Swimming in Slime

by

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Abstract

The purpose of this thesis is to study the problem of a low Reynolds number swimmer that is in very close proximity to a wall or solid boundary in a non-Newtonian fluid. We assume that it moves by propagating waves down its length in one direction, creating a thrust and therefore propelling it in the opposite direction. We model the swimmer as an infinite, inextensible waving sheet.

We consider two main cases of this swimming sheet problem. In the first case, the type of wave being propagated down the length of the swimmer is specified. We compare the swimming speeds of viscoelastic shear thinning, shear thickening and Newtonian fluids for a fixed propagating wave speed. We then compare the swimming speeds of these same fluids for a fixed rate of work per wavelength. In the latter situation, we find that a shear thinning fluid always yields the fastest swimming speed regardless of the amplitude of the propagating waves. We conclude that a shear thinning fluid is optimal for the swimmer. Analytical results are obtained for various limiting cases. Next, we consider the problem with a Bingham fluid. Yield surfaces and flow profiles are obtained.

In the second case, the forcing along the length of the swimmer is specified, but the shape of the swimmer is unknown. First, we solve this problem for a Newtonian fluid. Large amplitude forcing yields a swimmer shape that has a plateau region following by a large spike region. It is found that there exists an optimal forcing that will yield a maximum swimming speed. Next, we solve the problem for moderate forcing amplitudes for viscoelastic shear thickening and shear thinning fluids. For a given forcing, it is found that a shear thinning fluid yields the fastest swimming speed when compared to a shear thickening fluid and a Newtonian fluid. The difference in swimming speeds decreases as the bending stiffness of the swimmer increases.

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Dedication

To my parents, for always being there for me, my sister and brother, who try very hard find what I do interesting, and to Tyler, for putting up with me throughout.

Chapter 1

Introduction

Years ago, the question of how microscopic organisms propel themselves through fluid was asked. The standard mechanism of propulsion for larger bodies was understood to rely on the shedding of vortices in an inviscid fluid. It was something entirely new to consider an organism propelling itself through fluid with a very small Reynolds number [37]. The mechanism of vortex shedding no longer was valid, and so the field of low Reynolds number swimming was born.

Typically, the microorganism under specific analysis in this field is a spermatozoon. This is due to the biological importance of spermatozoa locomotion as well as the fact that the mechanisms of spermatozoa locomotion are significantly more amenable to mathematical modelling than other microorganisms. Many flagellated microorganisms are propelled by a rather complicated spiralling or corkscrew motion of their flagella. However, the flagellar motion of spermatozoa tends to be more planar. In certain situations it can also be low amplitude [21]. Thus, it is clear why spermatozoa are desirable to model from a mathematician's perspective. Contributions to this field have been made by people such as Taylor, Reynolds, Gray and Hancock, with continual contributions from leading researchers to this day [37, 33, 14, 15, 22, 25].

The purpose of this thesis is to study the problem of a low Reynolds number swimmer that is in very close proximity to a wall or solid boundary in a non-Newtonian fluid. We assume that it moves by propagating waves down its length in one direction, creating a thrust and therefore propelling it in the opposite direction. We model the swimmer as an infinite, inextensible waving sheet, a standard approach in this field [37, 33, 22, 25]. Of course, realistic microorganisms have a body or head and flagellum of finite length and width, but this simplification to a swimming sheet provides useful insight into the general problem of swimming. This formulation of the problem also proves to have a wider range of applications [8, 26, 39, 2, 40, 18].

We consider two main cases of the swimming sheet problem. In the first problem, we suppose that the type of wave being propagated down the length

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of the swimmer is specified. In the second problem, the forcing along the length of the swimmer is specified, but the shape of the swimmer is unknown.

As described above, these questions are related to many other similar problems that have previously been studied. We take a unique twist by considering the lubrication approximation, introducing a wall and a non-Newtonian fluid and eventually an elastic swimmer. In this chapter we will outline some previous work done on similar problems, setting the backdrop for the current topic. As well, we will briefly introduce the reader to viscoelastic fluids in preparation for the following discussions and mathematical analysis. We will then present the general problem under consideration, emphasizing the assumptions that are made and used in all following chapters, and the physical rationalizations for these assumptions. Finally, we will detail the problem of a two dimensional swimmer in the presence of a wall without the lubrication approximation to motivate the use of the thin gap theory.

1.1   Literature Review

In the literature, there are several trains of thought about how to approach the low Reynolds number swimming problem. In all considerations of the problem, it is the swimming speed of the organism that is of primary interest. One approach is to simplify the problem to the two dimensional problem of a swimming sheet, as we will be doing in our discussion of this topic. Typically, an asymptotic approach is used where one assumes the form of the swimmer to be that of a low amplitude wave. The solution is expanded as a Taylor series in powers of the non-dimensional wave amplitude [37, 33, 9, 22, 25]. A common result is that the leading order swimming speed is proportional to the amplitude of the swimmer squared.

G. I. Taylor, in [37], approached the swimming problem in this way. He considered an infinite, inextensible, two dimensional waving sheet in a Newtonian fluid. The sheet was fixed to be a sinusoid, which he gave dimensionally by

$$y_0 = b \sin(kx - \sigma t), \qquad (1.1)$$

where b is the amplitude, k the wave number, and σ is the frequency. No dynamical considerations were made for the sheet. He assumed that the waves were low amplitude and found that at first order, the swimming speed was zero. Continuing the calculation to second order, he found a non-zero swimming

speed, which is henceforth referred to as the leading order swimming speed. This swimming speed, dimensionally, was calculated to be

$$\frac{V}{c} = \frac{1}{2}b^2k^2, \tag{1.2}$$

where V is the non-dimensional swimming speed, c is the speed of the propagating waves, b is the amplitude of the waves, and k is the wave number. As is evident, this leading order swimming speed is proportional to the amplitude of the swimmer squared. In this paper, it was observed that real microscopic organisms are in contact with fluid on both sides. When adding a fluid to both sides of the swimmer, only the rate of dissipation of energy changed, from W to $2W$. The speed of the swimmer did not. When referring to Taylor's problem below, we are referring to the problem of an infinite, inextensible waving sheet with fluid on one side.

Another approach to the problem, in three dimensions, is to model the swimmer as an infinite cylinder. This method involves the use of what is commonly referred to as resistive force theory. Here, a distribution of stokeslets and doublets is placed along the center of the cylinder [15, 14, 7, 27]. Similar to resistive force theory is slender-body theory, which was introduced when incorporating the head of a microorganism into the analysis. It provides more accuracy in this instance than resistive force theory [16, 19]. In this swimming cylinder approach, the assumption of low amplitude waves is not necessarily made. However, when the assumption is made, the results agree qualitatively with those of the swimming sheet approach, that the swimming speed is proportional to the amplitude of the swimmer squared.

Of these above approaches, several include solid boundaries [33, 6, 22]. Katz, in [22], considered a swimmer in a channel that moves by propagating a prescribed wave down its length. This wave form is given by, letting (x_0, y_0) be a point on the sheet surface and $(x_m, 0)$ be the mean position of (x_0, y_0),

$$\begin{aligned} x_0 &= x_m + a\cos k[x_m + (c-V)t] + d\sin k[x_m + (c-V)t], \\ y_0 &= b\sin k[x_m + (c-V)t,] \end{aligned} \tag{1.3}$$

where c is the speed of the propagating wave and V the swimming speed. This wave is slightly more general than that considered by G. I. Taylor, in [37]. In [22], Katz found that the leading order velocity of the organism was proportional to the amplitude of the swimmer squared. Therefore, introducing this more general

wave form did not change the result qualitatively from those that assumed no walls. In the limit as one wall of the channel goes to infinity, Katz's results return the results of Blake, in [6], who considered a swimmer in the presence of one wall with a general wave form. One thing to note about this general wave form is that if the horizontal wave motion is large enough, the sheet can actually be propelled in the opposite direction.

In [22], Katz also considered the problem using lubrication theory. Lubrication theory is appropriate for the problem where the swimmer is very close to the wall. It was found that the leading order swimming speed, V_0, is larger than in the case with the wall being further away. The dimensional result was

$$\frac{V_0}{c} = \frac{3}{2 + \left(\frac{h}{b}\right)^2},\tag{1.4}$$

where h is the gap width. This implies that swimming speed is bounded by the speed of the propagating waves. For this lubrication problem, the velocities due to horizontal motion of the sheet are of a higher order and do not contribute to the leading order solution. Therefore, the sheet cannot reverse directions in this problem, as it could in the problem without the lubrication approximation applied.

The problem of swimmers in a channel has also been considered numerically. In [12], the authors look at the problem of one and multiple swimmers in a Newtonian fluid with walls that are both rigid and elastic. Here, the results were very comparable to real observations of sperm movement.

The problem of a swimmer in a viscoelastic fluid has also been discussed in various contexts. Chaudhury, in [9], modified Taylor's problem to include a second-order viscoelastic fluid. He found that viscoelasticity augments propulsion in low Reynolds numbers, but as Reynolds number increases, there is a range in which the viscoelasticity hampers it. More recently, a paper written by Fu et al. [13] considers the swimming infinite cylinder problem using an upper convected Maxwell model as the constitutive equation for the viscoelastic fluid. They found that viscoelasticity tends to decrease the swimming speed and that swimming speed can reverse direction. This negative effect of viscoelasticity is in direct contrast with the work done by Chaudhury in the two dimensional problem. Lauga, in [25], considered Taylor's problem in numerous different viscoelastic fluids. For all cases, he found that the swimming speed was dependent on the viscoelasticity of the fluid and that it was always less that the correspond-

ing Newtonian swimming speed. He also found that the result was unchanged if more general wave shapes, with both normal and tangential motion, were considered. In all the above viscoelastic problems, the assumption that the amplitude of the swimmer is small compared the wavelength of the swimmer was employed and so swimming speed is technically referring to leading order swimming speed.

In this thesis, we also consider the problem of an elastic swimmer. Here, both the the unknown swimming speed and swimmer shape are of primary importance. Some related work has been done in this area. In [10], the authors considered a single elastic swimmer in a channel with a Newtonian fluid. They solved the problem using Peskin's immersed boundary method. Their results were comparable to observed flagella and ciliary motion. There also exists some work done with elastic swimmers in slightly different situations. Lauga in [24] considered a lone, finite elastic swimmer in a Newtonian fluid. This problem of a low amplitude swimmer was solved using resistive force theory and analytical formulae were obtained. Work on finite elastic sheets in the presence of walls has been done as well, namely in the context of settling elastic sheets, or elastic sheets with one end clamped and oscillating vertically [2], [18]. Aside from the formulation of the force balances in these papers, particularly in [2], the results are relatively unrelated to our current discussions as they are for sheets and swimmers of finite length.

In this thesis, we use analytic methods, rather than numerics so that we may gain more understanding of the problem, elucidate the dependence on the physical parameters, and determine the importance of the non-Newtonian medium.

1.2 Overview of Non-Newtonian Fluids

When modelling fluids, we arm ourselves with equations that describe how the fluid flows. The Navier-Stokes equations governing Newtonian flow in vector form,

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \bigtriangledown \mathbf{u} \right) = - \bigtriangledown p + \mu \bigtriangledown^2 \mathbf{u} + \mathbf{F}. \qquad (1.5)$$

They represent the conservation of momentum.[1] Here ρ is the density of the fluid, $\mathbf{u} = (u, v)$ is the velocity field, p is the pressure within the fluid, μ is the

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[1] Note that we are considering the two dimensional equations here, not the full three dimensional ones.

viscosity of the fluid and **F** represents any body forces acting. These equations must be supplemented with continuity or conservation of mass,

$$\bigtriangledown \cdot \mathbf{u} = 0. \tag{1.6}$$

This is equivalent to assuming the fluid is incompressible or that density is constant. The question is, how do we describe the flow fluids that have behaviour that deviates from that of Newtonian fluid?

To begin with, we must consider the equations of motion in a more general form. Still assuming an incompressible fluid, the conservation of mass equation remains unchanged. However, let us write the conservation of momentum equations in a more general manner, referred to as Cauchy's equations of motion:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \bigtriangledown \mathbf{u} \right) = - \bigtriangledown p + \bigtriangledown \cdot \boldsymbol{\tau} + \mathbf{F}. \tag{1.7}$$

Here $\boldsymbol{\tau}$ is the deviatoric stress tensor. In the above Navier-Stokes equations for a Newtonian fluid, the relationship between the deviatoric stress and rate of strain is assumed to be

$$\boldsymbol{\tau} = \mu \dot{\boldsymbol{\gamma}}, \tag{1.8}$$

thus, giving the $\mu \bigtriangledown^2 \mathbf{u}$ term in equation (1.5). This constitutive relationship between stress and rate of strain is unable to describe the properties of non-Newtonian fluids. Let us put for now the constitutive relation between stress and rate of strain in a general form:

$$\mathbf{f}(\boldsymbol{\tau}, \dot{\boldsymbol{\gamma}}) = 0. \tag{1.9}$$

To get an idea of how a non-Newtonian fluid can differ from a Newtonian fluid, let us look at some standard examples of non-Newtonian fluids and related behaviours exhibited by these fluids.

To begin with, let us consider a cornstarch solution. This is an interesting example of what is called a shear thickening fluid.[2] Anyone that has ever made up their own home solution of cornstarch and water knows that when you play with it, in particular if you hit it or smack it with your palm, it immediately gets hard like a solid. But, when you move your hand through it gently, it behaves again like a liquid. Shear thickening means that the viscosity of the

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[2]It can be shown that cornstarch solutions have behaviour that is more complicated that that which would be expected of a simple shear thickening fluid [3]

fluid increases with the shear rate, or deformation rate. Therefore, when you smack the surface of your cornstarch solution, the rate of deformation increases, and so the viscosity increases, making the solution act momentarily like a solid, or a fluid of higher viscosity. Shear thickening behaviour is often observed in fairly concentrated suspensions of small particles.

Shear thickening is not the only interesting behaviour that non-Newtonian fluids can exhibit. The opposite effect has also been observed, that of shear thinning. In this instance, the viscosity decreases as the strain rate increases. Examples of fluids that exhibit shear thinning are molten polyethylene and polypropylene, as well as solutions of caryboxymethylcellulose in water and many others.

There also exists what are called viscoplastic fluids. These fluids exhibit a yield stress. If the yield stress of the fluid is exceeded, then the fluid flows. If the yield stress is not exceeded, then the material acts like a solid. Classic, everyday examples of viscoplastic fluids are mayonnaise and toothpaste.

The above behaviours are certainly not the only ones observed in non-Newtonian fluids that make them unique. An important phenomenon exhibited by visocelastic fluids is recoil, also referred to as fading memory or viscoelasticity, as the name would suggest. A good example to illustrate it is flow in a cylinder. Consider a viscoelastic fluid that starts at rest and is then subject to a pressure gradient, causing it to flow. This pressure gradient is then removed. What is observed is that the fluid actually recoils or moves backwards towards its original state. The amount of recoil in this situation is dependent on the specific viscoelastic fluid that is used. Some viscoelastic fluids exhibit very strong recoil, while with others it is less prominent. It is easy to see why this is often referred to as fading memory. A fully elastic material would return fully to its original state.

Two common examples of phenomena observed in viscoelastic fluids are jet swell and rod climbing. After exiting an orifice in a downward direction, a Newtonian fluid has a stream of a set diameter that is the same as that of the jet. In comparison, viscoelastic fluids have a diameter that increases twofold from the orifice diameter. This is also called the Barus or Merrington effect. Rod climbing, or the Weissenburg effect, refers to the situation where a thin solid cylinder is rotating in a viscoelastic fluid. It is observed that the fluid climbs up the rod, moving in the opposite direction to a Newtonian fluid in the same situation. A Newtonian fluid in this situation, besides moving in a different direction, does not climb the rod. In fact, a small indentation at the

base of the rod is observed.

Due to the composition of cervical mucous, namely that it is composed of various macromolecules, it is important to mention polymeric fluids.[3] Polymeric fluids are a specific type of viscoelastic fluid. Unlike Newtonian, fluids (and many non-Newtonian fluids), that are composed of micromolecules, polymeric fluids are composed of marcomolecules.[4] The rheological properties of a polymeric fluid are very dependent on the molecular weight distribution of the macromolecules that comprise it. That is, the rheological properties of a polymeric fluid depend not only on the number of molecules composing it, but the sizes and locations of those molecules as well. Polymeric fluids can exhibit many of the behaviours discussed above, including shear thinning and thickening, jet swell and rod climbing, and, obviously, viscoelasticity.[5]

1.2.1   Constitutive Models

There are many different ways to model non-Newtonian fluids ranging from relatively simple, to immensely complex, depending on the properties of the specific fluid one is interested in. On the simpler end of the scale there are generalized Newtonian fluid models and, for viscoelastic fluids, general linear viscoelastic fluid models. On the complex end of things there are corotational and codeformational models.

To begin with, let us look at generalized Newtonian fluids. This is the simple case when you take the constitutive model for a Newtonian fluid, as seen in equation (1.8), and relax the assumption that the viscosity, μ, is constant. The constant viscosity is replaced with a viscosity that varies with the shear rate, $\dot{\gamma}$, or even sometimes with $|\tau_{xy}|$. A standard of this type of constitutive law is the power law. This law states that

$$\mu = m\dot{\gamma}^{n-1},\tag{1.10}$$

where m and n are determined by the specific fluid under consideration.

Viscoplastic fluids exhibit a yield stress. A common example of a viscoplastic

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[3]Cervical mucous is a gel that consists of a network of glycoprotein macromolecules and a plasma component that contains soluble proteins, inorganic salts and other chemical species [21].

[4]Undiluted polymer solutions are typically referred to as melts. In any concentrated polymer solutions, the viscosity depends on concentration in a highly nonlinear fashion. In dilute solutions, the macromolecules are far apart from each other and therefore have negligible influence on each other.

[5]For a full, in depth discussion on polymeric fluids and the associated phenomena, see [5]

fluid is a Bingham fluid, which is modelled by

$$
\begin{aligned}
\mu &= \infty, && \tau \leq \tau_y, \\
\mu &= \mu_0 + \frac{\tau_y}{\dot{\gamma}}, && \text{otherwise.}
\end{aligned}
\tag{1.11}
$$

Another example of a viscoplastic fluid is a Herschel-Bulkley fluid, which also incorporates shear thinning, given by

$$
\begin{aligned}
\mu &= \infty, && \tau \leq \tau_y, \\
\mu &= (\tau_y + \mu|\dot{\gamma}|^n)\mathrm{sgn}(\dot{\gamma}), && \text{otherwise.}
\end{aligned}
\tag{1.12}
$$

Generalized linear viscoelastic fluid models are simple models of viscoelastic fluids that display some of the time dependent properties (i.e. recoil or viscoelasticity). These, however, are only applicable to situations that have low strains and shear rates. The most standard example of a generalized linear viscoelastic model is Maxwell's model

$$
\boldsymbol{\tau} + \lambda \frac{\partial}{\partial t} \boldsymbol{\tau} = \mu \dot{\boldsymbol{\gamma}},
\tag{1.13}
$$

where λ is a time constant and μ has units of viscosity. Another generalized linear viscoelastic model is the Jeffreys model,

$$
\boldsymbol{\tau} + \lambda_1 \frac{\partial}{\partial t} \boldsymbol{\tau} = \mu \left(\dot{\boldsymbol{\gamma}} + \frac{\partial}{\partial t} \dot{\boldsymbol{\gamma}} \right).
\tag{1.14}
$$

As you can see, this is just Maxwell's model with an extra term added in on the right-hand side, a time derivative of the strain rate and another time constant, λ_2. The Jeffreys model is important because it has been the starting point for other, more complicated rheological equations of state.

Both of these types of models are limited in scope. Ideally, we want a model that describes all flow phenomena, at least qualitatively. Two types of models that do this are corotational models and codeformational models. These are basically just versions of the linear viscoelastic models revised so that they are valid in situations when the fluid gets largely deformed from its initial position. Corotational models are formulated in a reference frame that corotates with the fluid. That is, the coordinates, call them θ_j, have a time rate of change given by

$$
\frac{\partial}{\partial t} \theta_j = -\frac{1}{2} \omega \cdot \theta_j,
\tag{1.15}
$$

where $\omega = \bigtriangledown \times \mathbf{u}$ is the fluid vorticity. Observers in this frame will not detect

rigid rotation. Codeformational models are formulated in codeforming coordinates, call them ϕ_j, with time rate of change

$$\frac{\partial}{\partial t}\phi_j = -\phi_j \cdot \bigtriangledown\mathbf{u}. \tag{1.16}$$

The models being used in this thesis are codeformational. For a more in depth discussion and analysis of these models, the reader is referred to [5] and [31].

Of specific interest are the Oldroyd models. The Oldroyd models can be both codeformational and corotational and are based on the Jeffreys model. We will restrict our discussion here to the codeformational type, since that is the type that will be used in this thesis. To begin with, let us look at the Oldroyd-B model. This model is in fact the Jeffreys model generalized in codeformational coordinates.[6] It is given by,

$$\boldsymbol{\tau} + \lambda_1\boldsymbol{\tau}^\bigtriangledown = \mu\left(\dot{\boldsymbol{\gamma}} + \lambda_2\dot{\boldsymbol{\gamma}}^\bigtriangledown\right). \tag{1.17}$$

Here, λ_1 is the polymer relaxation time, and λ_2, another relaxation time, represents the rate of decay of the residual rate of strain when the fluid is stress free, or the retardation time scale. The upper convected derivative of a tensor $\mathbf{A}$, is defined to be

$$\mathbf{A}^\bigtriangledown = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u}\cdot\bigtriangledown\mathbf{A} - \left((\bigtriangledown\mathbf{u})^T\cdot\mathbf{A} + \mathbf{A}\cdot\bigtriangledown\mathbf{u}\right). \tag{1.18}$$

It is possible to eliminate one relaxation time and write it in terms of the other by thinking of the fluid as being composed of a Newtonian solvent and a polymeric solute. This allows one to write the viscosities and deviatoric stresses as sums. We put $\mu = \mu_s + \mu_p$ and $\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_p$. In this situation, it is generally assumed that $\mu = \frac{\lambda_1}{\lambda_2}\mu_s$. This assumption allows us to remove one relaxation number and also allows us to get rid of the convected strain rate term. Applying these assumptions to the above Oldroyd-B constitutive equation simplifies it to the upper convected Maxwell model for the polymeric component,

$$\boldsymbol{\tau}_p + \lambda_1\boldsymbol{\tau}_p^\bigtriangledown = \mu_p\dot{\boldsymbol{\gamma}}. \tag{1.19}$$

---

[6]Technically, there are two possible generalizations of the Jeffreys model in codeformational coordinates, the Oldroyd-A model and the Oldroyd-B model. Oldroyd-A is very similar to Oldroyd-B, with the difference being in the convected derivative. Oldroyd-A is given by
$\boldsymbol{\tau} + \lambda_1\left(\frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau}\cdot(\bigtriangledown\mathbf{u})^T + \bigtriangledown\mathbf{u}\cdot\boldsymbol{\tau}\right) = \mu\left(\dot{\boldsymbol{\gamma}} + \lambda_2\left(\frac{D\dot{\boldsymbol{\gamma}}}{Dt} + \dot{\boldsymbol{\gamma}}\cdot(\bigtriangledown\mathbf{u})^T + \bigtriangledown\mathbf{u}\cdot\dot{\boldsymbol{\gamma}}\right)\right).$

Another Oldroyd model is the Oldroyd-8 model, aptly named as there are eight parameters in it. This model is also a generalization of the Jeffreys model, in which all possible quadratically nonlinear terms involving the deformation rates allowed by the symmetries of the problem are included. This model is given by,

$$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}^{\triangledown} + \frac{1}{2}\mu_0(\mathrm{tr}\boldsymbol{\tau})\dot{\boldsymbol{\gamma}} - \frac{1}{2}\mu_1\left(\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}}\cdot\boldsymbol{\tau}\right) + \frac{1}{2}\nu_1(\boldsymbol{\tau}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta} =$$
$$\mu\left(\dot{\boldsymbol{\gamma}} + \lambda_2\dot{\boldsymbol{\gamma}}^{\triangledown} - \mu_2(\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\gamma}}) + \frac{1}{2}(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}})\boldsymbol{\delta}\right). \qquad (1.20)$$

The eight constants and extra terms in this model allow for a much larger variety in rheological response compared to the Jeffreys model and the Oldroyd-B model [5]. This model is also convenient as it contains several other models with in it that can be obtained by merely setting some of the constants equal to zero, or multiples of each other. For example, it contains the Oldroyd-B model by setting all constants to zero except λ_1 and λ_2. It also contains the upper convected Maxwell model, the second-order fluid model, and the Oldroyd four and six constant models, to name a few.

Codeformational constitutive models can get significantly more complicated than those discussed above. However, a full discussion these rheological equations of state is not necessary here, as the only codeformational models we will be using are Oldroyd-B and Oldroyd-8 constitutive models.

1.3 Our Problem: Main Assumptions

Now that we are somewhat comfortable with viscoelastic fluids, we are in a position to set up our main problem mathematically.

Consider a two-dimensional, infinite periodic swimmer in the presence of a wall. We would like to consider the case when the fluid between the wall and the swimmer is viscoelastic. This is because the physical properties of cervical mucous are viscoelastic. We are assuming that the swimmer propagates waves down its length therefore propelling itself in the opposite direction. The shape of the swimmer is referred to as Y and is centered about $y = 0$. The wall is fixed at $y = h$, which is also the mean distance from the swimmer to the wall. The fact that the swimmer is periodic tells us that $\langle Y_x \rangle = 0$ and we assume that

$\langle Y \rangle = 0$, so that h is the mean gap width. Here

$$\langle \ldots \rangle = \frac{1}{2\pi} \int_0^{2\pi/k} (\ldots) dx. \tag{1.21}$$
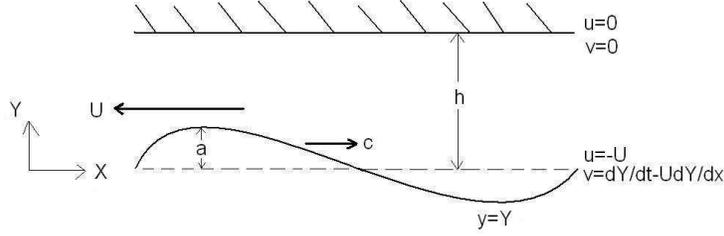
Figure 1.1: Geometry of Problem

The equations governing such a flow are

$$\bigtriangledown \cdot \mathbf{u} = 0, \tag{1.22}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \bigtriangledown \mathbf{u} \right) = -\bigtriangledown p + \bigtriangledown \cdot \boldsymbol{\tau} + \mathbf{F}, \tag{1.23}$$

$$\mathbf{f}(\boldsymbol{\tau}, \dot{\boldsymbol{\gamma}}) = 0. \tag{1.24}$$

Recall that the constitutive relation between stress and rate of strain in a general form. It is assumed that the swimmer propagates waves down its length in the positive x-direction, therefore propelling itself in the negative x-direction. The swimming speed is referred to as U, thus its velocity is $-U$. We assume that $U > 0$. Hence, we have boundary conditions as follows. For the horizontal velocity component,

$$u(x, h) = 0, \tag{1.25}$$

$$u(x, Y) = -U, \tag{1.26}$$

and for the vertical velocity component, assuming simple vertical motion rather

than complex elliptical trajectories of a material point on the swimmer we have,

$$v(x, h) = -UY_x + Y_t, \tag{1.27}$$

$$v(x, Y) = Y_t. \tag{1.28}$$

We will, however, not be solving this system in its entirety. We will make several simplifying assumptions to help ourselves along.

