REPLICATIVE NETWORK STRUCTURES: THEORETICAL DEFINITIONS AND ANALYTICAL APPLICATIONS

by

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Abstract

Among the techniques associated with the theory of musical transformations, network analysis stands out because of its broad applicability, demonstrated by the diverse examples presented in David Lewin’s seminal work *Musical Form and Transformation* and related articles by Lewin, Klumpenhouwer, Gollin, and others. While transformational theory can encompass a wide variety of analytical structures, objects, and transformations, two particular types of network postulated by Lewin are often featured: the product network and the network-of-networks. These structures both incorporate repetition, but in different ways.

This document will propose one possible definition for product networks and networks-of-networks that is consistent with Lewin’s theories as presented in *Generalized Musical Intervals and Transformations*. This definition will clarify how each of these two network formats may be generated from the same sub-graphs, which in turn will clarify the advantages and disadvantages of each structure for musical analysis, specifically demonstrating how analytical goals shape the choice of network representation.

The analyses of Chapters 3 and 4 examine works by contemporary Canadian composers that have not been the subject of any published analyses. Chapter 3 presents short examples from the works of contemporary Québécois composers, demonstrating the utility of these networks for depicting connections within brief passages that feature short, repeated motives. Chapter 4 presents an analysis of R. Murray Schafer’s *Seventh String Quartet*, demonstrating how these structures can be used to link small-scale events with longer prolongations and motivic development throughout a movement. Chapter 5
demonstrates through a wider repertoire how analytical goals shape the choice of network representation, touching on such factors as continuity, motivic return, and implied collections.
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Pages 2, 7 (last system), 10 (2nd system), 20 (3rd and 4th systems), 21 (1st and 2nd systems), 24 (last system) to 25 (1st system), 33 (1st and 2nd systems) from R. Murray Schafer, *Seventh String Quartet* (Indian River, ON: Arcana Editions, 1999). © R. Murray Schafer, by permission.

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Chapter 1: Introduction

Among the techniques associated with the theory of musical transformations, network analysis stands out because of its broad applicability. Analysts have found ways to use networks to model processes of pitch and rhythm in a wide variety of music. This document will develop one specific aspect of network analysis that I will call “replicative” networks, which are networks that feature a particular type of repetition; the objects and transformations in my examples will primarily include pitch classes and transpositions, respectively, in order to define and demonstrate the most basic structural aspects of these networks. One repertoire that often incorporates repetition consists of the instrumental works of Québécois composers active during the 1960s-1990s, namely Jacques Hétu, Clermont Pépin, Gilles Tremblay, and Serge Garant, and for this reason a selection of these works will form the basis for the analytical applications of my theory. Subsequent chapters will propose how replicative networks might be incorporated in the analyses of works less obviously centered on motive.

In the repertoire of interest, a particularly strong sense of musical unity is created when the repetition and development of short motives replicate the structure of the motives themselves, a relation Joseph Straus (in a basic textbook about this repertoire) names “composing-out.”¹ Take, for example, the music given in Figure 1.1. Consider the two boxed musical gestures that are isolated from the neighbouring music by rests, and are associated by similar articulations, rhythms, and contours on the figure.

Straus explains composing-out as a compositional technique, stating that “to organize the larger musical spans and draw together notes that may be separated in time, composers of post-tonal music sometimes enlarge the motives of the musical surface and project them over significant musical distances.”\(^2\) In **Figure 1.1**, let us consider the structure of the boxed motives to involve pitch classes participating in various pitch-class transpositions that change one collection to another. Such a view of structure is taken by David Lewin’s theory of *musical transformations*. Formally, a transformation is “A function from a family \(S\) into \(S\) itself.”\(^3\) In Lewin’s conceptualization, musical objects do not relate by distances; rather, one is transformed to generate the other. Well-known examples of transformations include transposition and inversion. Accordingly, one transformational analysis of the first boxed motive would be as follows: the transformation \(T_5\) (transposition by 5 semitones, ascending) appears repeatedly from the lower to the upper note of each dyad. It *also* determines the transposition from one dyad to the next and from one pair of dyads to the next. This type of musical organization suggests multiple musical strata or planes, each employing its own set of musical objects (pitch classes, dyads, and dyad pairs, respectively) and transformations. These strata could be considered to be “hierarchical” if we hear the small-scale events to “generate”

\(^2\) Ibid.  
larger-scale events in some sense. But in this analysis there is no suggestion that one stratum is conceptually prior to another; that is, we may also hear large-scale events generating smaller-scale ones. In Figure 1.1, for example, we could say “the pitch classes of each dyad relate by T₅, and this same transformation is then applied at a larger scale to generate one dyad from the other,” but we could also say “the dyads progress by T₅, and this transformation also determines the smaller-scale elaboration within each dyad.” Although this analysis is not hierarchical, strictly speaking, it does point out the replication of a particular transformation, T₅, at different levels of structure. Indeed we could say that the structure is recursive in the sense that T₅ transforms objects (pitch classes) to generate dyads; these dyads in turn are transformed by T₅ to form dyad pairs, and the dyad pairs are themselves also transformed as an ensemble under T₅. In this manner, T₅ operates at three distinct levels. This concept of composing-out need not be limited to pitch or pitch-class groupings, and accordingly this document will explore analytical methods for expressing several types of small- to large-scale replicative relations (as well as the reverse, large- to small-scale relations).

An analytical tool for music featuring such a “composing-out” is appropriate to the extent it can express these replicative structures. A particularly expressive tool is the network, theorized by David Lewin in Generalized Musical Intervals and Transformations (henceforth abbreviated as GMIT). Figure 1.2 gives an example of a network along with two related structures, a node-arrow system and a graph.
The simplest of these structures is the node-arrow system (1.2a). Formally, a node-arrow system is “an ordered pair (NODES, ARROW), where NODES is a family (i.e. set in the mathematical sense), and ARROW is a subfamily of NODES x NODES, i.e. a collection containing some ordered pairs (N_1, N_2) of NODES.” More informally, it is a collection of nodes in which pairs of nodes are connected by arrows. A graph, such as that given in Figure 1.2b, consists of a node-arrow system in which a specific transformation is associated with each arrow. Here the transformation is pitch-class transposition by 2 semitones. A network (1.2c) consists of a graph and a set of objects whose members are assigned to nodes of the graph; the transformation associated with each arrow maps the content of the node at the arrow’s tail to the content of the node at its head. So this network expresses the pitch class E being transposed by 2 semitones to F#. Informally speaking, two networks are “isographic” if they share the same graph.

The current discussion will be limited to informal definitions, deferring the formal definitions of graphs and networks until Chapter 2. Take, for example, the four networks

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4 Ibid., 193 (9.1.1).
5 Ibid., 198.
given in Figure 1.3a, which depict the dyads boxed in Figure 1.1. Each has an identical node-arrow system (two nodes connected by a single arrow), and the arrow of each network is labelled with T_5. Therefore the four networks are isographic since they share the same graph. Isographic networks and graphs are a good way of highlighting replication in a passage of music, particularly for elements at different hierarchical levels that may not be immediately evident to the listener. Understanding similar transformations at multiple musical levels can help the listener hear large-scale connections and continuity; perceiving commonalities among diverse objects contributes to a sense of aural unity. Isography can also structure the nodes within a single network.

Figure 1.3b gives an example. Each of the dyad networks of (a) occurs within (b) as a sub-network; the terms “sub-network” and “sub-graph” refer to a portion of a network or graph (that is, incorporating some but not all nodes and arrows of a larger graph or network). The two-node T_5 sub-graph is repeated four times within the larger network (each oriented vertically in this example), but it is not the only repeated sub-structure – the sub-graph that contains the four upper nodes (oriented horizontally) is also repeated among the four lower nodes.

Figure 1.3: Networks depicting the music of Pépin, Monade VI – Réseaux, Cahier 7, fourth system

a) Four dyad networks
b) One larger network with isographic sub-networks

Although analysis with networks may seem formal and absolute, in fact the opposite is true: the analyst must decide what combinations of objects and transformations will produce a meaningful interpretation, and what structures are most appropriate for depicting the aural experience. The analyst’s choice of network structure may prioritize specific elements or hierarchies in a particular passage.
Section 9.5.5 of *GMIT*, an analysis of a two-part vocal line in parallel perfect fourths, demonstrates how multiple interpretations are possible depending on the choice of object, transformation, and network structure. Lewin’s graphs and networks are given.
as Figure 1.4. The smallest-scale nodes of parts (c) through (g) contain diatonic scale members; the arrows are labelled by integers indicating mode-step transpositions within the diatonic collection. Graphs (b) and (e) can be taken to model the melody and simultaneous intervals of the passage, respectively. The structures of Lewin’s networks say a lot about the music. For example, network (d) is structured such that the lower-left node (with contents “E”, the first note of the Organalis) is an input node, that is, no arrows lead to this node, and the upper-right (with contents “F”, the final note of the Principalis) is an output node, that is, no arrows lead from this node. Multiple paths lead from the input to the output node; for example, the Bb can be reached from the input node through either F or A. This suggests that the Organalis’s initial E might be understood as the foundation of the passage and the Principalis’s final F, generated by both the Organalis’s final C and the Principalis’s penultimate G, as the point of rest. It also suggests that the mode-step transformation 3, corresponding to the perfect fourths between simultaneously-attacked notes, is just as important as the transformations -1, 0, and 1 in the melodic dimension.

Parts (d), (f), and (g) are of particular interest to this study because of the different ways they combine graphs (b) and (e). In (d), the repetition and convergence of two types of sub-networks (one a five-node network incorporating the series of transformations <1, -1, 0, -1, -1> and the other a two-node network incorporating the transformation 3) establishes this structure as a product network, a structure that will be defined in more detail in the next chapter. Networks (f) and (g), on the other hand, suggest a more hierarchical outlook, which Lewin explains as follows:

(f) is a network whose graph is (b); each node of (f) contains a network whose graph is (e). (f) is thus a network-of-networks; the arrows on (f) labelled 1, -1,
and 0 transpose entire (e)-networks. (f) models the thought, “We are singing (the graph of) ‘Nos qui vivimus,’ singing diatessera ((e)-networks) as we go.

(g) is a network whose graph is (e); each node of (g) contains a network whose graph is (b). (g) is thus, like (f), a network-of-networks. It reflects a way in which Organalis might think: “Principalis is singing ‘Nos qui vivimus’; I too am singing ‘Nos qui vivimus,’ and my relation to Principalis is governed by the Symphony of the Diatesseron (the symbol ‘3’ on the graph).

The manner in which the objects and transformations are combined significantly affects our interpretation of the music. In (f), the transformations 1, -1, and 0 are transformations of networks while the transformation 3 acts upon diatonic pitch classes, and vice-versa for network (g). Thus there are two levels and two types of objects (unlike network (d), whose objects all belong to a single group). Lewin’s network-of-networks (f) considers the perfect-fourth simultaneities between the two voices as a scale-step transposition of 3 steps between the two pitch classes; this network is a basic object which is in turn transformed by a series of scale steps, thus highlighting the voice simultaneities. His graph (g), on the other hand, identifies a melodic series of pitch classes as its most basic object which is simultaneously transposed into a second melodic voice. In network (d), a product network, there is no such structure to the basic objects – that is, there are no explicit groupings that correlate to voices or simultaneities except the horizontal/vertical layout, which is not a property of the network itself. In this network it makes sense to consider, for example, the three nodes {E, A, Bb} on the left side of the network together as a musical unit because they are linked into a chain by the same kinds of arrows. It does not make sense to associate the corresponding nodes in (f) or (g) because the {E, A} pair participates in a completely different nodal network than {Bb}; the transformation “1” in this case is conceptually between networks, not pitch classes.

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6 Ibid., 206.
When Lewin expresses the idea that “we are singing … networks”, he means that we are singing a repetition of the relationships (rather than the objects) expressed in the first network. This is an important distinction that will be emphasized throughout this document.

Although (d) as given is a network, we can also regard it as a graph by removing the contents of its nodes while retaining the node/arrow configuration and the arrow labels. The result could also be the graph of a different network, such as one that represented a transposition of the music that begins on \{D, G\} and ends on \{Bb, Eb\}. Likewise, (f) and (g) could also be regarded as graphs by removing their node contents. However, because each of these networks has two different types of node contents they have fundamentally different structures than that of (d). Lewin describes the graph of (d) “as a formal ‘product’ of graph (b) with graph (e),” while he describes graph (f) as “a network-of-networks; the arrows on (f) … transpose entire (e)-networks.” The former description suggests that although graph (d) has no explicit hierarchical structure – there is only one type of node (the type that can contain pitch classes) – it is nevertheless in the same sense composed of multiple copies of simpler graphs (b) and (e), connected together in a special way, and thus it is implicitly hierarchical. That is, when we navigate the arrows in graph (d), we have a sense that some moves are (e)-type moves, while the rest are (b)-type moves, even though all the moves are simply changes of pitch class. A product network could also be conceptualized as the repetition and convergence of two (or more) distinct processes on the same set of musical objects.

The description of (f) is rather different. This graph is explicitly hierarchical because it involves one type of node containing simple pitch classes, and a different type
of node containing *networks* of these nodes of pitch classes. Concomitantly, the arrows on (f) have two different meanings: one type of arrow represents a change of pitch class, and the other type represents a change from one *network* to another. Thus the visual similarity of the arrows is misleading. When we traverse the vertical arrows, we are simply changing pitch classes; but when we move horizontally we have to switch our objects from pitch classes to pitch-class networks.

Both networks could be described as replicative. Network (f) is *explicitly replicative* in the sense that it embeds networks within it, whereas network (d) is only *implicitly replicative* in the sense that it seems to be built entirely from simpler structures that are combined repeatedly according to some rule: it has two different types of sub-graphs (one that involves the transformation 3, and another that involves the transformations <1, -1, 0, -1, -1>) that are replicated to form the larger structure.

Although (f) and (g) are different from each other and from (d), Lewin’s critical insight is that (d) can be reconfigured with equal validity as network (f) or (g). Retaining this flexibility of interpretation allows us to discuss paths, groupings, and hierarchies in a variety of ways. In this context, it is important to have clear definitions of different sorts of replicative networks and procedures for deriving them from each other, where possible.

While Lewin anecdotally describes the differences between product networks and networks-of-networks in *GMIT*, he does not define them formally, which is surprising given the abundance of formal definitions for other elements of graph and network theory within the book. Neither do later theorists. Writings by Robert Peck and Julian Hook
indirectly examine similar constructions, but not as their main goal.\footnote{Robert Peck, “Aspects of Recursion in M-Inclusive Networks.” \textit{Intégral} 18-19 (2004-2005): 25-70; and Julian Hook, “Cross-Type Transformations and the Path Consistency Condition,” \textit{Music Theory Spectrum} 29/1 (Spring 2007): 1-39.} Robert Peck, for example, discusses how networks must be well-formed and how similarities among structural levels may link small-scale and large-scale elements in a piece of music, but his primary objective is to incorporate M-transformations into hierarchical networks of various sorts.\footnote{Peck 2004-5, 26-28.} Julian Hook’s discussion of ‘strong path consistency,’ while very relevant to replicative structures, does not explicitly link this property via an analytical example to product networks or networks of networks. Indeed, since his interest lies in defining cross-type transformations as transformations that accommodate a change of object and in presenting networks that incorporate them, his system strives to broaden rather than narrow transformational criteria.\footnote{Hook 2007, 3. A product-network-like structure that incorporates his cross-type transformations can be seen in his Example 24 (page 34).} A definition might help one to recognize and distinguish product networks and networks-of-networks, and to clarify situations in which one might prefer one structure over another, as well as the advantages and limitations of each.

There are many possible conceptions of product networks and networks-of-networks. For example, Rahn postulates one system that permits more than one transformation to be associated with a single arrow in a network, and consequently produces a methodology that more closely resembles mathematical category theory.\footnote{John Rahn, “The Swerve and the Flow: Music’s Relationship to Mathematics,” \textit{Perspectives of New Music} 42/1 (Winter 2004): 142-143.} However, in this document my product networks and networks-of-networks are restricted in a particular way – to isographies in which corresponding arrows have the same labels, as in Lewin’s examples 9.8(f) and (g) (my \textbf{Figure 1.4}). This type of isography is easiest
to hear because it involves an exact repetition of the same transformations, in the same
order or arrangement.

To understand this restriction, consider the music of Figure 1.5. Grey arrows on
the figure point out that a single inversion, I₄, consistently associates pitch-class pairs
throughout. This consistency can be represented, to a certain extent, with networks
similar to those in Figure 1.4. For instance, Figure 1.6a is a ‘flat’ network like Figure
1.4d, with the same transformation labelling every vertical arrow. Also, Figure 1.6b is a
network-of-networks like Figure 1.4g. In this case, remember, the vertical arrow label
does not transform pitch classes, but rather networks of pitch classes. In fact it instances
a particular way of creating two isographic networks, which Lewin calls a network
isomorphism:

Network isomorphism is thus a particular case of isography between networks.
To be isographic, two networks must have these features:

1. They must have the same configuration of nodes and arrows.
2. There must be some isomorphism F that maps the transformation-
   system used to label the arrows of one network, into the transformation-
   system used to label the arrows of the other.
3. If the transformation X labels an arrow of the one network, then the
   transformation F(X) labels the corresponding arrow of the other.¹¹

Specifically, the horizontally-aligned nodes can be partitioned into two sub-networks, one
involving the upper-row nodes <E, Eb, B, Bb, Ab, G> and another incorporating the
lower-row nodes <C, C#, F, Gb, Ab, A>. Every transformation between members of the
top-most network is replaced between the corresponding nodes of the bottom-most
network by its inverse (satisfying features 2 and 3, above). For instance, every T₁₁ in the
upper network is replaced by its inverse T₁ in the lower network.

¹¹ David Lewin,"Klumpenhouwer Networks and Some Isographies that Involve Them," Music Theory
Spectrum 12/1 (Spring 1990): 87.
Figure 1.5: Gilles Tremblay, *Phases*, $I_4$ relations in measures 1-4

![Figure 1.5](image)

Figure 1.6: An analysis of inversionally-based pitch-class relationships in Tremblay, *Phases*, mm. 1-3

a) A network depicting transformations at a single structural level

Each network says something different about the music. **Figure 1.6a** tells us that each pitch class is paired with its $I_4$ associate, but that at the same time it also participates in a series of transpositions. **Figure 1.6b** tells us that one six-node pitch class grouping, represented by a transpositional network, is inverted by $I_4$ to generate a second grouping. 

It should be noted that both networks invoke transformational relationships and graphic
layouts that in certain respects do not correspond to a strictly temporal hearing of the passage. In particular, the ordering of the initial notes $<$Eb, C, E, C#$>$ is modified on both networks in order to better emphasize the $I_4$ relations in the dyads $\{C\#, Eb\}$ and $\{C, E\}$; $I_4$ symmetry is also the reason why Ab is presented twice on both networks even though it only occurs once in the passage.

While these networks each accurately depict one aspect of the passage, neither meets the basic restriction specified earlier: that corresponding arrows have the same labels. In (a) and (b), as we saw, corresponding arrows are labelled by inverse transpositions, not identical ones. The horizontal or vertical sub-networks may relate to one another via an isomorphism, but they are not identical: they may indicate similarities, but not repetitions. To achieve our goal of identifying repetition as the highest degree of similarity possible among sub-networks, we must set aside isomorphisms like those in Figure 1.6 that do not generate such repetition.

Moreover, unlike Lewin’s (d), Figure 1.6a is not the kind of flat network that can be reconfigured as two different networks-of-networks like Lewin’s (f) and (g), each containing isographic sub-networks. To be sure, Figure 1.6a can be reconfigured as Figure 1.6b, the way Figure 1.4d can be rewritten as Figure 1.4g. But there is no way to rewrite Figure 1.6a as a network-of-networks like Figure 1.4f since the pitch classes within any of its isographic $I_4$ networks cannot be transposed by a single transformation to generate the pitch classes of any other $I_4$ sub-network.\(^\text{12}\)

\(^{12}\) $T_6$, which is its own inverse and can thus transpose $I_4$ networks to generate other $I_4$ networks, is the sole exception to this rule, a situation discussed in Lewin 1987, 55. However, it is not one of the transformations of this network. Figure 1.6a could be reconfigured to show $T_6$ between non-adjacent dyad pairs such as $\{C, E\}$ and $\{Gb, Bb\}$, but this reconfiguration would not accommodate the $\{Ab, Ab\}$ dyad since its $T_6$ transposition $\{D, D\}$ is not included within the network.
This example shows that in order to be able to interpret music at different levels of hierarchy, it is necessary to limit our conception of network isography to certain kinds of networks, and to transformations with special properties. One property is that these transformations will preserve intervals (an interval-preserving transformation is one that generates the same interval from its initial to its resultant object, regardless of what is chosen to be its initial object; in Lewin’s words, “The interval-preserving property does not depend on the choice of [referential object]"\(^\text{13}\)). Constructing network-of-networks and product networks based on the same sub-graphs will allow us to identify and compare our interpretations of motivic repetition, and thus outline the advantages of one analytical format over another.

Figure 1.7 gives another example to clarify the idea of interval preservation.

Three networks relate via two isomorphisms called \(<T_4>\) and \(<T_0>\), which change the arrow labels of each left-side network into those of the corresponding right-side network, in a consistent way (network pairs involved in these isomorphisms are linked by arrows on the figure).\(^\text{14}\) Observe that \(<T_4>\) maps the left-most network to the centre network by retaining the former’s transposition and adding 4 to each of its inversions, while \(<T_0>\) maps the centre network to the right-most network by keeping all the transformations of the former invariant in the latter (that is, they have the same graph). But neither the \(<T_4>\) nor the \(<T_0>\) transformation preserves intervals; that is, one cannot transpose the pitch classes \{G, C, D\} of the left-side network by \(T_4\) (or by any other single transformation) to generate those of the centre network, \{B, C, F#\}, nor can one transpose the pitch classes

\(^{13}\) Lewin 1989, 48-49.

\(^{14}\) For a more detailed discussion of this and other types of isomorphisms, see pages 86-90 of Lewin 1990.
of the centre network by a single transformation to generate those of the right-side network.

**Figure 1.7: A large-scale network of Klumpenhouwer networks**

![Diagram of Klumpenhouwer networks](image)

The limitations outlined herein are based on practical in addition to theoretical issues. Transpositions are relatively easy to hear, as are many other interval-preserving operations, and consequently the network repetitions that depict them correspond to our strongest perceptions within the music. Inversions and other more abstract transformations, on the other hand, while commonly audible among pitches, are less often audible among (more abstract) pitch classes or networks.\(^{15}\) In the repertoire I have

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\(^{15}\) The same is true for higher-level isomorphisms based on inversions; for example, hyper-inversion or dual inversion. Michael Buchler discusses both of these cases in his recent article “Reconsidering Klumpenhouwer Networks” ([Music Theory Online](https://www.mtswebsite.org) 13/2, June 2007, section 56). He states: “As challenging as it is to construct aural images of displaced pc inversion, as represented by \(<T_n>_\) transformations, the notion of dual pc inversion implicit in \(<I_n>_\) transformations seems far more problematic. Dual pc inversion can be hard enough to hear (or even to imagine hearing) when not realized as musical wedges, and when I think of one part of a chord being transformed by \(I_5\) and the other part moving by \(I_8\), I can only concede that I would have problems detecting that relation in most musical situations. When we further stipulate that the \(I_5/I_8\) dual transformation is only one manifestation of the \(<I_1>_\) hyper-transformation, I find myself wondering whether there is any distinctly \(<I_1>_\) (or \(<I_n>_\)) sound. Straus vividly encapsulated this problem: ‘… this poses severe perceptual problems, not just the old “can you hear it?” but rather “what are we even supposed to be listening for? How can we hope even to conceive the relationship mentally?”’. (Buchler cites Joseph Straus, “Three Problems in Transformational Theory,” (New York: talk delivered at the closing plenary session of the [Mannes Institute on Transformational Theory](https://www.mannes.com), 2003)).
selected for this study, replicative transposition in particular (that is, transpositions that are repeated, among objects of either the same or a different group) is significantly more common than replicative inversion; for instance, of the twenty or so Québécois works I analyzed in preparation for this dissertation, only two or three (including the Tremblay work examined in Figure 1.5) incorporated an inversion that might be described as replicative (the other examples will be discussed in Chapter 3).

Although we have seen in connection with Figure 1.6 that inversions can prohibit multiple interpretations and interval preservation, Lewin did invent a way of incorporating them in special situations, what he called ‘contextual inversions.’ For example, in Figure 1.8a and b the contextual inversion “K” is defined as an inversion on ordered forms of the SC 01346 pentachord that retains the pitch classes of the first and last dyads, but swaps their ordering. While we could label this transformation as a standard inversion, we would have to use a different inversion to describe the same effect on each pair of pentachords (a) and (b) (here $I_8$ and $I_0$). In his analysis, Lewin argues that $K$, in addition to $I_7$, $T_7/T_5$, and several other contextual inversions, is characteristic of Schoenberg’s op. 23, no. 3. He also observes that $K$ has a special property evident in Figure 1.9 (his Figures 9.1 and 9.2): it is commutative with both transpositions and inversions.  

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17 Ibid., 202-206.
Figure 1.8: An example of Lewin's contextual transformation "K"

a) An example derived from Lewin, page 199

b) A second example of K inversion
Figure 1.9: Figures from Lewin’s “Transformational Considerations in Schoenberg’s Opus 23, Number 3”

Lewin’s restriction to contextual inversions is appealing in several ways. In the discussion accompanying his Figure 9.1 (my Figure 1.9), he explains that “the two [pentachords] in the left hand [the bottom two nodes in 9.1] imitate the two from the right hand [the top two nodes of 9.1] ‘at the fifth.’” Employing contextual inversions allows us to depict this imitation as a replication from one part of the network to another, thereby creating a product graph rather than a simple graph. Figure 1.10 adds node contents to Lewin’s Figure 9.1, based on his analytical commentary. Had the inversions been labelled with standard fixed-do inversions, they would produce different index
numbers – $I_0$ between the top two pentachords and $I_2$ between the bottom two – and thus would not clearly illustrate this imitation.

Figure 1.10: A network, corresponding to Lewin’s analytical commentary, which incorporates the graph of his Figure 9.1

A second advantage of Lewin’s restriction is that it allows us to see the same passage in the three different ways evident in Figure 1.4d, f, and g. He states:

The graphs of figures 9.1 and 9.2 are thus “well formed” in the sense of my Generalized Musical Intervals and Transformations. They are, in fact, “product graphs” in the sense of that study. In that sense, we can think of figure 9.2 as portraying a vertical $I_7$-motif, horizontally K-transformed from left to right; or, we can think of the example as portraying a horizontal K-motif, vertically $I_7$-transformed from top to bottom.\(^{18}\)

Figure 1.11 demonstrates how we might re-interpret the graph of 9.2 based on Lewin’s description above. Part (a) gives Lewin’s original graph; part (b) reinterprets this as a graph-of-graphs that highlights the K-transformation of the $I_7$ motif, and part (c) reinterprets this as a different graph-of-graphs that highlights the $I_7$-transformation of the K-motif. The reinterpretation of the original product graph would not be possible if the graphs did not incorporate the special contextual inversions.

\(^{18}\) Ibid., 203.
Figure 1.11: Reinterpreting Lewin's Figure 9.2

a) Lewin’s original graph

As do Figure 1.4d, f, and g, the graphs in Figure 1.11b and c introduce an element of hierarchy to our discussion that is not present in a product graph analysis. Specifically, a graph-of-graphs interpretation identifies two different strata of musical events, represented analytically as two levels of objects (in this example, the smallest-scale nodes would contain pentachords, while the larger-scale nodes contain networks that represent the $I_7$-motif in graph (b) and the $K$-motif in graph (c)). The $K$-transformation in graph (b) and the $I_7$-transformation in graph (c) are understood to occur at a higher level among more complex objects. In contrast, no explicit hierarchy is evident in the product graph of (a) since all objects and all transformations occur at a
single level. The graph-of-graphs is therefore useful in situations where we might wish to distinguish musical processes at multiple levels. Much of the discussion in this document will do exactly that: my analyses will present several interpretations of a single passage, and the accompanying discussion will explain how a network-of-networks identifies elements that a product network does not (and vice-versa).

Figure 1.12 gives Lewin’s figures 9.10 and 9.11 from the same article in order to demonstrate one further analytical feature that I will incorporate into my analyses. His figure 9.11 analyzes the pentachords illustrated in figure 9.10. In order to complete the product network suggested by the characteristic pentachord pairings of measures 1-3, Lewin proposes a “missing” pentachord $\pi$. While incorporating this pentachord may seem like an arbitrary decision too heavily influenced by the network structure rather than the actual pentachord presentation within the excerpt, in reality this “hypothetical” pentachord helps us to understand the structure of a later passage. Lewin explains:

Particularly convincing in this connection is the later music of figure 9.11c (m. 14.2) … Here, as portrayed in figure 9.11d, the J-K-J chain, which implicates the succession $\psi$-$\kappa$-$\pi$-$\theta$, becomes manifest and explicit in the musical foreground. So does form $\pi$ itself … When figure 9.11d is compared with the top rank of figure 9.11b, the musical gesture of figure 9.11c seems a good deal less arbitrary or willful: not simply an amusing trick, it realizes a transformational potential already inherent in the opening measures of the piece.\(^{19}\)

Thus the repetition depicted within the product network clarifies, for the reader, replicative transformational processes that recur not only within the passage in question, but which also serve to organize later events in the piece. The “incomplete” network suggested that our transformational expectations for the passage were unfulfilled; the

\(^{19}\) Ibid., 216.
eventual appearance of $\pi$ completes the transformational process established in the opening measure, providing structural completion.

**Figure 1.12: Lewin’s Figures 9.10 and 9.11**

![Diagram](image1.png)

*Figure 9.10. $P$-Segmentations for mm. 1–3 in Op. 23, no. 3.*

![Diagram](image2.png)

*Figure 9.11 Transformational Network for figure 9.10.*

The concept of transformational networks expressing structural expectations is an idea that Lewin himself incorporates into his analysis of Stockhausen’s *Klavierstück III*.\(^{20}\)

**Figure 1.13**, which replicates his example 2.5, presents a network that associates the characteristic moves of the work with “consistent visual motifs”; for example, horizontal

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arrows within large-scale nodes always indicate J0, vertical arrows that connect the large-scale nodes always represent Te, and so forth. The small-scale objects of this network are SC 01236 pentachords, while its transformations combine mod12 transpositions (Tx) and contextual inversions (Jx). Specifically, J is a contextual inversion that keeps the 0123 subset of the 01236 pentachord invariant; Jx indicates the composition of operations (Tx)(J).

