Aspects of $SU(2|4)$ Symmetric Field Theories and the Lin-Maldacena Geometries

by

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Abstract

Gauge/gravity duality is an important tool for learning about strongly coupled gauge theories. This thesis explores a set of examples of this duality in which the field theories have $SU(2|4)$ supersymmetry and discrete sets of vacuum solutions. Specifically, we use the duality to propose Lagrangian definitions of type IIA Little String Theory on S^5 as double-scaling limits of the Plane-Wave Matrix Model, maximally supersymmetric Yang-Mills theory on $R \times S^2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $R \times S^3/Z_k$. We find the supergravity solutions dual to generic vacua of the Plane-Wave Matrix Model and maximally supersymmetric Yang-Mills theory on $R \times S^2$. We use the supergravity duals to calculate new instanton amplitudes for the Plane-Wave Matrix Model at strong coupling. Finally, we study a natural coarse-graining of the vacua, and find that the associated geometries are singular. We define an entropy functional that vanishes for regular geometries, is non-zero for singular geometries, and is maximized by the thermal state.

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Chapter 1

Introduction

The aim of physics is to provide a quantitative description of the phenomena we observe. Physics contains many theories and models, and in many cases mathematical techniques exist that allow us to extract detailed predictions from them. Some physical systems can be described using different sets of language, which can be very useful. Some insights are more apparent when the description of a phenomenon is formulated in one language, than in another. An area of physics in which this is particularly true is in understanding Quantum Chromodynamics (QCD), the theory that describes the physics of the strong nuclear interaction. The development of various techniques to study this theory is important because two its most interesting aspects, confinement and chiral symmetry breaking, occur in a regime in which the theory is not amenable to perturbative studies [5]. Developing nonperturbative techniques to study this theory is therefore particularly important.

In the case of chiral symmetry breaking, a common technique is to use phenomenological theories such as the Nambu–Jona-Lasinio model [6]. This model is an effective theory for the quarks that can be thought of as arising after integrating out the degrees of freedom of the gluons. This theory has a four fermion interaction with a dimensionful coupling constant. Since it does not have gluons, this theory cannot be used to study confinement. However, it does provide a setting for studying chiral symmetry breaking, a phenomenon that is responsible for giving masses to the quarks. For a detailed discussion, see [7]. Confinement, on the other hand can be studied using lattice gauge theory. This approach consists of finding an appropriate discretization of the field theory so that it can be simulated on a computer using Monte-Carlo methods. In the limit that the lattice spacing is taken to zero, the results of the simulations can be compared with experimental results. Lattice gauge theory is particularly well suited to studying gauge theories in thermal equilibrium, but it is challenging to apply it to processes that happen in real time, such as the strongly coupled quark-gluon plasma produced at the Relativistic Heavy Ion Collider (RHIC), as pointed out in, e.g. [8]. It is useful, therefore to look for other approaches to study gauge theories non-perturbatively.

The approach we will discuss in this thesis is called gauge/gravity duality, and comes from string theory. We will meet the duality in the next section. The key ingredients in string theory that lead to the duality proposal, supergravity and its connection to string theory, and the existence of branes in string theory will be described in the subsequent section.

1.1 AdS/CFT or Gauge/Gravity Duality

Since much of the interesting physics of non-Abelian gauge theories is non-perturbative, it is useful to search for approaches to allow for their study in this regime. One that has proven extremely useful is the so-called AdS/CFT correspondence [9], or more generally the gauge/gravity correspondence. Some time ago, 't Hooft noticed [10] that a perturbative expansion in a gauge theory with a large number of colours can be organized as an expansion in inverse powers of the number of colours. If the number of colours is N, the diagrams that contribute at each order in $1/N$ are those that can be drawn on a surface with a fixed number of handles. The leading term in the expansion comes from diagrams that can be drawn on a plane, the next term from diagrams that can be drawn on a torus or donut, and so on. 't Hooft noticed that this is similar to the diagrammatic expansion in string theory, in which the expansion is also in terms of surfaces with progressively more handles. In string theory, this expansion arises from the world sheets of strings as they join and split when strings scatter from one another. 't Hooft proposed, therefore, that gauge theories with a large number of colours should have an alternative description in string theory. An explicit realization of this, however, proved elusive for decades. The AdS/CFT correspondence provides such a realization.

These ideas arise from considering D-branes, which are extended objects that exist in string theory. In particular, D-branes in string theory can have different descriptions, and it is the existence of these multiple descriptions that motivates the duality. At low energies, D-brane degrees of freedom are described by supersymmetric gauge theories. The gauge theory associated with D3 branes is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. D-branes can also be associated with solutions of supergravity, and we will describe supergravity and this connection in the next section. In the case of D3 branes, the associated geometry is a solution of type IIB supergravity, which has a region that is the space $AdS_5 \times S^5$. It is the relationship between these pictures that gives rise to the AdS/CFT correspondence.

In its most studied form, the AdS/CFT correspondence [9], proposed by Maldacena, relates a maximally supersymmetric gauge theory in four-dimensions, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM), with type IIB string theory on the space $AdS_5 \times S^5$. [1] AdS_5 is five-dimensional Anti-De Sitter space, a maximally symmetric space with constant negative curvature, and S^5 is a five-dimensional sphere.[2] Part of the reasoning for why there should be a correspondence between these particular theories comes from symmetry considerations. $\mathcal{N} = 4$ SYM is a conformal field theory with six scalar fields. The conformal group in four dimensions includes the Lorentz group $SO(1, 3)$.[3] The generators of the scale transformations and so-called special conformal transformations can be combined with this to form the group $SO(2, 4)$. Furthermore, the scalar fields transform as a vector under $SO(6)$. However, $SO(6)$ is the isometry group of S^5, because $SO(6)$ is the group of rotations in six-dimensions, which maps an S^5 back to itself. Similarly, $SO(2, 4)$ is the isometry group of AdS_5. This is because AdS can be represented as an hyperboloid, and $SO(2, 4)$ is the group of rotations in a space with two time and 4 space dimensions that map a five-dimensional hyperboloid back to itself. A full discussion of these points is given in, e.g. [11]. The fact that these theories have identical symmetries, coupled with their related origin through two different pictures of D3-branes, as described above, suggests that this can provide an explicit realization of 't Hooft's proposal. Subsequently a dictionary that allowed the comparison of observables on both sides of the duality was found [15, 16], and since, a vast amount of evidence has accumulated to support the duality; an early review is given in [11]. The duality was also extended to field theories that describe the degrees of freedom living on branes of other dimensions [17]. This saw the extension of the duality to field theories without conformal symmetry, and, as result, to spaces that were not AdS. Indeed, 't Hooft's original proposal was motivated by general considerations, and so there should be many examples for which such a duality should exist. AdS/CFT has therefore given way to the more general concept of gauge/gravity duality, although the term AdS/CFT is sometimes used in the literature to refer to these dualities in general.

This duality has had a number of practical applications. It has been used, for example, to study confinement [18], construct models with both confinement and chiral symmetry breaking [19], suggest a universal bound

[1] Comprehensive reviews of this subject are given in [11], [12], [13].

[2] Throughout this thesis we will denote the n-sphere as S^n.

[3] We will use the standard notation for simple Lie groups given in, e.g. [14].

on the ratio of the shear viscosity to the entropy density of plasmas [20], and more recently to study the strongly coupled quark-gluon plasma at RHIC, e.g. [8].

1.2 Supergravity and Branes

A major development in string theory was the realization that the theory contains not only strings, but also other extended objects called branes [21]. Branes have descriptions in both supergravity and in string theory, and the relationship between these descriptions plays a key role in the gauge/gravity correspondence. In this section we will describe some relevant aspects of supergravity, and then discuss the role of branes in supergravity. We will then discuss the origin of branes in string theory, and the connection to the supergravity description.

Supergravity has been a subject that has received considerable interest. One key reason for this is that following the invention of supersymmetry, it was natural to consider a supersymmetric generalization of general relativity. Supergravity is this generalization. Another motivation comes from string theory (see, e.g. [22, 23] for a full discussion). Quantizing the string produces a spectrum of particles; there is a set of massless particles, and an infinite tower of massive particles. One can then write an effective field theory for these particles. If we consider just the massless particles, at very low energies, this field theory is supergravity. In ten dimensions there are two closed oriented superstring theories, the type IIA theory and the type IIB theory. Since they are closed string theories, vibrations can travel along the world sheet of the string in two directions. The supersymmetries of these theories can have two different chiralities. In the type IIA theory, the chiralities of the supersymmetries for the vibrations that travel to the left and the vibrations that travel to the right are different. For the type IIB theory, they are the same. Due to this difference their particle spectra are different. In the type IIA case they produce particles that correspond to antisymmetric tensor fields with an odd number of indices. In the type IIB case they correspond to an even number of indices. Type IIA supergravity, then, contains fields that have odd numbered antisymmetric tensor fields, and type IIB contains even numbered ones. These higher dimensional antisymmetric tensor fields, though, can be thought of as being like the gauge potential in electromagnetism, but for higher dimensional objects.

The reason for this is the following. Consider electromagnetism. A particle couples to the electromagnetic field through a term in the action

$$\int d\sigma A_\mu \frac{dx^\mu}{d\sigma}. \tag{1.1}$$

Here A_μ is the electromagnetic potential, and $x^\mu(\sigma)$ describes the motion of the particle, and σ parametrizes the path. This could also be written without the explicit dependence on σ as

$$\int A_\mu dx^\mu, \tag{1.2}$$

which expresses the fact that we can choose different parametrizations for the path, on which the result should not depend. To generalize this to higher dimensions it is useful to use the language of differential forms. In this language we can introduce a 1-form $A = A_\mu dx^\mu$ so that we can rewrite the interaction of a particle with the electromagnetic field as

$$\int A. \tag{1.3}$$

Suppose we wanted to generalize this to objects of higher dimension. If we had a two dimensional object, its worldsheet would be described by two coordinates. An analogous expression to (1.1) would be

$$\int d\sigma d\tau A_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \tau}. \tag{1.4}$$

If we were to parametrize the worldsheet using a different set of coordinates, which we should be free to do, this would introduce a Jacobian factor. Since the Jacobian is antisymmetric (e.g. [24]), the only way for this expression to remain invariant after this Jacobian factor is included is if $A_{\mu\nu}$ is antisymmetric. We see then that a two dimensional object naturally couples to an antisymmetric tensor field with two indices. This is directly analogous to the coupling of a particle to a field with a single index, because it can be thought of as being extended in one direction, time. The same thing would happen if we were to consider coupling to objects of higher dimension, and therefore higher dimensional antisymmetric tensor fields naturally carry an analogue of the electromagnetic field sourced by higher dimensional objects.

Now reconsider the type IIA and IIB supergravities. Within type IIA supergravity are antisymmetric tensor gauge fields with odd numbers of indices. As we have described above, antisymmetric tensor fields are the analogues of the electromagnetic field carried by higher dimensional objects.

We see then that these can be thought of as the electromagnetic potential sourced by objects that are extended in an odd number of dimensions. If one of these objects is extended in time, like particles are, then it will be extended in even numbers of space dimensions. For example, this theory contains an antisymmetric field with three indices, suggesting the existence of an object extended in three dimensions. If one of these extended dimensions is time, then it will be extended in two space dimensions, and therefore the theory should have objects extended in two spatial dimensions. In the case of the type IIB theory, the tensor fields have even numbers of indices, so they can be thought of as carrying the electromagnetic potential of objects extended in an odd number of space dimensions. Indeed, corresponding supergravity solutions that describe extended objects of the correct dimension have been known for some time [25]. These are called black p-brane solutions, because they are like higher dimensional black holes.

These p-branes, however, were later recognized to be of great importance in string theory [21], as a connection was understood to exist between these supergravity solutions and D-branes in string theory. Comprehensive treatments of D-branes can be found in [26–28]; we will summarize some of the relevant details here. Open strings have endpoints and therefore some boundary conditions must be imposed at these endpoints. Natural boundary conditions to impose are that the string endpoints are free to move, which, mathematically, are Neumann boundary conditions[4] (e.g. [22]). String theories have a symmetry called T-duality (e.g. [22]) that arises when considering a string theory on a space with a compact direction. A closed string can wrap around a compact direction in the same way that an elastic band can wrap around a pole. T-duality is a symmetry of string theory that exchanges a string wrapping a compact direction of size R with another string carrying momentum around a compact direction of size l_s^2/R, where l_s is a parameter called the string length. These two different pictures of the string are completely equivalent. For open strings, however, this has another effect, which is that it exchanges Neumann boundary conditions for Dirichlet boundary conditions. Dirichlet boundary conditions, in turn, mean that the string endpoints are fixed, rather than free to move. Dirichlet boundary conditions are therefore also natural, and by imposing them, we are creating a region in space for the endpoints of open strings to live. These regions can be thought of as objects in their own right, called D-branes after

[4]Neumann boundary conditions are conditions imposed on the value of the derivative of a function, whereas Dirichlet conditions are imposed on the value of the function itself. See e.g. [29].

the Dirichlet boundary conditions. Indeed, the open string spectrum in the presence of a D-brane contains states that encode fluctuations of the brane, implying that it is an object with its own dynamics. Due to the connection between string theory and supergravity, however, these extended objects should have some description in supergravity, and were identified as the the p-brane supergravity solutions [21].

These complementary points of view are very satisfying. We began this section by considering supergravity, which arose as an effective theory of the massless particles in the string spectrum. This theory had higher dimensional black hole solutions corresponding to higher dimensional objects. These higher dimensional objects provided the source for the higher form gauge fields in the supergravity theory. We also considered the spectrum of open strings. This led to the consideration of strings with ends fixed to some higher dimensional objects. These higher dimensional objects have supergravity descriptions themselves, which are the higher dimensional black holes we had met previously.

1.3 $SU(2|4)$ Symmetric Field Theories

We have mentioned that the gauge/gravity correspondence provides a connection between $\mathcal{N} = 4$ SYM and supergravity on $AdS_5 \times S^5$ via the different descriptions of D3 branes. However, there are other interesting gauge theories for which it would be interesting to have a similar dualities, and the work in this thesis considers a particular set, field theories that have $SU(2|4)$ supersymmetry.

To motivate the consideration of this particular set, we will describe each of theories, discuss some ways in which they can be derived, and list some of their properties. We will show that an interesting property of each of these theories is the existence of discrete sets of vacua.

1.3.1 Plane-Wave Matrix Model

A very interesting development in string theory was the understanding that as the strength of the coupling in type IIA string theory is increased, it appears to pass from a ten-dimensional theory into an eleven-dimensional description, which was named M-theory (e.g. [23]). The Plane-Wave Matrix Model was first proposed in [30] as a microscopic description of this theory on a plane-wave space-time. It is similar to the Matrix theory proposal [31], which provided a microscopic description of M-theory in flat space. In this section we will describe the Plane-Wave Matrix Model briefly.

The plane-wave solution of eleven dimensional supergravity can be found by taking an appropriate limit of either the $AdS_7 \times S^4$ or $AdS_4 \times S^7$ solutions, so that the metric and field strength are given by

$$\begin{aligned} ds^2 =& -4dx^- dx^+ + d\vec{x}^2 - \\ & \left(\left(\frac{\mu}{3}\right)^2 (x_1^2 + x_2^2 + x_3^2) + \left(\frac{\mu}{6}\right)^2 (x_4^2 + \cdots + x_9^2) \right) (dx^+)^2, \\ F_{+123} =& \mu. \end{aligned} \tag{1.5}$$

Here $x^\pm$ are lightcone coordinates, x^i, $i = 1, 2, 3$ and x^a, $a = 4, \ldots, 9$, are coordinates in the transverse directions, and μ is a mass parameter. A first approach to the Plane-Wave Matrix Model action then is to consider the action for a superparticle in this background. This action is most tractable in the light-cone gauge in which it takes the form

$$\begin{aligned} S = \int dt \Big(& (\dot{X}^i)^2 + (\dot{X}^a)^2 - \left(\frac{\mu}{3}\right)^2 (X^i)^2 - \left(\frac{\mu}{6}\right)^2 (X^a)^2 \\ & + \psi^T \dot{\psi} - \frac{\mu}{4} \psi^T \gamma_{123} \psi \Big) \end{aligned} \tag{1.6}$$

where X^i, $i = 1, 2, 3$, and X^a, $a = 4, \ldots, 9$, are bosonic scalars, Ψ is fermionic, μ is the same mass parameter above, dots denote time derivatives, and γ_{123} is a product of Dirac matrices. The bosonic part of the action here is similar to that of an ordinary particle of unit mass attached to a spring with unit spring constant

$$S = \frac{1}{2} \int dt \left(\dot{x}^2 - x^2 \right) \tag{1.7}$$

We can think of the bosonic part of (1.6) as being the action of a particle in a 9-dimensional anisotropic harmonic oscillator, with the fermionic part added to make it supersymmetric. Equation 1.6 is the action for a single D0 brane. To find the action for multiple D0 branes it was argued in [30] that an extension could be found by ensuring supersymmetry. The resulting action is, with some rescaling,

$$\begin{aligned} S = \int dt \, \mathrm{Tr} \Big(& \frac{1}{2} (\dot{X}^i)^2 + \frac{1}{2} (\dot{X}^a)^2 - \frac{1}{2} \left(\frac{\mu}{3}\right)^2 (X^i)^2 - \frac{1}{2} \left(\frac{\mu}{6}\right)^2 (X^a)^2 \\ & + \psi^T \dot{\psi} - \frac{\mu}{4} \psi^T \gamma_{123} \psi - i \psi^T \gamma^i [X^i, \psi] - i \psi^T \gamma^a [X^a, \psi] \\ & + \frac{1}{4} [X^i, X^j]^2 + \frac{1}{4} [X^a, X^b]^2 + \frac{1}{2} [X^i, X^a]^2 - i \frac{\mu}{3} \epsilon_{ijk} X^i X^j X^k \Big), \end{aligned} \tag{1.8}$$

where X^i, X^a and ψ have become $N \times N$ matrices and Tr denotes the corresponding trace.

Another approach to this model came from considering membranes in the plane-wave background. Since membranes are fundamental objects in M-theory, a microscopic description of M-theory in the plane-wave background should also arise from considering the action for membranes in that background. It was shown in [32] that the Plane-Wave Matrix Model action arises as a regularization of the membrane action. Thus the same action arises as a description of M-theory in a plane-wave background from two completely different perspectives. This is also the case for M-theory in flat space [31, 33]. One additional way to find the Plane-Wave Matrix Model is to consider $\mathcal{N} = 4$ SYM on $R \times S^3$ and truncate the theory to the modes that are constant on S^3 [34]. The equations of motion for these modes are equivalent to the equations of motion that derived from the Plane-Wave Matrix Model action.

A convenient way of expressing this model that will help to obviate some of its properties is the form [32]

$$\begin{aligned} S = \int dt\, \mathrm{Tr}\Big(& \frac{1}{2}(D_0 X^i)^2 + \frac{1}{2}(D_0 X^a)^2 + i\psi^{\dagger I\alpha} D_0 \psi_{I\alpha} \\ & - \frac{1}{2}\Big(\frac{1}{3}\Big)^2 (X^i)^2 - \frac{1}{2}\Big(\frac{1}{6}\Big)^2 (X^a)^2 - \frac{1}{4}\psi^{\dagger I\alpha}\psi_{I\alpha} \\ & - \frac{i}{3} g\epsilon_{ijk} X^i X^j X^k - g\psi^{\dagger I\alpha}\sigma^{i\ \beta}_{\alpha}[X^i, \psi_{I\beta}] \\ & + \frac{1}{2} g\epsilon_{\alpha\beta}\psi^{\dagger\alpha I}\mathsf{g}^a_{IJ}[X^a, \psi^{\dagger\beta J}] - \frac{1}{2} g\epsilon^{\alpha\beta}\psi_{\alpha I}(\mathsf{g}^{a\dagger})^{IJ}[X^a, \psi_{\alpha J}] \\ & + \frac{1}{4} g^2 [X^i, X^j]^2 + \frac{1}{4} g^2 [X^a, X^b]^2 + \frac{1}{2} g^2 [X^i, X^a]^2 \Big), \end{aligned} \tag{1.9}$$

where now $i = 1, 2, 3$, $a = 1, \ldots, 6$, $I = 1, \ldots, 4$, $\alpha = 1, 2$, and g is a constant that determines the strength of interaction or coupling between the matrices. The model contains three types of bosonic fields, three scalar fields X^i, six scalar fields X^a, and a gauge field A_0 that enters through the gauge covariant derivative

$$D_0 = \partial_t - i[A_0, \ ]. \tag{1.10}$$

The model also contains fermions $\psi_{\alpha I}$. In addition to gauge symmetry, there are other symmetries of the matrix model. Let us concentrate first on the bosonic terms in the action. If we think of the X^i matrices as being like a vector in three dimensions, then terms in the action that have $(X^i)^2$ are like the lengths of vectors in three dimensions. However, the length of a vector is preserved by rotations, and so such terms are invariant under transformations that preserve the lengths of vectors in three dimensions, which is the group called $SO(3)$. Similarly the matrices X^a are like vectors in six dimensions, and terms that have the form $(X^a)^2$ are invariant under $SO(6)$ transformations. Indeed, all the bosonic terms in the action are invariant under the group $SO(3) \times SO(6)$ because they all take the form of lengths of three- and/or six-dimensional vectors. In the usual language, each term is a singlet.

Now consider the fermionic terms. In ordinary quantum mechanics, fermions do not transform under the rotations $SO(3)$, but instead the covering group $SU(2)$, which allows for them to have half-integer spins. Likewise, in the Plane-Wave Matrix Model, the fermions carry an index α for the spin $\frac{1}{2}$ representation of $SU(2)$, rather than an $SO(3)$ index. In six dimensions, the group $SU(4)$ plays the same role for $SO(6)$ that $SU(2)$ does for $SO(3)$ in three dimensions (e.g. [35]). The fermions, therefore, also carry an index I in $SU(4)$. Like the bosonic part of the model, the terms in the action involving fermions also form singlets, in this case under the group $SU(2) \times SU(4)$.

The symmetries discussed so far are all examples of 'bosonic symmetries', even though they involve fermions, because they are symmetries that rotate bosons into bosons, and fermions into fermions. This model also has supersymmetries that rotate bosons into fermions and fermions into bosons [30], and turn out to be important for understanding a number of aspects of the model [36]. In considering systems with symmetries in quantum mechanics, it is useful to group states by how they transform under those symmetries. This is what is done, for example, when considering the hydrogen atom. A similar strategy can be applied here, and gives useful results [36]. We will return to this shortly.

The model itself has a number of very interesting properties. One interesting feature of this model, as first noticed in [30], is that the model possesses a large number of discrete vacua. To see why this is the case consider the potential for the scalars that transform under $SO(3)$, which is proportional to

$$V \sim \mathrm{Tr}((X^i + 3ig\epsilon^{ijk} X^j X^k)^2). \tag{1.11}$$

here X^i are again the three matrices that transform under $SO(3)$ and ϵ^{ijk} is the totally antisymmetric tensor with three indices, and g is the coupling constant. Vacua of the matrix model are field configurations that minimize this potential, and it was noticed in [30] that the minima occur when

$$X^i = \frac{1}{3g} J^i, \tag{1.12}$$

where the matrices J^i satisfy the algebra of $SU(2)$:

$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k. \tag{1.13}$$

These matrices are $N \times N$, and so there will be vacua of the matrix model that are given by any $N \times N$ dimensional representation of $SU(2)$. Different representations can be formed by taking the matrices to be block diagonal with blocks of different sizes as

$$X^i = \frac{1}{3g} \begin{pmatrix} J^i_{N_1 \times N_1} & & \\ & \ddots & \\ & & J^i_{N_k \times N_k} \end{pmatrix}, \tag{1.14}$$

where the sizes of the blocks satisfy

$$\sum_j N_j = N. \tag{1.15}$$

Supersymmetry has important consequences for these vacua. The bosonic and supersymmetries for the matrix model were shown in [36] to combine into a particular supergroup, called $SU(2|4)$, and an analysis of the representations of states in the model was made. The vacua of the matrix model were shown to sit in the trivial representation of this supersymmetry algebra. The analysis in [36] showed that there were certain states in the matrix model whose energies are the same, regardless of the strength of the coupling constant. The vacua are examples of these states, and therefore the classical vacua (1.12) are vacua of the full quantum theory.

What is the interpretation of these vacua? The matrix model should provide a description of M-theory in a plane-wave background. The vacua of the model, therefore, should correspond to objects in M-theory: M2 and M5 branes. To understand how these arise it is useful to introduce a shorthand notation for describing the vacua. Vacua are given by different block diagonal combinations of representations of $SU(2)$. To keep track of the vacua, we can record the number of times that a block of a given size occurs on the diagonal, and introduce a Young diagram to do so.[5] In this

[5] To avoid potential confusion, the Young diagrams represent the partition of N into a sum of integers, not the symmetrization and antisymmetrization of indices in a representation of $SU(2)$. For Young diagrams for partitions, see, e.g. [37]. Contrast this with the treatment for $SU(2)$ give in, e.g. [38].