Due to the physical situation under consideration, a low Reynolds number regime is considered

$$0 < Re \ll 1, \tag{1.29}$$

allowing the inertial terms or the left hand side of Cauchy's equation of motion to be neglected. It will be assumed that there are no external forces acting in the problem

$$\mathbf{F} = 0, \tag{1.30}$$

that the unknown swimming speed of the sheet is constant and positive,

$$U = \text{Const} > 0. \tag{1.31}$$

It will also be assumed that the amplitude of the propagating waves is small compared to the wavelength.[7] It has been observed that in low viscosity mediums, activated (rather than hyperactivated) sperm generate relatively low amplitude, symmetrical waves.[8] This low amplitude assumption can be formulated mathematically by stating that

$$0 < \left(\max_{x \in [0, 2\pi]} |Y|\right) k = \varepsilon \ll 1, \tag{1.32}$$

where k is the wavenumber associated with Y. It is also assumed that the problem in its entirety is periodic in $(kx - (\omega - Uk)t)$. Of specific use will be the periodicity of the pressure, $\langle p_x \rangle = 0$.

To simplify things, we will consider the frame moving with the swimmer.

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[7]Note that this is not the same as saying that the amplitude of the propagating wave is small compared to the mean distance from the swimmer to the wall, or that a/h is small.

[8]Generally, in oviduct, sperm are hyperactivated, thus producing larger amplitude, more asymmetrical waves [36].

That is, put

$$x = \tilde{x} - Ut, \tag{1.33}$$

$$\frac{dx}{dt} = \frac{d\tilde{x}}{dt} - U. \tag{1.34}$$

This implies that the solutions will now be periodic in $(k\tilde{x} - \omega t)$. For the remainder of this thesis, we will assume that it is understood we are in the moving frame, and so will drop the tildes on the above changed variables.
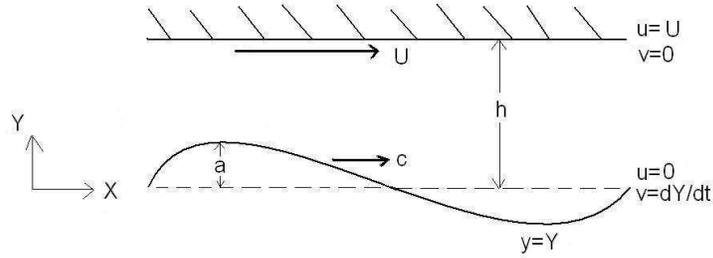
Figure 1.2: Geometry of Problem in Moving Frame of Reference

This simplifies the boundary conditions. In the new frame of reference, for the horizontal velocity component, we have

$$u(x, h) = U, \tag{1.35}$$

$$u(x, Y) = 0. \tag{1.36}$$

For the vertical velocity component,

$$v(x, h) = 0, \tag{1.37}$$

$$v(x, Y) = Y_t. \tag{1.38}$$

So far, the equations of motion, most assumptions of the problem and the new frame of reference have been specified.

1.3.1   Zero Force on Swimmer Condition

In addition to the above kinematical assumptions, we will also introduce a dynamical one. We assume that the average force per wavelength on the swimmer is zero. This is a reasonable assumption due to the fact that it has already been assumed that the speed of the swimmer is constant [22].

To formulate this condition precisely, we must establish what the force balances are in the problem.

We know that the force on the swimmer by the fluid must be equal to minus the total force on the fluid by the swimmer (per unit length). Thus, we can formulate this condition by considering the total force on the fluid.

It known that the total stress tensor, $\boldsymbol{\sigma}$, is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} = \begin{pmatrix} -p + \tau_{xx} & \tau_{xy} \\ \tau_{yx} & -p + \tau_{yy} \end{pmatrix},$$

and it is found that the unit outward normal to the swimmer is

$$\frac{1}{\sqrt{1 + Y_x^2}} \begin{pmatrix} Y_x \\ -1 \end{pmatrix}.$$

We can write the total force on the fluid per unit length as

$$\int_0^{2\pi} \begin{pmatrix} -p + \tau_{xx} & \tau_{xy} \\ \tau_{yx} & -p + \tau_{yy} \end{pmatrix} \begin{pmatrix} Y_x \\ -1 \end{pmatrix} dx.$$

Do to the force balance, we know that this is equal to minus the force on the swimmer, which is assumed to be zero. Thus, by writing things in component form, one can see that (recall the stress tensor is symmetric)

$$\begin{aligned} \langle (-p + \tau_{xx})Y_x - \tau_{xy} \rangle &= 0, \\ \langle \tau_{xy} Y_x - (-p + \tau_{yy}) \rangle &= 0, \end{aligned}$$

where the top equation is the tangential force, the bottom the normal force, recalling that $\langle \ldots \rangle$ denotes the x-average.

Now, consider the conservation of momentum equations in component form, slightly rearranged

$$\begin{aligned} (p - \tau_{xx})_x &= \tau_{xy,y}, \\ (p + \tau_{yy})_y &= \tau_{xy,x}. \end{aligned}$$

Concerning ourselves with only the first equation [9] and by applying the average in x while maintaining the assumed periodicity of the problem, we discover

$$\langle \tau_{xy} \rangle_y = 0,$$

and by integrating

$$\langle \tau_{xy} \rangle = \text{Const.}$$

Now, consider the same momentum equation. Rather than applying an average in x, integrate in y. This gives

$$\int_Y^{hk} (p - \tau_{xx})_x dy = \tau_{xy}(hk) - \tau_{xy}(Y).$$

Pulling the x-derivative out of the integral,

$$\partial_x \int_Y^{hk} (p - \tau_{xx}) dy + (p - \tau_{xx})\big|_{y=Y} Y_x = \tau_{xy}(hk) - \tau_{xy}(Y),$$

and again average in x to obtain,

$$\langle \tau_{xy}(hk) \rangle = \langle \tau_{xy}(Y) \rangle + \langle (p - \tau_{xx})\big|_{y=Y} Y_x \rangle.$$

But, notice that the right-hand side of this above equation is merely the tangential force balance equation evaluated at $y = Y$. Knowing then that this force is equal to zero leaves

$$\langle \tau_{xy}(hk) \rangle = 0,$$

and since it was calculated above that $\langle \tau_{xy} \rangle = \text{Const}$, we finally have

$$\langle \tau_{xy} \rangle = 0. \tag{1.39}$$

This is a constraint on the system that will be used in both Chapters Two and Three.

---

[9]At this time we will ignore the vertical force balance due to the fact that either the swimmer is assumed to be fixed vertically by some external force or a more complex vertical force balance is required utilizing elasticity theory, as will be seen in Chapter Three.

1.4 Preliminary Problem

In this section, we discuss the fundamental problem of a swimmer in the presence of a wall and a viscoelastic fluid. We do not apply the lubrication approximation here. We will assume that the amplitude of the waves is small in comparison to the mean distance to the wall and that the fluid between the swimmer and the wall is an Oldroyd-B viscoelastic fluid. This is a nice warm up problem as it is a combination of Katz's biharmonic problem and Lauga's viscoelastic version of the Taylor problem [22, 25]. The method of solution applied here follows closely with that used in [25]. See figure (1.2) for geometry.

1.4.1 Mathematical Formulation

We assume that the sheet has prescribed motion, that of waves propagating to the right, see figure (1.2). In the frame moving with the swimmer, these waves look like

$$y = a\sin(kx - \omega t). \tag{1.40}$$

The equations of motion for this problem are

$$\bigtriangledown \cdot \mathbf{u} = 0, \tag{1.41}$$

$$\bigtriangledown p = \bigtriangledown \cdot \boldsymbol{\tau}, \tag{1.42}$$

$$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}^{\bigtriangledown} = \mu\left(\dot{\boldsymbol{\gamma}} + \lambda_2 \dot{\boldsymbol{\gamma}}^{\bigtriangledown}\right), \tag{1.43}$$

which are the conservation of mass, momentum and the Oldroyd-B constitutive equation respectively. [10] Recall that the upper convected derivative of a tensor $\mathbf{A}$, is defined to be

$$\mathbf{A}^{\bigtriangledown} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \bigtriangledown \mathbf{A} - \left((\bigtriangledown \mathbf{u})^T \cdot \mathbf{A} + \mathbf{A} \cdot \bigtriangledown \mathbf{u}\right). \tag{1.44}$$

We also have the zero force condition[11]

$$\langle \tau_{xy} \rangle = 0. \tag{1.45}$$

---

[10]Recall that λ_1 is the polymer relaxation time, and λ_2 represents the rate of decay of the residual rate of strain when the fluid is stress free, or the retardation time scale.

[11]This equation still holds even though there is no lubrication approximation here due to the low amplitude compared to mean height approximation.

Non-dimensionalizing the equations as follows

$$
\begin{aligned}
t &= \tfrac{1}{\omega}\hat{t}, & (x,y) &= \tfrac{1}{k}(\hat{x},\hat{y}), \\
\mathbf{u} &= c\hat{\mathbf{u}}, & \dot{\boldsymbol{\gamma}} &= \omega\hat{\dot{\boldsymbol{\gamma}}}, \\
(p,\boldsymbol{\tau}) &= \mu\omega(\hat{p},\hat{\boldsymbol{\tau}}), & \hat{U} &= \tfrac{U}{c}
\end{aligned}
$$

where, $c = \frac{\omega}{k}$, is the speed of the travelling wave, and dropping the hat notation, yields

$$
\nabla p = \nabla \cdot \boldsymbol{\tau}, \tag{1.46}
$$

$$
\nabla \cdot \mathbf{u} = 0, \tag{1.47}
$$

which are unchanged, and

$$
\boldsymbol{\tau} + De_1 \boldsymbol{\tau}^{\nabla} = \dot{\boldsymbol{\gamma}} + De_2 \dot{\boldsymbol{\gamma}}^{\nabla}, \tag{1.48}
$$

the constitutive equation, where De_1 and De_2 are Deborah numbers. The Deborah number is the ratio of the characteristic time for the fluid over the characteristic time for the flow system. Here, we have two characteristic times for the fluid, and hence two Deborah numbers.

It remains to non-dimensionalize the boundary conditions. Using the low amplitude approximation, we put

$$
\varepsilon = ak \ll 1, \tag{1.49}
$$

and find that (dropping hats):

$$
y = \varepsilon \sin(x - t). \tag{1.50}
$$

The boundary conditions become, for the horizontal velocity component

$$
u(x, hk) = U, \tag{1.51}
$$

$$
u(x, Y) = 0, \tag{1.52}
$$

and for the vertical velocity component

$$
v(x, hk) = 0, \tag{1.53}
$$

$$
v(x, Y) = -\varepsilon \cos(x - t). \tag{1.54}
$$

1.4.2 Stream Function and Asymptotics

Since the problem at hand is an incompressible, two-dimensional problem, it is most useful to consider a stream function ψ such that [12]

$$u \;=\; \frac{\partial \psi}{\partial y}, \quad v \;=\; -\frac{\partial \psi}{\partial x}. \tag{1.55}$$

To solve this system, consider low amplitude solutions of the form

$$U \;=\; \varepsilon U_1 + \varepsilon^2 U_2 + ... \tag{1.56}$$

$$\psi \;=\; \varepsilon \psi_1 + \varepsilon^2 \psi_2 + ... \tag{1.57}$$

$$\boldsymbol{\tau} \;=\; \varepsilon \boldsymbol{\tau_1} + \varepsilon^2 \boldsymbol{\tau_2} + ... \tag{1.58}$$

That is, regular perturbation expansions in ε for the swimming speed, the stream function and the stress.

Since it is assumed that the speed of the swimmer is constant, it must be the case that each U_k in the expansion of U is also constant. Notice that, since the velocity field components can be written in terms of the stream function and $\dot{\boldsymbol{\gamma}}$ can be written in terms of the velocity field components, we can put

$$\dot{\boldsymbol{\gamma}} \;=\; \varepsilon \dot{\boldsymbol{\gamma}}_1 + \varepsilon^2 \dot{\boldsymbol{\gamma}}_2 + ... \tag{1.59}$$

$$\mathbf{u} \;=\; \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ... \tag{1.60}$$

Removing the pressure term from the conservation of momentum equations and then summarizing the equations of motion gives

$$\bigtriangledown \times \bigtriangledown \cdot \boldsymbol{\tau} \;=\; 0, \tag{1.61}$$

$$\boldsymbol{\tau} + De_1 \boldsymbol{\tau}^{\bigtriangledown} \;=\; \dot{\boldsymbol{\gamma}} + De_2 \dot{\boldsymbol{\gamma}}^{\bigtriangledown}. \tag{1.62}$$

The next step is to then substitute in the perturbation expansions and equate orders of ε.

---

[12]Conservation of mass holds automatically since
$\bigtriangledown \cdot \mathbf{u} = u_x + v_y = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0.$

1.4.3   Leading Order Equations

At $O(\varepsilon)$ we have:

$$\nabla \times \nabla \cdot \boldsymbol{\tau_1} \;=\; 0, \tag{1.63}$$

$$\boldsymbol{\tau_1} + De_1 \frac{\partial}{\partial t}\boldsymbol{\tau_1} \;=\; \dot{\boldsymbol{\gamma}}_1 + De_2 \frac{\partial}{\partial t}\dot{\boldsymbol{\gamma}}_1. \tag{1.64}$$

We find the boundary conditions at this order by substituting in the regular perturbation expansion for ψ and expanding about $y = 0$. This gives

$$
\begin{aligned}
\psi_{1x}\big|_{(x,0)} &= 0, & \psi_{1y}\big|_{(x,0)} &= 0, \\
\psi_{1x}\big|_{(x,hk)} &= \cos(x - t), & \psi_{1y}\big|_{(x,hk)} &= U_1.
\end{aligned}
\tag{1.65}
$$

The zero force constraint $\langle \tau_{xy} \rangle = 0$ yields that

$$\langle \tau_{1xy} \rangle = 0. \tag{1.66}$$

If one stares at these leading order equations for a moment, it becomes apparent that the easiest thing to do is to take the divergence and then the curl of equation (1.64),thus eliminating leading order stress. By doing this we obtain

$$\left(1 + De_2 \frac{\partial}{\partial t}\right) \nabla^4 \psi_1 = 0, \tag{1.67}$$

where $\nabla^4 \psi = \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}$.

To solve this equation, we simply use separation of variables with hyperbolic sines and cosines. Assume that the form of the solution is

$$\psi_1(y) = [(A + BH)\cosh(H) + (C + DH)\sinh(H)]\sin(x - t),$$

where $H = hk - y$. We find then that

$$U_1 = 0, \tag{1.68}$$

and

$$\psi_1 = \frac{W(y)}{W(0)}\sin(x - t), \tag{1.69}$$

where

$$\Psi_1(y) = (hk - y)\sinh(hk - y), \qquad (1.70)$$

$$\Psi_2(y) = \sinh(hk - y) - (hk - y)\cosh(hk - y), \qquad (1.71)$$

and

$$W(y) = \Psi_1'(0)\Psi_2(y) - \Psi_2'(0)\Psi_1(y). \qquad (1.72)$$

So, the leading order swimming speed is zero, a result that is precedented. Thus, we must look to second order to find the speed. To do this, we must calculate the leading order stress.

To solve for the leading order stress, we employ the method of 'complexification'. That is, we think of the stress and rate of strain as being the real portion of a larger, complex problem that satisfies the same equations.[13] We put

$$\boldsymbol{\tau_1} = \Re[\tilde{\boldsymbol{\tau}}_1 e^{i(x-t)}], \quad \dot{\boldsymbol{\gamma}}_1 = \Re[\tilde{\dot{\boldsymbol{\gamma}}}_1 e^{i(x-t)}]. \qquad (1.73)$$

By employing this method of solution, we can easily obtain

$$\tilde{\boldsymbol{\tau}}_1 = \frac{1 - iDe_2}{1 - iDe_1}\tilde{\dot{\boldsymbol{\gamma}}}_1, \qquad (1.74)$$

and upon substituting in $\tilde{\dot{\boldsymbol{\gamma}}}_1$, we get

$$\boldsymbol{\tau_1} = \Re\left\{ \frac{(1 - iDe_2)e^{i(x-t)}}{(1 - iDe_1)W(0)} \begin{pmatrix} 2W' & -i(W'' + W) \\ -i(W'' + W) & -2W' \end{pmatrix} \right\}. \qquad (1.75)$$

Notice that the condition,

$$\langle \tau_{1xy} \rangle = 0,$$

automatically holds due to the fact that $\tau_{1xy} \propto \dot{\gamma}_{1xy} \propto \sin(x - t)$.

1.4.4 Second Order Equations

The leading order problem has been solved and it has been shown that in order to maintain the assumption that the speed of the swimmer is constant, at leading order it must be equal to zero. Thus, in order to calculate the swimming speed of the organism, we must move on to the second order problem. This problem is

---

[13]Note the periodic dependence explicitly given due to the assumed periodicity of the problem

not quite as straightforward as the leading order problem due to the nonlinear terms of the upper convected derivative. At $O(\varepsilon^2)$ we have

$$\bigtriangledown \times \bigtriangledown \cdot \boldsymbol{\tau_2} = 0, \qquad (1.76)$$

$$\boldsymbol{\tau_2} + De_1 \frac{\partial}{\partial t} \boldsymbol{\tau_2} + De_1 \left(\mathbf{u_1} \cdot \bigtriangledown \boldsymbol{\tau_1} - \boldsymbol{\tau_1} \cdot \bigtriangledown \mathbf{u_1} - (\bigtriangledown \mathbf{u_1})^T \cdot \boldsymbol{\tau_1} \right) =$$

$$\dot{\boldsymbol{\gamma_2}} + De_2 \frac{\partial}{\partial t} \dot{\boldsymbol{\gamma_2}} + De_2 \left(\mathbf{u_1} \cdot \bigtriangledown \dot{\boldsymbol{\gamma_1}} - \dot{\boldsymbol{\gamma_1}} \cdot \bigtriangledown \mathbf{u_1} - (\bigtriangledown \mathbf{u_1})^T \cdot \dot{\boldsymbol{\gamma_1}} \right), \qquad (1.77)$$

with boundary conditions

$$\begin{aligned} \psi_{2x}\big|_{(x,0)} &= 0, & \psi_{2y}\big|_{(x,0)} &= -\sin^2(x-t)\frac{W''(0)}{W(0)}, \\ \psi_{2x}\big|_{(x,hk)} &= 0, & \psi_{2y}\big|_{(x,hk)} &= U_2. \end{aligned} \qquad (1.78)$$

The zero force condition states that

$$\langle \tau_{2xy} \rangle = 0. \qquad (1.79)$$

Inspecting the second order equations of motion reveals that the right-hand side of equation (1.77) is composed of leading order terms exclusively. To solve, let us put the right-hand side of the momentum equations in terms of the complexified variables and look only at the xy-component

$$\left(1 + De_1 \frac{\partial}{\partial t} \right) \tau_{2xy} - \left(1 + De_2 \frac{\partial}{\partial t} \right) \dot{\gamma}_{2xy} =$$

$$\Re \left\{ \frac{(De_2 - De_1)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1^* - (\bigtriangledown \tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] \right\}_{xy} +$$

$$+ \Re \left\{ \frac{(De_2 - De_1)}{2(1 - iDe_1)} e^{2i(x-t)} \left[\tilde{\mathbf{u}}_1 \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1 - (\bigtriangledown \tilde{\mathbf{u}}_1)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] \right\}_{xy}$$

$$(1.80)$$

This allows us to utilize equation (1.79) as a simplification. To apply this condition, let us take the x-average of equation (1.80). Then, by writing the rate of strain component in terms of the stream function, and observing the fact that $\langle \psi_{2xx} \rangle = \frac{1}{2\pi} \psi_{2x} \big|_0^{2\pi} = -\frac{1}{2\pi} v \big|_0^{2\pi} = 0$ due to the assumed periodicity of the solution, we obtain

$$\langle \psi_2 \rangle_{yy} = \Re \left\{ \frac{(De_1 - De_2)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1^* - (\bigtriangledown \tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} \right\}. \qquad (1.81)$$

To solve the problem and calculate the swimming speed we must integrate the above expression for $\langle\psi_2\rangle_{yy}$ once to obtain an expression for $\langle\psi_2\rangle_y$ and apply the boundary conditions. To be able to apply the boundary conditions, we must put them into the same averaged form as the equation. For the conditions on ψ_x, we obtain trivial averages due to the periodicity of the problem, $\langle\psi_{2x}\rangle = \frac{1}{2\pi}\psi_2\big|_0^{2\pi} = 0$, and thus no information. However, for the other two boundary conditions we obtain

$$\langle\psi_{2y}\rangle\big|_{y=0} = -\frac{W''(0)}{2W(0)}, \tag{1.82}$$

$$\langle\psi_{2y}\rangle\big|_{y=hk} = U_2. \tag{1.83}$$

By integrating we find that, writing $W(0) = W_0$ and $W''(0) = W_0''$ for simplicity,

$$\langle\psi_2\rangle_y = \frac{De_1(De_2 - De_1)}{2(1 + De_1^2)W_0^2}\left[WW'' + W'^2\right] - \frac{W_0''}{2W_0}\frac{(1 + De_1De_2)}{(1 + De_1^2)}. \tag{1.84}$$

Finally, the swimming speed is obtained,

$$U_2 = \langle\psi_2\rangle_y\big|_{y=hk} = -\frac{W_0''}{2W_0}\frac{(1 + De_1De_2)}{(1 + De_1^2)}. \tag{1.85}$$

Note that $W(hk) = 0$ and $W'(hk) = 0$.

1.4.5 Limiting Cases

Distance Between Wall and Swimmer Approaches Infinity

We would like to assure ourselves that this problem agrees with the infinite problem considered Lauga, in [25]. To do this, we take the limit as the distance between the wall and the swimmer approaches infinity. It is known that

$$\psi_1 = \frac{W(y)}{W(0)}\sin(x - t),$$

and so the limit as $h \longrightarrow \infty$ is easily calculated. We find that

$$\lim_{h\longrightarrow\infty}\frac{W(y)}{W(0)} = (1 + y)e^{-y}.$$

Thus,

$$\lim_{h\longrightarrow\infty}\psi_1 = (1 + y)e^{-y}\sin(x - t),$$

which is exactly the infinite solution as stated in [25].

It is obvious that if the limit of the streamfunction agrees with the infinite case that the limit of the swimming speed will as well, as $\psi_{2y}(hk) = U_2$. When calculated we find that

$$\lim_{h \longrightarrow \infty} U_2 = \frac{1}{2} \frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

which again agrees with the Lauga's result.

Similarly, we can calculate the limit for the averaged rate of work done by the swimmer. We find that

$$\lim_{h \longrightarrow \infty} \left\{ \frac{(1 + De_1 De_2)}{(1 + De_1^2)W_0^2} \int_0^{hk} \left[4W'^2 + (W + W'')^2 \right] dy \right\} = 2 \frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

which again agrees with the previous work done.[14]

So, it is clear that the above calculations for the problem of an organism swimming in an Oldroyd-B viscoelastic fluid near a wall are consistent with those of the problem of an organism swimming in an infinite fluid of the same type.

Swimmer Approaches the Wall

Next, we would like to consider the limit as the swimmer and the wall approach each other. There are two ways of thinking of this limit. The first is keeping the wall fixed at $y = h$ and increasing the amplitude of the swimmer until it reaches the wall. This line of thinking contradicts the assumption that $ak << 1$, as it implies that $O(a) \to O(h) = O(1/k)$. This would make our present solution invalid. The second way of thinking about this limit is to think of the wall approaching the swimmer, whose amplitude and wavelength remains the fixed. This implies that $O(h) \to O(a)$, still maintaining the assumption that $ak << 1$. In fact, in this manner of thinking we get that $hk << 1$ which is the same assumption that is applied when making the lubrication approximation. Taking this limit in this case, however, is not the same as applying the lubrication approximation fully. It is just a limiting case of the problem under discussion, and will suggest to us to fully apply the lubrication approximation to understand

---

[14]Please see Appendix A for a full derivation of this problem and the calculation of the work done.

this limit better. We have then, to leading order of this limiting case,

$$U = -\varepsilon^2 \frac{(1 + De_1 De_2)}{2(1 + De_1^2)} F(hk),$$

(1.86)

where

$$F(hk) = \frac{W_0''}{W_0} = \frac{(hk)^2 + \sinh^2(hk)}{(hk)^2 - \sinh^2(hk)}.$$

(1.87)

In our limit, $hk \to ak = \varepsilon$ and so we have that

$$F(\varepsilon) = \frac{\varepsilon^2 + \sinh^2 \varepsilon}{\varepsilon^2 - \sinh^2 \varepsilon}.$$

(1.88)

Expanding $\sinh \varepsilon = \varepsilon + \frac{1}{6}\varepsilon^3 + \ldots$ and simplifying gives us

$$F(\varepsilon) = \frac{2\varepsilon^2 + +\frac{1}{3}\varepsilon^4 + \cdots}{-\frac{1}{3}\varepsilon^4 - \cdots}.$$

(1.89)

Thus, in the limit when the wall approaches the swimmer, we find that the swimming speed is

$$U = 3\frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

(1.90)

which is significantly larger order than the original solution.[15] Thus, this problem suggests a more in depth analysis of the problem with the lubrication approximation specifically applied from the start of the formulation of the problem.

For a more comprehensive, detailed discussion of this problem please see Appendix A.

1.5   Thesis Outline

Chapter Two presents the problem of a infinite swimming sheet with the lubrication approximation applied. The sheet shape here is assumed. This problem is first solved for a Newtonian fluid in a preliminary sense, and is then solved for a viscoelastic fluid represented by the Oldroyd-8 constitutive model. Chapter Three presents the problem of a infinite swimming elastic sheet with the lubrication approximation applied. In this problem, there is a prescribed forcing

---

[15]Note that in this limit we do not take into account what is happening to the shear rates, which will be getting significantly larger. This is just a motivational example to show that when the swimmer approaches the wall, the swimming speed is not longer proportional to the amplitude of the swimmer squared and something more interesting is happening.

along the length of the sheet. This problem is solved for a Newtonian fluid for forcing of any strength, and then for a viscoelastic fluid, also represented by the Oldroyd-8 constitutive model, with forcing of moderate strength. The final chapter presents a summary of the findings and concluding remarks, including further work that may be done.

Chapter 2

Swimming with Lubrication Theory

As motivated by the preliminary problem discussed above, we now consider the problem of a two dimensional swimmer very near a wall. Particularly, we apply the lubrication approximation and assume that the mean distance of the swimmer from the wall, h, compared to the wavelength of the swimmer is very small.[16]
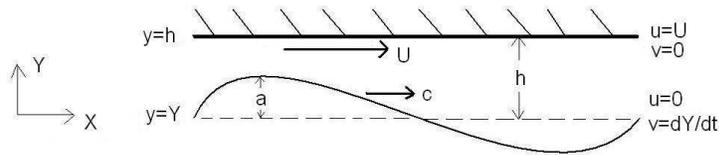
Figure 2.1: Lubrication geometry

2.1 Mathematical Formulation

Let us consider a swimmer in the presence of a wall with a fluid in between. The swimmer shape is assumed to be known. We put it to be

$$y = a \sin(kx - \omega t). \tag{2.1}$$

We consider this wave form and not a more general one because it has been shown by Katz in [22] that for the Newtonian lubrication problem, horizontal contributions to the wave shape only affect the higher orders of the solution. For the viscoelastic problem of no wall, Lauga in [25] considered horizontal

---

[16]For a mathematical discussion of viscoelastic lubrication theory, we refer the reader to [4]

oscillations of the same order as the amplitude in the sheet and found that they did not affect the leading order solution. Thus, it is very reasonable to assume that horizontal contributions to the wave form will not affect our leading order solution, and so they will be neglected.

Here, we are applying the lubrication approximation. Mathematically this means that

$$hk \ll 1. \tag{2.2}$$

The equations for this problem, using the general assumptions stated in Chapter One, are

$$\bigtriangledown \cdot \mathbf{u} = 0, \tag{2.3}$$

$$\bigtriangledown p = \bigtriangledown \cdot \boldsymbol{\tau}, \tag{2.4}$$

$$\mathbf{f}(\boldsymbol{\tau}, \dot{\boldsymbol{\gamma}}) = 0, \tag{2.5}$$

which are the conservation of mass, momentum and the general constitutive equations respectively.[17] This problem is also subject to the constraint that

$$\langle \tau_{xy} \rangle = 0, \tag{2.6}$$

from the zero force condition. As well, we require that the periodicity in pressure be considered explicitly for this problem, thus we will include

$$\langle p_x \rangle = 0, \tag{2.7}$$

in our formulation.

