The AdS/CFT Correspondence and String Theory on the pp-wave

by

Bojan Ramadanovic

B.Sc., Simon Fraser University, 2001
M.Sc., University of British Columbia, 2003

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Physics and Astronomy)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

October 2008

Abstract

Aspects of the AdS/CFT correspondence are studied in the pp-wave/BMN limit. We use the light cone string field theory to investigate energy shifts of the one and two impurity states. In the case of two impurity states we find that logarithmic divergences, in the sums of intermediate states, actually cancel out between the Hamiltonian and a Q-dependent " contact term". We show how non-perturbative terms, that have previously plagued this theory, vanish as a consequence of this cancelation. We argue from this that every order of internal impurities contributes to the overall energy shift and attempt to give a systematic way of calculating such sums for the case of the simplest 3-string vertex (one proposed by diVecchia).

We extend our analysis of the mass shift to the case of the most advanced 3-string vertex (proposed by Dobashi and Yoneya). We find agreement between our string field theory calculations and the leading order CFT result in the BMN limit. We also find strong similarities between our result and higher orders in the field theory, including, on the string side, the disappearance of the half-integer powers which generically do not exist in the field theory calculations.

We also study the orbifolding of the pp-wave background which results in the discrete quantization of the light-cone momentum. We present the string field theory calculation for such a discreet momentum case. We also observe how a particular choice of the orbifold, results in the string theory corresponding to the quantization of the finite size giant magnon on the CFT side. We study this theory in detail with particular emphasis on its super-algebra.

Contents

Acknowledgements

I would like to acknowledge here immense help of my supervisor, Gordon Semenoff in the research work leading to this thesis. I would like to thank Gordon for ideas, instruction, motivation, patience, funding and good cheer over almost seven years of collaboration. Likewise, I wish to acknowledge huge contributions of our other collaborators, Gianluca Grignani, Shirin Hadizadeh, Martha Orselli and, in particular, Donovan Young. Over the five years of our association Donovan's output was always inspiring in quality and humbling in its prodigious quantity. The core of this thesis owes a lot to his superior work [81]. Great thanks also go to the people of UBC physics department, both within the string group and beyond. In particular I would like to thank Don Witt and Kirstin Schleich for being extremely helpful and friendly people as well as great teachers; Mark Van Raamsdonk, Moshe Rosali and Joana Karczmarek for being the backbone of our group; Henry Ling, Brian Shieh, Thomas Klose, Phil DeBoer and Mark Laidlaw for many useful informal discussions and Janis McKenna for lots of valuable advice and, even more importantly, for her contagious optimism.

Not any less, I would like to thank my undergraduate teachers and colleagues at Simon Fraser University's mathematics and physics departments, in particular Igor Herbut, my undergraduate thesis supervisor, mentor and friend.

My thanks go in equal measure to all those important non-physicists in my life who have inspired and supported me, from parents and grandparents who ignited and preserved the spark of my curiosity about the world to the magnificent friends without whom I would have neither the mental acuity nor psychological energy for this work. Special thanks go to the people of St. Catherine's church of North Vancouver who have quite literally changed my life when they decided to help a family of refugees, with whom they had nothing in common except shared humanity, to immigrate to this wonderful country of Canada.

Finally I thank Hilary for the love of twelve years and many years to come.

Foreword

Since its inception in the 1960ies string theory has had a bewildering array of applications. Original version, intended to explain the "Regge slope " relationship between spins and masses of strongly interacting bosons, and thus provide theory of the strong nuclear force [1], fell by the wayside with the advent of quantum chromodynamics (QCD). In the 1980's a more elaborate string theory was developed, the main feature of which was the fact that the perturbations of the background metric of the theory were quantized objects within the theory itself. Quantization of gravity was an important goal in theoretical physics ever since the Einstein-Hilbert action was proposed for general relativity. String theory, in which gravity quanta are natural objects, seemed to offer the best ever hope for the unification of all known physical laws. To this day, the great popularity of string theory and its derivatives stems, to a large extent, from the idea that it is within its framework that such a "grand unified theory of physics" will be found.

The "modern" string theory of the 1980's had its own share of problems. Self-consistency of the theory constrains its background to the seemingly unphysical 10 dimensions. (26 in the case of a purely bosonic theory). Perhaps even more importantly, for a candidate to be the One Theory, string theory appeared to come in at least 5 very distinct flavors. The same underlying principles coupled with some significant differences, such as the number of supersymmetries and their relative chirality, resulted in a number of different physical theories, all of them with quantized gravity. The effort to resolve this " embarrassment of riches" was at the heart of the second superstring revolution that took place in the mid 1990's. At this time the mathematical transformations between the different string theories were discovered, indicating that the different theories can be considered as simply perturbative limits of a single underlying theory.

These transformations - or dualities - came to be of extreme importance to the development of string theory. In particular, a relationship was found, so called "strong-weak duality", that links the strong interaction behavior of one theory to the weak interaction behavior of another. Many physical theories rely on perturbative methods for making sensible calculations, a method which is limited to the weakly coupled sector of the theory in question. Therefore, a tool that could link "inaccessible" strongly coupled regimes to something that can be tackled by a perturbative method in a different theory, was potentially a ground breaking discovery in its own right.

While dualities did "unify" string theory into one systemic framework (sometimes referred to as M-theory) it remains plagued by the lack of unique connection to the 4-dimensional world of empirical experience. Specifically, the number of ways in which 6 "extra" dimensions can be compactified, or otherwise dealt with, remains so large as to give an almost infinite number of parameters to any effective 4 dimensional "new physics" coming out of string theory. While this remains an important problem in string theory, the discovery of dualities

also opened a different line of research with potentially more immediate benefits.

The important question in this line of research is: whether the dualities concept can be used beyond the limits of the string theory and, in particular, if it can be applied to the one physical theory that suffers most from the lack of perturbative method in the relevant regime - namely QCD itself.

One of the most important recent advancements in string theory came about in late 1990's when a duality was proposed that links a particular conformal field theory on 4 dimensions to a string theory on a specific curved background [2]. This conjecture, called the "Maldacena Conjecture" by the name of its original author or, more usually, AdS/CFT duality, represents a crucial step in our understanding of, and the ability to perform calculations in, both field theories and string theories, even if it does not yet lead to finding a perturbative dual of QCD itself.

While the Maldacena Conjecture has not been explicitly proven yet, it has acquired a considerable body of supporting evidence. Over the past ten years a large amount of work was done testing, expanding and qualifying the conjecture under various regimes. Our own research efforts were a small part of this expansive program.

This thesis is a result of over four years of work on the issues concerning the duality between conformal field theory and string theory on a ten dimensional space consisting of a five dimensional sphere and five dimensional anti-deSitter space ($AdS_5 \times S_5$). Even more specifically, most of our work was focused on the string side of this duality in the context of a particular particular Penrose limit of the $AdS_5 \times S_5$.

In the introduction chapter of this thesis we will provide the broad context for our work. We will give brief introduction to string theory in the context of Green-Schwarz quantization in the light-cone gauge, relying heavily on [6]. We will also attempt to explain the principle of dualities, in particular, that of AdS/CFT. Finally we will go over the limiting process by which we go from string theory on AdS - a very complex theory - to the solvable string theory on the plane-wave background.

In the second chapter we will discuss the string field theory and the "loop corrections" to the energy of the string states. We will discuss the efforts to obtain the energy shift using those "loop corrections", such that it corresponds to the known dual result in the CFT (that dual result being non-planar diagram contributions to scaling dimension of the operators in the conformal field theory). In the second half of this chapter we will present the results of our own work on this problem. Most important of those results is a generic cancelation of the divergencies that were previously present in the theory beyond a certain channel. Other results include the formula for the generic channel contributions to the energy shift (given particular vertex) and the conclusion that all channels appear to contribute in the same degree to the final result. Finally we also present the result of the single channel calculation which up to this point is the one in closest agreement with the desired CFT result.

The third chapter will focus on the consequences of the orbifolding of the background of both AdS and CFT theories. We will discuss the supersymmetry breaking that happens as a consequence of the orbifolding and its recovery in the plane-wave limit of the orbifolded theory. Also we will look at the magnon super-multiplet of the conformal field theory, its connection to the orbifolding, and its behavior in and near the pp-wave limit.

The scientific work that is at the basis of this thesis was a colaborative one. The work on the string field theory as discussed in Chapter 2. and published in [79] and [80] was

done jointly with Gordon Semenoff and Donovan Young. We met regularly and shared most aspects of the work with Gordon providing questions and ideas and Donovan and I doing most of the calculations. Our group worked in colaboration with Gianluca Grignani and Marta Orselli who were frequent visitors at UBC but otherwise worked independently on the same set of problems as us. There were frequent discussions between our two teams both in person and in correspondence that were crucial to the eventual results. Results concerning the "master formula" for the DiVecchia vertex were obtained after the joint publications and were largely done by this author alone with motivation and advice offered by Gordon Semenoff. Work published in [85] and refered to in Chapter 3. of this thesis was a colaboration between Gordon and I with Gordon providing context and most of the field-theoretic results and me doing the bulk of the work on the string theory side.

Chapter 1

Introduction

1.1 Perturbative string theory

1.1.1 Classical bosonic string

Particle path integral

The starting point for the discussion of string theory may well be the generalization of the action that describes the motion of a single massive particle in a background gravitational field. Such an action would be given by an invariant length of the world line:

$$S = -m \int ds; \quad ds^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu \tag{1.1}$$

where $g_{\mu\nu}(x)$ is the metric of the background field. If the particle trajectory is parametrized by some coordinate τ then we can write the above as:

$$S = -m \int d\tau \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^n} \tag{1.2}$$

with dots signifying the derivative with respect to τ.

We can write this more generically by using an auxiliary coordinate $e(\tau)$

$$S = \frac{1}{2} \int d\tau (e(\tau)^{-1} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^n - e(\tau)m^2) \tag{1.3}$$

which reduces to 1.2 if one solves the equation of motion for $e(\tau)$ and substitutes it back into 1.3.

Important to note is that in the original parameterization coordinate τ was chosen arbitrarily and that therefore 1.3 must be symmetric under the re-parametrization of $\tau \to \tau'$. This re-parametrization symmetry enables us to take a convenient choice of $e(\tau)$ such as $e(\tau) = 1/m$ making the solving of the 1.3 a simple matter. We would still have to satisfy the $e(\tau)$ equation of motion

$$g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^n + e(\tau)^2 m^2 = 0; \quad g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^n = -1 \tag{1.4}$$

keeping it as an on-shell condition for the particle.

In this basic framework one can then proceed to quantize the particle and construct the path integrals for its propagator, vertexes etc...

String action

It is possible in principle to generalize this framework to objects of higher dimensionality than a particle ($d = 1$ strings, $d = 2$ membranes etc...) however, as it turns out, the case of $d = 1$ has multiple advantages over the alternatives, including the symmetry under Weyl scaling that will prove important in the further discussion.

We will therefore focus, for now, on the theory of one dimensional objects - strings.

As the strings propagate they trace surfaces in space-time. These world-sheets play the same role as the world line does in the particle case. Specifically we can construct the action out of invariant surface area of the world sheet. This action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \det(\sqrt{g_{\mu\nu}(X)\partial_\alpha X^\mu(\sigma,\tau)\partial_\beta X^\nu(\sigma,\tau)}) \tag{1.5}$$

called the Nambu-Goto action is the equivalent of the 1.2. Once again we can introduce the auxiliary coordinate - metric of the world sheet: $h^{\alpha\beta}$ The resulting action, equivalent of the 1.3 and named after Polyakov, is arguably the foundational equation of the string theory.

$$S = -\frac{T}{2}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta} g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.6}$$

Where $h = \det(h^{\alpha\beta})$ and we introduce the parameter $T = (2\pi\alpha')^{-1}$, with the dimension of the inverse square of length, which can be though of as the tension of the string.

As in the case of the particle, the parameters σ and τ used to describe the world sheet are arbitrary. The action will therefore be symmetric under the re-parametrization of those variables. In addition, due to the fact that we are dealing with a $1+1$ dimensional world sheet, the action will also be symmetric under the Weyl scaling $\delta h^{\alpha\beta} = \Lambda h^{\alpha\beta}$ where Λ is an arbitrary infinitesimal function of the coordinates.

Again, like in the case of the particle, varying the action with the auxiliary parameter gives us the on-shell condition for the fields.

$$T_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}h_{\alpha\beta}h^{\alpha'\beta'}g_{\mu\nu}(X)\partial_{\alpha'}X^\mu\partial_{\beta'}X^\nu = 0 \tag{1.7}$$

Where the $T_{\alpha\beta}$ is an energy-momentum tensor of the world sheet. It can easily be shown that the condition 1.7 directly relates the expressions 1.5 and 1.6.

The three symmetries listed above (two re-parametrization ones and Weyl) give us enough freedom to manipulate the independent components of the world sheet metric and set it - for simplicity - to a two dimensional Minkowski metric.

It is important to note that even with the worldsheet metric gauged away to Minkowski, its equation of motion 1.7 remains a constraint on the theory; so called Virasoro constraint.

We can now proceed to write down the equations of motion for the Xs. They will clearly depend on the background geometry $g_{\mu\nu}$. For now we can simplify the problem by setting the background to be the flat Minkowski space.

With the Minkowski background, and the world sheet metric gauged away, the equations of motions for Xs are very simple:

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu(\sigma,\tau) = 0 \tag{1.8}$$

To fully ensure the invariance of the action under the variation of X we also need the boundary conditions. We can choose between

$$X^\mu(\sigma,\tau) = X^m(\sigma+\pi,\tau) \tag{1.9}$$

and

$$\partial_\sigma X(0,\tau) = \partial_\sigma X(\pi,\tau) = 0 \tag{1.10}$$

where in both cases σ ranges between 0 and π. The above two conditions correspond to closed and open strings respectively.

Mode expansion and the light cone gauge

Focusing on the closed strings we can easily solve 1.8

$$X^\mu(\sigma,\tau) = X^\mu_L + X^\mu_R \tag{1.11}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}$$
$$X^\mu_L = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}$$

Here we introduced the Fourier components α^μ_n which can be though of as oscillator coordinates. It is important to note that the reality of X functions implies:

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^\dagger, \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^\dagger \tag{1.12}$$

and also the fact that x^μ and p^μ are themselves real. We can get Poisson brackets of the X^μ and $\dot{X}^\mu$ from 1.6

$$\begin{aligned}[X^\mu(\sigma), X^\nu(\sigma')]_{P.B.} = [\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')]_{P.B.} = 0 \\ \left[\dot{X}^\mu(\sigma), X^\nu(\sigma')\right]_{P.B.} = T^{-1}\delta(\sigma-\sigma')\eta^{\mu\nu}\end{aligned} \tag{1.13}$$

which give the Poisson brackets of the oscillators:

$$\begin{aligned}[\alpha^\mu_m, \alpha^\nu_n]_{P.B.} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]_{P.B.} = im\delta_{m+n}\eta^{\mu\nu} \\ [\alpha^\mu_m, \tilde{\alpha}^\nu_n]_{P.B.} = 0\end{aligned} \tag{1.14}$$

Furthermore, Virasoro constraint 1.7 can be written as:

$$\dot{X}X' = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \tag{1.15}$$

A very similar procedure can give us the mode expansion of the open string.

At this point it is useful to depart slightly from [6] and introduce the light cone gauge - framework for most of the subsequent work.

What allows us to impose any new gauge on the theory is the fact that setting $h^{\mu\nu} = \eta^{\mu\nu}$ does not fully use up both re-parametrization symmetries and Weyl. Specifically, once we have a Minkowski world sheet metric, we can preserve it while still changing the coordinates though a particular combination of the re-parametrization and Weyl scaling. If the re-parametrization $\xi^0 = \frac{\partial\tau}{\partial\tau'}, \xi^1 = \frac{\partial\sigma}{\partial\sigma'}$ transforms $h^{\mu\nu}$ as

$$\delta h^{\mu\nu} = \xi^\phi \partial_\phi h^{\mu\nu} - \partial_\phi \xi^\mu h^{\phi\nu} - \partial_\phi \xi^\nu h^{\phi\mu} \tag{1.16}$$

then

$$\partial^\mu \xi^\nu + \partial^\nu \xi^\mu = \Lambda \eta^{\mu\nu} \tag{1.17}$$

combination leaves $h^{\mu\nu} = \eta^{\mu\nu}$ invariant. We can see the power of this symmetry best if we work in the coordinates $\sigma^\pm = \tau \pm \sigma$ Then

$$\partial_+ \xi^- = 0, \quad \partial_- \xi^+ = 0 \tag{1.18}$$

meaning that the all transformations of the form

$$\sigma^+ \to \sigma^{+'}(\sigma^+), \quad \sigma^- \to \sigma^{-'}(\sigma^-) \tag{1.19}$$

keep the metric intact.

The main advantage of the light cone gauge is that it makes the implementation of Virasoro constraint trivial and eliminates unphysical degrees of freedom. Its disadvantage is that it makes Lorentz invariance non-obvious.

To impose the light cone gauge we begin by singling out two directions and label them as $+$ and $-$

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}) \tag{1.20}$$

while keeping remaining the $d-2$ ones intact. We can then use 1.19 to simplify X^+ considerably.

$$\begin{aligned} \sigma^+ &= \frac{1}{\alpha' p^+}(X_L^+ - \frac{1}{2}x^+), \\ \sigma^- &= \frac{1}{\alpha' p^+}(X_R^+ - \frac{1}{2}x^+) \end{aligned} \tag{1.21}$$

This relates X^+ directly to the τ coordinate of the world sheet by:

$$X^+(\sigma, \tau) = x^+ + p^+ \tau \tag{1.22}$$

We use this simple form of the X^+ to re-write the Virasoro condition $(\dot{X} \pm X')^2 = 0$

$$(\dot{X}^- \pm X^{-'}) = \frac{1}{4\alpha' p^+}(\dot{X}^i \pm X^{i'})^2 \tag{1.23}$$

The upshot of this result is that the expression X^- becomes completely fixed in terms of X^i making it explicit that the system carries only $d-2$ physical degrees of freedom. $X^\pm$ are fully fixed by the gauge freedom and constraints.

Equation 1.23 does not cover all of the Virasoro constraints from 1.15. To do that we use the Fourier expansion of the energy-momentum tensor from equation 1.7.

$$L_m = \frac{T}{2}\int_0^\pi d\sigma e^{-2im\sigma}T_{--} = \alpha'\frac{p^2}{8} + \frac{1}{2}\sum_{n\neq 0}\alpha_{m-n}\alpha_n$$
$$\tilde{L}_m = \frac{T}{2}\int_0^\pi d\sigma e^{-2im\sigma}T_{++} = \alpha'\frac{p^2}{8} + \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{m-n}\tilde{\alpha}_n \tag{1.24}$$

with the requirement $T_{--} = T_{++} = 0$ translating into $L_m = \tilde{L}_m = 0$ for all m. We can then use combinations $L_0 + \tilde{L}_0 = 0$ and $L_0 - \tilde{L}_0 = 0$ to obtain:

$$m^2 = -p^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n) \tag{1.25}$$

and

$$\sum_{n\neq 0}\alpha_{-n}\alpha_n = \sum_{n\neq 0}\tilde{\alpha}_{-n}\tilde{\alpha}_n \tag{1.26}$$

These two are important results. 1.25 tells us how excitation of the oscillators "creates" their space-time mass. 1.26 is usually referred to as the "level matching condition" and represents the remaining constraint on the kind of states strings can form. Modifications of the level-matching condition will play an important role when we begin considering strings on the orbifolded backgrounds.

We are now ready to begin quantizing the light-cone string. The usual procedure is used, of promoting fourier modes into creation and annihilation operators and poison brackets into commutators. However, there are some potential pitfalls. Specifically one can directly test Lorentz invariance of the quantized theory in the light cone gauge. This leads to the additional set of requirements on the theory. In particular, the disappearance of the commutator $[J^{i-}, J^{i-}]$ in Lorentz algebra is required if one is to avoid the anomalies. It can be shown that this disappearance can only be affected, within the above framework, if the number of the background dimensions is set to 26. This is an interesting calculation and a harbinger of the important geometric requirements that string theory tends to impose upon itself. I will not pursue this calculation here.

The theory we have followed so far is purely bosonic and thus not very interesting for most physical applications. I will therefore follow with introduction of the new world-sheet symmetry relating bosonic degrees of freedom to their fermionic counterparts. The idea of such a "supersymmetry" was one of the most fruitful ones from the early years of "modern" string theory and have lead to a number of applications beyond string theory proper.

1.1.2 Supersymmetry

Superparticle

Returning to the action of the bosonic particle:

$$S = \frac{1}{2}\int (e^{-1}\dot{x}^2 - em^2)d\tau \tag{1.27}$$

and working in the $m \to 0$ limit we can extend the action to include more symmetries. Specifically we are interested in a symmetry relating the bosonic coordinates x^μ to a set of spinor coordinates θ^a. We can introduce $\mathcal{N}$ such supersymmetries with $\mathcal{N}$ corresponding sets of coordinates θ^{Aa} where $A = 1...\mathcal{N}$. The full description of the supersymmetry uses the Grassmann spinors ϵ^A:

$$\delta x^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A, \quad \delta\theta^A = \epsilon^A, \quad \delta e = 0 \tag{1.28}$$

where Γ^μ is the representation of the d-dimensional Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \tag{1.29}$$

A number of generalizations of 1.27 exist that are invariant under 1.28, one of the most straightforward being:

$$S = \frac{1}{2}\int \epsilon^{-1}(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A)^2 d\tau \tag{1.30}$$

This action maintains the Poincare invariance along with its super-symmetry. We can write down the equations of motion of the action 1.30

$$p^2 = 0, \quad \dot{p}^\mu = 0, \quad \Gamma\cdot p\dot{\theta} = 0 \tag{1.31}$$

where

$$p^m = \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A \tag{1.32}$$

From these it can be easily shown that the matrix $\Gamma\cdot p$ has only half of its maximum possible rank and that therefore half the components of θ^{Aa} decouple from the theory (because θ only ever appears in the action multiplying $\Gamma\cdot p$). This is related to the additional symmetry of 1.30 which gets its name - kappa symmetry - from another Grassmann spinor: $\kappa^{Aa}(\tau)$

$$\delta\theta^A = i\Gamma\cdot\partial\kappa^A, \quad \delta x^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A, \quad \delta e = 4e\dot{\bar{\theta}}^A\kappa^A \tag{1.33}$$

The reader should refer to [6] for proof that 1.33 is in fact a symmetry of 1.30. A peculiarity of the κ-symmetry is that it requires the equations of motion to close its algebra. Likewise, equations of motion make all the potential conserved charges of κ vanish. Thus the main effect of 1.33 is to keep half the components of θ^{Aa} decoupled from the theory.

Classical superstring

The most obvious generalization of the above principle to the 1.6 action in a flat background is:

$$S_1 = -\frac{1}{2\pi}\int \delta^2\sigma\sqrt{h}h^{\alpha\beta}\Pi_\alpha \cdot \Pi_\beta \tag{1.34}$$

where

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A \tag{1.35}$$

While this action does posses $\mathcal{N}$ supersymmetries as well as the usual re-parametrization invariance, it does not have κ-symmetry we have noticed in the discussion of superparticle. As a consequence, θ of this theory would have double the number of degrees of freedom leading to the very complex non-linear equations of motion. It is, however, possible to add a term S_2 to the action such that resulting action $S = S_1 + S_2$ has κ-symmetry and thus only half the degrees of freedom of θ leading to solvable equations of motion.

Introducing the S_2 term imposes a number of consistency conditions on the theory. First of all, it limits the number of supersymmetries to two or less, a construction of S_2 being impossible for the cases $\mathcal{N} > 2$. With that condition in mind we can write the action:

$$S_2 = \frac{1}{\pi}\int d^2\sigma\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_b\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_b\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\} \tag{1.36}$$

This term of the action must itself obey the $\mathcal{N} = 2$ supersymmetry. This requirement poses further conditions on the periodicity/handedness of the fermionic fields as well as the dimension of the background space-time. The possibilities left for the dimensionality of space-time left by the supersymmetry requirement are $D = 3, 4, 6$ or 10 with each dimension linked to a particular requirement for spinors being Majorana and/or Weyl. Further conditions imposed by the quantization will limit this to 10 dimensions with Weyl-Majorana fermions. [6] and other standard textbooks give comprehensive proof of the invariance of $S = S_1 + S_2$ under κ-symmetry as well as the detailed discussion of the supersymmetry dependance on the background dimension and properties of fermions.

Type I and II theories

The restrictions imposed by the supersymmetry: $\mathcal{N} \leq 2$, $D = 10$, θ - Mayorana-Weyl, still leave a number of distinct choices to be made when constructing the super-symmetric theory. The most obvious choice is between cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ corresponding to Type I and Type II string theories, respectively.

Furthermore, for the case of $\mathcal{N} = 2$, The Weyl condition requires that θ^1 and θ^2 each have definite handedness. That leaves two physically distinct possibilities: either they have the same or opposite handedness.

In the case of opposite handedness,the theory will necessarily exclude both open strings (because the boundary condition for open strings demands $\theta^1 = \theta^2$ on the string ends) and un-oriented closed strings (because the left and right propagating modes will be of opposite handedness). This is the Type IIA string theory.

The cases of same handedness can be divided into the cases where left and right moving modes are symmetrized to define a theory of unoriented strings - which also ends up allowing the open strings; and not imposing such restrictions thus ending with the theory of oriented closed strings of definite chirality. As it turns out, the first case loses one of the supersymmetries reducing to the Type I string theory whereas the later case become Type IIB string theory.

There are two further kinds of supersymmetric string theory which combine superstring modes with those of a bosonic string (taking left movers from one and right movers from the other), those Heterotic theories are not discussed here.

Superstring quantization and the superalgebra

The particular supersymmetry formalism discussed in this section is particularly well suited for use with light-cone gauge quantization, with κ-symmetry working along the re-parametrization symmetry. First we define:

$$\Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9) \tag{1.37}$$

and then use κ-symmetry to enforce the condition:

$$\Gamma^{+}\theta^{A} = 0 \tag{1.38}$$

for both. Seeing as exactly half of the eigenvalues of Γ^+ are non-zero, 1.38 amounts to setting exactly half of the θ components to zero. These are the same degrees of freedom that were already seen as decoupled from the theory due to κ-symmetry. Counting the fermionic degrees of freedom we start from the generic 32 complex d.o.f. The Weyl-Majorana condition reduces this to 16 real d.o.f and finally 1.38 reduces it further to 8 real d.o.f. for each θ^A

We have seen before that the gauge choice together with the Virasoro condition does something very similar for the bosonic degrees of freedom, fixing $X^{\pm}$ and leaving only 8 fields X^i as physical degrees of freedom.

The only remaining manifest spatial symmetry is therefore the rotational invariance $SO(8)$. The Dynkin diagram of the $SO(8) = D_4$ Lie algebra possesses a "triality" symmetry relating the in-equivalent irreducible representations of the same dimensionality. In particular, there are 3 irreducible representations of $SO(8)$ that are 8-dimensional: the vector one $\mathbf{8_v}$ and two spinorial ones $\mathbf{8_s}$ and $\mathbf{8_c}$. The vector representation is manifestly real and the reality of the spinor ones follows from triality.

It is then very easy to see the bosonic fields X^i as the vector representation $\mathbf{8_v}$ of the $SO(8)$ and the fermionic ones θ^{Aa} or $\theta^{A\dot{a}}$ as spinor representations $\mathbf{8_s}$ and $\mathbf{8_c}$. respectively, with the choice of the representation being governed by the chirality.

A further effect of 1.38 is to simplify action considerably by imposing

$$\bar{\theta}\Gamma^{\mu}\partial_a\theta = 0 \tag{1.39}$$

for all cases except when $\mu = -$. This can be seen by inserting the unity $1 = (\Gamma^+\Gamma^- + \Gamma^-\Gamma^+)/2$ between the two θs

As a consequence of this, the equations of motion take a very simple form:

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0, \quad (\partial_\sigma + \partial_\tau)\theta^{a1} = 0, \quad (\partial_\sigma - \partial_\tau)\theta^{a2} = 0 \tag{1.40}$$

The bosonic degrees of freedom are solved for and quantized in exactly the same way as in the bosonic theory above resulting in the oscillator modes given by 1.11 and 1.14. Solutions for the fermionic partners for closed strings are given by:

$$\begin{aligned} \theta^{1a}(\sigma\tau) &= \sum_{n=0}^{\infty} \beta_n^a e^{-2in(\tau-\sigma)} \\ \theta^{2a}(\sigma\tau) &= \sum_{n=0}^{\infty} \tilde{\beta}_n^a e^{-2in(\tau+\sigma)} \end{aligned} \tag{1.41}$$

where the usual Poisson brackets imply:

$$\begin{aligned} [\alpha_m^i, \alpha_n^j] = m\delta_{m+n}\delta^{ij} \quad [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta_{m+n}\delta^{ij} \quad [\alpha_m^i, \tilde{\alpha}_n^j] = 0 \\ \{\beta_m^a, \beta_n^b\} = m\delta_{m+n}\delta^{ab} \quad \{\tilde{\beta}_m^a, \tilde{\beta}_n^b\} = m\delta_{m+n}\delta^{ab} \quad \{\beta_m^a, \tilde{\beta}_n^b\} = 0 \end{aligned} \tag{1.42}$$

after the standard promotion of Poisson brackets into (anti)-commutators and oscillators into the creation/anihilation operators.

It is now possible to discuss the actual supersymmetries of the action and their relationship with other symmetries.

In order to preserve the gauge condition 1.38 ϵ and κ symmetries have to be combined in such a way as to ensure that the $\delta\theta$ is always annihilated by the Γ^+. The resulting transformation has the form:

$$\delta S^a = -i\rho \cdot \partial X^i \gamma_{a\dot{a}}^i \epsilon^{\dot{a}} \quad \delta X^i = 2\gamma_{a\dot{a}}^i \bar{\epsilon}^{\dot{a}} S^a \tag{1.43}$$

where $\gamma_{a\dot{a}}^i$ are Clebsch-Gordan coefficients for the coupling of the three representations. It can be seen that the anticommutation of two such transformation gives space-time translation. There is also 8 "trivial" transformations:

$$\delta S^a = \eta^a \quad \delta X^i = 0 \tag{1.44}$$

The 1.43 and 1.44 are generated by

$$Q^{\dot{a}} = \gamma_{\dot{a}a}^i \sum_{-\infty}^{\infty} \theta_{-n}^a \alpha_n^i \tag{1.45}$$

and

$$Q^a = \theta_0^a \tag{1.46}$$

respectively. With Qs components of a covariant Majorana-Weyl spinor satisfying the algebra:

$$\{Q^a, Q^b\} = \delta^{ab} \quad \{Q^a, Q^{\dot{a}}\} = \gamma_{a\dot{a}}^i p^i \quad \{Q^{\dot{a}}, Q^{\dot{b}}\} = \delta^{\dot{a}\dot{b}} H \tag{1.47}$$

it is possible to combine these generators with the generators of the rotations and boosts to create the full Super-Poincare algebra introducing commutators such as:

$$[J^{i-}, Q^a] = i\gamma^i_{a\dot{a}} Q^{\dot{a}} \tag{1.48}$$

Super-algrebra quantizes trivially with the usual exception of the $[J^{i-}, J^{i-}]$ commutator where $J^{\mu\nu}$ is given by:

$$J^{\mu\nu} = T \int_0^\pi d\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu + \bar{\theta}^A \Gamma^{\mu\nu} \theta^A) \tag{1.49}$$

As before, the demand for disappearance of this commutator in the quantum theory will select the dimensionality of the space-time background. In the supersymmetric case the required dimension is $D = 10$, compatible with the supersymmetry requirement in the Majorana-Weyl case.

Wider applications of supersymmetry

Supersymmetry was originally postulated in the context of string theory as discussed above. However,the idea of a symmetry relating particles of same quantum numbers but different spin-statistics had since found use in regular field theory, and is used as one of the possible modifications of the Standard Model [8] [9].

The main advantage of the super-symmetric theories over the ordinary kind is the fact that the ground state (vacuum) energy in a supersymmetric theory is generically zero. Assuming softly broken supersymmetry for our own universe would go long way towards explaining why is the observed vacuum energy (cosmological constant) is many orders of magnitude smaller than the formal sum of the zero-point energies of all the modes of all the fields of the standard model up to the Planck scale cutoff.

Another use of the supersymmetry is to naturally regulate mass of Higgs boson by providing cancelations to any mass corrections due to heavy particle loops in the field theory. This enables the Higgs boson to remain at the electro-weak scale in a theory containing particles whose own mass are significantly above that scale without resorting to fine-tuning of the bare Higgs mass.

Even more intriguingly, the introduction of supersymmetry corrects the extrapolated coupling constants for electromagnetic, weak and strong nuclear fields, so that they meet exactly at the sufficiently high energy scale - a result of significant mathematical elegance.

Furthermore, supersymmetry can be used in potential solutions to the problem of dark matter and the problem of matter/anti-matter symmetry in the universe.

Lack of supersymmetric pairings within the currently known particles of the Standard Model indicates unequivocally that any potential supersymmetry of the physical universe must be broken. However, considerations of vacuum energy and Higgs mass indicate that this breaking is likely to be soft and that the super-partners of the known particle may well be within the observational scope of the next generation of particle accelerators (including LHC). If that proves correct, within the next decade or so we will see the first experimental confirmation for a piece of "new physics" coming out of string theory.

1.1.3 Closed string spectrum and the background fields

The ground state of the bosonic string is a tachyon due to the presence of the normal ordering constant in the Hamiltonian. Zero mass states are then 8 excited states of the form $\alpha_1^{\dagger i}|0>$. In the supersymmetric case normal ordering constants for bosons and fermions cancel out, meaning that the ground state has zero mass. This was considered one of the early successes of superstring theory and the indication that the supersymmetry is somehow "natural" in string context.

The grounds states of the superstring must represent the algebra $\{S_0^a, S_0^b\} = \delta^{ab}$. Such representation is given by:

$$S_0^a = \begin{pmatrix} 0 & \gamma^a_{i\dot{a}} \\ \gamma^a_{\dot{a}i} & 0 \end{pmatrix} \tag{1.50}$$

The representation space is the $\mathbf{8_v} + \mathbf{8_c}$ (or alternatively $\mathbf{8_v} + \mathbf{8_s}$) 16-dimensional multiplet consisting of 8 fermions and 8 bosons.

In the case of closed strings, ground state will be a cross product of two such multiplets, one for right movers and one for left movers. The choice between Type IIA and IIB theories (different or same chiralities on left and right movers) is given by choice of different or same spinorial representation. We can thus write the full massless spectrum as either:

$$(\mathbf{8_v} + \mathbf{8_c}) \times (\mathbf{8_v} + \mathbf{8_s}) = (\mathbf{1} + \mathbf{28} + \mathbf{35_v} + \mathbf{8} + \mathbf{56_v})_B + (\mathbf{8} + \mathbf{56_s} + \mathbf{8} + \mathbf{56_c})_F \tag{1.51}$$

in the IIA case or

$$(\mathbf{8_v} + \mathbf{8_s}) \times (\mathbf{8_v} + \mathbf{8_s}) = (\mathbf{1} + \mathbf{28} + \mathbf{35_v} + \mathbf{1} + \mathbf{28} + \mathbf{35_c})_B + (\mathbf{8} + \mathbf{56_c} + \mathbf{8} + \mathbf{56_c})_F \tag{1.52}$$

in the IIB case. Where in either case B labels bosonic states and F fermionic ones. The rest of the string spectrum can be built by acting with non-zero mode creation operators on these ground states while obeying level matching conditions such as 1.26. Zero mode operators turn one ground state into another.

It turns out that the $\mathbf{35_v}$ in the Type II theory corresponds by its quantum number properties to the graviton $g_{\mu\nu}$. One of the most intriguing early results of string theory was that infinitesimal variations in the background space-time metric were shown to be exactly equivalent to the coupling with this element of the string spectrum, confirming that it is indeed a quantum of gravitational field. All other possible background fields were either shown to lead to anomalies or were themselves found within the spectrum of the string. This incredible self-consistency was part of the reason why string theory appeared, very early on, to hold promise of being an underlying theory of all physics.

Supergravity - low energy limit of the string theories

It is possible to construct an action whose fields satisfy the same equations of motion as the low energy states of the string theory. It turns out that this effective action was known already as the supersymmetrization of Einstein's gravity - often referred to as supergravity. My discussion of supergravity here is based primarily on [5].

The most "natural" supergravity is a 11-dimensional theory. This is a critical dimension because theories of dimensions $d > 11$ can be shown to contain massless *spin* > 2 particles

which can not couple consistently in a field theory and are therefore prohibited. There is an immediate relationship between 11 dimensional supergravity and the type II theories in that they share super-algebra. To actually recover low energy Type II theories we can use dimensional reduction of the original 11-d supergravity while keeping only the fields that are independent of the compact directions. This is actually related to the connection between Type II theories (in fact all string theories) and the 11-dimensional M(aster)-Theory. This is a fascinating subject which goes well outside the bounds of this thesis.

The bosonic content of the 11-dimensional supergravity is a metric G_{MN} and the 3-form potential $A_{MNP} = A_3$ and its field strength $F_4 = dA_3$. The action is given by:

$$S_{11}^{bos} = \frac{1}{2\kappa_{11}^2} \int d^1 1x(-G)^{1/2} \left(R - \frac{1}{2}|F_4|^2 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4 \tag{1.53}$$

where R is the Ricci scalar derived from G.

Here I skip over the details of the action itself including its fermionic component which can be obtained by supersymmetry, and details of the dimensional reduction. A key point of the latter is general metric under the compactification of the 11th dimension.

$$ds^2 = G_{MN}^{11} dx^M dx^N = G_{\mu\nu}^{10} dx^\mu dx^\nu + e^{2\phi(x^\mu)}(dx^1 0 + A_\nu(x^\mu)dx^\nu)^2 \tag{1.54}$$

mapping the original 11-dimensional G_{MN} to the 10-dimensional metric $G_{\mu\nu}$, gauge field A_1 and a scalar ϕ. 3-form A_3 either remains intact (if all three components are along the non compact directions) or losses an index becoming B_2. This spectrum actually corresponds to the one we have quoted for the low energy Type II string, with $\mathbf{1_V}$ corresponding to ϕ and $\mathbf{28_V}$, $\mathbf{8_V}$ and $\mathbf{56_V}$ to B_2, A_1 and A_3 respectively. After a fair bit of algebra, including a rescaling of the metric by the factor proportional to the dilaton ϕ we end up with the type IIA supergravity action:

$$S_{IIA} = S_{NS} + S_R + S_{CS} \tag{1.55}$$

$$\begin{aligned} S_{NS} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\phi} \left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2 \right) \\ S_R &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x(-G)^{-1/2} \left(|F_2|^2 + |\tilde{F}_4|^2 \right) \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \end{aligned} \tag{1.56}$$

where the overall factor $e^{-2\phi}$ can be made explicit in the S_R with further redefinitions.

Similar operations can be performed to produce the spectrum and the action of the type IIB supergravity.

Key point of the above process was to show that the coupling constant of the theory $\kappa_{10}^2 e^{2\phi}$ is set by the value of the dilaton - one of the "particles" of the string spectrum, meaning that it is set naturally by the theory itself in another example of amazing self-referential consistency of string theories.

1.2 T- Duality and Dirichlet branes

Even if the lack of a unique connection with our 4-dimensonal space-time prevents string theory from fulfilling its amazing promise, string theory will still contribute greatly to our understanding of physical laws through the concept of dualities. Basic notion of dualities is the idea that a certain regime of one theory corresponds to a different regime of a distinct theory. Most interesting dualities have to do with cases where the perturbative regime of one theory corresponds to the non-perturbative regime of a different theory. The concept is best introduced, however, through the duality between two different space-time geometries as seen by strings. This phenomenon, called T-duality [12] [13] [14], was one of the earliest string dualities discovered and a precursor of the revolution to come.

1.2.1 T-duality

T-duality of the closed strings

We have so far considered superstring in the flat 10-dimensional background. Many modifications of this background are possible, including compactifications of one or more space dimensions. Simplest possible compactification has one spatial dimension forming a circle - changing the background geometry from R^{10} to $R^9 \times S^1$ Considering only zero-modes of the string:

$$X(\sigma,\tau) = \frac{1}{2}(x_L + x_R) + \alpha'(p_L + p_R)\tau + \alpha'(p_L - p_R)\sigma \tag{1.57}$$

under the $\sigma \to \sigma + 2\pi$, $X(\sigma,\tau)$ transforms as: $X(\sigma,\tau) \to X(\sigma,\tau) + 2\pi\alpha'(p_L - p_R)$ which means that for all the non-compactified directions:

$$p_L^\mu = p_R^\mu = \frac{1}{2}p^\mu \tag{1.58}$$

In the compactified direction, however, $X^\nu(\sigma,\tau)$ is given by:

$$X^\nu(\sigma,\tau) = X^\nu(\sigma + 2\pi,\tau) + 2\pi R w \tag{1.59}$$

where R is the radius of the compactified direction and w is the number of times string winds around the S^1 before it closes back on itself. In this direction:

$$p_L - p_R = \frac{Rw}{\alpha'} \tag{1.60}$$

At the same time total momentum along the compactified direction will be quantized in the units of inverse R so

$$p_L + p_R = \frac{n}{R} \tag{1.61}$$

Combined these two equations give us:

$$\begin{aligned} p_L &= \frac{n}{R} + \frac{Rw}{\alpha'} \\ p_R &= \frac{n}{R} - \frac{Rw}{\alpha'} \end{aligned} \tag{1.62}$$

Seeing as n and w are simply integers this suggests the duality between this theory and the compactified theory where $R' = \frac{\alpha'}{R}$ with the trivial exchange of n and w; the only change in this new theory is the replacement $p_R \to -p_R$ or alternatively $X(\sigma, t) \to X_L(\sigma, \tau) - X_R(\sigma, \tau)$. With this coordinate change we have just demonstrated the equivalence between two theories with potentially considerably different geometries. This duality holds true for the higher oscilator modes of the string as well, with

$$\alpha_n \to -\alpha_n, \quad \tilde{\alpha}_n \to \tilde{\alpha}_n \tag{1.63}$$

being a general rule. What this means is that Type IIA and IIB string theories which we previously thought of as distinct actually represent two different geometries of the same theory [12]. Specifically, compactifying Type IIA string on a circle of radius R and then letting $R \to 0$, shifts the chirality of right movers and gives the Type IIB string on 10 dimensions. What is more, Type IIA and IIB theories come out as simply the limiting points in the full spectrum of theories governed by the value of R. A similar thing happens between two, hitherto unmentioned, Heterotic string theories. A movement to turn the 5 distinct string theories into one has begun.

T-duality of the open strings

When we try to apply T-duality to a theory that contains open strings, such as Type I, we run into another interesting result. For the closed strings in the $R \to 0$ limit, states with $n \neq 0$ become infinitely massive but the states with $n = 0$ form the continuum over all values of w because it is very cheap in energy terms to wind around a "small" dimension. Effectively, instead of decoupling from the theory, the compactified dimension re-appears.

Open strings, on the other hand, can not wrap around compactified direction. There is therefore no new continuum of states as R goes to 0, and the compactified dimension disappears from the theory. This leads to an apparent contradiction, because all open strings theories necessarily include closed strings and what is more, internal parts of the open strings are indistinguishable from those of the closed string (the only difference being boundary conditions).

Apparent paradox disappears when we observe what exactly happens with the open string under the duality transformations 1.63.

Originally:

$$\begin{aligned}
X(\sigma, \tau) &= X_L + X_R \\
X_L &= \frac{1}{2}x_L + \alpha'\frac{n}{R}(\tau + \sigma) + \frac{1}{\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{i}{n}\alpha_n e^{-in(\tau+\sigma)} \\
X_R &= \frac{1}{2}x_R + \alpha'\frac{n}{R}(\tau - \sigma) + \frac{1}{\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{i}{n}\alpha_n e^{-in(\tau-\sigma)} \\
X(\sigma, \tau) &= x_0 + 2\alpha' p\tau + \sqrt{\frac{2}{\alpha'}}\sum_{n\neq 0}\frac{i}{n}\alpha_n e^{-in\tau} cos(n\sigma)
\end{aligned} \tag{1.64}$$

T-dual coordinate $\tilde{X}(\sigma,\tau)$ is given by:

$$\tilde{X}(\sigma,\tau) = \tilde{X}_L + \tilde{X}_R = X_L - X_R$$
$$\tilde{X}(\sigma,\tau) = x_0 + 2n\tilde{R}\sigma + \sqrt{\frac{2}{\alpha'}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau} sin(n\sigma) \tag{1.65}$$

It can be seen from 1.65 that the boundary conditions on σ changed in the compactified direction from Neumann to Dirichlet (cos to sine) and that the actual position of the end points of the string got limited to a single plane in the compactified direction (because x_0 and $x_0 + 2\pi n\tilde{R}$ are identified).

Furthermore, one can carry the same argument to any path connecting two end-points thus showing that all end points of strings end on the same plane.

Thus, one dimension is indeed taken out from the degrees of freedom of the end-points of the open string and a new object is introduced to the theory - namely the surface on which the open strings may end, Dirichlet-brane or D-brane.

D-branes of different dimension can be constructed by compactifying (and T-dualizing) multiple dimensions of the original space. A much more detailed analysis than the one provided here can show that T-dualizing Type I string theory on an odd number of dimensions results in a D-brane of appropriate dimension and the Type IIA closed string theory away from the brane. Likewise, T-dualizing an even number of dimensions gives us the brane and the Type IIB closed strings away from it. A consequence of this, explained in more detail in [14] is that the states containing the D brane within Type II theories will lose exactly half of their supersymmetries making them 1/2 BPS states [15] [16].

1.2.2 Properties of D-branes

D-branes and gauge groups

The usual way of introducing gauge symmetry into (open) string theories is through Chan-Paton factors [6] [14]. Those are the degrees of freedom associated with the ends of open strings and characterized by vanishing Hamiltonian. Due to the latter they are strictly static terms (string prepared with one set of Chan-Paton states always retains those exact states). Labeling the end states of the strings i and j where i and j run between 1 and N we can write a generic string state as:

$$|p;a>= \sum_{i,j=1}^{N} |p;ij> \lambda^a_{ij} \tag{1.66}$$

with n×n matrices λ^a_{ij} being what is actually referred to as Chan-Paton factors. It can be shown that in the simplest case of oriented strings the non-dynamical nature of Chan-Paton degrees of freedom forces them into trace-like structures (because the two connecting ends of the string must always be in the same Chan-Paton state) creating factors such as:

$$\lambda^1_{ij}\lambda^2_{jk}...\lambda^n_{mi} = Tr(\lambda^1\lambda^2...\lambda^n) \tag{1.67}$$

in each open string amplitude.

All such amplitudes are invariant under the U(N) symmetry:

$$\lambda^i \to U\lambda^i U^{-1} \tag{1.68}$$

that transforms the end points of the string. Thus we introduce an additional gauge symmetry into the theory. More general symmetry groupscan be introduced if one considers the unoriented open strings

When we T-dualize a theory we can use Chan-Paton factors to introduce multiple D-branes. When compactifying a dimension d we can generically break $U(N) \to U(1)^N$ by introducing the matrix $A^d_{ij} = diag\{\theta_1, \theta_2, ..., \theta_N\}_{ij}/2\pi R$ acting on the Chan-Paton factors. Such a matrix can be thought of as a Wilson line in the compactified direction. Locally A^d_{ij} can be written down as a gauge:

$$A^d = -i\Lambda^{-1}\partial_d\Lambda, \quad \Lambda = diag\{e^{iX^d\theta_1/2\pi R}, e^{iX^d\theta_2/2\pi R}, ..., e^{iX^d\theta_N/2\pi R}\} \tag{1.69}$$

This would have no further effect on a non-compactified direction but in the compactified direction the string ends pick up phase depending on their Chan-Paton content:

$$\text{diag}\{e^{-i\theta_1}, e^{-i\theta_2}, ..., e^{-i\theta_N}\} \tag{1.70}$$

under the winding transformation $X^d \to X^d + 2\pi R$.

Due to phases 1.70 strings can now have fractional momentum along the compactified direction. This translates into fractional winding number and thus into the ability of open strings to end on different hyper-planes. Specifically, the string in the state $|ij>$ will pick up the phase $e^{i(\theta_i - \theta_j)}$ and thus its end points will end up being at

$$0 \quad \text{and} \quad (2\pi n + \theta_i - \theta_j)\tilde{R} = (\theta_i - \theta_j)\tilde{R} \tag{1.71}$$

In other words the arbitrary end point will be given by:

$$\tilde{X}^d = \theta_i\tilde{R} = 2\pi\alpha' A^d_{ii} \tag{1.72}$$

Generically then, there will be N hyper-planes at different positions at which open strings can end.

Dynamics of D-branes

It was noted before that string theory necessarily provides its own background fields. Most importantly, string theory contains gravity. In a theory with gravity it is unnatural for perfectly rigid objects to exist; therefore we expect to see D-branes fluctuate in shape and position as they interact with other branes and strings. Looking at the simple case of D 2-branes (one compactified direction) and using 1.25

$$M^2 = p^2 = \left(\frac{[2\pi n + (\theta_i - \theta_j)]\tilde{R}}{2\pi\alpha'}\right)^2 \tag{1.73}$$

massless states will clearly be the ones that are not winding and whose ends are both on a single brane. This makes sense as the string stretched between two branes has a tension that contributes to its energy. Furthermore, string excitations along the brane of the

form $\alpha^d_{-1}|p, ii>$ where d is a compactified direction, become, in dual theory, the transverse position of the D brane as we have already seen in the constant gauge case 1.72. More complicated gauge backgrounds will then correspond to the curved surfaces and the quanta of the gauge fields to the fluctuations. Much as the massless closed strings turn out to represent the fluctuations of the background geometry we find that certain massless open strings correspond to fluctuations in the shape of the D-branes.

The low-energy effective action of the brane fluctuation is the well known Dirac-Born-Infeld (DBI) action:

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi}\sqrt{\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} + ... \tag{1.74}$$

where ξ^a are the world-volume coordinates of the brane, $G_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$ and $B_{ab} = B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$ are the pull-back of space-time fields to the brane and ϕ is the dillaton. This action clearly interacts with the massless closed strings as described in 1.55

Considering the $U(N)$ symmetry breaking, as discussed above, the N separated D-branes end up with one massless vector each - $U(1)^N$. If m D-branes coincide then $\theta_1 = \theta_2 = ... = \theta_m$ and strings are allowed to end on any of the m D-branes while remaining massless. We thus have surviving $U(m)$ gauge group. If all N D-branes are coincident we recover the $U(N)$ gauge symmetry. Stacks of D-branes play important role in the later discussion of AdS/CFT.

One of the interesting properties of the stack of D-branes is that the (remaining) supersymmetry of the BPS state ensures that the gravitational attraction between the branes is exactly canceled out by the repulsion due to their form-potential charges. This means that the brane stacks are stable solutions.

The DBI action is obtained by promoting the gauge field $A^a(\xi)$ to an $N \times N$ matrix with components of the matrix corresponding to the end points of the open string and likewise for other fields. The action will be of the form:

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} Tr\left[\sqrt{\det\left(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}\right)} + ...\right] \tag{1.75}$$

We will soon return to this action.

1.3 AdS/CFT correspondence

1.3.1 Origin of the AdS/CFT

Black hole thermodynamics and holography

The core idea of the AdS/CFT correspondence goes back to attempts to understand the physical properties of black holes. The original problem was that the presence of black hole may locally break the second law of thermodynamics, if high entropy objects were to be "thrown" into the black hole. The problem was addressed in the mid 1970's by Bekenstein and Hawking [17], [18] who related the thermodynamic characteristics of a black hole to its surface properties (which are not shielded from the outside by the event horizon and are

thus part of the larger system). This concept became known as holography and it turned out to be very common in describing gravitational phenomena. At its core is the idea that complex gravitational phenomena, within a certain region, can be completely understood in terms of the properties of the boundary of that region. Useful as it was in understanding thermodynamics in the context of general relativity, holography was little more than an accounting mechanism until the advent of string theory. This is so because prior to string theory there was never an actual microscopic description of the gravitational system such as a black hole. String theory provided such a description by observing the fluctuations of the branes which were taken to form the bulk of the mass of a black hole. Strominger and Vafa [19] showed that the thermodynamics emerging from the state-counting of those fluctuations corresponds exactly to the holographic thermodynamics of Bekenstein and Hawking. This was the first insight into holography as a duality between two well understood theories, the gravitational one "inside" the black hole and field theory on the surface.

t' Hooft limit and the string gauge duality

The Relationship between string and gauge field theories was a foundational issue of string theory. Even after the discovery of QCD the hope remained that a connection will be made between this theory and some stringy equivalent. Very early it was shown by 't Hooft [20] that if such a relationship were to exist, free string theory would correspond to an $U(N)$ theory in $N \to \infty$ limit with the string coupling constant being given by $1/N$. t' Hooft's argument is very general, applying to all sorts of gauge theories. I do not replicate it here referring to the [22] for details.

We have seen before that string theoretic objects that carry the requisite $U(N)$ symmetry are stacks of N coinciding branes. Therefore the string theory that satisfies the t'Hooft requirement would be one that includes such a stack. Since the stack of branes is by definition a massive object, it is exactly the sort of system that would be engaged in the holographic duality as described above.

1.3.2 Black holes and stacks of branes

As usual, we begin by considering the low-energy (supergravity) limit of the string theory. In this limit the D-branes are described by the DBI action 1.75 with their metric, dilaton and the field form given by [21, 22, 25]:

$$ds^2 = H^{-1/2}(r)\left[-f(r)dt^2 + \sum_{i=1}^{p}(dx^i)^2\right] + H^{1/2}(r)\left[f^{-1}(r)dr^2 + r^2 d\Omega^2_{8-p}\right] \tag{1.76}$$

and

$$e^{\phi} = H^{\frac{3-p}{4}}(r) = g_s, \quad F_{ti_1\ldots i_p r} = \epsilon_{i_1\ldots i_p}\frac{1}{H^2(r)}\frac{Q}{r^{8-p}} \tag{1.77}$$

where

$$H(r) = 1 + (\frac{R}{r})^{7-p}, \quad f(r) = 1 - (\tfrac{r_0}{r})^{7-p} \tag{1.78}$$

and p is the dimension of the brane within the 10 dimensional background.

This metric corresponds to an extended black hole with horizon at $r = r_0$. Equations 1.76 - 1.78 correspond to the so called p-branes, objects in supergravity theory. To link those exactly to the low energy limit of the D-branes we require the condition of "extremality"; which is to say the equivalence of the mass and the charge Q of the branes, necessary for the BPS preservation of half supersymmetries. As both M and Q are functions of R and r_0 it can be shown that the extremality is achieved given $r_0 = 0$ condition.

It is also fairly obvious from the 1.76 - 1.78 that the $p = 3$ has a special status in the theory. It is both, the only dimensionality for which string coupling does not depend on the geometry and the only one in which it does not blow up in the $r \to 0$ limit. We are therefore interested in the $p = 3$, and specifically in the near-horizon limit of this theory given by $r \to r_0 = 0$. The metric of this limit is given by:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2 \tag{1.79}$$

where $z = \frac{R^2}{r}$

This is a geometry of the cross product between a 5-dimensional anti-de Sitter space AdS_5 and a 5-sphere of the same radius R. Over all, the geometry can be imagined as the semi infinite funnel which opens into the flat space in the $r >> R$ limit and is non-singular in the $r \to 0$ limit.

To connect this geometric picture back to the properties of the D-branes (which are the source of the geometry), we equate the tension of a given D3-brane (multiplied by N for an N-brane stack) with the total stress energy required to cause the above curving of the space-time. The later property is referred to as the ADM mass and is given by [30, 31]:

$$M_{ADM} = \frac{2\pi^3 R^4}{8\pi G_{10}} = \frac{R^4}{32\pi^4 g_s \alpha'^4} \tag{1.80}$$

where the relationship between the 10 dimensional gravitational constant G_{10} and the string coupling comes from the supergravity action 1.55. The tension of the N D-branes is inversely proportional to the string coupling and is given by:

$$T_{ND3} = \frac{N}{8\pi^3 g_s \alpha'^2} \tag{1.81}$$

In all of the above constants such as κ_{10} and T_p are written in terms of their explicit values which can be determined via the amplitudes of the closed string exchange. Equating 1.80 and 1.81 we get the first fundamental relation of the AdS/CFT correspondence, the one connecting the geometry of the background to the string coupling:

$$R^2 = \alpha' \sqrt{4\pi g_s N} \tag{1.82}$$

We have not yet defined the CFT part of the AdS/CFT correspondence. To do that we start with the low energy DBI action of the brane 1.74. The 3+1 coordinates of the D-brane can now be interpreted as dimensions in 4-dimensional space and the remaining 6 transverse

directions can be assigned fields on those dimensions. To do this we use the following kind of embedding of the brane on the target space:

$$X^a(\xi^a) = \xi^a, \quad X^I(\xi^a) = \sqrt{2}\pi\alpha'\Phi^I(\xi^a) \tag{1.83}$$

keeping in mind that the Φ^I field is different from the dilation ϕ which, being constant, can be expressed as the string coupling $g_s = e^{\phi}$. $a = 0...4$ corresponds to the worldvolume coordinates of the brane and the $I = 5..9$ to the transverse directions.