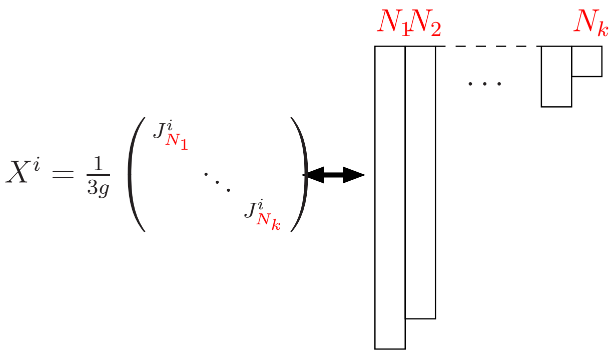
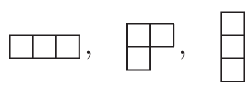


Figure 1.1: Correspondence between matrix model vacua and Young diagrams.

notation, each time an N_i-dimensional representation of $SU(2)$ appears, we add a column to the Young diagram with N_i boxes. For example, suppose the matrices were 3×3. The vacua would be: three copies of the trivial (singlet, or one-dimensional) representation; one copy of the doublet (two-dimensional) representation and a copy of the singlet representation; or one copy of the triplet (three-dimensional) representation. This expresses the fact that $3 = 1 + 1 + 1$, $3 = 2 + 1$ or simply $3 = 3$, where the corresponding Young diagrams are



(1.16)

The general case is depicted in figure 1.1. The matrix model provides a description of M-theory in a limit in which $N \to \infty$. In the Young diagram notation, this is a limit in which the number of boxes is taken to infinity, and the interpretation of the vacua in terms of M2 or M5 branes depends on the way in which this limit is taken [39]. For N finite, the vacua can be interpreted as collections of fuzzy spheres. The reason for this is that if we think again of the matrices X^i as being vectors, the length of the vector $(X^i)^2 \sim (J^i)^2$. However, from the form of $SU(2)$, $(J^i)^2$ is a matrix that commutes with each of the J^i, so it must be proportional to the identity. In a sense, then, $(X^i)^2$ is like a collection of pure numbers (one for each block) that gives the physical radii of a set of spheres. However, the X^i cannot be ordinary coordinates, since they are matrices. Whereas ordinary coordinates commute, matrices do not. In quantum mechanics the fact that position and momentum do not commute leads to the Heisenberg uncertainty relation, which implies that phase space is fuzzy: we cannot perform physical experiments to localize a particle in an arbitrarily small volume in phase space. When two coordinates do not commute, this leads to a similar uncertainty relation, implying that space itself is fuzzy. Therefore the spheres that are described by our vacua are called 'fuzzy spheres', since we cannot resolve arbitrarily small angles on them. The resolution on the sphere, however, is inversely proportional to N_i, the dimensions of the representations of $SU(2)$ [39]. If we take the large N limit that gives M-theory by allowing the $N_i \to \infty$, i.e. by allowing the Young diagram in figure 1.1 to get very tall, then the fuzziness of the spheres goes away, and we can interpret the vacua as spherical M2 branes. The picture for M5 branes is more subtle. Since all of the vacua can be interpreted as fuzzy spheres, it was initially unclear how to represent M5 branes. This interpretation was provided in [39].

The limit that was taken to find M2 branes was a large N limit that arose by taking the dimensions of the representations of $SU(2)$ or the height of the Young diagram to infinity. In the Young diagram notation, however, a large N limit simply means taking the number of boxes to infinity, and we could imagine doing this by taking the width of the Young diagram to infinity instead. How do we interpret this in terms of representations of $SU(2)$? Since in the Young diagram notation we add a column of fixed length for each representation of $SU(2)$ of a given dimension, to get a Young diagram that is very wide we see that we need to add a large number of representations of a given dimension. This is in contrast to what was done for M2 branes, which was to fix the number of times a representation occurred, but take the sizes to infinity. Support for this proposal of taking a large number of representations was provided in [39]. The analysis of [36] showed that in addition to the vacua, there are other states of the matrix model that have energies that are protected from perturbative corrections by supersymmetry. In particular, the spectrum of excitations about the putative M5-brane vacua are protected, and were shown to correspond precisely to the spectrum of fluctuations of the M5 branes [39].

We will mention one further limit of this theory here, which will lead us naturally into a discussion of the other $SU(2|4)$ symmetric field theories, although we will defer a full discussion until Chapter 2. We have learned so far that the Plane-Wave Matrix Model provides a regularization of the field theory on spherical M2 branes. The limit of the matrix model that gives the M2 branes is a large N limit, but it is also a limit in which the coupling constant is taken to infinity in a controlled way. Since this is a field theory that comes from considering D0 branes in type IIA string theory, it is natural to consider whether there is an appropriate limit in which this theory gives a regularization of the field theory on spherical D2 branes. This prescription was given in [39]. For the theory of k D2 branes, one can expand the Plane-Wave Matrix Model about a vacuum with N copies of the k dimensional representation and take N to infinity while keeping

$$g_{\mathrm{YM2}} = \frac{g_{\mathrm{YM0}}}{\sqrt{N}} \tag{1.17}$$

fixed. This provides a direct link between the Plane-Wave Matrix Model and the field theory on spherical D2 branes, which is maximally supersymmetric Yang-Mills theory on $R \times S^2$.

1.3.2 SYM on $R \times S^2$

Another interesting gauge theory with $SU(2|4)$ supersymmetry is maximally supersymmetric Yang-Mils theory on $R \times S^2$. The action for this theory can be written as [39]

$$\begin{aligned} S = \frac{1}{g_{\mathrm{YM}}^2} \int dt \frac{d\Omega_2}{\mu^2} \operatorname{Tr}\Big(& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(D_\mu X^a)^2 - \frac{1}{2}(D_\mu \phi)^2 + \frac{i}{2}\psi^\dagger D_0 \psi \\ & - \frac{i}{2}\epsilon^{ijk}\psi^\dagger \gamma^i x^j D_k \psi + \frac{1}{2}\psi^\dagger \gamma^i x^i [\phi, \psi] + \frac{1}{2}\psi^\dagger \gamma^a [X^a, \psi] \\ & + \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[\phi, X^a]^2 - \frac{\mu^2}{8}(X^a)^2 - \frac{\mu^2}{2}\phi^2 \\ & - \frac{3i\mu}{8}\psi^\dagger \gamma^{123}\psi + \frac{\mu}{2}\phi \epsilon^{ijk} x^i F_{jk} \Big), \end{aligned} \tag{1.18}$$

where x^i is a unit vector on S^2, $d\Omega_2$ measures the solid angle of an S^2 whose radius is inversely proportional to μ. Also, X^a, $a = 1, \ldots, 6$, and ϕ are scalar fields, F is the gauge field strength, ψ is fermionic, D is the gauge covariant derivative and γ^i are Dirac matrices. In addition to the limiting procedure mentioned in the previous subsection, this theory can be found by truncating $\mathcal{N} = 4$ SYM on $R \times S^3$ [40]. Fields on S^3 fall in representations of $SO(4)$, which is isomorphic to $SU(2) \times SU(2)$. If we take fields that are invariant under a $U(1)$ subgroup of one of the $SU(2)$ factors, the resulting theory is maximally supersymmetric Yang-Mills on $R \times S^2$. More recently, this theory has been derived by directly dimensionally reducing $\mathcal{N} = 1$ SYM in 10 dimensions to $R \times S^2$ [41].

This theory, like the Plane-Wave Matrix Model, has $SU(2|4)$ symmetry. The X^a are six scalar fields that transform as a vector under $SO(6)$, but here the $SO(3)$ is the group of rotations of S^2. The theory also has an additional scalar field, as well as fermions. Also, like the Plane-Wave Matrix Model, it has a discrete set of vacuum solutions [39]. These can be found by considering the potential for the spatial components of the gauge field and the scalar field ϕ. This potential can be written in the form

$$V \sim (F_{jk} - \mu\phi x^i \epsilon_{ijk})^2. \tag{1.19}$$

Solutions, which cause this potential to vanish, will simply be

$$F_{jk} = \mu\phi x^i \epsilon_{ijk}. \tag{1.20}$$

However, not all F's are allowed. Having the spatial components of $F \neq 0$, introduces a magnetic flux through the S^2, that is

$$\int_{S^2} \mathrm{Tr}(F). \tag{1.21}$$

This magnetic flux must be quantized, meaning $\mathrm{Tr}(F)$ must be an integer, and since F is a gauge field strength, independent configurations will be given by F having different sets of integers along its diagonal.

The limiting procedure of the Plane-Wave Matrix Model that yields this theory also allows for an interpretation of the vacua of this theory as limits of vacua of the Plane-Wave Matrix Model. We will see, when we discuss the gravity duals of these field theories, that this connection manifests itself nicely in the dual gravity picture.

1.3.3 SYM on $R \times S^3/Z_k$

The final $SU(2|4)$ symmetric field theory we will discuss at this stage is maximally supersymmetric Yang-Mills theory on $R \times S^3/Z_k$ [40], which is a close relative of the theory on $R \times S^2$. In the previous section, we have seen that SYM on $R \times S^2$ can be derived by considering $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, and keeping a set of states invariant under some $U(1)$ subgroup of $SO(4)$. Suppose that instead of keeping states that are invariant under $U(1)$, which are rotations of any amount around a circle, we keep states that are invariant under rotation by some fraction of the whole circle. This group of rotations by $2\pi/k$ is known as Z_k. This is the group of integers modulo k, since k rotations of $2\pi/k$, correspond to a rotation of 2π.

We will note briefly here that in the limit $k \to \infty$, arbitrarily small rotations are allowed. This means that SYM on $R \times S^3/Z_k$ when $k \to \infty$ is just SYM on $R \times S^2$. This relation is also manifest in the dual geometries as we will discuss later.

The vacua of this theory must be gauge configurations for which the field strength vanishes. Unlike the theory on S^3, however, the theory on S^3/Z_k may have nontrivial vacua. Consider taking coordinates on S^3 in which the metric takes the form

$$d\Omega_3^2 = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\phi^2 + (2d\psi + \cos\theta d\phi)^2\right) \tag{1.22}$$

where θ and ϕ are the usual coordinates on S^2, and ψ is an angular coordinate with period 2π. The space S^3/Z_k is obtained by identifying $\psi \equiv \psi + 2\pi/k$. If we consider gauge configurations of the form

$$A = -\mathrm{diag}(n_1, n_2, \ldots, n_N)d\psi, \tag{1.23}$$

the holonomy of the gauge field along the path $0 \leq \psi < 2\pi$

$$W = e^{i\int A} \tag{1.24}$$

will be trivial if the numbers n_i are suitably chosen. We might similarly consider the holonomy of the gauge field along the $0 \leq \psi < 2\pi/k$, which, due to the identification, is a topologically closed but non-trivial cycle. Non-trivial vacua of this field theory will be those that have non-trivial holonomy around the cycle $0 \leq \psi < 2\pi/k$, but that have trivial holonomy around the topologically trivial cycle $0 \leq \psi < 2\pi$. This means that the n_i must be integers so that $e^{2\pi i n_i/k}$ are k^{th} roots of unity. Since the holonomy is a gauge invariant quantity, these configurations will be gauge inequivalent for each unique set of integers $\{n_i\}$, and so we will have a discrete set of vacua parametrized by these sets of integers.

1.4 Gauge/Gravity Duality for $SU(2|4)$ Theories

The $SU(2|4)$ symmetric field theories, as we have described, are an interesting set of supersymmetric non-Abelian gauge theories with discrete vacua. Since gauge/gravity duality provides a way of studying field theories non-perturbatively, to better understand these theories it is useful to search for their gravity duals. One factor that both complicates this search, and also makes it more interesting, is that the field theories have discrete sets of vacua.

Before describing the supergravity duals in detail, it is useful to consider what we expect their properties to be. In the case of the Plane-Wave Matrix Model, this theory is a massive deformation of the theory describing D0 branes, and therefore we would expect that supergravity duals to this field theory would have an asymptotic region of the geometry that looks like the gravity dual for D0 branes. On the other hand, in the case of maximally supersymmetric Yang-Mills theory on $R \times S^2$, since this is the field theory describing spherical D2 branes, we would expect the dual geometries to have an asymptotic region that looks like the gravity dual for D2 branes. Also, as described in section 1.3, the representation theory of $SU(2|4)$ leads to a discrete set of protected vacuum states. This supersymmetric protection implies that the vacua are vacua of the interacting theory, and therefore if we are interested in finding a gravity dual, there should be a gravity solution dual to each vacuum. A key ingredient in such a search for gravity duals, as in the case described in section 1.1, is the symmetry of the field theory. Therefore dual gravity solutions should have the same $SU(2|4)$ symmetry as the field theories themselves. From these ingredients, we would expect that there should be discrete families of supergravity solutions identified with vacua of the $SU(2|4)$ symmetric field theories, and the discrete families associated with the Plane-Wave Matrix Model and maximally supersymmetric YM on $R \times S^2$ should differ in their asymptotics. This is what Lin and Maldacena did [40], and we will now discuss their work.

Lin and Maldacena [40] looked for all $SU(2|4)$ symmetric solutions of type IIA supergravity. These solutions are a special case of $SU(2|4)$ solutions of 11D SUGRA, which they sought in earlier work with Lunin [42]. The special case arises because type IIA supergravity can be found by compactifying 11D SUGRA on a small circle, and so solutions of 11D SUGRA with a translational symmetry are also solutions of type IIA SUGRA. Finding 11D solutions was difficult because the supergravity equations reduced to a non-linear partial differential equation. However, by taking an additional symmetry, it was found that the supergravity equations can be reduced to a simple elliptic linear partial differential equation. Let us outline how this comes about.

$SU(2|4)$ is a supergroup, meaning that it consists of taking a group with generators that satisfy a set of commutation relations, and adding in a set of generators that satisfy anti-commutation relations, in such a way that all (anti-) commutation relations among the generators close amongst themselves. In the case of $SU(2|4)$ this supergroup is an extension of the bosonic group $R \times SO(3) \times SO(6)$, or more precisely its universal covering group $R \times SU(2) \times SU(4)$, as we have mentioned in section 1.3. A starting point for looking for $SU(2|4)$ supergravity solutions, is then to search for solutions in which the isometry group of the solutions is $R \times SO(3) \times SO(6)$. This means that the metric in the space will have one translational symmetry, as well as the symmetries of two- and five-dimensional spheres. Together these symmetries mean that, in the case of 11D SUGRA, the metric can only depend on the three remaining coordinates, or in the case of type IIA supergravity, just two coordinates. This is already a drastic simplification of the supergravity equations. The remaining simplification comes from applying the fermionic part of the supergroup. In supergravity, supersymmetric solutions are sought by setting the fermionic fields to vanish. Because supersymmetry transformations mix the bosonic and fermionic fields, however, to consistently set the fermionic fields to zero we must also make sure that supersymmetry variations of the bosonic fields do not change the fermionic fields. To ensure that this is the case it is necessary to find what is called a "Killing spinor" (see, e.g. [43]). In differential geometry, when a space has a symmetry, there is a "Killing vector" associated with the symmetry. This is loosely analogous to the Noether theorem in classical mechanics relating symmetries with conserved quantities. A Killing vector satisfies the relation[6]

$$\nabla_a v_b + \nabla_b v_a = 0. \tag{1.25}$$

A Killing spinor in supergravity satisfies an equation that schematically takes the form

$$\nabla_a \eta + (\Gamma F)_a \eta = 0, \tag{1.26}$$

where Γ is an antisymmetrized product of Dirac matrices and F is some antisymmetric tensor supergravity field. Killing spinors are particularly useful in finding supersymmetric supergravity solutions because they can be combined with various Dirac matrices to form bilinears that can then be used to constrain the form of the metric in the remaining dimensions. Certain of these bilinears may be Killing vectors; hence the terminology Killing spinor.

In the present case, the analysis of [42] found the supergravity metric to have the form

$$\begin{aligned} ds_{11}^2 = &- 4e^{2\lambda}(1 + y^2 e^{-6\lambda})(dt + V_i dx^i)^2 + \frac{e^{-4\lambda}}{1 + y^2 e^{-6\lambda}}(dy^2 + e^D(dx_1^2 + dx_2^2)) \\ &+ 4e^{2\lambda} d\Omega_5^2 + y^2 e^{-4\lambda} d\Omega_2^2, \end{aligned} \tag{1.27}$$

where λ and V_i depend on a function D, which also appears explicitly. The solution is completely determined by this function D, which was shown to

[6]The defining condition for a Killing vector is that the Lie derivative of the metric along the vector field vanishes. That is equivalent to the form presented here if ∇ is the covariant derivative associated with the metric. We present the equation in this form so that the similarity with the Killing spinor is more apparent.

satisfy the differential equation

$$(\partial_1^2 + \partial_2^2)D + \partial_y^2 e^D = 0. \tag{1.28}$$

We will be interested in solutions that can be reduced to type IIA supergravity, which must have an additional isometry direction. By taking D to be independent of x_1, the change of variables [42]

$$y = \rho\partial_\rho V, \qquad x_2 = \partial_\eta V, \qquad e^D = \rho^2, \tag{1.29}$$

can be used to transform 1.28 into a simple linear form

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho V) + \partial_\eta^2 V = 0. \tag{1.30}$$

With this change of variables, the full supergravity solution now has the form

$$\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V} - 2\dot{V}}{-V''}\right)^{\frac{1}{2}} \left[\frac{-4\ddot{V}}{\ddot{V} - 2\dot{V}}dt^2 - \frac{2V''}{\dot{V}}(d\rho^2 + d\eta^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right], \\
e^{4\Phi} &= \frac{4(\ddot{V} - 2\dot{V})^3}{-V''\dot{V}^2\Delta^2}, \\
C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V} - 2\dot{V}}dt, \\
F_4 &= dC_3, \qquad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt \wedge d^2\Omega, \\
H_3 &= dB_2, \qquad B_2 = 2\left(\frac{\dot{V}\dot{V}'}{\Delta} + \eta\right)d^2\Omega, \\
\Delta &\equiv (\ddot{V} - 2\dot{V})V'' - (\dot{V}')^2 \, .
\end{aligned} \tag{1.31}$$

The $SU(2|4)$ symmetric solutions of type IIA supergravity are then specified by the unknown function V that satisfies the simple equation (1.30).

Some remarks about the supergravity solutions are necessary. One nice feature of equation (1.30) is that it is the Laplace equation for a function in three dimensional coordinates that does not depend on the angle of the axis about the cylinder. However, in three dimensional electrostatics the potential satisfies the Laplace equation, so supergravity solutions will be solutions to axially symmetric electrostatics problems. If we are interested in supergravity solutions that are dual to the Plane-Wave Matrix Model and maximally supersymmetric YM on $R \times S^2$, we have already pointed out that since these theories are related to the theories describing the degrees of freedom of D0 and D2 branes, respectively, we should look for solutions V that give these asymptotic forms. Lin and Maldacena noted [40] that for this to occur the solutions dual to the Plane-Wave Matrix Model must have, for large ρ, η, the form of an electric dipole plus

$$V_\infty = \rho^2 \eta - \frac{2}{3}\eta^3, \tag{1.32}$$

and in the case of SYM on $R \times S^2$, an electric monopole plus

$$V_\infty = \rho^2 - 2\eta^2. \tag{1.33}$$

It should also be noted that not all functions V will give rise to regular supergravity solutions. The solution (1.31) contains S^2 and S^5 factors that may shrink to zero size, and avoiding conical singularities requires imposing additional conditions. Lin and Maldacena showed [40] that in the case of the S^2 this meant that when the size of the S^2 shrinks, $\partial_\rho V = 0$ with η fixed, and that when the size of the S^5 shrinks, V must be regular at $\rho = 0$. In electrostatics $\partial_\rho V$ is just the electric field, and a vanishing electric field implies the presence of a conductor. Taking these facts together, regular supergravity solutions then correspond to regular electrostatics solutions with a set of coaxial circular conducting discs. One additional condition must be imposed at this stage. The change of variables included $x_2 = \partial_\eta V$; for this to be well defined, we must also require that at the edge of the conducting discs $\partial_\eta V = 0$, so that x_2 does not change discontinuously. In the electrostatics interpretation, this means that at the edge of the conducting disc there can be no charge.

Although the problem of finding the supergravity duals to the $SU(2|4)$ symmetric field theories of interest has been reduced to an electrostatics problem, there is still the task of matching the solutions to the vacua of the field theories. One remaining issue in doing this is that there appears to be a continuum of supergravity solutions, but discrete vacua. Lin and Maldacena noted, however, that the presence of the shrinking S^2s and S^5s in the geometry gives rise to topological cycles. See figure 1.2. In analogy with the usual Dirac quantization condition, this implies that some of the higher form fluxes in the geometry have to be quantized. The number of units, N_5, of H_3 flux on a 6-cycle[7] was found [40] to give, in the electrostatics language, the number of units of electrostatic charge on the conducting disc

[7] In electromagnetism the spatial components of the field strength F give the mag-

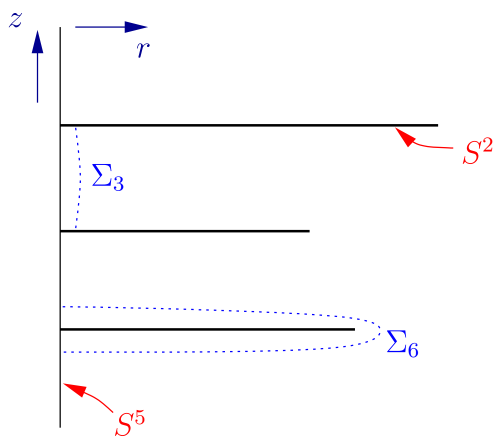


Figure 1.2: Topological cycles in the regular supergravity solutions. A path connecting two S^2s defines a 3-cycle Σ_3, and a path connecting two S^5s defines a 6-cycle Σ_6. We have drawn here only the (r, z)-plane and suppressed the other coordinates.

it surrounds. On the other hand it was found that the number of units, N_2, of $\star F_4$ on a 3-cycle gives the distance between the conducting discs in the electrostatics picture that it connects. By quantizing both the charges on the discs and their relative positions we find that on the gravity side there is also a discrete set of solutions. The fluxes H_3 and $\star F_4$ count the charges of NS5 and D2 branes, respectively, which is reflected in the subscripts N_5 and N_2.

To match the discrete set of supergravity solutions to the field theory vacua, note that solutions that have the same asymptotics as the D0-brane geometry had to have an asymptotic dipole charge, and so have no monopole charge. To implement this we can consider configurations of discs sitting above some infinite conducting plane. By the method of images, each charge above the plane will have a negative opposite charge below the plane, and will look like a dipole asymptotically, and these configurations will then be dual to vacua of the Plane-Wave Matrix Model. On the other hand we may consider configurations that have a monopole moment, which should

netic field (e.g. [44]), so that $\int F$ over a surface counts magnetic charge. There is an operation denoted $\star$ that maps the field strength to another anti-symmetric tensor whose spatial components are the electric field (e.g. [35]) such that $\int \star F$ counts electric charge. The charges we are speaking of here are simply higher dimensional analogues of those in electromagnetism.

correspond to some isolated set of discs. Such configurations will then correspond to vacua of SYM on $R \times S^2$. Finally, there may be configurations that asymptotically look like a line of charge along the axis. Lin and Maldacena [40] showed that configurations with periodic charge distributions correspond to solutions dual to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$.

In the previous section we have pointed out that the $SU(2|4)$ symmetric field theories are related to one another. Let us now show how these relations come about in the dual supergravity picture. SYM on $R \times S^2$ arises by taking a particular large N limit of the Plane-Wave Matrix Model in which the number of copies of the representations $SU(2)$ in the vacuum is taken to infinity. From the electrostatics picture of the gravity duals, we see that the analogous limit is to start with a set of discs above the infinite conducting plane and then take a limit which the set of discs moves infinitely far away from the plane. Similarly, SYM on $R \times S^2$ is related to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ in the limit $k \to \infty$. In the electrostatics picture, the configurations dual to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ are periodic, so to get isolated configurations dual to SYM on $R \times S^2$ we take the period to infinity.

Now that we have described the $SU(2|4)$ symmetric field theories and their supergravity duals, we will introduce two other topics that will be brought into the context of gauge/gravity duality for $SU(2|4)$ symmetric theories in the later chapters.

1.5 Little String Theory

Part of the goal of this thesis will be to provide a connection between the field theories we have just described, and one called Little String Theory. We will now turn to a description of some of the salient aspects of this theory.

