Analytical and Empirical Models of Online Auctions

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Abstract

This thesis provides a discussion on some analytical and empirical models of online auctions. The objective is to provide an alternative framework for analyzing online auctions, and to characterize the distribution of intermediate prices. Chapter 1 provides a mathematical formulation of the eBay auction format and background to the data used in the empirical analysis. Chapter 2 analyzes policies for optimally disposing inventory using online auctions. It is assumed a seller has a fixed number of items to sell using a sequence of, possibly overlapping, single-item auctions. The decision the seller must make is when to start each auction. The decision involves a trade-off between a holding cost for each period an item remains unsold, and a cannibalization effect among competing auctions. Consequently the seller must trade-off the expected marginal gain for the ongoing auctions with the expected marginal cost of the unreleased items by further deferring their release. The problem is formulated as a discrete time Markov Decision Problem. Conditions are derived to ensure that the optimal release policy is a control limit policy in the current price of the ongoing auctions. Chapter 2 focuses on the two item case which has sufficient complexity to raise challenging questions. An underlying assumption in Chapter 2 is that the auction dynamics can be captured by a set of transition probabilities. Chapter 3 shows with two fixed bidding strategies how the transition probabilities can be derived for a given auction format and bidder arrival process. The two specific bidding strategies analyzed are when bidders bid: 1) a minimal increment, and 2) their true valuation. Chapters 4 and 5 provides empirical analyzes of 4,000 eBay auctions conducted by Dell. Chapter 4 provides a statistical model where over discrete time periods, prices of online auctions follow a

zero-inflated gamma distribution. Chapter 5 provides an analysis of the 44,000 bids placed in the auctions, based on bids following a gamma distribution. Both models presented in Chapters 4 and 5 are based on conditional probabilities given the price and elapsed time of an auction, and certain parameters of the competing auctions. Chapter 6 concludes the thesis with a discussion of the main results and possible extensions.

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Notation

Time, decision epoch, and period	t
Length of planning horizon	T
Duration of an auction	τ
Starting price of an auction and bidders' lower bound on the valuation of item	p
Maximum price of an auction and bidders' upper bound on the valuation of the item	P
Bidders' valuation of the item	V
Index of auction or item	i
Current price of auction i	X_i
High-bid of auction i	H_i
Elapsed discrete time of auction i (used in Chapters 2 and 4)	Y_i
Elapsed continuous time of auction i (used in Chapters 3 and 5)	t_i
Current price of auction i after Y periods ($X_{i,t}$ in Chapters 3 and 5)	(X_i, Y_i) or $X_{i,Y}$
Number of ongoing auctions	Z
Cost per period of holding one unit of inventory	h
Indicator function; equals 1 if statement in brackets is true, 0 otherwise	$\mathbf{1}_{\{\cdot\}}$

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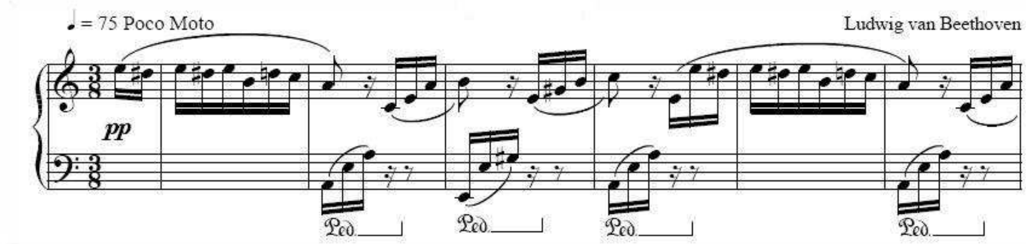
Dedication



CHAPTER 1

Introduction

1. Research Objective and Overview of Thesis

Auctions provide an important and integral part of commerce. One of the main appeals with auctions is that they can help solve the challenging *pricing* decision. If a seller sets the price too high he runs the risk of not selling the product, while if he sets it too low he might forfeit revenue. Similarly, though of a different nature, deciding how much to be willing to pay for a product is a difficult decision. Reasonably, there does not exist a price that is too low for a buyer, while clearly paying too much is either not feasible or not worthwhile. In order for a transaction to occur, the two parties' problems must be resolved to a mutual agreement. The trade mechanism of auctions provides a relatively easy to implement and often efficient solution. Generally speaking, an auction allocates a product to the buyer who values it the most, while generating the best possible revenue for the seller. This thesis aims to further our understanding of how auctions work, and provide a framework for sellers, buyers, and auctioneers to make better decisions.

The main objective is to provide a framework for analyzing the dynamic and stochastic nature of online auctions. There are two main departures from the traditional auction theory framework. First, online auctions are modeled as stochastic processes characterized by various parameters. In other words, the analysis does not follow the standard auction/game theory framework and derive properties of a bid strategy (Bayesian Nash) equilibrium. Second, rather than focusing on the distribution of the final price, the thesis centers on the distribution of intermediate prices of ongoing auctions. Specifically, the analysis mainly considers, the conditional intermediate price-transition probabilities, given certain parameters. It should be clear that the analysis of intermediate prices enables the analysis of the final price, but not necessarily vice versa.

An important managerial decision and the main research question addressed is: how should a seller release items for auction if he wishes to maximize his profit? That is, given an inventory of N items, and restricted to sell each item using a *single-item* auction, what is the optimal release policy? Should he release the N items for auctions 1) simultaneously, 2) as a series of N non-overlapping sequentially started auctions, or 3) according to a policy that depends on the ongoing auctions?

The objective of Chapter 2 is to address this issue. The problem is modeled as a discrete time Markov Decision Process (MDP), where each period auctions evolve according to a stochastic process. As a non-trivial constraint, a trade-off between a holding (or depreciation) cost and a 'cannibalization' effect among competing auctions is imposed. Though Chapter 2 only focus on the two item case ($N = 2$), the framework, analysis, and results give insight to the general N-item case. One of the main results is that given certain structural properties of the price-transition probabilities, the optimal release policy is of a threshold type. Specifically, for the two item case, in each period there exists a price such that if the first auction is above this price, then it is optimal to release the second item for auction. However, if the first auction is below the price threshold, then it is optimal to defer the release at least one more period. The insights and extensions for the general N item case are discussed in Chapter 6.

An underlying assumption of Chapter 2 is that the auction dynamics can be summarized by a set of conditional price-transition probabilities. Chapter 3 illustrates, with two examples, how these can be derived for a given auction format, bidder arrival process, and fixed bidding strategy. Although the two bidding strategies discussed, under certain conditions, result in a Bayesian Nash equilibrium, it is not argued that this is the case for the setting discussed in Chapter 2. In fact, in the implicit setting of Chapter 2 or eBay the two bidding strategies would not result in an equilibrium outcome. The objective of Chapter 3 is only to illustrate how the conditional price-transition probabilities can be derived from a given bidding behavior.

Since the framework for analyzing auctions presented in Chapter 2 is new, two empirical analyses for model validation are included in Chapter 4 and 5. The data for both empirical analyses come from the eBay auctions of Dell Financial Services, and consist of more than 4,000 auctions and 44,000 bids. More details regarding the data are presented in Section 4 below. The objective of the first empirical analysis is threefold. First, to present a statistical model that can characterize the stochastic process by which auctions evolve over discrete time periods. In other words, to provide a data driven or statistical methodology to characterize the stochastic process, and estimate the conditional intermediate price-transition probabilities. Second, to provide structural properties on the statistical model such that the main results from Chapter 2 hold. Third, to illustrate and validate the empirical model by fitting it to eBay auction data.

The second empirical analysis focuses on the individual bids. The objective is to propose and test a model regarding bidders' underlying bid strategies. Specifically, Chapter 5 provides a statistical analysis of bidders' bid-increments, i.e. the amount above the current price of an auction. Both Chapter 4 and 5 focus on the conditional probabilities given various auction parameters, and are based on Generalized Linear Models (GLM).

Chapter 6 concludes the thesis with an overall discussion and extensions for future work. The remainder of Chapter 1 provides a brief background to auctions and online auctions, a mathematical description of eBay's auction mechanism, and a description of the data used for the empirical analysis.

Comments Regarding Notation. For ease of discussion, sellers and bidders/buyers will be referred to as he, with no gender bias intended. The word 'seller' is used rather than 'bid-taker' (which is more common in the auction literature). Furthermore, the terms 'bidders' and 'buyers' are used interchangeably. The term 'auction' will be used instead of 'online auction'. Mathematical functions that are *non-decreasing* (*non-increasing*), are referred to as *increasing* (*decreasing*). *Non-overlapping sequentially* released auctions, are referred to as simply *sequentially* released auctions. Finally, throughout the thesis the pronoun 'we' is used.

2. Brief Background on Auctions and Online Auctions

Auctions as a formal commercial mechanism date back to antiquity [**29**, p.5],[**14**, p.1], and are today used for a wide variety of commodities, products, and services. Two of the more familiar products auctioned, or at least two that receive much attention in the news, are expensive art objects and radio (wireless) spectrum. Some interesting anecdotal stories, regarding extreme failures and successes of radio spectrum auctions, are provided in Tim Harford's *The Undercover Economist* (2005). Another type of auction that has received much attention over the last decade is the Internet auction. Despite their relatively short history, Internet based auctions, or online auctions, have quickly become an integral part of modern eCommerce. From having been mainly regarded as Internet based flea-markets for the *Consumer-to-Consumer* (C2C) markets, their importance and presence in the *Business-to-Consumer* (B2C) markets has and continues to grow rapidly. Today many well-established firms operate online auctions, not only as alternative sales channels, but also as strategic tools in pricing and product introduction decisions. Examples of large corporations using eBay include Sears, IBM, Fujitsu, and Dell.[1] Companies and organizations that host their own auctions include Dell, Major League Baseball, shopNBC, and Comet.[2] One common use of online auctions is as alternative salvage channels. For example, both Dell and Fujitsu use their online auction channel to sell refurbished products from returned and remaining inventory. Another important application of online auctions include the *Business-to-Business* (B2B) markets, such as online exchanges based on principles of *combinatorial* auctions (cf. [**14**, Ch.16]).

Though many web-sites that hosted online auctions no longer exist, including `auctions.yahoo.com` and `auctions.amazon.com`, there are still many online auction sites remaining, e.g. `ubid.com`, `bidz.com`, and `ebid.net`. However, the most dominant online auction 'house' was and still is `ebay.com`. Consequently, one of the most common yardsticks for measuring the importance and growth of online auctions are the annual figures of active users and sales volume on eBay. eBay defines *active users* to be those that at least once either placed a bid, bought, or listed something during the year. The sales volume is the value of all successfully closed listings and reported by eBay as *Gross Merchandise Volume* (GMV). Note that not all listings that make up the GMV figures are auctions. Over the last four years, the number of active users were: 83M (2007), 82M (2006), 72M (2005), 56M (2004). The sales volume over the same years were: \$56B (2007), \$52B (2006), \$44B (2005), \$34B (2004).[3] To put those figures in perspective, the *US Census Bureau* estimated

[1] `http://stores.ebay.com/Sears`; `http://stores.ebay.com/IbmFactoryOutlet`; `http://stores.ebay.com/Fujitsu-Scanner-Outlet`; `http://cgi3.ebay.com/ws/eBayISAPI.dll?ViewUserPage&userid=dell_financial_services`

[2] `http://www.dellauction.com`; `http://auction.mlb.com/`; `http://auctions.shopnbc.com/`; `http://www.clearance-comet.co.uk/`

[3] eBay Annual Report 2007, 2006, 2005, 2004. Available at `www.ebay.com`.

the 2007 eCommerce segment of US retail sales to account for close to \$127B (about 3.1% of total US retail sales).[4] Another comparison that may give additional perspective on eBay's sales volume, is with the annual revenue of the largest US retail stores. In 2006, the 6^{th} largest retail company Sears Holdings reported annual revenue of \$53B, followed by Walgreen with annual revenue of \$47B.[5]

Despite the fact that auctions have been used for centuries, it was not until the late 1950s that formal analysis of auctions started. Most people attribute the first auction theory paper with the 1961 seminal and Nobel Prize winning work of William Vickrey [**14**, p.ix].[6] However, Michael Rothkopf refers to Lawrence Friedman's paper 'A Competitive Bidding Strategy' from 1956 (*Operations Research*, vol.4), as the earliest formal analysis on auction and bidding theory [**26**, p.369]. In fact, according to Rothkopf, the first PhD in Operations Research was Lawrence Friedman's dissertation on competitive bidding from Case Institute of Technology in 1957 [**25**, p.1]. Since then, auction theory has flourished and resulted in an enormous body of literature. Two papers that deserve special attention, are the independent work from 1981 of Myerson (Optimal Auction Design, *Mathematics of Operations Research*, vol.6), and Riley and Samuelson (Optimal Auctions, *American Economic Review*, vol.71). Both of these papers generalize some of the ideas presented in Vickrey's original work from 1961. In particular, they prove the so-called *revenue equivalence principle* [**14**, p.36]. In addition, Myerson's paper includes the celebrated *revelation principle* [**14**, p.81].[7] For a formal and comprehensive account of auction theory, including the revenue equivalence and revelation principles, see V. Krishna's *Auction Theory* (2002). For a summary

[4] US Census Bureau News CB08-72

[5] http://www.stores.org/pdf/07TOP100Chart.pdf

[6] William S. Vickrey, born June 21, 1914, in Victoria, BC (Canada), was together with James A. Mirrlees awarded the Nobel prize on October 8, 1996. He passed away three days later on October 11, 1996.

[7] Roger B. Myerson, together with Leonid Hurwicz and Eric S. Maskin, received the 2007 Nobel prize.

and critique of some of the main auction theory results, see Rothkopf and Harstad (1994).

In the last ten years there has been an almost equally large proliferation of literature regarding online auctions. Despite some fundamental differences between traditional auctions and online auctions, most notably the context and time dimension, most researchers choose to analyze online auctions using the standard auction or game theoretical framework. As pointed out by Rothkopf this may or may not be the most useful or appropriate approach [**25**, p.8-9]. An advantage with online auctions is that they provide a great source of data for empirical analysis. This probably explains the huge proliferation of studies and PhD dissertations on online auctions, as predicted by Steven E. Landsburg in 1999.[8] Online auctions have also resulted in many experimental studies. Researchers can use the Internet as a laboratory, and run experiments and field tests to investigate various issues. Though a bit premature, given the infancy of online auctions at the time, the two early survey papers Pinker, Seidmann and Vakrat (2003), and Bajari and Hortacsu (2004), provide a good overview of some important issues regarding online auctions.

3. The eBay Single-Item Auction Format

In the western world, the term 'eBay' has become a household name. Most people are familiar with eBay, and know, for instance, that it is an online auction web-site. However, not everyone is aware of the exact price mechanism behind eBay auctions. In particular, there tends to be some confusion regarding the auction rules dictating the final price. At first it may seem that eBay auctions are first-price auctions, meaning that the bidder with highest bid wins and pays the amount he bid. This is not the case. In fact, eBay auctions

[8] `www.slate.com/id/22998`

are in effect more like second-price auctions, and almost seem to have been inspired by the following quote from the seminal 1961 auction paper by Vickrey,

> "An even more rapid procedure could be developed, with relatively little increase in the apparatus required, if each bidder were provided with a set of dials or switches which could be set to any desired bid, with the electronic or relay apparatus arranged to search out the two top bids and indicate the person making the top bid and the amount of the second bid." [**32**, p.23]

Note that Vickrey's paper precedes eBay by 35 years. Part of the confusion is that eBay does not provide a clear explanation for the auctions rules and what happens in certain specific situations. The objective of this section is to explicitly characterize the price mechanism of eBay's *single-item* auctions. The rules for *multi-item* auctions, auctions with multiple identical items, are a bit different. For a discussion on the differences and similarities between eBay and Vickrey auctions, see Chapter 2 of Stieglitz's *Snipers, Shills, & Sharks* (2007). A screen-shot of a typical eBay auction is displayed in Figure 1.1.

An eBay auction is characterized by five pieces of information:

(1) *Item* - Each auction includes a description of the item being auctioned and often pictures. Shown in Figure 1.1 is an auction for a 'DUAL SIM CNET IPhone Touch Screen PDA Mobile Phone.'

(2) *Seller* - The items are not sold by eBay but by a private seller. Information about each seller includes a user-id (proxy for the seller's name), their geographical location, a feedback rating score, and comments he has accumulated from previous transactions. The seller of the auction in Figure 1.1 is 'menzies1978,'who is located in the United Kingdom, and has a '99.9% Positive' feedback rating.

(3) *Time* - Each auction lasts for a pre-specified length of time. When a potential bidder visits the site he can see when the auction started, when it will end, and

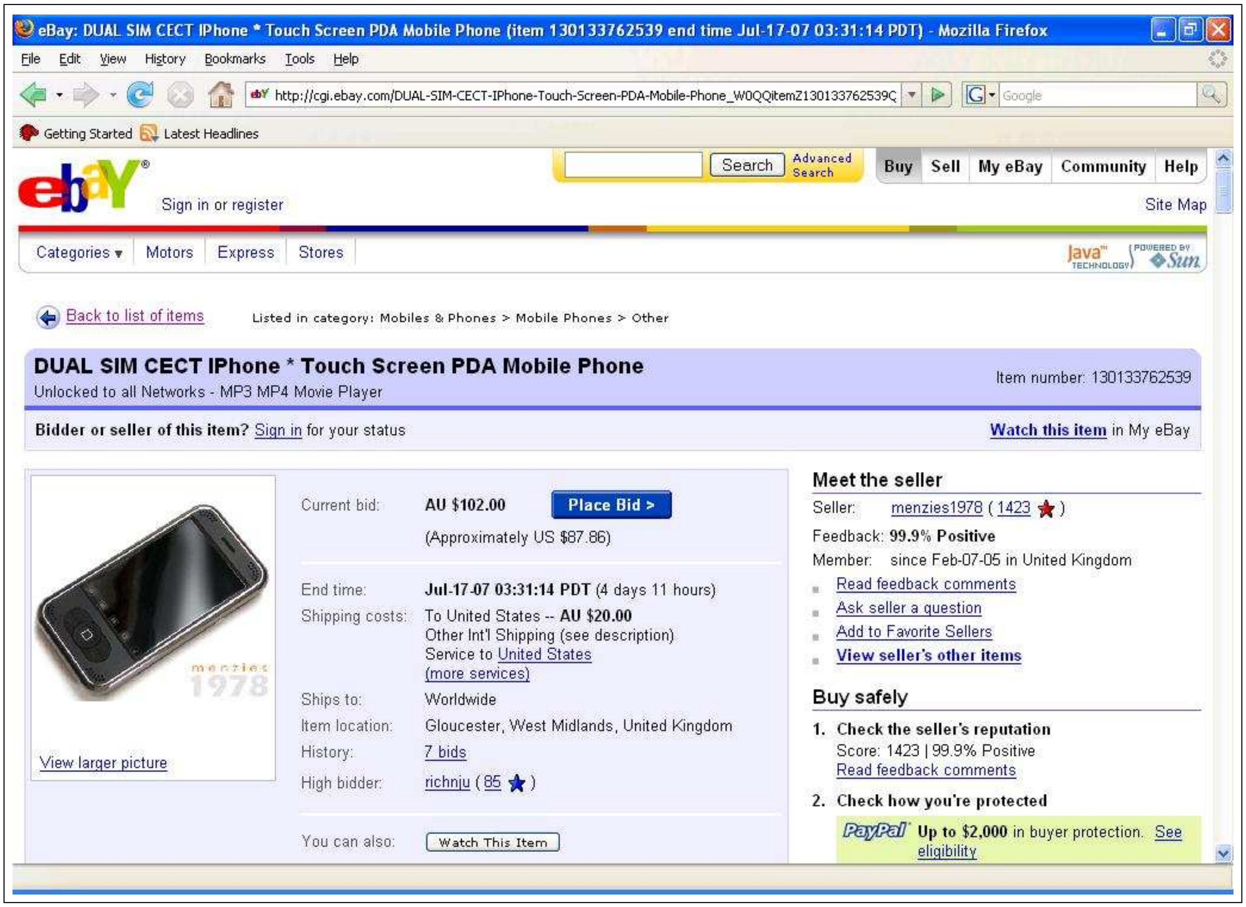


FIGURE 1.1. Screen-shot of an eBay auction.

how much time is remaining. The auction in Figure 1.1 ended on July 17, 2007, at 03:31:14 Pacific Daylight Time. At the time the screen-shot was taken there were 4 days and 11 hours remaining for the auction.

(4) *Price* - Each auction consists of a starting bid, current bid, and a minimal bid increment. In addition, some auctions also have a hidden reserve price and/or a Buy-it-Now price. The auction in Figure 1.1 has a current bid of AU\$102.00 (approximately US\$87.86).

(5) *Bid List* - Each auction also displays how many bids have been submitted, and a list of the corresponding time-stamps and the amount of each bid. The only exception is the highest bid placed, which is only shown as the minimum increment above the second highest bid. Prior to 2007, the list also included the user-id of

each bidder. However, in 2007 this changed and now only the seller is able to see the user-ids of the bidders. The bids are listed in ascending order. In case of a tie, the earlier bid take precedence. In Figure 1.1, the current high-bidder is 'richnju.'By clicking on the link '7 bids,'the history of bids is shown.

The information about the item and seller is fairly straight forward and requires no further discussion. A brief comment, however, is that many people not familiar with eBay are surprised that people would feel comfortable buying something that they cannot physically inspect, or is from someone that they have little information about. Today eBay supports auctions of almost everything and anything. This includes cars, baseball cards, jewelry, consumer electronics and real estate, to mention a few categories. Though fraud does exist on eBay, the overall sales volume and statistics speak for themselves. People have adapted to web-based shopping, and are not hesitant to buying something solely based on the description and picture a 'stranger' provides on a web-site. One method that eBay employs for building trust between sellers and buyers, is through their feedback rating system. After an auction has ended and the item and payment transactions have been made, the seller and buyer can report a feedback score and comment about each other. An early paper regarding how the feedback rating affects the final price is Lucking-Reiley et al. (2007). For a discussion on bidders' trust regarding a seller, see Chwelos et al. (2005) and Chwelos and Dhar (2005) .

Next we discuss the information regarding the listed bidders as it applied prior to January, 2007. Currently eBay handles the information regarding bidders differently as explained below. During an auction, the time-stamp and amount for all non-winning bids are disclosed. For the high-bidder only the time-stamp is displayed. In other words, the actual amount of the high-bidder's bid is not revealed (until of course he is out-bid). Information from expired auctions are available on eBay for a few weeks. This includes the amount of all non-winning bids, and prior to 2007, the user-id of the bidders. However, for privacy and anti-fraud purposes, in 2007, eBay changed the format and once an auction reached $200 the user-id was represented by a generic 'Bidder #'. Currently eBay uses a different disguise which takes in effect from the start of an auction. The data collected for this thesis is therefore unique, in that it tracks all individual bidders for almost all auctions offered by Dell Financial Services from December, 2005, until February, 2007. Section 4.2 provides more details.

The main difference with traditional auctions and online auctions, is that the latter lasts for a pre-determined length of time. When a seller starts an auction he must choose the auction length. The current options on eBay are 1, 3, 5, 7 or 10 days. We define the length of an auction as τ. There are three time-stamps provided by eBay. One for when the auction starts, one for when the auction ends, and one for the remaining time of the auction (which is continuously updated). Due to the speed of internet technology and a bidding strategy called *sniping*, the time-stamps are defined down to the second (Figure 1.1). Bids can only be submitted while the auction is ongoing. In particular, eBay auctions close firmly at the announced ending time regardless of any bidding activity. Other auctions sites, for instance, the former `auctions.amazon.com`, and `dellauction.com`, offer a *going, going, and gone* ending rule. There the auction end-time is extended by 10 minutes for every bid in the final 10 minutes. For an analysis on the impact of the two different ending rules, see Roth and Ockenfels (2002).

In this thesis, rather than focusing on the remaining time of an auction, we focus on the *elapsed time of an auction.* We define the elapsed auction time by t, and t^+ as the instantaneous moment after a bid has been placed. In other words, a potential bidder arrives at an auction after t time units have elapsed. If he decides to submit a bid then the moment immediately following his submission is denoted by t^+.

There are five different price variables in eBay auctions: *starting bid, current bid, bid increment, reserve price*, and *high bid*. In addition there are of course, the bids as well. We define the bid submitted at time t by B_t and refer to it as the *bid at time* t. It can be noted that eBay auctions are *standard* [**14**, p.15], [**26**, p.366], meaning the highest bid submitted wins the auction, and given the reserve price was met, is guaranteed the item.

Starting bid, defined by p_{min}, is the minimum allowable first bid as decided by the seller. An auction is initially priced at zero, and the first bidder must bid at or above p_{min}. With slight abuse of notation, $B_{1^{st}} \geq p_{min}$. This is not the same as *reserve price*, but rather the initial price the auction will jump to once a bid greater that it has been submitted. To illustrate, suppose $p_{min} = \$20$, and the first bidder bids $B_{1^{st}} = \$30$, then the current bid of the auction will jump to \$20.

The second price variable is the *current bid*, which we define by X_t. This is the amount that the high bidder would have to pay, if the auction were to end immediately. That is, if no more bids are submitted, then the high bidder only has to pay X_t. Though X_t is indexed by t, to indicate the price at time t, it is not a function of time but strictly a function of submitted bids. The dynamics of X_t will be discussed shortly.

The third price variable is the *bid increment* which we define by k_X. This is the minimum amount that a potential bidder must be willing to bid above X_t. In other words, if a potential bidder arrives at time t, and decides to place a bid B_t then,

$$B_t \geq X_t + k_X \qquad \forall t \in [0, \tau) \tag{1.1}$$

Though k_X is a function of the current price, which on eBay varies according to Table 1.1, we suppress the subscript X and simply write k.

Current Price - X_t (\$)	Bid Increment - k_X (\$)
0.01 - 0.99	0.05
1.00 - 4.99	0.25
5.00 - 24.99	0.50
25.00 - 99.99	1.00
100.00 - 249.99	2.50
250.00 - 499.99	5.00
500.00 - 999.99	10.00
1000.00 - 2499.99	25.00
2500.00 - 4999.99	50.00
5000.00 $\leq$	100.00

TABLE 1.1. Minimum bid increments on eBay (June 2008)

The *reserve price*, defined by v_r, is the minimum price for which the seller will award the item. Note that $v_r \geq p_{min}$. In other words if the auction ends below v_r, then the seller is not obligated to award the item. The seller chooses the reserve price. In auctions with no reserve price, the bidder who bids the most is guaranteed to be awarded the item. On eBay, the actual amount of v_r is not disclosed, instead there is a message stating whether the reserve price has been met or not.

Finally there is the *high bid*, defined by H_t, which is the amount of the *highest bid placed after t time units has elapsed.* Unlike p_{min} and X_t, the highest bid is never displayed as long as it remains the highest bid. That is, up to the time when a bid $B_t > H_t$, is submitted. Then H_t is revealed and the new high bid $H_{t+} = B_t$, remains hidden. Naturally the bidder who submitted the high bid knows the actual amount. However, potential bidders do have some information regarding H_t, since clearly $H_t \geq X_t$. In fact, due to the dynamics of the pricing mechanism, potential bidders have even more information about H_t, namely, if

$H_t \geq X_t + k$.

Next we discuss the price dynamics of eBay auctions. The most straightforward dynamic regards the *high bid*. When a bid, B_t, is submitted the new high bid, H_{t^+}, is simply the maximum of B_t and H_t, for $t \in [0, \tau)$,

$$H_{t^+} = \max\{B_t, H_t\} \tag{1.2}$$

In case of a tie, the current high-bidder will remain as high-bidder. The above relationship also holds true when the bidder is the current high-bidder, i.e. when the high-bidder revises his current high-bid. However, if a high-bidder revises his bid then nothing happens to the current bid, i.e. $X_{t^+} = X_t$.

The following discussion applies to cases when a bidder who is currently not the high-bidder arrives and places a bid. The dynamics for the *current bid* are also straightforward when $X_t = 0$ and the first bid arrives. For $X_t = 0$, $B_t \geq p_{min}$, $t \in [0, \tau)$,

$$X_{t^+} = p_{min}\mathbf{1}_{\{p_{min} \leq B_t < v_r\}} + v_r\mathbf{1}_{\{B_t \geq v_r\}} \tag{1.3}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function which equals 1 if the argument in the bracket is true, and 0 otherwise. Equation (1.3) states that if the first bid is less than the reserve price, then the current bid will jump to the minimum price (the bid has to, of course, be above the starting bid). If the bid is above the reserve price, then the current bid will jump to the reserve price. Note that equation (1.3) is well-defined even when $v_r = p_{min}$. When $X_t \geq p_{min}$, the dynamics are a bit more complicated. Therefore, we first consider the case with no reserve price ($v_r = p_{min}$). For $X_t \geq p_{min}$, $B_t \geq X_t + k$, $t \in [0, \tau)$,

$$X_{t^+} = \begin{cases} X_t + k & H_t < X_t + k \\ \begin{cases} \min\{B_t + k, H_t\} & B_t \leq H_t \\ \min\{H_t + k, B_t\} & B_t > H_t \end{cases} & H_t \geq X_t + k \end{cases}$$

which we write as, for $X_t \geq p_{min}$, $t \in [0, \tau)$,

$$(1.4)\quad X_{t^+} = (X_t + k)\mathbf{1}_{\{H_t < X_t + k\}} + (\max\{\min\{B_t + k, H_t\}, \min\{H_t + k, B_t\}\})\,\mathbf{1}_{\{H_t \geq X_t + k\}}$$

We now consider the situation when a seller has included a reserve price $v_r > p_{min}$. As mentioned above, the actual amount of v_r is not disclosed, and only a message stating whether v_r has been met is displayed. That is, if v_r has not been met, then $H_t < v_r$, i.e. $H_t < v_r$ if and only if $X_t < v_r$. Note that the dynamics for H_t are not affected by $v_r > p_{min}$ and still follow (1.2). For $X_t \geq p_{min}$, $B_t \geq X_t + k$, $t \in [0, \tau)$,

$$X_{t^+} = \begin{cases} \begin{cases} X_t + k & H_t < X_t + k \\ \min\{B_t + k, H_t\} & H_t \geq X_t + k, B_t \leq H_t \\ \min\{H_t + k, B_t\} & H_t \geq X_t + k, B_t > H_t \end{cases} & H_t < v_r, B_t < v_r \;\; or \;\; H_t \geq v_r \\ \begin{cases} \max\{X_t + k, v_r\} & H_t < X_t + k \\ \max\{H_t + k, v_r\} & H_t \geq X_t + k \end{cases} & H_t < v_r, B_t \geq v_r \end{cases}$$

which we write as, for $X_t \geq p_{min}$, $b_t \geq X_t + k$, $t \in [0, \tau)$,

(1.5)

$$\begin{aligned} X_{t^+} =& (1 - \mathbf{1}_{\{H_t < v_r \leq B_t\}})[\mathbf{1}_{\{H_t < X_t + k\}}(X_t + k) + \mathbf{1}_{\{H_t \geq X_t + k\}} \max\{\min\{B_t + k, H_t\}, \min\{H_t, B_t\}\}] \\ &+ \mathbf{1}_{\{H_t < v_r \leq B_t\}}[\mathbf{1}_{\{H_t < X_t + k\}} \max\{X_t + k, v_r\} + \mathbf{1}_{\{H_t \geq X_t + k\}} \max\{H_t + k, v_r\}] \end{aligned}$$

Note that, if $v_r = p_{min}$, then (1.5) simplifies to (1.4), since $H_t \geq p_{min}$, and $\mathbf{1}_{\{H_t < v_r \leq B_t\}} = 0$.

We summarize the main mechanisms of eBay as follows,

(1) An auction starts with a fixed time horizon τ, starting price p_{min}, and possibly a hidden reserve price v_r.

(2) The first bidder must bid at or above p_{min}.

(3) When a bidder arrives to the auction site he observes,[9]

 (i) How much time t has elapsed, $t \in [0, \tau]$.

 (ii) If he is the high-bidder.

 (iii) The current price X_t.

 (iv) The list of previous bids, i.e. all non-winning bids.

 (v) If the reserve price v_r has been met or not, i.e. is $X_t \geq v_r$.

(4) If he decides to bid, then $B_t \geq X_t + k$ (if the bidder is the high-bidder, then $B_t \geq H_t$).

(5) After a bid is submitted, H_t and X_t are updated accordingly.

(6) When the auction expires at $t = \tau$, the person with the high-bid, H_τ, pays X_τ and receives the item, provided $H_\tau \geq v_r$. However, if $H_\tau < v_r$, then the seller is not obligated to award the item.

Though the format of an eBay auction seems to resemble a mix of a *first-price sealed-bid* and *open English* (ascending) auction, since all non-winning bids are displayed and X_t is continuously updated. In effect, it is a *second-price* auction. Or more appropriately one could define eBay auctions as *second-price+k censored-English auctions.* We say 'second-price+k' since the highest bid wins, but only has to pay the second highest bid plus the minimum increment k. We say 'censored English' since although all non-winning bids are disclosed, the high-bid is never displayed. Another reason we say eBay auctions are second-price auctions is seen in the case of $k = 0$ for all X_t. If we ignore reserve price ($v_r = p_{min}$),

[9] There are of course many other variables a prospective bidder might consider in his decision whether to bid and/or the amount, but here we only point out the main time and price variables.

we notice that with $k = 0$, the following dynamics apply, for $X_t \geq p_{min}$, $B_t \geq X_t$, $t \in [0, \tau)$,

$$X_{t^+} = \min\{B_t, H_t\}$$

The above equation is exactly the price dynamic of a strictly second-price auction. Similarly if $v_r \geq p_{min}$ and $k = 0$, for $X_t \geq p_{min}$, $B_t \geq X_t$, $t \in [0, \tau)$,

$$X_{t^+} = (1 - \mathbf{1}_{\{H_t < v_r \leq B_t\}}) \min\{B_t, H_t\} + \mathbf{1}_{\{H_t < v_r \leq B_t\}} v_r$$

3.1. ***Buy-it-now*** **auctions.** There is one additional feature we have not included in the discussion above, namely the *Buy It Now* price, which we define by p_{buy}. As the name suggests, p_{buy} is a pre-set price at which the seller is willing to end the auction and award the item immediately. In other words, if someone bids p_{buy}, then the auction terminates. Naturally p_{buy} is shown and not hidden as v_r. However, p_{buy} is only available as long as $X_t < v_r$. That is, if 1) $p_{min} = v_r, X_t = 0$, and $B_t < p_{buy}$, or 2) $p_{min} < X_t < v_r$, and $v_r \leq B_t < p_{buy}$, then the *Buy It Now* option is removed at t^+. Note that $p_{buy} > v_r$ and therefore p_{buy} can be regarded as a maximum price or 'list price' of the item. However, this list price is only available until the first bid arrives, or until $B_t \geq v_r$ arrives. The requirement $B_t < p_{buy}$ in the second case above is a bit redundant, since if $B_t = p_{buy}$, then t^+ indicates the end of the auction and p_{buy} is also 'removed'. As a consequence of the *Buy It Now* feature the dynamics of the auction change slightly. Let us again consider the dynamics described earlier.

Case 1) $v_r = p_{min}$

For $t \in [0, \tau)$, the auction length is defined as follows,

$$\tau_{bin} = \min\{\tau, \inf\{t \mid X_t = 0, B_t = p_{buy}\}\}$$

and the dynamics of the high bid and current bid are given by,

$$
\begin{aligned}
H_{t^+} &= \min\{B_t, H_t\} \\
X_{t^+} &= \begin{cases} \begin{cases} p_{min} & B_t < p_{buy} \\ p_{buy} & B_t = p_{buy} \end{cases} & X_t = 0 \\ \mathbf{1}_{\{H_t < X_t + k\}}(X_t + k) + \mathbf{1}_{\{H_t \geq X_t + k\}} \max\{\min\{B_t + k, H_t\}, \min\{H_t + k, B_t\}\} & X_t \geq p_{min} \end{cases}
\end{aligned}
$$

Case 2) $v_r > p_{min}$ (recall that $H_t < v_r$ if and only if $X_t < v_r$.)

For $t \in [0, \tau)$, the auction length is,

$$\tau_{bin} = \min\{\tau, \inf\{t \mid X_t < v_r, b_t = p_{buy}\}\}$$

and the dynamics of the high bid and current bid are,

$$
\begin{aligned}
H_{t^+} &= \min\{B_t, H_t\} \\
X_{t^+} &= \begin{cases} \begin{cases} p_{min} & B_t < v_r \\ v_r & v_r \leq B_t < p_{buy} \end{cases} & X_t = 0 \\ \begin{cases} \max\{\min\{B_t + k, H_t\}, \min\{H_t + k, B_t\}\} & B_t < v_r \\ \max\{X_t + k, v_r\} & v_r \leq B_t < p_{buy} \\ p_{buy} & B_t = p_{buy} \end{cases} & X_t < v_r \\ \max\{\min\{B_t + k, H_t\}, \min\{H_t + k, B_t\}\} & v_r \leq X_t \end{cases}
\end{aligned}
$$

Therefore, for $X_t = 0$,

$$X_{t^+} = \mathbf{1}_{\{B_t < v_r\}} p_{min} + \mathbf{1}_{\{v_r < B_t < p_{buy}\}} v_r + \mathbf{1}_{\{B_t = p_{buy}\}} p_{buy}$$

and for $X_t > p_{min}$,

$$X_{t^+} = \mathbf{1}_{\{X_t < v_r, B_t = p_{buy}\}} p_{buy} + (1 - \mathbf{1}_{\{X_t < v_r, B_t = p_{buy}\}})[$$

$$(1 - \mathbf{1}_{\{H_t < v_r \leq B_t\}})[\mathbf{1}_{\{H_t < X_t + k\}}(X_t + k) + \mathbf{1}_{\{H_t \geq X_t + k\}} \max\{\min\{B_t + k, H_t\}, \min\{H_t + k, B_t\}\}]$$

$$+ \mathbf{1}_{\{H_t < v_r \leq B_t\}}(\mathbf{1}_{\{H_t < X_t + k\}} \max\{X_t + k, v_r\} + \mathbf{1}_{\{H_t \geq X_t + k\}} \max\{H_t + k, v_r\}]$$

Next we discuss the background for the data used in the empirical analysis of Chapter 4 and 5.

4. The Data and Dell Financial Services

The data we analyze in this thesis comes from the eBay listings of Dell Financial Services L.P. (DFS). DFS is a joint venture between Dell Inc. and CIT Group Inc. (CIT), that provides financing of Dell products to various customer groups, including home, education, small, medium, and large businesses. Dell, which was founded in 1984, is one of the largest computer system manufacturers and sellers in the world. CIT, which was founded in 1908, provides financial solutions for both commercial and consumer clients. For more information regarding Dell and CIT, visit `dell.com` and `cit.com`. Since founded in 1997, DFS has "originated more than $18 billion in finance transactions."[10]

After products return from leasing programs, DFS selects the highest quality products, refurbishes them and sells them via their private online channels, `dfsdirectsales.com` and `dellauction.com`, as well as on `ebay.com`. In other words, the three online channels provide DFS with alternative salvage channels. How long DFS has been using these channels or how DFS allocates the products to the different channels is not specified. However, from their eBay profile we know that DFS has been an eBay member since April 2001 (eBay user-id:

[10] `www.dfsdirectsales.com`

dell_financial_services). There are a few comments to make. First, DFS only sells to the US market. A customer must have a US based credit card and a US delivery address. Second, DFS does not operate an *eBay store*, but lists items on eBay as an individual member. Third, for unknown reasons, since February 26, 2007, DFS has ceased to list products on eBay. They still, however, use their own online channels. Finally, Dell, and not DFS, also employs an online channel called *Dell Outlet* for selling returned and refurbished products at fixed but discounted prices.[11]

From the perspective of the customers, the three channels provide opportunities to purchase used and refurbished Dell products at a discount. Issues like taxes, shipping and handling, warranties, and return policies vary for each product sold. In general, the products are covered under Dell's general warranty. However, since products are returned after various lease durations, the extent of the remaining warranty differs from product to product. The specifics of the remaining warranty is available online for a potential buyer to verify before having to commit to a purchase or bid. Moreover, DFS also provides a return policy of 30 days, for a credit of the purchase price (excluding shipping and handling). It should be noted, that unlike Dell's direct business model where customers tailor the product configuration according to their needs or budget, the configurations of items listed by DFS are fixed and DFS does not provide any upgrade or modification services.

The three channels each have different pricing mechanisms. At `dfsdirectsales.com` products are sold at a fixed list price. How DFS determines the list-price of each product or configuration is not specified. As a rough estimate, the list-price seems to be about half of the price of a new product, as sold on `dell.com`. At `dellauction.com` and `eBay.com`, products are predominantly sold using the online auction version of a second-price auction.

[11] `http://www.dell.com/content/segmenter.aspx?c=us\&l=en\&s=dfo`

The main difference between the two auction channels is that, unlike the hard auction end on `ebay.com`, at `dellauction.com`, DFS employs a *going, going, ..., gone* end. Similar to the former ending rule at `auctions.amazon.com`, `dellauction.com` auctions are automatically extended by 10 minutes for each arriving bid in the last 10 minutes.[12] This is presumably to provide less of an incentive for the strategy of sniping [**24**].

A common attribute among the three channels is that the picture and description of each product listing are standardized. For all listings, DFS uses a generic picture of the product that is being sold or auctioned. In other words, DFS does not provide digital images of the individual item that is sold or auctioned. In addition, DFS uses a standardized template to describe the product. The information provided includes a description of the main features, but does not give detailed description regarding, for instance, cosmetic appearance. It would seem reasonable to assume that by using generic pictures and standardized templates, DFS reduces the administrative cost of using online channels. The effect this has on the final price is not immediate. Some related research topics include how to allocate products to the different channels, and how the allocation affects the overall revenue and profit. A third research area is investigating how the fixed list-prices and auction prices affect each other, and the impact on overall revenue and profit. One paper that address the latter issue is Caldentey and Vulcano (2008).

4.1. Reasons for Selecting DFS. Besides eBay's huge commercial success and user popularity, eBay has also become one of the most popular sources of data for auction research. With millions of listings and data from completed auctions made public for a few weeks, eBay provides an unparalleled rich source for empirical investigations. However, though the data is available, it is not a trivial task to extract large data sets. Automating a

[12]cf. 'Auto Bid Extend', Glossary at `www.dellauction.com`

data extraction process requires Internet and eBay experience, a computer code or software technology, and an understanding of how to use the code or software. In the next section we give a brief explanation of how our data were extracted.

Most researchers tend to focus on a specific item, and then simply download as much auction data as possible for that particular item. We decided to take a different approach and only focus on a particular seller, namely DFS, and then download all auctions they listed on eBay over a given time-period. The main reason for this, was that we wanted to control for the effect of the seller's reputation, trustworthiness, and feedback rating. Studies have indicated that the trustworthiness and feedback rating may effect the final price of an auction [**16**]. Therefore, to control for this issue, we decided to only consider a specific seller, and to take a well-established company whose reputation and trustworthiness would in general not be considered suspect.

Another reason we chose DFS was that we wanted to maximize the chance that there would be an abundance of auctions over an extended period of time. In addition, we wanted to maximize the chance of having many auctions of identical or near identical products. An unforeseen benefit of the DFS auction data, was that the description and picture provided was standardized, thereby providing control for the effect wording, description and pictures may have on the auction dynamics. It should, however, be noted that over time DFS did make some changes to the template used.

4.2. Data Extraction and Handling. Extracting and analyzing the data involved both manual and automated steps. We will not provide a detailed account of each step, but instead give an overview regarding the main components. All data extraction, handling and analysis was executed on a standard PC with Internet connectivity. The first three steps of data extraction only needed to be done every three weeks.

(1) The first step was to log on to eBay and perform a manual search of all completed listing for *Dell_Financial_Services*. This step required an eBay user-id and password, and knowledge of how to use their search function.

(2) The next step was to save the entire list as a html-file using the 'Save As' function provided by the web-browser.

(3) The third step was to run a Perl script that automatically scanned the saved list and downloaded into flat files (text files) all the relevant information for each individual auction. This step produced eight files: two auction data files and one bid data file per PC category (laptop and desktop), one file with non-PC and recalled auction listings (monitors, docking stations, and test listing), and one file for any error messages. In the auction data files, each row corresponded to an individual auction and each column corresponded to a different variable. See Table 1.3 for the list of variables. In the bid data file each row corresponded to an individual bid for a specific auction.

    The script worked as follows. First, it searched through the saved list and picked out the individual listing ID for all new auctions since previous data extraction. It then queried `ebay.com` regarding each auction and collected the pre-specified information. Section 4.3 lists the specific information collected. Perl is a general purpose programming language, and to run the script requires a compiler. We used 'ActivePerl' available at no cost at `www.activestate.com`. The script was written by UBC student Andrew Gray and consists of 540 lines.

(4) After data had been extracted into flat files, it was populated into an Excel spreadsheet by a Visual Basic (VBA) macro. The same VBA macro was also used to

perform data cleaning, data selection as well as numerous computations. The script was also written by Andrew Gray and each user function was executed inside Excel.

(5) After a subset was selected and the variables of interest had been computed, the resulting Excel sheet was exported to a tab-delimited text-file.

(6) The resulting text-files was then imported into the statistical software 'R' [**22**], for various statistical and graphical analysis. 'R' is available at no charge at `www.r-project.org`, and is an extension to the statistical software 'S'.

4.3. The Data. Data were collected from mid December, 2005, until the end of February, 2007. More specifically we obtained complete information regarding 6,683 auctions, with start and end date between December 12, 2005, and February 26, 2007. As mentioned above, DFS has since ceased their activity on eBay. Only auction listings for PC desktops and PC laptops at *Dell_Financial_Services* eBay-site were collected. In Table 1.2 some aggregated statistics are summarized. There were 3,802 desktop auctions and 2,881 laptop auctions, which combined for a total sales of \$1,979,240. The average final price, average number of bids, and average number of bidders for the desktop auctions were \$205, 13.54, and 7.89, respectively. For laptop auctions the corresponding values were \$416, 17.09, and 9.37. In other words, on average a laptop PC sold for about twice that of desktop PC, and attracted about three more bids and one more bidder.

We also note the great variation in final price, bids and number of bidders. In particular, the standard deviation in final price for desktop and laptop auctions is quite large, at \$66.81 and \$86.24 respectively. This illustrates the huge uncertainty in final price a seller is faced with in selling via online auctions. It would not seem reasonable to assume that the variation is strictly a result of the variation in bidders' valuation. In Section 4.4 and Chapter 5 we analyze this issue further. The variation in bids and number of bidders is also interesting to note. Often in the auction theory literature it is assumed the number of bidders is fixed.

From Table 1.2 it is clear this is not the case here. Furthermore, this would intuitively seem to indicate at least one source of variation regarding the final price. Chapter 3 discusses a model for the distribution of final price that incorporates the variability in the number of bidders.

	Total			Mean (s.d.)		
Dell Product	Auctions	Bids	Sales	Final Price	Bids	Bidders
Desktop	3,802	51,495	$780,114	205.20 (66.81)	13.54 (5.53)	7.97 (2.32)
Laptop	2,881	49,242	$1,119,126	416.20 (86.24)	17.09 (6.79)	9.37 (2.68)
All	6,683	100,747	$1,979,240			

Table 1.2. Descriptive statistics for the all auctions.

Figure 1.2 provides a time series of the final price for all auctions coded by PC category Desktop and Laptop. The horizontal axis is time, and the vertical axis represents the final price of the auction. Each circle represents an individual auction, where black circles are laptop auctions, and grey circles are desktop auctions. There are at least three rather apparent features. First, the desktop auctions' final prices are about half those of laptop auctions, as noted above. Second, laptop auctions appear to have been sold at two levels. There is a clear divide of laptop final prices before and after July, 2006. During July, 2006, there is a clear decline in final prices, after which they seem to stabilize again. Desktop final prices seem to remain steady throughout the observation period. Third, as already noted, there is a considerable fluctuation in final prices for both laptops and desktops. Some of these fluctuations might be due to the difference in product configurations. In Section 4.4 below we briefly analyze this issue. However, the fluctuations also reflect the stochastic nature of the online auction revenue stream. The main motivation for this

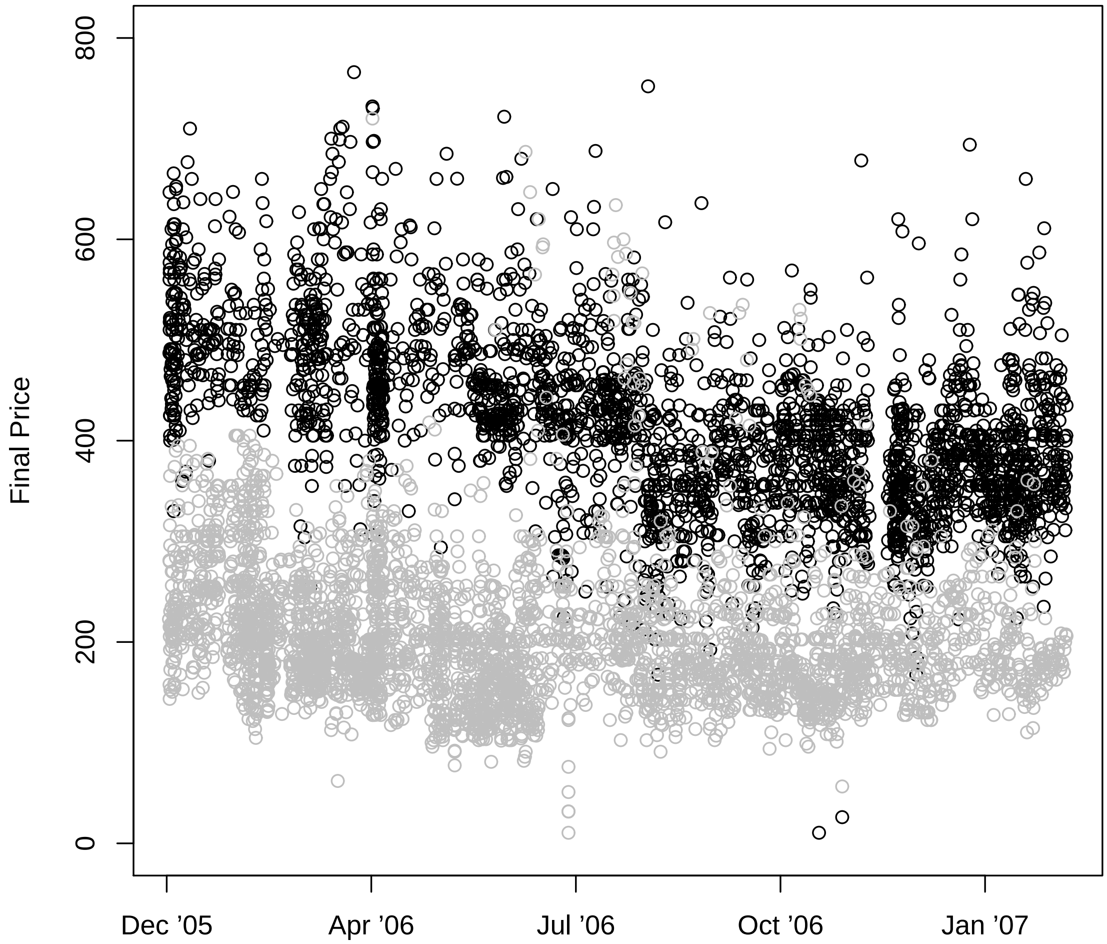


FIGURE 1.2. Final price of each auction for the entire data set (6,683). The horizontal axis represents time. The vertical axis represents the final price of an individual auction. Black circles represents laptop auctions (2,881), and grey circles desktop auctions (3,802).

thesis, is to provide insight for making better decisions given such a stochastic environment.

Table 1.3 below displays the auction and product information recorded for each product. The first ten variables are standard to all eBay auctions and provide information regarding the auction state and bidding history. Variables 8, 9, and 10, are recorded for each successful bid (registered bid) in the *bid history*. The last 11 variables are information that DFS decided to include in their listing. They provide the specifications of the product being auctioned. As mentioned above, DFS or bidders do not choose the specific product configuration. In other words, though DFS does have the option of not listing a particular product with a specific configuration, the last 11 variables are not decision variables. Therefore, the only decision variables are 'Description', 'Start time', 'End time', and 'Starting bid'. The 'Start time' is the time-stamp when DFS lists the auction, and therefore reflects the decision if and when to release an item for auction. The 'End time' is the time-stamp when an auction will end, and reflects the decision how long an auction should last.

In addition to the information listed in Table 1.3, DFS also includes a generic picture of a new product, some additional information regarding what is included (AC adapter, pointing device/mouse, keyboard, etc.), instructions how to verify any remaining warranty, and various shipping options. Shipping cost range from \$20 to \$80 depending on service requested by the auction winner. A few comments follow.

(1) The standardized template that DFS uses to list the last 11 variables was modified over time.

(2) It seemed that DFS was not consistent in always including the information regarding all 11 variables. More specifically the only variables consistently reported were: Category, Brand, Processor Type, Processor Model, Processor Speed, Memory (RAM), and Hard Drive Capacity. Therefore in the ensuing statistical analysis we only focus on these variables and ignore the others.

(3) Unlike the PC laptops, which naturally came with a screen, none of the PC desktops included a monitor.

(4) Though DFS also auctions items such as monitors and docking stations, we exclusively restricted our data extraction and analysis for PC desktops and laptops.

(5) All auctions that ended with the *Buy-It-Now* option were excluded. The reason for this is that once a bid arrives or the reserve price is met, the *Buy-It-Now* option disappears. Since the information, if an auction was initially listed with a *Buy-It-Now* price, is not recorded, we decided to ignore all *Buy-It-Now* transactions. It can, however, be noted that the number of *Buy-It-Now* transactions was less than one percent of all auction listings. With regard to the previous comment, it can also be noted that all auctions were listed without a *reserve price*, and almost exclusively had a *starting bid* of $.99. Though this might appear to be rather risky, both anecdotal and research evidence supports this use. The listing fees are less for auctions without a *reserve price* and a low *starting bid*. In addition, and more importantly, low reserve price and starting bid tends to lead to higher expected revenue [**4**].

(6) DFS only sells and ships to US based customers.

(7) Though data were extracted every 2-3 weeks, some auctions may have been missed. In other words, some completed auctions may have been removed by eBay before we had a chance to download the information. Therefore, the data set does not exclusively cover all DFS eBay desktop and laptop auctions. However, there is no reason to believe that the excluded auctions had special features or exhibited any unusual auction dynamics.

We now describe the data more in depth.

4.3.1. *Desktop Data.* In total there were 3,802 successful desktop auctions with a total value of $780,114. The mean final price over the entire study period was $205.20. As mentioned above, since not all variables listed in Table 1.3 were consistently reported, we chose to focus on the main characteristics that were reported: Category, Brand, Processor Type, Processor Model, Processor Speed, Memory (RAM), and Hard Drive Capacity. All

	Name	Description	Example
1.	Item ID	eBay listing ID number	6831091024
2.	Description	Brief description of item	
		DELL WINXP Latitude D600 1.6 GHz 1024MB CDRW DVD NR	
3.	Winning bid	Amount of final price	576.87
4.	Start time	Time stamp of auction start	12-Dec-05 10:07:39
5.	End time	Time stamp of auction end	15-Dec-05 10:07:39
6.	Starting bid	Amount first bid has to exceed	0.99
7.	Number of bids	Total number of bids that arrived in auction	19
8.	User ID	eBay User ID of bidder	frittikanada
9.	Bid Amount	Amount of bid	127.32
10.	Date of Bid	Time stamp of bid	11-Feb-06 16:46:23
11.	Category	Type of PC product (desktop or laptop)	Laptop
12.	Brand	Brand of PC product (all Dell)	Dell
13.	Processor Type	Brand of processor (all Intel)	Intel
14.	Processor Model	Model specification of processor	Pentium M
15.	Processor Speed	Speed specification of processor	1.6GHz
16.	Bundled Items	Included software (operating system)	WINXP
17.	Memory (RAM)	Specification of internal memory	1024MB
18.	Hard Drive Capacity	Specification of hard drive capacity	60GB
19.	Operating System	Description if/what operating system is included	Yes
20.	Primary Drive	Description of CD or DVD drive	CD-RW
21.	Condition	One word describing condition	Refurbished

TABLE 1.3. The auction and product variables collected.

auctions consisted of Dell PC with an Intel processor. In Table 1.4 the distribution of the other variables are listed. Out of the five processor models, the Intel Pentium 4 (IP4) was the most common, accounting for 87% of all listings. There were 18 different processor speeds, ranging from .866 to 3.2GHz. The most common was 2.0GHz which accounted for 32% of all listings, and the five most common, 1.8, 2.0, 2.2, 2.3, and 2.4GHz, accounted for 83% of all listings. There were eight different memory (RAM) sizes, where 256 and 512MB were the most common covering respectively 46% and 27% of all listings. There were 13 different hard drive capacities, ranging from 6 to 200GB. The two most common were 40 and 20GB, which accounted for 57% and 34% of sales each.

The last variable in Table 1.4 is the auction duration measured in days. Though DFS seemed to experiment with a few 1 day auctions, specifically 459 or 12% of all listings, they predominantly used an auction duration of 3 days, accounting for 88% of all listings. The eight auctions that do not have one of the standard eBay auction durations, i.e. the ones with the decimal extensions, are auctions that DFS for unknown reasons simply chose to end early and award to the high bidder at that time. It should be noted that these are not *Buy-It-Now* auctions, as these have been removed. They are also not auctions that were at a particular high price, such that it might seem unnecessary to let them proceed the full auction duration with little probability of seeing more bids. In fact almost the opposite, all eight auctions were at rather low prices. No further explanation as to why they were ended is provided. In the ensuing analysis these auctions will therefore be removed.

Processor Model	IP4	3,309	IP4 Xeon	2	IPM	34	IPIii	2	Celeron	455
Processor Speed	0.866GHz	2	1.3GHz	1	2.0GHz	1,231	2.6GHz	54	3.1GHz	2
			1.4GHz	6	2.2GHz	238	2.7GHz	102	3.2GHz	1
			1.5GHz	27	2.3GHz	308	2.8GHz	136		
			1.6GHz	15	2.4GHz	973	2.9GHz	18		
			1.7GHz	144	2.5GHz	142				
			1.8GHz	402						
Memory	128MB	144	256MB	1,760	512MB	1,021	1024MB	278		
	224MB	11	320MB	541	768MB	25				
					1000MB	22				
Hard Drive	6GB	1	20GB	1,279	40GB	2,179	120GB	9		
	10GB	15	30GB	64	60GB	78	160GB	2		
	18GB	4	33GB	2	80GB	162	200GB	1		
			36GB	6						
Duration (days)	0.51699d	1	1.68313d	1	1d	459				
	0.82015d	1	1.72839d	1	3d	3,334				
	1.00861d	1	1.91704d	1	5d	1				
	1.12295d	1	2.03098d	1						

TABLE 1.4. Counts of product configurations and auction duration for the 3,802 Dell PC desktop auctions.

4.3.2. *Laptop Data.* In total there were 2,881 successful laptop auctions for a total value of \$1,119,126. The mean final price over the entire study period was \$416.20. Similar to the desktop auctions we will only focus on the main characteristics that were consistently reported. All auctions were for a Dell PC with an Intel processor. In Table 1.5 the distribution of the laptop configurations are listed. The two most common processor models were Intel Pentium 4 (IP4) and Intel Pentium M (IPM), which accounted for 49% and 46% of all listings respectively. There were 22 different processor speeds, ranging from .866 to 3.2GHz. The five most common, which accounted for 87% of all listings, were: 1.4 (22%), 1.6 (15%), 1.7 (7%), 1.8 (18%), and 2.0GHz (25%). Out of the eight different memory (RAM) sizes the most common were 256, 512 and 1024MB, covering 18%, 57% and 20% respectively. There were 13 different hard drive capacities, ranging from 5 to 80 GB. The three most common were 20, 30, and 40GB, which accounted for 18%, 42% and 33% respectively.

Similar to the desktop auctions the most common auction duration was 3 days, which accounted for 91% of all listings. Furthermore, as with the desktop auctions, there are a few listings that have non-standard auction duration and are not *Buy-It-Now* auctions. The two special listings with an auction length of 2.95833 days, are auctions that were ended *exactly* one hour prior to the auction 'End time', and were released on the same date (but not at the same time). The reason and mechanism to do this is not known, but perhaps there was a glitch with the data extraction or error with the eBay listing. In total there were ten auctions released that day, and only two that did not elapse the full 3 days. In the ensuing analysis the four auctions with non-standard duration have been removed.

4.4. Data Selection and Some Descriptive Analysis. In order to control for the effect of product configuration on the analysis in Chapter 4 and 5, only six subsets are used. These are chosen by selecting the main categories of processor model, processor speed, memory, hard drive capacity, and auction length. More specifically, only the six product configurations listed in Table 1.6 were analyzed in Chapter 4 and 5. The products were chosen to limit the analysis to the cases with the most data. The aggregated 'products' D1 and L1 were selected by choosing the attributes listed in Table 1.4 and 1.5 with the most cases. The remaining products were chosen by looking at cross-tabulation counts from subsets D1 and L1. That is, D3 and D4 are subsets of D1, and L4 and L5 are subsets of L1.

Processor Model	IP4	1,422	IPM	1,327	IPIii	104	Celeron	28		
Processor Speed	0.866GHz	3	1.0GHz	34	1.5GHz	76	2.0GHz	708	2.5GHz	5
	0.9GHz	2	1.1GHz	2	1.6GHz	427	2.2GHz	42	2.6GHz	18
			1.2GHz	78	1.7GHz	203	2.3GHz	3	2.8GHz	8
			1.3GHz	58	1.8GHz	507	2.4GHz	75	2.9GHz	3
			1.4GHz	624	1.9GHz	3			3.0GHz	1
									3.2GHz	1
Memory	128MB	63	256MB	506	512MB	1,652	1024MB	563		
	224MB	10	320MB	14	768MB	18				
					1000MB	55				
Hard Drive	5GB	1	20GB	517	60GB	150				
	10GB	9	30GB	1,212	80GB	28				
	12GB	2	40GB	962						
Duration (days)	.73461d	1	1d	251						
	2.32205d	1	3d	2,626						
	2.95833d	2								

TABLE 1.5. Counts of product configurations and auction duration by category for the 2,881 Dell PC laptop auctions.

Category	Subset Name	Processor Model	Processor Speed (GHz)	Memory (MB)	Hard Drive (GB)	Duration (Days)	Count
Desktop	D1	IP 4	1.7-2.8	256, 512, 1024	20, 40	3	2,072
	D3	IP 4	2.4	256	40	3	274
	D4	IP 4	2.0	512	40	3	167
Laptop	L1	IP 4, IP M	1.4-2.0	256, 512, 1024	20, 30, 40	3	2,046
	L4	IP 4	1.8	512	30	3	172
	L5	IP M	1.4	512	40	3	163

TABLE 1.6. Product configuration for the subsets analyzed.

See Table 1.8 and 1.9 below. The last column in Table 1.6 includes only those listings with a 'Starting bid' of \$.99. For Laptop auctions there was one listing that started at \$501 and received one bid. For Desktop auctions there were four auctions that had a 'Starting bid' of \$227, \$304, \$374, and \$382 respectively, and where each received one bid. Note that these are not *Buy-It-Now* auctions. A notational comment is that throughout the thesis we refer to D1, D3, D4, L1, L4, and L5 as 'products', even though D1 and L1 span several product configurations.

Table 1.7 lists some descriptive statistics regarding the final price of each product. We note that the mean and median final price for product D3, is slightly below the mean and median for the aggregated subset D1, while for product D4 the reverse is true. This is most

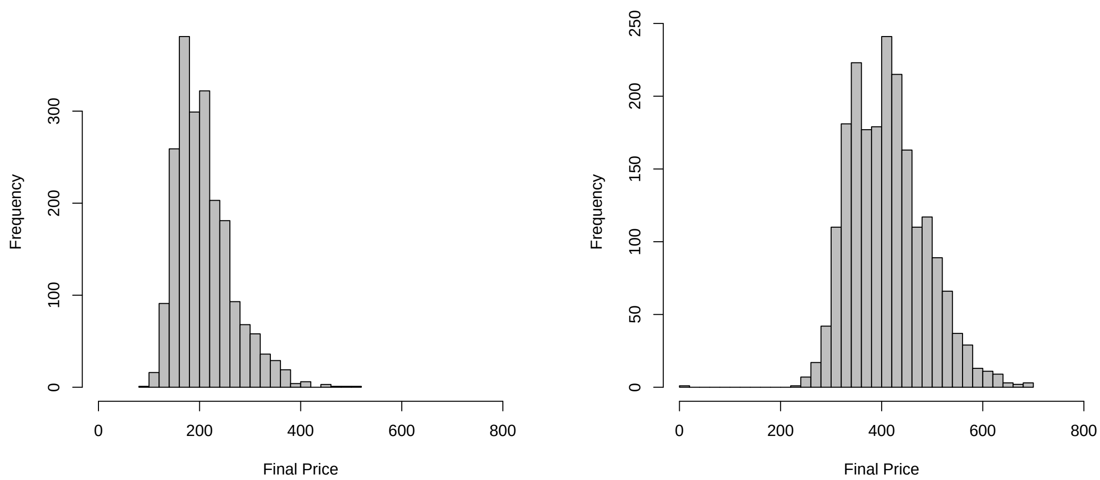
	D1	D3	D4	L1	L4	L5
Mean	208.20	193.65	230.59	412.20	427.98	409.59
(s.d.)	(55.74)	(36.90)	(36.78)	(74.33)	(68.45)	(73.63)
Median	200.00	190.71	230	405.00	424.44	405.00
Min	93.92	127.50	129.20	10.50	266.00	280.99
Max	501.00	338.50	348.26	700.00	613.00	596.99
Count	2,072	274	167	2,046	172	163

TABLE 1.7. Mean (st.dev), Median, Minimum, and Maximum Final Price for selected subsets.

likely because D4 has twice the memory (RAM) than D3 (though D3 has a faster processor). Similarly, we note that the mean and median for the product L5, is slightly less than for the aggregated subset L1, while L4 has values above L1. The only difference between L4 and L5, is with regard to the processor. L4 has a 1.8GHz Intel Pentium 4 processor, while L5 has an 1.4GHz Intel Pentium M. Another interesting observation is that the variance is less for the specific products D3, D4, L4, and L5, than the aggregated products D1 and L1. However, there is still considerable variation in final price for the four specific products. Figure 1.3 provides histogram of the final price for D1 and L1. Though both are skewed to the left, the distribution for L1 is more symmetrical than for D1. In the following sections the variation for the various product configurations are analyzed further.

A final comment, is that consecutive bids in a short time period by the same bidder were removed. Where ‘short time’ was defined as 10 minutes, meaning that if a bidder places another bid within 10 minutes of his previous bid, then the first bid is removed. In other words, only the last bid a bidder placed in one ‘session’ is considered. If a bidder waits more than 10 minutes to place another bid then this is defined as a second ‘session’ and two bids are recorded. The result was that 14,202 out of 100,747 bids were removed.

We conclude this chapter with a brief statistical analysis regarding the price variation for D1 and L1. Chapter 4 and 5 provides further analysis regarding the variation of final



Subset D1 - Desktop (N=2,072) Subset L1 - Laptop (N=2,046)

FIGURE 1.3. Distribution of the Final Price for subset D1 (Desktop) and subset L1 (Laptop).

price.

In each box-plot the lower and upper edge of the box represents the 25th and 75th percentiles. The line inside the box represents the median final price. The dashed lines, or 'whiskers', from each box, are drawn to the observation furthest away, but within a factor of $1.5 \times IQR$, from the edge of the box. The IQR is the inner quartile range. Circles outside the whiskers are observations that would be classified as extreme and potentially outliers. The notches inside the box indicates a range around the median. An informal statistical test if the median from two box-plots are different, is if the notches overlap.

Desktop - D1. For D1 the mean and standard deviation of the final price is \$208.20 and \$55.74, and the median, minimum, and maximum were \$200.00, \$93.92, and \$501.00 respectively. Table 1.8 displays the cross-tabulation counts for the selected categories. There is a clear cluster of products with a processor speed ranging from 1.8 to 2.4GHz, memory size of 256 or 512MB, and hard drive capacity of 20 or 40GB. Figure 1.4 shows the distributions of the final price by hard drive capacity, memory, and processor speed. There are two noteworthy observations. First, within each product specification there is considerable variation. For example, the median final price of D1, with hard drive capacity 40GB, is about \$200, with the 25^{th} and 75^{th} percentiles at about \$180 and \$250 respectively. Second, the median price is increasing in the product specification. This is most visible for hard drive capacity and memory, where the median price is clearly increasing in the respective specification. An informal test for a significant change in median price, is if the notches of two box-plots overlap. For memory, there clearly is no overlap in the notches, thus indicating a significant increase in median final price. For hard drive capacity the notches also do not overlap, though not by a great margin. For processor speed, overall the median final price seems to increase. However, the increase is not monotonic in the processor speed.

	Intel Pentium 4					
	256		512		1024	
	20	40	20	40	20	40
1.7	40	1	3	1	0	0
1.8	110	52	33	50	3	9
2.0	165	225	107	167	4	51
2.2	0	37	0	11	0	1
2.3	143	4	42	48	3	8
2.4	91	274	44	102	6	55
2.5	9	26	4	22	2	1
2.6	0	20	2	5	0	3
2.7	3	6	0	9	0	6
2.8	2	20	3	21	2	16

TABLE 1.8. Cross-Tabulation of selected Desktop product configurations.

Laptop - L1. For L1 the mean and standard deviation of the final price is \$412.20 and \$74.33, and the median, minimum, and maximum were \$405.00, \$10.50, and \$700.00 respectively. The minimum value of \$10.50 is the clear outlier as seen in both Figure 1.3 above and Figure 1.5 below. The outlier a ‘DELL WINXP Latitude C640 1.8 GHz 1024MB CDRW DVD’ listed between 25-Oct-06 21:29:47 and 28-Oct-06 21:29:47, and received three

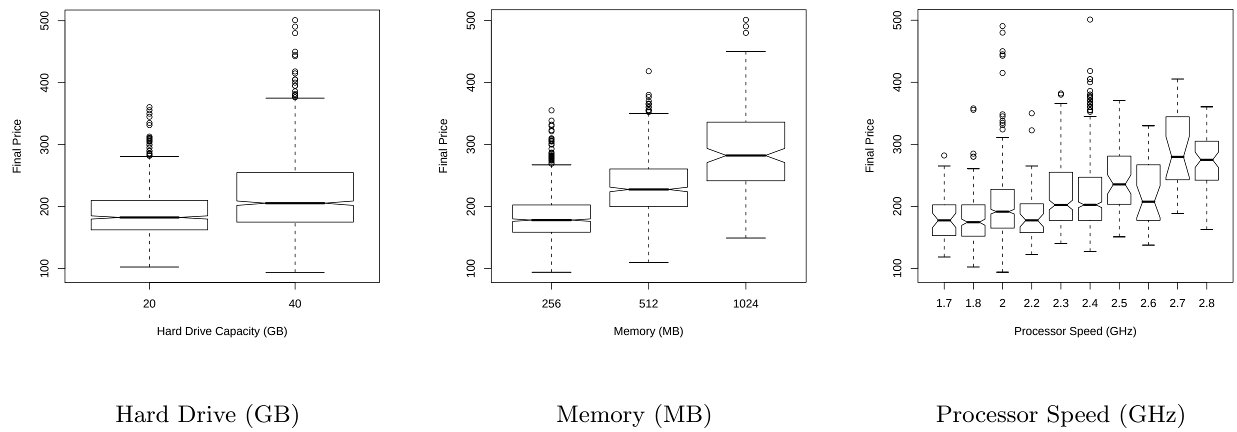


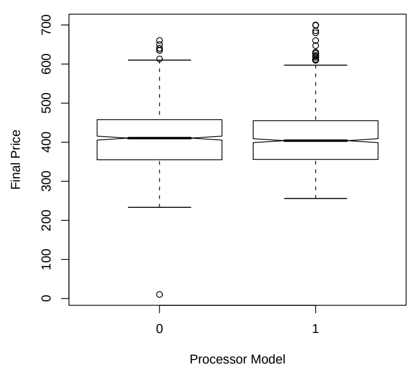
FIGURE 1.4. Distribution of the Final Price for D1 auctions by the three main categories.

bids from two bidders (eBay listing number: 160044957851). Removing the outlier only marginally changes the mean, median, and standard deviation, but drastically increases the minimum to \$233.50.

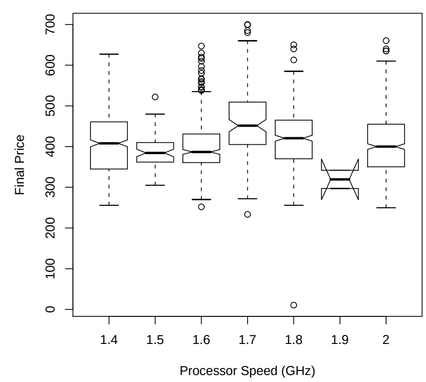
Table 1.9 displays the cross-tabulation counts for the selected categories. There are two clear clusters. One for Intel Pentium 4, 1.8-2.0GHz, 512MB, and 20, 30, or 40GB. And one for Intel Pentium M, 1.4-1.6GHz, 512MB, 20, 30, or 40GB. Figure 1.5 shows the distribution of the final price by processor model, processor speed, hard drive capacity, and memory. Similar to D1 we see that within each product specification there is considerable variation. For instance, the median final price of L1, with hard drive capacity of 40GB, is about \$400, with the 25^{th} and 75^{th} percentiles at about \$380 and \$450 respectively. However, unlike D1, the median final price does not increase as drastic in the product specification. In fact, for hard drive capacity and processor model, the median final price seems to be the same across the possible values. For memory, we see that auctions for L1 with 256MB, the median price is ‘significantly’ less than for L1 with 512 or 1024MB. However, between the two higher memory sizes, there does not seem to be any difference. For processor speed, the median final price, though fluctuating, is not monotonic. An interesting topic for further empirical

	Intel Pentium 4									Intel Pentium M								
	256			512			1024			256			512			1024		
	20	30	40	20	30	40	20	30	40	20	30	40	20	30	40	20	30	40
1.4	2	0	0	0	2	2	0	0	0	23	8	34	49	166	163	2	5	31
1.5	0	2	0	1	0	2	0	0	0	1	0	12	2	7	7	0	0	32
1.6	2	2	0	1	2	4	0	0	0	14	10	3	36	105	125	1	27	18
1.7	5	3	1	2	2	2	0	0	0	3	2	2	0	32	18	0	0	3
1.8	44	25	7	38	172	44	2	83	8	0	0	0	0	1	3	0	0	0
1.9	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2.0	62	52	27	40	169	106	9	93	33	0	0	0	0	0	0	0	0	0

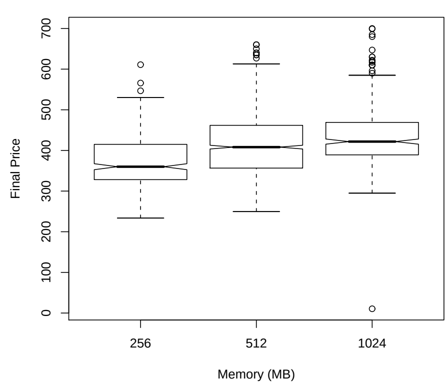
TABLE 1.9. Cross-Tabulation of selected Laptop product configurations.



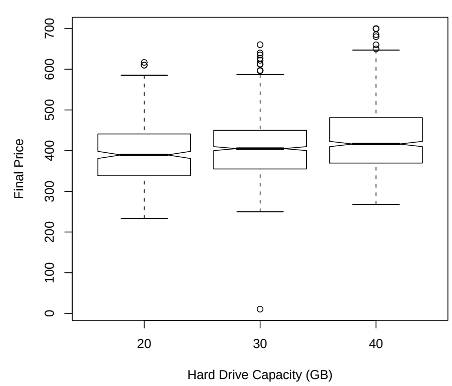
Processor Model (IP 4 - left, IP M - right)



Processor Speed (GHz)



Memory (MB)



Hard Drive (GB)

FIGURE 1.5. Distribution of the Final Price for L1 auctions by the four main specification categories.

research would be a hedonistic pricing analysis of the various product specifications. This will not be pursued in this thesis.

CHAPTER 2

Optimal Release of Inventory Using Online Auctions: The Two Item Case

1. Introduction

The objective of this chapter is to provide a model for selling a fixed inventory using a sequence of *single item* auctions, and to derive structural properties regarding the optimal release policy. More specifically, how should a seller, given N identical items, optimally release each individual item for auction in order to maximize total profit. We assume all auction parameters, such as auction duration, starting price, bid increment, etc., have been fixed and that the only decision to make is regarding the timing to release each item for auction. The problem is modeled as a discrete time *Markov Decision Problem* (MDP) with focus on sufficient conditions to ensure optimal monotone policies. The imposed trade-off to make the problem interesting is that, on the one hand, the seller incurs a holding cost for each period an item remains unsold, while on the other hand, the more ongoing auctions the seller has, the lower the expected final price in each of those auctions. In other words, we assume competing auctions 'cannibalize' on each other. The holding cost represents, in addition to the usual components, such as cost of capital, insurance, and space, the value depreciation of an item, and can therefore also be regarded as a depreciation factor. An illustration of the depreciation is provided in Figure 2.1, which depicts the final price for the D3 desktop (bottom circles) and L4 laptop (top solid circles) auctions. The vertical axis is the final price, and the solid and dashed line represents the least square linear regression for L4 and D3 respectively. We see that for the L4 laptops the average selling price decreased

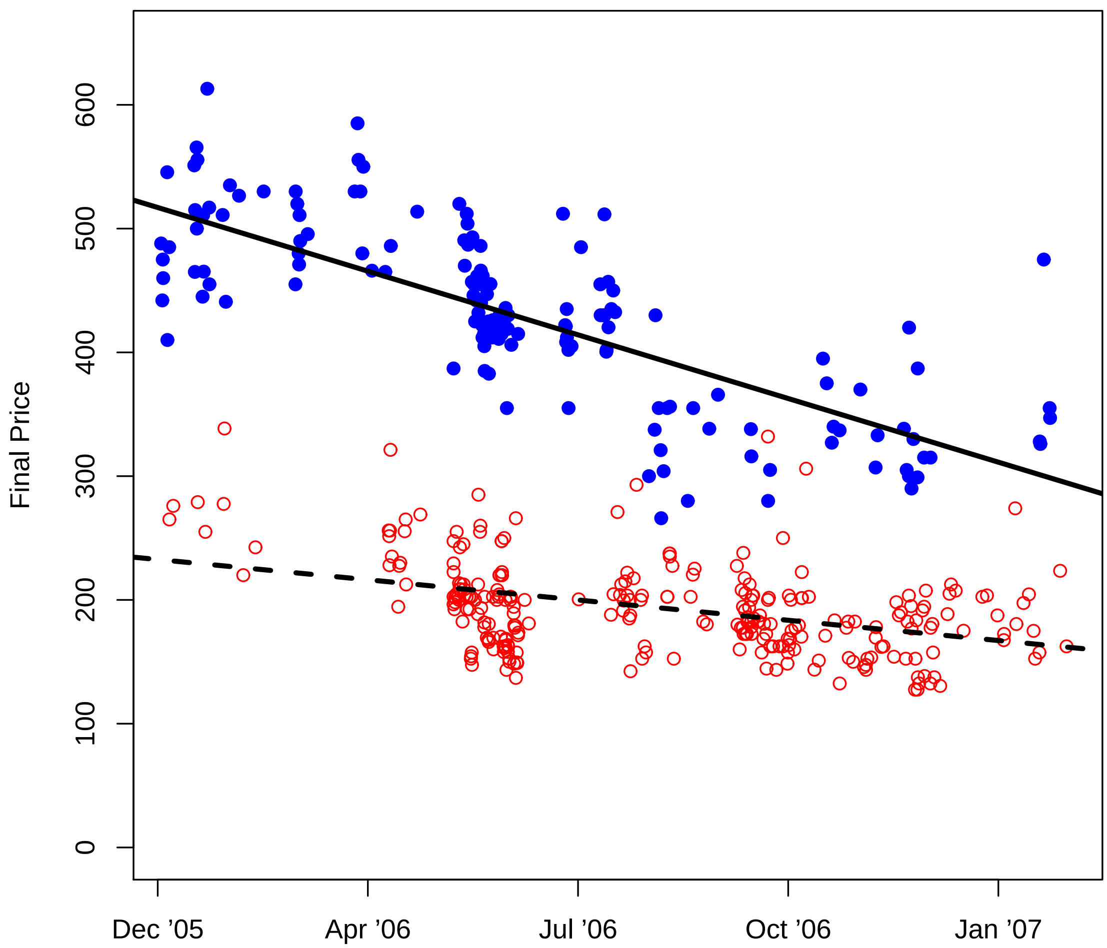


FIGURE 2.1. Final price as a function of the ending date, for the D3 (bottom circles) and L4 (top solid circles) auctions at the eBay site of *Dell_Financial_Services*. All auctions lasted for three days and took place between 15th of December, 2005, and 30th of June, 2006. The solid and dashed line represents the least square linear regression for L4 and D3 respectively.

more than \$200 over 2006. The decrease for D3 was not as drastic. More detail empirical analysis, including a validation of the cannibalization assumption, appears in Chapter 4.

The optimal release policy will mainly be driven by the effect from the holding cost and the cannibalization effect. If the holding cost is 'very low' then it will never be optimal to have more than one auction underway at any time. The reason for this is because the fewer the number of ongoing auctions the higher the expected final price for each of the auctions, due to the assumed cannibalization. That is, the optimal release policy is to wait until the current auction is completed before releasing the next item, i.e. to hold N non-overlapping *sequential* auctions. On the other hand, if the holding cost is 'very high' then it will never be optimal to delay the release of an item and instead all items should be released immediately. The reason for this is that the additional holding cost from deferring will exceed the gain in expected final price by having fewer ongoing auctions. The optimal release policy is to hold N *simultaneous* auctions (all overlapping and note that this is different from one N-item auction). Our main interest are situations where the holding cost has some strategic implication and the optimal policy is not one of the extreme policies. Furthermore, we will show that the optimal release policy is a state dependent or *closed loop* policy, in contrast to a state independent or *open loop* policy. More details regarding the definitions of open and closed loop policy will be given in Section 3.1. Note that the two extreme open loop policies are the sequential and simultaneous release policies, and that open loop policies are special cases of closed loop policies.

A numerical example illustrating the above discussion is provided in Figure 2.2. The figure depicts, for two items ($N = 2$) and auction length of three periods, the expected total profit (value) as a function of the per period per item holding cost. See Section 3.4 for details regarding formulation and computation of the example. The four dashed lines labeled A through D, represent the value for the four open loop policies; non-overlapping sequentially released (A), released with two day overlap (B), released with one day overlap (C), and simultaneously released (D). The solid line that lays above them represents the total expected profit for the optimal policy, a closed loop policy, which we describe how to compute below. We see that for 'low' holding cost the optimal policy is to release the auctions sequentially, while if the holding cost is 'high' the optimal policy is to release the

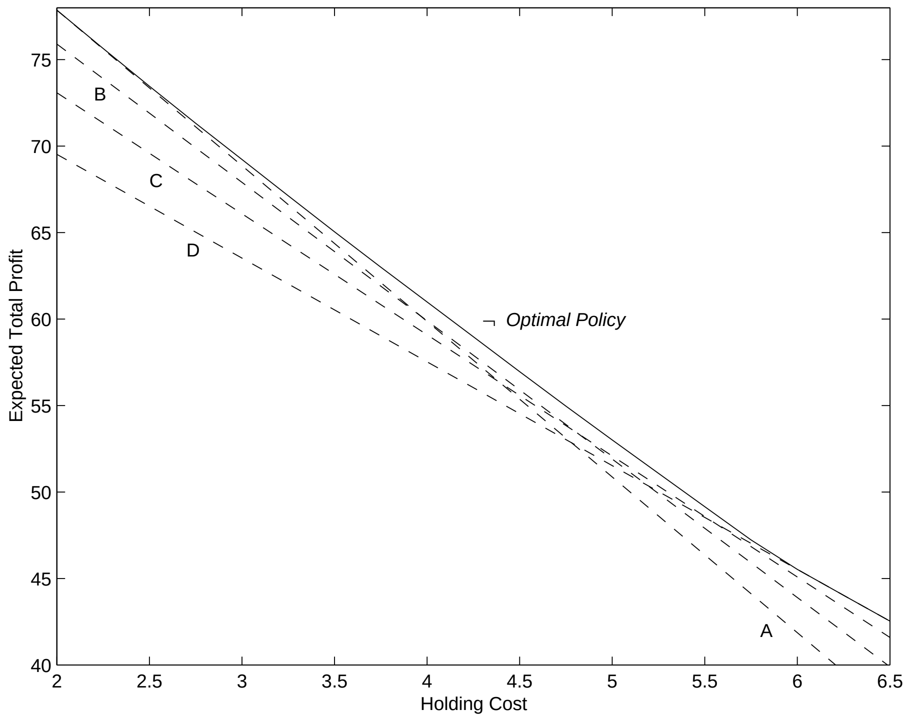


FIGURE 2.2. Two item, three period numerical example of expected total profit as a function of the per item per period holding cost. The four dashed lines represent the following open loop policies: (A) non-overlapping sequentially release, (B) release with two day overlap, (C) release with one day overlap, and (D) simultaneously release. The solid line that lays above them represents the total expected profit for the optimal policy, which is closed loop. See Section 3.4 for details regarding formulation and computation.

auctions simultaneously. For cases in between, the optimal closed loop policy is adaptive and depends on the current auction price.