Assuming flat background $G_{ab} = \eta_{ab}$ and the low energy limit $\alpha' \to 0$ we can re-write the equation 1.74:

$$S_{D3} = -\frac{1}{4\pi g_s}\int d^4\xi \left(\frac{1}{4}(F_{ab})^2 + \frac{1}{2}(\partial_a\Phi^I)^2\right) + ... \tag{1.84}$$

with a infinite constant corresponding to the volume of the brane being ignored.

This action describes the single brane. To go to a stack of branes we perform the same operation as the one leading to 1.75 promoting the fields to the matrices under the $U(N)$ group:

$$S_{ND3} = -\frac{1}{4\pi g_s}\int d^4\xi Tr \left(\frac{1}{4}(F_{ab})^2 + \frac{1}{2}(D_a\Phi^I)^2\right) + ... \tag{1.85}$$

where partial got promoted to a covariant derivative and "..." includes the interaction terms and fermions. From what we have written we can already see the relationship between the string coupling and the Yang-Mills coupling that represents the second fundamental relationship of the AdS/CFT:

$$4\pi g_s = g_{YM}^2 \tag{1.86}$$

Two fundamental relationships 1.82 and 1.86 together give:

$$R^4 = 2\alpha'^2 g_{YM}^2 N \tag{1.87}$$

It is very important to note in deriving the geometric description we relied on the limit in which $R >> \alpha'$ in order to suppress "stringy" corrections to the supergravity model. On the gauge-theory side this limit corresponds to the requirement that $g_{YM}^2 N >> 1$. In other words, the low-energy limit of string theory turned out to correspond to the strong coupling regime of the Yang-Mills theory.

1.3.3 Early equivalences - Entropy and absorption cross-section

While what we have described above is still not a full fledged correspondence, it is a construction with testable consequences. One of the early tests was the correspondence between Bekenstein-Hawking entropy in the gravitational picture and the entropy calculated statistically from the field theory [23]. The other was the absorption cross section for the closed string modes by the system [29]. We discuss both of them briefly.

Entropy of the 3-brane stacks

Relaxing the extremality limit temporarily we can write the metric of the near horizon $r << L$ region as:

$$ds^2 = \frac{r^2}{R^2}\left[-\left(1-\frac{r_0^4}{r^4}\right)dt^2 + d\vec{x}^2\right] + \frac{R^2}{r^2}\left(1-\frac{r_0^4}{r^4}\right)^{-1}dr^2 + R^2 d\Omega_5^2 \tag{1.88}$$

The above is a product of $\mathbf{S_5}$ with a limit of a Schwarzschild black hole. We can perform Euclidean continuation of the metric in order to get its periodicity (and thus temperature $\beta = 1/T$). For convenience we also change variables:

$$r = r_0(1 + R^{-2}\rho^2), \quad \tau = it \tag{1.89}$$

the relevant part of the Euclidean metric is then:

$$ds^2 = d\rho^2 + \frac{4r_0^2}{R^4}\rho^2 d\tau^2 \tag{1.90}$$

which implies, due to the need to avoid singularity at the horizon:

$$\beta = \frac{\pi R^2}{r_0} = \frac{1}{T} \tag{1.91}$$

To figure out the Bekenstein-Hawking entropy we can write down the Area of the horizon (8 dimensional - all dimensions except t and r) from the metric:

$$A_h = (r_0/L)^3 V_3 R^5 \Omega_5 = \pi^6 R^8 T^3 V_3 \tag{1.92}$$

where V_3 is the spatial volume of the D3-brane. Entropy is then given by [23]:

$$S_{AdS} = \frac{2\pi A_h}{\kappa^2} = \frac{\pi^2}{2} N^2 V_3 T^3 \tag{1.93}$$

The entropy on the CFT side is calculated using the usual statistical mechanics techniques for massless gas of N^2 scalars and Weyl fermions. Without going into details, which are presented in [29], it is given by:

$$S_{CFT} = \frac{2\pi^2}{3} N^2 V_3 T^3 \tag{1.94}$$

The exact scaling in N and T demonstrates that the two pictures in fact represent the same theory. The factor of 3/4 difference is an artifact of the different limits that the two pictures come from. Detailed calculations [24] conclude that the entropy is actually of the form:

$$S = \frac{2\pi^2}{3} N^2 f(g_{YM}^2 N) V_3 T^3 \tag{1.95}$$

where $f(g_{YM}^2 N)$ seems to vary monotonically between 1 at $g_{YM}^2 N = 0$ and 3/4 at $g_{YM}^2 N = \infty$.

Absorption cross-sections

Another important early test of the duality was the comparison of the two ways for calculating the cross-section for the absorption of the closed string modes by our system. On the CFT side (D-brane formalism) these cross section are calculated by the usual field-theoretic method, as in [3], starting from the interaction term of the Born-Infeld action. For dilaton, Ramond-Ramond scalar and graviton this term is given by [21, 29]:

$$S_{DBI}^{int} = \frac{\pi}{\kappa}\int d^4x \left[tr\left(\frac{1}{4}\phi F_{\alpha\beta}^2 - \frac{1}{4}CF_{\alpha\beta}\tilde{F}^{\alpha\beta}\right) + \frac{1}{2}h^{\alpha\beta}T_{\alpha\beta}\right] \tag{1.96}$$

where $T_{\alpha\beta}$ is the brane stress-energy tensor.

From the above we can see that the incoming dilaton would couple to the $\frac{1}{4g_{YM}^2}TrF_{\alpha\beta}^2$ which is to say it can be converted to a pair of the world volume bosons. Usual methods [29] yield cross-section for the low energy dilaton incoming onto the 3-brane stack:

$$\sigma_{CFT} = \frac{\kappa^2\omega^3N^2}{32\pi} \tag{1.97}$$

In the geometric picture the cross-section corresponds to the absorption of the waves incident from the $r >> R$ region by the throat region $r << R$. This is calculated by finding the equations of motion of the closed string mode (dilaton for comparison with the previous case) from the SUGRA action in the $AdS_5 \times S_5$ background described by a metric such as 1.88. Those turn out to be simple d'Alembertian equations for the relevant geometry:

$$\Box\phi(X) = \left[-\left(1+\frac{R^4}{r^4}\right)\partial_t^2 + \partial_r^2 + \frac{5}{r}\partial_r + r^2D_{\Omega_5}^2\right]\phi(X) = 0 \tag{1.98}$$

where D_{Ω_5} is the 5-sphere laplacian. Considering only perpendicular modes of the dilaton (in keeping with the D-brane formalism above) this can be reduced to a single-dimensional equation where the single dimension is related to the radial dimension of the geometry:

$$\left[\partial_z^2 + 2\omega^2R^2\cosh(2z)\right]\psi(z) = 0 \tag{1.99}$$

which is a regular barrier problem in quantum mechanics. This is then solved using the so called matching method whereby the equation is solved in $Z \to \infty$ and $Z \to -\infty$ limits and then matching the overlapping region. The end result is given by [29]:

$$\sigma_{AdS} = \frac{\pi^4}{8}\omega^3R^8 \tag{1.100}$$

which, taking into account 1.82 ends up exactly identical to the σ_{CFT} calculated above without even the numeric factor discrepancy as in the entropy calculation. It has been shown [21] that this agreement is not co-incidental, and that for all the absorption cross-sections in the large N limit have no perturbative corrections and should thus be identical between strong and weak interaction pictures. The agreement above was replicated for the case of gravitons and a number of other closed-string modes, giving impetus to the idea of the full-blown duality between the Yang-Mills theory and the IIB strings on a curved background.

1.3.4 Maldacena Conjecture

We have seen so far that the stack of N D3-branes embedded within the flat IIB string background can be thought of in terms of its geometric interaction with the background or in terms of the field theory given by the D-brane action itself. We have also seen how the perturbative regimes for the two "pictures" end up on the exactly opposite "sides" of the theory: with the supergravity limit of the strings on the AdS (defined by $R >> \alpha'$) requirement corresponding to the strong coupling limit of the field theory ($\lambda = g_{YM}^2 N >> 1$) and vice versa.

However, at this level of understanding the two pictures are still just different ways of describing the same phenomenon - a stack of D3-branes in a generically flat background. In 1997 Maldacena [2] proposed that it is possible to decouple the flat IIB string background from the brane-stack in both pictures while maintaining their correspondence. On the geometry side this decoupling takes place as we take a limit $R, \alpha' \to 0$ while keeping $R^4/\alpha'^2 = \lambda = 4\pi g_s N$ fixed. The region inside throat ($r < R$) is then described by:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_i^2) + \frac{R^2}{r^2}(dr^2 + r^2 d\Omega_5^2) \tag{1.101}$$

the cross-section 1.100 goes to zero in this limit making the 1.101 decoupled from the $R >> r$ region which is simple flat space type IIB string theory.

On the D-brane dynamics side the same limit likewise decouples the stack of N D3-branes from the IIB closed strings that propagate in the bulk. What is left is the $\mathcal{N} = 4$ supersymmetric gauge theory with $U(N)$ gauge group. The Maldacena Conjecture is that this theory is exactly dual to the IIB string theory propagating on the $AdS_5 \times S_5$ background 1.101.

The key corollary of this conjecture comes from the equation:

$$\frac{R^4}{\alpha'^2} = \lambda = g_{YM}^2 N \tag{1.102}$$

which is unaffected by the decoupling limit. It is easy to see that in the case of large λ,the ratio of the AdS curvature to the string length becomes large and thus the low energy of the string theory (supergravity) becomes sufficient to describe the dynamics of the system (provided N is taken to be large to avoid the higher genera of the string interactions). The long sought-after method for analytic handling of the strong interaction gauge theory is thus obtained.

1.3.5 Evidence of the AdS/CFT and holography

While there is - as of now - no actual proof of the full version of the Maldacena conjecture there are multiple reasons it is widely believed to be true. One is the identical symmetry groups of the two theories [22]. The second piece of evidence comes from the equivalence of the two (and higher) point functions between the operators in CFT and the fields on the boundary of the AdS [21, 22]. This later property suggests that the idea of *holography* is applicable to the AdS/CFT. This is further confirmed by comparing the degrees of freedom of CFT with the surface area of the AdS [21, 28].

Symmetry group of AdS/CFT

Conformal field theory is invariant under the usual Poincare group as well as the conformal transformations:

$$x^\mu \to \Lambda x^\mu \tag{1.103}$$

and

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2} \tag{1.104}$$

1.103 guarantees that the coupling of the theory does not run and is a free parameter which is necessary for the correspondence to hold. Taking Poincare generators to be $M_{\mu\nu}$ (lorentz rotations) and P_μ (translations) and the generators of 1.103 and 1.104 as D and K_μ respectively we can write the whole algebra [22]:

$$\begin{aligned} &[M_{\mu\nu}, P_\sigma] = -i(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu); \quad [M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} + permutations; \\ &[D, P_\mu] = -iP_\mu; \quad [D, K_\mu] = iK_\mu; \quad [D, M_{\mu\nu}] = 0 \\ &[K_\rho, M_{\mu\nu}] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu); \quad [K_\nu, P_\mu] = 2i\eta_{\mu\nu}D - 2iM_{\mu\nu} \end{aligned} \tag{1.105}$$

This algebra is isomorphic to the $SO(d,2)$ and can be put into the standard form of $SO(d,2)$ by defining:

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu}; \quad J_{\mu d} = \tfrac{1}{2}(K_\mu - P_\mu); \\ J_{\mu d+1} &= \frac{1}{2}(K_\mu + P_\mu); \quad J_{dd+1} = D \end{aligned} \tag{1.106}$$

Furthermore, the theory has SO(6) symmetry, called R-symmetry, that rotates 6 scalar fields into each other. Bosonic symmetry therefore is $SO(2,4) \times SO(6)$. By their very definition the AdS_5 and S_5 have the $SO(2,4)$ and $SO(6)$ symmetries respectively. To see this consider the embeding functions for AdS and the sphere:

$$x_0^2 + x_{d+1}^2 - (x_i)^2 = R, \quad (x_j)^2 = R \tag{1.107}$$

meaning that the two theories have exactly identical bosonic symmetries. Further investigation [22] shows that the supersymmetric extension of the algebra given schematically as:

$$\begin{aligned} &[D, Q] \simeq Q; \quad [D, S] \simeq S; \quad [K, Q] \simeq S; [P, S] \simeq Q; \\ &\{Q, Q\} \simeq P; \quad \{S, S\} \simeq K; \quad \{Q, S\} \simeq M + D + R \end{aligned} \tag{1.108}$$

where Q and S are generators of the supersymmetries, gives the full superalgebra as $SU(2,2|4)$ and is likewise identical between the two.

Correlation functions and the bulk-boundary correspondence

The core test of the AdS/CFT comes from the actual calculations of the corresponding quantities on the two sides. We need, however, to establish what quantities exactly are

corresponding to each other. A natural candidate on the CFT side is the operators. As a simple example we can take an operator within the $\mathcal{N}= 4$ super Yang-Mills which changes the value of the coupling constant. This is related by 1.86 to changing the coupling constant of the string theory and thus to the expectation value of the dilaton. The expectation value of the dilaton, for its part, is set by its boundary condition on the boundary of AdS (infinity). In other words adding the operator $\mathcal{O}$ to the Lagrangian of the CFT will change the boundary condition of the dilaton. Specifically we can write [22]:

$$\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{CFT} = \mathcal{Z}_{string}\left[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})\right] \tag{1.109}$$

where the left hand side is the generating function of the corelation functions in the field theory with arbitrary ϕ_0 and the right hand side is the full partition function of the string theory with the boundary condition for ϕ defined over all $d-1$ dimensions of the boundary. Similar equations hold in the general case for fields other then the dilaton as well with the interesting corollary of the scaling dimension Δ of the operator $\mathcal{O}$ being directly related to the mass of the string mode by:

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right) \tag{1.110}$$

because boundary condition in the region close to the boundary for massive fields becomes:

$$\phi(\vec{x}, \epsilon) = \epsilon^{d-\Delta}\phi_0(\vec{x}) \tag{1.111}$$

with attendant implications to the dimension of ϕ_0 and $\mathcal{O}$

The correlation functions of the gauge theory can be calculated from 1.109 by differentiation with respect to ϕ_0. Each differentiation brings down an insertion O and sends the ϕ particle into the bulk. The interactions in bulk can then be calculated by the Feynman diagrams of supergravity (whose external legs correspond to the boundary values ϕ_0). These could in principle be compared with the field theory except that in most general case the supergravity calculations in the bulk correspond to the strong coupling (and thus the non-perturbative) calculations in the field theory.

Nonetheless calculating the correlation functions is a valuable exercise which gives considerable insight in the relationship between the operators in CFT and strings on AdS. It is done explicitely in [22] for the case of 2 and 3 point functions and the discussion of the 4-point functions was presented. The topic is discussed extensively in literature ([26, 27] and many others).

Holography

One of the ways to think about the AdS/CFT duality is to go back to the idea of holography which was useful in understanding the entropy of black holes. The basic statement of the "holographic principle" is that for any quantum theory that includes gravity, all physics within a given volume can be described in terms of a different theory on the boundary of that volume, whose degrees of freedom are limited by the area of the boundary and the Bekenstein bound [22]. The boundary of the $AdS_5 \times S_5$ is naively 9-dimensional but the 5 spherical dimensions remain compact and thus small as the 4 remaining dimensions grow

(as we approach the boundary of the space). This makes any theory on the boundary of $AdS_5 \times S_5$ effectively 4-dimensional.

The idea behind the holographic treatment of AdS/CFT is that the CFT on 4 dimensions can be considered a theory on the boundary of the AdS that contains all of its physics. The intimate relationship between the operators of the CFT and the boundary conditions of the AdS fields discussed above support this idea but in order to satisfy the holographic principle AdS/CFT also has to meet the degrees of freedom requirement.

In the AdS/CFT case this requirement is hard to test directly because any conformal theory has an infinite number of degrees of freedom because it can go down to arbitrarily small scales, and the boundary of the Anti-deSitter space is likewise infinite. It is, however, possible to introduce a cutoff on the number of degrees of freedom of the CFT and see what effect this has on the gravitational theory [21, 28].

A convenient way to write the metric of the AdS_5 is:

$$ds^2 = R^2 \left[\frac{4dx^i dx^i}{(1-r^2)^2} - dt^2 \frac{1+r^2}{1-r^2} \right] \tag{1.112}$$

with the boundary set at $r = 1$. As we have discussed above, the correlators of the supergravity fields $A(x_1)A(x_2)$ should be equal to the super Yang-Mills correlators $Y(X_1)Y(X_2)$ in the limit in which the bulk coordinates x_i are brought to their boundary values $X_i = x_i|_{r=1}$ The way to do this limit is to specify the boundary condition at $r = 1 - \delta$ and then take $\delta \to 0$ Geodesic distance between two points X_1, X_2 on the sphere regulated by $r < 1 - \delta$ is of order $\log(\frac{|X_1 - X_2|}{\delta})$ and the propagator for the particle of mass m in the bulk will then go as:

$$\Delta(X_1, X_2) = e^{m \log(\frac{\delta}{|X_1 - X_2|})} = \frac{\delta^m}{|X_1 - X_2|^m} \tag{1.113}$$

This compares with the operator product in the CFT of the form:

$$Y(X_1)Y(X_2) = \left(\frac{1}{\mu |X_1 - X_2|} \right)^{-p} + ... \tag{1.114}$$

where μ is an arbitrary regulator mass scale making sure that the fields Y are dimensionless.

Comparing the two equations we see that there is a direct correspondence between the IR regulator δ in the bulk theory and the UV regulator in the Yang-Mills theory. The large boundary area corresponds to the short distance regulator on the CFT side.

The number of degrees of freedom of the super Yang-Mills with the UV cutoff δ is going to be

$$N_{dof} = \frac{N^2}{\delta^3} = \frac{N^2}{R^3} R^3 \delta^3 = \frac{R^5}{\alpha'^4 g_s^2} R^3 \delta^3 = \frac{A}{G_5} \tag{1.115}$$

Which is exactly the Bekenstein bound.

Further evidence of AdS/CFT

The joint symmetry group and satisfied holographic principle are both strong indicators that the AdS/CFT represents an actual duality. Further evidence comes from certain correlation

functions (usually related to anomalies) which do not depend on coupling constant and can thus be calculated on both sides. Further tests rely on equivalence in the spectrum of chiral operators, the moduli space of the theory and a number of qualitative tests such as the existence of confinement for the finite temperature theory. Discussing even a small section of those tests goes well beyond the scope of our research work so far, and of this thesis.

1.3.6 Summary

We end the introductory chapter with a brief introduction to AdS/CFT correspondence - one of the most important discoveries in modern string theory. With it, string theory has came full circle from its origin as potential theory of strong interactions. We also learn of the many tests that confirm AdS/CFT without explicitly proving it. In the remainder of this thesis we will focus on particular areas of AdS/CFT and the work, including our own research efforts, on extending the applicability of this amazing correspondence.

Chapter 2

AdS/CFT and the interacting strings

We have seen, in the introduction, how the Maldacena conjecture postulated a strong-weak duality between a string theory on an anti-deSitter space and the conformal super Yang-Mills theory on the 4-dimensional boundary of that space. None of the examples, however, dealt with the full string theory on AdS, but rather with a low-energy limit of it, called supergravity. Supergravity ignores all the oscillator degrees of freedom inherent to string theory and thus all the "stringy" components of it. As long as we are limited to supergravity, therefore, there is little hope of fully understanding the AdS/CFT correspondence by testing and expanding its applicability.

The actual string theory beyond supergravity on AdS is not solved. Fortunately, however, it is possible to take a particular Penrose limit [32] of the AdS space to obtain a background on which the non-interacting type IIB string theory can be fully solved [33, 36, 37]. Even more importantly, it has been shown by Berenstein, Maldacena and Nastase [35] that the equivalent limit can be taken on the field theory side and that the spectra of the two limits have the exact correspondence predicted by AdS/CFT.

This insight has led to considerable advancement in the understanding of the string/gauge duality and has been extended beyond the BMN limit [40–44] and into the non-perturbative sector [45, 46]. One further extension that would be particularly valuable would be to check the correspondence between the interacting strings and the non-planar corrections in the field theory. This has been a topic of vigorous research [54–62, 65–71, 79, 80] over the past years but the full correspondence is still elusive. Our own work [79, 80] provides some of the hitherto missing pieces.

In this chapter we cover briefly the plane-wave limit of $AdS_5 \times S_5$ and its BMN counterpart on the field theory side. We then introduce the ideas of string field-theory as a method for handling the string interactions, present the work leading to our research and, in final sections, give a detailed presentation of our own contribution.

2.1 Strings on the plane-wave and the BMN limit

2.1.1 Penrose limit of the $AdS_5 \times S_5$

To take a Penrose limit is to consider the trajectory of a particle moving very fast on the space (in this case along one of the S_5 geodesics) and focus on the geometry as seen by such a particle.

Starting from the $AdS_5 \times S_5$ metric:

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3'^2 \right] \tag{2.1}$$

We focus on the geodesic of S_5 defined by $\rho = \theta = 0$ and parametrized by ψ. To do that we introduce coordinates: $\tilde{x}^{\pm} = \frac{t \pm \psi}{2}$ and perform the rescaling:

$$x^+ = \tilde{x}^+ \ , \qquad x^- = R^2 \tilde{x}^- \ , \qquad \rho = \frac{r}{R} \ , \qquad \theta = \frac{y}{R} \ , \qquad R \to \infty \tag{2.2}$$

obtaining the metric:

$$ds^2 = -4dx^+dx^- - (\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{y}^{\,2} + d\vec{r}^{\,2} \tag{2.3}$$

Which can be written in the form of the pp-wave metric

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \mu^2 \vec{z}^{\,2} dx^{+2} + d\vec{z}^{\,2} \\ F_{+1234} &= F_{+5678} = \text{const} \times \mu \end{aligned} \tag{2.4}$$

Where μ is a mass term usually set to 1 but which can be re-introduced by scaling: $x^- \to x^-/\mu$ and $x^+ \to \mu x^+$.

This background has a well-defined Green-Schwarz superstring action in the light-cone gauge and solved string equations of motion. [36, 37].

As in our flat space discussion, we quantize the strings on this background in a light-cone gauge ending with 8 bosonic degrees of freedom corresponding to the 8 transverse directions. Coupling to the RR background gives 8 fermionic degrees of freedom. Out of the standard 32 supersymmetries, half end up being linear (again just as in the flat space case) and commuting with the Hamiltonian. This ensures the same mass between the bosons and the fermions. The Hamiltonian itself is of the form:

$$2p^- = -p_+ = H_{lc} = \sum_{n=-\infty}^{+\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \tag{2.5}$$

where N_n is the usual number operator:

$$N_n \simeq (\alpha_n^{i\dagger}\alpha_n^i + \beta_n^{i\dagger}\beta_n^i) \tag{2.6}$$

with α_n^i and β_n^i being the bosonic and fermionic oscillation modes. In this notation, the left moving modes are labeled with positive n and the right moving modes with negative n.

2.1.2 Strings on the pp-wave limit

Here we summarize the results concerning free string theory on the pp-wave background from the [36] and [37]. For detailed derivation of these results we refer the reader to those papers.

The Green-Schwarz Lagrangian for the superstring in the background 2.4 is obtained using the supercoset method in [36]. In the standard lightcone gauge

$$x^+ = p^+\tau \ , \qquad \Gamma^+\theta^{\mathcal{I}} = 0 \tag{2.7}$$

it is given by:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \ , \qquad \mathcal{L}_B = \frac{1}{2}(\partial_+ x^I \partial_- x^I - \mu^2 x_I^2) \ , \tag{2.8}$$

$$\mathcal{L}_F = i(\theta^1\bar{\gamma}^-\partial_+\theta^1 + \theta^2\bar{\gamma}^-\partial_-\theta^2 - 2\mu\theta^1\bar{\gamma}^-\Pi\theta^2)\,, \qquad \bar{\gamma}^+\theta^{\mathcal{I}} = 0 \tag{2.9}$$

where the x and θ bosonic and fermionic fields respectively. The former are given in vector notation and the later in the suppressed spinor notation. The same spinor notation is used for the γ and $\bar{\gamma}$ which are are 16×16 Dirac matrices.

2.8 and 2.9 give rise to the equations of motion:

$$\partial_+\partial_- x^I + \mu^2 x^I = 0\,, \tag{2.10}$$

$$\partial_+\theta^1 - \mu\Pi\theta^2 = 0\,, \qquad \partial_-\theta^2 + \mu\Pi\theta^1 = 0\,. \tag{2.11}$$

which, for the closed string boundary conditions, are solved by:

$$x^I(\sigma,\tau) = \cos\mu\tau\, x_0^I + \mu^{-1}\sin\mu\tau\, p_0^I + i\sum_{n\neq 0}\frac{1}{\omega_n}\Big(\varphi_n^1(\sigma,\tau)\alpha_n^{1I} + \varphi_n^2(\sigma,\tau)\alpha_n^{2I}\Big) \tag{2.12}$$

$$\theta^1(\sigma,\tau) = \cos\mu\tau\,\theta_0^1 + \sin\mu\tau\;\Pi\theta_0^2 + \sum_{n\neq 0} c_n\Big(\varphi_n^1(\sigma,\tau)\theta_n^1 + i\rho_n\varphi_n^2(\sigma,\tau)\Pi\theta_n^2\Big) \tag{2.13}$$

$$\theta^2(\sigma,\tau) = \cos\mu\tau\,\theta_0^2 - \sin\mu\tau\;\Pi\theta_0^1 + \sum_{n\neq 0} c_n\Big(\varphi_n^2(\sigma,\tau)\theta_n^2 - i\rho_n\varphi_n^1(\sigma,\tau)\Pi\theta_n^1\Big) \tag{2.14}$$

where the basis functions $\varphi_n^{1,2}(\sigma,\tau)$ are

$$\varphi_n^1(\sigma,\tau) = \exp(-i(\omega_n\tau - n\sigma))\,, \qquad \varphi_n^2(\sigma,\tau) = \exp(-i(\omega_n\tau + n\sigma)) \tag{2.15}$$

and

$$\omega_n = \sqrt{n^2 + \mu^2\alpha^2},\ \rho_n = \frac{\omega_n - |n|}{\mu\alpha}\,,\ c_n = \frac{1}{\sqrt{1+\rho_n^2}}\,, \qquad n = \pm 1, \pm 2, \ldots\,. \tag{2.16}$$

Before quantizing it makes sense to do the following change of basis:

$$a_0^I = \frac{1}{\sqrt{2\mu}}(p_0^I + i\mu x_0^I)\,, \qquad \bar{a}_0^I = \frac{1}{\sqrt{2\mu}}(p_0^I - \mathrm{i}\mu x_0^I)\,, \tag{2.17}$$

$$\alpha_{-n} = \sqrt{\frac{\omega_n}{2}}\,a_n\,, \qquad \alpha_n = \sqrt{\frac{\omega_n}{2}}\,\bar{a}_n\,, \qquad n = 1, 2, \ldots \tag{2.18}$$

for the bosons, and:

$$\begin{aligned}\theta_n &= \frac{1}{\sqrt{|\alpha|}}c_n\Big[(1+\rho_n\Pi)b_n + e(\alpha)(1-\rho_n\Pi)b_{-n}^\dagger\Big],\\ \theta_{-n} &= \frac{1}{\sqrt{|\alpha|}}c_n\Big[(1+\rho_n\Pi)b_{-n} - e(\alpha)(1-\rho_n\Pi)b_n^\dagger\Big]\end{aligned} \tag{2.19}$$

for the fermions. Furthermore, the choice leading to the zero vacuum energy breaks the apparent symmetry of the background from $SO(8)$ to $SO(4)\times SO(4)$ (which is in fact the real

symmetry of the background when the Ramond-Ramond field is taken into consideration). It will be easier to change the spinor-basis once more to account for this and thus give the fermionic oscillators the indices under $(SU(2)\times SU(2))_1\times(SU(2)\times SU(2))_2$; whereby $b^\dagger_{\alpha_1\alpha_2}$ and $b^\dagger_{\dot\alpha_1\dot\alpha_2}$ would transform in the $(1/2,0,1/2,0)$ and $(0,1/2,0,1/2)$ representations of $(SU(2)\times SU(2))_1\times(SU(2)\times SU(2))_2$, respectively.

One final change of variables is needed to reconcile the notation between [36, 37] and our work in [79, 80]. This final notation will be used in the remainder of this thesis.

$$\begin{aligned}
&\sqrt{2}a^i_n \equiv \alpha^i_n + \alpha^i_{-n}\,, \quad i\sqrt{2}a^i_{-n} \equiv \alpha^i_n - \alpha^i_{-n}\\
&\sqrt{2}a^{i'}_n \equiv \alpha^{i'}_n + \alpha^{i'}_{-n}\,, \quad i\sqrt{2}a^{i'}_{-n} \equiv \alpha^{i'}_n - \alpha^{i'}_{-n}\,,\\
&\sqrt{2}b^{\alpha_1\alpha_2}_n \equiv \beta^{\alpha_1\alpha_2}_n + \beta^{\alpha_1\alpha_2}_{-n}\,, \quad i\sqrt{2}b^{\alpha_1\alpha_2}_{-n} \equiv \beta^{\alpha_1\alpha_2}_n - \beta^{\alpha_1\alpha_2}_{-n}\,,\\
&i\sqrt{2}b^{\dot\alpha_1\dot\alpha_2}_n \equiv -\beta^{\dot\alpha_1\dot\alpha_2}_n + \beta^{\dot\alpha_1\dot\alpha_2}_{-n}\,, \quad \sqrt{2}b^{\dot\alpha_1\dot\alpha_2}_{-n} \equiv \beta^{\dot\alpha_1\dot\alpha_2}_n + \beta^{\dot\alpha_1\dot\alpha_2}_{-n}
\end{aligned} \tag{2.20}$$

for $n>0$, and

$$a^i_0 \equiv \alpha^i_0 \quad b^{\alpha_1\alpha_2}_0 \equiv \beta^{\alpha_1\alpha_2}_0 \tag{2.21}$$

for $n=0$.

The usual Poison brackets then yield, after promoting α and βs into operators:

$$[\alpha^I_m,\alpha^{J\dagger}_n] = \delta^{IJ}\delta_{mn}, \quad \left\{\beta_{n\,\alpha_1\alpha_2},\beta^{\beta_1\beta_2\dagger}_m\right\} = \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\delta_{mn}, \quad \left\{\beta_{n\,\dot\alpha_1\dot\alpha_2},\beta^{\dot\beta_1\dot\beta_2\dagger}_m\right\} = \delta^{\dot\beta_1}_{\dot\alpha_1}\delta^{\dot\beta_2}_{\dot\alpha_2}\delta_{mn} \tag{2.22}$$

The details of the spinor notation are given in Appendix A.

The Hamiltonian 2.5 can be written in this notation as:

$$H_2 = \sum_{n=-\infty}^{\infty}\frac{\omega_n}{\alpha}\left(\alpha^{(r)I\dagger}_n\alpha^I_n + \beta^{\alpha_1\alpha_2\dagger}_n\beta_{n\,\alpha_1\alpha_2} + \beta^{\dot\alpha_1\dot\alpha_2\dagger}_n\beta_{n\,\dot\alpha_1\dot\alpha_2}\right) \tag{2.23}$$

The equations of motion for x^- as usual lead to the level-matching condition which can be written as:

$$\sum_{n=-\infty}^{\infty} n\left(\alpha^{(r)I\dagger}_n\alpha^I_n + \beta^{\alpha_1\alpha_2\dagger}_n\beta_{n\,\alpha_1\alpha_2} + \beta^{\dot\alpha_1\dot\alpha_2\dagger}_n\beta_{n\,\dot\alpha_1\dot\alpha_2}\right)|\Psi\rangle = 0. \tag{2.24}$$

2.1.3 The Supersymmetry group of the pp-wave

Isometries of the pp-wave background are generated by H, P^+, J^{+I}, J^{ij} and $J^{i'j'}$ where $i,j=1,2,3,4$, $i'j'=5,6,7,8$. The latter two are angular momentum generators of the transverse $SO(4)\times SO(4)$ symmetry. There are 32 conserved supercharges Q^+, $\bar Q^+$ and Q^-, $\bar Q^-$. These generators are divided into two groups, kinematical generators:

$$P^I\ ,\ P^+\ ,\ J^{+I}\ ,\ J^{ij}\ ,\ J^{i'j'}\ ,\ Q^+\ ,\ \bar Q^+$$

which act locally and are thus not corrected when the string interactions are introduced, and the dynamical generators:

$$H\ ,\ Q^-\ ,\ \bar Q^-$$

which will get corrections from interactions.

The parts of the super-algebra that differ from those of the flat space are then given by:

$$\begin{aligned}[H,P^I] =& i\mu^2 J^{+I}, \qquad [P^I,Q^-] = \mu\Pi\gamma^I Q^+, \qquad [H,Q^+] = \mu\Pi Q^+, \\ \{Q^-,\bar{Q}^-\} =& 2H + i\mu\gamma_{ij}\Pi J^{ij} + i\mu\gamma_{i',j'}\Pi J^{i'j'}.\end{aligned} \tag{2.25}$$

It will prove convenient to define a linear combination of the free super-charges such as to separate Hamiltonian in the anti-commutator of the supercharges. In the notation of [79] this combination is given by:

$$\sqrt{2}\eta Q \equiv Q^- + i\bar{Q}^- \quad , \quad \sqrt{2}\bar{\eta}\widetilde{Q} \equiv Q^- - i\bar{Q}^- \tag{2.26}$$

where $\eta = e^{i\pi/4}$.

The dynamical constraints in this notation will then be equal to:

$$\begin{aligned}&\{Q_{\alpha_1\dot{\alpha}_2}, Q_{\beta_1\dot{\beta}_2}\} = \left\{\widetilde{Q}_{\alpha_1\dot{\alpha}_2}, \widetilde{Q}_{\beta_1\dot{\beta}_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H \\ &\left\{Q_{\alpha_1\dot{\alpha}_2}, \widetilde{Q}_{\beta_1\dot{\beta}_2}\right\} = -\mu\epsilon_{\dot{\alpha}_2\dot{\beta}_2}\left(\sigma^{ij}\right)_{\alpha_1\beta_1} J^{ij} + \mu\epsilon_{\alpha_1\beta_1}\left(\sigma^{i'j'}\right)_{\dot{\alpha}_2\dot{\beta}_2} J^{i'j'}\end{aligned} \tag{2.27}$$

The free dynamical supercharges are given by:

$$\begin{aligned}\sqrt{\frac{|\alpha|}{2}}Q^-_{\alpha_1\dot{\alpha}_2} =& -\frac{\sqrt{\mu|\alpha|}}{2\sqrt{2}}(1-e(\alpha))\left[\alpha_{0\,\alpha_1}^{\dot{\beta}_1}\beta^\dagger_{0\,\dot{\beta}_1\dot{\alpha}_2} + \alpha^{\dagger\,\beta_2}_{0\,\dot{\alpha}_2}\beta_{0\,\alpha_1\beta_2}\right] \\ &+ \sum_{k\neq 0}\Big[\sqrt{\omega_k+\mu\alpha}\,\alpha^{\dagger\,\dot{\beta}_1}_{k\,\alpha_1}\beta_{k\,\dot{\beta}_1\dot{\alpha}_2} - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha_{k\,\alpha_1}^{\dot{\beta}_1}\beta^\dagger_{k\,\dot{\beta}_1\dot{\alpha}_2} \\ &-e(\alpha)\left(\sqrt{\omega_k+\mu\alpha}\,\alpha^{\beta_2}_{k\,\dot{\alpha}_2}\beta^\dagger_{k\,\alpha_1\beta_2} - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha^{\dagger\,\beta_2}_{k\,\dot{\alpha}_2}\beta_{k\,\alpha_1\beta_2}\right)\Big], \\ \sqrt{\frac{|\alpha|}{2}}Q^-_{\dot{\alpha}_1\alpha_2} =& \frac{\sqrt{\mu|\alpha|}}{2\sqrt{2}}(1+e(\alpha))\left[\alpha_{0\,\dot{\alpha}_1}^{\beta_1}\beta^\dagger_{0\,\beta_1\alpha_2} + \alpha^{\dagger\,\dot{\beta}_2}_{0\,\alpha_2}\beta_{0\,\dot{\alpha}_1\dot{\beta}_2}\right] \\ &+ \sum_{k\neq 0}\Big[\sqrt{\omega_k+\mu\alpha}\,\alpha^{\dagger\,\dot{\beta}_2}_{k\,\alpha_2}\beta_{k\,\dot{\alpha}_1\dot{\beta}_2} - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha_{k\,\alpha_2}^{\dot{\beta}_2}\beta^\dagger_{k\,\dot{\alpha}_1\dot{\beta}_2} \\ &+e(\alpha)\left(\sqrt{\omega_k+\mu\alpha}\,\alpha_{k\,\dot{\alpha}_1}^{\beta_1}\beta^\dagger_{k\,\beta_1\alpha_2} - ie(\alpha k)\sqrt{\omega_k-\mu\alpha}\,\alpha^{\dagger\,\beta_1}_{k\,\dot{\alpha}_1}\beta_{k\,\beta_1\alpha_2}\right)\Big]\end{aligned} \tag{2.28}$$

2.1.4 BMN limit

The above background is a continuous limit of the $AdS_5 \times S_5$ background. It is therefore to be expected that the analogous limiting procedure exists on the CFT side of the correspondence. To identify it, we focus on the energy and angular momentum along the relevant geodesic in global AdS coordinates. The energy there is given by $E = i\partial_t$ and the angular momentum by $J = i\partial_\psi$ We have seen in the previous section that the angular momentum in AdS corresponds to charge under R-symmetry ($SO(6)$ symmetry identified with the symmetry of the S_5) which we can label J. Energy, as we have also seen is equal to the conformal dimension of the operator in the Yang-Mills theory.

We can then write [35]:

$$2p^- = -p_+ = \; i\partial_{x^+} = i\partial_{\tilde{x}^+} = i(\partial_t + \partial_\psi) = \Delta - J \tag{2.29}$$

$$2p^+ = -p_- = \ -\frac{\tilde{p}_-}{R^2} = \frac{1}{R^2} i\partial_{\tilde{x}^-} = \frac{1}{R^2} i(\partial_t - \partial_\psi) = \frac{\Delta + J}{R^2} \tag{2.30}$$

Focusing on the fast moving particle in the geodesic around ψ and taking the $R \to \infty$ limit is equivalent to setting: $J \sim R^2 \sim \sqrt{N}$ and sending $N \to \infty$. The operators corresponding to string spectrum will be ones with fixed $\Delta - J$.

We can then combine 2.29, 2.30 with 1.82 and 1.86 to get:

$$p^- = (\Delta - J) = \sum_n N_n \sqrt{1 + \frac{n^2 \lambda}{J^2}} \tag{2.31}$$

[35] then goes on to define the spectrum of operators with finite $\Delta - J$.

The single state operator corresponding to the vacuum $p^- = 0$ is given by $\Delta = J$, and the only single-trace operator satisfying this is:

$$\mathcal{O}_{vac} = \frac{1}{\sqrt{J} N^{J/2}} Tr[Z^J] \longleftrightarrow |0, p_+\rangle_{l.c.} \tag{2.32}$$

where $Z = \phi^5 + i\phi^6$. with plane $5 - 6$ being one rotated by J.

We start building the spectrum of the field theory by considering the super-multiplet of the $\mathcal{O}_{vac}$. We act on the $\mathcal{O}_{vac}$ with the generators of the symmetries. For example acting with the generators of SO(6) that are outside of SO(2) defined by J we get:

$$\frac{1}{\sqrt{J}} \sum_l \frac{1}{\sqrt{J} N^{J/2+1/2}} Tr[Z^l \phi^r Z^{J-l}] = \frac{1}{N^{J/2+1/2}} Tr[\phi^r Z^J] \tag{2.33}$$

where $r = 1...4$ are dimensions that do not rotate under J. Going through the Poincare symmetries and supersymmetries we find that all the other states of the super-multiplet can likewise be represented by the insertion of a new field or a derivative into $\mathcal{O}_{vac}$. It turns out that these inserted fields separate according to their $\Delta - J$ eigenvalue [35]. Modes contributing to the original supermultiplet are then ones with $\Delta - J = 1$ and they are: $\phi^1 ... \phi^4$, $D_i Z = \partial_i Z + [A_i, Z]$, with $i = 1..4$ and eight fermionic operators $\chi^a_{J=\frac{1}{2}}$. We can act with these generators multiple times every time turning one of the Zs into a $\Delta - J = 1$ mode.

On the gravity side, this is equivalent to multiplying a vacuum state with the zero-modes of the string oscillators $\alpha_0^{i\dagger}$ and $b_0^{j\dagger}$ thus once again affirming the correspondence between supergravity and CFT.

One of the principal contributions of [35] was to extend this correspondence to the higher "stringy" modes of the oscillators by associating the mode number n in α_n^i to a position dependent phase in such a way that, for example, $\alpha_n^{8\dagger}$ corresponds to:

$$\frac{1}{\sqrt{J}} \sum_{l=1}^{J} \frac{1}{\sqrt{J} N^{J/2+1/2}} Tr[Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \tag{2.34}$$

This, combined with the cyclicity of the trace assures that the level-matching conditions 1.26 are automatically satisfied - for example the one "impurity" states automatically disappear. More impurities can be added with the following identification:

$$\alpha^{\dagger i} \to D_i Z \quad \text{for } i = 1, \cdots, 4$$

$$\alpha^{\dagger j} \to \phi^{j-4} \quad \text{for } j = 5, \cdots, 8$$
$$\beta^a \to \chi^a_{J=\frac{1}{2}} \tag{2.35}$$

This covers the entire spectrum of the free theory on the string side.

2.1.5 Interactions in the BMN limit

Expanding the square root from 2.31 we can write:

$$(\Delta - J)_n = w_n = 1 + \frac{2\pi g_s N n^2}{J^2} + \cdots = 1 + \frac{\lambda n^2}{2J^2} + \cdots \tag{2.36}$$

where the $\lambda \sim R^4$ is the t'Hooft coupling and goes to infinity at the same rate as J^2. This suggests a new coupling $\lambda' = \frac{\lambda}{J^2}$ for the interactions between the BMN operators. Such a coupling would be a perturbative parameter in the expansion of the free string energy.

Field theoretic methods can be used to reproduce the entire square root from 2.31 [35, 47–51]. To give idea of those methods we glance briefly at the leading order calculation from [35]:

Consider the two impurity state / operator:

$$\alpha_n^{\dagger\,8}\alpha_{-n}^{\dagger\,7}|0, p_+\rangle_{l.c.} \longleftrightarrow \frac{1}{\sqrt{J}}\sum_{l=1}^{J}\frac{1}{N^{J/2+1}}Tr[\phi^3 Z^l \phi^4 Z^{J-l}]e^{\frac{2\pi i n l}{J}} = \mathcal{O}^{78} \tag{2.37}$$

and the quartic scalar interaction in the DBI action:

$$\sim g_{YM}^2 Tr([Z, \phi^j][\bar{Z}, \phi^j]) \tag{2.38}$$

If we look at the two point function $\langle \mathcal{O}^{78}(x), \mathcal{O}^{78}(0)\rangle$ the above interaction connects the term in $\mathcal{O}^{78}(x)$ with that in $\mathcal{O}^{78}(0)$ in which impurities ϕ are moved by one spot causing the change in conformal dimension:

$$\langle \mathcal{O}^{78}(x), \mathcal{O}^{78}(0)\rangle \sim \frac{1}{(x^2)^{J+2+n^2\lambda'+\cdots}} \tag{2.39}$$

Which, for $N_n = 2$ gives the appropriate leading order: $\Delta - J = 2 + n^2\lambda' + \ldots$. Details of this calculation are given in Appendix A. of [35] and are too involved to be reproduced here.

So far we have discussed one possible finite tunable coupling:

$$\frac{1}{(\mu\alpha' p^+)^2} = \frac{g_{YM}^2 N}{J^2} \equiv \lambda' \quad , \quad N, J \to \infty \tag{2.40}$$

and this is in fact the only one that is relevant in the free string theory, being connected only to the string tension. 2.5 is the equation of free string theory so the fact that it can be replicated with a perturbative expansion in λ' is not a surprise. Dimensional analysis, however, suggests that there is another possible coupling in the field theory that is also finite and tunable:

$$4\pi g_s \left(\mu\alpha' p^+\right)^2 = \frac{J^2}{N} \equiv g_2 \quad , \quad N, J \to \infty \tag{2.41}$$

This one depends on the string coupling as well and therefore - on the string side - corresponds to the full-blown interacting string theory.

The λ' coupling depends on the g_{YM} and N only through the t'Hooft coupling $g_{YM}^2 N$ and therefore the interactions that involve only that coupling correspond to the planar limit or large N t'Hooft limit of the Yang-Mills theory.

The theory involving interacting strings then corresponds to the non-planar diagrams of CFT. As mentioned in the preamble to this chapter, the planar limit has been thoroughly tested against the free string theory and their correspondence affirmed [35, 38, 39]. The non-planar limit has been studied extensively on the field theory side and a fairly reliable double expansion series in λ' and g_2 has been calculated for a number of examples [47–53]. The one that we will focus on in the remainder of this chapter describes a particular two impurity state of the string, and is given by:

$$\Delta - J = 2 + n^2\lambda' - \frac{1}{4}n^4\lambda'^2 + \frac{1}{8}n^6\lambda'^3 ... + \frac{g_2^2}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32n^2\pi^2}\right)\left(\lambda' - \frac{1}{2}\lambda'^2 n^2\right) + \dots \quad (2.42)$$

Confirming this series on the string side would be a major triumph for AdS/CFT correspondence but to this day despite the considerable effort [54–62, 65–71, 79, 80] this has not been fully accomplished.

The remainder of this chapter will focus on understanding this problem and on our own contribution towards its solution.

In order to even attempt the g_2 expansion on the string side, one needs a working theory of string interactions. For this we turn to string field theory.

2.2 String field theory on the pp-wave background

String theory as presented in chapter 1. and in the preceding sections is a first-quantized theory. Objects in the Lagrangian are not strings themselves but rather modes of oscillation within a string. This means that it is relatively difficult to construct robust theory of interacting strings from the materials we have provided so far. This is not to say that string interactions are not at all considered in the standard string theories. The idea of vertex operators uses the conformal symmetry of the strings to reduce the interaction to a point operator on the world-sheet of the interacting string [4–7]. This is a fruitful technique but ultimately a limited one and, as it happens, one that is hard to use in a light-cone gauge quantization. This makes it useless in our quest to find the interacting string correspondence to the non-planar aspects of the CFT.

The most successful alternative rests in fully second-quantizing the string by introducing the multi-string Hilbert space with operators that act to create/anihilate entire strings (contrary to the usual α and $\alpha^\dagger$ which only affect excitations on a given string).

Such a model was developed first in the 1980's by Green, Schwarz and Brink [10, 11] for the type IIB superstrings in a flat background. It was relatively dormant for two decades but got revived due to its relevance to AdS/CFT correspondence. In 2002 it was generalized by Spradlin and Volovich [54, 55, 65, 69] to a pp-wave background. In this context the energy shifts due to string interactions are equivalent to the anomalous dimension of their corresponding operators. Calculating these energy shifts exactly for the case of particular

2-impurity states was the subject of a number of papers, most notably by Pankiewitz [57, 60, 68, 70]. Our own work, which will be focus of the later sections of this chapter, continues the research from the papers listed above. Most of what will be presented in this section comes directly from one of the above references.

2.2.1 Bosonic particle in the plane wave background

Following [54, 55] we start by presenting the particle field theory on the pp-wave background and then generalize it to strings. This is a logical method because, contrary to the flat background case, there are no qualitative differences between the two cases (even a particle on pp-wave lives in a harmonic oscillator potential).

Writing again the pp-wave metric 2.4:

$$ds^2 = -2dx^+dx^- - \mu^2\vec{x}^2(dx^+)^2 + d\vec{x}^2. \tag{2.43}$$

we can write down the action of the free 'massless' field:

$$S = -\frac{1}{2}\int \partial_\mu\Phi\partial^\mu\Phi = \int dx^+dx^-dx\ \partial_+\Phi\partial_-\Phi - \int dx^+H, \tag{2.44}$$

with the free Hamiltonian

$$H = \frac{1}{2}\int dx^-dx\left[(\partial_x\Phi)^2 + \mu^2x^2(\partial_-\Phi)^2\right]. \tag{2.45}$$

To quantize this we find the canonical conjugate of the Φ, which turns out to be $\Psi = \partial_-\Phi$. We can then write:

$$[\Phi(x^-,x), \partial_-\Phi(y^-,y)] = i\delta(x^- - y^-)\delta(x-y). \tag{2.46}$$

and in the Fourier basis:

$$\Phi(x^-,x) = \frac{1}{2\pi}\int dp_-dp\ \Phi(p_-,p)e^{i(p_-x^- + px)}. \tag{2.47}$$

we get:

$$[\Phi(p^+,p), \Phi(q^+,q)] = \frac{1}{p^+}\delta(p^+ + q^+)\delta(p+q). \tag{2.48}$$

where

$$p^+ = -p_- \geq 0. \tag{2.49}$$

The latest inequality will not hold in the actual bosonic case, because it stems from the supersymmetries, but seeing as we are actually interested in generalizing to a supersymmetric case we can assume it. Since Φ is real (as a classical scalar field), the corresponding operator Φ is Hermitian, which means that

$$\Phi(p^+,p)^\dagger = \Phi(-p^+,-p). \tag{2.50}$$

The Hamiltonian can now be written as

$$H_2 = \frac{1}{2}\int dp_-dp\ \Phi^\dagger(p^2 + (\mu p^+x)^2)\Phi = \int dp^+dp\ p^+\Phi^\dagger h\Phi, \tag{2.51}$$

where h is the single-particle Hamiltonian

$$h = \frac{1}{2p^+}(p^2 + \omega^2 x^2), \qquad \omega = \mu|p^+|. \tag{2.52}$$

The single-particle Hamiltonian may be diagonalized in the standard way:

$$a = \frac{1}{\sqrt{2\omega}}(p - i\omega x), \qquad p = \sqrt{\frac{\omega}{2}}(a + a^\dagger), \qquad x = \frac{i}{\sqrt{2\omega}}(a - a^\dagger), \tag{2.53}$$

so that

$$h = e(p^+)\mu(a^\dagger a + \tfrac{1}{2}), \tag{2.54}$$

It is important to note that h is a Hamiltonian of the one particle Hilbert space and that the generators $a^\dagger$ and a create/anihilate excitations of a single particle. The states of a single particle can then be expressed as:

$$|N; p^+\rangle \equiv (a^\dagger)^N |0; p^+\rangle, \qquad N = 0, 1, \ldots. \tag{2.55}$$

In addition, there exists a multi-particle Hilbert space with operators $A_N(p^+)^\dagger / A_N(p^+)$ that create/anihilate particles in the state $|N; -p^+\rangle$. These operators satisfy $(A_N(p^+))^\dagger = A_N(-p^+)$ and

$$[A_M(p^+), A_N(q^+)] = e(p^+)\delta_{MN}\delta(p^+ + q^+). \tag{2.56}$$

Once we move to the string theoretic equivalent, the Hilbert space of the single particle (whose Hamiltonian is h) will be the world-sheet Hilbert space and the multi-particle one will become the "space-time" Hilbert space.

In the standard field-theoretic fashion we can write the expansion for Φ:

$$\Phi(p^+) = \frac{1}{\sqrt{|p^+|}} \sum_{N=0}^{\infty} |N; p^+\rangle A_N(p^+). \tag{2.57}$$

noting how it is an operator in "space-time" Hilbert space and the state in the "world-sheet" space. In this basis we can write the space-time Hamiltonian as:

$$H_2 = \int_0^\infty dp^+ \sum_{N=0}^{\infty} E_N A_N(-p^+) A_N(p^+), \qquad E_N = \mu(N + \tfrac{1}{2}). \tag{2.58}$$

An important generalization of 2.51 exists that relates symmetry generators on the "world-sheet" space with their equivalents on the "space-time":

$$G_2 = \int dp^+ dp\; p^+ \Phi^\dagger g \Phi. \tag{2.59}$$

To the above we can add interaction by introducing the cubic part of the Hamiltonian:

$$H_3 = g_s \int dx^- dx\; V, \tag{2.60}$$

where V is some cubic function of Φ and its derivatives.

In terms of modes this can be written as:

$$H_3 = \int dp_1^+ dp_2^+ dp_3^+ \delta(p_1^+ + p_2^+ + p_3^+) \sum_{N,P,Q=0}^{\infty} c_{NPQ}(p_1^+, p_2^+, p_3^+) A_N(p_1^+) A_P(p_2^+) A_Q(p_3^+). \quad (2.61)$$

here $c_{NPQ}(p_1^+, p_2^+, p_3^+)$ encode matrix elements of the interaction written in the basis of the harmonic oscillator wave-functions.

From here on we will use the convention that p_3^+ be the momentum whose sign is opposite of the remaining two. Thus index 3 will always be labeling the initial state of a splitting transition $3 \to 1 + 2$ or the final state of a joining transition $1 + 2 \to 3$.

In the Hilbert space of 3-particle states it is possible to identify the H_3 with a vertex state V:

$$|V\rangle = \sum_{N,P,Q=0}^{\infty} c_{NPQ}(p_1^+, p_2^+, p_3^+) |N; p_1^+\rangle |P; p_2^+\rangle |Q; -p_3^+\rangle. \quad (2.62)$$

The above construction fully describes the cubic interactions for this bosonic particle theory.

2.2.2 Bosonic string field theory

The analogy from the previous section is very direct. Instead of the field $\Phi(x)$ representing the particle we now have a functional $\Phi[x(\sigma)]$ of the string embedding $x(\sigma)$. Consequently the integrals over dx are replaced by functional integrals $Dx(\sigma)$. Delta functions are replaced by delta functions over all the Fourier modes of $x(\sigma)$ which are written as delta functionals $\Delta[x(\sigma)]$.

The action is then given by:

$$S = \int dx^+ dx^- Dx(\sigma)\ \partial_+ \Phi \partial_- \Phi - \int dx^+\ H, \quad (2.63)$$

where $H = H_2 + H_3 + \cdots$. The formula (2.59) is replaced by

$$G_2 = \int dp^+ Dp(\sigma)\ p^+ \Phi^\dagger g \Phi. \quad (2.64)$$

And the worldsheet Hamiltonian is then:

$$h = \frac{e(p^+)}{2} \int_0^{2\pi|p^+|} d\sigma \left[4\pi p^2 + \frac{1}{4\pi}((\partial_\sigma x)^2 + \mu^2 x^2) \right] = \frac{1}{p^+} \sum_{n=-\infty}^{\infty} \omega_n a_n^\dagger a_n, \quad (2.65)$$

where

$$\omega_n = \sqrt{n^2 + (\mu \alpha' p^+)^2} \quad (2.66)$$

which is very much in keeping with the bosonic part of the Hamiltonian on the pp-wave as discussed in the previous sections with $a^\dagger$ and a having usual stringy meanings. As before we have to impose the level-matching condition:

$$\sum_{n=-\infty}^{\infty} n N_n = 0. \quad (2.67)$$

States of the second quantized theory will now be labeled $|\vec{N}\rangle$ where the component N_n of the vector $\vec{N}$ gives the occupation number of oscillator n.

The second quantized Hilbert space $\mathcal{H}$ is then built from a vacuum $|0\rangle\rangle$, which is acted on by the operators $A_{\vec{N}}(p^+)$, which for $p^+ < 0$ create a string in the state $|\vec{N}; p^+\rangle$.

The free Hamiltonian is then:

$$H_2 = \int_0^\infty dp^+ \sum_{|\vec{N}\rangle} E_{\vec{N}} A^\dagger_{\vec{N}}(p^+) A_{\vec{N}}(p^+), \qquad E_{\vec{N}} = \frac{1}{p^+} \sum_{n=-\infty}^{\infty} \omega_n N_n \tag{2.68}$$

To introduce the interactions we need a vertex state equivalent of 2.62 in this formalism.

The main constraint on the interactions is the principle of continuity and conservation of momentum - that imply that the vertex must satisfy:

$$(p_1(\sigma) + p_2(\sigma) + p_3(\sigma))\,|V\rangle = (x_1(\sigma) + x_2(\sigma) - x_3(\sigma))|V\rangle = 0. \tag{2.69}$$

To construct such a vertex it is easiest to work in the basis of Fourier modes of one of the strings. We arbitrarily choose the string 3 and then construct the matrices $X^(r)_{mn}$ which express the Fourier basis of the string r in the basis of the string 3. These matrices are obtained by simple Fourier transforms,

$$X^{(1)}_{mn} = \frac{1}{\pi}(-1)^{m+n+1}\frac{\sin(\pi m\beta_1)}{n - m\beta_1}, \qquad X^{(2)}_{mn} = \frac{1}{\pi}(-1)^{n}\frac{\sin \pi m\beta_2}{n - m\beta_2}, \tag{2.70}$$

where $\beta_a = p^+_a/|p^+_3|$ is the ratio of the width of string a to the width of string 3. The $X^{(3)} = 1$ by definition.