As we described above, the massless degrees of freedom living on branes in string theory are described by gauge theories. For lower dimensional branes the descriptions in terms of gauge theories are sensible, since the gauge theories are renormalizable (see, e.g. [17] for a discussion). This is not true for gauge theories on higher dimensional branes. To see why this is the case, consider a pure gauge theory in d dimensions. The action for such a theory can be written as

$$S \sim \frac{1}{g^2} \int d^d x \, \mathrm{Tr}(F^2). \tag{1.34}$$

The $\mathrm{Tr}(F^2)$ term in the action has dimensions of length^{-4} so that for the action to be dimensionless, the coupling constant g must have dimensions of length$^{(d-4)/2}$. If we define the coupling constant at an energy scale, Λ, to be

$$g_\Lambda = g\Lambda^{(d-4)/2}, \tag{1.35}$$

then as we go to higher energies, the coupling becomes stronger if $d > 4$. In this case the gauge theory cannot be renormalized by adding a finite number of counter terms, implying that the theory does not make sense by itself and requires something else to be well understood at higher energies.

This is the situation that occurs for the gauge theories that describe the degrees of freedom living on higher dimensional branes. An interesting class of these theories are called Little String Theories. A review of these theories is given in [45]. Type IIA Little String Theory is the field theory that describes the degrees of freedom living on NS5 branes in type IIA string theory. NS5 branes are in some respects similar to D-branes. Like D-branes NS5 branes have corresponding supergravity solutions. However, D-branes couple to antisymmetric tensors that come from the Ramond-Ramond sector. The Ramond-Ramond sector is composed of bosonic states for which the left- and right-moving parts of the closed string state are fermionic (e.g. [23]). Conversely, the NS5 brane couples magnetically to the Neveu-Schwarz two-form. The NS 2-form is a rank two antisymmetric tensor field that comes from the sector of string states that are created using only bosonic operators (e.g. [23]). The type IIA and IIB theories have the same bosonic sector so they both have NS5 branes. This is in contrast to the states in the Ramond-Ramond sector, which are created using fermionic operators. The IIA and IIB theory have different fermionic sectors and this results in tensor fields of different rank, and therefore D-branes of different dimensions. The other difference is that NS5 branes and D-branes couple differently to the dilaton. [8]

The degrees of freedom that live on NS5 branes should be described by a gauge theory in $5+1$ dimensions. By the above discussion, however, such a gauge theory is not well defined. But Little String Theory turns out to be even more peculiar.

Type IIA or IIB Little String Theories are defined by considering the $g_s \to 0$ limit of a stack of NS5 branes. One interesting fact about Little String Theories is that even in the limit that the string coupling goes to zero, they have non-trivial dynamics. This is unusual, because usually the field theory describing the degrees of freedom living on a stack of branes

[8] The dilaton, denoted by Φ in (1.31), is a field that controls the strength of the coupling between strings (e.g. [22]). D-branes include a coupling to the dilaton $e^{-\Phi}$, whereas NS5 branes include a coupling $e^{-2\Phi}$.

becomes a free field theory in the limit that the string coupling is taken to zero with fixed α'. To see why this does not happen for Little String Theory, consider the case of type IIB Little String Theory. The following argument is due to Seiberg [46]. Consider NS5 branes in type IIB string theory. By S-duality, these are related to D5 branes. The coupling constant for the theory on the D5 branes goes like

$$g_{\mathrm{D5}}^2 \sim g_s \alpha'. \tag{1.36}$$

S-duality takes the combination $g_s\alpha' \to \alpha'$ so that the coupling constant for the theory on the NS5 branes goes like

$$g_{\mathrm{NS5}}^2 \sim \alpha'. \tag{1.37}$$

Now, whereas keeping α' fixed and taking $g_s \to 0$ in the D5-brane theory implies that the theory becomes weakly coupled, we see that in the case of the NS5-brane theory that is not true. Therefore, even when the string coupling vanishes, the theory of the NS5-brane degrees of freedom remains non-trivial.

Another interesting fact comes from T-duality. The action of T-duality maps NS5 branes in type IIB string theory to NS5 branes in type IIA string theory, and vice-versa. The action of the T-duality sends a circle with radius R to a circle with radius

$$R' \sim \frac{\alpha'}{R}. \tag{1.38}$$

Note that this does not involve g_s, so that T-duality should persist even in the limit that $g_s \to 0$ that defines Little String Theory. Field theories are usually thought of as theories of particles, which are not capable of winding around compact directions. Yet Little String Theory, the putative field theory describing the degrees of freedom living on NS5-branes, exhibits T-duality, a symmetry intrinsically tied to winding states. This is a peculiar feature of Little String Theory that means it is a rather interesting theory.

1.6 Gravitation and Entropy

One of the successes of string theory is the ability to account for the microstates of black holes. General considerations of black holes led to a comparison of the evolution of classical black holes to the laws of thermodynamics; for a summary see e.g. [47]. Part of this comparison required that the area of the horizon of a black hole be proportional to the thermodynamic entropy. The discovery that black holes radiate like thermal bodies suggested that this analogy could be taken more seriously, and that black holes do have an entropy. This position seems to lead to some difficulties, however.

Investigations into black holes in classical general relativity have led to the intuition that there is a unique black hole solution for a given set of asymptotically measurable charges (see, e.g. [47]). Statistical mechanics, however, dictates that entropy arises from counting the microstates of a particular macrostate. Black hole uniqueness would then seem to imply that such microstates should not exist. Understanding the microscopic origin of black hole entropy is therefore an important question.

In string theory this origin was first understood in [48] for D-branes wrapping cycles in the space $K3$, and later for the D1-D5-P system [49]. We will focus our discussion on subsequent work that is similar in spirit to the original analyses, and which will segue into work done in this thesis.

A similar study can be made of a gravitational singularity arising from a string wrapping a compact dimension. Lunin and Mathur [50] (see also the review [51]) considered IIB string theory on $T^4 \times S^1$. They considered a string wrapping a large number of times around the S^1 and carrying a large number of units of momentum around the compact circle. There is a singular solution of type IIB supergravity that carries the appropriate charges that arise from such a wrapped string. To carry momentum in the compact direction, there must be travelling waves on the string, and since strings do not carry longitudinal waves, the waves must be transverse. The microstates that give this singular solution an entropy are taken to be different ways of distributing the momentum in the compact direction among the modes of oscillation. However, whereas the supergravity solution indicates that the singularity is a point, the string that gives rise to the singularity must be spread out by the transverse oscillations. The singular supergravity solution does not, then, correspond to any string state, which appears to give a contradiction.

The singularity produced in this case was a naked singularity. Sen had argued previously [52] that such singularities should be given 'stretched horizons' at the location where the curvature becomes string scale. Supergravity should only be valid when the radius of curvature is large compared to the string scale, at which stringy effects become important. It is expected that these stringy effects would correct the geometry in regions with high curvature, and provide a horizon that cloaks the naked singularity–Sen's stretched horizon.

Lunin and Mathur [50] considered all of the different ways that the momentum could be distributed in transverse waves. They observed that for a string carrying a typical distribution of transverse waves, the profile of the string would fill out to the distance of the stretched horizon. Outside the stretched horizon the geometry would look like the singular solution, however inside the horizon the geometry would be quite different. This presents an interesting picture of the geometry inside the horizon. This geometry is frequently given the term 'fuzzball' in the literature because the string fills up the region inside the horizon [51].

In chapter 5 we will study an analogous phenomenon in the context of $SU(2|4)$ symmetric supergravity solutions and the dual field theories. In this case the singularity is replaced by a region of space with complicated topology.

1.7 Plan

In the remainder of this thesis we will apply and tie together the topics we have described above. The focus of chapters 2, 3 and 4 will be on taking information from gravity to learn about the dual $SU(2|4)$ symmetric field theories. In the next two chapters we will find a Lagrangian definitions of Little String Theory as limits of the $SU(2|4)$ supersymmetric field theories. The aim of these chapters is to make progress towards a better understanding of what this theory is, and to clarify physics that arises in the various limits that can be taken in the $SU(2|4)$ supersymmetric field theories. In chapter 4 we will find general $SU(2|4)$ supersymmetric solutions of type IIA supergravity by solving the problem formulated by Lin and Maldacena. The purpose of finding these solutions is to provide the information that will allow us to use duality to study the $SU(2|4)$ gauge theories at strong coupling in detail. As an example of this, we use some of the solutions to calculate new instanton[9] amplitudes in the Plane-Wave Matrix Model. In chapter 5 we will use information from the gauge theory to learn about the dual geometry in the gravitational theory. We use statistical mechanics to count the microstates of singular geometries that arise from taking a coarse-grained view of the regular supergravity solutions, and define an entropy function for coarse-grained geometries. Chapter 6 provides a summary of the main results and discusses future directions. Various technical details are collected in the appendices.

[9] See appendix A for a discussion of instantons.

Chapter 2

Little String Theory from the Double-Scaled Plane-Wave Matrix Model

2.1 Introduction

As we have described in the introduction, type IIA Little String Theory [46, 53] describes the degrees of freedom of NS5-branes in type IIA string theory in the decoupling limit in which $g_s \to 0$ with α' fixed (i.e. focusing on energies of order $(\alpha')^{-1/2}$). It is believed to be a six-dimensional interacting non-gravitational theory with a Hagedorn density of states.[10] In the infrared, the theory flows to the interacting (0,2) conformal field theory, but in general the theory does not have the properties of a local quantum field theory, as it exhibits T-duality. For a review of Little String Theory, see [45].

Since there is no direct Lagrangian description of the theory (though a DLCQ [11] formulation [54] and a description via deconstruction [55] have been proposed), the main tool for analyzing Little String Theory has been its gravity dual, the near-horizon NS5-brane solution of type IIA string theory. This is given for large r, where the IIA picture is valid, by

$$\begin{aligned} ds^2 &= N_5 \alpha' (-dt^2 + d\vec{x}_5^{\,2} + dr^2 + d\Omega_3^2) \\ e^{\phi} &= g_s e^{-r} \end{aligned}$$

with N_5 units of H flux through the S^3. However, even this description is problematic since the dilaton, which controls the string coupling, varies with the coordinate r, which sends the theory to strong coupling in the infrared part of the geometry. This, in turn, means that the description in that region must be in terms of M-theory.

[10] A Hagedorn density of states means that the density of states grows exponentially with energy.

[11] DLCQ, or discrete light-cone quantization involves formulating quantizing a theory on a space with a periodic light-like direction. The Hamiltonian operator evolves the theory in this periodic light-like direction.

Lin and Maldacena [40], however, have found a related supergravity solution in which the flat five-dimensional part of the geometry corresponding to the spatial NS5-brane worldvolume directions has been replaced by an S^5. We reproduce the (somewhat complicated) full supergravity solution in appendix B, but for large radius, the metric and dilaton become simply

$$\begin{aligned} ds^2 &= N_5\alpha' \left[2r\left(-dt^2 + d\Omega_5^2\right) + dr^2 + d\Omega_3^2\right] \\ e^\Phi &= g_s e^{-r} , \end{aligned}$$

again with N_5 units of H flux through the S^3. This new solution retains the linear dilaton behaviour and constant volume S^3 permeated by H-flux, so it is natural to associate this solution with NS5-branes on S^5. An advantage of this solution over its flat cousin is that, in this case, the full solution has a tunable maximum value for the dilaton at $r = 0$ and a tunable maximum curvature, so there is a regime in which a type IIA supergravity description is everywhere valid.[12] [13]

The main goal of this chapter will be to explicitly describe a field theory dual for string theory on this solution, and thus a Lagrangian field theory definition of Little String Theory on S^5.

The broader context for our story is a D0-brane quantum mechanics analogue of the model of Polchinski and Strassler [58] for D3-branes. The field theory we consider is the Plane-Wave Matrix Model [30], a mass-deformation of the maximally supersymmetric D0-brane quantum mechanics. In our case, the mass deformation is maximally supersymmetric, preserving 32 supercharges including an $SU(2|4)$ symmetry [36, 59, 60]. The theory has a discrete spectrum, a dimensionless parameter that acts as a tunable coupling constant [32], and a large number of degenerate supersymmetric vacua preserving the $SU(2|4)$ supersymmetry.

Before the mass-deformation, the D0-brane quantum mechanics is dual to string theory on the near-horizon D0-brane solution of supergravity [17]. As with the near-horizon NS5-brane solution discussed above, this becomes strongly coupled in the infrared region of the geometry, and we must go to an eleven-dimensional description. However, the mass-deformation provides an infrared cutoff for the theory, so we might expect that solutions dual to the various vacua of the Plane-Wave Matrix Model (for large enough mass) would have a IIA description that is valid everywhere.

[12] Unfortunately, the solution contains Ramond-Ramond fields, so string theory is difficult.

[13] A similar situation occurs in [56, 57], though in the present case, more of the R-symmetry is preserved.

This picture was verified by Lin and Maldacena [40]. As we have described above, following [42], they searched for type IIA supergravity solutions with $SU(4|2)$ symmetry, and showed that solutions of type IIA supergravity corresponding to all vacua of the Plane-Wave Matrix Model could be constructed in terms of the solutions to a class of axially symmetric electrostatics problems involving charged conducting discs in 3 dimensions. While they did not solve the electrostatics problem, except in certain limiting cases, they showed that solutions of this type generally contain throats with non-contractible 3-spheres carrying H-flux. These regions of the geometry may be associated with fivebrane degrees of freedom described by the matrix model.

In this chapter, we focus on the simplest class of vacua of the Plane-Wave Matrix Model, for which the dual geometry has only a single NS5-brane throat. We solve the appropriate electrostatics problem to find an exact supergravity solution, and determine the precise limit of this solution (which depends on three parameters) needed to decouple the throat region, yielding the explicit infinite-throat solution of Lin and Maldacena, corresponding to Little String Theory on S^5. By understanding how the parameters of the supergravity solution match up with the parameters of the gauge theory, we then see what this limit corresponds to in the matrix model. We find that the corresponding limit is a limit of large N with the 't Hooft coupling also taken to infinity in a particular way, roughly $\lambda \sim \log^4(N)$, while focusing on the excitations around a specific vacuum of the matrix model (the one corresponding to our original supergravity solution).

Since we are taking a strict large-N limit in the field theory, we might naively expect that the corresponding gravity dual should be a free string theory. This is true asymptotically, due to the linear dilaton background, but not for finite values of r, so there should still be a genus expansion[14] on the string theory side. We conjecture that this is reproduced in the gauge theory in a way very similar to the double-scaling limits used to describe low-dimensional string theories in terms of matrix models.[15] In our case, the 't Hooft coupling is scaled towards a critical coupling $\lambda_c = \infty$ in such a way that the various terms in the matrix model genus expansion all contribute, despite N being infinite. Assuming this picture is correct, we are able to make predictions for the large λ behaviour of the full set of genus n diagrams in perturbation theory (section 2.6.1).

[14]Genus refers roughly to the number of handles in the string worldsheet. Interacting strings are described by worldsheets with progressively larger genus, or more handles, as the interaction becomes more complicated.

[15]This conclusion was predicted by Herman Verlinde.

This chapter is organized as follows. In section 2.2, we review various aspects of the Plane-Wave Matrix Model, including decoupling limits that have been discussed in the past. In section 2.3, we review the Lin-Maldacena ansatz for gravity duals to the matrix model vacua and the electrostatics problems that need to be solved in order to find the solutions. We then provide an exact solution for the simplest such problem, which requires determining the potential due to parallel charged conducting discs in a specified background potential. In section 2.4, we discuss the matching of parameters between the gravity solutions and the matrix model. In section 2.5, we determine how to scale the parameters in our solution to obtain the Lin-Maldacena solution for Little String Theory on S^5, and then use the correspondence with the gauge theory to determine the matrix model description of Little String Theory on S^5. We also discuss the limit that gives a solution dual to the maximally supersymmetric theory of D2-branes on S^2, and the gravity interpretation of the 't Hooft limit of the matrix model. In section 2.6, we discuss the results. In 2.6.1, we explore the consequences of our result that the Little String Theory on S^5 is obtained as a double scaling limit of the matrix model. In section 2.6.2, we describe an infinite-parameter family of supergravity solutions similar to (and with the same symmetries as) the Lin-Maldacena solution, and speculate on the matrix model description of these. Finally in section 2.6.3, we describe an application of our results to calculating energies of near-BPS states[16] in the geometry. We collect some of the more technical details in the appendices.

2.2 Gauge Theory

Let us recall briefly some properties of the Plane-Wave Matrix Model that we have discussed in chapter 1. We have given the action for this theory in equation (1.9). There are two sets of scalar fields in this theory, those we denote by X^a, which transform as a vector under $SO(6)$, and those we denote by X^i, which transform as a vector under $SO(3)$. As we have described, the set of classical vacua for the model are given by $X^a = 0$, $X^i = \frac{1}{3g}J^i$, where J^i generate any N dimensional representation of the $SU(2)$ algebra. These vacua are in one-to-one correspondence with partitions of N, since we may have in general n_k copies of the k-dimensional irreducible representation such that $\sum_k kn_k = N$.

This model has several interesting large N limits (distinguished by which

[16]BPS states are states whose energies do not receive quantum corrections as a result of supersymmetry.

combination of g and N we hold fixed); we will describe these now, as promised in chapter 1.

The M-theory limit

According to the Matrix Theory conjecture [31], in the limit

$$N \to \infty, \qquad g^2/N^3 \text{ fixed },$$

this model should describe M-theory on the maximally supersymmetric plane-wave background of eleven-dimensional supergravity,[17] with

$$\mu p^+ l_p^2 = \frac{N}{g^{\frac{2}{3}}}, \qquad p^-/\mu = H .$$

Note that the quantities on the left are boost-invariant and dimensionless.

States of M-theory on the plane-wave with zero light-cone energy are BPS configurations involving concentric spherical membranes and/or concentric spherical fivebranes. These correspond to vacua of the Plane-Wave Matrix Model [30, 39].

We will be particularly interested in vacua involving only one type of irreducible representation, say N_2 copies of the N_5-dimensional irreducible representation,

$$X^i = \frac{1}{3g} \begin{pmatrix} J^i_{N_5} & & \{N_2 \text{ copies}\} \\ & \ddots & \\ & & J^i_{N_5} \end{pmatrix} . \tag{2.1}$$

In the M-theory large N limit with N_2 fixed, this configuration describes the state with N_2 coincident M2-branes. This is plausible, since classically, such a configuration corresponds to N_2 coincident fuzzy spheres. On the other hand, if we keep N_5 fixed in the limit, while taking N_2 to infinity, the configuration gives us N_5 coincident spherical M5-branes.[18]

In [39] it was pointed out that there are other interesting large N limits that do not describe all of the degrees of freedom of M-theory, but rather focus in on degrees of freedom associated with the spherical branes.

[17]This background has $ds^2 = ds^2_{flat} + (\frac{\mu^2}{9}x^ix^i + \frac{\mu^2}{36}x^ax^a)dx^+dx^+$ and $F_{123+} = \mu$.

[18]Evidence for this comes from the fact that the BPS excitations about this vacuum match the expected BPS excitations of coincident spherical M5-branes [39].

The D2-brane limit

To understand the next limit, consider the excitations around the vacuum (2.1) at finite N. Classically, this configuration corresponds to N_2 coincident (two-dimensional) fuzzy spheres, on which the modes have maximum angular momentum N_5. Fluctuations about this configuration are then described by noncommutative gauge theory on a fuzzy sphere with gauge group $U(N_2)$.

If we compare this action arising from the matrix model with an action written as a noncommutative field theory with noncommutativity parameter θ and coupling g_2 on a sphere of radius r, we find that the field theory parameters are related to the matrix model parameters via

$$\frac{\theta}{r^2} = \frac{1}{N_5}, \qquad g_2^2 r = \frac{g^2}{N_5}, \qquad Er = H \ .$$

Thus, by taking

$$N \to \infty, \qquad g^2/N \ \text{fixed}, \qquad N_2 \ \text{fixed} \ , \tag{2.2}$$

we obtain a commutative field theory on a sphere. This field theory, written down in [39, 40] is essentially the low-energy theory of D2-branes, with mass terms for the scalars and fermions and a coupling of the radial scalar to the magnetic field, such that the whole theory preserves $SU(2|4)$ supersymmetry. Note that we end up with a D2-brane theory instead of an M2-brane theory, since the limit (2.2) does not decompactify the M-theory circle.

We can also try to find a similar limit to describe decoupled fivebrane degrees of freedom:

The 't Hooft limit

For small values of g (with fixed N), we can study excitations about the various vacuum states perturbatively. In [32], it was found that the parameter controlling perturbation theory is different for different vacua. For the vacuum with N_2 copies of the N_5-dimensional irreducible representation, perturbation theory is controlled by the combination $g^2 N_2$, so for example, the vacuum with $N_2 = 1, N_5 = N$ is more weakly coupled than the vacuum with $N_5 = 1, N_2 = N$.

In particular, in the limit

$$N \to \infty \qquad g^2 N \ \text{fixed} \ ,$$

the coupling associated with the fivebrane vacua (with fixed N_5 with $N_2 \to \infty$) remains finite, while the coupling associated with the membrane vacua (fixed N_2 with $N_5 \to \infty$), or any other generic vacuum, goes to zero.

Again, this limit does not decompactify the M-theory circle, so it was suggested in [39] (also based on supergravity arguments) that this limit describes NS5-branes on a sphere. Below, we will see that it is a somewhat modified limit that is dual to the Lin-Maldacena gravity solution for NS5-branes on S^5.

2.3 Gravity

In the previous section, we have described the Plane-Wave Matrix Model, and various interesting large-N limits. At finite N, we can think of the matrix model as a massive deformation of the maximally supersymmetric quantum mechanics describing low-energy D0-branes in flat-space, similar in spirit to the deformation of $\mathcal{N} = 4$ SYM considered by Polchinski and Strassler [58]. The gravity dual of the undeformed theory is string theory on the near-horizon D0-brane geometry, so we expect that the gravity dual for the Plane-Wave Matrix Model should be some infrared modification of this.

As we have described in chapter 1, Lin and Maldacena [40] (following [42]) searched for type IIA supergravity solutions preserving the same $SU(2|4)$ symmetry as the vacua of the Plane-Wave Matrix Model. Using an ansatz with this symmetry (reproduced in appendix B), they were able to reduce the problem of finding supergravity solutions to the aforementioned problem of finding axially-symmetric solutions to the three-dimensional Laplace equation, with boundary conditions involving parallel charged conducting discs and a specified background potential. We will now describe the electrostatics problem in detail.

The Electrostatics Problem

Common to all vacua, we have in the electrostatics problem an infinite conducting plate at $z = 0$ (on which we may assume that the potential vanishes), and a background potential[19]

$$V_\infty = V_0(r^2 z - \frac{2}{3} z^3) \; . \tag{2.3}$$

[19] This potential results by taking point charges $\mp Z^4/3$ at $r = 0, z = \pm Z$ in a constant electric field $\vec{E} = -\frac{2}{3} Z^2 \hat{z}$ in the limit $Z \to \infty$.

In addition, corresponding to a matrix model vacuum with Q_i copies of the d_i-dimensional irreducible representation, we have conducting discs with charge Q_i parallel to the infinite plate and centred at $r = 0$, $z = d_i$. In order that the supergravity solution is non-singular, the radii R_i of the discs must be chosen so that the charge density at the edge vanishes.[20]

Thus, for each vacuum of the Plane-Wave Matrix Model, we have an electrostatics problem, whose solution (a potential $V(r, z)$) feeds into the equations (B.1) to give a supergravity solution.

Properties of the Supergravity Solutions

We briefly review some properties of the supergravity solutions [40]. It is straightforward to show that, as expected, all of these supergravity solutions approach asymptotically the near-horizon D0-brane solution. In the infrared region, the solutions have interesting topology, as we now recall.

The coordinates r and z in the electrostatics problem form two of the nine spatial coordinates in the geometry. In addition, for each value of r and z, we have an S^2 and an S^5 with varying radii. The S^5 shrinks to zero size on the $r = 0$ axis, while the S^2 shrinks to zero size at the locations of the conducting plates, so we have various non-contractible S^3s and S^6s corresponding to paths that terminate on different plates or on different segments of the vertical axis, respectively. This is illustrated in figure 2.1. As shown in [40], through an S^6, corresponding to a path surrounding plates with a total charge of Q, we have $N_2 = 8Q/\pi^2$ units of flux from the dual of the Ramond-Ramond four-form, suggesting the presence of N_2 D2-branes. Similarly, through an S^3, corresponding to a path between plates separated by a distance d, we have $N_5 = 2d/\pi$ units of H-flux, suggesting that this part of the geometry between the plates is describing the degrees of freedom of N_5 NS5-branes.

If we take large plates at a fixed separation, the region between the plates corresponds to a long throat in the geometry with NS5-brane flux. Below, we will understand how to take a limit where such a throat becomes infinite so that we recover the Lin-Maldacena geometry.

[20] To see that this should be possible, note that without a background field, the charge density on a conducting disc diverges as an inverse square root near the edge. On the other hand, we have an inward electric field coming from the background potential that increases linearly with the radius of the disc. Thus, for a large enough disc, the tendency for the charge on the disc to bunch up at the edge should be balanced by the action of the inward electric field so that the charge density vanishes.