Though it may appear to be an oversimplification, this chapter will only consider the case when $N = 2$. The reader will see that the two item case provides sufficient complexity to be both interesting and give rise to some surprising results. Furthermore, this will enable the discussion to focus on the governing trade-off, between releasing and deferring the release, and not become convoluted by the combinatorial complexity and curse of dimensionality of the N item case. It should also be noted that this problem has not yet been addressed in neither the existing auction theory or inventory literature. Previous research has mainly focused on the analysis of an isolated single auction, either *single-item* or *multi-item* auctions, and in the multiple auctions case only considered non-overlapping sequential auctions. In addition, most research has dealt with the (optimal) specification of various auction parameters, e.g. reserve price, bid increment, auction length, lot size, etc. The novelty of this chapter is that it provides a framework for analyzing the issue of strategic timing of auctions when auctions compete or cannibalize on each other. This problem falls into the third category of open research areas as outlined by Pinker, Seidmann and Vakrat (2003). Namely how could (or should) a firm integrate online auctions into their business model. Bajari and Hortacsu (2004) stated that more research needs to be done regarding "the analysis of markets with multiple simultaneous auctions." The ambition is that the ensuing discussion provides a structural framework, insights and results regarding this issue.

1.1. Literature Review. In recent years auction theory has come to play an important role in the management science and revenue management field, resulting in a wide spectrum of applications. However, given the voluminous literature on inventory management and dynamic pricing, relatively little has been written with regards to inventory management using online auctions. Two papers that consider the impact auctions have on the inventory re-ordering policy are Vulcano and van Ryzin (2004), and Huh and Janakiraman (2008). Vulcano and van Ryzin focus on how a seller should optimally choose the auction format and how this decision will affect the optimal inventory re-ordering policy. They formulate the problem as an infinite horizon dynamic program and show the optimal joint auction-format and replenishment policy. Huh and Janakiraman show that using auctions as a sales channel, conditions to ensure that (s, S) policies are optimal are satisfied. Vulcano, van Ryzin, and Maglaras (2002) have previously analyzed a problem that is similar to the one we address. There they consider a seller, who given a fixed inventory and fixed time-horizon, has to optimally 'auction' off the goods. The underlying 'auction' mechanism they consider is in the spirit of `www.priceline.com` where people place ‘bids’ and sellers can choose to accept or reject the offers. They model each *multi-item* ‘auction’ as a separate period and perform the symmetric equilibrium analysis for each period (auction). One of the main results is that the seller should not employ a *standard* auction format. An auction is standard if the highest bidder is guaranteed to be awarded the item [**14**, p.29]. Another related paper is by Pinker, Seidmann, and Vakrat (2001), who analyze the problem of disposing a given inventory using a sequence of non-overlapping *multi-item* online auctions. Based on the symmetric equilibrium analysis and uniform valuations, their objective is to categorize the optimal number of multi-item auctions and the optimal unit to release in each auction. In contrast to these papers, we permit the auctions to overlap and analyze the auction dynamics as a Markov chain.

The above papers all use a game theoretic approach. A paper which uses a different analysis methodology is Bertsimas, Hawkins and Perakis (2003). The problem they address is how a seller should optimally set the auction control parameters *starting price*, *reserve price* and *auction length*, in order to maximize revenue. They model the problem as a MDP and based on over 17,000 eBay auctions determine the optimal parameters. Bapna, Goes, and Gupta (2003) also address the issue of optimal auction control parameters in a revenue management context. The main focus of their analysis is to highlight the importance and structural implication of the *bid increment* in a *first-price multi-item* auction. Using data from 90 online auctions they empirically validate their findings. The two common elements of the above literature is that they focus on the optimal setting of auction parameters and analyze each auction in isolation. In contrast, we model the optimal release or timing of auctions given fixed auction parameters and a dynamic interaction between competing auctions.

A paper that analyzes the dynamics between competing auctions is Peters and Severinov (2006). They consider the case when all auctions are simultaneously released. They present a model with M bidders and N *single-item* auctions, where both M and N are fixed, and derive the Bayesian-Nash equilibrium for the final price of the N auctions. In particular they show that the final price will be the same for all auctions, namely one increment above the $M-N$ highest valuation. Meaning that, if there are 10 bidders and 5 items, the price in all 5 auctions will be one increment above the 6^{th} highest valuation. Though they are implicitly assuming an online setting, there is nothing explicit in their model that incorporates the special dynamics of online auctions, such as the arrival rate of bidders or fixed auction dead-line.

In contrast to the above papers, our framework is more in line with the model presented by Segev, Beam and Shantikumar (2001), where online auctions are modeled as Markov chains. The main focus of their paper is to characterize the distribution of the final price given a specific arrival rate and bidding strategy.

Overview of Chapter 2. The remainder of this chapter is organized as follows. In section 2 we formulate the problem and general model. In section 3 we discuss the case when the auctions are guaranteed to be successful, and hence the seller only has to list an item once. While in section 4 we discuss the case when there is a positive probability an auction receives zero bids and the seller has to re-list items from unsuccessful auctions. In section 5 we summarize our conclusions and provide ideas for future research.

2. Problem Formulation

We are considering a seller who, over a planning horizon T, intends to sell two identical items using a pair of single-item auctions. Each auction is assumed to have the same fixed and finite time-length τ. We divide τ into a sequence of discrete periods such that each *auction period* coincide with the length of the discrete *time period* that constitute T. The seller decides at the start of each period whether or not to release an item for auction. It is important to emphasize that an ongoing auction does not have to be completed before the next auction is started; auctions may overlap each other. We model the seller's problem as a discrete time *Markov Decision Problem* (MDP) with the objective of maximizing expected total profit. Two cases regarding the time-horizon will be considered. In the first case we assume the auctions are guaranteed to be successful and hence the seller only has to list an item once. Since the seller only has two items, the seller is faced with a finite planning horizon 2τ. In the second case, we assume there is a positive probability that an auction is unsuccessful, meaning that no bids arrived, and that the seller has to re-list unsold items. Consequently the seller is faced with an infinite planning horizon. The reason for separating the two cases is that they require different models and analysis. An important aspect to keep in mind is that we do not model the individual bidders or their bidding strategy. Each auction is modeled as a Markov chain, where the state of an auction evolves according to certain dynamics. Chapter 3 illustrates, with two fixed bidding strategies, how the Markov chain transition probabilities can be derived from the individual bidding behavior. In Chapter 4 we discuss an empirical model for how a seller can capture the Markov chain transition probabilities from real auction data.

We will throughout the chapter assume two fundamental aspects regarding the seller. The first is that the seller would only be interested in selling via auctions if the accumulated holding cost over the duration of an auction is compensated by the expected final price.

We summarize this in the following lemma and refer to it as the *positive expected profit* assumption.

Lemma *If the expected revenue from an auction does not exceed the holding cost accumulated over the auction duration, then it is optimal to immediately dispose of the items.*

The second assumption is that the seller is vigilant in keeping track of how many items he has released for auction and how many that are remaining, and that there will not be any reason to speculatively hold inventory. This is summarized as follows and referred to as the *vigilant seller* assumption.

Lemma *If the price dynamics of an auction are independent of time and the holding cost is positive, then it will always be optimal to have at least one auction underway while there still is remaining inventory.*

In other words, if there are no auctions underway but the seller still has items remaining he should always start at least one auction. Thus at the start of the planning horizon, he should always start at least one auction. This lemma is equivalent to Lemma 1 in Pinker, Seidmann and Vakrat (2001), where a proof is provided.

2.1. Markov Decision Problem Formulation. To formulate the seller's problem as an MDP, we require the following elements.

Decision Epochs, $t = 0, 1, \ldots, T$

We assume discrete time periods of equal length and that decisions are made at the beginning of each period. We are implicitly thinking of T as a fixed number of days and that decisions are made on a daily basis. However, for a general framework where, for instance, decisions are made more frequently, T could be increased to reflect the appropriate planning horizon. We will consider two cases: $T < \infty$ and $T = \infty$. The finite planning horizon case arises if the seller only lists an item once, while an infinite planning horizon formulation is required when there is a positive probability an item does not sell and the seller has to re-list it.

State Space

At each decision epoch t, the system state, $S = ([X_1, Y_1; X_2, Y_2], Z)$, consists of the state of each auction, $[X_i, Y_i]_{i=1,2}$, and the number of ongoing auctions Z. Each auction i, $i = 1, 2$, is defined by the pair of random variables *current price* (*bid*), X_i, and *elapsed auction time*, Y_i. We will consider both discrete and continuous prices. For the discrete case $X_i \in \{0, p, p+k, p+2k, \ldots, P\}$, where p, k and P are positive, finite integers. While for the continuous case $X_i \in \{0\} \cup [p, P]$, where $[p, P] \subset \Re_+$. In both cases, p is the starting price of the auction, P the upper limit of what any bidder would be willing to bid, and for discrete prices, k is the price-increment. We assume Y_i is discrete and finite, $Y_i \in \{0, 1, \ldots, \tau\} \cup \{\delta\}$, where $\tau < \infty$. The symbol δ is used to indicate that the auction is completed and the item awarded. We will interchangeably use the notation X_{i,Y_i} and (X_i, Y_i) to denote the state of auction i, $i = 1, 2$. For instance, $X_{i,\tau}$ is the final price of auction i. The notation X_Y is used to represent an auction that has elapsed for Y periods.

At the start of an auction $Y_i = 0$ and $X_i = 0$. For each additional period an auction is underway Y_i increases by one. When an auction has successfully been completed, that is $X_{i,\tau} \geq p$, the item is awarded and payment received. In this case, the state of the auction

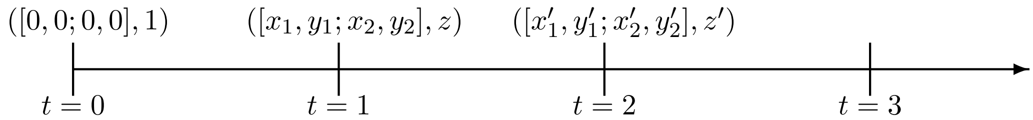


FIGURE 2.3. Time-line for two auctions. At $t = 0$ the first auction is started automatically, and the non-trivial decision is to decide whether to start auction 2. If auction 2 is started at $t = 0$, then at $t = 1$, $x_1, x_2 \geq 0$, $y_1 = y_2 = 1$, $z = 2$ (assuming $\tau > 1$), and there is no decision to make. If auction 2 is not started at $t = 0$, then at $t = 1$, $x_1 \geq 0$, $y_1 = 1$, $x_2 = 0$, $y_2 = 0$, $z = 1$, and the non-trivial decision is whether or not to start auction 2.

evolves as follows, for $p \leq x_i \leq x_i'$,

$$\ldots \Longrightarrow (x_i, \tau - 1) \Longrightarrow (x_i', \tau) \Longrightarrow \Delta_i \Longrightarrow \Delta_i \Longrightarrow \ldots$$

where $\Delta_i = (X_i, \delta)$. We let Δ denote the absorbing state when both items have been sold, $\Delta = ([\Delta_1; \Delta_2], 0)$. If an auction is unsuccessful, that is $X_{i,\tau} = 0$, the auction returns to the initial state $(0, 0)$, that is the transitions follow,

$$\ldots \Longrightarrow (0, \tau - 1) \Longrightarrow \begin{cases} (q, \tau) & \text{w. prob. } \Pr\{X_{i,\tau} = q \mid X_{i,\tau-1} = 0\} \\ \\ (0, \tau) \equiv (0, 0) & \text{w. prob. } \Pr\{X_{i,\tau} = 0 \mid X_{i,\tau-1} = 0\} \end{cases}$$

Though it may appear redundant we include a counter Z of the number of ongoing auctions. The number of ongoing auctions at time t will be defined by Z_t. In order to avoid issues with Z_t in decision epochs where an auction will be started by the vigilant seller assumption, we define Z_t to be the number of ongoing auctions in the *instantaneous moment* before decision epoch t, before any price jumps have occurred and before the seller has made a non-trivial or relevant decision. For instance, at the start of the planning horizon $Z_0 = 1$. See Figure 2.3 for an illustration of the time-line.

The reader familiar with auctions or auction theory, may notice that we have not included a *reserve price*. Section 3.6 discusses the implication of including a reserve price and shows that it imposes no change to the results. As a minor notational convention, we will avoid double parenthesis for functions where the state space is the only argument, that is we write $f([X_1, Y_1; X_2, Y_2], Z)$ instead of the strictly correct $f(([X_1, Y_1; X_2, Y_2], Z))$.

Actions

The only non-trivial decision facing the seller is to decide when to release an item provided that the current auction has not yet been successfully completed. In other words, non-trivial decision only pertain to states where $Y_i < \tau$ and $Y_j = 0$, $i \neq j$. Under all other conditions, the seller either does not have any decision to make or will release an item due to the vigilant seller assumption. At each decision epoch, the actions $a = 1$ corresponds to releasing the remaining item, and $a = 0$ not to release it. Furthermore, because the items are identical, one can without loss of generality, define the remaining item to be item 2. For the finite time-horizon this should be fairly obvious. However, for the infinite time-horizon, due to that when an auction is not successful and has to be re-listed, this may not be as obvious. We will revisit this issue in Section 4. Consequently the action space is, for $s = ([x_1, y_1; x_2, y_2], z)$,

$$A_s = \begin{cases} \{0,1\} & y_1 < \tau \text{ and } y_2 = 0 \\ \\ \{0\} & y_1 = \tau, \delta \text{ or } y_2 > 0 \end{cases}$$

In Figure 2.4 a simple example, with $X_i = 0, 1$ and $\tau = 2$, illustrates how the system state may evolve.States enclosed in a box indicate situations with non-trivial decisions. Transitions due to the non-trivial decision of releasing the second item are represented by the dashed lines. Transitions due to not releasing or releasing due to the vigilant seller assumption are represented by the solid lines. Note that there are four possible loops: $([0, 0; 0, 0], 1) \rightleftharpoons ([0, 1; 0, 1], 2)$, $([0, 0; 0, 0], 1) \rightleftharpoons ([0, 1; 0, 0], 1)$, $([0, 1; 0, 0], 1) \rightleftharpoons ([0, 1; 0, 0], 1)$, and $([1, \delta; 0, 0], 1) \rightleftharpoons ([1, \delta; 0, 1], 1)$. And that there is one absorbing state $\Delta = ([1, \delta_1; 1, \delta_2], 0)$, which is represented by the bold dashed line.

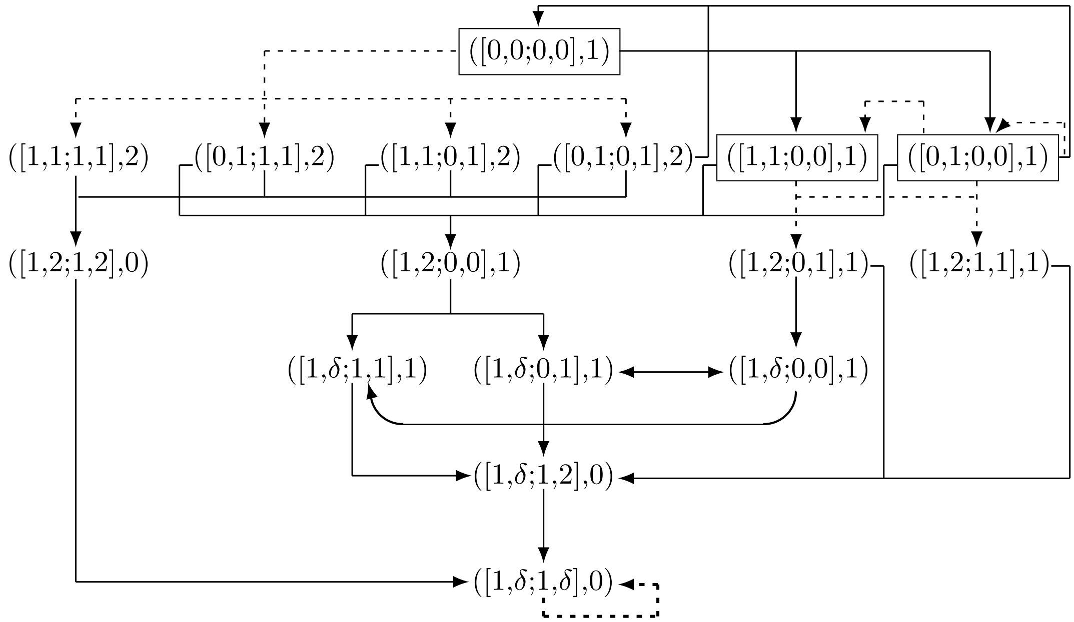


FIGURE 2.4. Example of system state transitions when $X_i = 0, 1$ and $\tau = 2$. States enclosed in a box indicate situations with non-trivial decisions. Solid lines represents transitions due to not releasing or release by vigilant seller assumption; dashed lines represents transitions due to non-trivial release decisions; bold dashed line represent absorbing cycle.

Rewards

For each period in which an item has not been sold, the seller incurs a positive holding cost h. When an auction is successfully completed the seller receives the payment and awards the item. After an item has been sold and the state $(X_i, Y_i) = \Delta_i$, $i = 1, 2$, the seller will in perpetuity neither incur any cost nor receive any payment for that item. Let $r_t(s)$ denote the reward in period t given a state $s \in S$. It is given by,

$$r_t([x_1, y_1; x_2, y_2], z) = x_1 \mathbf{1}_{\{y_1 = \tau\}} - h \mathbf{1}_{\{y_1 < \tau\}} + x_2 \mathbf{1}_{\{y_2 = \tau\}} - h \mathbf{1}_{\{y_2 < \tau\}}$$

Transition Probabilities

Each period in which an auction is underway the price transitions follow the dynamics of an *exogenously* given stochastic process. In other words, we assume that there is some underlying bidder arrival process and bidding behavior, which can be completely summarized by a probability distribution regarding the one period price transitions. For discrete prices, these are represented by the following transition probability matrices,

$$\Pi_1 = \begin{pmatrix} \pi_{0,0|1} & \pi_{0,p|1} & \cdots & \pi_{0,P|1} \\ 0 & \pi_{p,p|1} & \cdots & \pi_{p,P|1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi_{P,P|1} \end{pmatrix} \qquad \Pi_2 = \begin{pmatrix} \pi_{0,0|2} & \pi_{0,p|2} & \cdots & \pi_{0,P|2} \\ 0 & \pi_{p,p|2} & \cdots & \pi_{p,P|2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi_{P,P|2} \end{pmatrix}$$

Π_z, $z = 1, 2$, is the one-period transition probability matrix for an individual auction when there are z auctions underway, for $y < \tau$, $x \leq q$, and $z = 1, 2$,

$$\text{(2.1)} \qquad \Pr\{X_{i,y+1} = q \mid X_{i,y} = x, Z = z\} = \pi_{x,q|z} \qquad i = 1, 2$$

In the case of continuous prices, we assume the price transition dynamics can be represented by a conditional cumulative distribution function, for $y < \tau$, $x \leq x'$, and $z = 1, 2$,

$$\text{(2.2)} \qquad \Pr\{X_{i,y+1} \leq x' \mid X_{i,y} = x, Z = z\} = F^z_{X_{y+1}|X_y}(x'|x) = \int_x^{x'} f^z_{X_{y+1}|X_y}(q|x)dq$$

where $f^z_{X_{y+1}|X_y}(\cdot|x)$ is the one-period conditional transition probability density function for an auction which after y periods is at a price x, and there are z auctions underway.

Using the Chapman-Kolmogorov equations (cf. [**23**, Ch.4.2]), the n-period transition probabilities for a single auction can be derived. To illustrate, assume prices are discrete and we are interested in the two- and three-period transition probabilities, and that there are z_1, z_2, and z_3 auctions underway in the ensuing three periods respectively, for $y \leq \tau - 3$,

$$\Pr\{X_{y+2} = q \mid X_y = x, Z_t = z_1, Z_{t+1} = z_2\} = \sum_{j=x}^{q} \pi_{x,j|z_1}\pi_{j,q|z_2}$$

$$\Pr\{X_{y+3} = q \mid X_y = x, Z_t = z_1, Z_{t+1} = z_2, Z_{t+2} = z_3\} = \sum_{j=x}^{q}\sum_{k=j}^{q} \pi_{x,j|z_1}\pi_{j,k|z_2}\pi_{k,q|z_3}$$

Consequently, to derive the probability distribution of the final price we simply multiply the transition probability matrices accordingly. For instance, suppose $\tau = 3$ then the top row in $\Pi_{z_1}\Pi_{z_2}\Pi_{z_3} \equiv \Pi_{z_1 \cdot z_2 \cdot z_3}$ provides the unconditional probability distribution of the final price for an item with z_1, z_2, and z_3 auctions in the first, second, and third period respectively. For continuous prices and $y \leq \tau - 3$,

$$f^{z_1 \cdot z_2}_{X_{y+2}|X_y}(x'|x) = \int_x^{x'} f^{z_2}_{X_{y+2}|X_{y+1}}(x'|u) f^{z_1}_{X_{y+1}|X_y}(u|x) du$$

$$f^{z_1 \cdot z_2 \cdot z_3}_{X_{y+3}|X_y}(x'|x) = \int_x^{x'} \int_x^{v} f^{z_3}_{X_{y+3}|X_{y+2}}(x'|v) f^{z_2}_{X_{y+2}|X_{y+1}}(v|u) f^{z_1}_{X_{y+1}|X_y}(u|x) du dv$$

In order to simplify the notation we occasionally omit the subscript '$X_{y+1}|X_y$' and write $F^z(\cdot|\cdot)$ and $f^z(\cdot|\cdot)$, $i = 1, 2$, with the implicit assumption that $y < \tau$.

2.2. Assumptions Regarding the Transition Probabilities. We will next provide some additional assumptions regarding the transition probabilities. These assumptions, which can be seen as a reflection of the bidding behavior, will ensure that certain structural results will follow. The assumptions should not be regarded as categorical statements about all bidders, but rather as a statistical reflection of what the bidding behavior is like in the majority of auctions. In Chapter 4 a statistical model to derive the transition probabilities and validate the assumptions is provided. The validation is based on eBay auctions from *Dell_Financial_Services* (DFS) that ran between December 2005 to February 2007. More information regarding DFS and the data was discussed in Chapter 1.

When two auctions are underway we assume that the auction prices evolve independently. That is, the price in one auction does not affect the transition dynamics of the other auction. In other words, for discrete prices and $y_i < \tau$, $i = 1, 2$,

$$\begin{aligned}\Pr\{X_{i,y_i+1} = q \mid X_{1,y_1} = x_1, X_{2,y_2} = x_2, Z = 2\} &= \Pr\{X_{i,y_i+1} = q \mid X_{i,y_i} = x_i, Z = 2\} \\ &= \pi_{x_i,q|2}\end{aligned}$$

While for continuous prices and $y_i < \tau$, $i = 1, 2$,

$$\begin{aligned}\Pr\{X_{i,y_i+1} \le x' \mid X_{1,y_1} = x_1, X_{2,y_2} = x_2, Z = 2\} &= \Pr\{X_{i,y_i+1} \le x' \mid X_{i,y_i} = x_i, Z = 2\} \\ &= F^2(x'|x_i)\end{aligned}$$

Implicitly this assumes that bidders choose a bid-amount only based on the current price and elapsed auction time of the auction they are placing a bid in. Chapter 6 provides a discussion regarding extensions to correlated price-transitions, i.e. where the price-transitions also depend on the price of the competing auction. Consequently, with two auctions underway, the transition probability for the system state is the product of the individual transition probabilities. For discrete prices and $y_1, y_2 < \tau$,

$$\Pr\{S_{t+1} = ([q, y_1 + 1; r, y_2 + 1], z') \mid S_t = ([x_1, y_1; x_2, y_2], 2)\} = (\pi_{x_1,q|2})(\pi_{x_2,r|2})$$

Since the main interest pertains to the states $s \in S$ such that $A_s = \{0, 1\}$, we can explicitly state the *system state* transition probabilities. In the discrete case we define $\pi(s'|s, a)$ to be the one period system state transition probability, for $s = ([x_1, y_1; 0, 0], 1)$ and $y_1 < \tau$,

$$\pi(s'|s, a) = \begin{cases} \pi_{x_1,q|1} & a = 0 \\ (\pi_{x_1,q|2})(\pi_{0,r|2}) & a = 1 \end{cases}$$

where for $a = 0$, $s' = ([q, y_1 + 1; 0, 0], 1)$, and for $a = 1$, $s' = ([q, y_1 + 1; r, 1], z)$ with $z = 1, 2$. The extension to the continuous case is straight forward, though we need to define the notation a bit more carefully. We define $F(s'|s, a)$ to be the one period system state transition distribution function. That is, for $s = ([x_1, y_1; x_2, y_2], z)$, $s' = ([x'_1, y_1 + 1; x'_2, y_2 + 1], z')$,

$$F(s'|s, a) \equiv \Pr\{X_{1,y_1+1} \leq x'_1, X_{2,y_2+1} \leq x'_2, Z_{t+1} = z' \mid S_t = s\}$$

Similar to the discrete case we are mainly interested in states $s \in S$ such that $A_s = \{0, 1\}$, for $s = ([x_1, y_1; 0, 0], 1)$ and $y_1 < \tau$,

$$F(s'|s, a) = \begin{cases} F^1(q|x_1) & a = 0 \\ F^2(x'_1|x_1)F^2(x'_2|0) & a = 1 \end{cases}$$

where for $a = 0$, $s' = ([x'_1, y_1 + 1; 0, 0], 1)$, and for $a = 1$, $s' = ([x'_1, y_1 + 1; x'_2, 1], z)$ with $z = 1, 2$.

We assume bids are non-retractable, which for the case of discrete prices implies that Π_z, $z = 1, 2$, are upper-triangular ($\pi_{q,x_i|z} = 0$ for $q < x_i$). While for continuous prices, we require that $F^z(x'|x) = 0$, for $x' < x$, $z = 1, 2$. Consequently, the current price of an auction is increasing. Though strictly speaking on, for instance, eBay, bidders may retract a bid, it is very rare.

Transition probabilities are assumed to be *stationary* with respect to both: 1) calender time t, and 2) elapsed auction time Y_i, $i = 1, 2$. The former assumption, which was included in the vigilant seller assumption, is mainly for ease of notation and to ensure the model is tractable. In reality, the dynamics of X_i may depend on calender time. For instance, at night, weekends or certain weekdays there might be less bidding activity. We will for simplicity ignore this and strictly consider stationary transition probabilities with regard to calender time. Likewise we will ignore non-stationary transitions with regard to the elapsed auction time. A well-established phenomena of online auctions, is that the price dynamics or bidding behavior is dramatically different toward the end of auctions. One reason for this, is because some bidders try to place their bids as close as possible to the end of the auction, thereby leaving no time for others to counter-bid. This is referred to as *sniping.* Roth and Ockenfels (2002), and Shmueli, Russo, and Wolfgang (2004) analyze different aspects regarding non-stationary bidding activity. In Section 3.7 we will discuss how the change in auction dynamics over time can be incorporated. The next set of assumptions play a more crucial role in the ensuing analysis. Each is stated for discrete and continuous prices, and make use of Leibnitz Rule,

$$\frac{\partial}{\partial y}\int_{\alpha(y)}^{\beta(y)} f(x,y)dx = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f(x,y)}{\partial y}dx + f(\beta(y),y)\frac{\partial}{\partial y}\beta(y) - f(\alpha(y),y)\frac{\partial}{\partial y}\alpha(y)$$

Assumption 2.1. *The probability of making a jump to the higher prices is increasing in the current price.*

Discrete prices: for $x < P$, $z = 1, 2$,

$$\sum_{q=r}^{P} \pi_{x,q|z} \le \sum_{q=r}^{P} \pi_{x+1,q|z} \qquad \forall r \le P \tag{2.3}$$

Continuous prices: for $y < \tau$, $x \le x' \le P$, *and* $z = 1, 2$,

$$F^z(x'|x) = \int_x^{x'} f^z(u|x)du \qquad \text{is decreasing in } x \tag{2.4}$$

Equivalently,

$$\frac{\partial}{\partial x}F^z(x'|x) = \int_x^{x'} \frac{\partial}{\partial x} f^z(u|x)du - f^z(x|x) \le 0$$

Assumption 2.1 reflects that bids are increasing in the current price. In other words, the likelihood of placing a 'high' bid is increasing in the current price. This holds for example if bid increments were independent of the current price. In reality, however, bid increments tend to be decreasing in the current price, and it is therefore not immediate that Assumption 2.1 holds. Empirical evidence supporting Assumption 2.1 and showing that bid increments are decreasing in the current price, can be seen in Figure 2.5 below. They depict the price-jumps at 12 hour intervals for the L4 auctions. Each dot represents an individual auction. Auctions along the 45 degree line are auctions in which the price remained unchanged 12 hours later (no price-transition took place). Note that in the final period all auctions had strictly positive price-increments. The feature supporting our assumption is that in all figures the price-jumps form an upward sloping 'band'. A counter indication to our claim would be if there was a large number of auctions that at low prices ($\approx \$0 - 150$) made jumps to the high prices ($\approx \$500 - 600$). More details are provided in Chapter 4

Assumption 2.2. *The probability of making a jump to higher prices decreases when there are two ongoing auctions.*
Discrete prices: for $x \leq P$,

$$\sum_{q=r}^{P} \pi_{x,q|2} \leq \sum_{q=r}^{P} \pi_{x,q|1} \qquad \forall r \leq P \tag{2.5}$$

Continuous prices: for $y < \tau$ *and* $x \leq P$,

$$F^1(x'|x) \leq F^2(x'|x) \qquad \forall x' \leq P \tag{2.6}$$

This assumption formalizes how we model the cannibalization effect. In other words, with two ongoing auctions, each auction will experience more 'modest' price-transitions. For empirical support see Chapter 4.

An alternative to Assumption 2.2 is, for discrete prices, if $\sum_{q=r}^{P} \pi_{x,q|1} \leq \sum_{q=r}^{P} \pi_{x,q|2}$, for all $x, r \leq P$. That is, with two auctions you are more likely to see higher price jumps in each individual auction than when only one auction is underway. However, then the problem of releasing the second item becomes trivial. Since if it is better to have two auctions underway and the holding cost is positive, then it will always be optimal to release the second item

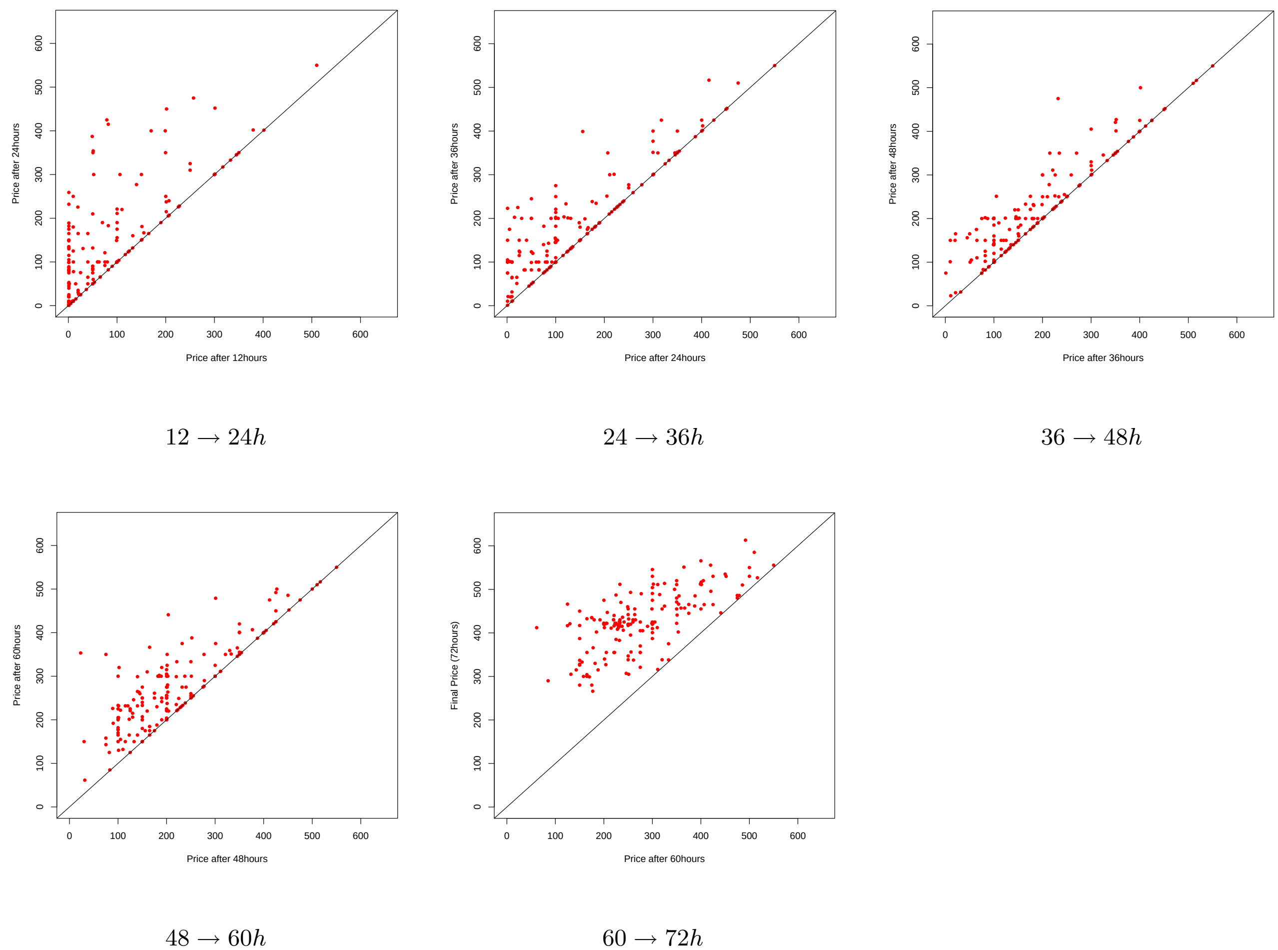


FIGURE 2.5. Price transitions at 12 hour intervals for the L4 laptop auctions. The horizontal axis represents the price at various 12 hour intervals, while the vertical axis represents the price 12 hours later. Each circle represents the price-transition for an auction. Observations on the 45 degree line represents auctions that received no bids for that period. Note that in the final period all auctions received bids.

immediately.

ASSUMPTION 2.3. *The difference, in probability of making jumps to the higher prices, between having one versus two ongoing auctions, is decreasing in the current price. Discrete prices: for $x < P$,*

$$\sum_{q=r}^{P}(\pi_{x,q|1} - \pi_{x,q|2}) \geq \sum_{q=r}^{P}(\pi_{x+1,q|1} - \pi_{x+1,q|2}) \qquad \forall r \leq P \tag{2.7}$$

Continuous prices: for $y < \tau$ and $x \leq P$,

$$F^2(x'|x) - F^1(x'|x) \qquad \textit{is decreasing in } x \tag{2.8}$$

Equivalently,

$$\frac{\partial}{\partial x} F^2(x'|x) \leq \frac{\partial}{\partial x} F^1(x'|x)$$

This states that the cannibalization effect is diminishing in the current price. In other words the closer the current price is to the upper bound P the less of a difference there will be between having one or two auctions underway. Qualitatively, we see in the graphs of Figure 2.5, that the closer the price is to $P \approx 600$ the less 'room' there is for the price-transitions, and hence the less cannibalization there can be. Again, a more rigorous empirical analysis is provided in Chapter 4.

2.3. Examples. At this point it may be natural to inquire about the existence of transition probability matrices and conditional cumulative distribution functions, that satisfy the above assumptions. We next provide conditions under which of some common probability distributions satisfy them. Namely, Uniform - discrete and continuous, Bernoulli, and Exponential. In addition, we later illustrate the assumptions and implications with numerical examples.

Discrete Uniform Distribution

Without loss of generality let $p = k = 1$. Suppose that in periods when there is only one auction underway there is an equal probability of jumping to any of the remaining prices, for $p_i \leq P$, $\pi_{p_i,q|1} = \pi_{p_i} = 1/(P + 1 - p_i)$ for all $q \in [p_i, P]$. Furthermore, suppose when two auctions are underway the probability of remaining at the same price increase with κ and that the probability of jumping to P decrease with κ, as shown in the transition probability matrices below.

$$
\Pi_1^{\mathrm{U}} = \begin{pmatrix} \frac{1}{P+1} & \frac{1}{P+1} & \cdots & \frac{1}{P+1} & \frac{1}{P+1} \\ 0 & \frac{1}{P} & \cdots & \frac{1}{P} & \frac{1}{P} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \Pi_2^{\mathrm{U}} = \begin{pmatrix} \frac{1}{P+1}+\kappa & \frac{1}{P+1} & \cdots & \frac{1}{P+1} & \frac{1}{P+1}-\kappa \\ 0 & \frac{1}{P}+\kappa & \cdots & \frac{1}{P} & \frac{1}{P}-\kappa \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}+\kappa & \frac{1}{2}-\kappa \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
$$

The next results summarizes that the above transition probability matrices support Assumptions 2.1, 2.2, and 2.3.

PROPOSITION 2.4. *If $0 \leq \kappa \leq \frac{1}{P+1}$ then Π_1^U and Π_2^U satisfies Assumptions 2.1, 2.2 and 2.3.*

Proof Proposition 2.4 - See Appendix A.

A modification to Π_2^{U} is to have κ be dependent on the price. In which case for Assumption 2.3 to hold we require $\frac{1}{P+1} \geq \kappa_0 \geq \kappa_1 \geq \ldots \geq \kappa_{P-1}$.

Continuous Uniform Distribution

An example with continuous prices and uniform distributed price-transitions, can be constructed as follows. Assume the starting price $p = 0$, and the maximum price $P = 1$, i.e. $X_i \in [0, 1]$. Assume that when there is only one ongoing auction that the price-transition is uniformly distributed between the current price and the upper limit 1. Furthermore, assume the 'cannibalization' effect is such that with two ongoing auctions, the price-transition is triangularly distributed between the current price and the upper limit. Specifically, let,

$$f^1(q|x) = \begin{cases} \frac{1}{1-x} & x \le q \le 1 \\ 0 & \text{o/w} \end{cases} \tag{2.9}$$

$$f^2(q|x) = \begin{cases} \frac{2-2q}{(1-x)^2} & x \le q \le 1 \\ 0 & \text{o/w} \end{cases} \tag{2.10}$$

PROPOSITION 2.5. *If prices are continuous and price transitions are distributed according to (2.9) and (2.10) then Assumptions 2.1, 2.2, and 2.3 holds.*

Proof Proposition 2.5 - See Appendix A.

Bernoulli Distribution

Suppose that for each period and every price level there are only two possible transitions - remain at same price or jump up by one increment. This bidding process is the core of the auction dynamics analyzed by Segev, Beam, and Shantikumar (2001). In this scenario the maximum price $P \equiv \tau$, and consequently the size of the transition probability matrices are $(\tau+1) \times (\tau+1)$. Let Π_1 and Π_2 be defined as follows,

$$\Pi_1^{\mathrm{Be}} = \begin{pmatrix} 1-\pi_0 & \pi_0 & \cdots & 0 & 0 \\ 0 & 1-\pi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-\pi_{\tau-1} & \pi_{\tau-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \Pi_2^{\mathrm{Be}} = \begin{pmatrix} 1-\rho_0 & \rho_0 & \cdots & 0 & 0 \\ 0 & 1-\rho_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-\rho_{\tau-1} & \rho_{\tau-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

In other words, $\Pr\{X_{y+1} = q+1 | X_y = q, Z = 1\} = \pi_q$, and $\Pr\{X_{y+1} = q+1 | X_y = q, Z = 2\} = \rho_q$. Due to the special structure of Bernoulli price transitions an adjustment to Assumption 2.3 is required: we assume, for $x < P-1$, $(\pi_{x,x+1|1} - \pi_{x,x+1|2}) \geq (\pi_{x+1,x+2|1} - \pi_{x+1,x+2|2})$. Without this modification Assumption 2.3 would not hold, since $\sum_{q=x+2}^{P}(\pi_{x,q|1} - \pi_{x,q|2}) = 0 \leq \sum_{q=x+2}^{P}(\pi_{x+1,q|1} - \pi_{x+1,q|2})$, while $\sum_{q=x+1}^{P}(\pi_{x,q|1} - \pi_{x,q|2}) \geq \sum_{q=x+1}^{P}(\pi_{x+1,q|1} - \pi_{x+1,q|2}) = 0$. The adjustment to the condition will not alter any of the structural properties for the Bernoulli distributed price transition. In order for Π_1^{Be} and Π_2^{Be} to satisfy Assumptions 2.1, 2.2, and 2.3(mod.), we require,

$$\pi_0 - \rho_0 \geq \pi_1 - \rho_1 \geq \ldots \geq \pi_{\tau-1} - \rho_{\tau-1} \quad \geq \quad 0 \tag{2.11}$$

Inequalities (2.11) reflects the diminishing 'cannibalization' effect, and ensures that Assumptions 2.2 and 2.3(mod.) holds. We summarize the result in the following proposition.

Proposition 2.6. *If price transitions are distributed according to Π_1^{Be} and Π_2^{Be}, and (2.11) holds, then Assumptions 2.1, 2.2, and 2.3(mod.) are satisfied.*

Proof Proposition 2.6 - See Appendix A.

Below a numerical example for the Bernoulli distributed price transitions is provided. It can be verified that (2.11), and hence that Assumptions 2.1, 2.2 and 2.3(mod.) holds.

$$\Pi_1^{\mathrm{Be}} = \begin{pmatrix} .4 & .6 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & 0 & .7 & .3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Pi_2^{\mathrm{Be}} = \begin{pmatrix} .5 & .5 & 0 & 0 \\ 0 & .6 & .4 & 0 \\ 0 & 0 & .8 & .2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A special case of Π_z^{Be}, $z = 1, 2$, is when the transition probabilities are independent of the current price, that is, when $\pi_q = \pi$ and $\rho_q = \rho$ for all $q = 0, 1, \ldots, \tau-1$. This special case has some interesting consequences which are discussed in Section 3.3.1.

Exponential Distribution

The following example will focus on the *price-increment* and not on the price-transitions. Assume prices are positive and unbounded, $X_i \in \Re_+$, and that the conditional within period price-increment C, given $X_i = x$, is exponentially distributed with rate $\lambda^z(x)$, $z = 1, 2$. That is, for $c = x' - x$, $x, x' \in \Re_+$, $\Pr\{X_{i,y+1} \leq x' | X_{i,y} = x, Z = z\} = \Pr\{C \leq c | X_{i,y} = x, Z = z\} =$

$$G_C^z(c|x) = \begin{cases} 1 - \exp(-\lambda^z(x)c) & c \geq 0 \\ 0 & o/w \end{cases} \tag{2.12}$$

The rate $\lambda^z(\cdot)$ is a function both of the current price and the number of ongoing auctions. The expected price increment is $1/\lambda^z(x)$, which it would seem natural to assume, is decreasing in the current price. Therefore, we require $\lambda^z(x)$ to be increasing in x. In other words, the higher the current price the smaller the expected price-increment. Though technical conditions on $\lambda^z(x)$ could be imposed, such that the three assumptions hold, they would make the problem both less intuitive and less informative. The main problem is due to the shape of the exponential distribution which, for instance, prevents Assumption 2.1 to hold. Therefore, we impose conditions to ensure that the expected price-transition has certain properties. Specifically, we assume, for $x \in \Re_+$,

$$1/\lambda^z(x) \quad \text{is decreasing in } x, z = 1, 2 \tag{2.13}$$

$$x + 1/\lambda^z(x) \quad \text{is increasing in } x, z = 1, 2 \tag{2.14}$$

$$1/\lambda^2(x) \leq 1/\lambda^1(x) \tag{2.15}$$

$$1/\lambda^1(x) - 1/\lambda^2(x) \quad \text{is decreasing in } x \tag{2.16}$$

With these conditions the ensuing structural results in Section 3.3 holds. An example of a rate function $\lambda^z(x)$ for which the above conditions hold includes,

$$1/\lambda^z(x) = \exp\left(\beta_0 - \beta_1 x - \beta_2 \mathbf{1}_{\{z=2\}}\right)$$

where $\beta_0, \beta_1, \beta_2 \geq 0$, and β_1 is such that,

$$\beta_1 \exp\left(\beta_0 - \beta_1 x - \beta_2 \mathbf{1}_{\{z=2\}}\right) \leq 1 \qquad \forall x, z$$

In Chapter 4 a more general version with gamma distributed price-increments is discussed and fitted to the eBay auction data from *DFS*.

3. Guaranteed Successful Auctions - Single Listing

The first case we consider is when the auctions are guaranteed to be successful, and hence the seller only has to list an item once. This could occur when the items are such that it is certain a positive bid will arrive (e.g. $\pi_{0,0|z} = 0,\ z = 1,2$), or when the seller decides in advance to immediately salvage items remaining from unsuccessful auctions. An illustration of the former includes the 6,000 laptop and desktop eBay auctions of DFS. Out of all auctions with a starting price of \$.99, not a single auction was unsuccessful. Due to the additional assumption that auctions are guaranteed to be successful, we can simplify the MDP model.

Decision Epochs As a consequence of the vigilant seller assumption there is no reason to consider a planning horizon beyond two sequential auctions, hence $T = 2\tau$. Furthermore, provided the second item has not been released, non-trivial decisions can only be made in periods $t = 0, 1, 2, \ldots, \tau - 1$. At $t = \tau$ the vigilant seller assumption requires that the second item is released immediately, if it has not already been released.

State Space Since we assume the items will at least sell for p, we omit the 0 state. Thus for the discrete case $X_i \in \{p, p+1, \ldots, P\}$ while for the continuous case $X_i \in [p, P]$.

Rewards In order to facilitate the 'accounting', and since we are not assuming discounting, we assume the seller receives the payment at $t = T$. Therefore, the reward $r_t(s)$ for a given $s \in S$ and period t is as follows,

$$r_t([x_1, y_1; x_2, y_2], z) = \begin{cases} -h\mathbf{1}_{\{y_1<\tau\}} - h\mathbf{1}_{\{y_2<\tau\}} & t = 0, 1, \ldots, T-1 \\ \\ x_1 + x_2 & t = T \end{cases}$$

Transition Probabilities

Since we assume $X_i \geq p > 0$ and the items are guaranteed to be awarded, we define the transition probability to start at p instead of 0, e.g. the entries in the top row of Π_z is $\pi_{p,q|z}$ for $q \in \{p, p+1, \ldots, P\}$, $z = 1, 2$. It should, however, be noted that from a behavioral point of view the bidding process may be different if the starting price or even current price is 0 rather than p. For instance, suppose we have two auctions, a and b, which both have elapsed for y_i periods and both with current price of \$$p$, but where auction b started at \$$p$ and still has not received any bids while auction a has reached \$$p$ after some bid activity. Then there is anecdotal 'evidence' to suggest that the bidding dynamics for the two auctions will be different. Auction a is more likely to receive more bids. To read accounts from eBay sellers on this issue, search the terms 'low starting price' at the eBay discussion boards for sellers,[1] and newcomers.[2] We will ignore such behavioral considerations.

3.1. Auction Release Policies. A Markov deterministic policy is a sequence of decision rules which determine what action to take in each decision epoch, possibly contingent on the state of the system but not on the past. Let $\gamma_t(s)$ be the decision rule in period t

[1] `http://forums.ebay.com/db2/forum.jspa?forumID=143`
[2] `http://forums.ebay.com/db1/forum.jspa?forumID=120`

given a state $s \in S$. As a consequence of the vigilant seller assumption, we only need to consider decision rules for $t = 0, 1, \ldots, \tau$, and hence, a policy γ is defined as follows,

$$\gamma = (\gamma_0(s), \gamma_1(s), \ldots, \gamma_\tau(s)) \qquad \gamma_t(s) \in \{0, 1\}, \forall s \in S, t = 0, 1, \ldots, \tau$$

If all the decision rules, $\gamma_t(s)$, are independent of the price components of state s we refer to the policy γ as an *open loop* policy, while if the decision rules depend on both the price and time components of state s the resulting policy is referred to as a *closed loop* policy. Note that there are only $\tau + 1$ open loop policies of interest. We write $V_{O(j)}$ to denote the total expected profit of releasing the second item j periods after the first, $j = 0, 1, 2, \ldots, \tau$. In Table 2.1 the four open loop policies and their respective total expected profit for discrete prices and $\tau = 3$ are provided. In the table we see that although we incur an additional unit of h for each additional period we hold the second item, the expected final price for both items increase since there is an additional period when both auctions evolve according to Π_1 instead of Π_2. The decision whether to release the second item or hold it one more period will depend on whether the increase in expected final price for both items will compensate the additional holding cost.

j	γ	Total expected profit - $V_{O(j)}$
0	(1,0,0,0)	$-6h + 2\sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|2})(\pi_{r,l\|2})$
1	(0,1,0,0)	$-7h + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|2})(\pi_{r,l\|2}) + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|2})(\pi_{r,l\|1})$
2	(0,0,1,0)	$-8h + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|1})(\pi_{r,l\|2}) + \sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|2})(\pi_{q,r\|1})(\pi_{r,l\|1})$
3	(0,0,0,1)	$-9h + 2\sum_{l=p}^{P}\sum_{q=p}^{l}\sum_{r=q}^{l} l(\pi_{p,q\|1})(\pi_{q,r\|1})(\pi_{r,l\|1})$

TABLE 2.1. Total expected profit for the four open loop policies of releasing item 2 j periods after item 1, for discrete prices and $\tau = 3$.

3.2. Optimality Equations. Before we discuss the optimality equations we introduce some notation. When *both items* have been released we define $E[X_{i,\tau}|S_t]$ to be the conditional expected final price of auction i, $i = 1, 2$, in period t,

Discrete prices:

(2.17)

$$E[X_{i,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)] = \sum_{q=x_i}^{P} q \Pr\{X_{i,\tau} = q | X_{i,y_i} = x_i, Z_t = z, Z_{t+1} = z', \ldots, Z_{t+(\tau-y_i)} = z''\}$$

Continuous prices:

$$E[X_{i,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)] = \int_{x_i}^{P} q f_{X_\tau|X_{y_i}}^{z \cdot z' \cdots z''}(q|x_i) dq \tag{2.18}$$

Where $\Pr\{X_{i,\tau} = q | X_{i,y_i} = x_i, Z_t = z, Z_{t+1} = z', \ldots, Z_{t+(\tau-y_i)} = z''\}$ and $f_{X_\tau|X_{y_i}}^{z \cdot z' \cdots z''}(q|x_i)$ are derived using the Chapman-Kolmogorov equations discussed in Section 2.1. The main issue regarding the expected final price is that it only depends on the current price of an auction, and how many auctions will be underway for the duration of the auction. Once both auctions have been released we know how many auctions there will be for the remainder of each individual auction. As a consequence of the assumptions that auctions progress independently and Assumption 2.1, $E[X_{i,\tau}|S_t]$ is increasing in x_i and independent of x_j, for $i \neq j$. We summarize this in the following result.

Corollary 2.7. *If auctions progress independently of price in other auctions and Assumption 2.1 holds, then the conditional expected final price, $E[X_{i,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)]$, is increasing in x_i and independent of x_j, $i = 1, 2$, $i \neq j$.*

Proof Corollary 2.7 - The result regarding independence of the price in the other auction is immediate by the assumption that price transitions do not depend on the price in the other auction. Proof by induction on the number of remaining periods $n = \tau - y_i$. Without loss of generality, consider auction 1. For $n = 0$, $E[X_{1,\tau}|S_t = ([x_1, \tau; x_2, y_2], z)] = x_1$, which is increasing in x_1. Assume the result holds for $n = 0, 1, \ldots, l-1$, i.e. for $y_1 =$

$\tau, \tau-1, \ldots, \tau-(l-1)$. Let $n = l$ then $y = \tau - l$,

$$E[X_{1,\tau}|S_t = ([x_1, \tau - l; x_2, y_2], z)] = \sum_{q=x_1}^{P} E[X_{1,\tau}|S_{t+1} = ([q, \tau-(l-1); x'_2, y'_2], z')]\pi_{x_1,q|z}$$

$$\leq \sum_{q=x_1+1}^{P} E[X_{1,\tau}|S_{t+1} = ([q, \tau-(l-1); x'_2, y'_2], z')]\pi_{x_1+1,q|z} = E[X_{1,\tau}|S_t = ([x_1+1, \tau-l; x_2, y_2], z)]$$

where inequality holds due to Lemma 4.7.2 in Puterman (1994), the induction assumption and Assumption 2.1. The proof for continuous prices is basically the same but with the summation replaced by an integration. Although Lemma 4.7.2 in Puterman (1994) is with respect to discrete variables and infinite sequences, it can be adapted to continuous variables and/or finite sequences. Alternatively the results from Lemma 9.1.1 and Proposition 9.1.2 in Ross (1996) can be applied. $\square$

Furthermore, when both items have been released, we define $R(S_t)$ to represent the total expected profit over the remainder of the planning horizon in period t,

(2.19)

$$R(S_t = ([x_1, y_1; x_2, y_2], z))$$

$$= -h(2\tau - y_1 - y_2) + E[X_{1,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)] + E[X_{2,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)]$$

There is a slight misuse of notation when $y_i = \delta$, $i = 1, 2$. In this case we implicitly define $\tau - \delta = 0$, since no holding cost will be incurred. Note that $R(S_t)$ is not necessarily increasing or decreasing in the elapsed time of the auctions. Though the incurred holding cost will decrease, the expected final price of the auctions will also decrease. It is this trade-off that is the crux of the problem regarding when to start the second auction. However, as summarized in the next result, $R(S_t)$ is increasing in x_1 and x_2.

Corollary 2.8. *If Assumption 2.1 holds then $R(S_t)$ is increasing in x_1 and x_2, for all y_1, y_2, and $z = 0, 1, 2$.*

Proof of Corollary 2.8 - Each auction progress independently of the price in the other auction, the result is therefore immediate by Corollary 2.7. $\square$

Lastly, we define $g_2(S_t)$ to be the gain in the expected final price of auction 2 by having delayed the release of item 2 for one period,

$$g_2(S_t = ([x_1, y_1; x_2, y_2], z)) \tag{2.20}$$

$$= E[X_{2,\tau}|S_{t+1} = ([x_1', y_1 + 1; x_2, y_2], z')] - E[X_{2,\tau}|S_t = ([x_1, y_1; x_2, y_2], z)]$$

where $z, z' = 0, 1, 2$ and by definition if $y_1 = \tau, \delta$ then $y_1 + 1 = \delta$. Due to Assumption 2.2 and that auctions progress independently, $g_2(S_t) \geq 0$ and independent of x_1. In other words the fewer periods remaining for the 1^{st} auction, and regardless of the price in auction 1, the higher the expected final price for the 2^{nd} auction. We summarize this in the following corollary.

Corollary 2.9. *If auctions progress independently of price in other auctions and Assumption 2.2 holds then $g_2(S_t) \geq 0$ and independent of x_1 and x_1'.*

Proof of Corollary 2.9 - See Appendix A.

Next we present the optimality equations. Let $V_t(s)$ denote the expected total future reward (expected total profit) given the system is in state $s \in S$ in period t. For discrete prices $V_t(s)$, satisfies the following optimality equations,

$$V_t(s) = \begin{cases} r_t(s) + \max\limits_{a \in A(s)} \sum\limits_{s' \in S} V_{t+1}(s')\pi(s'|s,a) & t = 0, 1, \ldots, T-1 \\ r_T(s) & t = T \end{cases} \tag{2.21}$$

For continuous prices $V_t(s)$ satisfies,

$$V_t(s) = \begin{cases} r_t(s) + \max\limits_{a \in A(s)} \int_S V_{t+1}(s') f(s'|s,a) ds' & t = 0, 1, \ldots, T-1 \\ r_T(s) & t = T \end{cases} \tag{2.22}$$

Due to the vigilant seller assumption, the structure of the transition probabilities, and that auctions are guaranteed to be successful, the value function (2.21) and (2.22) can be summarized and explicitly evaluated according to the three cases listed in the following lemma.

LEMMA 2.10. *If we assume a vigilant seller and that auctions are guaranteed to be successful, then the value functions of interest for discrete prices are as follows,*

$$V_t([x_1, y_1; x_2, y_2], 0) = x_1 + x_2 \qquad t = T$$

$$V_t([x_1, y_1; x_2, y_2], z) = R([x_1, y_1; x_2, y_2], z) \qquad t = \tau$$

$$V_t([x_1, y_1; x_2, y_2], 1) = \max\{-2h + \sum_{q=x_1}^{P} V_{t+1}([q, y_1 + 1; p, 0], 1)\pi_{x_1,q|1} \ , R([x_1, y_1; p, 0], 2)\} \quad t < \tau$$

If prices are continuous the only change is the final equation which becomes,

$$V_t([x_1, y_1; x_2, y_2], 1) = \max\{-2h + \int_{x_1}^{P} V_{t+1}([q, y_1 + 1; p, 0], 1) f^1(q|x_1) dq \ , R([x_1, y_1; p, 0], 2)\} \quad t < \tau$$

Proof of Lemma 2.10 - See Appendix A.

The above value functions can be computed using backward induction. In Table 2.2 the optimality equations for discrete prices and $\tau = 3$ are listed.

$$
\begin{aligned}
V_6([x_1,3;x_2,3],0) &= x_1 + x_2 \\
V_3([x_1,3;p,0],1) &= R([x_1,3;p,0],1) \\
V_2([x_1,2;p,0],1) &= \max\{-2h + \textstyle\sum_{q=x_1}^{P} V_3([q,3;p,0],1)\pi_{x_1,q|1}, R([x_1,2;p,0],2)\ \} \\
V_1([x_1,1;p,0],1) &= \max\{-2h + \textstyle\sum_{q=x_1}^{P} V_2([q,2;p,0],1)\pi_{x_1,q|1}, R([x_1,1;p,0],2)\ \} \\
V_0([p,0;p,0],1) &= \max\{-2h + \textstyle\sum_{q=p}^{P} V_1([q,1;p,0],1)\pi_{p,q|1}, R([p,0;p,0],2)\ \}
\end{aligned}
$$

TABLE 2.2. Optimality equations for discrete prices and $\tau = 3$.

3.3. Structural Results. Given the above MDP and the assumption that auctions are guaranteed to be successful, we derive three monotonicity properties: the optimal value function is increasing in the current price of the two auctions, the optimal policy is a threshold policy, and the threshold is decreasing in the holding cost. Note that though the proofs are for the case of discrete prices, the results hold for continuous prices as well.

PROPOSITION 2.11. *If Assumption 2.1 holds and auctions are guaranteed to be successful, then the optimal value function, $V_t^\star([x_1, y_1; x_2, y_2], z)$, is increasing in x_1 and x_2, for $t = 0, 1, \ldots, T$.*

Proof of Proposition 2.11 - By Lemma 2.10 there are only three cases to consider.

Case 1) If $t = T$, then by Lemma 2.10, $V_t^\star([x_1, y_1; x_2, y_2], z) = x_1 + x_2$, and the result is immediate.

Case 2) If $t < \tau$ and $z = 2$, or $\tau \le t < T$, then by Lemma 2.10, $V_t^\star([x_1, y_1; x_2, y_2], z) = R([x_1, y_1; x_2, y_2], z)$, and the result follows from Corollary 2.8.

Case 3) For $t < \tau$ and $z = 1$, by Lemma 2.10,

$$
V_t^\star([x_1, y_1; x_2, y_2], z) = \max\{-2h + \sum_{q=x_1}^{P} V_{t+1}^\star([q, y_1 + 1; p, 0], z)\pi_{x_1,q|1}\ , R([x_1, y_1; p, 0], 2)\}
$$

We establish the result using backward induction on t. Let $t = \tau - 1$ and hence $y_1 + 1 = \tau$, then by Lemma 4.7.2 in Puterman (1994), Case 2) above and Assumption 2.1, $\sum_{q=x_1}^{P} V_{t+1}^\star([q, \tau; p, 0], 1)\pi_{x_1,q|1}$ is increasing in x_1, and by Corollary 2.8, $R([x_1, y_1; p, 0], 2)$

is increasing in x_1. Since $V_t^\star([x_1, y_1; x_2, y_2], z)$ is the maximum of two increasing functions it is also increasing in x_1 and the result holds. Assume Proposition 2.11 holds for $t = \tau - (l-1), \ldots, \tau - 2, \tau - 1$. Let $t = \tau - l$ and hence $y_1 + 1 = \tau - (l-1)$, and again by Lemma 4.7.2 in Puterman (1994), the induction assumption and Assumption 2.1, $\sum_{q=x_1}^{P} V_{t+1}^\star([q, y_1 + 1; p, 0], 2)\pi_{x_1,q|1}$ is increasing in x_1, and by Corollary 2.8, $R([x_1, y_1; p, 0], 2)$ is increasing in x_1. Since $V_t^\star([x_1, y_1; x_2, y_2], z)$ is the maximum of two increasing functions it is also increasing in x_1 and the result holds. Similar to the proof of Corollary 2.7 the results from Lemma 9.1.1 and Proposition 9.1.2 in Ross (1996) can be applied. The proof for the continuous case is identical but with the summation replaced by an integration. $\square$

In other words an increase in the current price of either item 1 or item 2 will increase the optimal expected total reward. Though this might seem natural and 'obvious' it is a result of the assumptions made, most notably that at a higher price-level the auction is more likely to advance to the higher prices than at a low price-level. And as discussed if bid-increments are decreasing in price then it is not immediate that this assumption holds. Chapter 4 contains examples from DFS' eBay auctions where this result does not hold, as well as examples for which the result holds.

Theorem 2.12. *If Assumptions 2.1, 2.2, and 2.3 hold and auctions are guaranteed to be successful, then there exist optimal decision rules, $\gamma_t^\star([x_1, y_1; p, 0], 1)$, which are increasing in x_1, for $t = 0, 1, \ldots, \tau - 1$. Consequently, the optimal policy is a threshold policy in x_1.*

Proof of Theorem 2.12 - Let prices be discrete. Sufficient to show that $V_t^\star([x_1, y_1; p, 0], 1) - R([x_1, y_1; p, 0], 2)$ is decreasing in x_1, for all $t < \tau$. By Corollary 2.8 and Proposition 2.11, $R([x_1, y_1; p, 0], 2)$ and $V_t^\star([x_1, y_1; p, 0], 1)$ are increasing in x_1. We make use of the following relationship,

$$(2.23) \quad R([x_1,y_1;x_2,y_2],2) = -h + \sum_{q=x_1}^{P} R([q,y_1+1;x_2,y_2],z)\pi_{x_1,q|2} - g_2([x_1,y_1;x_2,y_2])$$

Proof by backward induction on t. Let $t=\tau-1$, then $y_1=\tau-1$ and by Lemma 2.10 and (2.23),

$$V_t^\star([x_1,y_1;p,0],1) - R([x_1,y_1;p,0],2)$$

$$= \max\{-h + \sum_{q=x_1}^{P} R([q,\tau;p,0],1)(\pi_{x_1,q|1}-\pi_{x_1,q|2}) \ + g_2([x_1,\tau-1;p,0]) \ ,0\}$$

$$= \max\{-h - h\tau + \sum_{q=x_1}^{P} q(\pi_{x_1,q|1}-\pi_{x_1,q|2}) \ + E[X_{2,\tau}|([x_1,\tau,p,0],1)] \ + g_2([x_1,\tau-1;p,0]) \ ,0\}$$

by Corollary 2.7 and 2.9, $E[X_{2,\tau}|([x_1,\tau,p,0],1)]$ respectively $g_2([x_1,\tau-1;p,0])$ are independent of x_1, and since,

$$\sum_{q=x_1}^{P} q(\pi_{x_1,q|1}-\pi_{x_1,q|2})$$

$$= x_1 \underbrace{\sum_{q=x_1}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})}_{=0} + \underbrace{\sum_{q=x_1+1}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})}_{\downarrow \text{ in } x_1 \text{by Ass.2.3}} + \underbrace{\sum_{q=x_1+2}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})}_{\downarrow \text{ in } x_1 \text{ by Ass.2.3}} + \ldots + \underbrace{\sum_{q=P}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})}_{\downarrow \text{ in } x_1 \text{ by Ass.2.3}}$$

the result holds for $t=\tau-1$. Assume the result holds for $t=\tau-(l-1),\tau-(l-2),\ldots,\tau-1$, and define $\Lambda_t^\star(x) = V_t^\star([x,y;p,0],1) - R([x,y;p,0],2)$. Let $t=\tau-l$. Then by Lemma 2.10 and (2.23),

$$V_t^\star([x_1,y_1;p,0],1) - R([x_1,y_1;p,0],2) =$$

$$= \max\{-h + \sum_{q=x_1}^{P} V_{t+1}^\star([q,y_1+1;p,0],1)\pi_{x_1,q|1} - R([q,y_1+1;p,0],2)\pi_{x_1,q|2} \ + g_2([x_1,y_1;p,0]) \ ,0\}$$

$$= \max\{-h + \sum_{q=x_1}^{P} \Lambda_{t+1}^\star(q)\pi_{x_1,q|1} + \sum_{q=x_1}^{P} R([q,y_1+1;p,0],2)(\pi_{x_1,q|1}-\pi_{x_1,q|2}) \ + g_2([x_1,y_1;p,0]) \ ,0\}$$

First show that $\sum_{q=x_1}^{P} R([q,y_1+1;p,0],2)(\pi_{x_1,q|1}-\pi_{x_1,q|2})$ is decreasing in x_1. By Corollary 2.8, $R([x,y;p,0],2)$ is increasing in x. Define $\alpha(x+1) = R([x+1,y;p,0],2) - R([x,y;p,0],2)$.

Therefore,

$$\sum_{q=x_1}^{P} R([q,y_1+1;p,0],2)(\pi_{x_1,q|1}-\pi_{x_1,q|2})$$

$$= R([x_1,y_1+1;p,0],2)\underbrace{\sum_{q=x_1}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})}_{=0}+\alpha(x_1+1)\sum_{q=x_1+1}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})$$

$$+\alpha(x_1+2)\sum_{q=x_1+2}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})+\ldots+\alpha(P)\sum_{q=P}^{P}(\pi_{x_1,q|1}-\pi_{x_1,q|2})$$

$$\geq +\alpha(x_1+1)\underbrace{\sum_{q=x_1+1}^{P}(\pi_{x_1+1,q|1}-\pi_{x_1+1,q|2})}_{=0}+\alpha(x_1+2)\sum_{q=x_1+2}^{P}(\pi_{x_1+1,q|1}-\pi_{x_1+1,q|2})$$

$$+\ldots+\alpha(P)\sum_{q=P}^{P}(\pi_{x_1+1,q|1}-\pi_{x_1+1,q|2})$$

$$= \sum_{q=x_1+1}^{P} R([q,y_1+1;p,0],2)(\pi_{x_1+1,q|1}-\pi_{x_1+1,q|2})$$

where the inequality holds by Assumption 2.3. Therefore, $\sum_{q=x_1}^{P} R([q,y_1{+}1;p,0],2)(\pi_{x_1,q|1}-\pi_{x_1,q|2})$ is decreasing in x_1. Next show that $\sum_{q=x_1}^{P}\Lambda^{\star}_{t+1}(q)\pi_{x_1,q|1}$ is decreasing in x_1. By the induction assumption $\Lambda^{\star}_{t+1}(x)$ is decreasing in x, therefore define $\beta(x+1) = \Lambda^{\star}_{t+1}(x)-\Lambda^{\star}_{t+1}(x+1)$, then,

$$\sum_{q=x_1}^{P}\Lambda^{\star}_{t+1}(q)\pi_{x_1,q|1}$$

$$= \Lambda^{\star}_{t+1}(x_1)\underbrace{\sum_{q=x_1}^{P}\pi_{x_1,q|1}}_{=1}-\beta(x+1)\sum_{q=x_1+1}^{P}\pi_{x_1,q|1}-\beta(x+2)\sum_{q=x_1+2}^{P}\pi_{x_1,q|1}-\ldots-\beta(P)\sum_{q=P}^{P}\pi_{x_1,q|1}$$

$$\geq \Lambda^{\star}_{t+1}(x_1)\underbrace{\sum_{q=x_1}^{P}\pi_{x_1+1,q|1}}_{=1}-\beta(x+1)\underbrace{\sum_{q=x_1+1}^{P}\pi_{x_1+1,q|1}}_{=1}-\beta(x+2)\sum_{q=x_1+2}^{P}\pi_{x_1+1,q|1}-\ldots-\beta(P)\sum_{q=P}^{P}\pi_{x_1+1,q|1}$$

$$= \sum_{q=x_1+1}^{P}\Lambda^{\star}_{t+1}(q)\pi_{x_1+1,q|1}$$

where the inequality holds due to Assumption 2.1. Therefore $\sum_{q=x_1}^{P}\Lambda^{\star}_{t+1}(q)\pi_{x_1,q|1}$ is decreasing in x_1. Since $g_2([x_1,t;p,0])$ is independent of x_1, the result holds for all $t<\tau$.

The proof for continuous prices follows the same logic but with slightly different arguments.

□

Theorem 2.12 implies that, for each t there exists a $p_t^\star$ such that if $X_1 \geq p_t^\star$ then it is optimal to release the second item for auction, while if $X_1 < p_t^\star$ it is optimal to hold the second item at least one more period. The reason we are only considering $t = 0, 1, \ldots, \tau - 1$ is because we are only interested in those periods where non-trivial decision can be made. For $t = \tau$ the decision to release is immediate by the vigilant seller assumption. Also note that if $Y_2 > 0$ the second item has already been released and no further decision needs to be made. The main assumption driving the result of Theorem 2.12 is the diminishing cannibalization effect of Assumption 2.3. Our next result summarizes the effect the holding cost has on $p_t^\star$.

Corollary 2.13. *For each $t < \tau$, the control limit in Theorem 2.12, $p_t^\star$, is decreasing in the holding cost h.*

Proof of Corollary 2.13 - For a given decision epoch t we know that for $X_1 \geq p_t^\star$ any additional holding cost by deferring the release is not compensated by the gain in expected final price for the two items. Therefore, if h increases and since the expected final prices remains the same, then any additional holding cost will still not be compensated (in fact it is even less compensated), and the result follows. □

Given these three properties it may be natural to ask if the threshold price, $p_t^\star$, is monotone in t or Y_1. With only the three assumptions stated above the answer is no. And it turns out that monotonicity over time will also depend on the holding cost h. In other words, depending on the holding cost $p_t^\star \leq p_{t+1}^\star$ or $p_t^\star \geq p_{t+1}^\star$. See the numerical example in

Section 3.4 for an illustration. In Section 3.5 some bounds on the holding cost are discussed.

3.3.1. *Examples.* In Section 2.3 we provided four common probability distributions and conditions on their parameters that support Assumptions 2.1, 2.2, and 2.3. Consequently, we have the following results.

COROLLARY 2.14. *If price increments are distributed as discrete or continuous Uniform random variables, as specified in Section 2.3, then Proposition 2.11 and Theorem 2.12 holds.*

COROLLARY 2.15. *If price increments are distributed as Bernoulli random variables as specified in Section 2.3 and (2.11) holds, then Proposition 2.11 and Theorem 2.12 holds.*

Proof of Corollary 2.14 and 2.15 - For each of the cases we have that Assumptions 2.1, 2.2, and 2.3 holds, it therefore follows that Proposition 2.11 and Theorem 2.12 holds. $\square$

In the Bernoulli case with price-independent transition probabilities, $\pi_q = \pi$ and $\rho_q = \rho$ for all $q = 0, 1, \ldots, \tau - 1$, the optimal policies simplify further. First note that due to the special structure of the transition probability matrices the n-period transition matrices are symmetric in the following sense, for $\tau = 3$, $\Pi_{1 \cdot 2 \cdot 2} = \Pi_{2 \cdot 2 \cdot 1}$ and $\Pi_{1 \cdot 1 \cdot 2} = \Pi_{2 \cdot 1 \cdot 1}$. As a result the expected value for the open loop policy of releasing the second item j periods after the first item has been released is given by,

$$V_{O(j)} = -(2\tau + j)h + 2(p + j\pi + (\tau - j)\rho) \qquad j = 0, 1, 2, \ldots, \tau \tag{2.24}$$

Furthermore, the total marginal gain by deferring the release one period, $2(\pi - \rho)$, is independent of X_1 and as a result closed loop policies are not required.

PROPOSITION 2.16. *In the case of price-independent Bernoulli increments, the optimal policy is to release both items simultaneously if and only if $h \geq 2(\pi - \rho)$. If $h < 2(\pi - \rho)$ then releasing the two auctions sequentially is the optimal policy.*

Proof of Proposition 2.16 - See Appendix A.

The two policies described in Proposition 2.16 are the only optimal policies - open and closed loop policies included. The interpretation of the condition $h \geq 2(\pi - \rho)$, is that if the holding cost exceeds the expected one-period gain for both auctions, by deferring the release it will never be optimal to defer the release of item 2.

3.4. Numerical Examples. Example 1 - $\tau = 2$

The purpose of this example is illustrate that a closed loop policy may be optimal, that the optimal threshold for releasing an item need not be monotone in t or Y_1, and that in order to determine the optimal policy, the seller must solve the dynamic program by backward induction. Consider a two period auction, $\tau = 2$, for an item valued at \$10, \$20 and \$30, with the following transition probability matrices.

$$\Pi_1 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .6 & .3 & .1 \\ 0 & .6 & .4 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Pi_2 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \end{array} \!\! \begin{array}{c} \begin{array}{ccc} 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .7 & .3 & 0 \\ 0 & .65 & .35 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note that Assumptions 2.1, 2.2, and 2.3 hold. At decision epoch $t = 2$, the 1^{st} auction has ended and the decision to release the 2^{nd} item is immediate irrespective of X_1 by the vigilant seller assumption. At time $t = 1$ the 1^{st} auction has one period remaining and the seller can choose to either release the 2^{nd} item or defer the release one period. If $X_1 = \$10$

then the seller's total expected profit is the following:

$$V_1^{\star}([10,1;10,0],1) = \max \begin{cases} -4h + 15 + 19.2 & a = 0 \text{ (defer release)} \\ -3h + 13 + 17.7 & a = 1 \text{ (release)} \end{cases}$$

We see that the marginal cost of deferring the release is h while the expected marginal gain is 3.5. If $h > 3.5$ it is optimal to release the 2^{nd} auction while if $h < 3.5$ it is optimal to defer the release. If $X_1 = \$20$ then the seller has the following total expected profit:

$$V_1^{\star}([20,1;10,0],1) = \max \begin{cases} -4h + 24 + 19.2 & a = 0 \text{ (defer release)} \\ -3h + 23.5 + 17.7 & a = 1 \text{ (release)} \end{cases}$$

In this case, the marginal cost is still h and the expected marginal gain is now only 2, i.e. if $h > 2$ then the optimal decision is to release else it is optimal to defer the release. If $X_1 = \$30$ then the seller has the following total expected profit:

$$V_1^{\star}([30,1;10,0],1) = \max \begin{cases} -4h + 30 + 19.2 & a = 0 \text{ (defer release)} \\ -3h + 30 + 17.7 & a = 1 \text{ (release)} \end{cases}$$

While the marginal cost is still h, the expected marginal gain is now only 1.5, i.e. if $h > 1.5$ then the optimal decision is to release else it is optimal to defer the release. Therefore when we roll back one period and analyze the optimal decision at $t = 0$, we have four cases to consider: 1) $h \geq 3.5$, 2) $3.5 > h \geq 2$, 3) $2 > h \geq 1.5$, and 4) $1.5 > h$. Let $t = 0$ and $X_1 = \$10$, the seller's expected profit is the following:

$$\begin{aligned} &V_0^{\star}([10,0;10,0],1) \\ &= \max \begin{cases} -2h + .6V_1^{\star}([10,1;10,0],1) + .3V_1^{\star}([20,1;10,0],1) + .1V_1^{\star}([30,1;10,0],1) & a = 0 \\ -4h + 2(16.15) & a = 1 \end{cases} \end{aligned}$$

Which for the four cases results in the following.

Case 1) $h \geq 3.5$:

$$V_0^\star([10,0;10,0],1) = \max\begin{cases} -5h + .6(30.7) + .3(41.2) + .1(47.7) & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

$$= \max\begin{cases} -5h + 35.55 & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

Case 2) $3.5 > h \geq 2$:

$$V_0^\star([10,0;10,0],1) = \max\begin{cases} -2h - .6(4h) - .4(3h) + .6(34.2) + .3(41.2) + .1(47.7) & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

$$= \max\begin{cases} -5.6h + 37.65 & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

Case 3) $2 > h \geq 1.5$:

$$V_0^\star([10,0;10,0],1) = \max\begin{cases} -2h - .9(4h) - .1(3h) + .6(34.2) + .3(43.2) + .1(47.7) & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

$$= \max\begin{cases} -5.9h + 38.25 & a=0 \\ \\ -4h + 32.3 & a=1 \end{cases}$$

Case 4) $1.5 > h$:

$$V_0^\star([10,0;10,0],1) = \max \begin{cases} -6h + .6(34.2) + .3(43.2) + .1(49.2) & a = 0 \\ -4h + 32.3 & a = 1 \end{cases}$$

$$= \max \begin{cases} -6h + 38.4 & a = 0 \\ -4h + 32.3 & a = 1 \end{cases}$$

In the three cases we can compare the expected marginal cost with the expected marginal gain for deferring the release. For case 1) the expected marginal cost is h while the expected marginal gain is 3.25. Since $h \geq 3.5$ the optimal decision is to release the 2^{nd} item immediately. For case 2) the expected marginal cost is $1.6h$ while the expected marginal gain is 5.35. Therefore we have two further sub-cases to consider: 2a)If $3.5 > h \geq 3.34$ then the expected marginal cost outweighs the expected marginal gain and it is optimal to release the 2^{nd} item immediately, 2b) If $3.34 > h \geq 2$ then the expected marginal gain by deferring the release compensates for the expected marginal cost, and therefore it is optimal to defer the release. For case 3) the expected marginal cost is $1.9h$ while the expected marginal gain is 5.95, therefore it is optimal to defer the release. And similarly for case 4) the expected marginal cost is $2h$ while the expected marginal gain is 6.1, and since $1.5 > h$ it is optimal to defer the release.

In Table 2.3 we summarize the above results. For instance, we see that if $h = \$3$ and $t = 0$ or 1 and $X_1 = 10$ it is optimal to defer the release. If, however, at $t = 1$ the price has jumped to either \$20 or \$30 then it is optimal to release the 2^{nd} item. The main point of this example is to illustrate that a closed loop policy may be optimal, and that for a given h and X_1 it may be optimal to release item 2 in one period though the optimal decision in a later period, at the same price, is to defer the release. Note, however, that for the cases when $h \geq 3.34$ a seller that follows an optimal policy would not have to make a decision

	$t=0$		$t=1$	
	$X_1=10$	$X_1=10$	$X_1=20$	$X_1=30$
$h \geq 3.5$	Release	Release	Release	Release
$3.5 > h \geq 3.34$	Release	Defer	Release	Release
$3.34 > h \geq 2$	Defer	Defer	Release	Release
$2 > h \geq 1.5$	Defer	Defer	Defer	Release
$1.5 > h$	Defer	Defer	Defer	Defer

TABLE 2.3. Example 1 - Optimal decision as a function of the holding cost h ($\tau = 2$)

in period $t = 1$, but in order to derive the optimal policy, the optimal decision for period $t = 1$ has to be evaluated.

Example 2 - $\tau = 3$

Our next example illustrates the potential gain by using an adaptive closed loop policy versus an non-adaptive open loop policy. It also shows that depending on the holding cost the price threshold may be increasing or decreasing over time. Assume the auction length is three periods, $\tau = 3$, the start price $p = \$10$, the upper bound $P = \$60$, the price increments are \$10, and the transition probabilities are as follows.

$$\Pi_1 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{pmatrix} .25 & .25 & .2 & .1 & .1 & .1 \\ 0 & .25 & .3 & .25 & .1 & .1 \\ 0 & 0 & .25 & .4 & .25 & .1 \\ 0 & 0 & 0 & .4 & .45 & .15 \\ 0 & 0 & 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Pi_2 = \begin{array}{c} \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \\ 60 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{pmatrix} .3 & .3 & .25 & .1 & .05 & 0 \\ 0 & .35 & .35 & .2 & .1 & 0 \\ 0 & 0 & .4 & .35 & .2 & .05 \\ 0 & 0 & 0 & .5 & .4 & .1 \\ 0 & 0 & 0 & 0 & .65 & .35 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

It can be verified that the matrices satisfy Assumptions 2.1, 2.2, and 2.3. The expected profit for the four open loop policies can explicitly be computed,

$$V_{O(0)} = -6h + 2 \times 40.76$$

$$V_{O(1)} = -7h + 43.69 + 43.40$$

$$V_{O(2)} = -8h + 46.09 + 45.82$$

$$V_{O(3)} = -9h + 2 \times 47.93$$

While for the closed loop policy we use backward induction to find the action that maximizes the value equations. In Table 2.4 below the value of each policy with respect to various holding costs are shown. The last two columns displays the difference between the closed loop policy and the best and worst open loop policy for a given h. As one can see if the holding cost is 'low' then it is better to release the two auctions sequentially, while if the holding cost is 'high' then it will never be worth holding the 2^{nd} item an additional period so that the optimal policy is to release both immediately. The interesting cases are in between where we see that the closed loop policy performs better than any of the open loop policies.

Though the gain at a given h for the optimal closed loop policy versus the best open loop policy is perhaps not that drastic, the gain versus the other open loop policy can be quite large. For instance, if $h = \$2.30$ then the difference between using the optimal closed loop policy and open loop policy of simultaneous release is more than \$7 (11% improvement).

In Table 2.5 the critical price thresholds $p_t^\star$ for various holding costs are summarized (in the table 'n/a' indicates that $p_t^\star > P$). In the table we see that if, for instance, $h = 4.00$ and $t = 1$ then if $X_1 < 40$ then it is optimal to defer the release of item 2, while if $X_1 \geq 40$

h	OP0	OP1	OP2	OP3	Optimal Policy	Max Gain(%)	Min Gain(%)
.10	80.93	86.39	91.10	**94.97**	**94.97**	17.9	0
1.00	75.53	80.09	83.90	**86.87**	**86.87**	15.0	0
2.00	69.53	73.09	75.90	**77.87**	**77.87**	12.0	0
2.30	67.73	70.99	73.50	**75.17**	**75.21**	11.0	.1
4.00	57.53	59.09	**59.90**	59.87	**60.99**	6.0	1.8
5.00	51.53	**52.09**	51.90	50.87	**53.01**	4.2	1.8
5.80	**46.73**	46.49	45.50	43.67	**46.85**	7.3	.3
6.00	**45.53**	45.09	43.90	41.87	**45.53**	8.7	0
9.00	**27.53**	24.09	19.90	14.87	**27.53**	85.1	0
10.00	**21.53**	17.09	11.90	5.87	**21.53**	266.8	0
15.00	**−8.47**	−17.91	−28.10	−39.14	**−8.47**	-	-

TABLE 2.4. Expected profit vs holding cost comparison for the *Open* and *Closed* Loop Policies

it is optimal to start the 2^{nd} auction. Note that when h is 'low' then $p_t^\star > P$ and hence it is always optimal to wait one more period before releasing the 2^{nd} auction . Similarly, if h is 'high' then it is never optimal to defer the release. Another thing to note is that depending on the holding cost, the price threshold could be either increasing or decreasing. To illustrate, consider the case when $h = 2.75$, then $p_0^\star = \$60$, while $p_1^\star = p_2^\star = \$50$. On the other hand, if, for instance, $h = 5.50$ then $p_0^\star = \$20$, while $p_1^\star = p_2^\star = \$30$. The managerial consequence is that depending on the holding cost, the manager may become more or less 'sensitive' as to when release the second item as the first auction evolves. For instance, when h is relatively low, such that $p_t^\star$ is decreasing in t, then the manager will lower his release threshold for each period and hence be less sensitive to the current price. On the other hand, when h is relatively high, such that $p_t^\star$ is increasing in t, then the manager becomes more sensitive to the current price and requires a higher release threshold.

We conclude by illustrating the optimal policy for two realizations. Let $h = 4.00$, at $t = 0$, since auction 1 starts at $x_1 = \$10$, which is less than the release threshold $p_0^* = \$40$, it is optimal to defer the release. Suppose that after one period, auction 1 has jumped to $x_1 = \$30$, since this is less than the release threshold $p_1^* = \$40$, it is optimal to yet again defer the start of auction 2. Suppose that after two periods, auction 1 has jumped to $x_1 = \$40$,

h	$t=0$	$t=1$	$t=2$
1.00	n/a	n/a	n/a
2.50	n/a	60	60
2.75	60	50	50
4.00	40	40	40
5.00	30	30	40
5.50	20	30	30
6.00	10	20	30
8.00	10	10	10

TABLE 2.5. Threshold price level $p_t^\star$ for various holding cost h

since this is equal to the release threshold $p_2^* = \$40$, it is now optimal to start the second auction. In other words, given the price-transitions of auction 1, it was optimal to wait until the start of the third period, or when auction 1 had elapsed 2 periods, before starting auction 2. Suppose instead that after one period, auction 1 had jumped to $x_1 = \$40$, then it would have been optimal to start auction 2 already at the start of the second period.