We can now write the following equations that must hold for each m:

$$\sum_{r=1}^{3}\sum_{n=-\infty}^{\infty} X^{(r)}_{mn} p_{n(r)}|V\rangle = 0, \qquad \sum_{r=1}^{3}\sum_{n=-\infty}^{\infty} e(p^+_r) X^{(r)}_{mn} x_{n(r)}|V\rangle = 0. \tag{2.71}$$

Those are most easily solved if we assume an ansatz solution:

$$|V\rangle = f(p^+_1, p^+_2, p^+_3)\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} \alpha^\dagger_{m(r)} \tilde{N}^{(rs)}_{mn} \alpha^\dagger_{n(s)}\right]|0_{(1)}\rangle|0_{(2)}\rangle|0_{(3)}\rangle, \tag{2.72}$$

and expand the x's and p's appearing in (2.71) into creation and annihilation operators. Solving the resulting matrix equations we get:

$$\tilde{N}^{(rs)}_{mn} = \delta^{rs}\delta_{mn} - 2\sqrt{\omega_{m(r)}\omega_{n(s)}}(X^{(r)\mathrm{T}}\Gamma^{-1}X^{(s)})_{mn}. \tag{2.73}$$

where

$$(\Gamma_a)_{mn} = \sum_{r=1}^{3}\sum_{p=-\infty}^{\infty} \omega_{p(r)} X^{(r)}_{mp} X^{(r)}_{np}. \tag{2.74}$$

as a unique solution modulo undetermined function $f(p^+_1, p^+_2, p^+_3)$.

Exact calculation of the Neumann matrices N_{ab} is a non-trivial exercise that is done in detail in [62]. We will return to it briefly in the next section.

2.2.3 Fermionic contribution to the vertex

Generalizing the procedure listed above to the superstring involves two major considerations. We address them in this section and the following one.

First of all, the continuity/conservation requirements given by

$$(p_1(\sigma) + p_2(\sigma) + p_3(\sigma))\,|V\rangle = (x_1(\sigma) + x_2(\sigma) - x_3(\sigma))|V\rangle = 0. \tag{2.75}$$

need to be supplemented by the equivalent requirements for the fermionic coordinates θ and their canonical conjugates λ

$$(\lambda_1(\sigma) + \lambda_2(\sigma) + \lambda_3(\sigma))\,|V\rangle = (\theta_1(\sigma) + \theta_2(\sigma) - \theta_3(\sigma))|V\rangle = 0. \tag{2.76}$$

We do so by introducing an additional factor to the vertex, to be annihilated by the fermionic operators. Thus: $|V\rangle = E_a E_b|0\rangle$ where E_a is given by the

$$E_a|0\rangle \sim \exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} \alpha^{I\dagger}_{m(r)}\tilde{N}^{(rs)}_{mn}\alpha^{I\dagger}_{n(s)}\right]|0\rangle, \tag{2.77}$$

as defined above and E_b is constructed to satisfy the 2.76.

As before we can work from an ansatz:

$$E_b|0\rangle = \exp\left[\sum_{r,s=1}^{3}\sum_{m,n=0}^{\infty} \beta^{\alpha_1\alpha_2\dagger}_{-m(r)}\tilde{Q}^{(rs)}_{mn}\beta^{\dagger}_{n(s)\alpha_1\alpha_2} - \beta^{\dot{\alpha}_1\dot{\alpha}_2\dagger}_{-m(r)}\tilde{Q}^{(rs)}_{mn}\beta^{\dagger}_{n(s)\dot{\alpha}_1\dot{\alpha}_2}\right]|0\rangle, \tag{2.78}$$

Taking into account the transformations 2.19 we can write:

$$\tilde{Q}^{(rs)}_{mn} = \frac{1}{\alpha_{(r)}}V^{(r)}_m\left[\delta^{rs}\delta_{mn}\left(1 + \frac{\mu\alpha_{(r)}}{\omega_{n(r)}}\Pi\right) - 2(\overline{Y}^{(r)\mathrm{T}}\Gamma_b^{-1}Y^{(s)})_{mn}\right]V^{(s)}_n. \tag{2.79}$$

where

$$Y^{(r)}_{mn} = P^{(3)}_{|m|}X^{(r)}_{mn}P^{(r)-1}_{|n|}, \qquad \overline{Y}^{(r)}_{mn} = \alpha_{(r)}P^{(3)}_{|m|}X^{(r)}_{-m,-n}P^{(r)-1}_{|n|}$$

$$V^{(r)}_n = \sqrt{|\alpha_{(r)}|}\frac{(1+\rho_{n(r)}\Pi)^2}{c_{n(r)}(1-\rho^2_{n(r)})^{3/2}} \qquad P_n = \frac{(1-\rho_n\Pi)}{\sqrt{1-\rho_n^2}} \qquad \Gamma_b = \sum_{r=1}^{3}Y^{(r)}\overline{Y}^{(r)\mathrm{T}} \tag{2.80}$$

and ρ is defined in 2.16.

A fair bit of information is skipped in this brief overview. Most specifically the method of Gaussian integrals which originally motivates the form of ansatz that we take. Also, methods for solving the matrix equations and certain subtleties involving the zero modes of the matrices N_{mn} and Q_{mn}. For all these details we refer the reader to the [54, 55, 69] from which the results presented above are taken.

2.2.4 Interaction super-algebra and the pre-factors

When we discussed the supersymmetry group of the pp-wave we made a distinction between the generators that act on a point on the string world-sheet (local or kinematical operators)

and are thus incapable of joining or separating strings, and those which act on the whole string and thus can include string interactions. The latter, called dynamical generators , are the Hamiltonian H and the half of the supercharges, labeled Q and $\bar{Q}$ (not to be confused with Neumann matrices $\tilde{Q}$). All the dynamical generators will be of the form:

$$G = G_2 + \kappa G_3 + \kappa^2 G_4 + \cdot \tag{2.81}$$

where κ is proportional to the string coupling constant.

All the G_3 terms will have to contain the $|V\rangle$ as derived above, because the Qs have the same relevant symmetries as the Hamiltonian. However, the Qs and H also have to satisfy their own relationship within super-algebra, namely:

$$\{Q_{\alpha_1\dot{\alpha}_2}, Q_{\beta_1\dot{\beta}_2}\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H \tag{2.82}$$

Note how we can ignore the potential contribution of the J^{ij} generators to this anti-commutator because they can be factored away for a given linear combination of Qs but also will give no contribution at all to the non-zero orders in κ due to being kinematical. In fact, the equation 2.82 is the only one in the entire super-algebra that will have higher κ corrections. We will be interested in it at orders κ and κ^2.

At order κ the 2.82 can be written schematically as $\{Q_2, Q_3\} \sim H_3$. In the state-language defined above it can be written as:

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot{\alpha}_2}|Q_{3\,\beta_1\dot{\beta}_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\beta_1\dot{\beta}_2}|Q_{3\,\alpha_1\dot{\alpha}_2}\rangle = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}|H_3\rangle\,, \tag{2.83}$$

$$\sum_{r=1}^{3} Q_{(r)\,\dot{\alpha}_1\alpha_2}|Q_{3\,\dot{\beta}_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot{\beta}_1\beta_2}|Q_{3\,\dot{\alpha}_1\alpha_2}\rangle = -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}|H_3\rangle\,, \tag{2.84}$$

$$\sum_{r=1}^{3} Q_{(r)\,\alpha_1\dot{\alpha}_2}|Q_{3\,\dot{\beta}_1\beta_2}\rangle + \sum_{r=1}^{3} Q_{(r)\,\dot{\beta}_1\beta_2}|Q_{3\,\alpha_1\dot{\alpha}_2}\rangle = 0 \tag{2.85}$$

where $Q_{(r)\,\beta_1\dot{\beta}_2}$ and $Q_{(r)\,\dot{\beta}_1\beta_2}$ are the quadratic, free string supercharges Q_2 as defined in 2.28.

Simply using $|V\rangle$ for the Hamiltonian and generators Q will obviously not suffice to satisfy the super-algebra. We do, however, have a free function of the momentum floating in front of $|V\rangle$ in each of the above cases (see 2.72). We can utilize that to introduce the pre-factors to $|V\rangle$ that would differ depending on the generator:

$$|H_3\rangle = h_3|V\rangle, \qquad |Q_3^-\rangle = q_3^-|V\rangle, \qquad |\bar{Q}_3^-\rangle = \bar{q}_3^-|V\rangle \tag{2.86}$$

Actually solving for the pre-factors is complicated and the result turns out to be non-unique. The first and most widely used result is due to Pankiewitz and that is the one will we be using in the following sections. We state result here and refer the reader to [68] for the derivation:

$$
\begin{aligned}
|H_3\rangle &= g_2\, f(\mu\alpha_3\,,\, \frac{\alpha_1}{\alpha_3})\frac{\alpha'}{8\,\alpha_3^3}\Big[\big(K_i\widetilde{K}_j - \frac{\mu\kappa}{\alpha'}\delta_{ij}\big)v^{ij} - \big(K_{i'}\widetilde{K}_{j'} - \frac{\mu\kappa}{\alpha'}\delta_{i'j'}\big)v^{i'j'} \\
&\quad - K^{\dot\alpha_1\alpha_1}\widetilde{K}^{\dot\alpha_2\alpha_2}s_{\alpha_1\alpha_2}(Y)s^*_{\dot\alpha_1\dot\alpha_2}(Z) - \widetilde{K}^{\dot\alpha_1\alpha_1}K^{\dot\alpha_2\alpha_2}s^*_{\alpha_1\alpha_2}(Y)s_{\dot\alpha_1\dot\alpha_2}(Z)\Big]|V\rangle\,, \\
|Q_{3\,\beta_1\dot\beta_2}\rangle &= g_2\,\eta\, f(\mu\alpha_3\,,\, \frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad\qquad + is_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,, \\
|Q_{3\,\dot\beta_1\beta_2}\rangle &= g_2\,\bar\eta\, f(\mu\alpha_3\,,\, \frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s^*_{\gamma_1\beta_2}(Y)t^*_{\dot\beta_1\dot\gamma_1}(Z)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad\qquad + is^*_{\dot\beta_1\dot\gamma_2}(Z)t_{\beta_2\gamma_2}(Y)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,.
\end{aligned}
\tag{2.87}
$$

Where a number of additional notational definitions are in order, starting with $\kappa \equiv \alpha_1\alpha_2\alpha_3$. Further

$$
K^{\dot\gamma_1\gamma_1} \equiv K^i\sigma^{i\,\dot\gamma_1\gamma_1}\,,\quad K^{\dot\gamma_2\gamma_2} \equiv K^{i'}\sigma^{i'\,\dot\gamma_2\gamma_2}\,,\quad \widetilde{K}^{\dot\gamma_1\gamma_1} \equiv \widetilde{K}^i\sigma^{i\,\dot\gamma_1\gamma_1}\,,\quad \widetilde{K}^{\dot\gamma_2\gamma_2} \equiv \widetilde{K}^{i'}\sigma^{i'\,\dot\gamma_2\gamma_2} \tag{2.88}
$$

where the σ-matrices are given in the appendices and are used to convert between vector and spinor notation. Furthermore,

$$
\begin{aligned}
v^{ij} &= \delta^{ij}\Big[1 + \frac{1}{12}(Y^4 + Z^4) + \frac{1}{144}Y^4Z^4\Big] \\
&\quad - \frac{i}{2}\Big[Y^{2ij}(1 + \frac{1}{12}Z^4) - Z^{2ij}(1 + \frac{1}{12}Y^4)\Big] + \frac{1}{4}\big[Y^2Z^2\big]^{ij}\,, \\
v^{i'j'} &= \delta^{i'j'}\Big[1 - \frac{1}{12}(Y^4 + Z^4) + \frac{1}{144}Y^4Z^4\Big] \\
&\quad - \frac{i}{2}\Big[Y^{2i'j'}(1 - \frac{1}{12}Z^4) - Z^{2i'j'}(1 - \frac{1}{12}Y^4)\Big] + \frac{1}{4}\big[Y^2Z^2\big]^{i'j'}\,.
\end{aligned}
$$

where

$$
Y^{2ij} \equiv \sigma^{ij}_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1}\,,\quad Z^{2ij} \equiv \sigma^{ij}_{\dot\alpha_1\dot\beta_1}Z^{2\dot\alpha_1\dot\beta_1}\,,\quad \big(Y^2Z^2\big)^{ij} \equiv Y^{2k(i}Z^{2j)k} \tag{2.89}
$$

and

$$
Y^2_{\alpha_1\beta_1} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_2}_{\beta_1}\,,\qquad Y^2_{\alpha_2\beta_2} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_1}_{\beta_2} \tag{2.90}
$$

$$
Y^3_{\alpha_1\beta_2} \equiv Y^2_{\alpha_1\beta_1}Y^{\beta_1}_{\beta_2} = -Y^2_{\beta_2\alpha_2}Y^{\alpha_2}_{\alpha_1}\,, \tag{2.91}
$$

$$
Y^4_{\alpha_1\beta_1} \equiv Y^2_{\alpha_1\gamma_1}Y^{2\gamma_1}_{\beta_1} = -\frac{1}{2}\epsilon_{\alpha_1\beta_1}Y^4\,,\qquad Y^4_{\alpha_2\beta_2} \equiv Y^2_{\alpha_2\gamma_2}Y^{2\gamma_2}_{\beta_2} = \frac{1}{2}\epsilon_{\alpha_2\beta_2}Y^4 \tag{2.92}
$$

where

$$Y^4 \equiv Y^2_{\alpha_1\beta_1} Y^{2\alpha_1\beta_1} = -Y^2_{\alpha_2\beta_2} Y^{2\alpha_2\beta_2} . \tag{2.93}$$

The spinorial quantities s and t are defined as

$$s(Y) \equiv Y + \frac{i}{3} Y^3 , \qquad t(Y) \equiv \epsilon + iY^2 - \frac{1}{6} Y^4 . \tag{2.94}$$

Analogous definitions can be given for Z.

Finally:

$$K^I = \sum_{s=1}^{3} \sum_{n\in\mathbf{Z}} K_{n(s)} \alpha^{I\dagger}_{n(s)} , \quad \widetilde{K}^I = \sum_{s=1}^{3} \sum_{n\in\mathbf{Z}} K_{n(s)} \alpha^{I\dagger}_{-n(s)} \tag{2.95}$$

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3} \sum_{n\in\mathbf{Z}} G_{|n|(s)} \beta^{\dagger\,\alpha_1\alpha_2}_{n(s)} , \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3} \sum_{n\in\mathbf{Z}} G_{|n|(s)} \beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)} , \tag{2.96}$$

where:

$$K_n = +\alpha_3 \sin(n\pi r) \sqrt{\frac{r(1-r)}{\pi\alpha'}} \frac{\Lambda^-_n - \Lambda^+_n}{\sqrt{\omega_n}}, \tag{2.97}$$

$$K_q = -\alpha_3 \sqrt{\frac{r(1-r)}{\pi\alpha'\beta_r}} \frac{\Lambda^+_q - \Lambda^-_q}{2\sqrt{\omega_q}}, \tag{2.98}$$

$$G_q = \frac{1}{\sqrt{4\pi\omega_q}}, \qquad G_n = -\frac{\sin(|n|\pi r)}{\sqrt{\pi\omega_n}} \tag{2.99}$$

and

$$\Lambda^+_q = \sqrt{\omega_q - \beta_r \mu\alpha_3}, \qquad \Lambda^-_q = e(q)\sqrt{\omega_q + \beta_r \mu\alpha_3}, \tag{2.100}$$

$$\Lambda^+_n = \sqrt{\omega_n - \mu\alpha_3}, \qquad \Lambda^-_n = e(n)\sqrt{\omega_n + \mu\alpha_3}. \tag{2.101}$$

It is also important to get the exact form of the Neumann matrices 2.73 and 2.79. This was done in [62]. The key element of that calculation is inverting the infinite matrix Γ_a as defined in 2.74. This is done by finding a differential equation for the function of this inverse with respect to the mass parameter μ and then solving with the initial conditions of $\mu = 0$ corresponding to the already known flat space solution. This solution requires a particular inverse integral transform but can and has been done successfully to all orders in μ.

For our purposes, however, we are interested primarily in the match of the string theory to the gauge theory expansion in λ' that we listed earlier. Seeing as $\lambda' \sim \frac{1}{\mu}$ (2.40) what we are really interested in is the large μ limit of the Neumann matrices. By the happy coincidence in this limit matrices take a rather simple form. We present these forms here and the full ones in the Appendices. For the details of the derivation we refer the reader to the [62].

In the large μ limit:

$$\widetilde{N}^{3\,r}_{n\,q} = -\frac{\sin(n\pi r)\sqrt{\beta_r}\left(\Lambda^+_n\Lambda^+_q + \Lambda^-_n\Lambda^-_q\right)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)}, \qquad \widetilde{N}^{r\,s}_{q\,p} = \frac{\sqrt{\beta_r\beta_s}\left(\Lambda^+_q\Lambda^+_p + \Lambda^-_q\Lambda^-_p\right)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)}, \tag{2.102}$$

$$\widehat{Q}^{3\,r}_{n\,q} = \frac{i\sin(|n|\pi r)\,(\omega_q+\beta_r\omega_n)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)}, \qquad \widehat{Q}^{r\,s}_{q\,p} = \frac{i\,(\beta_s q-\beta_r p)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)} \tag{2.103}$$

where $\widehat{Q}^{s\,r}_{n\,q} = \widetilde{Q}^{s\,r}_{n\,q} - \widetilde{Q}^{r\,s}_{q\,n}$ [1].

2.2.5 The contact interaction term

Equations 2.77 to 2.102 give a description of a three-string vertex in the large μ limit (and can be complemented with the equations in the Appendix for the full μ dependency). This vertex has weight of g_2. In order to calculate the one-loop energy shift we use a world-sheet diagram containing two such vertexes. One string propagates, splits at the first vertex, two string propagate from there and rejoin at the second vertex. This means that the lowest order in g_2 such a shift would exhibit would be g_2^2, in correspondence with the 2.42.

From 2.81 we can see that the quartic Hamiltonian (as well as quartic super-symmetry generators) also carries the factor of g_2^2. Therefore, it is to be expected that the single H_4 term will contribute to the same extent to the energy shift as the two H_3 term. Such a term is called contact interaction term or contact term and physically represents the limit in which the propagators of the two strings vanish and the two vertices are brought in contact with each other.

The contact term is the Achilles' heel of the calculation we are attempting to perform. There is to this point no generic quartic vertex state equivalent to $|V\rangle$ and the conservation laws such as we used in constructing $|V\rangle$ do not seem sufficient to construct one. What understanding we do have of the contact term comes from the super-algebra.

At the second order in g_2 the $\{Q,Q\} = H$ anti-commutator can be written as:

$$\{Q_{2\,\alpha_1\dot\alpha_2}, Q_{4\,\beta_1\dot\beta_2}\} + \{Q_{4\,\alpha_1\dot\alpha_2}, Q_{2\,\beta_1\dot\beta_2}\} + \{Q_{3\,\alpha_1\dot\alpha_2}, Q_{3\,\beta_1\dot\beta_2}\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H_4\,. \tag{2.104}$$

meaning that we can calculate at least one part of the contact term - specifically $\{Q_{3\,\alpha_1\dot\alpha_2}, Q_{3\,\beta_1\dot\beta_2}\}$ - using the available components.

The existence and contribution of the Q_4 components remains one of the most important unsolved problems in the string field theory. The current state of the art has it that Q_4 is not necessary to close the algebra and can thus be set to zero without causing inconsistencies [70] but that there is no physical reason why it would have to be set to zero.

In our work we follow the convention and set Q_4 to zero for the lack of another option, resulting in the contact term vertex given by:

$$H_4 = \frac{1}{4}Q_3^{\alpha_1\dot\alpha_2}Q_{3\,\alpha_1\dot\alpha_2}. \tag{2.105}$$

[1] These expressions are also valid for $q,p=0$, except in the case of $\widetilde{N}^{rs}_{00} = -\widetilde{N}^{rs}_{qp}|_{q,p=0}$, and in the case of $\widehat{Q}^{3r}_{00} = -ir^{-1}\beta_r\widehat{Q}^{3r}_{np}|_{n,p=0}$.

Luckily, the divergence cancelations that are one of the main results we are presenting in this thesis come fully from this part of the contact term, making our work relevant despite the possible incompleteness of the equation 2.105. Nonetheless, the lack of understanding of the Q_4 term is quite possibly the greatest stumbling block in establishing the correlation between the non-planar CFT and the interacting string. We will return to it briefly in the conclusion to this chapter.

2.2.6 Impurity conserving channel

We now have the entire machinery needed to perform the energy-shift calculations. We will conclude this section by presenting the single-impurity calculation both as an example of applying the above formalism and also because it was the state of the art calculation before our work which will be presented in the next section. This calculation and a number of other important insights presented here are due to Pankiewicz and collaborators and can be found in [57, 60, 68, 70].

The first question to be addressed is the string form of the two-impurity state whose energy shift is being calculated. The result 2.42 is valid for a generic two Φ^i-impurity operator independent of the space-time index of the impurities. This is a consequence of the conformal nature of the CFT. This means we have a choice between the scalar two-oscilator states on the string side. The natural choice and one that minimizes the calculation is the state:

$$|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left(\alpha_n^{\dagger i}\alpha_{-n}^{\dagger j} + \alpha_n^{\dagger j}\alpha_{-n}^{\dagger i} - \frac{1}{2}\delta^{ij}\alpha_n^{\dagger k}\alpha_{-n}^{\dagger k}\right)|\alpha\rangle \tag{2.106}$$

Its advantage is that it is a unique representation of the state under $SO(4)\times SO(4)$ and the one that can be expressed solely by the bosonic oscillators. It is fairly easy to see that while it is possible to write the fermionic representation of the state

$$|[\mathbf{1},\mathbf{1}]\rangle = \alpha_n^{\dagger k}\alpha_{-n}^{\dagger k}|\alpha\rangle \tag{2.107}$$

in the form:

$$|[\mathbf{1},\mathbf{1}]\rangle = \beta^\dagger_{n\,\alpha_1\alpha_2}\beta^{\dagger\,\alpha_1\alpha_2}_{-n}|\alpha\rangle \tag{2.108}$$

Fermi statistics actually prohibit such a dual representation for $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$. Using the later, therefore, prevents the mixing of states as we turn on the interactions, making for the much easier calculation of the energy shift.

The actual calculation of the shift starts from the standard perturbation theory:

$$\delta E_n^{(2)} = \langle[\mathbf{9},\mathbf{1}]|H_3\frac{\mathcal{P}}{E_n^{(0)} - H_2^{\text{int}}}H_3|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} + \langle[\mathbf{9},\mathbf{1}]|H_4|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} \tag{2.109}$$

where the $E_n^{(0)}$ is the free energy of the string, $\mathcal{P}$ is the projection operator on the space of the two-string states and H_2^{int} is a free Hamiltonian acting on the internal strings.

At this stage, it was customary to restrict the definition of $\mathcal{P}$ to the two string states which between them carry only oscillators. This was justified by the argument from gauge theory in which the number of impurities is in fact conserved. The principal reason it was actually

done though, was that the "impurity non-conserving" channels appeared to result in the non-perturbative terms and unphysical divergences. In the next section we will be addressing the non-impurity conserving channels and dealing with most of the problems associated with them. For now we use the impurity conserving channel and write the equation:

$$(1+\delta^{ij})\,\delta E_n^{(2)} = {}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|H_3\frac{\mathbf{1}_B}{E_n^{(0)}-H_2^{\text{int}}}H_3|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}+\frac{1}{4}{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|Q_3^\dagger\,\mathbf{1}_F\,Q_3|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} \quad (2.110)$$

with:

$$\begin{aligned}
\mathbf{1}_B &= \sum_{K,L=1}^{8}\int_0^1\frac{dr}{2\,r(1-r)}\Big(\sum_p \alpha_p^{\dagger K}\,\alpha_{-p}^{\dagger L}\,|\alpha_1\rangle\,|\alpha_2\rangle\langle\alpha_2|\,\langle\alpha_1|\,\alpha_{-p}^L\,\alpha_p^K \\
&\qquad\qquad + \alpha_0^{\dagger K}\,|\alpha_1\rangle\,\alpha_0^{\dagger L}\,|\alpha_2\rangle\langle\alpha_2|\,\alpha_0^L\langle\alpha_1|\,\alpha_0^K\Big) \\
\mathbf{1}_F &= \sum_{\Sigma_1,\Sigma_2}\int_0^1\frac{dr}{r(1-r)}\Big(\sum_p \alpha_p^{\dagger K}\,\beta_{-p}^{\dagger\,\Sigma_1\,\Sigma_2}\,|\alpha_1\rangle\,|\alpha_2\rangle\langle\alpha_2|\,\langle\alpha_1|\,\beta_{-p}^{\Sigma_1\,\Sigma_2}\,\alpha_p^K \\
&\qquad\qquad + \alpha_0^{\dagger K}\,|\alpha_1\rangle\,\beta_0^{\dagger\,\Sigma_1\,\Sigma_2}\,|\alpha_2\rangle\langle\alpha_2|\,\beta_0^{\Sigma_1\,\Sigma_2}\,\langle\alpha_1|\,\alpha_0^K\Big).
\end{aligned} \quad (2.111)$$

where Σ_1 and Σ_2 are a generalization of spinor indices to include both dotted and undotted ones. This distribution of the oscillators is a product of the level-matching condition, something that will prove important when we begin looking at the orbifolded backgrounds where the discrete quantization changes the level-matching.

Furthermore we note the normalization of the string vacuua:

$$\langle\alpha_1|\langle\alpha_2|\alpha_2\rangle|\alpha_1\rangle = r(1-r), \qquad \langle\alpha_3|\alpha_3\rangle = 1. \quad (2.112)$$

where, once again, $r \equiv -\alpha_1/\alpha_3$ and $1-r \equiv -\alpha_2/\alpha_3$.

In general terms there is also a double fermion term in $\mathbf{1}_B$ but it is irrelevant for traceless state such as $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ due to Fermi statistics.

We then need the matrix elements between the internal string states and $|H_r\rangle/|Q_3\rangle$. The calculation of those is relatively straightforward and the results are given by [70]:

$$\begin{aligned}
{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\alpha_0^l\,\langle\alpha_1|\,\alpha_0^k\,|H_3\rangle &= -2\,r(1-r)\Big(\frac{\omega_{n(3)}}{\alpha_3}+\mu\Big)\,\widetilde{N}_{n,0}^{31}\,\widetilde{N}_{n,0}^{32}\,\Delta^{ijkl} \\
{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\langle\alpha_1|\,\alpha_{-p}^l\,\alpha_p^k\,|H_3\rangle &= -2\,r(1-r)\Big(\frac{\omega_{n(3)}}{\alpha_3}-\frac{\omega_{p(1)}}{\alpha_3 r}\Big)\,\widetilde{N}_{n,p}^{31}\,\widetilde{N}_{n,-p}^{31}\,\Delta^{ijkl}
\end{aligned} \quad (2.113)$$

and

$$\begin{aligned}
&{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,(\beta_0)^{\dot\sigma_1\dot\sigma_2}\,\langle\alpha_1|\,\alpha_0^k\,|Q_{3\,\beta_1\dot\beta_2}\rangle = \\
&\qquad -2i\,\bar{C}\,G_{0(2)}\,\big(K_{n(3)}+K_{-n(3)}\big)\,\widetilde{N}_{n,0}^{31}\,\Delta^{ijkl}(\sigma^l)_{\beta_1}^{\dot\sigma_1}\,\delta_{\dot\beta_2}^{\dot\sigma_2} \\
&{}^{(ij)}\langle[\mathbf{9},\mathbf{1}]|\,\langle\alpha_2|\,\langle\alpha_1|\,(\beta_{-p})^{\dot\sigma_1\dot\sigma_2}\,\alpha_p^k\,|Q_{3\,\beta_1\dot\beta_2}\rangle = \\
&\qquad -2i\,\bar{C}\,G_{|p|(1)}\,\big(K_{n(3)}\widetilde{N}_{n,p}^{31}+K_{-n(3)}\widetilde{N}_{n,-p}^{31}\big)\,\Delta^{ijkl}(\sigma^l)_{\beta_1}^{\dot\sigma_1}\,\delta_{\dot\beta_2}^{\dot\sigma_2}
\end{aligned} \quad (2.114)$$

where $\Delta^{ijkl} \equiv \frac{1}{\sqrt{2}}\{\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac{1}{2}\delta^{ij}\delta^{kl}\}$ and $\bar{C} \equiv \frac{\bar{\eta}}{4}\sqrt{-\frac{\alpha'}{2\alpha_3^3}}\sqrt{r(1-r)}$. Some of the intermediate steps of this calculation are presented in the Appendices.

The energy denominator is given by:

$$\frac{-\alpha_3}{2\left(\omega_n - r^{-1}\omega_p\right)} \tag{2.115}$$

We have now all the elements from the equation 2.110. All that is left is the conceptually simple, although technically challenging task of performing a sum over the mode number p. This is accomplished through the technique of contour integrals whereby the the sum is calculated by the following substitution:

$$\sum_{p=-\infty}^{\infty} f(p) = -\frac{i}{2}\oint dz\, f(z)\, \cot(\pi z). \tag{2.116}$$

We then rotate and scale the integration variable through the substitution $z \to -i\mu\alpha_3 r z$, to turn the cotangent into $\coth(\pi\mu\alpha_1 z)$ which can be set to one in the large μ limit[2]. If the summand $f(z)$ has no poles on the real axis, the procedure simply replaces p by $p' = r\mu\alpha_3 p$ and integrates

$$\sum_{p=-\infty}^{\infty} f(p) = \int_{-\infty}^{\infty} dp' f(p') \tag{2.117}$$

yielding the large μ behaviour. If there are poles on the real axis, one must evaluate their residue using the integrand in (2.116) and then integrate along any cut which $f(z)$ may possess along the imaginary axis.

Specifically, the sums in our calculation will fall into two general forms:

$$F_1 = \sum_p \frac{P(p)}{Q(p)\sqrt{p^2 + (r\mu\alpha_3)^2}}, \qquad F_2 = \sum_p \frac{P(p)}{Q(p)} \tag{2.118}$$

where $P(p)$ and $Q(p)$ are polynomials in p. F_1 will therefore have both poles and a cut and F_2 just the poles. generically then, result will be:

$$\begin{aligned} F_1 = -\pi \sum_i \mathrm{Res}\left(\cot(\pi p_i)\frac{P(p_i)}{Q(p_i)}, p_i \in \{p \,|\, Q(p) = 0\}\right) \\ + \int_1^{\infty} dz \frac{[Q(ixrz)]^* P(ixrz) + \text{c.c.}}{|Q(ixrz)|^2 \sqrt{z^2-1}} \end{aligned} \tag{2.119}$$

$$F_2 = -\pi \sum_j \mathrm{Res}\left(\cot(\pi p_j)\frac{P(p_j)}{Q(p_j)}, p_j \in \{p \,|\, Q(p) = 0\}\right) \tag{2.120}$$

Obtaining the result is then just a matter of careful computation. The original result from [70] turns out to have a calculation error that was corrected in our paper [80].

[2]The terms neglected by this approximation are of order $\exp(-\mu|\alpha_3|)$.

The correct result is given by:

$$\begin{aligned}\delta E_n^{(2)} = \frac{g_2^2\mu}{4\pi^2}\Bigg[&\left(\frac{1}{24}+\frac{65}{64\pi^2n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} \\ &- n^2\left(\frac{1}{48}+\frac{89}{128\pi^2n^2}\right)\lambda'^2 - \frac{9n^2}{32}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{5/2} \\ &+ n^4\left(\frac{1}{64}+\frac{339}{512\pi^2n^2}\right)\lambda'^3 + n^4\left(\frac{59}{160\pi^2}+\frac{45}{256\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg]\end{aligned} \tag{2.121}$$

There are notable similarities between the result 2.121 and 2.42 but also some rather dramatic discrepancies. Most importantly, the half power terms in λ' appear to be native to string theory results and yet absent in principle from their gauge theory counterparts. Also, despite notable similarity, the actual numeric terms differed even in the full powers of λ'. Our own work, presented in the following sections was motivated in large part by the desire to reconcile these discrepancies.

2.3 Higher impurity channels

Two major assumptions were made on the string side to get the result 2.121. One was that the Q_4 supercharge can be ignored for which there is necessary, but by no means sufficient, justification. The second was that the only channel contributing to the energy shift will be one where the intermediate states have the same number of oscillators as the end states - so called "impurity conserving" channel. We could not do much about the first assumption but were interested in testing the second.

One of the primary motivations for excluding the higher impurity channels was that the contribution of the four impurity channel appeared to diverge if the large μ limit was taken before summing over the mode numbers. On the other hand, if the sum is taken before the limit then the divergence is regularized but the non-perturbative term $\sqrt{\lambda'}$ appeared in the energy shift causing the result to be fundamentally different to the leading order from the gauge-theory result.

In our paper [79] we analyze the relationship between the potential divergencies and the non-perturbative terms and then show that divergent terms actually cancel between the H_3 and the contact term, taking with them the $\sqrt{\lambda'}$ contributions. We generalize this to all impurity channels.

This section draws heavily on the work of the author and his collaborators in [79] and a substantial part of it is taken directly from that paper.

2.3.1 Trace state

The simplest example of the behavior we are interested in can be observed in the careful calculation of the two-impurity channel contribution to the mass shift of the normalized bosonic trace state

$$|[\mathbf{1},\mathbf{1}]\rangle = \frac{1}{2}\alpha_n^{i\dagger}\alpha_{-n}^{i\dagger}|\alpha\rangle \tag{2.122}$$

In [66], this calculation was performed by taking the large μ limit first, then summing over the mode numbers. That procedure found a finite result. If μ is kept finite, however, there are logarithmically divergent summations which must be dealt with before the large μ limit is taken.

We calculate the following matrix element for the state $|[\mathbf{1},\mathbf{1}]\rangle$

$$\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_p^K\alpha_{-p}^L|H_3\rangle = -g_2\,\frac{r\,(1-r)}{8}\left[8\left(\frac{\omega_n^{(3)}}{\alpha_3}+\frac{\omega_p^{(1)}}{\alpha_1}\right)\widetilde{N}^{31}_{-np}\widetilde{N}^{31}_{np}\,\delta^{kl}\right.$$
$$\left.+16\,\frac{\omega_n^{(3)}}{\alpha_3}\widetilde{N}^{33}_{nn}\widetilde{N}^{11}_{p-p}\,\delta^{KL}+16\,\frac{\omega_p^{(1)}}{\alpha_1}\widetilde{N}^{33}_{n-n}\widetilde{N}^{11}_{pp}\,\Pi^{KL}\right] \tag{2.123}$$

where the index $i=1,\ldots,4$ is summed over. Note that $K,L=1,\ldots,8$, while δ^{kl} is non-zero only for $k=l=1,\ldots,4$. The matrix Π^{KL} is given by

$$\Pi^{KL}=\mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$$

When calculating the H_3 contribution to the mass shift it is only the very last term in (2.123) which is divergent. Singling-out its contribution, one finds (using the two impurity channel as defined in 2.110 and 2.111):

$$\delta E^{\mathrm{div}}_{H_3}=\int_0^1 dr\,\left(g_2\frac{r(1-r)}{8}\right)^2\frac{-\alpha_3}{2\,r\,(1-r)}\sum_{KL}\sum_{p=-\infty}^{\infty}\frac{\left[16\,\frac{\omega_p}{-r\,\alpha_3}\widetilde{N}^{33}_{n-n}\,\widetilde{N}^{11}_{pp}\,\Pi^{KL}\right]^2}{2\,\omega_n-2\,r^{-1}\omega_p} \tag{2.124}$$

A quick inspection of the forms of the Neumann matrices (see Appendix B.) reveals that the numerator in (2.124) goes like a constant for large $|p|$, and thus the sum as a whole goes like $1/|p|$ for $|p|\gg|\mu\alpha_3|$. This is a logarithmically diverging sum. In [66] the strict large μ limit was taken for the energy denominator, leading to a convergent $1/p^2$ behavior instead. Here we will stick with the finite μ expressions and show that the divergence is removed by the contact term. Note that a double fermionic impurity intermediate state also contributes to the H_3 piece, however it does not display any divergent behavior. Further, the $\alpha_0^\dagger|\alpha_1\rangle\alpha_0^\dagger|\alpha_2\rangle$ intermediate state is unimportant to us as it does not contain the sum over mode numbers.

The contribution from the contact term stems from the following matrix element,

$$\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1}\langle\alpha_3|\frac{1}{2}\alpha_n^i\alpha_{-n}^i\,\langle\widetilde{\alpha}_2|\langle\widetilde{\alpha}_1|\alpha_p^K\beta_{-p}^{\Sigma_1\,\Sigma_2}|Q_{3\,\beta_1\dot{\beta}_2}\rangle$$
$$=\left(G^{(1)}_{|p|}\,K^{(3)}_n\widetilde{N}^{31}_{np}+G^{(1)}_{|p|}\,K^{(3)}_{-n}\widetilde{N}^{31}_{-np}\right)(\sigma^k)^{\dot{\sigma}_1}_{\beta_1}\delta^{\dot{\sigma}_2}_{\dot{\beta}_2}+4\,G^{(1)}_{|p|}\,K^{(1)}_{-p}\widetilde{N}^{33}_{n-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta} \tag{2.125}$$

along with a similar element with $|Q_{3\,\dot{\beta}_1\beta_2}\rangle$. Here $K=1,\ldots,8$ while the Σ and β indices are either dotted or undotted as required by the particular SO(4) representation indicated by K.

The last term in (2.125) gives rise to a log-divergent sum. For large positive p, $(K^{(1)}_{-p})^2$ goes as a constant, and so the sum is controlled by $(G^{(1)}_{|p|})^2$ which goes as $1/p$, and hence

diverges logarithmically. For p negative, the sum converges. Thus, the divergent contribution to $\delta E^{(2)}$ is found to be:

$$\delta E_{H_4}^{\text{div}} = 8 \int_0^1 dr \left(g_2 \frac{1}{4} \sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}} \right)^2 \frac{1}{r(1-r)} \sum_{p=1}^{\infty} \left(4\, G_{|p|}^{(1)}\, K_{-p}^{(1)} \widetilde{N}_{n\,-n}^{3\,3} \right)^2 \tag{2.126}$$

The leading factor of 8 comes from the sum over K. Note that two factors of 2 from the delta function (in Pauli indices) and the (squared) Pauli matrix trace cancel the two factors of 1/8 coming from the two terms of the contact term, $Q_{3\beta_1\dot{\beta}_2} Q_3^{\beta_1\dot{\beta}_2}$ and $Q_{3\dot{\beta}_1\beta_2} Q_3^{\dot{\beta}_1\beta_2}$. Again the intermediate state $\alpha_0^\dagger|\alpha_1\rangle\beta_0^\dagger|\alpha_2\rangle$ is unimportant to the convergence and is ignored here.

In taking the large p limits of the summands in (2.124) and (2.126), one finds,

$$\delta E_{H_3}^{\text{div}} \sim -\frac{1}{2} \int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2} \left(\widetilde{N}_{n\,-n}^{3\,3} \right)^2 \frac{1}{|p|} \tag{2.127}$$

$$\delta E_{H_4}^{\text{div}} \sim +\int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2} \left(\widetilde{N}_{n\,-n}^{3\,3} \right)^2 \frac{1}{p} \tag{2.128}$$

Noting that in the H_3 contribution the divergence is found for both positive and negative p, while in the H_4 contribution the divergence occurs only for positive p, and hence a relative factor of 2 is induced in the H_3 term, one sees that the logarithmically divergent sums cancel identically between H_3 and contact terms, leaving a convergent sum.

This cancellation fixes the relative weight of H_3 and contact terms to that employed in [70]. It differs by a factor of 1/2 from the weight originally given in [65], where it was argued to be a reflection symmetry factor.

This last statement is important because the purported factor of 1/2 provided for the leading and sub-leading order agreement between the two-impurity $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ energy shift and the gauge-theory. Despite this inviting coincidence, the above argument, together with the reasoning in [70] seems to exclude the possibility of such a factor.

2.3.2 Four impurity channel

We now consider the mass shift of the $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ string state due to intermediate states which contain four impurities.

In the explicit expression for the matrix element quoted below, we see that the parameter $\mu\alpha_3$ occurs only in combinations involving ω_p and there is a duality between the large p and the large $\mu\alpha_3$ limits. Therefore, since a logarithmic divergence in the sums indicates that the summands have as many (inverse) powers of the summation variables as there are summation variables, this translates into a vanishing $\mu\alpha_3$ dependence for this contribution to $\delta E^{(2)}$, leaving $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$. It is thus seen that $\sqrt{\lambda'}$ behavior is simply the result of log divergences, which should, if the pp-wave light-cone string field theory is to make any sense, cancel out entirely.

We begin with the H_3 contribution to the mass shift. We consider the following intermediate state,

$$\mathbf{1}_B = \int_0^1 \frac{dr}{4!\, r(1-r)} \sum_{p_1\, p_2\, p_3\, p_4} \alpha^{\dagger K}_{p_1}\, \alpha^{\dagger L}_{p_2}\, \alpha^{\dagger M}_{p_3}\, \alpha^{\dagger N}_{p_4}\, |\widetilde{\alpha}_1\rangle\, |\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\, \langle\widetilde{\alpha}_1|\, \alpha^N_{p_4}\, \alpha^M_{p_3}\, \alpha^L_{p_2}\, \alpha^K_{p_1} \tag{2.129}$$

where the sum over the mode numbers is restricted by the level matching condition $\sum_i p_i = 0$ and $\widetilde{\alpha}_1 \equiv -\alpha_3\, r$, $\widetilde{\alpha}_2 \equiv -\alpha_3(1-r)$.

Although there are many possible contractions of this state with the oscillators in $|H_3\rangle$, we will only be concerned with those which lead to log divergent sums. These are the ones where the $\alpha^\dagger$ in the prefactor of $|H_3\rangle$ contracts with one of the oscillators in $\mathbf{1}_B$. We find this contribution to $\delta E^{(2)}$ to be[3],

$$\delta E^{\text{div}}_{H_3} = \int_0^1 \frac{dr}{4!\, r\,(1-r)} \left(g_2 \frac{r(1-r)}{4}\right)^2 \sum_{p_2\, p_3\, p_4} \frac{-\alpha_3\, r}{2\,\omega_n\, r - \sum_{i=1}^4 \omega_{p_i}} \times$$
$$\left(2\,\frac{\omega_{p_1}+\omega_{p_2}}{-r\,\alpha_3}\,\widetilde{N}^{1\,1}_{-p_1\,p_2}\right)^2 \left\{8\cdot 12\,\left(\widetilde{N}^{3\,1}_{n\,p_3}\widetilde{N}^{3\,1}_{-n\,p_4}\right)^2 + 6\,\widetilde{N}^{3\,1}_{n\,p_3}\widetilde{N}^{3\,1}_{-n\,p_3}\widetilde{N}^{3\,1}_{n\,p_4}\widetilde{N}^{3\,1}_{-n\,p_4}\right\} \tag{2.130}$$

where $p_1 = -(p_2+p_3+p_4)$. The factors of 6 and 12 are combinatoric and count the number of ways equivalent contractions can be made. The factor of 8 comes from a sum over the spacetime indices of $\mathbf{1}_B$ and only affects squared terms. It is easy to see that in the above, the sum over p_2 is log divergent. In fact, it is the very same form as appears in (2.124).

The matrices with one leg on the external string have a common form:

$$\widetilde{N}^{3\,r}_{n\,p} \simeq e(n)\,\frac{\sin(|n|\pi r)}{2\pi\sqrt{\omega^{(3)}_n\,\omega^{(r)}_p}}\,\frac{\left(\omega^{(r)}_p + \beta_r\,\omega^{(3)}_n\right)}{p-\beta_r\,n} \tag{2.131}$$

$$\widetilde{Q}^{3\,r}_{n\,p} - \widetilde{Q}^{r\,3}_{p\,n} = -i\,\frac{\sin(|n|\pi r)}{2\pi\sqrt{\omega^{(3)}_n\,\omega^{(r)}_p}}\,\frac{\left(\omega^{(r)}_p + \beta_r\,\omega^{(3)}_n\right)}{p-\beta_r\,n} \tag{2.132}$$

The sums over mode numbers involved with these matrices will be dominated by the poles $p = \beta_r n$ and will be of the order zero in μ. We are therefore back exactly to the situation described in (2.124) as far as the power-counting of μ is concerned. The remaining summand, p_2 is replaced by $z = \mu|a_3|p'$ and the integral is taken over p' as per the instructions for the contour integral technique given above. μ dependance then disappears from the square of the matrix $\widetilde{N}^{11}_{-p_1 p_2}$ and the energy denominator cancels out the measure of the integration giving: $\delta E^{\text{div}}_{H_3} \sim$ constant, and therefore $\delta E^{(2)}/\mu \sim \sqrt{\lambda'}$.

There are also contributions from intermediate states which contain two bosonic and two fermionic impurities, however these produce convergent sums and $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$.

[3]The normalization $1 + \frac{1}{2}\delta^{ij}$ of the external state has been suppressed here.

We now show that the contact term contribution stemming from the following intermediate state,

$$\mathbf{1}_F = \int_0^1 \frac{dr}{3!\,r(1-r)} \sum_{p_1\,p_2\,p_3\,p_4} \beta^{\dagger\,a}_{p_1}\,\alpha^{\dagger\,L}_{p_2}\,\alpha^{\dagger\,M}_{p_3}\,\alpha^{\dagger\,N}_{p_4}\,|\widetilde{\alpha}_1\rangle\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\langle\widetilde{\alpha}_1|\,\alpha^N_{p_4}\,\alpha^M_{p_3}\,\alpha^L_{p_2}\,\beta^a_{p_1} \tag{2.133}$$

cancels the divergent piece coming from the H_3 contribution, leaving an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. In the above a is an SO(8) index and thus represents both dotted and undotted indices in the language of [70]. The log divergent piece comes from contractions where the $\alpha^\dagger$ in the prefactor of $|Q_3\rangle$ is joined with one of the bosonic oscillators in $\mathbf{1}_F$. One finds,

$$\begin{aligned}\delta E^{\text{div}}_{H_4} &= \int_0^1 \frac{dr}{3!\,r\,(1-r)} \left(g_2 \frac{1}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^2 \sum_{p_2\,p_3\,p_4} (2\,G_{p_1} K_{-p_2})^2 \\ &\quad \times \left\{8\cdot 6\,\left(\widetilde{N}^{3\,1}_{n\,p_3}\widetilde{N}^{3\,1}_{-n\,p_4}\right)^2 + 3\,\widetilde{N}^{3\,1}_{n\,p_3}\widetilde{N}^{3\,1}_{-n\,p_3}\widetilde{N}^{3\,1}_{n\,p_4}\widetilde{N}^{3\,1}_{-n\,p_4}\right\}\end{aligned} \tag{2.134}$$

In the above one sees the very same pattern as was seen in section 2.3.1. The sum over p_2 is divergent on the positive side, and cancels the divergence in (2.130). The remaining (convergent) expression gives an $\mathcal{O}(\lambda')$ contribution to $\delta E^{(2)}/\mu$. Again, there is a non-divergent contribution from the intermediate state with three fermionic and one bosonic impurity which is not considered here.

The cancellation exposed here is also found for the following remaining pairs of intermediate states,

$$\begin{aligned}\mathbf{1}_B &= \int_0^1 \frac{dr}{3!\,r(1-r)} \sum_{p_1\,p_2\,p_3} \alpha^{\dagger\,K}_{p_1}\,\alpha^{\dagger\,L}_{p_2}\,\alpha^{\dagger\,M}_{p_3}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,N}_0\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\alpha^N_0\,\langle\widetilde{\alpha}_1|\,\alpha^M_{p_3}\,\alpha^L_{p_2}\,\alpha^K_{p_1} \\ \mathbf{1}_F &= \int_0^1 \frac{dr}{2!\,r(1-r)} \sum_{p_1\,p_2\,p_3} \beta^{\dagger\,a}_{p_1}\,\alpha^{\dagger\,L}_{p_2}\,\alpha^{\dagger\,M}_{p_3}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,N}_0\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\alpha^N_0\,\langle\widetilde{\alpha}_1|\,\alpha^M_{p_3}\,\alpha^L_{p_2}\,\beta^a_{p_1}\end{aligned} \tag{2.135}$$

where $\sum_{i=1}^3 p_i = 0$ and,

$$\begin{aligned}\mathbf{1}_B &= \int_0^1 \frac{dr}{2\cdot(2!)^2\,r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\,\alpha^{\dagger\,L}_{-p_1}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,M}_{p_2}\,\alpha^{\dagger\,N}_{-p_2}\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\alpha^N_{-p_2}\,\alpha^M_{p_2}\,\langle\widetilde{\alpha}_1|\,\alpha^L_{-p_1}\,\alpha^K_{p_1} \\ \mathbf{1}_F &= \int_0^1 \frac{dr}{2!\,r(1-r)} \sum_{p_1\,p_2} \alpha^{\dagger\,K}_{p_1}\,\alpha^{\dagger\,L}_{-p_1}\,|\widetilde{\alpha}_1\rangle\,\alpha^{\dagger\,M}_{p_2}\,\beta^{\dagger\,a}_{-p_2}\,|\widetilde{\alpha}_2\rangle\langle\widetilde{\alpha}_2|\,\beta^a_{-p_2}\,\alpha^M_{p_2}\,\langle\widetilde{\alpha}_1|\,\alpha^L_{-p_1}\,\alpha^K_{p_1}\end{aligned} \tag{2.136}$$

and so we find that the entire contribution to $\delta E^{(2)}/\mu$ from the four impurity channel is convergent / leads as λ'. It is not hard to generalize the above argument to $\mathbf{1}_B$'s containing an arbitrary number of bosonic impurities and no fermionic impurities. The divergent expressions cancel against contact interactions with $\mathbf{1}_F$'s containing one fermionic and the same number (less-one) of bosonic oscillators as $\mathbf{1}_B$. Adding fermionic impurities is far less trivial because the full forms [68] of $|H_3\rangle$ and $|Q_3\rangle$, given in Appendix B. must be used for the calculation[4]. In the next section, however, a more elegant argument is presented which claims the absence of log divergences for arbitrary impurity intermediate states.

[4]We remind the reader that the $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ state receives no contributions to its energy shift from the zero impurity channel.

2.3.3 Generalizing to arbitrary impurities

It is possible to formally manipulate the contact term in such a way that the H_3 portion of the energy shift is cancelled entirely, leaving a convergent expression, which does not contain any $\sqrt{\lambda'}$ contributions to $\delta E^{(2)}/\mu$. The manipulation proceeds through supersymmetry algebra.

At order g_2 we have

$$\begin{aligned}\left\{Q_{2\alpha_1\dot{\alpha}_2}, Q_{3\beta_1\dot{\beta}_2}\right\} + \left\{Q_{3\alpha_1\dot{\alpha}_2}, Q_{2\beta_1\dot{\beta}_2}\right\} &= -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H_3\ ,\\ \left\{Q_{2\dot{\alpha}_1\alpha_2}, Q_{3\dot{\beta}_1\beta_2}\right\} + \left\{Q_{3\dot{\alpha}_1\alpha_2}, Q_{2\dot{\beta}_1\beta_2}\right\} &= -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}H_3\end{aligned} \tag{2.137}$$

analogously to order g_2^2 one has

$$\begin{aligned}\left\{Q_{3\alpha_1\dot{\alpha}_2}, Q_{3\beta_1\dot{\beta}_2}\right\} + \left\{Q_{2\alpha_1\dot{\alpha}_2}, Q_{4\beta_1\dot{\beta}_2}\right\} + \left\{Q_{4\alpha_1\dot{\alpha}_2}, Q_{2\beta_1\dot{\beta}_2}\right\} &= -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H_4,\\ \left\{Q_{3\dot{\alpha}_1\alpha_2}, Q_{3\dot{\beta}_1\beta_2}\right\} + \left\{Q_{2\dot{\alpha}_1\alpha_2}, Q_{4\dot{\beta}_1\beta_2}\right\} + \left\{Q_{4\dot{\alpha}_1\alpha_2}, Q_{2\dot{\beta}_1\beta_2}\right\} &= -2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}H_4.\end{aligned} \tag{2.138}$$

To get H_3 and H_4 the first of the equations in both (2.137) and (2.138) should be multiplied by $\epsilon^{\alpha_1\beta_1}\epsilon^{\dot{\beta}_2\dot{\alpha}_2}$ and the second by $\epsilon^{\dot{\alpha}_1\dot{\beta}_1}\epsilon^{\beta_2\alpha_2}$. On the left hand sides of the equations the epsilons just raise indices, on the right hand sides they give -4. We thus have:

$$\left\{Q_{2\beta_1\dot{\beta}_2}, Q_3^{\beta_1\dot{\beta}_2}\right\} = +4H_3, \quad \left\{Q_{2\dot{\beta}_1\beta_2}, Q_3^{\dot{\beta}_1\beta_2}\right\} = +4H_3 \tag{2.139}$$

and

$$\begin{aligned}H_4 = \frac{1}{8}Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2} + \frac{1}{8}Q_{4\beta_1\dot{\beta}_2}Q_2^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{4\dot{\beta}_1\beta_2}Q_2^{\dot{\beta}_1\beta_2}\\ + \frac{1}{8}Q_{2\beta_1\dot{\beta}_2}Q_4^{\beta_1\dot{\beta}_2} + \frac{1}{8}Q_{2\dot{\beta}_1\beta_2}Q_4^{\dot{\beta}_1\beta_2}.\end{aligned} \tag{2.140}$$

Using these formula, the contribution of H_4 to $\delta E^{(2)}$ can be rewritten as a sum of a term which cancels the H_3 contribution plus other pieces which all contain Q_2 acting on one of the external states. Taking the expectation value of part of (2.140), and introducing P as a representation of unity, we have

$$\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2} + Q_{3\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle = \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}P\frac{E_0 - H_2}{E_0 - H_2}Q_3^{\beta_1\dot{\beta}_2}\right\rangle \tag{2.141}$$

$$+\frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}P\frac{E_0 - H_2}{E_0 - H_2}Q_3^{\dot{\beta}_1\beta_2}\right\rangle. \tag{2.142}$$

It could be that the energy denominator which we have introduced here will have a zero. In that case, the projector P is a reminder to define the singularity using a principle value prescription[5]. Equation (2.141) can be written as

[5]There is one additional subtlety, the intermediate states must each obey the level-matching condition. This condition can be enforced by inserting a projection operator. For example, for two-string intermediate

$$= -\frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_2, Q_3^{\beta_1\dot{\beta}_2}\right]\right\rangle - \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_2, Q_3^{\dot{\beta}_1\beta_2}\right]\right\rangle \tag{2.145}$$

Up to order g_2 the following equation holds

$$\left[H_2, Q_3^{\beta_1\dot{\beta}_2}\right] = \left[Q_2^{\beta_1\dot{\beta}_2}, H_3\right] \tag{2.146}$$

so that (2.145) becomes

$$= \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}\left[H_3, Q_2^{\beta_1\dot{\beta}_2}\right]\right\rangle + \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}\left[H_3, Q_2^{\dot{\beta}_1\beta_2}\right]\right\rangle. \tag{2.147}$$

Since Q_2 commutes with H_2 one has

$$\begin{aligned} = \; &+ \frac{1}{8}\left\langle Q_{2\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3\right\rangle + \frac{1}{8}\left\langle Q_{2\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle \\ &+ \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3Q_2^{\beta_1\dot{\beta}_2}\right\rangle + \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\dot{\beta}_1\beta_2}\right\rangle \\ &- \left\langle H_3\frac{P}{E_0-H_2}H_3\right\rangle \end{aligned} \tag{2.148}$$

and the last term cancels the H_3 contribution to the energy shift. The final expression for the energy shift is

$$\begin{aligned} \delta E^{(2)} = \; &+ \frac{1}{8}\left\langle Q_{2\beta_1\dot{\beta}_2}Q_3^{\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3\right\rangle + \frac{1}{8}\left\langle Q_{2\dot{\beta}_1\beta_2}Q_3^{\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3\right\rangle \\ &+ \frac{1}{8}\left\langle Q_{3\beta_1\dot{\beta}_2}\frac{P}{E_0-H_2}H_3Q_2^{\beta_1\dot{\beta}_2}\right\rangle + \frac{1}{8}\left\langle Q_{3\dot{\beta}_1\beta_2}\frac{P}{E_0-H_2}H_3Q_2^{\dot{\beta}_1\beta_2}\right\rangle \\ &+ \frac{1}{4}\left\langle Q_{2\beta_1\dot{\beta}_2}Q_4^{\beta_1\dot{\beta}_2}\right\rangle + \frac{1}{4}\left\langle Q_{2\dot{\beta}_1\beta_2}Q_4^{\dot{\beta}_1\beta_2}\right\rangle \\ &+ \frac{1}{4}\left\langle Q_{4\beta_1\dot{\beta}_2}Q_2^{\beta_1\dot{\beta}_2}\right\rangle + \frac{1}{4}\left\langle Q_{4\dot{\beta}_1\beta_2}Q_2^{\dot{\beta}_1\beta_2}\right\rangle. \end{aligned} \tag{2.149}$$

---

states, we can combine such a projector with the energy denominator as

$$\frac{P}{E_0-H_2} = \int_0^\infty d\tau\, e^{E_0\tau}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\, e^{-H_2^{(1)}\tau+i\theta_1 N^{(1)}}\; e^{-H_2^{(2)}\tau+i\theta_2 N^{(2)}} \tag{2.143}$$

where

$$N^{(r)} = \sum_n n\left(a_n^{I(r)\dagger}a_n^{I(r)} + b_{an}^{(r)\dagger}b_{an}^{(r)}\right) \tag{2.144}$$

with $r = 1, 2$ are the level number operators for the two intermediate strings. The net effect of the operators in the above equation is to make the replacement $\left(a_n^{(r)\dagger}, b_n^{(r)\dagger}\right) \to \left(e^{-\omega_n\tau+in\theta_{(r)}}a_n^{(r)\dagger}, e^{-\omega_n\tau+in\theta_{(r)}}b_n^{(r)\dagger}\right)$ for all creation operators which lie to the right of the projector. Then, after the matrix element is computed, we multiply it by $e^{E_0\tau}$ and integrate over τ and θ_r. Any potential divergences come from the region near $\tau = 0$.