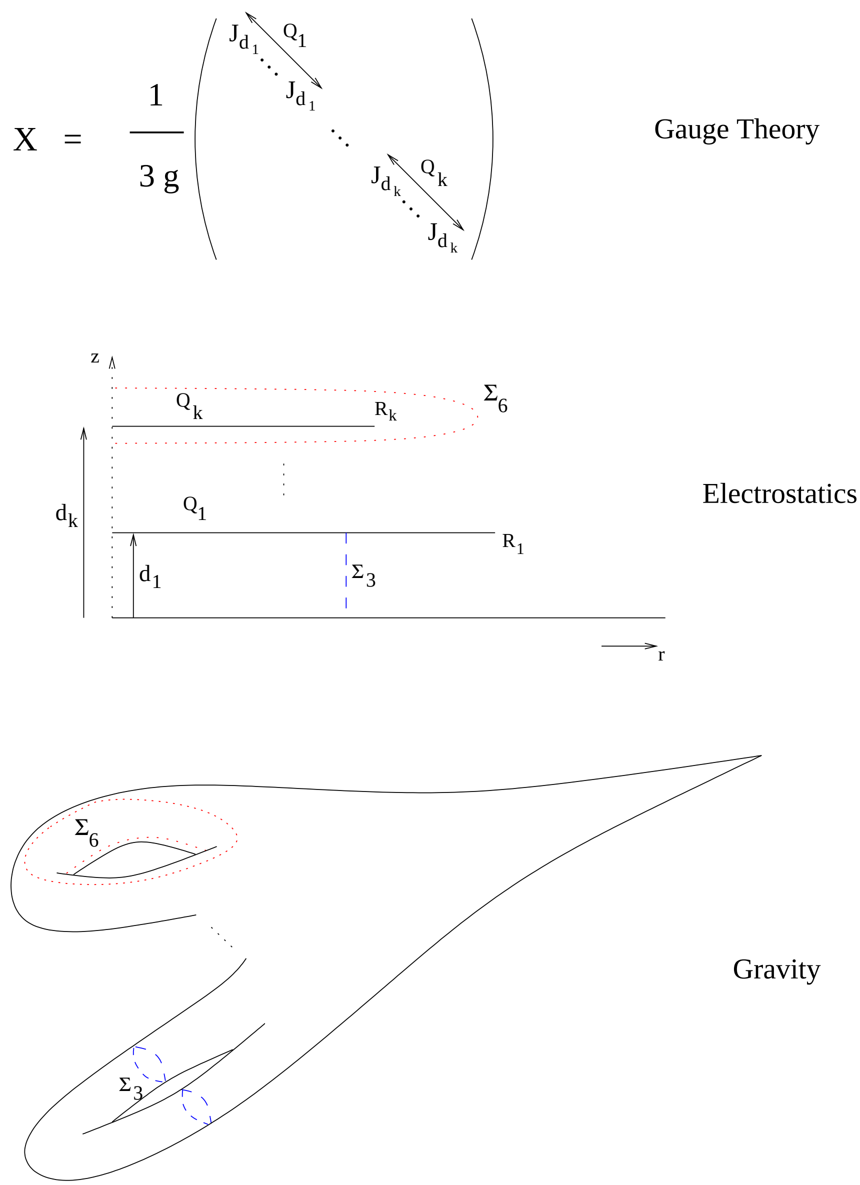


Figure 2.1: Mapping between matrix model vacua, electrostatics configurations, and geometries. For illustrative purposes, we have replaced the $S^2 \times S^5$s associated to each point (r, z) with $S^0 \times S^0$. In the full geometry, the dotted segment maps to a submanifold Σ_6 that is topologically $S^6 \times S^2$ (simply connected) rather than the $S^1 \times S^0$ shown here. Similarly, the dashed segment maps to a submanifold Σ_3 that is topologically $S^5 \times S^3$ rather than the $S^0 \times S^1$ here.

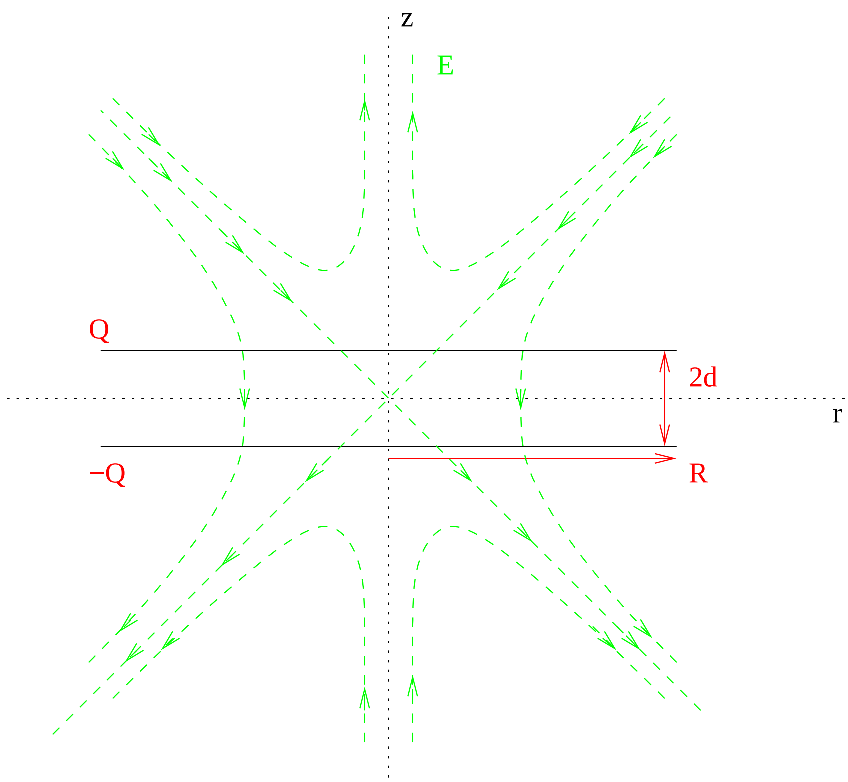


Figure 2.2: Electrostatics problem (cross section) corresponding to gravity solutions with single NS5 throat. The method of images has been used to replace the infinite conducting plate with an image disc.

Solution to the Electrostatics Problem (Simplest Case)

We will now solve the electrostatics problem above in the simplest case of a single disc above the infinite plate, with the space in between corresponding to a single NS5-brane throat. Thus, we would like to find the potential for a conducting disc of charge $Q = \pi^2 N_2/8$ a distance $d = \pi N_5/2$ above the infinite conducting plate, in the presence of the background field (2.3), as shown in figure 2.2. This will give the geometry dual to the matrix model vacuum with N_2 copies of the N_5 dimensional irreducible representation.

We can simplify the problem somewhat using the the fact that electrostatics is linear and scale-invariant. Thus, if

$$V = (r^2 z - \frac{2}{3} z^3) + \phi_\kappa(r, z)$$

is the solution to the electrostatics problem above with $V_0 = 1$, $R = 1$, and $d = \kappa$ (with $Q = q(\kappa)$ determined by the condition of vanishing charge density at the edge of the disc), then the solution to the general problem will be

$$V = V_0(r^2 z - \frac{2}{3}z^3) + R^3 V_0\ \phi_{\frac{d}{R}}(r/R, z/R)\ , \tag{2.4}$$

and the charge on the disc will be

$$Q = q(d/R)V_0 R^4\ . \tag{2.5}$$

Note that ϕ is the part of the potential that vanishes at infinity, arising from the charges on the discs. We would now like to determine $\phi_\kappa(r, z)$.

We may assume that the potential vanishes on the infinite conducting plate. Let us call the (constant) potential on the disc $V = \Delta$. This will be determined in terms of κ by the condition that the charge density vanishes at the edge of the discs, but for now, we will take it to be arbitrary and solve for the potential in general.

By the method of images, the potential will be the same as for a pair of conducting discs at $z = \pm\kappa$ with potentials $V = \pm\Delta$, with the potential going like $V \to zr^2 - \frac{2}{3}z^3$ at infinity. Without the background potential, this is a classic problem in electrostatics, considered by Maxwell, Claussius and Helmholtz, Kirchhoff, Polya and Szego, and eventually solved by Nicholson [61] and Love [62]. For these references and a nice summary, see [63]. Fortunately, the method of solution may easily be extended to our case with the background potential, as we describe in appendix C.1.

To give the solution, it is convenient to define

$$\beta = \Delta + \frac{2}{3}\kappa^3\ . \tag{2.6}$$

Then the potential is given by

$$\phi_\kappa(r, z) = \frac{\beta}{\pi}\int_{-1}^{1} G_\kappa(r, z, t) f_\kappa(t) dt\ , \tag{2.7}$$

where

$$G_\kappa(r, z, t) = -\frac{1}{\sqrt{r^2 + (z + \kappa + it)^2}} + \frac{1}{\sqrt{r^2 + (z - \kappa + it)^2}}$$

and

$$f_\kappa(t) = f_\kappa^{(0)}(t) - 2\frac{\kappa}{\beta} f_\kappa^{(2)}(t)\ . \tag{2.8}$$

The functions $f_\kappa^{(n)}(t)$ are special functions solving the integral equation

$$f_\kappa^{(n)}(t) - \int_{-1}^{1} K_\kappa(t,x) f_\kappa^{(n)}(x) = t^n, \tag{2.9}$$

with kernel

$$K_\kappa(t,x) = \frac{1}{\pi}\frac{2\kappa}{4\kappa^2 + (x-t)^2} \ .$$

This is a Fredholm integral equation of the second kind (see, e.g. [64]), and the solution may be written as a series

$$f_\kappa^{(n)}(t) = \sum_{m=0}^{\infty} K_\kappa^m \circ t^n \tag{2.10}$$

where

$$(K \circ g)(t) \equiv \int_{-1}^{1} K(x,t) g(x) \ .$$

It may be shown that the series converges for any value of $\kappa > 0$ to define a bounded continuous function $f_\kappa^{(n)}(t)$.

The function f is related to the charge density on the disc as

$$\begin{aligned} f_\kappa(t) &= \frac{2\pi}{\beta}\int_t^1 \frac{r\sigma_\kappa(r)dr}{(r^2-t^2)^{\frac{1}{2}}}, \\ \sigma_\kappa(r) &= \frac{\beta}{\pi^2}\left[\frac{f_\kappa(1)}{(1-r^2)^{\frac{1}{2}}} - \int_r^1 \frac{f_\kappa'(t)dt}{(t^2-r^2)^{\frac{1}{2}}}\right] \ , \end{aligned} \tag{2.11}$$

so that the total charge on the disc is

$$q(\kappa) = \frac{\beta}{\pi}\int_{-1}^{1} f_\kappa(t)dt \ . \tag{2.12}$$

Vanishing Charge Density Constraint

The solution in the previous section was for arbitrary potential Δ, and will generally have a charge density that is nonvanishing at the tip of the disc. We will now determine a formula for $\Delta(\kappa)$ for which the charge density vanishes.

From (2.11), it follows that $\sigma(1) = 0$ if and only if $f(1) = 0$ and $f'(1)$ is bounded. From the series solution, it is straightforward to prove that the latter condition is always satisfied for the functions $f_n(t)$, so our constraint comes from requiring $f(1) = 0$. Now, from (2.8) and the definition (2.6) of β, we see that the condition $f(1) = 0$ determines Δ to be

$$\Delta(\kappa) = 2\kappa \frac{f_\kappa^{(2)}(1)}{f_\kappa^{(0)}(1)} - \frac{2}{3}\kappa^3 . \tag{2.13}$$

Finally, the charge on the disc is given in terms of κ by

$$q(\kappa) = \frac{f_\kappa^{(2)}(1)}{f_\kappa^{(0)}(1)} \frac{2\kappa}{\pi} \int_{-1}^{1} f_\kappa^{(0)}(t)dt - \frac{2\kappa}{\pi} \int_{-1}^{1} f_\kappa^{(2)}(t)dt . \tag{2.14}$$

Via equation (2.5) this function q determines the radius of the disc in the original problem in terms of the charge Q, the potential V_0, and the separation d. The function q is plotted in figure 2.3. We show in appendix C.1.1 that its limiting behaviour for small and large κ is

$$\begin{array}{lcll} q(\kappa) & \rightarrow & \frac{1}{8}, & \kappa \rightarrow 0 , \\ q(\kappa) & \rightarrow & \frac{8}{3\pi}, \kappa & \kappa \rightarrow \infty. \end{array} \tag{2.15}$$

We will see in section 2.6.3 that the function $q(\kappa)$ is physically important since it computes the energies of certain near-BPS states in the theory.

Summary

In summary, to generate the supergravity solution dual to the vacuum of the Plane-Wave Matrix Model corresponding to N_2 copies of the N_5 dimensional irreducible representation, we:

- choose κ (ultimately related to a choice of coupling g), determine $\phi_\kappa(r, z)$ from (2.7) and $q(\kappa)$ from equation (2.14),
- take $R = (\pi N_5)/(2\kappa)$ so that $d = R\kappa = \pi N_5/2$,
- choose $V_0 = (2\kappa^4 N_2)/(q(\kappa)\pi^2 N_5^4)$ so that (using (2.5)) $Q = q(\kappa)V_0 R^4 = \pi^2 N_2/8$.

Then the gravity dual is given by equations (B.1), with V given by (2.4).

2.4 Matching Parameters with Gauge Theory

In discussing the various scaling limits of the theory, we will need to understand how the gauge theory parameters match with the parameters in

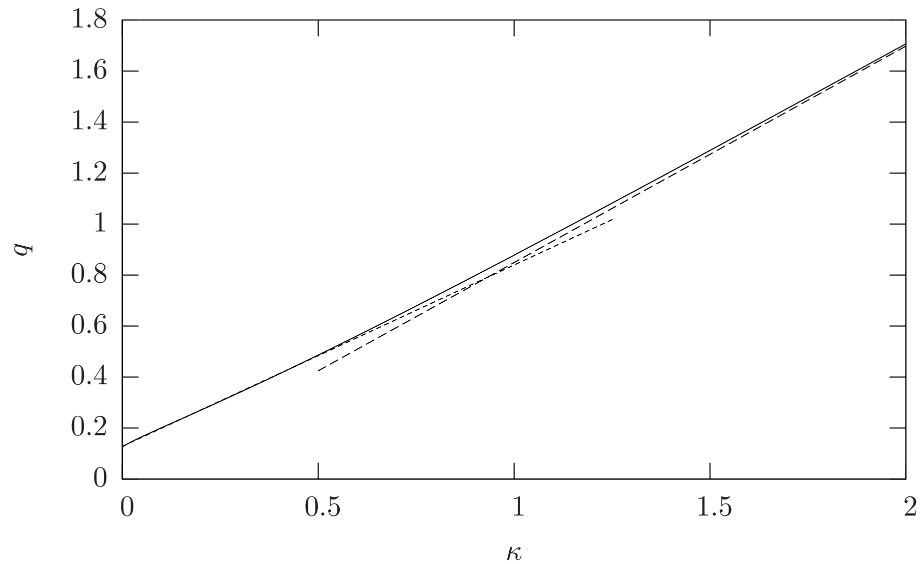


Figure 2.3: Plot of $q(\kappa)$. Dashed lines are the asymptotes $\frac{8}{3\pi}\kappa$ for large κ and $1/8 + 0.711\kappa$ for small κ.

the supergravity solution, or equivalently, with the parameters d, R, Q, and V_0 in the electrostatics problem. As we mentioned, the parameters d and Q are proportional to the number of units of NS5-brane and D2-brane flux through the noncontractible S^3 and S^6, respectively, in the geometry. As shown in [40], this allows us to associate [21]

$$d = \frac{\pi}{2} N_5$$

and

$$Q = \frac{\pi^2}{8} N_2 \ .$$

The remaining electrostatics parameter is V_0 (or equivalently, R), which we interpret in the gauge theory as follows. The asymptotic form of the geometry is determined by the background field in combination with the leading dipole fields arising from the charges on the plates. These depend, respectively, only on V_0 and the combination dQ, which is proportional to the dipole moment. These asymptotics should be the same for all vacua

[21] Note that we have taken all quantities in the electrostatics problem to be dimensionless.

of a given theory, so the parameters V_0 and dQ must depend only on g and $N = N_2N_5$, the parameters that determine which matrix model we are talking about. This is clearly true for dQ, but we must also have

$$V_0 = f_1(g^2, N_2N_5) \ . \tag{2.16}$$

To further constrain V_0 it is useful to note that in the gauge theory, the planar amplitudes depend only on N_5 and the combination g^2N_2. This suggests that the free string theory on the dual spacetime should be controlled by these two parameters, and in particular, that these parameters should control the metric. In terms of the electrostatics parameters, it is straightforward to see that the metric depends only on d and R (scaling V_0 while holding these fixed scales the dilaton and Ramond-Ramond fields, but leaves the metric fixed), so d and R should each be some function of N_5 and g^2N_2. This is clearly true for d, but we must also have

$$R = f_2(g^2N_2, N_5) \ . \tag{2.17}$$

Using the relation $Q = V_0R^4q(d/R)$, together with (2.16) and (2.17), we may conclude that

$$V_0 = \frac{1}{g^2}h(g^2N_2N_5) \tag{2.18}$$

for some function h.

In section 2.5.1, we will see that the D2-brane limit of the gauge theory discussed in section 2.2 matches with the corresponding limit of the supergravity solution only if the function h approaches some constant h_∞ at large values of its argument. Since we will mostly be interested in this regime ($g^2N_2N_5$ is always large when supergravity is valid), we get the identification

$$V_0 = \frac{h_\infty}{g^2} \ . \tag{2.19}$$

The electrostatics parameter R is a more complicated function of the gauge theory parameters, but follows from the other identifications via (2.5).

2.5 Scaling Limits

In this section, we consider various scaling limits of the gravity theory in which one of the three parameters is scaled to infinity, with the others scaled such that we end up with something nontrivial.

2.5.1 Large d: the D2-brane Limit

We begin by considering a limit of large d with fixed R. To understand how we should scale V_0 to leave us with a nontrivial supergravity solution, note that for large d/R, the formula (2.5) determining the charge on the discs becomes

$$Q = \frac{8}{3\pi} V_0 d R^3 \ , \tag{2.20}$$

where we have used the large κ behaviour of $q(\kappa)$. Also, the potential (2.3), taken near the position of the disc by replacing $z = d + \eta$, becomes

$$V = -\frac{2}{3} V_0 d^3 - 2 V_0 d^2 \eta + V_0 d (r^2 - 2\eta^2) + V_0 (\eta r^2 - \frac{2}{3}\eta^3) \ . \tag{2.21}$$

The first two terms here have no effect on the supergravity solution, since the solution (B.1) depends only on $\partial_z^2 V$ and $\partial_r V$. Thus, from (2.21) and (2.20), we see that in order to leave a finite nontrivial background potential and a finite non-zero charge on the disc, we must take V_0 to scale like $1/d$. Thus, our limit is

$$d \to \infty, \qquad Q \ \text{fixed}, \qquad V_0 d = W_0 \ \text{fixed} \ . \tag{2.22}$$

In this limit, we have a single charged conducting disc (with no infinite plate) in a background potential

$$V = W_0 (r^2 - 2\eta^2) \ . \tag{2.23}$$

In [40] Maldacena and Lin wrote down explicitly the geometry corresponding to this situation. In appendix C.2, we give an alternate derivation of the solution by explicitly solving the electrostatics problem, verifying that (2.20) correctly gives the charge necessary to ensure that the charge density vanishes at the edge of the plates. For large r, the solution approaches the solution for near-horizon D2-branes but with the flat directions along the D2-branes replaced by an S^2 [40]. Thus, we expect that this limit should correspond to the limit of the matrix model giving rise to D2 branes on S^2. Using the correspondence between matrix model parameters and electrostatics parameters (in particular, assuming that the function h in (2.18) is simply a constant at large argument), we find that the limit (2.22) becomes,

$$N_5 \to \infty, \qquad N_2 \ \text{fixed}, \qquad \frac{g^2}{N_5} \ \text{fixed},$$

which is precisely the D2-brane limit discussed in section 2.2.

2.5.2 Large V_0: the 't Hooft Limit

The next limit we consider is the 't Hooft limit discussed in section 2.2,

$$N_2 \to \infty, \qquad N_5 \text{ fixed}, \qquad g^2 N_2 \text{ fixed},$$

which appeared to be a decoupling limit retaining interacting fivebrane degrees of freedom. Using the correspondence between matrix model parameters and electrostatics parameters, we find that this is a limit with

$$Q \to \infty, \qquad d \text{ fixed}, \qquad R \text{ fixed} \qquad V_0 \to \infty$$

From the supergravity point of view, this is a limit in which the metric is held fixed with the maximum value of the dilaton going to zero. Thus, we have free string theory on the background corresponding to a single finite-sized disc above the infinite conducting plate.

This geometry contains a finite throat region with NS5-brane flux, but also a noncontractible S^3 with D2-brane flux. Thus, while we are describing fivebrane degrees of freedom, this limit of the gauge theory does not correspond to the infinite-throat Lin-Maldacena solution for NS5-branes on S^5.

2.5.3 Large R: Little String Theory on S^5

Finally, we would like to understand precisely what limit of our solution is required to obtain the Lin-Maldacena infinite-throat solution for NS5-branes on S^5. This corresponds to an electrostatics problem with two infinite conducting plates, with the potential between the plates equal to

$$V = \tilde{V}_0 \sin\left(\frac{\pi z}{d}\right) I_0\left(\frac{\pi r}{d}\right). \tag{2.24}$$

To obtain this from our solution, we certainly need to take a limit where R is going to infinity with d fixed. However, generically, we would simply end up with a constant vertical electric field between the plates. This does not give rise to any metric (the supergravity fields depend only on $\partial_z^2 V$ and $\partial_r V$) so we must take V_0 large enough so that the leading corrections to this constant electric field remain nonzero in the limit.

To understand the proper scaling, we start by considering the electrostatics solution for finite R. In the $r < R$ region between the plates, we have an axially-symmetric solution to the Laplace equation that is regular at $r = 0$, so we can write

$$V(r, z) = V_{z=d}\frac{z}{d} + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi z}{d}\right) I_0\left(\frac{n\pi r}{d}\right) .$$

Here, we have separated off a constant electric field term that does not affect the metric such that the remaining piece vanishes at $z = 0$ and $z = d$ for $r < R$. Now the c_ns are determined by the potential at $r = R$,

$$c_n = \left(I_0\left(\frac{n\pi R}{d}\right)\right)^{-1} V_0 R^3 \frac{2}{d} \int_0^d dz \, \sin\left(\frac{n\pi z}{d}\right) \left(\frac{1}{R^3}(R^2 z - \frac{2}{3}z^3) + \phi_{\frac{d}{R}}(1, \frac{z}{R}) - \Delta_{\frac{d}{R}} \frac{z}{d}\right) \tag{2.25}$$

where we have used the expression (2.4) for V. Using our solution for ϕ, it is simple to show that the integral here has only a power law dependence on R, so the large R behaviour of c_n is dominated by the exponential damping coming from the Bessel function at large argument,

$$(I_0(z))^{-1} \sim \sqrt{2\pi z} e^{-z} \ .$$

To compensate for this damping, we must scale V_0 exponentially in R,

$$V_0 \sim e^{\frac{\pi R}{d}}, \tag{2.26}$$

which allows us to keep c_1 finite in the limit. All the other coefficients $c_{n>1}$ still vanish in the limit, so we indeed end up with the Lin-Maldacena solution. To be more precise, we can evaluate the integral in (2.25) to find the prefactor in (2.26). Using the results of appendix C.1.1, we find the behaviour

$$c_1 \to V_0 \, C \, (Rd)^{\frac{3}{2}} \, e^{\frac{-\pi R}{d}}, \tag{2.27}$$

where we have numerically estimated C to be $C \approx 0.082$. Thus, the precise limit we need to take to recover (2.24) is

$$R \to \infty, \qquad d \ \text{fixed}, \qquad V_0 \to \tilde{V}_0 \, \frac{1}{C} \, (Rd)^{-\frac{3}{2}} \, e^{\frac{\pi R}{d}} \ ,$$

which also implies $Q \to \infty$.

In supergravity language, the limit we are taking is designed to take the NS5-brane throat infinite while holding the dilaton at the bottom of the throat fixed. The fact that V is exponentially damped as we go towards the middle of the plates gives rise to the linear dilaton behaviour of the final supergravity solution.