Figure 1.13: An analysis of Stockhausen's *Klavierstück III*, from Lewin’s *Musical Form and Transformation*

![Network Diagram]

- horizontal arrows within boxes = J0; between boxes = J3 or J9
- vertical arrows within boxes = T6; between boxes = Te or T1
- diagonal arrows within boxes = J6; between boxes = Je or J1

Example 2.5. A network that reflects a more spatial sense of pentachord organization.
As it stands, Lewin’s network is incomplete. For example, the operation J3 between the two upper dashed boxes seems to only invert two of the pentachords within the left-side network (p6 and P6 to generate p9 and P9). However, in this particular conceptualization Lewin is not implying that the right-side network is a complete inversion of the left-side one; rather, he is limiting his network to forms of P that appear within the piece. His example 2.5 therefore implies the larger network (a product network-of-networks) given in Figure 1.14. In this network, I have indicated Lewin’s original pentachords, arrows, and transformations in black, with unrealized objects and transformations in grey. I have also omitted some large-scale arrows and transformations to make the product network structure more apparent. This expansion of Figure 1.13 is not out-of-line with Lewin’s own analysis; he concedes that

It would be possible to augment example 2.5 by adjoining the P-forms necessary to complete the incomplete complexes. We could thus add more nodes on the map, to represent the missing forms P5, pe, p3, P3, and so forth. This recourse
would give a fuller sense of the space in the example as a space of “potentialities” rather than “presences.” That done, we could proceed to add more arrows and transformations as well. Indeed, we could already add more arrows to example 2.5 as it stands… Even if we do not intend the extra arrows to assert actual events in the music, we can surely assert them as theoretical potentialities in the P-form space through which the piece moves.21

He expresses concern, however, in deciding what criteria should determine what elements of the network to include or exclude:

Clearly, our network must portray some actualities about the piece, not only to define but also to place bounds upon its potentialities. For my taste, example 2.5 as it stands is about right in this respect. It indeed shows a certain abstractly structured space of possibilities through which the piece moves, but it also shows how the abstract structuring is suggested and bounded by actual transitions within the progress of the piece itself.22

Several of my analyses will take a similar approach. I will employ replicative network structures to demonstrate repeated transformations, and to express the “potential” moves that continue earlier processes. While Lewin attempts to restrict his networks to phenomenological elements of the music, my networks will take a slightly different approach, examining potentialities suggested by recursive structures. As such, my analyses will reflect a similar attitude. They will incorporate what I perceive to be the salient phenomena of a passage, but they will often incorporate other “potential” events when doing so clarifies a larger consistency that I perceive but that is incompletely manifested in the passage, or when I hear the actual events to imply the possibility of other events continuing the same processes.

In the following chapters, I will define these structures formally and present a number of examples in which I hope to illustrate their potential for analysis. Chapter 2

21 Ibid., 35.
22 Ibid., 36.
outlines theoretical definitions for product networks and networks-of-networks modeled after Lewin’s work in *GMIT* and makes the restrictions I have discussed technically explicit. The analytical chapters of this document will demonstrate several advantages that these network structures provide for analysis, combining original interpretations with more conventional ones.
Chapter 2: Theoretical Definitions

To help resolve what qualifies as a product network or network-of-networks, and to prepare for the analytical work in later chapters, I will describe these two structures formally by first presenting specific examples of them, and then extrapolating a formal definition from these cases. I will also describe a third replicative structure that incorporates the repetition of a single transformation which I will call a “sequential network”. This format does not manifest the same combination of sub-graphs associated with either product networks or networks-of-networks, but will be useful in depicting sequential elements discussed in the analytical chapters of this document. Throughout both the theoretical and analytical chapters, I will also answer other questions: for example, what aspects of a musical passage suggest analysis via a product network or via a network-of-networks, and when it is analytically appropriate to choose one over the other. Replication is certainly an important element, but other factors that I will soon discuss are also important.

Several definitions will be presented in this chapter. Section 1 will formally describe a particular product graph using Lewin’s terminology from GMIT. The particular subset relations and isomorphisms characteristic of product networks will also be defined through this example, leading to a general definition of product graphs and networks. Section 2 of the chapter will present an example of a network-of-networks; like section 1, this will be described via Lewin’s terminology, the subset relations and isographies that characterize graphs-of-graphs will be outlined, and a general definition will be extrapolated. Section 3 will briefly define a third type of replicative network, the
sequential network. Finally, Section 4 will explain how product networks and networks-of-networks are related, both formally and informally, and will provide an algorithm for deriving one from the other.

Section 1: Defining Product Graphs and Networks

Lewin refers to a “product graph” as a particular synthesis or combination of two simple graphs. The following discussion will illustrate this idea, presenting a specific example from which general definitions of product graphs and product networks will be extrapolated. Citations, unless specified otherwise, will refer to the numbered sections within *GMIT*.

Consider the following visualization of a graph:

**Graph B:**

Each circle represents a node, while each arrow represents an ordered pairing of nodes; the contents of the first node of each pair are mapped to those of the second via the transformation given above the arrow. Each node is labelled with one of {B1, B2, B3, B4} for reference (the letter is meant to evoke part of the corresponding graph in Lewin’s section 9.5.5, my Figure 1.4). This graph meets the formal definition of a transformational graph according to Lewin’s criteria 9.2.1:
Graph B is an ordered quadruple \((\text{NODES}_B, \text{ARROW}_B, \text{SGP}_B, \text{TRANSIT}_B)\) where:

- the set of nodes, \(\text{NODES}_B = \{B1, B2, B3, B4\}\)
- the set of arrows, \(\text{ARROW}_B = \{(B1, B2), (B2, B3), (B3, B4)\}\), so \((\text{NODES}_B, \text{ARROW}_B)\) is a node/arrow system (Lewin 9.1.1)
- the semigroup \(\text{SGP}_B\) is the group of (diatonic) scale-step transpositions denoted by integers mod 7 (the graph does not use all the members of this semigroup)
- the function \(\text{TRANSIT}_B\) associates each node-pair in \(\text{ARROW}_B\) with a member of \(\text{SGP}_B\) as follows: \((B1, B2) \rightarrow 1, (B2, B3) \rightarrow -1, (B3, B4) \rightarrow 0\)

Since there is only one arrow chain (that is, a pathway) between any two nodes, the quadruple trivially satisfies Lewin’s criterion 9.2.1 (D), that the semigroup product of every arrow chain between any two given nodes must be the same (in other words, that the same total transformation results, regardless of the path taken).\(^{23}\) Since this semigroup product is a composition of members of \(\text{SGP}_B\) it must also be a member of \(\text{SGP}_B\).

Now consider another graph, \(E\):

\(^{23}\) Julian Hook calls this a “strong path consistency,” which requires “the transformational products along the two paths … to be identical.” Hook 2007, 28. Hook relaxes this condition in order to incorporate non-operations into his networks (which allow him to transform one group of objects into another). Also, he feels that certain networks that do not exhibit path consistency can nevertheless express useful analytical insights about specific passages of music. Granting these insights, however, it is important to remember that path consistency is necessary in order for us to be able to say that two networks are isographic – a certain concern of Lewin’s, and a basic property of replicative networks. If we eliminate the path-consistency condition then we may not substitute node contents (while keeping the same graph) to produce another valid network. Since this isography is essential for creating product networks and networks-of-networks (a feature to be discussed shortly), we will thereby not be able to form valid networks for replicative structures in a consistent manner.
Graph E is likewise an ordered quadruple \((\text{NODES}_E, \text{ARROW}_E, \text{SGP}_E, \text{TRANSIT}_E)\) where:

- the set of nodes, \(\text{NODES}_E = \{E1, E2\}\)
- the set of arrows, \(\text{ARROW}_E = \{(E2, E1)\}\), so \((\text{NODES}_E, \text{ARROW}_E)\) is a node/arrow system
- the semigroup \(\text{SGP}_E = \text{SGP}_B\), the same semigroup as in Graph B
- the function \(\text{TRANSIT}_E\) associates each node-pair in \(\text{ARROW}_E\) with a member of \(\text{SGP}_E\) as follows: \((E2, E1) \rightarrow 3\)

Since there is only one arrow chain between the two nodes, this quadruple also trivially satisfies Lewin’s criterion 9.2.1 (D).

Now consider the following graph:
Graph D bears obvious visual similarities to Graphs B and E, with copies of both smaller graphs embedded within it (the horizontally-aligned nodes form copies of Graph B, while the vertically-aligned nodes form copies of Graph E). However, we cannot rely on the visual properties to establish this relationship because graphs can have many different visualizations. For instance, Graph D can with equal validity be visualized as follows:

Graph D:
which is not obviously similar to graphs B or E. Whatever its visual layout, Graph D may be referred to as a product graph of Graphs B and E in the manner of Lewin 9.5.5; its status as a product of B and E is a consequence of the formal definitions, not the visualization. While Lewin’s analyses involving these structures are clear in GMIT, his definition of the structure itself is not. He does not clarify what the similarities are between the larger graph and its embedded sub-graphs, what the term “product” means, nor even in general how we recognize such a graph to be a “product” of other graphs. The following formalism addresses these issues.

First, let us explicitly specify D as a transformation graph:

Graph D is an ordered quadruple (NODES_D, ARROW_D, SGP_D, TRANSIT_D) where:

- the set of nodes, NODES_D = {D1, D2, D3, D4, D1’, D2’, D3’, D4’}
- the set of arrows, ARROW_D = {(D1, D2), (D2, D3), (D3, D4), (D1’, D2’), (D2’, D3’), (D3’, D4’), (D1’, D1), (D2’, D2), (D3’, D3), (D4’, D4)}, so (NODES_D, ARROW_D) is a node-arrow system
- SGP_D = SGP_B
- the function TRANSIT_D associates each node-pair in ARROW_D with a member of SGP_D as follows: (D1, D2) → 1, (D2, D3) →-1, (D3, D4) → 0, (D1’, D2’) → 1, (D2’, D3’) →-1, (D3’, D4’) → 0, (D1’, D1) → 3, (D2’, D2) →3, (D3’, D3) → 3, (D4’, D4) → 3

Note that in this example, unlike that of Graph B, there are multiple possible arrow chains from some nodes to others. For instance, there are two paths from D1’ to D2 and four paths from D1’ to D4. In order to satisfy Lewin’s condition 9.2.1(D) for transformation graphs (that any two arrow chains between a pair of nodes produce the
same semigroup product of transformations), the members of $SGP_D$ must commute. That is, the series of scale-step transformations from $D_1'$ to $D_2$ through $D_1$ (3 then 1) has to yield the same result as the series of transformations in the opposite order, from $D_1'$ to $D_2$ through $D_2'$ (1 then 3). Among other things, the members of $SGP_D$ must be what Lewin calls ‘operations’ (and Hook calls ‘bijections’) in order to commute, in other words transformations that are 1-to-1 and onto, and thus have unique inverses.\textsuperscript{24}

Transformations that are not operations map a family of objects into (rather than onto) itself, and as a consequence these functions do not preserve intervals; rather, the effect of each transformation differs based on the object to which it is applied. For this reason the same series of transformations on a different object, or the series of transformations in the opposite order, may not produce the same result.\textsuperscript{25} As we shall see, the condition that the members of $SGP$ are commutative operations greatly constrains the kind of graphs we can recognize as product graphs; most of the examples given in this document will be limited to a single type of operation within each graph (usually pitch-class transpositions).\textsuperscript{26} While other theorists such as Rahn (as previously discussed), Klumpenhouwer, Stoecker, Buchler, and others have found networks with non-commutative operations to be analytically useful, such networks cannot be combined to produce path-consistent product networks.\textsuperscript{27}

\textsuperscript{24} Lewin 1987, 3 (1.3.1).
\textsuperscript{25} See Hook 2007, 13 for an alternate explanation.
\textsuperscript{26} We can label the ensemble of transformations a group rather than a semigroup since they involve operations (in other words, because every element has an inverse; see Lewin 1987, 4-6 for more information). To be clear, all operations are transformations, but not all transformations are operations; the same is true for groups and semigroups (all groups are also semigroups, but not all semigroups are groups). However, in the following discussion I will continue to refer to ‘transformations’ and ‘semigroups’ in an attempt to highlight the similarities between the given formalisms and Lewin’s definitions in GMIT, which employ these two terms.
Resuming our earlier discussion, comparing the formal properties of **Graph B** and **Graph D** is the first step in creating a definition of “product graph”. Ovals on **Graph D** suggest that its nodes can be partitioned into two sets $\alpha$ and $\beta$ with the properties:

- $\alpha \cup \beta = NODES_D$
- $\alpha \cap \beta = \{ \}$

Considering now the arrows in **Graph D**, we can form graphs from each of these sets of nodes with the following properties:

For the graph $(\alpha, ARROW_{\alpha}, SGP_{\alpha}, TRANSIT_{\alpha})$:

- the set of nodes, $NODES_{\alpha} = \{D1, D2, D3, D4\}$
- the set of arrows, $ARROW_{\alpha} = \{(D1, D2), (D2, D3), (D3, D4)\}$, so $(NODES_{\alpha}, ARROW_{\alpha})$ is a node/arrow system
- the semigroup $SGP_{\alpha}$ is the group of scale-step transpositions denoted by integers mod7
- the function $TRANSIT_{\alpha}$ associates each node-pair in $ARROW_{\alpha}$ with a member of $SGP_{\alpha}$ as follows: $(D1, D2) \rightarrow 1$, $(D2, D3) \rightarrow -1$, $(D3, D4) \rightarrow 0$

Since there is only one arrow chain between any two nodes, the quadruple trivially satisfies Lewin’s criterion 9.2.1 (D).

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For the graph ($\beta$, ARROW$_\beta$, SGP$_\beta$, TRANSIT$_\beta$):

- the set of nodes, NODES$_\beta$ = \{D1’, D2’, D3’, D4’\}
- the set of arrows, ARROW$_\beta$ = \{(D1’, D2’), (D2’, D3’), (D3’, D4’)\}, so
  (NODES$_\beta$, ARROW$_\beta$) is a node/arrow system
- the semigroup SGP$_\beta$ = SGP$_\alpha$
- the function TRANSIT$_\beta$ associates each node-pair in ARROW$_\beta$ with a
  member of SGP$_\beta$ as follows:  (D1’, D2’) $\rightarrow$ 1, (D2’, D3’) $\rightarrow$-1, (D3’, D4’) $\rightarrow$ 0

Since there is only one arrow chain between any two nodes, the quadruple trivially
satisfies Lewin’s criterion 9.2.1 (D).

The graphs ($\alpha$, ARROW$_\alpha$, SGP$_\alpha$, TRANSIT$_\alpha$) and ($\beta$, ARROW$_\beta$, SGP$_\beta$, TRANSIT$_\beta$) are “isomorphic”: there is a one-to-one mapping of the nodes of $\alpha$ to the
nodes of $\beta$, and a one-to-one mapping of the arrow labels of graph $\alpha$ to those of graph $\beta$.
(Indeed, the sub-graphs are trivially isomorphic – the corresponding arrow labels are
identical, and the isomorphism itself consists solely in relabelling the nodes. A
restriction to this type of isomorphism will be particularly important for the forthcoming
definitions of product networks and networks-of-networks. See chapter 1 for further
discussion of this constraint.) Informally speaking, we can obtain the latter graph simply
by substituting D1’ for D1, D2’ for D2, and so forth in the first graph. More importantly,
however, each of these two graphs is isomorphic to Graph B as there is a one-to-one
mapping of the nodes of $\alpha$ and the nodes of $\beta$ to the nodes of B, and a one-to-one
mapping of the arrow labels of graph $\alpha$ and graph $\beta$ to those of $B$. This demonstrates that

Graph $D$, of which $(\alpha, \text{ARROW}_\alpha, \text{SGP}_\alpha, \text{TRANSIT}_\alpha)$ and $(\beta, \text{ARROW}_\beta, \text{SGP}_\beta, \text{TRANSIT}_\beta)$ are sub-graphs, is a “product” of Graph $B$ in one sense: there are multiple “copies” of the latter embedded in the former. The sense of “copy” is captured by the following statements, which refer to Lewin’s definition 9.4.2:

The graphs $(\alpha, \text{ARROW}_\alpha, \text{SGP}_\alpha, \text{TRANSIT}_\alpha)$ and Graph $B$ are isomorphic because there exists a pair $(\text{NODEMAP}_\alpha B, \text{SGMAP}_\alpha B)$ with the following properties:

A) $\text{NODEMAP}_\alpha B$ is an isomorphism of $(\alpha, \text{ARROW}_\alpha)$ with $(\text{NODES}_B, \text{ARROW}_B)$.

B) $\text{SGMAP}_\alpha B$ is an isomorphism of $\text{SGP}_\alpha$ with $\text{SGP}_B$ (specifically, the identity)

as follows:

(A) The isomorphism NODEMAP$_\alpha B$ is specified by the following function table:

<table>
<thead>
<tr>
<th>NODES$_\alpha$</th>
<th>NODES$_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>B1</td>
</tr>
<tr>
<td>D2</td>
<td>B2</td>
</tr>
<tr>
<td>D3</td>
<td>B3</td>
</tr>
<tr>
<td>D4</td>
<td>B4</td>
</tr>
</tbody>
</table>

This is a one-to-one map of one set of nodes onto another, such that every pair of nodes $(N_1, N_2)$ in $\alpha$ is in the ARROW$_\alpha$ relation if and only if $(\text{NODEMAP}_\alpha B(N_1), \text{NODEMAP}_\alpha B(N_2))$ is also in the arrow relation, fulfilling Lewin’s definition of isomorphic node/arrow systems (9.4.1).
(B) Since $SGP_B = SGP_\alpha$, the two semigroups are identical and thus $SGMAP_{\alpha B}$ is the (identity) isomorphism that maps each transformation onto itself.

For the $NODEMAP_{\alpha B}$ given in (A) above, and for $TRANSIT_\alpha$ and $TRANSIT_B$ defined earlier, and considering that $SGP_\alpha = SGP_\beta$, we can see by the following table:

<table>
<thead>
<tr>
<th>Pairs in $ARROW_\alpha$ in the format $(N_1, N_2)$:</th>
<th>$NODEMAP_{\alpha B}(N_1)$: $NODEMAP_{\alpha B}(N_2)$:</th>
<th>$TRANSIT_B(NODEMAP_{\alpha B}(N_1))$: $NODEMAP_{\alpha B}(N_2)$:</th>
<th>$TRANSIT_\alpha(N_1, N_2)$:</th>
<th>$SGMAP_{\alpha B}(TRANSIT_\alpha(N_1, N_2))$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1, D2</td>
<td>B1, B2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>D2, D3</td>
<td>B2, B3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>D3, D4</td>
<td>B3, B4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

that for every pair $(N_1, N_2)$ in $ARROW_\alpha$,

$TRANSIT_B(NODEMAP_{\alpha B}(N_1), NODEMAP_{\alpha B}(N_2)) = SGMAP_{\alpha B}(TRANSIT_\alpha(N_1, N_2))$.

By exactly the same kind of demonstration we can show that the graphs ($\beta$, $ARROW_\beta$, $SGP_\beta$, $TRANSIT_\beta$) and Graph B are isomorphic because there exists a pair ($NODEMAP_{\beta B}$, $SGMAP_{\beta B}$).

Thus we have established that Graph D can be decomposed into two graphs, ($\alpha$, $ARROW_\alpha$, $SGP_\alpha$, $TRANSIT_\alpha$) and ($\beta$, $ARROW_\beta$, $SGP_\beta$, $TRANSIT_\beta$), of which both are isomorphic to B. Indeed, since both of these graphs are isomorphic to B, they are isomorphic to one another as well. Thus we can define an isomorphism ($NODEMAP_{\alpha \beta}$, $SGMAP_{\alpha \beta}$), where $NODEMAP_{\alpha \beta}$ maps each node of the set $\beta$ to the corresponding nodes in $\alpha$, and $SGMAP_{\alpha \beta}$ maps $SGP_\alpha$ onto $SGP_\beta$. This decomposition into (trivially) isomorphic graphs is one property of a product graph. A second property is evident when we consider the arrows of Graph D that are not members of $ARROW_\alpha$ or $ARROW_\beta$.

Observe that all these arrows associate exactly the same nodes as does $NODEMAP_{\alpha \beta}$.

For example, one of these arrows connects $D1'$ to $D1$, and $NODEMAP_{\alpha \beta}(D1') = D1$. All
of the transformations labelling these arrows are identical, namely the 3-scale-step transformation of **Graph D** that did not participate in **Graph B**.

In the same way that **Graph D** was demonstrated to embed multiple copies of **Graph B**, it also embeds multiple copies of **Graph E**:

**Graph D:**

![Graph D diagram](image)

Four sets of nodes with structures visually similar to those of **Graph E** have been circled above on **Graph D**. Labelled \( \chi \), \( \delta \), \( \epsilon \), and \( \psi \), these sets of nodes have the properties:

- \( \chi \cup \delta \cup \epsilon \cup \psi = \text{NODES}_D \)
- no two of \( \chi \), \( \delta \), \( \epsilon \), \( \psi \) intersect in any node

Graphs can be formed from each of these sets of nodes, each with the same properties.

For the graph (\( \chi \), ARROW\( _\chi \), SGP\( _\chi \), TRANSIT\( _\chi \)):

- \( \text{NODES}_{\chi} = \{D1, D1'\} \)
- \( \text{ARROW}_{\chi} = \{(D1', D1)\} \), so \( (\text{NODES}_{\chi}, \text{ARROW}_{\chi}) \) is a node/arrow system
- \( \text{SGP}_{\chi} \) is the group of scale-step transpositions denoted by integers mod 7
• TRANSIT\(\chi\): (D1', D1) \rightarrow 3

Since there is only one arrow chain between any two nodes, the quadruple trivially satisfies Lewin’s criterion 9.2.1 (D).

Node sets \(\delta\), \(\varepsilon\), and \(\psi\) can be incorporated into similarly defined graphs. By a demonstration just like the one for graphs \(\alpha\) and \(\beta\), we can show that the graphs (\(\chi\), ARROW\(\chi\), SGP\(\chi\), TRANSIT\(\chi\)), (\(\delta\), ARROW\(\delta\), SGP\(\delta\), TRANSIT\(\delta\)), (\(\varepsilon\), ARROW\(\varepsilon\), SGP\(\varepsilon\), TRANSIT\(\varepsilon\)), and (\(\psi\), ARROW\(\psi\), SGP\(\psi\), TRANSIT\(\psi\)) are each isomorphic with Graph E (and thus with each other). It is easy to see how to map the nodes of each set to the nodes of another; and that the SGMAP for any two of these graphs is the identity. Thus we see that Graph D partitions exhaustively into multiple “copies” of Graph E, just as it partitions exhaustively into multiple “copies” of Graph B. That is one of the senses in which it is a “product” of E (and a product of B).

Let us consider the arrows of Graph D that are not arrows in any of the E-isomorphic sub-graphs \(\chi\), \(\delta\), \(\varepsilon\), and \(\psi\). They associate exactly the same pairs of nodes as do NODEMAP\(\chi\), NODEMAP\(\delta\), NODEMAP\(\varepsilon\), and NODEMAP\(\psi\). The transformations that label these arrows are those that label the arrows in Graph B and its isomorphic sub-graphs (specifically, the union of TRANSIT\(\alpha\) with TRANSIT\(\beta\)) but which do not occur in Graph E and its isomorphic sub-graphs.

We are now prepared to define how Graph D is a product of B with E:

• Node pairs \{(D1', D1), (D2', D2), (D3', D3), (D4', D4)\} are the only members of ARROW\(\chi\) not in ARROW\(\alpha\) or ARROW\(\beta\). Each pair defines a graph previously
identified as isomorphic to $E$, the graphs $(\chi, \text{ARROW}_\chi, \text{SGP}_\chi, \text{TRANSIT}_\chi), (\delta, \text{ARROW}_\delta, \text{SGP}_\delta, \text{TRANSIT}_\delta), (\epsilon, \text{ARROW}_\epsilon, \text{SGP}_\epsilon, \text{TRANSIT}_\epsilon), \text{and} (\psi, \text{ARROW}_\psi, \text{SGP}_\psi, \text{TRANSIT}_\psi)$, respectively.\(^{28}\)

- Node pairs $\{(D1, D2), (D2, D3), (D3, D4), (D1', D2'), (D2', D3'), (D3', D4')\}$ are the only members of ARROW$_D$ not in ARROW$\chi$, ARROW$\delta$, ARROW$\epsilon$, or ARROW$\psi$. These can be exhaustively partitioned into two sets of node pairs $\alpha = \{(D1, D2), (D2, D3), (D3, D4)\}$ and $\beta = \{(D1', D2'), (D2', D3'), (D3', D4')\}$ whose graphs were earlier shown to be isomorphic to $B$, the graphs $(\alpha, \text{ARROW}_\alpha, \text{SGP}_\alpha, \text{TRANSIT}_\alpha)$ and $(\beta, \text{ARROW}_\beta, \text{SGP}_\beta, \text{TRANSIT}_\beta)$.

To summarize, all node pairs not involved in copies of $B$ form graphs that are copies of $E$, and all node pairs not involved in copies of $E$ can be grouped into sets of nodes whose graphs form copies of $B$. Observe that, as a consequence of the isomorphisms shown above,

- $(\text{ARROW}_\chi \cup \text{ARROW}_\delta \cup \text{ARROW}_\epsilon \cup \text{ARROW}_\psi)$ involves all the node pairs that NODEMAP$_{\alpha\beta}$ does, and no others.

- $(\text{ARROW}_\alpha \cup \text{ARROW}_\beta)$ involves all the node pairs that $(\text{NODEMAP}_{\delta\epsilon} \cup \text{NODEMAP}_{\delta\psi} \cup \text{NODEMAP}_{\epsilon\psi})$ does, and no others.

\(^{28}\) The SGMAP functions will not be discussed in this section since all of the semigroups involved are identical (all are the group of scale-step transpositions denoted by integers mod7), and thus any SGMAP$_x$ maps a semigroup onto itself.
By demonstrating these similarities between the members of NODEMAP and ARROW, we have shown that the transformations of $B$ map copies of $E$ onto one another (and vice-versa). Thus we have demonstrated that $D$ is a product of these two graphs.

Abstract Version:

Let us now define a product graph in all generality, based on the observations above.

Given:

- a graph $A$ on a set of $M$ nodes: $\text{NODES}_A = \{a_1, a_2, \ldots, a_M\}$, a semigroup $\text{SGP}_A$, the relation $\text{ARROW}_A \subseteq \text{NODES}_A \times \text{NODES}_A$, and the function $\text{TRANSIT}_A$:
  \[
  \text{ARROW}_A \rightarrow \text{SGP}_A
  \]

- a graph $B$ on a set of $N$ nodes: $\text{NODES}_B = \{b_1, b_2, \ldots, b_N\}$, a semigroup $\text{SGP}_B$, the relation $\text{ARROW}_B \subseteq \text{NODES}_B \times \text{NODES}_B$, and the function $\text{TRANSIT}_B$:
  \[
  \text{ARROW}_B \rightarrow \text{SGP}_B
  \]

We will not require $A$ or $B$ to be as simple as Lewin’s examples (b) and (e) from Figure 1.4; in particular, we will allow each node to function as the input for more than one ARROW relation (and likewise each may also function as the output for more than one ARROW relation). Consider a graph $P$ on a set of $M \times N$ nodes and a semigroup $\text{SGP}_P$, with structure $\text{ARROW}_P$ and $\text{TRANSIT}_P$ (modelling, for example, a musical texture of $M \times N$ distinct pitch events, between many pairs of which we hear intervals – we want to know whether our hearing can be simplified by a particular structuring of these events).\(^{29}\)

\(^{29}\) The following definition is not the only one possible for product graphs; indeed, previous work in the field of mathematicians has modeled these structures as Cartesian products. See V. G. Vizing, “The Cartesian Product of Graphs,” Vychislitel’nye sistemy 9 (1963): 30-43; and Gert Sabidussi, “Graph Multiplication,” Mathematische Zeitschrift 72 (1960): 446-457 for more information.
**DEFINITION:** Graph P is a product of graph A and graph B iff

a) \(SG_P \subseteq SG_A \subseteq SG_P, SG_B \subseteq SG_P,\) and \(SG_P\) is commutative.

b) There is a one-to-one map of NODES\(_A\) x NODES\(_B\) to the nodes \(p_{i\alpha}\) of P as follows:

\[PMAP(a_i, b_{\alpha}) \rightarrow p_{i\alpha}\] such that the \(p_{i\alpha}\)s satisfy conditions 1-3 below (see the diagram below for reference).

<table>
<thead>
<tr>
<th>BNODES:</th>
<th>Nodes of P:</th>
<th>column i</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_N)</td>
<td>(p_{1N})</td>
<td>(p_{2N})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(b_{\alpha})</td>
<td>(p_{i\alpha})</td>
<td>(p_{2\alpha})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(b_3)</td>
<td>(p_{13})</td>
<td>(p_{23})</td>
</tr>
<tr>
<td>(b_2)</td>
<td>(p_{12})</td>
<td>(p_{22})</td>
</tr>
<tr>
<td>(b_1)</td>
<td>(p_{11})</td>
<td>(p_{21})</td>
</tr>
</tbody>
</table>

| ANODES: | \(a_1\) | \(a_2\) | \(a_3\) | ... | \(a_i\) | ... | \(a_M\) |

For each \(\alpha\) denoting a node \(b_{\alpha}\) of Graph B, consider the M corresponding nodes \(P_{\alpha} = \{p_{i\alpha} = PMAP(a_i, b_{\alpha})\}\) for \(1 \leq i \leq M\). Let us call this set "row \(\alpha\)" of P, as indicated on the diagram. Let the arrows of P that are defined only between the nodes within row \(\alpha\) be denoted as ARROW\(_\alpha\); each of these arrows is labelled by TRANSIT\(_P\) (for clarity, arrows are not shown on the diagram). Similarly, for each \(i\) denoting a node \(a_i\) of Graph A, consider the N corresponding nodes \(P_i = \{p_{i\alpha} = PMAP(a_i, b_{\alpha})\}\) for \(1 \leq \alpha \leq N\). Let us call this set "column \(i\)" of P, also indicated on the diagram. Let the arrows of P that are defined
only between the nodes within column i be denoted as ARROW\textsubscript{i}; each of these arrows is labelled by TRANSIT\textsubscript{P}.

**Condition 1a:** for all $\alpha$ from 1 to N, the graph $(P_\alpha, \text{ARROW}_\alpha, \text{TRANSIT}_P, \text{SGP}_P)$ is isomorphic to Graph A.

In P there may also be arrows connecting nodes that belong to different rows. For rows $\beta$ and $\gamma$, call these ARROW\textsubscript{\beta\gamma}.

**Condition 1b:** for any two values of $\beta$ and $\gamma$ for which TRANSIT\textsubscript{B}(b_\beta, b_\gamma) is defined, TRANSIT\textsubscript{P}(p_\beta, p_\gamma) = TRANSIT\textsubscript{B}(b_\beta, b_\gamma) for all i from 1 to M.

In other words, the arrows in Graph P that connect nodes in the same column but different rows are labelled by the same transformations as are the arrows in Graph B that connect the nodes corresponding to those rows.