It is amusing to note that the vanishing energy correction for a supersymmetric external state is manifest in (2.149), since if Q_2 annihilates the external state, all of the terms are identically zero. Q_4 is unknown and it is consistent with the closure of the super-algebra to set it to zero.

Using the $\big|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ external state, we can check that what is left is manifestly convergent for the four impurity channel, and then show that the addition of impurities will not disturb this, leaving $\mathcal{O}(\lambda')$ contributions at every order in impurities. We have two sorts of terms in (2.149), which we can represent schematically as follows

$$\delta E_1 = \sum_I \frac{\Big(\langle\Phi|\langle I|Q_3\rangle\Big)\Big(\langle\Psi|\langle I|H_3\rangle\Big)^*}{E_\Phi - E_I} \qquad \delta E_2 = \sum_I \frac{\Big(\langle\Phi|\langle I|H_3\rangle\Big)\Big(\langle\Psi|\langle I|Q_3\rangle\Big)^*}{E_\Phi - E_I} \tag{2.150}$$

where $|\Phi\rangle$ is the $\big|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ external state, $|\Psi\rangle = Q_2|\Phi\rangle$, and $|I\rangle$ is a level-matched, two-string intermediate state. In order to evaluate the convergence and large μ behaviour of these terms, we can be entirely schematic. We take (see (A.22) for the expression of Q_2 in the BMN basis)

$$|\Psi\rangle \sim \sqrt{-\mu\alpha_3}\,\beta_n^\dagger\,\alpha_{-n}^\dagger|\alpha_3\rangle \qquad |\Phi\rangle \sim \alpha_n^\dagger\,\alpha_{-n}^\dagger|\alpha_3\rangle \tag{2.151}$$

while for the purpose of evaluating convergence we can take

$$G_p^{(1)} \sim \frac{1}{\sqrt{p}} \qquad K_{-p}^{(1)} \sim \text{constant} \qquad \widetilde{N}_{np}^{3\,r} \sim \frac{1}{p} \qquad \widetilde{N}_{qp}^{r\,s} \sim \frac{1}{p+q} \tag{2.152}$$

where we take all integers to be positive. Let us begin with δE_1 in (2.150), we have two choices for four impurity intermediate states

$$\begin{aligned} |I\rangle &\sim \alpha_{p_1}^\dagger\beta_{p_2}^\dagger\alpha_{p_3}^\dagger\alpha_{p_4}^\dagger|\alpha_1\rangle|\alpha_2\rangle \\ |I\rangle &\sim \alpha_{p_1}^\dagger\beta_{p_2}^\dagger\beta_{p_3}^\dagger\beta_{p_4}^\dagger|\alpha_1\rangle|\alpha_2\rangle. \end{aligned} \tag{2.153}$$

We can proceed with the first one, which will give

$$\begin{aligned} \delta E_1 \sim \sqrt{x} \sum_{p_1\,p_2\,p_3\,p_4} \frac{1}{2\,r\,\omega_n - \sum_{i=1}^4 \omega_{p_i}} &\langle\alpha_3|\alpha_n\,\alpha_{-n}\langle\alpha_2|\langle\alpha_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|Q_3\rangle \\ &\times\Big(\langle\alpha_3|\beta_n\,\alpha_{-n}\langle\alpha_2|\langle\alpha_1|\alpha_{p_1}\,\beta_{p_2}\,\alpha_{p_3}\,\alpha_{p_4}\,|H_3\rangle\Big)^* \end{aligned} \tag{2.154}$$

where $x = -\mu\alpha_3$ and $\sum_i p_i = 0$. There are two general ways in which we can contract the $\beta^{(r)}$'s. They can connect to factors of $\sum_m G_m\beta_m^\dagger$ in the prefactors of $|H_3\rangle$ and $|Q_3\rangle$, or they can pair-up to bring down a factor of $\widehat{Q}_{mp}^{r\,s}$ from the exponential. As far as convergence and large x power-counting is concerned however, $G_m^{(r)}G_p^{(s)}$ is equivalent to $\widehat{Q}_{mp}^{r\,s}$, and so we will simply use the former. When contracting $\beta^{(3)}$'s there is a fundamental difference between $G_n^{(3)}G_p^{(r)}$ and $\widehat{Q}_{np}^{3\,r}$, as far as large x behaviour is concerned, because of the pole in the latter.

In fact $\widehat{Q}^{3\,r}_{n\,p}$ is essentially equivalent to $\widetilde{N}^{3\,r}_{n\,p}$ and therefore the two can be interchanged in this analysis.

Because K_{-p} goes as a constant for large p, the worst convergence will always be realized by contracting the intermediate bosonic impurities with the prefactors of $|H_3\rangle$ and $|Q_3\rangle$. These contractions will yield

$$\delta E_1 \sim \sqrt{x} \sum_{p_1\,p_2\,p_3\,p_4} \frac{G^{(1)}_{p_2}\widetilde{N}^{3\,1}_{-n\,p_1}K^{(1)}_{-p_3}\widetilde{N}^{3\,1}_{n\,p_4} \times K^{(1)}_{-p_3}K^{(1)}_{p_4}\widetilde{N}^{3\,1}_{-n\,p_1}\begin{cases}\widetilde{Q}^{3\,1}_{n\,p_2} - \widetilde{Q}^{1\,3}_{p_2\,n}\\ G^{(3)}_{n}G^{(1)}_{p_2}\end{cases}}{2\,r\,\omega_n - \sum_{i=1}^{4}\omega_{p_i}} \tag{2.155}$$

Taking $p_4 = -(p_1 + p_2 + p_3)$, and using (2.152) we see that

$$\delta E_1 \sim \sum_{p_1\,p_2\,p_3} \frac{1}{(p_1+p_2+p_3)^2}\frac{1}{p_1^2}\begin{cases}\frac{1}{p_2^{3/2}}\\ \frac{1}{p_2}\end{cases} \tag{2.156}$$

where all p_i are considered absolute valued, or equivalently the sum considered over positive integers. This is manifestly convergent. Continuing on to evaluate the leading x dependence, for the top choice in (2.155) we have poles for all three summation variables, while in the large x limit the K's go as constants, $G \sim 1/\sqrt{x}$ and the energy denominator is linear in x, thus giving $\delta E_1 \sim 1/x$. For the bottom choice in (2.155), p_1 and p_3 have poles, while the sum over p_2 must be executed using (2.117). The scaling turns out identical however. Thus $\delta E_1/\mu$ is convergent and $\mathcal{O}(\lambda')$. One can repeat this argumentation for the second intermediate state in (2.153) and find the same behaviour. Also the entire exercise may be repeated for δE_2 in (2.150) using the following intermediate states

$$\begin{aligned}|I\rangle &\sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}\alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle\\ |I\rangle &\sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}\beta^\dagger_{p_3}\beta^\dagger_{p_4}|\alpha_1\rangle|\alpha_2\rangle\end{aligned} \tag{2.157}$$

and one discovers the same behaviour. The essential point is that we will always have at least 5 (inverse) powers of the summation variables, while the number of summation variables is 3. Alternate positionings of the oscillators in the intermediate states such as $|I\rangle \sim \alpha^\dagger_{p_1}\alpha^\dagger_{p_2}|\alpha_1\rangle\,\alpha^\dagger_{p_3}\alpha^\dagger_{p_4}|\alpha_2\rangle$ only improves the convergence, since level matching removes one more summation variable in these cases.

We can now consider adding additional pairs of fermionic and bosonic impurities to the intermediate state $|I\rangle$. This will add two factors of $\widetilde{N}^{1\,1}_{p_i\,p_j}$ or two factors of $G^{(1)}_{p_i}G^{(1)}_{p_j}$ (or equivalently two factors of $\widehat{Q}^{1\,1}_{p_i p_j}$). Either way the number of powers of summation variables increases in concert with the number of summation variables, preserving the convergence. Similarly the leading behaviour in λ' is unaffected. So it would seem that there are $\mathcal{O}(\lambda')$ contributions to $\delta E^{(2)}/\mu$ at every order in impurities, however any non-perturbative $\sqrt{\lambda'}$ behaviour is absent.

2.3.4 Conclusion

The principal result we have presented in this section is the cancelation of the logarithmic divergences that appear in the sum over the excitation numbers for the intermediate states with multiple oscillators. This cancelation is not unexpected as it is reminiscent of similar behavior in string field theory on Minkowski space.

We have also shown that the non-perturbative term $\sqrt{\lambda'}$ that appeared in the calculations involving these "impurity non-conserving" channels are an artifact of those divergences and thus disappear in a careful calculation leaving the $\mathcal{O}(\lambda')$ as the leading order in the expansion. These statements were generalized to an arbitrary number of impurities estabilishing generically both the absence of $\mathcal{O}(\sqrt{\lambda'})$ and the presence of $\mathcal{O}(\lambda')$ in the calculation at an arbitrary number of impurities.

The upshot of the later result is that, barring some further insights, it would be very difficult to perform the full calculation on the string theory side that would match the CFT result given in 2.42. Furthermore, our results seem to imply that the intermediate states of arbitrary energy would all contribute to the same order to the energy shift. This is a position that our physical intuition argues heavily against but that, so far at least, seems to be born out by the mathematics.

There are several possible ways in which this problem could be solved. The most obvious is the potential contribution from the hitherto ignored Q_4 factor. Unfortunately we know very little about this factor and so it remains the black box of this theory. The second potential solution is the existence of hidden cancelations that would suppress low order terms in λ' from the high impurity channels. We were interested enough in the possibility of such hidden cancelations that we actually performed the full 4-impurity calculation under two different formalisms (both will be presented in later sections and Appendices) as well as the rudimentary 6-impurity calculation. In neither of those cases are there any cancelation of the $\mathcal{O}(\lambda')$ terms. The final consideration of potential significance is the fact that the string vertex presented above is not a unique solution satisfying the required symmetries and conservations. Two other distinct vertices can be constructed which fulfill the same requirements. In the remainder of this chapter we will discuss the string field theory using these alternate vertices.

2.4 Alternate vertices

The vertex given by 2.87 (and referred to as SVPS vertex after Spradlin, Volovich, Pankiewicz and Stefanski) is not the only solution that satisfies the super-algebra and conservation laws. The alternative vertex becomes apparent if one gives up on smooth flat space limit as $\mu \to 0$ and instead focuses on the simplest possible pre-factors which still satisfy the super-algrebra. This vertex was first proposed by Di Vecchia, Petersen, Peterini, Russo and Tanzini [67] and is referred to as the DVPPRT vertex. It was then shown by Dobashi and Yoneya that the weighted averages of DVPPRT and SVPS vertices also satisfy the necessary conditions. They, in particular, argue that the equally weighted average of the two gives a vertex which is the most physically relevant of the three. This vertex is referred to as DY vertex.

The simple form of the DVPPRT vertex enabled us to develop a generic calculational

method for higher impurity channels which we used to test for "miraculous cancelations" of low order terms at higher impurities. This, previously unpublished work, will be presented in this section. The full 4-impurity calculation and cursory 6-impurity calculation have not yielded any indication of cancelation of low order terms or "dampening" of higher impurity contributions.

We have also confirmed the divergence cancelation - studied in the previous section - for the DVPPRT and DY vertices and calculated the 2-impurity DY energy shift which, for the reasons which may or may not have a physical basis, is in so far the closest result to the CFT one given by 2.42. These results were published in our paper [80] and will be repeated in this section as well.

2.4.1 DVPPRT vertex

The cubic term in the supercharge and Hamiltionian in the DiVecchia proposal are

$$|Q_3>=\sum_{r=1}^{3} Q_2^{(r)}|v> \quad , \quad |H_3>=\sum_{r=1}^{3} H_2^{(r)}|v>$$

Where

$$|v>= e^{\left(\frac{1}{2}\alpha^\dagger \tilde{N}\alpha^\dagger+\frac{1}{2}\beta^\dagger Q\beta^\dagger\right)}|\tilde{\alpha}_1>\otimes|\tilde{\alpha}_2>\otimes|\tilde{\alpha}_3> \tag{2.158}$$

where $\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3=0$ and the Neumann matrices N^{rs}_{mn}, Q^{rs}_{mn} are diagonal in the SO(4)×SO(4) indices, which we have suppressed in the above formula. These cubic terms are designed to satisfy the supersymmetry algebra

$$\left\{Q_{\alpha_1\dot{\alpha}_2}, Q_{\beta_1\dot{\beta}_2}\right\} = 2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}H \quad , \quad \left\{Q_{\dot{\alpha}_1\alpha_2}, Q_{\dot{\beta}_1\beta_2}\right\} = 2\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\alpha_2\beta_2}H \tag{2.159}$$

for the full supercharge and Hamiltonian operators which are defined as the series of string coupling constant, g_2,

$$Q = Q_2 + g_2Q_3 + g_2^2Q_4 + \dots \quad , \quad H = H_2 + g_2H_3 + g_2^2H_4 + \dots \tag{2.160}$$

It should also satisfy $[Q, H] = 0$.

We will consider the corrections to the energy of a state of two bosonic oscillators,

$$|e>\equiv \alpha_n^{i(3)\dagger}\alpha_{-n}^{j(3)\dagger}|\tilde{\alpha}_3> \tag{2.161}$$

Its energy at zeroth order in g_2 is

$$H_2|e>= E_0|e> \quad , \quad E_0 = 2\omega_{n(3)} = 2\sqrt{n_3^2+\mu^2\tilde{\alpha}_3^2} \tag{2.162}$$

We will compute the leading correction to this energy coming from string loops. First order perturbation theory vanishes. At second order, the contribution is

$$\delta^2 E = -g_2^2 < H_3|e> \frac{1}{H_2 - E_0} < e|H_3> + g_2^2 < e|H_4|e> \tag{2.163}$$

Here, we note that H_4 is an operator which can contain either two-string states or four-string states. In (2.163) above, we are using only its two-string part, where one of the strings is incoming, the other outgoing.

(2.163) can be written as

$$\delta^2 E = g_2^2(E_0 < v|e >< e|v > + \frac{1}{8}[< v|Q_2 e >. Q_2 < e|v > + \\ + < v|e > Q_2 < e|Q_2 v > + < v|Q_2 e >< Q_2 e|v >]) \tag{2.164}$$

The intermediate states in the above formula are not automatically level-matched and must be projected onto level-matched states. We will accomplish this by inserting a projection operator onto the level-matched states. The operator is

$$\mathcal{P} = \prod_{s=1}^{2} \int_{-\pi}^{\pi} \frac{d\theta_s}{2\pi} \exp\left(i\theta_s N^{(s)}\right) \tag{2.165}$$

where the level number operators for each string are

$$N^{(s)} = \sum_{n=-\infty}^{\infty} n\left(\alpha_n^{(s)\dagger}\alpha_n^{(s)} + \beta_n^{(s)\dagger}\beta_n^{(s)}\right)$$

and we have suppressed the SO(4)×SO(4) indices.

One might worry that the supersymmetry algebra is only valid once the level matching condition is applied, and we have used it to derive the equation above. It is easy to see that, if level matching is applied earlier, at the derivation of the equation for second order perturbation theory, since both the Hamiltonian H_2 and supercharge Q_2 commute with the constraint, and the external state $|e>$ is automatically level-matched, all steps remain valid and in the end we obtain (2.164) with the operator $\mathcal{P}$ inserted in intermediate states.

The calculational method of Appendix E can then be used to obtain the full formula for the second order energy shift:

$$\begin{aligned}
\frac{\delta E}{\mu} = \frac{4}{|\tilde{\alpha}_3|\mu} \int \frac{d\theta_a}{2\pi} \cdot \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \cdot \\
\cdot \Bigg(\delta^{jl}\delta^{ik} \left[NU\frac{1}{1-NUNU}N\right]^{33}_{-n,-n} \Bigg[E_0 NU\frac{1}{1-NUNU}N + \Omega QU\frac{1}{1+QUQU}Q\Omega \\
+\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\Bigg]^{33}_{nn} \\
+\delta^{jl}\delta^{ik}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{nn}\Bigg[E_0 NU\frac{1}{1-NUNU}N + \Omega QU\frac{1}{1+QUQU}Q\Omega \\
+\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\Bigg]^{33}_{-n,-n} \\
+\delta^{jk}\delta^{il}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-nn}\Bigg[E_0 NU\frac{1}{1-NUNU}N + \Omega QU\frac{1}{1+QUQU}Q\Omega
\end{aligned}$$

$$+\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\Big]^{33}_{n,-n}$$
$$+\delta^{jk}\delta^{il}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{n,-n}\left[E_0NU\frac{1}{1-NUNU}N+\Omega QU\frac{1}{1+QUQU}Q\Omega\right.$$
$$\left.+\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\Big]^{33}_{-nn}\right) \quad (2.166)$$

Where N and Q are usual Neumann matrices as defined in the appendix B and U are given by:

$$U^{sr}_{pq} = \delta^{sr}\delta_{pq}e^{ip\theta_s} \quad r,s=1,2 \quad (2.167)$$
$$U^{sr}_{pq} = 0 \quad r \text{ or } s=3 \quad (2.168)$$

.

2.166 is a "Master Formula" that holds exactly for the DVPPRT vertex. It is from the specific ways of solving this equation that we obtain particular impurity channels and the limits such as the large μ limit. The existence of this formula and its potential counterparts for the other vertices could potentially be very useful in investigating the behavior of the channels with the arbitrary number of impurities. 2.166 can also be used to calculate the higher impurity channels more easily then the usual methods like the ones in [80]. We give the 2-impurity calculation here and the 4-impurity one in the Appendix.

The two impurity truncation

To find the 2-impurity truncation, we keep all terms that are of the second order in the matrix U. Then, integration over the angles just enforces the level-matching condition, which we find more convenient to write explicitly in this case. A useful identity is

$$\Omega^{(3)2}_n = 2\omega^{(3)}_n - 2n \quad , \quad \Omega^{(3)}_n\Omega^{(3)}_{-n} = -2\mu\tilde{\alpha}_3$$

In the 2-impurity approximation, we get

$$\frac{\delta E_r}{\mu} = \frac{4}{|\tilde{\alpha}_3|\mu}\sum_{r=1}^{2}\sum_{p=-\infty}^{\infty}\frac{sin^4(|n|\pi r)}{16\pi^4\omega^{(3)2}_n\omega^{(r)2}_p}\left(\omega^{(r)}_p+\beta_r\omega^{(3)}_n\right)^4\cdot$$
$$\left[\frac{2n}{(p-\beta_r n)^4}-\frac{2n}{(p+\beta_r n)^4}+4\frac{\omega^{(3)}_n-\mu\alpha_3}{(p-\beta_r n)^2(p+\beta_r n)^2}\right] \quad (2.169)$$

Here, both of the impurities must be on one of the two internal strings and the summation over $r=1,2$ counts the cases where both impurities are on one string or the other string. The sum over p is the sum over the opposite world-sheet momenta of the pair of internal strings. Since the integration measure is symmetric under the replacement $r\to 1-r$ which interchanges the two internal strings. this sum can be replaced by a factor of 2. Also, note that the first two terms (which arise because of the averaging over n and $-n$ in the master formula) cancel when we change the sign of the summand, p.

To perform the sum over p we use the contour integral formula

$$\sum_{p=-\infty}^{\infty} f(p) = \oint_C \frac{dz}{2i} \cot \pi z f(z)$$

where the contour C is the sum of infinitesimal circles surrounding the integers on the real axis, z with counter-clockwise orientation.

$$\frac{\delta E_r}{\mu} = \frac{2}{|\tilde{\alpha}_3|\mu} \oint_C \frac{dz}{2i} \frac{\sin^4(|n|\pi r)}{\pi^4 \omega_n^{(3)2} \omega_z^{(r)2}} \left(\omega_z^{(r)} + \beta_r \omega_n^{(3)}\right)^4 \cot \pi z \frac{\omega_n^{(3)} - \mu\alpha_3}{(z-\beta_r n)^2 (z+\beta_r n)^2} \tag{2.170}$$

The contour can then be deformed to encircle (with clockwise orientation) the singularities of $f(z)$ and goes to zero sufficiently rapidly at $z \to \infty$ that there is no pole there. In our case, $f(z)$ has both pole and cut singularities. The cut singularities are square-root cuts which occur in the linear and cubic terms when the quartic in the numerator is expanded. The poles are double poles. Taking into account the reversal of the orientation of the contour, we get the following contribution from the poles.

$$\frac{\delta E_r}{\mu} = -\frac{4}{|\tilde{\alpha}_3|\mu} [\omega_n^{(3)} - \mu\alpha_3] \partial_z \left[\frac{\sin^4(|n|\pi r)}{\pi^3 \omega_n^{(3)2} \omega_z^{(r)2}} \frac{\left(\omega_z^{(r)} + \beta_r \omega_n^{(3)}\right)^4}{(z+\beta_r n)^2} \cot \pi z \right]_{z=\beta_r n} + \ldots \tag{2.171}$$

This is not an approximate expression. The three dots denote contributions from contour integrals around the cuts. It can be shown that the large μ limit of the cut integrals is smaller than the order of interest, $1/\mu^2$ and thus for our considerations, it can be neglected. (Another way to see this is to take the large μ limit, keeping the contour fixed.)

The dominant terms in the large μ limit are the terms produced by the derivative acting on the $\cot \pi z$ and the $1/(z+\beta_r n)^2$. The result is

$$\begin{aligned} \frac{\delta E_r}{\mu} = & \frac{4}{|\tilde{\alpha}_3|\mu} [\omega_n^{(3)} - \mu\alpha_3] \left[\frac{\sin^4(|n|\pi r)}{\pi^2 \omega_n^{(3)2} \omega_z^{(r)2}} \frac{\left(\omega_z^{(r)} + \beta_r \omega_n^{(3)}\right)^4}{(z+\beta_r n)^2} \frac{1}{\sin^2 \pi z} \right]_{z=\beta_r n} + \\ & + \frac{8}{|\tilde{\alpha}_3|\mu} [\omega_n^{(3)} - \mu\alpha_3] \left[\frac{\sin^4(|n|\pi r)}{\pi^3 \omega_n^{(3)2} \omega_z^{(r)2}} \frac{\left(\omega_z^{(r)} + \beta_r \omega_n^{(3)}\right)^4}{(z+\beta_r n)^3} \cot \pi z \right]_{z=\beta_r n} + \ldots \end{aligned} \tag{2.172}$$

Taking the large μ limit, we get

$$\frac{\delta E_r}{\mu} = \frac{8}{\pi^2 |\tilde{\alpha}_3|^2 \mu^2} \sin^2(|n|\pi r) + \frac{8}{\pi^3 |\tilde{\alpha}_3|^2 \mu^2} \frac{1}{rn} \sin^3(n\pi r) \cos(n\pi r) + \ldots \tag{2.173}$$

In the first term on the right-hand-side, a factor of π as well as a minus sign was produced by taking the derivative. The derivative produces an inverse of $\sin^2$ which cancels two powers of the sine. In the second term, the derivative of the denominator produces a minus sign

and a factor of 2. In taking all of the limits, we have assumed that $n << \beta\mu\alpha_3$. The final expression should be integrated over r from 0 to 1. The result is:

$$\frac{\delta E_{2,1}}{\mu} = g_2^2 \int_0^1 dr \ \frac{r(1-r)}{2} \ \frac{\delta E_r}{\mu} = \frac{8}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{1}{24} + \frac{11}{64\pi^2 n^2}\right)\ldots \tag{2.174}$$

This, however, ignores the channel whereby one impurity is on each string. In that case though, the level-matching condition limits the sum to the zero modes. Therefore the missing part of energy shift will be given by:

$$\begin{aligned}\frac{\delta E_{2,2}}{\mu} &= \frac{4}{|\tilde{\alpha}_3|\mu}\left[2N^{3p}_{n0}N^{p3}_{0-n}\right]\left[-2(\omega_n - \mu\alpha)N^{3s}_{-n0}N^{s3}_{0n}\right] = \\ &= \frac{-8n^2}{|\tilde{\alpha}_3|^2\mu^2}\left[N^{3p}_{n0}N^{p3}_{0-n}\right]\left[N^{3s}_{-n0}N^{s3}_{0n}\right]\end{aligned} \tag{2.175}$$

with p and s here being different strings and with $[..]_{nn}[..]_{-n-n}$ factors canceling each other as they do above. Taking the expression for the zero mode Neuman matrix we can write the above in the large μ limit as:

$$\frac{\delta E_{2,2}}{\mu} = g_2^2 \int_0^1 dr \ \frac{r(1-r)}{2}\frac{-8n^2}{|\tilde{\alpha}_3|^2\mu^2}\frac{sin^2(n\pi r)sin^2(n\pi(1-r))}{\pi^4 n^4 r(1-r)} = \frac{-8n^2}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{6}{64\pi^2 n^2}\right) \tag{2.176}$$

Therefore, the full value of two impurity contribution is:

$$\frac{\delta E_2}{\mu} = \frac{\delta E_{2,1}}{\mu} + \frac{\delta E_{2,2}}{\mu} = -\frac{8}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{1}{24} + \frac{5}{64\pi^2 n^2}\right)\ldots \tag{2.177}$$

Which is in keeping with the full 2-impurity result we used in [80]:

$$\begin{aligned}\delta E^{\text{DVPPRT}} = \frac{g_2^2}{4\pi^2}\Bigg[&-\left(\frac{1}{24} + \frac{5}{64\pi^2 n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2} + \frac{1}{2\pi}\right)\lambda'^{3/2} \\ &+ n^2\left(\frac{5}{48} + \frac{1}{128\pi^2 n^2}\right)\lambda'^2 - n^2\left(\frac{29}{32\pi^2} + \frac{21}{64\pi}\right)\lambda'^{5/2} \\ &+ n^4\left(-\frac{9}{64} + \frac{105}{512\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{303}{160\pi^2} + \frac{165}{256\pi}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\Bigg].\end{aligned} \tag{2.178}$$

Four impurity truncation

The result derived from the "master formula" confirms the absence of divergences and agrees with the usual method of calculation. However, this result is still different from the CFT one given in 2.42. We were therefore interested in performing the 4-impurity calculation with hope of either observing some cancelations or perhaps getting closer to the 2.42 result. The details of the 4-impurity calculation (which can also be used as a template for all the higher impurity calculations are given in the appendix E.2. The end result for the 4-impurity truncation is:

$$\frac{\delta E_4}{\mu} = \frac{1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{1}{12} + \frac{64}{64\pi^2 n^2}\right) + \cdots \tag{2.179}$$

$$\frac{\delta E_2}{\mu}+\frac{\delta E_4}{\mu}=\frac{2}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{5}{24}+\frac{57}{64\pi^2n^2}\right)+\cdots \tag{2.180}$$

The conclusion of this calculation is that the 4 impurity channel contributes to the same orders in μ as the 2 impurity one, and that it does not lead to any noticeable convergence to the 2.42 result. The skeletal 6-impurity calculation which we do not give in detail in this thesis appears to lead to the same conclusions.

Divergence cancelation

We have seen explicitly that the DVPPRT vertex does not carry $\sqrt{\lambda'}$ term. The reason for this is once again cancelation of the divergent terms between the H_3 and the contact term. This cancelation can be seen exactly in the derivations of 2.166 but it is useful to derive it explicitly in the language of [79]. We did that in [80]:

In the language of [79], DVPPRT vertex is given by the following expressions [67],

$$\begin{aligned}
|H_3^D\rangle &= -g_2\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{\alpha'}{16\,\alpha_3^3}\Big[K^2+\widetilde{K}^2-4Y^{\alpha_1\alpha_2}\widetilde{Y}_{\alpha_1\alpha_2}-4Z^{\dot{\alpha}_1\dot{\alpha}_2}\widetilde{Z}_{\dot{\alpha}_1\dot{\alpha}_2}\Big]|V\rangle\,,\\
|Q^D_{3\,\beta_1\dot{\beta}_2}\rangle &= g_2\,\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(Z_{\dot{\gamma}_1\dot{\beta}_2}K^{\dot{\gamma}_1}_{\beta_1}-iY_{\beta_1\gamma_2}K^{\gamma_2}_{\dot{\beta}2}\Big)|V\rangle\,,\\
|Q^D_{3\,\dot{\beta}_1\beta_2}\rangle &= g_2\,\bar{\eta}\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(Y_{\gamma_1\beta_2}K^{\gamma_1}_{\dot{\beta}_1}-iZ_{\dot{\beta}_1\dot{\gamma}_2}K^{\dot{\gamma}_2}_{\beta_2}\Big)|V\rangle\,.
\end{aligned} \tag{2.181}$$

Unlike the SVPS case, the H_3 divergence does not stem from the two-bosonic-impurity intermediate state. This can be traced to the substitution of $K^2+\widetilde{K}^2$ for $K\,\widetilde{K}$ in the H_3 prefactor. There is, however, another divergence that was not present in the SVPS case. It is due to the contribution coming from matrix elements with two fermionic impurities in the intermediate state. In particular, the relevant matrix elements are given by

$$\begin{aligned}
&\langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\beta^{\alpha_1\alpha_2}_{p(1)}\beta_{-p(1)\beta_1\beta_2}|H_3^D\rangle=\\
&4\,g_2 r\,(1-r)\left(\frac{\omega_n^{(3)}}{\alpha_3}+\frac{\omega_p^{(1)}}{\alpha_1}\right)\left(\widetilde{Q}^{1\,1}_{-p\,p}\;-\widetilde{Q}^{1\,1}_{p\,-p}\right)\widetilde{N}^{3\,3}_{-n\,n}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}
\end{aligned} \tag{2.182}$$

and similarly for the intermediate state with dotted indices. The divergent contribution to the energy shift coming from these matrix elements is found by taking the large p limits of the summands. One finds:

$$\delta E^{\rm div}_{H_3^D}\sim-\frac{1}{2}\int_0^1 dr\;\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2\sum_p\frac{1}{|p|} \tag{2.183}$$

The contribution from the contact term stems from the following matrix element,

$$\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1}\langle\alpha_3|\alpha^i_n\alpha^i_{-n}\,\langle\alpha_2|\langle\alpha_1|\alpha^{K\,(1)}_p\beta^{(1)\,\Sigma_1\,\Sigma_2}_{-p}|Q^D_{3\,\beta_1\dot{\beta}_2}\rangle=$$

$$2\left(G^{(1)}_{|p|}\,K^{(3)}_{-n}\widetilde{N}^{31}_{np}+G^{(1)}_{|p|}\,K^{(3)}_{n}\widetilde{N}^{31}_{-np}\right)(\sigma^k)^{\dot{\sigma}_1}_{\dot{\beta}_1}\delta^{\dot{\sigma}_2}_{\dot{\beta}_2}+8\,G^{(1)}_{|p|}\,K^{(1)}_{p}\widetilde{N}^{33}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}. \quad (2.184)$$

The divergent contribution to the energy shift is found to be,

$$\delta E^{\text{div}}_{H^D_4}\sim+\int_0^1 dr\,\frac{g_2^2\,r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{33}_{n\,-n}\right)^2\sum_{p>0}\frac{1}{p} \quad (2.185)$$

Noting that in the H^D_3 contribution the divergence is found for both positive and negative p, while in the H^D_4 contribution the divergence occurs only for negative p, and hence a relative factor of 2 is induced in the H^D_3 term, one sees that the logarithmically divergent sums cancel identically between the H^D_3 and contact terms, leaving a convergent sum. This result can be generalized to arbitrary impurity channels, as was done for the SVPS case in section 2.3.3

2.4.2 DY vertex

Seeing as both the SVPS and DVPPRT vertex satisfy the super-algebra it is to be expected that every linear combination of them will do likewise. We can thus create any number of vertices by taking weighted averages of the two. One of those, namely the one in which weights of the two vertices are equal is of special importance. This was noticed by Dobashi and Yoneya in [71] and we refer the reader to that paper for the exact details of the reasoning.

In summary, they claimed that the cubic Hamiltonian ought to only count excitations of the one $SO(4)$ of the $SO(4)\times SO(4)$ background, specifically the one that inherits from the geometry of the S_5. The reasons for this are subtle and have to do with isolating the contributions of various fields to the three point functions on the CFT side of the duality. The end result was that in order to maintain the AdS/CFT duality the H_3 part of the vertex must explicitly break the $\mathbb{Z}_2$ symmetry and ignore the contributions from one of the $SO(4)$s. This is accomplished by taking an average of the $\mathbb{Z}_2$-even prefactor of DVPPRT and the $\mathbb{Z}_2$-odd prefactor of SVPS. In this way the second $SO(4)$ zero modes cancel-out. The DY vertex is then given by:

$$\begin{aligned}|H_3^{\text{DY}}\rangle&=\frac{1}{2}\left(|H_3^{\text{DVPPRT}}\rangle+|H_3^{\text{SVPS}}\rangle\right)\\|Q_3^{\text{DY}}\rangle&=\frac{1}{2}\left(|Q_3^{\text{DVPPRT}}\rangle+|Q_3^{\text{SVPS}}\rangle\right)\end{aligned} \quad (2.186)$$

Divergence Cancelation

The argument from section 2.4.1 can be extended to the DY vertex. More generally it can be extended to any linear combination of the SVPS and DVPPRT vertices. We give this generalization in [80]:

An arbitrary combination of the SVPS and DVPPRT vertices:

$$H_3^N=\alpha\,H_3^{\text{SVPS}}+\beta\,H_3^{\text{DVPPRT}} \quad (2.187)$$

$$Q_3^N=\alpha\,Q_3^{\text{SVPS}}+\beta\,Q_3^{\text{DVPPRT}} \quad (2.188)$$

similarly yields a finite energy shift. We calculate the mass shift of the trace state. The divergence stemming from the H_3 term is simply α^2 times the SVPS H_3 divergence (2.124) plus β^2 times the DVPPRT divergence. The reason is simple - the SVPS divergence stems from an entirely bosonic intermediate state, while (2.183) results from an entirely fermionic one. This precludes any divergences arising from cross-terms. We note that the SVPS divergence is exactly equal to (2.183), therefore we have

$$\delta E^{\text{div}}_{H_3^N} \sim -(\alpha^2+\beta^2)\frac{1}{2}\int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{33}_{n\,-n}\right)^2 \sum_p \frac{1}{|p|}. \tag{2.189}$$

The pieces of the SVPS Q_3 relevant to a two-impurity channel calculation are exactly Q_3^{DVPPRT} with $K \leftrightarrow \widetilde{K}$, therefore, from (2.126)

$$\begin{aligned}
&\left(g_2\frac{\eta}{4}\sqrt{\frac{r\,(1-r)\,\alpha'}{-2\,\alpha_3^3}}\right)^{-1} \langle\alpha_3|\alpha_n^i\alpha_{-n}^i\,\langle\alpha_2|\langle\alpha_1|\alpha_p^{K\,(1)}\beta_{-p}^{(1)\,\Sigma_1\,\Sigma_2}|Q^{\text{DVPPRT}}_{3\,\beta_1\dot{\beta}_2}\rangle = \\
&2G^{(1)}_{|p|}\left(\left[\alpha\,(K^{(3)}_{-n}\widetilde{N}^{3\,1}_{-n\,p}+K^{(3)}_n\widetilde{N}^{3\,1}_{n\,p})+\beta\,(K^{(3)}_{-n}\widetilde{N}^{3\,1}_{n\,p}+K^{(3)}_n\widetilde{N}^{3\,1}_{-n\,p})\right](\sigma^k)^{\dot{\sigma}_1}_{\beta_1}\delta^{\dot{\sigma}_2}_{\dot{\beta}_2}\right. \\
&\left.+\,4\,(\beta\,K^{(1)}_p+\alpha\,K^{(1)}_{-p})\widetilde{N}^{3\,3}_{n\,-n}(\sigma^K)^{\Sigma}_{\beta}\delta^{\Sigma}_{\beta}\right)
\end{aligned} \tag{2.190}$$

The last term in (2.190) gives rise to a log-divergent sum, the large-p behaviour of which is

$$\delta E^{\text{div}}_{H_4^N} \sim +(\alpha^2+\beta^2)\int_0^1 dr\, \frac{g_2^2\, r(1-r)}{r\,|\alpha_3|\,\pi^2}\left(\widetilde{N}^{3\,3}_{n\,-n}\right)^2 \sum_{p>0} \frac{1}{p}. \tag{2.191}$$

Thus, by the usual arguments , the energy shift is finite for arbitrary α and β. The DY vertex uses $\alpha = \beta = 1/2$. Again, as for the DVPPRT vertex, the generalization of these arguments to the impurity non-conserving channels is a straightforward application of the treatment given in section 2.3.3.

2-impurity channel DY energy shift

In [80] we reported the results of the calculation of the 2 impurity channel contribution to the energy shift in the case of DY vertex. As it happens it is the closest result thus obtained to the one on the CFT side. While it is possible that this indicates that something is fundamentally "correct" about the DY vertex it bears remembering that the correspondence is still very much inexact and, perhaps more importantly, we have every reason to believe that the higher impurity channels contribute in equal order to the energy shift making every correspondence from a single channel potentially just a numerical coincidence.

On the topic of numerical coincidences, we also find that the unjustified factor of two change between H_3 and the contact term contributions still improves the result in the DY case making it even more closely aligned to 2.42

The calculations undertaken in [80] are practically identical to those in [70], using the DVPPRT and DY vertices in place of the SVPS vertices used there.

The external state for which we are calculating the energy shift is still

$$|[\mathbf{9},\mathbf{1}]\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left(\alpha_n^{\dagger i}\alpha_{-n}^{\dagger j} + \alpha_n^{\dagger j}\alpha_{-n}^{\dagger i} - \frac{1}{2}\delta^{ij}\alpha_n^{\dagger k}\alpha_{-n}^{\dagger k}\right)|3\rangle.$$

For this particular state, individual H_3 and contact terms are not divergent in the two impurity approximation. It should be further noted that for this state, and for the impurity conserving channel, we shall find that use of the DY vertex, rather than the SVPS vertex, is equivalent to making the replacements of the quantities $(K, \widetilde{K})$ as $K \to (K+\widetilde{K})/2$ and $\widetilde{K} \to (K+\widetilde{K})/2$ in the SVPS vertex. This is the simplest way of reproducing our results.

The separate H_3 and contact term contributions to the energy shift for each of the three vertices are given below. We find that the DY energy shift agrees with gauge theory only at the leading order, while also enjoying the vanishing of the 3/2 and 5/2 powers of λ'. The order-λ'^2 term is of the correct form, but suffers from an overall factor of 4/3. The SVPS and DVPPRT results do not agree with gauge theory at the leading order. By multiplying the contact terms by two (an unjustified operation), one can recover the correct gauge theory result up to λ'^2 order with the SVPS (including vanishing of its $\lambda'^{3/2}$ term) and DY vertices. Further, this operation does not spoil the vanishing 3/2 and 5/2 powers of λ' for the DY result.

H_3 terms

$$\begin{aligned}\delta E_{H_3}^{\text{SVPS}} &= \frac{g_2^2}{32\pi^2}\left[\frac{15}{2\pi^2 n^2}\lambda' + 3\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - \frac{27}{4\pi^2}\lambda'^2 - n^2\left(\frac{5}{\pi^2}+\frac{9}{4\pi}\right)\lambda'^{5/2}\right.\\ &+ \left.\frac{111n^2}{16\pi^2}\lambda'^3 + n^4\left(\frac{45}{16\pi}+\frac{33}{5\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \qquad (2.192)\end{aligned}$$

$$\begin{aligned}\delta E_{H_3}^{\text{DVPPRT}} &= \frac{g_2^2}{32\pi^2}\left[-\left(\frac{2}{3}+\frac{5}{4\pi^2 n^2}\right)\lambda' + 3\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + n^2\left(1-\frac{9}{8\pi^2 n^2}\right)\lambda'^2\right.\\ &- 5n^2\left(\frac{2}{\pi^2}+\frac{3}{4\pi}\right)\lambda'^{5/2} - 5n^4\left(\frac{1}{4}-\frac{21}{32\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{105}{16\pi}+\frac{94}{5\pi^2}\right)\lambda'^{7/2}\\ &+ \left.\mathcal{O}(\lambda'^4)\right] \qquad (2.193)\end{aligned}$$

$$\begin{aligned}\delta E_{H_3}^{\text{DY}} &= \frac{g_2^2}{4\pi^2}\left[\frac{3}{4}\left(\frac{1}{12}+\frac{35}{32\pi^2 n^2}\right)\lambda' - 5n^2\left(\frac{1}{96}+\frac{35}{256\pi^2 n^2}\right)\lambda'^2\right.\\ &+ \left. n^4\left(\frac{17}{384}+\frac{655}{1024\pi^2 n^2}\right)\lambda'^3 + n^4\left(\frac{3}{256\pi}+\frac{23}{640\pi^2}\right)\lambda'^{7/2} + \mathcal{O}(\lambda'^4)\right] \qquad (2.194)\end{aligned}$$

Contact terms

$$\begin{aligned}\delta E_{H_4}^{\text{SVPS}} &= \frac{g_2^2}{32\pi^2}\left[\left(\frac{1}{3}+\frac{5}{8\pi^2 n^2}\right)\lambda' - \frac{3}{2}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - n^2\left(\frac{1}{6}-\frac{19}{16\pi^2 n^2}\right)\lambda'^2\right.\\ &+ n^2\left(\frac{11}{4\pi^2}+\frac{9}{8\pi}\right)\lambda'^{5/2} + \frac{n^4}{8}\left(1-\frac{105}{8\pi^2 n^2}\right)\lambda'^3 - n^4\left(\frac{45}{32\pi}+\frac{73}{20\pi^2}\right)\lambda'^{7/2}\end{aligned}$$

$$+ \quad \mathcal{O}(\lambda'^4)] \tag{2.195}$$

$$\delta E_{H_4}^{\mathrm{DVPPRT}} = \delta E_{H_4}^{\mathrm{SVPS}} \tag{2.196}$$

$$\begin{aligned}\delta E_{H_4}^{\mathrm{DY}} &= \frac{g_2^2}{4\pi^2}\left[n^2\left(\frac{1}{96}+\frac{35}{256\pi^2n^2}\right)\lambda'^2 - \frac{5n^4}{128}\left(\frac{1}{3}+\frac{29}{8\pi^2n^2}\right)\lambda'^3\right.\\ &+ \left.\frac{n^4}{256}\left(\frac{3}{2\pi}+\frac{5}{\pi^2}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right]\end{aligned} \tag{2.197}$$

Energy shifts

The results for the complete energy shifts are as follows,

$$\begin{aligned}\delta E^{\mathrm{SVPS}} &= \frac{g_2^2}{4\pi^2}\left[\left(\frac{1}{24}+\frac{65}{64\pi^2n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2}\right.\\ &- n^2\left(\frac{1}{48}+\frac{89}{128\pi^2n^2}\right)\lambda'^2 - \frac{9n^2}{32}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{5/2}\\ &+ \left. n^4\left(\frac{1}{64}+\frac{339}{512\pi^2n^2}\right)\lambda'^3 + n^4\left(\frac{59}{160\pi^2}+\frac{45}{256\pi}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right]\end{aligned} \tag{2.198}$$

$$\begin{aligned}\delta E^{\mathrm{DVPPRT}} &= \frac{g_2^2}{4\pi^2}\left[-\left(\frac{1}{24}+\frac{5}{64\pi^2n^2}\right)\lambda' + \frac{3}{16}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2}\right.\\ &+ n^2\left(\frac{5}{48}+\frac{1}{128\pi^2n^2}\right)\lambda'^2 - n^2\left(\frac{29}{32\pi^2}+\frac{21}{64\pi}\right)\lambda'^{5/2}\\ &+ \left. n^4\left(-\frac{9}{64}+\frac{105}{512\pi^2n^2}\right)\lambda'^3 + n^4\left(\frac{303}{160\pi^2}+\frac{165}{256\pi}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right]\end{aligned} \tag{2.199}$$

$$\begin{aligned}\delta E^{\mathrm{DY}} &= \frac{g_2^2}{4\pi^2}\frac{3}{4}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2n^2}\right)\left(\lambda' - \frac{4}{3}\frac{n^2}{2}\lambda'^2\right) + \frac{n^4}{24}\left(1+\frac{255}{16\pi^2n^2}\right)\lambda'^3\right.\\ &+ \left.\frac{n^4}{384}\left(\frac{9}{\pi}+\frac{142}{5\pi^2}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right]\end{aligned} \tag{2.200}$$

Recall that the leading 3/4 is irrelevant and can be scaled away by fixing the overall f factor which multiplies the vertices (and which has not been written in the above formulae, where it would appear in each as an overall factor of $|f|^2$). We see that the gauge theory result 2.42 is matched only by the DY result, and only at leading order in λ', with the λ'^2 term being of the correct form but with an overall factor of 4/3. We also see the miraculous absence of the $\lambda'^{3/2}$ and $\lambda'^{5/2}$ terms which are clearly generic in the string field theory. The result (2.200) represents the best matching of this quantity to gauge theory so far, and thus is an indication that the DY vertex is an improvement over its predecessors.

Mysteriously, if the contact terms are scaled by a factor of 2, the agreement with gauge theory is enhanced for both the SVPS and DY results,

$$\begin{aligned}\delta E_{2H_4}^{\mathrm{SVPS}} &= \frac{g_2^2}{4\pi^2}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2n^2}\right)\left(\lambda'-\frac{n^2}{2}\lambda'^2\right)+\frac{n^2}{16\pi^2}\lambda'^{5/2}\right.\\ &+ \left. n^4\left(\frac{1}{32}+\frac{117}{256\pi^2n^2}\right)\lambda'^3-\frac{7n^4}{80\pi^2}\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{2.201}$$

$$\begin{aligned}\delta E_{2H_4}^{\mathrm{DY}} &= \frac{g_2^2}{4\pi^2}\frac{3}{4}\left[\left(\frac{1}{12}+\frac{35}{32\pi^2n^2}\right)\left(\lambda'-\frac{n^2}{2}\lambda'^2\right)+n^4\left(\frac{7}{288}+\frac{365}{768\pi^2n^2}\right)\lambda'^3\right.\\ &+ \left. n^4\left(\frac{1}{10\pi^2}+\frac{1}{32\pi}\right)\lambda'^{7/2}+\mathcal{O}(\lambda'^4)\right] \end{aligned} \tag{2.202}$$

however, the DY result is still superior in that the $\lambda'^{5/2}$ power is absent.

2.5 Conclusion

The string field theory program is very ambitious. In the best case scenario it could provide one of the most important remaining confirmations of the AdS/CFT duality - one that has to do with the interacting strings and non-planar limit of the CFT. In attempting to pursue this program, however, we have encountered a number of important obstacles.

Perhaps most importantly we still do not have any reason for setting Q_4 to zero and therefore may well be missing a fundamental factor in our calculations. There is currently no obvious way to address this issue.

Secondly, in the aftermath of our own work we are facing an unpalatable result which would have intermediate states of arbitrary energy contribute equally to the energy shift. Our proof of divergence cancelations removed the a-priori reasoning used in excluding them and the explicit calculations of the 4 and 6 impurity contributions for the case of DVPPRT vertex confirmed their existence at all relevant orders in λ'. This is a difficult problem but one that can possibly be tackled. Constructing a DY version of "master formula" 2.166 that enables us to work formally in all impurities, and working to develop a way of manipulating the inverse matrix products such as the ones that appear in that formula, is a possible line of approach. This author is quite interested in continuing that line of research.

The fact that, despite these serious problems, forms of the stringy result still closely resemble 2.42 is both tantalizing and fascinating.

Chapter 3

Orbifolding and the discrete light-cone quantization

There are a number of interesting modifications to the pp-wave/BMN limit. Most broadly, it is possible to investigate what happens as one goes to higher orders in the original coupling constant $\lambda = g_{YM}^2$ thus going beyond the strict pp-wave limit. This was done by Callan and collaborators and published in [44] as well as by a number of other authors [40–44].

Equally interesting is a change of geometry of the background space in the string theory whereby some of the dimensions get orbifolded by the periodic identification. Early work in this direction is due to Takayanagi and Terashima [82] and Mukhi, Rangamani and Verlinde [83] and was followed by extensive efforts including [73] and our work in [85].

The important point about orbifolding is that it, if it is done in particular directions, leads to discrete light-cone gauge quantization of the string and to changes to the level-matching conditions. One of the consequences of this is that the string field theory calculations of the sort discussed in the Chapter 2. become somewhat easier.

Starting from the observations of [83] where the gauge theory analogous to the orbifolded strings is described, De Risi, Grignani, Orselli and Semenoff calculate the generalizations of the double expansion energy shift formula 2.42 to the DLCQ case for various values of the total string momentum. As part of our string field theory research we performed string theoretic analogues of that calculation. The results, published in [81] suffer from most of the problems already discussed with regards to the string field theory calculations, but the methods employed may well prove valuable in later attempts to resolve those issues. We will discuss those methods and results in the following sections.

Another interesting result is that the DLCQ inducing orbifolding is an inevitable result of imposing the "magnon boundary condition" on the string, which is necessary for finding the string theoretic equivalents of a "single magnon" multiplet in CFT theory. This connection was the main result of [85]. In the last section of this chapter we will discuss the relationship between the magnon multiplet and the single impurity string multiplet in and near the pp-wave limit.

3.1 Orbifolding and the BMN limit

3.1.1 String theory on the pp-wave orbifold

We can take the plane-wave limit of the orbifold of the $AdS_5 \times S_5$. We begin from the metric given by [83]:

$$\begin{aligned} ds^2 &= R^2 \Big[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \\ &\quad d\alpha^2 + \sin^2\alpha\, d\theta^2 + \cos^2\alpha\, \left(d\gamma^2 + \cos^2\gamma\, d\chi^2 + \sin^2\gamma\, d\phi^2\right) \Big] , \end{aligned} \tag{3.1}$$

Here, the first line is the AdS metric in global coordinates and the second is the S_5 metric embedded in the 6-dimensional space. If we want the metric to be orbifolded we treat it as $AdS_5 \times S_5/Z_M$ where Z_M is contained in the same 6-dimensional space as S_5. Somewhat different form of the metric was used here from that in 2.1 in order to emphasise the periodicities of the AdS_5 and S_5. The coordinate frame given here is also one in which orbifolding is most natural. It is a simple exercise to perform the coordinate transformations relating 2.1 to 3.1

We identify the scalar fields acted on by the orbifold group with the coordinates of this space. At this point, all the directions of the embedding space are equivalent so our labeling is arbitrary. However, we will soon be focusing on a plane-wave limit of the $AdS_5 \times S_5$ at which point one direction will be singled out as a light-cone direction. It will then become important whether this direction was chosen for orbifolding or not. Throughout, we will point to the consequences of those two possible choices.

Picking two[1] arbitrarily labeled angles we can then write the periodicity condition for the orbifolded S_5:

$$\chi \to \chi + \frac{2\pi}{M}, \qquad \phi \to \phi - \frac{2\pi}{M} . \tag{3.2}$$

We now take a pp-wave limit.

As stated above, we can choose whether the light-cone direction will have orbifolded periodicity (light-cone in the direction of χ or ϕ) or not (light-cone in the direction of θ). The main consequence of this choice will be whether or not the x^- coordinate ends up being quantized and therefore whether the end theory is quantized with DLCQ or with the regular light cone gauge quantization. The choice, however, does not affect the number of supersymmetries in the near pp-wave limit as we shall show later. As we are interested mostly in the DLCQ case, we proceed with taking limit with the light cone in the direction of χ. It will be relatively easy to follow what happens in the alternative case. Still following [83] we introduce new coordinates:

$$r = \rho R, \quad w = \alpha R, \quad y = \gamma R . \tag{3.3}$$

[1]It turns out not to be possible to orbifold just one direction and end up with a supersymmetric theory, the gauge theory discussion will provide some indication as to why this is the case

however, seeing as we will be interested in near-pp wave limit and want to avoid periodicity in x^+ we choose slightly different light-cone coordinates then usual:

$$x^+ = t, \quad x^- = \frac{R^2}{2}(t - \chi) \ . \tag{3.4}$$

Making the substitutions the metric (3.1) becomes

$$\begin{aligned} ds^2 &= R^2\Bigg[-\cosh^2\frac{r}{R}\left(dx^+\right)^2 + \frac{dr^2}{R^2} + \sinh^2\frac{r}{R}\,d\Omega_3^2 + \frac{dw^2}{R^2} + \\ &\sin^2\frac{w}{R}\,d\theta^2 + \cos^2\frac{w}{R}\left(\frac{dy^2}{R^2} + \cos^2\frac{y}{R}\left(dx^+ - \frac{2}{R^2}dx^-\right)^2 + \sin^2\frac{y}{R}\,d\phi^2\right)\Bigg]. \end{aligned} \tag{3.5}$$

In the limit $R \to \infty$ the metric reduces to

$$ds^2 = -4dx^+dx^- - (r^2 + w^2 + y^2)\,dx^{+2} + dr^2 + r^2 d\Omega_3^2 + dw^2 + w^2 d\theta^2 + dy^2 + y^2 d\phi^2 \tag{3.6}$$

which is the standard pp-wave metric. To see that, we make substitions:

$$x^5 = \omega\cos\theta, \quad x^6 = \omega\sin\theta \tag{3.7}$$

$$x^7 = y\cos\phi, \quad x^8 = y\sin\phi \tag{3.8}$$

And similar for the x^1 through x^4 from r and Ω angles. The metric can then be written as usual:

$$ds^2 = -4dx^+dx^- - \sum_{i=1}^{8}(x^i)^2\,dx^{+2} + \sum_{i=1}^{8} dx^{i^2} \ , \tag{3.9}$$

It is important to note here, that we are interested in the near-pp wave limit and should therefore be keeping the next order in $\frac{1}{R^2}$ as well. However, a detailed analysis shows that for the purposes of analyzing the effects of orbifold on the spectrum of one impurity states and supersymmetries higher orders in the metric and the subsequent corrections to the action and Hamiltonian do not contribute and that the only relevant contribution to the order $\frac{1}{R^2}$ will come from orbifold identifications. To establish this we have re-done all the near-pp wave limit calculations, originally done in [44], but with the orbifolding condition imposed.

Looking at the periodicity equations coming from the orbifolding 3.2 and the definitions of the pp-wave coordinates. we can see first of all that the light cone gauge coordinate x^- acquires periodicity condition:

$$x^- \to x^- + \frac{\pi R^2}{M} \tag{3.10}$$

R^2, the radius of AdS_5 and S_5, is given by:

$$R^2 = \sqrt{4\pi g_s \alpha'^2 NM} \tag{3.11}$$

where MN is the total number of units of 5-form flux through the 5-sphere with M being the number of copies of fundamental domain that are identified by the orbifold group and

N the number of flux units per fundamental domain. The rules of the Penrose limit taken, demand that R^2 be made large by scaling both N and M to infinity while keeping g_s small but finite. Also, the ratio $\frac{N}{M}$ is kept fixed and finite so a finite quantity can be defined:

$$R^- = \frac{R^2}{2M} \tag{3.12}$$

It can be seen therefore, that even in the full pp-wave limit the null direction becomes periodic, resulting in the light-cone momentum $2p^+$ being quantized in the units of $\frac{1}{R^-}$. This is exactly the discrete light-cone gauge quantization (DLCQ) which will lead to the introduction of wrapped states and a change of the level-matching condition in comparison to the standard pp-wave string theory and ultimately to the introduction of the central extension into the superalgebra. All this has been discussed, elsewhere in our work, as well as in [73].