The boundary conditions are, for the horizontal velocity component

$$u(x, h) = U, \tag{2.8}$$

$$u(x, Y) = 0, \tag{2.9}$$

and for the vertical velocity component

$$v(x, h) = 0, \tag{2.10}$$

$$v(x, Y) = Y_t, \tag{2.11}$$

---

[17]Note that due to the low Reynold's number assumption, the inertial terms are neglected in the conservation of momentum equation.

as we can see in (2.1). Recall that Y is the shape of the swimmer.

Let us non-dimensionalize these equations and apply the lubrication approximation. We will use the following non-dimensional variables [42]:

$$
\begin{aligned}
t &= \tfrac{1}{\omega}\hat{t}, & \dot{\boldsymbol{\gamma}} &= \omega\hat{\dot{\boldsymbol{\gamma}}}, \\
x &= \tfrac{1}{k}\hat{x}, & y &= h\hat{y}, \\
\mathbf{u} &= c\hat{\mathbf{u}}, \\
(p, \boldsymbol{\tau}) &= \tfrac{\mu c}{h}(\tfrac{1}{\varepsilon}\hat{p}, \hat{\boldsymbol{\tau}}), & \hat{U} &= \tfrac{U}{c}
\end{aligned}
$$

where $c = \frac{\omega}{k}$ is the speed of the travelling wave.

Upon substituting in the above variables (dropping hats), we find that the conservation of mass equation remains unchanged

$$u_x + v_y = 0. \tag{2.12}$$

Substitution of the non-dimensional variables into the conservation of momentum yields (again, hats dropped),

$$
\begin{aligned}
p_x &= \varepsilon\tau_{xx,x} + \tau_{yx,y}, \\
p_y &= \varepsilon^2\tau_{xy,x} + \varepsilon\tau_{yy,y}.
\end{aligned}
$$

Thus, at leading order we have,

$$
\begin{aligned}
p_x &= \tau_{yx,y}, & (2.13) \\
p_y &= 0. & (2.14)
\end{aligned}
$$

This tells us that pressure is independent of y and that the xy-component of the stress tensor is the dominant component. The shape of the swimmer becomes

$$Y = \frac{a}{h}\sin(x - t). \tag{2.15}$$

We can then note that, $Y_t = -Y_x$, when in dimensionless variables.

Observe that if we integrate the equation $p_x = \tau_{yx,y}$

$$\int_Y^y p_x dy = \int_Y^y \tau_{yx,y} dy,$$

we get that

$$\tau_{xy} = \tau_0 + (y - Y)p_x, \tag{2.16}$$

where,

$$\tau_0 = \tau_{xy}|_{y=Y},$$

is the stress on the swimmer. If we observe then that

$$\langle \tau_{xy} \rangle = \langle \tau_0 + (y - Y)p_x \rangle = \langle \tau_0 - Y p_x \rangle + y \langle p_x \rangle$$

and apply the periodic pressure condition, $\langle p_x \rangle = 0$ which remains unchanged in dimensionless variables, the zero force condition simplifies to

$$\langle \tau_0 - p_x Y \rangle = 0. \tag{2.17}$$

Note that since we have

$$p_x = \frac{\tau_1 - \tau_0}{1 - Y},$$

where $\tau_1 = \tau_{xy}|_{y=1}$, it is the case that

$$\tau_0 - Y p_x = \tau_1 - p_x.$$

Thus, we have that

$$\langle \tau_0 - Y p_x \rangle = \langle \tau_1 \rangle = 0, \tag{2.18}$$

since we are assuming that $\langle p_x \rangle = 0$.

So, our final equations are then

$$u_x + v_y \;\; = \;\; 0, \tag{2.19}$$
$$\tau_{xy} \;\; = \;\; \tau_0 + (y - Y)p_x, \tag{2.20}$$
$$\mathbf{f}\left(\hat{\boldsymbol{\tau}}, \hat{\dot{\gamma}}\right) \;\; = \;\; 0, \tag{2.21}$$
$$\langle p_x \rangle \;\; = \;\; 0, \tag{2.22}$$
$$\langle \tau_1 \rangle \;\; = \;\; 0, \tag{2.23}$$

and non-dimensional boundary conditions

$$u(x, 1) \;\; = \;\; U, \tag{2.24}$$
$$u(x, Y) \;\; = \;\; 0, \tag{2.25}$$

and for the vertical velocity component

$$v(x, 1) \quad = \quad 0, \tag{2.26}$$

$$v(x, Y) \quad = \quad -Y_x. \tag{2.27}$$

2.2 Newtonian Problem

Due to this problem's complexity, let us first solve it with a Newtonian fluid. Not only will this give us some insight into the more general problem of a non-Newtonian fluid, but the results will be used as an initial guess for the numerical method used to solve the viscoelastic lubrication problem, detailed in the following section. For the viscous problem we have the following:

$$u_x + v_y \quad = \quad 0, \tag{2.28}$$

$$p_x \quad = \quad u_{yy}, \tag{2.29}$$

$$p_y \quad = \quad 0, \tag{2.30}$$

$$\langle u_y|_{y=Y} - p_x Y \rangle \quad = \quad 0, \tag{2.31}$$

$$\langle p_x \rangle \quad = \quad 0, \tag{2.32}$$

and boundary conditions given by equations (2.24), (2.25), (2.26), and (2.27).

To solve, let us first integrate $p_x = u_{yy}$ twice in y exploiting the fact that p_x is independent of y and apply the boundary conditions on u to obtain

$$u = \frac{1}{2} p_x (y - Y)(y - 1) + U \frac{(y - Y)}{(1 - Y)}. \tag{2.33}$$

Then, look at the continuity equation. Integrate in y first to obtain

$$\int_Y^1 u_x dy + Y_x = 0.$$

Pulling the x derivative out of the integral, noting that $u|_{y=Y} = 0$, and integrating in x gives

$$\int_Y^1 u \, dy + Y = q,$$

where q is the mass flux. Upon calculating this integral and solving for p_x we

find

$$p_x = 12\frac{(1-q)}{(1-Y)^3} + 6\frac{(U-2)}{(1-Y)^2}. \tag{2.34}$$

Thus, it is left only to determine q and U. To do this, we will simply apply the constraints $\langle p_x \rangle = 0$, $\langle u_y|_{y=Y} - p_x Y \rangle = 0$ and solve simultaneously. This involves solving integrals of the form

$$I_j^* = \int_0^{2\pi} \frac{dx}{(1-Y)^j} \qquad j = 1, 2, 3 \tag{2.35}$$

This gives us that

$$U = q = \frac{6\left(I_1^* I_3^* - I_2^{*2}\right)}{\left(4I_1^* I_3^* - 3I_2^{*2}\right)}. \tag{2.36}$$

Solving the integrals specifically gives

$$I_1^* = \frac{1}{\left(1 - \left(\frac{a}{h}\right)^2\right)^{1/2}}, \tag{2.37}$$

$$I_2^* = \frac{1}{\left(1 - \left(\frac{a}{h}\right)^2\right)^{3/2}}, \tag{2.38}$$

$$I_3^* = \frac{\left(\left(\frac{a}{h}\right)^2 + 2\right)}{\left(1 - \left(\frac{a}{h}\right)^2\right)^{5/2}}, \tag{2.39}$$

which yields

$$U = q = \frac{3\left(\frac{a}{h}\right)^2}{\left(2\left(\frac{a}{h}\right)^2 + 1\right)}. \tag{2.40}$$

Dimensionally,

$$\frac{U}{c} = \frac{3\left(\frac{a}{h}\right)^2}{\left(2\left(\frac{a}{h}\right)^2 + 1\right)}. \tag{2.41}$$

This is exactly the result found by Katz, in [22]. By writing this in the form

$$\frac{U}{c} = \frac{3}{2 + \left(\frac{h}{a}\right)^2}, \tag{2.42}$$

we can easily see that the swimming speed is bounded by the wave speed. Substituting equation (2.41) into the expression for p_x, u and τ_1 solves the problem entirely. Of particular interest are the formulas for the pressure gradient and the stress on the wall. The formula for is given in in equation (2.34), and

the formula for the stress is given by

$$\tau_1 = \frac{(4U - 6)}{(1 - Y)} - \frac{6(q - 1)}{(1 - Y)^2}.$$ (2.43)

Now that we have a basic procedure for solving the problem, let us apply it to the viscoelastic problem.

2.3 Viscoelastic Problem

With the Newtonian problem solved fully above, we are now in a position to consider the lubrication problem with a viscoelastic fluid in the gap between the swimmer and the wall. When substituting a viscoelastic fluid into the problem for a Newtonian fluid, we have to pick a constitutive model for the viscoelastic fluid. As detailed briefly in Chapter One, this is not necessarily a straight forward task. We use the Oldroyd-8 model because it is the most general weakly-nonlinear constitutive model that includes all possible quadratically nonlinear terms involving the deformation rates allowed by the symmetries of the problem [32]. It is given by,

$$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}^\triangledown + \frac{1}{2}\mu_0 (\mathrm{tr}\boldsymbol{\tau})\dot{\boldsymbol{\gamma}} - \frac{1}{2}\mu_1 \left(\boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\tau} \right) + \frac{1}{2}\nu_1 (\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \quad =$$
$$\mu \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \dot{\boldsymbol{\gamma}}^\triangledown - \mu_2 (\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}) + \frac{1}{2}(\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}})\boldsymbol{\delta} \right),$$ (2.44)

where $\boldsymbol{\delta}$ is the two-dimensional identity tensor. Recall the upper convected derivative of a tensor $\mathbf{A}$,

$$\mathbf{A}^\triangledown = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \partial \mathbf{A} - \left((\triangledown\mathbf{u})^T \cdot \mathbf{A} + \mathbf{A} \cdot \triangledown\mathbf{u} \right).$$ (2.45)

The substitution of the non-dimensional variables into the Oldroyd-8 constitutive model is non-trivial. The leading order equations component wise are:

$$\tau_{xx} - \left(2De_1 + \frac{\mu_1 c}{h} - \frac{\nu_1 c}{h} \right) u_y \tau_{xy} + \left(2De_2 + \frac{\mu_2 c}{h} - \frac{\nu_2 c}{h} \right) u_y^2 \quad = \quad 0,$$ (2.46)

$$\tau_{xy} - De_1 u_y \tau_{yy} + \frac{1}{2} \left(\frac{\mu_0 c}{h} - \frac{\mu_1 c}{h} \right) (\tau_{xx} + \tau_{yy}) u_y - u_y \quad = \quad 0,$$ (2.47)

$$\tau_{yy} - \left(\frac{\mu_1 c}{h} - \frac{\nu_1 c}{h} \right) u_y \tau_{xy} + \left(\frac{\mu_2 c}{h} - \frac{\nu_2 c}{h} \right) u_y^2 \quad = \quad 0.$$ (2.48)

Note that the time derivatives and advective terms are negligible here, just as

they are in the conservation of momentum equations.

The component of the stress tensor that we want to solve for is τ_{xy}, since this is the component that appears in the equations of motion. Solving the equations (2.46) through (2.48), we find that

$$\tau_{xy} = \frac{1 + \alpha u_y^2}{1 + \beta u_y^2} u_y, \tag{2.49}$$

where

$\alpha = \left(\frac{\nu_2 c}{h} - \frac{\mu_2 c}{h}\right)\left(De_1 - \frac{1}{2}\left(\frac{\mu_0 c}{h} - \frac{\mu_1 c}{h}\right)\right) + \frac{1}{2}\left(\frac{\mu_0 c}{h} - \frac{\mu_1 c}{h}\right)\left(2De_2 + \frac{\mu_2 c}{h} - \frac{\nu_2 c}{h}\right),$

and

$\beta = \left(\frac{\nu_1 c}{h} - \frac{\mu_1 c}{h}\right)\left(De_1 - \frac{1}{2}\left(\frac{\mu_0 c}{h} - \frac{\mu_1 c}{h}\right)\right) + \frac{1}{2}\left(\frac{\mu_0 c}{h} - \frac{\mu_1 c}{h}\right)\left(2De_1 + \frac{\mu_1 c}{h} - \frac{\nu_1 c}{h}\right).$

We are using this model only as an example. Much more general models can also be dealt with, so long as we know $\tau_{xy}(u_y)$, and that the details of the constitutive model do not interfere with the key simplification of the lubrication theory, namely that the law reduces to that for a steady-state shear flow for the thin-gap geometry with dominant shear stress.

2.3.1 Effective Viscosity

Observe that the deviatoric stress equation,

$$\tau_{xy} = \frac{1 + \alpha u_y^2}{1 + \beta u_y^2} u_y, \tag{2.50}$$

can be written as

$$\tau_{xy} = \tilde{\mu}(u_y; \alpha, \beta) u_y, \tag{2.51}$$

where we can think of

$$\tilde{\mu}(u_y; \alpha, \beta) = \frac{1 + \alpha u_y^2}{1 + \beta u_y^2}$$

as being an effective viscosity. In the Newtonian case we have that $\tilde{\mu}(u_y; \alpha, \beta) = 1$.

We can see in figure (2.2), detailed in table (2.1), that there are five fluid type regimes, including the Newtonian case, depending on the parameters α and β. Two of these regimes are physically unrealistic because the viscosity diverges and reverses sign for increasing shear rate. We will focus our subsequent analysis on

the more realistic regimes of Newtonian, shear thinning and thickening fluids.[18]
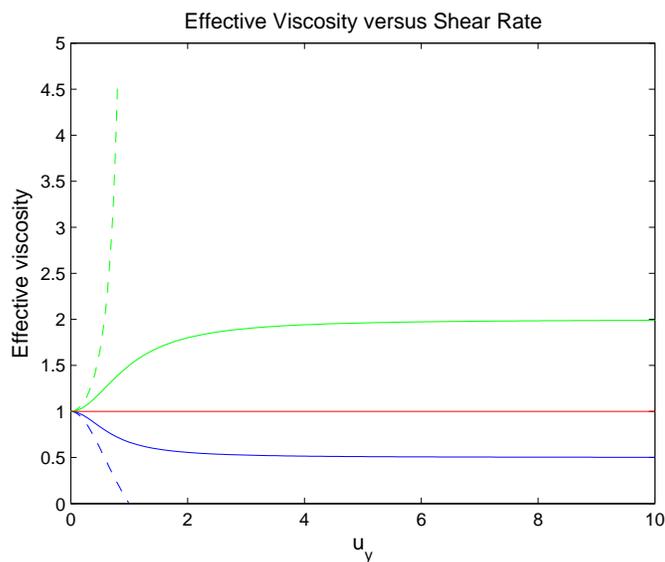
Figure 2.2: Effective Viscosities. Red: Newtonian, Solid Blue: Shear Thinning, Dashed Blue: Unrealistic Shear Thinning, Solid Green: Shear Thickening, Dashed Green: Unrealistic Shear Thickening

Fluid Types	
Newtonian	$\alpha = \beta$
Shear Thinning	$0 < \alpha < \beta$
Shear Thickening	$0 < \beta < \alpha$
Unrealistic Thinning	$\alpha < 0 < \beta$
Unrealistic Thickening	$\beta < 0 < \alpha$

Table 2.1: Fluid types and the corresponding parameter values.

The specific parameter values chosen to create figure (2.2) are listed below, in table (2.2). These are also the parameter values that are used in the numerical method detailed in the following section. Obviously, any parameter values that satisfy the restrictions in table (2.1) could be used. These are just chosen as a

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[18]Notice that if we reduce the Oldroyd-8 model to the Oldroyd-B model, we find that $\alpha = \beta = 0$, returning to us the Newtonian problem.

specific example.

Fluid Types	Parameters	
	α	β
Newtonian	1	1
Shear Thinning	1	2
Shear Thickening	2	1
Unrealistic Thinning	-1	1
Unrealistic Thickening	1	-1

Table 2.2: The specific parameter values chosen for the various fluid types.

2.3.2 Numerical Formulation

In this section, we will simplify the viscoelastic lubrication equations in a fashion that allows us to solve them numerically with relative ease. Once the equations are simplified, we will discretize them and outline the numerical method applied.

We will use the method employed in the Newtonian problem as a guideline. Since we cannot solve for u explicitly in the viscoelastic problem, let us first consider the continuity equation $u_x + v_y = 0$. As we did in the Newtonian case, let us integrate twice to obtain

$$\int_Y^1 u dy + Y = q. \tag{2.52}$$

We will exploit the fact that

$$\int_Y^1 u_y dy = U, \tag{2.53}$$

and so can write

$$\int_Y^1 u dy = U - \int_Y^1 y u_y dy,$$

which can be found using integration by parts. We have then

$$U + Y - \int_Y^1 y u_y dy = q.$$

However, we do not know u explicitly, so let us write it in terms of the deviatoric

stress. Define

$$T(u_y) = \tau_0 + (y - Y)p_x = \frac{1 + \alpha u_y^2}{1 + \beta u_y^2}. \tag{2.54}$$

Solving for y gives

$$y = \frac{T - \tau_0 + Yp_x}{p_x}$$

and so we have that

$$\frac{dy}{du_y} = \frac{T'}{p_x}. \tag{2.55}$$

For simplification, we will define

$$\dot{\gamma} = u_y, \tag{2.56}$$

and

$$\dot{\gamma}_1 = \dot{\gamma}|_{y=1}, \quad \dot{\gamma}_Y = \dot{\gamma}|_{y=Y}. \tag{2.57}$$

Substituting this gives

$$\int_Y^1 y u_y dy = \frac{1}{p_x^2} \left[\int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} TT'\dot{\gamma}d\dot{\gamma} - (\tau_0 - Yp_x) \int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} T'\dot{\gamma}d\dot{\gamma} \right]. \tag{2.58}$$

Let us call

$$I_1(\dot{\gamma}_Y, \dot{\gamma}_1) = \int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} TT'\dot{\gamma}d\dot{\gamma}, \tag{2.59}$$

and

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) = \int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} T'\dot{\gamma}d\dot{\gamma}. \tag{2.60}$$

Note that

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) = Up_x \tag{2.61}$$

Our equations are then as follows:

$$I_1(\dot{\gamma}_Y, \dot{\gamma}_1) - p_x^2 \left(U + Y - q - UY \right) - U\tau_0 p_x = 0, \tag{2.62}$$

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) - Up_x = 0, \tag{2.63}$$

$$\langle p_x \rangle = 0, \tag{2.64}$$

$$\langle \tau_0 - Yp_x \rangle = 0, \tag{2.65}$$

where we observe from $T(\dot{\gamma}) = \tau_0 + (y - Y)p_x$ that

$$\tau_0 = T(\dot{\gamma}_Y), \tag{2.66}$$

$$p_x = \frac{T(\dot{\gamma}_1) - T(\dot{\gamma}_Y)}{1 - Y}, \tag{2.67}$$

and the unknowns are $\dot{\gamma}_1$, $\dot{\gamma}_Y$, U, and q.

To solve this set of equations, we will discretize and apply Newton's Method. The full discretized system is

$$I_1(\dot{\gamma}_Y^j, \dot{\gamma}_1^j) - \left(\frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}\right)^2 \left(U + Y^j - q - UY^j\right) -$$

$$-UT(\dot{\gamma}_Y^j)\left(\frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}\right) = 0, \tag{2.68}$$

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) - U\left(\frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}\right) = 0, \tag{2.69}$$

for $j = 1, ..., N$, and

$$\sum_{j=1}^{N} \left[\frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}\right] \Delta\xi = 0, \tag{2.70}$$

$$\sum_{j=1}^{N} \left[T(\dot{\gamma}_Y^j) - Y^j \left(\frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}\right)\right] \Delta\xi = 0, \tag{2.71}$$

where

$$\Delta\xi = (\xi^{j+1} - \xi^j). \tag{2.72}$$

Thus there are $2N + 2$ unknowns in the system

$$U, \quad q, \quad \dot{\gamma}_1^{1..N}, \quad \dot{\gamma}_Y^{1..N},$$

from which we can easily calculate the pressure gradient and the stress on the swimmer.

Figure (2.3) shows the resulting non-dimensional swimming speeds for a fixed propagating wave speed.
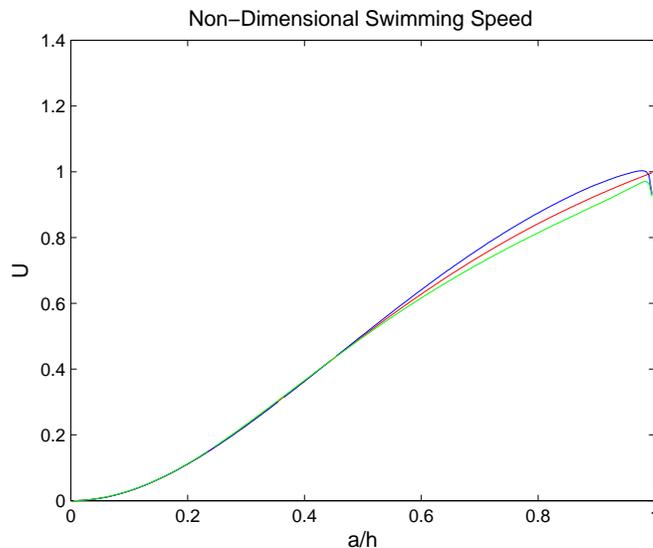
Figure 2.3: Non-Dimensional Swimming Speeds. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.

Solving this problem for a fixed propagating wave speed is not necessarily illuminating. It is not clear which fluid type is optimal for the swimmer, except for a small range of $\frac{a}{h}$. What we would like to do then, is formulate a different method of comparison that allows us to determine the optimal fluid type for the swimmer for the entire range of values of $\frac{a}{h}$. This method is detailed below.

2.3.3 Fixed Dimensional Work

To obtain a clear result of which fluid is optimal for the swimmer, let us compare the different realistic fluid regimes when the rate of work per wavelength is fixed. This allows the speed of the propagating wave to vary. We can calculate the rate of work per wavelength of the swimmer, W, by calculating the volume integral of

$$\langle \boldsymbol{\tau} : \dot{\boldsymbol{\gamma}} \rangle. \tag{2.73}$$

Non-dimensionally and at leading order, this gives us

$$W = \int_Y^1 \int_0^{2\pi} 2u_y \tau_{xy} dx dy. \tag{2.74}$$

This can be rewritten in a way that can utilize the numerics detailed above. That is, we can put

$$W = \frac{1}{p_x} \int_0^{2\pi} \int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} 2\dot{\gamma}T(\dot{\gamma})T'(\dot{\gamma})d\dot{\gamma}d\xi. \tag{2.75}$$

What we would like to do now is fix the dimensional work and look at the dimensional swimming speeds for different fluid types and compare. Particularly, we will be looking at Newtonian, shear thickening and shear thinning fluids.

We know that the dimensional swimming speed, let us call it $\tilde{U}$, is given by

$$\tilde{U}^{0,1,2} = c^{0,1,2}U^{0,1,2}, \tag{2.76}$$

where U is the non-dimensional swimming speed, c is the speed of the propagating waves and the superscripts 0, 1, 2, represent the fluid types Newtonian, shear thinning and shear thickening respectively. For a fixed dimensional work, the speed of the propagating waves has to vary in order to compensate. By putting things into dimensional form we know that

$$W^{\text{fixed}} = 2\frac{\mu c^{0,1,2}}{hk}\left(\frac{2}{p_x}\int_0^{2\pi}\int_{\dot{\gamma}_Y}^{\dot{\gamma}_1}\dot{\gamma}T(\dot{\gamma})T'(\dot{\gamma})d\dot{\gamma}d\xi\right)^{0,1,2}. \tag{2.77}$$

We can easily solve for the speed of the propagating waves and therefore the dimensional swimming speed.

From figure (2.4) we can see that the shear thinning fluid yields the fastest dimensional swimming speed for all values of $\frac{a}{h}$. From figure (2.4) we can also see that, contrary to what figure (2.3) might suggest, it is not optimal to swim near the wall.
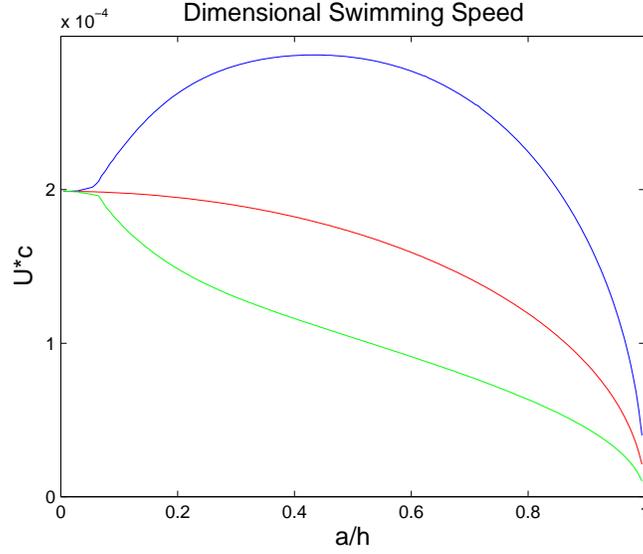
Figure 2.4: Dimensional Swimming Speeds. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.

2.3.4 Limiting Cases

In the above analysis, there are two cases that may be of analytical interest when thinking of $Y = \frac{a}{h}\sin\xi$, that when $a/h \longrightarrow 1$, or the swimmer approaches the wall and that when $a/h \longrightarrow 0$. To calculate these limits we assume that $\dot{\gamma} \longrightarrow \infty$ and that $\dot{\gamma} \longrightarrow 0$ respectively. Let us consider the former limit first.

With this assumption of the leading order component of the strain rate approaching infinity, we have that

$$\tau_{xy} = T(\dot{\gamma}) \sim \frac{\alpha}{\beta}\dot{\gamma}. \tag{2.78}$$

This simplifies the integrals I_0 and I_1 significantly

$$I_0 = \frac{\beta}{2\alpha}\left(\tau_1^2 - \tau_0^2\right), \tag{2.79}$$

$$I_1 = \frac{\beta}{3\alpha}\left(\tau_1^3 - \tau_0^3\right). \tag{2.80}$$

We can then solve the system of equations (3.23) through (3.26) explicitly for

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U. We find that

$$U = \frac{6\left(I_1^* I_3^* - I_2^{*2}\right)}{4 I_1^* I_3^* - 3 I_2^{*2}},\tag{2.81}$$

where I_j^* are defined above in equation (2.35). We can see that this swimming speed is independent of the parameters of the problem, and in fact agrees with the Newtonian solution. Since we are assuming that the swimmer is approaching the wall here, let us put

$$Y = (1 - \varepsilon^2)\sin(\xi).\tag{2.82}$$

If we assume that the dominant contribution to the integrals I_j^* are at the point along the swimmer that is nearest the wall, [8], that is at $\xi = \pi/2$, then we have

$$U \longrightarrow 1 \qquad \text{as} \qquad a/h \longrightarrow 1.\tag{2.83}$$

This result was not necessarily expected, due to figure (2.3). However, in figure (2.3), we can see that the general trend is, in fact, towards one. The behaviour near one in the figure can be attributed to a numerical singularity due to the presence of $1/(1 - Y)$.

Solving for the remainder of the variables gives that

$$q = \frac{(2U - 3)I_1^*}{3 I_2^*} + 1,\tag{2.84}$$

$$p_x = \frac{\alpha}{\beta}\frac{6(U - 2)}{(1 - Y)^2} - \frac{\alpha}{\beta}\frac{12(q - 1)}{(1 - Y)^3},\tag{2.85}$$

$$\tau_1 = \frac{\alpha}{\beta}\frac{4U - 6)}{(1 - Y)} - \frac{\alpha}{\beta}\frac{6(q - 1)}{(1 - Y)^2}.\tag{2.86}$$

Notice that U and q are independent of the parameters of the problem α and β, but p_x and τ_1 are not. Assuming the shape of the swimmer to be that as stated in equation (2.82) gives us to leading order

$$q = 1,\tag{2.87}$$

$$p_x = \frac{\alpha}{\beta}\frac{-6}{(1 - \sin(x - t))^2},\tag{2.88}$$

$$\tau_1 = \frac{\alpha}{\beta}\frac{-2}{(1 - \sin(x - t))} + \frac{\alpha}{\beta}\frac{12}{(1 - \sin(x - t))^2}.\tag{2.89}$$

Similarly, we have when $a/h \longrightarrow 0$, assuming this implies $\dot{\gamma} \longrightarrow 0$, that

$$\tau_{xy} = T(\dot{\gamma}) \sim \dot{\gamma}.\tag{2.90}$$

Thus the problem simplifies to the Newtonian low amplitude problem. Here, the integrals I_0 and I_1 simplify to

$$I_0 \;=\; \frac{1}{2}\left(\tau_1^2 - \tau_0^2\right),\tag{2.91}$$

$$I_1 \;=\; \frac{1}{3}\left(\tau_1^3 - \tau_0^3\right).\tag{2.92}$$

The solution to this Newtonian problem can be seen above in equations (2.41), (2.34) and (2.43).