**Condition 2a:** for all i from 1 to M, the graph $(P_i, \text{ARROW}_i, \text{TRANSIT}_P, \text{SGP}_P)$ is isomorphic to Graph B.

In P there may also be arrows connecting nodes that belong to different columns. For columns $j$ and $k$, call these ARROW\textsubscript{jk}.

**Condition 2b:** for any two values of $j$ and $k$ for which TRANSIT\textsubscript{A}(a_j, a_k) is defined,

$$\text{TRANSIT}_P(p_{j\alpha}, p_{k\alpha}) = \text{TRANSIT}_A(a_j, a_k)$$

for all $\alpha$ from 1 to N.
In other words, the arrows in Graph P that connect nodes in the same row but different columns are labelled by the same transformations as are the arrows in Graph A that connect the nodes corresponding to those columns.

\textbf{Condition 3}: \( \text{ARROW}_\alpha \cup \text{ARROW}_i \cup \text{ARROW}_{\beta\gamma} \cup \text{ARROW}_{j\kappa} = \text{ARROW}_P \).

That is, there are no arrows in Graph P other than those that map onto arrows of Graph A or Graph B under the isomorphisms mentioned by Conditions 1a and 2a.

This definition tells us some readily identifiable features of product graphs, and some ways that we can rule out some graphs as product graphs. The number of nodes must be non-prime since the number of nodes will equal \( MN \); therefore counting the nodes is one quick way to tell whether a graph is \textit{not} a product graph. (However, not every graph with a non-prime number of nodes is a product graph). Also, every node must be in the ARROW relation with at least two other nodes, and more than one arrow must be labelled with the same transformation (that is, no transformation can be unique).

So just by inspection we know that the network in Figure 2.1a is not a product network: it has a prime number of nodes, 7; one node (E) is in only one arrow relation; and the arrow label 1 appears only once. In contrast, the network in Figure 2.1b, which has very similar nodes and arrows, \textit{is} a product network: it has six nodes (a non-prime), every node is in the arrow relation with at least two others, each transformation is repeated at least twice, and each node participates in a copy of both sub-graphs given in Figure 2.1c. These conditions are necessary (although not sufficient) properties of product graphs and networks, so they they can help us identify possible candidates for such structures.
Section 2: Networks-of-Networks

Let us now examine the second structure discussed by Lewin in 9.5.5, the network-of-networks (parts (f) and (g) of Figure 1.4). This structure, like the product network, involves isomorphisms but it is explicitly rather than implicitly replicative, depicting hierarchies in the music by embedding networks within other networks.
Defining Networks-of-Networks:

Two structures are given below in Figure 2.2: (a), a network-of-networks NON whose lowest-level node contents are the ‘white-key’ pitch classes, and (b), an abstract graph-of-graphs GOG containing no lowest-level node contents.\(^{30}\) GOG is the graph of NON. In addition, the large-scale nodes in GOG, labelled GRAPH1 and GRAPH2, contain graphs that are isomorphic to one another. Despite appearances, the transformations of the large and small-scale graphs are different in kind: the transformation (-1) participating in the smaller graphs transposes pitch classes, while the transformation (-2) participating in the larger graph transposes networks/graphs. We can, however, understand these as analogous to the extent that we understand them to transform their respective objects in similar ways; this similarity, when it exists, can provide a means of relating the events of the larger and smaller structural levels. For this reason, various parts of this document will present special situations where it is beneficial to consider these as the same types of transformations.

Figure 2.2: Nested network and graph structures

\[ \begin{align*}
&\text{a) A network-of-networks, henceforth called “NON”} \\
&\text{b) An abstract graph-of-graphs, henceforth called “GOG”}
\end{align*} \]

\[ \begin{align*}
&\text{NETWORK1} \quad \text{NETWORK2} \\
&\text{GRAPH1} \quad \text{GRAPH2}
\end{align*} \]

---

\(^{30}\) While part (b) is technically a “network-of-graphs” since its large-scale nodes contain objects (the graphs GRAPH1 and GRAPH2), I will refer to it (and similar structures throughout this document) as a “graph-of-graphs” in order to emphasize that its lowest-level nodes (n1, n2, n1’, and n2’) do not contain objects.
Within the network-of-networks NON given above, each large-scale node has as its contents another network. These networks will be referred to by the labels NETWORK1 and NETWORK2, as shown on the figure above.

NETWORK1 is an ordered sextuple \((S, NODES, ARROW, SGP, TRANSIT, CONTENTS)\) where:

- \(S = \) the set of white-key pitch classes
- \(NODES = \{n_1, n_2\}\)
- \(ARROW = \{(n_1, n_2)\}\)
- \(SGP = \) the group of scale-step transpositions denoted by integers mod7
- \(TRANSIT\) associates each node-pair in \(ARROW\) with a member of \(SGP\) as follows: \((n_1, n_2) \rightarrow -1\)
- \(CONTENTS\) associates each node with a member of \(S\) as follows:
  \[n_1 \rightarrow A, \ n_2 \rightarrow G\]

The graph of this network, \(GRAPH1\), is the ordered quadruple \((NODES, ARROW, SGP, TRANSIT)\).

Similarly, NETWORK2 is an ordered sextuple \((S’, NODES’, ARROW’, SGP’, TRANSIT’, CONTENTS’)\) where:

- \(S’ = \) the white-key pitch classes
- \(NODES’ = \{n_1’, n_2’\}\)
- \(ARROW’ = \{(n_1’, n_2’)\}\)
• SGP’ is the group of scale-step transpositions denoted by integers mod7

• TRANSIT’ associates each node-pair in ARROW with a member of SGP as follows: \((n1’, n2’) \rightarrow -1\)

• CONTENTS’ associates each node with a member of S as follows:
  \(n1’ \rightarrow F, n2’ \rightarrow E\)

The graph of this network, GRAPH2, is the ordered quadruple (NODES’, ARROW’, SGP’, TRANSIT’).

Note especially our stipulation that the networks share the same families of objects and transformations. This will allow us to specify the relations of such networks-of-networks to product networks as we have just defined them.

In order for a network to be considered a network-of-networks, the contents of its nodes must belong to the “same family” of networks under a semigroup of transformations of networks. While we are quite familiar with families of objects that consist of pitch classes, chords, and other musical elements, it may not be readily apparent what constitutes a “family of networks.” In the simplest sense, a “family of networks” is a set of networks whose graphs are all isomorphic to one another, as per Lewin 9.4.2.\(^{31}\) Using the definition of isomorphism previously identified in Chapter 1, we can identify the isomorphism between small-scale networks for our specific example NON:

---

\(^{31}\) Lewin 1987, 199.
GRAPH1 and GRAPH2 are isomorphic because there exists a pair (NODEMAP, SGMAP) with the following properties (Lewin, 9.4.2):

A) NODEMAP is an isomorphism of (NODES, ARROW) with (NODES’, ARROW’).

B) SGMAP is an isomorphism of SGP with SGP’.

as follows:

(A) The isomorphism NODEMAP is specified simply by the following function table:

<table>
<thead>
<tr>
<th>NODES</th>
<th>NODES’</th>
</tr>
</thead>
<tbody>
<tr>
<td>n1</td>
<td>n1’</td>
</tr>
<tr>
<td>n2</td>
<td>n2’</td>
</tr>
</tbody>
</table>

This is a one-to-one map of one set of nodes onto the other, such that every pair of nodes (NODE₁, NODE₂) in GRAPH1 is in the ARROW relation if and only if (NODEMAP(NODE₁), NODEMAP(NODE₂)) is also in the ARROW relation, fulfilling Lewin’s criteria for isomorphic node/arrow systems. This in turn satisfies Lewin’s first condition for network isomorphism given above.

(B) SGMAP is a one-to-one mapping of the semigroup SGP to SGP’. Since we have stipulated that SGP’ = SGP, SGMAP maps the set of scale-step transpositions onto itself, that is, it is an “automorphism” of the scale-step-transposition group. Now consider the label γ on the arrow connecting NETWORK1 to NETWORK2 on Figure 2.2a. One might take this to signify SGMAP itself – that is, γ is the function that changes the arrow labels of NETWORK1 to the corresponding arrow labels of NETWORK2.

Such an interpretation, however, glosses over some important details. The label on the arrow from NETWORK1 to NETWORK2 means something different from the
arrow labels within these networks from node to node. Within each network, the arrow labels transform pitch classes. From NETWORK1 to NETWORK2, the arrow label evidently involves changes to the contents (pitch classes) of the small-scale nodes, but less apparently there is also a mapping that involves the labels of the small-scale arrows. To accommodate these details, let us understand the arrow label $\gamma$ of the network NON more broadly to change the entire contents of the NON node—the nodes, arrows, and contents of the network in that node, as well as the labels. Specifically, we will understand that SGMAP entails not only the isomorphism of SGP with SGP’, but also NODEMAP, an isomorphism of the node-arrow system (in the sense of Lewin 9.4.1), and a mapping of the contents of each node in NETWORK1 to the contents of the corresponding node in the isomorphic node-arrow system of NETWORK2.

This content mapping is especially intuitive in the special case in which not only SGP = SGP’, but also the automorphism SGMAP itself is a member of SGP. In Figure 2.2a, this would be the case if SGMAP were a scale-step transposition, for example, -2.

Lewin specifies the content mapping in this case as follows:

Supposing any network of pitch classes, then, and any pc operation $A$, there is a special relation between the given network and the new network obtained as follows:

1. Replace each pc $s$ of the given network by pc $A(s)$ in the new network.
2. Taking in turn each operation $X$ that labels an arrow of the old network, replace $X$ by the operation $AXA'$, to label the corresponding arrow of the new network. (Here $A'$ means the inverse operation of $A$.)

---

32 The mapping involves an automorphism that maps transpositions and other interval-preserving operations to themselves (that is, it acts as the identity), and thus in this example the transposition remains the same from NETWORK1 to NETWORK2.

Step 1 maps the contents of one network to another, the entailment mentioned above (to be discussed shortly). Step 2 details the automorphism mapping the group of transformations onto itself. To apply this to Figure 2.2, let $X = (-1)$, the transformation between the nodes of NETWORK1, and $A = (-2)$, the transformation $\gamma$ between the nodes of NON mapping the pitch classes of NETWORK1 to the corresponding pitch classes of NETWORK2. Then SGMAP changes the arrow label $(-1)$, in NETWORK1, as follows:

$$SGMAP(-1) = (-2)(-1)(2) = (5)(6)(2) = 13 \mod 7 = (-1)$$

This particular automorphism maps each transformation onto itself; however, this is not true for all automorphisms. Lewin calls this particular kind of network transformation a “network isomorphism,” distinct from other sorts of isographies such as positive and negative. He discusses these particular situations in detail, including their mathematical proofs, in "Klumpenhouwer Networks and Some Isographies that Involve Them."³⁴

Indeed, the automorphism presented above is not the only possible type of automorphism; those I employ in this document are examples of inner automorphisms. Outer automorphisms, featured prominently in Klumpenhouwer Network analysis, function quite differently. Both types “map a group onto itself so as to preserve group composition,”³⁵ but inner and outer automorphisms differ in what aspects of the network they affect. One difference is that outer automorphisms involve a mapping of a network’s transformations and arrows, while inner automorphisms additionally involve a mapping of node contents from one network to the other. Henry Klumpenhouwer describes this clearly when discussing the outer automorphism $<I_x>$; he states:

³⁴ Lewin 1990, 116-120.
³⁵ Klumpenhouwer 1998, 82.
[<Ix> automorphisms] cannot be shown to generate predictable relationships between the pitch-class contents of corresponding nodes in <Ix>-related networks. Rather, <Ix> automorphisms can only be conceived as operations on pitch-class operations or network arrow labels. Accordingly, they pose problems for those opposed to such high levels of abstraction in music theoretic work on the grounds that they fail to take into account important phenomenological considerations.  

Since the mapping of nodes and their contents is an important feature of replicative networks (particularly in the case of networks-of-networks), inner automorphisms – which can account for such object mappings – are more suitable to the theory presented in this document.

Returning to our specific case at hand, NODEMAP and SGMAP also determine how the objects (S and S’) of the networks and their node-mappings (CONTENTS and CONTENTS’) relate to one another. Recall that CONTENTS is a function mapping NODES into S (Lewin 9.3.1). Similarly, CONTENTS’ is a function mapping NODES’ into S’. The following tables specify these functions for the NON from Figure 2.2a.

<table>
<thead>
<tr>
<th>NODES</th>
<th>CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>n1</td>
<td>A</td>
</tr>
<tr>
<td>n2</td>
<td>G</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NODES’</th>
<th>CONTENTS’</th>
</tr>
</thead>
<tbody>
<tr>
<td>n1’</td>
<td>F</td>
</tr>
<tr>
<td>n2’</td>
<td>E</td>
</tr>
</tbody>
</table>

The change of contents entailed by SGMAP and NODEMAP in a network isomorphism can be construed as a function CONTENTSMAP that maps CONTENTS onto CONTENTS’:

---

Ibid., 89. Emphasis my own. For those interested in learning more about such automorphisms, the article also describes other ways in which inner and outer automorphisms differ, as well as the advantages and disadvantages of each.
Examining these results, one can see that CONTENTSMAP(CONTENTS(n1)) = CONTENTS’(n1’). Specifically, CONTENTSMAP(A) = F, moving the pitch class down two scale steps, that is, (-2). CONTENTSMAP(G) = E also moves the original pitch class down two scale steps. Expressed in another way,

\[
\text{CONTENTS'}(n1') = (-2)(\text{CONTENTS}(n1)) \text{ and } \text{CONTENTS'}(n2') = (-2)(\text{CONTENTS}(n2))
\]

That is, CONTENTSMAP has the same effect as the scale-step transposition (-2), which is what one assumed SGMAP to be. This is consistent with step 1 of the procedure Lewin describes for constructing a network isomorphism (given previously on page 52).

By construing \(\gamma\) in Figure 2.2 as such a network isomorphism, we understand the arrow labelled (-2) to say that NETWORK1 is transformed into NETWORK2 by applying the isomorphism (NODEMAP, SGMAP) such that the contents of each node are transformed by (-2) and the transformation X associated with each arrow undergoes the automorphism (-2)(X)(2). This fleshes out what Lewin means in 9.5.5 of GMIT when he says that “the arrows on [network] (f) labelled 1, - 1, and 0 transpose entire (e)-networks”.

In fact, further conclusions can be drawn from the same information. CONTENTSMAP maps the contents of each node in NETWORK1 to the contents of a distinct node in NETWORK2; one could alternatively understand this process as a transformation applied to the contents of NETWORK1 to generate the contents of NETWORK2. CONTENTSMAP and TRANSIT\text{\textsubscript{NON}}(N1, N2) have the same effect on

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>CONTENTS’</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>F</td>
</tr>
<tr>
<td>G</td>
<td>E</td>
</tr>
</tbody>
</table>
the contents of the nodes inside NETWORK1, even though they are conceptually
different functions (CONTENTSMAP applies a transformation to node contents, while
TRANSIT\textsubscript{NON} assigns a member of the semigroup of transformations to an arrow of the
large-scale graph). In both cases, the members of the semigroup of transformations act
on the small-scale node contents of NETWORK1, the elements (S, CONTENTS) of the
network.

To preserve intervals, the members of SGP, SGP', and SGP\textsubscript{NON} must commute.
While this is not as obvious as for product networks since there are no evident arrow
chains between the nodes of NETWORK1 and NETWORK2, the transformations from
node n1 through n2 to n2' of NON must produce the same result as the transformations
from node n1 through n1' to n2' for the network to be valid. That is,
\[(\text{TRANSIT}(n1, n2))(\text{CONTENTSMAP}) = (\text{CONTENTSMAP})(\text{TRANSIT}(n1', n2')).\]

Figure 2.3 illustrates these mappings on the network-of-networks NON.

Figure 2.3: CONTENTSMAP and TRANSIT mappings within the network-of-networks NON
The definitions above have helped to clarify the isomorphisms and transformational relationships between the nodes of NON. We are now prepared to define NON as a network-of-networks:

NON is an ordered sextuple \( (S_{\text{NON}}, \text{NODES}_{\text{NON}}, \text{ARROW}_{\text{NON}}, \text{SGP}_{\text{NON}}, \text{TRANSIT}_{\text{NON}}, \text{CONTENTS}_{\text{NON}}) \):

- \( S_{\text{NON}} \) = the set of all networks that can be produced by transforming NETWORK1 by the members of SGP_{\text{NON}} (these will be isographic to NETWORK1 and NETWORK2)
- \( \text{NODES}_{\text{NON}} = \{N1, N2\} \)
- \( \text{ARROW}_{\text{NON}} = \{(N1, N2)\} \), so \((\text{NODES}_{\text{NON}}, \text{ARROW}_{\text{NON}})\) is a node/arrow system
- \( \text{SGP}_{\text{NON}} \) is the group of scale-step transpositions denoted by integers mod7
- \( \text{TRANSIT}_{\text{NON}} \) associates each arrow with a member of SGP_{\text{NON}} as follows:
  \((N1, N2) \rightarrow -2\)
- \( \text{CONTENTS}_{\text{NON}} \) associates each node with a member of \( S_{\text{NON}} \) as follows:
  \( N1 \rightarrow \text{NETWORK1}, N2 \rightarrow \text{NETWORK2} \)

The graph of this network, GOG, is the ordered quadruple \((\text{NODES}_{\text{NON}}, \text{ARROW}_{\text{NON}}, \text{SGP}_{\text{NON}}, \text{TRANSIT}_{\text{NON}})\).
Abstract Version:

Let us now define a network-of-networks that will comport with our definition of product networks, based on the observations above.

**Definition:**

G, a network, is an ordered sextuple (S, NODES, ARROW, SGP, TRANSIT, CONTENTS). It is a **network-of-networks** iff:

a) The contents of NODES = {A, B, ..., Z}. {A, B, ..., Z} are networks.

b) The semigroups of networks {A, B, ..., Z} are identical.

c) The networks {A, B, ..., Z} are isomorphic to one another; that is, for any two networks X and Y (NODES\(_X\) = \{x\(_1\), x\(_2\), ..., x\(_M\)\} and NODES\(_Y\) = \{y\(_1\), y\(_2\), ..., y\(_M\)\}) there exists a pair (NODEMAP\(_{XY}\), SGMAP\(_{XY}\)) such that NODEMAP\(_{XY}(x\(_i\)) = (y\(_i\)) for 1 ≤ i ≤ M and SGMAP\(_{XY}\) is the identity.

d) For any two networks X and Y, CONTENTS\(_X\) is a function mapping NODES\(_X\) into S\(_X\) (the set of objects for network X) and CONTENTS\(_Y\) is a function mapping NODES\(_Y\) into S\(_Y\) (the set of objects for network Y). According to NODEMAP\(_{XY}\) as above, CONTENTSMAP\(_{XY}\) is a function mapping S\(_X\) onto S\(_Y\).

Given these constraints,

```
CONTENTSMAP\(_{XY}\) = TRANSIT(<X, Y>).
```

In other words, the transformation that maps the contents of a particular node in network X to its corresponding node in Y will be identical to the transformation associated with the arrow connecting nodes X and Y in the network-of-networks.

e) SGP and \{SGP\(_A\), SGP\(_B\), ..., SGP\(_Z\)\} must commute in order for a valid network to be produced under conditions (c) and (d) above.
To review, we have placed certain restrictions on networks-of-networks. We require all transformations that participate in a network-of-networks to be commutative (and consequently interval-preserving) so that any chain of transformations between two objects produces the same result, fulfilling the path-consistency condition. We also require that the transformation mapping the objects of one sub-network to its corresponding object in another sub-network be identical to the transformation linking these sub-networks in the larger-scale graph. And because we wish to identify repetition as the highest degree of similarity, we require that nodal sub-networks map to one another by an isomorphism that preserves their graphs (in other words, because the graphs are identical, corresponding arrows in each network have the same labels). These restrictions will allow us to reinterpret any network-of-networks as a product network and vice-versa.

Section 3: Other Replicative Structures

A third type of replicative network will be incorporated into several analyses in this document. This is the sequential network, which involves the repetition of a single transformation such that all transformations in the network are identical. We could conceptualize this network as having a single two-node sub-graph that is replicated and overlapped, as given in Figure 2.4b: the sub-graph incorporating nodes N₁ and N₂ is isomorphic to sub-graph (a); another sub-graph incorporating nodes N₂ and N₃ is isomorphic to sub-graph (a) and overlaps with the previous by one node, and so forth. The result depicts a sequence. This structure is much simpler than both the product
network and network-of-networks because it incorporates a single transformation, and as such is often employed as a sub-graph within other replicative structures.

**Figure 2.4: The derivation of a sequential graph**

a) **Sub-graph:**

b) **Sequential graph derived from the sub-graph in (a):**

Continuing the formalisms established for product networks and networks-of-networks, we can briefly define the structure of such a graph.

Given a sequential graph with \( n \) nodes and transformation \( \alpha \), its properties are as follows:

**NODES:** \( \{N_1, N_2, \ldots, N_n\} \)

**ARROW:** \( \{(N_1, N_2), (N_2, N_3), \ldots, (N_{n-1}, N_n)\} \)

**SGP:** \( \alpha \) and its powers \( (\alpha^2, \alpha^3, \ldots, \alpha^y) \) where \( y \) is any positive integer

**TRANSIT:** \( \{(N_1, N_2) \rightarrow \alpha, (N_2, N_3) \rightarrow \alpha, \ldots, (N_{n-1}, N_n) \rightarrow \alpha\} \)
Section 4: Relating Replicative Networks

The preceding discussion has outlined some general differences between product networks and networks-of-networks; for example, networks-of-networks have networks as their node contents at a minimum of one structural level, whereas product networks need not. As well, for a network-of-networks the CONTENTSMAP function from one small-scale network to another is the same transformation as the TRANSIT function between the nodes containing these networks within the large-scale network.

Another difference between these two structures is the manner by which each combines sub-graphs. Product networks embed at least two graph types; however, these two graph types may have the same structure (that is, they may be identical). Each node must participate in one copy of each graph type. The ARROW relation of one graph type will be equivalent to the NODEMAP relation of the other graph type. A network-of-networks must also have at least two sub-graph types (which may or may not be isographic to one another), but in this case one type functions as an object, the smaller-scale network, while the other functions as the ‘container’ for the objects, the larger-scale network.

These features notwithstanding, product networks and networks-of-networks also involve similar elements: replication, isomorphisms among their components, and commutative semigroups of transformations. Because of these commonalities, the two structures can be formed from one another via a simple algorithm.
To form two different networks-of-networks from a two-dimensional product network:\textsuperscript{37}

1. a) Partition each ‘row’ of the product network into mutually exclusive sub-networks, each of which should have the same number of nodes. Any sub-network $x$ should be isomorphic with any other sub-network $y$ via the relation NODEMAP$_{xy}$, a one-to-one map of one set of nodes onto another such that every pair of nodes $(N_1, N_2)$ in $x$ is in the ARROW$_x$ relation if and only if $(\text{NODEMAP}_{xy}(N_1), \text{NODEMAP}_{xy}(N_2))$ is also in the arrow relation. (The semigroups of all sub-networks will be identical since they are each identical to the semigroup of the original product network.)

b) Partition each ‘column’ of the original product network into a second form of isomorphic, mutually-exclusive networks, each of which should have the same number of nodes (analogous to the process described in (a)). These new sub-networks will include only the members of ARROW$_{PN}$ not in ARROW$_x$, ARROW$_y$, or any of their isomorphic sub-networks.

2. a) Assign the graph of the isomorphic networks derived in step 1a as the large-scale graph of the networks-of-networks.

b) Place each network of step 1b into a node of the large-scale graph such that the transformation assigned to the large-scale arrow matches the function required to map one sub-network to another (that is, the CONTENTSMAP function that maps one sub-network to another in the network-of-networks should be equal to the TRANSIT function between the same two nodes).

\textsuperscript{37} This algorithm can be adapted to a three- (or more) dimensional product network; however, given that three-dimensional product networks can be interpreted into networks-of-networks in a variety of different ways, the algorithm would be quite lengthy (and perhaps unwieldy). For this reason it is not presented here.
3. a) Assign the graph of the isomorphic networks derived in step 1b as the large-scale graph of the networks-of-networks.

b) Place each network of step 1a into a node of the large-scale graph such that the transformation assigned to the large-scale arrow matches the function required to map one sub-network to another (that is, the CONTENTSMAP function that maps one sub-network to another in the network-of-networks should be equal to the TRANSIT function between the same two nodes).

The possibility of interpreting a single product network as two different networks-of-networks is one of the main attractions of product networks. Consequently, I will demonstrate the analytical advantages of this flexibility throughout the analytical chapters.

Similarly, to form a product network from a network-of-networks:

1) Take the network inside a large-scale node ‘out of’ the node by constructing an identical network. Do the same for the remaining large-scale nodes, resulting in a disconnected set of isographic networks. Each node in one of these networks corresponds to the input or output of the NODEMAP function in the source network-of-networks.

2) Take each pair of nodes in this disconnected set of networks that corresponds to the node pair originally involved in NODEMAP (the mapping of the contents of a node from one nodal network to the corresponding node in a second nodal
network), and connect them with an arrow whose TRANSIT is equal to CONTENTSMAP in the source network-of-networks.

It should be noted that a definition for sequential networks is included in this chapter because these structures are often incorporated into product networks and networks-of-networks; they cannot participate in a similar reconfiguration since they do not involve two (or more) distinct sub-graph types.

As a final note, many of the analyses incorporating network-of-networks in this document will involve isomorphisms between the larger-scale network and its sub-networks. This is not a requirement for networks-of-networks, but it presents a means of formally describing replication at different structural levels in the music – that is, self-replication. Much of Lewin’s discussion of Klumpenhouwer Networks revolves around this idea, particularly the concept that a higher-level network acts prolongationally; he states:

In the course of this work I began to notice the recursive potentialities of the theoretical apparatus. When a lower-level Klumpenhouwer Network is interpreting a chord, and a higher-level network-of-networks is interpreting a progression of chords (more precisely, of chord-interpretations), I noted that one could conceive of the higher-level network as ‘prolonging’ the lower-level one, particularly when the given chord is part of the given progression.38

While K-nets will not be featured in this document due to the technical constraints established in this chapter, Lewin’s idea is nonetheless relevant for the analyses of the following chapters. Let us now present some analytical examples incorporating replicative and self-replicative networks.

38 Ibid., 115.
Chapter 3 : Analyses of Contemporary Québécois Works

The previous chapter formally defined three types of replicative network structures as a combination of identical sub-graphs: the sequential network, the product network, and the network-of-networks. The current chapter will show the practical consequences of these formal distinctions by clarifying what type of information each network type provides in an analysis, how the structures differ from one another, and why one is more suitable than another for any particular analysis. Through brief analyses, I will clarify the analytical differences between these replicative network structures, and propose reasons for choosing one network format over another.

Because these networks model repetition, they are well-suited to depicting repeated motives. The analytical examples of this chapter thus focus on a repertoire that clearly involves the repetition and development of smaller motives: a selection of works by Québécois composers dating from the 1960s to the present. In these examples, the transformations that link repetitions of smaller motives are often replicated between larger-scale motives, collections, sequences, and so forth; consequently, networks-of-networks and product networks will provide different means of discussing such issues as object groupings and hierarchy. This document is the first to analyse these Québécois works – in fact, there are few analyses of Québécois art music at all, particularly in English.  

The excerpt from Pépin’s *Monade VI – Réseaux*, Cahier 7 given in **Figure 3.1** (and also briefly examined in **Figure 1.1** of Chapter 1) demonstrates this type of small-to-large scale repetition. Its structure suggests, indeed, that a replicative network might be an appropriate analytical tool. Its $T_5$ recursion was discussed in Chapter 1. There are many possible ways to represent this transformational structure. **Figure 3.2** shows three of them: part (a) involves a network-of-networks, part (b) is a product network, and part (c) combines a network-of-networks with a product network. While the nodes of these networks appear to represent pitch classes, they in fact represent the pitch-classes of specific events; that is, certain pitch classes are repeated in order to more specifically detail how they appear in relation to other pitch classes (that is, pitch classes that are repeated in the score are repeated within the network). A simple sequential $T_5$ network is also a possible analytical structure, but as this would not depict the wealth of transformational paths or groupings suggested by the present analyses, it will not be further discussed here.

**Figure 3.1**: Pépin, *Monade VI – Réseaux*, Cahier 7, fourth system

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examine general stylistic traits of Pépin and Hétu, respectively. Pépin briefly discusses several of his own works in his autobiography *Piccoletta* (Montreal: Les éditions Triptyque, 2006), and Hétu explains his compositional techniques in the brief article “Pour un style composite,” *Vie Musicale* 11 (1969), 12-15. Several works of Garant have been analysed (see Alepin 2000 for a listing); however, none touch on the musical relationships presented in this chapter.
Figure 3.2: Network representations of the motive boxed in Figure 3.1

a) A network-of-networks

b) A product network

c) A combination product network/network-of-networks

Let us consider the distinct meanings these transformational structures convey.

The first, network (a), presents a large-scale two-node $T_5$ network whose nodes contain isomorphic copies of itself at two levels. This network-of-networks-of-networks asserts that a single process, transposition by 5 semitones, generates all the material of the boxed
motives – simultaneous pitch classes, consecutive dyads, and consecutive four-note
groups – by a replication at different structural levels of the music. It best expresses the recursive structure. Network (b), a product network, also represents the passage in terms of a repeated application of $T_5$, but not recursively because all three applications of $T_5$ occur at a single structural level (that is, on a single type of object, pitch classes).

Network-of-networks (c) is a hybrid of the two types, suggesting that all the notes in each tetrachord are on the same structural level, forming a product network, but that the later tetrachord network is isomorphic to the earlier one under $T_5$. All three analyses are well-formed networks, but each suggests a different interpretation of the passage in question. Network (a) asserts two large-scale objects, each containing two smaller-scale objects, which in turn each contain two smallest-scale objects; (b) asserts eight objects, all at a single structural level; and (c) asserts two large-scale objects each containing four smaller-scale objects. Because networks (a) and (c) involve objects at multiple levels, they specify a hierarchy among the events of the passage; while we might understand network (b) to imply a similar hierarchy through its three dimensions, there are no elements in the network structure that suggest the events (and objects) of one dimension occur at a higher level than those of another. If we wish to consider (b) as a hierarchical structure, we can only identify it as implicitly (rather than explicitly) hierarchical.\footnote{One perceptual difficulty of network (b) is that it seems to disregard many elements of temporality: whether pitch classes occur simultaneously or melodically, whether they occur consecutively or over a longer time span, and so forth is not indicated by the node-arrow structure nor by the transformations of the product network. The analyst may choose to indicate these elements in the visual layout of the network; however, this is not a requirement for producing product networks since the visual layout is not a formal element of the network’s structure. See pages 33-34 of Chapter 2 for a further discussion of this subject. Network (b) does reflect a hearing of the passage that identifies various perfect fourths in the passage, even those between pitch classes in different, temporally-separated structures. My thanks to William Benjamin for this observation.}
A further distinction between the three networks of Figure 3.2 is that some networks suggest paths that the others do not. Network (b), for example, asserts connections between Bb and Eb in the first tetrachord, but also between this Bb and the Eb of the second tetrachord – indeed, there is a path from the Bb to every other node of this network. Network (c), on the other hand, asserts a connection between Bb and Eb in the same tetrachord, but not, at the same level, between individual pitch classes from one tetrachord to the next. None of these connections occurs in Network (a), but the network nonetheless asserts a structural parallelism among the corresponding events: analogous arrows have the same orientation (that is, from the earlier to the later pitch for consecutive events and from the lower to the higher pitch for simultaneous ones) and are associated with the same transformations (T5). Moreover, Network (a) depicts motivic similarities between its structural levels that the others do not because it is constructed entirely from isomorphic graphs of a single type.