3.5. Holding Cost. As we have seen if the holding cost is 'too low' then it will never be optimal to release the 2^{nd} item until the 1^{st} item has been sold. And consequently, the optimal closed loop policy is same as the open loop policy of *sequentially* releasing the two auctions. A necessary and sufficient condition for this to hold, is if the holding cost is so low that the additional gain for the 2^{nd} item by deferring the release alone compensates for the additional holding cost.

PROPOSITION 2.17. *The optimal policy is to release the 2^{nd} item for auction after the 1^{st} auction is completed if and only if*

$$h \leq \min_{y_1 < \tau} \{g_2([x_1, y_1; p, 0])\} \tag{2.25}$$

Proof of Proposition 2.17 - ($\Leftarrow$) If (2.25) holds then the additional gain in expected final price for item 2, by deferring the release one period, alone compensates for the additional holding cost. Since that gain holds for all periods that the 1^{st} auction is still ongoing and independently of x_1, it is always optimal to release the 2^{nd} auction after the 1^{st} is finished.

(⇒) If (2.25) does not hold then there exist a period for which the gain in expected final price for item 2 does not compensate the additional holding cost. Consequently, in order for it to be optimal to defer the release in that period, there must be some gain in expected final price of item 1. However, this gain is dependent on x_1 and hence the optimal policy is not independent of x_1. Therefore, the optimal policy might not be to release the items sequentially. □

Similarly, if h is 'too high' then there will be no incentive to hold the 2^{nd} item any further since the expected total gain will not compensate for the additional holding cost. Consequently the optimal closed loop policy will be the same as the open loop policy of *simultaneous* release. However, the condition for h to be 'too high' is more complicated than the condition for 'too low'. For instance, it is not sufficient that h is such that the best open loop policy is to release them simultaneously to ensure that this is the optimal closed loop policy (see Example 2 above). Also recall that though it may be optimal to release the two items simultaneously, it does not necessarily mean that in every period it is always optimal to release the 2^{nd} item. It is possible that for $t = 0$ and $X_1 = p$ it is optimal to release the 2^{nd} auction while for $t = 1$ and $X_1 = p$ it would be optimal to defer the release given that it has not been released (see Example 1 above). A 'bound' for h to be 'too high ' and consequently the optimal policy can never be to release the two auctions simultaneously is if,

$$V_{O(1)} - V_{O(0)} \geq 0$$

The implication on the holding cost h in the above inequality is better illustrated as follows,

$$(V_{O(1)} + (2\tau + 1)h) - (V_{O(0)} + 2\tau h) \geq h$$

In other words, if the expected gain in revenue by deferring the release one period is greater than h, then the optimal closed loop policy can never be to release the 2^{nd} auction immediately. Therefore in order for h to be 'too high' it has to be large enough that the open loop policy of simultaneous release is better than the open loop policy of deferring the release by one period. Note though that this is only a necessary condition, it could be optimal to defer the release despite that the above inequality does not hold. To determine the threshold on the holding cost for which the optimal closed loop policy is to release the two items simultaneously one has to solve the dynamic program given that h is such that the above inequality holds.

Example 2 (continued) - $\tau = 3$

In the numerical example above we see that if

$$h \leq \min_{y_1 < \tau} \{g_2([x_1, y_1; p, 0])\} = 47.93 - 45.81 = 2.12$$

then the optimal closed loop policy is to release the 2^{nd} auction after the 1^{st} auction is over, i.e. if $h \leq 2.12$ then $p_t^\star > P$ for all $t < \tau$. To find the upper limit of h for which the optimal closed loop policy is to release the two auctions simultaneously we solve for h by backward induction given that $h \geq (43.69 + 43.40) - (2 \times 40.76) = 5.57$, this results in that if $h \geq 5.88$ then the optimal closed loop policy is to release the 2^{nd} item immediately, i.e. if $h \geq 5.88$ then $p_0^\star = p = 10$. Note too that if, for instance, $h = 6.00$ then $p_1^\star = 20$ and $p_2^\star = 30$.

3.6. Reserve Price. In most online auctions the seller has the option of imposing a reserve price v_r. If the final price $X_{i,\tau} < v_r$ then the seller is not obligated to award the item. In eBay auctions the value of v_r is not disclosed and the only information bidders have is if $X_{i,Y_i} \geq v_r$ and hence if the item is guaranteed to be awarded. A natural comment might be why not simply set the starting bid at the reserve price, i.e. set $p = v_r$. Anecdotal evidence suggest that the bidding behavior is different for auctions with low starting prices. The reasoning seems to be that most bidders prefer to participate in 'active' auctions, where the price has reached a level due to bidding activity, rather than auctions where no bidding has taken place though the price is exactly the same. In order for the seller to protect himself against the possibility of selling below his reservation value he can impose a reserve price. However, it is also anecdotally known that having a 'too high' reserve price deters bidding activity. Here the argument is that auctions that are guaranteed to be awarded, i.e. the reserve price has been met, attracts more bidding competition due to the guarantee that the highest bidder will win. Bertsimas, Hawkins and Perakis (2003) analyze some of these issues and determine the optimal start price and reserve price.

In the following discussion we still assume the seller will at most list an item once, but now with an alternative salvage channel with a guarantee of v_r (and not p as before). We will use the same MDP formulation as before but assume the value of an auction that ends at or below the reserve price is v_r. In other words, the expected final price of an item is defined as follows, for $i = 1, 2$,

Discrete prices:

(2.26)

$$E[X_{i,\tau}|S_t = ([x_1, y_1, x_2, y_2]z)] =$$

$$v_r \sum_{q=x_i}^{v_r} \Pr\{X_{i,\tau} = q | X_{i,y_i} = x_i, Z_t = z, Z_{t+1} = z', \ldots, Z_{t+(\tau-y_i)} = z''\}$$
$$+ \sum_{q=v_r+1}^{P} \Pr\{X_{i,\tau} = q | X_{i,y_i} = x_i, Z_t = z, Z_{t+1} = z', \ldots, Z_{t+(\tau-y_i)} = z''\}$$

Continuous prices:

$$E[X_{i,\tau}|S_t = ([x_1, y_1, x_2, y_2]z)] = v_r \int_{x_i}^{v_r} f_{X_\tau|X_{y_i}}^{z \cdot z' \cdots z''}(q|x_i)dq + \int_{v_r}^{P} q f_{X_\tau|X_{y_i}}^{z \cdot z' \cdots z''}(q|x_i)dq \quad (2.27)$$

Note that the behavior issue regarding increased bidding activity or higher price jumps due to the reserve price having been met is consistent in the current MDP framework. The main monotonicity assumption for each transition probability matrix Π_z, $z = 1, 2$, is Assumption 2.1. This assumption does not exclude experiencing higher price jumps once the current price has passed a given threshold $v_r \in [p, P]$. As the reader may notice, introducing a reserve price does not effect the main structural results.

Proposition 2.18. *If the auctions are guaranteed to be successful and the seller has imposed a reserve price v_r, and the value of an auction is defined according to (2.26) or (2.27), then Proposition 2.11, Theorem 2.12, and Corollary 2.13 holds.*

Proof of Proposition 2.18 - see Appendix A.

3.7. Dynamic Price Transition Probabilities. We have assumed that the price dynamics, i.e. transition probabilities, are *stationary* with respect to calender time t and elapsed auction time Y. There is, however, anecdotal and statistical evidence to suggest the opposite. For example, anecdotal evidence suggest that eBay auction activity for certain products vary between the weekdays, as well as between weekdays and weekends. Similarly it is well established that the bidding activity is much higher toward the end of an auction. This phenomena includes the well known bidding strategy *sniping*. Papers that analyze the phenomena of dynamic bid arrivals with respect to Y, and its implication include Roth and Ockenfels (2002) and Shmueli, Russo and Wolfgang (2004).

For ease of tractability and we ignore the issue of non-stationarity with respect to calender time t. Instead we focus on the implication of having non-stationary transition probabilities with respect to the elapsed auction time. That is when Π_z or F^z changes as the auction progresses. Let $\Pi_{z,y}$ be the transition probability matrix and let $F^{z,y}$ be the transition distribution function, for an individual item when there are z auctions underway and the auction has elapsed for y periods. For instance, suppose prices are discrete and the price dynamics are the same for all periods except the last one, i.e. $\Pi_{z,0} = \Pi_{z,1} = \ldots = \Pi_{z,\tau-2} \neq \Pi_{z,\tau-1}$. Even with dynamic transition probabilities, for the guaranteed successful auction case, the main structural results still hold which we summarize in the following proposition.

PROPOSITION 2.19. *For the single listing case, if for $y = 0, 1, \ldots, \tau - 1$ and $z = 1, 2$, $\Pi_{z,y}$ and $F^{z,y}$ supports Assumption 2.1, and the relation between $\Pi_{1,y}$ and $\Pi_{2,y}$, and between $F^{1,y}$ and $F^{2,y}$ supports Assumption 2.2 and 2.3, then Propositions 2.11, Theorem 2.12, and Corollary 2.13 still hold.*

Proof of Proposition 2.19 - None of the proofs involve arguments across different elapsed auction periods. Each argument is always with respect to a given elapsed auction period. Therefore including a subscript to indicate the elapsed auction period, on the transition probabilities or transition distribution functions does not impact the validity of any of the arguments in any of the proofs. Since the three assumptions are still valid for each elapsed auction period transition probability, there are no changes to the above results. $\square$

4. Possibly Unsuccessful Auction - Multiple Re-Listing

We now consider the case when there is a positive probability an auction is unsuccessful, meaning that there is some chance an auction receives no bids, and that the seller does not have any alternative salvage channel. For discrete prices this means that $\pi_{0,0|z} > 0$, $z = 1, 2$, while for continuous prices there is some point mass for having a zero price transition. An example of a distribution for continuous prices with this property is the 'zero-inflated gamma' distribution that will be discussed in Chapter 4. Reasons why an auction may not receive any bids are that the seller perhaps posted a too high starting price or that the items simply do not generate enough interest. For instance, a quick search on the completed listing of the eBay stores *Pokerstores* (poker chips), *The_Sharper_Image* (consumer electronics), *uptempoair* (Nike sportswear), *GlobalGolfUSA* (used golf clubs) reveal that a large quantity of their listings do not attract a single bid. In contrast, to the 6,000 auctions from *Dell_Financial_Services*'s eBay store we have data on, where not a single auction with a starting price of \$.99 was unsuccessful. Bertsimas, Hawkins and Perakis (2003) discuss and provide empirical evidence regarding optimal control of starting price and reserve price. For our purposes, even if the sample of companies listed above are doing something wrong in their administration of auction control parameters, we include this section for mathematical completeness. And it turns out that the managerial consequence is both important and interesting, since unlike the single listing case the optimal policy need not be a control limit policy.

By the positive expected profit assumption, even though the seller at the end of an unsuccessful auction has incurred a total cost of τh, it is still optimal to try to auction the item once again. Therefore the seller has to decide when to re-list items that remain from previous unsuccessful auctions. Since the items are identical we can classify the item that is waiting to be released as item 2, and the item which is currently up for auction as item 1. That is, the ongoing auction will be labeled as auction 1, and the auction that we are deciding whether to start or not as auction 2. This means that if both auctions are underway, auction 2 was started after auction 1, and auction 1 is unsuccessful, then auction 2 immediately becomes auction 1. Therefore $Y_1 \geq Y_2$ and the system state transitions may form loops. Specifically it may loop back to the initial starting state. See Figure 2.4 in Section 2.1. Consequently the time when an auction is successfully completed is not known. To address this issue, we formulate the problem as an *expected total-reward* MDP [**21**, Ch.7]. However, the problem involves some subtleties such that neither the properties of *positive* or *negative* dynamic programs directly apply. In particular, there need not be unique solutions to the optimality equations, and policy or value iteration need not converge without further assumptions [**21**, Ch.7], [**3**]. Our model has the following relevant structural properties,

(1) The state space and actions are finite.

(2) The expected one-period reward in each state is bounded above and below.

(3) There is a single absorbing state Δ under all policies.

The third property holds due to the vigilant seller assumption, without which there would be an additional absorbing state, $([0,0;0,0],0)$, with total reward of negative infinity, resulting from the policy of never releasing an item for auction. Given these three properties, the problem can be converted to a negative dynamic problem with the following transformation. In each transient state subtract $2P$ from the one-period reward. As a consequence all rewards in the transformed problem are less than or equal to zero [**21**, Proof of Theorem 8.10.1.]. Therefore, optimal solutions exists, and value iteration and policy iteration converges (though a modification to policy iteration may be required [**21**, Ch.10.4.2]). Alternatively *Assumption 1* and *2* of Bertsekas and Tsitsiklis (1991) holds (even without the vigilant seller assumption).

For simplicity of notation in the remaining of this section we only consider the case of discrete prices. All results presented will hold for continuous prices as well.

4.1. Auction Release Policy. For the multiple re-listing (infinite horizon) case a Markov deterministic policy γ is defined by,

$$\gamma = (\gamma_0(s), \gamma_1(s), \gamma_2(s), \ldots) \qquad \gamma_t(s) \in \{0,1\}, \forall s \in S, t \geq 0$$

If $\gamma_t(s) = \gamma_{t'}(s)$, for all $t \neq t'$, and for a given $s \in S$, the policy is referred to as *stationary.* In the multiple re-listing case, we only need to consider stationary policies and therefore use the notation $\gamma(s)$, and interchangeable refer to it as both the decision rule and policy [**21**, Theorem 7.3.6.], [**3**, Proposition 2]. In the multiple re-listing case, *open loop* policies do not apply. Instead we define two types of closed loop policies. We refer to a policy that only depends on Y_1 as *time-based closed loop*, and policies that depends on (X_1, Y_1) as *price-based closed loop*. The reason we need this distinction is because, unlike the single listing case, it is necessary to consider decisions even after the first decision to release item 2 has been made. The reason for this is that auctions may get out of 'sync' with each other. We illustrate with an example, suppose $\tau = 5$ and the seller decides to start the second auction 2 periods after the first. And suppose further that the first auction is unsuccessful, which means that the seller must now decide whether, after the second auction has elapsed three periods, to re-list item 1, and if the decision is not to release what to do when the second auction has elapsed four periods. This reasoning generalizes for any τ, and any policy which specify not to release if $Y_1 < y_r$ and release if $Y_1 = y_r$, where $y_r < \tau/2$, must also specify what to do when $Y_1 \geq \tau - y_r$. For policies defined such that the first release should occur when $Y_1 = y_r > \tau/2$, the issue of what decision to make if the first auction is unsuccessful has already been addressed. In other words, though there are $2^{\tau+1}$ time-based closed loop policies, many are infeasible. For example, if $\tau = 5$ then the policy to only release item 2 after auction 1 has elapsed 1 or 3 periods, is basically the same as the policy to only release item 2 after auction 1 has elapsed 1 period, i.e. if auction 2 is to be released when auction 1 has elapsed for 1 period, then there will never be an opportunity to release item 2 after auction 1 has elapsed 3 periods.

4.2. Optimality Equation. Before analyzing the optimality equations for two items we begin by only considering one item. Let the pair of random variables (X, Y) denote the state of an auction, where $X \in \{0, p, p+k, \ldots, P\}$ and $Y \in \{0, 1, \ldots, \tau\}$ or δ, and $E[X_\tau | X_y = x]$ denotes the expected final price given $X = x$ and $Y = y$. Let $v(x, y)$ denote the value (total expected future reward) given the system is in state (x, y). By the positive expected profit and vigilant seller assumptions, the item should be immediately re-listed following an unsuccessful auction. Therefore the value of an item in state (x, y) is,

$$v(x,y) = \begin{cases} -h(\tau - y) + E[X_\tau | X_y = x] + (\pi_{x,0|1})(\pi_{0,0|1})^{(\tau-(y+1))} v(0,0) & y \neq \delta \\ 0 & y = \delta \end{cases}$$

Note that $\pi_{x,0|1} = 0$ for all $x > 0$, and that we are not considering discounting. Consequently the expected value of an item continuously re-listed until the auction is successful is,

$$v(0,0) = \frac{-h\tau + E[X_\tau | X_0 = 0]}{1 - (\pi_{0,0|1})^\tau} \tag{2.28}$$

We now return to the two item case. In total there are 19 different cases for which the optimality equation needs to be evaluated. These appear in Table 2.6 below. Note that there are only non-trivial decisions in those periods for which $y_1 < \tau$ and $y_2 = 0$, namely cases 16, 17, and 19. Furthermore, note that under cases 13, 15 and 17 there is a positive probability of looping back to the initial state $([0, 0; 0, 0], 1)$. However, due to the positive expected profit assumption, we have that $\pi_{0,0|z} < 1, z = 1, 2$, and therefore with probability one the system state will eventually reach the recurrent state $\Delta = ([\Delta_1, \Delta_2], 0)$. Recall that $\Delta_i = (X_i, \delta)$ is the state of item i, $i = 1, 2$, when it has been awarded and hence will not incur any additional cost or generate any further revenue. From the discussion above, we note that any solution satisfying the optimality equations in Table 2.6 is an optimal solution [**21**, Proposition 7.3.4], [**3**, Proposition 2]. Though the optimal policy need in general not be a control limit policy (see the example below), under some of the cases the optimal policy is a control limit.

4.3. Structural Results. Similarly to the single listing case the optimality equations can be simplified and explicitly evaluated for some of the cases.

Case	Item 1 y_1	Item 2 y_2	z	$V([x_1, y_1; x_2, y_2], z)$
1)	δ	δ	0	$= 0$
2)	δ	τ	0	$= x_2 + V(\Delta)$
	δ	$\tau - 1$	1	
3)		$x_2 > 0$		$= -h + \sum_{r=x_2}^{P} V([x_1, \delta; r, \tau], 0)\pi_{x_2,r\|1}$
4)		$x_2 = 0$		$= -h + \sum_{r=p}^{P} V([x_1, \delta; r, \tau], 0)\pi_{0,r\|1} + V([x_1, \delta; 0, 0], 1)\pi_{0,0\|1}$
5)	δ	$< \tau - 1$	1	$= -h + \sum_{r=x_2}^{P} V([x_1, \delta; r, y_2 + 1], 1)\pi_{x_2,r\|1}$
6)	τ	τ	0	$= x_1 + x_2 + V(\Delta)$
	τ	$\tau - 1$	1	
7)		$x_2 > 0$		$= x_1 - h + \sum_{r=x_2}^{P} V([x_1, \delta; r, \tau], 0)\pi_{x_2,r\|1}$
8)		$x_2 = 0$		$= x_1 - h + \sum_{r=p}^{P} V([x_1, \delta; r, \tau], 0)\pi_{0,r\|1} + V([x_1, \delta; 0, 0], 1)\pi_{0,0\|1}$
9)	τ	$< \tau - 1$	1	$= x_1 - h + \sum_{r=x_2}^{P} V([x_1, \delta; r, y_2 + 1], 1)\pi_{x_2,r\|1}$
	$\tau - 1$	$\tau - 1$	2	
10)	$x_1 > 0$	$x_2 > 0$		$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau; r, \tau], 0)\pi_{x_1,q\|2}\pi_{x_2,r\|2}$
11)	$x_1 > 0$	$x_2 = 0$		$= -2h + \sum_{q=x_1}^{P} (\sum_{r=p}^{P} V([q, \tau; r, \tau], 0)\pi_{0,r\|2} + \pi_{0,0\|2} V([q, \tau; 0, 0], 1))\pi_{x_1,q\|2}$
12)	$x_1 = 0$	$x_2 > 0$		$= -2h + \sum_{r=x_2}^{P} (\sum_{q=p}^{P} V([q, \tau; r, \tau], 0)\pi_{0,q\|2} + \pi_{0,0\|2} V([r, \tau; 0, 0], 1))\pi_{x_2,r\|2}$
13)	$x_1 = 0$	$x_2 = 0$		$= -2h + \sum_{q=p}^{P} \sum_{r=p}^{P} V([q, \tau; r, \tau], 0)\pi_{0,q\|2}\pi_{0,r\|2} + \pi_{0,0\|2}\pi_{0,0\|2} V([0, 0; 0, 0], 1)$ $+ \sum_{q=p}^{P} V([q, \tau; 0, 0], 1)\pi_{0,q\|2}\pi_{0,0\|2} + \sum_{r=p}^{P} V([r, \tau; 0, 0], 1)\pi_{0,0\|2}\pi_{0,r\|2}$
	$\tau - 1$	$< \tau - 1$	2	
14)	$x_1 > 0$			$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau; r, y_2 + 1], 1)\pi_{x_1,q\|2}\pi_{x_2,r\|2}$
15)	$x_1 = 0$			$= -2h + \sum_{r=x_2}^{P} (\sum_{q=p}^{P} V([q, \tau; r, y_2 + 1], 1)\pi_{0,q\|2} + \pi_{0,0\|2} V([r, y_2 + 1; 0, 0], 1))\pi_{x_2,r\|2}$
	$\tau - 1$	0	1	
16)	$x_1 > 0$			$= -2h + \max\{\sum_{q=x_1}^{P} V([q, \tau; 0, 0], 1)\pi_{x_1,q\|1}\ , \sum_{q=x_1}^{P} \sum_{r=0}^{P} V([q, \tau; r, 1], 1)\pi_{x_1,q\|2}\pi_{0,r\|2}\}$
17)	$x_1 = 0$			$= -2h + \max\{\sum_{q=p}^{P} V([q, \tau; 0, 0], 1)\pi_{0,q\|1} + \pi_{0,0\|1} V([0, 0; 0, 0], 1)\ ,$ $\sum_{r=0}^{P} (\sum_{q=p}^{P} V([q, \tau; r, 1], 1)\pi_{p,q\|2} + \pi_{0,0\|2} V([r, 1; 0, 0], 1))\pi_{0,r\|2}\}$
18)	$< \tau - 1$	$< \tau - 1$	2	$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, y_1 + 1; r, y_2 + 1], 2)\pi_{x_1,q\|2}\pi_{x_2,r\|2}$
19)	$< \tau - 1$	0	1	$= -2h + \max\{\sum_{q=x_1}^{P} V([q, y_1 + 1; 0, 0], 1)\pi_{x_1,q\|1}\ ,$ $\sum_{q=x_1}^{P} \sum_{r=0}^{P} V([q, y_1 + 1; r, 1], 2)\pi_{x_1,q\|2}\pi_{0,r\|2}\}$

TABLE 2.6. Optimality equations for multiple re-listing case (infinte planning horizon)

LEMMA 2.20. *If we assume a vigilant seller and that the first auction has received a bid, then the value functions for those states can be evaluated as follows,*

(1) *If* $x_1 > 0$, $y_1 = \tau, \delta$ *and* $z = 0, 1$, *or* $x_1 > 0$, $y_1, y_2 < \tau$ *and* $z = 2$, *then*

$$V([x_1, y_1; x_2, y_2], z) = R_1([x_1, y_1; x_2, y_2], z)$$

(2) *If* $x_1 > 0$, $y_1 < \tau, y_2 = 0, z = 1$ *then*

$$V([x_1, y_1; x_2, y_2], z) = \max\{-2h + \sum_{q=x_1}^{P} V([q, y_1 + 1; 0, 0], 1)\pi_{x_1,q|1} \ , R_1([x_1, y_1; 0, 0], 2)\}$$

where $R_1(\cdot)$ *represents the value of having both items released and a positive price in the first auction, for* $x_1 > 0$, *and* $z = 0, 1, 2$,

$$R_1(S = ([x_1, y_1; x_2, y_2], z)) = -h(2\tau - y_1 - y_2) + E[X_{1,\tau}|S = ([x_1, y_1; x_2, y_2], z)] + E[X_{2,\tau}|S = ([x_1, y_1; x_2, y_2], z)] + (\pi_{x_2,0|z})(\pi_{0,0|z})^{\tau - y_1 - 1}(\pi_{0,0|1})^{y_1 - y_2} v(0, 0)$$

where $E[X_{i,\tau}|S = ([x_1, y_1; x_2, y_2], z)]$ *is defined by (2.17) and* $v(0, 0)$ *is defined by (2.28).*

Proof of Lemma 2.20 - see Appendix A.

The implication of Lemma 2.20 is that once a bid arrives in the first auction, there are no loops back to the initial state. Therefore, the number of auctions in each of the periods an auction is underway and the expected final price of each auction can be determined. Hence the problem is reduced to the case of guaranteed successful auctions. Consequently the optimal decision when $X_1 > 0$ and $Y_2 = 0$ follows a control limit policy. This result also holds when both auctions are underway but the second auction has received a bid as summarized in the following result.

Lemma 2.21. *If we assume a vigilant seller and that both auctions are underway but only the second auction has received a bid, then the value functions can be evaluated as follows, for* $y_1, y_2 < \tau, x_1 = 0, x_2 > 0, z = 2$,

$$V([x_1, y_1; x_2, y_2], z) = R_2([x_1, y_1; x_2, y_2], 2)$$

where $R_2(\cdot)$ represents the value of having both auctions underway and a positive current price in the second auction, for $y_1, y_2 < \tau$, $x_1 = 0, x_2 > 0$, and $z = 2$,

$$\begin{aligned} R_2(S = ([x_1, y_1; x_2, y_2], z)) = & -2h(\tau - y_1) + E[X_{1,\tau}|S = ([x_1, y_1; x_2, y_2], z)] \\ & + (1 - (\pi_{x_1,0|z})^{\tau - y_1})(-h(y_1 - y_2) + E[X_{2,\tau}|S = ([x_1, y_1; x_2, y_2], z)]) \\ & + (\pi_{x_1,0|z})^{\tau - y_1} E[V(S')|S = ([x_1, y_1; x_2, y_2], z)] \end{aligned}$$

where $S' = ([X_2, y_2 + \tau - y_1; 0, 0], 1)$, $X_2 > 0$, $V(S')$ is calculated according to the second case of Lemma 2.20 and the conditional expectation is with respect to X_2, and $E[X_{i,\tau}|S = s]$ is defined by (2.17).

Proof of Lemma 2.21 - see Appendix A.

The implication of Lemma 2.21 is that the *possible* decision to re-list item 1, which happens with probability $(\pi_{0,0|2})^{\tau - t_1}$, follows a control limit policy. For all other cases, in order to determine the optimal solution and policy, one has to solve the optimality equations either using value iteration or policy iteration. Below the resulting 8 cases of the optimality equation are summarized. The value function for those states where a positive bid has arrived have been separated from the states where neither auction has received a bid. The issue with the multiple re-listing case is exactly when no bid has arrived and the potential for looping back to the starting state exist.

(1) $y_1 = \tau, \delta, z = 0, 1$, or $y_1, y_2 < \tau, x_1 > 0, z = 2$

$$V([x_1, y_1; x_2, y_2], z) = R_1([x_1, y_1; x_2, y_2], z)$$

(2) $y_1 < \tau, y_2 = 0, x_1 > 0, z = 1$,

$$V([x_1, y_1; x_2, y_2], z) = \max\{-2h + \sum_{q=x_1}^{P} V([q, y_1 + 1; 0, 0], 1)\pi_{x_1,q|1}\ , R_1([x_1, y_1; 0, 0], 2)\}$$

(3) $y_1, y_2 < \tau, x_1 = 0, x_2 > 0, z = 2$

$$V([x_1, y_1; x_2, y_2], z) = R_2([0, y_1; x_2, y_2], 2)$$

(4) $y_1, y_2 = \tau - 1, x_1, x_2 = 0, z = 2,$

$$\begin{aligned} V([x_1, y_1; x_2, y_2], z) = &- 2h + \sum_{q=p}^{P} \sum_{r=p}^{P} R_1([q, \tau; r, \tau], 0)\pi_{0,q|2}\pi_{0,r|2} + \sum_{q=p}^{P} R_1([q, \tau; 0, 0], 1)\pi_{0,q|2}\pi_{0,0|2} \\ &+ \sum_{r=p}^{P} R_1([r, \tau; 0, 0], 1)\pi_{0,0|2}\pi_{0,r|2} + (\pi_{0,0|2})^2 V([0, 0; 0, 0], 1) \end{aligned}$$

(5) $y_1 = \tau - 1, y_2 < \tau - 1, x_1 = x_2 = 0, z = 2,$

$$\begin{aligned} V([x_1, y_1; x_2, y_2], z) = &- 2h + \sum_{q=p}^{P} \sum_{r=0}^{P} R_1([q, \tau; r, y_2 + 1], 1)\pi_{0,q|2}\pi_{0,r|2} + \sum_{r=p}^{P} V([r, y_2 + 1; 0, 0], 1)\pi_{0,0|2}\pi_{0,r|2} \\ &+ (\pi_{0,0|2})^2 V([0, y_2 + 1; 0, 0], 1) \end{aligned}$$

(6) $y_1 = \tau - 1, y_2 = 0, x_1 = 0, z = 1,$

$$\begin{aligned} V([x_1, y_1; x_2, y_2], z) = &- 2h + \max\{\sum_{q=p}^{P} V([q, \tau; 0, 0], 1)\pi_{0,q|1} + \pi_{0,0|1} V([0, 0; 0, 0], 1) \ , \\ &\sum_{q=p}^{P} \sum_{r=0}^{P} R_1([q, \tau; r, 1], 1)\pi_{p,q|2}\pi_{0,r|2} + \sum_{r=p}^{P} V([r, 1; 0, 0], 1)\pi_{0,0|2}\pi_{0,r|2} \\ &+ (\pi_{0,0|2})^2 V([0, 1; 0, 0], 1) \ \} \end{aligned}$$

(7) $y_1 < \tau - 1, y_2 < \tau - 1, x_1 = x_2 = 0, z = 2,$

$$\begin{aligned} V([x_1, y_1; x_2, y_2], z) = &- 2h + \sum_{q=p}^{P} \sum_{r=0}^{P} R_1([q, y_1 + 1; r, y_2 + 1], 2)\pi_{0,q|2}\pi_{0,r|2} \\ &+ \sum_{r=p}^{P} R_2([0, y_1 + 1; r, y_2 + 1], 2)\pi_{0,0|2}\pi_{0,r|2} + V([0, y_1 + 1; 0, y_2 + 1], 2)\pi_{0,0|2}\pi_{0,0|2} \end{aligned}$$

(8) $y_1 < \tau - 1, y_2 = 0, x_1 = 0, z = 1,$

$$V([x_1, y_1; x_2, y_2], z) = -2h + \max\{\sum_{q=p}^{P} V([q, y_1+1; 0, 0], 1)\pi_{x_1,q|1} + V([0, y_1+1; 0, 0], 1)\pi_{0,0|1} \;,$$

$$\sum_{q=p}^{P}\sum_{r=0}^{P} R_1([q, y_1+1; r, 1], 2)\pi_{0,q|2}\pi_{0,r|2} + \sum_{r=p}^{P} R_2([0, y_1+1; r, 1], 1)\pi_{0,0|2}\pi_{0,r|2}$$

$$+ \pi_{0,0|2}\pi_{0,0|2}V([0, y_1+1; 0, 1], 2)\;\}$$

4.4. Numerical Example. Next we illustrate this with a numerical example. This example shows that the optimal policy in the multiple re-listing case might not be a threshold policy. Let prices be discrete, $\tau = 2$, $p = k = 10$, $P = 30$ and the transition probability matrices be defined as follows,

$$\Pi_1 = \begin{array}{c} \\ 0 \\ 10 \\ 20 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .5 & .2 & .2 & .1 \\ 0 & .6 & .3 & .1 \\ 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \Pi_2 = \begin{array}{c} \\ 0 \\ 10 \\ 20 \\ 30 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 10 & 20 & 30 \end{array} \\ \begin{pmatrix} .6 & .2 & .2 & 0 \\ 0 & .7 & .3 & 0 \\ 0 & 0 & .65 & .35 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

In Table 2.7 below the optimal policy, derived using policy iteration with $V(\Delta) = 0$, for various holding costs is shown. Note in particular that there are instances when it may be optimal to release the second item when $X_1 = 0$ yet defer if $X_1 > 0$. For example, if $h = 2.75$ and the first auction elapsed one period, $Y_1 = 1$, then we see that it is optimal to release the item 2 if $X_1 = 0$ (or $X_1 \geq \$20$), but optimal to defer the release if $X_1 = \$10$. Note that this scenario can occur since at the start of the first auction it is optimal to defer the release of the item 2 and $\pi_{0,0|1} > 0$, but that this can not occur if, for instance, $h = 3$, since then the optimal decision at the start of the first auction is to release item 2, and hence there is no decision to be made when $Y_1 = 1$. If, however, both auctions are unsuccessful, which happens with probability $(.6)^{2*2} = .1296$ then the problem is back to its original state at which it was optimal to release both items.

	$Y_1 = 0$		$Y_1 = 1$		
	$X_1 = 0$	$X_1 = 0$	$X_1 = 10$	$X_1 = 20$	$X_1 = 30$
$3.6 \leq h$	Release	Release	Release	Release	Release
$2.9 \leq h \leq 3.5$	Release	Release	Defer	Release	Release
$2.7 \leq h \leq 2.8$	Defer	Release	Defer	Release	Release
$1.9 \leq h \leq 2.6$	Defer	Defer	Defer	Release	Release
$1.3 \leq h \leq 1.8$	Defer	Defer	Defer	Defer	Release
$h \leq 1.2$	Defer	Defer	Defer	Defer	Defer

TABLE 2.7. Optimal decision as a function of various holding costs for numerical example.

5. Discussion

This chapter has analyzed the problem of strategically releasing items for auction in order to maximize profit. The objective has been to provide a framework for modeling the dynamics of competing auctions and derive structural properties on the optimal auction release policy. The two main underlying assumptions that formed the basis for our analysis were: 1) each period an item remains unsold a holding cost is incurred, and 2) competing auctions 'cannibalize' each other and decrease the expected final price of each auction. Two scenarios were analyzed - guaranteed successful auctions and possibly unsuccessful auctions. For the first case the problem reduces to a finite horizon MDP, while the second case results in an infinite horizon negative dynamic program. Given certain structural assumptions on the transition probabilities, we were able to show that in the first case the optimal release policy is a control limit policy in the current price of the ongoing auction. Furthermore, we showed that the control limit is decreasing in the holding cost. However, for the case when there is a positive probability that an auction may be unsuccessful, the optimal policy does not have to be a control limit policy. The problem that arises is that the optimal decision when the ongoing auction has not received any bids may or may not be consistent with a control limit policy.

The main managerial insight and contribution of this chapter is that there is a significant value of understanding the special dynamics of online auctions, and that by using a price adaptive or closed loop policy a seller can improve his expected total profit. Because online auctions are rather inexpensive to conduct and administer, they are becoming more and more popular as alternative salvage channels. In industries where the value of 'old' items depreciates quickly, such as consumer electronics or fashion goods, being able to optimally sell excess inventory quickly can be of great importance. In addition, even though the gain on each individual item may be small, the overall impact can be quite substantial as the size of the inventory grows.

In the concluding Chapter 6, an overview and some insights regarding the two most obvious and perhaps important extension are provided. Namely the general N-item case and the case when price-transitions also depend on the price of the competing auction. Other extensions, that will not be covered in this thesis, include continuous time decision making and incorporating the decision regarding other auction parameters. In particular, the decision regarding how long an auction should last. That is, when should an auction be started and for how long should it last.

CHAPTER 3

Fixed Bidding Strategies: The Two Auctions Case

1. Objective

In the previous chapter the analysis focused on the within period price-transitions of auctions. We assumed auctions evolve according to a specified stochastic process, and that within a period, transitions of price follow a given probability distribution. Furthermore, structural properties on the distribution function were provided such that the optimal release policy is of a threshold type. In the succeeding Chapter 4, we describe a statistical model for analyzing the within period price-transitions based on the Dell auction data. The model there is based on that within a period, price-increments follow a zero-inflated gamma distribution. Chapter 4 also includes an empirical validation of the structural results of Chapter 2, based on data from 4,000 eBay auctions of *Dell Financial Services* (DFS). Neither Chapter 2 or 4 includes any specifics regarding what or how individual bidders behave. Since this approach is different from the traditional auction theory framework, the reader may be interested to know what underlying individual bidding behavior would result in stochastic processes such as the ones discussed. In particular, what game theoretic model might apply, and what would the within period price-transition probability distribution look like at a bid strategy *equilibrium.* In other words, does there exists a bid strategy (Bayesian Nash) equilibrium for a given set of auction rules and bidder attributes? And what is the resulting within period price-transition distribution function?

To the disappointment of some, this chapter will not address these questions. Instead we analyze two fixed bidding strategies and derive the *cumulative distribution function* (CDF) of the within period price-transitions. That is, the objective is not to provide an equilibrium analysis on the individual bidding strategies, but to discuss the resulting within period price-transitions from two specific bidding strategies: 1) bid the minimum increment above the lowest priced auction, 2) bid truthfully your valuation in the lowest priced auction. The former is the bid strategy proposed in Peters and Severinov (2006), while the latter is an extension of the bid strategy proposed in Vickrey (1961). The truthful bidding strategy is also what eBay promotes bidders to do.[1] More details and discussion follow. In other words, the objective is to illustrate how the CDF for the within period price-transitions can be derived from the individual bidding behavior. Depending on the auction rules, assumptions of how bidders arrive and their possible strategies, the transition probabilities might be very difficult to derive. For simple bidding strategies and/or simple properties due to certain assumptions, closed form solutions to the CDF may exist. The first strategy we analyze provides an example of this. For more complicated situations, such as the second strategy analyzed, upper and lower bounds on the CDF can be given. For even more complicated situations, one may have to revert to simulation in order to estimate the CDF.

One reason why the CDF of the within period price-transitions is of interest, is that it enables us to model how auctions progress, and consequently to make better informed decisions during an auction. For instance, a seller might be interested to know whether or not to start another auction, while a bidder might be interested to know if he should place a bid, and if so how much to bid. By understanding how auctions evolve both sellers and bidders are able to make better decisions. A second reason why we are interested in the CDF of the within period price-transitions, is that it provides a mechanism for analyzing the variation in final price of online auctions. In Figure 3.1 the distribution of the final price for six products auctioned by DFS on eBay from February 2005 to January 2007 are shown. In

[1] http://pages.ebay.com/help/buy/buying-ov.html

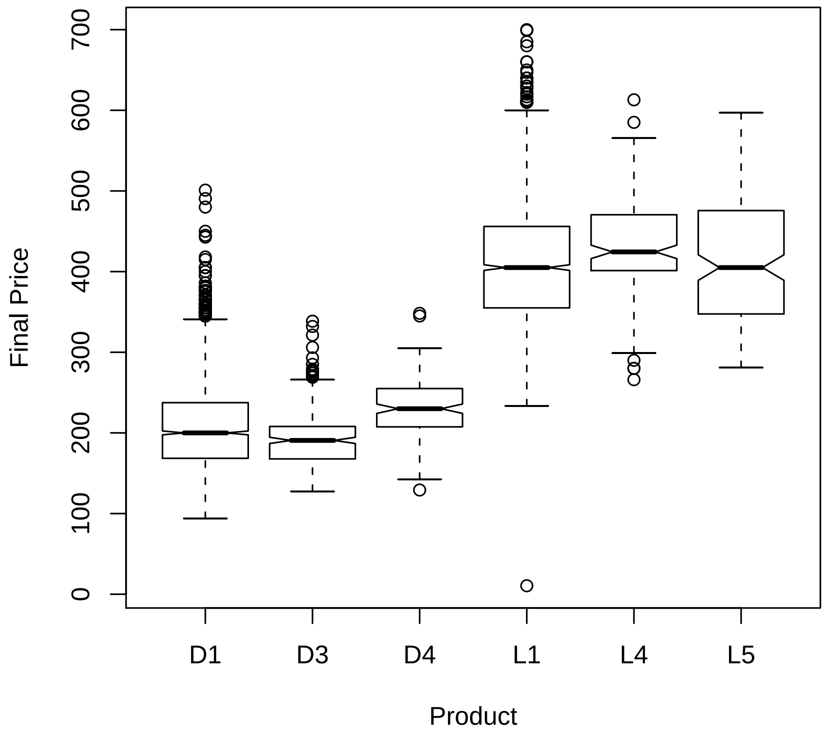


FIGURE 3.1. Distribution of final price for the six products analyzed. D1 and L1 represent a product line, while D3, D4, L4, and L5 represent a specific product. More information and descriptive statistics is provided in Chapter 1 and 5

each box-plot the lower and upper edge of the box represents the 25^{th} and 75^{th} percentiles. The line inside the box represents the median final price. The dashed lines ('whiskers') from each box, are drawn to the observation furthest away but within a factor of $1.5 \times IQR$, from the edge of the box. The IQR is the inner quartile range. Circles outside the whiskers are observations, which could be classified as extreme and potentially outliers. The notches inside the box indicates a range around the median. An informal statistical test if the median from two box-plots are different, is if the notches overlap. We note that for all six products the final price exhibits great variation. A natural question is: what drives this apparent variation? Furthermore, is it possible to characterize the conditional distribution of the final price given that an auction has elapsed for some time and is currently at some price.

The traditional auction theory answer is that the variation in final price is a direct result of the variation in bidders' valuation. For instance, in a single sealed-bid second-price auction, where bidders follow the Nash equilibrium strategy of bidding truthfully, the distribution of the final price follows the order statistics of the bidder with the second highest valuation [**14**, p.15]. In the online auction setting, there would seem to be at least one more source of variation. Namely, the number of bidders. A key difference between traditional auctions and online auctions, is that in the former the number of bidders is fixed, while in the latter the number of bidders varies between auctions and over time. This chapter describes a method for incorporating the stochastic number of bidders into the CDF of the within period price-transition as well as final price.

Though there are many papers that analyze the distribution regarding the final price of auctions, there are few that focus on the progression of prices or how auctions evolve. One paper that explicitly model the progression is Segev, Beam and Shantikumar (2001). They consider a single auction and model price as a Markov chain where the 'active' bidders form a queue. Their model is based on a *first-price* auction format, bidders arriving according to a Poisson process, and a bidding strategy identical to the minimal bid increment strategy. This chapter differs from their paper in that we consider a *second-price* auction format, and analyze the progression of price in two ongoing auctions.

Overview of Chapter 3. The chapter is organized as follows. In Section 2 an overview of the auction rules, bidders' valuation and arrival process is provided. Section 3 and 4 discuss the two bidding strategies. Section 5 summarizes the findings.

1.1. Notation. The notation in this chapter will be slightly different than the one used in the previous chapter. In Chapter 2, elapsed auction time was modeled as a discrete random variable and represented by Y. In contrast, we now define elapsed auction time as continuous and denote it by t. Suppose two auctions are underway simultaneously, then we index auctions by i, and define X_{i,t_i} as the current price of auction i that has elapsed t_i time units, $i = 1, 2$. The objective of the chapter is to derive the conditional cumulative distribution function of the price in auction i at the end of a time interval $[t, t + \Delta t]$,

$$(3.1) \quad F^z_{X_{i,t_i+\Delta t}|X_{i,t_i}}(q|x_i) = \Pr\{X_{i,t_i+\Delta t} \leq q|X_{i,t_i} = x_i, Z_{[t,t+\Delta t]} = z\} \qquad i, z = 1, 2$$

where Δt is the length of the interval and $Z_{[t,t+\Delta t]}$ is the number of ongoing auctions during the time interval (including the one under consideration). Similar to Chapter 2, this chapter only considers the case of at most two ongoing auctions. Though the results for the first bidding strategy extends to the general N auction case, the notation becomes more convoluted with no substantial gain. For the second bidding strategy the extension to general N auctions becomes a lot more complicated. In order to simplify notation we write $F^z_i(q|x_i)$ instead of $F^z_{X_{i,t_i+\Delta t}|X_{i,t_i+\Delta t}}(q|x_i)$, and $Z_{\Delta t}$ instead of $Z_{[t,t+\Delta t]}$.

2. Auction Rules and Bidder Attributes

We consider a simplified version of the eBay auction format, namely a *second-price* auction format. That is, the highest bidder wins but only has to pay the amount of the second highest bid, or in the case of only one bidder the starting price p. We do not include a reserve price, and although a minimum bid increment could be included, omitting it simplifies the notation. Therefore, suppose an auction has elapsed for t time units, then the price of the auction, X_t, is either: 1) the second highest bid placed so far given that two or more bids have been placed, 2) the starting price p given that only one bid has been placed so far, or 3) 0 if no bids have arrived. Let B be the bid amount placed by a bidder, and

$B_t^{(j)}$ be the j^{th} highest bid amount so far, e.g. $B_t^{(2)}$ is the second highest bid placed in an auction that has elapsed t time units. The price of an auction that has elapsed t time units is,

$$X_t = \begin{cases} 0 & \text{no bid has been placed} \\ p & \text{one bid has been placed} \\ B_t^{(2)} & \text{at least two bids have been placed} \end{cases} \tag{3.2}$$

If two bidders bid the same amount then the bidder that placed his bid first becomes the high bidder. We define X_{i,t_i} as the price of auction i after it has elapsed for t_i time units, and H_{i,t_i} as the hidden or censored value of the highest bid, $i = 1, 2$. Both bidders and the seller can only observe X_{i,t_i}, $i = 1, 2$. The state of the two auctions is defined by $[X_{1,t_1}, H_{1,t_1}, t_1; X_{2,t_2}, H_{2,t_2}, t_2]$. For example, suppose elapsed auction time is measured in hours, then $[15, 31.01, 36; 5, 7, 12]$ denotes the system state where auction 1 has elapsed for 36 hours, is currently priced at \$15 (the second highest bid placed so far in that auction), and has a current high-bid of \$31.01, and auction 2 has elapsed for 12 hours, current price of \$5, and a current high-bid of \$7.

We assume the seller does not keep track of who is currently the high-bidder or who the bidders have been. The information regarding who the high-bidder is, or rather if there has been a change of the high-bidder, does provide additional information that can be used to further derive information regarding the size of the high-bid in the two auctions. The derivations in Section 3 and 4 are strictly with respect to the current price of an auction.

Bidders have unit demand and private valuation, V, which is independent and identically distributed across bidders and over time, with support on the interval $[p, P]$. In other words, p is both the starting price of each auction and the lower bound of the valuation. We define

$G_V(v)$ as the distribution function of the valuation, for $v \in \Re$,

$$G_V(v) = \Pr\{V \leq v\} \tag{3.3}$$

Given that a bidder's valuation exceeds a threshold r, we define the *conditional* cumulative distribution function of a bidder's valuation by $G_V(v|r)$, for $v \in \Re$,

$$G_V(v|r) = \Pr\{V \leq v | V \geq r\} = \frac{G_V(v) - G_V(r)}{1 - G_V(r)} \tag{3.4}$$

The conditional CDF provides information regarding the amount of the censored high-bid, and is required to characterize the probability distribution of the price-transitions. The density function associated with each distribution function is symbolized by a lower case letter,

$$g_V(v) = \frac{\partial}{\partial v} G_V(v)$$
$$g_V(v|r) = \frac{\partial}{\partial v} G_V(v|r) = \frac{\partial}{\partial v} \frac{G_V(v) - G_V(r)}{1 - G_V(r)} = \frac{g_V(v)}{1 - G_V(r)}$$

For simplicity of notation we omit the subscript V and write $G(v)$ $(g(v))$ and $G(v|r)$ $(g(v|r))$ instead of $G_V(v)$ $(g_V(v))$ respectively $G_V(v|r)$ $(g_V(v|r))$. We define $V_{(j)}$ to be the j^{th} order statistic of the valuation of bidders that have arrived so far. For instance, the random variable $V_{(3)}$ denotes the third highest valuation of the bidders that have arrived. Next we describe the sequence of events depicted in Figure 3.2.

The sequence of events are as follows. Suppose at time t two auctions are underway, the system state is $([x_1, h_1, t_1; x_2, h_2, t_2], 2)$, and a bidder with valuation $V = v$ arrives. The bidder can only observe the price and elapsed time of the two auctions, i.e. he only observes x_1, t_1, x_2, t_2. If the lowest priced auction is above his valuation then he leaves and never returns, i.e. if $v \leq \min\{x_1, x_2\}$. If, on the other hand, his valuation is not below the lowest priced auction then he will bid in a given auction; the bid is prescribed by the

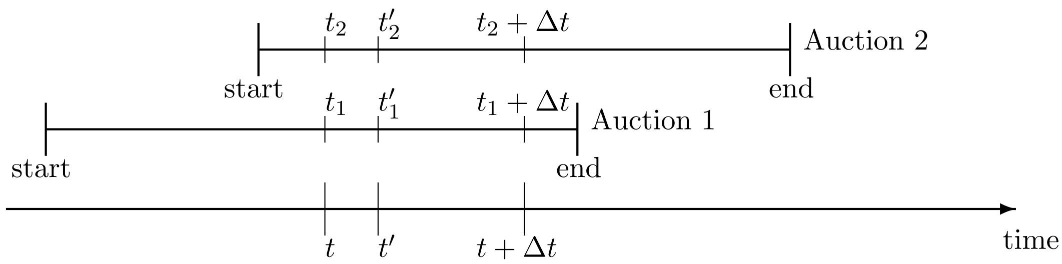


FIGURE 3.2. Time-line for the sequence of bidding events. At time t a bidder with valuation $V = v$ arrives and observes x_1, t_1, x_2, t_2. If $v \leq \min\{x_1, x_2\}$, then the bidder leaves, while if $v > \min\{x_1, x_2\}$, then he will place a bid according to a fixed bidding strategy. The bidder will remain at the auction site and keep bidding until he either wins an auction or the price in both auctions exceed his valuation v. At time t' another bidder, with valuation $V = v'$, arrives and observes x'_1, t'_1, x'_2, t'_2. He too will follow the same fixed bid strategy, and remain at the auction site until either he wins an auction or both auctions exceed his valuation v'. The objective of this chapter is to characterize the prices at time $t + \Delta t$ when the system state is $([X_1, U_1, t_1 + \Delta t; X_2, U_2, t_2 + \Delta t], 2)$, given an initial state $([x_1, U_1, t_1; x_2, U_2, t_2], 2)$ at time t.

specific *bidding strategy.* A bid strategy is a function that specifies the auction and amount a bidder should bid upon arriving at the auction site. We define B_i as the amount a bidder bids in auction i, $i = 1, 2$, and $\mathbf{B} = (B_1, B_2)$ as the pair of bid amounts for the two auctions. If a bidder chooses not to place a bid in auction i then $B_i = 0$, $i = 1, 2$. We assume no bidder will place a positive bid amount above P or below p, i.e. $B_i \in \{0\} \cup [p, P]$, $i = 1, 2$. Given that bidders have unit demand, bidders will not place positive bids in both auctions at the same time. The two bidding strategies discussed in this chapter only depend on how many auctions are underway, the price of the auctions, and the valuation of the bidder,

$$\mathbf{B} : \underbrace{\{1, 2\}}_{Z} \times (\underbrace{\{0\} \cup [p, P]}_{X_1}, \underbrace{\{0\} \cup [p, P]}_{X_2}) \times \underbrace{[p, P]}_{V} \longrightarrow (\underbrace{\{0\} \cup [p, P]}_{B_1}, \underbrace{\{0\} \cup [p, P]}_{B_2}) \tag{3.5}$$

Additional factors that make the bidding strategy more realistic include the elapsed time of the auctions, risk aversion, and bid history. In Chapter 5 an empirical analysis of 44,000 individual bids placed in 4,000 desktop and laptop auctions of DFS is provided. The objective there is to characterize the bid-increment and explore the relationship to various factors. For instance, how does the price of an auction and/or the price of a competing auction affect the mean bid-increment.

The bidder will remain at the auction site and keep bidding until he either wins an auction or the lowest priced auction exceeds his valuation v. That is, suppose the bidder placed a bid in auction 1, but that the amount he bid did not make him the current high-bidder, then he will again observe the prices of both auctions and follow the same logic once more, i.e. if the lowest priced auction exceeds v then he leaves, else he bids according to the given bid strategy. If a bidder does become the high-bidder then he remains at the auction site and observes the progression of the auctions. If at some point he is outbid, then he observes the prices in both auctions, and by the same logic as before, will leave if v is less than the lowest priced auction, and else bid according to the fixed bidding strategy. That a bidder 'remains at the auction site' does not necessarily mean that he is continuously observing the auction site. For instance, he could go about his affairs and when another bidder places a bid he receives notification. On eBay, for instance, you receive an email that you have been outbid.

To illustrate, label the bidder arriving at time t with valuation v as Bidder 1. Suppose that at time t, Bidder 1 become the high-bidder in auction 1, and that the former high-bidder (the bidder that Bidder 1 replaced) decided to leave. Furthermore, suppose that at time t' a second bidder (Bidder 2) with valuation $V = v'$ arrives. Bidder 2 observes x_1', t_1', x_2', t_2', and by the same logic, if $v' \leq \min\{x_1', x_2'\}$, then he will leave, and if $v' > \min\{x_1', x_2'\}$, then he will place a bid according to the fixed bid strategy. Suppose Bidder 2 bids in auction 1, and replaces Bidder 1 as the high-bidder. Bidder 1 will now observe the prices in both auctions, and follow the same procedure as before, i.e. if v is less than the lowest priced auction then he leaves, else he bids according to the given bid strategy.

We assume that when placing a bid no time elapses. This implies that no time elapses between when a bidder arrives and he is the high-bidder in an auction or the price in both auctions exceeds his valuation. And consequently, since a bidder will remain at the auction site until he either wins an auction or the price in both auctions exceeds his valuation, at any given time there will be at most one bidder per auction present. Namely the high bidder of each auction. The assumption that no time elapses in placing a bid is clearly unrealistic and purely for modeling purposes.

In addition, we assume that bidders do not speculate regarding how many auctions will be released in the future. More specifically, we assume bidders do not consider the possibility that additional auctions may start at a later time. That is, a bidder with valuation above the lowest priced auction, would never choose not to place a bid, and instead re-visit at a later time in hope for a lower priced auction. If the lowest priced auction is below a bidder's valuation then he bids, otherwise he leaves and never returns. Furthermore, if only one auction is underway and a bidder is outbid he leaves and never returns to see if a second auction has started. Next we discuss the arrival process of bidders.

We assume bidders arrive according to an exogenously given stochastic process that only depends on time. Let $M_{[t,t+\Delta t]}$ be the number of bidders that arrive in a time interval $[t, t+\Delta t]$, and,

$$\rho_M(m|[t,t+\Delta t]) = \Pr\{M_{[t,t+\Delta t]} = m\} \qquad m = 0, 1, 2, \ldots \tag{3.6}$$

For simplicity of notation we write $M_{\Delta t}$ and $\rho_M(m|\Delta t)$, instead of $M_{[t,t+\Delta t]}$ and $\rho_M(m|[t,t+\Delta t])$. Other variables that might affect the the arrival rate to the auction site include the price and elapsed time of the auctions, and the number of ongoing auctions. As an example of an arrival process, bidders may arrive according to a non-homogeneous

Poisson process with rate λ_t. The distribution of the number of bidders arriving in the interval $[t, t + \Delta t]$ is then given by,

$$\rho_M(m|\Delta t) = \frac{e^{-\gamma(t)}(\gamma(t))^m}{m!} \qquad\qquad m = 0, 1, 2, \ldots$$

where

$$\gamma(t) = \int_t^{t+\Delta t} \lambda_u du$$

Next we show how the CDF of the within period price-transitions can be evaluated given the above auctions rules and two specific bidding strategies.

3. Minimum Bid Increment Strategy

The first strategy we consider is when each bidder bids a minimal increment above the lowest priced auction. A bidder will stop bidding once he is the high-bidder in an auction or the price is above his valuation. If he is the high-bidder but subsequently is out-bid then he evaluates all auctions and follows the same bid strategy as before. In other words, the process is as follows,

(1) A bidder with valuation $V = v$ arrives to the auction site.

(2) He observes the price in all ongoing auctions.

(3) If the price in the lowest priced auction is below his valuation ($v > \min\{X_1, X_2\}$), then he bids in the lowest priced auction an amount that is the lesser of a minimal increment k above the price or his valuation ($B = \min\{X_1 + k, X_2 + k, v$). If the auctions are priced equally, then with probability .5 he chooses either one.

  (a) If he is successful and becomes the high-bidder, then he continuously observe how the auction progress and if he is ever outbid returns to step (2).

  (b) If he is not successful and his bid does not make him the high-bidder, then he returns to step (2).

(4) If the lowest priced auction is above his valuation ($v \leq \min\{X_1, X_2\}$), then he leaves the auction site and never returns.

Note that this implies that the amount of the high-bid in either auction at any given time is the minimal increment above the current price, i.e. $H_i = X_i + k$. However, the high-bidder will remain at the auction site until the auctions exceed his valuation or he wins an auction. We illustrate with a numerical example which is depicted in Figure 3.3.

Suppose at time t, the system state is $([10, 11, 2; 8, 9, 1], 2)$, Bidder 1 with valuation $V = 15$ is the high-bidder in auction 1, Bidder 2 with valuation $V = 17$ is the high-bidder in auction 2, and that Bidder 3 with valuation $V = 10$ arrives to the auction site. Bidder 3 observes the prices in both auctions and places a bid in auction 2 for \$9. This raises the price in auction 2 to \$9, but does not make Bidder 3 the high-bidder, since Bidder 2 placed his bid first. Bidder 3 then observes the prices again, and places a bid for \$10 in auction 2. Bidder 3 replaces Bidder 2 as the high-bidder, while the price in auction 2 remains \$9. Bidder 2 now observes the two auctions, and places a bid for \$10 in auction 2. The price in auction 2 jumps to \$10 but due to that Bidder 3 placed his bid first, he remains as the high-bidder. Bidder 2 observes that the price in both auctions is \$10, chooses one with probability .5, and places a bid of \$11. Suppose that Bidder 2 chose auction 2 and replaced Bidder 3 as the high-bidder. Bidder 3 now observes that the price in both auctions is at \$10, which is his valuation, and therefore decides to leave the auction site. We assume that no time elapsed from that Bidder 3 arrived at the auction site until he left the auction site. At time t' the next bidder arrives or auction 1 has ended.

This bidding strategy is discussed by Peters and Severinov (2006), who show that for M bidders and N simultaneously released auctions, this strategy leads to a Bayesian-Nash equilibrium. An implicit assumption in their model is that there is no 'friction' or 'cost' for

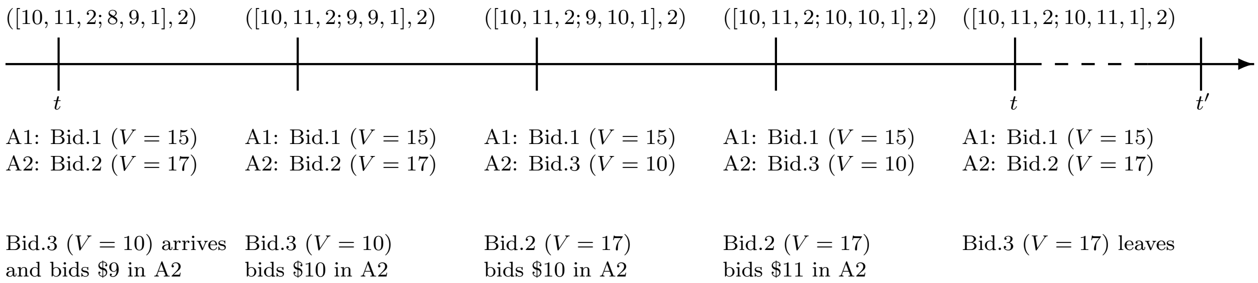


FIGURE 3.3. Illustration of the sequence of bidding events for the minimal bid increment strategy. It is assumed that no time elapses from Bidder 3 arrives until he leaves.

placing bids [**18**, p.223]. In other words, no time elapses when placing a bid. For instance, if placing a bid takes time and there is a positive probability that a bid will not get registered in the final moments (e.g. eBay auctions), then the above strategy would not result in an equilibrium. That is, a potential bidder who doubts that his valuation is or will be among the N highest, would be better off by not allowing the price of the auctions to raise to his valuation. Contrariwise, it could be more beneficial for such bidders to wait until the final moments of the auction and place one single bid.

Regardless whether the strategy results in an equilibrium, we assume all bidders follow this strategy. Similar to Peters and Severinov (2006) we assume placing a bid takes no time, i.e. the system is 'frictionless'. This results in that the current price follows the order statistics of the $N+1$ highest valuation of the bidders [**18**, p.229]. In other words, if there is only one auction underway then the price at any given time is $V_{(2)}$, i.e. the second highest valuation of the bidders that has visited so far. While if two auctions are underway, then the price of both auctions is $V_{(3)}$, i.e. the third highest valuation of the bidders that has visited so far.

An assumption that should be made explicit is that bidders are not time-sensitive. That is, a bidder's decision is strictly based on price and not on how much time is remaining. A major difference between the setting analyzed in this thesis and, for instance, Peters and Severinov (2006) is that we include the possibility that one auction will end before the other.

As a result it may seem more reasonable that the bidders would include the time aspect in their decision making. And this may further question whether this strategy results in an equilibrium.

To derive the distribution function for the within period price-transitions we consider the case of one and two ongoing auctions separately.

3.1. One Ongoing Auction. We start by considering the case when there is only one ongoing auction, which without loss of generality we index as auction 1. Since there is only one ongoing auction and bidders are non-speculative regarding the possibility that a second auction will start, the CDF of the within period price-transition follows the distribution of the second highest valuation that arrives in the time-interval $[t, t + \Delta t]$. Therefore, we can condition upon how many bidders arrive and determine the order statistics of the valuation of bidders. However, if there is a current high-bidder then he has a valuation above the current price and we first condition upon his valuation. Therefore, there are two possible starting states $([0, 0, t_1; 0, 0, 0], 1)$ and $([x, x + k; 0, 0], 1)$. Recall that the system state is defined by the price, high-bid, and elapsed auction time of each auction, $([X_1, H_1, t_1; X_2, H_2, t_2], Z)$, and implicitly it is assumed the second auction has not started and no information regarding it is available to potential bidders. The results for the two possible initial states are summarized in the following lemmas.

LEMMA 3.1. *If no bids have been placed and the system state is* $([0, 0, t_1; 0, 0, 0], 1)$*, and auction 2 will not be released during time-interval* $[t, t + \Delta t]$*, then the CDF of the within period price-transition of auction 1 is,*

$$F_1^1(q|0) = \sum_{m=0}^{\infty} \Pr\{X_{1,t+\Delta t} \leq q | X_{1,t} = 0, Z_{\Delta t} = 1, M_{\Delta t} = m\} \rho_M(m|\Delta t)$$

where $\Pr\{X_{1,t+\Delta t} \le q | X_{1,t} = 0, Z_{\Delta t} = 1, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < 0 \\ 1 & 0 \le q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - G(q)) & p \le q \end{cases} \qquad m \ge 1$$

Proof Lemma 3.1 - See Appendix B.

Lemma 3.2. *If at least one bid has been placed and the system state is* $([x, x+k, t_1; 0, 0, 0], 1)$, *and auction 2 will not be released during time-interval* $[t, t+\Delta t]$, *then the CDF of the within period price-transition of auction 1 is,*

$$\begin{aligned} F_1^1(q|x) &= \int_x^P \Pr\{X_{t+\Delta t} \le q | X_t = x, V_{(1)} = v, Z_{\Delta t} = 1\} g(v|x) dv \\ &= \int_x^P \sum_{m=0}^{\infty} \Pr\{X_{t+\Delta t} \le q | X_t = x, V_{(1)} = v, Z_{\Delta t} = 1, M_{\Delta t} = m\} \rho_M(m) g(v|x) dv \end{aligned}$$

where $\Pr\{X_{t+\Delta t} \leq q | X_t = x, V_{(1)} = v, Z_{\Delta t} = 1, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x \\ 1 & x \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x \\ G(q) & x \leq q < v \\ 1 & v \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v \\ (G(q))^m + m(G(q))^{m-1}(1 - G(q)) & v \leq q \end{cases} \qquad m \geq 2$$

Proof Lemma 3.2 - See Appendix B.

Note that if $x = p$ in Lemma 3.2, then $g(v|x) = g(v)$.

3.2. Two Ongoing Auctions. Next we consider when two auctions are underway. Using the same approach as the one auction case, we condition upon the number of arriving bidders and the current high-bidders' valuations. There are a two cases regarding when the two auctions were released.

Case 1: Auctions Released Simultaneously

If the auctions are released simultaneously then if at least two bidders arrive the price will be the same in both auctions [**18**, p.229]. This is a consequence that a bidder will bid up to his valuation and not leave the auction site until both auctions exceed his valuation. If on the other hand only one bidder arrives then the price in one auction will be p while in the other it will be zero. And if no bidder has arrived, then the price is of course zero in both. That is there are only three possible states at time t: $([0,0,t_1;0,0,t_2],2)$, $([p,p+k,t_1;0,0,t_2],2)$, $([x,x+k,t_1;x,x+k,t_2],2)$. Below the CDF of the within period price-transition for each auction and each possible starting state is provided.

LEMMA 3.3. *If no bids have been placed and the system state is* $([0,0,t_1;0,0,t_2],2)$*, and neither auction will expire during the time-interval* $[t,t+\Delta t]$*, then the CDF of the within period price-transition of auction* i *is, for* $i=1,2$*,*

$$F_i^2(q|0)=\sum_{m=0}^{\infty}\Pr\{X_{i,t_i+\Delta t}\leq q|X_{i,t_i}=0,Z_{\Delta t}=2,M_{\Delta t}=m\}\rho_M(m|\Delta t)$$

where $\Pr\{X_{i,t_i+\Delta t}\leq q|X_{i,t_i}=0,Z_{\Delta t}=2,M_{\Delta t}=m\}$

$$=\begin{cases}0 & q<0\\ 1 & 0\leq q\end{cases}\qquad m=0$$

$$=\begin{cases}0 & q<0\\ .5 & 0\leq q<p\\ 1 & p\leq q\end{cases}\qquad m=1$$

$$=\begin{cases}0 & q<p\\ (G(q))^m+m(G(q))^{m-1}(1-G(q))+\binom{m}{2}(G(q))^{m-2}(1-G(q))^2 & p\leq q\end{cases}\qquad m\geq 2$$

Proof Lemma 3.3 - See Appendix B.

LEMMA 3.4. *If only one bid has been placed and the system state is* $([p,p+k,t_1;0,0,t_2],2)$ *(or* $([0,0,t_1;p,p+k,t_2],2)$*), and neither auction will expire during the time-interval* $[t,t+$

$\Delta t]$, then the CDF of the within period price-transition of auction i is, for $i = 1, 2$,

$$F_i^2(q|x_i) = \int_p^P \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2\} g(v_1) dv_1$$
$$= \int_p^P \sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) g(v_1) dv_1$$

where $\Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x_i \\ 1 & x_i \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < p \\ 1 & p \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & p \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 2$$

Proof Lemma 3.4 - See Appendix B.

Lemma 3.5. *If at least one bid in each auction has been placed and the system state is $([x, x+k, t_1; x, x+k, t_2], 2)$, and neither auction will expire during the time-interval $[t, t+\Delta t]$, then the CDF of the within period price-transition of auction i is, for $i = 1, 2$,*

$F_i^2(q|x_i)$

$$= \int_x^P \int_x^{v_1} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2, Z_{\Delta t} = 2\} \phi(v_2|x, v_1) g(v_1|x) dv_2 dv_1$$
$$= \int_x^P \int_x^{v_1} \sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) \phi(v_2|x, v_1) g(v_1|x) dv_2 dv_1$$

where $g(v_1|x)$ *is the conditional density of the* $V_{(1)}$ *given that* $V_{(1)} \geq x$*, and* $\phi(v_2|x, v_1)$ *is the conditional density of* $V_{(2)}$*, given that* $x \leq V_{(2)} \leq v_1$*,*

$$\phi(v_2|x, v_1) = \frac{\partial}{\partial v_2} \frac{G(v_2) - G(x)}{(1 - G(x)) - (1 - G(v_1))} = \frac{\partial}{\partial v_2} \frac{G(v_2) - G(x)}{G(v_1) - G(x)}$$

and $\Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x \\ 1 & x \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x \\ G(q) & x \leq q < v_2 \\ 1 & v_2 \leq q < v_1 \\ 1 & v_1 \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & v_2 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 2$$

Proof Lemma 3.5 - See Appendix B.

Case 2: Auction 2 Started After Auction 1

If the auctions were not started simultaneously then there are two additional possible states. Namely the states where the price in auction 1 is higher than the price of auction 2. Since auction 1 started before auction 2, it could have reached a price $x > p$ by the time auction 2 started. In this case there are two possible starting points for auction 2 to consider. Recall that bidders are not time sensitive and strictly focus on the price in an auction. Furthermore, bidders who have been outbid in auction 1 do not return to see if another auction has started. Below the distribution functions for the possible starting states are given.

LEMMA 3.6. *If the system state is either* $([0,0,t_1;0,0,t_2],2)$, $([p,p+k,t_1;0,0,t_2],2)$ *(*$([0,0,t_1;p,p+k,t_2],2)$*), or* $([x,x+k,t_1;x,x+k,t_2],2)$*, where* $t_2 < t_1$*, and neither auction will expire in the time-interval* $[t,t+\Delta t]$*, then the CDF of the within period price-transitions are identical to the respective CDF when the auctions were released simultaneously.*

Proof Lemma 3.6 - See Appendix B.

LEMMA 3.7. *If the system state is* $([x,x+k,t_1;0,0,t_2],2)$*, where* $x > p$ *and* $t_2 < t_1$*, and if neither auction will expire in the time-interval* $[t,t+\Delta t]$*, then the CDF of the within period price-transition of auction* i*,* $i=1,2$*, is given by the following equations.*

For auction 1,

$$
\begin{aligned}
&F_1^2(q|x) \\
&= \int_x^P \Pr\{X_{1,t_1+\Delta t} \le q | X_{1,t_1} = x, V_{(1)} = v_1, Z_{\Delta t} = 2\} g(v_1|x) dv_1 \\
&= \int_x^P \sum_{m=0}^{\infty} \Pr\{X_{1,t_1+\Delta t} \le q | X_{1,t_1} = x, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) g(v_1|x) dv_1
\end{aligned}
$$

where $\Pr\{X_{1,t_1+\Delta t} \le q | X_{1,t_1} = x, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x \\ 1 & x \le q \end{cases} \qquad m = 0, 1$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & x \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \le q \end{cases} \qquad m \ge 2$$

For auction 2,

$$F_2^2(q|0)$$

$$= \begin{cases} \sum_{m=0}^{\infty} \Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, Z_{\Delta t} = 2, M_{\Delta t} = m\}\rho_M(m|\Delta t) & q < x \\ \int_x^P \Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, V_{(1)} = v_1, Z_{\Delta t} = 2\} g(v_1|x) dv_1 & x \le q \end{cases}$$

$$= \begin{cases} \sum_{m=0}^{\infty} \Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, Z_{\Delta t} = 2, M_{\Delta t} = m\}\rho_M(m|\Delta t) & q < x \\ \int_x^P \sum_{m=0}^{\infty} \Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, X_{1,t_1} = x, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}\rho_M(m|\Delta t) g(v_1|x) dv_1 & x \le q \end{cases}$$

where $\Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < 0 \\ 1 & 0 \le q < x \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & p \le q < x \end{cases} \qquad m \ge 1$$

and $\Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = 0, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x \\ 1 & x \le q \end{cases} \qquad m = 0, 1$$

$$= \begin{cases} (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & x \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \le q \end{cases} \qquad m \ge 2$$

Proof Lemma 3.7 - See Appendix B

LEMMA 3.8. *If the system state is* $([x_1, x_1+k, t_1; x_2, x_2+k, t_2], 2)$*, where* $p \le x_2 < x_1$ *and* $t_2 < t_1$*, and if neither auction will expire in the time-interval* $[t, t+\Delta t]$*, then the CDF of the within period price-transition of auction* i*,* $i = 1, 2$*, is given by,*

$$\begin{aligned} F_i^2(q|x_i) &= \Pr\{V_{(2)} < x_1\} F_i^2\left(q|x_i, V_{(2)} < x_1\right) + \Pr\{V_{(2)} \ge x_1\} F_i^2\left(q|x_i, V_{(2)} \ge x_1\right) \\ &= \frac{G(x_1) - G(x_2)}{1 - G(x_2)} F_i^2\left(q|x_i, V_{(2)} < x_1\right) + \frac{1 - G(x_1)}{1 - G(x_2)} F_i^2\left(q|x_i, V_{(2)} \ge x_1\right) \end{aligned}$$

where,

1) For $V_{(2)} < x_1$*,*

$F_1^2\left(q|x_1, V_{(2)} < x_1\right)$

$$= \int_{x_1}^{P} \Pr\{X_{1,t_1+\Delta t} \le q | X_{1,t_1} = x_1, V_{(1)} = v_1, V_{(2)} < x_1, Z_{\Delta t} = 2\} g(v_1|x_1) dv_1$$

$$= \int_{x_1}^{P} \sum_{m=0}^{\infty} \Pr\{X_{1,t_1+\Delta t} \le q | X_{1,t_1} = x_1, V_{(1)} = v_1, V_{(2)} < x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) g(v_1|x_1) dv_1$$

where

$$\Pr\{X_{1,t_1+\Delta t} \leq q | X_{1,t_1} = x_1, V_{(1)} = v_1, V_{(2)} < x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < x_1 \\ 1 & x_1 \leq q \end{cases} \qquad m = 0, 1$$

$$= \begin{cases} 0 & q < x_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & x_1 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 2$$

and

$$F_2^2\left(q | x_2, V_{(2)} < x_1\right)$$

$$= \int_{x_1}^{P} \int_{x_2}^{x_1} \Pr\{X_{2,y_2+\Delta t} \leq q | X_{2,y_2} = x_2, V_{(1)} = v_1, V_{(2)} = v_2 < x_2, Z_{\Delta t} = 2\} \phi(v_2 | x_2, x_1) g(v_1 | x_1) dv_2 dv_1$$

$$= \int_{x_1}^{P} \int_{x_2}^{x_1} \sum_{m=0}^{\infty} \Pr\{X_{2,y_2+\Delta t} \leq q | X_{2,y_2} = x_2, V_{(1)} = v_1, V_{(2)} = v_2 < x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) \phi(v_2 | x_2, x_1) g(v_1 | x_1) dv_2 dv_1$$

where

$$\Pr\{X_{2,t_2+\Delta t} \le q | X_{2,t_2} = x_2, V_{(1)} = v_1, V_{(2)} = v_2 < x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < x_2 \\ 1 & x_2 \le q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x_2 \\ G(q) & x_2 \le q < v_2 \\ 1 & v_2 \le q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < x_2 \\ (G(q))^m & x_2 \le q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_2 \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \le q \end{cases} \qquad m \ge 2$$

and

2) For $V_{(2)} \geq x_1,$ *and* $i = 1, 2$

$F_i^2\left(q|x_i, V_{(2)} \geq x_1\right)$

$$= \int_{x_1}^{P}\int_{x_1}^{v_1} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2 \geq x_2, Z_{\Delta t} = 2\}\phi(v_2|x_1, v_1)g(v_1|x_1)dv_2dv_1$$

$$= \int_{x_1}^{P}\int_{x_1}^{v_1}\sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2 \geq x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}\rho_M(m|\Delta t)\phi(v_2|x_1, v_1)g(v_1|x_1)dv_2dv_1$$

where

$\Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, V_{(2)} = v_2 \geq x_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$

$$= \begin{cases} 0 & q < x_i \\ 1 & x_i \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x_i \\ G(q) & x_i \leq q < v_2 \\ 1 & v_2 \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < x_i \\ (G(q))^m & x_i \leq q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_2 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 2$$

Proof Lemma 3.8 - See Appendix B

3.3. Numerical Example. We illustrate the above calculations with a numerical example. Let bidders valuations be uniformly distributed on $[.01, 1]$, and the number of arriving bidders in the interval $[t, t + \Delta t]$, be uniform distributed over 0, 1, 2, 3, and 4, i.e. $\rho_M(m|\Delta t) = .2$, for $m = 0, 1, 2, 3, 4$, and $\rho_M(m|\Delta t) = 0$, for $m \geq 5$. The minimal bid-increment is assumed to be very small ($< .00001$), and does not affect the derivations. Similarly, the value of Δt is irrelevant, but can be thought of as, for instance, 24 hours. The resulting within period price-transition CDF for different starting states are shown in Figure 3.4.

The upper left graph shows the CDF when only one auction (defined as auction 1), is underway with current price $X_{1,t_1} = 0, .01, .2, .4, .6, .8$, as labeled in the graph. For instance, we see that if $X_{1,t_1} = \$0$ (the top line), then the probability that $X_{1,t_1+\Delta t} \leq \$.4$, is slightly greater than .6. Whereas if $X_{1,t_1} = \$.2$ (the third line from the top), then the probability that $X_{1,t_1+\Delta t} \leq \$.4$, has decreased to slightly below .4.

The upper right graph shows the CDF for two simultaneously released auctions when the price in the auctions $(X_1, X_2) =$ (0,0), (.1,0), (.1,.1), (.2,.2), (.4,.4), (.6,.6), (.8,.8), as labeled. The second case when $(X_1, X_2) = (.1, 0)$, is slightly different from the others and therefore represented by a dotted line. The difference is that $F_1^2(q|.01) = 0$, while $F_2^2(q|0) > 0$, for $q \leq \$.01$. However, for $q > \$.01$, $F_1^2(q|.01) = F_2^2(q|0)$. We see that if the price in both auctions is \$0 (the top line), then the probability that $X_{i,t_i+\Delta t} \leq \$.4$, is about .85, $i = 1, 2$. While if both auctions are priced at \$.2, the probability that $X_{i,t_i+\Delta t} \leq \$.4$, is slightly less than .6, $i = 1, 2$.

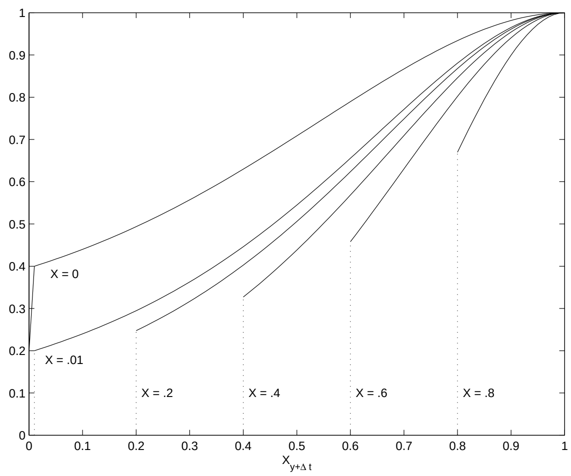
The two graphs in the bottom represent the CDF when the two auctions were started sequentially such that $t_2 < t_1$. The bottom left graph shows the CDF for auction 1 and

2 when the prices at time t, $(X_1, X_2) = (.2,0)$, $(.4,0)$, $(.6,0)$, $(.8,0)$. The CDF for auction 1 is represented by the dashed line, and is of course 0 for $X_{1,t_1+\Delta t} < \$x_1$. The CDF for auction 2 is represented by the dotted line for $X_{2,t_1+\Delta t} < \$x_1$, and coincide with auction 1's dashed line for $X_{2,t_1+\Delta t} \geq \$x_1$. That is, the CDF for both auctions are identical for $X_{i,t_i+\Delta t} > \$x_1$, but differ for $X_{i,t_i+\Delta t} < \$x_1$, $i = 1, 2$. For instance, if $(X_1, X_2) = (.2, 0)$, then the probability that either auction is priced $\leq \$.4$, at time $t + \Delta t$ is about .7. While the probability that $X_{i,t_i+\Delta t} \leq \$.1$, is 0 for auction 1, and slightly above .4 for auction 2.

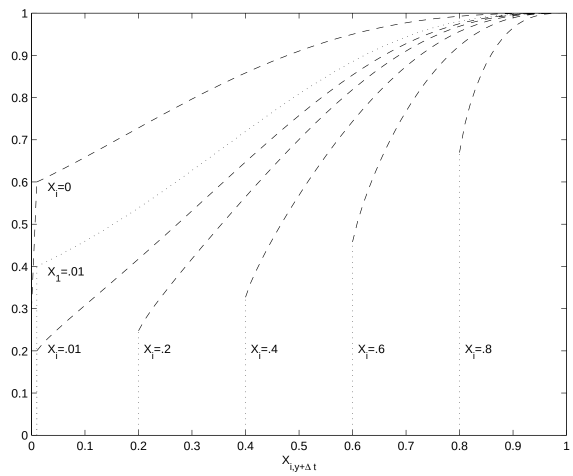
The bottom right graph shows the CDF for auction 1 and 2 when $(X_1, X_2) = (.2,.1)$, $(.4,.3)$, $(.6,.5)$, $(.8,.7)$. The features are similar to the previous case in that for $q \geq x_1$, the CDF for both auctions are identical, $F_1^2(q|x_1) = F_2^2(q|x_2)$. While for $x_2 \leq q < x_1$, the CDF for auction 1 is 0 while the CDF for auction 2 is positive. For $q < x_2$ both auctions' CDF is of course 0. For instance, if $(X_1, X_2) = (.2, .1)$, then the probability that $X_{i,t_i+\Delta t} \leq \$.4$ is about .55, $i = 1, 2$, and the probability that $X_{i,t_i+\Delta t} \leq \$.15$, is 0 for auction 1 $(i = 1)$ and about .25 for auction 2 $(i = 2)$.

From the graphs we can visually assess and confirm that Assumptions 2.1, 2.2, and 2.3 from Chapter 2 holds. Assumption 2.1 was that the CDF of the within period price-transitions is decreasing in price. That is, at a higher current price the CDF decreases for a given threshold. This can immediately be confirmed since in each of the four graphs the CDF for higher initial prices are always beneath the CDF for the lower initial prices. For instance, in the top two graphs, the top line represents when current price is \$0, and the bottom most line when current price is \$.8.

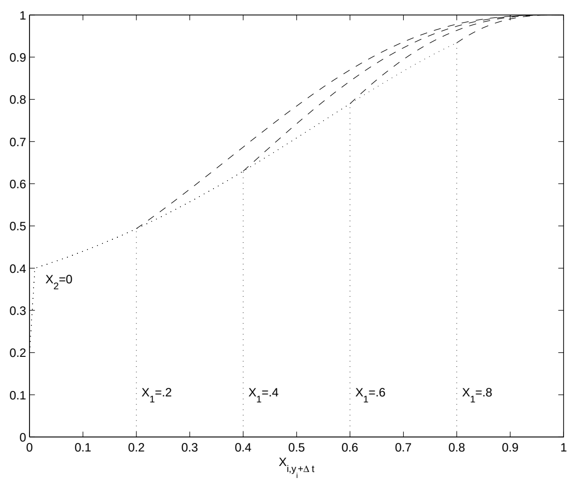
Assumption 2.2 stated that the CDF of the within period price-transitions increases when there are two rather than one ongoing auctions. In order to confirm this assumption,



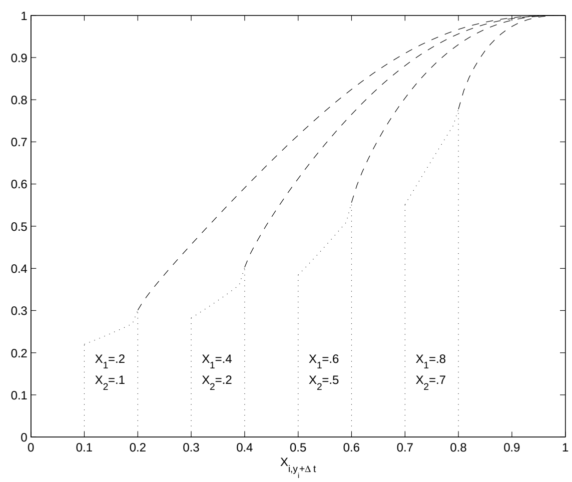
CDF for One Auction



CDF for Two Simultaneous Auctions



CDF for Two Sequential Auctions ($X_2 = 0$)



CDF for Two Sequential Auctions

FIGURE 3.4. The conditional CDF of the within period price-transitions when bidders valuation is uniform on $[.01, 1]$, the number of arriving bidders is uniform between 0 and 4, and each bidder follows the minimal bid increment strategy. The top left graph is when there is only one auction, the top right when there are two simultaneously started auctions, and the bottom when there are two overlapping auctions. In each graph the lines represent the distribution function given the initial prices (X_1, X_2) as labeled.

for a given current price $X_{i,t_i} = x_i$ and future price $X_{i,t_i+\Delta t}$, $i = 1, 2$, we compare the graphs when there are two auctions with the graph when there is only one auction. For instance, let $X_i = \$.4$, and suppose we are interested to evaluate the probability that $X_{i,t_i+\Delta t} \leq \$.5$. In the top left graph we see that when there is only one ongoing auction then the probability is slightly above .4. If there are two simultaneously released auctions (top right graph), then the probability has increased to about .6. While if the two auctions were started sequentially, we see that if $(X_1, X_2) = (.4, 0)$ (bottom left) or $(X_1, X_2) = (.4, .2)$ (bottom right), then the probability is about .75 and about .6 respectively. Since the probability increased for all cases when there were two ongoing auctions, this 'confirms' that there is a cannibalization effect and that the CDF is increasing in number of ongoing auctions.