However, in this situation it is the case that the amplitude of the swimmer is very small. Let us put

$$Y = \varepsilon \sin \xi.\tag{2.93}$$

We find that to leading order

$$U = 3\varepsilon^2.\tag{2.94}$$

This agrees quantitatively with other low amplitude results [37, 22, 25]. This result also agrees nicely with figure (2.3), in which the non-dimensional swimming speed appears to be approaching zero quadratically.

Solving for the remainder of the variables to leading order gives us that

$$q = U \;=\; 3\varepsilon^2,\tag{2.95}$$

$$p_x \;=\; 12Y,\tag{2.96}$$

$$\tau_1 \;=\; 6Y.\tag{2.97}$$

Thus, we can see that these two different limiting cases simplify into problems that are solvable analytically. In the latter limit, we found that the problem reduces to the Newtonian problem. In the former limit, we found that the problem reduced to one similar to the Newtonian, but with a simple dependence on the viscoelastic parameters, α and β.

2.4   Bingham Problem

As stated above, much more general models can be dealt with in this formulation of the lubrication problem, so long as we know $\tau_{xy}(u_y)$, and that the details of the constitutive model do not interfere with the key simplification of the lubrication theory, namely that the law reduces to that for a steady-state shear flow for the thin-gap geometry with dominant shear stress. Let us consider the

specific example of a Bingham fluid.

In the previous discussion of a Oldroyd fluid, the thin-layer approximation washed away many details of the more complicated viscoelastic rheological model, leaving only the effect of shear thinning or thickening. Considering a Bingham fluid leads to a more interesting lubrication problem from the rheological perspective. As well, the addition of yield stress is biologically relevant to moving organisms [8, 26].

A Bingham fluid has the following constitutive law

$$\tau_{xy} = \tau_y \text{sgn}(u_y) + \mu u_y, \qquad |\tau_{xy}| > \tau_y. \tag{2.98}$$

If $|\tau_{xy}| \leq \tau_y$ then it is the case that $u_y = 0$ and the fluid does not yield.

As stated above, the standard set of non-dimensional equations governing this lubrication problem are:

$$u_x + v_y = 0, \tag{2.99}$$

$$\tau_{xy} = \tau_1 - (1-y)p_x, \tag{2.100}$$

$$\langle p_x \rangle = 0, \tag{2.101}$$

$$\langle \tau_0 - p_x Y \rangle = \langle \tau_1 \rangle = 0, \tag{2.102}$$

and the shape of the swimmer is

$$Y = \frac{a}{h} \sin(x - t). \tag{2.103}$$

Boundary conditions are unchanged, given by equations (2.24), (2.25), (2.26), and (2.27). If we non-dimensionalize the Bingham constitutive model according to the lubrication approximation we obtain,

$$\tau_{xy} = B\text{sgn}(u_y) + u_y \qquad |\tau_{xy}| > B, \tag{2.104}$$

where $B = \frac{h}{\mu c}\tau_y$ is the Bingham number. If $|\tau_{xy}| \leq B$ then it is the case that $u_y = 0$ and the fluid does not yield. Note that this implies that

$$\tau_{xy,y} = u_{yy}, \tag{2.105}$$

which tells us, recalling that $p_x = \tau_{xy,y}$,

$$p_x = u_{yy}. \tag{2.106}$$

Let us begin solving this by equating our two expressions for τ_{xy}. This yields,

$$\tau_1 - (1-y)p_x = B\text{sgn}(u_y) + u_y. \tag{2.107}$$

Rearranging to solve for u_y gives,

$$u_y = \tau_1 - B\text{sgn}(u_y) - (1-y)p_x. \tag{2.108}$$

If $|\tau_{xy}| \leq B$ then it is the case that $u_y = 0$ and the fluid does not yield. As is evident by equation (2.108), the flow profile is going to be dependent on the sign of u_y. The different values of $\text{sgn}(u_y)$ represent different regions of the flow profile. Let us classify these regions as illustrated in table (2.3). We will define $y = Y_+$ as the upper yield surface, and $y = Y_-$ as the lower yield surface. It is the case that the fluid yields if $y > Y_+$ or $y < Y_-$. If $Y_- < y < Y_+$, it is the case that $u_y = 0$ and the fluid moves at a constant speed. Let us call this speed u_p. It is always true that,

$$Y_+ > Y_-. \tag{2.109}$$

	Outer Regions	Inner Region
Location	$y < Y_-$ or $y > Y_+$	$Y_- < y < Y_+$
Stress	$|\tau_{xy}| > B$	$|\tau_{xy}(Y_\pm)| = B$
Strain Rate	$|u_y| > 0$	$u_y = 0$
Flow Type	parabolic	plug-like

Table 2.3: Bingham Fluid Flow Profile Regions. In the outer regions, we see parabolic flow profiles typically characterized by Newtonian flows. In the inner region, we see what we refer to as pseudo-plug flow. In this region the velocity profile is constant.

Basically, what we expect is a flow profile that looks similar the Newtonian flow profile, see figure (2.5), but with a flattened region around where $u_y = 0$.
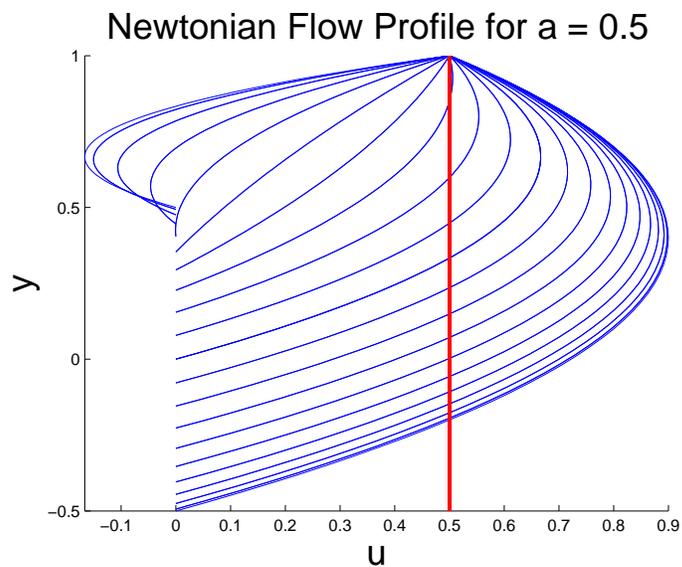
Figure 2.5: Newtonian Flow Profiles for $\frac{a}{h} = 0.5$. The speed of the swimmer, U, is depicted by the vertical red line.

We assume continuity of the flow profile. This implies that

$$u|_{(y=Y_-)} = u_p = u|_{(y=Y_+)} \tag{2.110}$$

where, $u_p = \text{Const}$, is the speed of the plug-like flow in the inner region. We call this region the pseudo-plug. Define

$$\sigma = \text{sgn}(u_y|_{y>Y_+}). \tag{2.111}$$

Following immediately from this is,

$$-\sigma = \text{sgn}(u_y|_{y<Y_-}). \tag{2.112}$$

As we can see from table (2.3), this implies that,

$$\tau_{xy}(Y_\pm) = \pm\sigma B \tag{2.113}$$

Applying this to equation (2.108) and rearranging gives us,

$$\tau_1 - \sigma B = (1 - Y_+)p_x, \qquad (2.114)$$

$$\tau_1 + \sigma B = (1 - Y_-)p_x. \qquad (2.115)$$

Combining these equations to remove σ gives,

$$\tau_1 = \frac{p_x}{2}\left(2 - Y_+ - Y_-\right), \qquad (2.116)$$

and combining them to remove τ_1 gives,

$$p_x = \frac{2\sigma B}{(Y_+ - Y_-)}. \qquad (2.117)$$

The fact that, $Y_+ > Y_-$, tells us that

$$\mathrm{sgn}(p_x) = \sigma. \qquad (2.118)$$

Integrating the continuity equation across the gap, applying the boundary conditions on v, and then integrating in x gives

$$q = Y + \int_Y^1 u\,dy, \qquad (2.119)$$

since we know that the conditions on u are independent of x.

Let us now concern ourselves with the location of Y_+ and Y_-. There are five possible cases. These are detailed in table (2.4).[19]

Case	Location	Pressure	Pseudo-Plug Speed
I	$Y_- < Y_+ < Y < 1$	$p_x > 0$	$u_p < 0$
II	$Y_- < Y < Y_+ < 1$	$p_x > 0$	$u_p = 0$
III	$Y < Y_- < Y_+ < 1$	No restrictions	No restrictions
IV	$Y < Y_- < 1 < Y_+$	$p_x < 0$	$u_p = U$
V	$Y_- < Y_+ < Y < 1$	$p_x < 0$	$u_p > 0$

Table 2.4: Location of Yield Surfaces.

We consider each one of these cases separately, and obtain different equations

---

[19] For cases I and V, it is shown belwo that $\mathrm{sgn}(u_p) = -\mathrm{sgn}(p_x)$. Due to the fact that the flow profile in the yielded regions is necessarily parabolic and that we are assuming $U > 0$, we find that $u_p < 0$ and $u_p > 0$ for cases I and V respectively.

for each. These equations are summarized below.[20] For case I,

$$Y_+ = \frac{1}{2}(1+Y) - \frac{U}{p_x(1-Y)}, \tag{2.120}$$

$$Y_- = Y_+ - \frac{2B}{|p_x|}, \tag{2.121}$$

$$q = Y + \frac{U}{2}(1-Y) - \frac{p_x}{12}(1-Y)^3. \tag{2.122}$$

For case II,

$$Y_+ = 1 - \sqrt{\frac{2U}{p_x}}, \tag{2.123}$$

$$Y_- = Y_+ - \frac{2B}{|p_x|}, \tag{2.124}$$

$$q = Y + \frac{p_x}{6}(1-Y_+)^3. \tag{2.125}$$

For case III,

$$Y_+ = \frac{\left(1 - Y^2 - \frac{2U}{p_x} - 4\frac{B}{|p_x|}\left(Y + \frac{B}{|p_x|}\right)\right)}{2(1 - Y - 2\frac{B}{|p_x|})}, \tag{2.126}$$

$$Y_- = Y_+ - \frac{2B}{|p_x|}, \tag{2.127}$$

$$q = Y + \frac{p_x}{6}\left((1-Y_+)^3 + (Y_- - Y)^3 - 3(1-Y)(Y_- - Y)^2\right). \tag{2.128}$$

For case IV,

$$Y_- = Y + \sqrt{\frac{-2U}{p_x}}, \tag{2.129}$$

$$Y_+ = Y_- + \frac{2B}{|p_x|} \tag{2.130}$$

$$q = Y + \frac{p_x}{6}(Y_- - Y)^3 + U(1-Y), \tag{2.131}$$

---

[20]For the full analysis of each case, the reader is referred to Appendix B.

and for case V,

$$Y_- = \frac{1}{2}(1+Y) - \frac{U}{p_x(1-Y)}, \tag{2.132}$$

$$Y_+ = Y_- + \frac{2B}{|p_x|}, \tag{2.133}$$

$$q = Y + \frac{U}{2}(1-Y) - \frac{p_x}{12}(1-Y)^3. \tag{2.134}$$

In each case, we have three equations and five unknowns, p_x, Y_+, Y_-, U, and q. But, we still have the constraints,

$$\langle \tau_1 \rangle = 0, \tag{2.135}$$

$$\langle p_x \rangle = 0. \tag{2.136}$$

We can now solve the system using Newton's Iteration. The trick is, at each step in the iteration, checking which case is relevant, and then using the appropriate equations. This is done by checking the locations of Y_+ and Y_-, as well as the sign of p_x (for cases I, II, IV and V). Please refer to the appendices for the full code used to solve this problem.

Figure (2.6) below shows the pressure gradient and the yield surfaces found for $\frac{a}{h} = 1/2$ and $B = 1$.
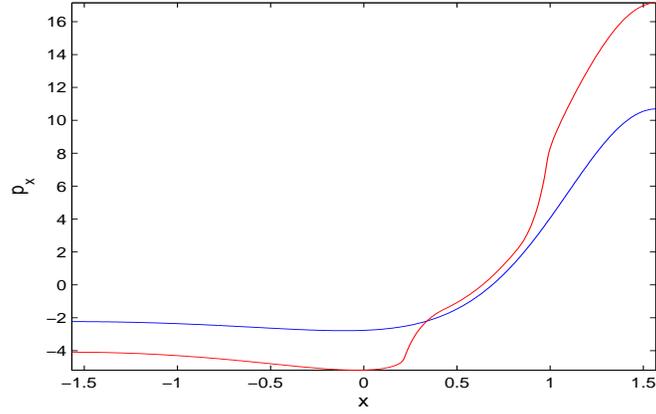
(a) Pressure Gradients
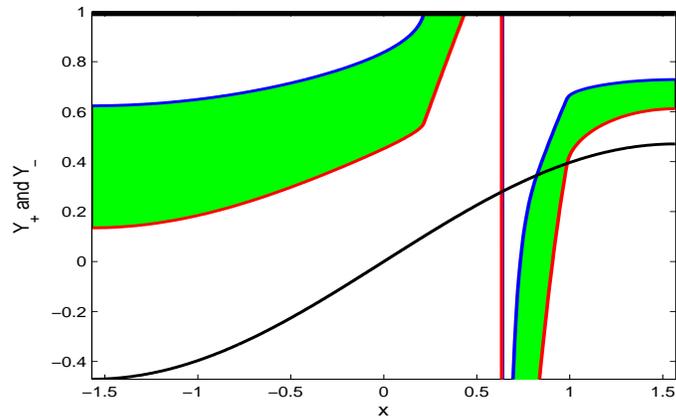
(b) Pseudo-Plug

Figure 2.6: Pressure Gradients and Yield Surfaces for $B = 1$ and $\frac{a}{h} = 1/2$. The Newtonian pressure gradient is blue, the Bingham red. In the lower figure, Y_+ is blue, Y_- is red, and the shaded green represents the pseudo-plug. The surface of the swimmer is represented by the black curve and the pink vertical line shows where the pressure gradient changes sign.

Figure (2.7) shows the velocity profiles characteristic to each case discussed above.

(a) Case I

(b) Case II

(c) Case III

(d) Case IV

(e) Case V
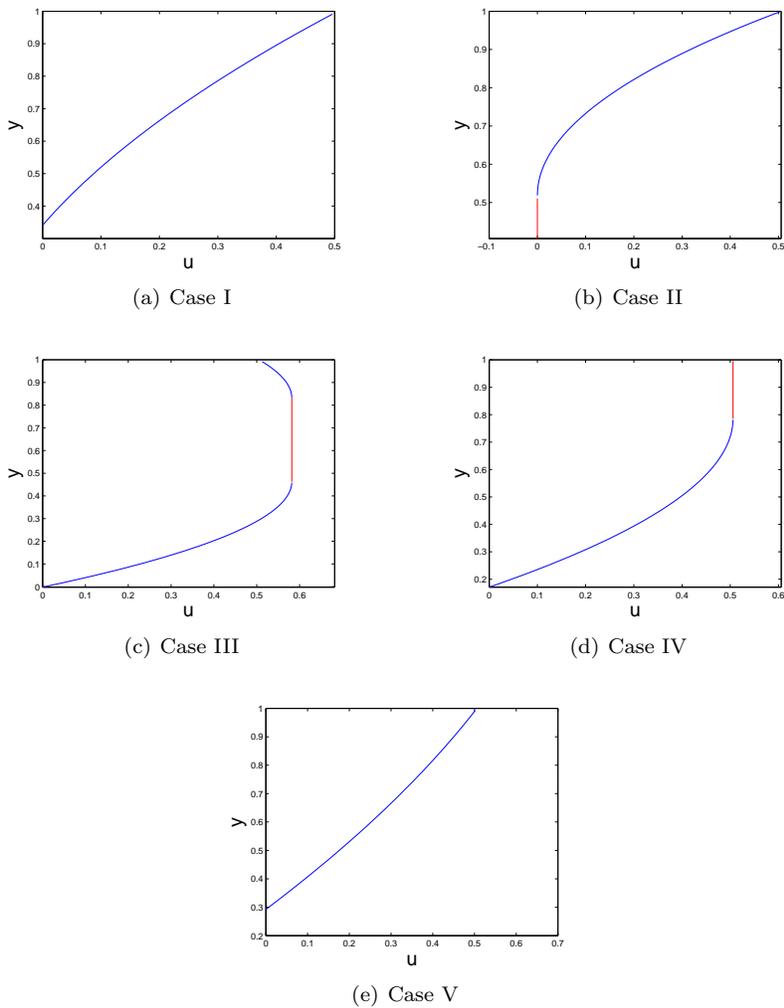
Figure 2.7: Characteristic Bingham Velocity Profiles for $B = 1$ and $\frac{a}{h} = 1/2$. Blue: Yielded Region. Red: Pseudo-plug Region.

As the Bingham number, $B = \frac{h}{\mu c}\tau_y$, is increased, the size of the pseudo-plug region increases until, eventually, it fills up most of the flow domain with viscous boundary layers next to the walls and around the location where the pressure gradient changes sign. As the Bingham number approaches zero, the size of the pseudo-plug region approaches zero and the solution approaches the Newtonian solution, where $Y_+ = Y_-$, see figure (2.8).

(a) Pressure Gradients for $B = 5$

(b) Pseudo-Plug for $B = 5$

(c) Pressure Gradients for $B = 0.1$

(d) Pseudo-Plug for $B = 0.1$
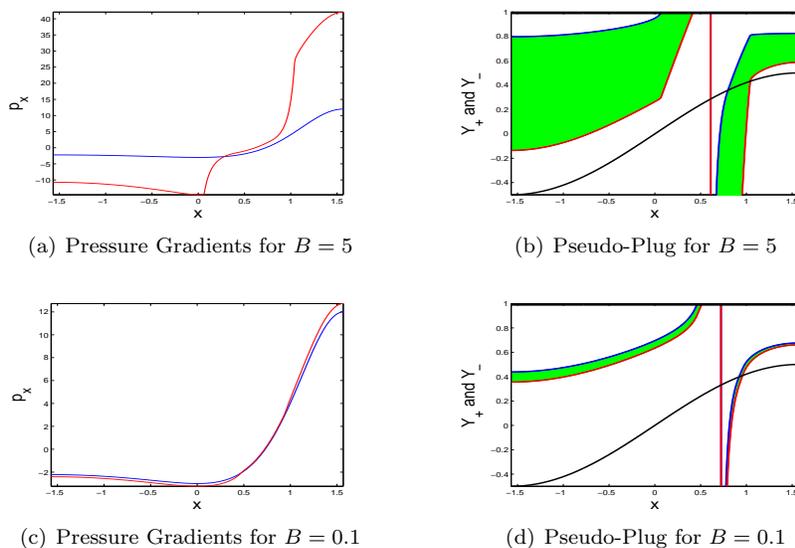
Figure 2.8: Pressure Gradients and Yield Surfaces for large and small B, $\frac{a}{h} = 1/2$. In the left figures, the Newtonian pressure gradient is blue, the Bingham red. In the right figures, Y_+ is blue, Y_- is red, and the shaded green represents the pseudo-plugs. The surface of the swimmer is represented by the black curve and the pink vertical line shows where the pressure gradient changes sign.

Figure (2.9) shows the how the swimming speed and mass flux vary with the Bingham number. The mass flux is a monotonically decreasing function of the Bingham number, which is expected due to the fact that the yielded region decreases in size as the yield stress increases. Whereas, the swimming speed is a non-monotonic function of the Bingham number with a clear maximum value, located at $B \simeq 1$. This suggests that is it beneficial for an organism to swim in a fluid that has a moderate yield stress.[21]

---

[21]It has been observed that snails and slugs move over a fluid that exhibits a yield stress [8].

(a) U versus B for $\frac{a}{h} = 1/2$ (b) q versus B for $\frac{a}{h} = 1/2$
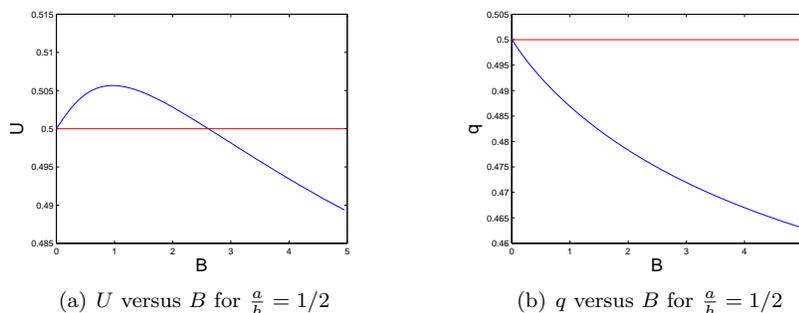
Figure 2.9: Swimming speed and mass flux for varying Bingham number. The corresponding Newtonian values are shown in red.

2.5 Discussion

In this chapter we presented the problem of a lubricated swimmer by a wall. We formulated the problem initially with a Newtonian fluid, and returned the results of previous work done by others [22]. Next, we considered the problem with a viscoelastic fluid in the gap.

We found that the speed of low amplitude swimmers close to walls is significantly larger than those away from walls. This is also observed experimentally [20]. We also found that a shear thinning fluid yields the optimal speed for the swimmer given a fixed rate of work when compared to a swimmer in a Newtonian fluid and a shear thickening fluid. It is the case that viscoelasticity can decrease or increase the swimming speed compared to the Newtonian case, depending on the properties of the specific viscoelastic fluid in question.

Motivated by the results shown in figure (2.3), we took the analytical limit as the swimmer approaches the wall. We found that the non-dimensional swimming speed approaches the value of one, regardless of the fluid it is in.[22] In the opposite limiting case, when the ratio of the amplitude of the swimmer on the gap width goes to zero, we have the Newtonian problem and its relevant

---

[22]This contradicts the limiting case of the introductory problem, where the limit was found to depend on the parameters of the problem. This is not of concern due to the fact that, in the introductory problem, no thought was given to what the strain rates were doing in the limit. The introductory problem was just a preliminary example used to motivate the use of lubrication theory. In fact, if we reduce the Oldroyd-8 constitutive model to the Oldroyd-B model above, the Newtonian problem is returned, hence no contradiction when the limit is taken properly.

results returned. That is, we found that the non-dimensional swimming speed approaches zero quadratically in a.

Due to the formulation of this problem, theoretically it should be easy to consider different constitutive models and wave shapes, thus allowing the range of applications of this problem to broaden.

For example, perhaps we would like to consider a different swimmer shape in Chapter Two. Let us consider the toy example of a swimmer shape given by $Y = \tanh(12\sin\xi)$, as shown in figure (2.10a). The analysis of the problem does not change. The resulting swimming speeds of such a swimmer are shown in figure (2.10b) and were found using the same code as was used for the results found above. It is evident that different swimmer shapes can yield different results, depending on the application one has in mind. The mathematical set up we chose here lends itself nicely to generality, and exploring these potential different applications it certainly something that should be looked into further.

(a) Wave Shape
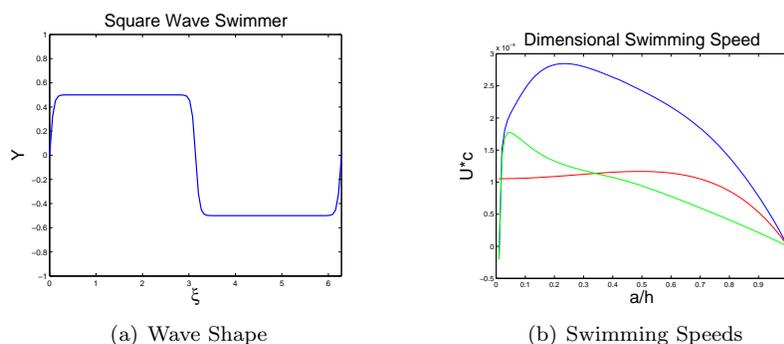
(b) Swimming Speeds

Figure 2.10: Example of alternate wave shape and resulting swimming speeds. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.

In the spirit of broadening the applications of this problem, we then considered the problem with a Bingham fluid. We found that in order to solve the problem, we had to consider five different possible cases that depended on the location of the yield surfaces. For a fixed swimmer amplitude, $\frac{a}{h} = 1/2$, we solved for the pressure gradient and the location of the yield surfaces. At this fixed amplitude, we found that it is potentially advantageous for an organism to swim in a Bingham fluid with a moderate yield stress, as it results in a faster swimming speed than a Newtonian fluid.

Chapter 3

Elastic Swimming with Lubrication Theory

In the previous chapter we looked at the swimming problem from the perspective of the swimmer having a known motion or shape. We then analyzed the resulting swimming speed with the swimmer subjected to a fixed dimensional work or rate of work per unit wavelength. There is, however, a different approach one could use to look at this problem. Rather than a prescribed shape, we can consider a prescribed force distribution along the length of the swimmer. In this situation we have to think more carefully about the swimmer itself. Previously, we only were really concerned with the swimming speed. Now, we must think about how this applied force affects the swimmer. What kind of properties does this swimmer have to govern how it will respond to this forcing? This is a question that has a large number of potential answers. The specifics of flagellum movement are very complex, and so if we were to model the swimmer as a realistic flagella the problem would be very involved [23, 22]. However, here we are more interested in the fluid dynamics and kinematics of the problem, and so the answer to the question posed above is that we will assume that the swimmer responds to the applied force as an elastic beam. This is a simpler, more practical choice than trying to incorporate all the intricacies of a realistic flagellum. In fact, this assumption makes our problem more general with a wider range of potential applications [2, 41, 8, 39].

Potentially, we could have considered a different mechanism of wave propagation along the swimmer. Rather than having forcing distributed along the entire length of the swimmer, we could consider the situation of a passive elastic swimmer with one fixed end that moves in a vertical fashion thus propagating waves down the length of the swimmer. This postulation is interesting from a mathematical perspective, and perhaps has some interesting alternate applications, but it has been shown that observed wave forms of spermatozoa cannot

be accounted for with this mechanism [29].

Formally then, this is the problem of an elastic swimmer subject to a known forcing along its length in a lubrication setting. Its shape and swimming speed are unknown.

Our goal in this chapter is to discover how an elastic swimmer reacts when subjected to a known distribution of applied torques whilst in a viscoelastic fluid. However, the problem of how such a swimmer acts in a Newtonian fluid proves to be interesting itself.

3.1 Mathematical Formulation

Let us consider the following: a thin, isotropic elastic beam in two-dimensions immersed in a fluid very close to a rigid wall. This beam is then subjected to known internal forces distributed along its length, $f(x,t) = a \sin k(x - ct)$ that cause it to bend into an unknown shape, Y. This shape then produces a thrust, propelling it through the fluid at an unknown, constant speed U. For the sake of simplicity, we will assume that the applied force is sinusoidal. We use a similar set up for our problem as was used in [2], where the forces applied are referred to as actively distributed torques.
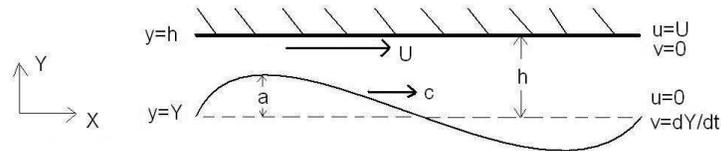
Figure 3.1: Geometry of Lubrication Problem. This time, the swimmer shape in unknown as well as the swimming speed

What we want to discover here is what the unknown swimmer shape is and the swimming speed it produces.