Later in the same work, a longer motive recalls the replicative application of T5 identified in Figure 3.1 and Figure 3.2, and its structure helps us to choose between Networks (a), (b), and (c). Figure 3.3 indicates repetitions of the longer motive (a restatement of the work’s opening motive) in the twelfth to fourteenth systems of the movement. Each repetition is generated by the same transformation heard in conjunction with the earlier passage, T5, but in reverse order, in the sense shown by the network in Figure 3.4a. It is constructed from the same two-node sub-graph as the networks in Figure 3.2; however, here these sub-graphs combine by sharing one node, forming a sequential network rather than a product network or a network-of-networks. In spite of

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41 Note that the arrow sequence and direction do not necessarily correspond to the temporal progression of the passage; rather, they indicate a particular transformational perspective of the music chosen by the analyst.
their different appearances, the two-fold repetition of T₅ presented in the new analysis is actually embedded in Networks (b) and (c) of the earlier analysis. The layout presented in **Figure 3.4b** models that of **Figure 3.2b and c** more closely in order to demonstrate this similarity. More specifically, the graph of **Figure 3.4** is identical to the sub-graphs of **Figure 3.2b** and **c** that involve the nodes <F, Bb, Eb> (beginning at F, moving up to Bb, and right to Eb) and <Bb, Eb, Ab> (beginning at the right-most Bb, moving up to Eb, and right to Ab).⁴² Thus the new passage suggests that Networks (b) and (c) are preferable over Network (a) since the former are capable of interpreting both passages.

**Figure 3.3: Pépin, Monade VI – Réseaux, Cahier 7, 12th-14th systems**

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⁴² Note that in spite of the visual re-positioning of the nodes in **Figure 3.4b**, the objects of this network still occur at a single structural level. The visual layout is selected by the analyst and does not necessarily reflect object groupings, transformation groupings, structural levels, or other elements of the music.
Passages involving such clear sequences are best suited to be represented with sequential networks; as examined in the previous discussion, similarities between these longer networks and the internal structure of smaller, related motives can be depicted through common sub-graphs. For example, the works of Jacques Hétu often present sequences whose transpositional levels are reminiscent of the structure of smaller-scale motives. An especially good example is the *Prélude*, op. 24, measures 16-20 of which are given in Figure 3.5. In this passage, several features are immediately apparent: each segment sequences the preceding one at $T_8$ (although some doubled pitch classes are omitted upon repetition), and each segment ends with an augmented triad (boxed on the
score). **Figure 3.6** illustrates the transformations of these two characteristic elements.\(^{43}\)

In network (a), the augmented triad is generated by a repeated application of \(T_8\) to any one of its pitch classes. In network (b), the \(T_8\) transformations between the segments of measures 16-20 are indicated. The identity of graphs (a) and (b) asserts that the progression of the main motivic group is modelled after the structure of the augmented triad. The association between these two structures is further substantiated by the repetition of the augmented triad at the end of each statement of the motivic group within measures 16-20: because the transposition that structures the augmented triad corresponds to the transposition that participates in the sequence, this triad is formed from the same pitch classes upon each sequential repetition.

**Figure 3.5: Hétu, Prélude, measures 16-20**

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\(^{43}\) Note, however, that the G and C that conclude segment 2 in the lowest voice are not \(T_8\) transposes of the corresponding pitch classes in segment 1; if the transposition were strictly followed, Bb and E would conclude segment 2. Since the given pitches G and C occur on the same staff lines in the treble clef as the expected pitches Bb and E do in the bass clef, it is likely this is an editorial error on the score.
Figure 3.6: Hétu, *Prélude*, similar network structures between representations of the augmented triad and instances of the main motivic group

a) The augmented triad

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>T₈</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G#/Ab</td>
<td>T₈</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td></td>
</tr>
</tbody>
</table>

b) The pcs of the main motivic groups

<table>
<thead>
<tr>
<th></th>
<th>pcs of mm. 16-17</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pcs of mm. 17-18</td>
</tr>
<tr>
<td></td>
<td>pcs of mm. 18-20</td>
</tr>
</tbody>
</table>

Because a similar process occurs in measures 25-28 of the work, given in Figure 3.7, it is also well-suited for representation as a sequential network. Three augmented triads featured in this passage are outlined on the figure: the recurring \{\text{Ab, C, E}\} and \{\text{Eb, G, B}\} on the last quarter duration of each measure, and the \{\text{D, F#, Bb}\} arpeggiated through whole-note durations in the bass. Each entire measure is transformed by \text{T₈} or \text{T₁₈} (shown with brackets above the figure) depending upon how we interpret the direction of each transformation; \text{T₁₈} is the same transformation that generates the trichord in Figure 3.6, while \text{T₄} is its inverse. As well, each sequential repetition gradually omits duplicated pitch classes, steadily thinning the overall texture. The augmented trichord \{\text{D, F#, Bb}\} results from the pitch classes on the first beat of each measure, highlighted musically through accents and duration and connected with a solid line below the figure. Figure 3.8 presents analyses of the augmented triads and sequential transpositions. These networks, like those of Figure 3.6, are trivially isomorphic to one another and to those of Figure 3.6. Because the transformations in
measures 25-28 occur in the opposite (temporal) direction to those of measures 16-20, the analysis of Figure 3.8 suggests that the later passage presents a reversal of the structural processes observed in the earlier passage (Figure 3.6).

**Figure 3.7: Hétu, Prélude, measures 25-28**

(gradually omits more and more pitches)

![Diagram of arpeggiation of augmented trichord]

**Figure 3.8: Hétu, Prélude, similar network structures between representations of the augmented triad and instances of the main motivic group in measures 25-28**

**a) The augmented triad**

C

\[ T_8 \]

G#/Ab

\[ T_8 \]

E

**b) The pitch classes of the main motivic groups**

pcs of m. 27

\[ T_8 \]

pcs of m. 26

\[ T_8 \]

pcs of m. 25
Sequential networks can also depict larger-scale events in Hétu’s *Toccata*, suggesting that similar processes occur among both the small- and large-scale elements. **Figure 3.9** gives the first eighteen measures of this work. Measures 1-4 are sequenced three times by T₅, with the last two statements abbreviated and preceded by new motivic material (deviations from the sequence are circled on the score). **Figure 3.10a** gives an analysis of this large-scale sequence. Each node represents the first two measures of each sequential re-statement (corresponding to the brackets on the score). **Figure 3.10b** gives the graph of this network. Portions of the passage have the same transformational structure as the large-scale sequence, and thus can be analyzed via the model presented in (b). For example, **Figure 3.10c** depicts transformations between unordered augmented trichords in the upper staff of measures 9-10, the new motivic material between the second and third legs of the sequence. As in the large-scale sequence, each object is transposed thrice by T₅.⁴⁴ The same process is further developed in measures 12-13 (depicted in **Figure 3.10d**), just prior to the entry of the third leg of the sequence. Here, both ordered interval-class 3 dyads and single pitch classes are transposed thrice by T₅, saturating this brief passage with the transformation characteristic of the larger sequence. The isography seen between the four examples depicts the same process of T₅ sequencing, but with a different choice of object for each.

---

⁴⁴ Because of the symmetrical structure of this set, three different transformations among unordered pitch classes are possible: T₅, T₉, and T₁. While T₅ does not correspond to the pitch transposition among these chords (which is T₃: its pitch-class equivalent is T₀), I am choosing T₅ for this analysis in order to demonstrate a connection between measures 9-10 and the surrounding music (which also articulate T₃). T₉ will be used in **Figure 3.11** to model the same passage.
Figure 3.9: Hétu, *Toccata*, op. 1, measures 1-18
Figure 3.10: Sequences and near-sequences in measures 1-18 of Hétu’s *Toccata*, op. 1

a) Large-scale sequence:

| mm. 1, 2 | T₅ | mm. 5, 6 | T₅ | mm. 13, 14 | T₅ | mm. 17, 18 |

b) Model:

|                | T₅ |                | T₅ |                | T₅ |                |

c) Mm. 9-10, right hand:

| {Eb, G, B}     | T₅ | {Ab, C, E}    | T₅ | {Db, F, A}    | T₅ | {Gb, Bb, D}   |

d) Abbreviated re-statement (mm. 12-13):

| Eb            | T₅ | Ab            | T₅ | Db            | T₅ | F#            |
| <G, E>        | T₅ | <C, A>        | T₅ | <F, D>        | T₅ | <Bb, G>      |

In addition to these obvious instances, T₅ occurs in more subtle ways. Grey arrows on Figure 3.9 show how it is established on a smaller scale in the first measure: in the lower system T₅ is heard from the lowest to highest pitch in each dyad, while in the upper system T₅ generates the second dyad from the first. It also has a distinctive
“signature,”\footnote{In other words, the characteristic effect of a particular transformation. Lewin uses the term ‘signature’ in his analysis of Stockhausen’s Klavierstück III (Chapter 2 of Musical Form and Transformation) to refer to the common chromatic tetrachord heard between J-related sets (this analysis was previously examined on page 23 of my introduction). See the discussion on page 32 of Lewin 1993 for further information.} keeping many pitch classes invariant between measures 1-4 and 5-8: the second dyad of the right hand in measure 1 becomes the first dyad of the right hand in measure 5, and three of the four pitch classes of the left hand in measure 1 return in measure 5, now in a different order.

In spite of the emphasis on $T_5$ throughout the passage, other network interpretations are possible within measures 1-18. For example, the pitch classes of the new material in the upper staff of measures 9-10, analysed in Figure 3.10c, are given a second interpretation in Figure 3.11. Whereas the previous analysis emphasized the $T_5$ transposition between trichords, the new analysis suggests hearing the passage as a series of $T_4$-generated augmented chords that are transposed by $T_9$, an interpretation more consistent with elements of pitch, register, and rhythm in the score. The network-of-networks formed from this combination is then transformed by $T_{10}$. The high-level $T_{10}$ two-node graph is attractive because it highlights transformations that are also characteristic of this piece, in the sense that they have been heard in close proximity to measures 9-10: between the left-hand dyads beginning in measure 1, and from one dyad to another in the left hand during measure 11. The augmented chords that are generated as simultaneities are much more aurally evident than the $T_5$ transformation between trichords, which does not take ordering or voice-leading into account.
In addition to sequential repetition, Hétu often employs motivic repetition at multiple levels, a procedure that is better represented with a network-of-networks than with a sequential network. The opening of his *Sonate pour Piano* (first movement), measures 1-6 of which are given in Figure 3.12, is a good example. Except for the material in the dotted boxes, the music in the two hands is pitch-symmetric first about D4 and then D3. The circled pitches are an instance of the movement’s main motive, a mirrored statement of the symmetric interval series \(<5, 1, 5>\). It is labelled (a), and two other motives are labelled (b) and (c).
Figure 3.12: Hétu, *Sonate pour Piano*, first movement, measures 1-6

I₄: (symmetric about D)

![Musical notation](image)

Figure 3.13: Analytical networks for Hétu, *Sonate pour piano*, first movement, mm. 1-6

a) motive in m. 2

\[
\begin{align*}
E \xrightarrow{T₁} A & \quad T₆ \xrightarrow{T₄} Bb \quad T₆ \xrightarrow{T₄} Eb \\
C \xrightarrow{T₆} G & \quad T₆ \xrightarrow{T₄} F# \quad T₄ \xrightarrow{T₄} C#
\end{align*}
\]

b) motive in m. 3

\[
\begin{align*}
Ab \xrightarrow{T₁} G & \quad T₃ \xrightarrow{T₄} C# \quad T₃ \xrightarrow{T₄} C \\
F \xrightarrow{T₃} E & \quad T₃ \xrightarrow{T₄} A \quad T₃ \xrightarrow{T₄} Bb \\
B \xrightarrow{T₃} C & \quad T₃ \xrightarrow{T₄} G \quad T₄ \xrightarrow{T₄} F#
\end{align*}
\]

c) motive in m. 5

\[
\begin{align*}
C \xrightarrow{T₄} G & \quad T₆ \xrightarrow{T₄} F# \quad T₄ \xrightarrow{T₄} C#
\end{align*}
\]

d) an alternate interpretation of the motives in mm. 2 and 5

\[
\begin{align*}
E \xrightarrow{T₁} A & \quad T₆ \xrightarrow{T₄} Bb \quad T₆ \xrightarrow{T₄} Eb \\
C \xrightarrow{T₄} G & \quad T₆ \xrightarrow{T₄} F# \quad T₄ \xrightarrow{T₄} C#
\end{align*}
\]

\[
\begin{align*}
F \xrightarrow{T₄} E & \quad T₃ \xrightarrow{T₄} A \quad T₃ \xrightarrow{T₄} Bb \\
B \xrightarrow{T₃} C & \quad T₃ \xrightarrow{T₄} G \quad T₄ \xrightarrow{T₄} F#
\end{align*}
\]
Figure 3.13 depicts these motives transformationally via networks-of-networks, an interpretation that shows similarities between two-node sub-graphs at different levels and that allows us to assert exactly how (b) and (c) are similar to (a) (the networks in (d) will be discussed shortly). Each of the first three networks feature at least two levels of sub-graphs – one incorporating pitch-class objects and the other dyad-network objects. Each also includes both a two-node $T_5$ network and a two-node $T_1$ or $T_6$ network (although the pitch classes are listed from left to right in chronological order, the arrow directions have been adjusted to emphasize the $T_5/T_1$ aspects of the motive rather than their temporal progression). Networks (a) and (c) bear a deeper resemblance in that each features a two-node large-scale transpositional network corresponding to the transposition from the upper to the lower voice. Because the motives both incorporate $T_5$ sub-graphs (albeit at different levels), they can be considered variants of one another.

Networks (b) and (c) feature the $T_1$ graph at the level of pitch class and the $T_5$ graph at the level of dyads; Network (a), on the other hand, incorporates the $T_5$ graph at the level of pitch class, but not the $T_1$ graph (although $T_1$ and its inverse, $T_{11}$, are implied by the combination of this network’s transformations, $T_5 + T_6 = T_{11}$). The motive here, then, is a two-node $T_5$ graph. $T_5$ is not merely a melodic interval between pitch classes, but rather a transformation heard at multiple structural levels, generating larger-scale units (such as the dyads of Network (a)). A secondary transformational motive, the two-node $T_1$ network, is often associated with the $T_5$ motive.

Such a comparison between multiple structural levels would not be possible if we were to map these sets via product networks. The objects of a product network occur at a single structural level, and thus do not explicitly specify node groupings; for example, a
product network would depict motive (a) as a collection of paths among individual pitch
classes rather than a repetition of the two-node T$_5$ motive depicted in Figure 3.13a. Nor
would a product network single out the two-node T$_5$ graph as the motive – this type of
structure often presents more possible paths than would an analogous network-of-
networks, and does not by design favour any one of its transformations over another.

Looking at the score, however, it is apparent that motives (a) and (c) present
contrary-motion versions of the same motive; the T$_9$ and T$_2$ transformations indicated
between the large-scale networks of (a) and (c) do not make this relationship clear.
Motives (a) and (c) could just as easily be depicted as in Figure 3.13d: the right and left
hands relate by I$_4$, and consequently each of the transpositional relationships in the right
hand is replaced by its inverse in the network representing the left hand music. Indeed,
the use of inversion in these networks better captures the overall symmetry of the passage
(although they no longer show the dyadic grouping structure present in (a) and (c)). The
difference between the two possible interpretations is that (a) and (c) understand the
right-to-left hand relationship as a retrograde in which each transformation is reversed,
whereas (d) understands the right-to-left hand relationship as an inversion in which each
pitch class is mirrored about a specific axis of symmetry. The former represents the
retrograde by retaining the same transformations but altering the arrow direction within
the node-arrow structure of the small- and middle-scale networks (not directly indicated
by the large-scale mappings T$_9$ and T$_2$). The latter, on the other hand, represents the
mirroring within its network-of-networks structure by retaining the node-arrow systems
of the small- and middle-scale sub-networks but replacing each transformation with its
inverse (generated by the I$_4$ mapping between large-scale node contents). It should be
noted that network (d) does not involve the isomorphisms nor the definitions established in Chapter 2: it can be flattened into a non-hierarchical network but not a product network. The choice of representation will therefore depend upon whether the analyst is more interested in expressing the symmetrical structure (in which case he will employ the non-commutative network (d)) or the motivic replication of the passage (in which case he will employ the commutative networks-of-networks (a) and (c)).

While Figure 3.13 depicts the main motive as generated from two-node dyads, this is not the only possible analysis. Indeed, for the return of this motive in measures 17-19 (given in Figure 3.14), a different interpretation better emphasizes the connections between small- and large-scale events. To demonstrate these connections, Figure 3.15 interprets the six intervals in this tetrachord as particular transformations (T₁, T₆, and T₇, and T₀), namely those that structure the ordered 0167 tetrachords of measures 17-18 that manifest the interval series of the main motive (<5, 1, 5>). Figure 3.16a presents a network that depicts the same transformations between successive tetrachords. Thus the transformation between pitch classes of the 0167 set act on a higher level between the tetrachords themselves. Another transformation, T₉, generates the tetrachords of the lower staff from those of the upper, recalling the similarly structuring T₉ of Figure 3.13a.

Figure 3.14: Hétu, Sonate pour Piano, first movement, measures 17-19
Figure 3.15: Transformations involved in SC 0167

```
6  \rightarrow  T_1 \rightarrow  7 \\
T_6 \downarrow      \downarrow \downarrow \downarrow \downarrow \downarrow \\
T_7 \downarrow   \downarrow   \downarrow   \downarrow   \downarrow \\
0  \rightarrow  T_1 \rightarrow  1
```

Figure 3.16: Transformations between tetrachords in measures 17-19 of Hétu's *Sonate pour piano*, first movement.

a) Transformations between ordered tetrachords

*Measure 17:*

\[
\begin{array}{c|c|c|c}
\text{Eb, Bb, A, E} & \text{G, D, C#, G#} & \text{Bb, E, B} & \text{D, A, Ab, Eb} \\
T_9 \downarrow & T_7 \downarrow & T_6 \downarrow & T_9 \downarrow \\
T_1 \downarrow & T_9 \downarrow & T_6 \downarrow & T_9 \downarrow \\
\text{C, G, F#, C#} & \text{E, B, Bb, F} & \text{G, D, C#, G#} & \text{B, F#, F, C} \\
\end{array}
\]

*Measure 18:*

\[
\begin{array}{c|c|c|c}
\text{B, F#, F, C} & \text{Eb, Bb, A, E} & \text{G, D, C#, G#} & \text{B, F#, F, C} \\
T_6 \downarrow & T_9 \downarrow & T_6 \downarrow & T_9 \downarrow \\
T_1 \downarrow & T_9 \downarrow & T_6 \downarrow & T_9 \downarrow \\
\text{Ab, Eb, D, A} & \text{C, G, F#, C#} & \text{E, B, Bb, F} & \text{Ab, Eb, D, A} \\
\end{array}
\]
b) Transformations between unordered tetrachords

The analysis of Figure 3.16a clearly shows a duplication of the transformations seen in Figure 3.15 (and thus a relation between small- and large-scale transformations). However, it is perhaps not the most concise means of depicting events in the passage, nor does it model other recurring transformations that occur within the passage. Figure 3.16b, a product network, is a more abstract interpretation on unordered SC 0167 tetrachords that thereby eliminates node-content duplication, combining a sub-graph whose three nodes are each transformed by $T_4$ (neighbouring tetrachords) with a two-node sub-graph whose nodes relate via $T_3$ (simultaneous tetrachords). The two-measure passage can be understood as two clockwise rotations around the prism: the rotation begins with the tetrachord pair $\{Eb, Bb, A, E\}/\{C, G, F\#, C\#\}$ on the rear edge of the rectangular base, continuing up the left legs of the triangles to the pair $\{G, D, C\#, G\#\}/\{E, B, Bb, F\}$ (repeated on the third quarter beat), and then down the right legs to $\{B, F\#, F, C\}/\{Ab, Eb, D, A\}$ (on the final quarter beat of the measure). This
interpretation has several advantages over the network in part (a): fewer nodes are required to depict the events of the passage, and the events of measure 17 can be interpreted via the same set of objects and transformations, along the same clockwise path around the prism, as those of measure 18. As well, since pairs of simultaneous 0167 tetrachords combine via $T_3$ to form complete octatonic collections, each of the moves around the prism described above corresponds to a change of octatonic collection (an effect easily heard in the passage).  

We have seen so far how product networks, networks-of-networks, and sequential networks are each appropriate for different musical situations. The combination of these structures can illustrate the occurrence of motivic transformations at several levels within the music, such as seen in the analyses in Figure 3.11 and Figure 3.13. An analysis of measures 18-23 of Clermont Pépin’s Toccate no. 3, given in Figure 3.17, further demonstrates how replicative structures can be combined to depict different levels of transformational motive. In this passage, $T_5$ transformations occur from one pitch class to the next, generating consecutive 027 trichords (outlined with boxes on the figure). Larger-scale 027 trichords can also be formed by associating the downbeat pitch class from each six-note gesture (shown with circles and dashed lines on the figure); these pitch classes generate the next corresponding pitch class in the same staff via $T_5$. Measures 18-19 are also transformed by $T_5$ to generate measures 20-21. Thus the same transformations heard from pitch to pitch are also heard between one- or two-measure passages, creating a “stretched” version of the motivic trichord over a longer passage.

---

46 OCT(C, C#) refers to the octatonic collection that contains the dyad {C, C#}.

47 The Db is not generated via the same melodic pattern; however, its presence in measure 23 echoes the completion of the large-scale SC 025 trichord in the left hand.
The network given in Figure 3.18 incorporates all three types of replicative network structures discussed thus far. The smallest-scale objects (pitch classes) are transformed repeatedly by $T_5$ in order to form a three-node sequential network, emphasizing the event-to-event reiteration of $T_5$ heard in the music. These networks, in turn, are incorporated into a product network combining a two-node $T_{11}$ sub-network with a two-node $T_2$ sub-network, reflecting the $T_{11}$ sequencing of consecutive trichords and the $T_2$ relation (between the right and left hands) that acts simultaneously on the same set of trichordal networks. Lastly, the network-of-networks at the largest-scale (incorporating entire product networks) represents measures 20-21 as an exact transposition, at $T_3$, of measures 18-19. The similarity between large-scale and small-scale object is demonstrated by the fact that $T_5$ generates both the sequential network at
the smallest level, and the network-of-networks at the largest level. Thus we can consider $T_5$ not only as a transformation, but as the characteristic motive of the passage.

**Figure 3.18: A network analyzing the pitch classes of Pépin, *Toccate no. 3*, measures 18-23**

By interpreting a *transformation* rather than an object as motive, we are identifying similarities between musical passages based on their gestures rather than their events. This allows us to consider manifestations of a motive at several different structural levels, and among a wider variety of objects. As a second example of this analytical perspective, consider measures 13-16 of the second movement of Hétu’s *String*.
Quartet no. 2, shown in Figure 3.19, another passage whose transformational relationships can be depicted via a replicative network. The four instruments are paired according to rhythm and melodic process: the violins move by $T_1$ and $T_{11}$ in contrary motion, completing the aggregate every two measures, while the viola and cello play a unison melody in $\text{OCT}(C, C\#)$ that alternates between leaps and semitone motion. The semitone motion in the violins could be understood as derived from both the structure of the octatonic scale (which alternates $T_1/T_{11}$ with $T_2/T_{10}$) and the melody in the lower strings (which alternates $T_{11}$ motion with leaps). The $T_{11}$ motion in the lower voices of measures 13-16 is emphasized by metric accent, durational accent, and repetition, highlighting the similarities between the two instrumental groups.

Figure 3.19: Hétu, String Quartet no. 2, second movement, measures 13-16

Figure 3.20: Networks interpreting the violins in Hétu, String Quartet no. 2, second movement, measures 13-16

a) A network of unordered pentachords
b) A product network of ordered pentachords

\[
\begin{align*}
<F\# , G\#, A , Bb> & \xrightarrow{T_9} <Eb, E, F, F\# , G> & \xrightarrow{T_7} <Bb, B, C, C\# , D> & \xrightarrow{T_9} <C\# , D, D\# , E, F> \\
\uparrow N & & \uparrow N & & \uparrow N \\
<Gb, F, E, Eb, D> & \xrightarrow{T_9} <Eb, D, Db, C, B> & \xrightarrow{T_7} <Bb, A, Ab, G, Gb> & \xrightarrow{T_9} <Db, C, B, Bb, A> \\
\end{align*}
\]


c) A network of product networks of ordered pentachords

\[
\begin{align*}
<F\# , G\#, A , Bb> & \xrightarrow{T_9} <Eb, E, F, F\# , G> & \xrightarrow{T_7} <Bb, B, C, C\# , D> & \xrightarrow{T_9} <C\# , D, D\# , E, F> \\
\uparrow N & & \uparrow N & & \uparrow N \\
<Gb, F, E, Eb, D> & \xrightarrow{T_9} <Eb, D, Db, C, B> & \xrightarrow{T_7} <Bb, A, Ab, G, Gb> & \xrightarrow{T_9} <Db, C, B, Bb, A> \\
\end{align*}
\]


d) A network of two-pentachord networks

\[
\begin{align*}
<F\# , G\#, A , Bb> & \xrightarrow{T_9} <Eb, E, F, F\# , G> & \xrightarrow{T_7} <Bb, B, C, C\# , D> & \xrightarrow{T_9} <C\# , D, D\# , E, F> \\
\uparrow N & & \uparrow N & & \uparrow N \\
<Gb, F, E, Eb, D> & \xrightarrow{T_9} <Eb, D, Db, C, B> & \xrightarrow{T_7} <Bb, A, Ab, G, Gb> & \xrightarrow{T_9} <Db, C, B, Bb, A> \\
\end{align*}
\]

e) A network of four-pentachord networks

\[
\begin{align*}
<F\# , G\#, A , Bb> & \xrightarrow{T_9} <Eb, E, F, F\# , G> & \xrightarrow{T_7} <Bb, B, C, C\# , D> & \xrightarrow{T_9} <C\# , D, D\# , E, F> \\
\uparrow N \\
<Gb, F, E, Eb, D> & \xrightarrow{T_9} <Eb, D, Db, C, B> & \xrightarrow{T_7} <Bb, A, Ab, G, Gb> & \xrightarrow{T_9} <Db, C, B, Bb, A> \\
\end{align*}
\]
f) A product network of two-pentachord networks

\[
\begin{align*}
\langle E_b, F, F\#, G \rangle & \quad \langle B_b, B, C, C\# \rangle \\
\langle E_b, D, D_b, C, B \rangle & \quad \langle B_b, A, A_b, G, G_b \rangle
\end{align*}
\]

\[
\begin{align*}
T_7 & & T_7 \\
T_9 & & T_9
\end{align*}
\]

\[
\begin{align*}
\langle F\#, G\#, A, B_b \rangle & \quad \langle C\#, D, D\#, E, F \rangle \\
\langle G_b, F, E, E_b, D \rangle & \quad \langle D_b, C, B_b, A \rangle
\end{align*}
\]

\[
\begin{align*}
T_7 & & T_7 \\
T_9 & & T_9
\end{align*}
\]

\[
\begin{align*}
\langle E_b, E, F, F\#, G \rangle & \quad \langle B_b, B, C, C\# \rangle \\
\langle E_b, D, D_b, C, B \rangle & \quad \langle B_b, A, A_b, G, G_b \rangle
\end{align*}
\]

\[
\begin{align*}
T_7 & & T_7 \\
T_9 & & T_9
\end{align*}
\]

\[
\begin{align*}
\langle F\#, G, G\#, A, B_b \rangle & \quad \langle C\#, D, D\#, E, F \rangle \\
\langle G_b, F, E, E_b, D \rangle & \quad \langle D_b, C, B_b, A \rangle
\end{align*}
\]

\[
\begin{align*}
T_7 & & T_7 \\
T_9 & & T_9
\end{align*}
\]

g) Another network of product networks
Figure 3.20 gives several possible network interpretations of this passage, and is useful not only for illustrating transformational motives but also for showing the relative merits of the various networks. The network of Figure 3.20a interprets the pitch classes of the violins within measures 13-16, representing their music in terms of a network-of-(product) networks. The small-scale objects, unordered 01234 (chromatic) pentachords, are transformed in two ways: the melodic dimension is characterized by $T_9$ transformations between members of this set class, while the interaction between the two violins is characterized by $T_4$ transformations between members of the same set class. Two product networks depict the ensemble of melodic and simultaneous transformations in each two-measure segment. These product networks are in turn the objects of the large-scale two-node network-of-networks whose sole transformation $T_7$ generates measures 15-16 from 13-14. This structure emphasizes that measures 15-16 are a transposition of measures 13-14. Like the previous example, the two types of replicative network structures depict sequential relationships between two different object groups.

Network (a), however, does not capture many of the obvious features of the passage. For instance, the simultaneous pentachords in the violin 1 and violin 2 are articulated in contrary motion starting on the same pitch, which suggests an inversional relationship between pitch-class series rather than a transpositional relationship between pitch-class sets. Network (b), a product network, attempts to capture this trait by

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48 The passage could have also been depicted with a network-of-networks whose large-scale nodes each contained a $T_4$ pentachord pair; however, this interpretation would imply groupings based on one-measure rather than two-measure units. While one might initially think this would better depict the measure-to-measure events, this interpretation would assign the measure-to-measure transpositions $T_9$, $T_7$, and $T_3$ (the inverse of $T_9$) to the same structural level. In network (a), on the other hand, $T_9$ is the only transposition between consecutive trichords, and thus the parallelism between measures 13-14 and 15-16 is made explicit.
substituting an inversion for the ‘vertical’ transformations of network (a). This contextual inversion N, which acts on ordered (rather than unordered) pentachords, is defined as the inversion about the first pitch class of the ordered SC 01234 pentachord. The advantage of incorporating a contextual inversion here is that only one inversion is required to demonstrate the same inversive process repeated between the pentachords of the violins in each measure. Since N commutes with the transpositions of network (b), we can reinterpret the product network as several different networks-of-networks, shown in parts c through g of Figure 3.20. Each of these networks presents a compromise between the interpretations of networks (a) and (b): (c), for example, indicates the same two-measure groupings as network (a) (interpreting the last two measures as a T₇ transposition of the first two), and also presents the transformations N and T₉ between ordered pentachords previously observed in (b).