The final Assumption 2.3, which states that the cannibalization effect is diminishing in price, is a bit more difficult to visually confirm. Again we compare, for a given initial price $X_{i,t_i} = x_i$ and future price $X_{i,t_i+\Delta t}$, $i = 1, 2$, the value of the CDF in the top left graph with the corresponding value in one of the other three graphs. For instance, suppose we are interested to compare the cannibalization effect when $X_{i,t_i} = \$.2$ and $X_{i,t_i} = \$.4$. If there is only one auction then the probability that $X_{i,t_i+\Delta t} \leq \$.6$, is about .62 and .59 when $X_{i,t_i} = \$.2$ and $\$.4$ respectively. When there are two simultaneously released auctions the probabilities have increased to about .82 and .74 for $X_{i,t_i} = \$.2$ and $\$.4$ respectively. We note that the difference in CDF at $X_{i,t_i} = \$.4$, about .15 (=.74-.59), is less than the difference in CDF at $X_{i,t_i} = \$.2$, about .2 (=.82-.62). A similar comparison can be made with the bottom graphs to the top left graph as well. This indicates that the cannibalization effect is diminishing in price, and provides support for Assumption 2.3

Next we discuss the bidding strategy when bidders bid truthfully in the lowest priced auction.

4. Truthful Bidding Strategy

A potential implication of the previous bidding strategy is that each bidder may end up placing several small bids and eventually still be outbid. Though we do not consider the effort or time it may take to submit a bid, in reality this might be a very tiresome and frustrating strategy. As an alternative to the previous strategy we consider when bidders simply bid their valuation in the lowest priced auction, i.e. truthfully bid their valuation. That is, the bidding behavior is as follows,

(1) A bidder with valuation $V = v$ arrives.

(2) He observes the price in all ongoing auctions.

(3) If the lowest priced auction is below his valuation ($v > \min\{X_1, X_2\}$), then he bids in the lowest priced auction his valuation ($B = v$). If the auctions are priced equally, then with probability .5 he chooses one.

  (a) If he is successful and becomes the high-bidder, then he continuously observe how the auction progress and if he is ever outbid returns to step (2).

  (b) If he is not successful and his bid does not make him the high-bidder, then he returns to step (2).

(4) If the lowest priced auction is above his valuation ($v \leq \min\{X_1, X_2\}$), then he leaves the auction site and never returns.

Similar to Section 3 we assume placing a bid takes no time. That is, no time elapses from a bidder arriving until he either is registered as a high-bidder or leaves the auction site. When there is only one ongoing auction, and no speculation regarding additional auctions released, this leads to an equilibrium outcome as originally shown by Vickrey (1961). Furthermore, this bidding strategy is consistent with what eBay promotes bidders to do, as seen in the following quote,

> "When you place a bid, we suggest that you enter the maximum amount that you're willing to pay for the item. (You won't necessarily pay the amount of your maximum bid.) eBay compares your bid to those of other bidders and increases your bid on your behalf using only as much of your bid as is necessary to maintain your high bid position."[2]

However, it is fairly clear why this is not an optimal bidding strategy if there are more than two auctions underway. The issue that can arise, as described below and unlike the previous bidding strategy, is that the bidders with the two highest valuations bid against each other. This results in that the bidder with the highest valuation has to pay a price equal to the second highest valuation, while the bidder with the second highest valuation only has to pay a price equal to the third highest valuation. Therefore, a bidder with a high valuation, who considers the chance that he has the highest valuation to be sufficient, will have an incentive not to bid truthfully. Instead, he should attempt to keep at par with the bidder with the second highest valuation, and make sure he does not end up in a bidding war with him. That is, he would be better of by following the minimal bid-increment strategy.

Numerical examples will illustrate possible sequence of events. Suppose that at time t, two auctions are underway, the system state is $[.25, .5, 2; .1, .25, 1]$, and a bidder with valuation $V = v$ arrives. Recall that the high-bids are censored and that the bidder will always participate in the lowest priced auction. There are then three possible transitions depending on the value of v.

$$[.25, .5, 2; .1, .25, 1] \Longrightarrow \begin{cases} [.25, .5, 2; .1, .25, 1] & v \leq .1 \\ [.25, .5, 2; v, .25, 1] & .1 < v \leq .25 \\ [.25, .5, 2; .25, v, 1] & .25 \leq v \end{cases}$$

[2] http://pages.ebay.com/help/buy/buying-ov.html

In the first case, when $v \leq .1$, the bidder simply leaves the auctions and the state of the auctions remain the same. In the second case, when $.1 < v \leq .25$, the bidder places a bid v in auction 2 and is immediately outbid and leaves the auction site. And in the third case, when $.25 < v$, the bidder places a bid v in auction 2 and becomes the high-bidder, while the previous high-bidder whose previous bid was .25 leaves the auctions.

Suppose instead the system state is $[.1, .5, 2; .1, .25, 1]$, and again a bidder with valuation $V = v$ arrives. There are now six possible transitions depending on v.

$$[.1, .5, 2; .1, .25, 2] \Longrightarrow \begin{cases} [.1, .5, 2; .1, .25, 1] & & v \leq .1 \\ [v, .5, 2; v, .25, 1] & & .1 \leq v \leq .25 \\ \begin{cases} [v, .5, 2; .25, v, 1] & w.\ Pr. = .5 \\ [.25, .5, 2; .25, v, 1] & w.\ Pr. = .5 \end{cases} & & .25 \leq v \leq .5 \\ \begin{cases} [.5, v, 2; .25, .5, 1] & w.\ Pr. = .5 \\ [.25, .5, 2; .25, v, 1] & w.\ Pr. = .5 \end{cases} & & .5 \leq v \end{cases}$$

In the first case the bidder simply leaves the auction and the state of the auctions remain the same. In the second case the bidder places a bid v first in one auction, and upon being outbid tries the other auction and then leaves (the order in which he bids in the auctions does not matter). In the third and fourth case, when $.25 \leq v \leq .5$, the bidder will with probability .5 choose one auction. If he chooses auction 1 first then he is immediately outbid and places a bid in auction 2 where he becomes the high-bidder. If he chooses auction 2 then he outbids the current high-bidder, who then tries to place a bid in auction 1. And in the fifth and sixth case, when $.5 \leq v$, the same logic applies. The bidder will choose one auction with equal probability and therefore either end up as the high-bidder in auction 1 or auction 2, and the high-bidder he replaces will try to bid in the other auction.

In order to derive the CDF of $X_{i,t_i+\Delta t}$, $i = 1, 2$, at the end of the time interval $[t, t+\Delta t]$, the same approach as in the previous bid strategy is employed. That is, first we condition on the valuation of the high-bidder in each auction, and then on the number of bidders that arrive in $[t, t + \Delta t]$. Based on this conditional information we derive the probability that $X_{i,t_i+\Delta t} \leq q$, for a given threshold q, $i = 1, 2$. However, due to that it is possible the bidder with the second highest valuation bids against the bidder with the highest valuation, there are situations in which the price is different in the two auctions. Therefore the calculations of the CDF are a bit more complicated. In Appendix C we outline how upper and lower bounds on the CDF can be derived for the case when there is one respectively two simultaneously released auctions. The case for sequentially released auctions further complicate the price-transitions as described in Appendix C. The underlying idea with the calculations in Appendix C is that given an initial state $[X_1, H_1, t_1; X_2, H_2, t_2]$, by conditioning on the value of H_1, H_2, and number of arriving bidders, for a given threshold q, upper and lower bounds on the probability that $X_{i,t_i+\Delta t} \leq q$ can be given. We illustrate the resulting CDF for the one and two simultaneously started auctions with a numerical example.

4.1. Numerical example. Similar to the numerical example in Section 3.3, let bidders valuation be uniformly distributed on $[.01, 1]$, and the number of arriving bidders in the interval $[t, t + \Delta t]$, be uniform distributed over 0, 1, 2, 3, and 4, i.e. $\rho_M(m|\Delta t) = .2$ for $m = 0, 1, 2, 3, 4$, and $\rho_M(m|\Delta t) = 0$ for $m \geq 5$. The value of Δt is irrelevant, but can be thought of as, for instance, 24 hours. The resulting within period price-transition CDF, for different starting states of one respectively two simultaneously started auctions, are shown in Figure 3.5. The upper left graph shows the CDF when there is only one ongoing auction and is identical to the graph for the minimum bid-increment strategy. See Section 3.3 for more comments.

The upper right graph shows the CDF when the auctions were started simultaneously, and the prices are identical as labeled, i.e. when $t_1 = t_2$ and $(X_1, X_2) =$ (.01,.01), (.2,.2), (.4,.4), (.6,.6), (.8,.8). The upper solid line represents the upper bound, and the lower dashed line represents the lower bounds. For instance, if (X_1, X_2)=(.2,.2), then the probability that $X_{i,t_i+\Delta t} \leq \$.4$, has an upper bound of about .55, and lower bound of about .48. A property with the auctions when they are simultaneously released and priced equally, is that the price-transitions are Markovian. Specifically, no additional information regarding which auction (auction 1 or 2) has the high-bid corresponding to the highest valuation $V_{(1)}$, is gained by keeping track of the auctions for each arriving bidder. This is summarized in Lemma C.1 of Appendix C.

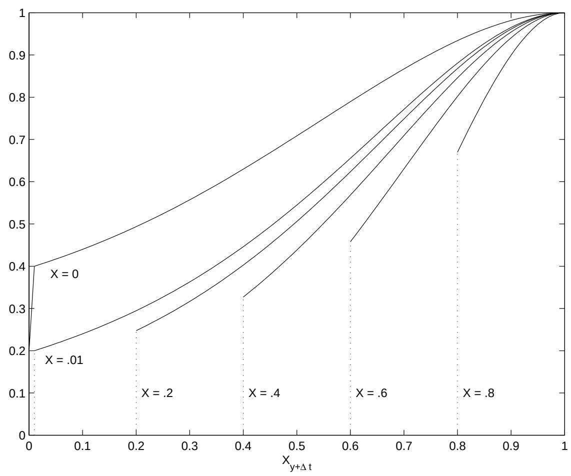
The two bottom graphs are the CDF for each individual auction when the auctions were started simultaneously but $X_2 < X_1$. The bottom left graph is for the CDF of auction 1, $F_1^2(q|x_1)$, when $X_{1,t} =$ 0, .01, .2, .4, .6, .8. Note that the bounds do not depend on the price of auction 2, only that it is priced below auction 1. For instance, suppose $X_{1,t_1} = .2$, then the probability that $X_{i,t_i+\Delta t} \leq \$.4$, is minimum .45 (lower bound), and maximum .65. A distinctive feature of the graphs is that for $X_{1,t_1} > .1$, the lower and upper bounds start apart. This is to be expected, and can be confirmed by the calculations shown in Appendix C.

The bottom right graph is for the CDF of auction 2, $F_1^2(q|x_2)$, when $(X_1, X_2) =$ (0,0), (.1,0), (.2,.3), (.5,.4), (.7,.6), (.9,.8). Note that in this case the price of auction 1 does effect the CDF, and that the lower and upper bounds for the probability that $X_{2,t_2+\Delta t} \leq x_1$, coincide. Another interesting feature is that, unlike the case of auction 1, the bounds for

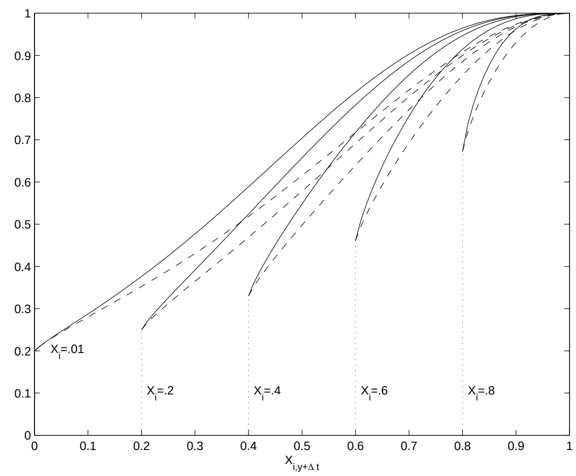
$X_{2,t_2+\Delta t} \geq x_1$, are much closer. That is, there is less difference between the upper and lower bounds for the CDF of auction 2, than there is for the CDF of auction 1. The reason for this is that the only difference in between the upper and lower bounds are for probabilities of $X_{2,t_2+\Delta t} \geq H_{1,t_1}$. Which can be seen in the calculations shown in Appendix C. To illustrate, suppose (X_1, X_2)=(.3,.2), then the upper and lower bound for the probability that $X_{2,t_2+\Delta t} \leq .4$, is about .64 and .59 respectively.

Although the figures only display the bounds, similar to Figure 3.4, Assumptions 2.1, 2.2, and 2.3 from Chapter 2 can be visually assessed. For instance, we see that in all graphs and for a given bound, the CDF associated with a higher initial price is always beneath the CDF for a lower initial price. The assumption regarding the cannibalization also seem to hold. For instance, suppose $X_{1,t_1} = .2$, and we are interested in the probability that $X_{1,t_1+\Delta} \leq .4$. When there is only one ongoing auction (top left), the probability is about .4. If the two auctions were started simultaneously and are priced equally (top right), then the upper bound is about .55, and lower bound about .48. While if the two auctions were started simultaneously but auction 2 is priced below auction 1 (bottom left), then the upper and lower bounds are about .65 and .45 respectively. Though this does not formally establish Assumption 2.2, it does provide some support that it would hold.

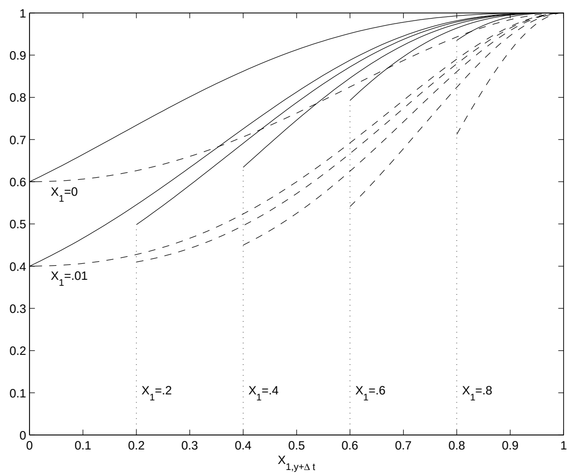
The final Assumption 2.3 is visually assessed same as with the previous bidding strategy. Suppose we are interested in comparing the cannibalization effect between $X_{1,t_1} = \$.2$ and \$.4. If there is only one ongoing auction, then the probability that $X_{1,t_1+\Delta t} \leq \$.6$, is about .62 and .59, for $X_{1,t_1} = \$.2$ and \$.4 respectively. If the two auctions were started simultaneously and are priced equally, then the probability that $X_{1,t_1+\Delta t} \leq \$.6$, has an upper/lower bound of about .79/.69 and .74/.64 for $X_{1,t_1} = \$.2$ respectively \$.4. In other words, the upper/lower difference between having one or two ongoing auctions is about .17/.07 when



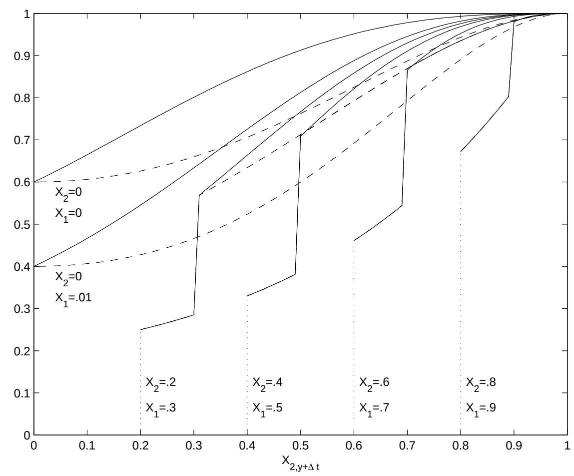
CDF for One Auction



CDF for Two Simultaneous Auctions, $X_1 = X_2$



CDF for Auction 1, $t_2 = t_1$, $X_2 < X_1$



CDF for Auction 2, $t_2 = t_1$, $X_2 < X_1$

FIGURE 3.5. The conditional CDF of the within period price-transitions when bidders valuation is uniform on $[.01, 1]$, the number of arriving bidders is uniform between 0 and 4, and each bidder follows the truthful bidding strategy. The top left graph is when there is only one auction ($Z_{\Delta t} = 1$), and the top right when there are two simultaneously started auctions and the prices are the same. The bottom graphs are for auction 1 (left) and auction 2 (right), when the auctions were started simultaneously but the prices are different. In each of the graph when there are two ongoing auctions, the solid lines indicates the upper bounds and the dashed lines the lower bounds.

$X_{1,t_1} = \$.2$, and about $.15/.05$ when $X_{1,t_1} = \$.4$. That is, the cannibalization seems to be diminishing in the price of an auction.

5. Discussion

This chapter has provided a discussion on how the conditional distribution function for the within period price-transitions can be computed. Although the focus has been with regard to two specific bidding strategies, the same methodology could be applied to other bidding strategies. Depending on the complexity of the auction rules, bidder attributes, and other assumptions, the within period price transitions might be challenging to evaluate. In some instance, like the minimal bid increment strategy, closed form solutions can be derived. In other cases, like the truthful bidding strategy, bounds can be evaluated. If the price transitions are much more complicated then a possible solutions is to estimate the conditional distribution function using simulation.

A comment regarding the truthful bidding strategy, is that it would have been possible to derive the exact CDF, by for instance evaluating the possible sample paths. That is, given an initial starting price X_{i,t_i}, number of bidders that arrive m, and threshold q, one could generate the paths for which $X_{i,t_i+\Delta t}$ ends above or below q. From this one could first calculate the probability of each path, and then determine the conditional distribution function given all possible paths. Though this might be a time consuming procedure (since it is exponentially growing in m), it is feasible and provides the exact conditional distribution function. The reason we did not show this was to illustrate how bounds can be evaluated.

The main objective of this chapter has been to show how the conditional distribution function for the within period price transitions can be derived based on the individual bidding behavior. The benefit with this is that it provides a mean for a seller to derive the CDF, based on particular bidding behavior, and verify if the assumptions from Chapter 2 holds. In other words, a seller that, rather than assume a particular distribution function of the within period price-transitions, prefers to assume certain aspects of the individual bidding behavior, can still verify if a threshold type policy is optimal based on the framework discussed in this chapter.

In addition, a seller can use the methodology presented to derive the distribution function of the final price. That is, if the length of the time-interval Δt coincides with the remaining time, then the conditional CDF of the final price is derived. Therefore, the chapter has also provided a method for deriving the conditional distribution function of the final price for an auction that has elapsed for some time t and is currently at a price level X_t. This information can then be used by a seller or buyer to make a better informed decision.

Although the two bidding strategies may, under certain conditions, result in an equilibrium outcome, the purpose has not been to analyze the equilibrium outcome. The bid strategic equilibrium analysis for auctions that are not started simultaneously is, to the best of our knowledge, still an open research topic and source for possible extensions to this chapter. In Chapter 5 we conduct an empirical analysis on how bidders actually behave.

CHAPTER 4

Empirical Analysis of Within Period Price-Increments

1. Introduction

In Chapter 2 a model for analyzing the optimal release of inventory for online auctions was discussed. The problem was formulated as a discrete time Markov Decision Process, where each period the seller has to decide whether or not to start a new auction. The framework of the analysis was that auctions evolve according to a stochastic process, and given certain properties the optimal release policy is of a threshold type. It might be natural to ask how a seller could go about and determine the process by which auctions evolve, and how to determine if the conditions in Chapter 2 holds. The objective of this chapter is three-fold. First, to present a statistical model that describes how auction prices progress. Second, to provide some structural properties on the statistical model such that the results of Chapter 2 hold. And thirdly, to empirically test and validate the proposed model based on the auction data from Dell Financial Services (DFS).

Overview of Chapter 4. This chapter is organized as follows. In Section 2 a statistical model for the within period price-increments is discussed. Section 3 provides some sufficient conditions under which the structural results of Chapter 2 hold. Section 4 gives an overview of Generalized Linear Models. Section 5 and 6 discuss the specific model formulation and the results of the empirical analysis pertaining to the DFS data. Section 7 concludes the chapter with an overall discussion.

2. Zero-Inflated Gamma Distributed Price-Increments

Let X_Y define the price of an auction that has elapsed Y periods, where X_Y is assumed continuous and non-negative and Y discrete and finite; $X_Y \geq 0$, $Y = \{0, 1, \ldots, \tau\}$. That is, each auction is divided into τ periods of equal length, and at the end of each period the price of the auction is observed. Note that there is a direct relationship between Y, the number of *elapsed* periods, and the period number. The period between Y and $Y + 1$ is defined as period Y, i.e. the first period is defined as period 0, the second period as period 1, and so on. Unlike the previous chapter, were we kept track of both items waiting to be released for auctions as well as ongoing auctions, we now only keep track of ongoing auctions. Which means that $Y = 0$ implies the auction in question has been released. Define Z_Y to be the discrete number of ongoing auctions after Y periods, $Z_Y = \{0, 1, 2, \ldots\}$. More specifically, an auction that has elapsed $Y < \tau$ periods will 'compete' with $Z_Y - 1$ additional auctions in the upcoming period Y. Since an auction is over after τ periods, and hence the number of ongoing auctions irrelevant, we define $Z_\tau \equiv 0$. The state of an auction is defined by the three variables, $S \equiv (X, Y, Z)$. A time-line for a three-period auction is provided in Figure 4.1.

We define C_Y as the price-increment in period Y, $C_Y = X_{Y+1} - X_Y$, which we assume follows a 'zero-inflated gamma distribution'. That is, the within period price-increments

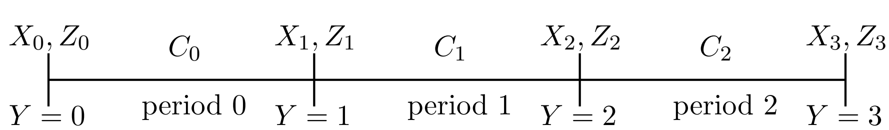


FIGURE 4.1. Time-line for an auction with 3 periods. The auction starts at $Y = 0$ and ends three periods later at $Y = 3$.

have the following density function,

$$f_{C_Y|S}(c|s) = \begin{cases} 1-\pi_s & c=0 \\ \pi_s \frac{1}{\Gamma(\nu_s)} \left(\frac{\nu_s}{\mu_s}\right)^{\nu_s} c^{\nu_s-1} e^{-\left(\frac{c\nu_s}{\mu_s}\right)} & c>0 \end{cases} \tag{4.1}$$

The above density indicates that with probability $1-\pi_s$, the within period price-increment is zero, i.e. no bids arrived and the price did not change. While if a positive price-transition occurs, then the price-increment is gamma distributed with mean μ_s and shape parameter ν_s. Note that the gamma distribution is a member of the exponential family, and that π_s, μ_s, and ν_s are functions of the auction state. More specifically we assume that transformations of π_s and μ_s are linear functions of the auction state as follows, for $s=(x,y,z)$,

(4.2)

$$\ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} \beta_{b0} + \beta_{b2} z & y=0 \\ \beta_{m0} + \beta_{m1} x + \beta_{m2} z + \beta_{m3} x \times z + \beta_{m4} \mathbf{1}_{\{y=2\}} + \cdots + \beta_{m\tau+1} \mathbf{1}_{\{y=\tau-2\}} & y \neq 0, \tau-1 \\ \beta_{e0} + \beta_{e1} x + \beta_{e2} z + \beta_{e3} x \times z & y=\tau-1 \end{cases}$$

(4.3)

$$\ln(\mu_s) = \begin{cases} \gamma_{b0} + \gamma_{b2} z & y=0 \\ \gamma_{m0} + \gamma_{m1} x + \gamma_{m2} z + \gamma_{m3} x \times z + \gamma_{m4} \mathbf{1}_{\{y=2\}} + \cdots + \gamma_{m\tau+1} \mathbf{1}_{\{y=\tau-2\}} & y \neq 0, \tau-1 \\ \gamma_{e0} + \gamma_{e1} x + \gamma_{e2} z + \gamma_{e3} x \times z & s=\tau-1 \end{cases}$$

The symbol $\mathbf{1}_{\{.\}}$ represents the indicator function, and is 1 if the argument in the brackets is true and 0 otherwise. The gamma shape parameter ν_s is assumed to only depend on the period of the auction as follows,

$$\nu_s = \nu_b \mathbf{1}_{\{y=0\}} + \nu_m \mathbf{1}_{\{y\neq 0,\tau-1\}} + \nu_e \mathbf{1}_{\{y=\tau-1\}} \tag{4.4}$$

The three cases and subscripts on the coefficients refer to the *beginning* ($Y = 0$), *middle* ($Y = 1, 2, \dots, \tau - 2$), and *end* ($Y = \tau - 1$) of an auction. The logit function (4.2) indicates that the log-odds of observing a positive price-increment is linear with respect to X_Y, Z_Y, and the interaction between X_Y and Z_Y. For the middle periods we also assume there is an additive effect depending on the period Y. Similarly, equation (4.3) means that the log of the average positive price-increment is linear with respect to the listed covariates. The reason we have separated three cases for each function is that we anticipate different dynamics for the three stages. For the first period this should be evident as all auctions start with $X_0 = 0$. That the final period might be different is due to the well-established observations that there is dramatically more bidding activity toward the end of an auction than during [**24, 28**].

2.1. Some Comments Regarding The Model. An immediate question one may have is why use the density function (4.1), and why base the parameters on the functions (4.2) and (4.3). The main attribute of the within period price-increments that we seek to model is that there is a 'high' chance of observing a 'low' price-increment, including a positive probability of a zero price-increment, and a 'low' chance of observing a high price-increment. This would seem a plausible assumption and supported by data shown in Figure 4.7, 4.8, and 4.9. In the graphs, with perhaps an exception for the final periods, most increments tend to be 'small' but there are also some observations with 'large 'increments. Therefore, the general characteristics of (4.1) seem to be supported by the data. However, the formal statistical analysis will reveal if (4.1) indeed is appropriate. Note that, if ν=1 then the positive price-increments are exponentially distributed and the model simplifies considerably.

The second part of the model regarding the linear functions, called *link* functions, defines how the parameters relate to the covariates of interest. The logit function for π_s and log of μ_s are standard choices in Generalized Linear Models (GLM) and are readily available in most statistics packages. The main benefits of using the two functions, are ease of interpretation and the restriction of the range of π_s and μ_s to the interval $[0, 1]$, respectively the non-negative real numbers. Some more details including a general overview of GLMs are discussed in Section 4.

The conditional expected within period price-increment, for $s = (x, y, z)$, is given by,

$$E[C_Y|S = s] = \pi_s \mu_s \tag{4.5}$$

From equations (4.2) and (4.3), we can solve for π_s and μ_s. To simplify the notation, we assume a given elapsed auction time $Y = y$, and write β_j (γ_j) instead of β_{bj}, β_{mj}, or β_{ej} ($\gamma_{bj}, \gamma_{mj}, \gamma_{ej}$), $j > 0$. In addition, we ignore the indicator functions $\mathbf{1}_{\{y=2\}}, \ldots, \mathbf{1}_{\{y=\tau-2\}}$, and define β_0 (γ_0) as a generic intercept, e.g. $\beta_0 = \beta_{m0} + \beta_{m4}$. Consequently, π_s and μ_s can be written as follows, for $s = (x, y, z)$,

$$\pi_s = \frac{e^{\beta_0+\beta_1 x+\beta_2 z+\beta_3 x\times z}}{1 + e^{\beta_0+\beta_1 x+\beta_2 z+\beta_3 x\times z}} = \frac{e^{\beta\mathbf{x}}}{1 + e^{\beta\mathbf{x}}} \tag{4.6}$$

$$\mu_s = e^{\gamma_0+\gamma_1 x+\gamma_2 z+\gamma_3 x\times z} = e^{\gamma\mathbf{x}} \tag{4.7}$$

Note that for $y = 0$, $\beta_1, \beta_3, \gamma_1, \gamma_3 = 0$.

Stern and Coe, 1984, fit a similar model to the analysis of daily rainfall. In their model they divide the analysis into two parts. In the first part they model the likelihood of a day having rain as a Bernoulli variable, i.e. a day is either wet or dry. While in the second part they model the amount of rainfall on wet days as being gamma distributed. Though their linear predictors are based on a harmonic series depending on previous days' rainfall the general link functions are the same. They use the logit function for the probability of observing rain/no-rain, and the log of the expected rainfall for the rate of rainfall. The paper includes a numerical illustration based on daily rainfall data from Morogoro, Tanzania. A summary of the paper appears in Section 8.4 of McCullagh and Nelder, 1989.

2.2. Maximum Likelihood Estimation. We use maximum likelihood to estimate the coefficients of the two link functions (4.2) and (4.3). The likelihood of (4.1) given n independent price-transitions, ordered such that the first k had a positive price-increment ($c > 0$), and the remaining $n-k$ had a zero price-increment ($c = 0$) is,

$$L = \prod_{i=1}^{k} \pi_{s_i} \frac{1}{\Gamma(\nu_{s_i})} \left(\frac{\nu_{s_i}}{\mu_{s_i}}\right)^{\nu_{s_i}} c_i^{\nu_{s_i}-1} e^{-\left(\frac{c_i \nu_{s_i}}{\mu_{s_i}}\right)} \prod_{j=k+1}^{n} (1-\pi_{s_j})$$

The log-likelihood is then,

(4.8)

$$\begin{aligned}
\ln(L) &= \sum_{i=1}^{k} \ln(\pi_{s_i}) + \sum_{j=k+1}^{n} \ln(1-\pi_{s_j}) + \sum_{i=1}^{k} \ln\left(\frac{1}{\Gamma(\nu_{s_i})} \left(\frac{\nu_{s_i}}{\mu_{s_i}}\right)^{\nu_{s_i}} c_i^{\nu_{s_i}-1} e^{-\left(\frac{c_i \nu_{s_i}}{\mu_{s_i}}\right)}\right) \\
&= \sum_{i=1}^{k} \ln(\pi_{s_i}) + \sum_{j=k+1}^{n} \ln(1-\pi_{s_j}) + \sum_{i=1}^{k} - \ln(\Gamma(\nu_{s_i})) + \nu_{s_i} \ln\left(\frac{\nu_{s_i}}{\mu_{s_i}}\right) + (\nu_{s_i}-1)\ln(c_i) - \frac{c_i \nu_{s_i}}{\mu_{s_i}}
\end{aligned}$$

Note that maximizing (4.8) is identical to separately maximizing a likelihood corresponding to n independent Bernoulli distributed random variables with $\Pr\{\text{success}\} = \pi_{s_i}$ and $\Pr\{\text{failure}\} = 1 - \pi_{s_i}$, with k 'success' ($n-k$ 'failure'), and k independent gamma distributed random variables with mean μ and shape parameter ν. We formally summarize this in the next lemma.

LEMMA 4.1. *The maximization of the log-likelihood function (4.8) is identical to separately maximizing the log-likelihood of n independent Bernoulli distributed random variables with $\Pr\{\mathit{success}\} = \pi_{s_i}$, with k 'success' and $n-k$ 'failure', and k independent gamma distributed random variables with parameters μ_{s_i} and ν_{s_i}.*

Proof Lemma 4.1 - The likelihood of n independent Bernoulli distributed random variables, $O_i = 0, 1,\ i = 1, \ldots, n$, with $\Pr\{\text{success}\} = \pi_{s_i}$, ordered such that the first k observations were 'success' and the remaining $n-k$ observations were 'failure', is given by, $L_B = \prod_{i=1}^{k} \pi_{s_i} \prod_{j=k+1}^{n} (1-\pi_{s_j})$. The log-likelihood is therefore, $\ln(L_B) = \sum_{i=1}^{k} \ln(\pi_{s_i}) + \sum_{j=k+1}^{n} \ln(1-\pi_{s_j})$.

The likelihood of k independent gamma distributed random variables, $C_i > 0$, $i = 1, \ldots, k$, with mean μ_{s_i} and shape parameter ν_{s_i} is given by, $L_E = \prod_{i=1}^{k} (1/\Gamma(\nu_{s_i}))(\nu_{s_i}/\mu_{s_i})^{\nu_{s_i}} c_i^{\nu_{s_i}-1} \exp(-c_i \nu_{s_i}/\mu_{s_i})$. The log-likelihood is therefore $\ln(L_E) = \sum_{i=1}^{k} \ln(1/\Gamma(\nu_{s_i})) + \nu_{s_i} \ln(\nu_{s_i}/\mu_{s_i}) + (\nu_{s_i} - 1)\ln(c_i) - c_i \nu_{s_i}/\mu_{s_i}$. $\square$

If all n observations were taken during the same stage, i.e. beginning, middle, or end, then the same instance of (4.2), (4.3), and gamma shape parameter ν_s applies to all observations, and (4.8) becomes,

$$
\begin{aligned}
&\ln(L) \\
&= \sum_{i=1}^{k} \ln(\pi_{s_i}) + \sum_{j=k+1}^{n} \ln(1-\pi_{s_j}) + k(\nu \ln(\nu) - \ln(\Gamma(\nu))) - \nu \sum_{i=1}^{k} \ln(\mu_{s_i}) - \frac{\nu-1}{\nu} \ln(c_i) + c_i/\mu_{s_i} \\
&= \sum_{i=1}^{k} \ln\left(\frac{e^{\beta \mathbf{x}_i}}{1+e^{\beta \mathbf{x}_i}}\right) - \sum_{j=k+1}^{n} \ln(1+e^{\beta \mathbf{x}_j}) + k(\nu \ln(\nu) - \ln(\Gamma(\nu))) - \nu \sum_{i=1}^{k} \gamma \mathbf{x}_i - \frac{\nu-1}{\nu} \ln(c_i) + c_i e^{-\gamma \mathbf{x}_i} \\
&= \sum_{i=1}^{k} \beta \mathbf{x}_i - \sum_{j=1}^{n} \ln(1+e^{\beta \mathbf{x}_j}) + k(\nu \ln(\nu) - \ln(\Gamma(\nu))) - \nu \sum_{i=1}^{k} \beta \mathbf{x}_i - \frac{\nu-1}{\nu} \ln(c_i) + c_i e^{-\beta \mathbf{x}_i}
\end{aligned}
$$

where $\beta \mathbf{x}_i$ and $\gamma \mathbf{x}_i$ are the linear predictors of observation i's vector of covariates. The objective therefore becomes to find the β and γ coefficients that maximizes the above expression. As a consequence of Lemma 4.1, we can estimate the β and γ coefficients using the framework of GLM in two steps. In the first step we create a new random variable as follows,

$$O_i = \begin{cases} 0 & c_i = 0 \\ \\ 1 & c_i > 0 \end{cases} \qquad\qquad i = 1, 2, \ldots, n$$

and then use a standard GLM algorithm to fit the β coefficients such that

$$\Pr\{O = o | S = s\} = \begin{cases} 1 - \pi_s & o = 0 \\ \\ \pi_s & o = 1 \end{cases} \tag{4.9}$$

and (4.2) holds. In the second step we limit the analysis to the auctions for which a positive price-increment occurred. In other words, we only consider the k auctions for which a positive price-increment occurred and use a GLM algorithm to fit the γ coefficients to the density function $g_{C|S}(c|s) = (1/\Gamma(\nu_s))(\nu_s/\mu_s)^{\nu_s} c^{\nu_s - 1} \exp(-c\nu_s/\mu_s)$ such that (4.3) holds.

Section 4 provides a discussion and references for the most common algorithm to estimate β and γ. Note that, due to Lemma 4.1, the proposed model with zero-inflated gamma distributed within period price-increments, is almost identical to the rainfall model proposed by Stern and Coe, 1984. Next we discuss what properties on the β and γ coefficients will support the main results from the Chapter 2.

3. Structural Properties For Optimal Auction Release Policy

In Chapter 2 we presented a model for the optimal release of inventory for online auctions. The main results, that the value function is increasing and that a threshold-type release policy is optimal, relied on three assumptions regarding the cumulative distribution function of the within period price-transitions. The assumptions were: 1) 'Monotonicity in price ', meaning that the probability of observing jumps to high prices is increasing in the current price of an auction, 2) 'Cannibalization effect', defined to be that for a specific auction, the more ongoing auctions there are, the less likely you are to observe price-transitions to high prices, and 3) 'Diminishing cannibalization', meaning that the gain for a specific auction, of having fewer ongoing auctions, is decreasing in the current price of the auction. The objective of this section is to provide structural properties on the β and γ coefficients under which the assumptions and consequently the results of Chapter 2 holds.

If the within period price-increment follow (4.1), then the cumulative distribution function of the within period price-transition is given by, for $s = (x, z, y)$, $c \geq 0$,

$$
\begin{aligned}
F_{X_{Y+1}|S}(x + c|s) &= 1 - \pi_s + \pi_s \int_0^c \frac{1}{\Gamma(\nu)} \left(\frac{\nu}{\mu_s}\right)^{\nu} u^{\nu-1} e^{-(u\nu/\mu_s)} du \\
&= 1 - \pi_s \left(1 - \frac{1}{\Gamma(\nu)} \int_0^{c\nu/\mu_s} u^{\nu-1} e^{-u} du\right) \\
&= 1 - \pi_s (1 - G_{C|S}(c|s))
\end{aligned} \tag{4.10}
$$

where,

$$
\ln\left(\frac{\pi_s}{1 - \pi_s}\right) = \beta_0 + \beta_1 x + \beta_2 z + \beta_3 x \times z
$$

$$
\ln(\mu_s) = \gamma_0 + \gamma_1 x + \gamma_2 z + \gamma_3 x \times z
$$

and $G_{C|S}(c|s)$ is the cumulative distribution of positive price-increments. The integral term in the second equality of (4.10) is the *lower incomplete gamma function*.[1] For simplicity of notation, the above equations are, similar to (4.6) and (4.7) in Section 2.1, based on a given period $Y = y$, and written with respect to generic β_j and γ_j coefficients. Furthermore, the gamma shape parameter ν in (4.10), has for simplicity of notation, been written without a subscript s. Implicitly, however, we assume that ν depends on the state of the auction s according to (4.4).

[1]cf. `http://mathworld.wolfram.com/IncompleteGammaFunction.html`

Though the assumptions in Chapter 2 were stated with respect to the distribution function of the within period price-transitions, in this section we focus on the *conditional expected within period price-transition* $E[X_{Y+1}|S]$, and provide structural conditions on the β and γ coefficients to ensure various monotone properties. The reason for this is two fold. First, it provides a more intuitive discussion by focusing on the effect on the expected or average price-transition, rather than the effect of the distribution function. Second, it simplifies the analysis, as might be evident by (4.10). The conditional expected within period price-transition is given by, for $s = (x, y, z)$,

$$E[X_{Y+1}|S = s] = x + \pi_s \mu_s = x + \frac{e^{\beta \mathbf{x}}}{1 + e^{\beta \mathbf{x}}} e^{\gamma \mathbf{x}} \tag{4.11}$$

where,

$$\ln\left(\frac{\pi_s}{1 - \pi_s}\right) = \beta_0 + \beta_1 x + \beta_2 z + \beta_3 x \times z$$

$$\ln(\mu_s) = \gamma_0 + \gamma_1 x + \gamma_2 z + \gamma_3 x \times z$$

To summarize, the objective with this section is to provide sufficient conditions such that (4.11) is increasing in x and decreasing in z. Next we discuss each of the three main assumptions in detail.

3.1. Monotonicity in price. The first assumption was that the probability of observing price-transitions to the 'high' prices is increasing in the current price. That is, we assumed, for $s = (x, y, z)$ and $c \geq 0$, $\Pr\{X_{Y+1} \leq c + x | S = s\}$ is decreasing in x. An implication of this assumption is that, for $s = (x, y, z)$, $E[X_{Y+1}|S = s]$ is increasing in x. As mentioned earlier, rather than providing structural properties such that (4.10) is decreasing in x, we focus on the latter consequence and provide sufficient conditions such that $E[X_{Y+1}|S = s]$ is increasing in x. One approach to characterize the conditions under which this holds, is to analyze the derivative of (4.11) with respect to x. A sufficient condition for $E[X_{Y+1}|S = s]$ to be increasing in x, is $\frac{\partial}{\partial x}E[X_{Y+1}|S = s] \geq 0$. The derivative of $E[X_{Y+1}|S = s]$ with respect to x is given by,

$$
\begin{aligned}
\frac{\partial}{\partial x}E[X_{Y+1}|S = s] &= \frac{\partial}{\partial x}x + \left(\frac{\partial}{\partial x}\pi_s\right)\mu_s + \pi_s\left(\frac{\partial}{\partial x}\mu_s\right) \\
&= 1 + \frac{(\beta_1 + \beta_3 z)e^{\beta \mathbf{x}}(1 + e^{\beta \mathbf{x}} - e^{\beta \mathbf{x}})}{(1 + e^{\beta \mathbf{x}})^2}\mu_s + \pi_s(\gamma_1 + \gamma_3 z)e^{\gamma \mathbf{x}} \\
&= 1 + (\beta_1 + \beta_3 z)\pi_s(1 - \pi_s)\mu_s + (\gamma_1 + \gamma_3 z)\pi_s\mu_s
\end{aligned}
\tag{4.12}
$$

where the last equality holds because $1 - \pi_s = 1/(1 + e^{\beta \mathbf{x}})$. Since $\mu_s, \pi_s, (1 - \pi_s) \geq 0$, it is clear that if either β_1 or γ_1 are negative then (4.12) is not necessarily non-negative. Note that $\beta_1 < 0$ implies at higher price you are less likely to see positive price-transition, and that $\gamma_1 < 0$ implies that the expected price-increment is smaller the higher the current price. Since it is reasonable to assume that β_1 and γ_1 are negative, some structural properties to ensure that (4.12) remains non-negative will be provided. This is sufficient for the value function in Chapter 2 to be increasing in x, and consequently for Proposition 2.11 to hold.

Note that $\beta_1, \gamma_1 < 0$ or convexity in x is neither sufficient nor necessary for $E[X_{Y+1}|S]$ to be increasing in x as illustrated in the left graph of Figure 4.2. In Figure 4.2, the left graph provides four examples of $E[X_{Y+1}|S]$ as a function of x for various combinations of negative β_1 and γ_1. The solid lines 'A' and 'B' are examples for which $E[X_{Y+1}|S]$ is increasing in x. The dashed lines 'C' and 'D' are examples for which $E[X_{Y+1}|S]$ is not monotone in x. All four examples are based on β and γ parameter settings which are close to the values observed in the empirical analysis later on. Specifically we used,

$$
\pi_s = \frac{e^{2+\beta_1 x - .01z}}{1 + e^{2+\beta_1 x - .01z}} \qquad\qquad \mu_s = e^{5+\gamma_1 x - .01z}
$$

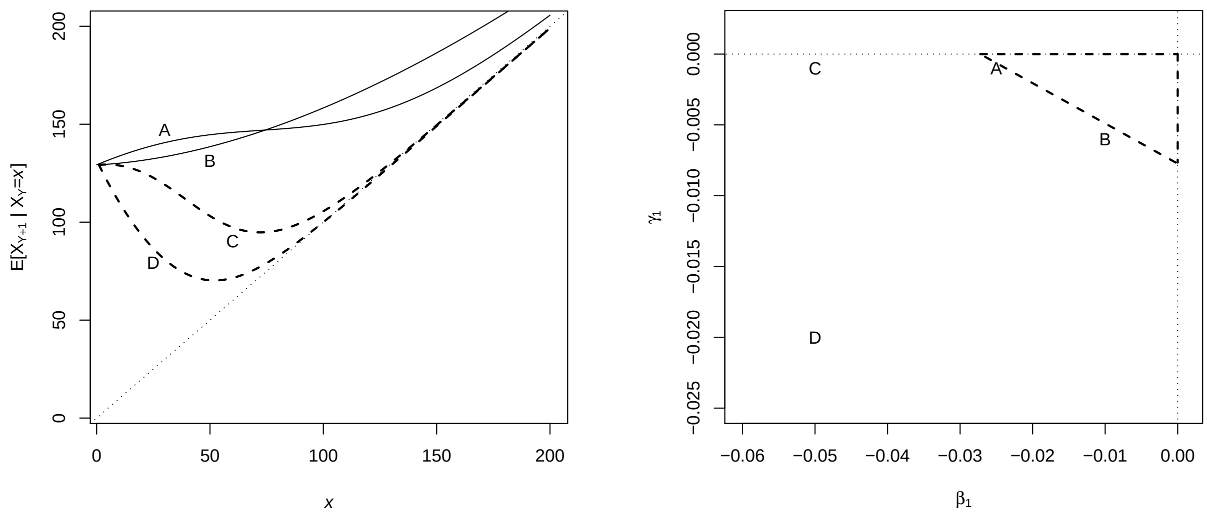


FIGURE 4.2. The graph to the left illustrates $E[X_{Y+1}|S = s]$ as a function of x. Each example is based on $\beta_0 = 2, \beta_2 = -.01, \beta_3 = 0, \gamma_0 = 5, \gamma_2 = -.01, \gamma_3 = 0$, and $z = 1$, but different β_1, γ_1 as follows: (A) $\beta_1 = -.025, \gamma_1 = -.001$, (B) $\beta_1 = -.01, \gamma_1 = -.006$, (C) $\beta_1 = -.05, \gamma_1 = -.001$, (D) $\beta_1 = -.05, \gamma_1 = -.02$. The solid lines A and B are increasing in x, while the dashed lines C and D are not monotone in x. Lines B and D are convex in x, while A and C are neither convex nor concave. The graph to the right displays the coordinates of (β_1, γ_1) for the four examples. The triangular region, enclosed by the dashed lines, represents the sufficient conditions for $E[X_{Y+1}|S = s]$ to be increasing in x, as specified by the second set of conditions in Lemma 4.2 and the values of $\beta_0, \beta_2, \gamma_0, \gamma_2$. The coordinates for A and B are slightly outside the region, which shows that the conditions are not necessary. The coordinates for C and D are, however, too far from the bounded region.

with $z = 1$, and β_1 and γ_1 as follows,

	A	B	C	D
β_1	-.025	-.01	-.05	-.05
γ_1	-.001	-.006	-.001	-.02

The following lemma gives some sufficient conditions for which $\frac{\partial}{\partial x} E[X_{Y+1}|S] \geq 0$, and consequently $E[X_{Y+1}|S]$ is increasing in x.

LEMMA 4.2. *Define* $\pi_0 \equiv \exp(\beta_0 + \beta_2)/(1 + \exp(\beta_0 + \beta_2))$ *and* $\mu_0 \equiv \exp(\gamma_0 + \gamma_2)$. *For* $s = (x, y, z)$*, sufficient conditions for* $E[X_{Y+1}|S]$ *given by (4.11) to be increasing in* x *include,*

(1) $\beta_1, \beta_3, \gamma_1, \gamma_3 \geq 0$

(2) (a) $\beta_2, \gamma_2 \leq 0$

(b) $\beta_3, \gamma_3 \geq 0$

(c) $-\frac{4}{\mu_0} - \beta_3 z \leq \beta_1 < 0$

(d) $-\frac{1}{\pi_0\mu_0} - \frac{\beta_1+\beta_3 z}{4\pi_0} - \gamma_3 z \leq \gamma_1 < 0$

(e) $z \leq \min\{\frac{-\beta_1}{\beta_3}, \frac{-\gamma_1}{\gamma_3}\}$

(3) (a) $\beta_2, \gamma_2 \leq 0$

(b) $\beta_3, \gamma_3 < 0$

(c) $-\frac{4}{\mu_0} - \beta_3 z \leq \beta_1 < 0$

(d) $-\frac{1}{\pi_0\mu_0} - \frac{\beta_1+\beta_3 z}{4\pi_0} - \gamma_3 z \leq \gamma_1 < 0$

(e) $z \leq \min\{\frac{\beta_1}{\beta_3}, \frac{\gamma_1}{\gamma_3}\}$

Proof Lemma 4.2 - Since $\pi_s, \mu_s \geq 0$, if $\beta_1, \beta_3, \gamma_1, \gamma_3 \geq 0$, then $\beta_1 + \beta_3 z \geq 0$ and $\gamma_1 + \gamma_3 z \geq 0$, and consequently, $1 + (\beta_1 + \beta_3 z)\pi_s(1 - \pi_s)\mu_s + (\gamma_1 + \gamma_3 z)\pi_s\mu_s > 0$. Therefore, if the first set of conditions holds then $\frac{\partial}{\partial x}E[X_{Y+1}|S] \geq 0$, where the derivative us given by (4.12), holds. The second set of conditions is the case when $\beta_1, \gamma_1 < 0$ and $\beta_3, \gamma_3 \geq 0$. Condition (a) ensures that π_s and μ_s are decreasing in z. Condition (e) ensures that $\beta_1 + \beta_3 z, \gamma_1 + \gamma_3 z \leq 0$, and hence π_s and μ_s are decreasing in x. Note that if $1 + (\beta_1 + \beta_3 z).25\mu_0 + (\gamma_1 + \gamma_3 z)\pi_0\mu_0 \geq 0$, then $\frac{\partial}{\partial x}E[X_{Y+1}|S = s] \geq 0$ for all s. By substituting the lower bounds for β_1 and γ_1, $1 + (\beta_1 + \beta_3 z).25\mu_0 + (\gamma_1 + \gamma_3 z)\pi_0\mu_0 = 0$, and $\frac{\partial}{\partial x}E[X_{Y+1}|S] \geq 0$ holds. Note that if $\beta_3, \gamma_3 = 0$ then the bounds on β_1 and γ_1 simplify and condition (e) holds vacuously (define $\min\{-\beta_1/0, -\gamma_1/0\} \equiv \infty$).

The third set of conditions is the case when $\beta_1, \beta_3, \gamma_1, \gamma_3 < 0$. The last condition (e) ensures that $\beta_1 + \beta_3 z \geq 2\beta_1$ and $\gamma_1 + \gamma_3 z \geq 2\gamma_1$. In other words, the effect of an increase in x is at most two times β_1 for π_s and two times γ_1 for μ_2. No changes to the previous proof is necessary, since the bound on β_1 and γ_1 adjusts according to β_3 and γ_3. $\square$

A few comments follow.

- The first condition in Lemma 4.2 does not have much intuitive appeal, since it implies auction price-increments are increasing in x, i.e. the higher the price, the more likely to see a larger positive price-increment.
- The lower bound on γ_1 in the last two set of conditions states that, for given values of $\beta_0, \beta_2, \beta_3, \gamma_0, \gamma_2, \gamma_3$, the space defined by γ_1 and β_1 defines a 'sufficient' region such that $E[X_{Y+1}|S]$ is increasing in x. See the right graph of Figure 4.2. Recall that the stated conditions are only sufficient, and hence outside the defined area, $E[X_{Y+1}|S]$ may or may not be increasing in x. Furthermore, the bounds on γ_1 are conservative, as it guarantees $\frac{\partial}{\partial x}E[X_{Y+1}|S] \geq 0$ in the worst case scenario when $\pi_s(1-\pi_s)\mu_s = .25\mu_0$ and $\pi_s\mu_s = \pi_0\mu_0$ (which only happens if $\pi_0 = .5$).
- Alternatively the lower bound on β_1 can be stated in terms of γ_1, for $\beta_3 \geq 0$,

$$\beta_1 \geq -\frac{1}{.25\mu_0} - \frac{\gamma_1\pi_0}{.25} - \beta_3 z$$

- The intuition behind the bound of γ_1 is as follows. A \$1 increase in x cannot result in a decrease of μ_s by more than the maximum average price-increment $\pi_0\mu_0$. That is, if γ_1 is 'small' (and $\gamma_3 = 0$) then $exp(\gamma_1 x) \approx 1 + \gamma_1 x$ and,

$$E[X_{Y+1}|S=s] = x + \pi_s\mu_s = x + \pi_s e^{\gamma_0+\gamma_2 z}e^{\gamma_1 x} \approx x + \pi_s e^{\gamma_0+\gamma_2 z}(1+\gamma_1 x)$$

$$\leq x + \pi_0\mu_0(1+\gamma_1 x) = x + \pi_0\mu_0 + \pi_0\mu_0\gamma_1 x$$

Therefore, since $\gamma_1 < 0$, unless $\gamma_1 > -1/\pi_0\mu_0$ an increase in x will result in a decrease of $E[X_{Y+1}|S]$. The lower bound on γ_1 is then adjusted up by a factor of

$.25/\pi_0$ given β_1. A similar though more convoluted argument leads to the lower bound of β_1, where .25 represents the maximum variance, $\pi_s(1-\pi_s)$, of a positive price-increment.

- If $\pi_s = 1$ the condition on γ_1 simplifies to the following, for $\gamma_3 \geq 0$,

$$\gamma_1 \geq -\frac{1}{\mu_0}$$

- The upper bound on z in the second condition ensures that π_s and μ_s are decreasing in x. While the upper bound on z in the third condition limits the negative effect an increase in x has on π_s and μ_s.
- The condition $\beta_2, \gamma_2 \leq 0$ ensures that π_s and μ_s are decreasing in z, but are not crucial for establishing lower bounds on β_1 and γ_1. However, if $\beta_2, \gamma_2 > 0$ then it would not seem reasonable that $\beta_3, \gamma_3 > 0$, since it would imply that π_s and μ_s are increasing in z. See Section 3.2. Therefore, if $\beta_2, \gamma_2 > 0$ then $\beta_3, \gamma_3 \leq 0$, and a simple amendment is to replace π_0, μ_0 with $\pi_0^\star, \mu_0^\star$, where $\pi_0^\star \equiv \exp(\beta_0 + \beta_2 z^\star)/(1+\exp(\beta_0 + \beta_2 z^\star))$, $\mu_0^\star \equiv \exp(\gamma_0 + \gamma_2 z^\star)$, and $z^\star = \min\{\beta_1/\beta_3, \gamma_1/\gamma_3\}$
- Since each coefficient can be positive, negative or zero, there are in total 81 possible combinations of $\beta_1, \beta_3, \gamma_1, \gamma_3$ to consider. However, the main combinations of interest are covered in the three cases listed in Lemma 4.2. In addition, most combinations are not of interest or acceptable, e.g. cases where $\beta_1, \gamma_1 = 0$ and $\beta_3, \gamma_3 \neq 0$.
- The conditions stated in Lemma 4.2 can be directly verified by data. See Section 5.

3.2. Cannibalization Effect. The second assumption imposed in Chapter 2 was that, for a given price, the seller is less likely to see transitions to the higher prices the more auctions are underway. That is, for $s = (x, y, z)$ and $c \geq 0$, $\Pr\{X_{Y+1} \leq c + x | S = s\}$ is increasing in z. This would imply that, for $s = (x, y, z)$, $E[X_{Y+1}|S]$ is decreasing in z.

Similar to the discussion with regard to price, rather than analyzing (4.10), we focus on the conditional expected price-transition $E[X_{Y+1}|S]$ given by (4.11). Since z is discrete the equivalent to analyzing the derivative is to look at differences due to increments in z. The difference in $E[X_{Y+1}|S]$ due to an unit increase in z is given by, for $s = (x, y, z)$, $s^+ = (x, y, z + 1)$,

$$
\begin{aligned}
E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z + 1)] &= (x + \pi_s\mu_s) - (x + \pi_{s^+}\mu_{s^+}) \\
&= \pi_s\mu_s - \pi_{s^+}\mu_{s^+}
\end{aligned}
\tag{4.13}
$$

where π_s, π_{s^+} are given by (4.6), and μ_s, μ_{s^+} are given by (4.7). Sufficient conditions for (4.13) to be positive, and hence $E[X_{Y+1}|S]$ decreasing in z, is $\beta_2, \gamma_2 < 0$. In other words, the more ongoing auctions there are, the less likely a positive price-increment occurs and the lower the average positive price-increment. This is formally summarized in the next lemma.

Lemma 4.3. *Sufficient conditions for*

$$E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z + 1)] \geq 0$$

to hold include,

(1) $\beta_2, \gamma_2 < 0, \beta_3, \gamma_3 \leq 0$

(2) $\beta_2, \gamma_2 < 0, \beta_3, \gamma_3 > 0$, *and* $x \leq \min\{-\beta_2/\beta_3, -\gamma_2/\gamma_3\}$

Proof Lemma 4.3 - $E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z + 1)] = \pi_s\mu_s - \pi_{s^+}\mu_{s^+} =$

$$\mu_s(\pi_s - \pi_{s^+}e^{\gamma_2+\gamma_3 x}) = \mu_s\left(\frac{1}{1 + e^{-\beta\mathbf{x}}} - \frac{e^{\gamma_2+\gamma_3 x}}{1 + e^{-\beta\mathbf{x}}e^{-\beta_2-\beta_3 x}}\right)$$

where $\exp(-\beta\mathbf{x}) = \exp(-(\beta_0 + \beta_1 x + \beta_2 z + \beta_3 x \times z))$. Therefore, if $\beta_2, \gamma_2 < 0, \beta_3, \gamma_3 \leq 0$ then $\mu_s \geq 0$, $e^{\gamma_2+\gamma_3 x} < 1$, and $e^{-\beta\mathbf{x}}e^{-\beta_2-\beta_3 x} > e^{-\beta\mathbf{x}}$. Consequently, $E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z + 1)] \geq 0$.

For the second set of conditions, the upper bound on x ensures that $e^{\gamma_2+\gamma_3 x} < 1$, and $e^{-\beta\mathbf{x}}e^{-\beta_2-\beta_3 x} > e^{-\beta\mathbf{x}}$, and the same proof as in the previous case holds. $\square$

An alternative to the 'strict' cannibalization effect, would be that the cannibalization effect effects the auctions at 'high prices'. That is $E[X_{Y+1}|S]$ is increasing in z for auctions below some price-level, and decreasing for auctions above the price-level. The following lemma summarizes this result.

LEMMA 4.4. *If* $\beta_2, \gamma_2 > 0, \beta_3, \gamma_3 \leq 0$ *and* $x \leq \min\{-\beta_2/\beta_3, -\gamma_2/\gamma_3\}$, *then* $E[X_{Y+1}|S]$ *is increasing in* z, *and if* $x \geq \max\{-\beta_2/\beta_3, -\gamma_2/\gamma_3\}$ *then* $E[X_{Y+1}|S]$ *is decreasing in* z.

Proof Lemma 4.4 - The proof is based on the same logic as in the previous proof. If x is below the lower bound then $e^{\gamma_2+\gamma_3 x} > 1$, and $e^{-\beta\mathbf{x}}e^{-\beta_2-\beta_3 x} < e^{-\beta\mathbf{x}}$. And consequently, $E[X_{Y+1}|S = s] - E[X_{Y+1}|S = s^+] \leq 0$. While if x is above the upper bound then the same conditions as in the proof of Lemma 4.3 holds. $\square$

A few comments follow.

- The result in Lemma 4.4 can be extended to the following. There exists a price-threshold $p^c \in [\min\{-\beta_2/\beta_3, -\gamma_2/\gamma_3\}, \max\{-\beta_2/\beta_3, -\gamma_2/\gamma_3\}]$, such that if $\beta_2, \gamma_2 < 0, \beta_3, \gamma_3 > 0$, and $x \leq p^c$, then $E[X_{Y+1}|S]$ is increasing in z, while for $x > p^c$, $E[X_{Y+1}|S]$ is decreasing in z.
- If $\beta_2, \gamma_2, \beta_3, \gamma_3 < 0$, then it implies that the effect of cannibalization is increasing in both the number of auctions as well as the current price, i.e. having more auctions underway is much worse at a higher current price than at a lower current price.
- Though there are other possible combinations of $\beta_2, \beta_3, \gamma_2, \gamma_3$, the three listed are the ones of most interest. Note that scenarios where $\beta_2, \gamma_2 = 0$ (non-significant) and $\beta_3, \gamma_3 \neq 0$ (significant) will not be considered in the ensuing data analysis.

- Recall that Z is the discrete number of ongoing auctions. However, in the ensuing data analysis we define Z as the *average* number of ongoing auctions. Which implies that Z is continuous. Consequently we may analyze the partial derivative with respect to z,

$$\frac{\partial}{\partial z} E[X_{Y+1}|S = (x, y, z)] = (\beta_2 + \beta_3 x)\pi_s(1 - \pi_s)\mu_s + (\gamma_2 + \gamma_3 x)\pi_s\mu_s$$

Note that Lemma 4.3 and 4.4 holds for z continuous.

3.3. Diminishing Cannibalization Effect in the Current Price. The third assumption in Chapter 2, which lead to the threshold policy of Theorem 2.12, is that the cannibalization effect is diminishing in the current price. That is the higher the current price, the less impact of cannibalization. This holds under the second set of conditions listed in Lemma 4.2 and the second set of conditions of Lemma 4.3. Note, that these conditions impose an upper bound on the current price. If, for instance, prices are not restricted, then there exist a price-level p^u such that, if $x < p^u$ then $E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z+1)] \geq 0$ and decreasing in x, while if $x > p^u$ then $E[X_{Y+1}|S = (x, y, z)] - E[X_{Y+1}|S = (x, y, z+1)] < 0$ and decreasing in x. That is higher priced auctions are better off with more ongoing auctions. Though this might seem a bit strange and counterintuitive, due to the decaying exponential shape of (4.6) and (4.7), this does not imply that high priced auctions are likely to see high price-increments. Contrariwise, the expected price-increments are still decreasing in price. See Figure 4.6 in Section **??** for examples.

Another interesting scenario is the third set of conditions of Lemma 4.2 and the first set of conditions in Lemma 4.3, where $\beta_2, \beta_3, \gamma_2, \gamma_3 < 0$. In this scenario the cannibalization effect is increasing. That is the exponential decline is steeper the more ongoing auctions there are. However, due to the shape of a negative exponential curve this cannot hold forever. And at some point the exponential curve flattens out and the cannibalization effect becomes diminishing. In other words, the cannibalization effect is increasing in x up to a price-level p^n, after which the cannibalization effect is decreasing in x. See Figure 4.6 on page 188 for an example.

Another possible scenario is when the interaction terms are countering the main effects, i.e. the conditions specified in Lemma 4.4. This scenario does have intuitive appeal as it displays that the 'cannibalization effect' is working against the 'high' priced auctions and for the 'low' priced auctions. In other words, rather than a cannibalization effect, there is a price competition effect favoring the low priced auctions. However, this would presumably reflect that the lower priced auctions benefit from competition only in the presence of higher priced auctions. And not categorically from the fact that there are more auctions underway. In other words, a low price auction would presumably not benefit from having more ongoing auctions if they also were at low prices. In this scenario the cannibalization effect would be negative for $x < p^i$, and positive for $x > p^i$, i.e. the expected price-increments are higher (lower) with more auctions underway if $x < p^i$ ($x > p^i$). In addition, there would exist a price-level $p^{ii} > p^i$ such that for $x < p^{ii}$ ($x > p^{ii}$) the cannibalization effect is increasing (diminishing) in x.

In Section 6 illustrations of the different cannibalization effects are provided in Figure 4.6. Next we provide an overview of generalized linear models (GLM).

4. Generalized Linear Models

The statistical models for analyzing the within period price-transitions of online auctions, is based on the theory of Generalized Linear Models (GLM). As the name implies GLM is a generalization of the normal (Gaussian) linear regression models. Linear models, which date back to the works by Gauss and Legendre, has been extensively developed over the last century, including the individual work by, for instance, Tukey, Fisher, and Cox [**17**, Ch.1],[**9**, p.56]. The term 'generalized linear model' was first introduced by Nelder and Wedderburn in 1972, who provided a unified theory and a general algorithm for computing the maximum likelihood estimates for a class of generalized linear models [**17**, p.19], [**9**, p.56]. A multivariate extension to GLM, labeled *Exponential Dispersion Models* (EDM), is proposed and discussed in Jørgensen (1987). Some of the underlying ideas of GLM and EDM, can be found in Tweedie (1947) [**13**, p.128, 145, 148].

One aspect of the 'generalization' is that instead of assuming a response variable Y to be normally distributed, we may assume it simply belongs to a distribution from the exponential family. That is, Y is derived from a distribution with a density that can be written as,

$$f_Y(y|\theta,\phi) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right)$$

The parameter θ is known as the *canonical* parameter and represents location, while ϕ is called the *dispersion* parameter and represents the shape. The functions $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are specific to the distribution in question. Illustrations based on the binomial and gamma distributions are provided in Section 4.2 and 4.3 below. The exponential family distributions have mean $\mu = E[Y] = \frac{\partial}{\partial\theta}b(\theta) = b'(\theta)$, and variance $\text{var}(Y) = a(\phi)\frac{\partial^2}{\partial\theta^2}b(\theta) = a(\phi)b''(\theta)$. It is interesting to note that the exponential family includes both continuous as well as discrete distributions. The most common are the normal, Poisson, binomial, gamma, and inverse Gaussian distribution.

In addition to specifying the distribution of the response variable, GLM requires a *link* function, $g(\mu)$, that describes how the mean response variable, μ, relates to the vector of known explanatory variables (covariates), $\mathbf{x}$, through a linear predictor,

$$\eta = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p = \beta \mathbf{x}$$

The most common algorithm to fit the β coefficients for a GLM is based on the Newton-Raphson method with Fisher scoring. This numerical iterative optimization method is equivalent to iteratively weighted least squares [**17**, p.42], [**30**, p.185], [**9**, p.64], [**8**, p.117], [**15**, p.200]. The reason why a numerical approach is used is that in general there is no exact analytical expression that maximizes the log-likelihood for the members of the exponential family. One member that does have an exact solution is the Gaussian [**8**, p.117]. We omit the details but note that the estimated variance of the β coefficients estimates are given by,

$$\hat{\mathrm{var}}(\hat{\beta}) = a(\hat{\phi}) \left(\mathbf{X}^T \mathbf{W}_{\hat{\beta}} \mathbf{X} \right)^{-1} \tag{4.14}$$

where $\mathbf{X}$ is the matrix of explanatory variables (covariates), $\mathbf{W}_{\hat{\beta}}$ is a diagonal matrix with the weights used in the final iteration, and $\hat{\phi}$ is the estimate of the dispersion parameter ϕ. The standard errors of $\hat{\beta}$, $\mathrm{se}(\hat{\beta})$, are the square-roots of the diagonal entries of $\hat{\mathrm{var}}(\hat{\beta})$. To evaluate the statistical significance of a particular $\hat{\beta}$ coefficient, we compare the ratio $\hat{\beta}/\mathrm{se}(\hat{\beta})$ to the standard normal distribution ($N(0, 1)$) or use a likelihood ratio test.

More details regarding the algorithm can be found in McCullagh and Nelder (1991, p.40), Firth (1991, p.62), Venables and Ripley (2002a, p.185), and Faraway (2006, p.117). Note that other methods of fitting a GLM are available. For instance, Venables and Ripley (2002a, p.445) provide a method for direct maximization of the likelihood for a binomial logistic regression. For the case of binary response variables see also McCullagh and Nelder (1991, p.115).

A second aspect of the 'generalization' is that the variance of Y, $\text{var}(Y) = a(\phi)b''(\theta)$, may vary with respect to the covariates. For instance, $\text{var}(Y)$ might be increasing in the mean μ. How the variance relates to the mean is captured by the *variance function* $V(\mu) \equiv b''(\theta)$. Recall from above that $\mu = b'(\theta)$. Examples of the variance function include: $V(\mu) = 1$ (normal), $V(\mu) = \mu$ (Poisson), $V(\mu) = \mu^2$ (gamma), and $V(\mu) = \mu(1-\mu)$ (binomial). Note that $\text{var}(Y)$ is also affected by the dispersion parameter ϕ through $a(\phi)$. Therefore in order to fully characterize the variance, ϕ either has to be given (fixed) or estimated. Examples for which $\phi = 1$ include the binomial, Poisson, and exponential distribution. Next follows a discussion how to estimate ϕ, two GLM examples based on the binomial and gamma distributions, and a discussion regarding model validation and residual analysis.

4.1. Estimation of Dispersion Parameter. The two most common estimates of ϕ are the maximum likelihood estimate (MLE) and the moment estimator. For distributions with a constant coefficient of variation,[2] such as the gamma distribution, McCullagh and Nelder (1991) argue for the use of the moment estimator. The two arguments they provide are: 1) the maximum likelihood estimate is very sensitive to rounding errors in observations with values close to zero, and 2) if the assumption regarding the distribution is false then the estimate does not consistently estimate the coefficient of variation [**17**, p.295]. Though the first comment most likely does not apply to our data set, we will nevertheless follow their recommendation and base our estimate on the moment estimator,

$$\hat{\phi} = \frac{X^2}{n-p}$$

where X^2 is the square-sum of Pearson residuals, n the number of observations, and p the number of fitted parameters. For the gamma distribution, $X^2 = \sum_{i=1}^{n}((y_i - \hat{\mu}_i)/\hat{\mu}_i)^2$ [**17**,

[2]Coefficient of variation = standard deviation/mean

p.296]. Note that $\hat{\phi}$ is derived for a given set of $\hat{\beta}$ coefficients, whose estimates are not based on ϕ or $\hat{\phi}$. In other words, the maximum likelihood estimate and GLM fitting algorithm of the β coefficients do not depend on ϕ or $\hat{\phi}$. However, as a consequence of (4.14) the inference regarding $\hat{\beta}$ does depend on $\hat{\phi}$. Therefore, even small changes in $\hat{\phi}$ may lead to different implications regarding the significance of the $\hat{\beta}$ coefficients.

The other method of estimating ϕ is to solve for the maximum log-likelihood function of the distribution. That is, for given β coefficients, $\hat{\phi}$ is determined by computing the MLE of ϕ. Note that the two estimates may result in different values and therefore reach different conclusions regarding the significance of the $\hat{\beta}$ coefficients. For more details regarding $\hat{\phi}$ refer to McCullagh and Nelder (1991, p.295), Firth (1991, p.64), and Venables and Ripley (2002a, p.186; 2002b, p.9). Firth (1991, p.65) also alludes to a third possible estimator. Though the moment estimator of ϕ is the default from the output summary in 'R', both estimates are available. For all estimates of ϕ in this chapter, the MLE estimates were always larger than the moment estimates.

4.2. Binomial Distribution as a GLM. To transform a binomial distribution, with $\Pr[success] = \pi$, to a member of the exponential family, let $\theta = \ln(\frac{\pi}{1-\pi})$, $b(\theta) = n\ln(1 + e^{\theta}) = -n\ln(1-\pi)$, and $c(y,\phi) = \ln\binom{n}{y}$. Note that for the binomial distribution the dispersion parameter $\phi = 1$ and $a(\phi) = 1$, and hence the issue of which estimator of ϕ to use does not apply. Though there are many options for the link function, the most common and the one we use, is the *logit* function,

$$g(\pi) = \ln\left(\frac{\pi}{1-\pi}\right) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p = \beta\mathbf{x} \tag{4.15}$$

The main benefit of using the logit function is that it ensures the range of π to be within the interval $[0, 1]$. Another benefit is the direct interpretation of the β coefficients. Namely, an unit increase in a covariate, while keeping the other variables fixed, results in a β increase of the log-odds. Alternatively, a unit increase in a covariate results in $\exp(\beta)$ increase of the odds.

4.3. Gamma Distribution as a GLM. The gamma distribution, $G(\lambda, \nu)$, has two parameters, a *scale* parameter λ and a *shape* parameter ν. The general density is,

$$f(y) = \frac{1}{\Gamma(\nu)} \lambda^{\nu} y^{\nu-1} e^{-\lambda y} \qquad\qquad y > 0$$

where $\Gamma(\cdot)$ is the gamma function.[3] The gamma distribution has mean ν/λ, and variance ν/λ^2. Note that the exponential distribution is the special case of the gamma distribution for which $\nu = 1$, and that the gamma distribution with $\nu < 1$ is steeper than an exponential. While a gamma distribution with $\nu > 1$ has the shape of a right skewed bell-shaped curve, where the skew becomes less dramatic the larger the value of ν. See Figure 4.3.

In order to write the gamma density as a member of the exponential family, reparametrize by defining $\mu = \nu/\lambda$ and re-write the gamma density as follows,

$$f(y) = \frac{1}{\Gamma(\nu)} \left(\frac{\nu}{\mu}\right)^{\nu} y^{\nu-1} e^{-\left(\frac{y\nu}{\mu}\right)} \qquad\qquad y > 0$$

Let the canonical parameter $\theta = -1/\mu$, the dispersion parameter $\phi = 1/\nu$, $b(\theta) = -\ln(-\theta) = -\ln(1/\mu)$, $a(\phi) = \phi = 1/\nu$, and $c(\phi, y) = \phi^{-1}\ln(\phi^{-1}) + (\phi^{-1} - 1)\ln(y) + \ln(1/\Gamma(\phi^{-1})) = \nu\ln(\nu) + (\nu - 1)\ln(y) + \ln(1/\Gamma(\nu))$. Similar to the binomial distribution there are many possible link functions. However, in order to ensure the range of λ, the main parameter of interest, is positive we use the log-link function,

$$g(\mu) = \ln(\mu) = \gamma_0 + \gamma_1 x_1 + \cdots + \gamma_p x_p = \gamma\mathbf{x} \tag{4.16}$$

[3] For a complex number z with a positive real part $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$. If z is a positive integer then $\Gamma(z) = (z-1)!$.

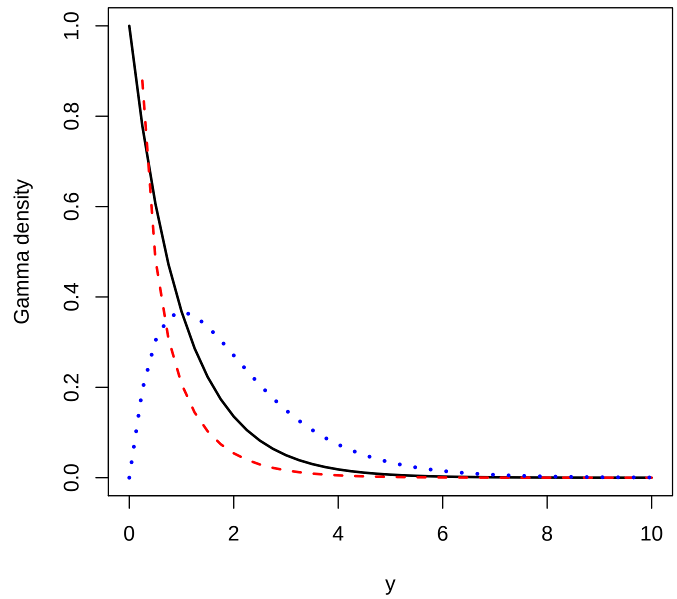


FIGURE 4.3. Illustration of the gamma distribution $G(1, \nu)$ for various values of ν. The solid line represents $\nu = 1$, the dashed line represents $\nu = 1/2$, and the dotted line represents $\nu = 2$. Note that the solid line depicts the exponential distribution with $\lambda = 1$.

Since $\exp(\gamma \mathbf{x}) > 0$, this ensures the range of μ, and hence λ, is positive. As mentioned above, the GLM estimation algorithm of the γ coefficients is independent of the dispersion parameter and its estimate. In other words, estimating the γ coefficients for a $G(\lambda, 1)$ or $G(\lambda, 2)$ results in exactly the same $\hat{\gamma}$. However, the standard errors, and resulting p-values, will be different depending on the estimated value of ϕ. Since $a(\phi) = \phi$, we see from (4.14) that the standard errors are increasing in ϕ. Therefore, if $\hat{\phi} < 1$ then, compared to an exponential distribution, the standard errors and associate p-values will be smaller. Consequently, by fitting an exponential distribution with fixed $\phi = 1$, a $\hat{\gamma}$ coefficient might be statistically insignificant, while by fitting the gamma distribution and deriving an estimate $\hat{\phi} < 1$ the $\hat{\gamma}$ coefficient might be significant.

4.4. Model Validation and Residual Analysis. With a statistical or probabilistic model, the question arises whether the model accurately reflects reality as represented by the data. That is, does the data support the proposed model. The test of model validation is referred to as *goodness-of-fit* test, and tests how probable it would be to observe the data *given* that the model is correct. A model 'fails' the test when the probability of observing the given data is very low. While a 'pass' implies that there is not enough evidence to refute the model, i.e. the probability of observing the data, as predicted by the model, is 'good' or 'large'. The advantage, or curse, of data validation is one aspect that separates statistical models from, for instance, normative economics or management science models.

The most common bases for GLM goodness-of-fit test are the *scaled deviance*, $D(\mathbf{y};\hat{\mu})/\phi$, and the generalized Pearson statistics, X^2. Both measures give a value of how close a model M fits the data, and are functions of the vector of observed values $\mathbf{y}$, and the vector of estimated means $\hat{\mu}$. The scaled deviance is defined as twice the difference in maximum log-likelihood between the *saturated* model and the model in question. Let S denote the saturated model, which has one parameter per observation and thus fits the data perfectly. Let $l_S(\mathbf{y},\phi|\mathbf{y})$ and $l_M(\hat{\mu},\phi|\mathbf{y})$ represent the maximum log-likelihoods for the two models respectively, then $D(\mathbf{y};\hat{\mu})/a(\phi) = 2[l_S(\mathbf{y},\phi|\mathbf{y}) - l_M(\hat{\mu},\phi|\mathbf{y})]$, which for GLM can be re-written as, for $a(\phi)=\phi$,

$$D(\mathbf{y};\hat{\mu})/\phi = 2\sum_{i=1}^{n}[y_i(\tilde{\theta}_{S,i}-\tilde{\theta}_{M,i}) - b(\tilde{\theta}_{S,i}) + b(\tilde{\theta}_{M,i})]/\phi$$

where $\tilde{\theta}_{S,i}$ and $\tilde{\theta}_{M,i}$ are the estimate of θ_i under S and M respectively. The generalized Pearson statistics is defined as,

$$X^2 = \sum_{i=1}^{n}\frac{(y_i-\hat{\mu}_i)^2}{V(\hat{\mu}_i)}$$

If model M has p parameters and n observations, then under the normal-theory linear models $D(\mathbf{y};\hat{\mu})$ and X^2 are χ^2_{n-p} distributed. For other distributions, though some asymptotic results are available, this is often not even approximately correct (even for large n). And it seems the following comment from McCullagh and Nelder (1991) still applies,

> "Further work on the asymptotic distribution of $D(\mathbf{Y};\hat{\mu})$ remains to be done." [**17**, p.36]

The main asymptotic results available seem to pertain to distributions that, in the limit of certain parameters, resemble the normal distribution. Examples of these include Poisson with 'large' means (μ_i), binomial with 'large' number of trials (m_i), and gamma with a 'large' shape parameter ν or alternatively a 'small' dispersion parameter ϕ. These results are referred to as *small-dispersion asymptotics*, and imply that the scaled deviance can be approximated by a χ^2_{n-p} distribution. Though there are no general asymptotic results for the deviance or Pearson X^2, there are results for the analysis of nested models. Similar to the normal-theory linear models, the difference in scaled deviance between nested GLM models is approximately χ^2 distributed. That is, if model 1 and model 2 have p_1 and p_2 parameters respectively, and the models are nested with $p_2 < p_1$, then $(D_2(\mathbf{y};\hat{\mu}) - D_1(\mathbf{y};\hat{\mu}))/\phi$ is approximately $\chi^2_{p_1-p_2}$ distributed. It should be noted that the difference in Pearson X^2 between nested models is not monotone and need not have a χ^2 distribution. For more details regarding goodness-of-fit and comparison of nested models refer to McCullagh and Nelder (1991, p.33), Firth (1991, p.68), Venables and Ripley (2002a, p.186), Faraway (2006, p.120), and Jørgensen (1987, p.134).

In addition to the goodness-of-fit test another form of model validation is *residual analysis*. Residual analysis can be used both to detect individual or clusters of observations that do not fit the overall pattern of the data, as well as the overall validation of model assumptions. The latter part, which will be the main focus of our analysis, can further be divided into the aspects that validate the structural part, and the stochastic part of the model. For example, for the positive price-increments the validation is with regards to that the log-link should be linear (structural part) and that the price-increments are gamma distributed (stochastic part). Though there are many different types of residuals, the most common include *deviance residuals*, r_d, and *Pearson residuals*, r_P. Deviance residuals are defined such that $\sum r_d^2 = D(\mathbf{y}; \hat{\mu})$, and Pearson residuals such that $\sum r_P^2 = X^2$. However, due to that the Pearson residuals tend to be skewed for non-normal distributions we limit our analysis to the deviance residuals [**17**, p.38], [**8**, p.123]. In addition, we will analyze the response residuals $y - \hat{\mu}$. Both the deviance residuals and response residuals will be depicted against the fitted linear predictor $\hat{\eta}$.

Two comments regarding the residual analysis follow. First, we will not investigate the residual plots for the Bernoulli distributed price-increments, as their residual plot are bound to be rather uninformative. The reason for this is that binary data will only generate two bands, one for $y = 0$ and one for $y = 1$. Second, when analyzing the residuals, note that we are assuming the positive price-increments are gamma distributed with a non-constant variance function. Specifically, we are assuming the variance is decreasing in the mean, e.g. the higher the current price, and thus smaller expected price-increments, the smaller the variance for the price-increments. As a result, the residual plots will look different than, for instance, the residual plots from a normal linear regression model. The patterns we should expect are as follows. For the deviance residuals, due to that the variance function has been scaled out, if the model is correct then there should not be any pattern in the data. That is we should expect to have the deviance residuals randomly distributed around zero with a constant range. If a pattern is observed it could indicate that an inappropriate link function and/or set of covariates was chosen. For the residual plot, on the other hand, due to that the variance is non-constant, we would expect to see a pattern. In particularly, we would expect the response residuals to be more spread for values of $\hat{\eta}$ for which the variance is larger. For more details, see Chapter 12.6 in McCullagh and Nelder (1991), and Chapter 6.4 in Faraway (2006).

4.4.1. *Goodness-of-fit for Bernoulli distributed observations.* In the case of Bernoulli distribution, a special case of the binomial, the goodness-of-fit test regarding the distributional assumption of the observed values becomes degenerate and vacuous. With only two possible outcomes and independent observations, there are no other plausible alternative distributions. Consider, for instance, a data set with n observations, out of which k were 'success' (1) and $n-k$ 'failure' (0). Since $\hat{\mu}_i = \hat{\pi} = k/n$ and $V(\hat{\mu}_i) = \hat{\pi}(1-\hat{\pi})$,

$$
\begin{aligned}
X^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{V(\hat{\mu}_i)} &= k\frac{(1-\hat{\pi})^2}{\hat{\pi}(1-\hat{\pi})} + (n-k)\frac{\hat{\pi}^2}{\hat{\pi}(1-\hat{\pi})} = k\frac{1-\hat{\pi}}{\hat{\pi}} + (n-k)\frac{\hat{\pi}}{1-\hat{\pi}} \\
&= k\frac{(n-k)/n}{k/n} + (n-k)\frac{k/n}{(n-k)/n} = (n-k) + k = n
\end{aligned}
$$

For the binomial distribution the deviance can be written as follows,

$$
D(\mathbf{y};\hat{\mu}) = 2\sum_{i=1}^{n} y_i \ln(y_i/\hat{\mu}_i) + (m_i - y_i)\ln((m_i - y_i)/(m_i - \hat{\mu}_i))
$$

Therefore, for the special case of Bernoulli distribution, where $m_i = 1$ and again assume there were k 'success'and $n-k$ 'failure', we have,

$$
\begin{aligned}
D(\mathbf{y};\hat{\mu}) &= 2\sum_{i=1}^{n} y_i \ln(y_i/\hat{\mu}_i) + (1 - y_i)\ln((1 - y_i)/(1 - \hat{\mu}_i)) \\
&= 2k\ln(1/\hat{\pi}) + (n-k)\ln(1/(1-\hat{\pi})) \\
&= 2n[-\hat{\pi}\ln(\hat{\pi}) - (1-\hat{\pi})\ln(1-\hat{\pi})]
\end{aligned}
$$

which does not depend on the observed vector $\mathbf{y}$ and therefore useless as a measure of goodness-of-fit [**17**, p.119], [**9**, p.69], [**8**, p.121]. Though the issue of testing the Bernoulli distribution might not be of concern, the choice of link function may be. In our analysis we use the *logit* or log-odds function, mainly due to ease of interpretation and wide use of applicability. Consequently for the Bernoulli distribution we only use $D(\mathbf{y}; \hat{\mu})$ for comparison of nested models.

4.4.2. *Goodness-of-fit for gamma distributed observations.* The deviance for the gamma distribution can be written as follows,

$$D(\mathbf{y}; \hat{\mu}) = 2 \sum_{i=1}^{n} - \ln(y_i/\hat{\mu}_i) + (y_i - \hat{\mu}_i)/\hat{\mu}_i$$

Despite that the goodness-of-fit analysis based on the deviance also has certain limitations, there are some asymptotic results. As mentioned above, though there are no general asymptotic results, there are asymptotic results pertaining to the case when the dispersion parameter ϕ is small [**13**, p.134], [**9**, p.69]. In these instances, labeled *small-dispersion asymptotics*, $D(\mathbf{y}; \hat{\mu})/\phi$ has approximately a χ^2_{n-p} distribution. However, a working definition of what defines a 'small' ϕ does not seem documented, which makes it hard to judge when the approximation might be good or OK. It appears that the *small-dispersion asymptotics* apply to when the shape of the gamma distributions resembles the normal distribution. Therefore, it seems a reasonable working definition would be that at a minimum $\phi < .5$, i.e. $\nu > 2$. See also discussion in Section 8.3.6 of McCullagh and Nelder (1991).