To begin with, let us consider the boundary conditions. In the frame moving with the swimmer, our boundary conditions remain unchanged for this problem from the last. That is, non-dimensionally they are for the horizontal velocity

component,

$$u(x, h) \;=\; U, \tag{3.1}$$

$$u(x, Y) \;=\; 0, \tag{3.2}$$

and for the vertical velocity component v we have

$$v(x, h) \;=\; 0, \tag{3.3}$$

$$v(x, Y) \;=\; -Y_x, \tag{3.4}$$

recalling that $Y_t = -Y_x$. It is interesting to note here that if we remained in the original frame with the wall fixed, the boundary condition on the swimmer would actually be a kinematic or free boundary condition due to the fact that the swimmer is now considered elastic. But, in the frame moving with the swimmer there is no horizontal velocity associated with the beam, therefore the kinematic condition reduces to the one stated above.

Now, let us consider the fluid. It can be modelled as was done previously, applying the general assumptions detailed in Chapter One, by

$$\nabla p \;=\; \nabla \cdot \boldsymbol{\tau}, \tag{3.5}$$

$$\nabla \cdot \mathbf{u} \;=\; 0, \tag{3.6}$$

and general constitutive model

$$\mathbf{f}(\boldsymbol{\tau}, \dot{\boldsymbol{\gamma}}) = 0. \tag{3.7}$$

Here we have an elastic beam, thus we must include another equation for the vertical force balance. This is the beam equation including an internally applied torque [2, 28]. In full dimensional terms this equation is, neglecting gravity,

$$0 = -\partial_{xx} \left(f + D\partial_{xx} Y \right) + p, \tag{3.8}$$

where D is the bending stiffness of the beam, and p, Y, f the pressure, beam shape and applied forcing respectively. We enforce periodicity in the swimmer up to and including its third derivative. Mathematically it is required for the higher derivatives due to the fact that, once the beam equation is incorporated, we end up with a fourth order ordinary differential equation, thus requiring the four conditions. Physically, only this condition and the continuity of the

swimmer are required.[23] In truth, we are interested in the pressure gradient only, thus we take the derivative of the above equation and use

$$p_x = \partial_{xxx} \left(f + D\partial_{xx} Y \right). \tag{3.9}$$

We also have the constraints

$$\langle p_x \rangle = 0, \tag{3.10}$$

$$\langle \tau_1 \rangle = 0, \tag{3.11}$$

$$\langle Y \rangle = 0. \tag{3.12}$$

When looking at equation (3.9), we can see that in order for $\langle p_x \rangle = 0$ to hold we require that $\langle Y_{xxxxx} \rangle = 0$. This is the same thing as saying Y is periodic in its fourth derivative. This requirement is due to the choice of $f(x,t)$. Thus we can enforce the constraint on the pressure gradient by enforcing periodicity on the swimmer shape up to and including its fourth derivative, and from here on out not include the constraint specifically. Observe that the condition $\langle p_x \rangle = 0$ is also utilized in the formulation of the zero force on swimmer condition when it is put into the form $\langle \tau_1 \rangle = 0$, as noted in Chapter One.

Non-dimensionalizing as we did in Chapter Two, using the following non-dimensional variables,

$$
\begin{aligned}
t &= \tfrac{1}{\omega}\hat{t}, & \dot{\boldsymbol{\gamma}} &= \omega\hat{\dot{\boldsymbol{\gamma}}}, \\
x &= \tfrac{1}{k}\hat{x}, & y &= h\hat{y}, \\
\mathbf{u} &= c\hat{\mathbf{u}}, & \hat{D} &= \tfrac{\varepsilon h^2 k^4}{\mu c} D \\
(p, \boldsymbol{\tau}) &= \tfrac{\mu c}{h}(\tfrac{1}{\varepsilon}\hat{p}, \hat{\boldsymbol{\tau}}), & \hat{a} &= \tfrac{\varepsilon k^2 h}{\mu c} a
\end{aligned}
$$

applying the lubrication approximation and simplifying algebraically yields, dropping hats,

$$I_1(\dot{\gamma}_Y, \dot{\gamma}_1) - p_x^2 \left(U + Y - q - UY \right) - U\tau_0 p_x = 0, \tag{3.13}$$

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) - Up_x = 0, \tag{3.14}$$

$$\langle Y \rangle = 0, \tag{3.15}$$

$$\langle \tau_1 \rangle = 0. \tag{3.16}$$

---

[23]This condition's specific use was not required previously because the shape of the swimmer was assumed.

Recall that

$$I_0 = \int_{\dot{\gamma}_Y}^{\dot{\gamma}_1} T' \dot{\gamma} d\dot{\gamma}, \tag{3.17}$$

$$I_1 = TT' \dot{\gamma} d\dot{\gamma}, \tag{3.18}$$

$$T = \tau_{xy}. \tag{3.19}$$

Similarly then, we also utilize the fact that

$$p_x = \frac{T(\dot{\gamma}_1) - T(\dot{\gamma}_Y)}{1 - Y}, \tag{3.20}$$

to remove p_x from the above equations. As well, recall that

$$\tau_1 = T(\dot{\gamma}_1). \tag{3.21}$$

Putting the beam equation into non-dimensional terms yields

$$p_x = \partial_{xxx} \left(f + D \partial_{xx} Y \right). \tag{3.22}$$

So, our unknowns are now $\dot{\gamma}_1$, $\dot{\gamma}_Y$, U, and q and the new unknown, the swimmer shape, Y. Summarizing the full set of non-dimensional equations relevant to solving the problem gives us

$$I_1(\dot{\gamma}_Y, \dot{\gamma}_1) - p_x^2 \left(U + Y - q - UY \right) - U\tau_0 p_x = 0, \tag{3.23}$$

$$I_0(\dot{\gamma}_Y, \dot{\gamma}_1) - U p_x = 0, \tag{3.24}$$

$$\mathbf{f}\left(\hat{\boldsymbol{\tau}}, \hat{\dot{\gamma}} \right) = 0, \tag{3.25}$$

$$p_x - \partial_{xxx} \left(f + D \partial_{xx} Y \right) = 0, \tag{3.26}$$

$$\langle Y \rangle = 0, \tag{3.27}$$

$$\langle \tau_1 \rangle = 0, \tag{3.28}$$

along with periodic conditions on the derivatives of Y up to and including the fourth derivative which can be written as

$$\langle Y_x \rangle = \langle Y_{xx} \rangle = \langle Y_{xxx} \rangle = \langle Y_{xxxx} \rangle = \langle Y_{xxxxx} \rangle = 0 \tag{3.29}$$

These equations are very similar to what we had before with only the addition of the beam equation and the periodicity conditions on the swimmer and its first five derivatives. Again, the conditions on the swimmer were true before

since we were imposing the shape of the swimmer, but now we must rigorously enforce it.

As we have seen above, the layout of this problem and the last one has the fluid properties included simply in the form of the equation $\tau_{xy} = T(\dot{\gamma})$. This is an elegant and general way of looking at the problem and allows us to look at different constitutive equations with relative ease. However, the complexity of the constitutive equation can still heavily affect the ease of solving the system both analytically and numerically.

As done above, let us first consider the Newtonian problem in order to gain some insight into the problem before considering more complex fluid models.

3.2   Newtonian Problem

Since, with the exception of the inclusion of the beam equation and the constraints on the swimmer, the equations are the same for this problem as they are for the Newtonian problem for a swimmer of prescribed shape, we can skip a few steps. We have already solved equations (3.23), (3.24), and (3.28) for a Newtonian fluid in the previous chapter. However, since Y is unknown, we can only utilize the equations for the pressure gradient and stress on the wall, as the calculation of the swimming speed and mass flux relied on calculating the integrals I_j^*. Not all is lost though, we can use still use the following equations for p_x and τ_1

$$p_x = \frac{6(U-2)}{(1-Y)^2} - \frac{12(q-1)}{(1-Y)^3}, \tag{3.30}$$

$$\tau_1 = \frac{(4U-6)}{(1-Y)} - \frac{6(q-1)}{(1-Y)^2}. \tag{3.31}$$

These are most useful since the pressure gradient and unknown swimmer shape appear in the beam equation and the stress on the swimmer appears in a constraint. First, let us substitute the algebraic equation for p_x, equation (3.30), into the beam equation, equation (3.26). This gives us

$$\partial_{xxx}\left(a\sin(x-t) + D\partial_{xx}Y\right) = \frac{6(U-2)}{(1-Y)^2} - \frac{12(q-1)}{(1-Y)^3}. \tag{3.32}$$

If we introduce a new variable,

$$\zeta = (1-Y), \tag{3.33}$$

and utilize the dependence on $\xi = (x - t)$ in the problem, we obtain the equation

$$\zeta^{(5)} = -\frac{1}{D}\left(a\cos\xi + \frac{6(U-2)}{\zeta^2} - \frac{12(q-1)}{\zeta^3}\right), \tag{3.34}$$

where derivatives are with respect to ξ. We can then solve for ζ and U and q by applying the constraints. To do this, we must first put the constraints into terms of ζ. The first constraint, $< Y >= 0$ is simple. It becomes

$$\langle\zeta\rangle = 1. \tag{3.35}$$

The second, $\langle\tau_1\rangle = 0$ is less straight forward. However, due to our above algebraic manipulation, we already know what τ_1 is in terms of Y, U, and q. Thus it is a simple matter of putting this in terms of ζ

$$\tau_1 = \frac{(4U-6)}{\zeta} - \frac{6(q-1)}{\zeta^2}, \tag{3.36}$$

and so the constraint becomes

$$(4U-6)\langle\zeta^{-1}\rangle - 6(q-1)\langle\zeta^{-2}\rangle = 0. \tag{3.37}$$

To solve this analytically requires solving a cubic, so rather than doing that, let us convert the problem to something which we can apply numerics.

3.2.1 Numerical Formulation

We can think of equation (3.34) as an ordinary differential equation with eigenvalues U and q and constraints given by equations (3.35) and (3.37). To solve this numerically, let us define a new variable r such that $r' = \xi$, thus giving us

$$r = \int_0^\xi \zeta d\xi. \tag{3.38}$$

Similarly define the variable s so that $s' = \left((4U-6)\xi^{-1} - 6(q-1)\xi^{-2}\right)$, which gives us

$$s = \int_0^\xi \left((4U-6)\xi^{-1} - 6(q-1)\xi^{-2}\right)d\xi. \tag{3.39}$$

We have then that

$$r(0) = 0, \qquad r(2\pi) = 2\pi, \tag{3.40}$$

and

$$s(0) = 0, \qquad s(2\pi) = 0, \tag{3.41}$$

as conditions on these new variables.

Thus, our system of equations to solve is

$$\zeta^{(5)} = -\frac{1}{D} \left(a \cos \xi + \frac{6(U-2)}{\zeta^2} - \frac{12(q-1)}{\zeta^3} \right), \tag{3.42}$$

$$r = \int_0^\xi \zeta d\xi, \tag{3.43}$$

$$s = \int_0^\xi \left((4U - 6)\xi^{-1} - 6(q-1)\xi^{-2} \right) d\xi, \tag{3.44}$$

where we think of U and q as eigenvalues, with conditions

$$\zeta(0) = \zeta(2\pi), \qquad \zeta'(0) = \zeta'(2\pi), \qquad \zeta''(0) = \zeta''(2\pi),$$
$$\zeta^{(3)}(0) = \zeta^{(3)}(2\pi), \quad \zeta^{(4)}(0) = \zeta^{(4)}(2\pi), \tag{3.45}$$

and

$$r(0) = 0, \qquad r(2\pi) = 2\pi, \tag{3.46}$$

$$s(0) = 0, \qquad s(2\pi) = 0, \tag{3.47}$$

for unknowns ζ, ζ', ζ'', $\zeta^{(3)}$, $\zeta^{(4)}$, $\zeta^{(5)}$, r and s with eigenvalues U and q. To do this we use the MATLAB solver bvp4c. To see the code please refer to appendices.

Figure (3.2) shows the resulting shape of the swimmer as the amplitude of the forcing increases. It shows us that as the amplitude gets larger the swimmer shape develops a plateau region followed by a sharp spike region. By looking at the phase portraits of this large amplitude solution shown in figure (3.3) we can see that there exists a inner region that reminiscent of a fixed point. This explains the section of the swimmer shape that is nearly constant that exists near $\xi \sim \pi/2$.

(a) Newtonian Solution for D = 0.2

(b) Newtonian Solution for D = 1.5
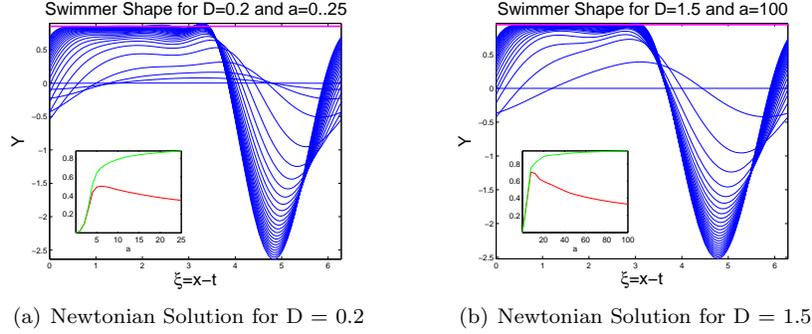
Figure 3.2: Newtonian Solution of Elastic Lubrication Problem. In the bottom left hand corner of each figure we see the swimming speed (red) and the mass flux (green) versus the forcing amplitude. The magenta horizontal lines are the calculated plateau values for each case.

(a) Phase Portrait for D=0.2

(b) Phase Portrait for D=1.5
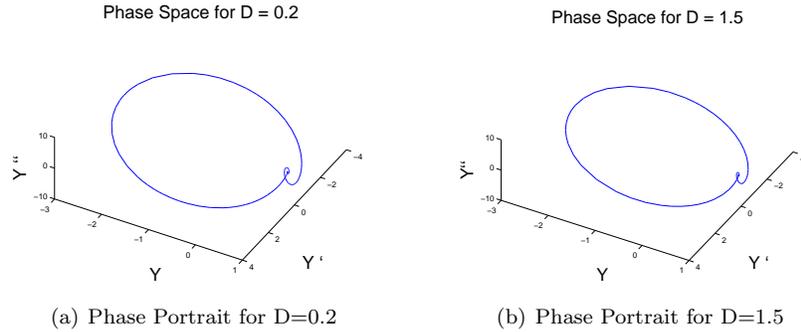
Figure 3.3: Phase portraits for Newtonian Solution of Elastic Lubrication Problem with large amplitude forcing.

We would like to get an approximation of this plateau value. However, it is not immediately obvious how we may scale things in such a way that gives us a rigorous asymptotic solution to this inner region. Let us assume that we are inside this inner region for ζ, near $\xi = \pi/2$, and expand in the standard asymptotic fashion

$$\zeta = \zeta_p + \zeta_1 + \dots \tag{3.48}$$

where ζ_p is the plateau constant and $|\zeta_1| << 1$. Substituting this into equation

(3.34), and rearranging slightly we obtain

$$\frac{D}{a}\frac{d^5}{d\xi^5}\zeta_1 = -\cos\left(\frac{\pi}{2}+\sigma\right) - \frac{6(U-2)}{a(\zeta_p+\zeta_1)} + \frac{12(q-1)}{a(\zeta_p+\zeta_1)}, \qquad (3.49)$$

where we are assuming that σ is some small number. It is easy to see then that at leading order we obtain

$$\zeta_p = \frac{2(q-1)}{U-2}. \qquad (3.50)$$

To solve this fully analytically is certainly a challenge, but if we take this formula for the plateau value, recalling that $\zeta = 1 - Y$, and plot it using our numerical values for U and q we obtain a very remarkable fit, as can be seen as the magenta line in figure (3.2). Note that the plateau value is independent of the bending stiffness. These results suggest that there are some truly interesting dynamics at work in this large amplitude regime which most certainly should be analyzed at greater depth, including a full analysis of the outer region.

We can also see from figure (3.2b) that the plateau approaches $y = 1$. This observation is substantiated by the formula

$$Y_p = 1 - \frac{2(q-1)}{U-2}, \qquad (3.51)$$

if we observe that $q \to 1$ as the amplitude gets large. As well, we can note from figure (3.2) that there is an obvious optimal forcing amplitude for the swimming speed and after that, as the swimmer gets too close to the wall and the stresses get large, the speed decreases.

3.2.2 Low Amplitude Solution

Let us consider the situation when the amplitude of the applied forcing or torque on the swimmer is very small. That is, put

$$a \ll 1, \qquad (3.52)$$

where the forcing is given by $f = a\sin(x - t)$. We can then assume that this implies that the amplitude of the swimmer shape is very small

$$|Y| \ll 1. \qquad (3.53)$$

At this point we can then notice that this is a similar situation to the one discussed in Chapter Two. Therefore we can use the low amplitude approximations for p_x and τ_1 as we found in Chapter Two. We found initially that

$$p_x = \frac{6(U-2)}{(1-Y)^2} - \frac{12(q-1)}{(1-Y)^3}, \tag{3.54}$$

$$\tau_1 = \frac{(4U-6)}{(1-Y)} - \frac{6(q-1)}{(1-Y)^2}. \tag{3.55}$$

Except here, Y is unknown. However, this does not stop us from being able to get a low amplitude approximation for these equations. We obtain the same leading order approximations as we did before

$$p_x = 12Y, \tag{3.56}$$

$$\tau_1 = 6Y. \tag{3.57}$$

Notice that the constraint on τ_1 holds automatically if we enforce the constraint on Y. Applying the low amplitude approximation to equation $(3.9)^{24}$ and substituting into it equation (3.56) gives, after some simple rearranging,

$$DY_{xxxxx} - 12Y = f_{xxx}, \tag{3.58}$$

with constraint

$$\langle Y \rangle = 0, \tag{3.59}$$

and periodicity in Y up to and including its fourth derivative. Solving this we find that the homogeneous solutions must be ignored due to the fact that they do not satisfy the periodicity conditions. Thus it is only the particular solution that is relevant. We find the particular solution to be

$$Y = \frac{a}{D^2 + 12^2} \left(-12 \cos \xi + D \sin \xi\right). \tag{3.60}$$

From this low amplitude solution we can see that for a moderate value of D all terms will balance, but for large or small values of D we will have different terms balancing. For small amplitude forcing and a small bending stiffness we

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[24]Note that the low amplitude approximation does not change the beam equation since all terms in it become small and therefore all balance

would expect the shape of the swimmer to look like

$$Y = \frac{-12a}{D^2 + 12^2} \cos \xi \qquad (3.61)$$

and for small amplitude forcing and large bending stiffness we expect the shape of the swimmer to look like

$$Y = \frac{aD}{D^2 + 12^2} \sin \xi \qquad (3.62)$$

3.3   Viscoelastic Results

Let us now tackle the elastic lubrication problem with a viscoelastic fluid within the gap. Here again, we choose to use the Oldroyd-8 constitutive model. Non-dimensionalizing the constitutive model, simplifying, and solving for τ_{xy}, as we did in the previous chapter, yields

$$\tau_{xy} = T(\dot{\gamma}) = \frac{(1 + \alpha \dot{\gamma}^2)}{(1 + \beta \dot{\gamma}^2)} \dot{\gamma}. \qquad (3.63)$$

To solve this numerically, we employ a similar method as was used in Chapter Two. That is, we discretize the system of equations and apply Newton's Method. The system of discretized equations is as follows:

$$I_0(\dot{\gamma}_Y^j, \dot{\gamma}_1^j) - U p_x^j = 0, \qquad (3.64)$$

$$I_1(\dot{\gamma}_Y^j, \dot{\gamma}_1^j) - p_x^{j2} \left(U + Y^j - q - U Y^j \right) - U T(\dot{\gamma}_Y^j) p_x^j = 0, \qquad (3.65)$$

$$p_x^j - f_{xxx}^j - D Y_{xxxxx}^j = 0, \qquad (3.66)$$

$$\Sigma_{j=1}^N T(\dot{\gamma}_1^j) = 0, \qquad (3.67)$$

$$\Sigma_{j=1}^N Y^j = 0. \qquad (3.68)$$

There are $3N + 2$ unknowns, $\dot{\gamma}_Y^{1..N}$, $\dot{\gamma}_1^{1..N}$, $Y^{1..N}$, U, and q. And we have that, $p_x^j = \frac{T(\dot{\gamma}_1^j) - T(\dot{\gamma}_Y^j)}{1 - Y^j}$. Note that f_{xxx} is given, as f is given. The Y_{xxxxx} term is found using a difference formula. Please refer to appendices for the full numerical method used to solve this problem.

Figure (3.4) shows the swimmer shape varying for changing $\frac{a}{h}$ values and the resulting swimming speeds. We see that for low bending stiffness, the differences between the three fluid types is noticeable. Particularly the swimming speed, where you can see that a shear thinning fluid yields the fastest speed and a shear

thickening fluid yields the slowest. But, as we increase the bending stiffness, as seen in figures (3.5) and (3.6), both the swimmer's shape and speed becomes less differentiated between the different fluid types. Notice the asymmetry here for certain values of the bending stiffness and the similarities between the shape of the swimmer for Newtonian, shear thinning, shear thickening.
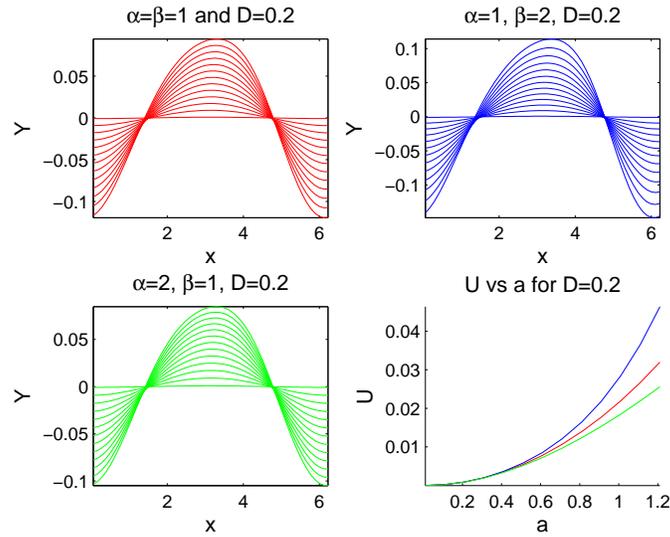
Figure 3.4: Solution for viscoelastic elastic lubrication problem for various fluid types with D=0.2. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.
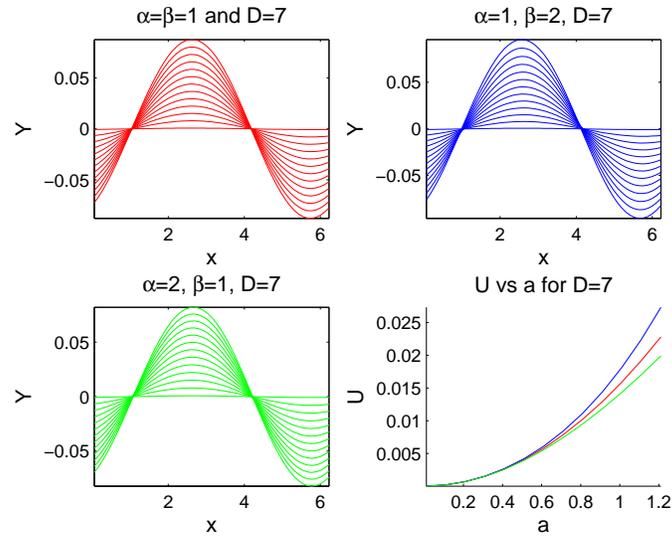
Figure 3.5: Solution for viscoelastic elastic lubrication problem for various fluid types with D=7. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.
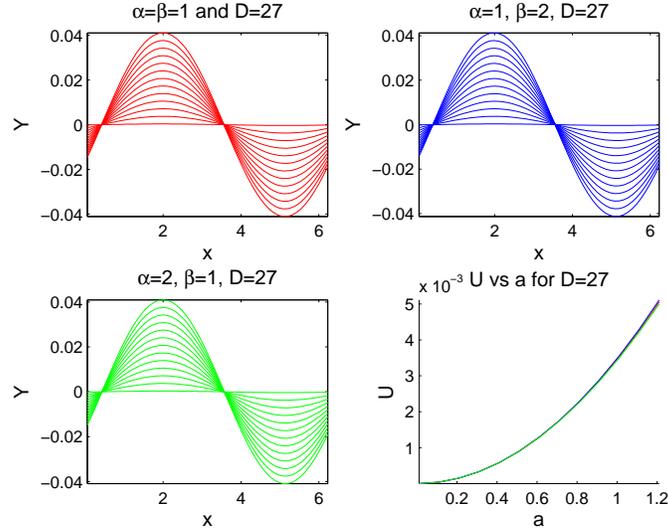
Figure 3.6: Solution for viscoelastic elastic lubrication problem for various fluid types with D=27. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.

3.3.1   Low Amplitude Solution

Let us again consider the situation when the amplitude of the applied forcing or torque on the swimmer is very small. Above we assumed that the small forcing amplitude yields a small swimmer amplitude. Here, we will assume the same, but also assume that this very small forcing and resulting small swimmer amplitude implies a very small shear rate

$$|\dot{\gamma}| \ll 1. \tag{3.69}$$

This small shear rate affects the form of the constitutive equation for the stress. In particular, we have that the constitutive equation becomes

$$T(\dot{\gamma}) = \dot{\gamma}, \tag{3.70}$$

reducing the problem to the Newtonian one. Therefore, the solution found above for the low amplitude Newtonian problem is also the solution for the low

amplitude viscoelastic problem. Recall that we found

$$Y = \frac{a}{D^2 + 12^2} \left(-12\cos\xi + D\sin\xi\right).\tag{3.71}$$

Therefore, we can see that for small values of the bending stiffness the swimmer will look like a negative cosine curve, and for large values of the bending stiffness it will look like a sine curve. As we can see from figure (3.7) these analytical results agree very nicely with the numerical viscoelastic results for small amplitude forcing.
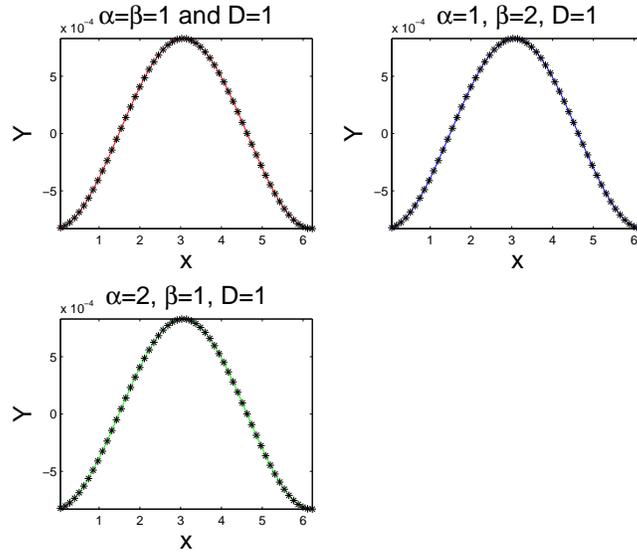
Figure 3.7: Comparison between analytical and numerical results for low amplitude swimming problem. Numerical results are solid lines, analytical results are shown by *. Red: Newtonian. Blue: Shearing Thinning. Green: Shear Thickening.