Much like the examples presented by Lewin in his analysis of ‘Nos qui vivimus’ (my Figure 1.4), each interpretation says something different about the passage. Network (a) says that each pentachord is involved in both a T₉ and a T₄ transformation (representing the melodic and simultaneous relationships, respectively) to complete the transformational motive. The last two measures repeat the first at T₇. Network (b) says that in the two violins, the pentachords are transposed by T₉ and T₇, but at the same time each pentachord also mirrors that of the opposing violin (via N), beginning on the same pitch class. Network (c) says that the pentachords in each melodic voice change by T₉, while simultaneously maintaining an inversive relationship N to those of the opposing voice, and that the last two measures repeat the first at T₇. Network (d) groups the

---

49 If we had incorporated standard inversions into this network, the index of inversion would be different for each pairing of simultaneous pentachords.
pentachords of each measure into small-scale networks, and consequently identifies that the violin pentachord simultaneities are transposed as a unit by $T_7$ and $T_9$. On the other hand, network (e) articulates the opposite grouping: each violin plays a four-pentachord melodic line, which is inverted as a unit via $N$. Networks (f) and (g) present less temporal representations of these measures, although both reinforce the repetition of the melodic transformations $T_9$ and $T_7$ via a product network: (f) depicts this product network at the highest level with two-pentachord networks (the one-measure violin simultaneities) as objects, while (g) designates the product network as the middle-level object (each representing one of the violin parts) of a two-node $N$ network. The former says that the one-measure violin pairings are repeatedly transposed by two different processes, while the latter says that the smallest-scale objects (the individual pentachords) are repeatedly transposed, and that each four-pentachord network related to the other by $N$. For each interpretation, groupings are specified by a secondary (tertiary, etc.) level of network – in other words, hierarchically – and therefore the ‘most suitable’ representation will be that which most clearly expresses the analyst’s perception of grouping structure.\(^{50}\)

The analyses presented thus far have employed replicative networks to show motivic continuity and gestural repetition within short excerpts with simple textures. Let us now examine a passage featuring a more detailed texture, using these network structures to interpret a recurring motive. Serge Garant’s *Plages* (1981) incorporates serial pitch techniques into rhythmically aleatoric passages. As described in the forward to the work, “The music is derived from a matrix, a group of five notes, which by successive inversions and transpositions generates a network of intervals extending over

---

\(^{50}\) My personal view is that (c) best expresses the passage – I hear the $T_7$ transposition of one two-measure unit to the next as a significant motive, and this is the only network I have provided that depicts this action at this level of hierarchy.
six octaves, using the augmented fourth F-B as a center." The five-note motive is lengthened to six notes by the addition of the augmented fourth at the beginning, and this six-note set combines with its inversion to form a twelve-tone row whose interval series is used throughout the work.

Particular combinations of row forms exploit the symmetric structure of the row; indeed, inversive symmetry is instrumental in a large-scale composing-out of the row, as we shall see. In the strings’ music in measures 204-206, given in Figure 3.21, the interval series of the row specifies the register of the pitches, but the ordering is determined by the performer. Each row is followed by a quarter-tone distortion of itself; I will consider these to be variations of the more structural equal-tempered rows and thus will not analyze them here. The twelve-tone row is interpreted in two musical ‘dimensions’: the intervals, in semitones, between consecutive pitches in the row become the interval of transposition from one string part to another. An analysis of this passage is given in Figure 3.22; abbreviations on each node identify row forms for each instrument (the cell doubles rows in the viola and contrabasses, filling in the missing pitch classes – these pitch classes are indicated in brackets).

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Figure 3.21: Garant, *Plages*, measure 204-206, strings
Figure 3.22: A network analysing row forms in Garant’s *Plages*, measure 204
A transformational interpretation of this music more clearly demonstrates how melodic and simultaneous relationships are closely linked. In the network of Figure 3.22, the small-scale objects are pitch classes transformed by T_1, T_2, T_3, T_5, and T_6, forming a twelve-tone row. Each one of these networks is an object of a large-scale node. The large- and small-scale networks are isomorphic to one another; that is, the series of transformations generating each twelve-tone row replicates the series of transformations relating the instrumental parts from low to high, forming a recursive network.

The analyses presented in this chapter have shown how three types of replicative network structures accomplish various analytical goals. Figure 3.2a, Figure 3.6, and Figure 3.22 demonstrated via isomorphisms how small-scale transformations mirror larger-scale ones. Figure 3.4 and Figure 3.8 illustrated how network isomorphisms can indicate similarities between two passages based on their transformational structures, while Figure 3.10 and Figure 3.13 employed isomorphisms to demonstrate motivic similarities between different objects in a single passage. Figure 3.2b and Figure 3.20 showed how product networks involve several transformations acting on a single object to create multiple paths, often reflecting the melodic or harmonic realization of pitches in the music. In Figure 3.2, Figure 3.11, and Figure 3.16, multiple network interpretations of a single passage provided different perspectives on the choice of objects and transformations, as a result isolating different repeated generative processes. Several analyses featured common sub-networks.

The chapter has presented only a brief sample of repertoire by Québécois composers. Because of their daily interactions and similar educations, this group of
composers strongly influence one another in their compositional approaches and techniques. In addition, many Québécois composers active during the 1970s and 1980s had similar educational backgrounds: most had completed undergraduate study in Montreal or the city of Québec, followed by further study in the province of Québec or in Paris. Of those who traveled to Paris from 1945 through the late 1960s, a large proportion studied with Olivier Messiaen, Henri Dutilleux, or both. Messiaen, in particular, was an enormous influence not only on these composers, but on many of the best-known composers in Europe. As a direct result of Messiaen’s teachings, many Québécois composers integrate common elements into their works; each composer creates a fundamentally different style, but from the same constituent components. Specifically, they often emphasize the repetition and development of smaller motives (reminiscent of Messiaen’s use of birdsong), incorporate modality into their music by using novel, non-diatonic scales (often in atonal contexts), and superimpose layers of contrasting material (Jacques Hétu, for example, discusses this technique explicitly in their writings, describing it as a layering of “sonic planes,” a topic that will be discussed further in the conclusion of this document). The analyses in this chapter have focused on one of these traits, the use of characteristic gestures at multiple structural levels in order to better integrate short, repeated motives. In the next chapter, a different approach to replicative networks will be undertaken: common network sub-graphs will be used to relate passages throughout a single work, emphasizing the transformational gesture rather than the objects as the characteristic motive of the work.

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Chapter 4: An Analysis of Schafer’s *Seventh String Quartet*

The previous chapter demonstrated how network analysis involving isomorphisms can represent repeated elements in short passages, and how the choice of one type of network structure over another in such an analysis can highlight a specific voice-leading, musical line, recurring element, hierarchy, and so forth. The current chapter applies similar analytical techniques but on a larger scale, comparing passages that are not immediately successive. These analyses will demonstrate how similarities in network structure can indicate motivic recurrence and development between passages; in other words, that replicative networks are particularly useful in interpreting, organizing, and linking elements throughout a work.

The focus of the analysis will be the *Seventh String Quartet* (1999) of R. Murray Schafer. This work incorporates several unusual elements including an obligato soprano part, detailed theatrics including colour and costume motifs associated with each performer, and a text based on the writings of a schizophrenic patient. In spite of the sporadic use of aleatoric elements, the pitch structure of the quartet is strongly motivic and is primarily based on the octatonic collection in its various transpositions. Rather than relying on traditional variation techniques, Schafer creates and develops motives by re-combining their characteristic gestures. For example, a gesture governing melodic motion in one motive may be transformed into one generating the level of transposition between two melodic lines in a second motive.

To clarify the repetition of characteristic gestures within a motive, I will employ networks incorporating isomorphic sub-graphs, the replicative network structures
previously defined. By interpreting each motive as a replicative network that clearly expresses its characteristic gestures, similarities between motives can be understood in terms of sub-graphs that their networks have in common. The analyses in this chapter incorporate three methods of expressing recurring transformations and their combinations: by repetition, creating an extended sequence; by combination, forming a product network from multiple sub-networks; and by embedding, creating a hierarchical network-of-networks.

The goal of this chapter is to demonstrate, using replicative networks, how elements that incorporate similar characteristic gestures form motivic groups that help the analyst and listener to structure their hearing of the movement. As a consequence of this approach, the analysis of these passages will indirectly examine issues of unity, continuity, and cohesion within the movement. Since Schafer’s motivic material generally incorporates repeated relationships rather than repeated objects undergoing traditional sequences, transpositions, and so forth, understanding structure in his works requires a conceptualization that emphasizes the gestures between events – in other words, how passages or motives are transformed – in addition to the events themselves. The repetition of these gestures will suggest several possible network structures; one focus of this chapter will be to demonstrate some principles for choosing a single network representation among the many possible.

Within the Quartet, the semitone is ubiquitous. Rather than understanding the semitone as an interval (an object), the analyses presented herein will instead interpret the semitone as a repeated gesture (an action or transformation), most often realized in the analytical networks as the mod12 transposition $T_1$ or its inverse, $T_{11}$. This approach is
appropriate given that the semitone transformation occurs in more than just a pitch-melodic context: it is heard among pitch classes, dyads, and larger sub-motives, in both the ‘vertical’ (that is, simultaneous or harmonic) and ‘horizontal’ (in other words, melodic) dimensions. For these reasons, I will refer to this gesture as the characteristic gesture or transformation of the quartet.

The opening passage, given in Figure 4.1, introduces several elements that will prevail throughout the quartet. First, as outlined on the figure, the viola plays a rapid gesture whose pitches belong entirely to OCT(C, D). Just before this gesture peaks at the end of the first system, it segments into a motive that is repeated, each time extended by one note. The process of repetition is taken up immediately by the violins, who upon their entry in the second system repeat a motive incorporating interval classes 1, 4, and 5. Interval class 1, heard within this passage in the gradual pitch-semitone descents of the viola and cello, becomes especially prominent later on.
Figure 4.1: Schafer, Seventh String Quartet, page 2

SEVENTH STRING QUARTET
(WITH OBLIGATO SOPRANO)

R. Murray Schafer

AT THE BEGINNING, ALL PLAYERS ARE OFFSTAGE. AS THE CELLO AND VIOLA BEGIN TO PLAY, THE TWO VIOLINISTS ENTER FROM EITHER SIDE. THEY BOW CEREMONIOUSLY TO ONE ANOTHER, THEN TURN TO FACE THE AUDIENCE.

OCT (C, D)

semitone glissando

(emphasized by ic1 difference between viola and violin 1)
Later, at the soprano’s first entrance (seen in Figure 4.2), many of the same elements recur. Like the earlier viola motive, the repeated melodic fragments in the strings present octatonic material, with violin 1 and cello in OCT(C, D) and the viola in OCT(C#, D). The semitone heard in the earlier passage also returns: the reiteration of the word "moan" in the soprano, in conjunction with breath marks, parses her line into dyads that are highly saturated with the semitone transformation, from one note to the next as well as from one dyad to the next. Figure 4.3 reinterprets her music transformationally, asserting a gestural similarity at multiple structural levels – in other words, through a network that is explicitly self-replicative, as defined in the theoretical chapter of this document. To reflect the dyad interval-class structure established by the soprano, the network interprets her opening gesture as a series of dyads both generated and transformed by $T_{11}$: beginning at the central node of the figure, G is transformed by $T_{11}$ into F#, and this dyad is subsequently transformed by $T_1$ (to the left on the network) to form <Ab, G>. The soprano returns to the initial dyad by the next transformation, $T_{11}$, and then continues via another $T_{11}$ transformation to the dyad <Gb, F>. This passage thus establishes $T_{11}$ and its inverse, $T_1$, as characteristic transformations. Many passages throughout the work apply a single transformation to different structural levels, a process that will be illustrated through isomorphic networks.
Three types of replicative networks will analyze, relate, and comment on passages throughout the quartet. **Figure 4.4** builds an example of each from a two-node graph representing the $T_{11}$ transformation shown to be prominent at the beginning of the quartet: part (a) gives the original two-node graph, while parts (b), (c), and (d) of the figure incorporate this smaller graph into three more complex structures, a graph-of-graphs (where $T_{11}$ occurs recursively at two structural levels), a sequential graph (where an object is repeatedly transformed by $T_{11}$), and a product graph (the combination of two smaller networks), respectively. Each says something different about a musical passage.

The sequential graph involves repeated transformations. The graph-of-graphs can emphasize hierarchical elements within the music; in cases where the node contents of the corresponding network are isomorphic both to each other and to the larger graph, this
type also suggests that a single process influences multiple structural levels. The product
graph suggests (as mentioned in Chapter 1) the repetition and convergence of two or
more distinct processes on the same set of musical objects. Figure 4.5 gives a passage
that can be analyzed with all three types of structures. With reference to a network-of-
networks, this music can be thought of (in addition to other ways) as three chromatic
lines that each generate the next-higher line by $T_5$ (Figure 4.5b), or as four SC 027
trichords that each generate the succeeding trichord by $T_{11}$ (Figure 4.5c). In either
network-of-networks, there is only one gesture from each pitch class ($T_{11}$ in (b) and $T_5$ in
(c)); there are other gestures, but they move from one entire network to another ($T_5$ in (b)
and $T_{11}$ in (c)). A product network interpretation (Figure 4.5d), on the other hand,
combines both perspectives, modelling passage as a web of objects, nearly every one of
which is transformed repeatedly by both $T_5$ and $T_{11}$. Moreover, in this example each of
the repeated sub-graphs of the product network forms a sequential graph, as do the large-
scale graphs for the two networks-of-networks; this immediate replication of a
transformation on the same level creates continuity.

Figure 4.4: Graphs and networks incorporating $T_{11}$

a) The simplest form: one structural level

\[
\begin{array}{c}
\text{Input} \xrightarrow{T_{11}} \text{Output}
\end{array}
\]

b) At two structural levels (a recursive or self-replicative network-of-networks)

\[
\begin{array}{c}
\text{Input} \xrightarrow{T_{11}} \text{Input} \xrightarrow{T_{11}} \text{Output} \xrightarrow{T_{11}} \text{Output}
\end{array}
\]
c) A sequential graph

\[ \begin{array}{c}
  & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} \\
  \text{□} & \rightarrow & \text{□} & \rightarrow & \text{□} & \rightarrow \\
\end{array} \]

\[ \begin{array}{c}
  & \text{T}_{N} & \text{T}_{N} & \text{T}_{N} & \text{T}_{N} \\
  \text{□} & \rightarrow & \text{□} & \rightarrow & \text{□} & \rightarrow \\
\end{array} \]

d) A product graph

\[ \begin{array}{c}
  & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} \\
  \text{□} & \rightarrow & \text{□} & \rightarrow & \text{□} & \rightarrow \\
\end{array} \]

\[ \begin{array}{c}
  & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} & \text{T}_{11} \\
  \text{□} & \rightarrow & \text{□} & \rightarrow & \text{□} & \rightarrow \\
\end{array} \]

**Figure 4.5: An example involving linear T_{11} transformations and simultaneous T_{5} transformations**

a) A short musical passage

\[ \text{\textit{\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet}} \]

b) A networks-of-networks grouping the semitone lines

\[ \begin{array}{c}
  \text{C} & \text{T}_{11} & \text{B} & \text{T}_{11} & \text{Bb} & \text{T}_{11} & \text{A} \\
  \uparrow \text{T}_{5} \\
  \text{G} & \text{T}_{11} & \text{F\#} & \text{T}_{11} & \text{F} & \text{T}_{11} & \text{E} \\
  \uparrow \text{T}_{5} \\
  \text{D} & \text{T}_{11} & \text{C\#} & \text{T}_{11} & \text{C} & \text{T}_{11} & \text{B} \\
\end{array} \]
c) A network-of-networks grouping the 027 trichords

![Diagram of network-of-networks grouping the 027 trichords]

d) A product network that does not explicitly specify groupings

![Diagram of product network]

The pertinence of these various $T_{11}$ networks is evident when we consider the music on page 10 of the score, beginning in the second system (given as Figure 4.6), which recalls several elements heard at the beginning of the quartet. Semitone transformations generate the series of chromatically-descending dyads and trichords in the cello (outlined with dashed boxes on the score), as well as the music of the soprano, which repeats a descending semitone on the word “Hell”. This $T_{11}$ material alternates...
with another type of material based on symmetrical collections, the octatonic passage (outlined with a solid box on the figure) introduced at the beginning of the quartet. The three pairs of T₅-simultaneities heard at the end of the system (outlined with a dotted box) also subtly suggest symmetrical collections: their pairs form members of SC 0167 (a subset of the octatonic collection), and the total collection forms the complementary, symmetrical SC 01236789.

Figure 4.6: Schafer, Seventh String Quartet, page 10, second system

Figure 4.7: Transformations within the chromatically-descending trichords in the cello, page 10, second system, of Schafer’s Seventh String Quartet (marked on the previous figure)

a) Interpreted via a network-of-networks:

b) Interpreted via a product network:
Through network representations of both the chromatically-descending trichords and the T₃-simultaneity dyads, we can hear how the two types of material are similarly structured. Figure 4.7 analyses the first of these materials, the chromatically-descending trichords, in two different ways (as was discussed in connection with the similar Figure 4.5), each of which says something different about the music. Network (a) is a network-of-networks, identifying the highest-level objects of the passage as 016 trichords (generated by T₅, T₆, and T₁₁ transformations among pitch classes). These objects in turn are transformed repetitively by T₁₁, just as the dyads of the voice’s opening music were transformed in the music of Figure 4.2 (another reason for considering T₁₁ the characteristic transformation of the movement). In this representation there is no explicit “voice-leading” between registrally-corresponding members of the chords. T₁₁ exists as a higher-level manifestation of the T₅ + T₆ path within each trichord simultaneity, but does not occur linearly except implicitly in the mapping of the F# node to the F node as part of the T₁₁ isomorphism.

Network (b) identifies the highest-level objects as pitch classes, with multiple possible paths from most of them. Because its structure satisfies the definition of a product network, it asserts both T₅ + T₆ = T₁₁ simultaneities (the “columns” of the network) and three parallel chromatic lines (the “rows” of the network), suggesting

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54 To clarify, in a network-of-networks there is an isomorphism that maps the contents of one node to the next, but this is a mapping of a larger-scale object (the network) and not a direct mapping between its smallest-scale objects. When I speak of ‘transformational paths’ between nodes, I am referring to an arrow path that links two nodes, where the transformations associated with the arrows occur at the same level as the objects upon which they act. Therefore the F# and F seen at the top-left of network (a) are not connected via a transformational path since the T₁₁ transformation that links F#’s network to F’s network is a transformation of networks, not pitch classes. This distinction was made in Chapter 1; see pages 10-11 for more information.
voices that the network-of-networks does not. The polyvalent approach values multiple processes equally, and implies that events in one ‘dimension’ must co-ordinate with those in another. For example, the initial low G, the input node of the network, is transformed by $T_{11}$ both to a simultaneity, F#, and to a successive pitch class, Gb; we could interpret the melodic G-to-Gb transformation as an echo of the previous G-to-F# simultaneity. Network (b) also allows multiple paths between the same pairs of objects; for example, the $T_{11}$ transformation from G to Gb is one possible path between these pitch classes, but the path $T_{6}+T_{11}+T_{6}$ (through the nodes <G, C#, C, Gb>) is also possible. Analytically, we might desire this longer path if we wanted to observe that these transformations are prevalent in surrounding music (for instance, $T_{11}$ is heard in the soprano, in the preceding octatonic motive, and the chromatically-descending dyads, and $T_{6}$ melodically in the subsequent dyad motive). Thus the choice of network type determines what observations are made about the passage.

Networks (a) and (b) make additional assertions about the music that assist in linking the motive to surrounding elements in the cello and the voice. The large-scale transformations of network (a) correspond temporally to a $T_{11}$ sequence within the music, emphasizing the importance of this transformation. The upper pitch classes of each SC 016 trichord network can be interpreted, according to their networks, as transformational derivatives of the lowest pitch class. Accordingly, one might understand each of them as a G-generated network, an F#-generated network, and so forth. Similar transformations occur between the lowest pitch classes and the surrounding music: the move from a G-object to an E-object in Network (a) recalls the cello’s move at the beginning of the passage.

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55 The combination of the transformations $T_{5} + T_{6} = T_{11}$ seen in the vertical sub-graph of this product network, means that the lowest node of each functions as the input for more than one arrow relation. This is the situation discussed on page 40 of Chapter 2.
system from G to E, and the move from a G-object to an F#-object is like the soprano’s move \(<G, F#>\). In network (b), the structure of the product network was chosen to suggest for certain objects roles that also associate them with the surrounding music. For example, the input node of the network, G (on the lower-left), can be considered the source of the entire pitch-class complex. This special role is perhaps signalled by the long G that ends the preceding cello passage. Similarly, the output node, Eb (at the top-right of the network), is the conclusion of the entire complex. The \(T_5\) transformation by which it is generated from Bb is immediately repeated in the cello, suggesting a link between the end of this complex and the beginning of the next motivic group based on \(T_5\) dyads.

These two analyses are not mutually exclusive, however, because they are constructed from the same two sub-graphs in the manner demonstrated in the theoretical chapter of this document. The ‘row’ graph, labelled \(Graph R\) in Figure 4.8a, consists of four nodes \(\{R1, R2, R3, R4\}\) each generated by \(T_{11}\) from the preceding node. The ‘column’ graph, labelled \(Graph C\) in Figure 4.8b, consists of three nodes \(\{C1, C2, C3\}\) where \(C1\) generates \(C2\) by \(T_6\), and \(C2\) generates \(C3\) by \(T_5\). Figure 4.7a is a network-of-networks whose sub-graphs are \(R\) and \(C\): its large-scale network has the same graph as \(R\), and the small-scale networks within the nodes of \(Graph R\) are isomorphic both to one another and to \(Graph C\). In addition, there is a CONTENTSMAP function by which the small-scale node contents of \(R1\) (\(C1, C2, C3\)) are mapped to the small-scale node contents of \(R2\) (\(C1, C2, C3\)) by \(T_{11}\), the large-scale transformation (this is the “implicit voice leading” to which I refer on page 110). Moreover, Figure 4.7b is a product network whose sub-graphs are also \(R\) and \(C\): its nodes are mapped one-to-one from
NODES_R x NODES_C such that the three sub-networks incorporating the node sets whose contents are {F#, F, E, Eb} (i.e. the ‘top row’), {C#, C, B, Bb} (the ‘middle row’), and {G, Gb, F, E} (the ‘bottom row’), respectively, are each isomorphic to Graph R, and the four networks that incorporate the node sets whose contents are {G, C#, F#} (the ‘leftmost column’), {Gb, C, F} (the ‘second column from the left’), {F, B, E} (the ‘third column from the left’), and {E, Bb, Eb} (the ‘rightmost column’), respectively, are each isomorphic to Graph C.⁵⁶

**Figure 4.8: Sub-graphs of Figure 4.7**

a) *Graph R*, the ‘row graph’

b) *Graph C*, the ‘column graph’

Notwithstanding their common formal origins, the two distinctive combinations of sub-graphs represented by **Figure 4.7a and b** isolate different features within the music. They bring out different aspects of continuity, development, contrast, and so forth, within the excerpt. Elements of the network structure such as input and output nodes, repeated sub-graphs, and object types can create links with the surrounding material, and will in turn influence later hearings of the piece (and thus later network

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⁵⁶ Other product networks and networks-of-networks may also be formed by different combinations of R and C.
structures as well). Take, for example, the cello’s T₅ dyad simultaneities at the end of the excerpt. These can be interpreted as a product network involving the same transformations as the previous passage, T₁₁, T₅, and T₆, but recombined in a different manner; this analysis is given in Figure 4.9. The network in Figure 4.9a first builds a product network from two dyadic sub-networks (those involving T₅ and T₆). These four-node product networks are in turn incorporated into another product network involving T₁₁.⁵⁷ For both views, the input node is Bb, which immediately generates Eb via the T₅ transformation heard at the end of the previous passage (as discussed in conjunction with Figure 4.7b, the previous passage concluded with a T₅ transformation of Bb to Eb). The output node is G, the same pitch class given as Figure 4.7b’s input node. This return suggests a governing role for G (the extended pitch class heard immediately prior to these two gestures in the cello), supported by the larger-scale continuity of this pitch class through the {Bb, Eb} dyad and back again.

---

⁵⁷ Figure 4.9 (parts (a) and (b)) satisfy the definition of a product network given in Chapter 2. The ‘vertical’ graph of (b) consists of two nodes generated by T₅; the ‘horizontal’ graph consists of two nodes generated by T₆. The ‘diagonal’ graph consists of three nodes each generated by T₁₁ from the preceding node. Figure 4.9a and b are two representations of a three-dimensional product network whose sub-graphs are the vertical, horizontal, and diagonal graphs identified above: its nodes are a one-to-one map of NODESVERTICAL x NODESORIZONTAL x NODESDIAGONAL such that the six networks that incorporate the node sets {Bb, Eb}, {E, A}, {A, D}, {Eb, Ab}, {G#, C#}, and {D, G}, respectively (i.e. the “T₅ networks”), are each isomorphic to the vertical graph, the six networks that incorporate the node sets {Bb, E}, {Eb, A}, {A, Eb}, {D, Ab}, {G#, D}, and {C#, G}, respectively (i.e. the “T₆ networks”), are each isomorphic to the horizontal graph, and the four networks that incorporate the node sets {A, Ab, G}, {E, Eb, D}, {Bb, A, G#}, and {Eb, D, C#}, respectively (i.e. the “T₁₁ networks”), are each isomorphic to the diagonal graph.
Figure 4.9: A network indicating transformations among the $T_5$ simultaneity dyads (the last six dyads in the cello) of Schafer’s *Seventh String Quartet*, page 10, second system

a) One view of a product network combining three sub-networks:

```
\begin{tikzpicture}[>=latex]
\node (input) at (0,0) {Input node};
\node (output) at (6,0) {Output node};
\node (A) at (1.5,1) {A};
\node (Ab) at (1.5,2) {Ab};
\node (D) at (3,1) {D};
\node (C#) at (4.5,1) {C#};
\node (G) at (4.5,2) {G};
\node (G#) at (4.5,0) {G#};
\node (E) at (0,1) {E};
\node (Eb) at (0,0) {Eb};
\node (Bb) at (-1.5,0) {Bb};

\path
(input) edge [dotted] node [above] {$T_5$} (Bb)
(input) edge [dotted] node [below] {$T_6$} (Eb)
(input) edge [dotted] node [right] {$T_{11}$} (A)

(A) edge [dotted] node [above] {$T_{11}$} (Ab)
(A) edge [dotted] node [right] {$T_6$} (D)

(E) edge [dotted] node [above] {$T_5$} (E)
(E) edge [dotted] node [right] {$T_6$} (A)

(Eb) edge [dotted] node [below] {$T_5$} (E)

(D) edge [dotted] node [right] {$T_{11}$} (C#)
(D) edge [dotted] node [right] {$T_6$} (G)

(C#) edge [dotted] node [right] {$T_5$} (G)
(G#) edge [dotted] node [right] {$T_5$} (D)

(Bb) edge [dotted] node [above] {$T_6$} (A)

(Eb) edge [dotted] node [below] {$T_6$} (E)

\end{tikzpicture}
```

b) A second view of the same product network:

```
\begin{tikzpicture}[>=latex]
\node (input) at (-2,0) {Input node};
\node (output) at (2,0) {Output node};
\node (A) at (0,1) {A};
\node (Ab) at (0,2) {Ab};
\node (D) at (1,1) {D};
\node (C#) at (2,1) {C#};
\node (G) at (2,2) {G};
\node (G#) at (2,0) {G#};
\node (E) at (-1,1) {E};
\node (Eb) at (-1,0) {Eb};

\path
(input) edge [dotted] node [below] {$T_5$} (Bb)
(input) edge [dotted] node [below] {$T_6$} (Eb)
(input) edge [dotted] node [right] {$T_{11}$} (A)

(A) edge [dotted] node [above] {$T_{11}$} (Ab)
(A) edge [dotted] node [right] {$T_6$} (D)

(E) edge [dotted] node [above] {$T_5$} (E)
(E) edge [dotted] node [right] {$T_6$} (A)

(Eb) edge [dotted] node [below] {$T_5$} (E)

(D) edge [dotted] node [right] {$T_{11}$} (C#)
(D) edge [dotted] node [right] {$T_6$} (G)

(C#) edge [dotted] node [right] {$T_5$} (G)
(G#) edge [dotted] node [right] {$T_5$} (D)

(Bb) edge [dotted] node [above] {$T_6$} (A)

(Eb) edge [dotted] node [below] {$T_6$} (E)

\end{tikzpicture}
```

The visualization given in Figure 4.9b represents each of the three sub-graph types in a distinct dimension in order to clarify how they relate. The dotted lines on the figure do not participate in the dyad-to-dyad voice-leading of the music, but are required to complete the product network. This network format also highlights certain processes we may hear within the music. Arrows along the top plane of network (b) correspond to motion in the upper voice, as arrows in the bottom plane do with motion in the lower
voice, and the dyad simultaneities are indicated as $T_5$ dyad networks along the vertical axis, corresponding to their orientation in the printed music. $T_{11}$ is heard between the consecutive dyad pairs $\{E, A\}$ to $\{Eb, Ab\}$ and $\{D, A\}$ to $\{C\#, G\#\}$, but is also present from $\{Bb, Eb\}$ to $\{A, D\}$ and $\{Eb, Ab\}$ to $\{D, G\}$; thus the path from $\{Bb, Eb\}$ to $\{A, D\}$ may be understood as either a single transformation of $T_{11}$ or as the combination $(T_6 + T_{11} + T_6)$, heard among consecutive pitch class pairs in the music. The input and output nodes occur at the beginning and end, respectively, of the solid-line path, corresponding to their temporal location within the motive.

Both Figure 4.7 and Figure 4.9 involve sub-graphs incorporating $T_5$, $T_6$, and $T_{11}$, but in different combinations. Figure 4.7 combines $T_5$ and $T_6$ into a single sub-graph type in the product network, whereas Figure 4.9 keeps them in separate sub-graphs.\(^{58}\) The $T_{11}$ sub-graph associated with Figure 4.9b is an abbreviated version of that associated with Figure 4.7. The voice part (the two-note motive supporting the word “Hell”) can also be depicted by the same sub-graph, creating motivic links to the instrumental music heard at the same time. Figure 4.10 illustrates that the soprano’s two repeated pitch classes relate by $T_{11}$. Like the sub-graph identified in Figure 4.8a, $T_{11}$ is heard three times, but in this case the transformation is not replicative (in other words, it does not get applied to the pitch class previously generated by $T_{11}$). Instead, the soprano returns to the G for the beginning of each statement, recalling the repetition established at the beginning of the work (Figure 4.1).