Field theory description

Using the correspondence of parameters between field theory and the electrostatics we find that the limit of the Plane-Wave Matrix Model that defines the dual of the Lin-Maldacena solution, i.e. the field theory description of Little String Theory on S^5, is

$$N_2 \to \infty, \qquad N_5 \ \text{fixed}, \qquad \frac{1}{g^2}\lambda^{\frac{3}{8}} e^{-a\lambda^{\frac{1}{4}}/N_5} \ \text{fixed} . \tag{2.28}$$

where a is a numerical coefficient related to the constant in (2.19) by $a = 2(\pi^2/h_\infty)^{\frac{1}{4}}$. Thus, rather than holding the 't Hooft coupling fixed, we scale it to infinity in a controlled way

$$\lambda \sim N_5^4 \ln^4(N_2) . \tag{2.29}$$

Unfortunately, because the coupling constant must be large, this implies that perturbation theory is not useful on the field theory side, though perhaps there are some near-BPS sectors of the theory where the expansion parameter is not the naive 't Hooft coupling.

2.6 Discussion

2.6.1 The Double-Scaling Limit

We have seen that to obtain the field theory dual of the Lin-Maldacena supergravity solution for Little String Theory on S^5, we need to take a large N_2 limit while scaling the 't Hooft coupling to infinity in a controlled way. This double scaling limit is reminiscent of limits used to define low-dimensional string theories in old matrix models (see for example [65]). There, the 't Hooft coupling is scaled to some critical value in a controlled way as N goes to infinity, such that all terms in the genus expansion continue to contribute even though N becomes infinite. We suspect that this is also the situation here, except that in our case, the "critical" value of the 't Hooft coupling is infinity. The fact that N_2 becomes infinite is consistent with the fact that the dilaton vanishes asymptotically in the supergravity solution. On the other hand, we still have a string genus expansion in the bulk of the supergravity solution, so we can understand the scaling of λ to infinity as necessary for the gauge theory to reproduce nontrivial string interactions in the bulk.

To understand this in more detail, consider the matrix model genus expansion for some physical observable. It takes the form

$$F = \sum_n \frac{f_n(g^2 N_2, N_5)}{N_2^n}, \tag{2.30}$$

where $f_n(\lambda, N_5)$ gives the sum of genus n diagrams. In the 't Hooft limit with $\lambda = g^2 N_2$ fixed and N_2 taken to infinity, only the planar $n = 0$ term contributes. What we are suggesting is that the scaling (2.29) is such that all terms in the expansion (2.30) contribute. If this is true, it predicts that the large λ behaviour of f_n is

$$f_n(\lambda) \to a_n \left[\lambda^{\frac{5}{8}} e^{\frac{a\lambda^{\frac{1}{4}}}{N_5}} \right]^n . \tag{2.31}$$

The quantity in square brackets divided by N_2, which we can call $\tilde{g}$, is the inverse of the quantity being held fixed in (2.28), so the genus expansion (2.30) becomes

$$F = \sum_n a_n \tilde{g}^n .$$

Thus, the constant $\tilde{g}$ serves as the effective string coupling.

The behaviour (2.31) is a nontrivial prediction of our results and the assumption that the string theory genus expansion is still related to the gauge theory genus expansion. It should apply to the behaviour of any physical observable that survives the scaling limit, for example the energy of any state in the matrix model corresponding to some excitation in the NS5-brane throat. It would be interesting to understand more precisely from the gauge theory point of view which set of observables remain in the limit.

Unfortunately, it seems difficult to check the behaviour (2.31) directly from the gauge theory, since it would involve summing infinite sets of diagrams. However, it may be that this is possible for certain BPS or near-BPS observables, as for the circular Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, where for example the full contribution at the planar level is given by an infinite set of ladder diagrams that can be summed explicitly [66]. Intriguingly, that result,

$$\langle W \rangle_{N=\infty} = \sqrt{\frac{2}{\pi}} \lambda^{-\frac{3}{4}} e^{\sqrt{\lambda}},$$

takes a rather similar form to our prediction here.[22] A specific limit in which

[22]Note that the $\sqrt{\lambda}$ in this example and the $\lambda^{\frac{1}{4}}$ in our case both represent the squared radius of the respective S^5s in string units.

some matching similar to [30] might be possible is in a Penrose limit of the geometry [67], associated with geodesics around the S^2 at $r = 0$.

2.6.2 An Infinite Parameter Family of NS5-brane Solutions

Starting from our exact supergravity solution for the simplest class of vacua, we have found a specific limit that gives the Lin-Maldacena solution corresponding to the region between two infinite conducting plates. It is interesting to note that this solution is actually the simplest in an infinite-parameter family of solutions. In the electrostatics language, we can have any function

$$V = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi z}{d}\right) I_0\left(\frac{n\pi r}{d}\right),$$

with c_n chosen to fall off fast enough so that the sum converges for all r. The supergravity solution corresponding to any such potential will have an infinite throat with noncontractible S^3 carrying fivebrane flux.

While it does not seem possible to obtain these more general solutions as limits of the solution we considered in this chapter, it is plausible that we could obtain them as limits of solutions corresponding to more general vacua of the Plane-Wave Matrix Model. Specifically, we could imagine starting with a solution containing some arbitrarily large number of discs, taking the size of the lowest disc to infinity as before, but now tuning all of the parameters V_0, d_2, Q_2, $\ldots$, d_n, Q_n in such a way that all of the upper discs have some non-trivial influence on the potential between the plates. It would be interesting to understand better the physical interpretation of these more general solutions.

2.6.3 Energies of Near-BPS States

A useful result coming from our solution is the formula (2.5) that determines the radius of the disc in terms of the other parameters d, Q, and V_0. From the supergravity solution (B.1), it is straightforward to show [40] that R determines the radius of the S^5 at the point $(r = R, z = d)$ corresponding to the edge of the disc as[23]

$$R^2_{S^5}/\alpha' = 4R.$$

As described in [40] section 2.2, this in turn determines the energies of certain near-BPS states with large angular momentum on the S^5 (see equation

[23]To see this, we use the Laplace equation to rewrite V'' and note that $\partial_r V$ vanishes on the discs.

(2.42)).[24] Thus, our function $q(\kappa)$ in (2.5), defined in (2.14) and plotted in figure 3, determines the near-BPS energies for large R and d but arbitrary d/R, interpolating between the small d/R and large d/R results given in [40], equations (2.84) and (2.57) respectively.

2.6.4 Other Definitions of Little String Theory on S^5

Finally, we note that while the limit we have defined may be the simplest description of the Little String Theory on S^5, there should be many other field theoretic definitions. In the electrostatics picture, it should arise any time two nearby discs are scaled to infinite radius at fixed separation, with the background potential scaled so that the potential between the plates remains nontrivial. Thus, we could start with more general vacua of the Plane-Wave Matrix Model, vacua of maximally supersymmetric YM on $R \times S^2$, or vacua of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$.[25] In the next chapter we will show precisely how this occurs for for simple vacua of the latter two theories.

[24] These are the states of string theory on a plane-wave obtained by a Penrose limit associated with geodesics going around the S^5 at fixed r and z.

[25] This latter possibility was noted in [40].

Chapter 3

Little String Theory from Double-Scaled Field Theories

3.1 Introduction

In the preceding chapter we have shown that type IIA Little String Theory on S^5arises as a double-scaling limit of the Plane-Wave Matrix Model. The argument we presented was based on understanding the scaling of the supergravity solutions dual to each of these theories. We found that by taking a region of the geometry containing a throat carrying NS5-brane flux, and taking a limit isolating this throat, we could start with a geometry dual to a vacuum of the Plane-Wave Matrix Model and end up with a vacuum dual to little string theory on S^5. This led us to a corresponding scaling argument on the field theory side that gave us a Lagrangian definition of Little String Theory. We noted in the previous chapter, however, that in the electrostatics picture of the dual geometries, the limit isolating a throat with NS5-brane flux involved taking the size of adjacent charged conducting discs to infinity, and that, as the supergravity duals of maximally supersymmetric YM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ generically contain charged conducting discs, it is reasonable to think that a similar limiting procedure could be used to find Lagrangian definitions of Little String Theory based on these other theories.

According to the proposal in [40], the supergravity solutions arising from configurations with a finite number of discs correspond to the (classically degenerate) vacua of maximally supersymmetric YM on $R \times S^2$. Configurations with an infinite number of discs, arranged in a periodic fashion, correspond to the vacua of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. Finally, configurations with one infinitely large disc, and a finite number of discs above it, correspond to the vacua of the Plane-Wave Matrix Model (see figure 3.1). The relations among these field theories have been discussed in chapters 1, 2 and [39, 68, 69].

In this chapter, we extend the argument given in 2 that Little String

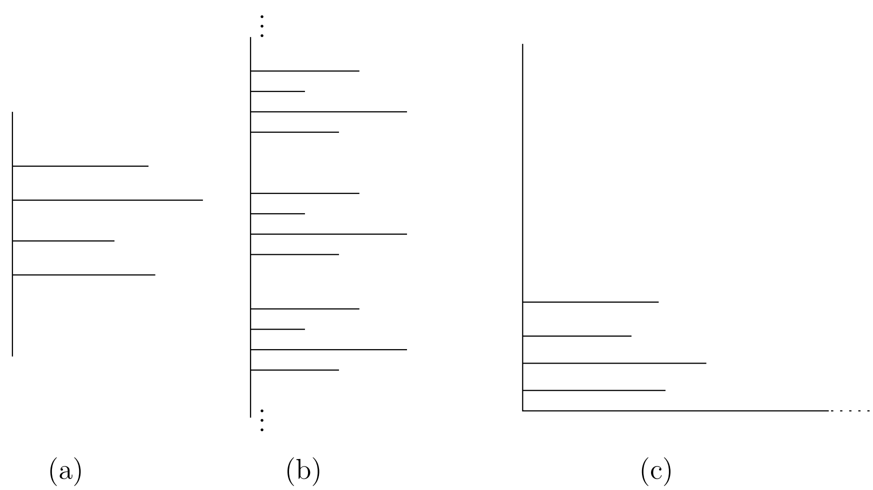


Figure 3.1: The three generic types of electrostatics configurations. The isolated set of discs in (a) is a configuration dual to a vacuum of SYM theory on $R \times S^2$ with sixteen supercharges. The periodic configuration in (b) is dual to a vacuum of $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$. The set of discs above an infinite conducting plane in (c) is dual to a vacuum of the Plane-Wave Matrix Model.

Theory arises as a double-scaling limit of the Plane-Wave Matrix Model to find similar limits for maximally supersymmetric YM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. We solve the electrostatics problems corresponding to specific simple vacua of these field theories and determine the scaling of parameters in the supergravity solutions that is required to obtain the Lin-Maldacena solution for NS5-branes on S^5. By considering the matching between the parameters in the field theories and those in the corresponding electrostatics problems, we thereby determine the precise scaling of the gauge theory parameters that is required to obtain Little String Theory on S^5. The proposed prescriptions are found to be double-scaling limits, similar to the one found in the case of the Plane-Wave Matrix Model [1]. Whereas in the Plane-Wave Matrix Model case it was found that the 't Hooft coupling must be scaled like $\ln^4 N$ [1], we will show below that for the SYM theories on $R \times S^2$ and $R \times S^3/Z_k$ the 't Hooft coupling must be scaled like $\ln^3 N$ and $\ln^2 N$ respectively.

3.2 The Gauge Theories and Their Dual Supergravity Solutions

In this section we will recall some facts about the gauge theories in question and their supergravity duals, that we have described in chapter 1.

In [40], Lin and Maldacena found a class of solutions of type IIA supergravity with $SU(2|4)$ symmetry depending on one single function V. This function V solves the three dimensional Laplace equation and satisfies the same boundary conditions as the electrostatic potential of an axisymmetric arrangement of charged conducting discs in a background electric field. By specifying the positions and sizes of the conducting discs, the charges on the discs, and the asymptotic form of V at infinity, V is determined uniquely. Each different specification of these parameters leads to a different V, however not all such choices give rise to physically acceptable supergravity solutions. Flux quantization in the supergravity solution tells us that the charges on the discs and the spacing between discs are quantized. Positive-definiteness of various metric components in the supergravity solutions imposes constraints on the form of the asymptotic potential. Finally, the regularity of the supergravity solutions tells us that the surface charge density on the discs must vanish at the edge of the discs. This final condition suggests that for a fixed asymptotic potential, the positions, charges, and sizes of the discs cannot be independently specified. For example, the sizes of the discs may be fixed once the other parameters are freely specified.

For an extensive discussion of the general properties of these supergravity solutions see [40].

Here we are interested in the supergravity solutions dual to the vacua of the SYM theory on $R \times S^2$ and $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$. For all vacua of these two field theories, Lin and Maldacena determined the asymptotic form of V to be $W_0(r^2 - 2z^2)$, where $W_0 > 0$. The choice of vacuum is then given by specifying the charges, positions, and sizes of the discs. Let us review in some detail the connection between these parameters for the supergravity solutions and the parameters defining the field theory vacua.

First consider $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. The space S^3/Z_k can be described most directly by choosing coordinates on the unit S^3 such that the metric takes the form

$$d\Omega_3^2 = \frac{1}{4}\left[(2d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{3.1}$$

where the ψ coordinate is 2π periodic, and θ, ϕ are the usual coordinates for S^2. Then the orbifold S^3/Z_k is obtained by identifying $\psi \sim \psi + 2\pi/k$. The vacua of this field theory are given by the space of flat connections on S^3/Z_k. Up to gauge transformations, these are of the form $A = -\mathrm{diag}(n_1, n_2, \ldots, n_N)\, d\psi$, where $e^{2\pi n_i/k}$ are k-th roots of unity (clearly, to label the vacua uniquely, we should restrict the values of the integers n_i to be in some fixed interval of length k). To understand intuitively how these vacua map to configurations of discs in the electrostatics problem, consider the field theory as a theory of D3-branes wrapped on an S^3/Z_k. Now apply a T-duality transformation in the isometry direction ψ. The T-dual coordinate $\tilde{\psi}$ is periodic $\tilde{\psi} \sim \tilde{\psi} + 2\pi k$, and the background gauge field is mapped to an arrangement of D2-branes located at the positions $\tilde{\psi} = 2\pi n_1, 2\pi n_2, \ldots, 2\pi n_N$ (along with their images under translations by integer multiples of $2\pi k$). Naturally, this suggests that the dual supergravity solution is obtained by considering a periodic configuration of discs with period proportional to k. The integers n_i that specify the gauge theory vacuum now determine the positions and charges of the discs within one period in the obvious manner. Presumably, the sizes of the discs are then fixed by demanding regularity of the supergravity solution. In rest of this chapter, we will be interested in the simplest vacuum state of the theory, given by the trivial gauge field $n_i = 0$ for all i. In the normalization conventions of Lin and Maldacena [40], the dual supergravity solution is generated by the axisymmetric electrostatic potential $V(r, z)$ for an arrangement of equal-sized discs at $z = (\pi/2)km$ for all integers m, where the charge on each disc is

$Q = (\pi^2/8)N$.

Now we consider the case of the SYM theory on $R \times S^2$. As discussed in [40], we can think of this theory as $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, in the limit where $k \to \infty$ and $g^2_{YM3} \to 0$ while keeping $g^2_{YM3}k$ fixed. Up to a numerical constant, the limiting value of $g^2_{YM3}k$ is the coupling g^2_{YM2}. If we start with a vacuum in the S^3/Z_k theory with background gauge field $A = -\text{diag}(n_1, n_2, \ldots, n_N)d\psi$ and take $k \to \infty$ with the integers n_i fixed, then we obtain a vacuum of the S^2 theory with a vacuum expectation value for one of the adjoint scalars $\Phi = -\text{diag}(n_1, n_2, \ldots, n_N)$ and a background gauge field with associated flux $F = dA = \Phi \sin\theta d\theta d\phi$. All of the vacua of $\mathcal{N} = 4$ SYM on $R \times S^2$ discussed in [40] can be obtained in this way. This limit has a clear interpretation in the T-dual picture. We start with a configuration of a finite number of D2-branes, repeated periodically by translating the whole arrangement by integer multiples of $2\pi k$. In the limit $k \to \infty$, we are left with only one copy of the configuration of D2-branes, the images being pushed off to infinity. This naturally suggests that the dual supergravity solution is obtained by considering a configuration of a finite number of discs. It is clear that the integers n_i determine the positions and charges of the discs in a manner analogous to the situation in the $R \times S^3/Z_k$ theory. Again, the sizes of the discs are presumably fixed by demanding regularity of the supergravity solutions. Note that the total sum of the charge on the discs must equal the rank of the gauge group N. In the rest of this chapter, we consider non-trivial vacua of the form $\Phi = (n, \ldots, n, -n, \cdots - n)$, where the integers n and $-n$ each appear $N/2$ times. In this case the dual supergravity solution is generated by the potential $V(r, z)$ corresponding to two equal-sized discs at $z = \pm(\pi/2)n$ with charge $(\pi^2/8)(N/2)$ on each disc.

The final issue we need to discuss in this section is the normalization of the asymptotic potential at infinity. For the SYM theory on $R \times S^2$, we can relate W_0 to g^2_{YM2} by using the results in [1]. As discussed in [1, 39, 69] the SYM theory on $R \times S^2$ can be obtained as a limit of the Plane-Wave Matrix Model. This statement, together with the matching of parameters in the Plane-Wave Matrix Model discussed in [1], tell us that we must have

$$W_0 = \frac{h_2}{g^2_{YM2}}, \tag{3.2}$$

where the positive constant h_2 does not depend on the parameters N, g^2_{YM2}, which define the gauge theory, and the eigenvalues of Φ, which label its vacua. For SYM theory on $R \times S^3/Z_k$, the above mentioned relation between this theory and the theory on $R \times S^2$ suggests that we make the identification

$$W_0 = \frac{h_3}{g_{YM3}^2 k}, \tag{3.3}$$

where h_3 is a positive constant that does not depend on N, k, g_{YM3}^2 and the integers that label the vacua of the gauge theory.

3.3 Little String Theory from SYM on $R \times S^2$

In this section, we consider in detail the supergravity solution corresponding to the electrostatics problem for two identical discs of radius R located at $z = \pm d$ with charge Q on each disc and a background potential $W_0(r^2 - 2z^2)$. We wish to solve the electrostatics problem explicitly and determine the required scaling to obtain the Lin-Maldacena NS5-brane solution.

The Electrostatics Problem for the Case of two Identical Discs

Following the approach of [1], we first solve the electrostatics problem for the specific case $W_0 = 1$, $R = 1$, $d = \kappa$ (the solution for the general case is then obtained by linear rescaling of the coordinates and an overall rescaling of the potential). In this case the solution must have the form

$$V(r, z) = (r^2 - 2z^2) + \phi_\kappa(r, z), \tag{3.4}$$

where ϕ_κ is an axisymmetric solution of the Laplace equation that vanishes at infinity. We can expand ϕ_κ in terms of Bessel functions, and in the region between $z = -d$ and $z = d$, this expansion takes the form

$$\phi_\kappa(r, z) = \int_0^\infty \frac{du}{u} e^{-u\kappa} A(u)\big(e^{-uz} + e^{uz}\big) J_0(ru). \tag{3.5}$$

The potential on the two conducting discs, Δ, must be constant, and the electric field must be continuous at all points not on the discs. Imposing these boundary conditions leads to the following dual integral equations

$$\begin{aligned} \int_0^\infty \frac{du}{u}(1 + e^{-2\kappa u}) J_0(ru) A(u) = \Delta - r^2 \qquad 0 < r < 1 \\ \int_0^\infty du J_0(ru) A(u) = 0 \qquad r > 1. \end{aligned} \tag{3.6}$$

Following [63] we find that the solution of these integral equations can be given in terms of the solution to a Fredholm integral equation of the second kind. The problem in this case is very similar to the one considered in [1]. We have

$$A(u) = \frac{2u}{\pi}\int_0^1 dt\cos(ut)f(t), \tag{3.7}$$

where $f(t)$ satisfies the integral equation

$$f(t) + \int_{-1}^1 dx K(t,x)f(x) = \Delta - 2t^2, \tag{3.8}$$

and

$$K(t,x) = \frac{1}{\pi}\frac{2\kappa}{4\kappa^2 + (t-x)^2}. \tag{3.9}$$

For each value of Δ, the integral equation for f can be solved numerically. From the resulting electrostatics potential, we can compute the surface charge density on the discs

$$\sigma(r) = \frac{1}{\pi^2}\left[\frac{f(1)}{\sqrt{1-r^2}} - \int_r^1 dt \frac{f'(t)}{\sqrt{t^2-r^2}}\right]. \tag{3.10}$$

We can adjust the constant Δ until we find the value Δ_κ for which the corresponding solution f_κ satisfies $f_\kappa(1) = 0$. Then the surface charge distribution $\sigma_\kappa(r)$ for this solution vanishes at the edge of the discs. This final condition ensures the regularity of the corresponding supergravity solutions. The total charge on each disc is given by

$$q_\kappa = \frac{2}{\pi}\int_0^1 dt f_\kappa(t). \tag{3.11}$$

Figure (3.3) shows a plot of q_κ. For large κ the charge on each disc approaches $8/3\pi$, and for small κ the charge on each disc approaches $4/3\pi$.

Finally the solution for the general case is obtained by rescaling. The electrostatics potential is given by

$$V(r,z) = W_0(r^2 - 2z^2) + W_0 R^2 \phi_{d/R}(r/R, z/R), \tag{3.12}$$

and the total charge on each disc is given by

$$Q = W_0 R^3 q_{d/R}. \tag{3.13}$$

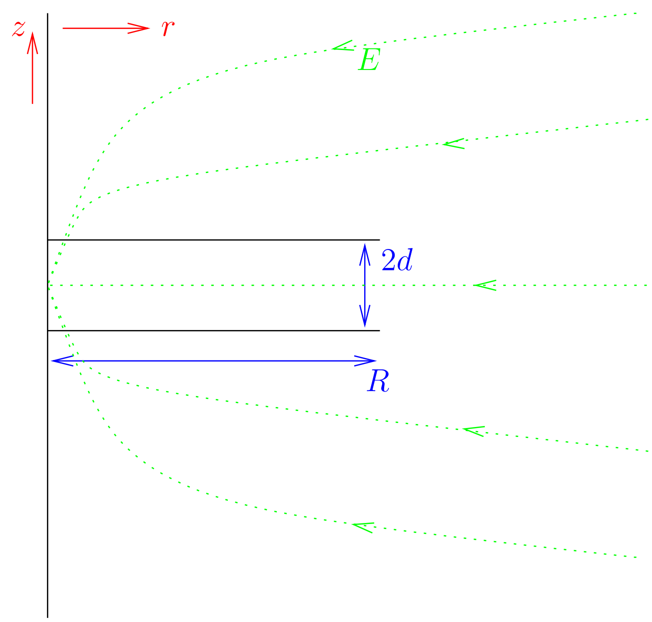


Figure 3.2: The electrostatics problem for two identical discs. The dotted lines show the background electric field configuration.

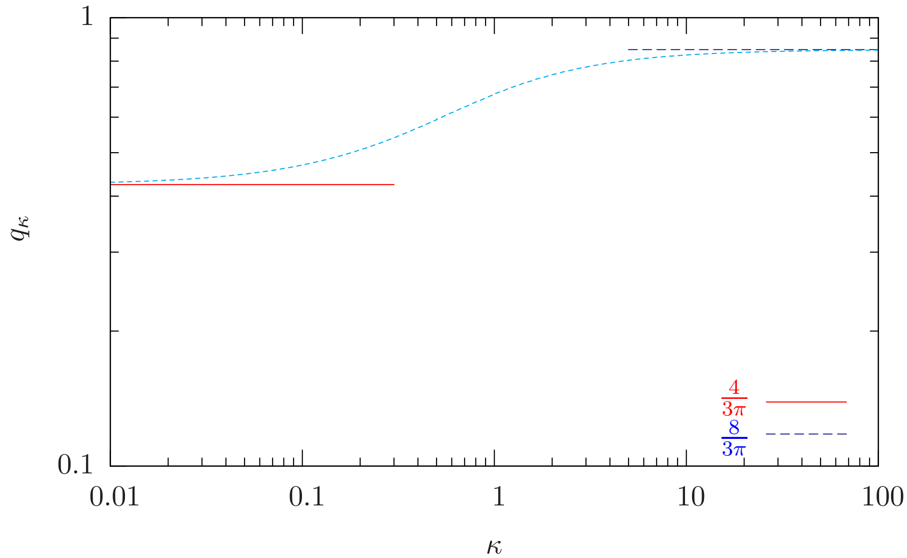


Figure 3.3: The charge on each disc in the two-disc case. The solid and dashed lines show the asymptotes for small and large κ respectively.