5. Model Formulation and Selection for DFS Data

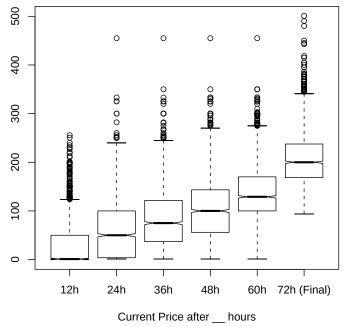
Statistical analysis is based on the six data subsets described in Chapter 1, and for convenience summarized in Table 4.1 below. All auctions lasted for three days ($\tau = 72$ hours), and for the purpose of analysis divided into 12 hour intervals. Therefore, there are in total seven observations per auction, and six periods for the three stages for which we estimate π_s and λ_s. To provide a more intuitive index of the elapsed time we count elapsed

Product	PC Model	Processor Model	Processor Speed (GHz)	Memory (MB)	Hard Drive (GB)	Count
D1	Desktop	Intel Pentium 4	1.7-2.8	256, 512, 1024	20, 40	2,072
L1	Laptop	Intel Pentium 4, M	1.8-2.4	256, 512	20, 30, 40	2,046
D3	Desktop	Intel Pentium 4	2.4	256	40	274
D4	Desktop	Intel Pentium 4	2.0	512	40	167
L4	Laptop	Intel Pentium 4	1.8	512	30	172
L5	Laptop	Intel Pentium M	1.4	512	40	163

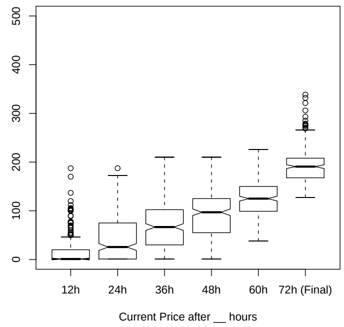
TABLE 4.1. Characteristics of the six products analyzed.

periods in increments of 12, $Y = 0, 12, 24, 36, 48, 60, 72$.

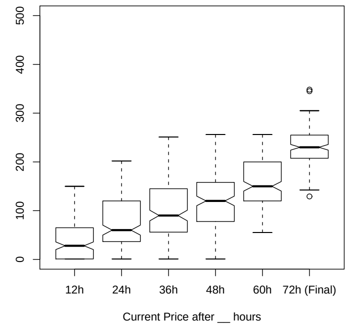
In Figure 4.4 and Table 4.2 below, boxplots of price and descriptive statistics, for the 12 hour intervals are shown. Note that the information regards the distribution of price at the specific intervals, and not the distribution of the price-increments. The leftmost columns in Table 4.2 represents descriptive statistics for the *average number* of ongoing auctions in each 12 hour period. In other words, in each period an auction is underway the average number of ongoing auctions is recorded. For instance, for product D1 in the first period (0h), the smallest and largest *average* number of ongoing auctions observed were 1.00 and 83.88 respectively. More details are given below. An immediate observation for the boxplots in Figure 4.4, is the steady increase of the median price over time (the line inside each box). Informal statistical support to the observation that the median price is increasing, is that the notches of the boxes do not overlap. In Table 4.2 we see that for each product line, the median price-increments are almost identical for the first four periods, while in the final period the increase is almost twice as large as the previous periods' price-increments. For desktops, the median increase for the first period is almost zero, while in the next four periods the price increases between \$25-30 per period. And in the last period the increase is about \$75-80. For laptops, the median increase for the first five periods is about \$50, while for the last period the increase is over \$100. For the mean number of ongoing auctions, since there is no reason the numbers should fluctuate systematically between time-periods,



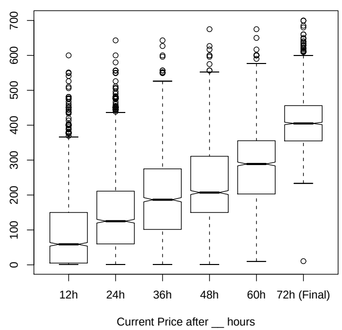
D1



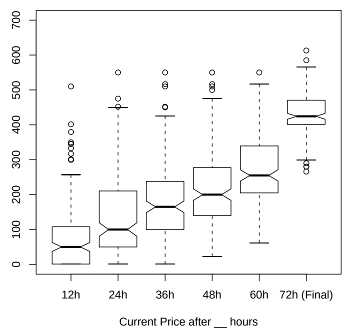
D3



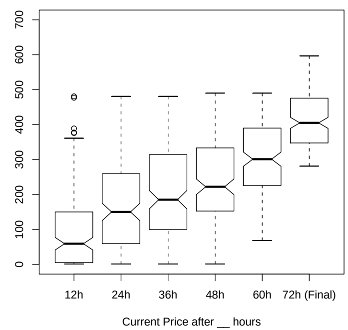
D4



L1



L4



L5

FIGURE 4.4. Distribution of the current price at the 12hour periods. The upper and lower edge of each box represents the 75th respectively 25th percentile of the observations. The line inside each box represents the median. Non-overlapping notches indicate significant difference in median.

the consistency of the values is to be expected.

Besides the observation that the price-increment in the last period is larger than the previous period, there is an additional interesting aspect to the dynamics in the last period. Namely that, almost all auctions have a positive price-increment. See Table 4.3 below. With the exception of a handful of auctions for the two aggregated categories D1 and L1, all auctions have a positive price-increment in the final 12 hours. Therefore, we assume $\pi_e = 1$ and that price-increments in the final 12 hours are strictly gamma distributed. The

Product	Time Period	Price, X				Average # of Auctions, Z			
		Min	Median	Mean	Max	Min	Median	Mean	Max
D1	0h					1.00	20.73	24.14	83.88
	12h	0.99	1.04	29.70	255.60	1.00	20.97	24.69	83.88
	24h	0.99	50.00	59.20	455.00	1.00	21.23	25.07	83.88
	36h	0.99	75.00	83.46	455.00	1.00	21.36	25.07	83.88
	48h	0.99	100.00	104.60	455.00	1.00	21.04	24.69	83.82
	60h	0.99	129.00	136.10	455.00	1.00	20.95	24.14	83.82
	72h (τ)	93.92	200.00	208.20	501.00				
D3	0h					1.00	5.000	5.933	14.700
	12h	0.99	0.99	16.18	187.40	1.00	5.051	6.009	14.590
	24h	0.99	25.62	42.10	187.40	1.00	5.240	6.041	14.700
	36h	0.99	66.60	68.85	210.00	1.00	5.154	6.041	14.510
	48h	0.99	97.00	92.72	210.00	1.00	5.033	6.009	14.700
	60h	38.00	125.00	123.20	226.00	1.00	5.018	5.933	14.700
	72h (τ)	127.50	190.70	193.60	338.50				
D4	0h					1.00	4.157	5.764	19.150
	12h	0.99	28.00	41.05	150.00	1.00	4.249	6.220	19.510
	24h	0.99	60.10	75.77	202.00	1.00	4.660	6.505	19.270
	36h	0.99	90.05	100.50	251.00	1.00	4.666	6.505	19.520
	48h	0.99	120.00	122.00	256.10	1.00	4.389	6.220	19.540
	60h	55.00	150.00	154.50	256.10	1.00	4.071	5.764	19.560
	72h (τ)	129.20	230.00	230.60	348.30				
L1	0h					1.00	19.09	19.17	53.91
	12h	0.99	58.76	94.51	600.00	1.027	19.57	19.52	53.91
	24h	0.99	125.00	154.40	643.00	1.00	19.55	19.77	53.90
	36h	0.99	186.50	200.60	643.00	1.00	19.57	19.77	53.90
	48h	0.99	207.00	236.40	675.00	1.00	19.39	19.52	53.91
	60h	10.00	288.90	289.80	675.00	1.00	19.16	19.17	53.91
	72h (τ)	10.50	405.00	412.20	700.00				
L4	0h					1.00	4.279	5.549	17.650
	12h	0.99	50.01	84.51	510.00	1.00	4.002	5.632	17.570
	24h	0.99	100.00	144.10	550.00	1.00	4.183	5.742	17.580
	36h	0.99	165.00	187.50	550.00	1.00	4.350	5.742	17.610
	48h	23.00	200.00	219.00	550.00	1.00	4.232	5.632	17.620
	60h	61.44	255.00	276.50	550.00	1.00	4.236	5.549	17.650
	72h (τ)	266.00	424.40	428.00	613.00				
L5	0h					1.00	3.00	3.148	8.000
	12h	0.99	59.00	100.10	481.00	1.00	3.00	3.239	7.884
	24h	0.99	150.00	163.40	481.00	1.00	3.00	3.268	8.000
	36h	0.99	185.00	208.30	481.00	1.00	3.00	3.268	7.871
	48h	0.99	222.00	241.70	490.00	1.00	3.00	3.239	8.000
	60h	68.30	301.00	302.50	490.00	1.00	3.00	3.148	8.000
	72h (τ)	281.00	405.00	409.60	597.00				

TABLE 4.2. Descriptive statistics for price X and mean number of ongoing auctions Z.

proposed model is therefore modified as follows, for $s = (x, y, z)$ and $y = 0, 12, 24, 36, 48$,

$$f_{C_Y|S}(c|s) = \begin{cases} 1 - \pi_s & c = 0 \\ \pi_s \frac{1}{\Gamma(\nu_s)} \left(\frac{\nu_s}{\mu_s}\right)^{\nu_s} c^{\nu_s - 1} e^{-\left(\frac{c\nu_s}{\mu_s}\right)} & c > 0 \end{cases}$$

	D1		D3		D4		L1		L4		L5	
Time Period	Zero	Pos.	Zero	Pos.	Zero	Pos.	Zero	Pos.	Zero	Pos.	Zero	Pos.
0h → 12h	330	1,742	63	211	8	159	93	1,953	6	166	4	159
12h → 24h	887	1,185	120	154	54	113	685	1,361	62	110	62	100
24h → 36h	939	1,133	117	157	78	89	832	1,214	72	100	80	83
36h → 48h	1,022	1,050	125	149	82	85	948	1,098	80	92	89	74
48h → 60h	689	1,383	95	179	59	108	670	1,376	46	126	55	108
60h → 72h	1	2,071	0	274	0	167	6	2,040	0	172	0	163

TABLE 4.3. Number of zero vs positive price-increment transitions.

and for $y = 60$,

$$f_{C_Y|S}(c|s) = \frac{1}{\Gamma(\nu_s)} \left(\frac{\nu_s}{\mu_s}\right)^{\nu_s} c^{\nu_s - 1} e^{-\left(\frac{c\nu_s}{\mu_s}\right)} \qquad c > 0$$

where

$$\ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} \beta_0 \qquad + \beta_2 z & y = 0 \\ \beta_0 + \beta_1 x + \beta_2 z + \beta_3 x \times z + \beta_4 1_{\{y=24\}} + \beta_5 1_{\{y=36\}} + \beta_6 1_{\{y=48\}} & y = 12, 24, 36, 48 \\ n.a. & y = 60 \end{cases}$$

$$\ln(\mu_s) = \begin{cases} \gamma_0 \qquad + \gamma_2 z & y = 0 \\ \gamma_0 + \gamma_1 x + \gamma_2 z + \gamma_3 x \times z + \gamma_4 1_{\{y=24\}} + \gamma_5 1_{\{y=36\}} + \gamma_6 1_{\{y=48\}} & y = 12, 24, 36, 48 \\ \gamma_0 + \gamma_1 x + \gamma_2 z + \gamma_3 x \times z & y = 60 \end{cases}$$

The above models are referred to as the *unabridged models*, to reflect that all parameters are incorporated into the analysis. The *final models* are then derived by iteratively eliminating non-significant parameters as follows. First the β (γ) coefficients for the unabridged models are estimated by maximum likelihood. The parameter with the most non-significant β (γ) coefficient, i.e. largest p-value, is then removed and the coefficients for the reduced model are re-estimated. The process of iteratively eliminating one parameter is repeated until all coefficients are found to be significant. However, if the coefficient for X or Z is found non-significant but the coefficient for the interaction $X \times Z$ is significant, then the interaction term is also removed. For instance, if β_2 is non-significant while β_3 is significant, then in the next iteration both Z and $X \times Z$ are removed. Note that, due to Lemma 4.1 the β and γ coefficients are estimated independently, which implies that a parameter might, for instance, be found non-significant with respect to π_s and significant with respect to λ_s.

The term *unabridged* is to contrast with the general statistics terminology of the *null* and *full* (or saturated) models. The *null* model refers to the model with intercept only, i.e. no dependence on the covariates. And the *full* model usually refers to the model with one parameter per observation, i.e. by fitting a function that perfectly matches all observations. The model that remains after all non-significant coefficients have been eliminated is referred to as the *final* model. The analysis was run using the 'glm' function in the statistical software R [**22**].

Though the data for estimating the β and γ coefficients came from the same products, there was one difference between the two estimation procedures. In estimating the γ coefficients only the observations with positive price-increments were used. That is, for each product, when estimating the γ coefficients for λ_b, λ_m, and λ_e, all zero price-increments were filtered out. To illustrate, suppose a product consists of 100 auctions, which means that for each of the six periods ($0 \to 12h$, $12 \to 24h$, $24 \to 36h$, $36 \to 48h$, $48 \to 60h$, $60 \to 72h$,) there are 100 observations. Suppose further that for each of the periods there were 50, 60, 70, 80, 90, and 100 auctions with positive price-increments, i.e. out of the 100 auctions, 50 observed a positive price-increment in the first period, 60 in the second period, and so on. Note that it is not necessarily the same set of auctions with positive price-increments in each period. Then when estimating the γ_b coefficients in the first period only the 50 observations were used, when estimating γ_m for the four middle periods only the 300 (=60+70+80+90) observations were used, and when estimating γ_e for the final period the 100 observations were used. In contrast, for estimating π_b and π_m all the within period observations were used. Recall that we assume $\pi_e = 1$. This means that when estimating the β_b coefficients for the first period all 100 observations were used, and when estimating β_m for the four middle periods 400 ($= 100 \times 4$) observations were used.

The implicit assumption behind the above set-up is that auctions' within period price-transitions are independent, both with respect to competing auctions as well as with respect to the previous price-transitions. In other words, besides the cannibalization effect, auctions are assumed to be independent of each other, and the price-transitions are memoryless or Markovian.

Since the data was not collected using a designed study, the number of ongoing auctions in each 12-hour interval varies. That is, the auctions analyzed were not released in synchronized 12-hour intervals, but instead with various time overlap. Consequently, for almost all auctions the number of ongoing auctions in a given 12-hour period fluctuates. Therefore, the arithmetic *mean number of ongoing auctions* in a time-period is used as a proxy for Z.

There are of course other possible variables that could have been included. In particularly, for the aggregated products D1 and L1, information regarding the product specifications such as processor speed, hard drive memory, and RAM, could have been included as covariates. We will leave these and other variables for possible future extensions to the above model. Before discussing the results we examine the individual products a bit closer.

5.1. Hypothesis Testing of β and γ Coefficients. In Figures 4.7, 4.8 and 4.9 at the end of this chapter, the within period price-increments for the six products are displayed. Each individual graph has price at the beginning of a period on the x-axis, and the 12 hour within period price-increment on the y-axis. Each observation represents an individual auction. Observations along the y-axis are auctions that did not change price during the period, i.e. received no bids in the 12 hour period. The line in each graph represents the fitted values of the conditional expected price-increment for the final model, and is discussed in Section 6. For the products and periods where z or $x \times z$ are significant, the median value of z was used in evaluating the fitted values. Note the following distinction. Z is the average number of ongoing auctions in a period. In the figures, the median of the observed values of Z was used to fit the line. in A few comments follow.

First, there is clear variability in the price-increments. For a given price, or a small interval around a given price, the 12 hour within period price-increment varies. Most auctions tend to have 'small' price-increments and a few auctions have 'large' price-increments. The main motivation of the proposed model is to model this feature. Second, one can see that the price-increments are decreasing in the price. In other words, as the price increases the within period price-increments tend to decrease. Though this would seem intuitive, the objective of the analysis is to capture the rate at which the mean within period price-increment decreases with respect to price, and assess if the structural properties discussed in Section 3.1 hold. We summarize our observations regarding price in the following testable hypothesis.

HYPOTHESIS 4.5. *The within period log-odds of observing a positive price-increment are on average decreasing in price.*

HYPOTHESIS 4.6. *The within period log-mean of the positive price-increments are on average decreasing in price.*

A third comment is, it seems that the middle four 12 hour periods, i.e. $12 \rightarrow 24h$, $24 \rightarrow 36h$, $36 \rightarrow 48h$, and $48 \rightarrow 60h$, all exhibit similar though distinct dynamics. As noted earlier, the last period is different in that almost all auctions exhibit a positive price-increment. Furthermore, though price-increments are still clearly decreasing in price, for auctions at 'low' prices there are no 'small' price-increments. That is, there are no observations in the lower left quadrant of the graphs for the final period. The main take-away is that elapsed time seems to have an effect on the auctions dynamics, which we do not state in terms of formal hypotheses.

However, an important implication regarding the price-dynamics in the final period, is that the price-transitions for the first five periods might not affect the final price. In other words, if the expected price-increment in the final period decreases *linearly* at a rate of \$1 per unit increase in price, then the expected final price would be constant. And consequently the previous periods' dynamics may be irrelevant. Note that the proposed model with (4.3), is not based on the mean price-increment being linearly related to price. It assumes that the log mean price-increment is linearly related to price. If a *normal linear regression* relationship was used there would be a price, beyond which, all within period price-increments are negative, which would seem as a rather restrictive implication. A benefit of (4.3) is that it ensures price-increments remain positive. However, since running an additional regression analysis does not entail much effort, in Appendix E a normal linear regression model for the last period will be discussed. The results of which, can be used to support or refute that the mean price-increment is decreasing at a rate of \$1, and consequently the previous periods dynamics irrelevant.

In Figure 4.10, 4.11 and 4.12 at the end of this chapter, the six within period price-increments as a function of the average number of ongoing auctions is displayed. On the x-axis is the average number of ongoing auctions for a particular period, and on the y-axis is the within period price-increment. Each observation represents an individual auction.

Similar to the previous set of graphs the price-increments again display variability with regard to the average number of ongoing auctions. However, unlike the previous graphs there does not seem to be any clear or obvious trend. Intuitively, it would seem reasonable that the within period price-increments are negatively correlated with the number of ongoing auctions. The formal statistical analysis and hypothesis testing will reveal if this is the case.

HYPOTHESIS 4.7. *The within period log-odds of observing a positive price-increment are on average decreasing in the number of ongoing auctions.*

HYPOTHESIS 4.8. *The within period log-mean of the positive price-increments are on average decreasing in the number of ongoing auctions.*

The final hypothesis testing is with regard to the interaction term '$x \times z$'. From the discussion in Section 3 we would expect to see the interaction term counter the main effects. That is, for the structural properties to hold we expect β_3 and γ_3 to be positive. Therefore, the final set of hypothesis are as follows.

HYPOTHESIS 4.9. *The expected decrease of the within period log-odds of observing a positive price-increment due to an increase in the number of ongoing auctions, is diminishing in price.*

HYPOTHESIS 4.10. *The expected decrease of the within period log-mean of the positive price-increments due to an increase in the number of ongoing auctions, is diminishing in price.*

A few comments follow.

- The stated hypothesis are meant to represent the interpretation of the actual hypothesis testing. That is, the actual null hypothesis is, of course, $\beta_k = 0$ or $\gamma_k = 0$, and the alternative hypothesis is $\beta_k \neq 0$ or $\gamma_k \neq 0,\ k = 1, 2, \ldots, 6$. In order to provide a more interesting and intuitive discussion the hypothesis have been stated to be consistent with our observations.

- Though our goal is to derive general conclusions regarding the various variables, the hypothesis testing will be conducted for each product individually. The reason for this is that the final model for each product is likely to differ, and therefore we will not conduct a simultaneous hypothesis testing for all products. Recall that the model selection is such that non-significant variables are sequentially removed for each product. However, based on the individual products' test results we attempt to derive some general conclusions. A different statistical approach would have been to conduct a simultaneous hypothesis test for all products.
- Finally, we report the exact p-value (a posteriori probability) as the significance level of the hypothesis test. All p-values less than .0001 will be reported as '$<$ .0001'. Note that the reported p-values from the hypothesis tests that $\beta = 0$ and $\gamma = 0$ are used for the stated hypotheses.

6. Results for DFS Data

To assess the proposed model we first look at the fitted means and qualitatively evaluate the fit. In Figures 4.7, 4.8 and 4.9 at the end of this chapter, the line in each graph represents the fitted values for the conditional expected price-increments for the final model. Note that the line is the conditional expected price-increment as specified by (4.5), and not the conditional expected price-increment for the positive price-increments, i.e. the line represents $\pi_s \mu_s$ and not μ_s. For products and periods where either z or $x \times z$ were found to be significant, the median value of z was used to determine the fitted values. Recall that in the last period $\pi_s = 1$. For the first period the fitted value of the expected price-increment is displayed by the dashed line and is of course independent of price. Overall the lines fit the data and are in accord with the trends exhibited by the graphs. In particular, for the final period, with perhaps the exception of D1, we see that the line follows the trends very well and seems to represent the mean price-increments perfectly. A possible problem with D1 in the final period might be the outlier observed around $450. For the middle four periods the fit is also good and seems to follow the overall trends very well. This is perhaps best seen in the four product specific products D3, D4, L4, and L5, for which the fitted values provide a very good fit. For the first period it is hard to comment, even qualitatively, regarding the proposed model.

Table 4.4 shows the resulting equations for the fitted means. The first, second and third line for each equation represents the first, middle and last periods respectively. A '*n.s.*' entry represents *not significant*, meaning that none of the variables were found to be significant, while '*n.a.*' represents *not applicable*. Detail 'R' output for each product is provided in Appendix D. Next we discuss the quantitative results and formal measures of goodness-of-fit.

6.1. Effect of Price. The first thing to note is that price, x, is always significant and always represented by a negative coefficient. In other words, the higher the price, the less likely it is for a positive price-transition to occur. And if there is a positive price-transition then the expected price-increment is smaller. Though this might be expected, and consistent with the trends observed in Figures 4.7, 4.8 and 4.9, the issue arises if the β_1 and γ_1 coefficients are 'too negative'. That is, are the decreasing rates such that $E[X_{Y+1}|S]$ for each product is not monotone in the current price. Before we discuss more details we summarize the formal hypotheses results.

$$\text{(D1)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} n.s. \\ .841 - .010x - .009z - .00005x \times z + (.228)\mathbf{1}_{\{y=24\}} + (.327)\mathbf{1}_{\{y=36\}} + (1.307)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} 4.100 - .004z \\ 4.029 - .002x - (.131)\mathbf{1}_{\{y=24\}} - (.149)\mathbf{1}_{\{y=36\}} \\ 4.699 - .004x + .003z \end{cases}$$

$$\text{(D3)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} .759 - .180z \\ .931 - .021x - .059z + (.603)\mathbf{1}_{\{y=24\}} + (1.056)\mathbf{1}_{\{y=36\}} + (2.116)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} 3.824 - .064z \\ 4.000 - .007x + (.305)\mathbf{1}_{\{y=48\}} \\ 4.681 - .005x + .108z - .0008x \times z \end{cases}$$

$$\text{(D4)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} n.s. \\ 1.375 - .012x - .027z + (.948)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} n.s. \\ 4.241 - .005x - .023z + (.294)\mathbf{1}_{\{y=48\}} \\ 5.442 - .008x \end{cases}$$

$$\text{(L1)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} 2.198 - .044z \\ 1.867 - .006x - .020z - .00008x \times z + (.144)\mathbf{1}_{\{y=24\}} + (.250)\mathbf{1}_{\{y=36\}} + (1.197)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} 5.183 - .023z \\ 4.838 - .002x - .011z - (.057)\mathbf{1}_{\{y=24\}} - (.119)\mathbf{1}_{\{y=36\}} + (.107)\mathbf{1}_{\{y=48\}} \\ 6.000 - .004x \end{cases}$$

$$\text{(L4)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} 1.481 - .090z \\ 1.331 - .007x + (1.320)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} 5.057 - .065z \\ 4.620 - .001x - (.170)\mathbf{1}_{\{y=24\}} - (.328)\mathbf{1}_{\{y=36\}} \\ 5.736 - .003x + .079z - .0002x \times z \end{cases}$$

$$\text{(L5)}\quad \ln\left(\frac{\pi_s}{1-\pi_s}\right) = \begin{cases} 2.421 - .295z \\ 1.671 - .007x - .140z + (1.299)\mathbf{1}_{\{y=48\}} \\ n.a. \end{cases}$$

$$\ln(\mu_s) = \begin{cases} n.s. \\ 5.155 - .003x - .123z + .0004x \times z - (.186)\mathbf{1}_{\{y=36\}} \\ 5.959 - .005x \end{cases}$$

TABLE 4.4. The resulting equations for the final model. In each set of equations, the first corresponds to the first period, the second to the middle four periods, and the third to the final period. All coefficients shown are significant, '$n.a.$' represents not-applicable, and '$n.s.$' represents cases with all covariates non-significant.

Test of Hypothesis 4.5 - For all products during the middle periods we reject the hypothesis that $\beta_1 \geq 0$ ($p < .0001$ for all products), and conclude that the within period log-odds of observing a positive price-increment are on average decreasing in price.

Test of Hypothesis 4.6 - For all products during the middle and final periods we reject the hypothesis that $\gamma_1 \geq 0$ ($p < .005$ for all products), and conclude that the within period price-increments are on average decreasing in price.

In Lemma 4.2 of Section 3.1 sufficient conditions to ensure that $E[X_{Y+1}|S]$ is increasing were provided. The illustration of these conditions are provided in Figure 4.5. The top six graphs of Figure 4.5 represents the coordinates for β_1 (x-axis) and γ_1 (y-axis) for each product. The first row are for D1, D3 and D4, while the second row are for L1, L4 and L5 (from left to right). The triangular area enclosed by the dashed lines, display the region for which the sufficient conditions of Lemma 4.2 hold for the four middle periods. The 'x' in each graph is the coordinate for the point estimate of (β_1, γ_1). For all six products we see that the 'x' falls inside the 'sufficient' region and hence support that $E[X_{Y+1}|S]$ is increasing in the current price. For the final 12 hour period, since $\pi_s = 1$, there is no estimate for β_1. Consequently the condition on γ_1 simplifies and results in a lower bound. See the discussion at the end of Section 3.1. The line on the y-axis labeled '60h' indicates the lower bound. The circle on the y-axis is the point estimate for γ_1 for the final period. We see that, with the exception of product D1, the circle is below the line. This does not, however, necessarily mean that $E[X_{Y+1}|S]$ is non-monotone in price as the conditions are only sufficient. Therefore to further assess if $E[X_{Y+1}|S]$ is increasing in the final period, the graph of $E[X_{Y+1}|S]$ for the final period is analyzed.

The bottom six graphs of Figure 4.5 display the conditional expected price transition for the final period. On the x-axis is X_{60}, and on the y-axis is the final price X_τ. The first row of graphs are for D1, D3 and D4, while the second row are for L1, L4 and L5 (left to right). The 45 degree line represents the line at which auctions remain unchanged in the final 12 hours. The bold curve represents $E[X_\tau|S] = x + \mu_s$. The main feature of interest is if this curve is monotone. We see that for the desktop products $E[X_\tau|S]$ is indeed increasing, although for D3 and D4 there seems to be a slight downward trend at the lower prices. However, for the laptop products the curve is clearly non-monotone. The implication is that, for laptop products, there exist a price $p^\star$ such that $E[X_\tau|S]$ is decreasing in price for $x < p^\star$ and increasing in price for $x > p^\star$. Which in turn implies that the result of Proposition 2.11 of Chapter 2 does not hold. In other words, it is not necessarily true that the seller is always better off the higher the price. However, since for all the previous periods, the expected price transitions are increasing in price, overall the seller is better off the higher the price. Meaning that, though at the start of the final period, there exist a 'low' price which has the same expected final price as a 'high' price, the seller would be even better off above the 'high' price. And at the extreme end, the seller would want to be above the 'high' price that has an equivalent expected final price as the 'lowest' priced auction. Therefore, we conclude that, in general, the seller is always better of the higher the price of an auction.

6.2. Effect of Number of Ongoing Auctions and Interaction Term. The second and perhaps more interesting results regards the cannibalization effect. In the equations in Table 4.4 this is represented by the coefficients for z and $x \times z$. We observe that the probability of observing a positive price-transition is decreasing in the average number of ongoing auctions, z. This is seen by observing that the coefficients for z in the equations pertaining to $\ln(\pi_s/1 - \pi_s)$ are either negative or non-significant. A third observation is that, though the interaction term $x \times z$ is rarely significant, it does exhibit some influence. This means that, with the exception of D4, the dynamics between price and number of

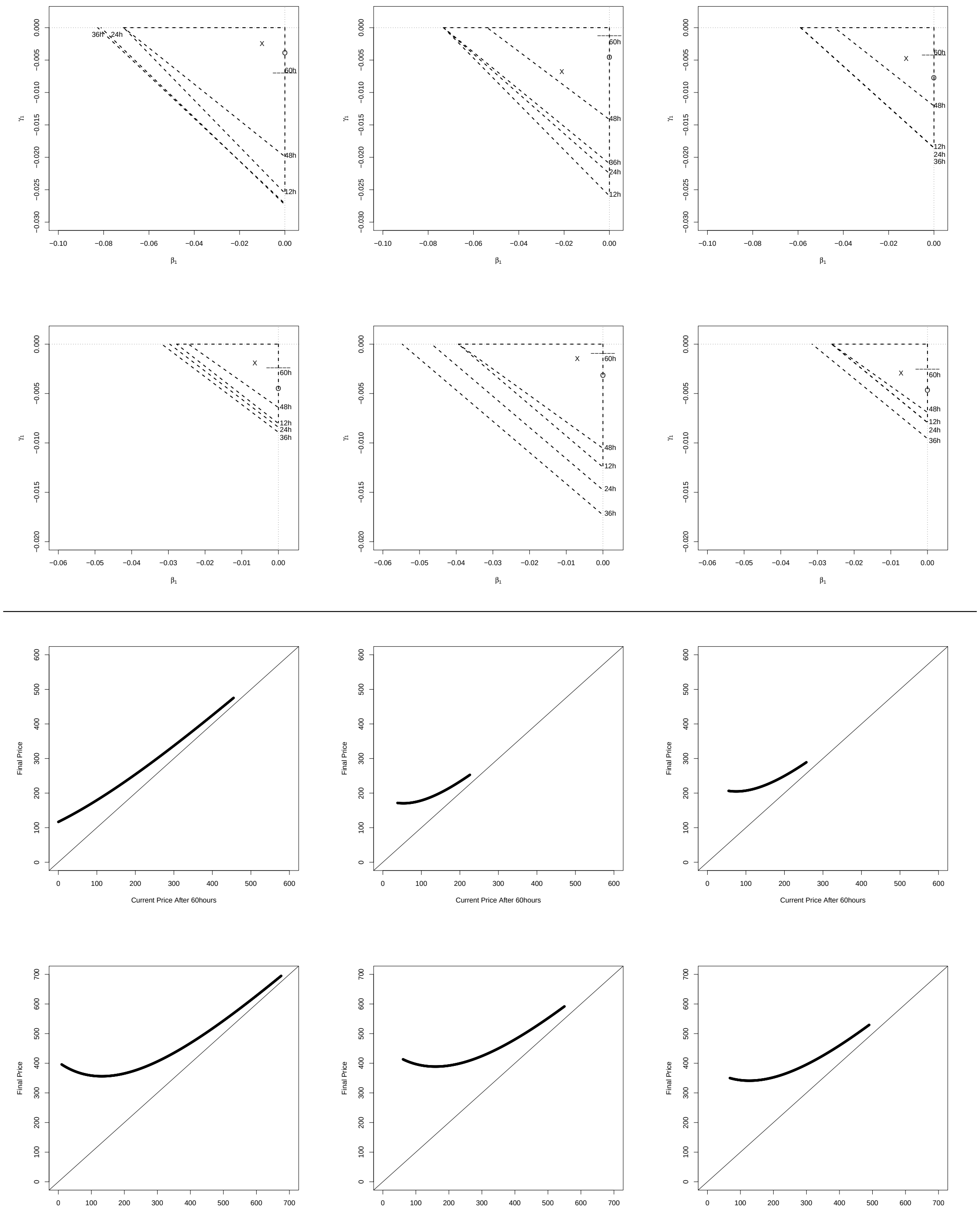


FIGURE 4.5. The top six graphs display the feasible region for which the sufficient conditions of Lemma 4.2 holds (first row: D1, D3, D4, second row: L1, L4, L5; left to right). Each graph has β_1 on the x-axis and γ_1 on the y-axis. The triangular area enclosed by the dashed lines provides the region such that Lemma 4.2 holds for the middle four periods. The ‘x’ represents the point estimate of (β_1, γ_1). The line labeled ‘60h’ on the y-axis is the lower bound for γ_1, and the circle represents the point estimate of γ_1 for the final period. If the ‘x’ or circle is outside the region respectively lower bound it implies Lemma 4.2 does not hold.
The bottom six graphs show the conditional expected price transitions for the final period (first row: D1, D3, D4, second row: L1, L4, L5; left to right). The x-axis represents price after 60 hours, and the y-axis represent the final price. The 45 degree line indicates auctions that received no bids. The bold curve line shows $E[X_\tau|S] = x + \mu_s$. The main feature of interest is to see if $E[X_\tau|S]$ is increasing.

ongoing auctions will at some point during an auction come into effect. More details follow but first we summarize the formal hypothesis results regarding the number of competing auctions and the interaction term.

Test of Hypothesis 4.7 - For D3, L1, L4, L5 in the first period we reject the hypothesis that $\beta_2 \geq 0$ ($p < .02$ for all products), while for D1 and D4 in the first period we fail to reject the hypothesis that $\beta_2 \leq 0$ ($p > .42$ for D1 and $p > .75$ for D4).

For D1, D3, D4, L1, L5 during the middle periods we reject the hypothesis that $\beta_2 \geq 0$ ($p < .005$ for D1, D3, L1, L5, and $p < .091$ for D4), while for L4 we fail to reject the hypothesis that $\beta_2 \leq 0$ ($p > .66$). We therefore conclude that the within period log-odds of observing a positive price-increment are on average decreasing in the number of competing auctions.

Test of Hypothesis 4.8 - For D1, D3, L1, L4 in the first period we reject the hypothesis that $\gamma_2 \geq 0$ ($p < .04$ for all products), while for D4 and L5 we fail to reject the hypothesis that $\gamma_2 \leq 0$ ($p > .18$ for D4 and $p > .33$ for L5).

For D4, L1, L5 during the middle periods we reject the hypothesis that $\gamma_2 \geq 0$ ($p < .002$ for all products), while for D1, D3, L4 we fail to reject the hypothesis that $\gamma_2 \leq 0$ ($p > .53$ for D1, $p > .22$ for D3, and $p > .83$ for L4).

For D1, D3, L4 in the final period we reject the hypothesis that $\gamma_2 \leq 0$ ($p < .002$ for all products), while for D4, L1, L5 we fail to reject the hypothesis that $\gamma_2 \geq 0$ ($p > .17$ for D4, $p > .98$ for L1, $p > .78$ for L5).

Therefore, for the first and middle periods we conclude that the within period price-increments are on average decreasing in the number of competing auctions, while for the final periods we conclude that the within period price-increments are on average not decreasing in the number of competing auctions. In other words, Hypothesis 4.8 only holds for the first and middle periods, and not for the final period.

Test of Hypothesis 4.9 - For D1 and L1 during the middle periods we reject the hypothesis that $\beta_3 \geq 0$ ($p < .075$ for both), while for D3, D4, L4, L5 we fail to reject the hypothesis that $\beta_3 = 0$ ($p > .77$ for D3, $p > .94$ for D4, $p > .66$ for L4, and $p > .18$ for L5). We therefore conclude that the decreasing rate of the within period log-odds of observing a positive price-increment due to an increase in the number of ongoing auctions, is not diminishing in price. That is, Hypothesis 4.9 does not hold.

Test of Hypothesis 4.10 - For L5 during the middle periods we reject $\gamma_3 \leq 0$ ($p < .03$), while for D1, D3, D4, L1, L4 we fail to reject $\gamma_3 = 0$ ($p > .64$ for D1, $p > .22$ for D3, $p > .27$ for D4, $p > .50$ for L1, and $p > .83$ for L4).
For D3 and L4 during the final period we reject $\gamma_3 \geq 0$ ($p < .004$ for both products), while for D1, D4, L1, L5 we fail to reject $\gamma_3 = 0$ ($p > .39$ for D1, $p > .43$ for D4, $p > .36$ for L1, and $p > .51$ for L5). We therefore conclude that during both the middle and final periods the expected decrease of the within period price-increments due to an increase in the number of ongoing auctions, is diminishing in price. In other words, Hypothesis 4.10 holds.

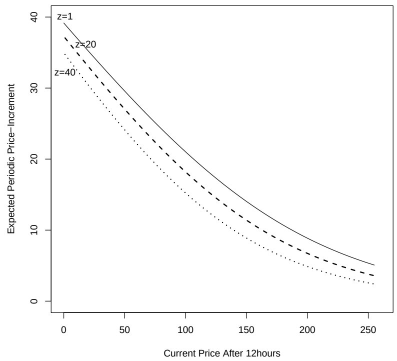
A comment regarding the test of Hypothesis 4.10 is that although the sign of the γ_3 coefficient for the final period is different than during the middle periods, it is also different from the sign of the γ_2 coefficient for the final period. In other words, we note that for both the middle and final periods, the main effect of the number of ongoing auctions is 'diminished' by the interaction effect from price.

Similar to the effect of price, there seems to be a different dynamics between the first five periods and the final period. In the first five periods, though z is not always significant and $x \times z$ hardly ever is, the coefficients are always such that Lemma 4.3 holds. For example, for D4 we see that an unit increase of z, decreases the log-odds of observing a positive price-increment with .027, and decreases the log-mean of the positive price-increment with .023. Note, however, that for the middle periods of L5, there is a slight violation regarding the condition on price, as $-\gamma_2/\gamma_3 = .123/.0004 = 308$, and there are several L5 auctions during the middle periods at prices higher than \$308. Consequently, we conclude that during the first five periods, the more ongoing auctions the lower the conditional expected price-increment $E[C_{Y+1}|S]$. Though this might not clearly be visible in Figures 4.10, 4.11, and 4.12, it has intuitive appeal and should not be too surprising.

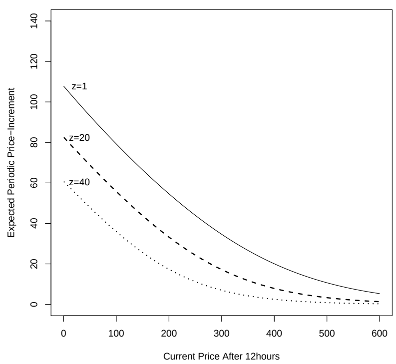
In the final period, for the three instances where z is significant, namely D1, D3, and L4, γ_2 is positive. This would imply that the main effect of having more ongoing auctions, is that it increases the conditional expected price-increment. Though this might seem a bit counter-intuitive and paradoxical, it ignores the interaction effect $x \times z$. For both D3 and L4, since γ_3 is negative the conditions for Lemma 4.4 are satisfied. Specifically, the price-level $p^i = -\gamma_2/\gamma_3$, is the point such that if $x > p^i$ ($x < p^i$) then the conditional expected price-increment decreases (increases) the more ongoing auctions there are. As mentioned in Section 3.2, the 'cannibalization effect', or rather price competition effect, benefits the lower priced auctions and works against the higher priced auctions. With the interpretation that this only happens in the presence of both low and high priced auctions, this has intuitive appeal and seem natural to expect. The specific price levels p^i are as follows. For D3, $p^i = \$135$, which is slightly above the median of X_{60} and about \$50 below the median X_τ. And for L4, $p^i = \$395$, which though it is almost \$100 above the median of X_{60}, is just slightly below the median of X_τ. Therefore, the loose interpretation would be that auctions for D3 or L4 products, which are priced below the median of X_τ and only has 12 hours remaining, are considered 'good deals', and consequently will either attract more bidders and/or higher bids. In the bottom two graphs of Figure 4.6 the 'cannibalization effect' in the last period for D3 and L4 is shown. On the x-axis is the price after five periods (X_{60}) and on the y-axis is the conditional expected within period price-increment. The solid, dashed and dotted lines, represents the scenario with 1, 5 and 10 ongoing auctions. We see that for D3 and L4, if prices are below \$135 respectively \$395, then the more ongoing auctions, the expected within period price-increment increases. And for prices above the thresholds the expected price-increment decreases. The only anomaly regarding a positive γ_2 is the aggregated product D1. There is no intuitive reason why this should be the case, and the graph in Figure 4.10 does not indicate any strong positive trend. One possibility might be that, since the expected price transitions are decreasing in the number of ongoing auctions for the first five periods, in the final period there is some compensation. Regardless, we conclude that, in general, each auction will experience a lower price transition the more ongoing auctions there are.

The third result of interest is regarding the diminishing cannibalization effect, which would imply that a threshold type policy is optimal. For the middle four periods only β_3 for D1 and L1, and γ_3 for L5 were found to be significant. In the former cases, β_3 is negative, which implies that the cannibalization effect is increasing in price. That is, the higher the price and the more ongoing auctions the less likely there will be a positive within period price-increment. While in the latter case, γ_3 is positive, which implies that the cannibalization effect is diminishing in price. Illustration of the two different cannibalization effects are shown in the top three graphs of Figure 4.6. Each graph has the expected within period price-increment for the second period ($12 \rightarrow 24h$) on the y-axis, and price after 12 hours on the x-axis. Each line represents the expected within period price-increments for a given number of ongoing auctions as labeled. For L5 we see that as price increases the lines get closer to each other, i.e. the cannibalization effect is diminishing in price. While for D1 and L1 we see that if there are more ongoing auctions then the exponential curve becomes steeper, and hence the cannibalization effect is increasing in price. However, we also see that, as discussed in Section 3.3, due to the shape of the negative exponential curve, there is a price-level at which the difference between the lines is maximized and after which the difference diminishes. That is, the cannibalization effect is first increasing and then decreasing in price. For D1 and L1 the cannibalization effect seems to be increasing up to about \$125 respectively \$250, and thereafter decreasing or diminishing. The other noteworthy observation to point out is the difference between D1 and L1 regarding the cannibalization effect. For D1 there is a much smaller shift in going from 1 to 20, or from 20 to 40, ongoing auctions, as compared with L1. In other words, the laptop auctions are much more sensitive to an increase in the number of ongoing auctions. One explanation for this might be that laptops are approximately twice as expensive, and hence there is more room for price variability (in particularly downward). While for desktops the prices might in general already be so low that there is not much room for them to decrease to.

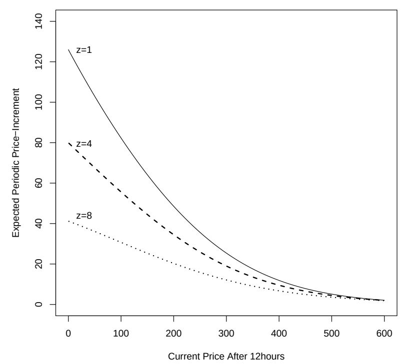
For the final period, as mentioned above and illustrated in Figure 4.6, only D3 and L4 have significant interaction terms. Although, as discussed in Section 3.3, there exist a price-level p^{ii}, such that if $x > p^{ii}$ ($x < p^{ii}$) then the cannibalization effect is decreasing (increasing) in price, it is hard to see if this price-level is within range of values displayed. The graphs in Figure 4.6 are limited to the range of values displayed by the data. Consequently, we conclude that the cannibalization effect is not diminishing but rather increasing in price (note that at first the cannibalization effect is negative). The implication on the optimal release policy for all six products will be discussed in Section 7.



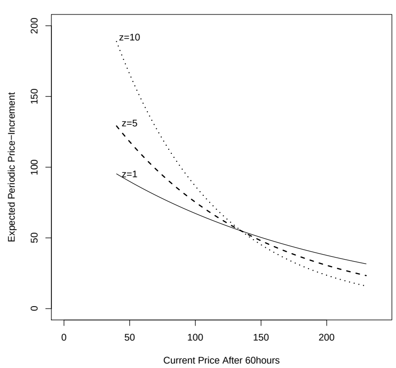
D1 - Y=12h ($\beta_2, \beta_3, \gamma_2 < 0, \gamma_3 = 0$)



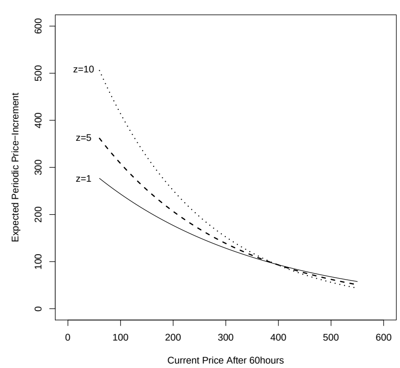
L1 - Y=12h ($\beta_2, \beta_3, \gamma_2 < 0, \gamma_3 = 0$)



L5 - Y=12h ($\beta_2 < 0, \beta_3 = 0, \gamma_2 < 0, \gamma_3 > 0$)



D3 - Y=60h ($\gamma_2 > 0, \gamma_3 < 0$)



L4 - Y=60h ($\gamma_2 > 0, \gamma_3 < 0$)

FIGURE 4.6. Each graph shows the expected within period price-increment (y-axis) as a function of price (x-axis). The top three graphs are for D1, L1 and L5 in the second period (12 to 24h), while the bottom two graphs are for D3 and L4 in the final period (60 to 72h). Each line represents the conditional expected price-increment given the number of ongoing auctions as labeled. The solid line is the base case when there is only one ongoing auction.

6.3. Effect of Elapsed Auction Time. Another observation to note is that the individual periods exhibit different dynamics. Clearly there is a difference between the first, middle and final periods, since they all have different coefficient values. In addition, for the middle periods we note that, for all products, at least one of the elapsed period, y, indicator functions is significant. This is true both with regard to $\ln(\pi_s/1-\pi_s)$ as well as $\ln(\mu_s)$.

6.4. Estimation of Gamma Shape Parameter. The other result of interest concerns the shape parameter of the gamma distribution for the positive price-increments. Recall from Figure 4.3 in Section 4.3, that depending on the shape parameter ν, the gamma distribution will look and behave differently. If $\nu > 1$ then the gamma distribution resembles a right skewed unimodal curve, and if $\nu < 1$ then it resembles a steeper exponential curve. In the last two columns of Tables 4.5, 4.6, and 4.7 below, the resulting estimates of ϕ and ν for the different periods of the auctions are shown (recall $\phi = 1/\nu$). First thing to note is that all products have very similar parameters and hence similar shapes. The only striking exception is for the first period of D3, where the distribution resembles a steeper exponential distribution due to that $\hat{\nu} < 1$. In all other instances, the distribution behaves like a heavily skewed unimodal curve. That is, there is a long tail to the right (large price-increments) and almost no tail to the left (small price-increments). This is expected and natural since price-increments cannot be negative. Moreover, we see that as the auctions progress the unimodal curve shifts more and more to the right, i.e. to higher price-increments. In particular, in the final period $\hat{\nu}$ is approximately twice as large as during the middle periods. This further reflects the difference in auction dynamics in the final period, where the low priced auctions do not exhibit small price-increments. In other words, the 'large' values of $\hat{\nu}$ in the final period, reflects the change in pattern as displayed in Figures 4.7, 4.8 and 4.9, and discussed in Section 5.1. This further attest to the applicability and appropriateness of the proposed model. Not only does it adjust the mean rate according to changes in the

	$\ln(\pi_s/1-\pi_s)$		$\ln(\mu_s)$			
	D	df	D	df	$\hat{\phi}$	$\hat{\nu}$
D1	*n.a.*	*n.a.*	1915	1080	.805	1.242
D3	347	272	278	117	1.243	.804
D4	*n.a.*	*n.a.*	*n.a.*	*n.a.*	*n.a.*	*n.a.*
L1	2058	2044	1879	1607	.747	1.339
L4	198	170	141	122	.702	1.425
L5	154	161	*n.a.*	*n.a.*	*n.a.*	*n.a.*

TABLE 4.5. Summary of residual deviance D, df, $\hat{\phi}$ and $\hat{\nu}$ for the final models in the first period.

	$\ln(\pi_s/1-\pi_s)$				$\ln(\mu_s)$					
	D	df	ΔD	Δdf	D	df	ΔD	Δdf	$\hat{\phi}$	$\hat{\nu}$
D1	10430	8281	-	-	5075	4747	2.094	3	.700	1.428
D3	1311	1090	.083	1	619	636	7.744	4	.582	1.719
D4	833	664	1.034	3	334	391	.940	3	.555	1.801
L1	9638	8177	-	-	4580	5043	.294	1	.659	1.516
L4	806	685	7.265	4	373	424	1.030	3	.584	1.712
L5	777	648	1.880	3	287	360	1.923	2	.603	1.660

TABLE 4.6. Summary of residual deviance D, df, $\hat{\phi}$ and $\hat{\nu}$ for the final models during the middle periods. The columns ΔD and Δdf represents the increase in D respectively df in the final models versus the unabridged models.

	$\ln(\mu_s)$					
	D	df	ΔD	Δdf	$\hat{\phi}$	$\hat{\nu}$
D1	1048	2068	.302	1	.350	2.85
D3	124	270	-	-	.325	3.078
D4	56	165	.557	2	.232	4.312
L1	828	2038	1.603	2	.279	3.589
L4	44	168	-	-	.161	6.220
L5	76	161	.434	1	.342	2.924

TABLE 4.7. Summary of residual deviance D, df, $\hat{\phi}$ and $\hat{\nu}$ for the final models in the final periods. The columns ΔD and Δdf represents the increase in D respectively df in the final models versus the unabridged models.

covariates, but it can also adjust the shape of the price-increment distribution to shifts over time.

6.5. Goodness-of-Fit and Residual Analysis. As mentioned earlier, formal statistical model validation is a bit problematic. There are no general asymptotic results regarding the deviance D or the Pearson X^2. Nevertheless some comments regarding the 'goodness-of-fit' based on the deviance for the two components of the proposed model can be made. In Table 4.5, 4.6 and 4.7, a summary of D and degrees of freedom (df) for the final models are provided. Table 4.6 and 4.7 also includes the difference in D and df between the unabridged models and the final models. In the first period the final model is either the unabridged model or simply the null model. Cases where the final model is the null model are represented by $n.a.$ Cases where the unabridged model is the final model has no entry for ΔD and Δdf. Detail 'R' output for each product is provided in Appendix D.

The deviance and degrees of freedom for the first component, the probability of a positive price-increment, is summarized in the first set of columns of Table 4.5 and 4.6. The first observation to make, is that D is fairly close to the df for almost all cases. Ordinarily, this would have been ideal and indicated a good fit. However, for Bernoulli distributed observations, as shown in Section 4.4.1, this should not come as a surprise. In fact, the resulting values of D illustrates the arguments of Section 4.4.1. For X^2 the evidence is even stronger (see output in Appendix D). Therefore, we make no statistical claims regarding the fit and simply leave it for the reader to judge whether the Bernoulli model with a logit-function is appropriate or not. On the other hand, since the difference in D of nested models does follow a χ^2 distribution, some formal assessment of the final model for the middle periods is possible. An informal assessment is to see if the gain in df compensates the increase of D (the final model has a larger D but also more df). With perhaps the exception of L4, we see that the gain in df more than compensates for the loss in D. And the formal test statistics, with the appropriate χ^2 distribution, does in fact indicate that the increase in D is not significant. Hence we conclude that the final models are more appropriate than the unabridged models.

The second set of columns of Table 4.5 and 4.6, and Table 4.7 summarizes the results for the second component, the positive price-increments. The first thing to note is that the deviance D (unscaled) measures are close to the degrees of freedom (with exception of D1 in the first period). Recall though, that the 'proper' comparison is to analyze the scaled deviance D/ϕ. We see that $D/\hat{\phi}$ is almost consistently a factor of 1.5 times greater than df. Although this is perhaps not excessive it would indicate a 'poor' fit. However, bare in mind that the available asymptotic results only apply when ϕ is small. If we define $\phi < .5$ as small, then only for the values in the final period would the χ^2 approximation hold. Formal statistical test, in the final period, of $D/\hat{\phi}$ compared with χ^2 with the appropriate df are as follows: D1) $p < .0001$, D3) $p < .0001$, D4) $p = .0001$, L1) $p < .0001$, L4) $p < .0001$, L5) $p = .001$.Therefore, rather than using any formal statistical significance test based on the deviance, we base the 'goodness-of-fit' on the ensuing analysis of difference in deviance of the nested models and residuals plots.

Similar to the earlier case, formal statistical assessment can be made with regard to the nested models. Here the 'proper' assessment is with regard to the scaled ΔD.[4] In all cases, with one exception, the increase in ΔD is not significant. That is, for all products in the last five periods, $\Delta D/\hat{\phi}$ as compared to a χ^2 distribution with Δdf degrees of freedom, has 'large' p-values. The only exception is for D3 during the middle periods. The associated p-values for each of the periods are (middle/final): D1) .393/.353, D3) .01/$n.a$, D4) .638/.301, L1) .504/.057, L4) .623/$n.a.$, L5) .203/.260. We therefore conclude that the final models fit the data better than the unabridged models, but note that for D3 the excluded variables might be influential.

In Figures D.1, D.2, and D.3 of Appendix D, two sets of residuals for each product and every period are shown. The top row for each product shows the deviance residual r_d, while the bottom row shows the response residuals $y - \hat{\mu}$. Each residual type is depicted against

[4]More correct would be to have the difference in scaled deviance tested $(D_{fin}/\hat{\phi}_{fin} - D_{unabr.}/\hat{\phi}_{unabr.})$, and not the scaled difference in deviance $((D_{fin} - D_{unabr.})/\hat{\phi}_{fin})$.

the fitted linear predictor $\hat{\eta}$. In each row, the first, second, and third graph represents the first, middle, and final periods respectively. It is important to remember that the linear predictor is, for instance, decreasing in price. Consequently the larger values of $\hat{\eta}$ correspond to the lower prices. For all products, the deviance residuals, for the first and middle periods, appear to be randomly distributed without any obvious trend. Thus indicating a good model fit. However, in the final period, for all products, the deviance residuals display a clear funnel shaped pattern. This would for a normal linear regression model, potentially indicate that additional covariates or a transformation of one of the covariates be included. Though various combinations of quadratic and log transforms of X and Z were tested, none resulted in a drastic change of the deviance residual plot. Furthermore, recall that in the final period the dynamics appeared to be a bit different, as lower priced auctions do not have 'small' price-increments (see Figures 4.7, 4.8 and 4.9). This might be further reflected by the apparent funnel shape of the deviance. Nevertheless, we do not conclude a poor model fit.

In the bottom row for each product, the response residuals versus the fitted linear predictor are shown. For products L1, D3 and L4, in the first period, one might argue that a funnel shape pattern exist. Since γ_2 is significant and negative, it would imply that the variance of the price-increments is decreasing in Z. Similarly, for all products, in the middle periods, there is a clear funnel shaped pattern. In these instances, since γ_1 is significant and negative, the variance of the price-increments is decreasing in X. In both situations, the funnel shape does not indicate a poor model fit. Contrariwise, since the variance of the gamma distribution is $\nu\mu^2$, and the mean is decreasing in Z and X for the first respectively middle periods, the response residuals indicates a good model fit. In the final period, for all products, the response residuals appear to be evenly and randomly distributed, thus indicating a constant variance. The graph for L1 is clearly distorted due to the outlier.

7. Discussion

This chapter has provided a statistical framework for analyzing the progression of online auctions. The objective was to provide a model that characterizes the expected within period price-increments conditional upon certain variables. Although the general framework would allow for almost any information, the variables that were the focus of this chapter included, price and elapsed time of an auction, and the number of ongoing or competing auctions. The main reason for analyzing these variables, was to determine if the structural properties and implications of Chapter 2 can be empirically justified in a real setting. Sufficient conditions on the parameters of the proposed model to support the structural properties in Chapter 2 were derived and discussed. Finally, the model was applied to six data-sets for Dell desktop and laptop auctions. There were two main results. First, the proposed model fit the data well, and auctions' within period price-increments appear to follow a zero-inflated gamma distribution. Second, though some exceptions exist, the main results of Chapter 2 hold.

The main trade-off considered in Chapter 2 was if the additional incurred holding cost would be compensated by the gain in expected price-increment by deferring the release one period. To illustrate consider Tables 4.8 and 4.9 below. The columns represents the price at the start of a period, while the rows represents the increase in number of ongoing auctions. Each entry is the decrease in the expected within period price-increment as a result of having more auctions underway as specified. The values are computed by substituting the values for x and z as specified, into the equations for the final model as listed in Table 4.4. Positive entries represents a loss while negative entries represents a gain. For instance, for the aggregated desktop data set D1, if an auction is priced at \$0 at the start of the second period, then the seller would lose \$.96 in expected within period price-increment if there were 10 rather than only 1 ongoing auction. Note that all values are positive in Table 4.8, which means that the seller will always be worse of with more ongoing auctions. While in Table 4.9 some of the values are negative, which indicates that the seller would benefit by having more ongoing auctions. In addition, note that the remaining middle periods will be similar but not identical to the values shown in Table 4.8, due to the indicator functions for the elapsed period y effecting the intercept term.

Two interesting and illustrative examples are products D3 and L5. We see that, for the second period, an auction for a D3 product currently priced at \$0, will lose \$2.80 in expected price-increment by having 5 rather than 1 ongoing auction, lose \$3.79 by having 10 rather than 5 ongoing auctions, and lose \$3.98 by having 15 instead of 10 ongoing auctions underway. In contrast, an auction for a L5 product currently priced at \$0, will lose \$32.93 by having 3 rather than 1 ongoing auction, lose \$35.52 by having 6 rather than 3 ongoing auctions, and lose \$23.27 by having 9 rather than 6 ongoing auctions. Furthermore, we see that the gain by having fewer ongoing auctions diminishes the higher the price. For instance, if a D3 auction is priced at \$200 at the start of the second period, then the decrease in expected within period price-increment by having 5 rather than 1 ongoing auction is \$.10, the decrease is \$.10 if there are 10 rather than 5 ongoing auctions, and the decrease is \$.07 if there are 15 rather than 10 ongoing auctions. While for a L5 auction currently priced at \$200, the decrease in expected price-increment is \$10.03 if there are 3 rather than 1 ongoing auction, the decrease is \$12.24 if there are 6 rather than 3 ongoing auctions, and the decrease is \$9.02 if there are 9 rather than 6 ongoing auctions. In the final period, we see that for L5 there is neither a gain or loss in having more auctions underway, while for D3 we have the situation where 'low' priced auctions gain from having more auctions and 'high' priced auctions are worse of. More specifically, we see that a D3 auction currently priced at \$50 at the start of the final period, will gain \$28.41 in expected price-increment if there are 5 ongoing auctions rather than 1, gain \$48.15 if there are 10 rather than 5 ongoing auctions, and gain \$67.38 if there are 15 instead of 10 ongoing auctions. While if a D3 auction priced at \$200 at the start of the final period, will lose \$8.07 in expected price-increment when there are 5 rather than 1 ongoing auction, lose \$7.92 if there are 10 instead of 15 ongoing auctions, and lose \$6.04 if there are 15 instead of 10 ongoing auctions. However, as discussed earlier, this presumably only holds given that there are both 'low' and 'high' priced auctions. In other words, if a D3 auction is priced at \$0-100 and only has 12 hours remaining, then it would presumably not gain the amounts indicated by starting the additional four or five D3 auctions. The main point is that if the additional incurred holding cost is not compensated by the gain in expected within period price-increment then it is optimal to release more items for auction.

(D1)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 10$	.96	1.23	1.32	.96
$10 \to 20$	1.09	1.38	1.44	.97
$20 \to 40$	2.28	2.77	2.74	1.62

(D3)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 5$	2.80	2.25	1.08	.10
$5 \to 10$	3.79	2.66	1.14	.10
$10 \to 15$	3.98	2.41	0.93	.07

(D4)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 5$	5.81	4.33	3.00	1.05
$5 \to 10$	6.51	4.80	3.26	1.10
$10 \to 15$	5.75	4.18	2.79	.89

(L1)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 10$	12.62	12.20	11.90	11.12
$10 \to 20$	12.73	12.35	12.01	10.72
$20 \to 40$	21.81	21.05	20.00	15.79

(L4)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 5$	0	0	0	0
$5 \to 10$	0	0	0	0
$10 \to 15$	0	0	0	0

(L5)	X_{12}			
Z_{12}	\$0	\$50	\$100	\$200
$1 \to 3$	32.93	25.27	19.04	10.03
$3 \to 6$	35.52	28.34	22.09	12.24
$6 \to 9$	23.27	19.30	15.53	9.02

TABLE 4.8. Illustration of cannibalization effect in the second period for four different prices. The columns represents X at the start of the second period, while the rows represents the decrease in Z for the second period. Each entry is the loss in expected price-increment. For example, a D1 auction priced at \$0 will loose \$.96 in expected price-increment if Z increases from 1 to 10.

(D1) Z_{60}	X_{60} \$0	\$50	\$100	\$200
$1 \to 10$	-3.02	-2.49	-2.06	-1.40
$10 \to 20$	-3.45	-2.85	-2.35	-1.60
$20 \to 40$	-7.23	-5.96	-4.92	-3.35

(D3) Z_{60}	X_{60} \$50	\$100	\$200
$1 \to 5$	-28.41	-7.99	8.07
$5 \to 10$	-48.15	-11.27	7.92
$10 \to 15$	-67.38	-12.89	6.04

(D4) Z_{60}	X_{60} \$50	\$100	\$200
$1 \to 5$	0	0	0
$5 \to 10$	0	0	0
$10 \to 15$	0	0	0

(L1) Z_{60}	X_{60} \$0	\$50	\$100	\$200
$1 \to 10$	0	0	0	0
$10 \to 20$	0	0	0	0
$20 \to 40$	0	0	0	0

(L4) Z_{60}	X_{60} \$50	\$100	\$200
$1 \to 5$	-86.79	-58.63	-22.40
$5 \to 10$	-146.86	-93.95	-32.19
$10 \to 15$	-205.23	-123.65	-37.58

(L5) Z_{60}	X_{60} \$50	\$100	\$200
$1 \to 3$	0	0	0
$3 \to 6$	0	0	0
$6 \to 9$	0	0	0

TABLE 4.9. Illustration of cannibalization effect in the final period for four different prices. The columns represents X at the start of the final period, while the rows represents the decrease in Z for the final period. Each entry is the difference in expected price-increment. Positive values represents a loss and negative values represents a gain. For example, a D1 auction priced at \$0 will gain \$3.02 in expected price-increment if Z increases from 1 to 10.

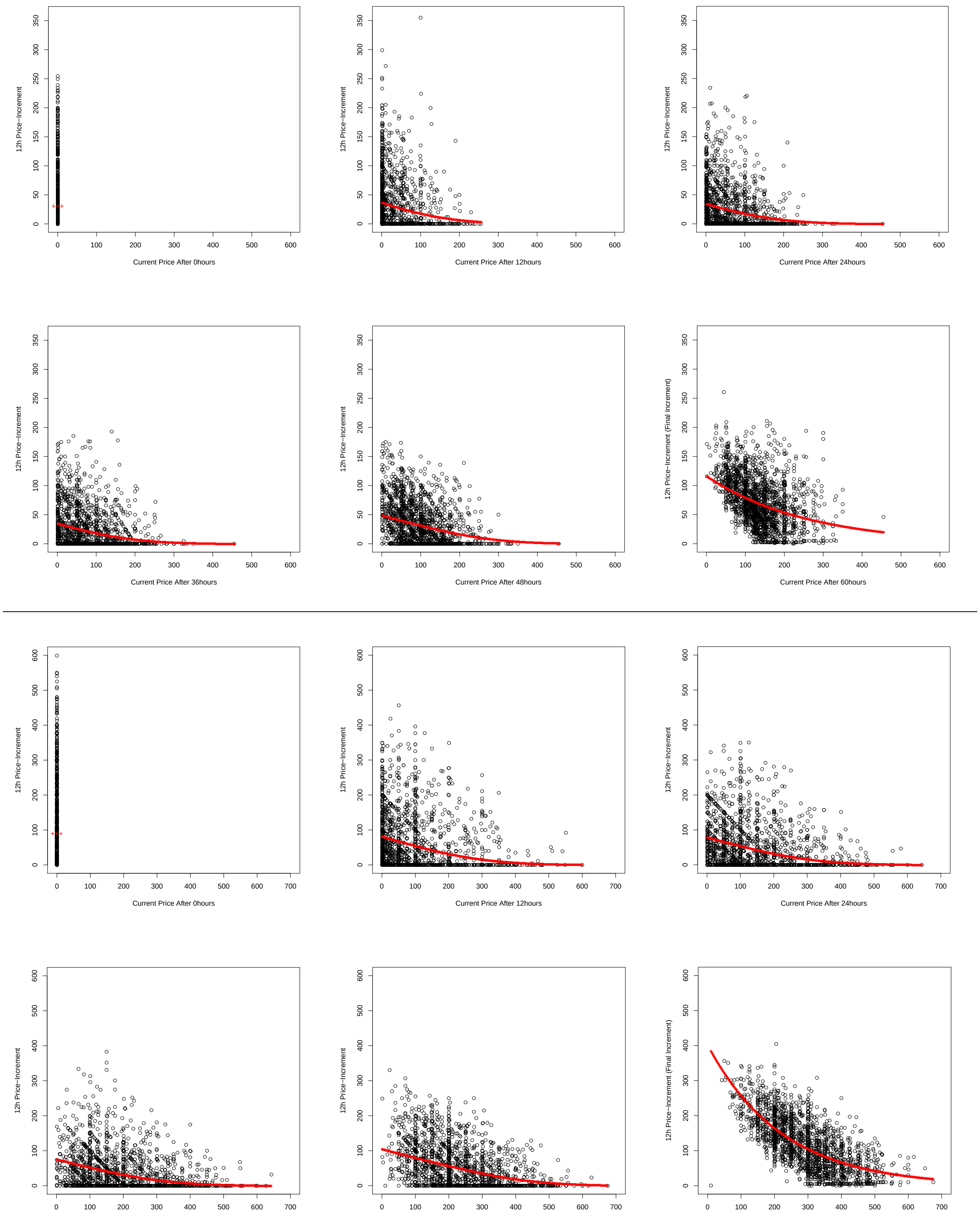


FIGURE 4.7. Within period price-increments for products D1 (top) and L1 (bottom) as a function of price. The line in each graph represents the conditional expected price-increment for the $0G$ model with z set at the median value (D1 $z = 20$, L1 $z = 20$).

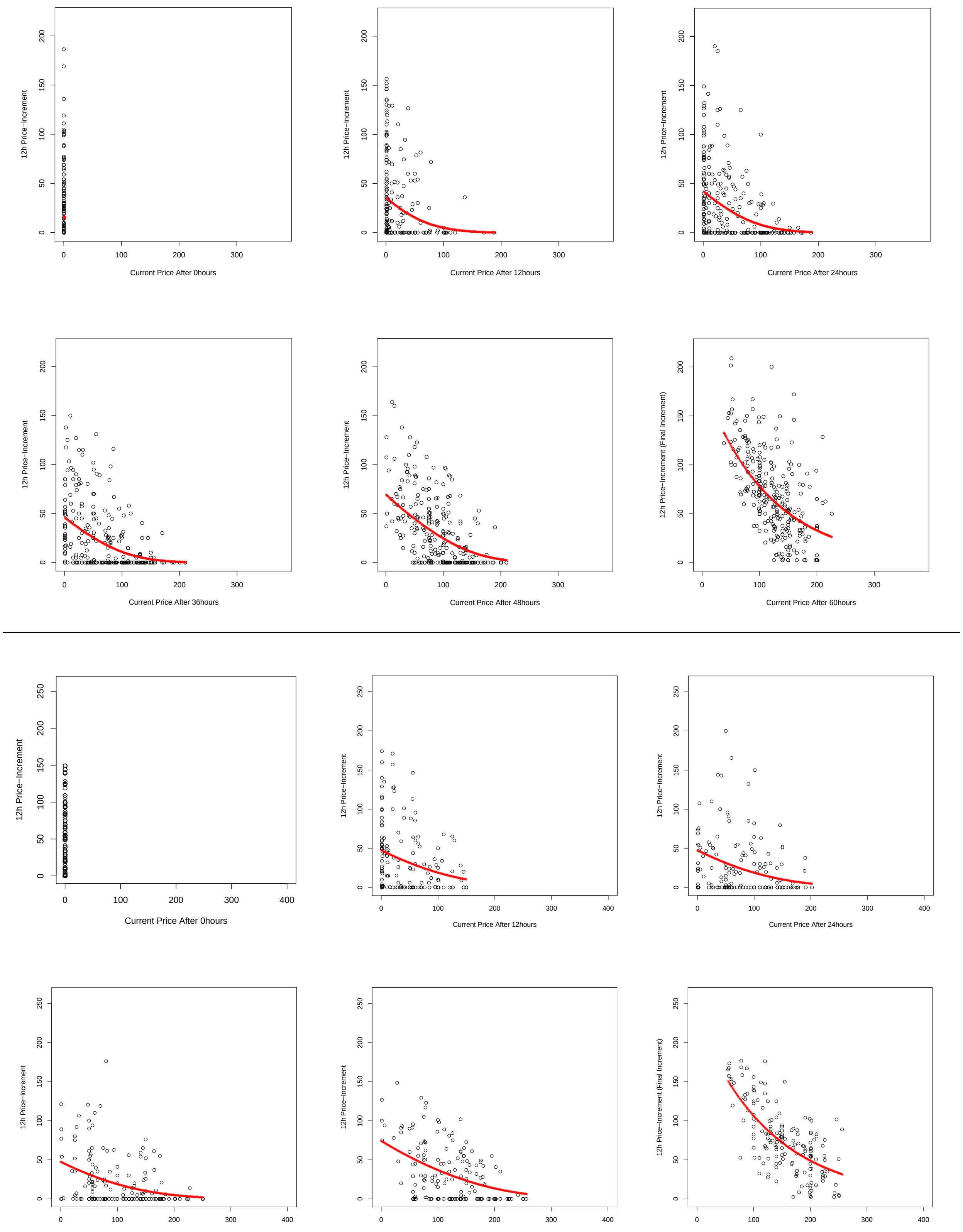


FIGURE 4.8. Within period price-increments for products D3 (top) and D4 (bottom) as a function of price. The line in each graph represents the conditional expected price-increment for the $0G$ model with z set at the median value (D3 $z = 5$, D4 $z = 5$).

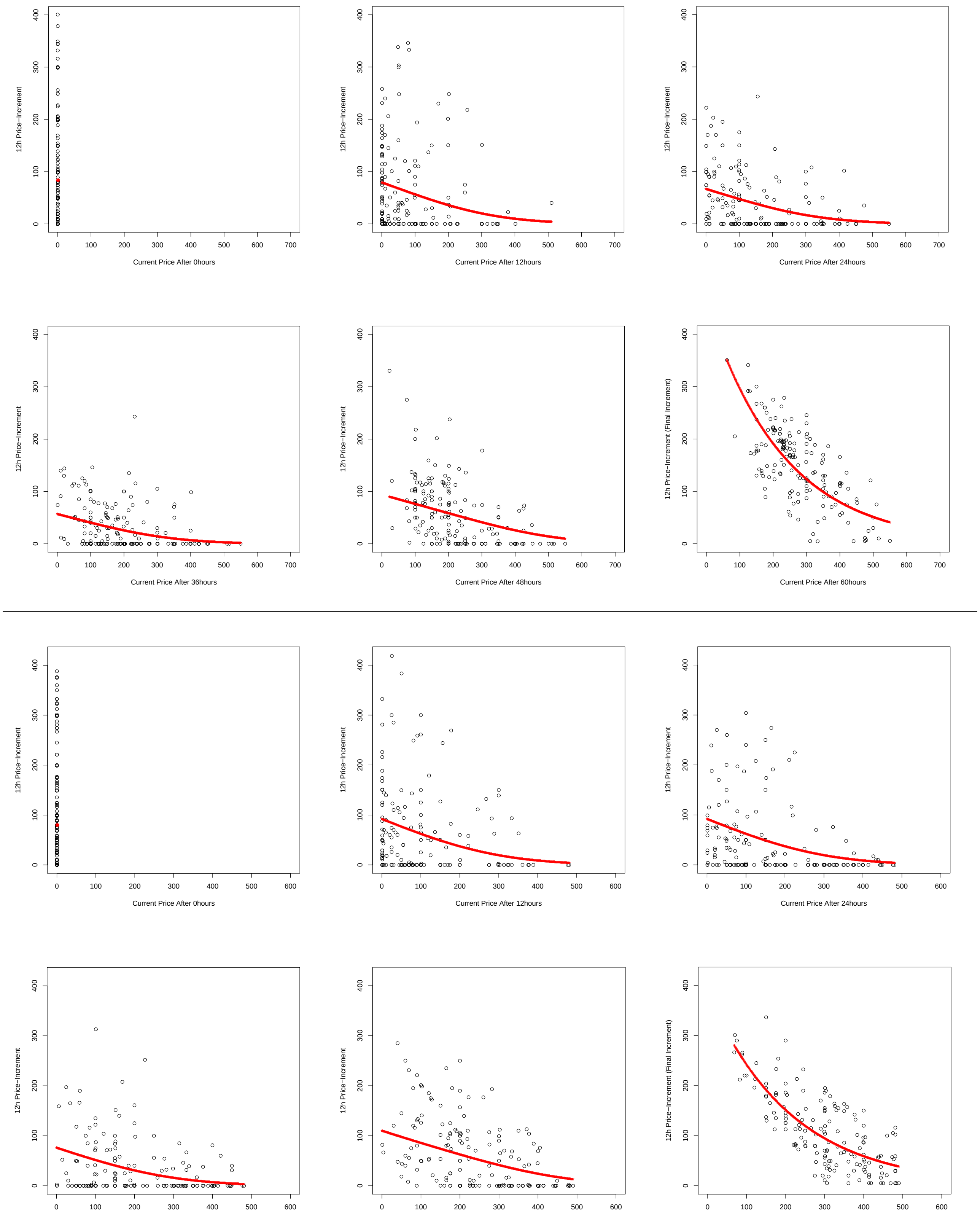


FIGURE 4.9. Within period price-increments for product L4 (top) and L5 (bottom) as a function of price. The line in each graph represents the conditional expected price-increment for the $0G$ model with z set at the median value (L4 $z = 5$, L5 $z = 3$).

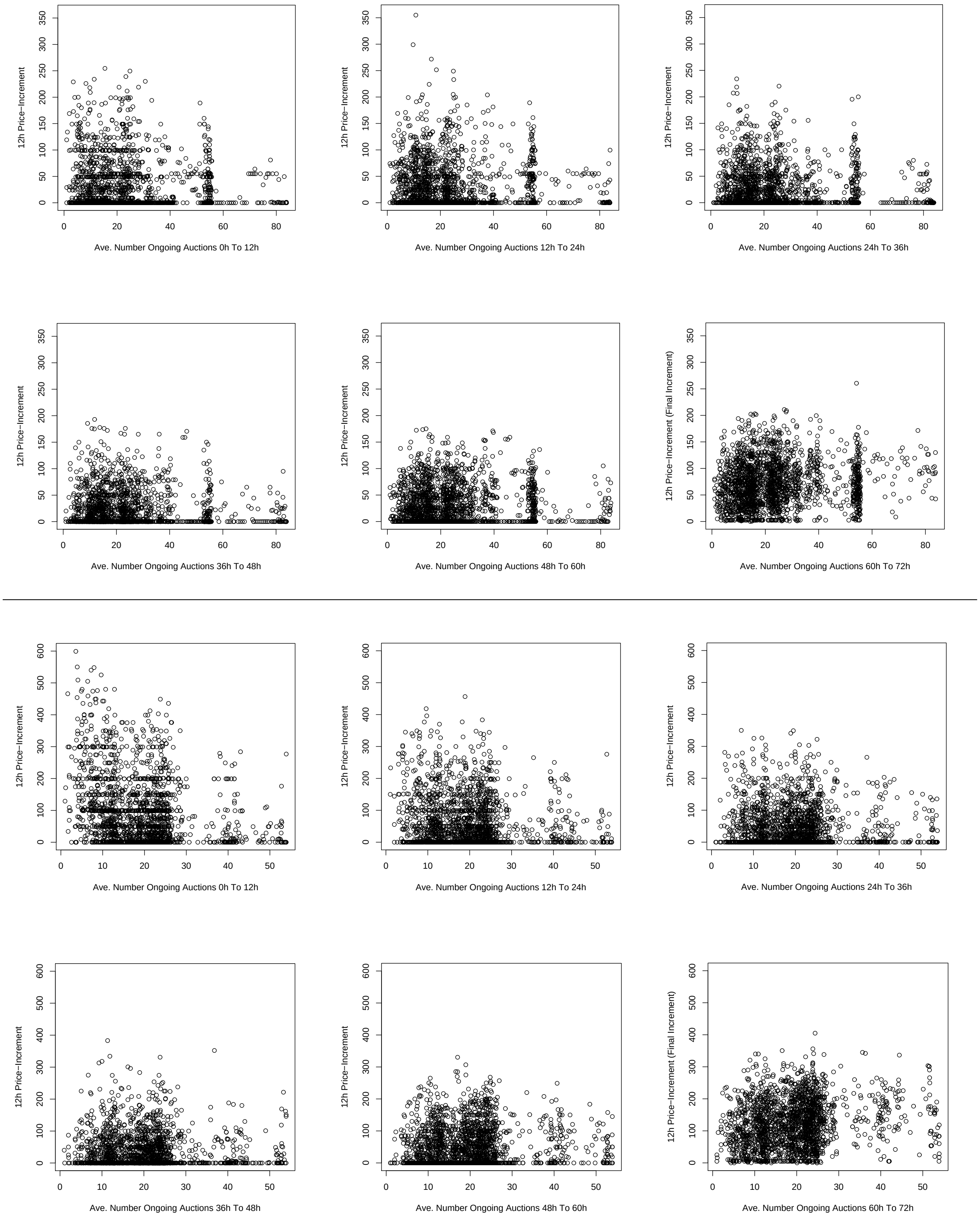


FIGURE 4.10. Within period price-increments for products D1 (top) and L1 (bottom) as a function of the average number of ongoing auctions.

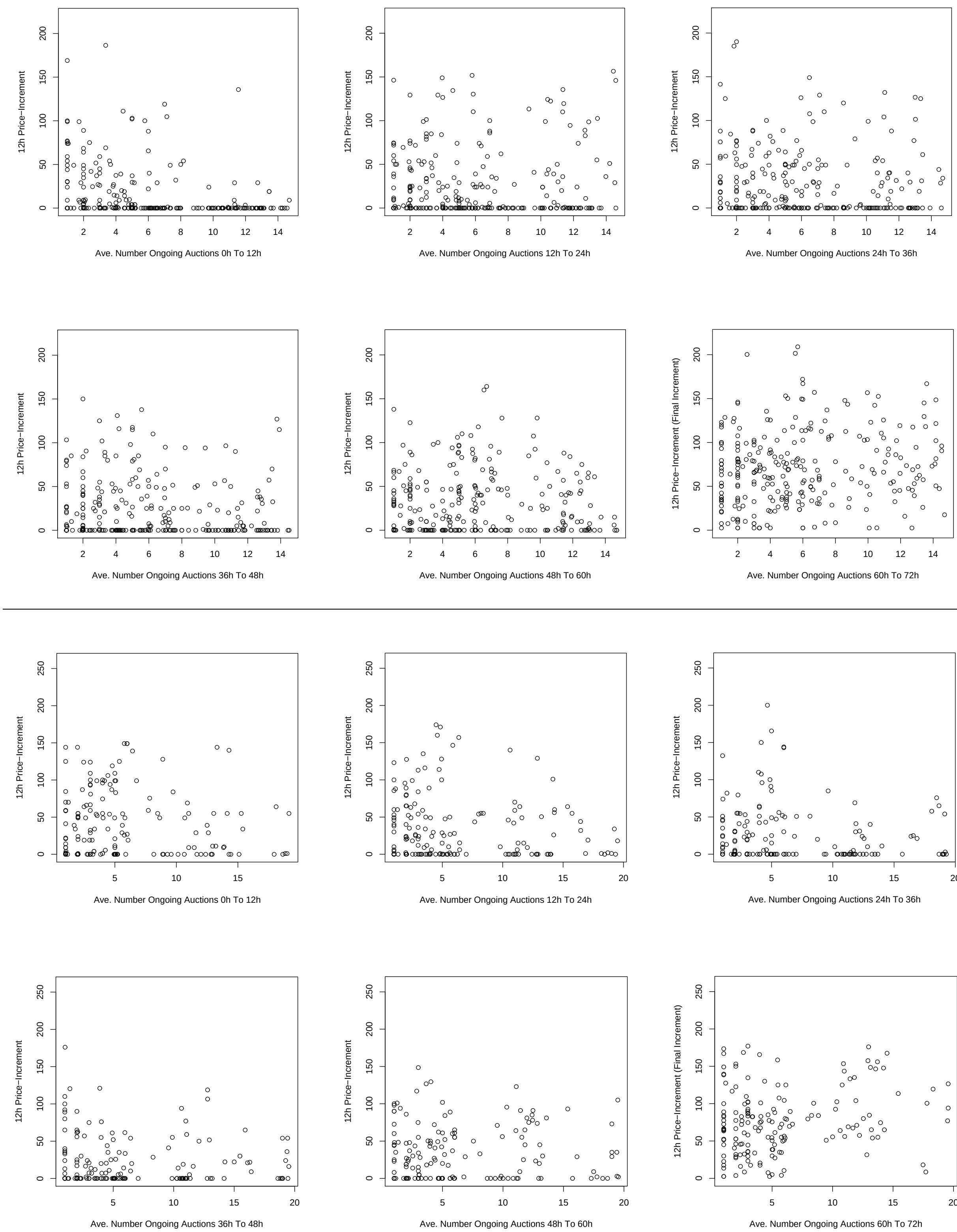


FIGURE 4.11. Within period price-increments for products D3 (top) and D4 (bottom) as a function of the average number of ongoing auctions.

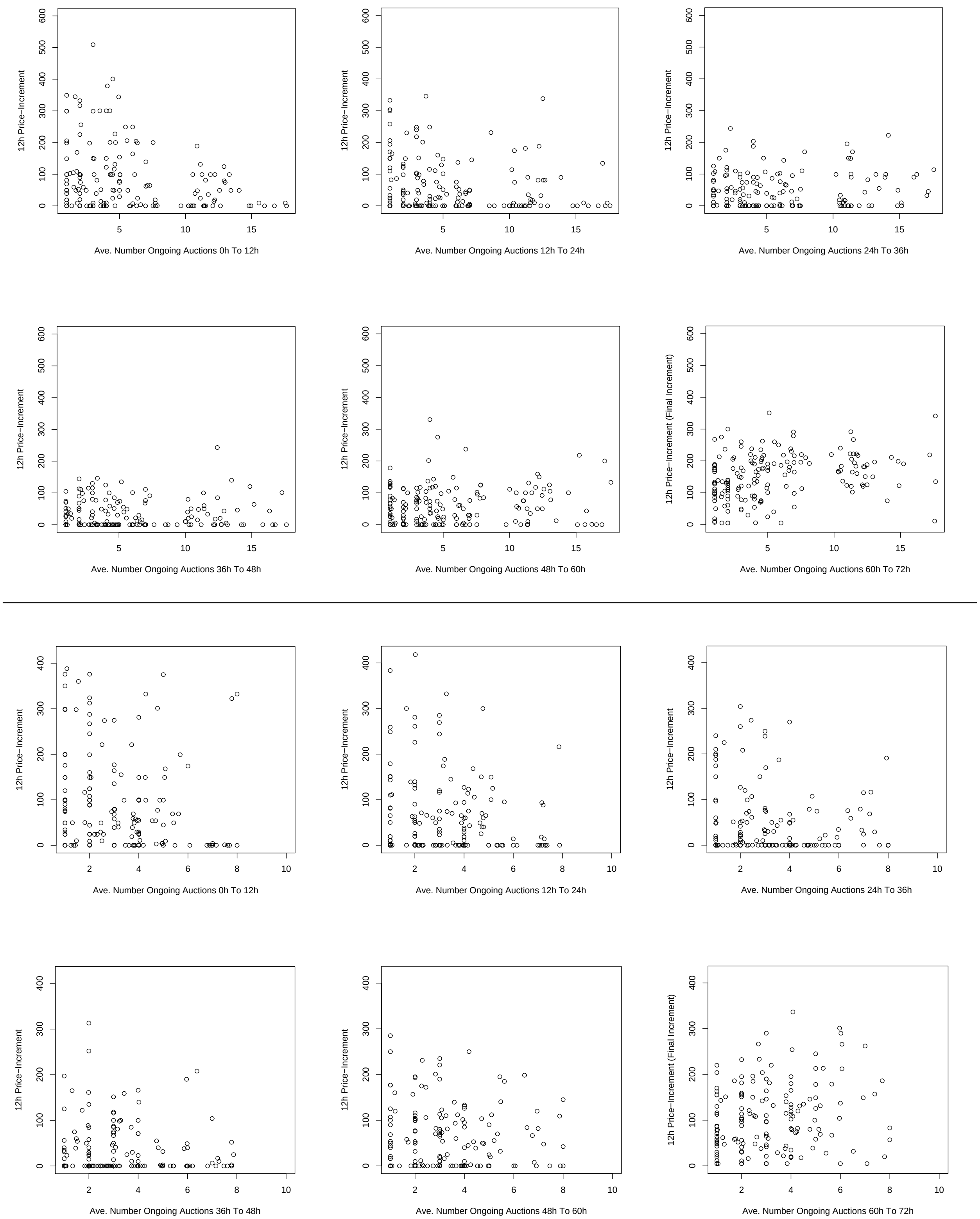


Figure 4.12. Within period price-increments for product L4 and L5 as a function of the average number of ongoing auctions.