\(^{58}\) Although $T_5$ is implied by the combination $(T_6)(T_{11}) = T_5$. 
While previous passages have featured $T_{11}$ in combination with other prominent transformations, the musical material of page 20, third and fourth systems (Figure 4.11), presents a climax of sorts by developing the recursive possibilities of $T_{11}/T_1$, possibilities exposed most clearly by networks that involve self-replication. One clear indication of the structural importance of this transformation is the interval of imitation between the viola and violins, represented transformationally as $T_1$. This transformation also dominates local details on two levels. Figure 4.12 shows, on one level, that $T_{11}$ sequences occur within short pitch-class motives grouped into segments (labelled alphabetically on the figure for future reference). At a second level, the figure also shows that the octatonic collections heard in the first system of the passage progress by $T_1$ and $T_{11}$.\footnote{While it is true that any pair of octatonic collections will be in a $T_1$ or $T_{11}$ relationship, the presence of alternating octatonic collections at this point in the work suggests that this property is being deliberately exploited in order to further emphasize these transformations.} Figure 4.13 illustrates that some of these segments vary previously-heard pitch-class motives. Specifically, the first three notes of the passage (the first trichord shown in Figure 4.13) reprise the opening material of the violins discussed in conjunction with Figure 4.1. Subsequently, this pitch-class motive’s component transformations, $T_5$ and $T_1$ (indicated on the figure), are recombined to yield the music of segments c, d, e, and f. In other words, the transformations characteristic of the pitch-class motive are retained:
the opening trichord combines $T_1$ and $T_5$ to form SC 015, while the remaining pitch-class motives, representing segments c/d, and e/f, respectively, combine these to form SC 016.

Figure 4.11: Schafer, Seventh String Quartet, page 20, third and fourth systems (rehearsal O)

Figure 4.12: Schafer, Seventh String Quartet, page 20, third and fourth systems, viola only
Figure 4.13: Pitch-class motives within page 20, third and fourth systems, of Schafer’s Seventh String Quartet

\[ T_1 \text{ and } T_{11} \text{ transformations structure these motives within each segment. Consider first the segments (a) and (b), which present fragments of changing octatonic collections. Considering their objects as members of the 12-pitch-class aggregate, they feature a mix of } T_1/T_{11} \text{ and } T_2/T_{10}. \text{ But considering their objects as members of an 8-member octatonic collection, they are identical, scale-step transformations. Figure 4.14 demonstrates this identity, using the notation } 8 \text{T}^{-1} \text{ to indicate a transposition “down” one mode step in an 8-pitch-class collection, much like } T_1^{-1} = T_{11} \text{ is a transposition “down” a semitone within the chromatic collection. The networks also incorporate a specific kind of mod12 transformation, which I will notate as } ^*T_x, \text{ that changes the modular system as a result of transforming its pitch class members. For example, } ^*T_{10} \text{ is a mod12 transposition that transforms a pitch class (or pitch class set) and transposes the octatonic collection associated with that pitch class (or set).}^{60} \text{ For this reason, these will be referred to as mode-altering mod12 transformations.}^{61} \text{ The group formed by the combination of these}

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60 The asterisk is used here to indicate that this is not a standard mod12 transposition (even though in some ways it behaves as such), but rather a contextual transposition that modifies elements in addition to the pitch class or pitch-class set. In order to know how this transformation affects a pitch class, for example, we must also know to what octatonic collection the pitch class belongs when it appears in the music.

61 Since the mod12 transpositions do not maintain the octatonic collection, it is important to keep in mind that an octatonic set transformed via a mode-altering mod12 transposition may shift collections. Specifically, \{’T_0, ’T_3, ’T_6, ’T_9\} keep the set within the same octatonic scale; \{’T_1, ’T_4, ’T_7, ’T_{10}\} shift the set to an octatonic collection a semitone higher; and \{’T_2, ’T_5, ’T_8, ’T_{11}\} shift the set to an octatonic collection a semitone lower than the original collection.
transformation types can be applied to a pitch class in any order and still produce the same result; in other words, the mod8-transformations and the mode-altering mod12 transformations commute. For example, in OCT(C, C#):

\[ {}^*T_{10}(sT_{1}^{-1}(D#)) = {}^*T_{10}(C#) = B \]

(transposes D# down one scale step within OCT(C, C#))

(transposes C# and its octatonic collection up 10 semitones, changing the collection to OCT(Bb, B))

\[ sT_{1}^{-1}( {}^*T_{10}(D#)) = sT_{1}^{-1}(C#) = B \]

(transposes D# and its octatonic collection up 10 semitones, changing the collection to OCT(Bb, B))

(transposes C# down one scale step within OCT(Bb, B))

The transformations \( {}^*T_{10} \) and \( sT_{1}^{-1} \) can be applied to the input pitch class, D#, in either order and still produce the same result. This is true for all objects and transformations within the (mode-altering mod12 transpositions + mod8 transpositions) group.

Commutativity is a requirement for both product networks and networks-of-networks, and in conjunction with the large- and small-scale networks defined earlier satisfy the conditions required for these structures.\(^{62}\)

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\(^{62}\) This combination of mod8 and mod12 transpositions is not without precedent: Julian Hook combines mod8 and mod12 transformations in Hook 2007. His analysis of interscalar transposition in Wagner’s *Tristan und Isolde*, in particular (p. 17), presents several similarities to the current analysis, although his analysis focuses on mappings from one collection type to another rather than mappings within a particular collection.
Figure 4.14: A network analysis of motivic groups within Figure 4.12 (outlined with dotted boxes on the previous figure)

a)

```
\[ G \rightarrow_{8T_1^{-1}} F\# \rightarrow_{8T_1^{-1}} E \rightarrow_{^*T_{10}} F \rightarrow_{8T_1^{-1}} E \rightarrow_{8T_1^{-1}} D \]
```

```
\[ F\# \rightarrow_{8T_1^{-1}} E \rightarrow_{8T_1^{-1}} D\# \rightarrow_{^*T_{10}} E \rightarrow_{8T_1^{-1}} D \rightarrow_{8T_1^{-1}} C\# \]
```

b)

```
\[ E \rightarrow_{8T_1^{-1}} Eb \rightarrow_{8T_1^{-1}} Db \rightarrow_{8T_1^{-1}} C \rightarrow_{8T_1^{-1}} Bb \]
```

```
\[ Eb \rightarrow_{8T_1^{-1}} Db \rightarrow_{8T_1^{-1}} C \rightarrow_{8T_1^{-1}} Bb \]
```

f)

```
\[ A\# \rightarrow_{8T_1} B \rightarrow_{8T_3} E \rightarrow_{^*T_{10}} G\# \rightarrow_{8T_1} A \rightarrow_{8T_3} D \]
```

```
\[ A \rightarrow_{8T_1} A\# \rightarrow_{8T_3} D\# \rightarrow_{^*T_{10}} G \rightarrow_{8T_1} G\# \rightarrow_{8T_3} C\# \]
```
Thus networks (a), (b), and (f) of Figure 4.14 differ from those presented so far; the transformations given in these networks are not pitch-class transpositions in the traditional sense, but rather contextual transpositions on “pitch classes as members of an octatonic collection”.

With this set of transformations in hand, we can hear the octatonic passages of Figure 4.12 as tightly structured. In segment (a), analysed in Figure 4.14a, each trichord is generated via a repeated application of $gT_1^{-1}$ to its pitch classes. The four trichord
networks are isomorphic under \( sT_1^{-1} \) or \( ^*T_{10} \), forming a network-of-networks. That is, \( sT_1^{-1} \) transforms the network of the first trichord into that of the second and the network of the third trichord into that of the fourth, while \( ^*T_{10} \) transforms the network of the first trichord into that of the third and the network of the second trichord into that of the fourth. This particular combination of \( ^*T_{10} \) (in ‘rows’ of the network) and \( sT_1^{-1} \) (in ‘columns’) defines a product network. This network manifests both explicit and implicit replication; that is, it embeds networks within its large-scale nodes, but these nodes build a structure based on the repetition and combination of isomorphic sub-graphs, as defined in the theory chapter. The implicit replication of the product network suggests the manner in which the entire segment is constructed, by the initial trichord being transposed in two different ways (down a mode step to produce the second trichord of the group, and by \( T_{10} \) to produce the second pair of the group), whereas the explicit replication of the network-of-networks suggests an explanation for the choice of transformation (the transformation generating the three-note melodic motive determines the choice of larger-scale transformation).

Segment (b) continues this series of octatonic scale-step transpositions, as shown by Figure 4.14b. Like Figure 4.14a, this example gives a network-of-networks involving \( sT_1^{-1} \) at two structural levels: repeated from one pitch-class to the next, and between instances of the four-note motive.\(^{63}\) The network structure asserts the four-note group as the salient motive at this point, with the motive’s generating transformation playing an important role because of its appearance at two structural levels.

Unexpectedly, the final pitch class of segment (b) does not satisfy our transformational

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\(^{63}\) Whereas part (a) involved a three-note motive generated by repeated instances of \( sT_1^{-1} \), part (b) depicts a four-note motive; this extension by one is similar to that seen in the music of page 10, second system, examined in Figure 4.7 and Figure 4.9.
expectations: B substitutes for the anticipated Bb, perhaps foreshadowing the descending chromatic scale-step transformation that is developed within the next motivic segment.

While segments (c) through (e) do not involve the octatonic collection (these will be analysed shortly), motive group (f) does. This group is split into two sub-groups via a rest, and each group of six notes presents sub-groups comprising a three-note motive followed by its (mod12) transposed retrograde. Figure 4.14f interprets the passage as a product network whose objects are isomorphic networks of pitch classes generated by mod8 transformations. The product network involves elements similar to the network of Figure 4.14a: the small-scale networks involve single pitch classes transformed by mod8 transpositions; the two left-side large-scale nodes relate by a descending scale step ($^9T_{1}^{-1}$ in (f), instead of $^8T_{1}^{-1}$ as in (a)) as do the two right-side nodes; and the left-side nodes generate the right-side nodes by $^9T_{10}$. In spite of these similarities, the networks in Figure 4.14a and f are not isomorphic since there is no one-to-one mapping of the transformations of network (a) onto those of network (f): in other words, while mod8 and mode-altering mod12 transformations can perform similar actions, they cannot be mapped onto one another because they do not contain the same number of members, nor do they function the same way on their objects. There is instead a mapping of the large-scale transformations of network (a) (that is, the transformations between trichord networks) into those of network (f) which is not one-to-one, suggesting a transition back to musical processes dominated by mod12 transformations initiated by similar transformations in two different modular systems.

Like motive groups (a) and (b), motive groups (c), (d), (e), and (f) also feature an inverse step transformation, but within the chromatic rather than the octatonic collection.
The network in Figure 4.14g interprets these remaining motivic groups as a series of interval class 5 dyads generated by the repeated application of $T_{11}$ (the chromatic inverse step transformation). Each motive group involves a portion of the network, and for each the relevant dyad pairs are indicated by various dashed- and solid-line groupings on the figure. All twelve pitch-class manifestations of interval class 5 are heard within this passage; the completion of all possible forms closes the passage.

We have seen so far that $T_{11}$ is prominent in several passages throughout the Quartet, employed in the initial motives of the work and returning later in motivic material that may not initially be heard as related. Another passage, the first and second systems of page 21 (given in Figure 4.15), gives a good example of how the characteristic transformation $T_{11}$ generates objects at multiple levels of the music. Two measures are particularly useful for illustrating these transformations: the last measure of the first system and the first measure of the second system. These are modeled transformationally as networks-of-networks in Figure 4.16 and Figure 4.17. The network-of-network structure of these two analyses represents the texture of the passage better than would an analysis of it by product networks. The dyad grouping suggested by register and rhythm in the passage disrupts any sense of chromatic line that would be implied in a product network featuring the same transformations. And while the viola could at first be interpreted as a pairing of chromatically-descending voices, distinguished by register, the rests that begin to intrude in the first measure of the second system instead reinforce its dyadic groupings.
Figure 4.15: Schafer, *Seventh String Quartet*, page 21, first and second systems

Figure 4.16: Networks interpreting the pitch classes of the violins and viola on page 21, first system, last measure, of Schafer’s *Seventh String Quartet*

**violin 1:**

\[
\begin{array}{c}
C \downarrow T_{11} \\
B
\end{array} \quad \begin{array}{c}
B \downarrow T_{11} \\
Bb
\end{array} \quad \begin{array}{c}
B \downarrow T_{11} \\
Bb
\end{array} \quad \begin{array}{c}
Bb \downarrow T_{11} \\
A
\end{array}
\]

**violin 2:**

\[
\begin{array}{c}
A \downarrow T_{11} \\
G#
\end{array} \quad \begin{array}{c}
\downarrow T_{11} \\
G
\end{array}
\]

**viola:**

\[
\begin{array}{c}
E \downarrow T_{11} \\
Eb
\end{array} \quad \begin{array}{c}
Eb \downarrow T_{11} \\
D
\end{array} \quad \begin{array}{c}
D \downarrow T_{11} \\
Db
\end{array} \quad \begin{array}{c}
Db \downarrow T_{11} \\
C
\end{array}
\]
In Figure 4.16, three networks-of-networks analyse the music of the violin 1, violin 2, and viola in the final measure of the first system of Figure 4.15. The structure of the networks reflects the groupings heard in the music; for example, the first violin’s dyad pairs are clearly distinguished via register, and dyad groups are also established in the other two strings by repeated rhythms. Repetition in rhythm and contour also suggests a four-note grouping in the first violin. All three networks involve pitch classes as their most basic object and T_{11} transformations between objects. In the first violin network, T_{11} occurs recursively at three structural levels (between pitch classes, dyads, and tetrachords), whereas in the second violin and viola T_{11} occurs at two structural levels (between pitch classes and dyads). The small-scale dyad network occurring at the most local level of each network, the dyad-to-dyad network of the violin 1, and the large-scale violin 1 and violin 2 networks are all isomorphic to one another; these are in turn a subset of the viola’s large-scale network. These features, reflected in the formats of the networks, suggest that while pitch classes are the most basic object, the transformation T_{11} itself (rather than the objects upon which it acts) is characteristic of the passage, appearing at multiple levels.
Network structures involving isomorphisms can be used to depict structural similarities between passages, voices, motives, or other musical elements, while implicit and explicit replication can indicate structural similarities between different musical levels within a single passage. Both features are important aspects of the theory developed in this document since they are a means of illustrating continuity, unity, and motivic development. Let us examine such issues in the music of the second system, first measure, of Figure 4.15. This passage, like the previous example, features dyads both generated and transformed by $T_{11}$; an analysis of the measure is given in Figure 4.17. The networks for the violin 2 and viola are isomorphic to the corresponding networks in Figure 4.16, as might be expected given that second excerpt continues processes established in the previous measure. The music of the violin 1, on the other hand, is modified to eliminate the $T_{11}$-related tetrachord groupings established in the previous
measure, instead presenting consecutive $T_{11}$-dyads. This is supported by the rhythmic and motivic sequencing of the violin 1 in these two measures: the four-note repeated pattern in the first system, last measure is no longer repeated in the second system, first measure. Because of this change, the analytical structures given for both the violin 1 and violin 2 in Figure 4.17 can be interpreted as sub-graphs of the viola’s network, implying that the viola music is the principal melodic motive (which manifests the $T_{11}$ transformational motive) of the passage from which the violins’ music is derived. The analysis continues to suggest that the dyads are the characteristic object of the passage, even though they are not the smallest-scale object, and that their internal structure and external organization are closely linked.

An analysis of the music of page 24, last system, gives one example of how identities and other similarities in network structure can identify motivic development. This passage, given within Figure 4.18, continues the repeated application of $T_{11}$, combining this with word-painting of the text “contrapuntal,” sung by the soprano. Two types of material occur here: the boxed notes in the soprano, violin 2, and viola are involved in a $T_{11}$ counterpoint, to be examined shortly, while the bracketed music of the violin 2 reprises the opening motive examined in Figure 4.1 and its continuation (which had previously occurred on page 3 of the score, not discussed in this chapter). In order to capture all the relationships among the voices and their recall of previous material, the network of Figure 4.19 analyses this passage into what is the most elaborate product network we have yet encountered. Below the complete structure in the figure are given sub-networks that correspond to the soprano, violin 2, and viola, respectively. They all share a common sub-network whose nodes {F, E, Eb, D} are highlighted in bold on the
figure; the sequential graph of this common sub-network, which transforms an object three times by $T_{11}$, incorporates the characteristic gesture of the quartet. More specifically, it recalls the three-fold repetitions of $T_{11}$ depicted in the analyses of Figure 4.6 and Figure 4.15. The product network structure best represents the “contrapuntal” interrelations of voices that paints the soprano’s text (to be examined shortly), and the elements of the network structure reflect several features heard in the passage. For example, the vertical arrows are oriented upward in order to indicate that the members of the common sub-network $<F, E, Eb, D>$ are the source for each instrument. F was chosen as the input to reflect that it is the first pitch class of the soprano, violin 2, and viola in this passage. The line $<Gb, F, E, D#$ is replicated enharmonically in two parts of the structure. Each pitch class of this line frames one of the voice’s statements of “contrapuntal”; while we might consider indicating the return to the initial pitch class by a return to the original object within the network, the shift in register in the music from the first instance of each pitch class to its repetition disrupts the sense of melodic continuity and repetition that might otherwise be heard. To express this idea another way, the semitone motion between consecutive pitches associates them much more strongly (via register) than the non-consecutive octave leap between pitch-class repetitions.
Figure 4.18: Schafer, *Seventh String Quartet*, page 24, last system to page 25, beginning of the first system

(opening motive – see page 3 of score)

Figure 4.19: A network incorporating the pitch classes of the soprano, violin 2, and viola during the text "contrapuntal" on pages 24-25 of Schafer’s *Seventh String Quartet*

**Complete structure:**

Legend:

- shaded square = input node(s)
- shaded circle = output node(s)
Soprano:

Violin 2:

Viola:

Extended structure:
This common sub-network isolates a melodic skeleton from which the rest of the passage is derived. The basic continuity of the passage (manifested in this melodic skeleton) is the three-fold repetition of $T_{11}$ in the viola. The product network of that instrument shows F as the input node. From each node in this line, another node is generated by $T_6$, shown by diagonal arrows, and the resulting nodes are also represented as connected into a three-$T_{11}$ network; in other words, the two lines have identical sub-graphs. Thus the viola’s parallel tritones are literally the product, under the definitions presented in the theoretical chapter, of the two simpler networks. The output node of this graph – the node from which no arrows proceed – contains Ab, suggesting a special role for that pitch class. Indeed, it may be heard to motivate the next events of Figure 4.18, which reprises the opening motive of the quartet that strongly emphasizes Ab.

The violin 2 music can also be heard as a two-dimensional product. One component of the product is a three-fold $T_{11}$-network from F, duplicating that of the viola. From each of these nodes, a new node is derived by a constant transformation, as in the viola, but the transformation is $T_1$ rather than $T_6$, making the second violin’s gestures distinctive. Another three-fold network results from these derivations, connecting Gb to D#. The latter node is the output node of this graph; while it does not immediately play an obvious structural role, it helps to establish closure to the passage by recurring as the final note of the violin 2’s subsequent flourish.

The soprano music can be heard as a product of three distinct sub-networks. One component is, as for the two strings, the three-fold $T_{11}$ network from F to D. Since the soprano begins with the dyad $<F, Gb>$, F is temporally prioritized; the soprano subsequently reverses this order for the remaining dyads. The network prioritizes the
notes of the <F, E, D#, D> line – which doubles the viola and violin 2 – by showing the other notes as transformational derivatives of them. From each node of this skeleton network, two separate nodes are derived by a single constant transformation, T₁ (shown by both the vertical and diagonal arrows). These two distinctive sets of nodes correspond to two different elements within the music: the top-front nodes represent the pitch classes immediately preceding each of the skeleton notes, whereas the back-rear nodes represent notes that are one eighth-beat later than their corresponding skeleton notes. The remaining three-fold T₁₁ network at the back-top (from G to E) corresponds to the notes that occur on every second eighth note. To clarify, the top-front, bottom-front, top-back, and bottom-back nodes contain the notes of the soprano’s syllables “con,” “tra,” “pun,” and “tal,” respectively. The product network structure of this analysis has several advantages a network-of-networks does not: it shows the canonic repetition of <Gb, F, E, D#> (and its enharmonic equivalent) in two parts of the structure, and the whole arrangement has E as its output node. This termination on E, starting from F, echoes the first move F to E in the skeleton line. As well, the back face of the figure can be understood as a canonic imitation at T₁ of the front face, emphasizing the contrapuntal nature of the passage.

The network not only depicts events within the boxed passage on Figure 4.18, but can also help to identify how subsequent music is derived from the same transformations. The extended network in Figure 4.19 applies T₁₁ twice more to the five right-most nodes of the original network; the original output nodes are indicated with an empty circle, while the new output nodes are indicated with a shaded circle. The new nodes that result incorporate the pitch classes of several motives within the last system of page 24: the
shaded ovals identify the nodes associated with the soprano’s slowed final iteration of the text “contrapuntal”, following a similar path through the network as seen for previous instances (the syllable “con” corresponds to the top-front D and “tal” to the bottom-back D); the grey boxes identify members of the viola’s last chord within the passage; and the dashed boxes identify the work’s opening motive that returns in the violin 2 music. In other words, the reprise of the opening motive and its accompaniment now sound like they are structured by products of the T
\[ T_{11} \]-recursion. The input to output for the whole network is the same as for the skeleton identified earlier.\(^{64}\)

The music of page 33, first and second systems (Figure 4.20), also develops previously-heard materials involving inverse-step transformations like T
\[ T_{11} \], now integrating them with changes of octatonic collections that we heard earlier. Three of its segments, each involving similar elements, are indicated in Figure 4.21: measures 2-3 of the first system, measure 4 of the first system, and measure 1 of the second system. Within the first segment, the music in measures 2 and 3 of the first system, interval classes 1 and 2 alternate, like the octatonic passage examined earlier in Figure 4.12. For this reason, the objects and transformations of the network in Figure 4.21a will be the same as those of the earlier example: the objects are “pitch-classes as members of an octatonic collection,” and the transformations are the mod8 (octatonic) and mode-altering mod12 transpositions. The violin 1 begins on F, which will be the “input node of the input node” of the network-of-networks in Figure 4.21a. It continues in OCT(C#, D) for the entire measure, repeating the octatonic scale-step transposition in the next dyad. The following measure presents similar motion, except the entire passage has been shifted

\(^{64}\) A second output is F#, seen in the portion of the network associated with the viola’s music. While this output does not reinforce the skeleton line, it does recall the {F, Gb} pairing that began both the soprano and violin 2 in this excerpt.
down two semitones, transposing the collection to OCT(C, D). The descending scale-step is ubiquitous in these two measures, heard between every beamed pair of notes. The two-node networks that incorporate the scale-step transposition are an appropriate choice for the objects of the large-scale network since they emphasize how the scale-step transformation is essential to the structure of the passage. Depicting these two-node networks as the objects of the larger-scale network reinforces the transformational similarities between dyads, while choosing pitch classes related by the octatonic descending scale-step transposition as the smallest-scale objects emphasizes the octatonic orientation of the passage.
In this passage, the previous simultaneity transformations now become the note-to-note transformations (see graph, below)
Figure 4.21: A network mapping the pitch classes of page 33, first system, Schafer’s *Seventh String Quartet*

a) first system, measures 2-3
The repetition of the descending scale-step dyad within the violin 1 asserts several canons, within both the octatonic scale and the chromatic collection. The immediate repetition of the descending scale-step dyad in the violin 1, shifted a scale-step lower, suggests a canon at $8T_{1}^{-1}$. This transformation is associated with the vertical arrows on the network. The reiteration of this relation in the next measure further reinforces this canon, but also establishes a second canon at $^{*}T_{10}$ between the dyad networks of the second and third measures, associated with the horizontal arrows of the network. Thus the entire violin 1 music can be understood as the product of an $8T_{1}^{-1}$ network with a $^{*}T_{10}$ network.

The pitch classes of the viola double those of the violin 1 and can be interpreted through the same portion of the network. The violin 2 and cello double one another, in addition to being an exact transposition (at $^{*}T_{7}$) of the violin 1’s music. These are
represented by the four lowest-left nodes of the network. The arrow directions between the upper and lower nodes suggest that the violin 2/cello dyads are derived via $^*T_7$ from those of the violin 1/viola. Since the four lowest-left and four upper-right nodes share the same graph, the dyad networks of the violin 2/cello pair can be described as having an identical transformational structure to those of the violin 1/viola.

While the network of Figure 4.19 interpreted the music via a strict product network, the decision to combine a product network with a network-of-networks in Figure 4.21a emphasizes several similarities and differences between this and earlier material. For example, the choice of objects and transformations for this analysis highlights further similarities between the networks of Figure 4.14a and Figure 4.21a: the large-scale networks are isomorphic, and both generate their pitch classes via $^8T_1^{-1}$. On the other hand, while the music of page 33, measures 2-4, could be described as a process of sequencing (as for the analysis of Figure 4.18), that interpretation does not take into account the situation depicted in Figure 4.21a, where different object types of the passage undergo the same transformations – a situation discussed in detail throughout this document.

The output of the network suggests a larger-scale continuity within the passage. The output node of the product network, on the lower-right, contains the dyad network involving A and G. The output node of that network, in turn, is G. Thus G can be interpreted as the output node for the entire structure. This choice of output prepares the following passage as it is immediately repeated in the same voices (the violin 2 and cello) at the beginning of the next measure. As will be examined shortly, this continuity is
depicted analytically by retaining the G as the input node of the network for the next measure.

While measures 2-3 and 4 of the first system, page 33, are similar in many ways, the change of rhythm and of simultaneous interval in measure 4 suggests that a different interpretation might be appropriate. Figure 4.21b gives a network that makes the difference clear. This measure is a compression of the previous two measures, both rhythmically (each two-node descending scale-step network lasted two beats in measures 2-3 but one beat in measure 4) and motivically (the recurring dyad motive of measure 4 can be understood to develop as the first descending scale-step of measures 2-3, rather than the entire two measures). The compression eliminates the octatonic transformations from the previous material, but retains the chromatic transformations – the octatonic context is no longer necessary for the new analysis. A limited number of transposition types occur within the passage; for example, the transposition from the first to second pitch class in a beamed pair is always $T_{11}$ (the descending scale-step of the chromatic rather than octatonic space), and the transposition from the violin 2/cello to the violin 1/viola is always $T_6$ (the lower part, to which the input node belongs, is heard as generating the upper part). These two particular transformations occur twice within each

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65 The ‘wedge’ progression depicted by this network could also be represented as seen below (imagine the product network given in part (b), but folded along a diagonal axis such that the two repeated large-scale nodes lie upon one another). In my opinion, however, this visualization makes the product network structure less apparent, which is why I chose to repeat a node (a reasonable choice given that the material depicted by the repeated node occurs twice within the measure anyhow).
beat, acting on the same pitch classes; this repetition and interweaving suggests an analysis via a product network. These smallest-scale product networks each analyze a single beat of the music. Interestingly, the two-node sub-networks involving $T_{11}$ and $T_6$ that form these product networks also form the product network of the viola in the analysis of Figure 4.19, suggesting a motivic connection between the two passages.

Transformations are also repeated from beat to beat, depicted in the larger-scale network: the music of the second beat is generated from the first by $T_2$, while the music of the first two beats generates that of the latter two by $T_{10}$. Because these two transformations are inverses, the input and output nodes of this large-scale network are identical: the path from the input to the output node will always involve a move of $T_2$ combined with a move of $T_{10}$, resulting in a total transformation of $T_0$. This return to the beginning suggests musical closure, and that the music of the inner two beats can be heard as an ornamentation of the initial four pitch classes. The beat-to-beat transformations present a second instance of transformational repetition and interweaving, motivating the decision to incorporate a product network as the largest-scale network. The overall network structure, a product network-of-product networks, suggests that instances of the four-note group heard from one beat to the next are simply transpositional variants of one another, and that the process that determines the progression between four-note groups, $T_2$, is reversed to generate the pairing of simultaneous tetrachords by $T_{10}$. The input node of the input node is the $G$ located within the top-left large-scale node, which as previously discussed suggests a continuation from Figure 4.21a. The output node of the output node, $C$ (located within the bottom-right large-scale node), does not provide an explicit connection to the following passage, yet
this lack of connection itself is appropriate given that the strings drop out in the second measure of the second system, replaced by the soprano (whose music no longer emphasizes the descending scale-step).

The analysis of the second system, first measure can also be understood via the network of Figure 4.21b, beginning with the A in the violin 1/viola corresponding to the same pitch class within the top-right small-scale network. The $T_6$ transformations that previously represented simultaneities now represent both simultaneities (between the violin 1/viola doubling and the violin 2/cello doubling) as well as beamed melodic dyads (for example, in the violin 1). The music of the second beat is generated from that of the first beat by $T_{11}$. This passage uses only a portion of the network in Figure 4.21b, omitting the bottom-right large-scale node as well as the E-to-Bb $T_6$ dyad within the bottom-left large-scale node. Specifically, the first beat of the music corresponds to the product network in the top-right node, the second beat corresponds to the product network in the top-left node, and the \{F, B\} dyad forms a portion of the product network in the bottom-left node.

Looking back across all the examples of this chapter, we see how they all share certain elemental structural features that are brought out, in somewhat various ways, by the network analyses I have presented. First, all networks involve step transformations (in chromatic or octatonic space) and their inverses, and many involve $T_6$, including Figure 4.7b, Figure 4.9b, Figure 4.19, and Figure 4.21b. Second, they all involve either single replication –the repetition of a transformation within one structural level to form a sub-network (especially involving $T_{11}$, repeated thrice in Figure 4.7, twice in Figure 4.9, twelve times in Figure 4.14c, a variety of ways in Figure 4.16 and Figure
4.17, and three times in Figure 4.19) – or self-replication – the action of a transformation at multiple structural levels (notably Figure 4.3, Figure 4.14a, b, and f, Figure 4.16, Figure 4.17, and Figure 4.21a). Later structures recall and elaborate earlier ones.

The given excerpts have demonstrated that $T_{11}$ occurs throughout the quartet at the surface level (from one note to another, such as in Figure 4.3, Figure 4.7b, Figure 4.9, Figure 4.10, Figure 4.16, Figure 4.17, Figure 4.19, and Figure 4.21b), but also between larger dyads (Figure 4.3, Figure 4.14g, Figure 4.16, and Figure 4.17), trichords (Figure 4.7a), tetrachords (Figure 4.16), and so forth; it is the transformation itself (incorporated into explicit and implicit network structures) and not the resulting pitch-class interval that is the motive throughout the work. Schafer develops this scale-step transformation by adapting it to octatonic (mod8) passages such as shown in Figure 4.14a,b, and f and Figure 4.21a.