Discrete light-cone quantization of the string on the pp-wave background is a slight generalization of [36]. One component of the light-cone momentum is quantized as

$$2p^+ = \frac{k}{R^-} \quad , \quad k = 1, 2, 3, ... \tag{3.13}$$

The other component is the light-cone-gauge Hamiltonian,

$$\begin{aligned} 2p^- &= \sum_{n=-\infty}^{\infty} \left(\sum_{i=1}^{8} \alpha_n^{i\dagger} \alpha_n^i + \sum_{\alpha=1}^{8} \beta_n^{\alpha\dagger} \beta_n^\alpha \right) \sqrt{1 + \frac{4n^2 (R^-)^2}{k^2 \alpha'^2}} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{i=1}^{8} \alpha_n^{i\dagger} \alpha_n^i + \sum_{\alpha=1}^{8} \beta_n^{\alpha\dagger} \beta_n^\alpha \right) \sqrt{1 + \frac{4\pi g_s N}{M} \frac{n^2}{k^2}} \end{aligned} \tag{3.14}$$

where $\alpha_n^i, \alpha_n^{i\dagger}$ and $\beta_n^\alpha, \beta_n^{\alpha\dagger}$ are the annihilation and creation operators for the discrete bosonic and fermionic transverse oscillations of the string, respectively. They obey the (anti-) commutation relation

$$\left[\alpha_{n_1}^i, \alpha_{n_j}^{j\dagger}\right] = \delta^{ij} \delta_{n_i n_j} \quad , \quad \left\{\beta_{n_1}^\alpha, \beta_{n_j}^{\beta\dagger}\right\} = \delta^{\alpha\beta} \delta_{n_i n_j} \tag{3.15}$$

In the last line of 3.14 we have written the compactification radius in terms of string background parameters.

There are also wrapped states. If the total number of times that the closed string wraps the compact null direction is m, the level-matching condition is

$$km = \sum_{n=-\infty}^{\infty} n \left(\sum_{i=1}^{8} \alpha_n^{i\dagger} \alpha_n^i + \sum_{\alpha=1}^{8} \beta_n^{\alpha\dagger} \beta_n^\alpha \right) \quad , \tag{3.16}$$

The states of the string are characterized by their discrete light-cone momentum k and their wrapping number m. The lowest energy state in a given sector is the string sigma model vacuum, $|k, m\rangle$ which obeys

$$\alpha_n^i |k, m\rangle = 0 = \beta_n^\alpha |k, m\rangle \quad , \quad \forall n, i, \alpha$$

Other string states are built from the vacuum by acting with transverse oscillators,

$$\prod_{j=1}^{L} \alpha_{n_j}^{i_j\dagger} \prod_{j'=1}^{L'} \beta_{n_{j'}}^{\alpha_{j'}\dagger} \mid k, m\rangle \tag{3.17}$$

The level matching condition reads

$$\sum_{j=1}^{L} n_j + \sum_{j'=1}^{L'} n_{j'} = k\,m. \tag{3.18}$$

There is, however, one more simultaneous periodicity condition for the bosonic variables. Namely, one coming from the periodicity of ϕ. One way to express it simply is to do the following change of variables:

$$z^1 = x^1 + ix^2, \quad z^2 = x^3 + ix^4 \tag{3.19}$$

$$y^1 = x^5 + ix^6, \quad y^2 = x^7 + ix^8 \tag{3.20}$$

which explicitly shows the breaking of the $SO(8)$ symmetry into $SO(4) \times SO(4)$ with the first $SO(4)$ coordinates (z) being descendent from the coordinates of the AdS_5 and (y)s - the second $SO(4)$ - coming from the S_5. The orbifolding periodicity condition is then simply:

$$y_2 \to e^{\frac{2\pi i w}{M}} y_2, \quad \bar{y}_2 \to e^{\frac{-2\pi i w}{M}} \bar{y}_2 \tag{3.21}$$

If we chose the light-cone to be in non-orbifolded direction, we do not get the DLCQ quantization but we obtain the same periodicity condition on y_1 as we do on y_2.

It should be noted that contrary to the light-cone direction periodicity, these periodicity conditions are of the order $\frac{1}{M}$ and therefore are not present in the full pp-wave limit. However, it is the contribution from them that will cause energy splitting and the change in the number of supersymmetries that occurs in the near pp-wave limit.

Fermionic fields also acquire periodicity from the orbifolding. The superstring on the pp-wave is described by the 8 worldsheet scalars and 8 worldsheet fermions which are free and massive. Bosonic fields are described by the functions y^i and z^i as a representation of $SO(4)_1 \times SO(4)_2$. Fermionic fields $\psi_{\alpha_1\alpha_2}, \psi_{\dot{\alpha}_1\dot{\alpha}_2}$ will likewise be a representation of those, with the first and second spinorial indices transformining under $SO(4)_1$ and $SO(4)_2$ respectively. The dot on the spinorial index represents the splitting of the $SO(4)$ into $SU(2) \times SU(2)$ as per the Appendix A.

The orbifold transformation will always affect exactly one half of the fermionic fields. This can be seen from the index analysis of bosons in the spinorial representation (once again, using the conventions of the Appendix A) or more generally, by analysis of the breaking of the $SO(4)_2$ under the orbifold group.

Half of the fermionic fields affected will split into quarters, each consisting of a pair of fields and those pairs will acquire $\psi \to e^{\frac{2\pi i}{M}}\psi$ and $\psi' \to e^{-\frac{2\pi i}{M}}\psi'$ periodicities respectively. Which fermionic fields end up acquiring the periodicity ends up depending on the choice of the orbifolding direction.

We will argue that in the DLCQ case two pairs of fermionic fields will have different chiralities and will therefore be from the different $SU(2)$s:

$$\psi_{\alpha_1 1_2} \to e^{\frac{2\pi i w}{M}} \psi_{\alpha_1 1_2}, \quad \psi_{\dot{\alpha}_1 \dot{2}_2} \to e^{-\frac{2\pi i w}{M}} \psi_{\dot{\alpha}_1 \dot{2}_2} \tag{3.22}$$

$$\psi_{\alpha_1 2_2} \to \psi_{\alpha_1 2_2}, \quad \psi_{\dot{\alpha}_1 \dot{1}_2} \to \psi_{\dot{\alpha}_1 \dot{1}_2} \tag{3.23}$$

To see that this is necessarily the case, consider the following argument: The effect of orbifolding on the $SO(4)_2$ will depend on the directions on the original $SO(6)$ being orbifolded. There is exactly three distinct choices there. Either the orbifolding directions will be two transverse directions (one option) or they will be a light-cone direction and one of the transverse directions (two options).

Assuming that one fermionic pair affected by orbifolding belongs to an arbitrary $SU(2)$ the other pair will then either belong to the same $SU(2)$ (one option because two pairs would constitute entire $SU(2)$) or to the other $SU(2)$ (two options). This "singlet" and "doublet" are theories with physically distinct symmetries and must match each other between the two cases.

The conclusion is that the periodicity conditions on the fermions differ, depending on whether or not we orbifold along the light-cone direction, even though no fermionic field "disappears" out of the action in the fashion of x^+ and x^-.

This argument will become even more clear when we write down explicit orbifold transformations in the gauge theory case.

The periodicity of the DLCQ fermions is the one given above in 3.22 and 3.23 whereas the other case of orbifolding in two transverse directions yields:

$$\psi_{\dot{\alpha}_1 \dot{1}_2} \to e^{\frac{2\pi i w}{M}} \psi_{\dot{\alpha}_1 \dot{1}_2}, \quad \psi_{\dot{\alpha}_1 \dot{2}_2} \to e^{-\frac{2\pi i w}{M}} \psi_{\dot{\alpha}_1 \dot{2}_2} \tag{3.24}$$

$$\psi_{\alpha_1 1_2} \to \psi_{\alpha_1 1_2}, \quad \psi_{\alpha_1 t 2_2} \to \psi_{\alpha_1 2_2} \tag{3.25}$$

The results 3.22, 3.23, 3.24 and 3.25 can also be obtained in rigorous but not particularly illuminating fashion by index analysis starting from the bosonic results.

We will return to the Hamiltonian and the super-charges of this theory later in the context of the near-pp-wave limit.

3.1.2 $\mathcal{N} = 2$ gauge theory

Most of the treatment in this subsection follows closely that in [73].

The dual gauge theory is constructed by replacing the N' coincident branes of the original $U(N')$ gauge theory with N coincident branes located at the $\mathbb{C}^3/\mathbb{Z}_M$ orbifold point. In this new theory we can therefore talk about M copies of N branes. This breaks the gauge group in the following way:

$$U(NM) \to U(N)^{(1)} \times U(N)^{(2)} \times \cdots U(N)^{(M)}. \tag{3.26}$$

$U(N)^{(M+1)}$ is identified with $U(N)^{(1)}$. The orbifold group will be the cyclic group Z_M whose generator γ acts on the six scalar fields of $\mathcal{N}=4$ theory as

$$\gamma : \left(\frac{\phi^1+i\phi^2}{\sqrt{2}}, \frac{\phi^3+i\phi^4}{\sqrt{2}}, \frac{\phi^5+i\phi^6}{\sqrt{2}}\right) = \left(\omega\frac{\phi^1+i\phi^2}{\sqrt{2}}, \omega^{-1}\frac{\phi^3+i\phi^4}{\sqrt{2}}, \frac{\phi^5+i\phi^6}{\sqrt{2}}\right), \quad \omega = e^{\frac{2\pi i}{M}} \tag{3.27}$$

If we want to express the orbifolding in terms of the R-symmetry we can assemble the scalar fields into the anti-symmetric bi-spinor

$$\varphi_{ab} = \begin{bmatrix} 0 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & 0 & \bar{\varphi}_3 & -\bar{\varphi}_2 \\ -\varphi_2 & -\bar{\varphi}_3 & 0 & \bar{\varphi}_1 \\ -\varphi_3 & \bar{\varphi}_2 & -\bar{\varphi}_1 & 0 \end{bmatrix} \tag{3.28}$$

which transforms as

$$\varphi_{ab} \to U_a{}^{a'} U_b{}^{b'} \varphi_{a'b'} \tag{3.29}$$

under the $SU(4)$ R-symmetry and which satisfies the self-dual constraint

$$\bar{\varphi}^{ab} = \frac{1}{2}\epsilon^{abcd}\varphi_{cd} \tag{3.30}$$

This constraint ensures that $\sum_{ab}\bar{\varphi}^{ab}\varphi_{ab} = 4\sum_i \varphi_i\varphi_i$ is invariant. In the same basis, the four Weyl spinors transform as

$$\chi_{\alpha a} \to U_a{}^{a'}\chi_{\alpha a'} \quad , \quad \bar{\chi}^a_{\dot{\alpha}} \to \bar{\chi}^{a'}_{\dot{\alpha}} U^{\dagger}_{a'}{}^{a} \tag{3.31}$$

The orbifold transformation is implemented with

$$U_{\text{orb}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^{-1} \end{bmatrix} \tag{3.32}$$

The $U(1)$ symmetry corresponding to the conserved charge J is implemented by

$$U_J = \begin{bmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{bmatrix} \tag{3.33}$$

The four Weyl supercharges transform in the conjugate representation. They are schematically like

$$\bar{Q}^a \sim \bar{\varphi}^{ab}\chi_b \quad , \quad Q_a = \varphi_{ab}\bar{\chi}^b \tag{3.34}$$

If χ_a, Q_a transform like a 4, $\bar{\chi}^a, Q^a$ transform like $\bar{4}$.

The orbifold projection is the constraint

$$U_{\text{orb}}Q = Q \quad , \quad \bar{Q}U^{\dagger}_{\text{orb}} = \bar{Q}$$

which will set $\bar{Q}^3, \bar{Q}^4 \sim 0$ and $Q_3, Q_4 \sim 0$. In this way the $\mathcal{N}=4$ supersymmetry will be reduced to $\mathcal{N}=2$ supersymmetry with eight supercharges $Q_1, Q_2, \bar{Q}^1, \bar{Q}^2$.

Importantly, however, in the double scaling limit BMN limit $M \to \infty$ and so $\omega \to 1$ and the $\mathcal{N}=4$ supersymmetry reappears.

3.1.3 Duality

Following the treatment of BMN we identify the energy in the string theory with the conformal dimension Δ of the operators in the gauge theory.

Likewise, we would like to identify the J angular momentum with the charge under under SO(6) R-symmetry. With the R-symmetry broken, however, where there was one angular momentum J, now there are two angular momenta [83]:

$$J = -\frac{i}{2M}\left(\partial_\chi - \partial_\phi\right), \qquad J' = -\frac{i}{2}\left(\partial_\chi + \partial_\phi\right) \tag{3.35}$$

of which J' generates $U(1)$ group which is in the remaining $SU(2)$ of the R-symmetry. There is also $U(1)$ generated by J whose eigenvalues are integer multiples of M.

Light-cone momenta are then expressed as

$$\begin{aligned} 2p^- &= i(\partial_t + \partial_\chi) = \Delta - MJ - J' \\ 2p^+ &= i\frac{(\partial_t - \partial_\chi)}{R^2} = \frac{\Delta + MJ + J'}{R^2}. \end{aligned} \tag{3.36}$$

It is useful to label the scalar fields of the gauge theory as:

$$A_I = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2), \quad B_I = \tfrac{1}{\sqrt{2}}(\phi^3 + i\phi^4), \quad \Phi_I = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)\ . \tag{3.37}$$

Where the fields are $N \times N$ matrix blocks and the index $I = 1...M$ refers to the corresponding gauge group .

The charges of the A_I, B_I, and Φ_I fields are [83]:

	Δ	MJ	J'
A_I	1	1/2	1/2
B_I	1	−1/2	1/2
Φ_I	1	0	0

With a similar table for the super-partners being given in full in [83].

In analogy with the BMN case, we would like to take Δ and $MJ + J'$ to infinity as R^2, while keeping their difference finite.

The desired operators are long chains of A_I's (for which $\Delta - (MJ + J') = 0$), with insertions of $\Phi_I, \bar{\Phi}_I, B_I, \bar{B}_I$ as the fundamental impurities which have $\Delta = 1$ while having $MJ + J' = 0$. The other $SO(4)$ impurities are constructed via insertions of derivatives of the A_I. In order that the operator be gauge invariant, the product must be over all M copies of $SO(N)$. The vacuum is then given by:

$$|k = 1, m = 0\rangle \leftrightarrow \frac{1}{\sqrt{N_1^M}}\,\mathrm{Tr}\left(A_1 A_2 \ldots A_M\right) \tag{3.38}$$

where for the string vacuum we must have $m = 0$ (a string must exist in order to wrap a direction). For general k, the operator is

$$|k, m=0\rangle \leftrightarrow \frac{1}{\sqrt{N_1^{kM}}} \mathrm{Tr}\Big((A_1 A_2 \ldots A_M)^k \Big) \tag{3.39}$$

so that k copies of the string $A_1 \ldots A_{N_2}$ are traced over. Adding impurities we see a novel feature as compared to the standard BMN picture. Consider the addition of a single impurity to the operator (3.38)

$$\left(a_n^{5\dagger} + i a_n^{6\dagger}\right) |k, m\rangle \leftrightarrow \sum_{I=1}^{kM} e^{2\pi i n I/(kM)} \mathrm{Tr}\left(A_1 \ldots A_{I-1} \Phi_I A_I \ldots A_M \left(A_1 \ldots A_M\right)^{k-1} \right) \tag{3.40}$$

We have superposed over positions at which the impurity could be inserted. The momentum in the insertion n coincides with the world-sheet momentum of the oscillator state. The level matching condition comes from realizing that the actual periodicity of the operator is $I \to I + M$, rather than $I \to I + kM$, which the plane waves anticipate. This requires that $n = km$, where m is an integer; this is the level matching condition. The integer m is identified with the wrapping number of the world-sheet on the compact coordinate.

In [73], the DLCQ analogue of (2.42) was computed for one and two-impurity operators built upon $k = 1, 2$, and 3 vacuua. The couplings λ' and g_2 may be expressed in terms of N, M, and k

$$\begin{aligned} \alpha' p^+ &= \frac{\alpha' k}{2R_-} = \frac{\alpha' k M}{R^2} = \frac{k}{g_{YM}} \sqrt{\frac{M}{N}} \quad \to \quad \lambda' = \frac{1}{(\alpha' p^+)^2} = \frac{g_{YM}^2 N}{k^2 M} \\ g_2 &= g_{YM}^2 (\alpha' p^+)^2 = \frac{k^2 M}{N} \end{aligned} \tag{3.41}$$

where μ has been scaled out of the metric. The results of [73] may be summarized as follows.

1. The single oscillator state is no longer a protected operator. Its dimension Δ should get radiative corrections beyond the tree level in Yang-Mills theory, even for planar diagrams. In fact, it must get such corrections if it is to match the string spectrum,

$$2p^- = \sqrt{1 + \frac{g_{YM}^2 N}{M} \frac{n^2}{k^2}} \tag{3.42}$$

   for planar diagrams. It produces this spectrum to one order in g_{YM}^2. However, the operator is quasi-protected in that, in the double scaling limit, all non-planar corrections to (3.42) vanish. Yang-Mills computation predicts that the spectrum of this state in string theory does not receive string loop corrections.

2. String states with one unit of light cone momentum and any number of oscillators are free states in that they do not get string loop corrections.

3. For string states with two units of light cone momentum and two impurities there are two possibilities: states for which both the world-sheet momenta are integer multiples

of $k = 2$, namely are even, have a free spectrum; states for which both the world-sheet momenta are odd get only the one string loop correction given by:

$$\Delta - N_2 J - J' = \left(2 + \frac{1}{2}(n_1^2 + n_2^2)\lambda' + \dots\right) + \begin{cases} g_2^2\left(\frac{1}{16\pi^2}\lambda' + \dots\right) & n_1, n_2 \text{ odd} \\ 0 & n_1, n_2 \text{ even} \end{cases} \tag{3.43}$$

which truncates at $\mathcal{O}(g_2^2)$. The states with even-odd world-sheet momenta are excluded by level matching.

4. String states with three units of light cone momentum and two impurities for which both the world-sheet momenta are integer multiples of $k = 3$, have a free spectrum. $k = 3$ states for which both the world-sheet momenta are not integer multiples of 3 get computable corrections to all orders:

$$\begin{aligned} \Delta - N_2 J - J' &= \left(2 + \frac{1}{2}(n_1^2 + n_2^2)\lambda' + \dots\right) \\ &+ \frac{g_2^2\lambda'}{16\pi^2}\left[1 + \frac{6}{\pi(n_1 - n_2)}\left(\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right) - \cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right)\right] \\ &+ \dots \end{aligned} \tag{3.44}$$

where **all** non-planar corrections (not just the leading term shown) vanish for n_1, n_2 multiples of three.

3.2 String field theory on pp-wave orbifold

The construction of the string field theory in the DLCQ case is parallel to the construction in the regular light-cone gauge, with two obvious but important distinctions. Firstly, momentum p^+ is quantized and secondly the new level-matching conditions take effect that, among other things, allow single impurity states to exist.

Quantization of the momentum imposes several constraints on the energy shift that correspond to the results of the previous section.

1. Every Neumann matrix with a "leg" on an external string carries the term proportional to $\sin(\frac{n_i k_1}{k})$ where k is the momentum of the external string. This means that the states with mode numbers which are a integer multiples of the external k will always have zero non-planar corrections. This corresponds to result 1. from the previous section and also explains riders on the results 3. and 4..

2. $k = 1$ string can not split because, by conservation of momentum, there is no lower momentum number for it to split into. It will therefore be described solely by the non-interacting string / plane level diagrams regardless of the number of impurities. This corresponds to result 2. from the previous section.

To calculate the actual energy-shift we apply exactly the same methods as in the regular light-cone gauge with the exception of the integral over p^+ being replaced by a sum. The results quoted here were calculated by our group after our work on [79] and [80] and were first published in [81].

3.2.1 $k = 2$ Impurity-conserving mass-shift

The mode numbers of the external $|[\mathbf{9},1]\rangle$ state have distinct, odd values n_1 and n_2 satisfying

$$n_1 + n_2 = 2m \tag{3.45}$$

where m is the external wrapping number. For the impurity-conserving channel, we may either place the two intermediate-state impurities on the same string (string #1), or one on each string. In the former case string #2 is in its vacuum state and necessarily has wrapping number $m_2 = 0$. The level-matching condition for the excited string gives $q_1+q_2 = m_1$, where q_i are the internal mode numbers; conservation of wrapping number then gives $m_1 = m$. In the latter case we have $q_1 = m_1$, and $q_2 = m_2$ while $m_1 + m_2 = m$. Thus the two choices for the distribution of intermediate state impurities are indistinguishable, both leading to the same condition which is introduced into the amplitudes via the factor $\delta_{q_1,m-q_2}$ where $m = (n_1 + n_2)/2 \in \mathbb{Z}$. We begin with the SVPS result for the H_3 term

$$\begin{aligned}\delta E^{\text{SVPS}}_{H_3} = \frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{64\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[&\left(L^{3\,r_1}_{n_1\,q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + L^{3\,r_1}_{n_1\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}\\ &+ L^{3\,r_1}_{n_2\,q_1}L^{3\,r_1}_{n_1\,q_1}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2} + L^{3\,r_1}_{n_2\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}\Bigg]\\ &\times\frac{-\alpha_3\,\delta_{q_1,m-q_2}}{\omega_{n_1}+\omega_{n_2}-\beta_{r_1}^{-1}\omega_{q_1}-\beta_{r_2}^{-1}\omega_{q_2}} + (n_1 \leftrightarrow n_2)\end{aligned} \tag{3.46}$$

where now instead of an integration over a continuous $r \in [0,1]$, r is fixed at $1/2$. The result is

$$\begin{aligned}\left(\delta E^{\text{SVPS}}_{H_3}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[&8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - \frac{1}{\pi}\left(\frac{13}{4}(n_1^2+n_2^2)+\frac{1}{2}n_1n_2\right)\lambda'^{5/2}\\ &-\frac{1}{\pi^2}\left(\frac{22}{3}(n_1^2+n_2^2)+\frac{4}{3}n_1n_2\right)\lambda'^{5/2}+\ldots\Bigg].\end{aligned} \tag{3.47}$$

The contact term contribution is as follows

$$\begin{aligned}\delta E^{\text{SVPS}}_{H_4} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[&(K_{-n_1})^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{-n_1}K_{-n_2}\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\Bigg]\delta_{q_1,m-q_2}\\ &+(n_1 \leftrightarrow n_2)\end{aligned} \tag{3.48}$$

giving

$$\left(\delta E_{H_4}^{\mathrm{SVPS}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[\lambda' + \left(\frac{n_1+n_2}{2} - 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} - \left(\frac{n_1^2+n_2^2}{4} + 2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 - \left(\frac{n_1^3+n_2^3}{2} - \frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right) - \frac{1}{\pi^2}\left(\frac{23}{6}(n_1^2+n_2^2)+\frac{1}{3}n_1n_2\right)\right)\lambda'^{5/2} + \ldots\Bigg]. \tag{3.49}$$

Combining the results we find

$$\left(\delta E^{\mathrm{SVPS}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[\lambda' + \left(\frac{n_1+n_2}{2} + 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} - \left(\frac{n_1^2+n_2^2}{4} + 2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 - \left(\frac{n_1^3+n_2^3}{2} + \frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right) + \frac{1}{\pi^2}\left(\frac{7}{2}(n_1^2+n_2^2)+n_1n_2\right)\right)\lambda'^{5/2} + \ldots\Bigg]. \tag{3.50}$$

This result does display a leading agreement with the gauge theory result (3.43). However, it also suffers maximally from half-integer powers of λ'. We will see that the DY vertex will do better, in analogy with the standard case. First, we present the results for the DVPPRT vertex. The expression for the H_3 term is

$$\delta E_{H_3}^{\mathrm{DVPPRT}} = \frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{64\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[\left(\widetilde{L}_{n_1\,q_1}^{3\,r_1}\right)^2\left(\widetilde{N}_{n_2\,q_2}^{3\,r_2}\right)^2 + \widetilde{L}_{n_1\,q_1}^{3\,r_1}\widetilde{L}_{n_1\,q_2}^{3\,r_2}\widetilde{N}_{n_2\,q_2}^{3\,r_2}\widetilde{N}_{n_2\,q_1}^{3\,r_1} + \widetilde{L}_{n_2\,q_1}^{3\,r_1}\widetilde{L}_{n_1\,q_1}^{3\,r_1}\widetilde{N}_{n_1\,q_2}^{3\,r_2}\widetilde{N}_{n_2\,q_2}^{3\,r_2} + \widetilde{L}_{n_2\,q_1}^{3\,r_1}\widetilde{L}_{n_1\,q_2}^{3\,r_2}\widetilde{N}_{n_1\,q_2}^{3\,r_2}\widetilde{N}_{n_2\,q_1}^{3\,r_1}\Bigg] \times \frac{-\alpha_3\,\delta_{q_1,m-q_2}}{\omega_{n_1}+\omega_{n_2}-\beta_{r_1}^{-1}\omega_{q_1}-\beta_{r_2}^{-1}\omega_{q_2}} + (n_1 \leftrightarrow n_2) \tag{3.51}$$

with result

$$\left(\delta E_{H_3}^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[-2\lambda' + 8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + \frac{3}{2}(n_1^2+n_2^2)\lambda'^2 - \frac{1}{\pi}\left(\frac{21}{4}(n_1^2+n_2^2)+\frac{1}{2}n_1n_2\right)\lambda'^{5/2} - \frac{1}{\pi^2}\left(\frac{38}{3}(n_1^2+n_2^2)-\frac{4}{3}n_1n_2\right)\lambda'^{5/2} + \ldots\Bigg] \tag{3.52}$$

while the contact term gives

$$\delta E_{H_4}^{\mathrm{DVPPRT}} = -\frac{g_2^2\,\alpha'}{16\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\left[(K_{n_1})^2\,(G_{q_1})^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{n_1}K_{n_2}\,(G_{q_1})^2\,\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right]\delta_{q_1,m-q_2} + (n_1 \leftrightarrow n_2) \tag{3.53}$$

with result

$$\begin{aligned}\left(\delta E_{H_4}^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[&\lambda' - \left(\frac{n_1+n_2}{2} + 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} \\ &- \left(\frac{n_1^2+n_2^2}{4} - 2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 \\ &+\left(\frac{n_1^3+n_2^3}{2} + \frac{1}{\pi}\left(\frac{13}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right) + \frac{1}{\pi^2}\left(\frac{23}{6}(n_1^2+n_2^2)+\frac{1}{3}n_1n_2\right)\right)\lambda'^{5/2} + \ldots\Bigg].\end{aligned} \tag{3.54}$$

Combining these results we obtain the mass-shift for the DVPPRT vertex

$$\begin{aligned}\left(\delta E^{\mathrm{DVPPRT}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\Bigg[&-\lambda' - \left(\frac{n_1+n_2}{2} - 4\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^{3/2} \\ &+ \left(\frac{5}{4}(n_1^2+n_2^2) + 2(n_1+n_2)\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\right)\lambda'^2 \\ &+\left(\frac{n_1^3+n_2^3}{2} - \frac{1}{\pi}\left(\frac{29}{8}(n_1^2+n_2^2)+\frac{1}{4}n_2n_1\right) - \frac{1}{\pi^2}\left(\frac{53}{6}(n_1^2+n_2^2)-\frac{5}{3}n_1n_2\right)\right)\lambda'^{5/2} + \ldots\Bigg]\end{aligned} \tag{3.55}$$

which fails to agree with the gauge theory result even at the leading order, as the sign is incorrect. Finally, we compute the extra cross-terms required to assemble the DY result. The H_3 cross-term is given by

$$\begin{aligned}\delta E_{H_3}^{\text{S-DV}} = 2\frac{\langle H_3^{\mathrm{DVPPRT}}|e\rangle\langle e|H_3^{\mathrm{SVPS}}\rangle}{\Delta E} = \\ \frac{2}{r(1-r)}\frac{g_2^2\,\alpha'^2}{32\,\alpha_3^6}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\Bigg[&\widetilde{L}^{3\,r_1}_{n_1\,q_1}L^{3\,r_1}_{n_1\,q_1}\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + \widetilde{L}^{3\,r_1}_{n_1\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1} \\ &+ \widetilde{L}^{3\,r_1}_{n_2\,q_1}L^{3\,r_1}_{n_1\,q_1}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2} + \widetilde{L}^{3\,r_1}_{n_2\,q_1}L^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_1}_{n_2\,q_1}\Bigg] \\ &\times\frac{-\alpha_3\,\delta_{q_1,m-q_2}}{\omega_{n_1}+\omega_{n_2}-\beta_{r_1}^{-1}\omega_{q_1}-\beta_{r_2}^{-1}\omega_{q_2}} + (n_1 \leftrightarrow n_2)\end{aligned} \tag{3.56}$$

resulting in

$$\left(\delta E_{H_3}^{\text{S-DV}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\left[8\lambda' - 16\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} - 4(n_1^2+n_2^2)\lambda'^2 + \frac{1}{\pi}\left(\frac{17}{2}(n_1^2+n_2^2)+n_1n_2\right)\lambda'^{5/2} + \frac{20}{\pi^2}(n_1^2+n_2^2)\lambda'^{5/2} + \ldots\right]. \tag{3.57}$$

The contact cross-term is given by

$$\delta E_{H_4}^{\text{S-DV}} = \frac{1}{2}\langle Q_3^{\text{DVPPRT}}|e\rangle\langle e|Q_3^{\text{SVPS}}\rangle = -\frac{g_2^2\,\alpha'}{8\,\alpha_3^3}\sum_{r_1\,r_2}\sum_{q_1\,q_2}\left[K_{n_1}K_{-n_1}\left(G_{q_1}\right)^2\left(\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right)^2 + K_{n_1}K_{-n_2}\left(G_{q_1}\right)^2\widetilde{N}^{3\,r_2}_{n_1\,q_2}\widetilde{N}^{3\,r_2}_{n_2\,q_2}\right]\delta_{q_1,m-q_2} + (n_1\leftrightarrow n_2) \tag{3.58}$$

with result

$$\left(\delta E_{H_4}^{\text{S-DV}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\left[-2\lambda' + 8\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{3/2} + \left(n_1^2+n_2^2\right)\lambda'^2 - \left(\frac{1}{\pi}\left(\frac{15}{4}(n_1^2+n_2^2)+\frac{3}{2}n_2n_1\right)+\frac{1}{\pi^2}\left(\frac{26}{3}(n_1^2+n_2^2)+\frac{8}{3}n_1n_2\right)\right)\lambda'^{5/2} + \ldots\right]. \tag{3.59}$$

Assembling the final result $\delta E^{\text{DY}} = (\delta E^{\text{DVPPRT}} + \delta E^{\text{S-DV}} + \delta E^{\text{SVPS}})/4$, we find

$$\left(\delta E^{\text{DY}}\right)_{k=2} = \frac{g_2^2}{16\pi^2}\frac{3}{2}\left[\lambda' - \frac{n_1^2+n_2^2}{3}\lambda'^2 - \frac{(n_1+n_2)^2}{6}\left(\frac{1}{\pi^2}+\frac{1}{2\pi}\right)\lambda'^{5/2} + \ldots\right] \tag{3.60}$$

This result matches the leading order gauge theory result (3.43) if we re-scale the undetermined function f (appearing in front of the vertices) by $\sqrt{2/3}$. The result is superior to the SVPS result (3.50) as it does not contain the 3/2's power of λ'. It would be interesting to know whether the λ'^2 term also agrees with gauge theory, however the gauge theory computation of this term has yet to be done.

3.2.2 $k = 3$ Impurity-conserving mass-shift

For the $k = 3$ string, the splitting and level-matching are more involved. There are two distinct cases, the first is when string #1 has $k_1 = 1$. We can then distribute the two

intermediate state impurities both on string #1, both on string #2, or one impurity per string (of which there are two equivalent configurations). The next case is when the assignments of light-cone momenta are reversed, so that string #1 has $k_1 = 2$ (and so string #2 has $k_2 = 1$). This just counts the $k_1 = 1$ case again, leading to a factor of two. The level-matching is therefore achieved via the insertion of the following operator

$$r = \frac{1}{3}, \qquad 2\left(\delta^{r_1,1}\delta^{r_2,1}\delta_{q_1,m-q_2} + \delta^{r_1,2}\delta^{r_2,2}\delta_{q_1,2m-q_2} + 2\delta^{r_1,1}\delta^{r_2,2}\delta_{q_1,2(m-q_1)}\right) \tag{3.61}$$

where the intermediate-state impurities have mode-number/string label configurations (q_1, r_1) and (q_2, r_2), and $m = (n_1 + n_2)/3 \in \mathbb{Z}$ is the external state winding number while n_1 and n_2 are integers and not multiples of three.

The expressions given for the $k = 2$ case in the previous subsection are equally valid here, however with the replacement of the $k = 2$ delta function with (3.61). The results are difficult to obtain for high order in λ', and so we present leading order results only. Since the calculations are straightforward, we will be brief and simply state the results

$$\left(\delta E^{\text{SVPS}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\left[1 + \frac{9}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right) - \cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1 - n_2)}\right] + \ldots \tag{3.62}$$

$$\left(\delta E^{\text{DVPPRT}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\left[-1 + \frac{3}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right) - \cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1 - n_2)}\right] + \ldots \tag{3.63}$$

$$\left(\delta E^{\text{DY}}\right)_{k=3} = \frac{g_2^2\lambda'}{16\pi^2}\frac{3}{2}\left[1 + \frac{3}{2}\frac{\left[\cos\left(\frac{\pi n_1}{3}\right)\sin\left(\frac{\pi n_1}{3}\right) - \cos\left(\frac{\pi n_2}{3}\right)\sin\left(\frac{\pi n_2}{3}\right)\right]}{\pi(n_1 - n_2)}\right] + \ldots \tag{3.64}$$

Comparing with the gauge theory result (3.44), we see that although the dependence on the external mode numbers is of the correct form, the coefficient of the second term is not matched by any of the vertices. Further, the first term of the DVPPRT does not match on account of the sign.

3.2.3 4 impurity channel for the k=2

The relative simplicity of the DLCQ calculations also makes it possible for the "standard" method calculations to be carried to the four impurity channel. The principal simplifications have to do with the fact that the $k = 2$ string splits necessarily into two $k = 1$ strings. Unfortunately the calculation still leads to an integral:

$$\delta E_1 = \frac{g_2^2\lambda'}{16\pi^4}\int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_3 \frac{\left(\Lambda_3^+ + \Lambda_3^-\right)\Lambda_1^+\left[\Lambda_1^+\Lambda_3^+ - \Lambda_1^-\Lambda_3^-\right]}{q_1\omega_1\omega_3\omega_4\left(1 - \omega_1 - \omega_3 - \omega_4\right)} \tag{3.65}$$

where

$$\omega_i = \sqrt{q_i^2 + 1}, \qquad \Lambda_i^+ = \sqrt{\omega_i + 1}, \qquad \Lambda_i^- = e(q_i)\sqrt{\omega_i - 1}. \tag{3.66}$$

which is not analytically solvable.

Our collaborator D. Young solved this integral numerically and the results are presented in [81]. His results confirm the higher-impurity behavior that we have observed in the master-formula 2.166 calculations for the DVPPRT vertex and generalize them to SVPS and DY indices. Specifically, there is still a contribution to all orders in λ, no miraculous cancelations are taking place and the added results are still functionally same but numerically different from the gauge theory ones.

3.3 Giant magnon and the single impurity multiplet

It is possible to draw a parallel between the string theory on the orbifold and the so called "giant magnon" states in the CFT. This was noticed in [86] and [87] and in particular in [84]. A strong version of the statement was made in our paper [85]. In this section we follow [84] to provide the context and background and then state results of our work as published in [85].

3.3.1 Giant magnon in AdS/CFT

The term "magnon" comes from solid state physics and refers to the collective excitation of the electron's spin structure in a crystal lattice. In the CFT context it refers to an impurity in the chain of Z fields in the large J limit. The name is chosen because the problem of diagonalizing the planar Hamiltionian can be reduced to a type of spin-chain.

The large J limit that we are interested in here is defined by:

$$J \rightarrow \infty , \qquad \lambda = g^2 N = \text{fixed} \tag{3.67}$$

$$p = \text{fixed} , \qquad E - J = \text{fixed} \tag{3.68}$$

This differs from the BMN/plane wave limit in two ways. First, here we are keeping λ fixed, while in BMN it was taken to infinity. Secondly, here we are keeping p fixed, while in BMN $n = pJ$ was kept fixed. The reason this limit is of interest in the first place is that it decouples the quantum effects which are governed by the λ from the finite effects governed by J. This distinction persists even after we eventually take the large λ limit.

Despite stated differences with BMN limit we proceed in a similar fashion as there by studying the states with finite $E - J$. The state with $E - J = 0$ corresponds to a long chain (or string) of Zs, namely to the operator $Tr[Z^J]$. We can also consider a finite number of other fields W that propagate along this chain of Zs. In other words we consider operators of the form

$$O_p \sim \sum_l e^{ilp}(\cdots ZZZWZZZ\cdots) \tag{3.69}$$

where the "magnon" field W is inserted at position l along the chain. Using supersymmetry, Beisert has shown [90] that these excitations have a dispersion relation of the form:

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \tag{3.70}$$

Note that the periodicity in p comes from the discreteness of the spin chain. The large 'tHooft coupling limit of this result is

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right| \tag{3.71}$$

3.71 is a strong coupling result and we therefore expect it to have a dual in the perturbative regime of the string theory. Hofman and Maldacena in [84] determine the conditions needed to recover the 3.71 result in the string theory using the usual strings in $AdS_5 \times S_5$.

They show that the strings corresponding to the "giant" magnon (which is to say a magnon on a long chain of Zs) will be a closed string with an open boundary condition, where the azimuth angle spanned by the two ends of the string corresponds to p_{mag} with the side result of introducing the central charges to the super-algebra that matched ones noticed by Beisert [90].

[86] argued that the open boundary condition led to a modification of the level-matching condition and gauge parameter dependence of the spectrum was a result. In [87] it was suggested that the single magnon is well-defined as the twisted state of a closed string on an orbifold – where the orbifold group acts in such a way that it identifies the ends of the string, resulting in a legitimate state of closed string theory.

In [85] we took this reasoning a bit further.

3.3.2 Magnon boundary conditions and the orbifold

The main observation of [85] was that if we consider the single magnon state in the Type IIB string theory, with the boundary condition that the string is open in the direction of magnon motion, we are inevitably led to an orbifold.

To get the gist of our argument, consider the following (drastically oversimplified) example of the closed bosonic string on flat Minkowski spacetime where we legislate that one of the string coordinates is not periodic, but obeys the "magnon" boundary condition $X^1(\tau, \sigma = 2\pi) = X^1(\tau, 0) + p_{\text{mag}}$ and all other variables, including $\partial_\sigma X^1(\tau, \sigma)$ are periodic. Then, a solution of the worldsheet equation of motion $(\partial_\tau^2 - \partial_\sigma^2)\, X^1 = 0$ with the appropriate boundary condition is [4] $X^1 = x^1 + \alpha' p^1 \tau + \frac{\sigma}{2\pi} p_{\text{mag}} +$ oscillators. One of the Virasoro constraints is the level matching condition $L_0 - \tilde{L}_0 = 0$ which takes the form

$$N - \tilde{N} + p^1 \frac{p_{\text{mag}}}{2\pi} = 0 \tag{3.72}$$

where $N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$ and $\tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$. Since the spectra of the operators N and $\tilde{N}$ are integers, there is no solution of the level-matching condition unless $p^1 p_{\text{mag}} = 2\pi \cdot \text{integer}$, i.e. the momentum p^1 is quantized in units of integer$\cdot 2\pi / p_{\text{mag}}$. This is identical to (and indistinguishable from) the situation where the dimension X^1 is compactified with radius $R = \frac{p_{\text{mag}}}{\text{integer}}$ and where we consider a wrapped string with fixed momentum which is then quantized in units of $\frac{2\pi}{R}$. We see that the magnon boundary condition leads us to string theory on a simple orbifold, a periodic identification of the direction in which the magnon boundary condition was taken. We shall observe a similar fact for the more complicated case of a single magnon on the $AdS_5 \times S^5$ background.

The reverse of the above argument is a simple exercise in T-duality, applied to the string sigma model in a direction where the space is not identified. The T-dual of a string with momentum p is one which is no longer closed, but has the magnon boundary condition.

The original giant magnon of [84] is a solition solution of a bosonic Type IIB sigma model on $AdS_5 \times S^5$. We can write the Lagrangian of this theory:

$$\mathcal{L} = -\frac{\sqrt{\lambda}}{4\pi}\left\{-\left(\frac{1+\frac{Z^2}{4}}{1-\frac{Z^2}{4}}\right)^2 \partial_a T\partial^a T + \left(\frac{1}{1-\frac{Z^2}{4}}\right)^2 \partial_a Z\cdot\partial^a Z \right.$$
$$\left. + \left(\frac{1-\frac{Y^2}{4}}{1+\frac{Y^2}{4}}\right)^2 \partial_a\chi\partial^a\chi + \left(\frac{1}{1+\frac{Y^2}{4}}\right)^2 \partial_a Y\cdot\partial^a Y\right\} \tag{3.73}$$

supplemented by Virasoro constraints. The eight fields $\vec{Z}$ and $\vec{Y}$ transform as 4-vectors under $SO(4)\times SO(4) \sim SU(2)^4$. We will impose the magnon boundary condition on the angle coordinate

$$\chi(\tau,\sigma=2\pi) = \chi(\tau,\sigma=0) + p_{\text{mag}} \tag{3.74}$$

If $\chi(\tau,\sigma) = \tilde{\chi}(\tau,\sigma) + p_{\text{mag}}\sigma/2\pi$ with $\tilde{\chi}$ periodic,

$$\mathcal{L}[T,\vec{Z},\chi,\vec{Y}] = \mathcal{L}[T,\vec{Z},\tilde{\chi},\vec{Y}] - \frac{\sqrt{\lambda}}{4\pi}\left(\left(\frac{p_{\text{mag}}}{2\pi}\right)^2 + \frac{p_{\text{mag}}}{\pi}\tilde{\chi}'\right)\left(\frac{1-\frac{Y^2}{4}}{1+\frac{Y^2}{4}}\right)^2 \tag{3.75}$$

The effect of the magnon boundary condition is to add terms to the action. These, as well as similar terms which appear in the Virasoro constraints, will break some of the (super-)symmetries of the background. The last term in 3.75 has the symmetries $SU(2)^2\times SU(2)^2\times R^2$ where the R^2 are translations of T and $\tilde{\chi}$. The bosonic part of the level-matching condition is

$$0 = \int_0^{2\pi} d\sigma\,\{\Pi_T T' + \Pi_Z Z' + \Pi_{\tilde{\chi}}\tilde{\chi}' + \Pi_Y Y'\} + \frac{p_{\text{mag}}}{2\pi}J \tag{3.76}$$

where $\Pi_\mu \equiv \partial\mathcal{L}/\partial\dot{X}^\mu$ are the canonical momenta conjugate to coordinates X^μ and the charge J is the generator of translations of $\tilde{\chi}$, $\chi\to\chi+\text{const}$.

$$J = \int_0^{2\pi} d\sigma\Pi_{\tilde{\chi}} = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\left(\frac{1-\frac{Y^2}{4}}{1+\frac{Y^2}{4}}\right)^2\dot{\tilde{\chi}} \tag{3.77}$$

Since $\chi\sim\chi+2\pi$, the eigenvalues of J must be integers.[2] Furthermore, being generators of translations of the worldsheet σ-argument of the fields, and the fields involved being periodic in σ, the first four terms in (3.76) must be integers plus a possible constant.[3] Since

[2]When fermions are included, they could be half-integers.

[3]Consider the operator ξ which has the property $[\xi,\varphi(\sigma)] = i\frac{d}{d\sigma}\varphi(\sigma)$. Consider eigenstates $|\alpha>$ and $|\alpha'>$ where $\xi|\alpha>=\alpha|\alpha>$. If $<\alpha'|\varphi(\sigma)|\alpha>=<\alpha'|e^{-i\sigma\xi}\varphi(0)e^{i\sigma\xi}|\alpha>=e^{i(\alpha-\alpha')\sigma}<\alpha'|\varphi(0)|\alpha>$, the matrix element obeys $<\alpha'|\varphi(\sigma)|\alpha>=<\alpha'|\varphi(\sigma+2\pi)|\alpha>$ only when $\alpha-\alpha'=$ integers. The eigenvalues are equal to integers plus a constant which is common to all eigenvalues. If, there is a reflection symmetry $\sigma\to 2\pi-\sigma$ under which $\xi\to-\xi$, the constant must be either an integer or half-integer.

the theory has a symmetry under $\sigma \to 2\pi - \sigma$, the constant must be either zero or one-half. Thus, the spectrum of the first terms in (3.76) is either integers or integers+$\frac{1}{2}$. To eliminate the second possibility, we shall see that, in the plane wave limit, we can solve for the spectrum explicitly and there we find that it is integers. Then, since the spectrum should not change discontinuously as the plane wave limit is taken, we conclude that it should always be integers.

Since J comes in units of integers, and the first four terms in (3.76) are integers, (3.76) will only have a solution if $\frac{p_{\rm mag}}{2\pi}$ is a rational number, $\frac{m}{M}$. Then, J is quantized in units of M. This is identical to what should occur for a m-times wrapped string on a Z_M orbifold of $AdS_5 \times S^5$ where the orbifold group Z_M makes the identification $\chi \to \chi + 2\pi\frac{m}{M}$.

To get the superstring, we must include the fermions. For this, we must decide what their boundary conditions will be. It is clear that, at large J, we will obtain the correct magnon supermultiplet if we add them in such a way that, in the modification of the Virasoro constraint (3.76), J also contains the appropriate fermionic contribution $J \to \tilde{J} = \int(\Pi_{\tilde{\chi}}\tilde{\chi}' + \Pi_\psi \Sigma \psi')$. This gives the magnon boundary condition for the fermions

$$\psi(\tau, \sigma = 2\pi) = e^{ip_{\rm mag}\tilde{J}}\psi(\tau, \sigma = 0)e^{-ip_{\rm mag}\tilde{J}} = e^{ip_{\rm mag}\Sigma}\psi(\tau, \sigma = 0) \tag{3.78}$$

where $\Sigma = \text{diag}\left(\frac{1}{2}. - \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ and the orbifold identification is

$$(\chi, \psi) \sim \left(\chi + p_{\rm mag}, e^{ip_{\rm mag}\Sigma}\psi\right) \tag{3.79}$$

All of the fermions have a twist in their boundary condition. With this identification, all supercharges transform non-trivially under the orbifold group and all of the supersymmetries will be broken (in fact, the supercharges are set to zero by the obtifold projection). This twist in the fermion boundary condition and concomitant breaking of supersymmetry is well known from orbifold constructions in string theory [88] and was outlined in detail in a context similar to ours in [89].

Some supersymmetry can be saved if we impose a slightly more elaborate identification:

$$(\chi, Y_1 + iY_2, \psi) \sim \left(\chi + p_{\rm mag}, e^{-ip_{\rm mag}}(Y_1 + iY_2), e^{ip_{\rm mag}\tilde{\Sigma}}\psi\right) \tag{3.80}$$

where, now $\tilde{\Sigma} = \text{diag}\,(0, 0, 1, -1)$. This contains the previous identification of the angle χ as well as a simultaneous rotation of the transverse Y-coordinates. Half of the fermions are un-twisted and this identification preserves half of the supersymmetries. The giant magnon can still be considered a wrapped state of this orbifold where the identified Y-coordinates are not excited.

This double orbifolding geometry turns out to be exactly the one whose plane-wave limit leads to the DLCQ quantization of the string, as described early in this chapter.

3.3.3 Orbifold in gauge theory

The gauge theory dual is likewise one described above. It is obtained by beginning with the parent theory, $\mathcal{N} = 4$ super Yang-Mills with gauge group $SU(MN)$ and coupling constant g_{YM}. Then, we consider a simultaneous R-symmetry transformation by a generator of the Z_M orbifold group and a gauge transform by a constant $SU(MN)$ matrix

$\gamma = \mathrm{diag}(1, \omega, \omega^2, ..., \omega^{M-1})$ where ω is the M-th root of unity. Each diagonal element of the $MN \times MN$-matrix γ is multiplied by the $N \times N$ unit matrix. The projection throws away all fields which are not invariant under the simultaneous transformation. This reduces a typical field which was an $MN \times MN$ matrix in the parent theory to $M \quad N \times N$ blocks embedded in that matrix in the orbifold theory.

For example, consider a field Z of the parent theory which is charged under the orbifold group and transforms as $Z \to \omega Z$. The orbifold projection reduces it to a matrix which obeys

$$Z\gamma = \omega\gamma Z \tag{3.81}$$

By similar reasoning, a field Φ which was neutral in the parent theory commutes with γ once the orbifold projection is imposed,

$$\Phi\gamma = \gamma\Phi \tag{3.82}$$

Given any single-trace operator of the parent $\mathcal{N} = 4$ theory, for example, a single magnon state such as $\mathrm{Tr}Z^J\Phi$, there are a family of M states of the orbifold theory $\mathrm{Tr}\gamma^m Z^J\Phi$ with $m = 0, 1, ..., M-1$. The operator must be neutral under the orbifold group transformation in the parent theory. To see this: we could insert $1 = \gamma^{M-1}\gamma$ into the trace and use the commutators such as 3.81 and 3.82 and cyclicity of the trace to show that the trace of any operator which is not a singlet under the orbifold group must vanish. In our example, if Φ is neutral, this requires quantization of J in units of M, $J = kM$, in the state $\mathrm{Tr}\gamma^m Z^J\Phi$. This gauge dual of the quantization of the momentum J in units of M·integers, rather than integers after the orbifold projection, is imposed in the sigma model, discussed above after 3.77. In addition, the single-trace operator of the parent theory descends to a family of M operators which are distinguished by an additional quantum number, m. It is easy to see that moving the position where Φ was inserted into $\mathrm{Tr}\gamma^m Z^J\Phi$ changes the operator by an overall factor of ω^m. This implies that this trace is already an eigenstate of magnon momentum, $p_{\mathrm{mag}} = 2\pi\frac{m}{M}$. The integer m is the gauge theory dual of the wrapping number of the string state on the orbifold cycle.

There is a theorem to the effect that, in the planar limit of the orbifold gauge theory, untwisted operators (with $m = 0$ in the above examples) have the same correlation functions with each other as those in the planar parent $\mathcal{N} = 4$ gauge theory – with the only difference being a re-scaling of the coupling constant by the order of the orbifold group [91]. For this reason, in the planar limit, the gauge theory resulting from either of the orbifold projections (3.79) or (3.80) is a conformal field theory. In the non-supersymmetric case (3.79) non-planar corrections would give a beta-function, whereas in the $\mathcal{N} = 2$ supersymmetric case (3.80) the beta function would vanish in the full theory.

On the orbifold, the spectrum of states in the $\mathcal{N} = 4$ magnon super-multiplet are expected to be split according to the residual symmetries. In the two cases we considered, the first (3.79) has no supersymmetry but has $SU(2)^4 \times R^2$ bosonic symmetry. We would expect that the fermionic states gain different energies than the bosonic states and that the $SU(2)$ multiplets within the bosonic states also split. In the other case (3.80), there remains $\mathcal{N} = 2$ supersymmetry and the spectrum should represent the super-algebra $SU(2|1)^2 \times R^2$. The

$\mathcal{N}=4$ magnon supermultiplet becomes

$$\mathrm{Tr}\gamma^m D_\mu Z Z^{kM-1} \tag{3.83}$$

$$\mathrm{Tr}\gamma^m \Phi Z^{kM}\ ,\ \mathrm{Tr}\gamma^m \bar{\Phi} Z^{kM}\ ,\ \mathrm{Tr}\gamma^m \bar{\Psi} Z^{kM+1}\ ,\ \mathrm{Tr}\gamma^m \Psi Z^{kM-1} \tag{3.84}$$

$$\mathrm{Tr}\gamma^m \chi_{1\alpha} Z^{kM}\ ,\ \mathrm{Tr}\gamma^m \chi_{3\alpha} Z^{kM-1}\ ,\ \mathrm{Tr}\gamma^m \bar{\chi}^2_{\dot{\alpha}} Z^{kM}\ ,\ \mathrm{Tr}\gamma^m \bar{\chi}^4_{\dot{\alpha}} Z^{kM+1} \tag{3.85}$$

Here m gives the number of units of magnon momentum $p_{\mathrm{mag}} = \frac{2\pi}{M} m$ and k is the number of units of space-time momentum $J = kM$. There are two limits where the operators in the set (3.83)-(3.85) are degenerate and have energies $\Delta - J = 1$: One is when we turn off the 'tHooft coupling $\lambda = g^2_{YM} MN \to 0$ so that the operators have their classical conformal dimension. The other is when magnon momentum vanishes, $m = 0$. In the latter, the "untwisted operator" with $m = 0$ is known to have identical correlation functions with the operators in the parent $\mathcal{N} = 4$ theory and therefore have an exact conformal dimension $\Delta = J + 1$. The spectrum away from these limits will depend on both λ and m.

3.3.4 Orbifold in plane-wave limit

We re-define the string coordinates as: $T = X^+$, $\chi = \frac{1}{\sqrt{\lambda}} X^- - X^+$. This has been chosen so that $\Delta - J = \frac{1}{i}\left(\frac{\partial}{\partial T} - \frac{\partial}{\partial \chi}\right) = \frac{1}{i}\frac{\partial}{\partial X^+}$. In addition we re-scale the transverse coordinates $\vec{Y} \to \vec{Y}/\lambda^{\frac{1}{4}}$, $\vec{Z} \to \vec{X}/\lambda^{\frac{1}{4}}$. The appropriate plane-wave limit [35] then takes $\lambda \to \infty$ simultaneously with $\Delta \to \infty$ and $J \to \infty$ with $\Delta - J$ and $\frac{J}{\sqrt{\lambda}}$ finite. From (3.76) we see that the limit should be taken so that $p_{\mathrm{mag}} J$ is finite. This implies that

$$p_{\mathrm{mag}} \sim \frac{1}{\sqrt{\lambda}} \tag{3.86}$$

The magnon boundary condition (3.74) implies

$$X^-(\sigma = \pi) = X^-(\sigma = 0) + p_{\mathrm{mag}}\sqrt{\lambda} \tag{3.87}$$

The scaling (3.86) then gives a finite radius for X^-.

We have already argued that $J = \frac{1}{i}\frac{\partial}{\partial \chi} = \sqrt{\lambda}\frac{1}{i}\frac{\partial}{\partial X^-}$ should be quantized in integral units. In fact, in the magnon sector, we have argued that the level-matching condition (3.76) has a solution only when $p_{\mathrm{mag}} = 2\pi\frac{m}{M}$ where m and M are integers and J is quantized in units of M, $J = kM$ with k an integer. To get the correct scaling of p_{mag} we must therefore take the plane wave limit by taking M to be large so that $\frac{M}{\sqrt{\lambda}}$ is held finite.

What is effectively the same limit was discussed in [83] where it was shown to result in a plane-wave background with a periodically identified null direction, $X^- \sim X^- + 2\pi R^-$ where $R^- = \frac{\sqrt{\lambda}}{M}$. (To be consistent with (3.87)), the integer m which appears in p_{mag} is interpreted is a wrapping number.) The resulting discrete light-cone quantization of the string on the plane wave background is a simple generalization of Metsaev's original solution [36]. Here, we are interested in a wrapped sector where $X^-(\sigma = 2\pi) = X^-(\sigma = 0) + 2\pi R^- m$. In [83] the spectrum of the IIB string theory in this plane wave limit was matched with the appropriate generalization of the BMN limit of the $\mathcal{N} = 2$ Yang-Mills theory which is

obtained from $\mathcal{N}=4$ by the orbifold projection corresponding to (3.80). It was also used to study non-planar corrections [123] and finite-size corrections at weak coupling [117].