The Limit of the Lin-Maldacena Solution

Now we can determine the limit of this solution that gives the Lin-Maldacena solution for NS5-branes on S^5. In the region between the discs with $0 < r < R$, our solution is an axisymmetric solution of the Laplace equation that is regular at $r = 0$, so we can expand the solution in terms of modified Bessel functions

$$V(r,z) = V_{z=d} + \sum_{n=1}^{\infty} c_n \cos\left(\tfrac{(2n+1)\pi z}{2d}\right) I_0\left(\tfrac{(2n+1)\pi r}{2d}\right). \tag{3.14}$$

The coefficients c_n may be determined by using the potential at $r = R$. This gives

$$c_n = \left(I_0\left(\tfrac{(2n+1)\pi R}{2d}\right)\right)^{-1} 2W_0R^2 \int_0^1 dz \cos\left(\tfrac{(2n+1)\pi z}{2}\right) \left(1 - 2(\kappa z)^2 - \Delta_\kappa + \phi_\kappa(1,\kappa z)\right). \tag{3.15}$$

Using our numerical solution for ϕ_κ, the above integral can be performed numerically. In the limit $d \ll R$, this gives

$$c_1 \approx 1.56 W_0 R d \left(I_0 \left(\tfrac{\pi R}{2d} \right) \right)^{-1}. \tag{3.16}$$

For large R/d this expression will be dominated by the Bessel function, which takes the asymptotic form

$$(I_0(z))^{-1} \sim \sqrt{2\pi z} e^{-z}$$

To preserve some non-trivial geometry, we must then scale W_0 exponentially. Doing so keeps c_1 finite in the limit, but sends all the other coefficients to zero so that we recover the Lin-Maldacena solution. More precisely, the Lin-Maldacena solution is obtained in the limit

$$R \to \infty \qquad\qquad d \text{ fixed} \qquad\qquad W_0 \sim R^{-1}(Rd)^{-1/2} e^{\frac{\pi R}{2d}}. \tag{3.17}$$

The Gauge Theory Interpretation

Having understood the correct scaling on the gravity side, we can translate this into a condition on the gauge theory parameters. This amounts to

$$N \to \infty \qquad n \text{ fixed} \qquad \frac{1}{g_{YM2}^2} \lambda^{1/2} n^{1/2} e^{-b\lambda^{1/3}/n} \text{ fixed}, \tag{3.18}$$

where the 't Hooft coupling is $\lambda = g_{YM2}^2 N$ and b is a numerical coefficient related to the constant appearing in (3.2) by $b = (\pi/4)(3/h_2)^{1/2}$. We see that this is a large N limit, where the 't Hooft coupling is also scaled to infinity in a controlled way, and is very similar to the limit that was found in the case of the Plane-Wave Matrix Modelin [1]. Note that the number of NS5-branes is $N_5 = 2n$.

3.4 Little String Theory from $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$

Now we wish to perform a similar detailed analysis for the supergravity solution corresponding to a periodic array of discs of radius R, where the discs are located at $z = (2m+1)d$ (m is any integer), the charge on each disc is Q, and the background electric field is given by the potential $W_0(r^2 - 2z^2)$. Again we first solve the electrostatics problem, then find the limit that recovers the NS5-brane solution of Lin and Maldacena.

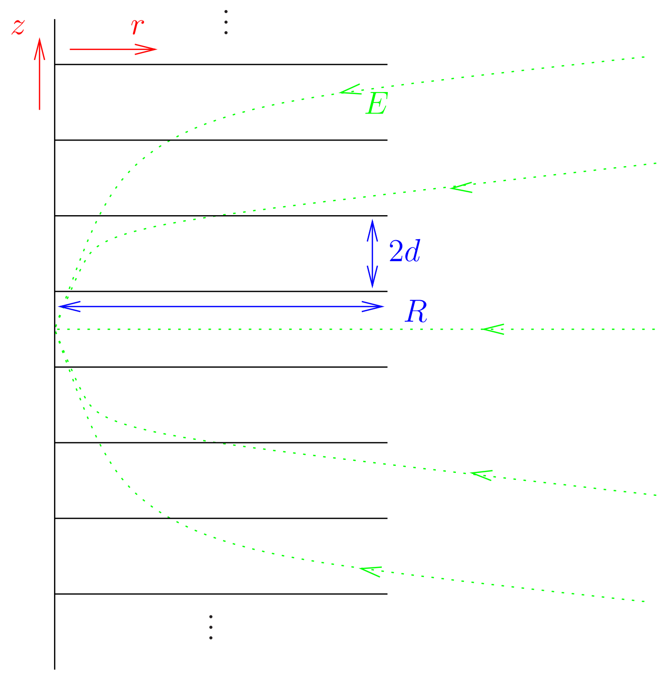


Figure 3.4: The electrostatics problem in the case a periodic array of discs. The dotted lines show the background electric field configuration.

The Electrostatics Problem for a Periodic Array of Discs

As in the previous section, we solve the electrostatics problem for the special case $W_0 = 1$, $R = 1$, $d = \kappa$, and obtain the solution for the general case by rescaling. In the absence of the background potential the charge distribution on each disc will be the same. Adding the background field will affect the charge distribution on each disc, but since the radial part of the electric field it creates is identical on each disc, the charge distribution will remain the same on each disc (see figure 3.4).

We can separate the potential into the sum of the background field and the part due to the charge on the discs.

$$V = r^2 - 2z^2 + \phi_\kappa(r, z), \tag{3.19}$$

where ϕ_κ is periodic in z because the charge on each disc is identical. Formally, we can expand $\phi_\kappa(r,z)$ in terms of Bessel functions as

$$\phi_\kappa(r,z) = \int_0^\infty \frac{du}{u} J_0(ru)A(u) \sum_{n=-\infty}^{\infty} e^{-u|(2n+1)\kappa - z|}, \tag{3.20}$$

and then try to determine the function $A(u)$ by the imposing the boundary conditions. If we take the value of the potential V to be $\Delta - 2\kappa^2$ on the disc at $z = \kappa$, then by imposing the boundary conditions we obtain the following dual integral equations

$$\begin{aligned} \int_0^\infty \frac{du}{u} \left(1 + \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}\right) J_0(ru)A(u) = \Delta - r^2 \qquad 0 < r < 1 \\ \int_0^\infty du J_0(ru)A(u) = 0 \qquad r > 1. \end{aligned} \tag{3.21}$$

However, direct attempts to solve these equations are met with divergences and various difficulties. The reason is that these equations hold only formally, because the sum in the expression for the potential (3.20) actually diverges. Physically, there is no divergence because the electric field remains finite. This is the same type of situation encountered for an infinite number of equally spaced point charges (or an infinite line of charge) on the z-axis, which occurs simply because we try to express the potential as a sum of the Coulomb potential for each charge. If we consider the potential difference between any two points, there is no divergence, so we can regularize (3.20) by subtracting the potential at any fixed reference point.

In this case, it is more convenient to consider the first integral equation (3.21) as a condition on the electric field rather than the electric potential

$$\int_0^\infty du \left(1 + \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}\right) J_1(ru)A(u) = 2r \qquad 0 < r < 1. \tag{3.22}$$

The dual integral equations can then be solved by introducing a function satisfying a Fredholm integral equation of the section kind,

$$f_\kappa(x) + \int_0^1 du K(x,u) f_\kappa(u) = -\frac{8x}{\sqrt{\pi}}, \tag{3.23}$$

where

$$A(u) = -\frac{1}{\sqrt{\pi}} \int_0^1 d\xi \, \sin(u\xi) f_\kappa(\xi). \tag{3.24}$$

The kernel is given by

$$K(x,u) = \frac{1}{\pi} \int_0^\infty dt k(t)(-\cos(u+x)t + \cos|u-x|t), \tag{3.25}$$

where

$$k(u) = \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}. \tag{3.26}$$

These integrals can be evaluated and the result is

$$\begin{aligned} K(x,u) = \frac{1}{2\pi\kappa}\Big(\Psi\Big(1 + \frac{i(x+u)}{2\kappa}\Big) + \Psi\Big(1 - \frac{i(x+u)}{2\kappa}\Big) \\ - \Psi\Big(1 + \frac{i|x-u|}{2\kappa}\Big) - \Psi\Big(1 - \frac{i|x-u|}{2\kappa}\Big)\Big), \end{aligned} \tag{3.27}$$

where Ψ is the digamma function. We solved (3.23) numerically using the Nyström method (e.g. [70]). In contrast to the two disc case, since we considered the integral equation corresponding to a condition on the electric field, there is no Δ to adjust to ensure that the surface charge density at the edge of the disc vanishes. In fact, for the form of the solution given in (3.24), this condition is automatically satisfied as long as f_κ is bounded. In terms of f_κ, the charge on each disc is

$$q_\kappa = -\frac{1}{\sqrt{\kappa}} \int_0^1 dt\ t f_\kappa(t). \tag{3.28}$$

Using our numerical solution for f_κ we found that q_κ approaches $8/3\pi$ for large κ and approximately 1.99κ for small κ (see figure 3.4).

In principle, it is possible to determine the regularized potential completely from this solution for f_κ (however, the integrals involved are rather computationally expensive). Then the potential for the case of general W_0, R and d is obtained by a linear rescaling of coordinates and an overall rescaling of the potential. Specifically, we note that the charge on each disc in the general case is

$$Q = W_0 R^3 q_{d/R}. \tag{3.29}$$

The limit of the Lin-Maldacena Solution

To determine how the Fourier coefficients of the potential scale with κ, we found it was most efficient to use the method of conformal mapping. Near the edge of the discs, when their radial size is much larger than their separation, the electrostatics problem becomes two-dimensional. By defining the complex coordinates $\zeta = (r - R) + iz$, and $w = 2\partial_\zeta V$ any holomorphic function $w(\zeta)$ will be a solution of the Laplace equation. As described in [40] the appropriate mapping in this case is

$$\partial_w \zeta = \alpha \tanh\big(\frac{\pi w}{\beta}\big) \tag{3.30}$$

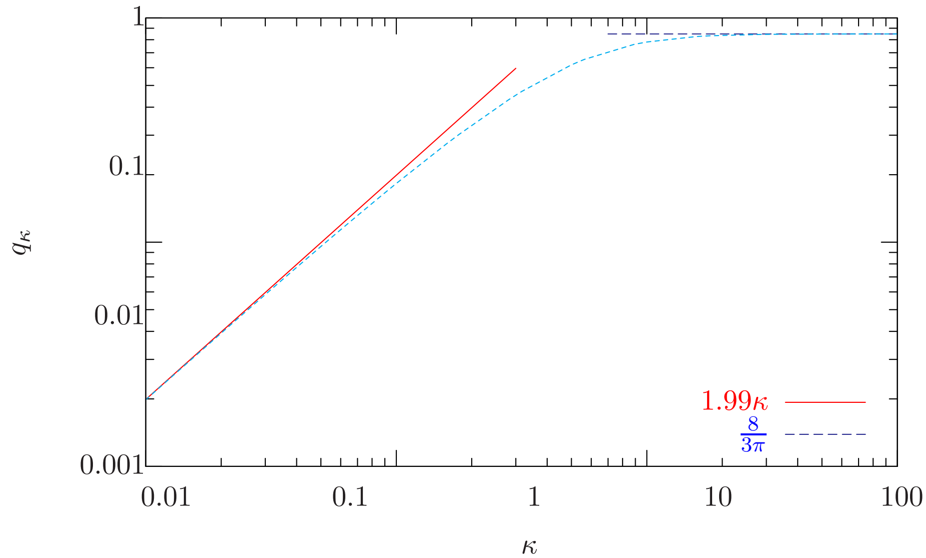


Figure 3.5: The charge on a disc as a function of the spacing between discs. The numerical result is given by the dashed line. The solid line is the asymptotic behaviour for small κ, $q \sim 1.99\kappa$. For large κ the charge approaches $\frac{8}{3\pi}$.

and so

$$\zeta = \frac{\alpha\beta}{\pi} \log\cosh\left(\frac{\pi w}{\beta}\right), \tag{3.31}$$

where α, β are constants. Inverting this we find

$$w = \frac{\beta}{\pi} \cosh^{-1}\left(e^{\frac{\pi\zeta}{\alpha\beta}}\right). \tag{3.32}$$

If we fix the positions of the discs to be at $\zeta = i(2md)$, where m is an integer, we have $d = \alpha\beta/2$. The vertical electric field at any disc should be $-4W_0\Im(\zeta)$, so that $\beta = 8W_0 d$ and $\alpha = 1/4W_0$.

Expanding the potential in terms of modified Bessel functions, as in the two-disc case, we find that

$$c_1 \approx \frac{16W_0 d^2}{\pi}(I_0(\tfrac{\pi R}{2d}))^{-1}(0.659). \tag{3.33}$$

Again, therefore, to preserve non-trivial geometry we must scale W_0 exponentially. The precise scaling form to obtain the Lin-Maldacena solution is

$$R \to \infty \qquad d \text{ fixed} \qquad W_0 \sim R^{-1/2} d^{-3/2} e^{\frac{\pi R}{2d}}. \tag{3.34}$$

The Gauge Theory Interpretation

In terms of the gauge theory parameters, we have

$$N \to \infty \qquad k \text{ fixed} \qquad \frac{1}{g^2_{YM3}} \lambda^{1/4} k^{1/2} e^{-c\lambda^{1/2}/k} \text{ fixed}, \tag{3.35}$$

where the 't Hooft coupling is $\lambda = g^2_{YM3} N$ and c is a numerical coefficient related to the constant appearing in (3.3) by $c = (2\pi/1.99h_3)^{1/2}$. This is again a double-scaling limit in which the 't Hooft coupling is scaled to infinity in a controlled way. Note that the number of NS5-branes in this case is $N_5 = k$.

3.5 Discussion

We have given an explicit prescription for taking double-scaling limits of maximally supersymmetric YM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ to obtain Little String Theory on S^5. These limits were obtained by using the family of supergravity solutions found by Lin and Maldacena [40]. With the similar result in 2, we have demonstrated that it is possible to take such a limit in each of the three generic examples of this family of solutions, and in each of the three field theories to which they are dual.

In each case, the precise form of the double-scaling limit is similar. Whereas in the Plane-Wave Matrix Model it was found the correct limit was [1]

$$N_2 \to \infty \qquad N_5 \text{ fixed} \qquad N_2 \sim \lambda^{5/8} e^{a\lambda^{1/4}/N_5} \,, \tag{3.36}$$

we found above that for the SYM theory on $R \times S^2$ we have

$$N \to \infty \qquad n \text{ fixed} \qquad N \sim \lambda^{1/2} n^{-1/2} e^{b\lambda^{1/3}/n} \,, \tag{3.37}$$

and for $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$ we have

$$N \to \infty \qquad k \text{ fixed} \qquad N \sim \lambda^{3/4} k^{-1/2} e^{c\lambda^{1/2}/k} \,. \tag{3.38}$$

As noted in [1], it is sensible that the correct limit to obtain Little String Theory from these field theories is a double-scaling limit as opposed to a strict 't Hooft limit. If the correct limit was the 't Hooft limit, then it would seem strange that the field theory could produce string loop interactions. That the 't Hooft coupling should also be scaled to infinity in a controlled way allows the field theory to reproduce the string genus expansion.

Suppose we consider the genus expansion for some physical observable in one of these theories

$$F = \sum_g N^{2-2g} f_g(\lambda, \alpha), \tag{3.39}$$

where α represents the other parameters. The double-scaling limit should be such that all terms in this expansion contribute. For this to occur, the terms in the expansion would have to take a particular form when λ is large. In the case of the Plane-Wave Matrix Model, this form was found to be [1]

$$f_g(\lambda) \to a_g \left(\lambda^{5/8} e^{a\lambda^{1/4}/N_5}\right)^{2g-2}, \tag{3.40}$$

where the bracketed expression divided by N_2 serves as the effective coupling constant. Here we find for the SYM theory on $R \times S^2$ we must have

$$f_g(\lambda) \to a_g \left(\lambda^{1/2} e^{b\lambda^{1/3}/n}\right)^{2g-2}, \tag{3.41}$$

and for $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$

$$f_g(\lambda) \to a_g \left(\lambda^{3/4} e^{c\lambda^{1/2}/k}\right)^{2g-2}. \tag{3.42}$$

Interestingly, although these field theories live in different numbers of dimensions, it is possible to recover Little String Theory from each of them by similar double-scaling limits.

Obvious difficulties arise in checking these predictions. One might hope that there are some BPS observables for which such a check might be feasible. In the case of the circular Wilson loop in $\mathcal{N} = 4$ SYM the full set of planar diagrams can be summed [66]. The result in that case took the form

$$\langle W \rangle_{N=\infty} = \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}}. \tag{3.43}$$

This result has been extended to all orders in [71], where it was shown that the asymptotic behaviour goes like $e^{\sqrt{\lambda}}$ at each order. That behaviour also arises from modified Bessel functions. It would be interesting to calculate the circular Wilson loop in $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, and to compare it with our results here.

Other open questions remain. For example, as noted in chapter 1, the solution for Little String Theory on S^5 given by Lin and Maldacena [40] is the simplest of an infinite family of solutions that have an infinite throat with H-flux. It would be interesting to understand if these solutions could arise from limits of more general disc configurations. It would also be interesting to understand more about the vacua of Little String Theory dual to these solutions.

Chapter 4

General Lin-Maldacena Solutions and Plane-Wave Matrix Model Instantons from Supergravity

4.1 Introduction

In the preceding two chapters, we have been able to solve the electrostatics problem that arises in finding regular $SU(2|4)$ symmetric solutions of type IIA supergravity that are dual to vacua of the Plane-Wave Matrix Model and maximally supersymmetric YM on $R \times S^2$. We have also found that these solutions can be used to find definitions of Little String Theory as double-scaling limits of the $SU(2|4)$ symmetric field theories. Buoyed by our success, it is interesting to attempt to find more general solutions, and to determine what other aspects of the field theories at strong coupling might be elucidated using knowledge of the dual gravity side.

The solutions that we have found so far, and the ones found by others [40] are all special cases. One particular solution given by Lin and Maldacena [40] corresponds to two infinitely large discs held at fixed separation. This is the solution dual to Little String Theory that has played a key role in chapters 2 and 3. The gravity dual was used in [67] to argue that Little String Theory on S^5 has interesting features that differ from the theory in flat space. An explicit solution has also been given in the case of a single isolated disc, dual to a vacuum of the maximally supersymmetric Yang-Mills theory on $R \times S^2$ [1, 40]. Also, in the region very close to the tip of a disc, the problem becomes two dimensional, and it is possible to solve it by conformal mapping [40]. Solutions in the case of two discs were given above. More general solutions, however, are not known.

In general this has prevented this set of dualities from being used to study the $SU(2|4)$ symmetric field theories at strong coupling. It is interesting to consider what questions can be addressed from the information we do know on the gravity side. Recently, Lin [72] has made some progress in applying this correspondence to instanton calculations in the Plane-Wave Matrix Model and maximally supersymmetric Yang-Mills theory on $R \times S^2$. In the case of the matrix model, this question had been studied directly in the field theory [73]. Lin gave explicit results for weak coupling from the supergravity side, and found precise agreement with the gauge theory analysis.

It would certainly be desirable to be able to perform other gauge theory calculations using the dual gravity description. It is, therefore, quite interesting to obtain more general supergravity solutions that would allow this to be done.

In this chapter we will demonstrate that it is possible to reduce the generic electrostatics problem to a simple linear system that can be solved very simply using numerical methods. We will then use this technique to find some explicit results using Lin's prescription for instanton calculations on the dual gravity side. For a simple example electrostatics configuration, dual to a vacuum of the Plane-Wave Matrix Model, we will give an explicit expression for the superpotential at strong coupling, and also the leading correction to Lin's result at weak coupling.

4.2 Supergravity Solutions

In this section we will review the Lin-Maldacena formulation of supergravity solutions in terms of electrostatics problems, full details can be found in [40]. Then in subsections 4.2.1 and 4.2.2 we will discuss the solution of these problems.

To find the supergravity duals to field theories with $SU(2|4)$ symmetry, Lin and Maldacena looked for similarly symmetric supergravity solutions. In particular, the bosonic part of this symmetry group is $R \times SO(3) \times SO(6)$ so the supergravity solutions should contain an S^2 and an S^5. Interestingly, with this restriction all of the supergravity fields can be expressed in terms of a single function of the two remaining coordinates. For the supergravity equations to be satisfied, this function must be an axisymmetric solution to the Laplace equation in three dimensions.

The full supergravity solution in terms of this function, in the string frame, is

$$
\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V}-2\dot{V}}{-V''}\right)^{\frac{1}{2}} \left\{\frac{-4\ddot{V}}{\ddot{V}-2\dot{V}}dt^2 - \frac{2V''}{\dot{V}}(d\rho^2+d\eta^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right\} \\
e^{4\Phi} &= \frac{4(\ddot{V}-2\dot{V})^3}{-V''\dot{V}^2\Delta^2} \\
C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V}-2\dot{V}}dt \\
F_4 &= dC_3, \qquad\qquad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt\wedge d^2\Omega, \\
H_3 &= dB_2, \qquad\qquad B_2 = 2\left(\frac{\dot{V}\dot{V}'}{\Delta}+\eta\right)d^2\Omega, \\
\Delta &\equiv (\ddot{V}-2\dot{V})V'' - (\dot{V}')^2 \,.
\end{aligned} \tag{4.1}
$$

where V is the electrostatics potential, and dots and primes indicate derivatives with respect to $\log r$ and z, respectively. To avoid conical singularities in (4.1), when the size of the S^2 or S^5 shrinks, requires that either V is regular at $r = 0$ (S^5 shrinks) or $\partial_r V = 0$ (S^2 shrinks). Different supergravity solutions can, therefore, be specified by inserting some conducting discs of various radii R_i at positions z_i. See figure 4.1. Inserting a disc will create separate two regions on the z-axis on which the S^5 shrinks and will therefore mean adding a non contractible 6-cycle, which will carry NS5-brane flux. Similarly, the region between two discs, on which the S^2 shrinks, will be a non-contractible 3-cycle carrying D2-brane flux.

Two additional constraints on the electrostatics solution come from ensuring that all of the metric components are positive definite and that the transformation to these coordinates is well defined. Positive definiteness requires that the electrostatics potential takes a definite asymptotic form, and the coordinate transformation requires that the charge density vanishes at the edge of each disc.

We will now describe a method for solving the electrostatics problems for generic configurations.

4.2.1 General Solutions Dual to SYM on $R \times S^2$

One of the field theories for which Lin and Maldacena found the corresponding electrostatics configurations is maximally supersymmetric $SU(N)$

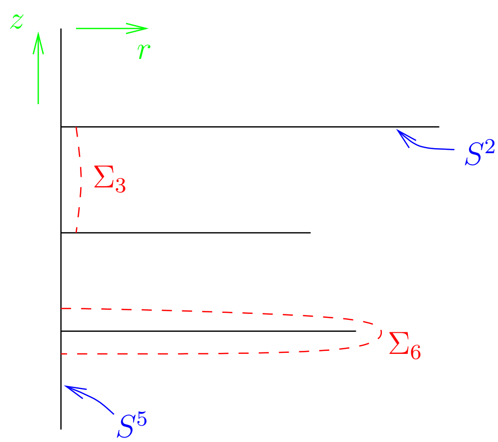


Figure 4.1: An example electrostatics configuration. The conducting discs are the horizontal solid lines. Fibred above the rz-plane are an S^2 and an S^5. The size of the S^2 shrinks on the discs, whereas the S^5 shrinks on the z axis. The dashed lines indicate topological 3- and 6-cycles in the geometry.

Yang-Mills theory on $R \times S^2$. This theory is related to $\mathcal{N} = 4$ SYM on $R \times S^3$ by dimensionally reducing that theory on the Hopf fibre of S^3 [40]. The field content is similar to $\mathcal{N} = 4$, however the theory on $R \times S^2$ admits vacuum configurations with non-trivial Φ, the scalar field resulting from the dimensional reduction. The vacua of this theory are parametrized by a set of integers where $\Phi = \mathrm{diag}(n_1, n_2, \ldots, n_N)$ [39]. General discussion of the relations among the vacua in the $SU(2|4)$ symmetric field theories can be found in [68, 69].

The set of electrostatics configurations in question are given by an arbitrary set of positively charged conducting discs. Each disc will be associated with some D2 brane charge, so it is natural to think of the supergravity solutions as being dual to vacua of maximally supersymmetric Yang-Mills on $R \times S^2$ [40]. The integers in the vacuum configuration for Φ are related to positions of the discs by $z = n\pi/2$, and the number of units of charge on the disc is related to the number of times that each integer appears by $Q = \pi^2 N/8$ [40].

The solutions to these electrostatics problems have been given in some specific cases [1, 2, 40]. For example the limit that the discs are very large, or only the geometry near the tip of a disc is of interest, the problem becomes two dimensional and it is possible to treat it with conformal mapping [1, 2, 40, 72]. In the case that there is a single disc it is possible to find an exact solution [1, 40]. For two equally sized discs a formal solution can be found [1, 2]. We will show that the techniques of [1, 2, 63] to solve the electrostatics problem for two identical discs can be extended to more arbitrary disc configurations. We will discuss how these more general solutions may be found using these techniques.

