^\nu \theta_\nu(b)| > \varepsilon \quad \text{for all} \quad b \in \Gamma^\# \setminus \{0\} \right\}$$

and is contained in

$$\left\{ z \in \mathbb{C} \mid |z + (-1)^\nu \theta_\nu(b)| > \frac{\varepsilon}{2} \quad \text{for all} \quad b \in \Gamma^\# \setminus \{0\} \right\},$$

where $\theta_\nu(b) = \frac{1}{2}((-1)^\nu b_2 + ib_1)$. Furthermore,

$$pr^{-1} : \text{Image}(pr) \longrightarrow T_\nu(0),$$

$$y \mapsto (-i(-1)^\nu y - r(y),y),$$

where

$$|r(y)| \leq \frac{C}{\rho}$$

and $C = C_{\Lambda,\varepsilon,q,A}$ is a constant.

This theorem provides a simpler picture than Theorem 3.4.1 because here $|r(y)|$ decays with respect to $\rho$. 
Bibliography