Together with the limit, we take the light-cone gauge, $X^+ = p^+\tau$. Periodicity of X^- quantizes $p^+ = k/R^-$. We obtain the sigma model as a free massive worldsheet field theory

$$\mathcal{L} = -\frac{1}{4\pi}\left\{\partial_a\vec{Y}\cdot\partial^a\vec{Y} + \partial_a\vec{Z}\cdot\partial^a\vec{Z} + (p^+)^2(Y^2+Z^2)\right\} \\ -\frac{ip^+}{2\pi}\left(\bar{\psi}\partial_-\bar{\psi} + \psi\partial_-\psi + 2ip^+\bar{\psi}\Pi\psi\right) \tag{3.88}$$

with $\Pi = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$. In this limit, the magnon parameter p_{mag} does not appear in the Lagrangian or the mass-shell condition which determines the light-cone Hamiltonian:

$$p^- = \frac{1}{p^+}\sum_{n=-\infty}^{\infty}\sqrt{n^2+(p^+)^2}\left(\alpha_n^{\alpha_1\dot{\alpha}_1\dagger}\alpha_{n\alpha_1\dot{\alpha}_1} + \alpha_n^{\alpha_2\dot{\alpha}_2\dagger}\alpha_{n\alpha_2\dot{\alpha}_2}\right. \\ \left. +\beta_n^{\alpha_1\dot{\alpha}_2\dagger}\beta_{n\alpha_1\dot{\alpha}_2} + \beta_n^{\alpha_2\dot{\alpha}_1\dagger}\beta_{n\alpha_2\dot{\alpha}_1}\right) \tag{3.89}$$

Its only vestige is in the level-matching condition.

$$km = \sum_{n=-\infty}^{\infty} n\left(\alpha_n^{\alpha_1\dot{\alpha}_1\dagger}\alpha_{n\alpha_1\dot{\alpha}_1} + \alpha_n^{\alpha_2\dot{\alpha}_2\dagger}\alpha_{n\alpha_2\dot{\alpha}_2} + \beta_n^{\alpha_1\dot{\alpha}_2\dagger}\beta_{n\alpha_1\dot{\alpha}_2} + \beta_n^{\alpha_2\dot{\alpha}_1\dagger}\beta_{n\alpha_2\dot{\alpha}_1}\right) \tag{3.90}$$

where k are the number of units of $J = kM$ and m is the wrapping number. The bosonic $\alpha_{n..}$ and fermionic $\beta_{n..}$ oscillators have the non-vanishing brackets

$$[\alpha_{m\alpha_1\dot{\alpha}_1}, \alpha_n^{\beta_1\dot{\beta}_1\dagger}] = \delta_{mn}\delta_{\beta_1}^{\alpha_1}\delta_{\dot{\beta}_1}^{\dot{\alpha}_1} \quad , \quad \{\beta_{m\alpha_1\dot{\alpha}_2}, \beta_n^{\beta_1\dot{\beta}_2\dagger}\} = \delta_{mn}\delta_{\beta_1}^{\alpha_1}\delta_{\dot{\beta}_2}^{\dot{\alpha}_2} \tag{3.91}$$

$$\left[\alpha_{m\alpha_2\dot{\alpha}_2}, \alpha_n^{\beta_2\dot{\beta}_2\dagger}\right] = \delta_{mn}\delta_{\beta_2}^{\alpha_2}\delta_{\dot{\beta}_2}^{\dot{\alpha}_2} \quad , \quad \{\beta_{m\alpha_2\dot{\alpha}_1}, \beta_n^{\beta_2\dot{\beta}_1\dagger}\} = \delta_{mn}\delta_{\beta_2}^{\alpha_2}\delta_{\dot{\beta}_1}^{\dot{\alpha}_1} \tag{3.92}$$

and bi-spinors of $SO(4)\times SO(4) \sim SU(2)^4$.[4] We confirm in (3.90), which is the plane wave limit of (3.76), there is solution of the level matching constraint unless $\frac{p_{\mathrm{mag}}}{2\pi}J$ =integer. Here, we can think of the null identification as the vestige of the orbifold identification.

The level-matching condition (3.76) allows 1-oscillator states and the magnon supermultiplet is the sixteen states

$$\alpha^\dagger_{km\alpha_1\dot{\alpha}_1}|p^+> \ , \ \alpha^\dagger_{km\alpha_2\dot{\alpha}_2}|p^+> \ , \ \beta^\dagger_{km\alpha_1\dot{\alpha}_2}|p^+> \ , \ \beta^\dagger_{km\alpha_2\dot{\alpha}_1}|p^+> \tag{3.93}$$

These states are degenerate with spectrum given by

$$p^- = \frac{1}{p^+}\sqrt{(km)^2+(p^+)^2} = \sqrt{1+(R^-)^2m^2} = \sqrt{1+\frac{\lambda'}{M^2}m^2} \tag{3.94}$$

The degeneracy of the states in (3.93) can be attributed to an enhancement of the supersymmetry which is well known to occur in the Penrose limit. One would expect, and we shall

[4]Indices are raised and lowered with $\epsilon^{\alpha_i\beta_i}$ and $-\epsilon_{\alpha_i\beta_i}$, respectively, always operating from the left.

confirm, that the supersymmetry is broken when corrections to the Penrose limit are taken into account. Before that, we recall that in [90], [98] Beisert argued magnon states form a sixteen dimensional short multiplet of an extended super-algebra $SU(2|2)\times SU(2|2)\times(R^1)^3$ where the spectrum (3.70) is the shortening condition. The superalgebra $SU(2|2)$ has generators $\mathcal{R}^{\alpha_1}_{\ \beta_1}$ and $\mathcal{L}^{\dot\alpha_2}_{\ \dot\beta_2}$ of $SU(2)\times SU(2)$, supercharges $\mathcal{Q}^{\dot\alpha_2}_{\ \alpha_1}$ and $\mathcal{S}^{\alpha_1}_{\ \dot\alpha_2}$ and the algebra

$$\begin{aligned}
\left[\mathcal{R}^{\alpha_1}_{\ \beta_1},\mathcal{J}^{\gamma_1}\right] &= \delta^{\gamma_1}_{\beta_1}\mathcal{J}^{\alpha_1}-\frac{1}{2}\delta^{\alpha_1}_{\beta_1}\mathcal{J}^{\gamma_1} \quad , \quad \left[\mathcal{L}^{\dot\alpha_2}_{\ \dot\beta_2},\mathcal{J}^{\dot\gamma_2}\right] = \delta^{\dot\gamma_2}_{\dot\beta_2}\mathcal{J}^{\dot\alpha_2}-\frac{1}{2}\delta^{\dot\alpha_2}_{\dot\beta_2}\mathcal{J}^{\dot\gamma_2}\\
\left\{\mathcal{Q}^{\dot\alpha_2}_{\ \alpha_1},\mathcal{S}^{\beta_1}_{\ \dot\beta_2}\right\} &= \delta^{\beta_1}_{\alpha_1}\mathcal{L}^{\dot\alpha_2}_{\ \dot\beta_2}+\delta^{\dot\alpha_2}_{\dot\beta_2}\mathcal{R}^{\beta_1}_{\ \alpha_1}+\delta^{\beta_1}_{\alpha_1}\delta^{\dot\alpha_2}_{\dot\beta_2}\mathcal{C}\\
\left\{\mathcal{Q}^{\dot\alpha_2}_{\ \alpha_1},\mathcal{Q}^{\dot\beta_2}_{\ \beta_1}\right\} &= \epsilon^{\dot\alpha_2\dot\beta_2}\epsilon_{\alpha_1\beta_1}\mathcal{P} \quad , \quad \left\{\mathcal{S}^{\alpha_1}_{\ \dot\alpha_2},\mathcal{S}^{\beta_1}_{\ \dot\beta_2}\right\} = \epsilon_{\dot\alpha_2\dot\beta_2}\epsilon^{\alpha_1\beta_1}\mathcal{K}
\end{aligned}$$

$\mathcal{J}^{\cdots}$ represents any generator with the appropriate index, $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{C}$ are central charges. In our application, $\mathcal{C}=\Delta-J=p^-$ and

$$\begin{aligned}
\mathcal{R}^{\alpha_1}_{\ \beta_1} &= \sum_n\left\{\alpha_n^{\dagger\alpha_1\dot\gamma}\alpha_{n\beta_1\dot\gamma_1}+\beta_n^{\dagger\alpha_1\gamma_2}\beta_{\beta_1\gamma_2}\right\}-\frac{1}{2}\delta^{\alpha_1}_{\beta_1}\sum_n\left\{\alpha_n^{\dagger\gamma_1\dot\gamma_1}\alpha_{n\gamma_1\dot\gamma_1}+\beta_n^{\dagger\gamma_1\gamma_2}\beta_{\gamma_1\gamma_2}\right\}\\
\mathcal{L}^{\dot\alpha_2}_{\ \dot\beta_2} &= \sum_n\left\{\alpha_n^{\dagger\gamma_2\dot\alpha_2}\alpha_{n\gamma_2\dot\beta_2}+\beta_n^{\dagger\dot\alpha_2\dot\gamma_1}\beta_{\dot\gamma_1\dot\beta_2}\right\}-\frac{1}{2}\delta^{\dot\alpha_2}_{\dot\beta_2}\sum_n\left\{\alpha_n^{\dagger\gamma_2\dot\gamma_2}\alpha_{n\gamma_2\dot\gamma_2}+\beta_n^{\dagger\dot\gamma_1\dot\gamma_2}\beta_{\dot\gamma_1\dot\gamma_2}\right\}\\
\mathcal{Q}^{\dot\beta_2}_{\ \alpha_1} &= \frac{\bar\eta}{\sqrt{8p^+}}\sum_n\Big\{-e(n)\sqrt{\omega_n+p^+}\alpha^\dagger_{n\alpha_1\dot\gamma_1}\beta_n^{\dot\gamma_1\dot\beta_2}+i\sqrt{\omega_n-p^+}\alpha_{n\alpha_1\dot\gamma_1}\beta_n^{\dagger\dot\gamma_1\dot\beta_2}\\
&\qquad -i\sqrt{\omega_n-p^+}\beta^\dagger_{n\alpha_1\gamma_2}\alpha_n^{\gamma_2\dot\beta_2}+e(n)\sqrt{\omega_n+p^+}\beta_{n\alpha_1\gamma_2}\alpha_n^{\dagger\gamma_2\dot\beta_2}\Big\}\\
\mathcal{S}^{\alpha_1}_{\ \dot\beta_2} &= \frac{\bar\eta}{\sqrt{8p^+}}\sum_n\Big\{\sqrt{\omega_n-p^+}\alpha_n^{\dagger\alpha_1\dot\gamma_1}\beta_{n\dot\gamma_1\dot\beta_2}-ie(n)\sqrt{\omega_n+p^+}\alpha_n^{\alpha_1\dot\gamma_1}\beta^\dagger_{n\dot\gamma_1\dot\beta_2}+\\
&\qquad +ie(n)\sqrt{\omega_n+p^+}\beta_n^{\dagger\alpha_1\gamma_2}\alpha_{n\gamma_2\dot\beta_2}-\sqrt{\omega_n-p^+}\beta_n^{\alpha_1\gamma_2}\alpha^\dagger_{n\gamma_2\dot\beta_2}\Big\}
\end{aligned} \tag{3.95}$$

where $\omega_n=\sqrt{(p^+)^2+n^2}$ and $e(n)=\frac{n}{|n|}$. We have used Metsaev's [36] conventions for the supercharges (those called Q^- and $\bar Q^-$). Computing their algebra, we find that the plane wave background supercharges indeed satisfy Beisert's extended superalgebra with the central extensions set to the plane-wave limits of those found by Beisert [90]

$$\mathcal{P}=-i\frac{\sqrt{\lambda}p_{\rm mag}}{4\pi}\leftarrow\frac{\sqrt{\lambda}}{4\pi}\left(e^{-ip_{\rm mag}}-1\right)\ ,\ \mathcal{K}=i\frac{\sqrt{\lambda}p_{\rm mag}}{4\pi}\leftarrow\frac{\sqrt{\lambda}}{4\pi}\left(e^{ip_{\rm mag}}-1\right) \tag{3.96}$$

The existence of the central extension follows directly from the fact that the unextended algebra closes up to the level matching condition and the level-matching condition (3.76) contains the term with $km=\frac{1}{2\pi}2\pi\frac{m}{M}\cdot kM=\frac{1}{2\pi}p_{\rm mag}J$.

A derivation of Beisert's superalgebra in the context of the $AdS_3\times S^5$ sigma model was first given in [118] and developed in [119]. They worked with the un-orbifolded theory by "relaxing" the level-matching condition. Then, there is a central charge in the superalgebra which depends on the level miss-match. The idea is that, once the resulting algebraic structure is used to study magnon and multi-magnon states, the level-matching condition

should be re-imposed so as to get a physical state of the string theory. They work in the "magnon limit", where $J \to \infty$, but magnon momentum is not necessarily small (in our case it relaxes the plane-wave limit by taking M not necessarily large). They obtain the full central extension, rather than the form linearized in p_{mag} that we have found in (3.96). In their work, they use a generalized light-cone gauge $x^+ = \tau = (1-a)T + a\chi$, $x_- = \chi - T$ with a a parameter. They also use the identification, $x_-(\tau, \sigma = 2\pi) - x_-(\tau, \sigma = 0) = p_{\text{ws}}$ with p_{ws} an eigenvalue of the level operator and $x_+ = \tau$ trivially periodic in σ. For the variables in (3.73), this amounts to using the boundary condition $\chi(\tau, \sigma = 2\pi) - \chi(\tau, \sigma = 0) = -(1-a)p_{\text{ws}}$ and $T(\tau, \sigma = 2\pi) - T(\tau, 0) = ap_{\text{ws}}$ which is different from the one which we use when $a \neq 0$ (they primarily use $a = \frac{1}{2}$) - where $T(\tau, \sigma = 2\pi) = T(\tau, \sigma = 0)$ and $\chi(\tau, \sigma = 2\pi) - \chi(\tau, \sigma = 0) = p_{\text{mag}}$. This makes no difference at infinite J where the effect of a is diluted by scaling. However, it matters at finite size. In fact, the same gauge fixing was used in Ref. [86] and the a-dependence of the one-magnon spectrum found there (away from the infinite J limit) can be attributed to this a-dependence of boundary conditions, rather than the gauge variance which is claimed there.

3.3.5 Near pp-wave limit

To see how the spectrum will be split in the near plane-wave limit, we must include corrections to the Lagrangian and the Virasoro constraints that are of order $\frac{1}{\sqrt{\lambda}}$. A systematic scheme for including these corrections in the usual $p_{\text{mag}} = 0$ sector are outlined in the series of papers [120]-[44] and nicely summarized in [122]. There they find that the corrections terms to the Hamiltonian add normal ordered terms which are quartic in oscillators. They also adjust the gauge by adjusting the worldsheet metric in such a way that the level-matching condition remains unmodified. We have shown, and will present elsewhere, that the modification of at procedure in the magnon sector are minimal. The corrections to the free field theory light-cone Hamiltonian are of two types, quartic normal ordered pieces from near-plane-wave limit corrections to the sigma model identical in form to those found in [120], [122] and terms such as the last one in 3.75 which arise from the orbifolding.

To leading order in perturbation theory, the normal ordered quartic interaction Hamiltonian cannot shift the spectrum of 1-oscillator states. Furthermore, none of the extra terms displayed in Eq. (3.75) contribute in the leading order in $1/\sqrt{\lambda}$. However, recall that, to preserve some supersymmetry, the orbifold identification (3.80) that we have been discussing also acts on the transverse direction and this action must also be taken into account. This generates simple correction terms in the Hamiltonian to order $\frac{1}{\sqrt{\lambda}}$. The derivation of the interaction Hamiltonian is given in the appendix F.

The relevant part of the interaction Hamiltonian is

$$H_{\text{int}} = i\frac{p_{\text{mag}}}{2\pi}\frac{1}{2\pi}\int_0^2 \pi d\sigma \left(Y_{1_1\dot{2}_1}Y'_{2_1\dot{1}_1} + ip^+\left(\psi\tilde{\Sigma}\psi + \bar{\psi}\tilde{\Sigma}\bar{\psi}\right)\right) \tag{3.97}$$

With this orbifold identification exactly half of the supersymmetries are preserved in the near plane-wave limit. Specifically, out of the 16 supersymmetries $\mathcal{Q}^{\dot{\alpha}_2}_{\alpha_1}, \mathcal{S}^{\alpha_1}_{\dot{\alpha}_2}$ only $\mathcal{S}^{\alpha_1}_{\dot{1}_2}/\mathcal{Q}^{\dot{1}_2}_{\alpha_1}$ and $\mathcal{S}^{2_2}_{\dot{\alpha}_1}/\mathcal{Q}^{\dot{\alpha}_1}_{2_2}$ survive. This leads to a splitting of the energies of the single impurity states.

The original multiplet had 16 states (8 bosons - $\alpha^\dagger_{\alpha_1\dot{\alpha}_1}|0>, \alpha^\dagger_{\alpha_2\dot{\alpha}_2}|0>$ and 8 fermions - $\beta^\dagger_{\alpha_1\alpha_2}|0>, \beta^\dagger_{\dot{\alpha}_1\dot{\alpha}_2}|0>$). In the near plane-wave, it breaks up into 4 super-multiplets of

the residual superalgebra: one with 9 elements (5 bosons and 4 fermions) and two with 3 elements (2 fermions and a boson in each) and one boson singlet.

The following table illustrates the breaking of the original super-multiplet:

$$
\begin{array}{cccccccc}
 & & \mathcal{S}^{2_1}_{\dot{1}_2}/\mathcal{Q}^{\dot{1}_2}_{2_1} & & \mathcal{Q}^{\dot{1}_2}_{1_1}/\mathcal{S}^{1_1}_{\dot{1}_2} & & \mathcal{S}^{1_1}_{\dot{2}_2}/\mathcal{Q}^{\dot{2}_2}_{1_1} & & \mathcal{Q}^{\dot{2}_2}_{2_1}/\mathcal{S}^{2_1}_{\dot{2}_2} \\
 & \alpha^\dagger_{2_1\dot{2}_1}|0> & - & \beta^\dagger_{2_1\dot{1}_2}|0> & - & \alpha^\dagger_{1_1\dot{2}_1}|0> & & \beta^\dagger_{\dot{2}_1\dot{2}_2}|0> \\
\mathcal{Q}^{\dot{2}_1}_{2_2}/\mathcal{S}^{2_2}_{\dot{2}_1} & | & & | & & | & & | \\
 & \beta^\dagger_{2_1 2_2}|0> & - & \alpha^\dagger_{2_2\dot{1}_2}|0> & - & \beta^\dagger_{1_1 2_2}|0> & & \alpha^\dagger_{2_2\dot{2}_2}|0> \\
\mathcal{S}^{2_2}_{\dot{1}_1}/\mathcal{Q}^{\dot{1}_1}_{2_2} & | & & | & & | & & | \\
 & \alpha^\dagger_{2_1\dot{1}_1}|0> & - & \beta^\dagger_{\dot{1}_1\dot{1}_2}|0> & - & \alpha^\dagger_{1_1\dot{1}_1}|0> & & \beta^\dagger_{\dot{1}_1\dot{2}_2}|0> \\
\mathcal{Q}^{\dot{1}_1}_{1_2}/\mathcal{S}^{1_2}_{\dot{1}_1} & & & & & & & \\
 & \beta^\dagger_{2_1 1_2}|0> & - & \alpha^\dagger_{1_2\dot{1}_2}|0> & - & \beta^\dagger_{1_1 1_2}|0> & & \alpha^\dagger_{1_2\dot{2}_2}|0> \\
\mathcal{S}^{1_2}_{\dot{2}_1}/\mathcal{Q}^{\dot{1}_2}_{2_1} & & & & & & &
\end{array}
\tag{3.98}
$$

Here, columns and rows with dashes represent the surviving supersymmetry transformations: $\mathcal{S}^{\alpha_1}_{\dot{1}_2}/\mathcal{Q}^{\dot{1}_2}_{\alpha_1}$ and $\mathcal{S}^{2_2}_{\dot{\alpha}_1}/\mathcal{Q}^{\dot{\alpha}_1}_{2_2}$. Columns and rows without dashes represent the broken supersymmetries: $\mathcal{S}^{\alpha_1}_{\dot{2}_2}/\mathcal{Q}^{\dot{2}_2}_{\alpha_1}$ and $\mathcal{S}^{1_2}_{\dot{\alpha}_1}/\mathcal{Q}^{\dot{\alpha}_1}_{1_2}$.

The energy degeneracy of the original multiplet is likewise broken by the interaction Hamiltonian in the near plane-wave limit. One of the triplets gets positive energy shift, its energy becoming:

$$\sqrt{1+\lambda\frac{m^2}{M^2}}+\frac{1}{2\sqrt{\lambda}}\frac{\lambda\frac{m^2}{M^2}}{\sqrt{1+\lambda\frac{m^2}{M^2}}}$$

The other triplet gets an equal but negative energy shift:

$$\sqrt{1+\lambda\frac{m^2}{M^2}}-\frac{1}{2\sqrt{\lambda}}\frac{\lambda\frac{m^2}{M^2}}{\sqrt{1+\lambda\frac{m^2}{M^2}}}$$

The s-inglet and a 9-multiplet are annihilated by the interaction Hamiltonian and thus retain the energy of the original multiplet:

$$\sqrt{1+\lambda\frac{m^2}{M^2}}$$

3.4 Conclusion

We have made an number of observations about the giant magnon solution of string theory. We noted that the previously noted resemblance of the magnon to a wrapped string on a Z_M orbifold of $AdS_5 \times S^5$ seems to be the only solution of the Virasoro constraints in the string sigma-model. We argued that this point of view is consistent with AdS/CFT duality as single magnons are physical states of the orbifold projections of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We also argued that this point of view is consistent with the plane-wave limit, where the sigma model is solvable. In that limit, the orbifold identification appears as a periodic identification of the null coordinate and the magnon is a wrapped

string. There, we can see explicitly how the wrapping modifies the supersymmetry algebra and is consistent with the magnon spectrum. The $\mathcal{N} = 2$ supersymmetry of the orbifold is enhanced to $\mathcal{N} = 4$ supersymmetry in the plane-wave limit, so that the full sixteen dimensional magnon supermultiplet appears there. We end with a question. We have shown that the supersymmetry is broken again by near plane-wave limit corrections to the sigma model by showing that the energies of the magnon multiplet are split. However, there is another limit, the "magnon limit" which is similar to the plane wave in that λ and J are taken to infinity but it differs in that $p_{\mathbf{mag}}$ remains of order one, rather than scaling to zero. It would be interesting to understand whether the supersymmetry is also enhanced in this limit so that the orbifold quantization of the infinite volume limit has more supersymmetry than the orbifold itself.

Chapter 4

Summary

This thesis is a result of five years of research in the area of string-gauge duality with the particular emphasis on the string theory in the plane-wave background. Two main lines of inquiry we pursued were the question of string interactions and the consequences of placing the theory on an orbifold. The following is the list of the most important original results we achieved which are presented in detail in the earlier chapters.

In the context of string interactions :

1. We investigated the impurity non-conserving channel of the interacting string, in the context of string field theory, and found that the divergences, which were previously thought to disqualify this channel from having a meaningful physical contribution to the energy shift, actually cancel out at all orders. This introduces an important new consideration for the theory of interacting strings.

2. This cancelation also settled the issue of symmetry factors, and relative weight between the interaction term and the contact term in the energy shift calculation.

3. We calculated the energy shifts of a particular string state of interest, under the different vertex regimes, up to a variable number of impurities. In doing so we have achieved hitherto closest agreement ever between the interacting string calculation and its gauge theory dual.

We were also interested in the string theory on the orbifolded plane-wave. In this context:

1. We applied the string field theory methods and have shown how features of the non-planar gauge theory, peculiar to the orbifold background, translate directly onto its string dual.

2. We studied the equivalence between the giant magnon and the string wrapped in the orbifolded direction, and have shown that, in and near the plane-wave limit, string spectrum matches the behaviour of the magnon super multiplet and the super-algebras of string and gauge theories are exactly identical.

Bibliography

[1] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," Nuovo Cim. A **57**, 190 (1968).

[2] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[3] M. E. Peskin and D. V. Schroeder, "An Introduction To Quantum Field Theory," Reading, USA, Addison-Wesley (1995)

[4] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge Unversity, Cambridge University Press, (1998)

[5] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge Unversity, Cambridge University Press, (1998)

[6] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," Cambridge Unversity, Cambridge University Press, (1987)

[7] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge Unversity, Cambridge University Press, (1987)

[8] S. Weinberg, "The Quantum Theory of Fields, Vol. 3: Supersymmetry" Cambridge Unversity, Cambridge University Press, (2000)

[9] G.Kane, "Supersymmetry" Helix Books, Perseus Publishing

[10] M. B. Green, J. H. Schwarz and L. Brink, "Superfield Theory Of Type Ii Superstrings," Nucl. Phys. B **219**, 437 (1983).

[11] M. B. Green and J. H. Schwarz, "Superstring Field Theory," Nucl. Phys. B **243**, 475 (1984).

[12] J. H. Schwarz, "Introduction to M theory and AdS/CFT duality," arXiv:hep-th/9812037.

[13] A. Giveon, M. Porrati and E. Rabinovici, "Target space duality in string theory," Phys. Rept. **244**, 77 (1994) [arXiv:hep-th/9401139].

[14] J. Polchinski, "TASI Lectures on D-branes," arXiv:hep-th/9611050.

[15] E. B. Bogomolny, "Stability Of Classical Solutions," Sov. J. Nucl. Phys. **24**, 449 (1976) [Yad. Fiz. **24**, 861 (1976)].

[16] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution For The 'T Hooft Monopole And The Julia-Zee Dyon," Phys. Rev. Lett. **35**, 760 (1975).

[17] J. D. Bekenstein, "Black Holes And The Second Law," Lett. Nuovo Cim. **4**, 737 (1972).

[18] S. W. Hawking, "Particle Creation By Black Holes," Commun. Math. Phys. **43**, 199 (1975) [Erratum-ibid. **46**, 206 (1976)].

[19] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," Phys. Lett. B **379**, 99 (1996) [arXiv:hep-th/9601029].

[20] G. 't Hooft, "A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS," Nucl. Phys. B **72**, 461 (1974).

[21] I. R. Klebanov, "TASI lectures: Introduction to the AdS/CFT correspondence," arXiv:hep-th/0009139.

[22] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[23] S. S. Gubser, I. R. Klebanov and A. W. Peet, "Entropy and Temperature of Black 3-Branes," Phys. Rev. D **54**, 3915 (1996) [arXiv:hep-th/9602135].

[24] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, "Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **534**, 202 (1998) [arXiv:hep-th/9805156].

[25] P. Di Vecchia, "An introduction to AdS/CFT correspondence," Fortsch. Phys. **48**, 87 (2000) [arXiv:hep-th/9903007].

[26] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Graviton exchange and complete 4-point functions in the AdS/CFT correspondence," Nucl. Phys. B **562**, 353 (1999) [arXiv:hep-th/9903196].

[27] H. Liu, "Scattering in anti-de Sitter space and operator product expansion," Phys. Rev. D **60**, 106005 (1999) [arXiv:hep-th/9811152].

[28] L. Susskind and E. Witten, "The holographic bound in anti-de Sitter space," arXiv:hep-th/9805114.

[29] I. R. Klebanov, "World-volume approach to absorption by non-dilatonic branes," Nucl. Phys. B **496**, 231 (1997) [arXiv:hep-th/9702076].

[30] R. Arnowitt, S. Deser and C. W. Misner, "The dynamics of general relativity," arXiv:gr-qc/0405109.

[31] J. X. Lu, "ADM masses for black strings and p-branes," Phys. Lett. B **313**, 29 (1993) [arXiv:hep-th/9304159].

[32] R. Penrose "Any space has a plane-wave as a limit" Differential Geometry and Relativity 271-275 (1976)

[33] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," JHEP **0201**, 047 (2002) [arXiv:hep-th/0110242].

[34] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "Penrose limits and maximal supersymmetry," Class. Quant. Grav. **19**, L87 (2002) [arXiv:hep-th/0201081].

[35] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[36] R. R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," Nucl. Phys. B **625**, 70 (2002) [arXiv:hep-th/0112044].

[37] R. R. Metsaev and A. A. Tseytlin, "Superstrings In Curved Ramond-Ramond Backgrounds," Russ. Phys. J. **45**, 719 (2002) [Izv. Vuz. Fiz. **2002N7**, 67 (2002)].

[38] D. J. Gross, A. Mikhailov and R. Roiban, "Operators with large R charge in N = 4 Yang-Mills theory," Annals Phys. **301**, 31 (2002) [arXiv:hep-th/0205066].

[39] A. Santambrogio and D. Zanon, "Exact anomalous dimensions of N = 4 Yang-Mills operators with large R Phys. Lett. B **545**, 425 (2002) [arXiv:hep-th/0206079].

[40] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "A semi-classical limit of the gauge/string correspondence," Nucl. Phys. B **636**, 99 (2002) [arXiv:hep-th/0204051].

[41] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003) [arXiv:hep-th/0212208].

[42] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, "Stringing spins and spinning strings," JHEP **0309**, 010 (2003) [arXiv:hep-th/0306139].

[43] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the AdS(5) x S**5 superstring," Phys. Rev. D **69**, 046002 (2004) [arXiv:hep-th/0305116].

[44] C. G. . Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, "Quantizing string theory in AdS(5) x S**5: Beyond the pp-wave," Nucl. Phys. B **673**, 3 (2003) [arXiv:hep-th/0307032].

[45] M. B. Green, A. Sinha and S. Kovacs, "Non-perturbative effects in the BMN limit of N = 4 supersymmetric Yang-Mills," arXiv:hep-th/0506200.

[46] M. B. Green, S. Kovacs and A. Sinha, "Non-perturbative contributions to the plane-wave string mass matrix," JHEP **0505**, 055 (2005) [arXiv:hep-th/0503077].

[47] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "A new double-scaling limit of N = 4 super Yang-Mills theory and PP-wave strings," Nucl. Phys. B **643**, 3 (2002) [arXiv:hep-th/0205033].

[48] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, "PP-wave string interactions from perturbative Yang-Mills theory," JHEP **0207**, 017 (2002) [arXiv:hep-th/0205089].

[49] M. Bianchi, B. Eden, G. Rossi and Y. S. Stanev, "On operator mixing in N = 4 SYM," Nucl. Phys. B **646**, 69 (2002) [arXiv:hep-th/0205321].

[50] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "BMN correlators and operator mixing in N = 4 super Yang-Mills theory," Nucl. Phys. B **650**, 125 (2003) [arXiv:hep-th/0208178].

[51] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, "Operator mixing and the BMN correspondence," JHEP **0210**, 068 (2002) [arXiv:hep-th/0209002].

[52] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, "BMN gauge theory as a quantum mechanical system," Phys. Lett. B **558**, 229 (2003) [arXiv:hep-th/0212269].

[53] N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," Nucl. Phys. B **664**, 131 (2003) [arXiv:hep-th/0303060].

[54] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background," Phys. Rev. D **66**, 086004 (2002) [arXiv:hep-th/0204146].

[55] M. Spradlin and A. Volovich, "Superstring interactions in a pp-wave background. II," JHEP **0301**, 036 (2003) [arXiv:hep-th/0206073].

[56] C. S. Chu, V. V. Khoze, M. Petrini, R. Russo and A. Tanzini, "A note on string interaction on the pp-wave background," Class. Quant. Grav. **21**, 1999 (2004) [arXiv:hep-th/0208148].

[57] A. Pankiewicz, "More comments on superstring interactions in the pp-wave background," JHEP **0209**, 056 (2002) [arXiv:hep-th/0208209].

[58] S. Dobashi, H. Shimada and T. Yoneya, "Holographic reformulation of string theory on AdS(5) x S**5 background in Nucl. Phys. B **665**, 94 (2003) [arXiv:hep-th/0209251].

[59] J. Gomis, S. Moriyama and J. w. Park, "SYM description of SFT Hamiltonian in a pp-wave background," Nucl. Phys. B **659**, 179 (2003) [arXiv:hep-th/0210153].

[60] A. Pankiewicz and B. . J. Stefanski, "pp-wave light-cone superstring field theory," Nucl. Phys. B **657**, 79 (2003) [arXiv:hep-th/0210246].

[61] C. S. Chu, M. Petrini, R. Russo and A. Tanzini, "String interactions and discrete symmetries of the pp-wave background," Class. Quant. Grav. **20**, S457 (2003) [arXiv:hep-th/0211188].

[62] Y. H. He, J. H. Schwarz, M. Spradlin and A. Volovich, "Explicit formulas for Neumann coefficients in the plane-wave geometry," Phys. Rev. D **67**, 086005 (2003) [arXiv:hep-th/0211198].

[63] J. Lucietti, S. Schafer-Nameki and A. Sinha, "On the plane-wave cubic vertex," Phys. Rev. D **70**, 026005 (2004) [arXiv:hep-th/0402185].

[64] J. Lucietti, S. Schafer-Nameki and A. Sinha, "On the exact open-closed vertex in plane-wave light-cone string field theory," Phys. Rev. D **69**, 086005 (2004) [arXiv:hep-th/0311231].

[65] R. Roiban, M. Spradlin and A. Volovich, "On light-cone SFT contact terms in a plane wave," JHEP **0310**, 055 (2003) [arXiv:hep-th/0211220].

[66] J. Gomis, S. Moriyama and J. w. Park, "SYM description of pp-wave string interactions: Singlet sector and arbitrary impurities," Nucl. Phys. B **665**, 49 (2003) [arXiv:hep-th/0301250].

[67] P. Di Vecchia, J. L. Petersen, M. Petrini, R. Russo and A. Tanzini, "The 3-string vertex and the AdS/CFT duality in the pp-wave limit," Class. Quant. Grav. **21**, 2221 (2004) [arXiv:hep-th/0304025].

[68] A. Pankiewicz, "An alternative formulation of light-cone string field theory on the plane wave," JHEP **0306**, 047 (2003) [arXiv:hep-th/0304232].

[69] M. Spradlin and A. Volovich, "Light-cone string field theory in a plane wave," arXiv:hep-th/0310033.

[70] P. Gutjahr and A. Pankiewicz, "New aspects of the BMN correspondence beyond the planar limit," Nucl. Phys. B **704**, 583 (2005) [arXiv:hep-th/0407098].

[71] S. Dobashi and T. Yoneya, "Impurity non-preserving 3-point correlators of BMN operators from pp-wave Nucl. Phys. B **711**, 54 (2005) [arXiv:hep-th/0409058].

[72] E. D'Hoker and S. B. Giddings, "Unitary Of The Closed Bosonic Polyakov String," Nucl. Phys. B **291**, 90 (1987).

[73] G. De Risi, G. Grignani, M. Orselli and G. W. Semenoff, "DLCQ string spectrum from N = 2 SYM theory," JHEP **0411**, 053 (2004) [arXiv:hep-th/0409315].

[74] M. M. Sheikh-Jabbari, "Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture," JHEP **0409**, 017 (2004) [arXiv:hep-th/0406214].

[75] M. B. Green and N. Seiberg, "Contact Interactions In Superstring Theory," Nucl. Phys. B **299**, 559 (1988).

[76] E. Cremmer and . L. Gervais, "Combining And Splitting Relativistic Strings," Nucl. Phys. B **76**, 209 (1974).

[77] B. Chandrasekhar and A. Kumar, "D-branes in pp-wave light cone string field theory," JHEP **0306**, 001 (2003) [arXiv:hep-th/0303223].

[78] S. Dobashi and T. Yoneya, "Resolving the holography in the plane-wave limit of AdS/CFT correspondence," Nucl. Phys. B **711**, 3 (2005) [arXiv:hep-th/0406225].

[79] G. Grignani, M. Orselli, B. Ramadanovic, G. W. Semenoff and D. Young, "Divergence cancellation and loop corrections in string field theory on a plane wave background," JHEP **0512**, 017 (2005) [arXiv:hep-th/0508126].

[80] G. Grignani, M. Orselli, B. Ramadanovic, G. W. Semenoff and D. Young, "AdS/CFT vs. string loops," JHEP **0606**, 040 (2006) [arXiv:hep-th/0605080].

[81] D. Young, "The AdS/CFT Correspondence: Classical, Quantum, and Thermodynamical Aspects," arXiv:0706.3751 [hep-th].

[82] T. Takayanagi and S. Terashima, "Strings on orbifolded pp-waves," JHEP **0206**, 036 (2002) [arXiv:hep-th/0203093].

[83] S. Mukhi, M. Rangamani and E. P. Verlinde, "Strings from quivers, membranes from moose," JHEP **0205**, 023 (2002) [arXiv:hep-th/0204147].

[84] D. M. Hofman and J. M. Maldacena, J. Phys. A **39**, 13095 (2006) [arXiv:hep-th/0604135].

[85] B. Ramadanovic and G. W. Semenoff, "Finite Size Giant Magnon," arXiv:0803.4028 [hep-th].

[86] G. Arutyunov, S. Frolov and M. Zamaklar, "Finite-size effects from giant magnons," Nucl. Phys. B **778**, 1 (2007) [arXiv:hep-th/0606126].

[87] D. Astolfi, V. Forini, G. Grignani and G. W. Semenoff, "Gauge invariant finite size spectrum of the giant magnon," Phys. Lett. B **651**, 329 (2007) [arXiv:hep-th/0702043].

[88] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," arXiv:hep-th/9603167.

[89] L. F. Alday, G. Arutyunov and S. Frolov, "Green-Schwarz strings in TsT-transformed backgrounds," JHEP **0606**, 018 (2006) [arXiv:hep-th/0512253].

[90] N. Beisert, arXiv:hep-th/0511082.

[91] M. Bershadsky and A. Johansen, "Large N limit of orbifold field theories," Nucl. Phys. B **536**, 141 (1998) [arXiv:hep-th/9803249].

[92] N. Beisert and R. Roiban, "The Bethe ansatz for Z(S) orbifolds of N = 4 super Yang-Mills theory," JHEP **0511**, 037 (2005) [arXiv:hep-th/0510209].

[93] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[94] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[95] N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," Nucl. Phys. B **670**, 439 (2003) [arXiv:hep-th/0307042].

[96] G. Arutyunov, S. Frolov and M. Staudacher, "Bethe ansatz for quantum strings," JHEP **0410**, 016 (2004) [arXiv:hep-th/0406256].

[97] M. Staudacher, "The factorized S-matrix of CFT/AdS," JHEP **0505** (2005) 054 [arXiv:hep-th/0412188].

[98] N. Beisert, "The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2—2) Symmetry," J. Stat. Mech. **0701**, P017 (2007) [arXiv:nlin/0610017].

[99] R. A. Janik, "The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry," Phys. Rev. D **73**, 086006 (2006) [arXiv:hep-th/0603038].

[100] R. Hernandez and E. Lopez, "Quantum corrections to the string Bethe ansatz," JHEP **0607**, 004 (2006) [arXiv:hep-th/0603204].

[101] N. Beisert, R. Hernandez and E. Lopez, "A crossing-symmetric phase for AdS(5) x S**5 strings," JHEP **0611**, 070 (2006) [arXiv:hep-th/0609044].

[102] N. Beisert, B. Eden and M. Staudacher, "Transcendentality and crossing," J. Stat. Mech. **0701** (2007) P021 [arXiv:hep-th/0610251].

[103] L. Freyhult and C. Kristjansen, "A universality test of the quantum string Bethe ansatz," Phys. Lett. B **638**, 258 (2006) [arXiv:hep-th/0604069].

[104] M. K. Benna, S. Benvenuti, I. R. Klebanov and A. Scardicchio, "A test of the AdS/CFT correspondence using high-spin operators," Phys. Rev. Lett. **98**, 131603 (2007) [arXiv:hep-th/0611135].

[105] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "Wrapping at four loops in N=4 SYM," arXiv:0712.3522 [hep-th].

[106] C. A. Keeler and N. Mann, "Wrapping Interactions and the Konishi Operator," arXiv:0801.1661 [hep-th].

[107] J. Ambjorn, R. A. Janik and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain / string duality," Nucl. Phys. B **736**, 288 (2006) [arXiv:hep-th/0510171].

[108] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, "Dressing and Wrapping," J. Stat. Mech. **0710**, P10003 (2007) [arXiv:0704.3586 [hep-th]].

[109] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling – the giant magnon," Phys. Rev. D **76**, 126008 (2007) [arXiv:0708.2208 [hep-th]].

[110] R. Ishizeki and M. Kruczenski, "Single spike solutions for strings on S2 and S3," Phys. Rev. D **76**, 126006 (2007) [arXiv:0705.2429 [hep-th]].

[111] Y. Hatsuda and R. Suzuki, "Finite-Size Effects for Dyonic Giant Magnons," arXiv:0801.0747 [hep-th].

[112] J. A. Minahan and O. Ohlsson Sax, "Finite size effects for giant magnons on physical strings," arXiv:0801.2064 [hep-th].

[113] H. Hayashi, K. Okamura, R. Suzuki and B. Vicedo, "Large Winding Sector of AdS/CFT," JHEP **0711**, 033 (2007) [arXiv:0709.4033 [hep-th]].

[114] N. Gromov, S. Schafer-Nameki and P. Vieira, "Quantum Wrapped Giant Magnon," arXiv:0801.3671 [hep-th].

[115] T. Klose and T. McLoughlin, "Interacting finite-size magnons," arXiv:0803.2324 [hep-th].

[116] A. Solovyov, "Bethe Ansatz Equations for General Orbifolds of N=4 SYM," arXiv:0711.1697 [hep-th].

[117] D. Astolfi, V. Forini, G. Grignani and G. W. Semenoff, "Finite size corrections and integrability of N = 2 SYM and DLCQ strings on a pp-wave," JHEP **0609**, 056 (2006) [arXiv:hep-th/0606193].

[118] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, "The off-shell symmetry algebra of the light-cone AdS(5) x S**5 superstring," J. Phys. A **40**, 3583 (2007) [arXiv:hep-th/0609157].

[119] G. Arutyunov, S. Frolov and M. Zamaklar, "The Zamolodchikov-Faddeev algebra for AdS(5) x S**5 superstring," JHEP **0704**, 002 (2007) [arXiv:hep-th/0612229].

[120] C. G. . Callan, T. McLoughlin and I. Swanson, "Higher impurity AdS/CFT correspondence in the near-BMN limit," Nucl. Phys. B **700**, 271 (2004) [arXiv:hep-th/0405153].

[121] C. G. . Callan, T. McLoughlin and I. Swanson, "Holography beyond the Penrose limit," Nucl. Phys. B **694**, 115 (2004) [arXiv:hep-th/0404007].

[122] T. McLoughlin, "The near-Penrose limit of AdS/CFT," . UMI-31-97357, 2005. Caltech Ph.D. Thesis

[123] G. De Risi, G. Grignani, M. Orselli and G. W. Semenoff, JHEP **0411**, 053 (2004) [arXiv:hep-th/0409315].

Appendix A

Fermion representations

In greater part of this thesis we work in the representations of $SO(4)_1 \times SO(4)_2$ labeled by $(SU(2) \times SU(2))_1 \times (SU(2) \times SU(2))_2$ spinor indices. Fermionic creation operators are then given by $\beta^\dagger_{\alpha_1\alpha_2}$ and $\beta^\dagger_{\dot\alpha_1\dot\alpha_2}$ which transform in the $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$ representations of $(SU(2) \times SU(2))_1 \times (SU(2) \times SU(2))_2$, respectively; α_k,$\dot\alpha_k$ being two-component Weyl indices of $SO(4)_k$.

The $SO(8)$ vector index I splits into two $SO(4) \times SO(4)$ vector indices (i, i') so that we use vector index $i = 1, \ldots, 4$ and bi-spinor indices $\alpha_1, \dot\alpha_1 = 1, 2$ for the first $SO(4)$ and $(i', \alpha_2, \dot\alpha_2)$ for the second $SO(4)$. Vectors are constructed in terms of bi-spinor indices as $(\alpha_n)_{\alpha_1\dot\alpha_1} = \sigma^i_{\alpha_1\dot\alpha_1}\alpha^i_n/\sqrt{2}$, $(\alpha_n)_{\alpha_2\dot\alpha_2} = \sigma^{i'}_{\alpha_2\dot\alpha_2}\alpha^{i'}_n/\sqrt{2}$ and transform as $(1/2, 1/2, 0, 0)$ and $(0, 0, 1/2, 1/2)$, respectively.

Here the σ-matrices consist of the usual Pauli-matrices together with the 2d unit matrix

$$\sigma^i_{\alpha\dot\alpha} = \left(i\tau^1, i\tau^2, i\tau^3, -1\right)_{\alpha\dot\alpha} \tag{A.1}$$

and satisfy the reality properties $\left[\sigma^i_{\alpha\dot\alpha}\right]^\dagger = \sigma^{i\dot\alpha\alpha}$, $\left[\sigma^{i\dot\alpha}_{\alpha}\right]^\dagger = -\sigma^{i\alpha}_{\dot\alpha}$. These properties are also satisfied by the fermionic oscillators, so that $(\beta_{n\,\alpha_1\,\alpha_2})^\dagger = \beta^{\dagger\,\alpha_1\,\alpha_2}_n$ and $\left(\beta^{\alpha_2}_{n\,\alpha_1}\right)^\dagger = -\beta^{\dagger\,\alpha_1}_{n\,\alpha_2}$; the same relations are obeyed for the dotted-index fermions.

Spinor indices are raised and lowered with the two-dimensional Levi-Civita symbols, $\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(\epsilon^{\alpha\beta})^\dagger = \epsilon_{\beta\alpha}$, for example

$$A^\alpha = A_\beta\,\epsilon^{\alpha\beta} \qquad A_\alpha = A^\beta\,\epsilon_{\alpha\beta} \tag{A.2}$$

and

$$\sigma^i_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\sigma^{i\dot\beta\beta} \equiv \epsilon_{\alpha\beta}\,\sigma^{i\beta}_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\,\sigma^{i\dot\beta}_{\alpha}\,. \tag{A.3}$$

The σ-matrices satisfy the relations

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\dot\alpha\beta} + \sigma^j_{\alpha\dot\alpha}\sigma^{i\dot\alpha\beta} = 2\delta^{ij}\delta^\beta_\alpha\,, \qquad \sigma^{i\dot\alpha\alpha}\sigma^j_{\alpha\dot\beta} + \sigma^{j\dot\alpha\alpha}\sigma^i_{\alpha\dot\beta} = 2\delta^{ij}\delta^{\dot\alpha}_{\dot\beta}\,. \tag{A.4}$$

Some other properties satisfied by these matrices are

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = \delta_\alpha^\delta\delta_\beta^\gamma - \delta_\alpha^\gamma\delta_\beta^\delta\,, \tag{A.5}$$

$$\sigma^i_{\alpha\dot\beta}\sigma^{j\dot\beta}_{\ \ \beta} = -\delta^{ij}\epsilon_{\alpha\beta} + \sigma^{ij}_{\alpha\beta}\,, \qquad (\sigma^{ij}_{\alpha\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\dot\alpha}_{\ \ \beta} = \sigma^{ij}_{\beta\alpha}) \tag{A.6}$$

$$\sigma^i_{\alpha\dot\alpha}\sigma^{j\alpha}_{\ \ \dot\beta} = -\delta^{ij}\epsilon_{\dot\alpha\dot\beta} + \sigma^{ij}_{\dot\alpha\dot\beta}\,, \qquad (\sigma^{ij}_{\dot\alpha\dot\beta} \equiv \sigma^{[i}_{\alpha\dot\alpha}\sigma^{j]\alpha}_{\ \ \dot\beta} = \sigma^{ij}_{\dot\beta\dot\alpha}) \tag{A.7}$$

$$\sigma^k_{\alpha\dot\alpha}\sigma^k_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \tag{A.8}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\gamma\delta} = 4(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})\,, \tag{A.9}$$

$$\sigma^{kl}_{\alpha\beta}\sigma^{kl}_{\gamma\dot\delta} = 0\,, \tag{A.10}$$

$$2\sigma^i_{\alpha\dot\alpha}\sigma^j_{\beta\dot\beta} = \delta^{ij}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \sigma^{k(i}_{\alpha_1\beta_1}\sigma^{j)k}_{\dot\alpha_1\dot\beta_1} - \epsilon_{\alpha\beta}\sigma^{ij}_{\dot\alpha\dot\beta} - \sigma^{ij}_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,. \tag{A.11}$$

$$(\sigma^{ij}_{\dot\alpha\dot\beta})^\dagger = \sigma^{ij\dot\alpha\dot\beta} \tag{A.12}$$

$$\sigma^{k\dot\beta}_{\gamma}\sigma^{ij}_{\dot\alpha\dot\beta} = \delta^{ik}\sigma^j_{\gamma\dot\alpha} - \delta^{jk}\sigma^i_{\gamma\dot\alpha} \tag{A.13}$$

In this basis the gamma matrices have the following representation

$$\gamma^i_{a\dot a} = \begin{pmatrix} 0 & \sigma^i_{\alpha_1\dot\beta_1}\delta^{\beta_2}_{\alpha_2} \\ \sigma^{i\dot\alpha_1\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} & 0 \end{pmatrix}, \qquad \gamma^i_{\dot a a} = \begin{pmatrix} 0 & \sigma^i_{\alpha_1\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \\ \sigma^{i\dot\alpha_1\beta_1}\delta^{\beta_2}_{\alpha_2} & 0 \end{pmatrix}, \tag{A.14}$$

$$\gamma^{i'}_{a\dot a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'}_{\alpha_2\dot\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'\dot\alpha_2\beta_2} \end{pmatrix}, \qquad \gamma^{i'}_{\dot a a} = \begin{pmatrix} -\delta^{\beta_1}_{\alpha_1}\sigma^{i'\dot\alpha_2\beta_2} & 0 \\ 0 & \delta^{\dot\alpha_1}_{\dot\beta_1}\sigma^{i'}_{\alpha_2\dot\beta_2} \end{pmatrix}. \tag{A.15}$$

and the projector reads

$$\Pi_{ab} = \begin{pmatrix} \left(\sigma^1\sigma^2\sigma^3\sigma^4\right)^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & \left(\sigma^1\sigma^2\sigma^3\sigma^4\right)^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix} = \begin{pmatrix} \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} & 0 \\ 0 & -\delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\dot\alpha_2}_{\dot\beta_2} \end{pmatrix}, \tag{A.16}$$

so that $(1 \pm \Pi)/2$ projects onto $(1/2, 0, 1/2, 0)$ and $(0, 1/2, 0, 1/2)$, respectively.

The supercharge $Q^-_{\alpha_1\dot\beta_2}$ is a $(1/2, 0, 0, 1/2)$ and $Q^-_{\dot\alpha_1\beta_2}$ is a $(0, 1/2, 1/2, 0)$ representation. In this notation it is convenient to define the linear combinations of the free supercharges

$$\sqrt{2}\eta Q \equiv Q^- + i\bar{Q}^- \quad , \quad \sqrt{2}\bar\eta\widetilde{Q} \equiv Q^- - i\bar{Q}^- \tag{A.17}$$

where $\eta = e^{i\pi/4}$, and $\bar{Q}^\pm = e(\alpha)(Q^\pm)^\dagger$. On the space of physical states they satisfy the dynamical constraints

$$\begin{aligned} \left\{Q_{\alpha_1\dot\alpha_2}, Q_{\beta_1\dot\beta_2}\right\} &= \left\{\widetilde{Q}_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} = -2\epsilon_{\alpha_1\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}H \\ \left\{Q_{\alpha_1\dot\alpha_2}, \widetilde{Q}_{\beta_1\dot\beta_2}\right\} &= -\mu\epsilon_{\dot\alpha_2\dot\beta_2}\left(\sigma^{ij}\right)_{\alpha_1\beta_1}J^{ij} + \mu\epsilon_{\alpha_1\beta_1}\left(\sigma^{i'j'}\right)_{\dot\alpha_2\dot\beta_2}J^{i'j'} \end{aligned} \tag{A.18}$$

and similarly for $Q_{\dot\alpha_1\alpha_2}$ and $\widetilde{Q}_{\dot\beta_1\beta_2}$. The free supercharge with raised indices is understood as

$$Q_2^{\alpha_1\dot{\alpha}_2} \equiv e(\alpha)\,(Q_{2\,\alpha_1\dot{\alpha}_2})^\dagger\,, \qquad Q_2^{\dot{\alpha}_1\alpha_2} \equiv e(\alpha)\,(Q_{2\,\dot{\alpha}_1\alpha_2})^\dagger \tag{A.19}$$

and this gives

$$Q_2^{\alpha_1\dot{\alpha}_2} Q_{2\,\alpha_1\dot{\alpha}_2} = +4H_2 = Q_{2\,\alpha_1\dot{\alpha}_2} Q_2^{\alpha_1\dot{\alpha}_2} \tag{A.20}$$

for these operators in the single string Hilbert space $\mathcal{H}_1$. For states in the three-string Hilbert space $\mathcal{H}_3$, i.e. $|Q_3\rangle$, the $e(\alpha)$ is already encoded into the construction so that it should be dropped in the adjoint

$$Q_{2\,\alpha_1\dot{\alpha}_2}|Q_3^{\alpha_1\dot{\alpha}_2}\rangle = Q_2^{\alpha_1\dot{\alpha}_2}|Q_{3\alpha_1\dot{\alpha}_2}\rangle \equiv (Q_{2\,\alpha_1\dot{\alpha}_2})^\dagger\,|Q_{3\alpha_1\dot{\alpha}_2}\rangle = +4|H_3\rangle \tag{A.21}$$

and similarly $Q_3^{\alpha_1\dot{\alpha}_2} \equiv (Q_{3\,\alpha_1\dot{\alpha}_2})^\dagger$. In the BMN basis, the full expression for the quadratic supercharge $Q_{2\,\alpha_1\,\dot{\alpha}_2}$ is[1]

$$\begin{aligned} Q_{2\,\alpha_1\,\dot{\alpha}_2} = \frac{\bar{\eta}}{\sqrt{|\alpha|}} \sum_{k\neq 0} \Omega_k \Big(& \alpha_{k\alpha_1}^{\dagger\,\dot{\beta}_1}\,\beta_{k\dot{\beta}_1\,\dot{\alpha}_2} + i\,e(\alpha)\,\alpha_{k\alpha_1}^{\dot{\beta}_1}\,\beta_{k\dot{\beta}_1\,\dot{\alpha}_2}^\dagger \\ & + i\,\alpha_{k\dot{\alpha}_2}^{\dagger\,\beta_2}\,\beta_{k\alpha_1\,\beta_2} + e(\alpha)\,\alpha_{k\dot{\alpha}_2}^{\beta_2}\,\beta_{k\alpha_1\,\beta_2}^\dagger \Big) \\ + \bar{\eta}^{\,e(\alpha)}\,\sqrt{2\,\mu}\,\Big(& \alpha_{0\alpha_1}^{\dagger\,\dot{\beta}_1}\,\beta_{0\dot{\beta}_1\,\dot{\alpha}_2} + i\,\alpha_{0\alpha_1}^{\dot{\beta}_1}\,\beta_{0\dot{\beta}_1\,\dot{\alpha}_2}^\dagger \\ & + i\,e(\alpha)\,\alpha_{0\dot{\alpha}_2}^{\dagger\,\beta_2}\,\beta_{0\alpha_1\,\beta_2} + e(\alpha)\,\alpha_{0\dot{\alpha}_2}^{\beta_2}\,\beta_{0\alpha_1\,\beta_2}^\dagger \Big) \end{aligned} \tag{A.22}$$

where Ω_k is defined in (C.6).

Among states that are created by two oscillators, the state with quantum numbers $(1,1,0,0)$ and $(0,0,1,1)$ which are created by two bosons have no analogues amongst the two oscillator states containing either one or two fermions. Thus, they are not mixed with other members of the supermultiplet. These states in the main text are denoted $|[\mathbf{9},\mathbf{1}]\rangle^{(ij)}$ and $|[\mathbf{1},\mathbf{9}]\rangle^{(i'j')}$ in SO(8) notation.

[1]Note that $\alpha_{k\alpha_1}^{\dagger\,\dot{\beta}_1} = -\sigma^{i\dot{\beta}_1}_{\alpha_1}\alpha_k^{i\,\dagger}/\sqrt{2}$, and similarly for the other $SO(4)$ since $[\sigma^{i\alpha_1}_{\dot{\beta}_1}]^\dagger = -\sigma^{i\dot{\beta}_1}_{\alpha_1}$.