Consider the case of a collection of k charged conducting discs in the case of maximally supersymmetric Yang-Mills theory on $R \times S^2$. This problem is similar to the one for two discs considered in [2], however we will allow the discs here to sit at arbitrary positions, d_i and have arbitrary sizes, R_i. We can take the potential to be

$$V = W_0 \left(r^2 - 2z^2 + \sum_i \phi_i(r,z) \right) , \tag{4.2}$$

where the first two terms ensure the correct asymptotic conditions, and the third is an asymptotically vanishing contribution that comes from the charges on the discs. It takes the form

$$\phi_i(r,z) = \int_0^\infty \frac{du}{u} J_0(ru) A_i(u) e^{-u|z-d_i|} . \tag{4.3}$$

Each function A_i will be shown to determine the charge density on the i^{th} disc. To fix the form of these functions we impose the conducting boundary conditions on the discs. In particular, if the discs are held at fixed potentials Δ_i, then we will find a set of dual integral equations similar in form to those in [1, 2, 63]. The conditions at the i^{th} disc are that for $r < R_i$

$$\int_0^\infty \frac{du}{u} J_0(ur) \left[A_i(u) + \sum_{j \neq i} A_j(u) e^{-u|d_j - d_i|} \right] = \Delta_i + 2d_i^2 - r^2 , \tag{4.4}$$

and for $r > R_i$

$$\int_0^\infty du J_0(ur) A_i(u) = 0 . \tag{4.5}$$

We can make the ansatz

$$A_i(u) = \frac{2u}{\pi} \int_0^{R_i} dt \, \cos(ut) f_i(t) , \tag{4.6}$$

so that the conditions in (4.5) are automatically satisfied, and the conditions (4.4) become

$$f_i(r) + \sum_{j \neq i} \int_0^{R_j} dx \, \bar{K}_{ij}(x,r) f_j(x) = g_i(r) , \tag{4.7}$$

where

$$\bar{K}_{ij}(x,r) = \frac{|d_i - d_j|}{\pi}\left[\frac{1}{(x+r)^2 + |d_i - d_j|^2} + \frac{1}{(x-r)^2 + |d_i - d_j|^2}\right], \quad (4.8)$$

$g_i(r) = \beta_i - 2r^2$, and $\beta_i = \Delta_i + 2d_i^2$. Since the g_i are all symmetric functions, it is simpler to take the system as

$$f_i(r) + \sum_{j\neq i}\int_{-R_j}^{R_j} dx\, K_{ij}(x,r) f_j(x) = g_i(r)\,, \quad (4.9)$$

where

$$K_{ij}(x,r) = \frac{1}{\pi}\frac{|d_i - d_j|}{(x-r)^2 + |d_i - d_j|^2}\,. \quad (4.10)$$

It is straightforward to show that the charge densities on the discs are given in terms of the f_i as

$$\sigma_i(r) = \frac{W_0}{\pi^2}\left[\frac{f_i(R_i)}{\sqrt{R_i^2 - r^2}} - \int_r^{R_i} du \frac{f_i'(u)}{\sqrt{u^2 - r^2}}\right], \quad (4.11)$$

and that the total charges are

$$Q_i = \frac{W_0}{\pi}\int_{-R_i}^{R_i} du\, f_i(u)\,. \quad (4.12)$$

To find the f_i we must solve the set of linear equations in (4.9), schematically this takes the form

$$\begin{pmatrix} 1 & K_{12} & \cdots \\ K_{21} & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 - 2r^2 \\ \beta_2 - 2r^2 \\ \vdots \end{pmatrix}. \quad (4.13)$$

Due to the complicated form of the kernels K_{ij} this system is not easy to solve analytically, however, it is straightforward to solve it numerically using the Nyström method (see, e.g. [70]). This consists of discretizing the interval and solving the resulting linear system. An additional set of constraints comes from ensuring that the charge densities vanish at the edges of the discs. This amounts to enforcing that $f_i(R_i) = 0$. We define $f_i^{(j)}$ as the set of solutions to (4.9) with $g_i(r) = \delta_i^j$, where if there are N discs $j = 1, \ldots, N$, and $f_i^{(0)}$ as the set with $g_i(r) = 2r^2$. The condition that the charge density vanishes at the edge of the disc is then that

$$\sum_j f_i^{(j)}(R_i)\beta_j = f_i^{(0)}(R_i)\,. \tag{4.14}$$

There will be a unique solution for β_i if $\det(f_i^{(j)}(R_i)) \neq 0$. The full solution f_i is then

$$f_i(r) = -f_i^{(0)}(r) + \sum_j \beta_j f_i^{(j)}(r)\,, \tag{4.15}$$

with the potentials ϕ_i given by

$$\phi_i = \int_{-R_i}^{R_i} dt\, G_i(r,z,t) f_i(t)\,, \tag{4.16}$$

where

$$G_i(r,z,t) = \frac{1}{\pi}\frac{1}{\sqrt{(|z-d_i|+it)^2+r^2}}\,. \tag{4.17}$$

We have therefore reduced the electrostatics problem to a very simple linear system. In the case that there are only two discs, the problem is very straightforward and solution has been used to understand the relationship between SYM on $R \times S^2$ and Little String Theory [2].

4.2.2 General Solutions Dual to the PWMM

Another theory for which Lin and Maldacena found the corresponding electrostatics configurations is the Plane-Wave Matrix Model [30]. The Plane-Wave Matrix Model can be found by a consistent truncation of $\mathcal{N} = 4$ SYM on $R \times S^3$ to the set of constant modes on the sphere [34]. The vacua of matrix model are given by the scalars that come from the former $\mathcal{N} = 4$ gauge field taking values in a representation of $SU(2)$ [30]. Lin and Maldacena [40] associated these vacua with configurations of charged conducting discs above an infinite conducting plane.

The method of solution is very similar to the case above. For the sake of brevity we will give the final solution. We will write the potential as

$$V = V_0\left(r^2 z - \frac{2}{3}z^3 + \sum_i \phi_i(r,z)\right)\,, \tag{4.18}$$

where the first two terms are the background field and the ϕ_i arise from the charged discs as

$$\phi_i(r,z) = \int_{-R_i}^{R_i} dt\, G_i(r,z,t) f_i(t)\,. \tag{4.19}$$

The Green function is

$$G_i(r,z,t) = \frac{1}{\pi}\Big(\frac{1}{\sqrt{(|z-d_i|+it)^2+r^2}} - \frac{1}{\sqrt{(|z+d_i|+it)^2+r^2}}\Big), \quad (4.20)$$

and f_i is a solution of the integral equation

$$f_i(r) + \sum_j \int_{-R_j}^{R_j} dx\, K_{ij}(r,x) f_j(x) = g_i(r)\,, \quad (4.21)$$

in which the kernel is given by

$$K_{ij}(x,r) = \frac{1}{\pi}\left[\frac{|d_i-d_j|}{(x-r)^2+|d_i-d_j|^2} - \frac{|d_i+d_j|}{(x-r)^2+|d_i+d_j|^2}\right], \quad (4.22)$$

and $g_i(r) = \beta_i - 2d_i r^2$, where $\beta_i = \Delta_i + \frac{2}{3}d_i^3$. The differences between this solution and the one presented in section 4.2.1 arise from the presence of the infinite conducting plane that, via the method of images, implies the presence of oppositely charged conducting discs below the image plane. However, the conditions on the charges on the discs (4.11),(4.12), still hold, so the requirement that the charge density vanishes at the edge of the disc is that $f_i(R_i) = 0$. We may again consider solutions to (4.21) in which $g_i(r) = \delta_i^j$, which we will call $f_i^{(j)}$, and $f_i^{(0)}$ for which $g_i(r) = 2d_i r^2$. The condition that the charge density vanishes at the edge of each disc is again (4.14), and f_i will be given by (4.15).

As in the case of SYM on $R \times S^2$, the electrostatics problem has been reduced to a very simple linear system. We will now show how we can solve this system to study instantons at strong coupling on the field theory side using this method.

4.3 Instanton Calculations

Recently, Lin [72] has considered tunnelling between vacua in the Plane-Wave Matrix Modeland in maximally supersymmetric YM on $R \times S^2$. It is possible to study this on both the gauge theory and gravity sides. In the gauge theory case, this can be approached by directly studying the instanton solutions [73]. Lin [72] has also shown that it is possible to introduce a superpotential that gives a bound for the instanton action according to

$$S_{inst} = -\frac{1}{g^2}\Delta W\,. \quad (4.23)$$

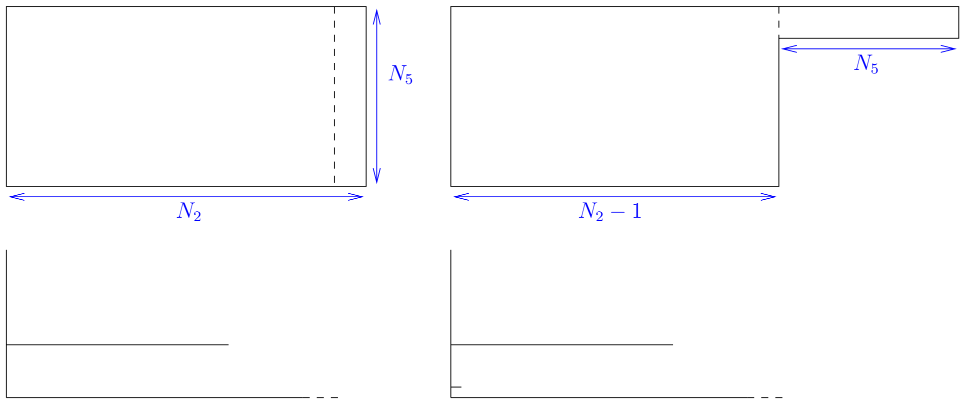


Figure 4.2: The Young diagrams associated with the initial and final vacua in the Plane-Wave Matrix Model, and the electrostatics problems for the dual supergravity solutions.

Lin [72] further studied this on the gravity side. Explicit answers were given in the case that the discs in the corresponding electrostatics problem were small and could be approximated by point charges. Moreover, Lin [72] gave a prescription for how the instanton action could be expressed in terms of the electrostatics potential in more general cases, but was not able to give explicit expressions.

For completeness, we will briefly review Lin's prescription for finding the instanton action from the gravity side [72], and then we will demonstrate the use of the techniques developed above to calculate the instanton action for some non-trivial electrostatics configurations.

4.3.1 Instantons on the Gravity Side

Since Lin and Maldacena [40] have found electrostatics configurations corresponding to $SU(2|4)$ symmetric field theory vacua, it is interesting to understand how instantons in the field theories can be described in the gravity picture. Lin [72] has studied this question by first considering vacua in the field theories for which the electrostatics configurations do not differ drastically. These instantons can be addressed by calculating the action for a Euclidean D2-brane wrapping a non-contractible Σ_3 in the geometry. As discussed in [40], since the brane will be wrapping a cycle carrying some N_5 units of flux, it should have N_5 D0-branes ending on it, and therefore describe the creation of N_5 D0-branes in the throat, see figure 4.2.

Using the mapping to an electrostatics configuration, the action for the

Euclidean D2-brane can be expressed in terms of the solution to the electrostatics problem. Consider the case of a charged conducting disc at a position z_0 above an infinite conducting plane, as shown in figure 4.2. Lin [72] has shown that the action for such a configuration takes the form

$$S_E = -\frac{2}{\pi}[V(z_0) - V(0) - z_0 V'(0)]\,, \tag{4.24}$$

where V is the electrostatics potential evaluated along $r = 0$, and prime denotes differentiation with respect to z. This expression, however, is proportional to the change in energy of the electrostatics configuration, $S_E = 8\Delta U/\pi^3$. This led Lin [72] to identify the superpotential at strong coupling as

$$W \equiv -\frac{8g^2}{\pi^3}U = -\frac{16g^2}{\pi^3}\sum_i Q_i V_i\,, \tag{4.25}$$

where U is the energy of the electrostatics configuration.

In the case that the discs are small relative to their separation, which is at weak coupling in the gauge theory, the superpotential is given by the energy of a system of point charges. Using the prescription (4.25), Lin found [72]

$$W = \frac{1}{3}\sum_i N_2^{(i)} N_5^{(i)3}\,, \tag{4.26}$$

in perfect agreement with the weak coupling gauge theory results [72, 73]. In the case that the coupling is not weak, the charges arrange to form extended discs. We will find the superpotential at strong coupling by solving the electrostatics problem for a set of extended discs.

4.3.2 Instantons in the PWMM

In this section we will consider instantons in the simplest non trivial electrostatics configuration, that of a single conducting disc carrying $\pi^2 N_2/8$ units of charge at a distance $\pi N_5/2$ above a conducting plane. This corresponds to a field theory vacuum with N_2 copies of the N_5 dimensional representation. We will determine the superpotential for arbitrary N_2 and N_5, and calculate the action for Euclidean D2-brane wrapping the non-contractible Σ_3.

In the case of small changes to the background mediated by the Euclidean D2-brane, this will compute the action between a vacuum of the Plane-Wave Matrix Model with N_2 copies of the N_5 dimensional representation, and a vacuum with $N_2 - 1$ copies of the N_5 dimensional representation and N_5 copies of the trivial representation. See figure 4.2.

The electrostatics problem in this case can be approached using the technique outlined in section 4.2.2 applied to the case of a single disc above the conducting plane. We consider a disc of radius R, at a distance d from the plane, which is a generalization of the approach in [1]. The solution to the electrostatics problem will be

$$V = V_0 \left(r^2 z - \frac{2}{3} z^3 + \phi \right) , \tag{4.27}$$

where ϕ is

$$\phi(r, z) = \int_{-R}^{R} dt \, G(r, z, t) f(t) , \tag{4.28}$$

with Green function

$$G(r, z, t) = \frac{1}{\pi} \left(\frac{1}{\sqrt{(|z - d|^2 + it)^2 + r^2)}} - \frac{1}{\sqrt{(|z + d|^2 + it)^2 + r^2)}} \right) . \tag{4.29}$$

Here f satisfies the integral equation

$$f(r) + \int_{-R}^{R} dx \, K(r, x) f(x) = g(r) , \tag{4.30}$$

with $g(r) = \beta - 2dr^2$, $\beta = \Delta + \frac{2}{3} d^3$, and kernel

$$K(r, x) = -\frac{1}{\pi} \frac{2d}{(x - r)^2 + 4d^2} . \tag{4.31}$$

We will solve the problem by finding a numerical solution to the integral equation (4.30).

Solving this integral equation is straightforward. As a check on our numerical results, we ensured that the asymptotic form for the superpotential in the limit that the number of units of charge on the discs was small is given by (4.26). Indeed, we found that for $N_5 \gg \lambda^{\frac{1}{3}} \gg 1$ [26]

$$W \approx \frac{1}{3} N_2 N_5^3 + a\sqrt{\lambda} N_2 N_5^{\frac{3}{2}} , \tag{4.32}$$

where the numerical constant $a \approx 1.4$.

The solution of the electrostatics problem when the discs are large gives the superpotential at strong coupling. A plot of the superpotential for $N_5 =$

[26] Here $\lambda \equiv g^2 N_2$, where g is the Yang-Mills coupling of the matrix model.

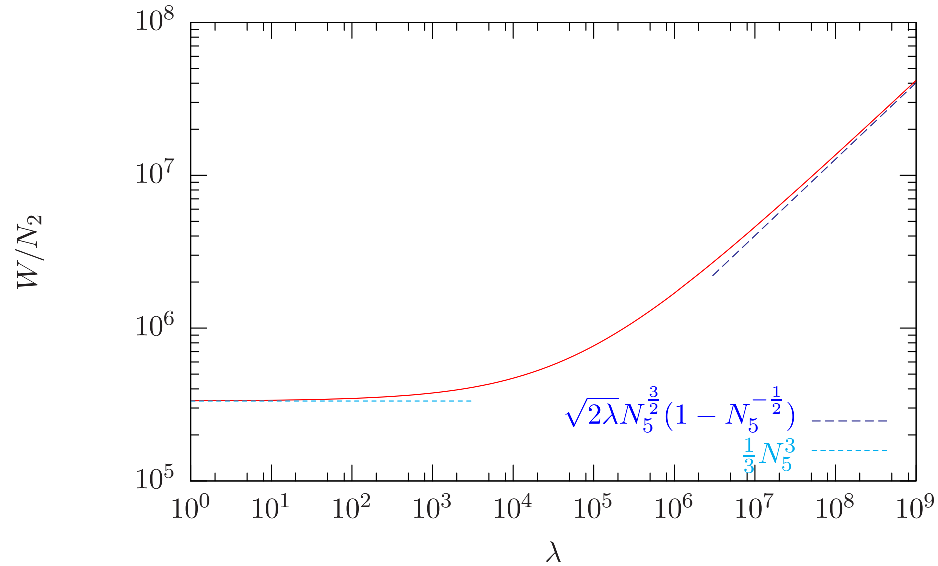


Figure 4.3: The superpotential for $N_5 = 100$. The dashed lines indicate the asymptotic values for small and large λ compared to N_5.

100 is given in figure 4.3. It is possible to extract the asymptotic form for the superpotential in the limit that $\lambda^{\frac{1}{4}} \gg N_5 \gg 1$. We find the result

$$W \approx b\sqrt{\lambda}N_2N_5^{\frac{3}{2}}(1-N_5^{-\frac{1}{2}}) + c\lambda^{\frac{1}{4}}N_2N_5^2(1-N_5^{-\frac{3}{4}}), \tag{4.33}$$

where the numerical constants $b \approx 1.4142 \approx \sqrt{2}$, and $c \approx 0.7$. In defining the superpotential, there was the freedom to choose an overall constant factor. Here we have defined the superpotential to be zero for the vacua given by N_2N_5 copies of the trivial representation.

We can also use the electrostatics solution to determine the action according to (4.24) for the instanton shown in figure 4.2. A plot of the result for $N_5 = 100$ is shown in figure 4.4. When λ is small compared to N_5,

$$S_E \approx \frac{1}{3}\frac{N_2N_5^3}{\lambda}, \tag{4.34}$$

and it falls off faster than any power of λ when λ is large. When λ is small compared to N_5, the potential on the disc in the electrostatics problem is negative, which is due to the form of the background potential. As the size of the disc increases the potential on the disc increases. The instanton action

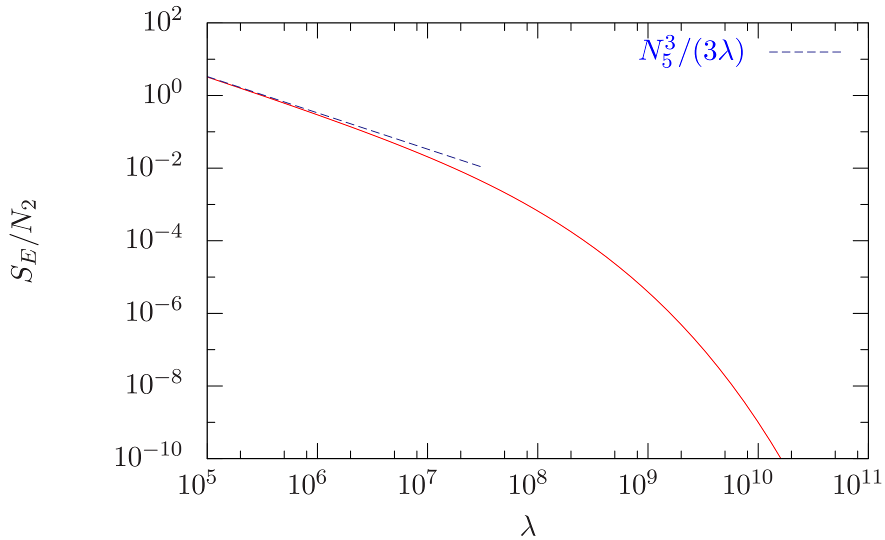


Figure 4.4: The action for a euclidean D2 brane for the situation shown in figure 4.2. Here $N_5 = 100$. The dashed line shows the asymptotic behaviour for λ small compared to N_5.

according to (4.24) begins to fall off from the behaviour at weak coupling near where the potential on the disc crosses zero. It is sensible that it should vanish when the coupling is infinite, since in that case we would expect the electric field to become constant between the discs near the origin, and so the potential difference and dipole contributions should cancel out. Let us briefly mention when these results should be valid. The euclidean brane approximation will be valid when the number of units of charge on the disc is large, the potential from the dipole at the origin is small at the surface of the disc, and the curvature is small. Therefore λ, N_2 and N_5 must all be large.

It would be helpful to better understand the scaling behaviour that was found in (4.32) and (4.33). We have taken the normalization of the superpotential thus far to allow for direct comparison of vacua with N_2 copies of the N_5 dimensional representation to ones with N_2N_5 copies of the trivial representation. The asymptotic behaviours we found in (4.32) and (4.33) using this prescription have some interesting features. In particular, at a fixed order in λ, we found that the coefficients for the subleading terms in N_5 are one (i.e. the factors of $(1 - N_5^{-\alpha})$, where α is some positive number). The reason for this is as follows. If we took the superpotential to be normalized to zero for the empty background instead, it would have been advantageous to take out a further scaling factor of $N_5^3 \sim d^3$ in (4.27). In that case, the potential would be of the form $V = V_0 N_5^3 \bar{V}(R/d)$. Likewise, the charge on the disc would then have the form $Q = V_0 N_5^4 \bar{q}(R/d)$. Since the total charge is proportional to N_2, we must have that

$$Q \sim N_2 \sim \frac{N_5^4}{g^2}\bar{q}(R/d)\,, \tag{4.35}$$

or

$$\bar{q}(R/d) \sim \frac{g^2 N_2}{N_5^4} = \frac{\lambda}{N_5^4}\,. \tag{4.36}$$

We see, then, that functions of R/d in the scaled electrostatics problem can only depend on the combination of gauge theory parameters λ/N_5^4. Therefore the superpotential with this alternative normalization must have the form

$$\bar{W}(\lambda, N_2, N_5) = N_2 N_5^3 \bar{w}\left(\frac{\lambda}{N_5^4}\right). \tag{4.37}$$

Our numerical results confirm this. We find the following asymptotic behaviour for $\bar{w}$:

$$\bar{w}(x) \approx \frac{1}{3}\left(1 - \frac{10}{3}x^{\frac{2}{3}}\right), \qquad x \ll 1, \tag{4.38}$$
$$\bar{w}(x) \approx -\sqrt{2}x^{\frac{1}{2}} + 0.7x^{\frac{1}{4}}, \qquad x \gg 1.$$

The superpotential with the original normalization is given in terms of $\bar{W}$ by

$$W(\lambda, N_2, N_5) = \bar{W}(\lambda, N_2, N_5) - \bar{W}(\lambda N_5, N_2 N_5, 1). \tag{4.39}$$

If we combine the contributions from the asymptotic behaviour of each of the terms in this expression that come from the behaviour found in (4.38), then we will recover the asymptotic behaviour that was found in (4.32) and (4.33). The factors of $(1 - N_5^{-\alpha})$ occur as a result of those combinations.

4.4 Discussion

In this chapter we have given a prescription for finding the supergravity solutions dual to general vacua of the Plane-Wave Matrix Model and maximally supersymmetric YM on $R \times S^2$ by using the mapping of Lin and Maldacena [40] on to axisymmetric electrostatics problems.

The prescription extends the technique developed in the previous two chapters and [1, 2] to arbitrary electrostatics configurations. The electrostatics problems are reduced to a set of integral equations that can be solved quite straightforwardly using the Nyström method.

We have shown that an application of the prescription to a specific case can be used to study instantons at strong coupling in the Plane-Wave Matrix Model. In particular we found that the instanton action for a transition between a vacuum described by N_2 copies of the N_5 dimensional representation and one by $N_2 - 1$ copies of the N_5 dimensional representation and N_5 copies of the trivial representation falls off faster than any power of λ at strong coupling (see figure 4.4). We also found that at strong coupling the superpotential for a vacuum with N_2 copies of the N_5 dimensional representation behaves like $\sqrt{2\lambda} N_2 N_5^{3/2}$, when N_5 is large (see figure 4.3). This demonstrates that the techniques developed above are useful for obtaining strong coupling results in the field theory.

One question that would be interesting to address using this method is to calculate the superpotential explicitly for more general electrostatics configurations. For example, studying the vacuum of maximally supersymmetric Yang-Mills theory on $R \times S^2$ in which $\Phi = \mathrm{diag}(n, \ldots, n, -n, \ldots, -n)$, a similar scaling could be applied as was done for the Plane-Wave Matrix Model. In that case we would expect that the corrections to the weak coupling results given by Lin [72] would depend on the parameter λ/n^3. It would certainly be interesting to study that case in detail, as well as other more general vacua.

One open question is to prove that requiring the charge density vanish at the edge of each disc implies that there is a unique solution to the electrostatics problem. The condition for a unique solution to exist is given above by requiring $\det(f_i^{(j)}(R_i)) \neq 0$. We have not been able to prove that this is true in general. It would be interesting to do so.

Finally, it would be very interesting to use this method for finding the dual geometry to study other strong coupling phenomena on the gauge theory side.