CHAPTER 5

Empirical Analysis of Online Bidding Behavior

1. Introduction

In Chapter 3 we discussed how the within period price-transition probabilities can be derived given an underlying bidding strategy. Specifically we discussed the bidding strategies when bidders bid: 1) a minimum increment in the lowest priced auction, and 2) their true valuation in the lowest priced auction. The objective was not to establish that those are optimal strategies, or that in an eBay type setting they would lead to an (Bayesian Nash) equilibrium. Instead the objective was to show how the conditional price-transitions of an auction can be characterized given a specific bidding strategy. In contrast to Chapter 4, were the empirical analysis focused on the within period price-transition probabilities, this chapter will focus on the individual bidders and their actual bidding behavior. The objective is to propose and fit a statistical model for analyzing the underlying bid strategy. Similar to Chapter 3 we define bid strategy to be the amount a bidder decides to bid. In other words, the bidding strategy does not include the decision of when to place a bid. Therefore, the inter-arrival time of bids and the bid amount will be analyzed separately.

In Figure 5.1 histograms of the *bids* and *bid-increments* for the six products analyzed in Chapter 4 are shown. The top graphs depict the distribution of the bids, that is the amount of the bids, while the bottom graphs shows the distribution of the bid-increments, that is the amount above the current price that a bidder bid. To illustrate, suppose the current price of an auction is \$100, and that a bidder bids \$150. In this case, (the amount of) the bid is \$150, and the bid-increment is \$50. Though the histograms of bids do not seem to follow any of the common distributions, there is one striking feature. We see that bids in even \$100 amounts are much more frequent that other amounts, i.e. there are clear spikes around \$0, \$100, \$200, \$300, and so on. In other words, bidders seem to prefer bids in \$100 amounts. In addition, there are slightly shorter spikes around the \$50 amounts, i.e. around \$50, \$150, \$250, and so on. And after that the spikes are around the \$25 amounts, i.e. around \$25, \$75, \$125, \$175, and so on. Note that it is not possible to register a bid for \$0, and that the left most spike represents bids less than \$5.

The histograms of the bid-increments seem to tell a slightly different story. First of all, there is a clear pattern which resembles the exponential distribution. Or more specifically, with the exception of product D1, since the left most bar is shorter than the subsequent bars the distribution resembles a gamma distribution with shape parameter slightly above one. Recall from Section 4.3 of Chapter 4, that a gamma distribution with shape parameter greater than one, resembles a heavily skewed uni-modal curve. In other words, there seems to be a 'high' probability of observing 'small' increments, and a 'low' probability of observing 'large' increments. Furthermore, there is a similar feature as observed in the histograms of the bids. Namely, spikes around the \$50 increments, i.e. there are spikes around \$50, \$100, \$150, \$200, and so on. Given the spikes observed in the bid histograms, this should not come as a surprise. Clearly the spike features in one histogram would imply spike features in the other. However, it does beg the question: How do bidders decide on their bid? Do they decide on a given bid amount, or do they decide on a given bid-increment?

It is of course possible, and probably likely, that there are some bidders who choose a bid amount and others who choose a bid-increment. And the data shown in Figure 5.1 reflects bids from both types. The objective of this chapter is not to analyze the issue of *how* bidders decide, but rather to statistically characterize the bids. In particular, the objective is to propose a model that characterizes the bid-increments. Furthermore, the analysis will focus on the conditional bid-increment given various variables. Important information that is lacking in Figure 5.1 includes, for instance, the price of the auction when the bids were placed. How the distribution of the bid-increments depend on, for instance, price is the main motivation behind this chapter.

Overview of Chapter 5. The remaining chapter is organized as follows. In Section 2 we provide some descriptive statistics of the data analyzed. Section 3 proposes a model where bid-increments follow a gamma distribution, and Section 4 discusses the results of fitting the model to the data. Section 5 provides a brief analysis of the inter-arrival time of bids. Finally, Section 5 concludes the chapter with a discussion on the findings.

An important issue to bare in mind is that this chapter only considers bids that are strictly greater than the current price of an auction. In other words, potential bidders who show up but choose not to bid will not be registered, and therefore not included in the analysis. This implies that there is no information regarding which auctions a bidder contemplated between. The only available information is that a bidder placed a bid in a given auction.

2. Descriptive Statistics of Bids

The data for which the ensuing analysis is based on comes from the bid history of the products analyzed in Chapter 4. For each auction complete information on all non-winning bids is available. This includes the user-id of the bidder, and the time and amount of the bid. For the winning bid, however, the bid amount is censored and is only shown as the

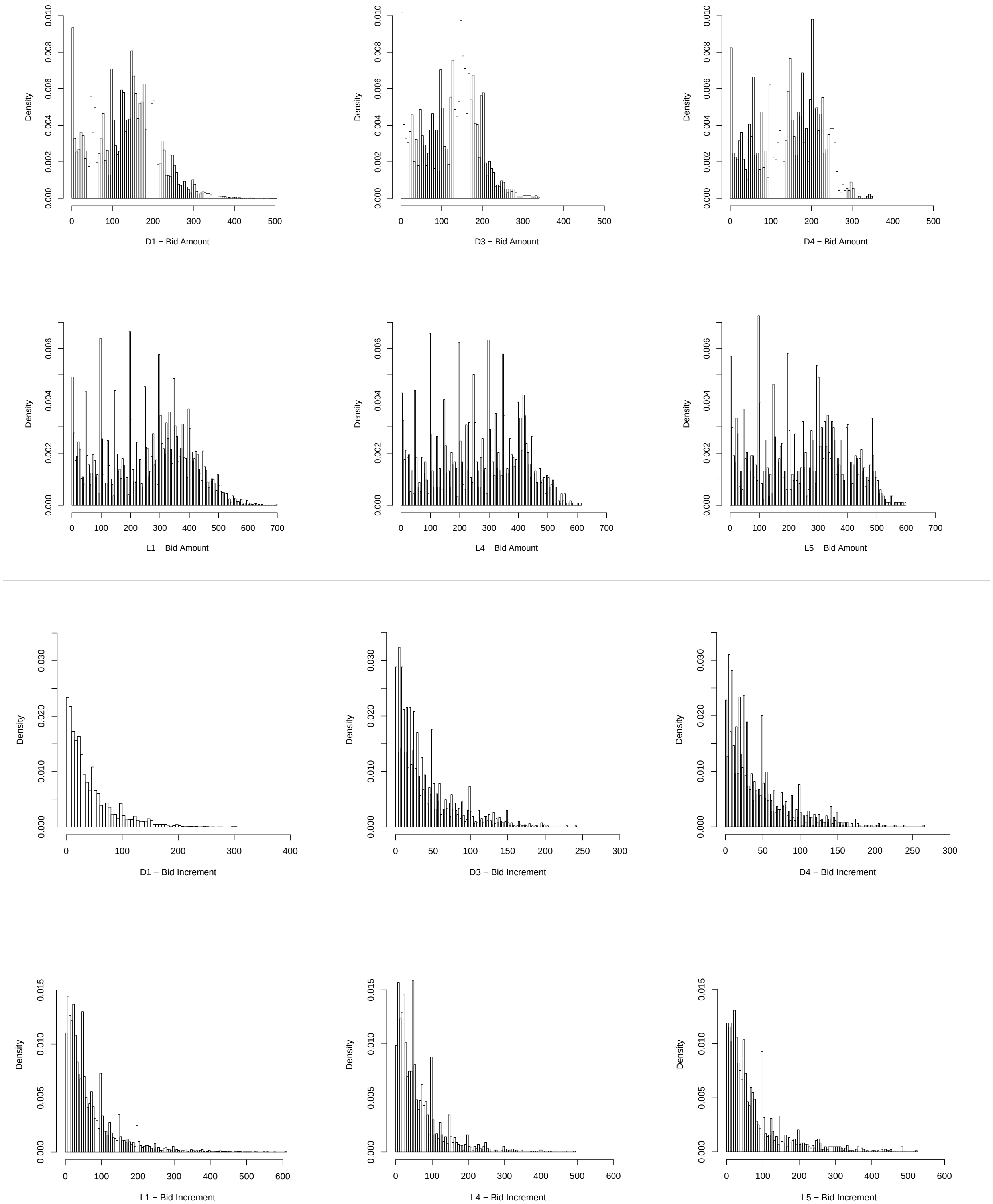


FIGURE 5.1. Histogram of the bids (top) and bid-increments (bottom). The top row of each set of histograms is D1, D3, and D4 (left to right), while the bottom row is L1, L4, and L5 (left to right),

minimum increment above the second highest bid. The minimum increments, which depend on the current price, are listed in Chapter 1. Despite the fact that the winning bid is censored, since a bidder does not with certainty know his bid will be the highest, there is no reason to suspect that the winning bids are the result from a different bidding strategy than the non-winning bids.

Table 5.1 provides some descriptive statistics for the bids and bidders of each product. The second and third column lists the total number of auctions and bids respectively. The seventh column lists the percentage of bids that are placed in the final hour. Column 11 and 12 lists the percentage of winning bids that arrive in the final hour and as the last bid respectively. On average about 17% of all bids and about 60% of the winning bids arrive in the final hour of an auction. Furthermore, about 60% of the winning bids are the last bid of each auction. Note that the last bid may or may not fall within the final hour of an auction. The mean, standard deviation, median, min and max for the number of bids and bidders per auction are also listed. On average there are about 10 bids and 8 bidders per auction. In other words, a bidder most likely places only one bid per auction.

An important note regarding the bid history and reported statistics, is that we treated consecutive bids within 10 minutes from an individual bidder as a single bid. That is, there were instances where a bidder would immediately place another bid upon not being registered as the high-bidder. For instance, a bidder might arrive, bid \$1 above the current price and see that this is insufficient to become the high-bidder. Immediately following this, he may try a second time and bid \$1 above the new current price. This process might continue until either the bidder gives up or becomes the high-bidder. If the time between an individual bidder's bids were less than 10 minutes, then only the last bid counted as the bid from the bidder. If on the other hand, another bidder placed a bid in between,

Product	# of Auctions	# of Bids	# of Bids/Auction Average (St.Dev.)	Med.	Min. /Max.	% Bids Final Hr.	# of Bidders/Auction Average (St.Dev.)	Med.	Min. /Max.	% Winners Final Hr.	% Winners Last Bid
D1	2,075	20,479	9.86 (3.08)	10.0	1/25	17.2	8.1 (2.3)	8.0	1/17	59.0	61.0
L1	2,047	23,842	11.64 (3.54)	11.0	1/29	17.4	9.4 (2.6)	9.0	1/19	64.4	61.8
D3	274	2,669	9.73 (2.97)	9.0	4/21	16.5	7.9 (2.3)	8.0	3/15	55.1	58.4
D4	167	1,773	10.61 (3.27)	10.0	4/25	16.5	8.6 (2.2)	9.0	4/16	58.7	59.3
L4	172	2,274	13.21 (3.60)	13.0	4/23	19.0	10.7 (2.8)	10.5	4/19	69.2	55.8
L5	163	1,680	10.30 (3.37)	10.0	4/19	15.5	8.5 (2.5)	8.0	3/16	62.0	66.3

TABLE 5.1. Descriptive statistics of the bids and bidders per auction.

then both bids from the first bidder were included in the analysis. The reason 10 minutes was chosen was arbitrary, and only meant to give the bidder enough time to judiciously decide the amount to bid. With the 10 minute threshold there still were instances where an individual bidder accounted for two consecutive bids. An empirical analysis on how bidders react and bid based on the timing of the counter-bid can be found in Haubl and Popkowski Leszczyc (2003), who define the reaction and mental state to counter-bids as *bidding frenzy*. Here, however, all statistics reported are based on that consecutive bids within 10 minutes from an individual bidder are truncated to a single bid.

It should also be noted that prior to January 4, 2007, eBay included the user-id of all bidders in the bid history of each auction. After January 4, 2007, eBay started concealing the bidders' user-id with a generic 'Bidder #', once the price of an auction reaches \$200.[1] In other words, for the auctions that started prior to January 4, 2007, the user-id of each individual bidder is available. Consequently, information regarding the number of auctions a particular bidder participated in and how many bids he placed is available, while after January 4, 2007, only the bid amount or amounts for the bidders in a given auction is available. In Table 5.2 descriptive statistics regarding the bidders from the auctions with known bid history is provided. The second column is the number of auctions for which the user-id of the bidders are known. The third and fourth column lists how many bidders and winners there were for each of the products. These are counted based on the user-id.

[1]Currently eBay uses a different method to disguise the bidders' user-id.

Product	# of Auctions	# of Bidders	# of Winners	# of Single Unit Winners	Most Auctions Won by Single Bidder
Total	3,726	9,424	1,942	1,496	75
D1	2,051	4,933	1,154	868	41
L1	1,675	5,334	882	719	75
D3	269	1,032	201	163	7
D4	167	766	139	123	5
L4	167	984	112	95	15
L5	125	654	90	75	10

TABLE 5.2. Aggregate descriptive statistics for auctions with complete bid history. The fifth column is the number of winners that only won one auction. The sixth column is the maximum number of auctions won by a single bidder.

In other words, the third column is the number of unique user-id that placed bids in each of the products. It is possible that an individual bidder has more than one user-id. The fifth column lists the number of bidders who won only a single auction, e.g. 163 out of the 201 D3 winners won only one auction. The final column lists the most number of auctions won by a single bidder, e.g. there was one bidder who won 75 L1 auctions (out of 1,675 auctions). We see that only about half of all auctions are won by bidders who only win one auction. In other words, about half the auctions are awarded to bidders with more than unit demand. In the auction theory literature, it is often assumed that bidders have unit demand. This provides further support to the points raised by Rothkopf and Harstad (1994), regarding the gap between bidding theory and bidding phenomena. More specifically, when developing normative models for analyzing multiple auctions, whether released with or without overlap, it may be noteworthy to address how the results would change if bidders have more than unit demand.

2.1. Variables. For a given product, there are several variables that can, either directly or indirectly, be derived from the bid history of the auctions. Below we list the main variables that are recorded for each bid in each auction (with respect to the product analyzed).

- Bidder ID

  For auctions that started before January 4, 2007, or auctions below $200 the user-id of each bidder is listed.

- Auction ID

  Each auction is represented by a unique registration number.

- Date of Bid

  A time stamp that indicates the date and time when the bid was placed.

- Bid Number

  The order of the bid in the auction, i.e a counter for how many bids have been placed in the auction.

- B - Bid Amount

  For all bids the dollar amount of the bid. If the bid is the winning bid, i.e. highest bid placed, then this amount is censored and only shows as the minimum increment above the second highest bid.

- X - Current Price

  Since eBay auctions are second-price auctions, the price of an auction when each bid was placed can be derived. Namely, at any given point in time the price is the second highest bid placed so far. Strictly speaking, this is not exactly correct since eBay enforces minimum increments. In other words, the real 'current price' of an auction at any given point in time, is the second highest bid plus the minimum increment. See Chapter 1 for more details. As a result X is slightly underestimated, which in turn slightly overestimates the bid-increment.

- C - Bid-Increment

  This variable is calculated as the amount above the current price that the bidder bid, $C \equiv B - X$. As noted above it ignores the minimum increments imposed by eBay and therefore slightly overestimates the real bid-increment.

- Current High Bid

  Similar to X once the bid history is revealed the highest bid at any given point in time can be calculated. Note, however, that this information is not available for the bidder at the time he placed his bid. Furthermore, the information for the final high bidder remains censored.

- t - Elapsed Auction Time

  The elapsed auction time in minutes. Note that we use the notation from Chapter 3, and define t as the elapsed time, instead of the notation in Chapter 2 and 4 where elapsed auction time was defined by Y. The reason for this is to distinguish elapsed auction time as a continuous random variable, rather than discrete as in Chapter 2 and 4.

- T_{int} - Inter-Arrival Time

  The time, in minutes, since the previous bid in the auction was placed, i.e. the inter-arrival time in minutes.

- Study Time

  The time, in days, since the first auction in the product was started.

- Z - Number Auctions

  This is the number of ongoing auctions for the product at the time of the bid.

- P_{low} - Low Price

  The price in the lowest priced competing auction at the time of the bid.

- P_{high} - High Price

  The price in the highest priced competing auction at the time of the bid.

- T_{low} - Time Low Price

  The time remaining in the auction with price P_{low}.

- T_{high} - Time High Price

  The time remaining in the auction with price P_{high}.

- B_{bid} - Bidder Bids

  A counter for the total number of bids the bidder has placed (including the bid under consideration). For all auctions prior to January 4, 2007, each individual bidder can be tracked and the number of bids he has placed for a given product counted. This variable is different from B_{num}, which counts how many bids have been placed in the auction. B_{bid} counts the number of bids the bidder has placed so far (with respect to the product under consideration).

- B_{auc} - Bidder Auctions

  A counter for the total number of auctions the bidder has placed bids in (including the auction under consideration).

- B_{fb} - First Bid

  This is a dummy variable to indicate if this is the bidder's first bid in the auction. It is 1 if it is the first bid in the auction for the bidder, and 0 otherwise.

There are of course additional variables that might be of interest, such as the number of auctions previously won and at the price at which they were won. From Table 5.2 we saw that approximately half the auctions are won by previous winners. Other possible variables could also include time of day when bid was placed, and for the aggregated products D1 and L1, the various configuration attributes, e.g. memory, hard disc, processor speed, etc. However, we leave those variables for future analyzes. The above list captures the main aspects of interest both regarding the auction itself, the competing auctions, and some bidder attributes.

A comment regarding notation is that capitalized (upper case) letters denote variables, while lower case represent their manifestation. For instance, X is the variable representing current price, while x refers to a realization of the current price. The only exception is elapsed auction time, where the variable is represented by t, and a realized value of elapsed auction time is represented by t'.

3. Analysis of Bid-Increments

There are conceivably many factors that might influence the bid-increment, both with respect to the auction itself, the competing auctions, as well as the bidder. The main variables of interest from the perspective of this thesis, however, are the current price and elapsed time of an auction, and the number of competing auctions, i.e. X, t, Z. These three variables provided the basis for the optimal release policy of Chapter 2.

Intuitively, it would seem reasonable to expect that the higher the price, the lower the bid-increment. Figure 5.2 shows the bid-increments as a function of current price. On the horizontal axis is the current price, and on the vertical axis is the bid-increment. Each observation represents a bid. The lines represents the fitted values and will be discussed in Section 4. The graphs have been divided into two sets. The top row for each product line is for the first 71 hours of the auction (out of 72), while the bottom row displays the bid-increments for the final hour. Note that the vertical scale is different for two sets of graphs. The motivation to separate the final hour is because almost 20% of the bids and 60% of the winning bids arrive in the final hour. In addition, it is well established that the bidding activity toward the end of an auction is different [**24, 28**]. Therefore, a separate analysis for the final hour will be conducted to see if there is a difference in underlying bidding strategy. The reason for choosing one hour as the final time frame was arbitrary, but such that a 'sufficient' number of bids are available for analysis. For the first 71 hours, there is a clear trend displaying bid-increments as decreasing in the current price. This should not come as a surprise and is consistent with the observations made in Chapter 4. It should be noted that excluding the observations from the final hour does not alter or amplify this trend.

Interestingly, the trend does not seem to hold for the final hour. With the possible exception of product L1 and D3, it seems that in the final hour bid-increments are evenly distributed across the current price. In other words, both 'high' and 'low' priced auctions are equally likely to see 'large'/'small' bid-increments. To see this from a different perspective, Figure 5.3 shows the distribution of the bid-increments in the final hour for 'low', 'medium', and 'high' price-ranges. For the desktop products the price-ranges are defined as follows, low: $X \leq \$150$, medium: $\$150 < X \leq \$250$, high: $\$250 < X$, while for the laptop products the price-ranges are defined to be, low: $X \leq \$300$, medium: $\$300 < X \leq \$450$, high: $\$450 < X$. An informal statistical test for the median bid-increment (the line inside each box) to be 'significantly' different, is if the notches of the boxes do not overlap.[2] Overall it seems that there is no difference in median price due to that the notches of each box seem to overlap. The only exception would be for product D1, where it seems that the median bid-increment in the final hour for bids placed when $X > \$250$ is 'significantly' larger than for the two other price-ranges. Therefore, we hypothesize that price does not significantly affect the bid-increments in the final hour. We formally state the two findings in the following testable hypotheses.

HYPOTHESIS 5.1. *During the first 71 hours of an auction (out of 72), bid-increments are on average decreasing in the current price of an auction.*

HYPOTHESIS 5.2. *In the final hour of an auction, bid-increments are independent of the current price of an auction.*

Section 3.2 provides definitions and details regarding the testing procedure of the hypotheses.

[2]The reason the boxplot for D3 and 'high' prices appears disorted, is that the definition of the notches are based on a fixed interval around the median. A possibility, as evident with D3, is that the 25^{th} or 75^{th} percentile falls within the range of the notches.

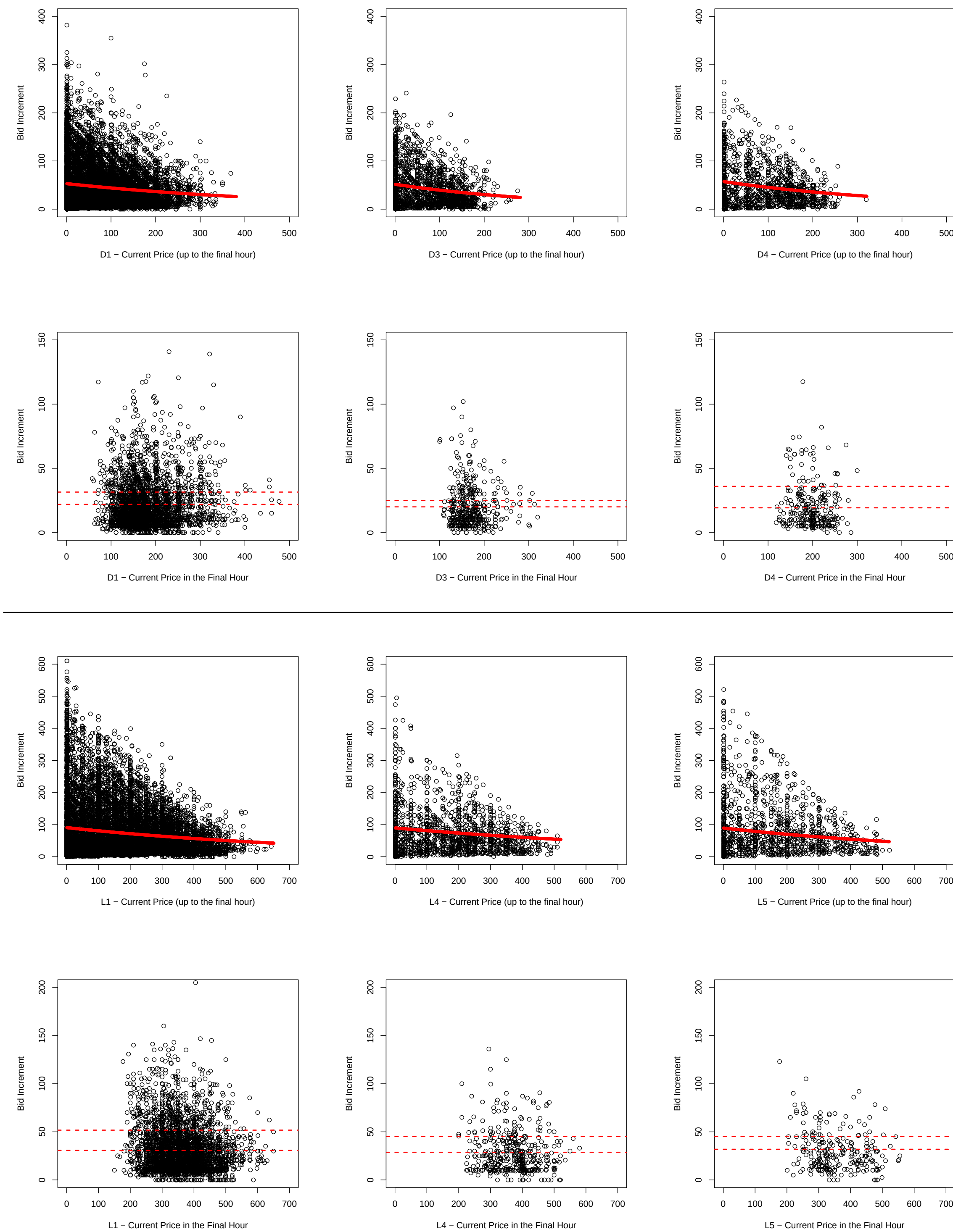


FIGURE 5.2. Bid-increments versus current price for D1, D3, D4 (top, left to right) and L1, L4, L5 (bottom, left to right). The first row of each product line is for the first 71 hours of the auction (out of 72), while the second row of each product line is for bids placed in the final hour. The horizontal axis in each graph represents the current price, and the vertical axis is the bid-increment. Each point represents a single bid. The lines represent the fitted values of the final model and are discussed in Section 4.

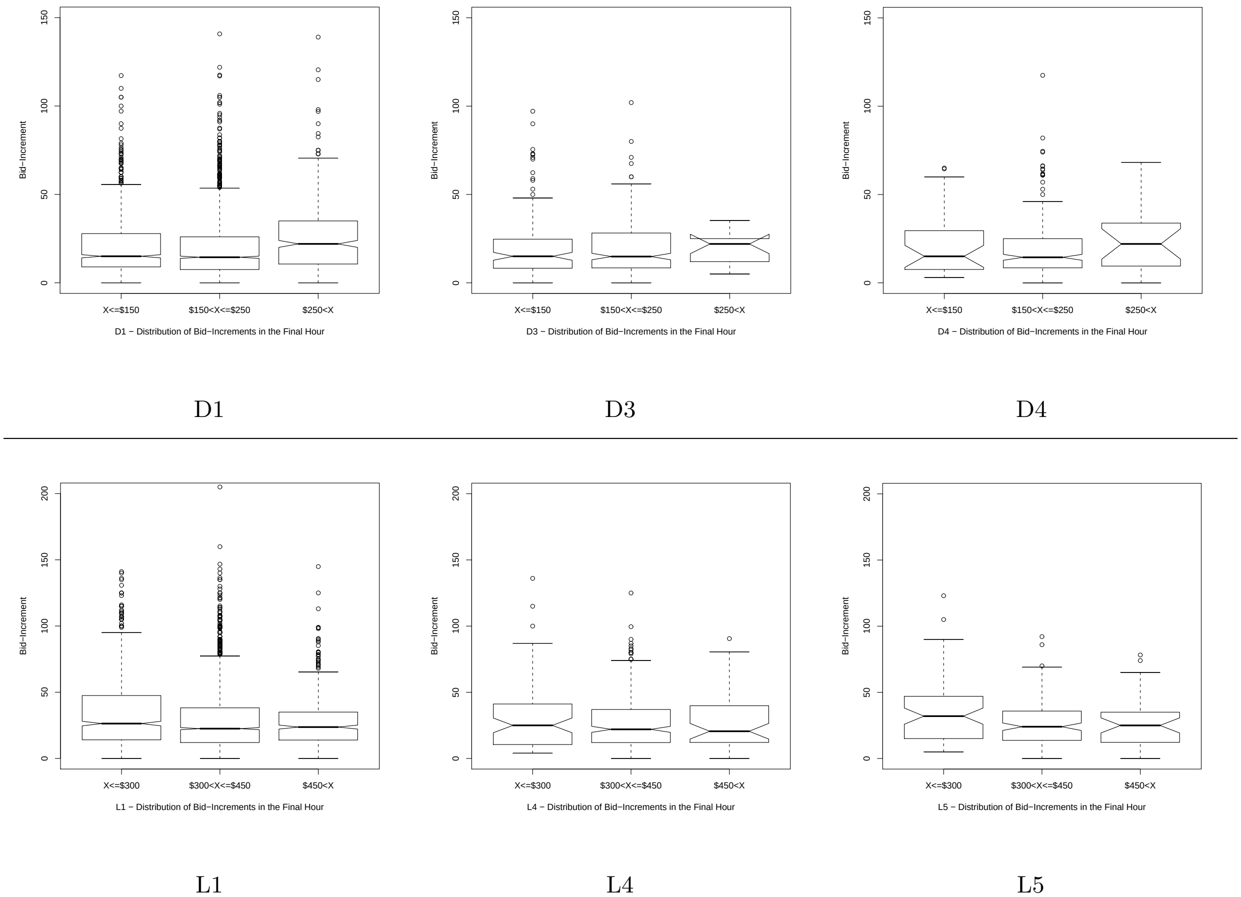


FIGURE 5.3. Distribution of bid-increments in the final hour. Each product shows the distribution in the final hour for three price ranges - 'low', 'medium', 'high'. For the desktop products, low: $X \leq \$150$, medium: $\$150 < X \leq \$250$, high: $\$250 < X$. For the laptop products, low: $X \leq \$300$, medium: $\$300 < X \leq \$450$, high: $\$450 < X$.

For elapsed auction time, the correlation with the bid-increments could presumably be argued both ways. In other words, the more time that has elapsed, or the less time that is remaining, bidders could become more or less 'aggressive' in their bidding, i.e. make smaller or larger bid-increments. However, in the final hour of an auction it would seem more reasonable to expect a positive trend between elapsed time and bid-increment. The reason for this would be that the more time that has elapsed (less time remaining), there is less opportunity to submit a counter-bid if needed. This could arguably lead to that bidders place bids based on larger increments. A problematic issue with analyzing the effect elapsed auction time has on bid-increments is the positive correlation between the current price and the elapsed auction time.

Figure F.1 in Appendix F shows the current price versus the elapsed auction time. On the horizontal axis is elapsed auction time, and on the vertical axis is the current price. Each observation represents a bid, and hence Figure F.1 shows the price at the time of each bid. In other words, Figure F.1 shows the aggregated evolution of all auctions, or the realized sample paths for each product. Similar to Figure 5.2 the final hour is shown in a separate graph, i.e. the top row for each product line represents time up to the final hour, while the bottom row represents the final hour. Note that the horizontal axis is scaled differently for the two sets of graphs. The purpose of these graphs is to illustrate the positive correlation between the current price and elapsed auction time. Needless to say as an auction progresses and bids arrive, the price of an auction increases. Which is clearly visible for the graphs depicting the first 71 hours of each product. However, this does not seem to hold for a given hour, and in particular not for the final hour. In other words, zooming in on a 'short' time interval for the graphs in the top rows, such as one hour, the positive trend is not present. In the graphs for the final hour this is more visible. The reason why this is important is that if current price and elapsed auction time are both included as independent variables, then it might be difficult to de-couple the main effect of each one in the presence of the other, i.e. there might be an issue with multi-collinearity. This is another reason for analyzing the final period separately.

In Figure 5.4 the bid-increment as a function of the elapsed time for the final hour is shown. On the horizontal axis is elapsed auction time in minutes, and on the vertical axis is the bid-increment. Each observation represents a bid. The lines represents the fitted values and will be discussed in Section 4. There are two main observable features. The first is that predominantly most bids arrive in the last minutes of the auction. Which is consistent with the analysis of, for instance, Roth and Ockenfels (2002), and Shmueli et al (2004), as well as the bid strategy of *sniping*. The objective of sniping is to place a bid as close to the end of an auction as possible, thereby not leaving any time for others to counter-bid. Recall that about 60% of the wining bids come in as the last bid. This may suggest that sniping is not as dominating or advantageous as it might seem. The second feature is that most bid-increments are rather small, particularly for the bids in the final minutes. This might mean that with sniping, bidders also try to bid as small an increment as possible. Though there does not seem to be a strong negative trend, due to the concentration of small bid-increments in the final minutes, a significant negative relation between bid-increment and elapsed time might be found. This would seem a bit counter-intuitive and not supportive of the informal reasoning above. We formally summarize our observations and reasoning in the following two hypotheses.

HYPOTHESIS 5.3. *During the first 71 hours of an auction (out of 72), bid-increments are independent of the elapsed auction time.*

HYPOTHESIS 5.4. *In the final hour of an auction, bid-increments are on average increasing in the elapsed auction time.*

For the number of competing auctions Z, it would seem intuitive that the more ongoing auctions there are, the smaller the bid-increment. The reason for this would be that if there are more ongoing auctions, then there are more auctions available to participate in and hence less reason for a bidder to commit to a higher price. That is, there is no reason to place a high bid-increment, since if a bidder is outbid, then there might be other lower priced auctions available. Other competing variables of interest include the price of the competing auctions. Rather than focusing on price of all competing auctions, only the highest and lowest price of the competing auctions have been included, i.e. P_{low}, P_{high}. Though it may be argued that the higher these prices are, the larger the bid-increment, it does not seem

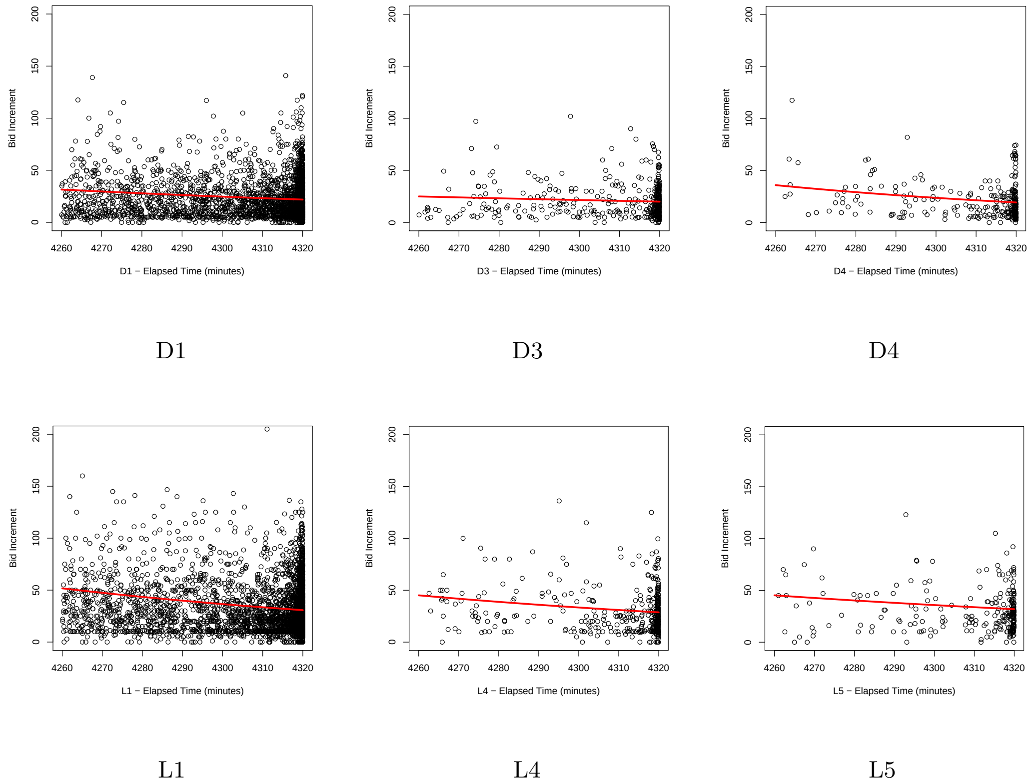


FIGURE 5.4. Bid-increments versus elapsed time in the final hour for desktop (top) and laptop (bottom) auctions. The horizontal axis in each graph represents the elapsed time in minutes, and the vertical axis is the bid-increment. Each point represents a single bid. The lines represent the fitted values of the final model and are discussed in Section 4.

overwhelmingly intuitive that there would be any relation between the bid-increment and the prices of the competing auctions. For instance, one reason why people buy things from auctions is to be able to buy something at a 'good' price. Therefore, just because there is someone who currently has committed to an auction at a 'high' price does not provide an incentive for a bidder to place a increase his bid-increment. We will nevertheless initially analyze how the prices of the competing auctions, and the time of those auctions, i.e. T_{low} and T_{high}, affect the bid-increment. The only formal hypothesis test we will conduct regarding the competing variables is the following.

HYPOTHESIS 5.5. *Bid-increments are on average decreasing in the number of ongoing auctions.*

The final set of variables included in the analysis, pertain to the individual bidder and are the number of bids he has bid, the number of auctions he has participated in, and a dummy variable indicating whether the bid is his first for the auction, i.e. B_{bid}, B_{auc}, B_{fb}. For these it is not certain how they might affect the bid-increment. It would seem equally intuitive to expect an increase or decrease in the bid-increment as a result of an increase in each of the three variables. Therefore, no formal hypotheses testing will be conducted with regard to the bidder variables.

Although there are many possible interaction terms, we only include the interaction between current price and elapsed auction time. The reason we include this interaction term is to gain further insight if current price or elapsed time is more influential on the bid-increments. To summarize, the objective is to analyze the bid-increments as a function of the the state of the auction and the bidder, $S_B \equiv (X, t, T_{int}, Z, P_{low}, T_{low}, P_{high}, T_{high}, B_{bid}, B_{auc}, B_{fb})$. Specifically the following relation will be evaluated,

$$C \sim \underbrace{X + Y + X \times t + T_{int}}_{\text{auction variables}} + \underbrace{Z + P_{low} + P_{high} + T_{low} + T_{high}}_{\text{competing variables}} + \underbrace{B_{bid} + B_{auc} + B_{fb}}_{\text{bidder variables}}$$

3.1. Gamma Distributed Bid-Increments. The formal analysis will be based on the Generalized Linear Model (GLM) concept discussed in Chapter 4. Motivated by the histograms in Figure 5.1, we assume that the bid-increments, given a state $s \in S_B$, follow a gamma distribution with mean μ_s and shape parameter ν,

$$f_{C|S_B}(c|s) = \frac{1}{\Gamma(\nu)} \left(\frac{\nu}{\mu_s}\right)^{\nu} c^{\nu-1} e^{-\left(\frac{c\nu}{\mu_s}\right)} \qquad c > 0 \tag{5.1}$$

where,

$$(5.2) \qquad \ln(\mu_s) = \gamma_0 + \gamma_1 x + \gamma_2 t' + \gamma_3 t_{int} + \gamma_4 z + \gamma_5 p_{low} + \gamma_6 p_{high} + \gamma_7 t_{low} + \gamma_8 t_{high}$$
$$+ \gamma_9 b_{bid} + \gamma_{10} b_{auc} + \gamma_{11} \mathbf{1}_{\{b_{fb}=1\}} + \gamma_{12} x \times t'$$

In other words, it is assumed that the log of the average bid-increment is a linear function of the variables. The symbol $\mathbf{1}_{\{\cdot\}}$ represents the indicator function, which equals 1 if the argument inside the brackets is true and 0 otherwise. Some comments about the model follows. First, since consecutive bids within 10 minutes are excluded, and most bidders only bid once per auction, it would seem reasonable to assume that the bids are independent. Parameter estimation of (5.2) uses maximum likelihood assuming independent observations. Second, as mentioned earlier, the basis for assuming gamma distributed bid-increments is due to the shape displayed in the histograms of Figure 5.1. However, since the gamma distribution is of course a smooth curve, equation (5.1) will not re-create the spike features that are displayed in Figure 5.1. In order to accommodate for the spikes, a model that allocates point mass to the 'even' bid-increments is needed. One possibility might be to include Poisson distributed bid-increments together with gamma distributed bid-increments. We leave this as a potential future extension. Third, similar to the model in Chapter 4, the log of the mean was chosen as the link function. The main reasons for choosing the log-link were due to the ease of interpretation and that it ensures the bid-increments are strictly positive. Finally, the relationship represented by equation (5.1) is the starting point of the analysis. In order to derive a final model, terms which are found to be non-significant will be removed. Rather than eliminating for each individual product the non-significant variables, a given set of variables will be selected for all products. This will facilitate the comparison of results across the products, as well as provide more support for the findings. We will refer to the model based on (5.1) and (5.2) as the *base* model, and the resulting model with the non-significant terms removed as the *final* model.

3.2. Hypothesis Testing. In Section 3 we stated five hypotheses regarding the relationship between the bid-increments and three of the covariates, namely current price, elapsed auction time, and number of ongoing auctions. The objective of these hypotheses is to establish general insights to the main factors determining the bid-increment. Related to the discussion at the end of the previous section, the purpose is not to draw specific conclusions about each of the products, but more to draw general inferences of the individual bidding behavior. We note that though Hypothesis 5.1-5.5 clearly relate to the coefficients in (5.2), they do not represent formal statistical hypotheses. The formal hypothesis testing will be as follows. For each product, and for each time-period (first 71 hours and final hour), the main null and alternative hypotheses of interest are,

$$H_0 : \gamma_1 = 0 \text{ vs. } H_a : \gamma_1 \neq 0 \qquad \text{(Current Price)}$$

$$H_0 : \gamma_2 = 0 \text{ vs. } H_a : \gamma_2 \neq 0 \qquad \text{(Elapsed Auction Time)}$$

$$H_0 : \gamma_4 = 0 \text{ vs. } H_a : \gamma_4 \neq 0 \qquad \text{(Number of Ongoing Auctions)}$$

That is, when deriving estimates for the γ coefficients of (5.2) for each product, the associated p-values for each coefficient, represents the test of $H_0 : \gamma = 0$ vs. $H_a : \gamma \neq 0$. If the p-value is 'small' then we reject H_0, and if it is 'large' then we fail to reject H_0, for each product and each of the time-periods analyzed. The reported significance level is based on the resulting p-value (a posteriori probability). A p-value less than .0001, is reported as $p < .0001$. Based on the tests, we conclude if Hypothesis 5.1-5.5 hold by considering the overall outcome of each of the formal hypothesis test regarding the gamma coefficients. If the outcome for a specific γ coefficient is consistent across all products and given time-period, then it would seem reasonable to draw a general inference regarding the variable in question. If the outcome is inconsistent then some more discussion might be required.

An alternative approach would have been to conduct a simultaneous test across all products. This could have been achieved by, for instance, implementing only one regression analysis for all products. In order to derive the product specific regression coefficients, we could introduce dummy variables for each product and each variable. We leave this as a potential for future extension and consistency check of the conclusions reached in this chapter.

3.3. Variable Selection. Table F.1 and F.2 in Appendix F shows the 'R' output [**22**], and resulting parameter estimates for the base model. The set of columns to the left are for the analysis of the first 71 hours, while the set of columns to the right are for the final hour. The columns labeled 'Estimates' represent the estimates of the gamma parameters of equation (5.2), and the columns labeled 'Pr($> |t|$)' are the associated p-values for each estimate. We see that for most products, many of the regression parameter estimates are non-significant, thus implying that the variable might not influence the bid-increment. Though few of the variables are consistently significant/non-significant across all six products, there are some important general findings. We discuss the findings related to the first 71 hours and final hour separately.

3.3.1. *Variable Selection - First 71 Hours.* Table 5.3 summarizes the information of interest of Table F.1 and F.2 in Appendix F. It shows the associated p-values of each variable for the base model. We first discuss the main auction variables. It is rather straight forward to exclude the inter-arrival time, since with the exception of one instance, it is always non-significant. For current price and elapsed time, the results are not as consistent. For products D1 and L1, we see that both terms as well as the interaction is highly significant. On the other hand, for the other products, only one of the main effects and

Variable		L1	L4	L5	D1	D3	D4
Current Price	γ_1	1.35e-05	.073522	.00196	1.23e-07	.83467	.16075
Elapsed Auction Time	γ_2	2.76e-055	.733728	.54060	1.34e-07	.03245	.00223
Inter-Arrival Time	γ_3	.7950	.411824	.62884	.0262	.18381	.96687
Number of Auctions	γ_4	5.18e-08	.009513	.00329	.5899	.17807	1.28e-05
Low Price	γ_5	.0135	.714115	.96119	.3997	.35542	.04518
High Price	γ_6	<2e-16	.249867	.18341	1.24e-1	.02534	.02521
Time Low Price	γ_7	.7898	.629439	.62001	.7221	.42682	.54395
Time High Price	γ_8	4.54e-07	.719061	.91662	.5680	.81052	.41907
Bidder Bids	γ_9	.8716	.863089	.42080	3.52e-13	.00671	.03115
Bidder Auctions	γ_{10}	2.92e-07	.285790	.60536	2.74e-06	2.76e-05	.42748
First Bid	γ_{11}	<2e-16	<2e-16	3.85e-09	<2e-16	<2e-16	8.17e-14
Curr Price× Elap Auc Time	γ_{12}	<2e-16	.000852	9.24e-08	<2e-16	.00256	.00666

TABLE 5.3. The resulting p-values associated with variables in base model for bid-increments in the first 71 hours.

interaction are significant. For the laptop products, the current price is significant, while for the desktop products, the elapsed time is significant. As previously discussed this is most likely due to the issue of multi-collinearity. Since it makes more sense that price affects the bid-increment, rather than elapsed auction time, we chose to include current price, and exclude both elapsed auction time as well as the interaction term. Consequently we fail to reject that $\gamma_3 = 0$, and accept Hypothesis 5.3. We formally summarize this.

Test of Hypothesis 5.3 - During the first 71 hours of an auction, in the presence of current price, for D3, L4, L5 we fail to reject that $\gamma_3 = 0$ ($p = .03$), while for D1, D4, L1 we reject that $\gamma_3 = 0$ ($p = .01$). Overall due to likely effect regarding multi-colinerarity, we conclude that bid-increments are independent of the elapsed auction time and fail to reject Hypothesis 5.3.

Next we consider the variables related to the competing auctions. The two variables that, with one exception, are not significant are T_{low} and T_{high}. These are therefore excluded. Regarding P_{low} and P_{high}, we see that for all cases these are predominantly not significant. Therefore, since there does not seem to be any overwhelmingly support for the significance of the competing prices these will also be removed. The remaining competing variable is the number of competing auctions. Here we also see some mixed results. For the laptop products Z is significant, while for the desktop products it is only significant for product D4. However, since Z was one of the main variables in Chapter 2 and 4, and we wish to include at least one variable regarding the competing auctions we will include it for the final model.

The final variables pertain to the bidders. The variable that, with two exceptions, is always significant is the dummy variable B_{fb}. Since it is intuitive that a bidder might behave differently on his first bid, and since the analysis warrants keeping it, we will continue to include it. For B_{bid} and B_{auc} the results are a bit mixed. This might not come as a complete surprise considering the strong positive correlation between the two variables. This can be seen in Figure F.4 in Appendix F, which has B_{bid} on the vertical axis and B_{auc} on the horizontal axis. Each observation represents a bid, and thus show how many bids the bidder has placed as a function of the number of auction he has participated in. The solid 45 degree line represents bids from bidders who have placed only one bid per auction. Note that there are no observations below this line. In the figure we see the almost perfect positive correlation between the two variables. Since consecutive bids in 10 minutes were excluded and as discussed earlier, bidders tend to bid only once per auction. Therefore, B_{bid} and B_{auc} are basically capturing the same essence of bidding or auction experience. Though it can presumably be argued which of the two variables is the most important, we will exclude B_{bid} and keep B_{auc}. The main reason for choosing B_{auc} is that it has a bit more intuitive interpretation with regard to capturing a bidder's auction experience.

Based on these findings, the following model for the first 71 hours will be tested, and referred to as the final model. For $s = (x, z, b_{auc}, b_{fb})$, bid-increments are still assumed to follow a gamma distribution with mean μ_s and shape parameter ν as specified by equation

Variable		L1	L4	L5	D1	D3	D4
Current Price	γ_1	.27862	.09798	.2514	.3795	.0785	.3699
Elapsed Auction Time	γ_2	.00221	.03584	.1662	.2361	.0593	.7622
Inter-Arrival Time	γ_3	.02241	.31747	.3717	2.90e-12	.9582	.0218
Number of Auctions	γ_4	.28972	.88026	.0789	.0486	.0292	.8953
Low Price	γ_5	.30122	.57388	.8854	.2400	.0559	.3914
High Price	γ_6	1.50e-07	.14562	.2807	.0895	.9991	.4528
Time Low Price	γ_7	.35010	.06325	.1343	.0465	.4424	.2698
Time High Price	γ_8	.79726	.46328	.9179	.3091	.7978	.3375
Bidder Bids	γ_9	.46805	.90244	.5653	.0295	.8389	.2193
Bidder Auctions	γ_{10}	.24396	.84775	.3538	.1493	.3925	.0499
First Bid	γ_{11}	$<$2e-16	.00632	.2332	1.40e-15	.7110	.0251
Curr Price$\times$ Elap Auc Time	γ_{12}	.28840	.09867	.2547	.3882	.0793	.3691

TABLE 5.4. The resulting p-values associated with variables in base model for bid-increments in the final hour.

(5.1). However, the log-link function of μ_s is now defined by,

$$\ln(\mu_s) = \gamma_0 + \gamma_1 x + \gamma_4 z + \gamma_{10} b_{auc} + \gamma_{11} \mathbf{1}_{\{b_{fb}=1\}} \tag{5.3}$$

3.3.2. *Variable Selection - Final Hour.* Table 5.4 shows the associated p-values of the variables in the final hour. For more details see Table F.1 and F.2 in Appendix F. Although the inter-arrival time has a few more instances where it is significant, similar to the case for the first 71 hours, we exclude it from the final model. For current price and elapsed auction time, the findings are more inconsistent than for the first 71 hours. For D1 and D4, there is neither a main nor interaction effect from X and t. And for D3 the main and interaction effect, is only significant at the .1 level. For the laptop products, the only significant term is the main effect of elapsed time for L1 and L4. One reason for these results, may be the issue of multi-collinearity between X and t as discussed above. Therefore, one of the variables as well as the interaction term will be excluded. Since the graphs in Figure 5.2 do not indicate any strong correlation between X and the bid-increments, and that it would seem a bit more intuitive that elapsed time (or rather remaining time) would effect the bid-increment, we exclude the current price and interaction term from the analysis of the final hour. We formally summarize our findings as follows.

Test of Hypothesis 5.2 - In the final hour, in the presence of elapsed auction time, for all products we fail to reject that $\gamma_2 = 0$ ($p = .07$), and therefore conclude that bid-increments are independent of the current price.

Next we consider the variables related to the competing auctions. The results are similar to the case for the first 71 hours. The variables P_{low}, P_{high}, T_{low}, and T_{high}, are pre-dominantly not significant and will be excluded. For the number of competing auctions, the results are again mixed. For the laptop products, only L5 is significant. While for the desktop products, only D4 is not significant. However, for the same reason as above, since Z was one of the main variables analyzed in Chapter 2 and 4, and we wish to include at least one variable regarding the competing auctions we include it.

The final set of variables pertain to the bidders. The findings here are also similar to the previous case. The dummy variable regarding if the bid is the bidder's first for the auction, is with two exceptions, significant. Therefore we continue to include it. The other variables are predominantly not significant. However, in order to have the model for the final hour more consistent with the model for the first 71 hours, we will continue to include the B_{auc}. This is despite that it is only significant for D4 at the .05 level.

Based on these findings, the following model for the final hour will be tested, and referred to as the final model. For $s = (x, t', b_{auc}, b_{bf})$, bid-increments are still assumed to follow a gamma distribution with mean μ_s and shape parameter ν as specified by equation (5.1).

However, the log-link function of μ_s is now defined by,

$$\ln(\mu_s) = \gamma_0 + \gamma_2 t' + \gamma_4 z + \gamma_{10} b_{auc} + \gamma_{11}\mathbf{1}_{\{b_{fb}=1\}} \tag{5.4}$$

4. Results

4.1. Results - First 71 Hours. We first discuss the results for the first 71 hours. The resulting equations for the first 71 hours are given below, with partial output from 'R' displayed in Table 5.5. In the equations below, the gamma parameter estimates that are non-significant at .1 have been italicized. Recall that x represents a realization of the current price of an auction, and z is a realized value for the number of ongoing auctions. The 'R' output in Table 5.5 has been limited to the gamma parameter estimates and their associated p-values, the estimated gamma dispersion parameter, as well as the null and residual deviance. In Figures 5.2 and 5.4 the lines represents the fitted means from the final models. The value for the number of ongoing auctions, was in each graph set to the median: D1) $z = 20$, D3) $z = 5$, D4) $z = 5$, L1) $z = 20$, L4) $z = 4$, L5) $z = 4$.[3] The bidder parameters were set such that the lines represent the first bid of the first auction a bidder participates in, i.e. $b_{auc} = 1$ and $b_{fb} = 1$.

[3] The median for D4 and L5 were actually 4 and 3 respectively. The reason the median from D3 and L4 were used, is to make comparison across products easier.

$$\text{D1} \qquad \ln(\mu_s) = 3.4567 - .00186x + \mathit{.00015}z + .00249b_{auc} + (.50627)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{D3} \qquad \ln(\mu_s) = 3.3696 - .00267x - .00102z + .01555b_{auc} + (.56103)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{D4} \qquad \ln(\mu_s) = 3.6146 - .00235x - .00865z + .01753b_{auc} + (.45623)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L1} \qquad \ln(\mu_s) = 4.0940 - .00117x - .00756z + .00597b_{auc} + (.55880)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L4} \qquad \ln(\mu_s) = 4.0489 - .00098x - .01599z + .01645b_{auc} + (.49573)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L5} \qquad \ln(\mu_s) = 4.1596 - .00123x - .04474z + .04871b_{auc} + (.50935)\mathbf{1}_{\{b_{fb}=1\}}$$

With the exception of one instance, z for D1, we see that the variables for the final model are significant. Specifically, we see that current price is highly significant and negatively related with the bid-increment. In other words, the higher the price, the smaller the bid-increment. This is of course to be expected and in accordance with the trends observed in Figure 5.2. It is interesting to note that the decrease in expected value is not as strong as one might have expected. Recall that according to (5.3) with $\gamma_1 < 0$, we assume the expected bid-increment to be exponentially decreasing in price. Furthermore, we notice that the gamma estimates for the laptops is about half that of the desktops. That is, the expected bid-increment for the desktops decreases almost twice as fast as for laptops. With different scale for the vertical axis, this is a bit difficult to see when comparing the graphs of Figure 5.2. Nonetheless, in addition to that the expected bid-increment for laptops are larger, due to a larger intercept term, it also remains higher for higher prices. This should not be too surprising since laptop auctions on average end at about twice the price of desktop auctions. We formally conclude the discussion regarding price by testing Hypothesis 5.1.

Up To Final Hour	D1		D3		D4	
	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)
(Intercept)	3.4566588	< 2e-16	3.3696290	<2e-16	3.6145544	<2e-16
Current Price	-.0018634	< 2e-16	-.0026689	6.78e-10	-.0023547	3.41e-14
Number Auctions	.0001528	.721	-.0010201	.000496	-.0086465	.0432
Bidder Auctions	.0024892	< 2e-16	.0155531	7.41e-05	.0175289	3.45e-09
First Bid	.5062723	< 2e-16	.5610335	<2e-16	.4562285	8.28e-14
Dispersion para.	.7633826		.7081784		.659622	
Null Dev.	18551 on 16358 *d.f.*		2499.6 on 2113 *d.f.*		1558.0 on 1455 *d.f.*	
Residual Dev.	17429 on 16354 *d.f.*		2315.5 on 2109 *d.f.*		1432.4 on 1451 *d.f.*	

	L1		L4		L5	
	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)
(Intercept)	4.094	< 2e-16	4.0489336	<2e-16	4.1595911	<2e-16
Current Price	-.001171	< 2e-16	-.0009847	4.44e-10	-.0012296	2.25e-09
Number Auctions	-.007560	< 2e-16	-.0159946	.000647	-.0447418	.00394
Bidder Auctions	.005971	< 2e-16	.0164525	.000217	.0487058	<2e-16
First Bid	.5588	< 2e-16	.4957287	<2e-16	.5093492	2.86e-11
Dispersion para.	.7580986		.7155694		.7482342	
Null Dev.	17275 on 15891 *d.f.*		1726.1 on 1741 *d.f.*		1222.4 on 1082 *d.f.*	
Residual Dev.	15432 on 15887 *d.f.*		1622.6 on 1737 *d.f.*		1074.1 on 1078 *d.f.*	

TABLE 5.5. The gamma parameter estimates of equation (5.4) with associated p-values for the six products.

Test of Hypothesis 5.1 - For all products we reject that $\gamma_2 \geq 0$ ($p < .0001$). We conclude that bid-increments are on average decreasing in the current price of an auction.

The results for the number of ongoing auctions also support our initial reasoning. In particular, we see that in all cases where $\hat{\gamma}_4$ is significant it is also negative, and the only instance when $\hat{\gamma}_4 > 0$ (D1) the estimate is non-significant. We therefore conclude that the number of ongoing auctions either either negatively effects the bid-increment or has no impact, i.e. bid-increments are decreasing in the number of ongoing auctions. Regarding the difference between the laptops and desktops, we see that there is a clear difference in the magnitude of the gamma parameter estimate. The estimates for the laptops is much larger than the estimates for the desktops. That is, the expected bid-increment decreases much faster for laptop auctions than for desktop auctions as Z increases. In other words, laptop auctions are much more sensitive to competition than desktop auctions. We note in particular that for L5 there is almost a 4.5% decrease per extra auction. We formally summarize our finding regarding the number of ongoing auctions in the first 71 hours by testing Hypothesis 5.5.

Test of Hypothesis 5.5 - During the first 71 hours, for D3, D4, L1, L4, L5 we reject that $\gamma_4 \geq 0$ ($p = .05$), while for D1 we fail to reject $\gamma_4 = 0$ ($p = .721$). Therefore, we conclude that bid-increments are on average decreasing in the number of ongoing auctions.

For the bidder attributes B_{bid} and B_{fb}, it is interesting to note that they are always highly significant and positively related with the bid-increment. Which means that the more auctions a bidder has participated in the more likely he is to place a larger bid-increment. That is, it appears bidders start out conservative in their bidding, and as they participate in more and more auctions, become more and more 'aggresive'. In other words, it seems that bidders start by 'testing the waters' and want to ensure they do not end up with *winner's curse* [**14**, p.85]. The magnitude of the effect for the two variables are about the same for the laptop and desktop auctions.

Table 5.5 also gives the estimates of the GLM dispersion parameter ϕ. Recall from Section 4.3 of Chapter 4 that the GLM dispersion parameter for the gamma distribution is $1/\nu$. Furthermore, depending on if ν is greater than or less than one, the gamma distribution will have a different shape. If $\nu < 1$ then the gamma distribution resembles a steep exponential distribution, while if $\nu > 1$ then the gamma distribution resembles a skewed uni-modal distribution (for $\nu = 1$ the gamma distribution is the exponential distribution). We see that for the first 71 hours the estimated dispersion parameter is about .7, and thus $\hat{\nu}$ is about 1.4 ($\approx 1/.7$). That is the shape of the gamma distribution bid-increments for the first 71 hours is a right skewed uni-modal curve, which is consistent with the histograms of Figure 5.1.

4.2. Results - Final Hour. Next we discuss the results for the final hour. The resulting equations for the final hour are given below, with partial output from 'R' displayed in Table 5.6. Similar to above, in the equations below, the $\hat{\gamma}$ that are non-significant at .1 are italicized. Recall that t' represents a realization of the elapsed time of an auction. In Figures 5.2 and 5.4 the lines represents the fitted means from the final models. The value for the other variables were set as above, i.e. $z = 20$ (D1), 5 (D3), 5 (D4), 20 (L1), 4 (L4), 4 (L5), and $b_{auc} = 1$ and $b_{fb} = 1$. In Figure 5.2, the fitted values for the bid-increments are represented by dashed lines to indicate that they do not depend on the current price. In each of the graphs the upper and lower dashed lines represents the expected bid-increment at the start ($t' = 4260$) and end ($t' = 4320$) of the final hour respectively.

$$\text{D1} \qquad \ln(\mu_s) = 28.9892 - .00606t' - .00176z - .00303b_{auc} + (.31234)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{D3} \qquad \ln(\mu_s) = 18.8979 - \mathit{.00371}t' + .02030z - .02495b_{auc} + (\mathit{.06694})\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{D4} \qquad \ln(\mu_s) = 47.2088 - .01037t' + \mathit{.00590}z + .04827b_{auc} + (.47429)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L1} \qquad \ln(\mu_s) = 40.5986 - .00867t' - \mathit{.00165}z - .00075b_{auc} + (.32270)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L4} \qquad \ln(\mu_s) = 35.5540 - .00752t' - \mathit{.00278}z - \mathit{.00234}b_{auc} + (.30507)\mathbf{1}_{\{b_{fb}=1\}}$$

$$\text{L5} \qquad \ln(\mu_s) = 28.4080 - .00584t' + \mathit{.02873}z - \mathit{.01984}b_{auc} + (\mathit{.16656})\mathbf{1}_{\{b_{fb}=1\}}$$

For the final hour, with the exception of D3, we see that elapsed time is significant and negatively related to the bid-increment. For product D3 the relation is also negative but only significant at the .13 level. Overall this implies that the longer an auction has elapsed, or the closer an auction is to the end, the smaller the expected bid-increment,

Final Hour	D1		D3		D4	
	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)
(Intercept)	28.9891502	5.12e-13	18.897922	.072171	47.208838	.000562
Elapsed Time	-.0060593	7.50e-11	-.003714	.127808	-.010371	.001071
Number Auctions	-.0017608	.05353	.020296	.026358	.005882	.529945
Bidder Auctions	-.0030266	.00744	-.024952	.000774	.048271	.033290
First Bid	.3123438	< 2e-16	.066940	.490226	.474289	5.97e-06
Dispersion parameter	.71784		.5995077		.5728611	
Null Deviance	2062.4 on 3330 d.f.		239.49 on 411 d.f.		169.20 on 281 d.f.	
Residual Deviance	1973.7 on 3326 d.f.		229.93 on 407 d.f.		148.36 on 277 d.f.	

	L1		L4		L5	
	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)	Estimate	Pr(> $\|t\|$)
(Intercept)	40.5986431	< 2e-16	35.554000	.000451	28.407963	.0466
Elapsed Time	-.0086718	< 2e-16	-.007520	.001374	-.005835	.0775
Number Auctions	-.0016538	.2002	-.002781	.719325	.028729	.2693
Bidder Auctions	-.0007487	.0825	-.002342	.580631	-.019838	.1387
First Bid	.3226979	< 2e-16	.305066	.000719	.166555	.1343
Dispersion parameter	0.5188403		0.4612364		0.4235749	
Null Deviance	1620.6 on 3241 d.f.		171.54 on 392 d.f.		84.181 on 195 d.f.	
Residual Deviance	1514.7 on 3237 d.f.		162.60 on 388 d.f.		80.457 on 191 d.f.	

TABLE 5.6. The gamma parameter estimates of equation (5.4) with associated p-values for the six products.

meaning that bidders are more conservative towards the end of the auction. However, we note that price has been included and therefore it may be that the effect of elapsed time is due to the price of an auction. On the other hand, in Figure F.1 of Appendix F, there was little evidence for a positive relationship between the current price and elapsed time of an auction in the final hour. Furthermore, in Figure 5.2 we did not detect a strong trend between bid-increments and current price for the final hour. Consequently, we conclude that as the auction end is getting closer, bidders tend to make smaller bid-increments.

Similar to the case with current price in the first 71 hours, the decrease due to elapsed time is not too steep. Again, note that according to (5.4) with $\gamma_2 < 0$, the expected bid-increment is assumed to be exponentially decreasing in elapsed time. However, as seen in Figure 5.4, and discussed in Section 3, the relation between bid-increments and elapsed time does not seem to be strong. Therefore, unlike the previous discussion regarding the effect of price, it is not too surprising that the fitted mean in Figure 5.4 is only slightly decreasing in elapsed time. Unlike the gamma estimates for current price, there does not seem to be a clear separation between the desktops and laptops. The magnitude of $\hat{\gamma_2}$ for the D1 and L1 are about the same, and for the specific products there is no categorical difference. Consequently, it does not appear that laptop auctions are more or less 'sensitive' to the elapsed time compared with desktop auctions.

In the graphs for the final hour of Figure 5.2, the upper and lower dashed lines represent the expected bid-increment at the start respectively end of the final hour, i.e. at $t' = 4260$ and $t' = 4320$. The fitted values are constant due to that current price is not included as a covariate. For this reason they are represented by dashed lines rather than solid lines. As already mentioned there does not seem to be any strong relationship between the bid-increments and current price in the final hour. Overall the fitted means for the final model in the final hour seems to correspond well with the observations. We conclude the discussion regarding elapsed auction time by formally testing Hypothesis 5.4

Test of Hypothesis 5.4 - For D1, D4, L1, L4, L5 we reject that $\gamma_3 \geq 0$ ($p = .08$), and for D3 we fail to reject that $\gamma_3 = 0$ ($p = .1$). We therefore conclude that bid-increments are on average decreasing in the elapsed auction time and that Hypothesis 5.4 does not hold.

The results for the number of ongoing auctions is a bit more mixed than in the previous case. First, note that for D1, L1, and L4, the gamma estimates are negative, while for D3, D4, and L5, the gamma estimates are positive. Second, it is only the gamma estimates for D1 and D3 that are significant at a reasonable level. Third, there does not appear to be any clear difference between the laptop and desktop auctions. That is, the magnitude of the gamma estimate are fairly similar for the two types of computers. Based on this we conclude that the number of ongoing auctions in the final hour does not effect the bid-increment. That is, in the final hour, bidders are less sensitive to the competing variables, in particular the number of ongoing auctions. One explanation for this, might be due to that eBay auctions are by default listed in descending order of remaining time. In other words, auctions that are about to expire are listed first, and auctions that just started are listed last. Consequently, it may be that the bidders who arrive to the auction site simply joins the auction that is about to expire, and do not have time or interest to further investigate what other auctions may be underway. We formally summarize our findings.

Test of Hypothesis 5.5 - In the final hour, for all products we fail to reject that $\gamma_4 = 0$ ($p = .02$). We therefore conclude that bid-increments are independent of the number of ongoing auctions.

For the bidder attributes B_{auc} and B_{fb}, there is one main difference between the final hour and the first 71 hours of the auction. Namely that in the final hour $\hat{\gamma}$ for B_{auc} is predominantly negative. This would imply that in the final hour, bidders with more auction experience, tend to make smaller bid-increment. It is hard to think of an intuitive explanation why bidders with experience would reverse their behavior in the final hour. For B_{fb} the findings are in line with what was observed earlier. Namely that the first bid from a bidder tends to be larger.

Table 5.6 also shows the estimates of the GLM dispersion parameter ϕ. For the final hour the dispersion parameter estimate is about .5, which is lower than for the first 71 hours and results in a $\hat{\nu}$ around 2. In other words, the curve for bid-increments in the final hour is more variable.

4.3. Goodness-of-fit and Residual Analysis. In Chapter 4 we discussed the limitations of formal goodness-of-fit test when fitting the gamma distribution in a GLM framework. Recall that the only asymptotic results regarding the residual deviance D were the *small-dispersion asymptotics* [**17, 9, 8, 13**]. In Chapter 4 we informally defined a 'small' dispersion parameter to mean less than .5. The other measure of goodness-of-fit discussed in Chapter 4 was the Pearson X^2, which also does not have any asymptotic properties in the setting analyzed. On the other hand, the difference in D of nested models can formally be evaluated using a χ^2 distribution with the appropriate degrees of freedom. Due to these circumstances the goodness-of-fit analysis will follow the one in Chapter 4. That is, for the final models based on (5.3) and (5.4), we informally compare the scaled residual deviance $D/\hat{\phi}$ with the degrees of freedom *df*, and formally test the scaled difference of residual deviance of the nested models with the χ^2 distribution with appropriate degrees of freedom.

In Table 5.7 the deviance and gamma shape parameter estimate for the final model appear. The columns labeled D and *df* represents the residual deviance and degrees of freedom respectively. The columns labeled ΔD lists the difference in residual deviance between the final model based on (5.3) or (5.4), and the original model based on (5.2). Recall that the original model with eight more variables has a smaller deviance but also fewer degrees of freedom. Specifically eight fewer degrees of freedom as stated in parenthesis below ΔD. The final two columns labeled $\hat{\phi}$ and $\hat{\nu}$ are the estimated dispersion and shape parameters respectively. The set of columns to the left are for the analysis up to the final hour, while the set of columns to the right are for the analysis pertaining to the final hour.

For bids up to the final hour the goodness-of-fit results are similar to the goodness-of-fit for the zero-inflated gamma distributed within period price-transitions of Chapter 4. Namely, while the residual deviance values are close to the values for the degrees of freedom,

Up To Final Hour						*Final Hour*				
product	D	$d.f.$	ΔD ($\Delta df = 8$)	$\hat{\phi}$	$\hat{\nu}$	D	$d.f.$	ΔD ($\Delta df = 8$)	$\hat{\phi}$	$\hat{\nu}$
D1	17429.05	16354	241.6715	.763	1.310	1973.73	3326	65.462	.718	1.393
D3	2315.46	2109	24.76920	.708	1.412	229.93	407	8.382	.600	1.668
D4	1432.39	1451	20.32077	.660	1.516	148.36	277	7.681	.573	1.746
L1	15431.67	15887	382.0654	.758	1.319	1514.69	3237	26.202	.519	1.927
L4	1622.63	1737	25.20366	.716	1.397	162.60	388	4.610	.461	2.168
L5	1074.06	1078	34.17948	.748	1.336	80.46	191	5.118	.424	2.361

TABLE 5.7. Deviance and shape parameter estimates for the six products.

the *scaled* residual deviance values are much larger than the degrees of freedom. Recall from Chapter 4 that scaled deviance $= D/\hat{\phi}$. In other words, while the residual deviance values would indicate a 'decent' goodness-of-fit, the scaled residual deviance values indicate a poor fit. For the difference in residual deviance between the final model and original model, we see that even without the scaling factor the increase is far larger than the gain of 8 degrees of freedom. The associated p-values for $\Delta D/\Delta df$ are as follows: D1) $< .0001$, D3) .0017, D4) .0092, L1) $< .0001$, L4) .0014, L5) $< .0001$. This implies that the increase in deviance is not compensated by the gain in degrees of freedom.

The goodness-of-fit results for the bids in the final hour are a bit more consistent and promising. The first thing to observe is that the scaled residual deviance values are close to the values of the degrees of freedom, and thus indicating a good fit. Second, we see that in general, the difference in residual deviance are in fact compensated by the gain in degrees of freedom. Specifically, we see that for products D3, D4, L4 and L5, that the $\Delta D/\hat{\phi}$ values are close to the 8 degrees of freedom gained. The associated p-values for $\Delta D/\Delta df$ are as follows: D1) $< .0001$, D3) .3971, D4) .4653, L1) .0010, L4) .7984, L5) .7449. In other words, with the exception of the D1 and L1, the increase in deviance is compensated by the gain in degrees of freedom. Therefore, the proposed model does seem to fit the data well, and we can with fair confidence accept the model.

Figure F.2 and F.3 in Appendix F depicts the deviance residual and response residual plots. The top row for each product line represents the deviance residuals r_d, while the bottom row shows the response residuals $y - \hat{\mu}$. Both residuals are plotted as a function of the linear predictor $\hat{\eta}$. Note that for the first 71 hours, the linear predictor is decreasing in price and consequently the larger values of $\hat{\eta}$ correspond to the lower prices. Similarly for the final hour, the linear predictor is decreasing in time and thus the larger values of $\hat{\eta}$ corresponds to the start of the final hour. Overall the deviance residuals, for both the first 71 hours and final hour, appear to be randomly distributed without any obvious trend. The only noticeable issue is with D1 for the final hour, were there seems to be three outliers.

For the response residuals there is a clear difference between the two models. For the model of fitting bids up to the final hour, each product clearly depicts a funnel shaped pattern. Recall that the smaller value of $\hat{\eta}$ corresponds to the bids placed at higher prices. In other words, we see that the difference between the observed bid-increment c and the estimated mean $\hat{\mu}$, is smaller at the larger prices. This should not come as a surprise given our observation of bid-increments in Figure 5.2. For the analysis of the final hour the residual plots seem to tell a different story. We note that there is no clear pattern in the plot, and in particular, no distinctive funnel shape pattern. Instead the residuals seem to be randomly and evenly scattered. The only striking feature is with regard to product D1, where three outliers are clearly present. These three observations clearly distort the residual plot all the other observations become highly concentrated.

Based on the analysis of the goodness-of-fit, and deviance and response residuals, we conclude that the data supports our models. That is, bid-increments appears to follow the gamma distribution according to (5.1), with the mean linearly dependent on the covariates according to (5.3) for the first 71 hours, and (5.3) for the final hour. We make a reservation though, that for the first 71 hours, there are some mixed results regarding the goodness-of-fit. However, since there are no formally established goodness-of-fit test for GLM with gamma distributed observations, we leave it to the reader to decide whether the proposed model is supported by the data. Next we briefly discuss the inter-arrival time of bids.

5. Analysis of Timing of Bids

The top six graphs of Figure 5.5 show the histogram of the timing of bids for each product. The horizontal axis represents the elapsed auction time, and each bar represents the fraction of bids placed in given 50 minute time interval. The only exception is for the right most bar, which depicts the number of bids placed in the final 20 minutes. For all six products the distribution of when the bids arrive is the same. In the first few hundred minutes, or first few hours, the auctions attract a number of bids, after which the fraction of arriving bids drops and remains constant until about there is six hours remaining. In the remaining six hours we see that the fraction of bids dramatically increases, and that in the final 20 minutes there is an order of magnitude more bids placed. That most bids arrive towards the end is a well-known and well-studied phenomena [**24, 28, 35**].

The bottom six graphs of Figure 5.5 show the histogram of the inter-arrival time of bids for each product. The horizontal axis represents the time between bids, i.e. inter-arrival time, and each bar represents the fraction of bids with an inter-arrival time given by each 50 minute interval. In other words, the left most bar depicts the fraction of bids that arrive within 50 minutes from the previous bid. We see that most bids come 'shortly' after a previous bid has been placed, and the general shape seems to imply that the inter-arrival time is exponentially distributed. This is again nothing new and previous papers have modeled the arrival of bids as a Poisson process [**28, 27**]. The choice of choosing 50 minute intervals was arbitrary, but changing it does not change the shape of the histograms. With shorter intervals the general pattern is the same.

In the top six graphs of Figure F.5 in Appendix F, the inter-arrival time as a function of the elapsed time is shown. On the horizontal axis is the elapsed time, and on the vertical axis is the time since the previous bid. Each observations represents a bid. Note that the graph is 'lower-triangular' since it is not possible that the time until the next bid exceeds the length of time since the auction started. For D1 and L1, it is hard to detect any clear trend. For the specific products, it seems as though there is a higher concentration of bids with 'short' inter-arrival time for the bids arriving in the beginning and the end of the auction. However, this interpretation of the conditional distribution given the elapsed auction time is a little distorted. First, recall that there is a larger, respectively much larger, number of bids arriving in the beginning and end of an auction. Second, due to the physical limitations of time, those bids can only have a 'short' inter-arrival time. The graphs are therefore what to expect.

In the bottom six graphs of Figure F.5 in Appendix F, the inter-arrival time as a function of the elapsed time is shown. On the horizontal axis is the current price when a bid arrived, and on the vertical axis is the time since the previous bid. Each observation represents a bid. The main purpose of these graphs is to see if the price of an auction has an effect on the inter-arrival time. From the graphs it does not seem to be any strong relationship between the two variables. Both 'low' and 'high' priced auctions seem to exhibit a similar distribution regarding the inter-arrival time. In other words, the proportion of bids with 'short' and 'long' inter-arrival time, appears to be fairly even distributed among the current price.

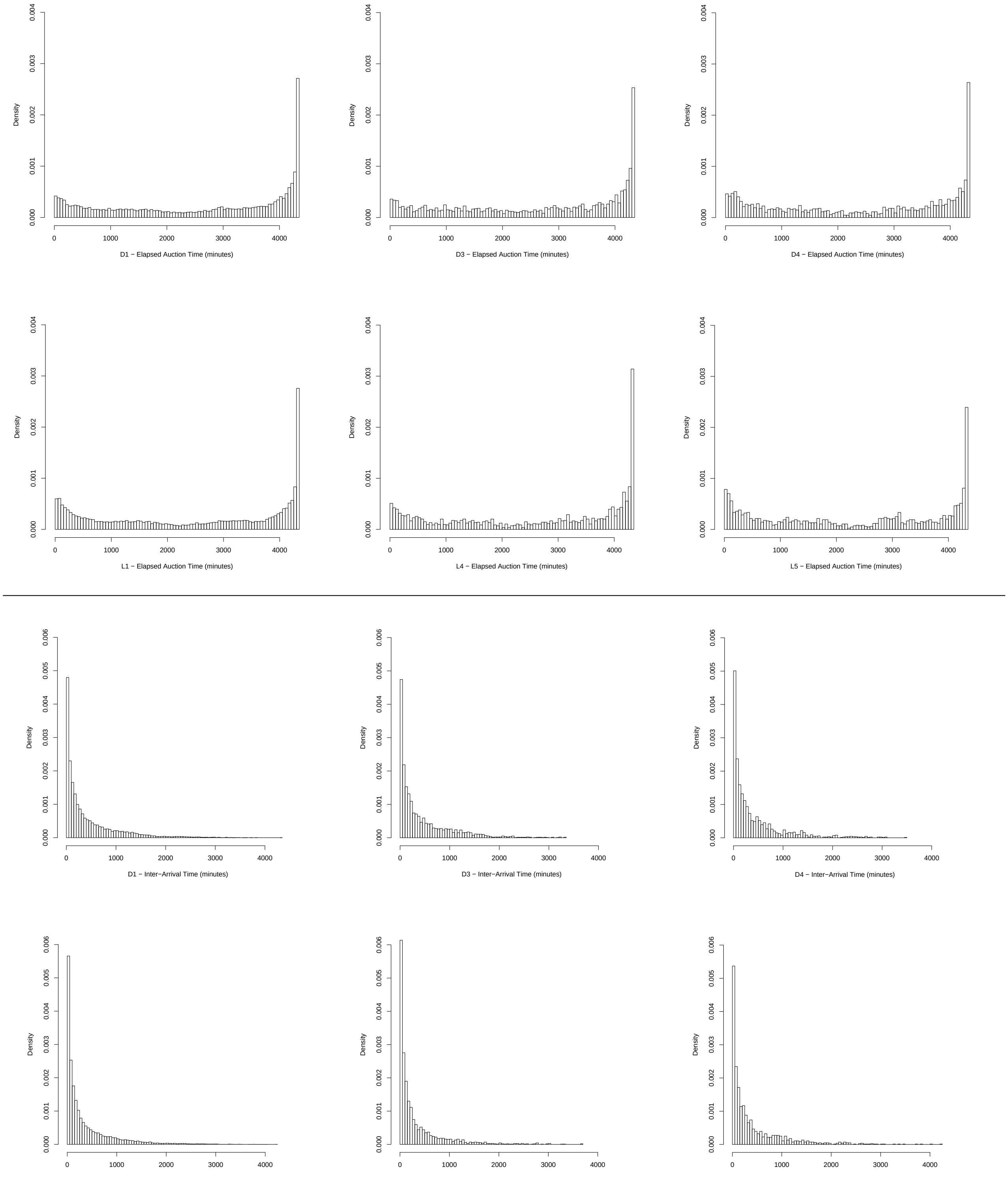


FIGURE 5.5. Histogram of the timing of bids (top) and inter-arrival time of bids (bottom) for the six products. For the top graphs the horizontal axis represents the elapsed time of an auction, while for the bottom the horizontal axis represents the inter-arrival time of a bid. In all graphs the vertical axis is the density (fraction) of bids placed in each interval. The first row for each set of histograms is D1, D3, and D4, while the second row is L1, L4, and L5 (left to right).

6. Discussion

This chapter has analyzed the individual bidding behavior for the six products D1, D3, D4, L1, L4, and L5. The main objective has been to propose and fit a model for the underlying bidding strategy. Specifically we discussed a model where bid-increments follow a gamma distribution, where the mean bid-increment is exponentially related to various covariates. The model and analysis was further divided into two time-periods. One for the bids in the first 71 hours (out of 72), and one for the bids in the final hour. The set of covariates for the first set included the current price of an auction, number of ongoing auctions, the number of auctions the bidder has participated in, and a dummy variable if the bid is the first bid the bidder placed in the auction. For the analysis of bids in the final hour, the only difference was that current price was exchanged with the elapsed auction time. Overall the proposed models fitted well with the data, and evidence suggest that bidders bid an increment above the current price which depends on the variables listed.

In particular, we statistically confirmed that in the first 71 hours, the current price of an auction and the number of ongoing auctions, are significantly and negatively correlated with the expected bid-increment. Furthermore, in the first 71 hours, the elapsed auction time, in the presence of current price, does not significantly affect the expected bid-increment. For the final hour, the statistical findings were as follows. The elapsed auction time was found to be statistically significant, while current price and number of ongoing auctions are not statistically significant.

Another interesting observation, was that the prices in the lowest and highest priced competing auctions, do not affect the expected bid-increment. In other words, bidders do not seem to base their bid-increment on the prices in the competing auctions. One reason why this might be the case, is that bidders do not have the ability, time or interest to compare auctions. On eBay, auctions are by default listed in descending order of remaining auction time. Therefore, bidders may simply choose to participate in the first auction they observe, and base their bid-increment on the available information in each auction.

The insight from the proposed model and statistical analysis is of great relevance to both bidders and sellers. By understanding the underlying bidding behavior both parties are able to make better decisions. For instance, a bidder may use it to further his chances of winning an auction. To illustrate, suppose a bidder is the high-bidder with only 10 minutes remaining, and that his bid is only $10 above the current price. Then he may wish to revise and increase his current bid (which is the high-bid), such that if a bidder or bidders arrive and bid, in expectation he is still likely to win the auction. Similarly, a bidder that arrives may wish to estimate the distribution of the final price, and decide whether it is worthwhile to participate. He can do that by estimating the expected bid-increment and apply the methodology discussed in Chapter 3. The same applies for a seller. This chapter has provided an alternative approach to analyze the auction dynamics, which can be used to decide on when to release another item for auction. That is, based on the analysis presented in this chapter, together with the methodology presented in Chapter 3, a seller can determine if the conditions presented in Chapter 2 holds, and if the optimal release policy is of a threshold type.

CHAPTER 6

Conclusion

Auctions have been used for centuries and most likely will continue to be used for many years to come. Over the last 50 years the theory of auctions has become an important and integral part of Economics, Management Science/Operations Research, and Computer Science. In the past, auction research mainly focused on normative studies regarding bid strategic equilibrium analysis. In recent years, partly due to the introduction of online auctions, additional streams of auction theory have emerged. The two most prevalent streams are empirical and experimental auction research. Online auctions provide a wealth of data and a great source for empirical analysis. Furthermore, the Internet and computer labs have enabled researchers to investigate various behavioral aspects of auctions. The objective with this thesis have been to contribute to auction research in two ways. First, to provide an alternative framework for researchers and practitioners to analyze online auctions. Second, to enable the analysis of intermediate prices of ongoing auctions and not just the final prices. The main motivation has been to provide an analysis of ongoing auctions, such that sellers, buyers, and auctioneers can make better decisions. Time will have to judge its success. We conclude with the main take-away and possible extensions from each chapter.

1. Chapter 2 Conclusions

The main research question of Chapter 2 was to investigate how should a seller, with a fixed inventory, release each item for auction if he wishes to maximize his profit. The problem was formulated as a discrete time Markov Decision Process, where auctions evolved according to a stochastic process. In order to make the problem interesting and non-trivial, two conflicting constraints were imposed. On the one hand, items incurred a holding or depreciation cost over time. This provides an incentive for the seller to release all items for auction immediately. On the other hand, ongoing auctions 'cannibalized' each other. This provides an incentive for the seller to release the items in a series of non-overlapping sequential auctions. One of the main results shown in Chapter 2 was that, given certain assumptions on the price-transition probabilities, the optimal release policy is of a threshold type. That is, a seller with two items should release an item for auction and observe its progression. If the ongoing auction is above a certain price-threshold, then it is optimal to release the second item. And if the ongoing auction is below the price-threshold, then it is optimal to defer the release. The threshold in each period is not necessarily constant over time. In fact, it may not even be monotone over time. Furthermore, the threshold policy is guaranteed to be optimal as long as there is no chance an auction is unsuccessful. If there is a positive probability that an auction will receive no bids, then the analysis is a bit more complicated and does not necessarily imply the optimal policy is of a threshold type.