Appendix B

Neumann matrices and associated quantities

In this section we present the explicit expressions for the quantities appearing in the prefactors and exponential part of $|H_3\rangle$ and $|Q_3\rangle$ (2.87). Following the notation of [70], the Neumann matrices can be written as

$$\widetilde{N}^{st}_{mn} = \begin{cases} \frac{1}{2}\bar{N}^{st}_{|m||n|}\left(1 + U_{m(s)}U_{n(t)}\right) & , m,n \neq 0 \\ \frac{1}{\sqrt{2}}\bar{N}^{st}_{|m|0} & m \neq 0 \\ \bar{N}^{st}_{00} \end{cases} \tag{B.1}$$

with[1]

$$\bar{N}^{st}_{mn} = -(1-4\mu\kappa K)^{-1}\frac{\kappa}{\alpha_s\omega_{n(t)} + \alpha_t\omega_{m(s)}}\left[CU^{-1}_{(s)}C^{1/2}_{(s)}\bar{N}^s\right]_m\left[CU^{-1}_{(t)}C^{1/2}_{(t)}\bar{N}^t\right]_n \tag{B.2}$$

$$\bar{N}^{st}_{m0} = \sqrt{-2\mu\kappa(1-\beta_t)}\sqrt{\omega_{m(s)}}\bar{N}^s_m\,, \qquad t \in \{1,2\} \tag{B.3}$$

$$\bar{N}^{st}_{00} = (1-4\mu\kappa K)\left(\delta^{st} + \sqrt{\beta_s\beta_t}\right)\,, \qquad s,t \in \{1,2\} \tag{B.4}$$

$$\bar{N}^{s3}_{00} = -\sqrt{\beta_s}\,, \qquad s \in \{1,2\} \tag{B.5}$$

while

$$\widetilde{Q}^{rs}_{mn} = \begin{cases} \frac{i}{2}e(m)\bar{Q}^{rs}_{|m||n|}, & m,n \neq 0 \\ \frac{i}{\sqrt{2}}e(m)\bar{Q}^{rs}_{|m|0}, & m \neq 0 \\ \bar{Q}^{rs}_{00} \end{cases} \tag{B.6}$$

where [68]

$$\begin{aligned} \bar{Q}^{rs}_{mn} &= e(\alpha_r)\sqrt{\left|\frac{\alpha_s}{\alpha_r}\right|}\left[U^{1/2}_{(r)}C^{1/2}\bar{N}^{rs}C^{-1/2}U^{1/2}_{(s)}\right]_{mn}, \qquad m,n > 0 \\ \bar{Q}^{sr}_{m0} &= -\alpha_3(1-\beta_r)\sqrt{\alpha_r}\frac{e(\alpha_s)}{\sqrt{|\alpha_s|}}\left[\left(U_{(s)}C_{(s)}C\right)^{1/2}\bar{N}^s\right]_m, \qquad m > 0 \\ \bar{Q}^{3r}_{00} &= -\bar{Q}^{r3}_{00} = \frac{1}{2}\sqrt{-\frac{\alpha_r}{\alpha_3}}, \qquad \bar{Q}^{rs}_{00} = 0, \qquad r,s = \{1,2\} \end{aligned} \tag{B.7}$$

[1]To have a manifest symmetry in $1 \leftrightarrow 2$ we additionally redefined the oscillators as $(-1)^{s(n+1)}\alpha_{n(s)} \to \alpha_{n(s)}$ for $n \in \mathbb{Z}$, $s = 1,2,3$ and analogously for the fermionic oscillators.

and we note that $\bar{Q}^{sr}_{0m} = 0$, while

$$C_n = n\,, \qquad C_{n(s)} = \omega_{n(s)} \equiv \sqrt{n^2 + (\mu\alpha_s)^2}\,, \qquad \kappa \equiv \alpha_1\alpha_2\alpha_3 \tag{B.8}$$

$$U_{n(s)} = \frac{1}{n}(\omega_{n(s)} - \mu\alpha_s)\,, \qquad U^{-1}_{n(s)} = \frac{1}{n}(\omega_{n(s)} + \mu\alpha_s) \tag{B.9}$$

and [62]

$$1 - 4\mu\kappa K \approx -\frac{1}{4\pi r(1-r)\mu\alpha_3} \tag{B.10}$$

$$\alpha_3 \bar{N}^3_n \approx -\frac{\sin(n\pi r)}{\pi r(1-r)} \frac{1}{\omega_{n(3)}\sqrt{-2\mu\alpha_3(\omega_{n(3)} + \mu\alpha_3)}} \tag{B.11}$$

$$\alpha_3 \bar{N}^s_n \equiv \alpha_3 \bar{N}_n(\beta_s) \approx -\frac{\sqrt{\beta_s}}{2\pi r(1-r)} \frac{1}{\omega_{n(s)}\sqrt{-2\mu\alpha_3(\omega_{n(s)} - \mu\alpha_3\beta_s)}} \tag{B.12}$$

up to exponential corrections $\sim \mathcal{O}(e^{-\mu\alpha_3})$ [2]. For the bosonic constituents of the prefactor one has

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} K_{n(s)}\alpha^{I\dagger}_{n(s)}\,, \quad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} K_{n(s)}\alpha^{I\dagger}_{-n(s)} \tag{B.13}$$

where

$$K_{0(s)} = (1 - 4\mu\kappa K)^{1/2}\sqrt{-\frac{2\mu\kappa}{\alpha'}(1-\beta_s)}\,, \qquad K_{0(3)} = 0 \tag{B.14}$$

and

$$K_{n(s)} = -\frac{\kappa}{\sqrt{2\alpha'}\alpha_s}(1 - 4\mu\kappa K)^{-1/2}(\omega_{n(s)} + \mu\alpha_s)\sqrt{\omega_{n(s)}}\bar{N}^s_{|n|}\big(1 - U_{n(s)}\big) \tag{B.15}$$

For the fermionic constituents of the prefactor one has

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{B.16}$$

where

$$G_{0(s)} = (1 - 4\mu\kappa K)^{1/2}\sqrt{1-\beta_s}\,, \qquad G_{0(3)} = 0 \tag{B.17}$$

and

$$G_{n(s)} = \frac{e(\alpha_s)}{\sqrt{2|\alpha_s|}} \frac{\sqrt{-\kappa}}{(1 - 4\mu\kappa K)^{1/2}}\sqrt{(\omega_{n(s)} + \mu\alpha_s)\omega_{n(s)}}\bar{N}^s_{|n|} \tag{B.18}$$

where in the above expressions we have used $\beta_1 \equiv r$ and $\beta_2 \equiv 1 - r$ (with $\beta_t \equiv -\alpha_t/\alpha_3$ and $\alpha_3 < 0$).

[2] To compare with the definition used in [62] note that $\bar{N}^s_{n\,\text{here}} = (-1)^{s(n+1)}U_{n(s)}C^{-1/2}_{n(s)}\bar{N}^s_{n\,\text{there}}$.

Appendix C

More Neumann matrices and relations

The above matrices can also be written in the following compact form:

$$\widetilde{N}^{3\,r}_{n\,q} = -\frac{\sin(n\pi r)\sqrt{\beta_r}\,(\Lambda^+_n\Lambda^+_q + \Lambda^-_n\Lambda^-_q)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)} \qquad \widetilde{N}^{r\,s}_{q\,p} = \frac{\sqrt{\beta_r\beta_s}\,(\Lambda^+_q\Lambda^+_p + \Lambda^-_q\Lambda^-_p)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)} \tag{C.1}$$

$$\widehat{Q}^{3\,r}_{n\,q} = \frac{i\sin(|n|\pi r)\,(\omega_q+\beta_r\omega_n)}{2\pi\sqrt{\omega_n\omega_q}\,(q-\beta_r n)} \qquad \widehat{Q}^{r\,s}_{q\,p} = \frac{i\,(\beta_s q-\beta_r p)}{4\pi\sqrt{\omega_q\omega_p}\,(\beta_s\omega_q+\beta_r\omega_p)} \tag{C.2}$$

where $\widehat{Q} = \widetilde{Q} - \widetilde{Q}^T$. We also find

$$K_n = +\alpha_3\sin(n\pi r)\sqrt{\frac{r(1-r)}{\pi\alpha'}}\frac{\Lambda^-_n - \Lambda^+_n}{\sqrt{\omega_n}} \tag{C.3}$$

$$K_q = -\alpha_3\sqrt{\frac{r(1-r)}{\pi\alpha'\beta_r}}\frac{\Lambda^+_q - \Lambda^-_q}{2\sqrt{\omega_q}} \tag{C.4}$$

$$G_q = \frac{1}{\sqrt{4\pi\omega_q}} \qquad G_n = -\frac{\sin(|n|\pi r)}{\sqrt{\pi\omega_n}} \tag{C.5}$$

$$\Omega_q = \Lambda^+_q - \Lambda^-_q \qquad \Omega_n = e(n)(\Lambda^-_n - \Lambda^+_n) \tag{C.6}$$

where,

$$\Lambda^+_q = \sqrt{\omega_q - \beta_r\mu\alpha_3} \qquad \Lambda^-_q = e(q)\sqrt{\omega_q + \beta_r\mu\alpha_3} \tag{C.7}$$

$$\Lambda^+_n = \sqrt{\omega_n - \mu\alpha_3} \qquad \Lambda^-_n = e(n)\sqrt{\omega_n + \mu\alpha_3} \tag{C.8}$$

We will also find use for

$$L^{3\,r}_{n\,q} \equiv K_n K_{-q} + K_{-n}K_q \qquad \widetilde{L}^{3\,r}_{n\,q} \equiv K_n K_q + K_{-n}K_{-q}. \tag{C.9}$$

The following relations may also be proven

$$K^{(s)}_p\,K^{(r)}_q + K^{(s)}_{-p}\,K^{(r)}_{-q} = \frac{2\,\alpha_3^2\,r\,(1-r)}{\alpha'}\left(\frac{\omega^{(r)}_q}{\beta_r} + \frac{\omega^{(s)}_p}{\beta_s}\right)\widetilde{N}^{r\,s}_{q\,p} \tag{C.10}$$

$$K^{(3)}_n\,K^{(r)}_q + K^{(3)}_{-n}\,K^{(r)}_{-q} = \frac{2\,\alpha_3^2\,r\,(1-r)}{\alpha'}\left(\frac{\omega^{(r)}_q}{\beta_r} - \omega^{(3)}_n\right)\widetilde{N}^{3\,r}_{n\,q} \tag{C.11}$$

$$\Omega_q^{(r)}\, G_q^{(r)} = \sqrt{\frac{\beta_r\, \alpha'}{r\,(1-r)}}\, \frac{1}{-\alpha_3}\, K_q^{(r)} \tag{C.12}$$

$$\Omega_n^{(3)}\, G_n^{(3)} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\, \frac{1}{-\alpha_3}\, K_n^{(3)} \tag{C.13}$$

$$i\, \frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\, \widehat{Q}_{q\,p}^{r\,s} + \frac{\Omega_p^{(s)}}{\sqrt{\beta_s}}\, \widetilde{N}_{q\,p}^{r\,s} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\, \frac{1}{-\alpha_3} K_q^{(r)} G_p^{(s)} \tag{C.14}$$

$$i\, \Omega_n^{(3)}\, \widehat{Q}_{n\,q}^{3\,r} + \frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\, \widetilde{N}_{n\,q}^{3\,r} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\, \frac{1}{-\alpha_3} K_n^{(3)} G_q^{(r)} \tag{C.15}$$

$$-i\, \frac{\Omega_q^{(r)}}{\sqrt{\beta_r}}\, \widehat{Q}_{n\,q}^{3\,r} + \Omega_n^{(3)}\, \widetilde{N}_{n\,q}^{3\,r} = \sqrt{\frac{\alpha'}{r\,(1-r)}}\, \frac{1}{-\alpha_3} K_q^{(r)} G_n^{(3)} \tag{C.16}$$

$$\left(\widetilde{N}_{q\,p}^{r\,s}\right)^2 - \left(\widehat{Q}_{q\,p}^{r\,s}\right)^2 = \left(G_{|q|}^{(r)}\, G_{|p|}^{(s)}\right)^2 \tag{C.17}$$

$$\left(\widetilde{N}_{n\,q}^{3\,r}\right)^2 + \left(\widehat{Q}_{n\,q}^{3\,r}\right)^2 = -\left(G_{|q|}^{(r)}\, G_{|n|}^{(3)}\right)^2 \tag{C.18}$$

Appendix D

Calculational method

While most of the work presented in this thesis was done in collaboration, particular credit for systematizing the calculational methods presented in this appendix goes to D. Young who was one of our co-authors in [79] and [80]. This appendix itself is taken largely intact from [81].

D.1 Vertices and definitions

Beginning with the known construction of $|H_3\rangle$ and $|Q_3\rangle$ in (2.87):

$$|V\rangle = |E_\alpha\rangle|E_\beta\rangle\delta(\sum_{r=1}^{3}\alpha_r) \tag{D.1}$$

where $|E_\alpha\rangle$ and $|E_\beta\rangle$ are exponentials of bosonic and fermionic oscillators respectively

$$|E_\alpha\rangle = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\alpha^{\dagger K}_{m\,(s)}\widetilde{N}^{st}_{mn}\alpha^{\dagger K}_{n\,(t)}\right)|\alpha\rangle_{123} \tag{D.2}$$

and

$$|E_\beta\rangle = \exp\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}(\beta^{\alpha_1\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\alpha_1\alpha_2} - \beta^{\dot\alpha_1\dot\alpha_2\,\dagger}_{m(r)}\beta^{\dagger}_{n(s)\,\dot\alpha_1\dot\alpha_2})\widetilde{Q}^{rs}_{mn}\right)|\alpha\rangle_{123} \tag{D.3}$$

where $|\alpha\rangle_{123} = |0;\alpha_1\rangle\otimes|0;\alpha_2\rangle\otimes|0;\alpha_3\rangle$. We then have

$$\begin{aligned}
|H_3\rangle &= g_2\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{\alpha'}{8\,\alpha_3^3}\Big[(K_i\widetilde{K}_j - \frac{\mu\kappa}{\alpha'}\delta_{ij})v^{ij} - (K_{i'}\widetilde{K}_{j'} - \frac{\mu\kappa}{\alpha'}\delta_{i'j'})v^{i'j'} \\
&\quad - K^{\dot\alpha_1\alpha_1}\widetilde{K}^{\dot\alpha_2\alpha_2}s_{\alpha_1\alpha_2}(Y)s^*_{\dot\alpha_1\dot\alpha_2}(Z) - \widetilde{K}^{\dot\alpha_1\alpha_1}K^{\dot\alpha_2\alpha_2}s^*_{\alpha_1\alpha_2}(Y)s_{\dot\alpha_1\dot\alpha_2}(Z)\Big]|V\rangle\,, \\
|Q_{3\,\beta_1\dot\beta_2}\rangle &= g_2\,\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad + is_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,, \\
|Q_{3\,\dot\beta_1\beta_2}\rangle &= g_2\,\bar\eta\, f(\mu\alpha_3\,,\,\frac{\alpha_1}{\alpha_3})\frac{1}{4\,\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}}\Big(s^*_{\gamma_1\beta_2}(Y)t^*_{\dot\beta_1\dot\gamma_1}(Z)\widetilde{K}^{\dot\gamma_1\gamma_1} \\
&\qquad + is^*_{\dot\beta_1\dot\gamma_2}(Z)t_{\beta_2\gamma_2}(Y)\widetilde{K}^{\dot\gamma_2\gamma_2}\Big)|V\rangle\,.
\end{aligned} \tag{D.4}$$

where

$$K^I = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} K_{n(s)}\alpha^{I\dagger}_{n(s)}\,, \qquad \widetilde{K}^I = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} K_{n(s)}\alpha^{I\dagger}_{-n(s)} \tag{D.5}$$

$$Y^{\alpha_1\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} G_{|n|(s)}\beta^{\dagger\,\alpha_1\alpha_2}_{n(s)}\,, \qquad Z^{\dot\alpha_1\dot\alpha_2} = \sum_{s=1}^{3}\sum_{n\in\mathbf{Z}} G_{|n|(s)}\beta^{\dagger\,\dot\alpha_1\dot\alpha_2}_{n(s)}\,, \tag{D.6}$$

and

$$\widetilde{K}^{\dot\gamma_1\gamma_1} \equiv \widetilde{K}^i\sigma^{i\dot\gamma_1\gamma_1}\,, \qquad \widetilde{K}^{\dot\gamma_2\gamma_2} \equiv \widetilde{K}^{i'}\sigma^{i'\dot\gamma_2\gamma_2}\,, \tag{D.7}$$

where the σ-matrices are defined in appendix A. We also have

$$\begin{aligned}
v^{ij} &= \delta^{ij}\Big[1+\frac{1}{12}(Y^4+Z^4)+\frac{1}{144}Y^4Z^4\Big] \\
&\quad -\frac{i}{2}\Big[Y^{2ij}(1+\frac{1}{12}Z^4)-Z^{2ij}(1+\frac{1}{12}Y^4)\Big]+\frac{1}{4}\big[Y^2Z^2\big]^{ij}\,, \\
v^{i'j'} &= \delta^{i'j'}\Big[1-\frac{1}{12}(Y^4+Z^4)+\frac{1}{144}Y^4Z^4\Big] \\
&\quad -\frac{i}{2}\Big[Y^{2i'j'}(1-\frac{1}{12}Z^4)-Z^{2i'j'}(1-\frac{1}{12}Y^4)\Big]+\frac{1}{4}\big[Y^2Z^2\big]^{i'j'}\,.
\end{aligned}$$

Here we defined

$$Y^{2ij} \equiv \sigma^{ij}_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1}\,, \quad Z^{2ij} \equiv \sigma^{ij}_{\dot\alpha_1\dot\beta_1}Z^{2\dot\alpha_1\dot\beta_1}\,, \quad \big(Y^2Z^2\big)^{ij} \equiv Y^{2k(i}Z^{2j)k} \tag{D.8}$$

and analogously for the primed indices. We have also introduced the following quantities quadratic and cubic in Y and symmetric in spinor indices

$$Y^2_{\alpha_1\beta_1} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_2}_{\beta_1}\,, \qquad Y^2_{\alpha_2\beta_2} \equiv Y_{\alpha_1\alpha_2}Y^{\alpha_1}_{\beta_2}\,, \tag{D.9}$$

$$Y^3_{\alpha_1\beta_2} \equiv Y^2_{\alpha_1\beta_1}Y^{\beta_1}_{\beta_2} = -Y^2_{\beta_2\alpha_2}Y^{\alpha_2}_{\alpha_1}\,, \tag{D.10}$$

and quartic in Y and antisymmetric in spinor indices

$$Y^4_{\alpha_1\beta_1} \equiv Y^2_{\alpha_1\gamma_1}Y^{2\gamma_1}_{\beta_1} = -\frac{1}{2}\epsilon_{\alpha_1\beta_1}Y^4\,, \qquad Y^4_{\alpha_2\beta_2} \equiv Y^2_{\alpha_2\gamma_2}Y^{2\gamma_2}_{\beta_2} = \frac{1}{2}\epsilon_{\alpha_2\beta_2}Y^4\,, \tag{D.11}$$

where

$$Y^4 \equiv Y^2_{\alpha_1\beta_1}Y^{2\alpha_1\beta_1} = -Y^2_{\alpha_2\beta_2}Y^{2\alpha_2\beta_2}\,. \tag{D.12}$$

The spinorial quantities s and t are defined as

$$s(Y) \equiv Y+\frac{i}{3}Y^3\,, \qquad t(Y) \equiv \epsilon + iY^2 - \frac{1}{6}Y^4\,. \tag{D.13}$$

Analogous definitions can be given for Z. The normalization of the dynamical generators is not fixed by the superalgebra at order $\mathcal{O}(g_2)$ and can be an arbitrary (dimensionless) function $f(\mu\alpha_3\,,\frac{\alpha_1}{\alpha_3})$ of the light-cone momenta and μ due to the fact that P^+ is a central element of the algebra.

D.2 Commutation relations

Rules for (anti)commutation of β annihilation operator with Y and Z elements in the prefactor:

$$Y^{\alpha_1\alpha_2} = \sum_{r=1}^{3}\sum_{n} G_{|n|\,(r)}\beta^{\dagger\alpha_1\alpha_2}_{n\,(r)} \tag{D.14}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^{\alpha_1\,\alpha_2}\} = G_{|m|\,(s)}\delta^{\alpha_1}_{\gamma_1}\delta^{\alpha_2}_{\gamma_2} \tag{D.15}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y_{\alpha_1\,\alpha_2}\} = G_{|m|\,(s)}\epsilon_{\alpha_1\,\gamma_1}\epsilon_{\alpha_2\,\gamma_2} \tag{D.16}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^2_{\alpha_1\,\beta_1}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_1\,\alpha_1}Y_{\beta_1\,\gamma_2} + \epsilon_{\gamma_1\,\beta_1}Y_{\alpha_1\,\gamma_2}\right) \tag{D.17}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^2_{\alpha_2\,\beta_2}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_2\,\alpha_2}Y_{\gamma_1\,\beta_2} + \epsilon_{\gamma_2\,\beta_2}Y_{\gamma_1\,\alpha_2}\right) \tag{D.18}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^3_{\alpha_1\,\beta_2}\} = G_{|m|\,(s)}\left(\epsilon_{\gamma_1\,\alpha_1}Y^2_{\gamma_2\,\beta_2} - \epsilon_{\gamma_2\,\beta_2}Y^2_{\alpha_1\,\gamma_1} + Y_{\alpha_1\,\gamma_2}Y_{\gamma_1\,\beta_2}\right) \tag{D.19}$$

$$\{\beta_{\gamma_1\,\gamma_2\,m\,(s)}, Y^4\} = -4\,G_{|m|\,(s)}Y^3_{\gamma_1\,\gamma_2} \tag{D.20}$$

And exactly the same for Z and the dotted indices

D.3 Matrix elements

Some useful matrix elements are (where $\langle 3| \equiv \langle\alpha_3|$)

$$\langle 3|\,\alpha^{(3)\,i}_n\,\widetilde{K}^{\dot\gamma_1\gamma_1}|V\rangle = \left(K^{(3)}_{-n}\,\sigma^{i\dot\gamma_1\gamma_1} + \widetilde{K}^{\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n\,p}\,\alpha^{\dagger\,(s)\,i}_p\right)\langle 3|V\rangle \tag{D.21}$$

Where s and any other internal string index is restricted to run over $1, 2$ only. We also have,

$$\langle 3|\,\alpha^{(3)\,i}_n\,K_k\widetilde{K}_l|V\rangle = \left(K^{(3)}_n\,\widetilde{K}_l\,\delta^{ik} + K^{(3)}_{-n}\,K_k\,\delta^{il} + K_k\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n\,p}\,\alpha^{\dagger\,(s)\,i}_p\right)\langle 3|V\rangle \tag{D.22}$$

$$\begin{aligned}\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,K_k\,\widetilde{K}_l|V\rangle = \Bigg(&K^{(3)}_{n_1}\,K^{(3)}_{-n_2}\,\delta^{ik}\,\delta^{jl} + K^{(3)}_{-n_1}\,K^{(3)}_{n_2}\,\delta^{il}\,\delta^{jk}\\ &+ K^{(3)}_{n_1}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{ik} + K^{(3)}_{n_2}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jk}\\ &+ K^{(3)}_{-n_1}\,K_k\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,j}_p\,\delta^{il} + K^{(3)}_{-n_2}\,K_k\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,\delta^{jl}\\ &+ K_k\,\widetilde{K}_l\left(\widetilde{N}^{3\,3}_{n_1\,n_2}\delta^{ij} + \widetilde{N}^{3\,s}_{n_1\,p}\,\widetilde{N}^{3\,r}_{n_2\,q}\,\alpha^{\dagger\,(s)\,i}_p\,\alpha^{\dagger\,(r)\,j}_q\right)\Bigg)\langle 3|V\rangle\end{aligned} \tag{D.23}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,t_{\beta_1\gamma_1}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{np} - \widetilde{Q}^{s\,3}_{pn}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(\epsilon_{\beta_1\gamma_1} + iY^2_{\beta_1\gamma_1} + \frac{1}{12}\,\epsilon_{\beta_1\gamma_1}Y^4\right) + iG^{(3)}_{|n|}\left(\epsilon_{\sigma_1\beta_1}\,Y_{\gamma_1\sigma_2} + \epsilon_{\sigma_1\gamma_1}\,Y_{\beta_1\sigma_2}\right) - \frac{1}{3}G^{(3)}_{|n|}\,\epsilon_{\beta_1\gamma_1}\,Y^3_{\sigma_1\sigma_2}\Bigg\}\langle 3|V\rangle \tag{D.24}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,t_{\beta_2\gamma_2}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{np} - \widetilde{Q}^{s\,3}_{pn}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(\epsilon_{\beta_2\gamma_2} + iY^2_{\beta_2\gamma_2} - \frac{1}{12}\,\epsilon_{\beta_2\gamma_2}Y^4\right) + iG^{(3)}_{|n|}\left(\epsilon_{\sigma_2\beta_2}\,Y_{\sigma_1\gamma_2} + \epsilon_{\sigma_2\gamma_2}\,Y_{\sigma_1\beta_2}\right) + \frac{1}{3}G^{(3)}_{|n|}\,\epsilon_{\beta_2\gamma_2}\,Y^3_{\sigma_1\sigma_2}\Bigg\}\langle 3|V\rangle \tag{D.25}$$

$$\langle 3|\,\beta^{(3)}_{n\,\sigma_1\sigma_2}\,s_{\beta_1\gamma_2}(Y)\,|V\rangle = \Bigg\{\left(\widetilde{Q}^{3\,s}_{np} - \widetilde{Q}^{s\,3}_{pn}\right)\,\beta^{\dagger\,(s)}_{p\,\sigma_1\sigma_2}\left(Y_{\beta_1\gamma_2} + \frac{i}{3}Y^3_{\beta_1\gamma_2}\right) + G^{(3)}_{|n|}\,\epsilon_{\sigma_1\beta_1}\,\epsilon_{\sigma_2\gamma_2} + \frac{i}{3}G^{(3)}_{|n|}\left(\epsilon_{\sigma_1\beta_1}\,Y^2_{\sigma_2\gamma_2} - \epsilon_{\sigma_2\beta_2}\,Y^2_{\sigma_1\beta_1} + Y_{\beta_1\sigma_2}\,Y_{\sigma_1\gamma_2}\right)\Bigg\}\langle 3|V\rangle \tag{D.26}$$

$$\langle\alpha_3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,Q_{2\,\alpha_1\dot\alpha_2} = \frac{\bar\eta}{\sqrt{2\,|\alpha_3|}}\langle\alpha_3|\,\left(\Omega_{n_2}\,\sigma^{j\dot\beta_1}_{\alpha_1}\,\alpha^i_{n_1}\,\beta_{n_2\,\dot\beta_1\dot\alpha_2} + \quad (i\leftrightarrow j,\,n_1\leftrightarrow n_2)\right) \tag{D.27}$$

We now construct the matrix elements we need in later calculations, for instance,

$$\begin{aligned}
\epsilon^{\alpha_1\beta_1}\,\epsilon^{\dot\alpha_2\dot\beta_2}\,\langle 3|\,\alpha^{(3)\,i}_{n_1}\,\alpha^{(3)\,j}_{n_2}\,Q_{2\,\alpha_1\dot\alpha_2}\,|Q_{3\,\beta_1\dot\beta_2}\rangle &= \frac{g_2}{4\,\alpha_3^3}\sqrt{\frac{-\alpha'\,\kappa}{-4\,\alpha_3}}\,\Omega_{n_2}\,\sigma^{j\dot\gamma_1}_{\alpha_1}\epsilon^{\alpha_1\beta_1}\,\epsilon^{\dot\alpha_2\dot\beta_2}\\
&\times\Bigg[\Big\{K^{(3)}_{-n_1}\,\sigma^{i\dot\sigma_1\sigma_1} + \widetilde{K}^{\dot\sigma_1\sigma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\Big\}\,t_{\beta_1\sigma_1}(Y)\,\Big\{\left(Z_{\dot\sigma_1\dot\beta_2} + \frac{i}{3}Z^3_{\dot\sigma_1\dot\beta_2}\right)\,\widehat{Q}^{3\,r}_{n_2\,q}\,\beta^{\dagger\,(r)}_{q\,\dot\gamma_1\dot\alpha_2}\\
&\qquad + G^{(3)}_{|n_2|}\,\epsilon_{\dot\gamma_1\dot\sigma_1}\,\epsilon_{\dot\alpha_2\dot\beta_2} + \frac{i}{3}G^{(3)}_{|n_2|}\left(\epsilon_{\dot\gamma_1\dot\sigma_1}Z^2_{\dot\alpha_2\dot\beta_2} - \epsilon_{\dot\alpha_2\dot\beta_2}\,Z^2_{\dot\gamma_1\dot\sigma_1} + Z_{\dot\sigma_1\dot\alpha_2}\,Z_{\dot\gamma_1\dot\beta_2}\right)\Big\}\\
&+ i\,\widetilde{K}^{\dot\sigma_2\sigma_2}\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha^{\dagger\,(s)\,i}_p\,s_{\beta_1\sigma_2}(Y)\Big\{\widehat{Q}^{3\,r}_{n_2\,q}\,\beta^{\dagger\,(r)}_{q\,\dot\gamma_1\dot\alpha_2}\left(\epsilon_{\dot\beta_2\dot\sigma_2} - iZ^2_{\dot\beta_2\dot\sigma_2} - \frac{1}{12}\,\epsilon_{\dot\beta_2\dot\sigma_2}\,Z^4\right) - \frac{1}{3}\,G^{(3)}_{|n_2|}\,\epsilon_{\dot\beta_2\dot\sigma_2}\,Z^3_{\dot\gamma_1\dot\alpha_2}\\
&\qquad + i\,G^{(3)}_{|n_2|}\,\left(\epsilon_{\dot\alpha_2\dot\beta_2}\,Z_{\dot\gamma_1\dot\sigma_2} + \epsilon_{\dot\alpha_2\dot\sigma_2}\,Z_{\dot\gamma_1\dot\beta_2}\right)\Big\}\Bigg]\langle 3|V\rangle \quad + \quad (i\leftrightarrow j, n_1\leftrightarrow n_2)
\end{aligned} \tag{D.28}$$

where $\widehat{Q} = \widetilde{Q} - \widetilde{Q}^T$.

$$
\begin{aligned}
\langle 3|\, \alpha^{(3)\,i}_{n_1}\, \alpha^{(3)\,j}_{n_2}\, |H_3\rangle = \frac{g_2\,\alpha'}{8\,\alpha_3^3}\Bigg[& \left(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{ij} + \widetilde{N}^{3\,s}_{n_1\,p}\,\widetilde{N}^{3\,r}_{n_2\,q}\,\alpha_p^{\dagger\,(s)\,i}\,\alpha_q^{\dagger\,(r)\,j}\right) \\
& \times \left(\left[K_k\widetilde{K}_l - \frac{\mu\,\alpha}{\alpha'}\,\delta_{kl}\right] v^{kl} - \left[K_{k'}\widetilde{K}_{l'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{k'l'}\right] v^{k'l'}\right. \\
& \left. - K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z) - \widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\right) \\
+ \Big(& K^{(3)}_{n_1}\,K^{(3)}_{-n_2}\,\delta^{ik}\,\delta^{jl} + K^{(3)}_{-n_1}\,K^{(3)}_{n_2}\,\delta^{il}\,\delta^{jk} + K^{(3)}_{n_1}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j}\,\delta^{ik} + K^{(3)}_{n_2}\,\widetilde{K}_l\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i}\,\delta^{jk} \\
& + K^{(3)}_{-n_1}\,K_k\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j}\,\delta^{il} + K^{(3)}_{-n_2}\,K_k\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i}\,\delta^{jl}\Big)\, v^{kl} \\
- \Big(& \sigma^{i\dot\rho_1\rho_1}\,K^{(3)}_{n_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j} + \sigma^{j\dot\rho_1\rho_1}\,K^{(3)}_{n_2}\,\widetilde{K}^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i}\Big)\, s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z) \\
- \Big(& \sigma^{i\dot\rho_1\rho_1}\,K^{(3)}_{-n_1}\,K^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j} + \sigma^{j\dot\rho_1\rho_1}\,K^{(3)}_{-n_2}\,K^{\dot\rho_2\rho_2}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i}\Big)\, s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\Bigg]\langle 3|V\rangle
\end{aligned}
\tag{D.29}
$$

D.4 More matrix elements

Consider

$$
\begin{aligned}
\langle 3|\, \alpha^{(3)\,i}_{n_1}\, \alpha^{(3)\,j}_{n_2}\, \widetilde{K}^{\dot\gamma_1\gamma_1}|V\rangle = (& K^{(3)}_{-n_1}\,\sigma^{i\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j} + K^{(3)}_{-n_2}\,\sigma^{j\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i} \\
& + \widetilde{K}^{\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j})\langle 3|V\rangle
\end{aligned}
\tag{D.30}
$$

and therefore

$$
\begin{aligned}
\langle 3|\, \alpha^{(3)\,i}_{n_1}\, \alpha^{(3)\,j}_{n_2}\, |Q_{3\,\beta_1\dot\beta_2}\rangle = & \frac{g_2\eta}{4\alpha_3^3}\sqrt{-\frac{\alpha'\kappa}{2}} \\
\times \Big\{ & s_{\dot\gamma_1\dot\beta_2}(Z)t_{\beta_1\gamma_1}(Y)\Big[K^{(3)}_{-n_1}\,\sigma^{i\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j} + K^{(3)}_{-n_2}\,\sigma^{j\dot\gamma_1\gamma_1}\,\widetilde{N}^{3\,s}_{n_1\,p}\,\alpha_p^{\dagger\,(s)\,i} \\
& + \widetilde{K}^{\dot\gamma_1\gamma_1}\left(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j} + \widetilde{N}^{3\,r}_{n_1\,q}\,\alpha_q^{\dagger\,(r)\,i}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j}\right)\Big] \\
& + i s_{\beta_1\gamma_2}(Y)t^*_{\dot\beta_2\dot\gamma_2}(Z)\widetilde{K}^{\dot\gamma_2\gamma_2}\left(\widetilde{N}^{3\,3}_{n_1\,n_2}\,\delta^{i\,j} + \widetilde{N}^{3\,r}_{n_1\,q}\,\alpha_q^{\dagger\,(r)\,i}\,\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha_p^{\dagger\,(s)\,j}\right)\Big\}
\end{aligned}
\tag{D.31}
$$

We will need the following expressions:

$$
\begin{aligned}
w^{i\,j}_{n\,\dot\beta_1\dot\alpha_2} \equiv [\beta_{n\,\dot\beta_1\dot\alpha_2}, v^{i\,j}] = G^{(3)}_{|n|}\Bigg\{ & \delta^{i\,j}\left(-\frac{1}{3}Z^3_{\dot\beta_1\dot\alpha_2} - \frac{1}{36}Y^4 Z^3_{\dot\beta_1\dot\alpha_2}\right) \\
& + \frac{i}{2}\left[\frac{1}{3}Y^{2\,i\,j}Z^3_{\dot\beta_1\dot\alpha_2} - \sigma^{i\,j}_{\dot\rho_1\dot\gamma_1}\left(\delta^{\dot\rho_1}_{\dot\beta_1}Z^{\dot\gamma_1}_{\dot\alpha_2} + \delta^{\dot\gamma_1}_{\dot\beta_1}Z^{\dot\rho_1}_{\dot\alpha_2}\right)\left(1 + \frac{1}{12}Y^4\right)\right] - \frac{1}{2}Y^{2\,k\,(i}\sigma^{j)\,k}_{\dot\beta_1\dot\gamma_1}Z^{\dot\gamma_1}_{\dot\alpha_2}\Bigg\}
\end{aligned}
\tag{D.32}
$$

$$w^{i'\,j'}_{n\,\dot\beta_1\dot\alpha_2} \equiv [\beta_{n\,\dot\beta_1\dot\alpha_2}, v^{i'\,j'}] = G^{(3)}_{|n|}\Bigg\{\delta^{i'\,j'}\left(\frac{1}{3}Z^3_{\dot\beta_1\dot\alpha_2} - \frac{1}{36}Y^4 Z^3_{\dot\beta_1\dot\alpha_2}\right)$$
$$+\frac{i}{2}\left[-\frac{1}{3}Y^{2\,i'\,j'}Z^3_{\dot\beta_1\dot\alpha_2} - \sigma^{i'\,j'}_{\dot\rho_2\dot\gamma_2}\left(\delta^{\dot\rho_2}_{\dot\alpha_2}Z^{\dot\gamma_2}_{\dot\beta_1} + \delta^{\dot\gamma_2}_{\dot\alpha_2}Z^{\dot\rho_2}_{\dot\beta_1}\right)\left(1-\frac{1}{12}Y^4\right)\right] - \frac{1}{2}Y^{2\,k'\,(i'}\sigma^{j')\,k'}_{\dot\alpha_2\dot\gamma_2}Z^{\dot\gamma_2}_{\dot\beta_1}\Bigg\} \tag{D.33}$$

We'll also need the following

$$Q_{2\,\alpha_1\,\dot\alpha_2}\,\alpha^{\dagger\,(3)\,k}_{n_1}\,\alpha^{\dagger\,(3)\,l}_{n_2}\,|\alpha_3\rangle = \frac{-\eta}{\sqrt{2\,|\alpha_3|}}\left(\Omega_{n_1}\,\sigma^{k\,\dot\gamma_1}_{\alpha_1}\,\alpha^{\dagger\,l}_{n_2}\,\beta^{\dagger}_{n_1\,\dot\gamma_1\dot\alpha_2} + \quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2)\right)|\alpha_3\rangle \tag{D.34}$$

or

$$|\lambda\rangle = Q_2^{\beta_1\,\dot\beta_2}\,\alpha^{\dagger\,(3)\,k}_{n_1}\,\alpha^{\dagger\,(3)\,l}_{n_2}\,|\alpha_3\rangle = \frac{-\eta}{\sqrt{2\,|\alpha_3|}}\left(\Omega_{n_1}\,\sigma^{k\,\dot\gamma_1\,\beta_1}\,\alpha^{\dagger\,l}_{n_2}\,\beta^{\dagger\,\dot\beta_2}_{n_1\,\dot\gamma_1} + \quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2)\right)|\alpha_3\rangle \tag{D.35}$$

allowing us to calculate,

$$\begin{aligned}
\langle\lambda|H_3\rangle = {} & \frac{g_2\,\alpha'}{8\,\alpha_3^3}\frac{\bar\eta}{\sqrt{2|\alpha_3|}}\sigma^k{}_{\beta_1\,\dot\gamma_1}\,\Omega_{n_1} \\
& \times\Bigg[\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)\widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,l}_p\Bigg\{\left[K_i\widetilde{K}_j - \frac{\mu\,\alpha}{\alpha'}\,\delta_{ij}\right]v^{ij} - \left[K_{i'}\widetilde{K}_{j'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{i'j'}\right]v^{i'j'} \\
& \qquad - K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\,s^*_{\dot\rho_1\dot\rho_2}(Z) - \widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\,s_{\dot\rho_1\dot\rho_2}(Z)\Bigg\} \\
& + \widetilde{N}^{3\,s}_{n_2\,p}\,\alpha^{\dagger\,(s)\,l}_p\Bigg\{\left[K_i\widetilde{K}_j - \frac{\mu\,\alpha}{\alpha'}\,\delta_{ij}\right]\left(w^{ij}_{n_1}\right)^{\dot\gamma_1}_{\dot\beta_2} - \left[K_{i'}\widetilde{K}_{j'} - \frac{\mu\,\alpha}{\alpha'}\,\delta_{i'j'}\right]\left(w^{i'j'}_{n_1}\right)^{\dot\gamma_1}_{\dot\beta_2} \\
& \qquad + G^{(3)}_{|n_1|}\,K^{\dot\rho_1\rho_1}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} - \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right] \\
& \qquad + G^{(3)}_{|n_1|}\,\widetilde{K}^{\dot\rho_1\rho_1}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} + \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\} \\
& + \left(K^{(3)}_{n_2}\,\widetilde{K}_j\,\delta^{li} + K^{(3)}_{-n_2}\,K_i\,\delta^{lj}\right)\Bigg\{\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right)v^{ij} + \left(w^{ij}_{n_1}\right)^{\dot\gamma_1}_{\dot\beta_2}\Bigg\} \\
& - \sigma^{l\,\dot\rho_1\,\rho_1}\,K^{(3)}_{n_2}\,\widetilde{K}^{\dot\rho_2\rho_2}\,s_{\rho_1\rho_2}(Y)\Bigg\{s^*_{\dot\rho_1\dot\rho_2}(Z)\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right) \\
& \qquad\qquad - G^{(3)}_{|n_1|}\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} - \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\} \\
& - \sigma^{l\,\dot\rho_1\,\rho_1}\,K^{(3)}_{-n_2}\,K^{\dot\rho_2\rho_2}\,s^*_{\rho_1\rho_2}(Y)\Bigg\{s_{\dot\rho_1\dot\rho_2}(Z)\left(-\beta^{\dagger\,\dot\gamma_1}_{q\,(r)\,\dot\beta_2}\,\widehat{Q}^{3\,r}_{n_1\,q}\right) \\
& \qquad\qquad - G^{(3)}_{|n_1|}\left[\delta^{\dot\gamma_1}_{\dot\rho_1}\,\epsilon_{\dot\beta_2\,\dot\rho_2} + \frac{i}{3}\left(\delta^{\dot\gamma_1}_{\dot\rho_1}\,Z^2_{\dot\beta_2\,\dot\rho_2} - \epsilon_{\dot\beta_2\,\dot\rho_2}\,Z^{2\,\dot\gamma_1}_{\dot\rho_1} + Z_{\dot\rho_1\,\dot\beta_2}\,Z^{\dot\gamma_1}_{\dot\rho_2}\right)\right]\Bigg\}\Bigg] \\
& + \quad (k\leftrightarrow l,\, n_1\leftrightarrow n_2)
\end{aligned} \tag{D.36}$$

Appendix E

Energy Shift for DVPPRT Vertex

This appendix contains the hitherto unpublished work by the author. It is therefore somewhat more detailed then the earlier appendices.

E.1 Derivation of the "Master Formula"

E.1.1 The two-string state $< e|v >$ and the number $< v|e >< e|v >$

The simplest quantity that we shall need to compute is

$$< e|v > \;=\; < \tilde{\alpha}_3|\alpha_n^{(3)i}\alpha_{-n}^{(3)j}|v >= \left[\tilde{N}^{33}_{-nn}\delta^{ij} + \tilde{N}^{3r}_{-np}\alpha_p^{(r)i\dagger}\alpha_q^{(s)j\dagger}\tilde{N}^{s3}_{qn}\right] < \tilde{\alpha}_3|v > \tag{E.1}$$

In the last expression, p and q are summed over all integers and r and s are summed from 1 to 2. This is a state in the two-string sector of the Hilbert space. The state $< e|v >$ has Neumann matrices with creation operators for the two strings.

Now, from Eqn. (E.1)

$$< e|v > \;=\; < \tilde{\alpha}_3|\alpha_n^{(3)i}\alpha_{-n}^{(3)j}|v >= \left[\tilde{N}^{33}_{-nn}\delta^{ij} + \tilde{N}^{3r}_{-np}\alpha_p^{(r)i\dagger}\alpha_q^{(s)j\dagger}\tilde{N}^{s3}_{qn}\right] < \tilde{\alpha}_3|v >$$

we have

$$\begin{aligned}
< v|e > \mathcal{P} < e|v > =& < v|\tilde{\alpha}_3 > \left[\tilde{N}^{33}_{-nn}\delta^{ij} + \tilde{N}^{3r}_{-np}\alpha_p^{(r)i\dagger}\alpha_q^{(s)j\dagger}\tilde{N}^{s3}_{qn}\right]\cdot \\
&\cdot\mathcal{P}\left[\tilde{N}^{33}_{-nn}\delta^{kl} + \tilde{N}^{3\bar{r}}_{-n\bar{p}}\alpha_{\bar{p}}^{(\bar{r})k\dagger}\alpha_{\bar{q}}^{(\bar{s})l\dagger}\tilde{N}^{\bar{s}3}_{\bar{q}n}\right] < \tilde{\alpha}_3|v > \\
&= \left[\tilde{N}^{33}_{-nn}\right]^2\delta^{ij}\delta^{kl} < v|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v > + \\
&+\tilde{N}^{33}_{-nn}\tilde{N}^{nr}_{-np} < v|\tilde{\alpha}_3 > \left(\mathcal{P}\delta^{ij}\alpha_p^{(r)k\dagger}\alpha_q^{(s)l\dagger} + \alpha_p^{(r)i}\alpha_q^{(s)j}\delta^{kl}\mathcal{P}\right) < \tilde{\alpha}_3|v > \tilde{N}^{s3}_{qn} + \\
&+\tilde{N}^{3r}_{-np}\tilde{N}^{3\bar{r}}_{-n\bar{p}} < v|\tilde{\alpha}_3 > \alpha_p^{(r)i}\alpha_q^{(s)j}\mathcal{P}\alpha_{\bar{p}}^{(\bar{r})k\dagger}\alpha_{\bar{q}}^{(\bar{s})l\dagger} < \tilde{\alpha}_3|v > \tilde{N}^{s3}_{qn}N^{\bar{s}3}_{\bar{q}n}
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
< v|e > \mathcal{P} < e|v > =& \int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} < v_{NQ}|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v_{MP} > \Bigg\{\left[\tilde{N}^{33}_{-nn}\right]^2\delta^{ij}\delta^{kl} + \\
&+\tilde{N}^{33}_{-nn}\tilde{N}^{3r}_{-np}\left(\delta^{ij}e^{i(q\theta_s+p\theta_r)} < \alpha_q^{(s)k\dagger}\alpha_p^{(r)l\dagger} > + < \alpha_p^{(r)i}\alpha_q^{(s)j} > \delta^{kl}\right)\tilde{N}^{s3}_{qn} + \\
&+\tilde{N}^{3r}_{-np}\tilde{N}^{3\bar{r}}_{-n\bar{p}}e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{s}})} < \alpha_p^{(r)i}\alpha_q^{(s)j}\alpha_{\bar{p}}^{(\bar{r})k\dagger}\alpha_{\bar{q}}^{(\bar{s})l\dagger} > \tilde{N}^{s3}_{qn}\tilde{N}^{\bar{s}3}_{\bar{q}n}\Bigg\}
\end{aligned} \tag{E.3}$$

Now, we have the formulae for contractions:

$$< \alpha_p^{(s)i} \alpha_q^{(r)j} >= \delta^{ij} \left(e^{ip\theta_s} \tilde{N}_{pq}^{sr} e^{iq\theta_r} + e^{ip\theta_s} \tilde{N}_{pl_1}^{st_1} e^{il_1\theta_{t_1}} \tilde{N}_{l_1 l_2}^{t_1 t_2} e^{il_2\theta_{t_2}} \tilde{N}_{l_2 q}^{t_2 r} e^{iq\theta_r} + ... \right) \tag{E.4}$$

$$< \alpha_p^{(s)i\dagger} \alpha_q^{(r)j\dagger} >= \delta^{ij} \left(\tilde{N}_{pq}^{sr} + \tilde{N}_{pl_1}^{st_1} e^{il_1\theta_{t_1}} \tilde{N}_{l_1 l_2}^{t_1 t_2} e^{il_2\theta_{t_2}} \tilde{N}_{l_2 q}^{t_2 r} + ... \right) \tag{E.5}$$

If we define

$$U_{pq}^{sr} = \delta^{sr} \delta_{pq} e^{ip\theta_s} \quad r, s = 1, 2 \tag{E.6}$$

$$U_{pq}^{sr} = 0 \quad r \text{ or } s = 3 \tag{E.7}$$

The first two lines of the equation above combined with the $< \alpha\alpha >$ and $< \alpha^\dagger \alpha^\dagger >$ contractions of the third line give the result:

$$\delta^{ij}\delta^{kl} \left\{ [N + NUNUN + NUNUNUNUN + ...]_{-nn}^{33} \right\}^2 \frac{\det^8(1 + UQUQ)}{\det^8(1 - UNUN)} =$$

$$= \delta^{ij}\delta^{kl} \left\{ \left[N \frac{1}{1 - UNUN} \right]_{-nn}^{33} \right\}^2 \frac{\det^8(1 + UQUQ)}{\det^8(1 - UNUN)}$$

The other two contractions of the last term produce

$$\delta^{ik}\delta^{jl} \left[N \frac{1}{1 - UNUN} UN \right]_{nn}^{33} \cdot \left[N \frac{1}{1 - UNUN} UN \right]_{-n,-n}^{33} \cdot \frac{\det^8(1 + UQUQ)}{\det^8(1 - UNUN)}$$

and

$$\delta^{il}\delta^{jk} \left[N \frac{1}{1 - UNUN} UN \right]_{-nn}^{33} \cdot \left[N \frac{1}{1 - UNUN} UN \right]_{n,-n}^{33} \cdot \frac{\det^8(1 + UQUQ)}{\det^8(1 - UNUN)}$$

Combined, these give

$$\begin{aligned} & < v|e > \mathcal{P} < e|v >= \\ = & \int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \frac{\det^8(1 + UQUQ)}{\det^8(1 - UNUN)} \left(\delta^{ij}\delta^{kl} \left[N \frac{1}{1 - UNUN} \right]_{-nn}^{33} \left[N \frac{1}{1 - UNUN} \right]_{-nn}^{33} \right. \\ & + \delta^{il}\delta^{jk} \left[N \frac{1}{1 - UNUN} UN \right]_{n,-n}^{33} \left[N \frac{1}{1 - UNUN} UN \right]_{-nn}^{33} \\ & \left. + \delta^{ik}\delta^{jl} \left[N \frac{1}{1 - UNUN} UN \right]_{nn}^{33} \left[N \frac{1}{1 - UNUN} UN \right]_{-n,-n}^{33} \right) \end{aligned} \tag{E.8}$$

This can be decomposed into a contribution to the singlet

$$
\begin{aligned}
&< v|e > \mathcal{P} < e|v >|_{(1,1)} = \\
&= 4\int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \frac{\det^8(1+UQUQU)}{\det^8(1-UNUN)} \left(\left[N\frac{1}{1-UNUN}\right]^{33}_{-nn} \left[N\frac{1}{1-UNUN}\right]^{33}_{-nn} + \right. \\
&\qquad \left[N\frac{1}{1-UNUN}UN\right]^{33}_{n,-n} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-nn} \\
&\qquad \left. + \left[N\frac{1}{1-UNUN}UN\right]^{33}_{nn} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-n,-n} \right)
\end{aligned}
\tag{E.9}
$$

the anti-symmetric state

$$
\begin{aligned}
&< v|e > \mathcal{P} < e|v >|_{(6,1)} = \\
&= \int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \left(\frac{1}{2}\left[N\frac{1}{1-UNUN}UN\right]^{33}_{n,-n} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-nn} \right. \\
&\qquad \left. -\frac{1}{2}\left[N\frac{1}{1-UNUN}UN\right]^{33}_{nn} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-n,-n} \right)
\end{aligned}
\tag{E.10}
$$

and the symmetric traceless state

$$
\begin{aligned}
&< v|e > \mathcal{P} < e|v >|_{(9,1)} = \\
&= \int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \left(\frac{1}{2}\left[N\frac{1}{1-UNUN}UN\right]^{33}_{n,-n} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-nn} \right. \\
&\qquad \left. +\frac{1}{2}\left[N\frac{1}{1-UNUN}UN\right]^{33}_{nn} \left[N\frac{1}{1-UNUN}UN\right]^{33}_{-n,-n} \right)
\end{aligned}
\tag{E.11}
$$

E.1.2 The two-string state $< Q_2 e|v >$ and the number $< v|Q_2 e >< Q_2 e|v >$

Now, we must consider the contribution of the states of the form

$$
\begin{aligned}
< Q_2 e|v > &=< \tilde{\alpha}_3|\alpha_n^{i(3)}\alpha_{-n}^{j(3)}Q^{(3)}_{2\alpha_1\dot{\alpha}_2}|v > \\
&= \frac{\bar{\eta}}{\sqrt{|\tilde{\alpha}_s|}} \sum_{k=-\infty}^{\infty} \Omega_{k(3)} < \tilde{\alpha}_3|\alpha_n^{i(3)}\alpha_{-n}^{j(3)} \left(\alpha^{(3)\dot{\beta}_1\dagger}_{k\alpha_1}\beta^{(3)}_{k\dot{\beta}_1\dot{\alpha}_2}\right)|v > \\
&= \frac{\bar{\eta}}{\sqrt{|\tilde{\alpha}_s|}} \Omega_{n(3)}\epsilon^{\dot{\gamma}_1\dot{\beta}_1}\sigma^{i*}_{\alpha_1\dot{\beta}_1} < \tilde{\alpha}_3|\alpha_{-n}^{j(3)}\beta^{(3)}_{n\dot{\gamma}_1\dot{\alpha}_2}|v > +(n\to -n, i\to j) \\
&= -\frac{\bar{\eta}}{\sqrt{|\tilde{\alpha}_s|}} \tilde{N}^{3s}_{-np}\alpha_p^{j(s)}\epsilon^{\dot{\gamma}_1\dot{\beta}_1}\sigma^{i*}_{\alpha_1\dot{\beta}_1}\beta^{(r)}_{q\dot{\gamma}_1\dot{\alpha}_2}Q^{r3}_{qn}\Omega_{n(3)} < \tilde{\alpha}_3|v > +(n\to -n, i\to j)
\end{aligned}
\tag{E.12}
$$

where this expression is summed over $p, q, r, s, \dot{\beta}_1$. The minus sign comes from transposition of Q.