As mentioned previously, the choice between multiple network interpretations should reflect the musical features of the excerpt. For example, the contents of the input and output nodes of a passage may indicate continuities with the surrounding music, while the choice of object in a network will indicate what element the analyst considers to be most fundamental to the passage; the choice of transformations within this chapter has been used to identify and emphasize repeated processes. Large-scale versus small-scale elements will reflect longer continuities or local connections, respectively. Product networks often suggest there are multiple paths at a single level of the music, while networks-of-networks often imply distinct musical processes at multiple levels of the music, sometimes suggesting a specific hierarchy of events. These two different structures also conceive of musical motives in contrasting ways: the former may
consider a motive as a combination of small-scale objects derived by a repeated pattern of transformations, whereas the latter may consider a motive as the most basic object of the analysis. For either option, the various elements of the network structure can create links with the surrounding material, and will influence later hearings of the piece.
Chapter 5: Analytical Goals and a Broader Range of Analyses

The previous chapters have provided formal definitions for replicative network structures, shown how these can be applied locally to works featuring motivic repetition, and depicted how a common transformational motive, represented as a network, can link gestural recurrences throughout a work. This chapter, which broadens the repertoire of the analyses, will demonstrate how analytical goals shape the choice of network representation and consequently suggest a particular hearing of a passage. The analyses will touch on such factors as continuity (how it is suggested by common structures, choice of input/output nodes, and so forth), motivic return, and implied collections (in other words, how diatonic and octatonic collections can be suggested through a particular organization of motives not belonging to these sets). The final analyses of this chapter will propose replicative network analyses for excerpts from one Baroque and one Romantic work, demonstrating in each case how these network structures can present a perspective that both supports and adds to traditional harmonic analyses.

By now we have seen how product networks and networks-of-networks represent motives and transformational paths in different ways. Several different goals may guide choosing a representation that is best for a given passage. By comparing the network structures, objects, and transformations of one network-type with those of another, we can select a representation that best identifies and relates the significant events of a passage. The choice of representation may also reflect the analytical goal of showing how later passages recall and elaborate earlier ones.
As a first illustration of how these goals shape the choice of representation, consider Bartók’s *String Quartet no. 2*, second movement, measures 1-7 (Figure 5.1). Figure 5.2a interprets measures 5-7 as a network-of-networks. The most basic objects are pitch classes. They are paired off so that the first pitch class of each pair is elaborated by tritone transposition an eighth note later. Each resultant two-node $T_6$ network is transformed by $T_5$ to generate the network of the following beat. The sequence terminates in measure 7 following the repeated pitch class D, but without the $T_6$ elaboration of D (Ab, which is indicated in grey in order to complete the network-of-networks). In this reading, there are four large-scale events (the $T_6$ dyad networks), each with an identical internal structure, and each of the first three are transposed by $T_5$.

**Figure 5.1**: Bartók, *String Quartet no. 2*, second movement, measures 1-7
Figure 5.2: Networks analyzing measures 1-7 of Bartók’s String Quartet no. 2, second movement

a) A network-of-networks:

```
F  \[T_6\]  B  \[T_5\]  Bb  \[T_5\]  E  \[T_6\]  Eb  \[T_5\]  Ab  \[T_6\]
| \[T_6\] | \[T_6\] | \[T_6\] | \[T_6\] |
B   E   A   D
```

b) A product network plus one extra node:

```
F  \[T_5\]  Bb  \[T_5\]  Eb
| \[T_6\] | \[T_6\] | \[T_6\] |
B   E   A   D
```

The passage could also be interpreted as a product network plus one extra node, given in Figure 5.2b. Like Figure 5.2a, it features the transformations $T_5$ and $T_6$, but here they both transform pitch classes, not more complex structures. It consists of two isomorphic sub-graph types: a three-node sequential $T_5$ graph, and a two-node $T_6$ graph. The combination of these two sub-graphs may be heard to create pitch-class continuity in the passage. For example, the eighth notes that pass between the 1st violin, viola, and 2nd violin in measures 5-6 articulate the $T_6$ dyad sub-networks, while the progression of quarter notes links the first member of each $T_6$ sub-network into a sequential $T_5$ sub-network. This is most clearly heard in the 2nd violin during measures 5-7, and is indicated with arrows on the score in Figure 5.1. We can thus hear a $(T_5)$ path from one
T₆ sub-network to the next. These two co-existing graph types suggest multiple concurrent pitch-class continuities.⁶⁶

Although the product network makes no distinction between ‘large-scale’ and ‘small-scale’ objects, its node/arrow structure suggests privileging certain paths over others. For example, the formal priority of the network’s input, B, is reflected by its repetition and metric accent in measures 1-4, suggesting an important role for this pitch class that is fulfilled by its persistence (through repetition) from measure 4 to measure 5. The formal priority of D, which is the output node that concludes the lower T₅ progression, reflects its status as the final pitch class of the passage and the most accented pitch class of measure 7 (through its metric position and instrumental doublings).

Another difference between Figure 5.2a and b is the degree to which they specify event succession, which also determines how flexible they are for explaining events. Figure 5.2a asserts a completely determined path, with only one possible move from each node. Figure 5.2b, however, gives us the option of following multiple paths through the network. For example, in measure 7 we hear Eb moving to D. There is no direct path between these two pitch classes in either Figure 5.2a or b. Figure 5.2a explains the Eb purely as a low-level transformation of A, consistent with the way it follows A in measure 6. In Figure 5.2b, Eb results by T₆ from A, but it also results via T₅ from Bb; like D, it is an output node of this network. Therefore Figure 5.2b attributes to Eb a role as the potential completion of this passage that Figure 5.2a does not.

Likewise, the <E, F, A, Bb> motive in measures 2 and 4 can be understood as a sub-

⁶⁶ Note that these are not the only two possible interpretations of this passage. For example, a third interpretation incorporating T₁₁ rather than T₅ would retain similar dyad network objects and pitch class objects to networks (a) and (b), respectively (although in the former case the dyad networks would require multi-directional T₆ arrows). The network reconfiguration presented above, however, assumes the same two component sub-graphs (and thus the T11 networks would not be reconfigurations in the same sense).
network of Figure 5.2b in which the first pitch class E generates A and Bb, and the last pitch class Bb is also generated by F. These continuities are not all possible in the interpretation of Figure 5.2a since they involve different structural levels.

Such an analysis can also influence our hearing of later events. Figure 5.3 gives measures 216-229 of the same movement, a passage that incorporates transformational motives similar to those in measures 1-7. Its analysis, Figure 5.4, has a graph that reproduces the graph of Figure 5.2b (with different node contents) and extends it by two iterations of T$_5$, and as a result the nodes of the new network constitute the entire aggregate. The contents of the input node and its T$_6$-associate are now D and Ab (the pitch classes repeated in the cello and viola during measures 216-220). D was the pitch class of the output node in Figure 5.2b, and its role as input node here suggests a strong connection between structural processes in the earlier and later passages. For instance, the events of measures 221-224 continue the presentation of T$_6$ sub-networks with their alternation of G and C#, while the melodic line of the second violin in measures 225-229 can be represented by a different sub-network that incorporates C and the two nodes it generates, F# and F. The right-most ‘extension’ nodes of the graph (that is, those that have no counterparts in Figure 5.2) account for most of the accompaniment material in measures 225-229: in the network B and A# in the viola both generate the E of the first violin, and A and D# in the cello are represented by another T$_6$ sub-network. Hearing the graph of Figure 5.2b embedded within that of Figure 5.4 calls attention to the node containing F in the latter figure because it corresponds to the output node of Figure 5.2

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67 The G of the viola remains unaccounted for, but perhaps represents an octave doubling of its previous instance in measures 221-224.
(which previously contained D). Like the D in Figure 5.1, the F ends the motive of the second violin in measure 229 of Figure 5.3.

**Figure 5.3:** Bartók, *String Quartet no. 2*, second movement, measures 216-229

![Figure 5.3: Bartók, String Quartet no. 2, second movement, measures 216-229](image)

**Figure 5.4:** An analysis of measures 216-229, Bartók, *String Quartet no. 2*, second movement

D, Ab = mm. 216-220  
G, Db = mm. 221-224  
C, F#, F = vl. 2, mm. 225-229  
accompanimental events in mm. 225-229: B, A# = vla., E = vl. 1, A, D# = vc.

![Figure 5.4: An analysis of measures 216-229, Bartók, String Quartet no. 2, second movement](image)

Input node of this network has the same content, D, as the output node of the **Figure 5.2b** network.

Embeds graph of **Figure 5.2b**.

Corresponds to the output node of **Figure 5.2b**; not an output node here, but functions as a completion for the vl. 2 motive.

Extends the underlying T₄ progression observed in **Figure 5.2b** by two full iterations (three iterations of the lower T₄ progression).
**Figure 5.4** offers a rich interpretation of **Figure 5.3** that highlights its similarities to measures 1-7. The two analyses demonstrate similar transformational processes that structure their pitch classes (T₆ and T₅) and similarities between their inputs and outputs. We also observed that nodes with an important functional role in the earlier network maintained a correspondingly important role in the second network. Since these advantages arise from the product network features of **Figure 5.4**, they suggest that the product network of **Figure 5.2b** – whose graph **Figure 5.4** embeds – be chosen as the better representation of measures 1-7. By doing so, we show how the later passage is based on the earlier one, and that they both feature the same kind of pitch-class associations. An added advantage is that the product network format of **Figure 5.2b** and **Figure 5.4** demonstrates melodic continuities within measures 216-229 that are not possible in a network-of-networks. For example, the T₆ dyad networks featured in the network-of-networks do not always form logical pitch-class groupings: the \{B, F\} T₆ dyad is shared between the violin 2 and viola, while the \{E, A#\} dyad is distributed among the violin 1 and the second eighth note of the viola (indicated with arrows on the score). There are no obvious musical features linking these pairs of pitch classes. The product network interpretation of **Figure 5.4**, however, incorporates both T₆ and T₅ pitch-class transformations. Thus we can consider the second violin’s melodic line as a combination of these two transformations: F# generates C by T₆, and C generates F by T₅. Again, these melodic continuities are only implicit in a network-of-networks.

This Bartók analysis demonstrates that transformational similarities between non-adjacent passages within a work can influence the network structures chosen for analysis. It also suggests further questions to consider: what elements suggest one interpretation
over another? Do certain layouts of nodes, arrows, and transformations suggest groupings and hierarchies that others do not? Do some types of replicative structures account for surface-level voice-leading events better than others? An analysis of “Portent,” the first movement of George Crumb’s *Zeitgeist* (1988), can address some of these questions.

“Portent” features several passages in which transpositions replicated among its short motives suggest larger-scale groupings and hierarchies. The movement is heavily saturated with small subsets of the whole-tone collection, and other symmetrical collections are indirectly suggested through different combinations of smaller sets. SC 024 is especially prominent, as can be observed in the first system of the movement (Figure 5.5) where groupings of 024 clusters (boxed on the figure) alternate with chromatic material; the groupings are numbered 1-4 for future reference. The four 024 trichords in each grouping combine in various ways: consecutive trichords in the same register form whole-tone collections, simultaneous trichords form six-note diatonic subsets, and the four trichords together form the aggregate. The transformations between these trichords establish a pattern suggestive of a product network, shown in Figure 5.6a for the first boxed grouping in the piano 1. The trichords of the second attack are generated by T₆ from those of the first, and the left-hand material is generated by T₅ from the right. This choice of layout reflects the temporal presentation of each trichord, and also designates the highest trichord of the accented (forzando) attack within its boxed grouping as the input of each network. This ensemble of trichords could also be interpreted as a network-of-networks (Figure 5.6b) in which the right-hand trichord pair forms a network that in turn generates the left-hand pair via T₅. However, the way
that this interpretation privileges the connection between consecutive trichords in each hand seems inconsistent with the various discontinuities between those trichords. First, in the third 024 grouping of the first system the large leap of eighteen semitones between consecutive trichords reduces the impression of linear continuity. Second, the moves between consecutive 024 trichords within one grouping involve different directions and pitch intervals than the moves in the other groupings, lessening the sense that consecutive 024 trichord pairs form a single unit.

Figure 5.5: Crumb, "Portent" from Zeitgeist, first system

Figure 5.6: The recurring 024 network in "Portent"

a) A network for the first boxed motive

\[
\begin{align*}
\{\text{Db, Eb, F}\} & \xrightarrow{T_5} \{\text{Gb, Ab, Bb}\} \\
\{\text{Gb, Ab, Bb}\} & \xrightarrow{T_6} \{\text{G, A, B}\}
\end{align*}
\]

b) A second possible interpretation incorporating a network-of-networks

\[
\begin{align*}
\{\text{Db, Eb, F}\} & \xrightarrow{T_5} \{\text{Gb, Ab, Bb}\} \\
\{\text{Gb, Ab, Bb}\} & \xrightarrow{T_6} \{\text{C, D, E}\}
\end{align*}
\]
Similar aggregate-grouping processes occur in Crumb’s *Processional*, and are described by Ciro Scotto in his article “Transformational Networks, Transpositional Combination, and Aggregate Partitions in *Processional* by George Crumb.” He states:

In *Processional*, the completion of one process and the initiation of a new process often determine the change from one aggregate to the next … I would also like to extend the concept of an aggregate compositional space to include transformations as well pitch classes. Object and process are inextricably intertwined in *Processional*, so aggregate transformational structures often mirror aggregate pitch-class structures. Furthermore, the concept of an aggregate transformational space is one source of the work’s coherence, since the space is closed with regard to the particular transformation. Partitioning the aggregate transformational space generates many of the work’s formal divisions. As I will demonstrate, these surface formations are the result of deeper or hidden processes.

Scotto describes how SC 024 is an important element within *Processional*; it acts not only as the smallest-scale object of his networks, but generates many of the networks’ transformations as well. In addition, he describes an exchange of object and process, where the transformations characteristic of SC 024 (T\textsubscript{2} and T\textsubscript{4}, which in combination can generate this set class from any given pitch class) are swapped with the transpositions commonly heard between instances of this set (in his example T\textsubscript{0}, T\textsubscript{1}, T\textsubscript{6}, and T\textsubscript{7}, the transpositions characteristic of SC 0167). This creates, in his words, “a bond between two set classes that do not rate very highly on any of the conventional similarity scales;” in other words, these two set classes have no intervals (or characteristic transformations) in common, but each is transposed by the transformations characteristic of the other.

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69 Ibid., section 5.
70 Ibid., section 9.
71 Ibid., section 13.
Scotto employs the aggregate partitions defined by his matrices to establish a transformational space. Much of his analysis describes how the musical surface either realizes or does not realize the expectations defined by this space; in the analyses to follow, I will similarly discuss how expectations are realized (or not), although in my case this will be depicted through replicative networks.\footnote{Ibid., section 23. Scotto employs networks-of-networks throughout his analyses, but does not identify them as such; he does not incorporate product networks.} Scotto also discusses grouping pitch classes at analogous positions within his transformational models in order to highlight collections at a deeper level of the music. I will again incorporate similar analytical techniques in order to highlight common larger collections that occur throughout “Portent”.\footnote{Ibid. See his example 16 for further information.}

Returning to the two analyses presented in Figure 5.6, one reason for preferring the product network representation of Figure 5.6a over the network-of-networks representation of Figure 5.6b is that members of 024 that can be interpreted via this structure often participate in larger-scale replicative relationships that recur throughout the work. Figure 5.7 demonstrates how the four boxed 024 groupings in the first system combine to form a multi-tiered product network: at a local level each 024 trichord fills a node of both a two-node \(T_6\) network and a two-node \(T_5\) network (as shown in the product network of Figure 5.6a), while on a larger scale these networks fill the nodes of an identical node/arrow system that combines repetitions of a two-node \(T_9\) graph with those of a two-node \(T_1\) graph (forming another product network). \(T_9\) is the transformation from boxed grouping 1 to boxed grouping 4 in Figure 5.5 and from boxed grouping 2 to boxed grouping 3, while \(T_1\) is that from grouping 1 to 2 and from grouping 4 to 3. This large-
scale transpositional structure involving $T_1$ and $T_9$ will recur at a small scale later in the piece (as will be discussed in connection with Figure 5.15).

Figure 5.7: Larger-scale transformational relations in the first system of "Portent"

![Diagram](image)

Figure 5.7 highlights some features shared by boxes 1 and 2 and boxes 3 and 4, features that articulate different musical functions. The contents of the input node of box 1 are the same as the contents of the output node of box 2, \{Db, Eb, F\}. Thus the last trichord of box 2 provides a sense of return or closure. The rightmost boxes also feature nodes with the same content, \{Bb, C, D\}, also occupying input and output positions, respectively. But here, since the contents of the lower box, 3, temporally precede those of the upper box, 4, in the actual music, the shared node content reflects a musical
continuity or connection, not closure. The opposing functions that we attribute to the same structural feature of the first pair and the second pair suggest that we hear the second pair as a reversal or contrast to the first.

Figure 5.8: The 'aggregate complex' graph

![Diagram of the 'aggregate complex' graph]

The transformational relationships observed in Figure 5.6a return throughout the quartet, often generating a pitch-class aggregate from SC 024. They can all be heard to have the same transformational structure, the graph given in Figure 5.8. When networks with this graph contain the aggregate, it will be referred to as the ‘aggregate complex’. This graph also occurs among other sets. For example, Figure 5.9 gives the music at rehearsal 1, which repeats and transposes a motive based on SC 015 (instances of which are outlined on the score). Figure 5.10 gives an incomplete network that analyzes the repeated motive according to the transpositions observed in the previous passage (sets in grey are not evident in the passage, but are necessary to complete the small-scale product networks). This particular interpretation resembles the network analyzing the opening music in that the small-scale pitch class collections are transposed by both $T_5$ and $T_6$, while the product networks formed by this process are in turn transposed by $T_9$. Further

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74 In some cases an extra pitch class is added to form SC 0157, but for the purposes of the present analysis the smaller set will be considered the object of the passage.
75 In this interpretation, the $T_{11}$ transposition heard between the two sets of the piano 2 results from the combination $T_5 + T_6 = T_{11}$. This transformation could also be depicted via a diagonal path on the network.
resemblances demonstrate that, from a textural viewpoint, the transformations between sets can be heard as analogous to the previous passage: both passages feature a process of sequencing between the piano 1 and piano 2, and both are featured alongside the same glissando motive (which occurs just prior to the beginning of the system but is not shown on the score of Figure 5.9).

Figure 5.9: Rehearsal 1, "Portent"

Figure 5.10: A partial network depicting interactions between instances of SC 015 at rehearsal 1 of Crumb's "Portent"

The ‘missing’ sets of the product networks (those in grey) suggest that several processes of transposition within the passage are incomplete. The piano 1 begins the passage with \{E, D#, B\}, and subsequently moves to a \{Bb, F\} dyad, suggesting a partial
transposition of the trichord by $T_6$. The piano 2 then realizes the $T_6$ transposition in its entirety by stating \{Bb, A, F\}, the complete SC 015 trichord. However, at the same time the piano 2 presents another SC 015 trichord, \{A, G#, E\}, suggesting a second transposition, $T_5$. Thus the music of the piano 1 is transformed via two different paths to generate that of the piano 2. The piano 1 reasserts its original \{E, D#, B\} trichord, while the piano 2 hints at a completion of the established $T_6$ and $T_5$ transformations by its \{Eb, D\} dyad (partially corresponding to the grey trichord in the left-most box of Figure 5.10). The double-forzando chord in the piano 1 seems to interrupt the final transposition of SC 015 to the boxed \{Db, C, Ab\}; however, this chord in combination with its immediate 015 neighbours contains all the unrealized pitch classes of the right-most box of Figure 5.10, \{Gb, F, Db\}, \{C, B, G\}, and \{G, F#, D\} (shown in grey on the network). Thus the $T_5$ and $T_6$ transpositional processes are completed, even though this may not be immediately obvious to the listener.

Other incomplete returns of the graph from Figure 5.10 and its large-scale transformational relations suggest a repeat of the initial motive. Figure 5.11 gives the music surrounding rehearsal 2. At the beginning of the excerpt we hear a series of SC 024 trichords, each pair of which generates its second member by $T_5$ from its first (resulting in two overlapping diatonic collections – an E major scale generated by the ensemble of \{B, C#, D#\}, \{E, F#, G#\}, and \{A, B, C#\} in the first five trichords, and an A major scale generated by the ensemble of \{E, F#, G#\}, \{A, B, C#\}, and \{D, E, F#\} in the second to sixth trichords). These lead to an aggregate complex (box 1 on the figure), whereupon the rhythm, registral layout, and transformational combinations of the opening measures’ 024 motive (Figure 5.5) are reprised at rehearsal 2. The culmination
of the series of $T_5$-generated trichords with an aggregate complex reinforces this transformation within the complex itself. Instances of the aggregate are boxed and numbered on Figure 5.11, and depicted graphically in Figure 5.12. Since the opening passage was comprised of a small-scale product network involving two-node $T_5$ and $T_6$ networks and a larger-scale product network involving two-node $T_1$ and $T_9$ networks, the same structure will be employed here; however, since there are only three aggregate complexes, the analogy to the earlier analysis implies that the large-scale structure is incomplete.\textsuperscript{76} The direction of the arrows on the large-scale network implies that the last aggregate complex, box 3 (represented by the top-left node), is the source of the first two: it generates complex 1 (in the piano 2) by $T_1$ and complex 2 (in the piano 1) by $T_9$. Thus complex 3 fulfills a similar formal role to the first complex of Figure 5.5, thereby suggesting that the music of rehearsal 2 is a reversal of the earlier passage, creating closure by returning, so to speak, to the source.

\textbf{Figure 5.11: Events surrounding rehearsal 2 of George Crumb’s “Portent”}

\textsuperscript{76} A similar graphical device is used by Lewin in chapter 2 of \textit{Musical Form and Transformation} (Lewin 1993).
Figure 5.12: An incomplete network depicting the return of the aggregate complex at rehearsal 2 of George Crumb’s “Portent”

The music around rehearsal 3, given in Figure 5.13, further reinforces the transpositions of the aggregate complex as structurally important. Each statement of the complex is boxed and labelled numerically for future reference. Figure 5.14 analyzes the passage in two ways. Network (a) interprets the passage as a large-scale product network incorporating several previously-described structures, specifically smaller-scale
product networks involving two-node $T_5$ and $T_6$ networks on members of SC 024 and larger-scale product networks involving $T_9$. These boxed larger-scale networks are once again labelled numerically in order to show how they correspond to the score (musical instances at 1 and 3 are identical). $T_5$ is now heard at the largest scale (in place of $T_1$); the emphasis on $T_5$ just before rehearsal 2 discussed above may be understood to prepare $T_5$ for this larger-scale structural role. The columns of Figure 5.14a depict how aggregate complexes are transformed by $T_5$ within each instrument, but the rows also show how the (mostly) piano 1 complexes generate the piano 2 complexes by $T_9$, the large-scale transformation heard earlier among the first aggregate complexes of the movement. This interpretation matches the temporality of the passage in one regard since its input node depicts the first aggregate complex of both instruments (since it is repeated), and the output node depicts the final aggregate complex of the passage in the piano 2.  

Figure 5.13: The music leading to rehearsal 3 of George Crumb’s “Portent”

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77 The temporality, however, does not correspond to all $T_9$ transpositions: for example, complex 5 generates complex 4. Given the close proximity of these complexes and rapid tempo of the passage, however, this disruption is negligible.
Figure 5.14: A network interpreting the material leading to rehearsal 3 of George Crumb’s “Portent”:

a) interpreted as a large-scale product network (a product network-of-product networks)

Piano 2

1. (and 3 in the piano 1)

\{Ab, Bb, C\} \xrightarrow{T_6} \{D, E, F\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Db, Eb, F\} \xrightarrow{T_6} \{G, A, B\}
\end{align*}

Piano 1

2.

\{F, G, A\} \xrightarrow{T_6} \{B, C#, D#\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Bb, C, D\} \xrightarrow{T_6} \{E, F#, G#\}
\end{align*}

5.

\{Db, Eb, F\} \xrightarrow{T_6} \{G, A, B\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Gb, Ab, Bb\} \xrightarrow{T_6} \{C, D, E\}
\end{align*}

4.

\{Bb, C, D\} \xrightarrow{T_6} \{E, F#, G#\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Eb, F, G\} \xrightarrow{T_6} \{A, B, C#\}
\end{align*}

6.

\{Gb, Ab, Bb\} \xrightarrow{T_6} \{C, D, E\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Cb, Db, Eb\} \xrightarrow{T_6} \{F, G, A\}
\end{align*}

7.

\{Eb, F, G\} \xrightarrow{T_6} \{A, B, C#\}
\begin{align*}
&\xrightarrow{T_5} \\
&\{Ab, Bb, C\} \xrightarrow{T_6} \{D, E, F\}
\end{align*}
b) interpreted through a sequential $T_5$ transformation

**Piano 2**

1. \[\{Ab, Bb, C\} \xrightarrow{T_6} \{D, E, F\}\]
   \[\xrightarrow{T_5} \{Db, Eb, F\} \xrightarrow{T_6} \{G, A, B\}\]

2. \[\{F, G, A\} \xrightarrow{T_6} \{B, C, D\}\]
   \[\xrightarrow{T_5} \{Bb, C, D\} \xrightarrow{T_6} \{E, F, G\}\]

3. \[\{Ab, Bb, C\} \xrightarrow{T_6} \{D, E, F\}\]
   \[\xrightarrow{T_5} \{Db, Eb, F\} \xrightarrow{T_6} \{G, A, B\}\]

4. \[\{Bb, C, D\} \xrightarrow{T_6} \{E, F, G\}\]
   \[\xrightarrow{T_5} \{Eb, F, G\} \xrightarrow{T_6} \{A, B, C\}\]

5. \[\{Gb, Ab, Bb\} \xrightarrow{T_6} \{C, D, E\}\]

6. \[\{Gb, Ab, Bb\} \xrightarrow{T_6} \{C, D, E\}\]
   \[\xrightarrow{T_5} \{Cb, Db, Eb\} \xrightarrow{T_6} \{F, G, A\}\]

**Piano 1**
A different interpretation, however, can also demonstrate how Crumb uses familiar large collections to organize larger-scale relationships among aggregates.\textsuperscript{78} Network (b) considers the entire cluster of aggregate complexes in Figure 5.13 to be generated by a sequential application of $T_5$ ($T_0$ is included between complexes 1 and 3 in order to show that this complex occurs in both instruments). The structure of the network suggests that the reiterative process generating the entire series begins with the second complex heard in the piano 2. The passage obviously features the full chromatic collection because of its saturation with aggregates, but also suggests the diatonic collection between corresponding pitch classes in each aggregate complex. This representation may not be as efficient as network (a), but its $T_5$ sequentiality evokes the generating process of the diatonic collection. This enhances the prominence of the diatonic collection that we first notice in the lead-up to rehearsal 2, Figure 5.11.

At rehearsal 12, structures appear that bear transformational similarities to the aggregate complexes, and that imply other familiar large collections (see Figure 5.15). In this passage, SC 015 trichords (pairs of which are boxed on the figure) generate one another via the same combination of transformations heard earlier among the SC 024 aggregate complexes, $T_1$ and $T_9$. Figure 5.16 interprets these events graphically.

Observe that the small-scale networks of this structure are isomorphic to the large-scale network of the first measures of the piece, as shown in Figure 5.5 and analyzed in Figure 78.

\textsuperscript{78} This is discussed by Richard Bass in his article "Models of Octatonic and Whole-Tone Interaction: George Crumb and His Predecessors" (Journal of Music Theory 38 (1994): 155-186). He states that “The ascendancy of aggregate-based atonal and serial methods during the mid-twentieth century may have temporarily relegated these symmetrical referential collections to a subordinate role, but recent works by a number of composers provide evidence that the compositional opportunities offered through interaction between octatonic, whole-tone, and related sonorities…were not exhausted in the early part of the century. Crumb in particular has developed clear and aurally accessible models for the integration of two, and sometimes more, non-diatonic reference sets that stand on a par with diatonic and chromatic writing within the broader spectrum of his eclectic harmonic language” (186). Scotto also cites this passage from Bass (Scotto 2002, section 1).
5.7. In other words, the same system of transformations (T_9 and T_1) occurs on a smaller scale. The large-scale transformations in this passage are now T_2 and T_6, transformations characteristic of the whole-tone collection. Whereas in previous networks whole-tone collections were suggested at a small scale via SC 024, now the large-scale transformations suggest the whole-tone collection. This is the procedure Scotto describes as a transformation from object to process.\(^7^9\)

Figure 5.15: Instances of SC 015 at rehearsal 12 in George Crumb’s “Portent”

\(^7^9\) Scotto 2002, section 9.
Figure 5.16: A network depicting the events of rehearsal 12 in George Crumb’s “Portent”

Just as we heard the music of Figure 5.13 to be structured diatonically on a large scale, we can hear this music structured octatonically. The transformational graph in the left column of Figure 5.16, when applied to the family of pitch classes, suggests the familiar transpositional combination by tritone of 0134s into an octatonic collection. For example, the first pitch class of each 015 statement in boxes 1 and 2 (identified with shaded circles on the score), which occur at analogous positions within each motive and repetition, combine to form the OCT(Db, D) collection. Likewise, the first pitch class of each 015 statement in boxes 3 and 4 combine to form OCT(C, Db). Although the entire pitch-class collection during this passage is chromatic, Crumb’s repetition of each motive statement accents the initial pitch class of each and thus enables the listener to hear larger-scale relationships.

The combinations of SC 024 at rehearsal 13 undergo a similar structuring. The passage is given in Figure 5.17. A network given in Figure 5.18 illustrates the
transformations between instances of SC 024 within the music. Through this network, we can see that the passage is analogous, both in its small- and large-scale transformational structure, to earlier events. Specifically, instances of SC 024 are once again structured via the aggregate complex, a product network incorporating $T_5$ and $T_6$, and these aggregate complexes are transformed by $T_9$ at a larger scale. However, in this case the large-scale network is an ‘incomplete’ version of the graph presented in Figure 5.7. Each set is stated and then retrograded, a different setting than usually heard for the aggregates, perhaps suggesting closure (this is, in fact, the final aggregate complex of the movement).

Figure 5.17: Rehearsal 13 of Crumb’s “Portent”
Overall, similar transformational repetitions organize instances of SC 024, but the same transformational relationships structure motives from other set classes as well (Figure 5.10). This is true for both smaller-scale structures featuring $T_5$ and $T_6$ as well as larger-scale structures featuring $T_9$ and $T_1$. The analyses demonstrated that larger collections are suggested by motivic repetition and rhythmic organization (Figure 5.14 and Figure 5.16). In addition, the network structures presented in these analyses may indicate continuity and closure via repetition or choice of input/output (such as in Figure 5.7).

Furthermore, Figure 5.10 and Figure 5.12 demonstrated that the replicative processes depicted by product networks and networks-of-networks establish specific transformational patterns, suggesting expectations that may or may not be fulfilled in the course of the piece (and whose analyses may be interpreted according to these expectations).  