3.4 Discussion

In this chapter we presented the problem of a lubricated elastic swimmer by a wall. We considered first the Newtonian problem and discovered that it yields very interesting results. For large amplitude forcing we found that a plateau region develops along the swimmer. This plateau approaches the wall as the

amplitude of the forcing becomes large. We referred to this region as the inner region as it corresponds to a region in the phase portrait that is reminiscent of a fixed point. Upon expanding around this inner region, we found a value of this plateau analytically that agreed very well with numerical results. Following this plateau region is a large spike type region. Unfortunately, solving for this region analytically is beyond the scope of this thesis. These results are not entirely irrelevant to the physical situation under consideration as large amplitude, asymmetrical beat patterns are observed in hyperactivated spermatozoa [35], [17], [36], [34]. These results are also extremely interesting from a mathematical perspective and it is the hope of this author that at some point, in the not so distant future, they will be analyzed in greater depth. As well, we saw that there is a definite forcing amplitude that yields an optimal swimming speed. The shape of the optimal swimmer at this amplitude can be seen in figure (3.8).
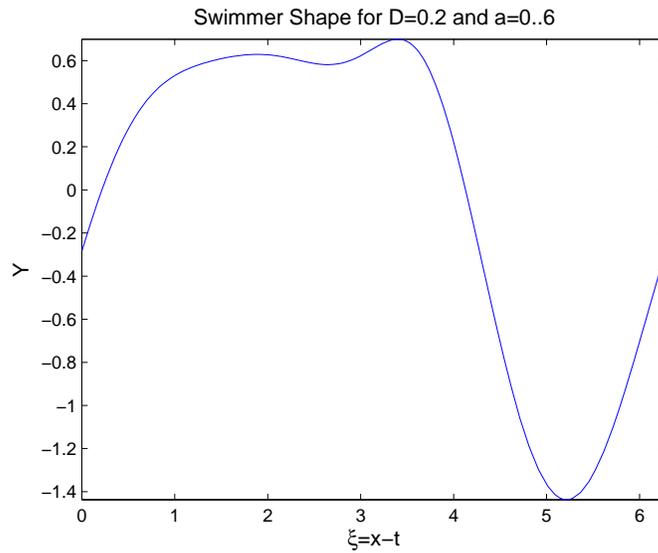
Figure 3.8: Optimal swimmer shape for D=0.2.

As we can see in figure (3.8), this optimal shape is neither a true sinusoid, nor does have the flattened, plateau region characteristic of the large amplitude limit.

We next calculated the low amplitude analytical solution of the Newtonian

problem. We found that both the pressure gradient and the stress on the wall are proportional to the swimmer shape, and that the swimmer shape is

$$Y = \frac{a}{D^2 + 12^2} \left(-12 \cos \xi + D \sin \xi \right), \tag{3.72}$$

a solution that is valid for any value of the bending stiffness D. This result was returned to us when we calculated the low amplitude solution for a elastic swimmer in a viscoelastic fluid. It was found to agree with the numerical results for the viscoelastic problem extremely well.

We then took the next logical step and considered a viscoelastic fluid. We found that for moderate bending stiffness, a shear thinning fluid yielded the fastest swimming speed and that as the bending stiffness was increased, the shear thinning and shear thickening solutions collapsed towards the Newtonian solution. When considering this fact and the Newtonian solution, it seems that the most interesting parameter regime for problem as a whole is moderate or small bending stiffness coupled with a large amplitude forcing.

It would be most desirable for us to find a numerical or analytical solution for the viscoelastic problem with a large amplitude forcing. This is not straightforward, as the code that currently solves the viscoelastic problem cannot handle large amplitude forcing. Finding an analytical solution is also not straightforward. To see why this is first note that from the Newtonian problem we have that

$$p_x = u_{yy}, \tag{3.73}$$

giving us that

$$u_y = \tau_1 - (1 - y)p_x. \tag{3.74}$$

We can see from figure (3.9) that the strain rates in the Newtonian problem are getting larger, but it is not obvious that we can make the blanket assumption that they get large enough to simplify the constitutive relation.[25] Even if the constitutive equation can be simplified, we would assume that the solution still has dependence on the viscoelastic parameters α and β, as did the solution for the large amplitude swimmer in chapter two.

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[25]Please recall that in the Newtonian problem, strain rates and stresses are equal non-dimensionally, $\tau = \dot{\gamma}$
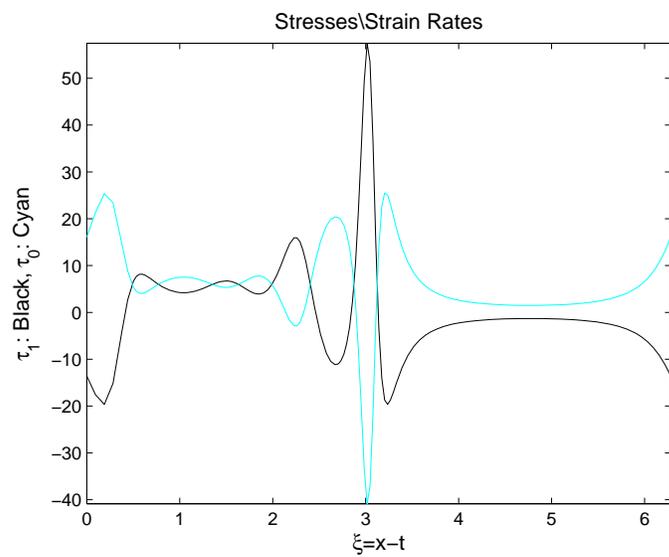
Figure 3.9: Strain rates on the swimmer (cyan) and on the wall (black) for a=100 and D=1.5. Note that the largest strain rates occur in the region where the plateau ends and the spike begins.

Chapter 4

Conclusions

This thesis has presented an in depth discussion of swimmers near walls. Several different problems were analyzed and within these problems, different fluid types were considered. The mathematical setup of these problems can be applied to a broad spectrum of applications as it easily adapts to many similar situations. For example, similar problems that have different boundary conditions, wall locations, shapes, or constitutive models all require only simple changes to the basic formulation.[26] Viscoelastic lubrication problems arise in many places, such as physiological and industrial applications, as well as other types of swimming problems [30, 1, 40, 41].

4.1   Summary of Results

To begin with, we motivated the lubrication problem by looking at the preliminary problem of a swimmer in the presence of a wall. We then considered the problem of a lubricated swimmer with a prescribed shape. It was concluded that shear thinning fluids are optimal for swimmers in this situation when compared to shear thickening and Newtonian fluids. This was found by considering the problem with a fixed rate of work and looking at the resulting dimensional swimming speeds. The two limiting cases of the distance between the wall and the swimmer approaching zero and infinity were considered analytically. It was found that the Newtonian swimming speed was returned in both situations. In the former limit, the Newtonian problem was returned entirely. In the latter limit, the solution, aside from the swimming speed and mass flux, was found to depend on the viscoelastic parameters α and β in a simple fashion.

We then considered the lubrication problem with a swimmer of prescribed shape in a Bingham fluid. Here we fixed the amplitude of the swimmer and

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[26]There are some restrictions to the scope of these differences, for example with changes in the constitutive model, we have to know $\tau_{xy}(u_y)$, and that the law reduces to a steady-state shear flow for the thin-gap geometry with dominant shear stress.

solved for the location of the yield surfaces in the problem. The yield surfaces, $y = Y_+$ and $y = Y_-$, were defined to be the upper and lower bounds of the region of the fluid where $|\tau_{xy}| \leq \tau_y$. We found that there exist regions along the swimmer where the plug-flow reaches the swimmer (case II), reaches the wall (case IV), and regions where the fluid is entirely yielded (cases I and V). When varying the Bingham number, it was found that there exists a maximum swimming speed at $B \simeq 1$, which is greater than the Newtonian swimming speed.

Combining the results from Chapter Two, we may speculate that it is the most advantageous for a swimmer to be moving through a fluid that exhibits a yields stress, and once yielded, behaves in the manner of a shear thinning fluid. An example of such a fluid is a Herschel-Bulkley fluid with $n < 1$.

The problem of an elastic lubricated swimmer was then considered. The swimmer was modelled with the beam equation with an addition of an internal applied torque as was done in [2]. When looking at the problem with a Newtonian fluid, it was found that there exists an amplitude of forcing that yields an optimal swimming speed. As the amplitude of the forcing was increased, it was found that the swimmer shape began to develop a plateau region that approached the wall, followed by a large downward spike region. The value of the plateau region was successfully calculated analytically. We then introduced a viscoelastic fluid. It was again found that a shear thinning fluid is optimal for a swimmer as it yields the fastest swimming speeds. As the bending stiffness of the swimmer was increased, it was observed that the problem collapsed into the Newtonian problem.

In general, we found that a shear thinning fluid or a fluid that exhibits a yield stress is theoretically optimal for swimmers near walls. Supporting this theory is the fact that polymeric fluids, the type of fluid found in most biological settings, are not often shear thickening and often exhibit a yield stress [5].

In the case of the elastic swimmer, we found that it is not optimal for a swimmer to continue increasing its energy output, as it does not continue to increase the swimmer's speed. This is because as the forcing along the swimmer increases, so does the stress within the fluid and so there exists a point in which the stresses in the fluid are so great that they begin to hamper movement. This suggests that in nature, swimmers do not exert themselves to their maximum, but rather moderately in such a way that maximizes speed.

4.2   Further Work

The results of this thesis suggest further work. Due to the formulation of the problem, and in particular, the simplification of the constitutive model when non-dimensionalized, it is theoretically straightforward to consider numerous different constitutive equations and fluid types. This suggests developing a more robust numerical method of solution.

From a biological perspective, there are many extensions of the analysis that can be done. For example, it is the case that in a biological setting, swimmers are of finite length. Thus, it may be of interest to consider the problems with a finite swimmer and relax the periodicity conditions.

There are numerous extensions that can be done with the elastic swimmer problem. Due to the interesting results of the Newtonian elastic swimmer problem, this author suggests a full analysis of the dynamics of this problem. It is unfortunate that such an analysis is beyond the scope of this thesis. As well, due to the large amplitude of the spike region encountered in this problem, one might consider the swimmer in a channel, therefore restricting the amplitude of the spike. There is potential for things to get quite interesting in this case.

One might also be interested in looking into the large amplitude forcing viscoelastic problem, as it is not entirely clear that this will reduce into the Newtonian problem as it did in the similar case considered in Chapter Two.

As mentioned throughout this thesis, we are by no means limited to the specific choices we have made. This is true for the choice of constitutive models, swimmer shapes (Chapter Two) and applied torques (Chapter Three). The mathematical set up we chose here lends itself nicely to generality, and exploring these potential alternate applications, via different choices of the constitutive model, swimmer shape and applied torques, is certainly something that should be looked into further.

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Appendix A

Full Analysis of Parent Problem

In this section we discuss a fundamental problem, that of a swimmer in the presence of a wall and a viscoelastic fluid without the lubrication approximation. Here we will assume that the amplitude of the waves is small in comparison to the mean distance to the wall. This is a nice warm up problem as it is a combination of Katz's biharmonic problem and Lauga's viscoelastic Taylor problem [22], [25].

A.1 Mathematical Formulation

We assume that the sheet has prescribed motion, that of waves propagating to the right. In the frame moving with the swimmer, these waves look like

$$y = a\sin(kx - \omega t). \tag{A.1}$$

The equations of motion for this problem are

$$\bigtriangledown \cdot \mathbf{u} = 0, \tag{A.2}$$

$$\bigtriangledown p = \bigtriangledown \cdot \boldsymbol{\tau}, \tag{A.3}$$

$$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}^{\bigtriangledown} = \mu \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \dot{\boldsymbol{\gamma}}^{\bigtriangledown} \right), \tag{A.4}$$

which are the conservation of mass, momentum and the Oldroyd-B constitutive equation respectively. We also have the zero force condition[27]

$$\langle \tau_{xy} \rangle. = 0 \tag{A.5}$$

---

[27]This equation still holds even though there is no lubrication approximation here due to the low amplitude compared to mean height approximation

The next step is then to non-dimensionalize the equations as follows

$$
\begin{aligned}
t &= \tfrac{1}{\omega}\hat{t}, & (x,y) &= \tfrac{1}{k}(\hat{x},\hat{y}), \\
\mathbf{u} &= c\hat{\mathbf{u}}, & \dot{\boldsymbol{\gamma}} &= \omega\hat{\dot{\boldsymbol{\gamma}}}, \\
(p,\boldsymbol{\tau}) &= \mu\omega(\hat{p},\hat{\boldsymbol{\tau}}),
\end{aligned}
$$

where $c = \frac{\omega}{k}$ is the speed of the travelling wave. The conservation of momentum and conservation of mass equations do not change:

$$
\triangledown p = \triangledown \cdot \boldsymbol{\tau}, \tag{A.6}
$$

$$
\triangledown \cdot \mathbf{u} = 0, \tag{A.7}
$$

and the constitutive equation becomes:

$$
\boldsymbol{\tau} + De_1 \boldsymbol{\tau}^{\triangledown} = \dot{\boldsymbol{\gamma}} + De_2 \dot{\boldsymbol{\gamma}}^{\triangledown}, \tag{A.8}
$$

where De_1 and De_2 are Deborah numbers.

It remains to non-dimensionalize the boundary conditions. Using the low amplitude approximation, we put

$$
\varepsilon = ak \ll 1, \tag{A.9}
$$

and find that (dropping hats):

$$
y = \varepsilon \sin(x - t). \tag{A.10}
$$

From which we can easily take the time derivative to find

$$
dy/dt = -\varepsilon \cos(x - t) \tag{A.11}
$$

The boundary conditions become, on the wall $y = hk$ [28]

$$
u = U, \tag{A.12}
$$

$$
v = 0, \tag{A.13}
$$

---

[28] note that we just write U rather than $(1/c)U$ for simplicity

and on the swimmer $y = \varepsilon \sin(x - t)$,

$$u \;=\; 0, \tag{A.14}$$

$$v \;=\; -\varepsilon \cos(x - t). \tag{A.15}$$

A.2 Stream Function and Asymptotics

Since the problem at hand is an incompressible, two dimensional problem, it is most useful to consider a stream function ψ such that

$$u \;=\; \frac{\partial \psi}{\partial y}, \tag{A.16}$$

$$v \;=\; -\frac{\partial \psi}{\partial x}. \tag{A.17}$$

Automatically we have conservation of mass holding since

$$\nabla \cdot \mathbf{u} = u_x + v_y = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0. \tag{A.18}$$

It is convenient to observe that the boundary conditions can be written concisely as:

$$\nabla \psi \big|_{(x,\varepsilon \sin(x-t))} \;=\; \varepsilon \cos(x - t)\mathbf{i} \tag{A.19}$$

$$\nabla \psi \big|_{(x,hk)} \;=\; U\mathbf{j} \tag{A.20}$$

where $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ are the unit vectors in the x and y directions respectively. This form of the boundary conditions is only practical for substituting in the below perturbation expansions and Taylor expanding.

To solve this system, consider low amplitude solutions of the form

$$U \;=\; \varepsilon U_1 + \varepsilon^2 U_2 + \dots \tag{A.21}$$

$$\psi \;=\; \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \dots \tag{A.22}$$

$$\boldsymbol{\tau} \;=\; \varepsilon \boldsymbol{\tau_1} + \varepsilon^2 \boldsymbol{\tau_2} + \dots \tag{A.23}$$

That is, regular perturbation expansions in ε for the swimming speed, the stream function and the stress.

Since it is assumed that the speed of the swimmer is constant, it must be

the case that each U_k in the expansion of U is also constant. That is

$$U_k = \text{Const} \qquad \forall k = 1, 2, 3...\tag{A.24}$$

Notice that since

$$\dot{\boldsymbol{\gamma}} = \begin{pmatrix} 2u_x & u_y + v_x \\ u_y + v_x & 2v_y \end{pmatrix} = \begin{pmatrix} 2\psi_{yx} & \psi_{yy} - \psi_{xx} \\ \psi_{yy} - \psi_{xx} & -2\psi_{xy} \end{pmatrix},\tag{A.25}$$

it is convenient to write

$$\dot{\boldsymbol{\gamma}} = \varepsilon\dot{\boldsymbol{\gamma}}_{\mathbf{1}} + \varepsilon^2\dot{\boldsymbol{\gamma}}_{\mathbf{2}} + ...\tag{A.26}$$

Similarly, we write

$$\mathbf{u} = \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + ...\tag{A.27}$$

In the spirit of asymptotics, the next step is to substitute the perturbation expansions into the equations of motion and the boundary conditions and sort out the first and second order equations. Before this is done, it is most useful to observe that

$$\nabla \times \nabla p = 0,\tag{A.28}$$

and so pressure can be removed from the conservation of momentum equation. Summarizing the equations of motion then gives

$$\nabla \times \nabla \cdot \boldsymbol{\tau} = 0 \tag{A.29}$$

$$\boldsymbol{\tau} + De_1\boldsymbol{\tau}^{\nabla} = \dot{\boldsymbol{\gamma}} + De_2\dot{\boldsymbol{\gamma}}^{\nabla}.\tag{A.30}$$

Substituting in the perturbation expansions yields at $O(\varepsilon)$:

$$\nabla \times \nabla \cdot \boldsymbol{\tau}_{\mathbf{1}} = 0,\tag{A.31}$$

$$\boldsymbol{\tau}_{\mathbf{1}} + De_1\frac{\partial}{\partial t}\boldsymbol{\tau}_{\mathbf{1}} = \dot{\boldsymbol{\gamma}}_{\mathbf{1}} + De_2\frac{\partial}{\partial t}\dot{\boldsymbol{\gamma}}_{\mathbf{1}},\tag{A.32}$$

and at $O(\varepsilon^2)$:

$$\nabla \times \nabla \cdot \boldsymbol{\tau}_{\mathbf{2}} = 0,\tag{A.33}$$

$$\boldsymbol{\tau}_{\mathbf{2}} + De_1\frac{\partial}{\partial t}\boldsymbol{\tau}_{\mathbf{2}} + De_1\left(\mathbf{u}_1 \cdot \nabla\boldsymbol{\tau}_{\mathbf{1}} - \boldsymbol{\tau}_{\mathbf{1}} \cdot \nabla\mathbf{u}_1 - (\nabla\mathbf{u}_1)^T \cdot \boldsymbol{\tau}_{\mathbf{1}}\right) =$$

$$\dot{\boldsymbol{\gamma}}_{\mathbf{2}} + De_2\frac{\partial}{\partial t}\dot{\boldsymbol{\gamma}}_{\mathbf{2}} + De_2\left(\mathbf{u}_1 \cdot \nabla\dot{\boldsymbol{\gamma}}_{\mathbf{1}} - \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \cdot \nabla\mathbf{u}_1 - (\nabla\mathbf{u}_1)^T \cdot \dot{\boldsymbol{\gamma}}_{\mathbf{1}}\right).\tag{A.34}$$

Now it is time to look at the boundary conditions. Substituting in the regular

perturbation expansion for ψ and expanding about $y = 0$ yields at $O(\varepsilon)$:

$$\triangledown \psi_1 \big|_{(x,0)} \;=\; \cos(x - t)\mathbf{i}, \tag{A.35}$$

$$\triangledown \psi_1 \big|_{(x,hk)} \;=\; U_1 \mathbf{j}, \tag{A.36}$$

and at $O(\varepsilon^2)$:

$$\triangledown \psi_2 \big|_{(x,0)} + \triangledown \left(\frac{\partial \psi_1}{\partial y} \right) \bigg|_{(x,0)} \sin(x - t) \;=\; 0, \tag{A.37}$$

$$\triangledown \psi_2 \big|_{(x,hk)} \;=\; U_2 \mathbf{j}. \tag{A.38}$$

Substituting the expansions into the constraint $\langle \tau_{xy} \rangle = 0$ yields that

$$\langle \tau_{kxy} \rangle = 0 \qquad k = 1, 2, 3... \tag{A.39}$$

A.3   Leading Order Equations

If one stares at the leading order equations for a moment,

$$\triangledown \times \triangledown \cdot \boldsymbol{\tau}_1 \;=\; 0, \tag{A.40}$$

$$\boldsymbol{\tau}_1 + De_1 \frac{\partial}{\partial t} \boldsymbol{\tau}_1 \;=\; \dot{\boldsymbol{\gamma}}_1 + De_2 \frac{\partial}{\partial t} \dot{\boldsymbol{\gamma}}_1, \tag{A.41}$$

it becomes apparent that the easiest thing to do is to take the divergence and then the curl of the second equation thus eliminating leading order stress. By doing this we obtain

$$\left(1 + De_2 \frac{\partial}{\partial t} \right) \triangledown \times \triangledown \cdot \dot{\boldsymbol{\gamma}}_1 = 0. \tag{A.42}$$

When substituting in

$$\dot{\boldsymbol{\gamma}}_1 = \dot{\boldsymbol{\gamma}}_1 (\psi_{1xx}, \psi_{1xy}, \psi_{1yy}),$$

we actually get

$$\left(1 + De_2 \frac{\partial}{\partial t} \right) \triangledown^4 \psi_1 = 0 \tag{A.43}$$

where

$$\triangledown^4 \psi = \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}.$$

Writing down the boundary conditions in a more practical fashion gives

$$\begin{aligned}
\left.\frac{\partial \psi_1}{\partial x}\right|_{(x,0)} &= 0, & \left.\frac{\partial \psi_1}{\partial y}\right|_{(x,0)} &= 0, \\
\left.\frac{\partial \psi_1}{\partial x}\right|_{(x,hk)} &= \cos(x-t), & \left.\frac{\partial \psi_1}{\partial y}\right|_{(x,hk)} &= U_1.
\end{aligned} \tag{A.44}$$

One way to solve this is to simply use separation of variables, cleverly guessing an initial form of the solution based on the boundary conditions

$$\psi_1 = f(y)sin(x-t), \tag{A.45}$$

where $f(y)$ is to unknown. Substituting this into the leading order equation for ψ yields

$$\frac{d^4 f}{dy^4} - 2\frac{d^2 f}{dy^2} + f = 0. \tag{A.46}$$

It is easily found that

$$f(y) = (a + by)e^{-y} + (c + dy)e^{y}, \tag{A.47}$$

where constants a, b, c, and d are found by applying the boundary conditions and will depend on h. When applying the boundary condition $\psi_{1y}\big|_{(x,hk)} = U_1$ it becomes apparent that in order for the assumption that $U = constant$ to hold, it must be the case that

$$U_1 = 0 \tag{A.48}$$

or else U_1 would depend on x and t, which is unacceptable.

This is a result that we could have easily predicted based on the work done previously by others, who all found the leading order swimming speed to be zero [37], [22], [25].

Alternately, one can solve this using hyperbolic sines and cosines.

The procedure is the same as above, only this time a different form for $f(y)$ will be used. Put

$$\psi_1(y) = [(A+BH)\cosh(H) + (C+DH)\sinh(H)]\sin(x-t).$$

This gives

$$\begin{aligned}
\psi_{1x} &= [(A+BH)\cosh(H)+ \\
&\quad +(C+D(H))\sinh(H)]\cos(x-t), \\
\psi_{1y} &= -[(A+BH+D)\sinh(H)+ \\
&\quad +(B+C+DH)\cosh(H)]\sin(x-t),
\end{aligned}$$

where $H = (hk - y)$. Upon applying the boundary conditions, for $y = hk$:

$$
\begin{aligned}
A\cos(x-t) &= 0, \\
-(B+C)\sin(x-t) &= U_1.
\end{aligned}
$$

Clearly, in order for the assumption, $U_1 = \text{Const}$, to hold, it must be the case that $U_1 = 0$, thus implying that $B = -C$. Also, it is clear from the first condition that $A = 0$. Thus so far,

$$
\psi_1 = [B[\sinh(hk-y)-(hk-y)\cosh(hk-y)]+D(hk-y)\sinh(hk-y)]\sin(x-t).
$$

Or, to simplify the x-derivatives, one could consider the complex analogy of the problem ('complexification', see below). That is put

$$
\psi_1 = \frac{1}{2}\left(\tilde{\psi}_1 e^{i(x-t)} + \tilde{\psi}_1^{\,*} e^{-i(x-t)}\right), \tag{A.49}
$$

and consider only

$$
\tilde{\psi}_1 e^{i(x-t)}.
$$

One then gets:

$$
\tilde{\psi}_1 = [\tilde{B}[\sinh(hk-y)-(hk-y)\cosh(hk-y)]+\tilde{D}(hk-y)\sinh(hk-y)].
$$

Define

$$
\begin{aligned}
\Psi_1(y) &= (hk-y)\sinh(hk-y), & \text{(A.50)} \\
\Psi_2(y) &= \sinh(hk-y)-(hk-y)\cosh(hk-y), & \text{(A.51)}
\end{aligned}
$$

thus giving

$$
\tilde{\psi}_1 = [\tilde{D}\Psi_1 + \tilde{B}\Psi_2].
$$

Now, the unknown coefficients $\tilde{B}$ and $\tilde{D}$ must be determined using the other boundary conditions. On $y = 0$:

$$
\begin{aligned}
\left(\tilde{\psi}_1 e^{i(x-t)}\right)_x\Big|_{y=0} &= ie^{i(x-t)}[\tilde{D}\Psi_1(0)+\tilde{B}\Psi_2(0)] &= e^{i(x-t)}, \\
\left(\tilde{\psi}_1 e^{i(x-t)}\right)_y\Big|_{y=0} &= e^{i(x-t)}[\tilde{D}\Psi_1'(0)+\tilde{B}\Psi_2'(0)] &= 0.
\end{aligned}
$$

Solving these gives

$$
\tilde{D} = i\frac{\Psi_2'(0)}{W(0)}, \quad \tilde{B} = -i\frac{\Psi_1'(0)}{W(0)},
$$

where

$$W(y) = \Psi_1'(0)\Psi_2(y) - \Psi_2'(0)\Psi_1(y). \tag{A.52}$$

Putting this altogether one has

$$\tilde{\psi}_1 = -i\frac{W(y)}{W(0)} \tag{A.53}$$

or writing down the real component,

$$\psi_1 = \frac{W(y)}{W(0)}\sin(x - t) \tag{A.54}$$

Although this second method of solving for the leading order stream function and swimming speed is longer, it provides a more concise final form. This is useful since the leading order swimming speed is zero, so the calculation must be done to second order. This aesthetically pleasing form of the leading order solution will make the second order problem much easier and less algebraically challenging to solve.

To solve for the leading order stress, we employ the method of 'complexification' (referred to as Fourier notations in [25]). That is, consider the leading order equation

$$\boldsymbol{\tau_1} + De_1\partial_t\boldsymbol{\tilde{\tau}_1} = \tilde{\boldsymbol{\gamma}}_1 + De_2\partial_t\tilde{\boldsymbol{\gamma}}_1,$$

and think of the stress and rate of strain as being the real portion of a larger, complex problem (note the periodic dependence explicitly given due to the assumed periodicity of the problem)

$$\boldsymbol{\tau_1} = \Re[\tilde{\boldsymbol{\tau}}_1 e^{i(x-t)}], \quad \dot{\boldsymbol{\gamma}}_1 = \Re[\tilde{\dot{\boldsymbol{\gamma}}}_1 e^{i(x-t)}]. \tag{A.55}$$

Alternately, one could write

$$\boldsymbol{\tau_1} = \tfrac{1}{2}\left(\tilde{\boldsymbol{\tau}}_1 e^{i(x-t)} + \tilde{\boldsymbol{\tau}}_1{}^* e^{-i(x-t)}\right), \quad \dot{\boldsymbol{\gamma}}_1 = \tfrac{1}{2}\left(\tilde{\dot{\boldsymbol{\gamma}}}_1 e^{i(x-t)} + \tilde{\dot{\boldsymbol{\gamma}}}_1{}^* e^{-i(x-t)}\right), \tag{A.56}$$

observing that $\tilde{\boldsymbol{\tau}}_1$ and $\tilde{\dot{\boldsymbol{\gamma}}}_1$ are functions of y only, and are independent of x and t (note that this is also true for $\tilde{\psi}_1$ defined above).