Then:

$$
\begin{aligned}
&< v|Q_2^{(3)}e > \mathcal{P} < Q_2^{(3)}e|v >= \\
&= \epsilon^{\alpha_1\bar{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\bar{\alpha}}_2}\frac{1}{|\tilde{\alpha}_3|}\Big\{< v|\tilde{\alpha}_3 > \Big[\sigma^{j\dot{\beta}_1}_{\alpha_1}\tilde{N}^{(3r)}_{-np}\alpha^{(r)i}_p\beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\tilde{Q}^{(3s)}_{nq}\Omega_n + (n\to -n, i\to j)\Big]\cdot \\
&\cdot\mathcal{P}\Big[\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2}\tilde{Q}^{(\bar{s}3)}_{\bar{q},n}\Omega_n + (n\to -n, k\to l)\Big] < \tilde{\alpha}_3|v >\Big\} \\
&= \epsilon^{\alpha_1\bar{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\bar{\alpha}}_2}\frac{1}{|\tilde{\alpha}_3|}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} < v_{NQ}|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v_{MP} > \cdot \\
&\cdot[\sigma^{j\dot{\beta}_1}_{\alpha_1}\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{3\bar{r}}_{-n\bar{p}}e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{s}})} < \alpha^{(r)i}_p\beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \tilde{Q}^{3s}_{nq}\tilde{Q}^{\bar{s}3}_{\bar{q}n} \\
&+\sigma^{i\dot{\beta}_1}_{\alpha_1}\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{np}\tilde{N}^{3\bar{r}}_{-n\bar{p}}e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{s}})} < \alpha^{(r)j}_p\beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \tilde{Q}^{3s}_{-nq}\tilde{Q}^{\bar{s}3}_{\bar{q}n} \\
&+\sigma^{j\dot{\beta}_1}_{\alpha_1}\sigma^{k\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{3\bar{r}}_{n\bar{p}}e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{s}})} < \alpha^{(r)i}_p\beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})l\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \tilde{Q}^{3s}_{nq}\tilde{Q}^{\bar{s}3}_{\bar{q}-n} \\
&+\sigma^{i\dot{\beta}_1}_{\alpha_1}\sigma^{k\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{np}\tilde{N}^{3\bar{r}}_{n\bar{p}}e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{s}})} < \alpha^{(r)j}_p\beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})l\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \tilde{Q}^{3s}_{-nq}\tilde{Q}^{\bar{s}3}_{\bar{q}-n}]
\end{aligned}
\tag{E.13}
$$

Fermionic contractions are:

$$
< \beta^{(s)}_{p\alpha_1\alpha_2}\beta^{(r)}_{q\beta_1\beta_2} >= \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\left(e^{ip\theta_s}\tilde{Q}^{sr}_{pq}e^{iq\theta_r} - e^{ip\theta_s}\tilde{Q}^{st_1}_{pl_1}e^{il_1\theta_{t_1}}\tilde{Q}^{t_1t_2}_{l_1l_2}e^{il_2\theta_{t_2}}\tilde{Q}^{t_2r}_{l_2q}e^{iq\theta_r} + ...\right) \tag{E.14}
$$

$$
< \beta^{(s)\dagger}_{p\alpha_1\alpha_2}\beta^{(r)\dagger}_{q\beta_1\beta_2} >= \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\left(\tilde{Q}^{sr}_{pq} - \tilde{Q}^{st_1}_{pl_1}e^{il_1\theta_{t_1}}\tilde{Q}^{t_1t_2}_{l_1l_2}e^{il_2\theta_{t_2}}\tilde{Q}^{t_2r}_{l_2q} + ...\right) \tag{E.15}
$$

Using those and the properties

$$
\sigma^{i\dot{\alpha}\alpha}\sigma^{j}_{\alpha\dot{\beta}} + \sigma^{j\dot{\alpha}\alpha}\sigma^{i}_{\alpha\dot{\beta}} = 2\delta^{ij}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{E.16}
$$

we get:

$$
\begin{aligned}
&< v|Q_2^{(3)}e >< Q_2^{(3)}e|v >= \frac{4}{|\tilde{\alpha}_3|}\int\frac{d\theta_a}{2\pi}\cdot\frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)}\cdot \\
&\cdot\Bigg(\delta^{jl}\delta^{ik}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-n,-n}\left[\Omega QU\frac{1}{1+QUQU}Q\Omega\right]^{33}_{nn} \\
&+\delta^{jk}\delta^{il}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-nn}\left[\Omega QU\frac{1}{1+QUQU}Q\Omega\right]^{33}_{n,-n} \\
&+\delta^{il}\delta^{jk}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{n,-n}\left[\Omega QU\frac{1}{1+QUQU}Q\Omega\right]^{33}_{-nn} \\
&+\delta^{jl}\delta^{ik}\left[NU\frac{1}{1-NUNU}N\right]^{33}_{nn}\left[\Omega QU\frac{1}{1+QUQU}Q\Omega\right]^{33}_{-n,-n}\Bigg)
\end{aligned}
\tag{E.17}
$$

Remaining terms are:

$$
< v|e > Q_2\mathcal{P} < Q_2e|v > + < v|Q_2e > Q_2\mathcal{P} < e|v > \tag{E.18}
$$

First of those is:

$$
\begin{aligned}
&< v|e > Q_2\mathcal{P} < Q_2 e|v >= \\
&\epsilon^{\alpha_1\bar{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\bar{\alpha}}_2}\frac{1}{\sqrt{|\tilde{\alpha}_s||\alpha_3|}} < v|\tilde{\alpha}_3 > \left[\tilde{N}^{33}_{-nn}\delta^{ij} + \tilde{N}^{3r}_{-np}\alpha^{(r)i}_p\alpha^{(s)j}_q\tilde{N}^{s3}_{qn}\right]\cdot \\
&\cdot\sum_{m=-\infty}^{\infty}\Omega_{m(t)}\left(\alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} + i\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\right)\cdot \\
&\cdot\mathcal{P}\left[-\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2}\tilde{Q}^{(\bar{s}3)}_{\bar{q},n}\Omega_n + (n\to -n, k\to l)\right] < \tilde{\alpha}_3|v >
\end{aligned}
\tag{E.19}
$$

Disregarding for now the trace component, we are left with:

$$
\begin{aligned}
&< v|e > Q_2\mathcal{P} < e|Q_2 v >= \epsilon^{\alpha_1\bar{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\bar{\alpha}}_2}\frac{1}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} < v_{NQ}|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v_{MP} > \cdot \\
&\cdot\Big[-\Omega_{m(t)}\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},n}\Omega_n e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} < \alpha^{(r)i}_p\alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\beta^{(T)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&-i\Omega_{m(t)}\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},n}\Omega_n e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} < \alpha^{(r)i}_p\alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})k\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&-\Omega_{m(t)}\sigma^{k\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},-n}\Omega_{-n} e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} < \alpha^{(r)i}_p\alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})l\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&-i\Omega_{m(t)}\sigma^{k\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},-n}\Omega_{-n} e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} < \alpha^{(r)i}_p\alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\alpha^{(\bar{r})l\dagger}_{\bar{p}}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} >\Big]
\end{aligned}
$$

Taking traceless Wick Contractions:

$$
\begin{aligned}
&< v|e > Q_2\mathcal{P} < Q_2 e|v >= \epsilon^{\alpha_1\bar{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\bar{\alpha}}_2}\frac{1}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} < v_{NQ}|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v_{MP} > \cdot \\
&\cdot\Big[-\Omega_{m(t)}\sigma^{l\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},n}\Omega_n e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})}\cdot\Big(< \alpha^{(r)i}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1}_{m\alpha_1} >< \beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+ < \alpha^{(r)i}_p\alpha^{(t)\dot{\beta}_1}_{m\alpha_1} >< \alpha^{(s)j}_q\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+i < \alpha^{(r)i}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1} >< \beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+i < \alpha^{(r)i}_p\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1} >< \alpha^{(s)j}_q\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} >\Big) \\
&-\Omega_{m(t)}\sigma^{k\dot{\bar{\beta}}_1*}_{\bar{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{N}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{n\bar{p}}Q^{(\bar{s}3)}_{\bar{q},-n}\Omega_{-n} e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})}\Big(< \alpha^{(r)i}_p\alpha^{(\bar{r})l\dagger}_{\bar{p}} >< \alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1}_{m\alpha_1} >< \beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+ < \alpha^{(r)i}_p\alpha^{(t)\dot{\beta}_1}_{m\alpha_1} >< \alpha^{(s)j}_q\alpha^{(\bar{r})l\dagger}_{\bar{p}} >< \beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+i < \alpha^{(r)i}_p\alpha^{(\bar{r})l\dagger}_{\bar{p}} >< \alpha^{(s)j}_q\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1} >< \beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} > \\
&+ i < \alpha^{(r)i}_p\alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1} >< \alpha^{(s)j}_q\alpha^{(\bar{r})l\dagger}_{\bar{p}} >< \beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2}\beta^{(\bar{s})\dagger}_{\bar{q}\dot{\bar{\beta}}_1\dot{\bar{\alpha}}_2} >\Big)\Big]
\end{aligned}
$$

Which is equal to:

$$
\begin{aligned}
< v|e > Q_2\mathcal{P} < e|Q_2 v >= \frac{4}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}} \int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \cdot \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \cdot \\
\cdot \Bigg[\delta^{jl}\delta^{ik} \Bigg(\left[NU \frac{1}{1-NUNU} N \right]^{33}_{-n-n} \left[NUNU \frac{1}{1-NUNU} \Omega Q \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{nn} \\
-i \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-n-n} \left[N \frac{1}{1-UNUN} \Omega \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{nn} \\
+ \left[NU \frac{1}{1-NUNU} N \right]^{33}_{nn} \left[NUNU \frac{1}{1-NUNU} \Omega Q \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{-n-n} \\
-i \left[NU \frac{1}{1-NUNU} N \right]^{33}_{nn} \left[N \frac{1}{1-UNUN} \Omega \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{-n-n} \Bigg) \\
\delta^{jk}\delta^{il} \Bigg(\left[NU \frac{1}{1-NUNU} N \right]^{33}_{n-n} \left[NUNU \frac{1}{1-NUNU} \Omega Q \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{-nn} \\
-i \left[NU \frac{1}{1-NUNU} N \right]^{33}_{n-n} \left[N \frac{1}{1-UNUN} \Omega \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{-nn} \\
+ \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-nn} \left[NUNU \frac{1}{1-NUNU} \Omega Q \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{n-n} \\
-i \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-nn} \left[N \frac{1}{1-UNUN} \Omega \frac{1}{1+UQUQ} UQ\Omega \right]^{33}_{n-n} \Bigg) \Bigg]
\end{aligned}
\tag{E.20}
$$

Similarly:

$$
\begin{aligned}
< v|Q_2 e > Q_2\mathcal{P} < e|v >= \\
\epsilon^{\alpha_1\tilde{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\tilde{\alpha}}_2} \frac{1}{\sqrt{|\tilde{\alpha}_s||\alpha_3|}} < v|\tilde{\alpha}_3 > \left[-\sigma^{j\dot{\beta}_1}_{\alpha_1} \tilde{N}^{(3r)}_{-np} \alpha^{(r)i}_{p} \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2} \tilde{Q}^{(3s)}_{nq} \Omega_n + (n \to -n, i \to j) \right] \cdot \\
\cdot \sum_{m=-\infty}^{\infty} \Omega_{m(t)} \left(i\alpha^{(t)\dot{\beta}_1}_{m\alpha_1} \beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} \alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1} \beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2} \right) \cdot \\
\cdot \mathcal{P} \left[\tilde{N}^{33}_{-nn} \delta^{kl} + \tilde{N}^{3\bar{r}}_{-n\bar{p}} \alpha^{(\bar{r})k\dagger}_{\bar{p}} \alpha^{(\bar{s})l\dagger}_{\bar{q}} \tilde{N}^{\bar{s}3}_{\bar{q}n} \right] \cdot < \tilde{\alpha}_3|v >
\end{aligned}
$$

For traceless part this is:

$$
\begin{aligned}
< v|Q_2 e > Q_2\mathcal{P} < e|v > &= \epsilon^{\alpha_1\tilde{\alpha}_1}\epsilon^{\dot{\alpha}_2\dot{\tilde{\alpha}}_2}\frac{1}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} < v_{NQ}|\tilde{\alpha}_3 >< \tilde{\alpha}_3|v_{MP} > \cdot \\
\cdot \Big[\Omega_{m(t)}\sigma^{j\dot{\tilde{\beta}}_1 *}_{\tilde{\alpha}_1}\tilde{N}^{3r}_{-np}\tilde{Q}^{s3}_{qn}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}N^{(\bar{s}3)}_{\bar{q},n}\Omega_n e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} \cdot \Big(& i < \alpha^{(r)i}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ i < \alpha^{(r)i}_p\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ < \alpha^{(r)i}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ < \alpha^{(r)i}_p\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2} > \Big) \\
+\Omega_{m(t)}\sigma^{i\dot{\tilde{\beta}}_1 *}_{\tilde{\alpha}_1}\tilde{N}^{3r}_{np}\tilde{Q}^{s3}_{q-n}\tilde{N}^{(3\bar{r})}_{-n\bar{p}}N^{(\bar{s}3)}_{\bar{q},n}\Omega_{-n} e^{i(\bar{q}\theta_{\bar{s}}+\bar{p}\theta_{\bar{r}})} \cdot \Big(& i < \alpha^{(r)j}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ i < \alpha^{(r)j}_p\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \alpha^{(t)\dot{\beta}_1}_{m\alpha_1}\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)\dagger}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ < \alpha^{(r)j}_p\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2} > \\
&+ < \alpha^{(r)j}_p\alpha^{(\bar{s})l\dagger}_{\bar{q}} >< \alpha^{(t)\dot{\beta}_1\dagger}_{m\alpha_1}\alpha^{(\bar{r})k\dagger}_{\bar{p}} >< \beta^{(s)}_{q\dot{\beta}_1\dot{\alpha}_2}\beta^{(t)}_{m\dot{\beta}_1\dot{\alpha}_2} > \Big)\Big]
\end{aligned}
\tag{E.21}
$$

Or:

$$
\begin{aligned}
< v|Q_2 e > Q_2\mathcal{P} < e|v > &= \frac{4}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}}\int_{-\pi}^{\pi}\frac{d\theta_a}{2\pi}\cdot\frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)}\cdot \\
\cdot\Bigg[\delta^{jl}\delta^{ik}\Bigg(&-i\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-n-n}\left[\Omega Q\frac{1}{1+UQUQ}\Omega\frac{1}{1-UNUN}UN\right]^{33}_{nn} \\
&+\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-n-n}\left[\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\right]^{33}_{nn} \\
&-i\left[NU\frac{1}{1-NUNU}N\right]^{33}_{nn}\left[\Omega Q\frac{1}{1+UQUQ}\Omega\frac{1}{1-UNUN}UN\right]^{33}_{-n-n} \\
&+\left[NU\frac{1}{1-NUNU}N\right]^{33}_{nn}\left[\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\right]^{33}_{-n-n}\Bigg) \\
\delta^{jk}\delta^{il}\Bigg(&-i\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-n-n}\left[\Omega Q\frac{1}{1+UQUQ}\Omega\frac{1}{1-UNUN}UN\right]^{33}_{-nn} \\
&+\left[NU\frac{1}{1-NUNU}N\right]^{33}_{n-n}\left[\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\right]^{33}_{-nn} \\
&-i\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-nn}\left[\Omega Q\frac{1}{1+UQUQ}\Omega\frac{1}{1-UNUN}UN\right]^{33}_{n-n} \\
&+\left[NU\frac{1}{1-NUNU}N\right]^{33}_{-nn}\left[\Omega QUQU\frac{1}{1+QUQU}\Omega N\frac{1}{1-UNUN}UN\right]^{33}_{n-n}\Bigg)\Bigg]
\end{aligned}
\tag{E.22}
$$

Taking into account that the expressions in square brackets which are odd in Q are anti-

symetric in change of indices and that ones that are even in Q are symmetric we can add the two cross terms to get:

$$
\begin{aligned}
< v|Q_2 e > Q_2\mathcal{P} < e|v > + < v|e > Q_2\mathcal{P} < Q_2 e|v >= & \frac{8}{\sqrt{|\tilde{\alpha}_3||\tilde{\alpha}_s|}} \int_{-\pi}^{\pi} \frac{d\theta_a}{2\pi} \cdot \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \cdot \\
& \cdot \Bigg[\delta^{jl}\delta^{ik} \Bigg(\left[NU \frac{1}{1-NUNU} N \right]^{33}_{-n-n} \left[\Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \right]^{33}_{nn} \\
& + \left[NU \frac{1}{1-NUNU} N \right]^{33}_{nn} \left[\Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \right]^{33}_{-n-n} \Bigg) \\
& \delta^{jk}\delta^{il} \Bigg(\left[NU \frac{1}{1-NUNU} N \right]^{33}_{n-n} \left[\Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \right]^{33}_{-nn} \\
& + \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-nn} \left[\Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \right]^{33}_{n-n} \Bigg) \Bigg]
\end{aligned}
\tag{E.23}
$$

We can then combine all the above to get the full formula for traceless part of energy shift.

$$
\begin{aligned}
\frac{\delta E}{\mu} = & \frac{4}{|\tilde{\alpha}_3|\mu} \int \frac{d\theta_a}{2\pi} \cdot \frac{\det^8(1+UQUQ)}{\det^8(1-UNUN)} \cdot \\
& \cdot \Bigg(\delta^{jl}\delta^{ik} \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-n,-n} \Bigg[E_0 NU \frac{1}{1-NUNU} N + \Omega QU \frac{1}{1+QUQU} Q\Omega \\
& \qquad + \Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \Bigg]^{33}_{nn} \\
& + \delta^{jl}\delta^{ik} \left[NU \frac{1}{1-NUNU} N \right]^{33}_{nn} \Bigg[E_0 NU \frac{1}{1-NUNU} N + \Omega QU \frac{1}{1+QUQU} Q\Omega \\
& \qquad + \Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \Bigg]^{33}_{-n,-n} \\
& + \delta^{jk}\delta^{il} \left[NU \frac{1}{1-NUNU} N \right]^{33}_{-nn} \Bigg[E_0 NU \frac{1}{1-NUNU} N + \Omega QU \frac{1}{1+QUQU} Q\Omega \\
& \qquad + \Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \Bigg]^{33}_{n,-n} \\
& + \delta^{jk}\delta^{il} \left[NU \frac{1}{1-NUNU} N \right]^{33}_{n,-n} \Bigg[E_0 NU \frac{1}{1-NUNU} N + \Omega QU \frac{1}{1+QUQU} Q\Omega \\
& \qquad + \Omega QUQU \frac{1}{1+QUQU} \Omega N \frac{1}{1-UNUN} UN \Bigg]^{33}_{-nn} \Bigg)
\end{aligned}
\tag{E.24}
$$

E.2 4-impurity calculation

At the 4-impurity level we have to take into account the determinants. Two posibilities for the 4-impurity contributions are one where two factors of U come from determinant expansion and two from the rest of the formula and the one where all four come from non-determinant part of the formula. To second order in U the determinant factor can be written as:

$$\frac{det^8(1+UQUQ)}{det^8(1-UNUN)} = 1 + 8[Tr(UQUQ) + Tr(UNUN)] + ... \tag{E.25}$$

so 4the order in U contribution of $\frac{\delta E}{\mu}$ will be of the form:

$$\begin{aligned}\frac{\delta E_4}{\mu} = \frac{4}{|\tilde{\alpha}_3|\mu}\int\frac{d\theta_a}{2\pi}\cdot \\ 8[Tr(UQUQ)+Tr(UNUN)]\cdot(\delta^{ik}\delta^{jl}[NUN]^{33}_{-n-n}[E_0NUN+\Omega QUQ\Omega]^{33}_{nn}+...) \\ +(\delta^{ik}\delta^{jl}[NUNUNUN])^{33}_{-n-n}[E_0NUN+\Omega QUQ\Omega]^{33}_{nn}+...) \\ +[NUN]^{33}_{-n-n}[E_0NUNUNUN+\Omega QUQUQUQ\Omega+\Omega QUQU\Omega NUN]^{33}_{nn}+...)\end{aligned} \tag{E.26}$$

where once again, integration over θ is substituted for by direct level-matching condition that all indices on Us in in any given product add up to 0.

To simplify the above lets make the following substitutions:

$$A^{3r}_{np} = \frac{1}{p-\beta_r n}\frac{sin(|n|\pi r)(\omega^r_p+\beta_r\omega^3_n)}{2\pi\sqrt{\omega^3_n\omega^r_p}} = \frac{1}{p-\beta_r n}\tilde{A}^{3r}_{np} \tag{E.27}$$

$$B^{r_1r_2}_{st} = \frac{1}{4\pi\sqrt{\omega^{r_1}_s\omega^{r_2}_t}(\beta_{r_2}\omega^{r_1}_s+\beta_{r_1}\omega^{r_2}_t)} \tag{E.28}$$

then

$$N^{3r}_{np} = e(n)A^{3r}_{np} \tag{E.29}$$

$$Q^{3r}_{np} = -iA^{3r}_{np} \tag{E.30}$$

$$N^{r_1r_2}_{st} = \sqrt{\beta_{r_1}\beta_{r_2}}\cdot(\sqrt{\omega_s+\mu\alpha_{r_1}}\sqrt{\omega_t+\mu\alpha_{r_2}}+e(st)\sqrt{\omega_s-\mu\alpha_{r_1}}\sqrt{\omega_t-\mu\alpha_{r_2}})B^{r_1r_2}_{st} \tag{E.31}$$

$$Q^{r_1r_2}_{st} = i(\beta_{r_2}s-\beta_{r_1}t)B^{r_1r_2}_{st} \tag{E.32}$$

In cases where both s and t are much smaller then μ this can be written more compactly as:

$$A^{3r}_{np} = \frac{1}{p-\beta_r n}\frac{-sin(|n|\pi r)\sqrt{\beta_r}}{\pi} \tag{E.33}$$

$$B^{r_1r_2}_{st} = \frac{1}{8\pi(\mu\alpha)^2(\beta_{r_1}\beta_{r_2})^{\frac{3}{2}}} \tag{E.34}$$

$$N^{3r}_{np} = e(n)A^{3r}_{np} \tag{E.35}$$

$$Q^{3r}_{np} = -iA^{3r}_{np} \tag{E.36}$$

$$N_{st}^{r_1r_2} = -2\mu\alpha_3\beta_{r_1}\beta_{r_2}B_{st}^{r_1r_2} = \frac{-1}{4\pi\mu\alpha\sqrt{\beta_{r_1}\beta_{r_2}}} \tag{E.37}$$

$$Q_{st}^{r_1r_2} = i(\beta_{r_2}s - \beta_{r_1}t)B_{st}^{r_1r_2} \tag{E.38}$$

We can then generalize the cancelation from 2-impurity calculation by noting:

$$[E_0NUN + \Omega QUQ\Omega]_{nn}^{33} = (E_0 - \Omega_n^2)[AUA]_{nn}^{33} = 2n[AUA]_{nn}^{33} \tag{E.39}$$

$$[E_0NUN + \Omega QUQ\Omega]_{-n-n}^{33} = (E_0 - \Omega_{-n}^2)[AUA]_{-n-n}^{33} = -2n[AUA]_{-n-n}^{33} \tag{E.40}$$

$$[E_0NUN + \Omega QUQ\Omega]_{n-n}^{33} = (-E_0 - \Omega_{-n}\Omega_n)[AUA]_{n-n}^{33} = -2(\omega_n - \mu\alpha)[AUA]_{n-n}^{33} \tag{E.41}$$

$$[E_0NUN + \Omega QUQ\Omega]_{-nn}^{33} = (-E_0 - \Omega_{-n}\Omega_n)[AUA]_{-nn}^{33} = -2(\omega_n - \mu\alpha)[AUA]_{-nn}^{33} \tag{E.42}$$

In all cases (to all impurities), the above are multiplied by a term proportional to $[AUA]_{mm}$ so the sum can be written as:

$$2n([AUA]_{nn}^{33}[AU...A]_{-n-n}^{33} - [AUA]_{-n-n}^{33}[AU...A]_{nn}^{33}) + \frac{1}{\mu\alpha}([AUA]_{n-n}^{33}[AU..A]_{-nn}^{33} + [AUA]_{-nn}^{33}[AU..A]_{n-n}^{33}) \tag{E.43}$$

It is obvious that the parts represented by 3 dots can be moved from one square bracket to the other so the first two terms cancel out and the second two can be writen compactly:

$$\frac{2}{\mu\alpha}[AUA]_{n-n}^{33}[AU..A]_{-nn}^{33} \tag{E.44}$$

Therefore all the terms proportional to $[E_0NUN + \Omega QUQ\Omega]$ will be of order $\frac{1}{\mu\alpha}$ rather then of order $\mu\alpha$ In 2-impurity case this brought the leading term from being of order 1 to being of order $\lambda = \frac{1}{(\mu\alpha)^2}$. In 4-impurity, as well as all higher impurity cases, this makes all terms containing $[E_0NUN + \Omega QUQ\Omega]$ term subleading.

To see this and to identify the leading order behavior we need to look into the $\mu\alpha$ dependance of the individual terms. We also have to take into account the indices that are being summed over that could in principle blow up to rival $\mu\alpha$ terms for any μ. To see that at 4-impurities, all indices are controled and kept finite by the poles we can look at a generic term:

$$\begin{aligned}
&[AUA]_{nn'}^{33}[AUBUBUA]_{mm'}^{33} = \\
&\sum_p\sum_q\sum_s\sum_t A_{np}^{3r_1}A_{pn'}^{r_13}A_{mq}^{3r_2}B_{qs}^{r_2r_3}B_{st}^{r_3r_4}A_{tm'}^{r_43}|p+q+s+t=0 = \\
&= \sum_p\sum_q\sum_t \frac{1}{p-\beta_{r_1}n}\frac{1}{p-\beta_{r_1}n'}\frac{1}{q-\beta_{r_2}m}\frac{1}{t-\beta_{r_4}m'}\cdot \\
&\cdot\tilde{A}_{np}^{3r_1}\tilde{A}_{pn'}^{r_13}\tilde{A}_{mq}^{3r_2}B_{q(-p-q-t)}^{r_2r_3}B_{(-p-q-t)t}^{r_3r_4}\tilde{A}_{tm'}^{r_43}
\end{aligned} \tag{E.45}$$

Clearly the poles will controll the sum and the term will be of the form:

$$[AUA]_{nn'}^{33}[AUBUBUA]_{mm'}^{33} = \tilde{A}_{np}^{3r_1}\tilde{A}_{pn'}^{r_13}\tilde{A}_{mq}^{3r_2}B_{q(-p-q-t)}^{r_2r_3}B_{(-p-q-t)t}^{r_3r_4}\tilde{A}_{tm'}^{r_43} \tag{E.46}$$

Where p,q and t are constants set by the values of the poles.

With all the sums thus controlled only factors of λ in the large μ limit will come from factors of $\alpha\mu$ and $\omega = \sqrt{n^2 + (\alpha\mu)^2}$. Expanding ω in large μ limit we have:

$$\omega = \alpha\mu + \frac{1}{\alpha\mu} + ... \tag{E.47}$$

We can then count the highest possible order (baring any cancelations) in λ coming from any term in the expansion so as to identify those terms which may contribute at order λ.

$\tilde{A}^{3r}_{np}$ is clearly of order 1 and so are therefore N^{3r}_{np} and Q^{3r}_{np} which are its multiples. $B^{r_1r_2}_{st}$ is of the order $\frac{1}{(\alpha\mu)^2}$ and so is $Q^{r_1r_2}_{st}$ (as long as the sums are controled) but the $N^{r_1r_2}_{st}$ has the $(\sqrt{\omega + \alpha\mu})^2$ term which makes it overall of order $\frac{1}{\alpha\mu}$. Furthermore, E_0 is of order $\alpha\mu$ and Ω is of order $\sqrt{\alpha\mu}$. Lastly, as we have seen above $[E_0NUN + \Omega QUQ\Omega]$ will be of the order $\frac{1}{\alpha\mu}$.

Therefore: The highest order possible contribution from the various terms will be

$$\frac{1}{\alpha\mu}8[Tr(UQUQ)+Tr(UNUN)]\cdot(\delta^{ik}\delta^{jl}[NUN]^{33}_{-n-n}[E_0NUN+\Omega QUQ\Omega]^{33}_{nn}+...) \leq O(\frac{1}{(\alpha\mu)^3}) \tag{E.48}$$

$$\frac{1}{\alpha\mu}(\delta^{ik}\delta^{jl}[NUNUNUN])^{33}_{-n-n}[E_0NUN + \Omega QUQ\Omega]^{33}_{nn} + ...) \leq O(\frac{1}{(\alpha\mu)^4}) \tag{E.49}$$

$$\frac{1}{\alpha\mu}[NUN]^{33}_{-n-n}[\Omega QUQUQUQ\Omega]^{33}_{nn} + ...) \leq O(\frac{1}{(\alpha\mu)^4}) \tag{E.50}$$

$$\frac{1}{\alpha\mu}[NUN]^{33}_{-n-n}[\Omega QUQU\Omega NUN]^{33}_{nn} + ...) \leq O(\frac{1}{(\alpha\mu)^3}) \tag{E.51}$$

$$\frac{1}{\alpha\mu}[NUN]^{33}_{-n-n}[E_0NUNUNUN]^{33}_{nn} + ...) \leq O(\frac{1}{(\alpha\mu)^2}) \tag{E.52}$$

Note how here fact that the internal N matrices are order in $\alpha\mu$ higher then internal Q matrices actually plays a pivotal role making it impossible for the two-impurity style cancelation to happen.

Therefore, to find the order λ contribution at the four impurity level we just have to calculate:

$$\begin{aligned}\frac{\delta E_4}{\mu} \approx \frac{4}{|\tilde{\alpha}_3|\mu}\int\frac{d\theta_a}{2\pi}\cdot(&[NUN]^{33}_{-n-n}[E_0NUNUNUN]^{33}_{nn}+\\ &+[NUN]^{33}_{nn}[E_0NUNUNUN]^{33}_{-n-n}+\\ &+[NUN]^{33}_{n-n}[E_0NUNUNUN]^{33}_{-nn}+\\ &+[NUN]^{33}_{-nn}[E_0NUNUNUN]^{33}_{n-n})\end{aligned} \tag{E.53}$$

This can be written as:

$$\frac{\delta E_4}{\mu} \approx \frac{4(2\omega_n)}{|\tilde{\alpha}_3|\mu} \cdot \sum_{pqt}$$

$$\frac{\beta_2\sqrt{\beta_1\beta_3}\sqrt{\omega_q^{r_1}+\mu\alpha_{r_1}}\sqrt{\omega_t^{r_3}+\mu\alpha_{r_3}}(\omega_s+\mu\alpha_{r_2})sin^4(|n|\pi r)(\omega_p^{r_4}+\beta_{r_4}\omega_n)^2(\omega_q^{r_1}+\beta_{r_1}\omega_n)(\omega_t^{r_3}+\beta_{r_3}\omega_n)}{256\pi^6\omega_n^2\omega_p^{r_4}\omega_q^{r_1}\omega_t^{r_3}\omega_s^{r_2}(\beta_{r_2}\omega_q^{r_1}+\beta_{r_1}\omega_s^{r_2})(\beta_{r_3}\omega_s^{r_2}+\beta_{r_2}\omega_t^{r_3})}$$

$$\cdot\left[(\frac{1}{p-\beta_{r_4}n})^2\frac{1}{q+\beta_{r_1}n}\frac{1}{t+\beta_{r_3}n}+(\frac{1}{p+\beta_{r_4}n})^2\frac{1}{q-\beta_{r_1}n}\frac{1}{t-\beta_{r_3}n}\right.$$

$$\left.+\frac{1}{p+\beta_{r_4}n}\frac{1}{p-\beta_{r_4}n}\frac{1}{q+\beta_{r_1}n}\frac{1}{t-\beta_{r_3}n}+\frac{1}{p+\beta_{r_4}n}\frac{1}{p-\beta_{r_4}n}\frac{1}{q-\beta_{r_1}n}\frac{1}{t+\beta_{r_3}n}\right] \quad \text{(E.5}$$

In the case when all the summands are bound by poles and/or level matching we can take a simple large μ limit:

$$\frac{\delta E_4}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6} \cdot \sum_{pqt} \frac{\beta_{r_4}}{\beta_{r_2}} sin^4(|n|\pi r)$$

$$\cdot\left[(\frac{1}{p-\beta_{r_4}n})^2\frac{1}{q+\beta_{r_1}n}\frac{1}{t+\beta_{r_3}n}+(\frac{1}{p+\beta_{r_4}n})^2\frac{1}{q-\beta_{r_1}n}\frac{1}{t-\beta_{r_3}n}\right.$$

$$\left.+\frac{1}{p+\beta_{r_4}n}\frac{1}{p-\beta_{r_4}n}\frac{1}{q+\beta_{r_1}n}\frac{1}{t-\beta_{r_3}n}+\frac{1}{p+\beta_{r_4}n}\frac{1}{p-\beta_{r_4}n}\frac{1}{q-\beta_{r_1}n}\frac{1}{t+\beta_{r_3}n}\right] \quad \text{(E.55)}$$

With level matching condition to be imposed by hand. When all the impurities are on the same internal string then

$$\beta_{r_1} = \beta_{r_2} = \beta_{r_3} = \beta_{r_4} = r \quad \text{(E.56)}$$

and the level matching condition is

$$s = -p - q - t \quad \text{(E.57)}$$

Furthermore, at the poles,

$$\omega_s = \sqrt{(\pm rn)^2 + (r\mu\alpha)^2} = r\omega_n = \omega_p = \omega_q = \omega_t \quad \text{(E.58)}$$

Using contour integral method we can then write it in large μ limit as as:

$$\frac{\delta E_{4,1}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2}\frac{sin^4(n\pi r)}{\pi^3}\left[\partial_p[cot(p\pi)]_{pr}cot^2(-n\pi r)+\partial_p[cot(p\pi)]_{p=-nr}cot^2(n\pi r)\right.$$

$$\left.+\frac{2}{-2rn}cot^2(-n\pi r)cot(n\pi r)+\frac{2}{2rn}cot^2(n\pi r)cot(-n\pi r)\right] \quad \text{(E.59)}$$

or, even more simply:

$$\frac{\delta E_{4,1}(r)}{\mu} \approx \frac{1}{\tilde{\alpha}_3^2\mu^2\pi^3}\left[-cos^2(n\pi r)\pi - \frac{1}{rn}sin(n\pi r)cos^3(n\pi r)\right] \quad \text{(E.60)}$$

Taking the large μ limit and integrating over r we get:

$$\frac{\delta E_{4,1}}{\mu} = g_2^2\int_0^1 dr\ \frac{r(1-r)}{2}\ \frac{\delta E_{4,1}(r)}{\mu} = \frac{1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{-1}{24}+\frac{3}{64\pi^2n^2}\right)... \quad \text{(E.61)}$$

We also have to consider the cases where the impurities split between two strings. There are three such combination one being:

$$\beta_{r_1} = \beta_{r_2} = r; \beta_{r_3} = \beta_{r_4} = r' \tag{E.62}$$

where

$$r + r' = 1 \tag{E.63}$$

In that case, level matching gives us:

$$s = -q; t = -p \tag{E.64}$$

In the same way then:

$$\omega_s = \omega_q = r\omega_n; \omega_t = \omega_p = r'\omega_n \tag{E.65}$$

Using this and taking large μ limit we get:

$$\frac{\delta E_{4,2}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2}\frac{sin^4(n\pi r)}{\pi^6}\frac{r'}{r}\left[-\frac{1}{(p-r'n)^3(q+rn)} - \frac{1}{(p+r'n)^3(q-rn)} - \frac{1}{(p-r'n)^2(p+r'n)(q-rn)} - \frac{1}{(p+r'n)^2(p-r'n)(q+rn)}\right] \tag{E.66}$$

Integrating over p and q and being carefull of the signs of r we get:

$$\frac{\delta E_{4,2}(r)}{\mu} \approx \frac{r'}{r}\frac{1}{(\tilde{\alpha}_3\mu\pi)^2}\left[-2cos^2(n\pi r) + \frac{1}{2r'n\pi}sin(n\pi r)cos(n\pi r)\right] \tag{E.67}$$

with the single-pole contributions canceling out the contributions from $\partial\frac{1}{p+r'n}$.

Finaly integrating over r:

$$\frac{\delta E_{4,2}}{\mu} = g_2^2\int_0^1 dr\ \frac{r(1-r)}{2}\ \frac{\delta E_{4,2}(r)}{\mu} = \frac{-1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{2}{12} + \frac{27}{128\pi^2n^2}\right) \tag{E.68}$$

remaining two combinations are:

$$\beta_{r_1} = \beta_{r_4}; \beta_{r_2} = \beta_{r_3} \tag{E.69}$$

and

$$\beta_{r_1} = \beta_{r_3}; \beta_{r_2} = \beta_{r_4} \tag{E.70}$$

First of those is identical to the one evaluated above and the second gives:

$$\frac{\delta E_{4,3}}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6}\cdot\sum_{pq} sin^4(|n|\pi r) \cdot\left[(\frac{1}{p-r'n})^2\frac{1}{q+rn}\frac{1}{q-rn} + (\frac{1}{p+r'n})^2\frac{1}{q-rn}\frac{1}{q+rn} + \frac{1}{p+r'n}\frac{1}{p-r'n}(\frac{1}{q+rn})^2 + \frac{1}{p+r'n}\frac{1}{p-r'n}(\frac{1}{q-rn})^2\right] \tag{E.71}$$

$$\frac{\delta E_{4,3}(r)}{\mu} \approx \frac{1}{(\tilde{\alpha}_3\mu\pi)^2}\left[\frac{-1}{2r'n\pi}sin(n\pi r)cos(n\pi r) + \frac{1}{2rn\pi}sin(n\pi r)cos(n\pi r)\right] \tag{E.72}$$

$$\frac{\delta E_{4,3}}{\mu} = g_2^2\int_0^1 dr\ \frac{r(1-r)}{2}\ \frac{\delta E_{4,3}(r)}{\mu} = \frac{1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{8}{64\pi^2n^2}\right) \tag{E.73}$$

As in the two impurity case we still have to account for 0 modes, specifically for the case where one 0 mode exists on one string and 3 remaining ones are left on the other. Consider first case where:

$$\beta_{r_1} = \beta_{r_2} = \beta_{r_3} = r\ \ ;\ \ \beta_{r_4} = 1-r \tag{E.74}$$

$$s = -q-t\ \ ;\ \ p = 0 \tag{E.75}$$

It should be noted that for some of the poles

$$\omega_s = \sqrt{(\pm 2rn)^2 + (r\mu\alpha)^2} \neq r\omega_n \tag{E.76}$$

However, as long as the both t and q are limited by the poles

$$\omega_s = r\omega_n = r\mu\alpha \tag{E.77}$$

will still hold in the large μ limit so we can continue to use it. We can then write:

$$\frac{\delta E_{4,4}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6}\cdot\sum_{qt}\frac{1}{r(1-r)n^2}sin^4(|n|\pi r)$$
$$\cdot\left[\frac{1}{q+rn}\frac{1}{t+rn} + \frac{1}{q-rn}\frac{1}{t-rn} + \frac{1}{q+rn}\frac{1}{t-rn} + \frac{1}{q-rn}\frac{1}{t+rn}\right] = 0 \tag{E.78}$$

Second 0-mode case is one with:

$$\beta_{r_1} = \beta_{r_4} = \beta_{r_3} = r\ \ ;\ \ \beta_{r_2} = 1-r \tag{E.79}$$

$$p = -q-t\ \ ;\ \ s = 0 \tag{E.80}$$

In this case we have:

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6}\cdot\sum_{qt}\frac{r}{1-r}sin^4(|n|\pi r)$$
$$\cdot\left[(\frac{1}{q+t+rn})^2\frac{1}{q+rn}\frac{1}{t+rn} + (\frac{1}{q+t-rn})^2\frac{1}{q-rn}\frac{1}{t-rn}\right.$$
$$+\frac{1}{q+t+rn}\frac{1}{q+t-rn}\frac{1}{q+rn}\frac{1}{t-rn} +$$
$$\left.+\frac{1}{q+t+rn}\frac{1}{q+t-rn}\frac{1}{q-rn}\frac{1}{t+rn}\right] \tag{E.81}$$

We can simplify:

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6}\cdot\sum_{qt}\frac{r}{1-r}sin^4(|n|\pi r)$$
$$\cdot\left[(\frac{1}{q+t+rn})^2\frac{1}{q+rn}\frac{1}{t+rn} + (\frac{1}{q+t-rn})^2\frac{1}{q-rn}\frac{1}{t-rn}\right.$$
$$\left.+\frac{1}{q+t+rn}\frac{1}{q+t-rn}\left(\frac{1}{q+rn}\frac{1}{t-rn} + \frac{1}{q-rn}\frac{1}{t+rn}\right)\right] \tag{E.82}$$

then using partial fractions:

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6} \cdot \sum_{qt} \frac{r}{1-r} sin^4(|n|\pi r)$$
$$\cdot \left[+\frac{1}{t}\left(\frac{1}{q+rn} - \frac{1}{q+t+rn}\right)\frac{1}{q+t+rn}\frac{1}{t+rn} \right.$$
$$+\frac{1}{t}\left(\frac{1}{q-rn} - \frac{1}{q+t-rn}\right)\frac{1}{q+t-rn}\frac{1}{t-rn}$$
$$\left. +\frac{1}{2rn}\left(\frac{1}{q+t-rn} - \frac{1}{q+t+rn}\right)\left(\frac{1}{q+rn}\frac{1}{t-rn} + \frac{1}{q-rn}\frac{1}{t+rn}\right)\right] \tag{E.83}$$

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6} \cdot \sum_{qt} \frac{r}{1-r} sin^4(|n|\pi r)$$
$$\cdot \left[\frac{1}{t^2}\left(\frac{1}{q+t+rn} - \frac{1}{q+rn}\right)\frac{1}{t+rn} - \frac{1}{t}(\frac{1}{q+t+rn})^2\frac{1}{t+rn} \right.$$
$$+\frac{1}{t^2}\left(\frac{1}{q+t-rn} - \frac{1}{q-rn}\right)\frac{1}{t-rn} - \frac{1}{t}(\frac{1}{q+t-rn})^2\frac{1}{t-rn}$$
$$\left. +\frac{1}{2rn}\left(\frac{1}{q+t-rn} - \frac{1}{q+t+rn}\right)\left(\frac{1}{q+rn}\frac{1}{t-rn} + \frac{1}{q-rn}\frac{1}{t+rn}\right)\right] \tag{E.84}$$

It can already be seen that with some shifting of variables like $q' = q+t$ and such large parts of the above dissapear. Only thing that is left is:

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6} \cdot \sum_{qt} \frac{r}{1-r} sin^4(|n|\pi r)$$
$$\cdot \frac{-1}{t}\left[(\frac{1}{q+rn})^2\frac{1}{t+rn} + (\frac{1}{q-rn})^2\frac{1}{t-rn}\right] \tag{E.85}$$

The pole at $t = 0$ cancels out and we are left with

$$\frac{\delta E_{4,5}(r)}{\mu} \approx \frac{-1}{(\tilde{\alpha}_3\mu)^2\pi^3 n}\frac{1}{1-r} sin(n\pi r)cos(n\pi r) \tag{E.86}$$

$$\frac{\delta E_{4,5}}{\mu} = g_2^2 \int_0^1 dr \ \frac{r(1-r)}{2} \ \frac{\delta E_{4,5}(r)}{\mu} = \frac{1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{8}{64\pi^2 n^2}\right) \tag{E.87}$$

Finaly there is two cases where q or t is equal to zero and is on the separate string. There:

$$\frac{\delta E_{4,6}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^6} \cdot \sum_{pq} e(q) sin^4(|n|\pi r)$$
$$\cdot \left[(\frac{1}{p-rn})^2\frac{1}{q+rn}\frac{1}{(1-r)n} + (\frac{1}{p+rn})^2\frac{1}{q-rn}\frac{1}{(1-r)n} \right.$$
$$\left. +\frac{1}{p+rn}\frac{1}{p-rn}\frac{1}{q+rn}\frac{1}{(1-r)n} + \frac{1}{p+rn}\frac{1}{p-rn}\frac{1}{q-rn}\frac{1}{(1-r)n}\right] \tag{E.88}$$

$$\frac{\delta E_{4,6}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^5}\frac{1}{(1-r)n}\cdot\sum_p sin^3(|n|\pi r)cos(|n|\pi r)$$
$$\cdot\left[(\frac{1}{p-rn})^2+(\frac{1}{p+rn})^2+\frac{1}{p+rn}\frac{1}{p-rn}+\frac{1}{p+rn}\frac{1}{p-rn}\right] \tag{E.89}$$

$$\frac{\delta E_{4,6}(r)}{\mu} \approx \frac{1}{2(\tilde{\alpha}_3\mu)^2\pi^5}\frac{1}{(1-r)n}\cdot\sum_p sin^3(|n|\pi r)cos(|n|\pi r)$$
$$\cdot\left[(\frac{1}{p-rn})^2+(\frac{1}{p+rn})^2+\frac{1}{p+rn}\frac{1}{p-rn}+\frac{1}{p+rn}\frac{1}{p-rn}\right] \tag{E.90}$$

$$\frac{\delta E_{4,6}(r)}{\mu} \approx \frac{1}{(\tilde{\alpha}_3\mu)^2\pi^3}\frac{1}{(1-r)n}\left[sin(|n|\pi r)cos(|n|\pi r)+\frac{2}{n\pi r}sin^2(|n|\pi r)cos^2(|n|\pi r)\right] \tag{E.91}$$

$$\frac{\delta E_{4,6}}{\mu} = g_2^2\int_0^1 dr\ \frac{r(1-r)}{2}\ \frac{\delta E_{4,6}(r)}{\mu} = 0 \tag{E.92}$$

We can then put it all together:

$$\frac{\delta E_4}{\mu} = 2\frac{\delta E_{4,1}}{\mu}+2\frac{\delta E_{4,2}}{\mu}+\frac{\delta E_{4,3}}{\mu}+\frac{\delta E_{4,4}}{\mu}+\frac{\delta E_{4,5}}{\mu}+2\frac{\delta E_{4,6}}{\mu} = \frac{1}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{1}{12}+\frac{64}{64\pi^2n^2}\right) \tag{E.93}$$

$$\frac{\delta E_2}{\mu}+\frac{\delta E_4}{\mu} = \frac{2}{\pi^2|\tilde{\alpha}_3|^2\mu^2}\left(\frac{5}{24}+\frac{57}{64\pi^2n^2}\right) \tag{E.94}$$

Appendix F

Hamiltonian and the supercharges of the orbifolded string

F.1 Hamiltonian of the orbifolded string

The action of the pp-wave string is derived using the formalism of Cartan forms [?]. This formalism uses vielbeins and spin connections associated with the metric 3.9. As the actual form of the metric remains unchanged by the orbifolding, the entire mechanism discussed in [?] applies for as long as no boundary conditions on the scalar and fermionic fields are used. As we are interested in the near pp-wave limit, we are in fact following formalism of the [?]. As far as the effects of orbifolding to the order $\frac{1}{M}$ are concerned, however, there is no difference between the results derived from [?] and [?]. In both cases we derive the same Hamiltonian density:

$$H = \int d\sigma \frac{1}{2}\left[(x^i)^2 + (p^i)^2 + (x^{i\prime})^2\right] + \frac{i}{2}\left[\psi\psi' + \psi^\dagger\psi^{\dagger\prime} + 2i\psi^\dagger\Pi\psi\right] \tag{F.1}$$

Where the prime indicates σ derivative and the addition over all possible fermionic indices is assumed. Furthermore, the bosonic and fermionic fields are governed by the usual equations of motion:

$$\ddot{x}^i - x''^i + x^i = 0 \tag{F.2}$$

$$i(\dot{\psi} + \psi'^\dagger) + \Pi\psi = 0 \tag{F.3}$$

However, the periodicity conditions of non orbifolded theory which demand that the xs and ψs are periodic over the σ coordinate are in our case replaced by the equations 3.21, 3.22 and 3.23. (Or 3.21, 3.24 and 3.25 if we are dealing with a case of two transverse directions being orbifolded). Before we proceed with solving the equations and integrating the Hamiltonian it is useful to note that for any $x(\sigma,\tau)$ and $\psi(\sigma,\tau)$ that solve equations F.2 and F.3 with standard periodicity conditions we can construct the $\tilde{x}(\sigma,\tau) = e^{\frac{2\pi i w}{M}\sigma}x(\sigma,\tau)$ and $\tilde{\psi}(\sigma,\tau) = e^{\frac{2\pi i w}{M}\sigma}\psi(\sigma,\tau)$ where the phase governs the periodicity of the boundary conditions. This lets us use the known solutions of the F.2 and F.3 and simply making sure to insert the $e^{\frac{2\pi i w}{M}\sigma}$ or $e^{-\frac{2\pi i w}{M}\sigma}$ phase where appropriate.

F.1.1 Bosonic interaction Hamiltonian

It is obvious that the only part of the Hamiltonian that will be affected by the orbifolding will be σ derivatives of the functions that have acquired the phase from the new periodicity condition. We can then write the interaction part of the bosonic Hamiltonian:

$$H^b_{int} = \int d\sigma \frac{1}{2}\left[\partial_\sigma(\tilde{x}^7 + i\tilde{x}^8)\partial_\sigma(\tilde{x}^7 - i\tilde{x}^8) - \partial_\sigma(x^7 + ix^8)\partial_\sigma(x^7 - ix^8)\right] \tag{F.4}$$

which, following the equation 3.21 can be re-written as:

$$H^b_{int} = \int d\sigma \frac{1}{2}\left[\partial_\sigma[e^{\frac{2\pi i w}{M}\sigma}(x^7+ix^8)]\partial_\sigma[e^{-\frac{2\pi i w}{M}\sigma}(x^7-ix^8)] - \partial_\sigma(x^7+ix^8)\partial_\sigma(x^7-ix^8)\right] \quad \text{(F.5)}$$

Which to the first order in $\frac{1}{M}$ is equal to:

$$H^b_{int} = \int d\sigma \frac{1}{2}\frac{2\pi i w}{M}\left[(x^7+ix^8)(x^{7\prime}-ix^{8\prime}) - (x^{7\prime}+ix^{8\prime})(x^7-ix^8)\right] \quad \text{(F.6)}$$

or:

$$H^b_{int} = \int d\sigma \frac{2\pi w}{M}\left[x^7x^{8\prime} - x^{7\prime}x^8\right] \quad \text{(F.7)}$$

We can now use well known solutions of the F.2:

$$x^i(\sigma,\tau) = \sum_{n=-\infty}^{\infty} x^i_n(\tau)e^{-in\sigma}, \quad x^i_n(\tau) = \frac{i}{\sqrt{2\omega_n}}(\alpha^i_n e^{-i\omega_n\tau} - \alpha^{i\dagger}_{-n}e^{i\omega_n\tau}) \quad \text{(F.8)}$$

Substituting solutions F.8 into the F.7 we get:

$$H^b_{int} = \int d\sigma \frac{2\pi w}{M}\sum_{n,m} i(n-m){x^7}_n{x^8}_m e^{-i(n+m)\sigma} \quad \text{(F.9)}$$

$$H^b_{int} = \frac{4\pi i w}{M}\sum_n n({x^7}_n{x^8}_{-n}) \quad \text{(F.10)}$$

$$H^b_{int} = \frac{2\pi i w}{M}\sum_n \frac{n}{\omega_n}\left[(\alpha^{\dagger 7}_n a^{\dagger 8}_{-n}e^{i\omega_n\tau} - \alpha^7_n a^8_{-n}e^{-i\omega_n\tau}) + (\alpha^{\dagger 7}_n\alpha^8_n - \alpha^{\dagger 8}_n\alpha^7_n)\right] \quad \text{(F.11)}$$

After quantization, $\alpha^\dagger$ are promoted to creation operators and the α into annihilation operators The part carrying the τ phase then mixes the states of various number of impurities, second part however acts on the single impurity multiplet (as well as any other multiplet of n impurities) and breaks the energy degeneracy of that multiplet. For the multiplet

$$\alpha^{\dagger i}_n|0>; i = 1..8 \quad \text{(F.12)}$$

we diagonalize H^b_{int} with the new set of eigenstates:

$$\alpha^{\dagger i}_n|0>; i = 1..6, (\alpha^{\dagger 7}_n + i\alpha^{\dagger 8}_n)|0>, (\alpha^{\dagger 7}_n - i\alpha^{\dagger 8}_n)|0> \quad \text{(F.13)}$$

The first 6 clearly remain degenerate with energy, $E^i = \omega_n$ whereas latter two get energy splitting of order $\frac{1}{M}$

$$E^7 = \omega_n - \frac{2\pi w n}{M\omega_n}, E^8 = \omega_n + \frac{2\pi w n}{M\omega_n} \quad \text{(F.14)}$$

In the case of orbifolding along the two transverse directions, an additional term would be added to the H^b_{int} of the same form but with the oscillators in directions 5 and 6 replacing the 7 and 8 ones respectively. In this case, each new energy level would get two states rather then one.

F.1.2 Fermionic interaction Hamiltonian

A very similar argument holds for fermions. Fermionic part of the interaction Hamiltonian can be written as:

$$H^f_{int} = \int d\sigma \frac{i}{2}\left[\tilde{\psi}^{\alpha_1 1_2}\partial_\sigma \tilde{\psi}_{\alpha_1 1_2} + \tilde{\psi}^{\dagger\alpha_1 1_2}\partial_\sigma \tilde{\psi}^\dagger_{\alpha_1 1_2} + \tilde{\psi}^{\dot{\alpha}_1 \dot{2}_2}\partial_\sigma \tilde{\psi}_{\dot{\alpha}_1 \dot{2}_2} + \tilde{\psi}^{\dagger\dot{\alpha}_1 \dot{2}_2}\partial_\sigma \tilde{\psi}^\dagger_{\dot{\alpha}_1 \dot{2}_2}\right] \tag{F.15}$$

$$H^f_{int} = \int d\sigma \frac{-\pi w}{M}\left[\psi^{\alpha_1 1_2}\psi_{\alpha_1 1_2} + \psi^{\dagger\alpha_1 1_2}\psi^\dagger_{\alpha_1 1_2} - \psi^{\dot{\alpha}_1 \dot{2}_2}\psi_{\dot{\alpha}_1 \dot{2}_2} - \psi^{\dagger\dot{\alpha}_1 \dot{2}_2}\psi^\dagger_{\dot{\alpha}_1 \dot{2}_2}\right] \tag{F.16}$$

Solutions of the F.3 are:

$$\psi^{\alpha_1\beta_2}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \psi_n^{\alpha_1\beta_2}(\tau)e^{-in\sigma} \tag{F.17}$$

$$\psi_n^{\alpha_1\beta_2}(\tau) = \frac{1}{2\sqrt{\omega_n}}(A_n\beta_n^{\alpha_1\beta_2}e^{-i\omega_n\tau} + B_n\beta_{-n}^{\dagger\alpha_1\beta_2}e^{i\omega_n\tau})$$

$$\psi^{\dagger\alpha_1\beta_2}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} \psi_n^{\dagger\alpha_1\beta_2}(\tau)e^{-in\sigma}$$

$$\psi_n^{\dagger\alpha_1\beta_2}(\tau) = \frac{1}{2\sqrt{\omega_n}}(\Pi B_n\beta_n^{\alpha_1\beta_2}e^{-i\omega_n\tau} - \Pi A_n\beta_{-n}^{\dagger\alpha_1\beta_2}e^{i\omega_n\tau})$$

$$A_n = (\sqrt{\omega_n - n} - \sqrt{\omega_n + n}\Pi), \quad B_n = (\sqrt{\omega_n + n} + \sqrt{\omega_n - n}\Pi)$$

Substituting back into the F.15

$$\begin{aligned} H^f_{int} = \frac{-\pi w}{4M}\sum_n \frac{1}{\omega_n}\Big[&(A_nA_{-n} + B_nB_{-n})(\beta_n^{\alpha_1 1_2}\beta_{-n\alpha_1 1_2} - \beta_n^{\dot{\alpha}_1\dot{2}_2}\beta_{-n\dot{\alpha}_1\dot{2}_2})e^{-2i\omega_n\tau} \\ &+(A_nA_{-n} + B_nB_{-n})(\beta_{-n}^{\dagger\alpha_1 1_2}\beta^\dagger_{n\alpha_1 1_2} - \beta_n^{\dagger\dot{\alpha}_1\dot{2}_2}\beta^\dagger_{-n\dot{\alpha}_1\dot{2}_2})e^{2i\omega_n\tau} \\ &+2(A_nB_{-n} - B_nA_{-n})(\beta_n^{\dagger\alpha_1 1_2}\beta_{n\alpha_1 1_2} - \beta_n^{\dagger\dot{\alpha}_1\dot{2}_2}\beta_{n\dot{\alpha}_1\dot{2}_2})\Big] \end{aligned} \tag{F.18}$$

$$\begin{aligned} H^f_{int} = \frac{-\pi w}{4M}\sum_n \frac{1}{\omega_n}\Big[&2(\beta_n^{\alpha_1 1_2}\beta_{-n\alpha_1 1_2} - \beta_n^{\dot{\alpha}_1\dot{2}_2}\beta_{-n\dot{\alpha}_2\dot{1}_2})e^{-2i\omega_n\tau} \\ &+2(\beta_{-n}^{\dagger\alpha_1 1_2}\beta^\dagger_{n\alpha_1 1_2} - \beta_n^{\dagger\dot{\alpha}_1\dot{2}_2}\beta^\dagger_{-n\dot{\alpha}_1\dot{2}_2})e^{2i\omega_n\tau} - 8n(\beta_n^{\dagger\alpha_1 1_2}\beta_{n\alpha_1 1_2} - \beta_n^{\dagger\dot{\alpha}_1\dot{2}_2}\beta_{n\dot{\alpha}_1\dot{2}_2})\Big] \end{aligned} \tag{F.19}$$

As with the bosonic case, there are two τ dependent parts that mix states of various number of impurities. For the energy splitting in the single impurity multiplet Hamiltonian is already diagonalized and we can see that 4 states break degeneracy into a pair of two-fold degenerate states:

$$E^5 = E^6 = \omega_n - \tfrac{2\pi wn}{M\omega_n}, \quad E^7 = E^8 = \omega_n + \tfrac{2\pi wn}{M\omega_n} \tag{F.20}$$

Where E^5 and E^6 correspond to states $\beta^{\dagger\dot{1}_1\dot{2}_2}|0>$ and $\beta^{\dagger\dot{2}_1\dot{2}_2}|0>$ and E^7 and E^8 to states $\beta^{\dagger 1_1 1_2}|0>$ and $\beta^{\dagger 2_1 1_2}|0>$

In the case of two transverse directions the fermionic result is very similar except that the index structure on the fermionic oscillators matches that of the boundary conditions - that is to say all indices are dotted.

F.2 Supersymmetries of the orbifolded string in near pp-wave limit

We can now examine the supersymmetries of the pp-wave and see which are broken by the orbifolding. Supercharges can be written as:

$$Q^{(s)}_{2\alpha_1\dot{\alpha}_2} = \frac{\bar{\eta}}{\sqrt{|\tilde{\alpha}_s|}} \sum_{k=-\infty}^{\infty} \Omega_{k(s)} \left(\alpha^{(s)\dot{\beta}_1\dagger}_{k\alpha_1} \beta^{(s)}_{k\dot{\beta}_1\dot{\alpha}_2} + ie(\tilde{\alpha}_s)\alpha^{(s)\dot{\beta}_1}_{k\alpha_1} \beta^{(s)\dagger}_{k\dot{\beta}_1\dot{\alpha}_2} - \right.$$
$$\left. -i\alpha^{(s)\beta_2\dagger}_{k\dot{\alpha}_2} \beta^{(s)}_{k\alpha_1\beta_2} - e(\tilde{\alpha}_s)\alpha^{(s)\beta_2}_{k\dot{\alpha}_2} \beta^{(s)\dagger}_{k\alpha_1\beta_2} \right) \tag{F.21}$$

$$Q^{(s)}_{2\dot{\alpha}_1\alpha_2} = \frac{\bar{\eta}}{\sqrt{|\tilde{\alpha}_s|}} \sum_{k=-\infty}^{\infty} \Omega_{k(s)} \left(\alpha^{(s)\beta_1\dagger}_{k\dot{\alpha}_1} \beta^{(s)}_{k\beta_1\alpha_2} + ie(\tilde{\alpha}_s)\alpha^{(s)\beta_1}_{k\dot{\alpha}_1} \beta^{(s)\dagger}_{k\beta_1\alpha_2} - \right.$$
$$\left. -i\alpha^{(s)\dot{\beta}_2\dagger}_{k\alpha_2} \beta^{(s)}_{k\dot{\alpha}_1\dot{\beta}_2} - e(\tilde{\alpha}_s)\alpha^{(s)\dot{\beta}_2}_{k\alpha_2} \beta^{(s)\dagger}_{k\dot{\alpha}_1\dot{\beta}_2} \right) \tag{F.22}$$

where we define

$$\Omega_k = \sqrt{\omega_k - \mu\tilde{\alpha}} - e(k)\sqrt{\omega_k + \mu\tilde{\alpha}} \quad , \quad e(x) = x/|x| \tag{F.23}$$

Obviously, in the strict pp-wave limit H_{int} disappears and thus all supersymmetries are preserved. To the first order in $\frac{1}{M}$ we have to consider which of the above commute with H_{int}. Ignoring the terms that carry the τ phase we can write:

$$H_{int} = \frac{2\pi i w}{M} \sum_n \frac{n}{\omega_n} \left[\alpha^{\dagger 3'}_n \alpha^{4'}_n - \alpha^{\dagger 4'}_n \alpha^{3'}_n - (\beta^{\dagger\alpha_1 1_2}_n \beta_{n\alpha_1 1_2} - \beta^{\dagger\dot{\alpha}_1\dot{2}_2}_n \beta_{n\dot{\alpha}_1\dot{2}_2}) \right] \tag{F.24}$$

Using the

$$\alpha^{(s)}_{n\alpha_1\dot{\alpha}_1} \equiv \frac{1}{\sqrt{2}} \sigma^i_{\alpha_1\dot{\alpha}_1} \alpha^{i(s)}_n \quad , \quad \alpha^{(s)}_{n\alpha_2\dot{\alpha}_2} \equiv \frac{1}{\sqrt{2}} \sigma^{i'}_{\alpha_2\dot{\alpha}_2} \alpha^{i'(s)}_n$$

It is relatively easy to show that

$$\begin{aligned} [H_{int}, Q_{\alpha_1\dot{1}_2}] = 0, \quad [H_{int}, Q_{\dot{\alpha}_1 2_2}] = 0 \\ [H_{int}, Q_{\alpha_1\dot{2}_2}] \neq 0, \quad [H_{int}, Q_{\dot{\alpha}_1 1_2}] \neq 0 \end{aligned} \tag{F.25}$$

And that therefore exactly half the supersymmetries survive.