Analyses that employ product networks and networks-of-networks need not be limited to contemporary works; indeed, many classical and romantic works can be interpreted through repeated transformational processes in their harmonic progressions, sequential passages, and melodic motives. An examination of a brief passage from

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80 As per Scotto 2002, section 23 (as discussed earlier).
Chopin’s *Impromptu in Ab Major*, op. 29 (measures 13-20 of which are given in Figure 5.19), will illustrate how such an analysis can explain a chromatic transition between diatonic harmonies.

**Figure 5.19: Chopin, *Impromptu in Ab Major*, op. 29, measures 13-20**

In this excerpt, several features are immediately apparent. On the downbeats of measures 13-14, the right-hand melody establishes a motive that presents two chord tones, C and Ab, bridged by the incomplete neighbour G. This motive is the basis for a chromatic sequence that, beginning in measure 15, connects the straightforwardly
conventional harmonic progression from measure 13 through the tonic on the downbeat of measure 15 to the ii chord in measure 18. Each harmony of this chromatic progression is a major chord in first inversion – in fact, each beat is an exact transposition of the previous one.

The chromatic progression may seem like a radical interruption to the harmonic progression established up until measure 14; however, the $T_{11}$ transformation that will shortly be demonstrated to structure the chromatic passage is anticipated by the music of measures 13-14. There is a cross-relational conflict between the D on the second beat of each measure in the left hand (a member of the secondary dominant harmony) and the Db on the fourth beat of each measure in the right hand (a member of the inverted $V^7$ harmony), foreshadowing the $T_{11}$ transformation that occurs throughout the chromatic descent. These are identified with shaded circles on the score, as are other instances of the D-Db relation. This D-Db conflict also returns at the conclusion of the chromatic passage in measures 17-18: D is the bass of the final chord in measure 17, and progresses by semitone to the Db in the bass on the first beat of measure 18. The D-Db conflict also acts as a link between the tonal and chromatic materials. The progression linking the Bb major harmony that ends the chromatic progression to the following Bb minor harmony is somewhat weak as it realizes neither the established chromatic descent in all voices, nor a functional diatonic progression. One might argue that the return of the D-Db conflict at this moment further highlights the missing output harmony since Db’s enharmonic equivalent, C#, should be the bass note of the first-inversion A Major harmony.
Three graphs given in Figure 5.20 interpret the chromatic progression of measures 15-17, each emphasizing a different aspect of the music. The objects in each case are pitch classes, representing the roots of the major chords heard on each beat. Mod12 transpositions serve as the transformational group.\textsuperscript{81} Network (a) depicts the music as a series of chord roots (indicated on the score) related by $T_{11}$ (arrow direction indicates whether $T_{11}$ is heard from the preceding or to the following chord root). The Eb harmony in grey does not, as expected, immediately precede the Fb harmony in measure 15, but is implied rather than heard; it does occur slightly earlier, however, as the final harmony of measure 14. On a small scale, each pair of harmonies creates a two-node $T_{11}$ network; on a larger scale, these $T_{11}$ networks are themselves transposed to create a six-node $T_{11}$ sequential network. The overall structure depicts the harmonic structure as saturated with $T_{11}$ at two different structural levels: among chord roots and pairs of chord roots. This interpretation is supported by the chromatic descent among the corresponding pitches of each beat.

\textsuperscript{81} The pitch class content of each beat could be equally well substituted for these objects; major chord roots have been chosen as the objects simply to make the analysis less cluttered.
Figure 5.20: Different interpretations of chromatic harmonic motion in Figure 5.19

a) Transpositions between pairs of consecutive chord roots in measures 14-17
b) A second network interpreting pairs of consecutive chord roots in measures 14-17

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Eb  \( T_1 \)  Fb  \( J \)  Eb  \( T_{11} \)  Ebb
    \( T_{10} \)                        \( T_{10} \)

Db  \( T_1 \)  Ebb  \( J \)  Db  \( T_{11} \)  C
    \( T_{10} \)                        \( T_{10} \)

B  \( T_1 \)  C  \( J \)  Cb  \( T_{11} \)  Bb
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c) A product network depicting transformations between consecutive harmonies and corresponding harmonies in measures 15-17

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Fb  \( T_{11} \)  Eb  \( T_{11} \)  Ebb  \( T_{11} \)  Db
    \( T_{10} \) \( T_{10} \) \( T_{10} \) \( T_{10} \)

Ebb  \( T_{11} \)  Db  \( T_{11} \)  C  \( T_{11} \)  B
    \( T_{10} \) \( T_{10} \) \( T_{10} \) \( T_{10} \)

C  \( T_{11} \)  Cb  \( T_{11} \)  Bb  \( T_{11} \)  (A)
```

While the interpretation of network (a) reinforces the $T_{11}$ recursion prominent within the passage, there is another way to model the passage that might better correspond to elements of pitch direction and temporality in the music. This analysis is given in network (b). While network (a) implied a reversal in the presentation of each dyad through a change of arrow direction, network (b) models the same reversal by a change of transformation, substituting an inversion in place of the earlier transposition. Two different contextual inversions are incorporated into network (b): J inverts each dyad network about its input pitch class (the chord root that is heard first), while K inverts each dyad about its output pitch class (the chord root that is heard last). This network no longer highlights the recursive presentation of $T_{11}$, but it does emphasize $T_{10}$.

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82 A similar distinction was made in the discussion accompanying Figure 3.13.
83 The inversion K is not necessary to form the product network, and is instead a by-product of the composition $T_{10}J$ (left orthography). For this reason, it has been indicated with dotted lines on the figure. If we were to re-interpret this structure as a network-of-networks, K should be omitted since the dyad-to-dyad paths will no longer exist (alternately, if we wish to include K, J should be omitted).
between measure beginnings (the measure-to-measure transformation) as well as the repetition of chord roots from one pair to the next.\textsuperscript{84}

Network (a) highlights the presence of chromatic transposition at multiple levels, but at the expense of other important elements within the passage: for example, the network prioritizes the barring over the indicated phrasing, ignoring the larger-scale four-beat repetition of the passage. Network (c) provides another alternative view. Beginning with the second beat of measure 15 (where the first leg of the sequence begins), this network depicts the transformation of chord roots in two different ways: between consecutive chords, creating a sequential $T_{11}$ network similar to the large-scale graph of network (a); and between the corresponding beats of each measure, creating a sequential $T_{10}$ network. The latter transformation is the inverse of the overall root motion from I to ii, the progression being elaborated by this chromaticism. The input and outputs of the network also suggest connections to the surrounding music. The pitch classes in the Fb Major chord whose root forms the network’s input are themselves generated from those of the preceding chord, Ab Major, by $T_{11}$ and its inverse $T_1$: the Ab is kept as a common tone, while C generates Cb by $T_{11}$ and Eb generates Fb by $T_1$. The anticipated output of the network is A, but this is not immediately realized in the music (and thus it is shaded in grey on the network) – the pitch class A is implied, as was the Eb input of network (a), but withheld until the last triplet-eighth note of measure 20, beat 3, at this point occurring too late to fulfill the earlier expectation. Instead, Bb and B act as outputs: the transpositional relationship between these two chord roots again suggests a chromatic

\textsuperscript{84} In network (b), we can also isolate diatonic fragments by grouping all the small-scale inputs within a row, all the small-scale outputs in a row, and so forth. We could further identify these fragments by creating a $4 \times 3$ product network incorporating $T_{11}$ and diatonic-$T_1$ (similar to my analyses of octatonic transformations in the previous chapter). However, I do not hear these fragments in a tonal context; rather, I hear them as SC 024 trichords generated from the measure-to-measure recursive application of $T_{10}$.
elaboration ($T_{11}$), and Bb is the root of the subsequent ii chord that re-initiates the diatonic progression. As well, the pairing of B and Bb is highlighted at the beginning and ending of the sequence, circled in measures 15 and 17 on the score (Figure 5.19). The following measures are still rich in $T_{11}$ transformations even though the chromatic progression has ended: for example, <Eb, D, Eb> is heard in the right hand on beat 1 of both measures 19 and 20, <Eb, D> is heard in the left hand between beats 1 and 2 of both measures 19 and 20, and {E, Eb} and {Eb, D} are heard as simultaneities from the third to the fourth beats of measure 19.

Product network (c) interprets the roots of the 5th, 6th, 7th, and 8th harmonies of the passage as $T_{10}$ transpositions of the 1st, 2nd, 3rd, and 4th, respectively, but does not make the sequential nature of the passage explicit. Network (d) reconfigures the combination of $T_{10}$ and $T_{11}$ as a network-of-networks in order to highlight this sequencing. Specifically, the new network depicts a four-object network that at a larger scale is repeatedly transposed by $T_{10}$. Once again, the anticipated output does not occur and is thus shaded in grey. In this interpretation, beat-to-beat transpositions are represented by the smaller-scale networks and measure-to-measure transpositions by the larger-scale network. Thus two ‘levels’ of musical activity are represented by two different types of objects.

In spite of their differences, all three networks of Figure 5.20 interpret this music as a chromatic elaboration of a diatonic step progression, a contrast with the surrounding music that is particularly rich in diatonic fifth relations. These elements may be somewhat reconciled via a slightly modified analysis of the chromatic passing chords. The chromatic elaboration is introduced with I moving to bVI (Ab+ to Fb+ on beats 1
and 2 of measure 15), and concludes on ii (Bb- on the first beat of measure 18). The motion from bVI (a borrowing from the parallel minor and the input node of the network of Figure 5.20c and d) to ii is generated by a diatonic fifth within the parallel minor, echoing the diatonic fifth in Ab Major that recurs as a transformation throughout the preceding measures. While the harmonic structure of the preceding measures is not strongly relevant to the present analysis, one might hear measures 15-17 as a chromatic elaboration of the root change from Fb (on beat 2 of measure 15) to Bb (on beat 1 of measure 18), rather than of the change from Ab to Bb. The former pair of harmonies share the same roots as the input and output nodes of networks (c) and (d), and thus networks (c) and (d) could depict a chromatic elaboration of a fifth rather than a second.

The multiple perspectives offered by these analyses of the Impromptu resemble the variety of approaches theorists take to diatonic sequences more generally. Take, for instance, the familiar circle of fifths sequence in Figure 5.21, the Menuet from Bach’s French Suite no. 3. From one point of view, this progression can be understood as the product of transposition by diatonic fifth with transposition by diatonic second. This view is represented by Figure 5.22a. The objects of this network are diatonic step classes that act as harmonic roots, symbolized by upper-case roman numerals, and its transformations are transpositions by diatonic scale-step (a negative integer represents a descending scale-step transposition, whereas a positive integer represents an ascending transposition; note that the scale-step transposition “4” is equivalent to a diatonic fifth, and “1” is equivalent to a diatonic second).85 From another point of view, the

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85 While Roman numerals usually refer to entire chords, I am instead symbolizing chord roots so that the network can be equally applicable to harmonies in the major and minor scales. I considered designating diatonic harmonies as my objects; however, this would not account for V in minor, which is technically a modal borrowing and thus not a diatonic chord. Note as well that the integers depicting scale-step
progression can be understood as a series of diatonic fifth-related chords that are transposed by descending step. This interpretation is depicted as a network-of-networks in Figure 5.22b, incorporating the same small-scale objects and transformations as for the product network in (a). A third interpretation, given in Figure 5.22c, models the progression as a descending-step progression that is subsequently transposed by diatonic fifth.

Figure 5.21: Bach, “Menuet” (French Suite no. 3), mm. 1-16, with harmonic analysis

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transpositions are assigned such that a unison is 0, one scale step is 1, and so forth. This labelling has been chosen so that multiple operations may compose accurately, and is borrowed from Lewin 1987, 16-17. This interpretation is that presented, for example, in Aldwell and Schachter, Harmony and Voice-Leading, 2nd ed. (San Diego: Harcourt Brace Jovanovich, 1989), 250: “In this type of [descending 5th] sequence, the chords are grouped in twos… and each repetition of the two-chord pattern is one step lower than the preceding statement.”
Figure 5.22: Networks modelling the descending fifths (circle-of-fifths) progression in Bach’s “Menuet”

a) as a product network

b) as a network-of-networks

c) another network-of-networks interpretation

There are several advantages to a product network interpretation of this tonal phenomenon. First, the input and outputs of the network have identical contents, the tonic of the key. This suggests that the entire progression is simply a prolongation or
elaboration of this tonic, and also highlights that the progression completes a full cycle around the possible chord roots of the key. The choice of arrows also says something specific in the context of the analysis: for example, the lack of a direct path from IV to VII suggests that VII does not directly result from the -4 transformation, but rather is generated via the -1 transformation from I; and the arrow direction implies that the progression I – VII – VI – V is the basis of the entire progression, elaborated with its -4 associate.

This chapter has presented a range of analyses spanning from the eighteenth through the twentieth centuries, depicting replications among elements of melody and harmony. I have demonstrated how multiple analyses can be equally applicable to a passage of music, each analysis highlighting different textural features, object groupings, or characteristic transformations. In presenting these analyses, I have accomplished several goals. The Bartók examples demonstrated that sharing of content between network inputs and outputs can demonstrate continuity or closure. It also illustrated how an analysis of a passage later in a piece might influence (or be influenced by) an earlier one – in Figure 5.4, for example, the node corresponding to the output of the earlier analysis marked a significant structural point in the music, even if it did not act as the output for the later analysis. Figure 5.4 also provided a representation that accounted for accompanimental events via paths that did not exist in a different network format. Thus our choice of network was determined not only by the similarities to the previous structure, but also by the type of events we wanted to incorporate.

Along the same lines, the similar structure of Figure 5.2 and Figure 5.4 illustrated the strong thematic link between their materials – the choice of objects, but
more importantly the choice of common repeated transformations. Such a similarity occurred in other analyses as well; for example, Figure 5.10 was able to postulate a transformational similarity between the events of the first system and the events surrounding rehearsal 1 even though the two passages featured different objects.

One advantage of these representations is the manner in which their objects may highlight certain groupings that in turn suggest objects or groupings at another structural level. For example, Figure 5.14 and Figure 5.16, two graphs that modelled repeated transpositions between trichords, could also generate the diatonic and octatonic collections when acting on pitch classes. These new collections grouped pitch classes that occurred in corresponding locations within each repeated motive. Similar groupings defined by network structure may also isolate specific aspects of a progression; for example, in the circle-of-fifths progression the three different networks interpreted the progression as a combination of transformations by diatonic second and diatonic fifth at a single hierarchical level (Figure 5.22a), as a series of chord pairs related by fifth (Figure 5.22b), and as a descending second progression that is elaborated via dominant (diatonic fifth) motion (Figure 5.22c). Together these interpretations suggest that we understand the circle of fifths progression as all three of these processes at once.

The Crumb analyses, like the Chopin, also showed how network structures may suggest continuations or expectations that are not realized in the music. Thus several of my analyses have commented on the motivation behind these ‘missing’ nodes – what factors of pitch-class organization, rhythm, instrumentation, and so forth motivate their omission from the actual music. The networks suggest possible transformations based on
previously-established processes within a passage, and the analyses discuss how these transformations are realized (or not).

The analyses of this chapter have dealt exclusively with pitch classes (or objects formed from pitch classes); nonetheless, elements of pitch such as register, direction, and so forth have been influential in choosing one interpretation over another. This is apparent in the analyses of Figure 5.6 and Figure 5.7. Metric elements have also influenced the choice of network in the discussion accompanying Figure 5.20a and b. Thus the analyses suggest not only an interpretation of their objects, but how their objects relate to one another via other criteria. One might also observe that these analyses for the most part involve transposition, something that might be considered a limitation of this system. Transformations incorporated into replicative structures need not be so limited, and may easily be expanded to include other transformations as long as commutativity is maintained (previously discussed in the theory chapter and demonstrated in Figure 5.20b). However, in this chapter I have attempted to include examples whose transformations and replicative processes are clearly audible (as is sometimes not the case with more abstract transformations). I will discuss the limitations of this system, as well as issues of audibility, in more detail in the conclusion of this document.
Chapter 6: Conclusion

Now that we have examined a variety of theoretical and analytical contexts for replicative network structures, it is time to make a few generalizations and to identify and clarify several unresolved matters. This document is by no means a complete examination of this topic, but each chapter has attempted to enlarge the perspective on one or more particular issues, including formal definitions, structural properties of these networks, and their advantages and limitations.

Limitations

Chapter 1 analyzed several short excerpts with replicative networks that asserted repetition, grouping, and hierarchy. By limiting attention to networks-of-networks that could also be reinterpreted as product networks, we were able to compare and contrast analyses of the same passages that incorporated different network structures. The limitations were practical as well, focusing on transformations that are easily audible (because they are commutative and interval-preserving). These limitations were also modeled on Lewin’s conception of product networks and networks-of-networks in *GMIT*, which, to judge from his analytical applications of it, were intended to maximize the ways of hearing a passage. For perspective on this decision, I have presented a few other analyses using networks-of-networks that could not be rewritten as product networks (for example, Figures 1.6, 1.7, and 3.13). Although these point out some interesting features of the music, they were not suitable for comparing the advantages and disadvantages of
both structures. The commutativity requirement need not limit our transformations to transposition. Contextual transformations provide a means of combining transposition and inversion while still maintaining commutativity. This was demonstrated in Figures 1.6-1.11, 3.13, 3.20b and c, and 5.20b.

Differences between Networks-of-Networks and Product Networks

Many of my analyses demonstrated the difference between transformations of small-scale objects (usually pitch classes) and transformations of networks. Transformations of networks are often understood as equivalent to their lower-level counterparts, but this is an over-simplification of the issue: as discussed in Chapters 1 and 2, transformations of networks involve a transformation not only of their objects but also an automorphism on the transformations that label their arrows. It is true that, by limiting our conception of product networks and networks-of-networks to those that involve repeated graphs, the automorphisms do not change the arrow-label transformations. But this does not mean that product networks and networks-of-networks, so constrained, have the same analytical significance.

In a network-of-networks, a network plays a dual role as both an object and a container of objects. Consider, for example, the four-note group and its inversion given in Figure 6.1. Typically one would hear this four-note unit and its repetitions to articulate a ‘motive’ (in the traditional sense). This is reinforced in Figure 6.2 (which duplicates Figure 3.14) where the repetition of this motive, coinciding with a change of contour, articulates a new beginning every quarter note. However, a repetition of dyads
from one eighth note to the next is also audible. If we hear both types of repetition, we
are hearing both the four-note unit and the two-note unit as objects – the former
corresponds to a product network conception of the passage as four-node (pitch-class)
networks transformed to generate the other subsequent and simultaneous four-node
networks, while the latter corresponds to a network-of-networks conception of the
passage as two-node networks, acting as nodes of a larger two-node network that is in
turn transformed (as for the product network) to generate the other subsequent and
simultaneous networks. The two interpretations need not be mutually exclusive: I hear
both object types concurrently.

Figure 6.1: A four-note pitch motive and its inversion, from Hétu’s *Sonate pour piano*, first
movement

![Four-note pitch motive and its inversion](image)

Figure 6.2: A duplication of Figure 3.14, Hétu, *Sonate pour piano*, first movement, measures 17-19

![Duplication of Figure 3.14](image)
Hierarchy

Hierarchy has figured prominently in the discussion and analyses presented here.

A distinction was made between explicit hierarchy, as manifested by network-of-networks, and implicit hierarchy, as manifested by product networks. Events that occur at different spans within the music can be represented hierarchically; we can understand a smaller-scale object grouping within a network-of-networks as a composing-out, in Straus’s terms, of a larger-scale process, and in a product network events that occur over one time span (which I have generally conceptualized as a ‘structural level’ throughout this document) may be represented with a different sub-network than those at another time span. The hierarchies discussed here are established via groupings of object-types. For example, in Chapter 5 I discussed how the progression in Chopin’s Impromptu in Ab Major represents events associated with distinct rhythmic interonsets as different object-types within the network, suggesting several hierarchical levels. The same is true for Figure 4.22, the network analyzing the first and second systems of Schafer’s Seventh String Quartet (a passage that combined octatonic and mode-altering transformations): the smallest-scale objects represented individual pitch classes, while the larger-scale objects represented beamed pairs of eighth notes. These groupings are manifested in the score as identical rhythmic patterns.
Network Structure and Analytical Meaning

This document has discussed how the analyst’s choice of objects, transformations, structure, input/outputs, and so forth crucially informs the reader about certain musical features within the passage. In Chapter 3, some of the factors that might suggest one particular type of replicative structure over another were noted, and we saw how product network and network-of-network interpretations could each express different relationships within the same excerpt. Chapter 4 emphasized how network structure can clarify the grouping of objects into new collections, groupings that were initially suggested through corresponding rhythms, registers, or accentuation in the passage. In both Chapter 3 and Chapter 4, we observed situations where, on the basis of common sub-graphs, we might consider two passages to manifest the same transformational gesture, even in cases where their objects might differ. The discussion of Chapter 5 detailed how the choice of input and output reflected musical objects that were either emphasized strongly via musical context, or that provided a link to previous and subsequent materials. The latter chapter also discussed the network as a combination of transformational paths – how some paths do not exist (explicitly) in some network representations, and thus affect what observations about the passage can be made in association with that representation. The absence of a particular path is itself an analytical observation about the music – that the analyst does not postulate a continuity corresponding to that path in that representation.

In Chapter 5, both the Bartók and Crumb examples presented several situations in which the network structure that analysed an earlier passage can be used to identify transformational/gestural similarities in later passages. Similarly, Chapter 4, in its
analyses of Schafer’s *Seventh String Quartet*, suggested that a single transformation permeates several structural layers of the music (this was particularly evident in the ‘contrapuntal’ excerpt, Figure 4.19, analyzed in 4.20). This saturation demonstrated that the repetition of a single transformational gesture can help to generate motivic cohesion and continuity.

**Expectation and Network Structure**

The transformational perspective of this document understands the relation between two objects as an action performed on one to generate the other, rather than as a measurement of the distance between two objects; the idea of transformation-as-motive (gesture) emphasizes this action-based approach. Because of this gestural orientation, and since product networks and networks-of-networks are so highly structured, they are good for expressing musical implications or expectations that listeners might build up. For example, if a passage begins obviously structured in ways that are characteristic of these types of networks (such as the $T_5$ recursion presented in Figure 1.1, suggestive of a self-replicative network-of-networks), then listeners may expect these structures to continue to unfold and may notice if elements are omitted or missing. Thus much of Chapter 5 discusses the fulfillment or denial of such transformational expectations. For example, Figures 4.19, 5.4, 5.10, 5.12, and 5.20 demonstrate how transpositional replication is suggested based on earlier material or pre-established processes, but that the music does not complete the anticipated network structures.

In some of the transformational literature, theorists construct ‘spaces’. These constructions (the most common example of which is the Tonnetz) present a
comprehensive conceptualization of all the objects in a transformational system – even those that do not appear in a particular composition to be analyzed – and of the transformations that link them. Specifically, a transformational space consists of an array of points (nodes) in a space lattice; along each dimension the arrow from one node to the next corresponds to one particular transformation. In other words, all segments from one node to the next on a particular axis will represent the same transformation. In some cases, these transformations can be associated with voice leading or with a formal distance between the objects.

In many ways, product networks and transformational spaces are similar: they are both constructed on a pattern of repeating transformations, where multiple transformations act on the same objects. In fact, a product network may be understood as a subset of a transformational space. However, there are several key differences between the two entities. First, in many cases (although not all, as evidenced by my analyses of Crumb’s “Portent” in Chapter 5) networks indicate actual musical events, whereas transformational spaces only indicate possible moves. This is reflected in the formal structure of the product network, which specifies input and output nodes as well as arrow direction. The product network is also a finite structure with a specific number of nodes,

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87 This definition is based on Edward Gollin’s introductory material (pages 196-197) in “Some Aspects of Three-Dimensional "Tonnetze"," Journal of Music Theory 42/2 (Autumn 1998): 195-206. Other theorists who discuss spaces include Fred Lerdahl in Tonal Pitch Space (New York: Oxford University Press, 2001); and Dmitri Tymoczko in “The Geometry of Musical Chords,” Science 313/72 (July 7, 2006): 72-74 and “Scale Networks in Debussy,” Journal of Music Theory 48/1 (Spring 2004): 219-294. While most of these articles deal primarily with tonal idioms (transforming triads, diatonic collections, and other tonal constructs), this theory can also accommodate atonal structures; Gollin states “One need not accept Hauptmann’s acoustical biases [towards perfect fifths and major/minor thirds] … in order to adopt a similar system of tokens that identifies tones based on their intervallic environment within tetrachords of some Tn/TnI class” (199).

whereas a transformational space may be infinite and does not necessarily specify directions for its transformations (a *path* through a transformational space will indicate inputs, outputs, and direction, but not the space itself). While the two structures are certainly closely related, networks provide significantly more information about a particular musical passage. Transformational spaces do not specify hierarchy or groupings, either explicitly or implicitly – they are more abstract than event networks.

**Scale and Choice of Networks**

While the analyses of this dissertation might suggest that I prefer product networks over networks-of-networks, this is more a result of the choice of excerpts presented within this document. Specifically, in the small-scale contexts I have examined it is the event-to-event transformations (such as ‘voice-leading’) that dominate our attention. According to the theory of Chapter 2, networks-of-networks are also possible for any smaller-scale passages representable as product networks. However, a more natural role for networks-of-networks might be the analysis of longer, more extended passages. They are best suited for clarifying grouping hierarchies and therefore for identifying composing-out.

**Issues for Future Research**

Some might find it difficult to co-ordinate their initial perceptions and intuitions about music with the analyses I have presented in this document. This theory does not in
itself specify significance based on acoustical factors such as consonance and dissonance, nor does it necessarily take into account elements of tonality. Yet, except for the Bach Menuet, my choice of repertoire presents examples whose structural organization is not immediately aurally apparent. Most are atonal, and thus do not suggest hearings in a particular tonal collection (except, perhaps, the octatonic elements in Crumb’s “Portent”).

When we begin to analyze tonal works, as seen in the Chopin and Bach examples, we have certain expectations established by our familiarity with standard tonal systems: for example, we expect certain functional progressions and resolutions, and therefore we interpret elements of pitch within these contexts. In situations where we cannot refer to our tonal knowledge, however, how might we organize our perceptions of a piece? Repetition, in my opinion, becomes an important feature. Consider the following scenario: Hétu's *Sonate pour piano*, 1st movement, is very clearly in sonata form because it initially presents two contrasting themes, develops these by repetition and fragmentation, and then concludes with a section in which the two themes are recalled in their initial forms. In spite of the work’s atonality, thematic repetition clearly establishes this form.\(^{89}\) It seems reasonable to suppose, then, that repetition is the fundamental organizing principle at smaller scales as well (manifested as groupings, for example).

This is what my network analyses assert. Further avenues of research could integrate this understanding of smaller-scale organization into an analysis of larger-scale forms, particularly for works that do not fall within a standard idiom; in particular, replicative

\[^{89}\text{Some theorists, such as William Caplin in } \textit{Classical Form: A Theory of Formal Functions for the Instrumental Music of Haydn, Mozart, and Beethoven} \textit{ }(New York: Oxford University Press, 1998), consider tonal organization to be an essential component of sonata form; however, they generally do not deny that motivic repetition is also characteristic of this formal model. A further discussion of this topic would be quite long and a significant divergence from the main topic of this document (and as such is not included here).\]
networks seem to be a promising analytical method for establishing grouping structure in atonal, gesture-based works.

The ideas, aired above, about the possible correspondence of network and grouping structure suggests some further avenues for exploration. For instance, “golden section” relationships manifest small-to-large correspondences that may be analogous to those examined in this document, so replicative networks may be useful for analyzing such works. A difficulty to be addressed in such an analysis, however, is that these relations occur between different object types, not particularly conducive to the formalisms established here. This may be resolved by allowing all sorts of isographies, not just the simple ones mentioned here, in order to demonstrate structural similarities in cases where different object types share the same graph. Julian Hook takes a similar approach, previously discussed in footnote 23 (page 31).

Another interesting line of inquiry would be to investigate how aware composers are of replication. For some indication of this awareness, let us return to comments made by Jacques Hétu, whose works were presented in Chapter 3. Hétu, like his mentor and teacher Olivier Messiaen, conceptualizes much of his composition as a process of layering. He states:

Personally, in regards to my compositional style, I see no use in completely abandoning the compositional techniques of the past; I am searching for a

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90 This direction was suggested to me by John Rea at the Perspectives on Music in Canada Symposium, University of Calgary, January 2006; this idea is further supported by John Schuster-Craig (as “John Schuster”) in “Compositional Process in Clermont Pepin’s Quasars,” Sonus: A Journal of Investigations into Global Musical Possibilities 12/2 (Spring 1992): 28-43. He explains that Pépin models formal divisions in his Quasars based on both the Golden Section and the Fibonacci series (36-38), although he does not use networks to model this structure. Golden Section analyses are not uncommon in published musical analyses, but they are not always considered convincing due to their often haphazard interpretations of proportion and structure. Ernő Lendvai, a Hungarian theorist, is the most well-known proponent of these theories (particularly in the works of Bartók); a good summary of his work – including an outline of its weaknesses – can be found in Roy Howat’s “Bartók, Lendvai and the Principles of Proportional Analysis,” Music Analysis 2/1 (March 1983): 69-95.
synthesis of elements from past and present, taking from each that which seems useful to me. In other words, I believe in the possible existence of a style encompassing several systems. A brief analysis of a fragment from one of my works will illustrate and clarify my thoughts.

The first four measures of my Variations for Piano state, in a contracted manner, the essential elements which generate the entire work. In total, one can recognize two sonic planes. On one hand, the extreme registers: these are the melodic declaration of the theme; on the other hand, the middle register: this is its harmonic declaration. The conjunction of these two planes creates the contrapuntal and rhythmic characteristics of this fragment.

First, the theme. Its declaration presents the twelve tones of the chromatic scale but only the first six will have a structural function. The last six tones are merely the transposed retrograde of the first six, at a close variation… Secondly, the harmony: the chords are constructed from a mode previously catalogued by Olivier Messiaen… there is a relationship between this mode and the theme: the last six notes of the latter are also part of the mode. The contrapuntal aspect of this passage is characterized by the imitative treatment of these two sonic planes.  

Hétu identifies several of the elements discussed in connection to the replicative network structures examined here: smaller-scale elements returning at a larger scale (“the first four measures … state, in a contracted manner, the essential elements which generate the entire work”), a separation of these elements into distinct tiers (his “sonic planes”), the conjunction of these distinct tiers (“The contrapuntal aspect of this passage is characterized by the imitative treatment of these two sonic planes”), and a generating
process incorporating both ‘harmonic’ and ‘melodic’ materials (‘there is a relationship between this mode and the theme’). It is not difficult to see how replicative networks would be useful in interpreting this music.

To the extent, then, that repetition is an essential feature of contemporary and classical art music (indeed, of most music), a theory that demonstrates this repetition will provide the analyst with powerful insights into a work’s organization. I hope I have clearly identified and clarified the main formal and analytical issues surrounding replicative networks, including both their advantages and limitations.
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