N **Item Case.** The most obvious extension is the general N item case. To ensure Proposition 2.11 holds requires only a minor adjustment of Assumption 2.1, and is straightforward to prove. That is, ensuring the seller is always better off the higher an auction is priced, simply requires the CDF be decreasing in price for any number of ongoing auctions z, $z = 1, 2, \ldots, N$. However, to solve the release problem and establish a version of Theorem 2.12, we must define higher orders of cannibalization and diminishing cannibalization. A natural assumption would be that the cannibalization effect is consistent and diminishing in the number of ongoing auctions z. By consistent we mean, for $x_i' \leq P$, $i = 1, 2, \ldots, N$,

$$F^{z}_{X_{i,t_i+1}|\mathbf{X}}(x_i'|\mathbf{x}) \leq F^{z+1}_{X_{i,t_i+1}|\mathbf{X}}(x_i'|\mathbf{x}) \qquad\qquad z = 1, 2, \ldots, N-1$$

where $\mathbf{X} = (X_{1,t_1}, X_{2,t_2}, \ldots, X_{N,t_N})$ is the vector of prices in each auction, and $\mathbf{x}$ is the vector of realized values. In other words, the probability that auction i will be priced less than x_i' is increasing in z. By diminishing in z, we mean, for $x_i \leq x_i' \leq P$, $i = 1, 2, \ldots, N$, $z = 1, 2, \ldots, N-1$,

$$F^{z+1}_{X_{i,t_i+1}|\mathbf{X}}(x_i'|\mathbf{x}) - F^{z}_{X_{i,t_i+1}|\mathbf{X}}(x_i'|\mathbf{x})$$

is decreasing in z, where $\mathbf{X}$ and $\mathbf{x}$ are defined as above. That is, the more ongoing auctions, the smaller the effect of starting an additional auction. In particular, it would seem intuitive that at some point the cannibalization effect would vanish. For instance, the same transition probabilities might apply if there are 30, 40, or 50 ongoing auctions.

In addition, we must define the release policy of interest, and in particular what we mean by a monotone release policy. Note that with N items, there could be several ongoing auctions and several items waiting to be released. Therefore, calculating the value function is increasingly challenging for increasing values of N, because of the curse of dimensionality. The objective would therefore be to simplify the problem and the resulting possible policies. One possibility that intuitively seem promising, is to focus on 'release-an-additional-item' policies. That is, the seller observes the system state of the current ongoing auctions, and then decides on whether to release one more item for auction. To illustrate, suppose in time period t the seller has five ongoing auctions and two items waiting to be released. He evaluates if it is worthwhile to release an additional item and have six ongoing auctions. If the decision is yes, then he evaluates if it is worthwhile to release an additional item and have seven ongoing auctions (given that he now has six). If the decision is no, then he defers the release at least one more period. The other, more complicated issue is to define the space over which the decision should be made. That is, which variables of the ongoing auctions to measure and how to define a policy over them. Note that there could be up to $N-1$ possible auctions underway. One policy to consider is to vary the price of one auction while keeping the prices of the other ongoing auctions fixed. In other words, suppose in period t there are z, $z = 1, 2, \ldots, N-1$, ongoing auctions, then we could investigate the effect of increasing the price in one auction, while keeping the prices in the remaining $z-1$ auctions fixed. Intuitively, it seems relatively straightforward to establish a threshold policy in this context. However, it is also not very interesting.

Another possibility is to sum up the expected final price of all ongoing auctions, given that no more auctions will be released, and define a 'release-an-additional-item' threshold, based on the number of ongoing auctions and the total expected final price. That is, to sequentially apply the methodology from the two-item case. To illustrate, suppose in time period t there are five auctions underway. The seller calculates the sum of the five expected final prices, given that no more auctions will be released, and if the sum is above a certain threshold, then he releases an additional item. If the sum is below the threshold then he defers the release at least one period. The main problem is that in order for this to be optimal, additional structural properties are required. The issue is that it is possible two vectors with different realized prices result in the same total expected final price, but the optimal 'release-an-additional-item' decisions are different. Other potential policies include a threshold policy in the maximum or minimum price of the ongoing auctions. Some preliminary work has been done, and the author continues the endeavor regarding N items.

Correlated Price-Transition Probabilities. The other extension to consider is when price-transitions are not independent of the price in the competing auction. An assumption that facilitated the analysis of Chapter 2 was that price-transition probabilities only depended on the price of an auction and the number of auctions underway. It might seem more appropriate that price-transitions of two auctions are correlated. In addition, that the correlation is such that the conditional price-transition probability to 'high' prices is increasing in the price of the competing auction. The first task is to define the correlation. There are at least two different ways this can be done. First, the marginal conditional distribution function of an auction could be defined. That is, to define for $x_1, x_2, x_i' \leq P$, $i = 1, 2$,

$$F^2_{X_{i,t_i+1}|\mathbf{X}}(x_i'|x_1, x_2) = \Pr[X_{i,t_i+1} \leq x_i'|(X_1, X_2) = (x_1, x_2)]$$

where $\mathbf{X} = (X_{1,t_1}, X_{2,t_2})$ is the vector of prices for the two auctions. Second, the bivariate conditional distribution function of the two auctions could be defined. That is, to define for $x_1, x_2, x_1', x_2' \leq P$,

$$F^2_{(X_{1,t_1+1}, X_{2,t_2+1})|\mathbf{X}}(x_1', x_2'|x_1, x_2) = \Pr[X_{1,t_1+1} \leq x_1', X_{2,t_2+1} \leq x_2'|(X_1, X_2) = (x_1, x_2)]$$

Note that a bivariate distribution function can always be constructed from any marginal distribution function. See, for instance, Gumbel (1960) for a discussion of the bivariate exponential distribution. However, the conceptual framework of the distribution function is of course different for the two approaches.

The second task is to impose structural properties on the distribution function, such that a threshold policy is still optimal. The crux in Chapter 2 was that the cannibalization on the expected final price of the ongoing auction was diminishing in the price of the auction. With correlated prices, in particular if the lowest-priced auction benefits, then the 'cannibalization' effect is not monotone. Therefore, the price-transitions to 'high' prices are more complicated. However, a reasonable assumption is to assume that the main effect of an auction dominates the cross effect of the competing auction. To illustrate, a \$1 increase in auction 1 is much better for auction 1, than a \$1 increase in auction 2. Recall that the higher the price in auction 2, the more likely auction 1 is to make a transition to 'high' prices. This assumption is standard in the Economics literature, and with the proper attributes might ensure that Theorem 2.12 holds. The author has done some preliminary work and is confident that an optimal threshold policy can be derived. However, a final comment is that from the empirical analysis there is little evidence that the competing auction prices matter. Although some further empirical analysis can be done, in Chapter 5 it was shown that the prices of the highest- and lowest-priced auction did not seem to effect the bid-increment.

Additional Application. In addition to the direct application of selling products using online auctions, the model and results could be extended to other settings. One setting that seems particular suitable is selling real estate. Suppose a real estate agent has N properties to sell, and wants to know how he should release the listings. That is, is it optimal to release all N properties immediately? Or would he and his clients be better off by not having all N properties compete with each other? Considering the interest and principal of a property, the 'holding' cost is considerable. Furthermore, in the wake of the current subprime mortgage crisis in the US, the depreciation cost of a property might be even greater. On the other hand, it would also seem intuitive that by flooding the real estate market with listings, the average selling price of a property would decrease. Since potential buyers place 'bids' (offers) on properties, the analysis and results presented in Chapter 2 might provide real estate buyers, sellers, and agents with better insights.

2. Chapter 3 Conclusions

The framework presented in Chapter 2 assumed that the dynamics of online auctions could be captured by a set of conditional price-transition probabilities. Two natural questions that might arise are: 1) What underlying bidding behavior would give rise to such transition probabilities? 2) How would you derive or estimate them? Chapters 3 and 4 provided some answers to these questions. Specifically, Chapter 3 illustrated with two fixed bidding strategies, how the conditional within period price-transition probabilities can be derived. The first bidding strategy is when bidders bid a minimal bid-increment, while the second bidding strategy is when bidders bid their true valuation. The objective was to show how the price-transition probabilities could be derived based on a given auction format, bidder arrival process, and fixed bidding strategy.

The most natural and definitely most challenging extension would be to develop a more 'realistic' auction/game theory model of eBay, and use the model to derive a bid strategy Bayesian Nash equilibrium. Two factors that makes this challenging, which would need to be resolved, are that auctions overlap and that towards the end of an auction there is a positive probability a bid does not get registered. Therefore, any bid strategy would need to address the issue that a potential bidder chooses when to bid and in which auction to bid, based on auction prices and remaining time of the ongoing auctions. To motivate further complexity, we note that from the empirical bid analysis of Chapter 5 many auctions were won by previous winners. That is, many bidders have more than unit demand.

3. Chapters 4 and 5 Conclusions

Chapter 4 presented a statistical model for estimating the price-transition probabilities based on real data. More specifically, it was proposed that within a period, *price-increments* follow a zero-inflated gamma distribution. That is, each period there is a positive probability that an auction will make a positive price-transition, and condition on that it does the price-increment is gamma distributed. The objective of Chapter 4 was threefold. First, to propose a statistical model for price changes over discrete periods. Second, to provide conditions of model parameters such that the results from Chapter 2 holds. Third, to fit the model to real auction data and estimate the conditional within period price-transition probabilities. From the empirical analysis in Chapter 4 we concluded that over discrete time periods, online auctions appear to follow a zero-inflated gamma distribution. And that the probability of a positive price-transition and price-increment are decreasing in the price of an auction, and decreasing in the number of competing auctions. Furthermore, overall the fitted price-transition probabilities exhibit properties such that the results from Chapter 2 holds.

Chapter 5 consisted of an empirical analysis of the actual bids. In other words, while Chapter 4 focused on an auction's price-transition over discrete time periods, Chapter 5 focused on the individual bidding behavior. More specifically, Chapter 5 proposed that bids follow a gamma distribution. For both Chapter 4 and 5 the proposed distribution function were conditional on certain auction parameters. Most notably the price and elapsed time of an auction, and number of competing auctions.

Extensions to enrich both empirical models are plentiful. One example is to include the information regarding the competing auctions differently. For the model in Chapter 4 only the average number of ongoing auctions was included, while in Chapter 5 only the price and time of the highest and lowest priced competing auction was included. Ideally, both models should incorporate the competing auctions more consistently. This might require the need for a controlled study. A second example is to include the product configuration as parameters. In other words, to combine a hedonistic pricing model with the price- and bid-increment models. A third example, which was briefly discussed in Chapter 5, is to accommodate for the fact that 'even' bid-increments are over represented. Recall that in Figure 5.1 on page 207, we observed spikes at 'even' bids and bid-increments. As we discussed, one possibility includes modeling bid-increments as a mix of a Poisson and gamma distribution.

Another stream of extension would be to analyze how the aggregation of the bids from Chapter 5 result in the model proposed in Chapter 4. In other words, what resulting properties does a random sum of gamma distributed variables have. In particular, what structural properties on the γ coefficients in Chapter 5 are needed, such that the structural properties of Chapter 4 and results of Chapter 2 holds. A final and much more general extension is to pursue the issue of goodness-of-fit and model validation for Generalized Linear Models.

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APPENDIX A

Proofs for Results in Chapter 2

Proof Proposition 2.4 - First validate that Π_1^{U} and Π_2^{U} are well-defined transition probability matrices. At any given price level $x_i \in [0, P]$ there are $P - x_i + 1$ levels the price can jump to. Under Π_1^{U} each jump has an equal probability of $1/(P + 1 - x_i)$, therefore 1) each transition probability is well-defined, $0 \leq \pi_{q,j|1} \leq 1$, and 2) the total probability of making a jump is $\sum_{q=x_i}^{P} 1/(P + 1 - x_i) = (P + 1 - x_i) * 1/(P + 1 - x_i) = 1$. Since under Π_2^{U} the only change is that the probability of jumping to P decreases with κ while the probability of remaining at x_i increases with κ, the total probability of making a jump remains constant. Furthermore, since $\kappa \leq \frac{1}{P+1} \leq 1/(P + 1 - x_i)$ for all $x_i \in [0, P]$, we still have $0 \leq \pi_{x_i,q|2} \leq 1$, and hence each transition probability is well-defined. Therefore, Π_1^{U} and Π_2^{U} are a well-defined transition probability matrices.

Next we validate that (2.3) holds. Let $x_i < P$, then, for $x_i < r \leq P$, $\sum_{q=r}^{P} \pi_{x_i,q|1} = \sum_{q=r}^{P} 1/(P+1-x_i) = (P-r+1)*1/(P+1-x_i) < (P-r+1)*1/(P-x_i) = \sum_{q=r}^{P} 1/(P-x_i) = \sum_{q=r}^{P} \pi_{x_i+1,q|1}$, while for $r \leq x_i$, $\sum_{q=r}^{P} \pi_{x_i,q|1} = 1 = \sum_{q=r}^{P} \pi_{x_i+1,q|1}$. And, for $x_i < r \leq P$, $\sum_{q=r}^{P} \pi_{x_i,q|2} = (P+1-r)*1/(P+1-x_i)-\kappa < (P+1-r)*(1/(P-x_i)-\kappa = \sum_{q=r}^{P} \pi_{x_i+1,q|2}$, while for $r \leq x_i$, $\sum_{q=r}^{P} \pi_{x_i,q|2} = 1 = \sum_{q=r}^{P} \pi_{x_i+1,q|2}$. Therefore Π_1^{U} and Π_2^{U} satisfies (2.3).

To validate (2.5), let $x_i < r \leq P$ then $\sum_{q=r}^{P} \pi_{x_i,q|2} = (P+1-r)*1/(P+1-x_i)-\kappa < (P+1-r)*1/(P+1-x_i) = \sum_{q=r}^{P} \pi_{x_i+1,q|1}$, while for $r \leq x_i \leq P$, $\sum_{q=r}^{P} \pi_{x_i,q|2} = 1 = \sum_{q=r}^{P} \pi_{x_i+1,q|1}$. Therefore Π_1^{U} and Π_2^{U} satisfies (2.5).

To validate (2.7), let $x_i < r \leq P$ and note that $\sum_{q=r}^{P} \pi_{x_i+1,q|1} - \pi_{x_i,q|2} = \kappa$, which is independent of x_i. And for $r \leq x_i \leq P$, $\sum_{q=r}^{P} \pi_{x_i+1,q|1} - \pi_{x_i,q|2} = 0$, which also is independent of x_i. Therefore Π_1^{U} and Π_2^{U} satisfies (2.7). $\square$

Proof Proposition 2.5 - First validate that $f^1(q|x)$ and $f^2(q|x)$ are well-defined probability density functions. We note that $\int_x^1 f^1(q|x)dq = \int_x^1 \frac{1}{1-x}dq = 1$, and, for all $x \le x' \le 1$, $\int_x^{x'} \frac{1}{1-x}dq = \frac{x'-x}{1-x} \le 1$ and increasing in x'. Similarly, $\int_x^1 f^2(q|x)dq = \int_x^1 \frac{2-2q}{(1-x)^2}dq = 1$, and, for all $x \le x' \le 1$, $\int_x^{x'} \frac{2-2q}{1-x}dq = \frac{2x'-(x')^2-2x+x^2}{(1-x)^2} \le 1$ and increasing in x'.

Next we validate that Assumption 2.1 holds. We note that, for $x \le x' \le 1$, $\frac{\partial}{\partial x}F^1(x'|x) = \frac{x'-1}{(1-x)^2} \le 0$. And, for $x \le x' \le 1$, $\frac{\partial}{\partial x}F^2(x'|x) = \frac{\partial}{\partial x}\frac{2x'-(x')^2-2x+x^2}{(1-x)^2} = \frac{-2(1-x')^2}{(1-x)^3} \le 0$.

To validate Assumption 2.2, for $x \le x' \le 1$, $F^1(x'|x) = \frac{x'-x}{1-x} \le \frac{x'(2-x')-x(2-x)}{(1-x)^2} = F^2(x'|x)$.

Finally, to validate Assumption 2.3, for $x \le x' \le 1$, $\frac{\partial}{\partial x}F^2(x'|x) = \frac{-2(1-x')^2}{(1-x)^3} \le \frac{x'-1}{(1-x)^2} = \frac{\partial}{\partial x}F^1(x'|x)$. $\square$

Proof Proposition 2.6 - In order for Π_1^{Be} and Π_2^{Be} to be well-defined transition probability matrices we assume that $0 \le \pi_x \le 1$ and $0 \le \rho_x \le 1$, for all $x < P$.

To validate Assumption 2.1, let $x < P$, then, (I) for $x+1 < r \le P$, $\sum_{q=r}^{P} \pi_{x,q|1} = 0 \le \sum_{q=r}^{P} \pi_{x+1,q|1} \le \pi_{x+1}$, (II) for $x+1 = r$, $\sum_{q=r}^{P} \pi_{x,q|1} = \pi_x \le 1 = \sum_{q=r}^{P} \pi_{x+1,q|1}$, and (III) for $r \le x < P$, $\sum_{q=r}^{P} \pi_{x,q|1} = \sum_{q=r}^{P} \pi_{x+1,q|1} = 1$. And similarly if we exchange π_x with ρ_x, and therefore Assumption 2.1 holds.

To validate Assumption 2.2, let $x \le P$, then (I) for $x+1 < r \le P$, $\sum_{q=r}^{P} \pi_{x,q|2} = 0 = \sum_{q=r}^{P} \pi_{x,q|1}$, (II) for $x+1 = r$, $\sum_{q=r}^{P} \pi_{x,q|2} = \rho_x \le \pi_x = \sum_{q=r}^{P} \pi_{x,q|1}$, where the inequality follows from (2.11), and (III) for $r \le x < P$, $\sum_{q=r}^{P} \pi_{x,q|2} = \sum_{q=r}^{P} \pi_{x,q|1} = 1$. Therefore Assumption 2.2 holds.

Assumption 2.3(mod.) follows immediately from (2.11). $\square$

Proof of Corollary 2.9 - Since by assumption each auction progress independently of the price in the other auction, the independence of x_1 is immediate. If $y_1 = \tau, \delta$, then $y_1 + 1 = \delta$, and

$E[X_{2,\tau}|S_{t+1} = s'] = E[X_{2,\tau}|S_t = s]$, since auctions dynamics are independent of calender time. Therefore assume $y_1 < \tau$. Proof by induction on y_1. If $y_1 = \tau - 1$,

$$E[X_{2,\tau}|S_t = ([x_1, \tau - 1; x_2, y_2], 2)] = \sum_{q=x_2}^{P} E[X_{2,\tau}|S_{t+1} = ([x_1', \tau; q, y_2 + 1], z)]\pi_{x_2,q|2}$$

$$\leq \sum_{q=x_2}^{P} E[X_{2,\tau}|S_{t+1} = ([x_1', \tau; q, y_2 + 1], z)]\pi_{x_2,q|1} = E[X_{2,\tau}|S_t = ([x_1', \tau; x_2, y_2], 1)]$$

$$= E[X_{2,\tau}|S_{t+1} = ([x_1', \tau; x_2, y_2], 1)]$$

where the inequality holds due to Lemma 4.7.2 in Puterman (1994), Corollary 2.7 and Assumption 2.2, and the last equality holds due to the assumption that price transitions are independent of calender time. Assume the result holds for $y_1 = \tau - 1, \tau - 2, \ldots, \tau - l$. Let $y_1 = \tau - (l + 1)$,

$$E[X_{2,\tau}|S_t = ([x_1, \tau - (l + 1); x_2, y_2], 2)] = \sum_{q=x_2}^{P} E[X_{2,\tau}|S_{t+1} = ([x_1', \tau - l; q, y_2 + 1], 2)]\pi_{x_2,q|2}$$

$$\leq \sum_{q=x_2}^{P} E[X_{2,\tau}|S_{t+1} = ([x_1', \tau - (l - 1); q, y_2 + 1], 2)]\pi_{x_2,q|2} = E[X_{2,\tau}|S_t = ([x_1', \tau - l; x_2, y_2], 2)]$$

$$= E[X_{2,\tau}|S_{t+1} = ([x_1', \tau - l; x_2, y_2], 2)]$$

where the inequality holds due to Lemma 4.7.2 in Puterman (1994), Corollary 2.7 and the induction assumption, and the last equality holds due to the assumption that price transitions are independent of calender time and the price of the other auction. Similar to Corollary 2.7 the extension to continuous prices is immediate, and the results from Lemma 9.1.1 and Proposition 9.1.2 in Ross (1996) could have been applied. $\square$

Proof of Lemma 2.10 - Due to the vigilant seller assumption and that we are considering the case when auctions are guaranteed to be successful we can explicitly write out the value function (2.21) according to Table A.1. Note that there are only non-trivial decisions to be made for $t < \tau$ and $z = 1$. Consequently once the second auction has started, we can evaluate the expected total future reward (= total remaining cost - expected final price for both items). The implication of this is summarized in following two lemmas which will facilitate the 'book keeping' and establish Lemma 2.10.

Period	Condition	$V_t([x_1, y_1; x_2, y_2], z)$
$t = T$	$z = 0$	$= x_1 + x_2$
$\tau \leq t < T$	$z = 0$	$= V_{t+1}([x_1, y_1; x_2, y_2], z)$
	$z = 1$	$= -h + \sum_{q=x_2}^{P} V_{t+1}([x_1, y_1; q, y_2 + 1], z')\pi_{x_2,q\|z}$
$t < \tau$	$z = 2$	$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V_{t+1}([q, x_1 + 1; r, t_2 + 1], z')\pi_{x_1,q\|2}\pi_{x_2,r\|2}$
	$z = 1$	$= -2h + \max\{ \sum_{q=x_1}^{P} V_{t+1}([q, y_1 + 1; x_2, y_2], z)\pi_{x_1,q\|1}$,
		$\sum_{q=x_1}^{P} \sum_{r=p}^{P} V_{t+1}([q, y_1 + 1; r, y_2 + 1], z')\pi_{x_1,q\|2}\pi_{x_2,r\|2}$ }

TABLE A.1. Optimality equations for the single listing case.

Lemma A.1. *If we assume a vigilant seller and each auction is guaranteed to be successful, then once item 2 has been released we can explicitly evaluate the value function, for 1) $\tau \leq t \leq T, z = 0, 1$, or 2) $t < \tau, z = 2$,*

$$V_t([x_1, y_1; x_2, y_2], z) = R([x_1, y_1; x_2, y_2], z) \tag{A.1}$$

Proof of Lemma A.1 - There are three cases to consider.

1) For $\tau \leq t \leq T$ and $z = 0$, proof by backward induction on t. Let $t = T$ then $y_1 = \delta$ and $y_2 = \tau$ or δ, and therefore $V_t([x_1, y_1; x_2, y_2], z) = -h0 + x_1 + x_2 = -h(2\tau - \delta - \delta) + E[X_{1,\tau}|([x_1, \delta; x_2, \delta], 0)] + E[X_{2,\tau}|([x_1, \delta; x_2, \delta], 0)]$, and the result holds. Assume the result holds for $t = l+1, l+2, \ldots, T$. Let $\tau \leq t = l$, then $y_1 = \tau$ or δ and $y_2 = \tau$ or δ, and therefore $V_t([x_1, y_1; x_2, y_2], z) = V_{t+1}([x_1, \delta; x_2, \delta], 0) = -h(2\tau - \delta - \delta) + E[X_{1,\tau}|([x_1, \delta; x_2, \delta], 0)] + E[X_{2,\tau}|([x_1, \delta; x_2, \delta], 0)]$, where the second equality holds due to the induction hypothesis. Therefore the result holds for all $\tau \leq t \leq T$ and $z = 0$. As noted earlier there is a slight abuse of notation when $y_i = \delta$, $i = 1, 2$, where we define $\tau - \delta = 0$.

2) For $\tau \leq t \leq T$ and $z = 1$, proof by backward induction on t. Let $t = T - 1$ and $z = 1$, then $t_1 = \delta$ and $t_2 = \tau - 1$, therefore $V_t([x_1, y_1; x_2, y_2], z) =$

$$\begin{aligned}
&= -h + \sum_{q=x_2}^{P} V_T([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} = -h + \sum_{q=x_2}^{P} x_1 \pi_{x_2,q|1} + \sum_{q=x_2}^{P} q\pi_{x_2,q|1} \\
&= -h + x_1 + \sum_{q=x_2}^{P} q\pi_{x_2,q|1} \\
&= -h(2\tau - \delta - (\tau - 1)) + E[X_{1,\tau}|([x_1, \delta; x_2, \tau - 1], 1)] + E[X_{2,\tau}|([x_1, \delta; x_2, \tau - 1], 1)]
\end{aligned}$$

And the result holds (note that if $t = T$ then due to the vigilant seller assumption all auctions are completed and $z \neq 1$). Assume the result holds for $t = l + 1, l + 2, \ldots, T$. Let

$\tau \leq t = l$ and $z = 1$, then $t_1 = \tau$ or δ, therefore $V_t([x_1, y_1; x_2, y_2], z) =$

$$= -h + \sum_{q=x_2}^{P} V_{t+1}([x_1, \delta; q, y_2 + 1], z')\pi_{x_2,q|1}$$

$$= -h - h(2\tau - \delta - (y_2 + 1)) + \sum_{q=x_2}^{P} E[X_{1,\tau}|([x_1, \delta; q, y_2 + 1], z')]\pi_{x_2,q|1} + \sum_{q=x_2}^{P} E[X_{2,\tau}|([x_1, \delta; q, y_2 + 1], z')]$$

$$= -h(2\tau - \delta - y_2) + E[X_{1,\tau}|([x_1, \delta; x_2, y_2], 1)] + E[X_{2,\tau}|([x_1, \delta; x_2, y_2], 1)]$$

Where the second equality holds due to the induction hypothesis when $z' = 1$, or case 1) above when $z' = 0$. Therefore the result holds for all $\tau \leq t \leq T$ and $z = 1$.

3) For $t < \tau$ and $z = 2$, proof by backward induction on t. Let $t = \tau - 1$ and $z = 2$, then $y_1 = \tau - 1$, therefore $V_t([x_1, y_1; x_2, y_2], z) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V_{t+1}([q, \tau; r, y_2 + 1], z')\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h - h(2\tau - \tau - (y_2 + 1)) + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} E[X_{1,\tau}|([q, \tau; r, y_2 + 1], z')]\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$+ \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} E[X_{2,\tau}|([q, \tau; r, y_2 + 1], z')]\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -h(2\tau - (\tau - 1) - y_2) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau; x_2, y_2 + 1], z')]\pi_{x_1,q|2} + \sum_{r=x_2}^{P} E[X_{2,\tau}|([x_1, \tau; r, y_2 + 1], z')]\pi_{x_2,r|2}$$

$$= -h(2\tau - (\tau - 1) - y_2) + E[X_{1,\tau}|([x_1, \tau - 1; x_2, y_2], 2)] + E[X_{2,\tau}|([x_1, \tau - 1; x_2, y_2], 2)]$$

Where the second equality follows from case 1) above when $z' = 0$, or case 2) above when $z' = 1$, while the third equality follows from that we assume each auction progress independently of the price of the other auction. Therefore the result holds for $t = \tau - 1$ and $z = 2$.

Assume the result holds for $t = \tau - (l-1), \tau - (l-2), \ldots, \tau - 1$. Let $t = \tau - l$ and $z = 2$, then $y_1 = \tau - l$ and $V_t([x_1, y_1; x_2, y_2], z) =$

$$
\begin{aligned}
&= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V_{t+1}([q, \tau-(l-1); r, y_2+1], z')\pi_{x_1,q|2}\pi_{x_2,r|2} \\
&= -2h - h(2\tau - (\tau-(l-1)) - (y_2+1)) \\
&\quad + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} E[X_{1,\tau}|([q, \tau-(l-1); r, y_2+1], z')]\pi_{x_1,q|2}\pi_{x_2,r|2} \\
&\quad + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} E[X_{2,\tau}|([q, \tau-(l-1); r, y_2+1], z')]\pi_{x_1,q|2}\pi_{x_2,r|2} \\
&= -h(2\tau - (\tau-l) - t_2) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau-(l-1); x_2, y_2+1], z')]\pi_{x_1,q|2} \\
&\quad + \sum_{r=x_2}^{P} E[X_{2,\tau}|([x_1, \tau-(l-1); r, y_2+1], z')]\pi_{x_2,r|2} \\
&= -h(2\tau - (\tau-l) - y_2) + E[X_{1,\tau}|([x_1, \tau-l; x_2, y_2], 2)] + E[X_{2,\tau}|([x_1, \tau-l; x_2, y_2], 2)]
\end{aligned}
$$

Where the second equality follows from the induction hypothesis when $z' = 2$, case 1) above when $z' = 0$, or case 2) above when $z' = 1$, and the third equality holds due to the assumption that each auction progress independently of the price in the other auction. Therefore the result holds for all $t < \tau$ and $z = 2$. $\square$

LEMMA A.2. *If we assume a vigilant seller and each auction is guaranteed to be successful then for $t < \tau$ and $z = 1$,*

(A.2)

$$V_t([x_1, y_1; x_2, y_2], z) = \max\{-2h + \sum_{q=x_1}^{P} V_{t+1}([q, y_1 + 1; p, 0], 1)\pi_{x_1,q|1}\ , R([x_1, y_1; p, 0], 2)\}$$

Proof of Lemma A.2 - Let $t < \tau$ and $z = 1$, then $y_1 = t$ and,

$$\begin{aligned}
&-2h + \sum_{q=x_1}^{P}\sum_{r=p}^{P} V_{t+1}([q, t+1; r, 1], z)\pi_{x_1,q|2}\pi_{x_2,r|2} \\
&= -2h + \sum_{q=x_1}^{P}\sum_{r=p}^{P}(-h(2\tau - (t+1) - 1) + E[X_{1,\tau}|([q, t+1; r, 1], z)] + E[X_{2,\tau}|([q, t+1; r, 1], z)])\pi_{x_1,q|2}\pi_{x_2,r|2} \\
&= -h(2\tau - t - 0) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, t+1; p, 1], z)]\pi_{x_1,q|2} + \sum_{r=p}^{P} E[X_{2,\tau}|([x_1, t+1; r, 1], z)]\pi_{x_2,r|2} \\
&= -h(2\tau - t - 0) + E[X_{1,\tau}|([x_1, t; p, 0], 2)] + E[X_{2,\tau}|([x_1, t; p, 0], 2)] \\
&= R([x_1, t; p, 0], 2)
\end{aligned}$$

where the first equality holds due to Lemma A.1 with $z = 1$ (if $t + 1 = \tau$) or with $z = 2$ (if $t + 1 < \tau$), and the second equality holds due to that each auction progress independently of the price in the other auction. □

Due to Lemma A.1 and A.2, and that we mainly are interested in states $s \in S$ such that $A(s) = \{0, 1\}$, we have the value functions listed in Lemma 2.10. The proof for the continuous case is identical but with an integral sign instead of summation □

Proof of Proposition 2.16 - From equation (2.24) we can compare various open loop policies and determine when each one dominates another. Let OPj and OP$(j + m)$ be the open loop policies of releasing the second auction j and $(j + m)$ periods respectively after the first auction. We then have, $V_{O(j)} \geq V_{O(j+m)} \Leftrightarrow -(2\tau + j)h + 2(p + j\pi + (\tau - j)\rho) \geq -(2\tau + j + m)h + 2(p + (j + m)\pi + (\tau - j - m)\rho) \Leftrightarrow h \geq 2(\pi - \rho)$.

Since this condition is independent of j and $j+m$ the result is that simultaneous release is optimal iff $h \geq 2(\pi - \rho)$. By symmetry (non-overlapping) sequential release is optimal iff $h < 2(\pi - \rho)$ and there are no other optimal Open Loop policies. $\square$

Proof of Proposition 2.18 - The proof of Proposition 2.11 is based on Assumption 2.1, Lemma 2.10, and Corollary 2.8, but not on any additional arguments involving the value of completed auctions. Therefore to show that introducing a reserve price impose no changes to Proposition 2.11, need to show that no changes to the listed assumption, lemmas and corollary occur. Assumption 2.1 does not depend on the value of a completed auction and so remains intact. Lemma 2.10, which is based on Lemmas A.1 and A.2, is only for book-keeping and does not depend on the actual value of completed auctions (same with Lemmas A.1 and A.2). Corollary 2.8 is a direct application of Corollary 2.7, therefore need to show that Corollary 2.7 holds.

Proof that Corollary 2.7 holds despite introducing v_r. By induction on the number of remaining periods n. For $n = 1$, $E[X_{1,\tau}|S_t = ([x_1, \tau - 1; x_2, y_2], z)] = v_r \sum_{q=x_1}^{v_r} \pi_{x_1,q|z} + \sum_{q=v_r+1}^{P} q\pi_{x_1,q|z} = v_r \sum_{q=x_1}^{P} \pi_{x_1,q|z} + \sum_{q=v_r+1}^{P} \pi_{x_1,q|z} + \sum_{q=v_r+2}^{P} \pi_{x_1,q|z} + \ldots + \sum_{q=P}^{P} \pi_{x_1,q|z} \leq v_r \sum_{q=x_1}^{P} \pi_{x_1+1,q|z} + \sum_{q=v_r+1}^{P} \pi_{x_1+1,q|z} + \sum_{q=v_r+2}^{P} \pi_{x_1+1,q|z} + \ldots + \sum_{q=P}^{P} \pi_{x_1+1,q|z} = E[X_{i,\tau}|S_t = ([x_1+1, \tau-1; x_2, y_2], z)]$, where the inequality holds due Assumption 2.1. Assume the result holds for $n = 1, 2, \ldots, l-1$ (with the introduction of v_r). For $n = l$, $y_1 + 1 = \tau - (l-1)$, $E[X_{1,\tau}|S_t = ([x_1, \tau - n; x_2, y_2], z)] = \sum_{q=x_1}^{P} E[X_{1,\tau}|S_t = ([x_1, \tau - n; x_2, y_2], z)]\pi_{x_1,q|z} \leq \sum_{q=x_1+1}^{P} E[X_{1,\tau}|S_t = ([x_1+1, \tau-n; x_2, y_2], z)]\pi_{x_1+1,q|z} = E[X_{1,\tau}|S_t = ([x_1+1, \tau-n; x_2, y_2], z)]$, where the inequality holds due to Lemma 4.7.2 of Puterman (1994), the induction assumption and Assumption 2.1. Therefore the expected final price is increasing in x_i, and Corollary 2.7 holds with the introduction of v_r.

Therefore Proposition 2.11 still holds when a reserve price has been imposed.

The proof of Theorem 2.12 is based on Assumptions 2.1, 2.2, 2.3, Lemma 2.10, equation

(2.23), Corollaries 2.8 and 2.9. As above all these results remain intact with the introduction of a reserve price. In addition the induction step in the proof involves the expected final value. Based on the same proof as above the induction step still holds even with a reserve price. The remaining part of the proof is not based on the actual value of an auction. Therefore Theorem 2.12 still holds when a reserve price has been imposed.

The proof of Corollary 2.13 does not involve the actual value of completed auctions, and therefore holds true when a reserve price has been imposed. $\square$

Proof of Lemma 2.20

Comment: Recall that $R_1([x_1, y_1; x_2, y_2], z) = -h(2\tau - y_1 - y_2) + E[X_{1,\tau}|([x_1, y_1; x_2, y_2], z)] + E[X_{2,\tau}|([x_1, y_1; x_2, y_2], z)] + (\pi_{x_2,0|z})(\pi_{0,0|z})^{\tau-y_1-1}(\pi_{0,0|1})^{y_1-y_2}v(0,0)$, and note that there is a slight abuse of notation for the cases when $t_i = \delta$. In these cases we implicitly assume that $\tau - t_i = 0$ and $E[X_{i,\tau}|([x_1, y_1; x_2, y_2], z)] = 0$, $i = 1, 2$.

1) $x_1 > 0, y_1 = \tau, \delta$ and $z = 0, 1$, or $x_1 > 0, y_1, y_2 < \tau$, and $z = 2$.

1a) For $x_1 > 0, y_1 = \delta$ and $z = 0, 1$, $R_1([x_1, \delta; x_2, y_2], z) = -h(\tau - y_2) + E[X_{2,\tau}|([x_1, \delta; x_2, y_2], z)]$ $+(\pi_{x_2,0|z})(\pi_{0,0|z})^{\tau-y_1-1}(\pi_{0,0|1})^{y_1-y_2}v(0,0)$.

Let $x_1 > 0, y_1 = \delta$, and $z = 0$. If $y_2 = \delta$ then $V([x_1, y_1; x_2, y_1], z) = 0 = R_1([x_1, \delta; x_2, \delta], 0)$. If $y_2 = \tau$ then $z = 0$ and $V([x_1, y_1; x_2, y_1], z) = x_2 + V(\Delta) = -h(\tau - \tau) + E[X_{2,\tau}|([x_1, \delta; x_2, \tau], 0)] = R_1([x_1, \delta; x_2, \tau], 0)$.

Let $x_1 > 0, y_1 = \delta$, and $z = 1$, that is $y_2 < \tau$. Proof by backward induction on y_2. Let $y_2 = \tau - 1$ and $x_2 > 0$ then $V([x_1, y_1; x_2, y_1], z) =$

$$= -h + \sum_{q=x_2}^{P} V([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} = -h + \sum_{q=x_2}^{P} R_1([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1}$$

$$= -h + \sum_{q=x_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([x_1, \delta; q, \tau], 0)])\pi_{x_2,q|1}$$

$$= -h(\tau - (\tau - 1)) + E[X_{2,\tau}|([x_1, \delta; x_2, \tau - 1], 1)]) = R_1([x_1, \delta; x_2, \tau - 1], 1)$$

Where the second equality holds due to the case above with $y_1 = \delta, z = 0$.

Let $y_2 = \tau - 1$ and $x_2 = 0$ then $V([x_1, y_1; x_2, y_1], z) =$

$$= -h + \sum_{q=p}^{P} V([x_1, \delta; q, \tau], 0)\pi_{0,q|1} + V([x_1, \delta; 0, 0], 1)\pi_{0,0|1}$$

$$= -h + \sum_{q=x_2}^{P} R_1([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} + v(0, 0)\pi_{0,0|1}$$

$$= -h + \sum_{q=x_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([x_1, \delta; q, \tau], 0)])\pi_{x_2,q|1} + v(0, 0)\pi_{0,0|1}$$

$$= -h(\tau - (\tau - 1)) + E[X_{2,\tau}|([x_1, \delta; x_2, \tau - 1], 1)]) + v(0, 0)\pi_{0,0|1}$$

$$= R_1([x_1, \delta; 0, \tau - 1], 1)$$

Where the second equality holds due to the case above with $y_1 = \delta$ and $z = 0$, and that $V([x_1, \delta; 0, 0], 1) = v(0, 0)$, since the first item has been awarded and by the vigilant seller assumption the second item will be continuously re-listed until the auction is successful.

Assume the result holds for $y_2 = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $y_2 = \tau - l$ then $V([x_1, y_1; x_2, y_1], z) =$

$$= -h + \sum_{q=x_2}^{P} V([x_1, \delta; q, \tau - (l - 1)], 1)\pi_{x_2,q|1} = -h + \sum_{q=x_2}^{P} R_1([x_1, \delta; q, \tau - (l - 1)], 1)\pi_{x_2,q|1}$$

$$= -h + \sum_{q=x_2}^{P} (-h(\tau - (\tau - (l - 1))) + E[X_{2,\tau}|([x_1, \delta; q, \tau - (l - 1)], 1)])\pi_{x_2,q|1}$$

$$= -h(\tau - (\tau - l)) + E[X_{2,\tau}|([x_1, \delta; x_2, \tau - l], 1)]) = R_1([x_1, \delta; x_2, \tau - l], 1)$$

Where the second equality holds due to induction hypothesis.

Therefore Lemma 2.20 holds for the case 1a) $x_1 > 0, y_1 = \delta$ and $z = 0, 1$.

1b) For $x_1 > 0, y_1 = \tau$ and $z = 0, 1$,

$$\begin{aligned} R_1([x_1, \tau; x_2, y_2], z) = &-h(2\tau - \tau - y_2) + E[X_{1,\tau}|([x_1, \tau; x_2, y_2], z)] + E[X_{2,\tau}|([x_1, \tau; x_2, y_2], z)] \\ &+ (\pi_{x_2,0|z})(\pi_{0,0|z})^{\tau - y_1 - 1}(\pi_{0,0|1})^{y_1 - y_2} v(0, 0) \end{aligned}$$

Let $x_1 > 0, y_1 = \tau$, and $z = 0$, that is $y_2 = \tau$. Therefore $V([x_1, y_1; x_2, y_1], z) = x_1 + x_2 + V(\Delta) = R_1([x_1, \tau; x_2, \tau], 0)$.

Let $x_1 > 0, y_1 = \tau$, and $z = 1$, that is $y_2 < \tau$. Proof by backward induction on y_2. Let $y_2 = \tau - 1$ and $x_2 > 0$ then $V([x_1, y_1; x_2, y_1], z)$

$$
\begin{aligned}
&= -h + x_1 + \sum_{q=x_2}^{P} V([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} = -h + x_1 + \sum_{q=x_2}^{P} R_1([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} \\
&= -h + E[X_{1,\tau}|([x_1, \tau; q, \tau - 1], 1) + \sum_{q=x_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([x_1, \delta; q, \tau], 0)])\pi_{x_2,q|1} \\
&= -h(2\tau - \tau - (\tau - 1)) + E[X_{1,\tau}|([x_1, \tau; x_2, \tau - 1], 1)]) + E[X_{2,\tau}|([x_1, \tau; x_2, \tau - 1], 1)]) \\
&= R_1([x_1, \tau; x_2, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to case 1a) above.

Let $y_2 = \tau - 1$ and $x_2 = 0$ then $V([x_1, y_1; x_2, y_1], z) =$

$$
\begin{aligned}
&= -h + x_1 + \sum_{q=p}^{P} V([x_1, \delta; q, \tau], 0)\pi_{0,q|1} + V([x_1, \delta; 0, 0], 1)\pi_{0,0|1} \\
&= -h + x_1 + \sum_{q=x_2}^{P} R_1([x_1, \delta; q, \tau], 0)\pi_{x_2,q|1} + v(0, 0)\pi_{0,0|1} \\
&= -h + E[X_{1,\tau}|([x_1, \tau; q, \tau - 1], 1)]) + \sum_{q=x_2}^{P} (-h(\tau - \tau) + E[X_{2,\tau}|([x_1, \delta; q, \tau], 0)])\pi_{x_2,q|1} + v(0, 0)\pi_{0,0|1} \\
&= -h(2\tau - \tau - (\tau - 1)) + E[X_{1,\tau}|([x_1, \tau; x_2, \tau - 1], 1)]) + E[X_{2,\tau}|([x_1, \tau; x_2, \tau - 1], 1)]) + v(0, 0)\pi_{0,0|1} \\
&= R_1([x_1, \tau; 0, \tau - 1], 1)
\end{aligned}
$$

Where the second equality holds due to case 1a) above and that $V([x_1, \delta; 0, 0], 1) = v(0, 0)$, which holds since the first item has been awarded and by the vigilant seller assumption the second item will be continuously re-listed until the auction is successful.

Assume the result holds for $y_2 = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $y_2 = \tau - l$ then

$V([x_1, y_1; x_2, y_1], z) =$

$$= -h + x_1 + \sum_{q=x_2}^{P} V([x_1, \delta; q, \tau - (l-1)], 1)\pi_{x_2,q|1}$$

$$= -h + x_1 + \sum_{q=x_2}^{P} R([x_1, \delta; q, \tau - (l-1)], 1)\pi_{x_2,q|1}$$

$$= -h + E[X_{1,\tau}|([x_1, \tau; q, \tau - (l-1)], 1)]$$

$$+ \sum_{q=x_2}^{P} (-h(\tau - (\tau - (l-1))) + E[X_{2,\tau}|([x_1, \delta; q, \tau - (l-1)], 1)])\pi_{x_2,q|1}$$

$$= -h(2\tau - \tau - (\tau - l)) + E[X_{1,\tau}|([x_1, \tau; x_2, \tau - l], 1)]) + E[X_{2,\tau}|([x_1, \tau; x_2, \tau - l], 1)])$$

$$= R_1([x_1, \tau; x_2, \tau - l], 1)$$

Where the second equality holds due to case 1a) above. Therefore Lemma 2.20 holds for the case 1b) $x_1 > 0, y_1 = \tau$ and $z = 0, 1$.

Note that for $x_1 > 0, y_1, y_2 < \tau$ and $z = 2$, $R_1([x_1, y_1; x_2, y_2], z) = -h(2\tau - y_1 - y_2) + E[X_{1,\tau}|([x_1, y_1; x_2, y_2], 2)] + E[X_{2,\tau}|([x_1, y_2; x_2, y_2], 2)] + (\pi_{x_2,0|z})(\pi_{0,0|z})^{\tau - y_1 - 1}(\pi_{0,0|1})^{y_1 - y_2} v(0, 0)$.

1c) Let $y_1 = \tau - 1$. Proof by backward induction on y_2. Let $y_2 = \tau - 1$ and $x_2 > 0$, then

$V([x_1, y_1; x_2, y_1], z) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau; r, \tau], 0)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} R_1([q, \tau; r, \tau], 0)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau; x_2, \tau], 2)]\pi_{x_1,q|2} + \sum_{q=x_2}^{P} E[X_{2,\tau}|([x_1, \tau; q, \tau], 2)]\pi_{x_1,q|2}$$

$$= -h(2\tau - (\tau - 1) - (\tau - 1)) + E[X_{1,\tau}|([x_1, \tau - 1; x_2, \tau - 1], 2)] + E[X_{2,\tau}|([x_1, \tau - 1; q, \tau - 1], 2)]$$

$$= R_1([x_1, \tau - 1; x_2, \tau - 1], 2)$$

Where the second equality holds due to case 1b) above, and the third equality holds due to that each auction progress independently of the price in the other auction.

Let $y_2 = \tau - 1$ and $x_2 = 0$, then $V([x_1, y_1; x_2, y_1], z) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=p}^{P} V([q, \tau; r, \tau], 0)\pi_{x_1,q|2}\pi_{x_2,r|2} + \sum_{q=x_1}^{P} V([q, \tau; 0, 0], 1)\pi_{x_1,q|2}\pi_{0,0|2}$$

$$= -2h + \sum_{q=x_1}^{P} \pi_{x_1,q|2}(\sum_{r=p}^{P} R_1([q, \tau; r, \tau], 0)\pi_{0,r|2} + R_1([q, \tau; 0, 0], 1)\pi_{0,0|2})$$

$$= -2h + \sum_{q=x_1}^{P} \pi_{x_1,q|2}(\sum_{r=p}^{P}(E[X_{1,\tau}|([q, \tau; r, \tau], 0)] + E[X_{2,\tau}|([q, \tau; r, \tau], 0)])\pi_{0,r|2}$$

$$+ (-h(2\tau - \tau - 0) + E[X_{1,\tau}|([q, \tau; 0, 0], 1)] + E[X_{2,\tau}|([q, \tau; 0, 0], 1)] + (\pi_{0,0|1})^{\tau} v(0, 0))\pi_{0,0|2})$$

$$= -h(2\tau - (\tau - 1) - (\tau - 1)) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau; x_2, \tau], 0)]\pi_{x_1,q|2} + \sum_{r=p}^{P} E[X_{2,\tau}|([x_1, \tau; r, \tau], 0)])\pi_{0,r|2}$$

$$+ ((1 - (\pi_{0,0|1})^{\tau})v(0, 0) + (\pi_{0,0|1})^{\tau} v(0, 0))\pi_{0,0|2}$$

$$= -h(2\tau - (\tau - 1) - (\tau - 1)) + E[X_{1,\tau}|([x_1, \tau - 1; x_2, \tau - 1], 2)] + E[X_{2,\tau}|([x_1, \tau - 1; 0, \tau - 1], 2)]$$

$$+ v(0, 0)\pi_{0,0|2}$$

$$= R_1([x_1, \tau - 1; 0, \tau - 1], 2)$$

Where the second equality holds due to case 1b) above, and the fourth equality holds due to (2.28).

Assume the result holds for $y_2 = \tau - (l-1), \tau - (l-2), \ldots, \tau - 1$. Let $y_2 = \tau - l$ then

$V([x_1, y_1; x_2, y_1], z) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau; r, \tau - (l-1)], 1)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} R_1([q, \tau; r, \tau - (l-1)], 1)\pi_{x_1,q|1}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} (-h(2\tau - \tau - (\tau - (l-1))) + E[X_{1,\tau}|([q, \tau; r, \tau - (l-1)], 1)]$$

$$+ E[X_{2,\tau}|([q, \tau; r, \tau - (l-1)], 1)] + (\pi_{r,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))-1}v(0,0))\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -h(2\tau - (\tau - 1) - (\tau - l)) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau; x_2, \tau - (l-1)], 1)]\pi_{x_1,q|2}$$

$$+ \sum_{r=x_2}^{P} E[X_{2,\tau}|([x_1, \tau; r, \tau - (l-1)], 1)]\pi_{x_2,r|2} + (\pi_{x_2,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))}v(0,0)$$

$$= -h(2\tau - (\tau - 1) - (\tau - 1)) + E[X_{1,\tau}|([x_1, \tau - 1; x_2, \tau - l], 2)] + E[X_{2,\tau}|([x_1, \tau - 1; q, \tau - l], 2)]$$

$$+ (\pi_{x_2,0|1})(\pi_{0,0|1})^{\tau-(\tau-(l-1))}v(0,0)$$

$$= R_1([x_1, \tau - 1; x_2, \tau - l], 2)$$

Where the second equality holds due to the case above with $y_1 = \tau$, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{x_2,r|z} = 0$ for $r < x_2$.

Therefore Lemma 2.20 holds for the case 1c) $x_1 > 0$, $y_1 = \tau - 1$, $y_2 < \tau$, and $z = 2$.

1d) Let $y_1 < \tau - 1$. Proof by backward induction on y_1. Let $x_1 > 0$, $y_1 = \tau - 2$, and $z = 2$, then $V([x_1, y_1; x_2, y_2], 2) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau - 1; r, y_2 + 1], 2)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} R_1([q, \tau - 1; r, y_2 + 1], 2)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} (-h(2\tau - (\tau - 1) - (y_2 + 1)) + E[X_{1,\tau}|([q, \tau - 1; r, y_2 + 1], 2)]$$

$$+ E[X_{2,\tau}|([q, \tau - 1; r, y_2 + 1], 2)] + (\pi_{r,0|2})(\pi_{0,0|1})^{\tau-1-y_2+1}v(0, 0)\ )\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -h(2\tau - (\tau - 2) - y_2) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau - 1; x_2, y_2 + 1], 2)]\pi_{x_1,q|2}$$

$$+ \sum_{r=x_2}^{P} E[X_{2,\tau}|([x_1, \tau - 1; r, y_2 + 1], 2)]\pi_{x_2,r|2} + (\pi_{x_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1-y_2+1}v(0, 0)$$

$$= -h(2\tau - (\tau - 2) - y_2) + E[X_{1,\tau}|([x_1, \tau - 2; x_2, y_2], 2)] + E[X_{2,\tau}|([x_1, \tau - 2; x_2, y_2], 2)]$$

$$+ (\pi_{x_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1-y_2+1}v(0, 0)$$

$$= R_1([x_1, \tau - 2; x_2, y_2], 2)$$

Where the second equality holds from case 1c) above, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{x_2,r|z} = 0$ for $r < x_2$.

Assume the result holds for $y_1 = \tau-(l-1), \tau-(l-2), \ldots, \tau-2$. Let $y_1 = \tau - l$, $x_1 > 0$, and $z = 2$, then $V([x_1, y_1; x_2, y_2], 2) =$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} V([q, \tau-(l-1); r, y_2+1], 2)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} R_1([q, \tau-(l-1); r, y_2+1], 2)\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=x_1}^{P} \sum_{r=x_2}^{P} (-h(2\tau-(\tau-(l-1))-(y_2+1)) + E[X_{1,\tau}|([q, \tau-(l-1); r, y_2+1], 2)]$$

$$+ E[X_{2,\tau}|([q, \tau-(l-1); r, y_2+1], 2)] + (\pi_{r,0|2})(\pi_{0,0|1})^{\tau-(l-1)-y_2+1}v(0,0)\ )\pi_{x_1,q|2}\pi_{x_2,r|2}$$

$$= -h(2\tau-(\tau-l)-y_2) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau-(l-1); x_2, y_2+1], 2)]\pi_{x_1,q|2}$$

$$+ \sum_{r=x_2}^{P} E[X_{2,\tau}|([x_1, \tau-(l-1); r, y_2+1], 2)]\pi_{x_2,r|2} + (\pi_{x_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-(l-1)-y_2+1}v(0,0)$$

$$= -h(2\tau-(\tau-l)-y_2) + E[X_{1,\tau}|([x_1, \tau-l; x_2, y_2], 2)] + E[X_{2,\tau}|([x_1, \tau-l; x_2, y_2], 2)]$$

$$+ (\pi_{x_2,0|2})(\pi_{0,0|2})(\pi_{0,0|1})^{\tau-l-y_2+1}v(0,0)$$

$$= R_1([x_1, \tau-l; x_2, y_2], 2)$$

where the second equality holds from the induction hypothesis, and the third equality holds due to that each auction progress independently of the price in the other auction and that $\pi_{x_2,r|z} = 0$ for $r < x_2$.

Therefore Lemma 2.20 holds for the case 1d) $x_1 > 0$, $y_1 < \tau-1$, $y_2 < \tau$, and $z = 2$.

And consequently Lemma 2.20 holds for 1) $x_1 > 0, y_1 = \tau, \delta$ and $z = 0, 1$, or $x_1 > 0, y_1, y_2 < \tau$ and $z = 2$.

It remains to show that Lemma 2.20 also holds for 2) $x_1 > 0, y_1 < \tau, y_2 = 0, z = 1$. Proof by backward induction on y_1. Let $y_1 = \tau - 1$ and $z = 1$, then

$-2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} V([q, \tau; r, 1], 1)\pi_{x_1,q|2}\pi_{0,r|2} =$

$$
\begin{aligned}
&= -2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} R_1([q, \tau; r, 1], 1)\pi_{x_1,q|2}\pi_{0,r|2} \\
&= -2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} \{-h(2\tau - \tau - 1) + E[X_{1,\tau}|([q, \tau; r, 1], 1)] + E[X_{2,\tau}|([q, \tau; r, 1], 1)] + (\pi_{r,0|1})^{\tau-1} v(0, 0)\}\pi_{x_1,q|2}\pi_{0,r|2} \\
&= -h(2\tau - (\tau - 1) - 0) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau; 0, 1], 1)]\pi_{x_1,q|2} + \sum_{r=0}^{P} (E[X_{2,\tau}|([x_1, \tau; r, 1], 1)] + (\pi_{r,0|1})^{\tau-1} v(0, 0))\pi_{0,r|2} \\
&= -h(2\tau - (\tau - 1) - 0) + E[X_{1,\tau}|([x_1, \tau - 1; 0, 0], 2)] + E[X_{2,\tau}|([x_1, \tau - 1; 0, 0], 2)] + (\pi_{0,0|2})(\pi_{0,0|1})^{\tau-1} v(0, 0) \\
&= R([x_1, \tau - 1; 0, 0], 2)
\end{aligned}
$$

where the first equality holds due to case 1) above, and the second equality holds due to that each auction progress independently of the price in the other auction. Assume the result holds for $y_1 = \tau - (l - 1), \tau - (l - 2), \ldots, \tau - 1$. Let $y_1 = \tau - l$ and $z = 1$, then

$-2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} V_{t+1}([q, \tau - (l - 1); r, 1], 2)\pi_{x_1,q|2}\pi_{0,r|2} =$

$$
\begin{aligned}
&= -2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} R_1([q, \tau - (l - 1); r, 1], 2)\pi_{x_1,q|2}\pi_{0,r|2} \\
&= -2h + \sum_{q=x_1}^{P} \sum_{r=0}^{P} \{-h(2\tau - (\tau - (l - 1)) - 1) + E[X_{1,\tau}|([q, \tau - (l - 1); r, 1], 2)] \\
&\qquad\qquad + E[X_{2,\tau}|([q, \tau - (l - 1); r, 1], 2)] + \pi_2(0|[q, \tau - (l - 1); r, 1], 2) v(0, 0)\ \}\pi_{x_1,q|2}\pi_{0,r|2} \\
&= -h(2\tau - (\tau - l) - 0) + \sum_{q=x_1}^{P} E[X_{1,\tau}|([q, \tau - (l - 1); 0, 1], 2)]\pi_{x_1,q|2} \\
&\qquad + \sum_{r=0}^{P} E[X_{2,\tau}|([x_1, \tau - (l - 1); r, 1], 2)]\pi_{0,r|2} + (\pi_{0,0|2})^{l-1}(\pi_{0,0|1})^{\tau-(l-1)} v(0, 0) \\
&= -h(2\tau - (\tau - l) - 0) + E[X_{1,\tau}|([x_1, \tau - l; 0, 0], 2)] + E[X_{2,\tau}|([x_1, \tau - l; 0, 0], 2)] + (\pi_{0,0|2})^{l-1}(\pi_{0,0|1})^{\tau-(l-1)} v(0, 0) \\
&= R([x_1, \tau - l; 0, 0], 2)
\end{aligned}
$$

where the first equality holds due to case 1) above, and the second equality holds due to that each auction progress independently of the price in the other auction. Therefore

Lemma 2.20 holds for the case 2) $x_1 > 0, y_1 < \tau, y_2 = 0$ and $z = 1$. $\square$

Proof of Lemma 2.21 -

Comment: Recall that

$$R_2([x_1,y_1;x_2,y_2],z) = -h(\tau-y_1) + E[X_{1,\tau}|([x_1,y_1;x_2,y_2],z)]$$
$$+ (1-(\pi_{x_1,0|2})^{\tau-y_1})(-h(y_1-y_2) + E[X_{2,\tau}|([x_1,y_1;x_2,y_2],z)])$$
$$+ (\pi_{x_1,0|2})^{\tau-y_1} E[V([X_2, y_2+\tau-y_1;0,0],1)|([x_1,y_1;x_2,y_2],2)]$$

Let $y_1 = \tau - 1$, $x_1 = 0$, $x_2 > 0$ and $z = 2$. Proof by backward induction on y_2. Let $y_2 = \tau - 1$, then $V([x_1,y_1;x_2,y_2],z) =$

$$= -2h + \sum_{r=x_2}^{P}(\sum_{q=p}^{P} V([q,\tau;r,\tau],0)\pi_{0,q|2} + \pi_{0,0|2}V([r,\tau;0,0],1))\pi_{x_2,r|2}$$
$$= -2h + \sum_{r=x_2}^{P}\sum_{q=p}^{P} R_1([q,\tau;r,\tau],0)\pi_{0,q|2}\pi_{x_2,r|2} + \pi_{0,0|2}\sum_{r=x_2}^{P} V([r,\tau;0,0],1)\pi_{x_2,r|2}$$
$$= -2h + \sum_{r=x_2}^{P}\sum_{q=p}^{P}(E[X_{1,\tau}|([q,\tau;r,\tau],0)]) + E[X_{2,\tau}|([q,\tau;r,\tau],0)]))\pi_{0,q|2}\pi_{x_2,r|2}$$
$$+ \pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;x_2,\tau-1],2)]$$
$$= -2h + \sum_{r=x_2}^{P}\sum_{q=p}^{P} q\pi_{0,q|2}\pi_{x_2,r|2} + \sum_{r=x_2}^{P}\sum_{q=p}^{P} r\pi_{0,q|2}\pi_{x_2,r|2} + \pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;x_2,\tau-1],2)]$$
$$= -2h + \sum_{q=p}^{P} q\pi_{0,q|2} + \sum_{q=p}^{P}\pi_{0,q|2}\sum_{r=x_2}^{P} r\pi_{x_2,r|2} + \pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;x_2,\tau-1],2)]$$
$$= -2h + E[X_{1,\tau}|([0,\tau-1;x_2,\tau-1],2)] + (1-\pi_{0,0|2})E[X_{1,\tau}|([x_1,\tau-1;x_2,\tau-1],2)]$$
$$+ \pi_{0,0|2}E[V([X_2,\tau;0,0],1)|([0,\tau-1;x_2,\tau-1],2)]$$
$$= R_2([0,\tau-1;x_2,\tau-1],2)$$

Where the second equality holds due to Lemma 2.20 above, the fifth equality holds because $\sum_{r=x_2}^{P}\pi_{x_2,r|2} = 1$, and the sixth equality holds because $\sum_{q=p}^{P}\pi_{0,q|2} = 1 - \pi_{0,0|2}$.

Assume the result holds for $y_2 = \tau - (l-1), \tau - (l-2), \ldots, \tau - 1$. Let $y_2 = \tau - l$, then

$$V([x_1, y_1; x_2, y_2], z) =$$

$$= -2h + \sum_{r=x_2}^{P} (\sum_{q=p}^{P} V([q, \tau; r, y_2+1], 1)\pi_{0,q|2} + \pi_{0,0|2} V([r, y_2+1; 0, 0], 1))\pi_{x_2,r|2}$$

$$= -2h + \sum_{r=x_2}^{P} \sum_{q=p}^{P} R_1([q, \tau; r, y_2+1], 1)\pi_{0,q|2}\pi_{x_2,r|2} + \pi_{0,0|2} \sum_{r=x_2}^{P} V([r, y_2+1; 0, 0], 1)\pi_{x_2,r|2}$$

$$= -2h$$

$$+ \sum_{r=x_2}^{P} \sum_{q=p}^{P} \{-h(2\tau - \tau - (y_2+1) + E[X_{1,\tau}|([q, \tau; r, y_2+1], 1)]) + E[X_{2,\tau}|([q, \tau; r, y_2+1], 0)])\}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$+ \pi_{0,0|2} E[V([X_2, y_2+1; 0, 0], 1)|([0, \tau-1; x_2, y_2], 2)]$$

$$= -2h + E[X_{1,\tau}|([0, \tau-1; x_2, y_2], 2)]) + \sum_{q=p}^{P} \{-h(2\tau - \tau - (y_2+1) + E[X_{2,\tau}|([0, \tau-1; x_2, y_2], 2)])\}\pi_{0,q|2}$$

$$+ \pi_{0,0|2} E[V([X_2, y_2+1; 0, 0], 1)|([0, \tau-1; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau-1)) + E[X_{1,\tau}|([0, \tau-1; x_2, y_2], 2)]) + (1 - \pi_{0,0|2})(-h(\tau - 1 - y_2)$$

$$+ E[X_{2,\tau}|([0, \tau-1; x_2, y_2], 2)]) + \pi_{0,0|2} E[V([X_2, y_2+1; 0, 0], 1)|([0, \tau-1; x_2, y_2], 2)]$$

$$= R_2([0, \tau-1; x_2, y_2], 2)$$

Where the second equality holds due to Lemma 2.20 above, the fourth equality holds because $\sum_{r=x_2}^{P} \pi_{x_2,r|2} = 1$, and the fifth equality holds because $\sum_{q=p}^{P} \pi_{0,q|2} = 1 - \pi_{0,0|2}$. Therefore Lemma 2.21 holds for $y_1 = \tau - 1, x_1 = 0, x_2 > 0$, and $z = 2$.

For $y_1 < \tau - 1, x_1 = 0, x_2 > 0, z = 2$, proof by backward induction on y_1. Let $y_1 = \tau - 2$, then $V([x_1, y_1; x_2, y_2], z) =$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} V([q, \tau - 1; r, y_2 + 1], 2)\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} R_2([q, \tau - 1; r, y_2 + 1], 2)\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} \{-2h(\tau - (\tau - 1)) + E[X_{1,\tau}|([q, \tau - 1; r, y_2 + 1], 2)]$$

$$+ (1 - (\pi_{q,0|2})^{\tau-(\tau-1)})(-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - 1; r, y_2 + 1], 2)])$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-1)}(E[V([X_2, y_2 + 1 + \tau - (\tau - 1); 0, 0], 1)|([q, \tau - 1; r, y_2 + 1], 2)]) \}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h(\tau - (\tau - 2))) + E[X_{1,\tau}|([0, \tau - 2; x_2, y_2], 2)]$$

$$+ \sum_{q=0}^{P} \sum_{r=x_2}^{P} \{(1 - (\pi_{q,0|2})^{\tau-(\tau-1)})(-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - 1; r, y_2 + 1], 2)]) \}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$+ \pi_{0,0|2}(\pi_{0,0|2})^{\tau-(\tau-1)} \sum_{r=x_2}^{P} E[V([X_2, y_2 + 1 + \tau - (\tau - 1); 0, 0], 1)|([0, \tau - 1; r, y_2 + 1], 2)]\pi_{x_2,r|2}$$

$$= -2h(\tau - (\tau - 2)) + E[X_{1,\tau}|([0, \tau - 2; x_2, y_2], 2)]$$

$$+ \sum_{q=p}^{P} \sum_{r=x_2}^{P} \{-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - 1; r, y_2 + 1], 2)])\}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$+ \sum_{r=x_2}^{P} \{(1 - (\pi_{0,0|2})^{\tau-(\tau-1)})(-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([0, \tau - 1; r, y_2 + 1], 2)])\}\pi_{0,0|2}\pi_{x_2,r|2}$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2, y_2 + \tau - (\tau - 2); 0, 0], 1)|([0, \tau - 2; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau - 2)) + E[X_{1,\tau}|([0, \tau - 2; x_2, y_2], 2)]$$

$$+ (1 - \pi_{0,0|2})(-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([0, \tau - 2; x_2, y_2], 2)])$$

$$+ \pi_{0,0|2}(1 - (\pi_{0,0|2})^{\tau-(\tau-1)})(-h(\tau - 1 - (y_2 + 1)) + E[X_{2,\tau}|([0, \tau - 2; x_2, y_2], 2)])$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2, y_2 + \tau - (\tau - 2); 0, 0], 1)|([0, \tau - 2; x_2, y_2], 2)]$$

$$
\begin{aligned}
=& -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;x_2,y_2],2)] \\
& + (1-\pi_{0,0|2})(-h(\tau-2-y_2) + E[X_{2,\tau}|([0,\tau-2;x_2,y_2],2)]) \\
& + \pi_{0,0|2}(1-\pi_{0,0|2})(-h(\tau-2-y_2) + E[X_{2,\tau}|([0,\tau-2;x_2,y_2],2)]) \\
& + (\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,y_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;x_2,y_2],2)] \\
=& -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;x_2,y_2],2)] \\
& + (1+\pi_{0,0|2})(1-\pi_{0,0|2})(-h(\tau-2-y_2) + E[X_{2,\tau}|([0,\tau-2;x_2,y_2],2)]) \\
& + (\pi_{q,0|2})^{\tau-(\tau-2)}(E[V([X_2,y_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;x_2,y_2],2)]) \\
=& -2h(\tau-(\tau-2)) + E[X_{1,\tau}|([0,\tau-2;x_2,y_2],2)] \\
& (1-\pi_{0,0|2})^{\tau-(\tau-2)})(-h(\tau-2-y_2) + E[X_{2,\tau}|([0,\tau-2;x_2,y_2],2)]) \\
& + (\pi_{q,0|2})^{\tau-(\tau-2)}E[V([X_2,y_2+\tau-(\tau-2);0,0],1)|([0,\tau-2;x_2,y_2],2)] \\
=& R_2([0,\tau-2;x_2,y_2],2)
\end{aligned}
$$

Where the second equality holds due to the case above with $y_1 = \tau - 1$, the fourth equality holds because the expected final price of the first auction is independent of the price in the second auction and that $\pi_{q,0|2} = 0$ for $q > 0$, the fifth equality holds because $\pi_{q,0|2} = 0$ for $q > 0$ and the second auction progress independently of the price in the first auction, and the sixth equality holds due to that $\sum_{q=p}^{P} \pi_{0,q|2} = 1 - \pi_{0,0|2}$ and that the second auction progress independently of the price in the first auction.

Assume the result holds for $y_1 = \tau - (l-1), \tau - (l-2), \ldots, \tau - 2$. Let $y_1 = \tau - l$, then

$V([x_1, y_1; x_2, y_2], z) =$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} V([q, \tau - (l-1); r, y_2 + 1], 2)\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} R_2([q, \tau - (l-1); r, y_2 + 1], 2)\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h + \sum_{q=0}^{P} \sum_{r=x_2}^{P} \{-2h(\tau - (\tau - (l-1))) + E[X_{1,\tau}|([q, \tau - (l-1); r, y_2 + 1], 2)]$$

$$+ (1 - (\pi_{q,0|2})^{\tau-(\tau-(l-1))})(-h(\tau - (l-1) - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - (l-1); r, y_2 + 1], 2)])$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-(l-1))}(E[V([X_2, y_2 + 1 + \tau - (\tau - (l-1)); 0, 0], 1)|([q, \tau - (l-1); r, y_2 + 1], 2)])\}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; x_2, y_2], 2)]$$

$$+ \sum_{q=0}^{P} \sum_{r=x_2}^{P} \{(1 - (\pi_{q,0|2})^{\tau-(\tau-(l-1))})(-h(\tau - (l-1) - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - (l-1); r, y_2 + 1], 2)])\}\pi_{0,q|2}\pi_{x_2,r|2}$$

$$+ \pi_{0,0|2}(\pi_{0,0|2})^{\tau-(\tau-(l-1))} \sum_{r=x_2}^{P} \{E[V([X_2, y_2 + 1 + \tau - (\tau - (l-1)); 0, 0], 1)|([0, \tau - (l-1); r, y_2 + 1], 2)]\}\pi_{x_2,r|2}$$

$$= -2h(\tau - (\tau - (l-1))) + E[X_{1,\tau}|([0, \tau - (l-1); x_2, y_2], 2)]$$

$$+ \sum_{q=p}^{P} \sum_{r=x_2}^{P} (-h(\tau - (l-1) - (y_2 + 1)) + E[X_{2,\tau}|([q, \tau - (l-1); r, y_2 + 1], 2)])\pi_{0,q|2}\pi_{x_2,r|2}$$

$$+ \sum_{r=x_2}^{P} \{(1 - (\pi_{0,0|2})^{\tau-(\tau-(l-1))})(-h(\tau - (l-1) - (y_2 + 1)) + E[X_{2,\tau}|([0, \tau - (l-1); r, y_2 + 1], 2)])\}\pi_{0,0|2}\pi_{x_2,r|2}$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, y_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; x_2, y_2], 2)] + (1 - \pi_{0,0|2})(-h(\tau - (l-1) - (y_2 + 1))$$

$$+ E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)]) + \pi_{0,0|2}(1 - (\pi_{0,0|2})^{\tau-(\tau-(l-1))})(-h(\tau - (l-1) - (y_2 + 1))$$

$$+ E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)]) + (\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, y_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau - l))) + E[X_{1,\tau}|([0, \tau - l; x_2, y_2], 2)] + (1 - \pi_{0,0|2})(-h(\tau - l - y_2)$$

$$+ E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)]) + \pi_{0,0|2}(1 - \pi_{0,0|2})(-h(\tau - l - y_2)$$

$$+ E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)]) + (\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, y_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; x_2, y_2], 2)]$$

$$+ (1 + \pi_{0,0|2})(1 - \pi_{0,0|2})(-h(\tau - l - y_2) + E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)])$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, y_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; x_2, y_2], 2)]$$

$$= -2h(\tau - (\tau - l)) + E[X_{1,\tau}|([0, \tau - l; x_2, y_2], 2)]$$

$$(1 - \pi_{0,0|2})^{\tau-(\tau-l)})(-h(\tau - l - y_2) + E[X_{2,\tau}|([0, \tau - l; x_2, y_2], 2)])$$

$$+ (\pi_{q,0|2})^{\tau-(\tau-l)} E[V([X_2, y_2 + \tau - (\tau - l); 0, 0], 1)|([0, \tau - l; x_2, y_2], 2)]$$

$$= R_2([0, \tau - l; x_2, y_2], 2)$$

Where the second equality holds due to the induction hypothesis, and the other equalities due to the same reasoning as above. Therefore Lemma 2.21 holds for $y_1 < \tau - 1, x_1 = 0, x_2 > 0$, and $z = 2$. $\square$

APPENDIX B

Proofs for Results to Minimal Bid-Increment Strategy in Chapter 3

Proof Lemma 3.1 - Due to the law of total probability $F_1^1(q|0) = \sum_{m=0}^{\infty} \Pr\{X_{1,t+\Delta t} \leq q | X_{1,t} = 0, Z_{\Delta t} = 1, M_{\Delta t} = m\} \rho_M(m|\Delta t)$, holds. Next we prove the various cases regarding $\Pr\{X_{1,t+\Delta t} \leq q | X_{1,t} = 0, Z_{\Delta t} = 1, M_{\Delta t} = m\}$. If no bidders arrive, $m = 0$, then the result holds trivially. If more than one bidder arrives, $m \geq 1$, then since for each bidder $V \geq p$, it is impossible that $X_{1,t_1+\Delta t} < p$. For $q \geq p$, the only possibility that $X_{1,t_1+\Delta t} \leq q$ is false is if more than two bidders bid above q. Since, a bidder will only stop bidding once the auction exceeds his valuation, this means that if two or more bidders with valuation above the threshold $q \geq p$ arrive, then $X_{1,t_1+\Delta t} > q$. The probability that all m bidders have $V \leq q$ is given by $(G(q))^m$, and the probability that exactly one bidder has $V > q$ and the remaining $m-1$ bidders have $V \leq q$, is given by $m(G(q))^{m-1}(1-G(q))$. $\square$

Proof Lemma 3.2 - The proof is based on the same principles as in the proof of Lemma 3.1. However, since there is a high-bidder present, who will continue to counter-bid any bidder that arrives in the interval $[t, t + \Delta t]$, until the price in the auction exceeds his valuation, we condition on a the high-bidder's valuation V and derive the CDF of $X_{1,t_1+\Delta t}$ given that $V = v$. That is, we derive the CDF for $X_{1,t_1+\Delta t}$ given the valuation v, number of arriving bidders m, and the threshold q. If there no bidders arrive, $m = 0$, then the result for $\Pr\{X_{1,t_1+\Delta t} \leq q | X_{1,t_1} = x, V_{(1)} = v, Z_{\Delta t} = 1, M_{\Delta t} = m\}$ holds trivially. If only one bidder arrives, $m = 1$, and $x < q \leq v$, then the only possibility that $X_{1,t_1+\Delta t} \leq q$, is if the arriving bidder's valuation is $\leq q$. This holds since the present high-bidder will continue to bid to his valuation which is above x. The probability of this event is $G(q)$. If $m = 1$ and $v \leq q$, then no matter what the arriving bidder's valuation is, the price cannot exceed q, since it will at most end at v (the amount high-bidder will stop bidding at). If more than one bidder arrives, $m \geq 2$, then based on the same logic as before, if $x \leq q < v$, then in order for $X_{1,t_1+\Delta t} \leq q$, all the arriving bidders must bid less than q. Therefore, since each bidder will bid up to his valuation before leaving, the probability that no one bids above q is given by $(G(q))^m$. On the other hand, for $v \leq q$, the only possibility that $X_{t+\Delta t} \leq q$, is if either all bidders have valuation $\leq q$, or at most one bidder has a valuation above q and the remaining $m-1$ are below q. The probability of this event is $(G(q))^m + m(G(q))^{m-1}(1-G(q))$. □

Proof Lemma 3.3 - The proof is based on the same reasoning as in the Proof of Lemma 3.1. If no bidders arrive, $m = 0$, then the result holds trivially. If one bidder arrives, $m = 1$, then he chooses an auction with equal probability, and hence the result holds. If two or more bidders arrive, $m \geq 2$, then since the bidder with the third highest valuation will continue to raise the price until both auctions exceed this price, this will leave the two bidders with the highest valuation as high-bidder in each of the auctions. Therefore, for $q \geq p$, the only possibility that $X_{i,t_i+\Delta t} \geq q$, is if at least three bidders with $V \geq$ arrive. The probability that, for $q \geq p$, at most two bidders with $V \geq q$ arrive, is if either all arriving bidders have $V \leq q$, or at most one bidder has $V > q$ and the remaining $m-1$ have $V \leq q$, or at most two bidders have $V \geq q$ and the remaining $m-2$ bidders have $V \leq q$. The probability of this is given by the expression written. □

Proof Lemma 3.4 - The proof is based on the same reasoning as in the previous proofs. The only difference with the case given in Lemma 3.3, is that we first condition on the possible valuation that the present high-bidder may have. Therefore, in the case of two or more arriving bidders, $m \geq 2$, there are three cases with respect to the threshold q.

For a given valuation v of the present high-bidder, if $q \leq v$, then there can at most be one arriving bidders which has valuation $\geq v$. Because if two ore more bidders arrive with $V \geq q$, then the price in both auctions would exceed the threshold. However, similar to the previous case if the threshold $q \geq v$, i.e. above the valuation of the current high-bidder, then $X_{i,t_i+\Delta t} \geq q$, as long as at most two of the arriving bidders' valuation is $\leq q$. $\square$

Proof Lemma 3.5 - The final case we consider, with the two auctions released simultaneously, is when there are two high-bidders present (one for each auction). By extending the logic for the one high-bidder case, we condition on the valuation of both bidders. The reasoning is identical to the previous proofs, and we condition on the two high-bidders' valuation v_1 and v_2, the number of arriving bidders m, and the threshold q. If no bidders arrive, $m = 0$, the result holds trivially. If only one bidder arrives, $m = 1$, and the threshold is less than the high-bidder with the lowest valuation, $q \leq v_2$, then the only possibility that $X_{i,t_i+\Delta t} \leq q$, is if the arriving bidder's valuation $\leq q$. The probability of this event is $G(q)$. On the other hand if $q \geq v_2$, then regardless of the arriving bidder's valuation, the price in both auctions will at most reach v_2. If more than one bidder arrives, $m \geq 2$, then depending on the threshold q with respect to the valuations of the two high-bidders, v_1 and v_2, either all arriving bidders' valuation has to be $\leq q$, or at most one of the arriving bidders can have valuation $> q$ while the remaining $m-1$ arriving bidders have valuation $\leq q$, or at most two of the arriving bidders can have valuation $\geq q$ while the remaining $m-2$ arriving bidders have valuation $\leq q$. The probability of each of the cases for q is given by the expressions stated. $\square$

Proof Lemma 3.6 - Since bidders are assumed not to be time-sensitive and bid in an auction based on the elapsed auction time, and since it is assumed that auction 1 will not end before the end of the time-interval $[t, t + \Delta t]$, the bidding dynamics is identical to the case when auctions were released simultaneously. $\square$

Proof Lemma 3.7 - In this case when auction 1 has elapsed for some time and reached a price $x > p$, the dynamics for the two auctions are different while the price in auction 2 is $< x$. Recall that we assume all arriving bidders will always bid in the lowest priced auction. Consequently, as long as auction 2 is priced $< x$, all bidding will take place in auction 2. Note too, that the present high-bidder in auction 1, has committed to a bid of $x + k$ in auction 1, and can only observe auction 2 but not retract his own bid and place a lower bid in auction 2. As a result, to calculate the CDF for auction 1, we condition on the valuation of the present high-bidder and as with the previous cases, derive the conditional CDF depending on the number of arriving bidders m and threshold q. For $m \geq 2$ and $x \leq q \leq v_1$, if at most one of the arriving bidders has valuation $\geq q$, then $X_{1,t_1+\Delta t} \leq q$. In this case, the present high-bidder and the arriving bidder with the highest valuation, each become the high-bidder in the two auctions, at the price in the two auctions will equal the second highest valuation among the arriving bidders. If $v_1 < q$, then in order for $X_{1,t_1+\Delta t} \leq q$, at most two of the arriving bidders valuation can exceed q.

For auction 2, the CDF is different depending on the threshold q. If we are interested to know the probability that, for $q < x$, $X_{2,t_2+\Delta t} \leq q$, then there is no need to condition on the valuation of the present high-bidder in auction 1. The calculations for the CDF is identical to the case when there is only one ongoing auction priced at 0. For threshold $q > x$, the same reasoning as with auction 1 apply. We first condition on the valuation of the high-bidder in auction 1, and then derive the conditional CDF given the valuation v_1, number of arriving bidders m, and threshold q. $\square$

Proof Lemma 3.8 - The last possible case is if auction 2 was started after auction 1, and has received bids but has not yet reached the same price as auction 1. Note that once auction 2 reaches the price in auction 1, x_1, then thereafter the auctions will be priced equally after each arriving bidder. This holds since each bidder will remain until the price in both auctions exceed his valuation. In order to derive the conditional CDF for each auction, $F_i^2(q|x_i)$, $i = 1, 2$, we first condition on the two possibilities regarding the valuation of the high-bidder with the second highest valuation. Note that we do not know which of the high-bidder that has the highest valuation. It is not necessarily the high-bidder in auction 1 that has the highest valuation (though it is of course more likely, since the price is higher in auction 1). However, given that the present high-bidder in auction 2 has a valuation $\leq x_1$ (since if $V_{(2)} < x_1$, then it can only be the high-bidder in auction 2), then the calculations for the CDF of $X_{1,t_1+\Delta t}$ follows the same arguments as before. And similarly the calculations for the CDF of $X_{2,t_2+\Delta t}$ follows the procedure as described in the proof of Lemma 3.5. If on the other hand, $V_{(2)} > x_1$ then based on the same argument as in the previous proofs, we condition on the valuations of the two present high-bidders v_1 and v_2, the number of arriving bidders m, and the threshold q that is of interest. For the various scenarios, either none, at most one, or at most two of the arriving bidders may have valuation $\geq q$. $\square$

APPENDIX C

Supplement to Truthful Bidding Strategy in Chapter 3

This appendix will summarize the conditional distribution function of the within period price-transitions given than bidders follow the truthful bidding strategy. The derivations are based on the same approach as with the previous bidding strategy. Namely we condition upon the possible high-bid in each auction and on the number of bidders that arrive in the time-interval $[t, t+\Delta t]$. Similar to the previous bidding strategy we consider the transitions when there is one and two ongoing auctions separately.

One Ongoing Auction

When there is only one ongoing auction the within period price transactions are identical to the previous bidding strategy. Namely after each bidder arrives to the auction site, the price in the auction is the second highest valuation of all bidders that has visited so far. Therefore, see Section 3.1 in Chapter 3 for the resulting distribution function.

Two Ongoing Auctions

When there are two ongoing auctions, due to that there are cases when the two bidders with the highest valuation end up bidding against each other, there are some additional possible states that were not possible with the previous bidding strategy. This results in that the transitions are a bit more complicated, and instead of closed form solution to the conditional CDF, lower and upper bounds are provided. Similar to the previous discussion there are two cases regarding when the two auctions were started.