This 'complexification' of the problem implies that

$$\partial_t = -\partial_x = -i.$$

Therefore, by exploiting this simplification of time derivatives, we can easily obtain

$$\tilde{\boldsymbol{\tau}}_{\mathbf{1}} = \frac{1 - iDe_2}{1 - iDe_1}\tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}. \tag{A.57}$$

Upon substituting in $\tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}} = \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}(\tilde{\psi}_{1xx}, \tilde{\psi}_{1xy}, \tilde{\psi}_{1yy})$ as defined above, we get

$$\tilde{\boldsymbol{\tau}}_{\mathbf{1}} = \frac{(1 - iDe_2)e^{i(x-t)}}{(1 - iDe_1)W(0)}\begin{pmatrix} 2W' & -i(W'' + W) \\ -i(W'' + W) & -2W' \end{pmatrix}. \tag{A.58}$$

Notice that the condition,

$$\langle \tau_{1xy} \rangle = 0,$$

automatically holds due to the fact that

$$\tau_{1xy} \propto \dot{\gamma}_{1xy} \propto \sin(x - t).$$

A.4   Second Order Equations

The leading order problem has been solved and it has been shown that in order to maintain the assumption that the speed of the swimmer is constant, at leading order it must be equal to zero as predicted. Thus, in order to calculate the swimming speed of the organism, one must move on to the second order problem. This problem is not quite as straightforward as the leading order problem due to the nonlinear terms of the upper convected derivative that crop up. Let us restate the second order problem:

$$\nabla \times \nabla \cdot \boldsymbol{\tau}_{\mathbf{2}} = 0,$$

$$\left(1 + De_1\frac{\partial}{\partial t}\right)\boldsymbol{\tau}_{\mathbf{2}} - \left(1 + De_2\frac{\partial}{\partial t}\right)\dot{\boldsymbol{\gamma}}_{\mathbf{2}} = De_2\left(\mathbf{u}_{\mathbf{1}} \cdot \nabla\dot{\boldsymbol{\gamma}}_{\mathbf{1}} - \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \cdot \nabla\mathbf{u}_{\mathbf{1}} - (\nabla\mathbf{u}_{\mathbf{1}})^T \cdot \dot{\boldsymbol{\gamma}}_{\mathbf{1}}\right) -$$
$$- De_1\left(\mathbf{u}_{\mathbf{1}} \cdot \nabla\boldsymbol{\tau}_{\mathbf{1}} - \boldsymbol{\tau}_{\mathbf{1}} \cdot \nabla\mathbf{u}_{\mathbf{1}} - (\nabla\mathbf{u}_{\mathbf{1}})^T \cdot \boldsymbol{\tau}_{\mathbf{1}}\right),$$

with boundary conditions:

$$\nabla\psi_2\big|_{(x,0)} + \nabla\left(\frac{\partial\psi_1}{\partial y}\right)\Big|_{(x,0)}\sin(x - t) = 0,$$
$$\nabla\psi_2\big|_{(x,hk)} = U_2\mathbf{j}.$$

The zero force condition states that

$$\langle \tau_{2xy} \rangle = 0. \tag{A.59}$$

To start things off slowly, consider first the boundary conditions. Writing things in a more user friendly fashion gives on $y = 0$:

$$\psi_{2x}\big|_{y=0} = -\sin(x-t)\psi_{1yx}, \quad \psi_{2y}\big|_{y=0} = -\sin(x-t)\psi_{1yy}.$$

But we know ψ_1 from solving the leading order problem, which gives $\psi_{1yx} = \frac{W'(y)}{W(0)}cos(x-t)$ recalling that $W(y) = \Psi_1'(0)\Psi_2(y) - \Psi_2'(0)\Psi_1(y)$ and $\Psi_1(y) = (hk-y)\sinh(hk-y), \Psi_2(y) = \sinh(hk-y) - (hk-y)\cosh(hk-y)$. Thus, we obtain

$$\psi_{2x}\big|_{y=0} = 0, \tag{A.60}$$

$$\psi_{2y}\big|_{y=0} = -\sin^2(x-t)\frac{W''(0)}{W(0)}. \tag{A.61}$$

On $y = hk$:

$$\psi_{2x}\big|_{y=hk} = 0, \tag{A.62}$$

$$\psi_{2y}\big|_{y=hk} = U_2. \tag{A.63}$$

Inspecting the second order equations of motion as written above reveals that the right-hand side of the momentum equation is composed of leading order terms exclusively. Since it is the case that the leading order stress can be simply represented in terms of the leading order rate of strain when complexified, let us put the right-hand side of the momentum equations in terms of the complexified variables

$$\mathbf{u}_1 = \tfrac{1}{2}(\tilde{\mathbf{u}}_1 e^{i(x-t)} + \tilde{\mathbf{u}}_1^* e^{-i(x-t)}), \quad \dot{\boldsymbol{\gamma}}_1 = \tfrac{1}{2}(\tilde{\dot{\boldsymbol{\gamma}}}_1 e^{i(x-t)} + \tilde{\dot{\boldsymbol{\gamma}}}_1^* e^{-i(x-t)}),$$

and of course

$$\tilde{\boldsymbol{\tau}}_1 = \frac{1 - iDe_2}{1 - iDe_1}\tilde{\dot{\boldsymbol{\gamma}}}_1.$$

We obtain

$$RHS = \frac{(De_2 - De_1)}{4(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \nabla\tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \nabla\tilde{\mathbf{u}}_1^* - (\nabla\tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] + C.C. +$$

$$+ \frac{(De_2 - De_1)}{4(1 - iDe_1)} e^{2i(x-t)} \left[\tilde{\mathbf{u}}_1 \cdot \nabla\tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \nabla\tilde{\mathbf{u}}_1 - (\nabla\tilde{\mathbf{u}}_1)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] + C.C.,$$

$$(A.64)$$

where $C.C.$ refers to the complex conjugate of the term directly preceding it.[29]

To solve this rather long, complicated looking equation, make use of the periodicity of the problem and look at the x-average of the equation. Notice at this point one could take the curl and divergence of the problem to remove the $\boldsymbol{\tau_2}$ terms, but that is rather cumbersome. Rather, only consider the xy-component and utilize the zero force condition to simplify the problem.

Let us first look at the boundary conditions. Note that $\langle\psi_{2x}\rangle = \frac{1}{2\pi}\psi_2\big|_0^{2\pi} = 0$ always, so the boundary conditions for ψ_{2x} yield no information. However, the other two boundary conditions have non-trivial averages:

$$\langle\psi_{2y}\rangle\big|_{y=0} = -\frac{W''(0)}{2W(0)}, \tag{A.66}$$

$$\langle\psi_{2y}\rangle >\big|_{y=hk} = U_2. \tag{A.67}$$

Now let us average the second order constitutive equation and look at the xy-component (recall that $\partial_t = -\partial_x$ and that $\langle e^{\pm 2i(x-t)}\rangle = 0$), subscripts here referring to components:

$$\langle\tau_{2xy}\rangle - \langle\dot{\gamma}_{2xy}\rangle = \Re\left\{ \frac{(De_2 - De_1)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \nabla\tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \nabla\tilde{\mathbf{u}}_1^* - (\nabla\tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} \right\}. \tag{A.68}$$

Applying the zero force condition and writing the rate of strain component in

---

[29] Alternately, one could write

$$RHS = \Re\left\{ \frac{(De_2 - De_1)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \nabla\tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \nabla\tilde{\mathbf{u}}_1^* - (\nabla\tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] \right\} +$$

$$+ \Re\left\{ \frac{(De_2 - De_1)}{2(1 - iDe_1)} e^{2i(x-t)} \left[\tilde{\mathbf{u}}_1 \cdot \nabla\tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \nabla\tilde{\mathbf{u}}_1 - (\nabla\tilde{\mathbf{u}}_1)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right] \right\}$$

$$(A.65)$$

terms of the stream function gives:

$$\langle \psi_{2yy} - \psi_{2xx} \rangle = \Re \left\{ \frac{(De_1 - De_2)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1^* - (\bigtriangledown \tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} \right\}. \tag{A.69}$$

But, $\langle \psi_{2xx} \rangle = \frac{1}{2\pi} \psi_{2x} \big|_0^{2\pi} = -\frac{1}{2\pi} v \big|_0^{2\pi} = 0$ due to the assumed periodicity of the solution. Since the bounds of the average are not dependent on y, we can put

$$\langle \psi_2 \rangle_{yy} = \Re \left\{ \frac{(De_1 - De_2)}{2(1 - iDe_1)} \left[\tilde{\mathbf{u}}_1^* \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 - \tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1^* - (\bigtriangledown \tilde{\mathbf{u}}_1^*)^T \cdot \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} \right\}. \tag{A.70}$$

Thus, to solve the problem and calculate the swimming speed we must simply integrate once to obtain an expression for $\langle \psi_2 \rangle_y$ and apply the boundary conditions. To to this, the xy-component of the right-hand side must be calculated. By using the fact that both $\tilde{\mathbf{u}}_1$ and $\tilde{\boldsymbol{\gamma}}_1$ are functions of $\tilde{\psi}_1 = -i\frac{W(y)}{W(0)} = -i\frac{W}{W_0}$, we can write down the terms on the right-hand side in a relatively concise manner. Namely, we obtain

$$\left[\tilde{\mathbf{u}}_1^* \cdot \bigtriangledown \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} = \frac{i}{W_0^2} \left(W'W'' + 2WW' + WW''' \right), \tag{A.71}$$

$$\left[\tilde{\boldsymbol{\gamma}}_1 \cdot \bigtriangledown \tilde{\mathbf{u}}_1^* \right]_{xy} = \frac{i}{W_0^2} \left(W'W'' + 3WW' \right), \tag{A.72}$$

$$\left[\bigtriangledown \tilde{\mathbf{u}}_1^{*T} \cdot \tilde{\boldsymbol{\gamma}}_1 \right]_{xy} = \frac{-i}{W_0^2} \left(3W'W'' + WW' \right). \tag{A.73}$$

Substituting these into the governing equation gives

$$\langle \psi_2 \rangle_{yy} = \Re \left\{ \frac{(De_1 - De_2)}{2(1 - iDe_1)} \frac{i}{W_0^2} \left[WW''' + 3W'W'' \right] \right\}. \tag{A.74}$$

But, notice that $[WW''' + 3W'W'']$ can be written as $[WW'' + W'^2]'$ and so we find that

$$\langle \psi_2 \rangle_{yy} = \frac{De_1(De_2 - De_1)}{2(1 + De_1^2)W_0^2} \left[WW'' + W'^2 \right]'. \tag{A.75}$$

Integrating and applying the first boundary condition gives

$$-\frac{W_0''}{2W_0} = \frac{De_1(De_2 - De_1)}{2(1 + De_1^2)W_0^2} \left[W_0 W_0'' + W_0'^2 \right] + \text{Const.}$$

Using the fact that $W_0' = 0$ the constant of integration can be easily solved for,

$$\text{Const} = -\frac{W_0''}{2W_0} \frac{(1 + De_1 De_2)}{(1 + De_1^2)}.$$

Thus,

$$\langle \psi_2 \rangle_y = \frac{De_1(De_2 - De_1)}{2(1 + De_1^2)W_0^2} \left[WW'' + W'^2 \right] - \frac{W_0''}{2W_0} \frac{(1 + De_1 De_2)}{(1 + De_1^2)}, \qquad (A.76)$$

and so finally one has the swimming speed

$$U_2 = \langle \psi_2 \rangle_y \big|_{y=hk} = -\frac{W_0''}{2W_0} \frac{(1 + De_1 De_2)}{(1 + De_1^2)}, \qquad (A.77)$$

since $W(hk) = 0$ and $W'(hk) = 0$.

A.5 Work

The leading order rate of work (per unit wavelength) is given by the volume integral of

$$\langle \boldsymbol{\tau_1} : \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \rangle, \qquad (A.78)$$

and is equal to the rate of viscous dissipation. To simplify this calculation, we look again to the 'complexified' variables. That is, put

$$\boldsymbol{\tau_1} : \dot{\boldsymbol{\gamma}}_{\mathbf{1}} = \frac{1}{4} \left(\tilde{\boldsymbol{\tau}}_{\mathbf{1}} + \tilde{\boldsymbol{\tau}}_{\mathbf{1}}^* \right) : \left(\tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}} + \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}^* \right).$$

Let us calculate the integral in x first. It is the case that

$$\begin{aligned}
\langle \tilde{\boldsymbol{\tau}}_{\mathbf{1}} : \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}} \rangle &\propto \langle e^{2i(x-t)} \rangle &= 0, \\
\langle \tilde{\boldsymbol{\tau}}_{\mathbf{1}}^* : \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}^* \rangle &\propto \langle e^{-2i(x-t)} \rangle &= 0.
\end{aligned}$$

Since both $\boldsymbol{\tau_1}$ and $\dot{\boldsymbol{\gamma}}_{\mathbf{1}}$ are symmetric, we can put

$$\langle \boldsymbol{\tau_1} : \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \rangle = \frac{1}{2} \Re \left\{ \langle \tilde{\boldsymbol{\tau}}_{\mathbf{1}} : \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}^* \rangle \right\}$$

Knowing the form of $\tilde{\boldsymbol{\tau}}_{\mathbf{1}}$ in terms of $\tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}$ gives

$$\langle \boldsymbol{\tau_1} : \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \rangle = \frac{1}{2} \Re \left\{ \left(\frac{1 - iDe_2}{1 - iDe_1} \right) \langle \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}} : \tilde{\dot{\boldsymbol{\gamma}}}_{\mathbf{1}}^* \rangle \right\}$$

Using the dependence of $\dot{\gamma}_1$ on ψ_1, and the fact that $\tilde{\psi}_1 = -i\frac{W}{W_0}$, gives us

$$\langle \boldsymbol{\tau_1} : \dot{\boldsymbol{\gamma}}_{\mathbf{1}} \rangle = \frac{1 + De_1 De_2}{1 + De_1^2} \left(\frac{4W'^2 + (W + W'')^2}{W_0^2} \right). \qquad (A.79)$$

Thus, leading order rate of work per unit wavelength, is given by

$$\frac{(1 + De_1 De_2)}{(1 + De_1^2)W_0^2} \int_0^{hk} \left[4W'^2 + (W + W'')^2 \right] dy. \tag{A.80}$$

A.6 Limiting Cases

A.6.1 Distance Between Wall and Swimmer Approaches Infinity

To make sure that this problem agrees with the infinite case considered by Lauga in [25], we must take the limit as the distance between the wall and the swimmer approaches infinity. It is known that

$$\psi_1 = \frac{W(y)}{W(0)} \sin(x - t),$$

recalling that

$$W(y) = \Psi_1'(0)\Psi_2(y) - \Psi_2'(0)\Psi_1(y),$$

and

$$\begin{aligned} \Psi_1(y) &= (hk - y)\sinh(hk - y), \\ \Psi_2(y) &= \sinh(hk - y) - (hk - y)\cosh(hk - y). \end{aligned}$$

The limit as $h \longrightarrow \infty$ is easily calculated. It is found that

$$\lim_{h \longrightarrow \infty} \frac{W(y)}{W(0)} = (1 + y)e^{-y}$$

Thus

$$\lim_{h \longrightarrow \infty} \psi_1 = (1 + y)e^{-y}\sin(x - t)$$

which is exactly the infinite solution as stated in [25].

It is obvious that if the limit of the stream function agrees with the infinite case that the limit of the swimming speed will as well. But to illustrate this clearly, recall that

$$U_2 = -\frac{W''(0)}{2W(0)}\frac{(1 + De_1 De_2)}{(1 + De_1^2)}.$$

So, the limit

$$\lim_{h \longrightarrow \infty} U_2 = \frac{1}{2}\frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

which again agrees with the infinite case.

Similarly, we can calculate the limit for the averaged rate of work done by the swimmer. It is found that

$$\lim_{h \to \infty} \left\{ \frac{(1 + De_1 De_2)}{(1 + De_1^2)W_0^2} \int_0^{hk} \left[4W'^2 + (W + W'')^2 \right] dy \right\} = 2\frac{(1 + De_1 De_2)}{(1 + De_1^2)}.$$

We cannot directly compare this limit with the form of the solution found in [25] as it is stated as

$$\frac{Work_\infty}{W_N} = \frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

and W_N is not stated. However, we can easily see that the calculation of work in the infinite case is exactly the same with the exception of the W function. In the infinite case one has

$$W_\infty = (1 + y)e^{-y},$$

and so it is found that

$$\frac{(1 + De_1 De_2)}{(1 + De_1^2)W_\infty(0)^2} \int_0^\infty \left[4W'^2_\infty + (W_\infty + W''_\infty)^2 \right] dy = 2\frac{(1 + De_1 De_2)}{(1 + De_1^2)},$$

exactly matching the limit calculated above.

So, it is clear that the above calculations for the problem of an organism swimming in an Oldroyd-B viscoelastic fluid near a wall are consistent with those of the problem of an organism swimming in an infinite fluid of the same type.

A.6.2 Swimmer Approaches the Wall

Next, we would like to consider the limit as the swimmer and the wall approach each other. There are two ways of thinking of this limit. The first is keeping the wall fixed at $y = h$ and increasing the amplitude of the swimmer until it reaches the wall. This line of thinking contradicts the assumption that $ak << 1$, as it implies that $O(a) \to O(h) = O(1/k)$. This would make our present solution invalid. The second way of thinking about this limit is to think of the wall approaching the swimmer, whose amplitude and wavelength remains the fixed. This implies that $O(h) \to O(a)$, still maintaining the assumption that $ak << 1$. In fact, in this manner of thinking we get that $hk << 1$ which is exactly the

lubrication approximation. We have that, to leading order

$$U = -\varepsilon^2 \frac{(1 + De_1 De_2)}{2(1 + De_1^2)} F(hk),$$ (A.81)

where

$$F(hk) = \frac{W_0''}{W_0} = \frac{(hk)^2 + \sinh^2(hk)}{(hk)^2 - \sinh^2(hk)}.$$ (A.82)

In our limit, $hk \to ak = \varepsilon$ and so we have that

$$F(\varepsilon) = \frac{\varepsilon^2 + \sinh^2 \varepsilon}{\varepsilon^2 - \sinh^2 \varepsilon}.$$ (A.83)

Expanding $\sinh \varepsilon = \varepsilon + \frac{1}{6}\varepsilon^3 + \dots$ and simplifying gives us

$$F(\varepsilon) = \frac{2\varepsilon^2 + + \frac{1}{3}\varepsilon^4 + \dots}{-\frac{1}{3}\varepsilon^4 - \dots}.$$ (A.84)

Thus, in the limit when the wall approaches the swimmer, we find that the swimming speed is

$$U = 3\frac{(1 + De_1 De_2)}{(1 + De_1^2)}.$$ (A.85)

Appendix B

Details of Bingham Analysis

B.1 Case I

Beginning with the first case, $Y_- < Y_+ < Y < 1$, let us look at equation (2.108). Considering the region where $y > Y_+$, we can simplify this equation using equation (2.114). Integrating,

$$\int_Y^y u_y dy = \int_Y^y (y - Y_+) p_x dy, \tag{B.1}$$

gives

$$u = \frac{p_x}{2} \left((y - Y_+)^2 - (Y - Y_+)^2 \right). \tag{B.2}$$

In this region, $y > Y_+$, so we can see that this flow profile is given by a concave-up parabola in y. Applying the boundary condition at $y = 1$ and rearranging to solve for Y_+ gives,

$$Y_+ = \frac{1}{2}(1 + Y) - \frac{U}{p_x(1 - Y)}. \tag{B.3}$$

Technically, the plug region is outside the boundaries we are considering, but, for the purpose of solving this set of equations, let us imagine that the flow profile continues through $y = Y$. Since the flow profile is a concave-up parabola, we can see that $u_p < 0$. As well, applying the condition on $y = Y_+$ gives us that

$$u_p = -\frac{p_x}{2}(Y - Y_+)^2. \tag{B.4}$$

In turn, this tells us

$$\text{sgn}(u_p) = -\text{sgn}(p_x). \tag{B.5}$$

Thus, we find that

$$\text{sgn}(p_x) > 0, \tag{B.6}$$

and so $\sigma = +1$. We can now look at equation (2.119). Substituting in u and simplifying, gives

$$q = Y + \frac{U}{2}(1 - Y) - \frac{p_x}{12}(1 - Y)^3. \tag{B.7}$$

B.2 Case II

For case II, we have $Y_- < Y < Y_+ < 1$. Let us begin with the region where $Y_- < y < Y_+$. Since we have that, $u|_{(y=Y)} = 0$, and $Y_- < Y < Y_+$, we know automatically that

$$u_p = 0. \tag{B.8}$$

Now, consider the region where $Y_+ < y < 1$. Taking equation (2.108), simplifying using equation (2.114), and integrating,

$$\int_{Y+}^{y} u_y dy = \int_{Y+}^{y} (y - Y_+) p_x dy, \tag{B.9}$$

gives us

$$u = \frac{p_x}{2}(y - Y_+)^2. \tag{B.10}$$

Applying the boundary condition at $y = 1$ and solving for Y_+ gives,

$$Y_+ = 1 - \sqrt{\frac{2U}{p_x}}. \tag{B.11}$$

Since we are assuming that $U > 0$, the above equation implies

$$p_x > 0. \tag{B.12}$$

This implies that $\sigma = +1$. Moving on to equation (2.119),

$$q = Y + \int_{Y+}^{1} u \, dy. \tag{B.13}$$

Substituting in our expression for u and simplifying, gives

$$q = Y + \frac{p_x}{6}(1 - Y_+)^3. \tag{B.14}$$

B.3 Case III

Case III is the case when the pseudo-plug is contained within the boundaries of the swimmer and wall, $Y < Y_- < Y_+ < 1$. In the pseudo-plug region, $Y_- < y < Y_+$, we have that $u = u_p$, where u_p is currently unknown. In the upper region, $Y_+ < y < 1$, we can again simplify equation (2.108) using equation (2.114) and integrate,

$$\int_{Y+}^y u_y dy = \int_{Y+}^y (y - Y_+) p_x dy. \tag{B.15}$$

This gives us that

$$u = u_p + \frac{p_x}{2} (y - Y_+)^2. \tag{B.16}$$

Applying the boundary condition on $y = 1$ and solving for u_p yields,

$$u_p = U - \frac{p_x}{2} (1 - Y_+)^2. \tag{B.17}$$

Now, consider the lower yielded region, $Y < y < Y_-$. In this region, we simplify equation (2.108) using equation (2.115). Integrating,

$$\int_y^{Y-} u_y dy = \int_y^{Y-} (y - Y_-) p_x dy, \tag{B.18}$$

gives

$$u = u_p + \frac{p_x}{2} (y - Y_-)^2. \tag{B.19}$$

Applying the boundary condition at $y = Y$ and solving for u_p gives,

$$u_p = -\frac{p_x}{2} (Y - Y_-)^2. \tag{B.20}$$

Equating these two equations for u_p and solving for Y_+ gives,

$$Y_+ = \frac{\left(1 - Y^2 - \frac{2U}{p_x} - 4\frac{B}{|p_x|}\left(Y + \frac{B}{|p_x|}\right)\right)}{2(1 - Y - 2\frac{B}{|p_x|})} \tag{B.21}$$

recalling that, $Y_- = Y_+ - 2\frac{B}{|p_x|}$, from equation (2.117). We can now look at equation (2.119),

$$q = Y + \int_{Y+}^1 u dy + u_p(Y_+ - Y_-) + \int_Y^{Y-} u dy. \tag{B.22}$$

Substituting in u and simplifying, gives

$$q = Y + \frac{p_x}{6}\left((1 - Y_+)^3 + (Y_- - Y)^3 - 3(1 - Y)(Y_- - Y)^2\right). \tag{B.23}$$

B.4 Case IV

For case IV, we have $Y < Y_- < 1 < Y_+$. Let us begin with the region where $Y_- < y < Y_+$. Since we have that, $u|_{(y=1)} = U$, and $Y_- < 1 < Y_+$, we know automatically that

$$u_p = U. \tag{B.24}$$

Now, consider the region where $Y < y < Y_-$. Taking equation (2.108), simplifying using equation (2.115), and integrating,

$$\int_y^{Y-} u_y dy = \int_y^{Y-} (y - Y_-)p_x dy, \tag{B.25}$$

gives us

$$u = U + \frac{p_x}{2}(y - Y_-)^2. \tag{B.26}$$

Applying the boundary condition at $y = Y$ and solving for Y_- gives,

$$Y_- = Y + \sqrt{\frac{-2U}{p_x}}. \tag{B.27}$$

Since we are assuming that $U > 0$, we know that, in this case,

$$p_x < 0. \tag{B.28}$$

This implies that $\sigma = -1$. Looking at equation (2.119),

$$q = Y + U(1 - Y_-) + \int_Y^{Y-} u dy, \tag{B.29}$$

substituting in u and simplifying, gives

$$q = Y + \frac{p_x}{6}(Y_- - Y)^3 + U(1 - Y). \tag{B.30}$$

B.5 Case V

Finally, the last case, $Y < 1 < Y_- < Y_+$. Look again at equation (2.108) and consider the region where $y < Y_-$. Simplifying using equation (2.115) and integrating,

$$\int_Y^y u_y dy = \int_Y^y (y - Y_-) p_x dy, \tag{B.31}$$

gives

$$u = \frac{p_x}{2} \left((y - Y_-)^2 - (Y - Y_-)^2 \right). \tag{B.32}$$

In this region, $y < Y_-$, so we can see that this flow profile is given by a concave-down parabola in y. Applying the boundary condition at $y = 1$ and rearranging to solve for Y_- gives,

$$Y_- = \frac{1}{2}(1 + Y) - \frac{U}{p_x(1 - Y)}. \tag{B.33}$$

As in case I, the plug region is technically outside the boundaries we are considering. But, for the purpose of solving this set of equations, let us again imagine that the flow profile continues through $y = 1$. Since the flow profile is a concave-down parabola, we can see that $u_p > 0$. As well, may applying the condition on $y = Y_-$ gives us that

$$u_p = -\frac{p_x}{2}(Y - Y_-)^2. \tag{B.34}$$

Again, this tells us,

$$\text{sgn}(u_p) = -\text{sgn}(p_x), \tag{B.35}$$

and so we find that

$$\text{sgn}(p_x) < 0, \tag{B.36}$$

which gives us that $\sigma = -1$. We can now look at equation (2.119). Substituting in u and simplifying, gives

$$q = Y + \frac{U}{2}(1 - Y) - \frac{p_x}{12}(1 - Y)^3. \tag{B.37}$$

Note that for all the above cases, equation (2.117) allows us to put Y_- in terms Y_+, or vice versa,

$$Y_- = Y_+ - \frac{2B}{|p_x|}. \tag{B.38}$$

Appendix C

Code For Chapter Two

Below is the MATLAB code used to obtain the results in Chapter Two.