Case 1: Auctions Started Simultaneously

Since the auctions are released simultaneously we, for simplicity of notation, omit the elapsed time of each auction, and write $[X_1, H_1; X_2, H_2]$ instead of $[X_1, H_1, t_1; X_2, H_2, t_2]$. Recall that H_i is censored, $i = 1, 2$. We define B_j as the amount of the j^{th} bid, $j = 1, 2, \ldots$. If both auctions are started at the same time then the following transitions are possible,

$$[0,0;0,0] \overset{bidder1}{\longrightarrow} \begin{cases} [p, V_{(1)}; 0, 0] \\ \\ [0, 0; p, V_{(1)}] \end{cases}$$

$$[p, V_{(1)}; 0, 0] \ ([0, 0; p, V_{(1)}]) \overset{bidder2}{\longrightarrow} \begin{cases} [p, V_{(1)}; p, V_{(2)}] & \{B_2 \leq B_1\} \ (\{B_1 < B_2\}) \\ \\ [p, V_{(2)}; p, V_{(1)}] & \{B_1 < B_2\} \ (\{B_2 \leq B_1\}) \end{cases}$$

$$[p, V_{(1)}; p, V_{(2)}] \overset{bidder3}{\longrightarrow} \begin{cases} [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{B_3 \leq B_2\} \text{ or } .5\{B_2 < B_3 \leq B_1\} \\ \\ [V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}] & .5\{B_2 < B_3 \leq B_1\} \text{ or } .5\{B_1 < B_3\} \\ \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & .5\{B_1 < B_3\} \end{cases}$$

$$[p, V_{(2)}; p, V_{(1)}] \overset{bidder3}{\longrightarrow} \begin{cases} [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{B_3 \leq B_2\} \text{ or } .5\{B_2 < B_3 \leq B_1\} \\ \\ [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] & .5\{B_2 < B_3 \leq B_1\} \text{ or } .5\{B_1 < B_3\} \\ \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & .5\{B_1 < B_3\} \end{cases}$$

$$[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] \overset{bidder4}{\longrightarrow} \begin{cases} [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{B_4 \leq V_{(2)}\} \text{ or } .5\{V_{(2)} < B_4 \leq V_{(1)}\} \\ [V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}] & .5\{V_{(2)} < B_4 \leq V_{(1)}\} \text{ or } .5\{V_{(1)} < B_4\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & .5\{V_{(1)} < B_4\} \end{cases}$$

$$[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}] \overset{bidder4}{\longrightarrow} \begin{cases} [V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{B_4 \leq V_{(2)}\} \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{V_{(2)} < B_4 \leq V_{(1)}\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{V_{(1)} < B_4\} \end{cases}$$

$$[V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] \overset{bidder4}{\longrightarrow} \begin{cases} [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] & \{B_4 \leq V_{(2)}\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{V_{(2)} < B_4 \leq V_{(1)}\} \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{V_{(1)} < B_4\} \end{cases}$$

$$[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] \overset{bidder4}{\longrightarrow} \begin{cases} [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{B_4 \leq V_{(2)}\} \text{ or } .5\{V_{(2)} < B_4 \leq V_{(1)}\} \\ [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] & .5\{V_{(2)} < B_4 \leq V_{(1)}\} \text{ or } .5\{V_{(1)} < B_4\} \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & .5\{V_{(1)} < B_4\} \end{cases}$$

After four bidders the state of the auctions repeat, and the above dynamics generalizes to the n^{th} arriving bidder, $n > 2$. Therefore, if both auctions are started simultaneously and more than three bidders has arrived, then there are only four possible states,

(1) $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$

(2) $[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}]$

(3) $[V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}]$

(4) $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$

The objective is to determine the conditional distribution function of the within period price-transitions, i.e. the CDF for $X_{i,t_i+\Delta t}$, $i = 1, 2$, and therefore given a threshold q there are four cases in each of the four states above, which we label as a, b, c, and d,

	(1) $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$	(2) $[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}]$	(3) $[V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}]$	(4) $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$
(a)	$q \le V_{(3)} \le V_{(2)} \le V_{(1)}$	$q \le V_{(3)} \le V_{(2)} \le V_{(1)}$	$q \le V_{(3)} \le V_{(2)} \le V_{(1)}$	$q \le V_{(3)} \le V_{(2)} \le V_{(1)}$
(b)	$V_{(3)} < q \le V_{(2)} \le V_{(1)}$	$V_{(3)} < q \le V_{(2)} \le V_{(1)}$	$V_{(3)} < q \le V_{(2)} \le V_{(1)}$	$V_{(3)} < q \le V_{(2)} \le V_{(1)}$
(c)	$V_{(3)} \le V_{(2)} < q \le V_{(1)}$	$V_{(3)} \le V_{(2)} < q \le V_{(1)}$	$V_{(3)} \le V_{(2)} < q \le V_{(1)}$	$V_{(3)} \le V_{(2)} < q \le V_{(1)}$
(d)	$V_{(3)} \le V_{(2)} \le V_{(1)} < q$	$V_{(3)} \le V_{(2)} \le V_{(1)} < q$	$V_{(3)} \le V_{(2)} \le V_{(1)} < q$	$V_{(3)} \le V_{(2)} \le V_{(1)} < q$

Case (a) is when the three bidders with the highest valuations exceed the threshold q of interest. Case (b) is when only the two bidders with the highest place bids exceeding the threshold q. Case (c) is when only the bidder with the highest valuation ends up exceeding the threshold. And finally, case (d) is when none of the arriving bidders' valuation, and hence bids, exceed the threshold of interest. In the ensuing analysis upper (U) and lower (L) bounds on the CDF, $F_i^2(q|x_i)$, $i = 1, 2$, are provided. The lower bound will consist of those events where either all bidders are below the given threshold q, or at most the highest bidder exceeded the given threshold q, i.e. states 1c-4c and 1d-4d. The upper bound will be all events except when the three highest bidders exceeded the given threshold q, i.e. states 1b-4b, 1c-4c, and 1d-4d. The reason this is an upper bound is due to that there are instances when the two highest bidders end up bidding against each other, namely states 2b and 3b. Therefore, the true value of $F_1^2(q|x_i)$, would only capture the probability of ending up in states 1b,3b,4b, 1c-4c, and 1d-4d, while for $F_2^2(q|x_i)$, to only consider the probability of ending up in states 1b,2b,4b, 1c-4c, and 1d-4d. That is,to derive the CDF of $X_{i,t_i+\Delta t}$ we, given an initial state $[X_1, H_1, t_1; X_2, H_2, t_2]$ and given threshold q, derive the lower and upper bounds on the probability that $X_{i,t_i+\Delta t} \le q$. The lower bound consist of those cases when only the bidder with the highest valuation may exceed the threshold

q. While the upper bound also includes the cases when the two bidders with the highest valuation exceeds the threshold q.

Note that due to the specific bidding dynamics, the lower bound is overly conservative and that the true value is closer to the upper bound. In fact, a less conservative lower bound would be half the distance between the upper and lower bound discussed above. Similar to the previous bidding strategy we provide the distribution function given various starting states.

In all cases below we assume that the numbers of ongoing auctions remains fixed for the duration of the time-interval $[t, t + \Delta t]$.

State of Auctions: $[0, 0; 0, 0]$

If no bids have been placed so far then the CDF is, for $i = 1, 2$,

$$F_i^2(q|0) = \sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = 0, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t)$$

where

$$\Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = 0, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < 0 \\ 1 & 0 \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < 0 \\ .5 & 0 \leq q < p \\ 1 & p \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < p \\ 1 & p \leq q \end{cases} \qquad m = 2$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & p \leq q \end{cases} \qquad m \geq 3\ (L)$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & p \leq q \end{cases} \qquad m \geq 3\ (U)$$

State of Auctions: $[p, V_{(1)}; 0, 0]$ $([0, 0; p, V_{(1)}])$

If only one bid has been placed so far then the CDF is, for $[p, V_{(1)}; 0, 0]$, and $i = 1, 2$,

$$\begin{aligned}
F_i^2(q|x_i) &= \int_p^P \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2\} g(v_1) dv_1 \\
&= \int_p^P \sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) g(v_1) dv_1
\end{aligned}$$

where

$$\Pr\{X_{1,y+\Delta t} \leq q | X_{1,y} = x_1, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < p \\ 1 & p \leq q \end{cases} \qquad m = 0, 1$$

$$= \begin{cases} 0 & q < p \\ (G(q))^2 + 2(G(q))(1 - (G(q))) & p \leq q < v_1 \\ 1 & v_1 \leq q \end{cases} \qquad m = 2$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m & p \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & v_1 \leq q \end{cases} \qquad m \geq 3 \ (L)$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & p \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 3 \ (U)$$

and

$$\Pr\{X_{2,t+\Delta t} \le q | X_{2,t} = x_2, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < 0 \\ 1 & 0 \le q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < p \\ 1 & p \le q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < p \\ (G(q))^2 + 2(G(q))(1-(G(q))) & p \le q \end{cases} \qquad m = 2$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m & p \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_1 \le q \end{cases} \qquad m \ge 3 \ (L)$$

$$= \begin{cases} 0 & q < p \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & p \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \le q \end{cases} \qquad m \ge 3 \ (U)$$

By symmetry if the state is $[0, 0; p, V_{(1)}]$ then the transition probabilities are reversed.

State of Auctions: $[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}]$ $([V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}])$

If the prices are different in the two auctions, then we know what the value of the highest bid is in the lower priced auction. And the distribution function is, for $[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}]$, and $i = 1, 2$,

$$
\begin{aligned}
F_i^2(q|x_i) &= \int_{x_1}^{P} \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2\} g(v_1|x_1) dv_1 \\
&= \int_{x_1}^{P} \sum_{m=0}^{\infty} \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x_i, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\} \rho_M(m|\Delta t) g(v_1|x_1) dv_1
\end{aligned}
$$

where

$$
\Pr\{X_{1,y+\Delta t} \le q | X_{1,y} = x_1, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}
$$

$$
= \begin{cases} 0 & q < x_1 \\ 1 & x_1 \le q \end{cases} \qquad m = 0, 1
$$

$$
= \begin{cases} 0 & q < x_1 \\ (G(q))^m & x_1 \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & v_1 \le q \end{cases} \qquad m \ge 2 \ (L)
$$

$$
= \begin{cases} 0 & q < x_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) & x_1 \le q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1 - (G(q))) + \binom{m}{2}(G(q))^{m-2}(1 - (G(q)))^2 & v_1 \le q \end{cases} \qquad m \ge 2 \ (U)
$$

and

$$\Pr\{X_{2,t+\Delta t} \leq q | X_{2,t} = x_2, V_{(1)} = v_1, Z_{\Delta t} = 2, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < x_2 \\ 1 & x_2 \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x_2 \\ G(q) & x_2 \leq q < x_1 \\ 1 & x_1 \leq q \end{cases} \qquad m = 1$$

$$= \begin{cases} 0 & q < x_2 \\ (G(q))^m & x_2 \leq q < x_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & x_1 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_1 \leq q \end{cases} \qquad m \geq 2\ (L)$$

$$= \begin{cases} 0 & q < x_2 \\ (G(q))^m & x_2 \leq q < x_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & x_1 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \leq q \end{cases} \qquad m \geq 2\ (U)$$

By symmetry if $[V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}]$ then the functions for auction 1 and auction 2 are simply reversed.

State of Auctions: $[p, V_{(1)}; p, V_{(2)}]$, $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$, $[p, V_{(2)}; p, V_{(1)}]$, $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$ Unlike the previous cases the seller has the added uncertainty which state he is in. The seller does not know if the state is $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$ or $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$. However, given the bidding dynamics (that a bidder will with probability .5 choose an auction when they are equally priced), by symmetry, with probability .5 a seller is either in state $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$ or in state $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$. We summarize this result in the following proposition.

LEMMA C.1. *If the auctions were started simultaneously and* $X_{1,t_1} = X_{2,t_2}$*, then* $\Pr\{H_{1,t_1} > H_{2,t_2}\} = .5$. *In other words, within period price transitions are Markovian.*

Proof Lemma C.1 - Since bidders valuation are *independent and identically distributed*, it is equally likely that the second bid is greater than the first bid or less than the first bid. After this it follows by symmetry, that as long as after each arriving bidder the two auctions are equally priced, then it is equally likely that $\{H_{2,t_2} \leq H_{1,t_1}\}$ and $\{H_{2,t_2} > H_{1,t_1}\}$. Therefore, assume there is a time when the auctions are priced differently. Let $t + \delta t$ be the time when a bidder with valuation $V_{[bid]}$ arrives, and $X_{2,t_2+\delta t} = x_2 < x_1 = X_{1,t_1+\delta t}$, i.e. the state at time $t + \delta t$ is $[x_1, V_{(1)}; x_2, x_1]$. And that after the bidder has placed his bid(s) the two auctions are priced equally. Since $V_{(1)} > x_1$, $V_{[bid]} \geq x_1$ (since the arriving bidder leveled the auctions) and bidders' valuation is *iid*,

$$
\begin{aligned}
\Pr\{V_{[bid]} > V_{(1)} | x_1 \le V_{[bid]}\} = \frac{\Pr\{V_{[bid]} > V_{(1)}\}}{\Pr\{x_1 \le V_{[bid]}\}} &= \int_{x_1}^{P} \frac{1-G(x)}{1-G(x_1)} g(x|x_1)dx \\
&= \int_{x_1}^{P} \frac{1-G(x)}{1-G(x_1)} \frac{\partial}{\partial x} \frac{G(x)-G(x_1)}{1-G(x_1)} dx \\
&= \frac{1}{(1-G(x_1))^2} \int_{x_1}^{P} \frac{\partial}{\partial x} G(x)dx - \frac{1}{(1-G(x_1))^2} \int_{x_1}^{P} G(x) \frac{\partial}{\partial x} G(x)dx \\
&= \frac{1-G(x_1)}{(1-G(x_1))^2} - \frac{1}{(1-G(x_1))^2} \int_{x_1}^{P} G(x) \frac{\partial}{\partial x} G(x)dx \\
&= \frac{1}{1-G(x_1)} - \frac{(1-(G(x_1))^2)}{2(1-G(x_1))^2} \\
&= \frac{2(1-G(x_1)) - (1-(G(x_1))^2)}{2(1-G(x_1))^2} \\
&= \frac{1-2G(x_1) + (G(x_1))^2}{2(1-G(x_1))^2} = \frac{(1-G(x_1))^2}{2(1-G(x_1))^2} = \frac{1}{2}
\end{aligned}
$$

where the sixth equality holds due to that,

$$
\begin{aligned}
\int_{x_1}^{P} G(x) \frac{\partial}{\partial x} G(x)dx &= (G(x))^2|_{x_1}^{P} - \int_{x_1}^{P} G(x) \frac{\partial}{\partial x} G(x)dx \\
&= (1-(G(x_1))^2) - \int_{x_1}^{P} G(x) \frac{\partial}{\partial x} G(x)dx \\
&= \frac{(1-(G(x_1))^2)}{2}
\end{aligned}
$$

Therefore $\Pr\{V_{[bid]} \le V_{(1)} | V_{[bid]} \ge x_1\} = \Pr\{V_{[bid]} > V_{(1)} | V_{[bid]} \ge x_1\}$, and the seller is equally likely to be in either of the equally priced states. $\square$

Therefore, for $i = 1, 2$,

$$
\begin{aligned}
&F_i^2(q|x) \\
&= \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}\} \Pr\{H_{1,t_1} > H_{2,t_2}\} \\
&\qquad + \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} \le H_{2,t_2}\} \Pr\{H_{1,t_1} \le H_{2,t_2}\} \\
&= .5 \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}\} + .5 \Pr\{X_{i,t_i+\Delta t} \le q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} \le H_{2,t_2}\}
\end{aligned}
$$

By symmetry,

$$\Pr\{X_{1,t_1+\Delta t} \leq q | X_{1,t_1} = x, Z_{\Delta t} = 2, H_{1,t_1} \leq H_{2,t_2}\} = \Pr\{X_{2,t_2+\Delta t} \leq q | X_{2,t_2} = x, Z_{\Delta t} = 2, H_{1,t_1+\Delta t} > H_{2,t_2}\}$$

Therefore sufficient to derive $\Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}\}$

$$= \int_x^P \int_x^{v_1} \Pr\{X_{i,t_i+\Delta t} \leq q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}\} \phi(v_2|x, v_1) g(v_1|x) dv_2 dv_1$$

$$= \int_x^P \int_x^{v_1} \sum_{m=0}^{\infty} \Pr\{X_{i,t+1} \leq q | X_{i,t_i} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}, M_{\Delta t} = m\} \rho_M(m|\Delta t) \phi(v_2|x, v_1) g(v_1|x) dv_2 dv_1$$

where $g(v_1|x)$ and $\phi(v_2|x, v_1) = \frac{\partial}{\partial v_2} \frac{G(v_2)-G(x)}{(1-G(x))-(1-G(v_1))} = \frac{\partial}{\partial v_2} \frac{G(v_2)-G(x)}{G(v_1)-G(x)}$, are the conditional density function for the highest and second highest valuation respectively,

and

$$\Pr\{X_{i,t_i+\Delta t} \leq q | X_{1,t_1} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}, M_{\Delta t} = m\}$$

$$= \begin{cases} 0 & q < x \\ 1 & x \leq q \end{cases} \qquad m = 0$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_2 \\ .5(G(q))^m + .5 & v_2 \leq q < v_1 \\ 1 & v_1 \leq q \end{cases} \qquad m \geq 1$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_1 \leq q \end{cases} \qquad m \geq 2\ (L)$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_2 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \leq q \end{cases} \quad m \geq 2\ (U)$$

and $\Pr\{X_{2,t_2+\Delta t} \leq q | X_{2,t_2} = x, Z_{\Delta t} = 2, H_{1,t_1} > H_{2,t_2}, M_{\Delta t} = m\} =$

$$= \begin{cases} 0 & q < x \\ 1 & x \leq q \end{cases} \quad m = 0$$

$$= \begin{cases} 0 & q < x \\ G(q) & x \leq q < v_2 \\ 1 & v_2 \leq q \end{cases} \quad m = 1$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_2 \leq q \end{cases} \quad m \geq 2\ (L)$$

$$= \begin{cases} 0 & q < x \\ (G(q))^m & x \leq q < v_2 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) & v_2 \leq q < v_1 \\ (G(q))^m + m(G(q))^{m-1}(1-(G(q))) + \binom{m}{2}(G(q))^{m-2}(1-(G(q)))^2 & v_1 \leq q \end{cases} \quad m \geq 2\ (U)$$

Case 2: Auction 2 Started After Auction 1

If auction 2 is started after auction 1 then the transitions become a bit more complicated. Suppose auction 1 has elapsed for some time t_1 and $X_1 = x_1$. Note that the bidder with valuation and bid x_1 has left the auction site and does not know a second auction has started. Furthermore, recall bidders choose which auction to participate in strictly based on the prices of the auctions, and not price and time until an auction is completed. We re-start the counting of the valuations and bids, and indicate values that are below x_1 with an underline. For simplicity of notation, we continue to omit the elapsed auction time and define the system state as $[X_1, U_1; X_2, U_2]$. Furthermore, it will be assumed that the number of ongoing auctions in the time-interval $[t, t + \Delta t]$ remains fixed. The following are the possible transitions,

$$[x_1, V_{(1)}; 0, 0] \overset{bidder 1}{\longrightarrow} \begin{cases} [x_1, V_{(1)}; p, \underline{V_{(2)}}] & \{B_1 \leq x_1\} \\ [x_1, V_{(1)}; p, V_{(2)}] & \{x_1 < B_1 \leq V_{(1)}\} \\ [x_1, V_{(2)}; p, V_{(1)}] & \{V_{(1)} < B_1\} \end{cases}$$

$$[x_1, V_{(1)}; p, \underline{V_{(2)}}] \xrightarrow{bidder2} \begin{cases} [x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}] & \{B_2 \le x_1\} \\ [x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] & \{x_1 < B_2 \le V_{(1)}\} \\ [x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] & \{V_{(1)} < B_2\} \end{cases}$$

$$[x_1, V_{(1)}; p, V_{(2)}] \xrightarrow{bidder2} \begin{cases} [x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] & \{B_2 \le x_1\} \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{x_1 < B_2 \le V_{(1)}\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{V_{(1)} < B_2\} \end{cases}$$

$$[x_1, V_{(2)}; p, V_{(1)}] \xrightarrow{bidder2} \begin{cases} [x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] & \{B_2 \le x_1\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{x_1 < B_2 \le V_{(2)}\} \\ [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] & \{V_{(2)} < B_2\} \end{cases}$$

$$[x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}] \xrightarrow{bidder3} \begin{cases} [x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}] & \{B_3 \le x_1\} \\ [x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] & \{x_1 < B_3 \le V_{(1)}\} \\ [x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] & \{V_{(1)} < B_3\} \end{cases}$$

$$[x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] \xrightarrow{bidder3} \begin{cases} [x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] & \{B_3 \le x_1\} \\ [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] & \{x_1 < B_3 \le V_{(1)}\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{V_{(1)} < B_3\} \end{cases}$$

$$[x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] \xrightarrow{bidder3} \begin{cases} [x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] & \{B_3 \le x_1\} \\ [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}] & \{x_1 < B_3 \le V_{(2)}\} \\ [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] & \{V_{(2)} < B_3\} \end{cases}$$

The state of the auctions now either repeat or are the same as in the case when the auctions were released simultaneously. Therefore, the above dynamics generalizes to the n^{th} arriving bidder, $n > 2$. Consequently, if auction 2 is started after auction 1 and more than two bidders arrives, then there are only seven possible states of the auctions,

$$
\begin{array}{rl}
(0^I) & [x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}] \\
(0^{II}) & [x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}] \\
(0^{III}) & [x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}] \\
(1) & [V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}] \\
(2) & [V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}] \\
(3) & [V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}] \\
(4) & [V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]
\end{array}
$$

As before, our interest is to determine the conditional distribution of the within period price-transitions. Given a threshold q that is of interest there are the same four cases for states 1, 2, 3, and 4, as listed above, and five cases for the three new states 0^I, 0^{II}, and 0^{III}, which we categorize as a, b, c, d, and e,

	(0^I) $[x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}]$	(0^{II}) $[x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}]$	(0^{III}) $[x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}]$
(a)	$q \leq \underline{V_{(3)}} \leq \underline{V_{(2)}} \leq x_1 \leq V_{(1)}$	$q \leq \underline{V_{(3)}} \leq x_1 \leq V_{(2)} \leq V_{(1)}$	$q \leq \underline{V_{(3)}} \leq x_1 \leq V_{(2)} \leq V_{(1)}$
(b)	$\underline{V_{(3)}} < q \leq \underline{V_{(2)}} \leq x_1 \leq V_{(1)}$	$\underline{V_{(3)}} < q \leq x_1 \leq V_{(2)} \leq V_{(1)}$	$\underline{V_{(3)}} < q \leq x_1 \leq V_{(2)} \leq V_{(1)}$
(c)	$\underline{V_{(3)}} \leq \underline{V_{(2)}} < q \leq x_1 \leq V_{(1)}$	$\underline{V_{(3)}} \leq x_1 < q \leq V_{(2)} \leq V_{(1)}$	$\underline{V_{(3)}} \leq x_1 < q \leq V_{(2)} \leq V_{(1)}$
(d)	$\underline{V_{(3)}} \leq \underline{V_{(2)}} \leq x_1 < q \leq V_{(1)}$	$\underline{V_{(3)}} \leq x_1 \leq V_{(2)} < q \leq V_{(1)}$	$\underline{V_{(3)}} \leq x_1 \leq V_{(2)} < q \leq V_{(1)}$
(e)	$\underline{V_{(3)}} \leq \underline{V_{(2)}} \leq x_1 \leq V_{(1)} < q$	$\underline{V_{(3)}} \leq x_1 \leq V_{(2)} \leq V_{(1)} < q$	$\underline{V_{(3)}} \leq x_1 \leq V_{(2)} \leq V_{(1)} < q$

Using the same logic as when the auctions were started simultaneously, upper and lower bounds on ending up in the various states can be derived. However, due to the specifics of the truthful bidding strategy there are some states that become a bit more complicated. Next we provide an overview of the conditional distribution functions from the seven possible starting states. The explicit bounds will not be listed. An implicit assumption in the following scenarios is that the seller remembers the price of auction 1 when auction 2 started, and knows if the prices in both auctions are below or above that price.

State of Auctions: $[0, 0; 0, 0]$, $[V_{(3)}, V_{(2)}; V_{(2)}, V_{(1)}]$, or $[V_{(2)}, V_{(1)}; V_{(3)}, V_{(2)}]$

Under these three scenarios the same bounds as above apply.

State of Auctions: $[x_1, V_{(1)}; 0, 0]$

If the second auction has been underway but no bidder has yet arrived, then similar to above, we condition upon the high-bidder's valuation and derive the upper and lower bounds accordingly.

State of Auctions: $[x_1, V_{(1)}; \underline{V_{(3)}}, \underline{V_{(2)}}]$, $[x_1, V_{(1)}; \underline{V_{(3)}}, V_{(2)}]$, or $[x_1, V_{(2)}; \underline{V_{(3)}}, V_{(1)}]$

In these scenarios the seller does not know which state he is in. The only information he has is that $X_2 < X_1$, and that X_1 has not changed since auction 2 was started. Consequently we first condition on the probability of being in each of the possible states. The conditional probability of being in a specific state, given that he is in one of three, is straight forward to derive. Based on each specific state the lower and upper bounds can be derived. And with this information the lower and upper bounds of the distribution function can then be evaluated.

State of Auctions: $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$ or $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$

The final scenario is when $x_1 < X_2 = X_1$. Similar to the previous case the seller does not know which of the two states he is in. However, unlike the case when the auctions were released simultaneously, the transitions are not Markovian and it is not equally likely to be in each of the two states. Instead there is additional information to be gained knowing the path the auctions took to arrive at this state. If it is known that at some point during the previous period that prices were different but still above the starting price of auction 1, i.e. that there was a time when $x_1 < X_2, X_1$ and $X_2 \neq X_1$, then it is equally likely to be state $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(2)}]$ or state $[V_{(3)}, V_{(2)}; V_{(3)}, V_{(1)}]$.

If, on the other hand, during the previous period the transitions always ended such that the two prices were the same, i.e. $X_2 = X_1$, then it is more likely the seller is in state $[V_{(3)}, V_{(1)}; V_{(3)}, V_{(1)}]$. Therefore, to derive the conditional distribution function the seller must first determine the conditional probability of being in each of the two states given that the prices have always been the same (which is straight forward).

This scenario shows that the seller can gain additional information knowing the exact transitions and maintain information how the auctions are progressing. The conditional distribution function for the within period price-transitions are, however, derived using the same logic as before. Namely, first condition on the possible state, then condition on the valuation of the high bidders, and finally condition on the number of arriving bidders.

APPENDIX D

GLM Output and Residual Plots for the Models in Chapter 4

D1 - Binomial Analysis

Coefficients	Estimate	Std. Error	z value	Pr($> \lvert z \rvert$)
β_0	.140926	.079054	1.783	.0746
$\#AUC$	-.002156	.002719	-.793	.4278
Null deviance: 2868.3 on 2071 *df*				
Residual dev.: 2867.7 on 2070 *df*				
Pearson X^2: 2071.99				
β_0	.8411	.07184	11.708	< 2e-16
$Price$	-.01007	.0007278	-13.842	< 2e-16
$\#AUC$	-.008813	.002147	-4.104	$4.07e-05$
$Price \times \#AUC$	-.00004713	.00002647	-1.781	.074947
$1_{y=24}$	.2288	.06646	3.444	.000574
$1_{y=36}$	.3265	.06911	4.724	$2.31e-06$
$1_{y=48}$	1.307	.07727	16.913	< 2e-16
Null deviance: 11311 on 8287 *df*				
Residual dev.: 10430 on 8281 *df*				
Pearson X^2: 8269.699				

D1 - Gamma Analysis

Coefficients	Estimate	Std. Error	z value	Pr($> \lvert z \rvert$)
γ_0	4.099977	.047331	86.623	<2e-16
$\#AUC$	-.003983	.001620	-2.458	.0141
Null deviance: 1919.6 on 1081 *df*				
Residual dev.: 1915.1 on 1080 *df*				
Pearson X^2: 869.9023				
Dispersion parameter: .805463				
γ_0	4.0288680	.0207884	193.804	< 2e-16
$Price$	-.0024607	.0002173	-11.326	< 2e-16
$\#AUC$	*n.s.*			
$Price \times \#AUC$	*n.s.*			
$1_{y=24}$	-.1312671	.0300084	-4.3740	1.24e-05
$1_{y=36}$	-.1490882	.0307015	-4.856	1.24e-06
Null deviance: 5182.2 on 4750 *df*				
Residual dev.: 5074.7 on 4747 *df*				
Pearson X^2: 3325.072				
Dispersion parameter: .7004573				
γ_0	4.6989211	.0394284	119.176	< 2e-16
$Price$	-.0038386	.0002232	-17.199	< 2e-16
$\#AUC$	.0030054	.0008106	3.708	.000215
Null deviance: 1172.5 on 2070 *df*				
Residual dev.: 1047.5 on 2068 *df*				
Pearson X^2: 724.7486				
Dispersion parameter: .3504665				

TABLE D.1. GLM Final Models - Desktop D1

D3 - Binomial Analysis				
Coefficients	Estimate	Std. Error	z value	$\Pr(>\lvert z \rvert)$
β_0	.7588	.2351	3.228	.00125
$\#AUC$	-.1800	.0366	-4.919	8.72e-07
Null deviance: 375.10 on 273 *df*				
Residual dev.: 347.23 on 272 *df*				
Pearson X^2: 277.4396				
β_0	.930908	.170490	5.460	4.76e-08
$Price$	-.020951	.001815	-11.546	< 2e-16
$\#AUC$	-.058955	.017426	-3.383	.000717
$1_{y=24}$	.603141	.190326	3.169	.001530
$1_{y=36}$	1.0559417	.210022	5.028	4.96e-07
$1_{y=48}$	2.115657	.246791	8.5730	< 2e-16
Null deviance: 1489.0 on 1095 *df*				
Residual dev.: 1310.5 on 1090 *df*				
Pearson X^2: 1087.464				

D3 - Gamma Analysis				
Coefficients	Estimate	Std. Error	z value	$\Pr(>\lvert z \rvert)$
γ_0	3.82447	.17328	22.072	<2e-16
$\#AUC$	-.06371	.03066	-2.078	.0399
Null deviance: 283.18 on 118 *df*				
Residual dev.: 278.18 on 117 *df*				
Pearson X^2: 145.4840				
Dispersion parameter: 1.243185				
γ_0	4.0006267	.0426792	93.737	< 2e-16
$Price$	-.0067343	.0007897	-8.527	< 2e-16
$1_{y=48}$	.3050728	.0765937	3.983	7.59e-05
Null deviance: 655.93 on 638 *df*				
Residual dev.: 618.90 on 636 *df*				
Pearson X^2: 370.0578				
Dispersion parameter: .5818503				
γ_0	4.6807511	.2145286	21.819	< 2e-16
$Price$	-.0045027	.0015865	-2.838	.00488
$\#AUC$	.1076435	.0331477	3.247	.00131
$Price \times \#AUC$	-.0008084	.0002566	-3.151	.00181
Null deviance: 159.15 on 273 *df*				
Residual dev.: 123.59 on 270 *df*				
Pearson X^2: 87.73738				
Dispersion parameter: .3249284				

TABLE D.2. GLM Final Models - Desktop D3

D4 - Binomial Analysis				
Coefficients	Estimate	Std. Error	z value	$\Pr(>\lvert z \rvert)$
β_0	.94465	.26933	3.507	.000452
$\#AUC$	-.01131	.03561	-.318	.750736
Null deviance: 202.13 on 166 *df*				
Residual dev.: 202.03 on 165 *df*				
Pearson X^2: 166.9976				
β_0	1.375155	.198706	6.921	4.50e-12
$Price$	-.012220	.001577	-7.747	9.38e-15
$\#AUC$	-.027210	.016086	-1.692	0.0907
$1_{y=48}$	.948437	.213150	4.450	8.60e-06
Null deviance: 903.64 on 667 *df*				
Residual dev.: 833.15 on 664 *df*				
Pearson X^2: 665.3135				

D4 - Gamma Analysis				
Coefficients	Estimate	Std. Error	z value	$\Pr(>\lvert z \rvert)$
γ_0	4.14498	.10818	38.317	<2e-16
$\#AUC$	-.01943	.01466	-1.326	.188
Null deviance: 139.35 on 117 *df*				
Residual dev.: 138.58 on 116 *df*				
Pearson X^2: 64.99357				
Dispersion parameter: .5602768				
γ_0	4.241299	.080579	52.635	< 2e-16
$Price$	-.004727	.000747	-6.327	6.84e-10
$\#AUC$	-.022634	.007164	-3.159	.00170
$1_{y=48}$	.293589	.091471	3.210	.00144
Null deviance: 357.12 on 394 *df*				
Residual dev.: 333.51 on 391 *df*				
Pearson X^2: 217.0827				
Dispersion parameter: .5551945				
γ_0	5.4422324	.1207055	45.09	< 2e-16
$Price$	-.0076826	.0007432	-10.34	< 2e-16
Null deviance: 81.782 on 166 *df*				
Residual dev.: 56.143 on 165 *df*				
Pearson X^2: 38.26583				
Dispersion parameter: 0.2319067				

TABLE D.3. GLM Final Models - Desktop D4

L1 - Binomial Analysis

Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
β_0	2.198050	.128609	17.091	$<$ 2e-16
$\#AUC$	-.044271	.005509	-8.037	9.24e-16
Null deviance: 2122.4 on 2045 *df*				
Residual dev.: 2057.6 on 2044 *df*				
Pearson X^2: 2017.104				
β_0	1.867	.1056	17.679	$<$ 2e-16
$Price$	-.006390	4.278e-04	-14.938	$<$ 2e-16
$\#AUC$	-.02037	.004252	-4.790	1.67e-06
$Price \times \#AUC$	-7.932e-05	2.138e-05	-3.710	.000207
$1_{y=24}$	.1436	.07132	2.013	.044144
$1_{y=36}$	.2504	.07355	3.405	.000663
$1_{y=48}$	1.197	.08158	14.671	$<$ 2e-16
Null deviance: 10893.6 on 8183 *df*				
Residual dev.: 9638.3 on 8177 *df*				
Pearson X^2: 8126.968				

L1 - Gamma Analysis

Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
γ_0	5.182720	.048540	106.771	<2e-16
$\#AUC$	-.0234323	.00238	-9.842	<2e-16
Null deviance: 1955.6 on 1608 *df*				
Residual dev.: 1878.8 on 1607 *df*				
Pearson X^2: 1200.379				
Dispersion parameter: .7469764				
γ_0	4.8381624	.036487	132.597	$<$ 2e-16
$Price$	-.0018918	.0001221	-15.488	$<$ 2e-16
$\#AUC$	-.0108160	.0013100	-8.256	$<$ 2e-16
$Price \times \#AUC$	*n.s.*			
$1_{y=24}$	-.0567224	.0326789	-1.736	.082669
$1_{y=36}$	-.1187386	.0351421	-3.379	.000734
$1_{y=48}$	.1068323	.035291	3.027	.002481
Null deviance: 4785.0 on 5048 *df*				
Residual dev.: 4579.6 on 5043 *df*				
Pearson X^2: 3325.579				
Dispersion parameter: .6594416				
γ_0	6.0006166	.0347937	172.46	$<$ 2e-16
$Price$	-.0044562	.0001132	-39.37	$<$ 2e-16
Null deviance: 1233.67 on 2039 *df*				
Residual dev.: 827.98 on 2038 *df*				
Pearson X^2: 567.8304				
Dispersion parameter: .2786254				

TABLE D.4. GLM Final Models - Laptop L1

L4 - Binomial Analysis

Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
β_0	1.48097	.29501	5.020	5.17e-07
$\#AUC$	-.09040	.03867	-2.338	.0194
Null deviance: 203.67 on 171 *df*				
Residual dev.: 198.21 on 170 *df*				
Pearson X^2: 172.3741				
β_0	1.3313782	.1458051	9.131	$<$ 2e-16
$Price$	-.0069890	.0007865	-8.887	$<$ 2e-16
$1_{y=48}$	1.3201964	.2240970	5.891	3.83e-09
Null deviance: 912.33 on 687 *df*				
Residual dev.: 805.77 on 685 *df*				
Pearson X^2: 677.0064				

L4 - Gamma Analysis

Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
γ_0	5.05686	.12259	41.249	<2e-16
$\#AUC$	-.06518	.01911	-3.411	.000878
Null deviance: 147.73 on 123 *df*				
Residual dev.: 141.38 on 122 *df*				
Pearson X^2: 85.61325				
Dispersion parameter: .7017345				
γ_0	4.6201571	.0671803	68.772	$<$ 2e-16
$Price$	-.0014305	.0003517	-4.067	5.68e-05
$1_{y=24}$	-.1698937	.0914717	-1.857	.063956
$1_{y=36}$	-.3283783	.0943592	-3.480	.000553
Null deviance: 390.36 on 427 *df*				
Residual dev.: 372.83 on 424 *df*				
Pearson X^2: 247.6141				
Dispersion parameter: .5839949				
γ_0	5.736e+00	1.421e-01	40.365	$<$ 2e-16
$Price$	-3.159e-03	4.737e-04	-6.668	3.58e-10
$\#AUC$	7.891e-02	2.239e-022	3.524	.000548
$Price \times \#AUC$	-2.397e-04	8.056e-05	-2.976	.003355
Null deviance: 70.250 on 171 *df*				
Residual dev.: 44.088 on 168 *df*				
Pearson X^2: 27.01011				
Dispersion parameter: .1607744				

TABLE D.5. GLM Final Models - Laptop L4

L5 - Binomial Analysis				
Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
β_0	2.4214	.4397	5.507	3.66e-08
$\#AUC$	-.2947	.1056	-2.792	.00524
Null deviance: 161.45 on 162 *df*				
Residual dev.: 153.53 on 161 *df*				
Pearson X^2: 169.1283				
β_0	1.6710325	.2415296	6.919	4.56e-12
$Price$	-.0071794	.0007637	-9.401	$<$ 2e-16
$\#AUC$	-.1407179	.0492062	-2.860	.00424
$1_{y=48}$	1.2992085	.2242116	5.795	6.85e-09
Null deviance: 894.51 on 651 *df*				
Residual dev.: 776.65 on 648 *df*				
Pearson X^2: 647.6267				

L5 - Gamma Analysis				
Coefficients	Estimate	Std. Error	z value	Pr($>\|z\|$)
γ_0	4.95651	.17048	29.074	$<$ 2e-16
$\#AUC$	-.04956	.05079	-.976	.331
Null deviance: 204.92 on 130 *df*				
Residual dev.: 203.90 on 129 *df*				
Pearson X^2: 113.2176				
Dispersion parameter: .8776013				
γ_0	5.1553123	.13232029	38.961	$<$ 2e-16
$Price$	-.0029357	.0006981	-4.205	3.29e-05
$\#AUC$	-.1234271	.0326537	-3.780	.000184
$Price \times \#AUC$	.0003948	.0001843	2.142	.032828
$1_{y=36}$	-.1862469	.1023950	2.142	.032828
Null deviance: 307.50 on 364 *df*				
Residual dev.: 286.52 on 360 *df*				
Pearson X^2: 216.9356				
Dispersion parameter: .602584				
γ_0	5.9592695	.1369126	43.53	$<$ 2e-16
$Price$	-.0046622	.0004266	-10.93	$<$ 2e-16
Null deviance: 116.477 on 162 *df*				
Residual dev.: 75.783 on 161 *df*				
Pearson X^2: 55.06233				
Dispersion parameter: .3419806				

TABLE D.6. GLM Final Models - Laptop L5

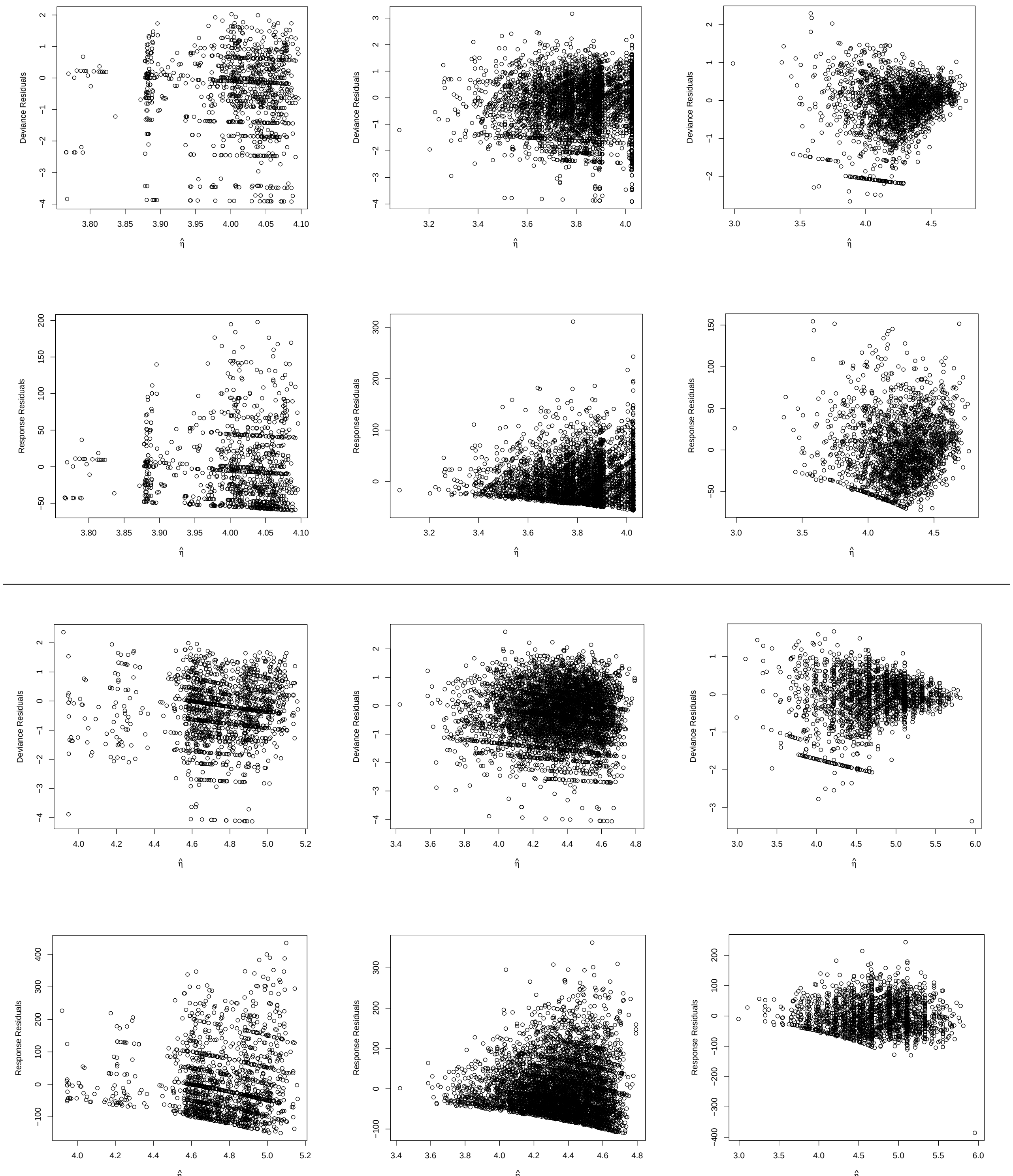


FIGURE D.1. Residual plots for D1 (top) and L1 (bottom). Each row is for the first, middle, and final periods (left to right). The first row are the deviance residuals, and the second row are the response residuals.

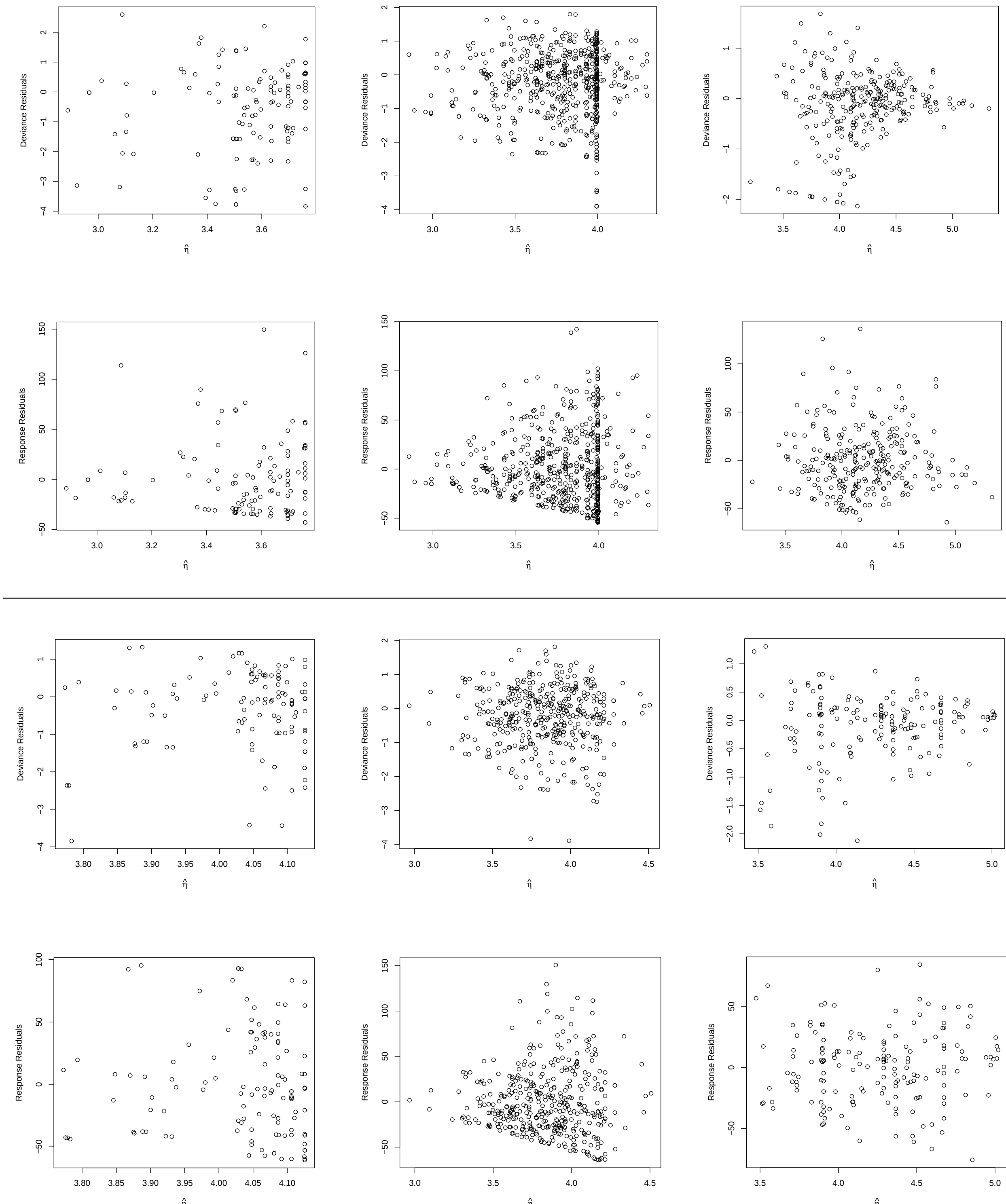


FIGURE D.2. Residual plots for D3 (top) and D4 (bottom). Each row is for the first, middle, and final periods (left to right). The first row are the deviance residuals, and the second row are the response residuals.

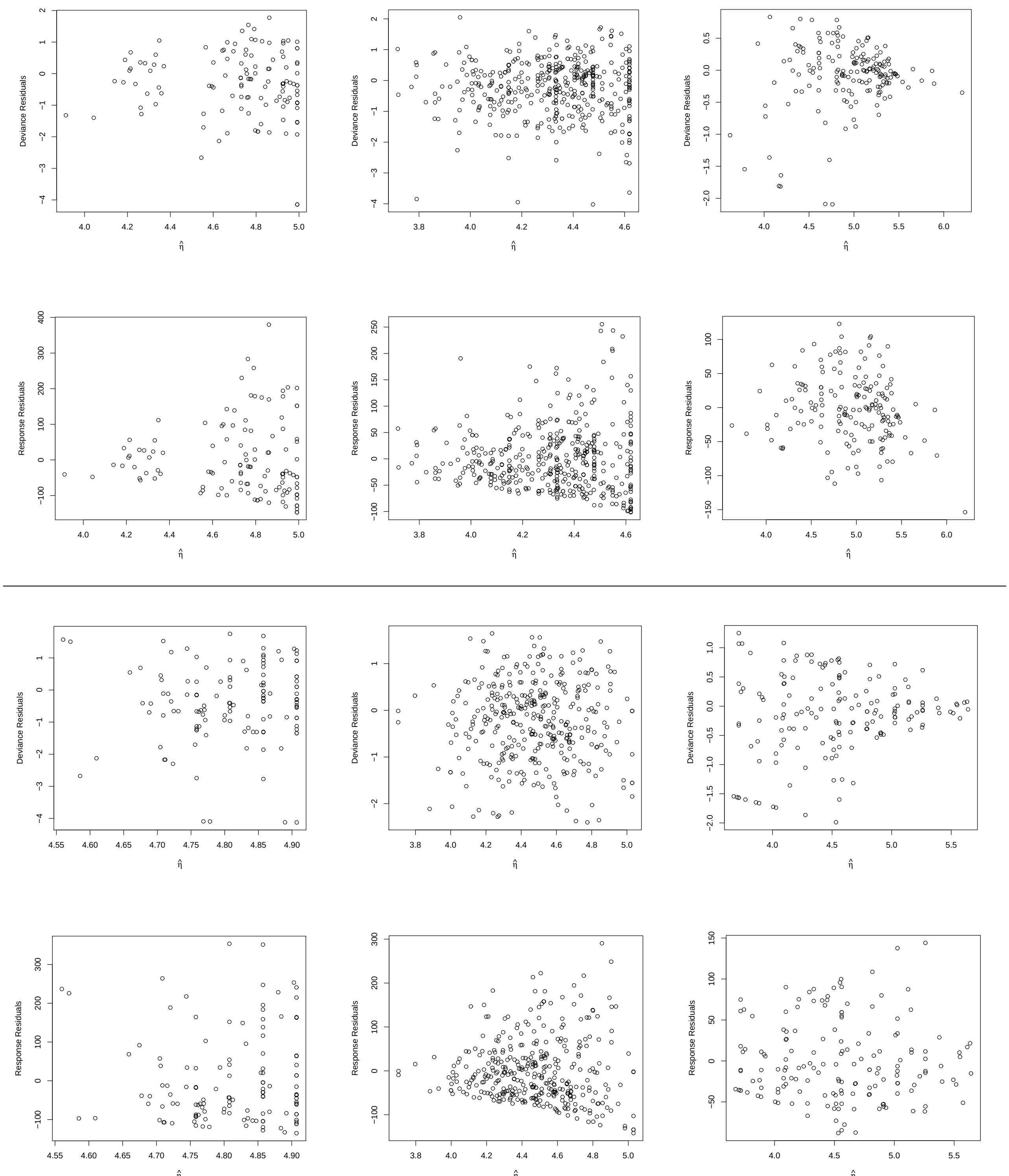


FIGURE D.3. Residual plots for L4 (top) and L5 (bottom). Each row is for the first, middle, and final periods (left to right). The first row are the deviance residuals, and the second row are the response residuals.

APPENDIX E

Linear Regression Model of Price-Increments in Final 12hours of an Auction

From Figures 4.7, 4.8 and 4.9 it might seem that a normal linear regression model could apply. Furthermore if the fitted regression line has a slope of 1 for the price variable, then it would imply that the expected final price conditional on price is constant and hence independent of the previous periods transitions. Therefore a multiple linear regression was fitted for the final period. Detail summary output from 'R' is provided in Table E.1 below. Below the resulting equations are listed, for $s = (x, 60, z)$,

$$\text{(D1)} \qquad E[C_{60}|S_{60} = s] = 102.00 - .257x + .476z - .0019x \times z$$

$$\text{(D3)} \qquad E[C_{60}|S_{60} = s] = 103.42 - .315x + 8.390z - .0618x \times z$$

$$\text{(D4)} \qquad E[C_{60}|S_{60} = s] = 151.80 - .526x + 2.442z - .0099x \times z$$

$$\text{(L1)} \qquad E[C_{60}|S_{60} = s] = 249.23 - .411x + 1.126z - .0056x \times z$$

$$\text{(L4)} \qquad E[C_{60}|S_{60} = s] = 231.75 - .350x + 12.294z - .0340x \times z$$

$$\text{(L5)} \qquad E[C_{60}|S_{60} = s] = 219.05 - .370x + 10.858z - .0381x \times z$$

The first thing to note is that x, z, and the interaction $x \times z$ are always significant (except for D4 where both z and $x \times z$ are non-significant). Second we see that the slope coefficient for x is substantially smaller than 1. This would imply that a \$1 increase in price will decrease the expected price-increment, and hence final price, by less than \$1. Consequently the seller is always better off the higher the price. And more importantly the previous periods transitions do matter and the expected final price is not constant.

However, the interaction term also has a negative slope and therefore there exist a z^s such that a unit increase in x will result in a decrease in expected price-increment of exactly \$1. Consequently for $Z = z^s$ the expected final price is independent of price and the previous periods. The specific values for z^s are as follows: D1) 391, D3) 11, D4) 48, L1) 105, L4) 19, and L5) 17. In other words, for each subset, if x takes on the listed values then \$1 increase in price will result in \$1 decrease in expected price-increment and the expected final price conditional upon price is constant. Comparing these values with the maximum z observed for each of the data sets (D1) 84, D3) 15, D4) 20, L1) 54, L4) 18, L5) 8; see Table 4.2), we see that, with one exception, z^s is far beyond these values. The exception is D3 for which we see that the largest average number of auctions observed was 15 while $z^s = 11$. Meaning that if the seller has 11 auctions underway in the final period, then the expected final price will be constant. On the other hand if he has more than 11 then the expected final price is decreasing in price. Therefore, we conclude that the effect of price is such that the expected final price does depend on the price at the start of the final period.

D1: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	102.0	3.902e+00	26.132	< 2e-16
$Price$	-.2571	2.597e-02	-9.898	< 2e-16
$\#AUC$	.4755	1.371e-01	3.469	.000533
$Price \times \#AUC$	-.001947	9.439e-04	-2.062	0.039293

Res std err: 38.08 on 2067 *df*
Mtpl R-Sq: .1874, Adj R-sq: .1863
F-stat: 158.9 on 3 and 2067 *df*, p-value: < 2.2e-16

L1: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	249.230326	7.551410	33.00	< 2e-16
$Price$	-.410907	.023015	-17.85	< 2e-16
$\#AUC$	1.125848	.338060	3.33	.000883
$Price \times \#AUC$	-.005572	.001156	-4.82	1.54e-06

Res std err: 52.53 on 2036 *df*
Mtpl R-Sq: .4901, Adj R-sq: .4894
F-stat: 652.4 on 3 and 2036 *df*, p-value: < 2.2e-16

D3: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	103.41637	12.49513	8.277	5.89e-15
$Price$	-.31525	.09241	-3.412	.000745
$\#AUC$	8.38895	1.93068	4.345	1.97e-05
$Price \times \#AUC$	-0.06175	.01494	-4.132	4.80e-05

Res std err: 33.2 on 270 *df*
Mtpl R-Sq: .3868, Adj R-sq: .38
F-stat: 56.78 on 3 and 270 *df*, p-value: < 2.2e-16

D4: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	151.795464	12.264153	12.377	< 2e-16
$Price$	-.525925	.076618	-6.864	1.33e-10
$\#AUC$	2.442041	1.477201	1.653	.100
$Price \times \#AUC$	-.009932	.009957	-0.998	.320

Res std err: 30.55 on 163 *df*
Mtpl R-Sq: .5063, Adj R-sq: .4972
F-stat: 55.71 on 3 and 163 *df*, p-value: < 2.2e-16

L4: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	231.75170	16.65124	13.918	< 2e-16
$Price$	-.34987	.05551	-6.303	2.48e-09
$\#AUC$	12.29419	2.62409	4.685	5.75e-06
$Price \times \#AUC$	-.03487	.00944	-3.694	.000298

Res std err: 46.99 on 168 *df*
Mtpl R-Sq: .5681, Adj R-sq: .5604
F-stat: 73.65 on 3 and 168 *df*, p-value: < 2.2e-16

L5: Linear Regression

Coefficients	Estimate	Std. Error	t value	$\Pr(>\lvert z\rvert)$
$Intercept$	219.05116	25.82991	8.481	1.46e-14
$Price$	-.36970	.07638	-4.840	3.05e-06
$\#AUC$	10.85792	6.20063	1.751	.0819
$Price \times \#AUC$	-.03808	.01954	-1.949	.0531

Res std err: 49.79 on 159 *df*
Mtpl R-Sq: 0.5462, Adj R-sq: 0.5377
F-stat: 63.8 on 3 and 159 *df*, p-value: < 2.2e-16

TABLE E.1. Normal linear regression models for the final period

APPENDIX F

Supplementary Material for Empirical Analysis in Chapter 5

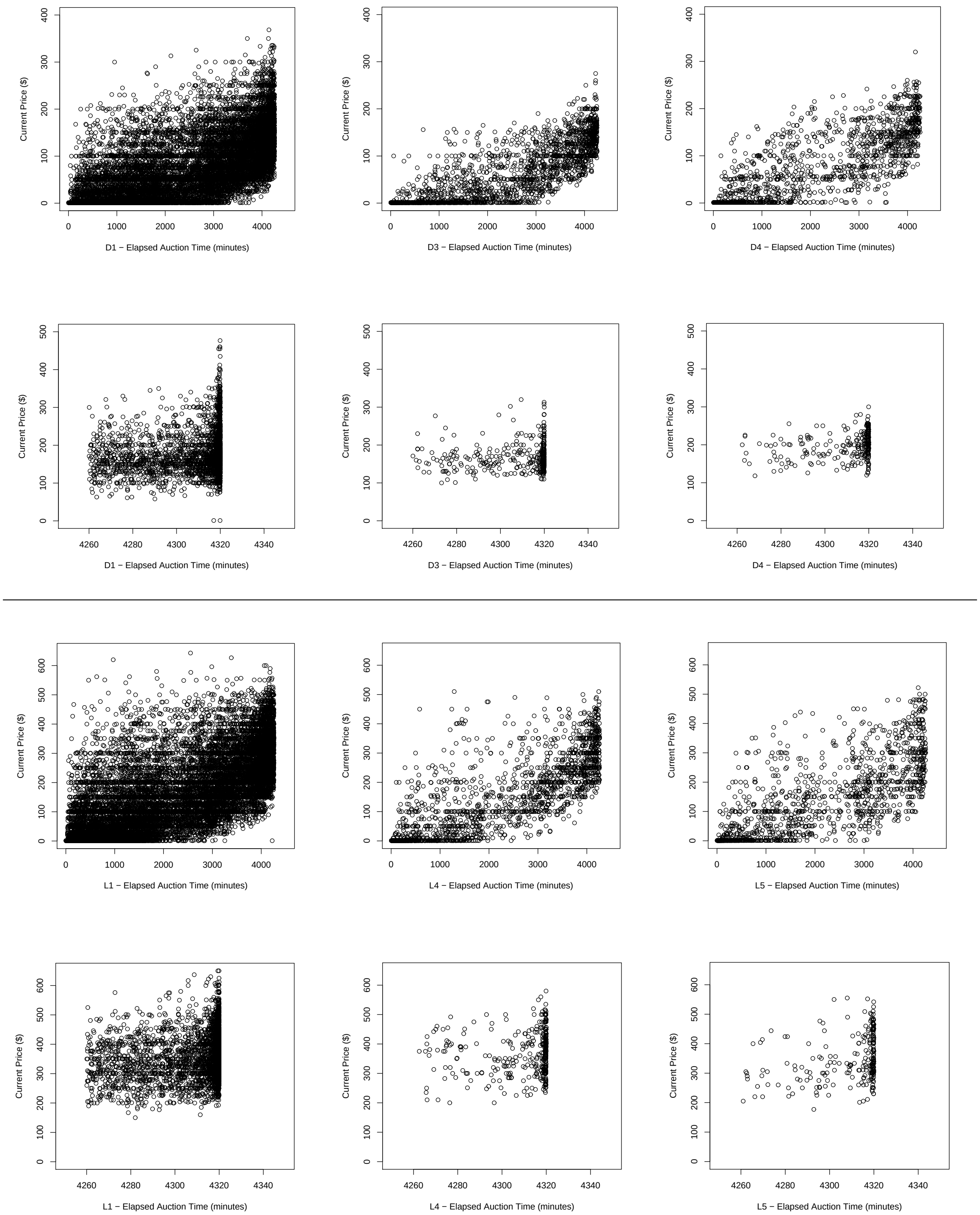


FIGURE F.1. Current price versus elapsed auction time for D1, D3, D4 (top, left to right) and L1, L4, L5 (bottom, left to right). The first row of each product line is for the first 71 hours of the auction, while the second row of each product line is for bids placed in the final hour. The horizontal axis in each graph represents the elapsed time of the auction, and the vertical axis is the current price when each bid was placed. Each point represents a single bid.

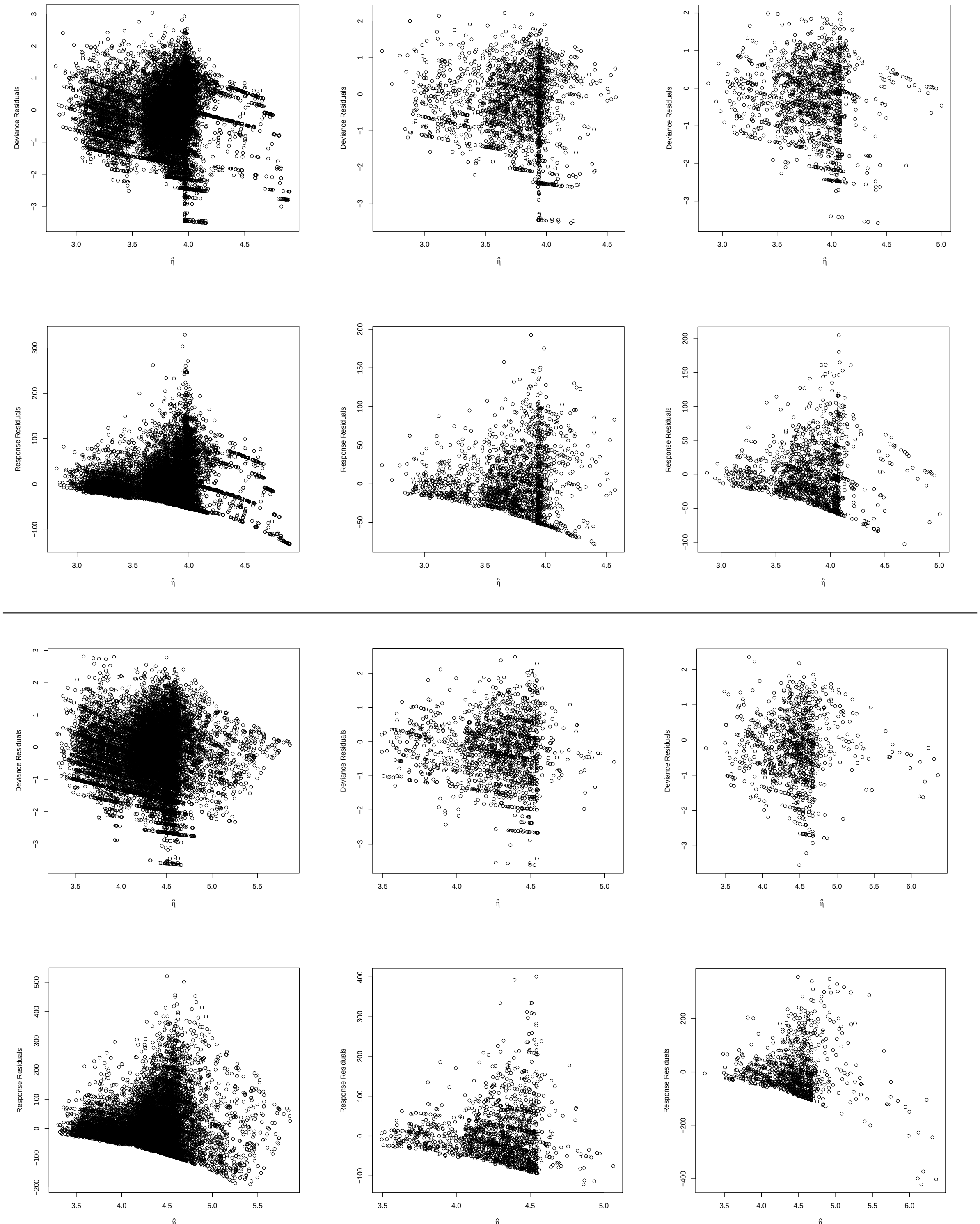


FIGURE F.2. Residual plot for D1, D3, D4 (top, left to right) and L1, L4, L5 (bottom, left to right). The first row shows the deviance residual plots and the second row shows the response residual plots for the analysis of bids up to the final hour.

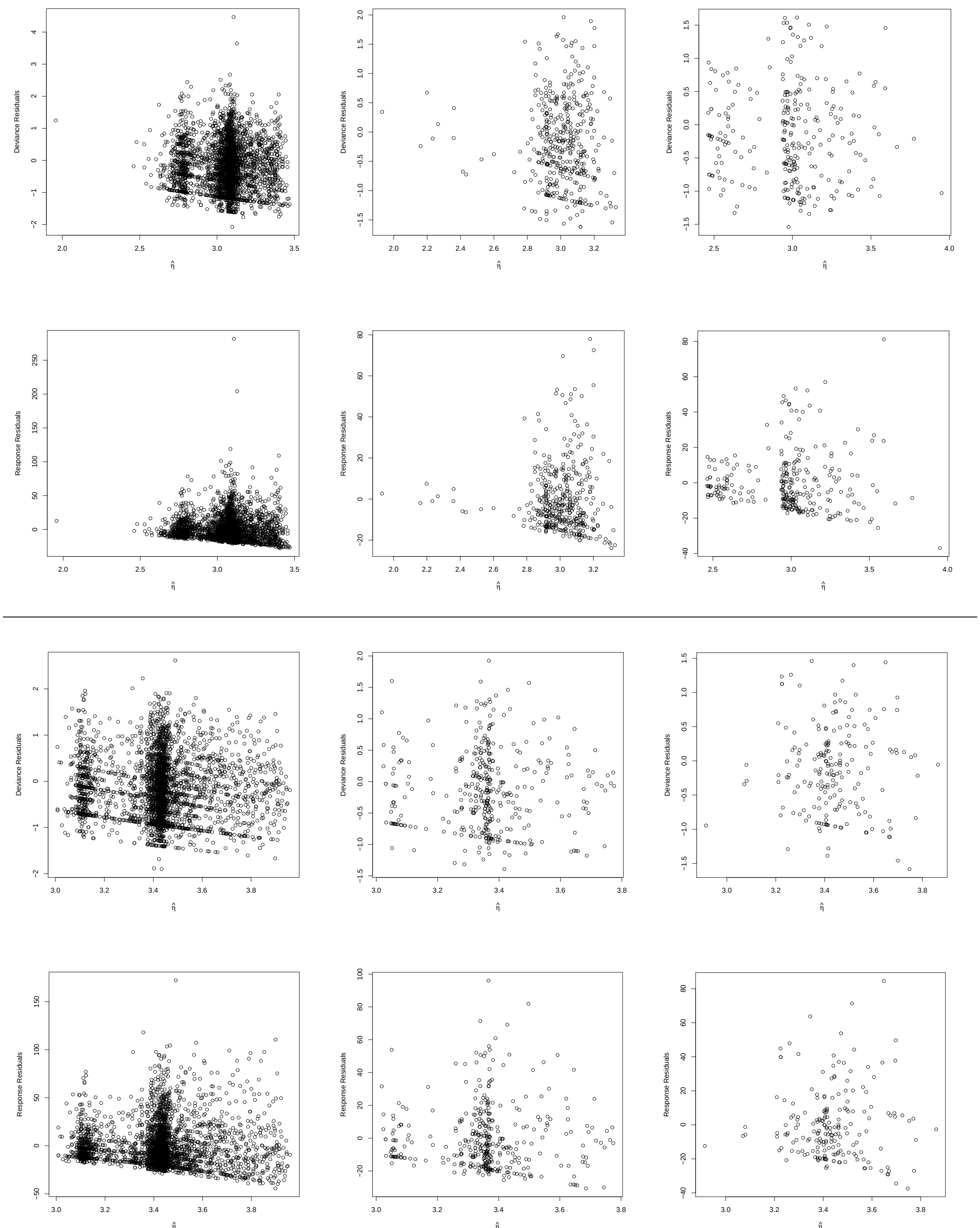
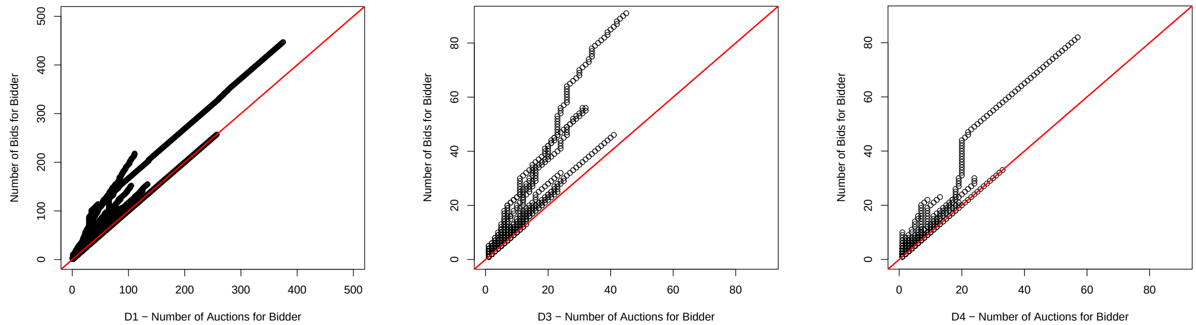
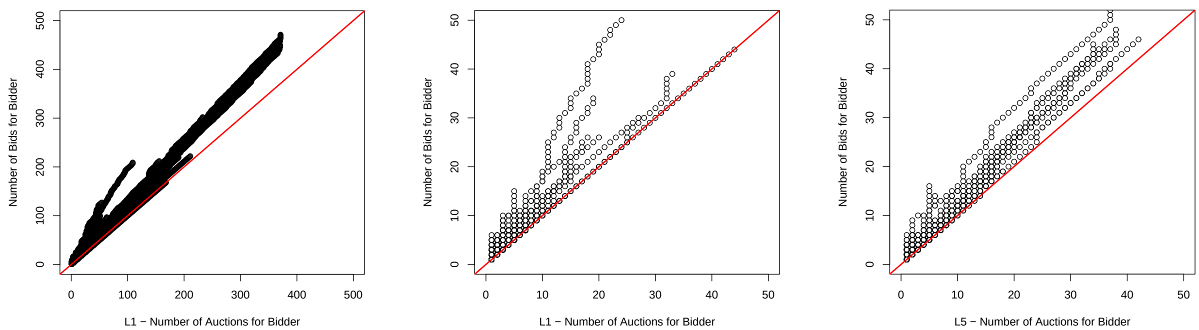


FIGURE F.3. Residual plot for D1, D3, D4 (top, left to right) and L1, L4, L5 (bottom, left to right). The first row shows the deviance residual plots and the second row shows the response residual plots for the analysis of bids in the final hour.



D1 D3 D4



L1 L4 L5

FIGURE F.4. The number of bids a bidder has placed versus the number of auctions a bidder has participated in. Each observation represents a bid. The solid line represents bids for which bidders have placed exactly one bid per auction.

L1	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	3.467e+00	5.755e-02	60.235	<2e-16	5.264e+01	1.610e+01	3.269	0.00109
Current Price	6.089e-04	1.399e-04	4.354	1.35e-05	-4.935e-02	4.554e-02	-1.084	0.27862
Elapsed Time	4.143e-05	9.880e-06	4.193	2.76e-05	-1.145e-02	3.737e-03	-3.063	0.00221
Inter-Arrival Time	-3.613e-06	1.391e-05	-0.260	0.7950	7.509e-05	3.287e-05	2.285	0.02241
Number Auctions	-4.555e-03	8.362e-04	-5.448	5.18e-08	-1.505e-03	1.421e-03	-1.059	0.28972
Low Price	3.216e-04	1.301e-04	2.471	0.0135	2.660e-04	2.573e-04	1.034	0.30122
High Price	1.098e-03	8.439e-05	13.015	<2e-16	8.671e-04	1.647e-04	5.264	1.50e-07
Time Low Price	1.816e-04	6.813e-04	0.267	0.7898	-1.078e-03	1.154e-03	-0.935	0.35010
Time High Price	2.312e-03	4.581e-04	5.047	4.54e-07	-2.244e-04	8.733e-04	-0.257	0.79726
Bidder Bids	-1.559e-04	9.644e-04	-0.162	0.8716	1.197e-03	1.650e-03	0.726	0.46805
Bidder Auctions	6.258e-03	1.220e-03	5.131	2.92e-07	-2.228e-03	1.912e-03	-1.165	0.24396
First Bid	5.519e-01	1.889e-02	29.223	<2e-16	3.020e-01	3.117e-02	9.691	< 2e-16
Cur. Price× Elaps.Time	-6.274e-07	4.430e-08	-14.161	<2e-16	1.122e-05	1.057e-05	1.062	0.28840
Dispersion Para.	0.7266593				0.5033141			
Null Deviance	17275 on 15891 $d.f.$				1620.6 on 3241 $d.f.$			
Residual Deviance	15050 on 15879 $d.f.$				1488.5 on 3229 $d.f.$			

L4	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	3.901e+00	8.426e-02	46.298	<2e-16	1.126e+02	5.187e+01	2.170	0.03062
Current Price	8.223e-04	4.592e-04	1.791	0.073522	-2.300e-01	1.386e-01	-1.659	0.09798
Elapsed Time	-1.021e-05	3.001e-05	-0.340	0.733728	-2.536e-02	1.204e-02	-2.106	0.03584
Inter-Arrival Time	-3.378e-05	4.115e-05	-0.821	0.411824	1.675e-04	1.673e-04	1.001	0.31747
Number Auctions	-1.928e-02	7.428e-03	-2.596	0.009513	-2.026e-03	1.344e-02	-0.151	0.88026
Low Price	1.174e-04	3.203e-04	0.366	0.714115	-3.531e-04	6.273e-04	-0.563	0.57388
High Price	3.206e-04	2.785e-04	1.151	0.249867	8.601e-04	5.899e-04	1.458	0.14562
Time Low Price	7.797e-04	1.616e-03	0.483	0.629439	-4.971e-03	2.668e-03	-1.863	0.06325
Time High Price	6.139e-04	1.706e-03	0.360	0.719061	2.085e-03	2.840e-03	0.734	0.46328
Bidder Bids	1.709e-03	9.907e-03	0.172	0.863089	1.907e-03	1.555e-02	0.123	0.90244
Bidder Auctions	1.485e-02	1.391e-02	1.068	0.285790	-3.310e-03	1.723e-02	-0.192	0.84775
First Bid	5.090e-01	5.998e-02	8.486	<2e-16	2.753e-01	1.002e-01	2.746	0.00632
Cur. Price× Elaps.Time	-4.601e-07	1.377e-07	-3.341	0.000852	5.326e-05	3.217e-05	1.655	0.09867
Dispersion Para.	0.6995354				0.4668382			
Null Deviance	1726.1 on 1741 $d.f.$				171.54 on 392 $d.f.$			
Residual Deviance	1597.4 on 1729 $d.f.$				157.99 on 380 $d.f.$			

L5	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	4.010e+00	1.107e-01	36.233	< 2e-16	8.407e+01	5.779e+01	1.455	0.1474
Current Price	1.707e-03	5.500e-04	3.103	0.00196	-2.100e-01	1.826e-01	-1.151	0.2514
Elapsed Time	2.156e-05	3.522e-05	0.612	0.54060	-1.863e-02	1.340e-02	-1.390	0.1662
Inter-Arrival Time	2.426e-05	5.018e-05	0.484	0.62884	6.299e-05	7.034e-05	0.895	0.3717
Number Auctions	-7.790e-02	2.644e-02	-2.946	0.00329	-7.692e-02	4.353e-02	-1.767	0.0789
Low Price	-1.876e-05	3.854e-04	-0.049	0.96119	1.004e-04	6.952e-04	0.144	0.8854
High Price	4.937e-04	3.709e-04	1.331	0.18341	6.932e-04	6.407e-04	1.082	0.2807
Time Low Price	1.077e-03	2.171e-03	0.496	0.62001	6.034e-03	4.012e-03	1.504	0.1343
Time High Price	-2.256e-04	2.154e-03	-0.105	0.91662	-4.122e-04	3.991e-03	-0.103	0.9179
Bidder Bids	2.459e-02	3.053e-02	0.805	0.42080	2.184e-02	3.791e-02	0.576	0.5653
Bidder Auctions	1.798e-02	3.479e-02	0.517	0.60536	-4.261e-02	4.584e-02	-0.930	0.3538
First Bid	5.327e-01	8.968e-02	5.940	3.85e-09	1.658e-01	1.386e-01	1.196	0.2332
Cur. Price× Elaps.Time	-8.870e-07	1.649e-07	-5.378	9.24e-08	4.836e-05	4.233e-05	1.143	0.2547
Dispersion Para.	0.7228855				0.3994007			
Null Deviance	1222.4 on 1082 $d.f.$				84.181 on 195 $d.f.$			
Residual Deviance	1039.9 on 1070 $d.f.$				75.339 on 183 $d.f.$			

TABLE F.1. Results for the base model of L1, L4, L5.

D1	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	3.279e+00	4.857e-02	67.498	<2e-16	1.862e+01	1.346e+01	1.383	0.1668
Current Price	1.661e-03	3.139e-04	5.291	1.23e-07	6.731e-02	7.658e-02	0.879	0.3795
Elapsed Time	-4.427e-05	8.390e-06	-5.276	1.34e-07	-3.700e-03	3.122e-03	-1.185	0.2361
Inter-Arrival Time	2.731e-05	1.228e-05	2.223	0.0262	2.692e-04	3.842e-05	7.009	2.90e-12
Number Auctions	-2.478e-04	4.598e-04	-0.539	0.5899	-1.852e-03	9.389e-04	-1.973	0.0486
Low Price	-3.066e-04	3.641e-04	-0.842	0.3997	-6.894e-04	5.866e-04	-1.175	0.2400
High Price	8.987e-04	1.211e-04	7.419	1.24e-13	4.208e-04	2.477e-04	1.699	0.0895
Time Low Price	-2.069e-04	5.816e-04	-0.356	0.7221	-2.235e-03	1.122e-03	-1.992	0.0465
Time High Price	2.558e-04	4.480e-04	0.571	0.5680	-9.459e-04	9.299e-04	-1.017	0.3091
Bidder Bids	5.280e-03	7.254e-04	7.279	3.52e-13	-5.632e-03	2.586e-03	-2.178	0.0295
Bidder Auctions	-4.297e-03	9.161e-04	-4.691	2.74e-06	5.276e-03	3.658e-03	1.442	0.1493
First Bid	5.291e-01	2.031e-02	26.051	<2e-16	2.832e-01	3.529e-02	8.024	1.40e-15
Cur. Price× Elaps.Time	-8.026e-07	9.352e-08	-8.582	<2e-16	-1.533e-05	1.776e-05	-0.863	0.3882
Dispersion Para.	0.6401172				0.6401172			
Null Deviance	2062.4 on 3330 $d.f.$				2062.4 on 3330 $d.f.$			
Residual Deviance	1908.3 on 3318 $d.f.$				1908.3 on 3318 $d.f.$			

D3	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	3.250e+00	8.531e-02	38.093	<2e-16	9.811e+01	5.023e+01	1.953	0.0515
Current Price	2.507e-04	1.201e-03	0.209	0.83467	-5.275e-01	2.990e-01	-1.764	0.0785
Elapsed Time	5.513e-05	2.576e-05	2.140	0.03245	-2.205e-02	1.166e-02	-1.892	0.0593
Inter-Arrival Time	4.764e-05	3.583e-05	1.330	0.18381	-4.665e-06	8.891e-05	-0.052	0.9582
Number Auctions	-1.002e-02	7.441e-03	-1.347	0.17807	3.634e-02	1.660e-02	2.189	0.0292
Low Price	-6.036e-04	6.530e-04	-0.924	0.35542	2.675e-03	1.395e-03	1.918	0.0559
High Price	1.115e-03	4.983e-04	2.238	0.02534	1.329e-06	1.196e-03	0.001	0.9991
Time Low Price	-1.129e-03	1.421e-03	-0.795	0.42682	-2.012e-03	2.617e-03	-0.769	0.4424
Time High Price	-3.604e-04	1.503e-03	-0.240	0.81052	-7.469e-04	2.914e-03	-0.256	0.7978
Bidder Bids	-1.433e-02	5.281e-03	-2.714	0.00671	-2.634e-03	1.295e-02	-0.203	0.8389
Bidder Auctions	3.978e-02	9.468e-03	4.202	2.76e-05	-2.152e-02	2.514e-02	-0.856	0.3925
First Bid	5.280e-01	5.633e-02	9.372	<2e-16	3.801e-02	1.025e-01	0.371	0.7110
Cur. Price× Elaps.Time	-9.983e-07	3.306e-07	-3.020	0.00256	1.221e-04	6.939e-05	1.759	0.0793
Dispersion Para.	0.7151947				0.5756648			
Null Deviance	2499.6 on 2113 $d.f.$				239.49 on 411 $d.f.$			
Residual Deviance	2290.7 on 2101 $d.f.$				221.55 on 399 $d.f.$			

D4	≤ 4260 minutes				> 4260 minutes			
	Estimate	Std. Error	t value	Pr(> $\|t\|$)	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	3.472e+00	8.976e-02	38.684	<2e-16	-2.030e+01	7.628e+01	-0.266	0.7903
Current Price	-1.265e-03	9.016e-04	-1.403	0.16075	3.608e-01	4.018e-01	0.898	0.3699
Elapsed Time	9.858e-05	3.217e-05	3.064	0.00223	5.362e-03	1.770e-02	0.303	0.7622
Inter-Arrival Time	-1.722e-06	4.144e-05	-0.042	0.96687	3.392e-04	1.470e-04	2.308	0.0218
Number Auctions	-2.760e-02	6.303e-03	-4.378	1.28e-05	1.895e-03	1.439e-02	0.132	0.8953
Low Price	-1.206e-03	6.018e-04	-2.005	0.04518	-1.108e-03	1.291e-03	-0.859	0.3914
High Price	1.215e-03	5.421e-04	2.241	0.02521	-9.641e-04	1.282e-03	-0.752	0.4528
Time Low Price	1.024e-03	1.688e-03	0.607	0.54395	4.356e-03	3.939e-03	1.106	0.2698
Time High Price	-1.415e-03	1.750e-03	-0.808	0.41907	-4.099e-03	4.266e-03	-0.961	0.3375
Bidder Bids	1.844e-02	8.550e-03	2.157	0.03115	-5.425e-02	4.406e-02	-1.231	0.2193
Bidder Auctions	-1.038e-02	1.308e-02	-0.794	0.42748	1.104e-01	5.604e-02	1.970	0.0499
First Bid	4.830e-01	6.404e-02	7.542	8.17e-14	3.138e-01	1.393e-01	2.253	0.0251
Cur. Price× Elaps.Time	-7.452e-07	2.742e-07	-2.717	0.00666	-8.384e-05	9.320e-05	-0.900	0.3691
Dispersion Para.	0.6618309				0.5483629)			
Null Deviance	1558.0 on 1455 $d.f.$				169.20 on 281 $d.f.$			
Residual Deviance	1412.1 on 1443 $d.f.$				140.68 on 269 $d.f.$			

TABLE F.2. Results for the base model of D1, D3, D4.

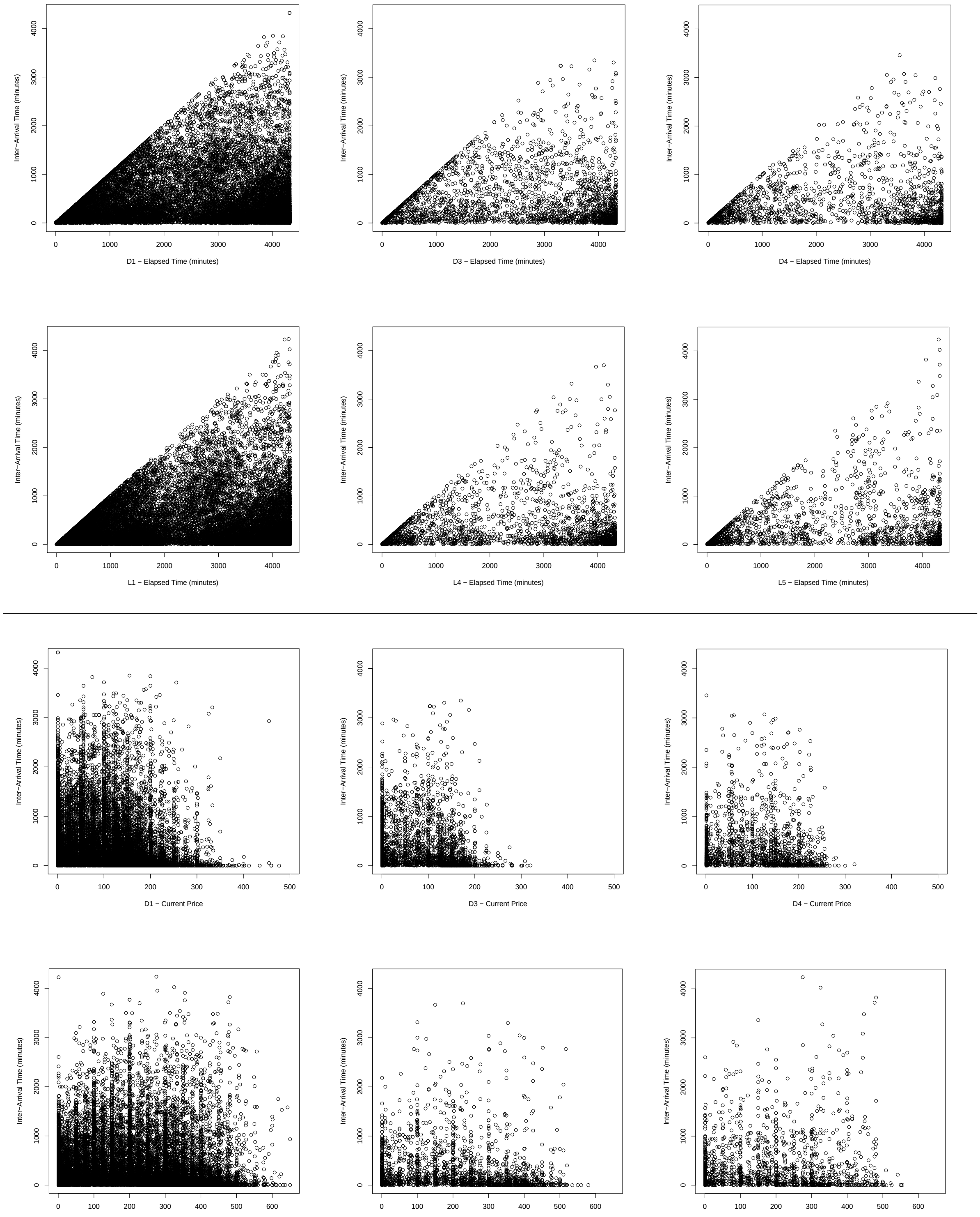


FIGURE F.5. The inter-arrival time of bids versus the elapsed auction time (top), and the inter-arrival time of bids versus the current price (bottom). Each observation represents a bid. The first row is D1, D3, D4, and the second row is L1, L4, L5 (left to right).