C.1   Viscoelastic Problem Code

```
% NEWTON'S METHOD
clear

N = 50; h = 2*pi/N;
xi = 0:h:2*pi;        % uniform mesh

DimWork=1;  % Fixed Dimensional Work
mu=1;        % viscosity
hh=0.1;      % mean depth

for k=1:199

    ah = k*0.005;  % a/h < 1;
    disp(['a/h = ',num2str(ah)])
    Y = ah*sin(xi);     % REGULAR
    %Y = ah*tanh(12*sin(xi));   % SQUARE WAVE

    a0=0; b0=0; % Newtonian - could have a=b=anything....
    a1=1; b1=2; % Thinning
    a2=2; b2=1; % Thickening

    [U1, q1, p_x1, g11, gY1, tau_01, G]=SolveSystem(N,xi,ah,Y,a1,b1);
    [U2, q2, p_x2, g12, gY2, tau_02, G]=SolveSystem(N,xi,ah,Y,a2,b2);

    U0=G(1);
```

```
q0=G(2);
g10=G(N+4:2*N+4);
gY0=G(2*N+5:3*N+5);

u0(k)=U0;    u1(k)=U1;    u2(k)=U2;
MF0(k)=q0;   MF1(k)=q1;   MF2(k)=q2;
AH(k)=ah;
% must calculate dimensionless work....
% wi = <<2ui_z\taui_{xz}>>
% Newtonian ;    T=g ;
p_x0 = (g10-gY0)./(1-Y);
W0=@(g) 2*g.^2;
% Tprime0 = 1;
I0 = (2/3)*(g10.^3-gY0.^3)./(p_x0);
II0 = I0(1:N);
w0(k)=sum(II0.*diff(xi));;


% Thinning
W1Tprime1=@(g) 2.*g.^2.*(1+a1*g.^2)./(1+b1*g.^2).*(1-b1*g.^2+
3*a1*g.^2+a1*b1*g.^4)./(1+b1*g.^2).^2;
for j=1:N+1
    Int1(j)=quadl(W1Tprime1,gY1(j),g11(j));
end
I1=Int1./(p_x1);
II1=I1(1:N);
w1(k)=sum(II1.*diff(xi));


% Thickening
W2Tprime2=@(g) 2.*g.^2.*(1+a2*g.^2)./(1+b2*g.^2).*(1-b2*g.^2+
3*a2*g.^2+a2*b2*g.^4)./(1+b2*g.^2).^2;
for j=1:N+1
    Int2(j)=quadl(W2Tprime2,gY2(j),g12(j));
end
I2=Int2./(p_x2);
II2=I2(1:N);
w2(k)=sum(II2.*diff(xi));
```

```
    % must calculate wave speed ...

    c0(k)=DimWork*hh^2/(2*mu*w0(k));      % Newtonian
    c1(k)=DimWork*hh^2/(2*mu*w1(k));      % Thinning
    c2(k)=DimWork*hh^2/(2*mu*w2(k));      % Thickening

    uu0(k)=c0(k)*U0;
    uu1(k)=c1(k)*U1;
    uu2(k)=c2(k)*U2;

end

figure plot(AH,uu0,'r',AH,uu1,'b',AH,uu2,'g') title(['Dimensional
Swimming Speed'],'FontSize',12) xlabel('a/h','FontSize',12)
ylabel('U*c','FontSize',12)
```

```
function [U, q, p_x, gam1, gamY, tau_0, G] =
SolveSystem(N,xi,ah,Y,a,b)

E = 10^(-4); maxit = 60;

% Initial guess ...
G = InitialApproxNewt(N,xi,ah,Y);

T = @(x) x.*(1+a*x.^2)./(1+b*x.^2); Tprime = @(x)
(1-b*x.^2+3*a*x.^2+a*b*x.^4)./(1+b*x.^2).^2;

I1prime = @(g)
g.^2.*(1+a*g.^2).*(1-b*g.^2+3*a*g.^2+a*b*g.^4)./(1+b*g.^2).^3;
% Observing that -dI1/Dg1 = dI1/dgY

I0prime = @(g) g.*(1-b*g.^2+3*a*g.^2+a*b*g.^4)./(1+b*g.^2).^2;
% Observing that -dI0/Dg1 = dI0/dgY

J = zeros(2*N+4);    % most entries are zero

U=G(1); q=G(2); gam1=G(N+4:2*N+4); gamY=G(2*N+5:3*N+5);

Guess = [U q gam1 gamY];

for i=1:maxit
    % Function evaluated at current guess
    PP = (feval(T,gamY)-((feval(T,gam1)-feval(T,gamY))./(1-Y)).*Y);
    P = PP(1:N);
    F(1) = sum(P.*diff(xi));
    QQ = ((feval(T,gam1)-feval(T,gamY))./(1-Y));
    Q = QQ(1:N);
    F(2) = sum(Q.*diff(xi));
    for j=1:N+1
        I1 = quadl(I1prime,gamY(j),gam1(j));
        I0 = quadl(I0prime,gamY(j),gam1(j));
        % F3's
        F(j+2) = U*(feval(T,gam1(j))
```

```
        - feval(T,gamY(j)))./(1-Y(j))*feval(T,gamY(j))
        - ((feval(T,gam1(j))-feval(T,gamY(j)))./(1-Y(j)))^2*(q-U-Y(j)+U*Y(j))
        - I1;
        % F4's
        F(j+N+3) =I0 - U*(feval(T,gam1(j))-feval(T,gamY(j)))./(1-Y(j));

        % JACOBIAN  made gam1 = x1, gamY = x2, q = x3, U = x4
        % dF3(1..N+1)/dxi
        J(j+2,1) = (feval(T,gam1(j))
        - feval(T,gamY(j)))./(1-Y(j))*feval(T,gamY(j))
        - ((feval(T,gam1(j))-feval(T,gamY(j)))./(1-Y(j)))^2*(Y(j)-1);
        J(j+2,2) = -((feval(T,gam1(j))-feval(T,gamY(j)))./(1-Y(j)))^2;
        J(j+2,j+2) = U*feval(T,gamY(j))*feval(Tprime,gam1(j))/(1-Y(j))
        - feval(I1prime,gam1(j)) - (q-U*Y(j)+U*Y(j))*2*(feval(T,gam1(j))
        - feval(T,gamY(j)))*feval(Tprime,gam1(j))/(1-Y(j))^2;
        J(j+2,j+N+3) = U*(feval(T,gam1(j))*feval(Tprime,gamY(j))
        - 2*feval(T,gamY(j))*feval(Tprime,gamY(j)))/(1-Y(j))
        + feval(I1prime,gamY(j)) + (q-U*Y(j)+U*Y(j))*2*(feval(T,gam1(j))
        -feval(T,gamY(j)))*feval(Tprime,gamY(j))/(1-Y(j))^2;
        % dF4(1..N+1)/dxi
        J(j+N+3,1) = -(feval(T,gam1(j))-feval(T,gamY(j)))./(1-Y(j));
        J(j+N+3,j+2) = feval(I0prime,gam1(j))
        -U*feval(Tprime,gam1(j))/(1-Y(j));
        J(j+N+3,j+N+3) = -feval(I0prime,gamY(j))
        +U*feval(Tprime,gamY(j))/(1-Y(j));
    end
    for j=1:N
        % Constraints
        % dF1/Dxi
        J(1,j+2) = -Y(j)/(1-Y(j))*feval(Tprime,gam1(j))*(xi(j+1)-xi(j));
        J(1,j+N+3) = feval(Tprime,gamY(j))+Y(j)/(1-Y(j))
        *feval(Tprime,gamY(j))*(xi(j+1)-xi(j));
        % dF2/Dxi
        J(2,j+2) = 1/(1-Y(j))*feval(Tprime,gam1(j))*(xi(j+1)-xi(j));
        J(2,j+N+3) = -1/(1-Y(j))*feval(Tprime,gamY(j))*(xi(j+1)-xi(j));
    end
    y = -J\F';
```

```
    Guess = Guess + y';
    U=Guess(1);
    q=Guess(2);
    gam1=Guess(3:N+3);
    gamY=Guess(N+4:2*N+4);
    if norm(y,inf) < E
        disp(['Method converging in ',num2str(i),' iterations'])
        break
    end
end if i==maxit
    disp(['NEWTONS METHOD NOT CONVERGING IN: ',num2str(maxit),
    ' ITERATIONS - ERROR'])
end tau_0 = feval(T,gamY); p_x =
(feval(T,gam1)-feval(T,gamY))./(1-Y);
```

C.2   Bingham Problem Code

```
%ssdriv
a = 0.5; B = 0.0;

N=400; x=([0:N-1]+1/2)/N*pi-pi/2; px=a*sin(x); X=zeros(N+2,1);
X(1:N)=px;

for niter=1:30 ssrhs dX=-EJ\E; conno=max(abs(dX)) X=X+dX; if
conno<1e-12 break end end

subplot(211) plot(x,px)

UU=X(N+1); qq=X(N+2); [UU qq]

a0=a; aa=a0;

%for na = 1:1
%a = a0+na/20;

B=1;

for niter=1:30 ssrhs dX=-EJ\E; conno=max(abs(dX)) if niter==1,
  dX=dX/2;
end X=X+dX; if conno<1e-8 break end end

%UU=[UU X(2*N+1)];
%qq=[qq X(2*N+2)];
%aa=[aa a];

hold on, plot(x,px,'r'), hold off axis tight xlabel('x',
'Fontsize', 22) ylabel('p_x', 'Fontsize', 22)

%end
%  figure(2), plot(aa,UU)
%  figure(1)
```

```
subplot(212)

xx = [x' x(end:-1:1)']; YY = [Yp' Ym(end:-1:1)']; fill(xx,YY,'g')
hold on,
plot(x,Yp,'b',x,Ym,'r',x,Y,'k',x,x./x,'LineWidth',2),axis([-pi/2
pi/2 min(Y) 1]) xlabel('x', 'Fontsize', 22) ylabel('Y_+ and Y_-',
'Fontsize', 22) hold off

legend('Pseudo-plug','Y_+','Y_-','Y')
```

```
% ssrhs

x=([0:N-1]+1/2)'/N*pi-pi/2; Y=a*sin(x);

px = X(1:N); U=X(N+1); q=X(N+2);

BB=B./abs(px); %BB=B./sqrt(px.^2+1e-6);

Yp1 = (1+Y)/2-U./px./(1-Y); Ym1 = Yp1-2*BB; Is1 =
(sign(Y-Yp1)+1)/2.*(1+sign(px))/2; Q1  =
U/2*(1-Y)-px/12.*(1-Y).^3+Y;

Yp2 = 1-sqrt(2*U./abs(px)); Ym2 = Yp2-2*BB; Is2 =
(sign(Y-Ym2)+1)/2.*(sign(Yp2-Y)+1)/2.*(1+sign(px))/2; Q2  =
px/6.*(1-Yp2).^3+Y;

Ym5 = (1+Y)/2-U./px./(1-Y); Yp5 = Ym5+2*BB; Is5 =
(sign(Ym5-1)+1)/2.*(1-sign(px))/2; Q5  =
U/2*(1-Y)-px/12.*(1-Y).^3+Y;

Ym4 = Y+sqrt(2*U./abs(px)); Yp4 = Ym4+2*BB; Is4 =
(sign(Yp4-1)+1)/2.*(sign(1-Ym4)+1)/2.*(1-sign(px))/2; Q4  =
U*(1-Y)+px/6.*(Ym4-Y).^3+Y;

Yp3 = (1-Y.^2-2*U./px-4*(Y+BB).*BB)/2./(1-Y-2*BB); Ym3 = Yp3-2*BB;
Is3 = (sign(1-Yp3)+1)/2.*(sign(Ym3-Y)+1)/2; Q3  =
px/6.*((1-Yp3).^3+(Ym3-Y).^3-3*(1-Y).*(Ym3-Y).^2)+Y;

Yp  = Yp1.*Is1+Yp2.*Is2+Yp3.*Is3+Yp4.*Is4+Yp5.*Is5; Ym  =
Ym1.*Is1+Ym2.*Is2+Ym3.*Is3+Ym4.*Is4+Ym5.*Is5; QQ  =
Q1.*Is1+Q2.*Is2+Q3.*Is3+Q4.*Is4+Q5.*Is5;

tau1 = (2-Yp-Ym).*px/2;

E=zeros(size(X)); E = QQ-q; E(N+1)=sum(tau1); E(N+2)=sum(px);

EJ=zeros(N+2,N+2);
```

```
for n=1:N+2 XX = X; XX(n) = X(n)+1e-4; px = XX(1:N); U=XX(N+1);
q=XX(N+2);

BB=B./abs(px); %BB=B./sqrt(px.^2+1e-6);

Yp1 = (1+Y)/2-U./px./(1-Y); Ym1 = Yp1-2*BB; Is1 =
(sign(Y-Yp1)+1)/2.*(1+sign(px))/2;; Q1  =
U/2*(1-Y)-px/12.*(1-Y).^3+Y;

Yp2 = 1-sqrt(2*U./abs(px)); Ym2 = Yp2-2*BB; Is2 =
(sign(Y-Ym2)+1)/2.*(sign(Yp2-Y)+1)/2.*(1+sign(px))/2; Q2  =
px/6.*(1-Yp2).^3+Y;

Ym5 = (1+Y)/2-U./px./(1-Y); Yp5 = Ym5+2*BB; Is5 =
(sign(Ym5-1)+1)/2.*(1-sign(px))/2; Q5  =
U/2*(1-Y)-px/12.*(1-Y).^3+Y;

Ym4 = Y+sqrt(2*U./abs(px)); Yp4 = Ym4+2*BB; Is4 =
(sign(Yp4-1)+1)/2.*(sign(1-Ym4)+1)/2.*(1-sign(px))/2; Q4  =
U*(1-Y)+px/6.*(Ym4-Y).^3+Y;

Yp3 = (1-Y.^2-2*U./px-4*(Y+BB).*BB)/2./(1-Y-2*BB); Ym3 = Yp3-2*BB;
Is3 = (sign(1-Yp3)+1)/2.*(sign(Ym3-Y)+1)/2; Q3  =
px/6.*((1-Yp3).^3+(Ym3-Y).^3-3*(1-Y).*(Ym3-Y).^2)+Y;

Yp  = Yp1.*Is1+Yp2.*Is2+Yp3.*Is3+Yp4.*Is4+Yp5.*Is5; Ym  =
Ym1.*Is1+Ym2.*Is2+Ym3.*Is3+Ym4.*Is4+Ym5.*Is5; QQ  =
Q1.*Is1+Q2.*Is2+Q3.*Is3+Q4.*Is4+Q5.*Is5;

tau1 = (2-Yp-Ym).*px/2;

EE=zeros(size(X)); EE = QQ-q; EE(N+1)=sum(tau1); EE(N+2)=sum(px);

EJ(:,n)=(EE-E)*1e4;

end
```

Appendix D

Code For Chapter Three

Below is the MATLAB code used to obtain the results in Chapter Three.

D.1 Newtonian Problem Code

First, we show the code used to solve the Newtonian problem.

```
function sbvp global aa DD sol aa=0.001; DD = 1.5; U=aa^2/48 q=U;
lambda(1)=q; lambda(2)=U; x = [0:100]/100*pi*2;

options=bvpset('Vectorized','on','RelTol',1e-4,'AbsTol',1e-7);

soli = bvpinit(x,@solinit,lambda); sol =
bvp4c(@solode,@solbc,soli,options);

figure(1) plot(sol.x,1-sol.y(1,:)) sol.parameters
qqq=sol.parameters(1); uuu=sol.parameters(2); aaa=aa; for na=1:25
aa=na*4 sol = bvp4cc(@solode,@solbc,sol,options); hold
on,plot(sol.x,1-sol.y(1,:)), hold off qqq=[qqq sol.parameters(1)];
uuu=[uuu sol.parameters(2)]; aaa=[aaa aa]; end plateau =
12*(1-qqq)./(12-6*uuu); figure(1) hold
on,plot(sol.x,1-ones(1,length(sol.x))*plateau(na),'m','LineWidth',2),
hold off figure (1) title(['Swimmer Shape for D=',num2str(DD),'
and a=', num2str(aa)],'FontSize',22)
xlabel('\xi=x-t','FontSize',22) ylabel('Y','FontSize',22) axis
tight axes('position',[0.2 0.2 0.28 0.28])
plot(aaa,uuu,'r',aaa,qqq,'g') xlabel('a')
axis tight
figure(2)
plot3(1-sol.y(1,:),-sol.y(2,:),-sol.y(3,:))
```

```
  view([-29,-64])
  xlabel('Y','FontSize',22)
  ylabel('Y ''','FontSize',22)
  zlabel('Y''','FontSize',22)
  title(['Phase Space for D = ',num2str(DD)],'FontSize',22)


px = 6*(sol.parameters(2)-2)./(sol.y(1,:)).^2 -
12*(sol.parameters(1)-1)./(sol.y(1,:)).^3;


t1 = (4*sol.parameters(2)-6)./(sol.y(1,:)) -
6*(sol.parameters(1)-1)./(sol.y(1,:)).^2;


t0 = t1 - (sol.y(1,:)).*px;


figure(3) plot(sol.x,t1,'k',sol.x,t0,'c')
xlabel('\xi=x-t','FontSize',12) ylabel('\tau_1: Black, \tau_0:
Cyan','FontSize',12) title('Stresses\Strain Rates', 'Fontsize',
12) axis tight


function yinit = solinit(x) global aa
 yinit = [  1+aa/12*cos(x)
           -aa/12*sin(x)
           -aa/12*cos(x)
            aa/12*sin(x)
            aa/12*cos(x)
       zeros(size(x))
       zeros(size(x))];


function res = solbc(ya,yb,lambda)
  res = [  ya(1)-yb(1)
       ya(2)-yb(2)
       ya(3)-yb(3)
       ya(4)-yb(4)
       ya(5)-yb(5)
           ya(6)
           ya(7)
           yb(6)-pi*2
```

```
          yb(7) ];


function dydx = solode(x,y,lambda)
global aa DD
  q=lambda(1);
  U=lambda(2);
dydx = [                  y(2,:)
              y(3,:)
              y(4,:)
              y(5,:)
     -(aa*cos(x)+(12*(1-q)-y(1,:)*(12-6*U))./y(1,:).^3)/DD
               y(1,:)
        6*(1-q)./y(1,:).^2+(4*U-6)./y(1,:)];
```

D.2 Viscoelastic Problem Code

Next, we list the code for the Viscoelastic problem with moderate amplitude.

```
% siniC
D =0.2;      %7;%27;
% Newtonian
alf = 1; bet = 1; I1prime = @(g)
g.^2.*(1+alf*g.^2).*(1-bet*g.^2+3*alf*g.^2+...
                alf*bet*g.^4)./(1+bet*g.^2).^3;
I0prime = @(g)
g.*(1-bet*g.^2+3*alf*g.^2+alf*bet*g.^4)./(1+bet*g.^2).^2;


figure(1), subplot(221) sdrivC figure(1), subplot(221)
  figure(2), hold on, figure(1), hold on, axis tight
  title(['\alpha=\beta=1 and D=',num2str(D)],'FontSize',12)
  xlabel('x','FontSize',12)
  ylabel('Y','FontSize',12)


% Shear Thinning
alf = 1; bet = 2; I1prime = @(g)
g.^2.*(1+alf*g.^2).*(1-bet*g.^2+3*alf*g.^2+...
                alf*bet*g.^4)./(1+bet*g.^2).^3;
I0prime = @(g)
g.*(1-bet*g.^2+3*alf*g.^2+alf*bet*g.^4)./(1+bet*g.^2).^2;


figure(1), subplot(222) sdrivC figure(1), subplot(222)
  figure(2), hold on, figure(1), hold on, axis tight
  title(['\alpha=1, \beta=2, D=',num2str(D)],'FontSize',12)
  xlabel('x','FontSize',12)
  ylabel('Y','FontSize',12)


% Shear Thickening
alf = 2; bet = 1; I1prime = @(g)
g.^2.*(1+alf*g.^2).*(1-bet*g.^2+3*alf*g.^2+...
                alf*bet*g.^4)./(1+bet*g.^2).^3;
I0prime = @(g)
g.*(1-bet*g.^2+3*alf*g.^2+alf*bet*g.^4)./(1+bet*g.^2).^2;
```

```
figure(1), subplot(223) sdrivC figure(1), subplot(223)
  figure(1), hold on, axis tight
  title(['\alpha=2, \beta=1, D=',num2str(D)],'FontSize',12)
  xlabel('x','FontSize',12)
  ylabel('Y','FontSize',12)


  figure(1), subplot(224)
  %hold on,plot([0 DD],[0 DD].^2/48,'r:'),hold off
  xlabel('a','FontSize',12)
  ylabel('U','FontSize',12)
  title (['U vs a for D=',num2str(D)],'FontSize',12)
axis tight
```

```
%sdrivC
a=0.01 N=55; x=([0:N-1]+1/2)/N*pi*2;


sdrivCC for niter=1:10
    srhsC
    dX=-EJ\E;
    conno=max(abs(dX));
    X=X+dX;
    if conno<1e-12
        disp('Converging 1')
        break
    end
end disp(['conno = ', num2str(conno)])


if (alf == 1) & (bet == 1)
    plot(x,Y,'r')
end
if (alf == 1) & (bet == 2)  %shear thinning
    plot(x,Y,'b')
end
if (alf == 2) & (bet == 1)  %shear thickening
    plot(x,Y,'g')
end
%%For comparison between Numerical & Analytical Solutions:
%plot(x,Y)
% FOR MEDIUM D SMALL A
%hold on,
%plot(x,D*a*sin(x)/(D^2+12^2)-a*12*cos(x)/(D^2+12^2),'k*'),
%hold off
% FOR LARGE D SMALL A
%hold on,plot(x,D*a*sin(x)/(D^2+12^2),'k*'),hold off
% FOR SMALL D SMALL A
%hold on,plot(x,-a*12*cos(x)/(D^2+12^2),'k*'),hold off


UU=X(3*N+1); qq=X(3*N+2); [UU qq] a0=a; aa=a0; D0=D; DD=D0; for na
= 1:12
    a = a0 +na/10
```

```
    for niter=1:15
        srhsC
        dX=-EJ\E;
        conno=max(abs(dX));
        X=X+dX;
        if conno<1e-8
            disp('converging 2')
            break
        end
    end
    disp(['conno = ', num2str(conno)])


    UU=[UU X(3*N+1)];
    qq=[qq X(3*N+2)];
    aa = [aa a];
    hold on
    if (alf == 1) & (bet == 1)
        plot(x,Y,'r'), axis tight
    end
    if (alf == 1) & (bet == 2)  %shear thinning
        plot(x,Y,'b'), axis tight
    end
    if (alf == 2) & (bet == 1)  %shear thickening
        plot(x,Y,'g'), axis tight
    end
    hold off

end figure(1), subplot(224) if (alf == 1) & (bet == 1)
    figure(2),plot(aa,qq,'r')
    figure(1), subplot(224),hold on, plot(aa,UU,'r'), hold off
end
if (alf == 1) & (bet == 2)  %shear thinning
    figure(2),plot(aa,qq,'b')
    figure(1), subplot(224),hold on, plot(aa,UU,'b'), hold off
end
if (alf == 2) & (bet == 1)  %shear thickening
    figure(2),plot(aa,qq,'g')
```

```
    figure(1), subplot(224), hold on, plot(aa,UU,'g') , hold off
end
figure(1)
```

```
% srhsC

gamY = X(1:N); gam1 = X(N+1:2*N); Y = X(2*N+1:3*N); U=X(3*N+1);
q=X(3*N+2);


x=([0:N-1]+1/2)'/N*pi*2;


D3f = -a*cos(x);


TY = gamY.*(1+alf*gamY.^2)./(1+bet*gamY.^2); T1 =
gam1.*(1+alf*gam1.^2)./(1+bet*gam1.^2); TpY =
(1+3*alf*gamY.^2)./(1+bet*gamY.^2)-...
        2*bet*gamY.^2.*(1+alf*gamY.^2)./(1+bet*gamY.^2).^2;
Tp1 = (1+3*alf*gam1.^2)./(1+bet*gam1.^2)-...
        2*bet*gam1.^2.*(1+alf*gam1.^2)./(1+bet*gam1.^2).^2;


I0 = gam1.^2/2-gamY.^2/2; I1 = gam1.^3/3-gamY.^3/3;


px = (T1-TY)./(1-Y); pg1 = Tp1./(1-Y); pgY = -TpY./(1-Y);
pY =
(T1-TY)./(1-Y).^2;


DiffY


E=zeros(size(X)); EJ=zeros(3*N+2,3*N+2);
% F4
E(3*N+1)=sum(T1);
% F5
E(3*N+2)=sum(Y);
for n=1:N
I0(n)=quadl(I0prime,gamY(n),gam1(n));
%F1
E(n)=I0(n)-U*px(n); I1(n)=quadl(I1prime,gamY(n),gam1(n));
%F2
E(N+n)=I1(n)-px(n)^2.*(U+Y(n)-q-U*Y(n))-U*TY(n)*px(n);
%F3
E(2*N+n) = px(n)-D3f(n)-D*DY5(n);
```

```
% DF1
EJ(n,n)=-gamY(n)*TpY(n)-U*pgY(n); EJ(n,n+N)=
gam1(n)*Tp1(n)-U*pg1(n);
EJ(n,2*N+n)=-U*pY(n);
EJ(n,3*N+1)=-px(n);
% DF2
EJ(N+n,n) =
-TY(n)*TpY(n)*gamY(n)-2*px(n)*pgY(n)*(U+Y(n)-q-U*Y(n))-
U*TpY(n)*px(n)-U*TY(n)*pgY(n);
EJ(N+n,N+n) =
T1(n)*Tp1(n)*gam1(n)-2*px(n)*pg1(n)*(U+Y(n)-q-U*Y(n))-
U*TY(n)*pg1(n);
EJ(N+n,2*N+n) =
-px(n)^2*(1-U)-2*px(n)*pY(n)*(U+Y(n)-q-U*Y(n))- U*TY(n)*pY(n);
EJ(N+n,3*N+1) = -px(n)^2.*(1-Y(n))-px(n)*TY(n);
EJ(N+n,3*N+2) =
px(n)^2;
% DF3
EJ(2*N+n,n) = pgY(n); EJ(2*N+n,N+n) = pg1(n);
EJ(2*N+n,2*N+n) = pY(n);    % Must add DY5 derivatives, see below
% DF4
EJ(3*N+1,N+n) = Tp1(n);
% DF5
EJ(3*N+2,2*N+n) = 1;
end

% Must do dF3/dY separately due to discretization
% A is differentiation matrix for DY5
EJ(2*N+1:3*N,2*N+1:3*N) =EJ(2*N+1:3*N,2*N+1:3*N)-D*A;
```

This code calculated the fifth derivative of a function given by a specific set of points numerically. It assumes periodicity of the function. It is assumed that the mesh used does not include its endpoints.

```
%DiffY

h = x(2)-x(1); %uniform mesh

A =
5*diag(ones(N-1,1),1)-4*diag(ones(N-2,1),2)+diag(ones(N-3,1),3);

A = A-A';

A(1,N-2) = -1; A(1,N-1) = 4; A(1,N) = -5; A(2,N-1) = -1; A(2,N) =
4; A(3,N) = -1;

A(N,1) = 5; A(N,2) = -4; A(N,3) = 1; A(N-1,1) = -4; A(N-1,2) = 1;
A(N-2,1) =1;

A = (1/(2*h^5))*A; DY5 = A*Y;
```

This is the code that formulates the initial guess for the Newton's Iteration used to solve the viscoelastic problem.

```
% srdrivCC
x=([0:N-1]+1/2)/N*pi*2; px=-a*cos(x); gamY=-px/2; gam1=px/2;
X=zeros(2*N+2,1); X(1:N)=gamY; X(N+1:2*N)=gam1;


for niter=1:10 srhsCC dX=-EJ\E; conno=max(abs(dX)); X=X+dX; if
conno<1e-12 break end end


for niter=1:10 srhsCC dX=-EJ\E; conno=max(abs(dX)); X=X+dX; if
conno<1e-8 break end end
% Updating X into a 4*N+2 length vector.
X(1:N)=gamY; X(N+1:2*N)=gam1; X(2*N+1:3*N)=Y; X(3*N+1)=U;
X(3*N+2)=q;
```

```
% srhsCC

gamY = X(1:N); gam1 = X(N+1:2*N); U=X(2*N+1); q=X(2*N+2);

x=([0:N-1]+1/2)'/N*pi*2; px=-a*cos(x);

TY = gamY.*(1+alf*gamY.^2)./(1+bet*gamY.^2); T1 =
gam1.*(1+alf*gam1.^2)./(1+bet*gam1.^2); TpY =
(1+3*alf*gamY.^2)./(1+bet*gamY.^2)-...
        2*bet*gamY.^2.*(1+alf*gamY.^2)./(1+bet*gamY.^2).^2;
Tp1 = (1+3*alf*gam1.^2)./(1+bet*gam1.^2)-...
        2*bet*gam1.^2.*(1+alf*gam1.^2)./(1+bet*gam1.^2).^2;
I0 = gam1.^2/2-gamY.^2/2; I1 = gam1.^3/3-gamY.^3/3;

Y = 1-(T1-TY)./px; Yp1 = -Tp1./px; YpY =  TpY./px;

E=zeros(size(X)); EJ=zeros(2*N+2,2*N+2);

E(2*N+1)=sum(T1); E(2*N+2)=sum(Y);

for n=1:N
I0(n)=quadl(I0prime,gamY(n),gam1(n));
E(n)=I0(n)-U*px(n); I1(n)=quadl(I1prime,gamY(n),gam1(n));
E(N+n)=I1(n)-px(n)^2.*(U+Y(n)-q-U*Y(n))-U*TY(n)*px(n);
EJ(n,n)=-gamY(n)*TpY(n); EJ(n,n+N)= gam1(n)*Tp1(n);
EJ(n,2*N+1)=-px(n); EJ(N+n,n)    =
-TY(n)*TpY(n)*gamY(n)-px(n)^2*(1-U)*YpY(n)-U*TpY(n)*px(n);
EJ(N+n,N+n)  =  T1(n)*Tp1(n)*gam1(n)-px(n)^2*(1-U)*Yp1(n);
EJ(N+n,2*N+1) = -TY(n)*px(n)-px(n)^2.*(1-Y(n)); EJ(N+n,2*N+2) =
px(n)^2; EJ(2*N+1,N+n) = Tp1(n); EJ(2*N+2,n) = YpY(n);
EJ(2*N+2,N+n) = Yp1(n); end
```