Chapter 5

Coarse-Graining the Lin-Maldacena Solutions

5.1 Introduction

In the preceding chapters we have seen a class of examples of gauge/gravity duality in which the field theories have discrete, degenerate bases of vacuum states, each of which corresponds to a non-singular geometry.

The field theories are those we have met in the the foregoing chapters: the Plane-Wave Matrix Model [30], maximally supersymmetric YM on $R \times S^2$ [39], and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ [40]. The theories have degenerate vacua and their $SU(2|4)$ supersymmetry may be used to argue that the classical vacua are degenerate in the quantum theory and are present at strong coupling [36]. Lin and Maldacena [40] looked for dual supergravity solutions and found solutions in one-to-one correspondence with the natural bases of vacuum states in the field theories. [27] Most of the discussion in this chapter will concentrate on the Plane-Wave Matrix Model, however in 5.6 we will generalize our analysis to the other theories.

We have seen so far that although the geometries corresponding the basis of vacuum states all share the same asymptotics. In the Plane-Wave Matrix Model, for example, they are asymptotically the near-horizon D0 brane solutions. In the infrared region, however, they differ greatly, even in their topology: they contain different sets of 3- and 6-cycles carrying different amounts of flux. Since the generic vacuum state in the field theory is a linear superposition of basis elements, such a state cannot be dual to a single non-singular supergravity solution with fixed topology (assuming there are observables that can detect topology), but must simply be dual to a quantum superposition of the topologically different geometries. Simi-

[27] The construction is completely analogous to the construction of gravity duals to half BPS states of $\mathcal{N} = 4$ SUSY Yang-Mills theory [42]. As in that case, the smooth supergravity solutions corresponding to particular states can have large curvatures, and thus are only approximations to the true dual geometries which should minimize the α'-corrected low-energy effective action.

larly, generic mixed states in the field theory, such as the zero-temperature limit of the thermal state, involve microstates corresponding to many different topologies so we might expect that a gravitational dual description in terms of a single geometry is impossible. On the other hand, there are many examples of geometries believed to be dual to thermal states of field theories, and these thermal states involve enormous numbers of microstates that can be very different in the infrared. Mathur and collaborators have advocated (see [74] for a review) that we should interpret the thermal state geometry as a coarse-grained description of the underlying microstates, just as the homogeneous magnetization that we use to describe the thermal state of a spin system is a coarse-grained description of the true microstates of the spins. Specifically, the macroscopic description of almost any state in the underlying ensemble of microstates is extremely close to one particular coarse-grained configuration, the thermal equilibrium state. We will see that in our case also, there is a natural way to coarse-grain (i.e. give a macroscopic description of) geometries corresponding to typical microstates, and that most of the microstates have a coarse-grained description that is very close to a particular geometry, which we propose is the correct dual to the zero temperature limit of the thermal state. In this geometry, the complicated topological features that distinguish the individual microstate geometries are replaced by a singularity.[28]

The details of the coarse graining procedure are described in section 5.3 below, but we give the essential idea here. The supergravity fields in the Lin-Maldacena geometries are determined in terms of the potential for an axially symmetric electrostatics problem involving a certain number of parallel coaxial charged conducting discs in a background electric field. The number, locations and charges of the discs are determined by the data specifying the field theory vacuum.[29] We will find that typical field theory vacua correspond to electrostatics configurations with a large number of closely spaced discs whose radii are very small compared with the separation between the discs. At large N, such a configuration has a natural coarse-grained description as a smoothly varying charge distribution on the axis. Inserting the potential arising from this coarse-grained configuration into the Lin-Maldacena supergravity solution, one finds a singular geometry. Since all of the nontrivial topological features are associated with the regions between the discs in the electrostatics configurations (these regions map to topo-

[28]For a general discussion of conditions under which field theory states can be associated with semiclassical geometries, see [75] in the LLM context and [76] in the D1-D5 context.

[29]The radii of the discs are determined by the other information via a constraint.

logically non-trivial throats in the supergravity solutions), we see that the complicated topologies that characterize individual microstates are replaced by a singularity in the coarse-grained description.[30]

A completely analogous coarse-graining has been discussed [77–80] for the half-BPS sector of $\mathcal{N} = 4$ SUSY Yang-Mills theory. There, the microstate geometries are the type IIB LLM geometries [42], constructed in terms of droplets of a two-dimensional incompressible fluid, and the coarse-grained description allows for configurations with arbitrary density of the fluid between zero and the maximal density. One significant difference is that all of the states we consider are ground states for the field theory, whereas the LLM discussion relates to a special class of excited states with energy equal to an R-charge.

As emphasized in [77], a given coarse-grained configuration provides an approximation to a very large number of microstates, just as in the thermodynamic description of ordinary physical systems. Further, there is one preferred coarse-grained configuration, analogous to the thermal equilibrium state, which is very close to the coarse-grained description of almost any randomly chosen microstate. For the type IIB LLM geometries, the geometry corresponding to this preferred state was determined in [77] and dubbed the "hyperstar" geometry. In section 5.3 of this chapter, we determine the corresponding geometry for the Plane-Wave Matrix Model. In our case, the ensemble of microstates we consider is just the set of vacuum states, or alternately the set of states that contribute (each with equal weight) to the $T \to 0$ thermal state density matrix. Thus, we propose that our preferred geometry is the $T \to 0$ limit of the geometry dual to the thermal state of the field theory. In section 5.5, we also derive geometries corresponding to the preferred states in other restricted ensembles, analogous to the type IIB superstar [81], and discuss thermal geometries for the remaining $SU(2|4)$ symmetric theories in section 5.6.

As for an ordinary thermodynamic system, the thermal states we derive should maximize an entropy functional that measures the number of microstates nearby an arbitrary coarse-grained configuration. In section 5.4, we derive such an entropy functional, and find that it may be written simply in terms of the data that specify the geometry. We find that this functional is indeed maximized by the thermal state geometry of section 5.4. Further, we note that for all the coarse-grained configurations, those

[30] We should note however, that for the case of closely spaced discs, the supergravity approximation is not valid for the region between the discs, so the classical topological features that we are discussing should be understood to be replaced by some stringy analogue.

for which the entropy functional vanishes are the ones that coincide the original non-singular microstate geometries. On the other hand, configurations with non-zero entropy are necessarily singular.

In the general proposal by Mathur and collaborators, black hole geometries with horizons are to be understood as coarse-grained descriptions of underlying horizon-free microstate geometries. In the present setup, the coarse graining leads to geometries with naked singularities uncloaked by horizons, but this is to be expected since the number of microstates in our case is not large enough to give a classical finite-area horizon in the supergravity limit. It may be that a horizon develops as we move from the supergravity approximation to solutions minimizing the full low-energy effective action, but, as we will see, realizing this would necessarily involve understanding both α' and string loop corrections.

5.2 The $SU(2|4)$ Symmetric Matrix Quantum Mechanics and the Dual Lin-Maldacena Geometries

In this section, we review the Plane-Wave Matrix Model, its vacua, and the dual geometries constructed by Lin and Maldacena. The other $SU(2|4)$ symmetric field theories are discussed in section 5.6. We will see that each of these theories has a large degeneracy of vacuum states at the classical level. This degeneracy remains at the quantum level, since the representation theory of $SU(2|4)$ does not allow for states with arbitrarily small non-zero energies, and therefore does not allow the zero-energy states in the classical limit of the theory to receive corrections to their energy [36, 59].

5.2.1 The Plane-Wave Matrix Model

The Plane-Wave Matrix Model [30] is a massive deformation of the supersymmetric matrix quantum mechanics describing decoupled low-energy D0-branes in flat space.[31] It is described by a dimensionless Hamiltonian

$$\begin{aligned} H &= \mathrm{Tr}\left(\frac{1}{2}P_A^2 + \frac{1}{2}(X_i/3)^2 + \frac{1}{2}(X_a/6)^2 + \frac{i}{8}\Psi^\top\gamma^{123}\Psi \right. \\ &\qquad \left. + \frac{i}{3}g\epsilon^{ijk}X_iX_jX_k - \frac{g}{2}\Psi^\top\gamma^A[X_A,\Psi] - \frac{g^2}{4}[X_A,X_B]^2\right), \end{aligned} \tag{5.1}$$

[31] This is similar to the Polchinski-Strassler deformation of $\mathcal{N}=4$ SUSY Yang-Mills theory [58], but in this case, we preserve all 32 supersymmetries.

where $A = 1, \ldots, 9$, $i = 1, \ldots, 3$, and $a = 4, \ldots, 9$. Here, the scalars X_A and 16-component fermions Ψ are hermitian $N \times N$ matrices, and P_A is the matrix of canonically conjugate momenta. Apart from N, the size of the matrices, the theory has one dimensionless parameter g, such that the theory is weakly coupled for small enough g.[32]

For this theory, the classical vacua, each with zero energy, are described by

$$X^a = 0 \qquad a = 4, \ldots, 9 \qquad\qquad X^i = \frac{1}{3g} J^i \qquad i = 1, 2, 3,$$

where J^i give any reducible representation of the $SU(2)$ algebra. These vacua are in one-to-one correspondence with partitions of N, since we may have in general n_k copies of the k-dimensional irreducible representation such that $\sum_k kn_k = N$. Below, it will be convenient to represent such a partition by a Young diagram with N boxes, containing n_k columns of length k.

In the D0-brane picture, a block-diagonal configuration with n_k copies of the k-dimensional irreducible representation is associated classically with concentric D2-brane fuzzy spheres, with n_k spheres at radius proportional to k. On the other hand, it was argued in [39] that at sufficiently strong coupling, such a configuration is better described as a collection of concentric fivebranes, with multiplicities and radii given in terms of the numbers and lengths of columns in the dual Young diagram.[33] For general values of parameters, we can interpret the solution as a fuzzy configuration with both D2-brane and NS5-brane characteristics. This will be apparent from the dual gravitational solutions, which include throats carrying D2-brane flux and throats carrying NS5-brane flux in the infrared part of the geometry.

5.2.2 Electrostatics

The vacua of the matrix model each preserve $SU(2|4)$ symmetry. In [40], Lin and Maldacena searched for type IIA supergravity solutions preserving the same $SU(2|4)$ symmetry (more precisely, with isometries given by the bosonic subgroup $SO(6) \times SO(3) \times U(1)$ of $SU(2|4)$). Using an ansatz with this symmetry (reproduced in B.1), they were able to reduce the problem of

[32]The model was introduced originally as a matrix model for M-theory on the maximally supersymmetric eleven-dimensional plane-wave. For this we are required to take a limit $N \to \infty$ with $g^2 N \sim N^4$. In the present work, we will mainly be concerned with the usual 't Hooft large N limit with λ fixed.

[33]In [39], the matrix model was discussed in the context of its conjectured description of M-theory on a plane-wave background. There, the fivebranes were M5-branes, while here we are considering a limit with fixed λ, dual to a IIA background, so the fivebranes are NS5 branes.

finding supergravity solutions to the problem of finding axially-symmetric solutions to the three-dimensional Laplace equation, with boundary conditions involving parallel charged conducting discs and a specified background potential. Corresponding to each classical vacuum and choice of parameters, we have a specific electrostatics problem, whose solution (a potential $V(r,z)$) feeds into the equations (B.1) to give the dual supergravity solution. Further, the smooth supergravity solutions for which fluxes through non-contractible cycles are quantized appropriately are in one-to-one correspondence with the vacua.

For the other $SU(2|4)$ symmetric theories described in section 5.6, the construction differs only by a choice of boundary conditions (background potential or the presence/absence of infinite-sized conducting plates). The solution to these electrostatics problems has been discussed in [3].

We now describe the electrostatics problem in detail and then review some general features of the dual supergravity solutions. Common to all vacua, we have in the electrostatics problem an infinite conducting plate at $z = 0$ (on which we may assume that the potential vanishes), and a background potential

$$V_\infty = V_0(r^2 z - \frac{2}{3}z^3) \ . \tag{5.2}$$

In addition, corresponding to a matrix model vacuum with Q_i copies of the d_i-dimensional irreducible representation, we have conducting discs with charge Q_i parallel to the infinite plate and centred at $r = 0$, $z = d_i$.[34] In order that the supergravity solution is non-singular, the radii R_i of the discs must be chosen so that the charge density at the edge vanishes.

The parameters of the matrix model are related to the parameters in the electrostatics problem as $N = \sum Q_i d_i$ and $g^2 \propto 1/V_0$.

5.2.3 Gravity Duals

The coordinates r and z in the electrostatics problem form two of the nine spatial coordinates in the geometry. In addition, for each value of r and z, we have an S^2 and an S^5 with radii that depend on (r,z). The S^5 shrinks to zero size on the $r = 0$ axis, while the S^2 shrinks to zero size at the locations of the conducting plates, so we have various non-contractible S^3s and S^6s corresponding to paths that terminate on different plates or on different segments of the vertical axis respectively. This is illustrated in figure 2.1. As shown in [40], through an S^6 corresponding to a path surrounding plates with a total charge of Q, we have $N_2 = Q$ units of flux from the dual of

[34] Our conventions here are slightly different from the ones in [40]

the Ramond-Ramond four-form, suggesting the presence of N_2 D2-branes. Similarly, through an S^3 corresponding to a path between plates separated by a distance d, we have $N_5 = d$ units of H-flux, suggesting that this part of the geometry between the plates is describing the degrees of freedom of N_5 NS5-branes.

Since the matrix model is a massive deformation of the maximally supersymmetric quantum mechanics describing low-energy D0-branes in flat-space, we should expect that the dual supergravity solutions correspond to infrared modifications of the near-horizon D0-brane geometry. Indeed, the solutions are asymptotically the same as the near-horizon D0-brane solution, with the strong-coupling region in the infrared replaced by smooth topological features that depend on the choice of vacuum.

5.3 Coarse-Graining the Lin-Maldacena Geometries

For large N, the Plane-Wave Matrix Model has of order $\exp(\sqrt{6N}/\pi)$ independent vacua labelled by reducible dimension N representations of $SU(2)$. In this section, we will argue that as for standard thermodynamic systems (e.g. particles in a box), if we use coarse-grained, macroscopic variables to describe the states, then despite the large number of possible microscopic states, the description of a randomly chosen microstate will, with very high probability, be extremely close to the average or "thermal equilibrium" state. We will see explicitly what the coarse-grained description of this average state is in our case, and see that there is a natural way to associate a geometry to this (and more general) coarse-grained configurations. We will interpret the resulting geometry as the zero-temperature limit of the thermal state, since this state has a density matrix with equal contributions from each basis vacuum state. Much of the discussion in this section follows ideas in [77] for the LLM geometries.

5.3.1 Macroscopic Variables

We begin by understanding the macroscopic variables appropriate in our case. As we will see, typical gauge theory states for large N will correspond to electrostatics configurations with large numbers of charged discs at unit separation. The microstate configurations are specified by giving the (integer) charge at each discrete location on the vertical axis. Since the extent of the disc configurations on this axis will be much larger than the disc separations (typically by a factor of $\sqrt{N}$ as we will see), it is sensible to characterize configurations by a macroscopic charge density $Q(z)$. This, we can define by averaging the microscopic charge over a distance much larger than the disc separations, but much smaller than the vertical extent of the disc configuration. Thus, in the coarse-grained description of states, $Q(z)$ should be a smooth function.

We still need to understand how the charge $Q(z)$ should be arranged in the directions perpendicular to z (recall that for the microstates it spreads out dynamically on the charged conducting discs), but first it will be helpful to see what $Q(z)$ looks like for typical states.

5.3.2 Typical States

In the microscopic description, the charges Q_n at position $z = n$ label how many times the irreducible representation of dimension n appears, and are subject to the constraint

$$\sum_{n=1}^{\infty} nQ_n = N \ . \tag{5.3}$$

We would now like to ask what a typical randomly chosen representation looks like. To do this, we first note that the independent vacuum states of the matrix model are in one-to-one correspondence with the quantum states of a free massless boson on an interval (a.k.a. a quantum guitar string) with energy $E - E_0 = \hbar\omega N$, where ω is the frequency of the lowest mode. In this analogy, Q_n give the number of particles of frequency $n\omega$. For large N, where the energy and number of particles are large, we know that a thermodynamic description is appropriate, and that any macroscopic quantities evaluated for a randomly chosen microstate are extremely likely to be extremely close to the average values.

For our discussion, we will be interested in the average coarse-grained charge distribution defined above, so we start by computing the expected value of Q_n for each n. This is equivalent to calculating the expected particle numbers for our gas of free bosons in the microcanonical ensemble at energy $E = N$ (setting $\hbar = \omega = 1$). For large N, this should agree up to tiny corrections with the result as computed in the canonical ensemble, so long as we choose the temperature such that the expected value of the energy is N. The calculation is much simpler in the canonical ensemble, since now we can sum over all states without a constraint.

To study the canonical ensemble, we write a partition function [82]

$$\begin{aligned} Z &= \sum_{Q_n} e^{-\beta \sum n Q_n} \\ &= \prod_n \sum_{Q_n} e^{-\beta n Q_n} \\ &= \prod_n \frac{1}{1 - e^{-\beta n}} \ . \end{aligned} \tag{5.4}$$

From this, the expectation value of Q_n is found (for example by changing the β in front of Q_n to α, differentiating $\ln(Z)$ with respect to $-\alpha n$, and setting $\alpha = \beta$) to be

$$\langle Q_n \rangle = \frac{1}{e^{\beta n} - 1} \ . \tag{5.5}$$

The expected value of energy is

$$\begin{aligned} \langle N \rangle = -\partial_\beta \ln(Z) &= \sum_n \frac{n}{e^{\beta n} - 1} \\ &\approx \frac{\pi^2}{6\beta^2} \ , \end{aligned}$$

where the last line assumes that the sum can be approximated by an integral (valid for large N). Solving for β in terms of N and plugging in to (5.5), we find

$$\langle Q_n \rangle = \frac{1}{e^{\frac{\pi n}{\sqrt{6N}}} - 1} \ . \tag{5.6}$$

Thus, the coarse-grained approximation to a typical microstate will have a linear charge density very close to

$$\langle Q(z) \rangle = \frac{1}{e^{\frac{\pi z}{\sqrt{6N}}} - 1} \ . \tag{5.7}$$

Or, defining $x = z/\sqrt{N}$ and $\sqrt{N} q(x)$ to be the charge density in terms of x, we have

$$\langle q(x) \rangle = \frac{1}{e^{\frac{\pi x}{\sqrt{6}}} - 1} \ . \tag{5.8}$$

5.3.3 Supergravity Solution for the Average State

We would now like to understand the supergravity solution corresponding to the average coarse-grained configuration we have found. To do this, we first need to understand precisely how the charge $Q(z)$ should be distributed in the horizontal directions. For the microstates, the actual distribution of charge is determined dynamically, since the charges are free to move on conducting discs whose radii are determined by the constraint that the charge density at the edge vanishes. However, we will now see that the typical configurations for large N with fixed λ have discs whose radii are much smaller than the separation between the discs. Thus, in the coarse-grained picture for typical states, we can take the charge distribution to sit on the vertical axis.

To understand how large the discs should be, we note that for the microstates, having conducting discs with the correct radii is necessary in order to avoid singularities in the supergravity solution. If we simply place all the charge on the axis, singularities should appear (wherever $\partial_r V = 0$). These cannot be at radii much larger than the original radii of the discs, since at these large radii, the electrostatics potential should be modified only slightly when we move all the charge to the axis. Thus, the distance scale defined by the sizes of the discs should be the same as the typical coordinate distance from the axis where singularities appear in the modified configuration. We will now use this to estimate the radii of the discs for the typical configurations.

For a charge distribution $Q(z)$ on the vertical axis, the corresponding potential will be given by [82]

$$\begin{aligned} V(r,z) =& V_0\left(r^2 z - \frac{2}{3}z^3\right) \\ & + \int_0^\infty dz' Q(z') \left\{ \frac{1}{\sqrt{r^2+(z-z')^2}} - \frac{1}{\sqrt{r^2+(z+z')^2}} \right\} , \end{aligned} \tag{5.9}$$

where the second term arises from the image charges below the infinite conducting plate. It is straightforward to check that such a potential for smooth $Q(z)$ always gives rise to a singular supergravity solution [82]. The singularity appears at the locus of points where the radial component of the electric field vanishes [40]. To estimate this radius, we note that for slowly varying $Q(z)$, the radial electric field near the axis is given by

$$E_r(r) = -2rzV_0 + 2\frac{Q(z)}{r} ,$$

so the singularity is located at[35]

$$r = \sqrt{\frac{Q(z)}{zV_0}} . \tag{5.10}$$

[35] This should be a good approximation so long as r is small compared with Q/Q'.

From (5.7), we see that for z of order $\sqrt{N}$, the typical value of the charge on each disc is of order one, while for z of order one, the typical charge is of order $\sqrt{N}$. Recalling that $V_0 \sim 1/g^2$, we estimate that the typical radii of the discs will be

$$r \sim \sqrt{\frac{\lambda}{N^{\frac{3}{2}}}} \qquad z = \mathcal{O}(\sqrt{N}) \; ,$$
$$r \sim \sqrt{\frac{\lambda}{N^{\frac{1}{2}}}} \qquad z = \mathcal{O}(1) \; .$$

In either case, for large N and fixed λ the typical radii go to zero. Thus, in the coarse-grained description of typical states in the 't Hooft limit, we can take all the charge to be located on the z-axis. This leads us to the following conclusion: *the geometry dual to the $T = 0$ thermal state of the Plane-Wave Matrix Model at large N is given by the Lin-Maldacena solution (B.1), with potential (5.9) determined in terms of the charge distribution (5.7).* It may be that for some coordinate choice, the solution takes a simpler, more explicit form, but we have not investigated this.

5.3.4 Coarse-Grained Solutions for Large Discs

For large N and fixed λ, we have seen that the typical states have electrostatics configurations for which the discs are small relative to their separations, so that the charge can simply be taken to lie on the vertical axis in the coarse-grained description. However, it is also useful to have a coarse-grained description of states in cases where the radii of the discs is larger than their separations. This is relevant, for example, if we allow λ to scale as a power of N, or for fixed λ in restricted ensembles for which we restrict the number of fivebranes (as in section 5.5).

In such cases, the coarse-grained picture will have the closely spaced discs replaced by a uniform material that conducts only in the directions perpendicular to the z-axis. This material will have some smooth profile described by a radius function $R(z)$ and carry charges such that total charge on the conductor between heights z and $z + dz$ is $Q(z)$. Just as the radii of the discs in the original setup are determined by the charges, we should expect that $R(z)$ in the coarse-grained situation will be determined by $Q(z)$. Specifically, it turns out that the shape $R(z)$ of the conductor must be chosen such that the surface charge density vanishes. This $R(z)$ gives the coordinate location of the singularity in the supergravity solution corresponding to a given coarse-grained $Q(z)$. The details of this coarse-graining procedure

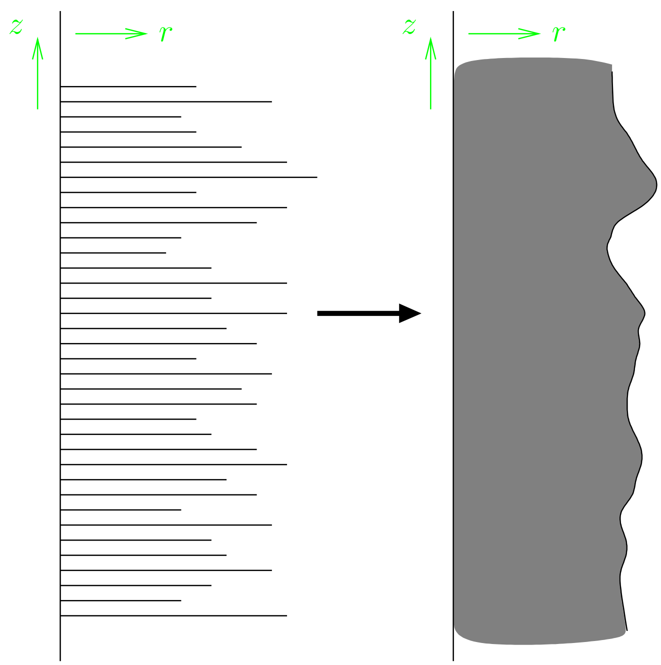


Figure 5.1: Coarse-graining for large discs. The shaded region represents a solid conductor that conducts only in the horizontal directions.

and the mathematical procedure that determines $R(z)$ in terms of $Q(z)$ are described in appendix D.1.

5.4 An Entropy Functional

In thermodynamic systems, we can often associate an entropy with coarse-grained configurations that are more general than the state of thermal equilibrium for the whole system. In this section, we give a functional that associates an entropy to a general coarse-grained Lin-Maldacena geometry and discuss its properties. A similar entropy functional has been derived recently for the LLM geometries in [75, 83].

5.4.1 A Familiar Example

To motivate our definition of an entropy functional for the Lin-Maldacena geometries, we will first concentrate on a more familiar system: a collection of spins on a line. We will work this example out in detail so that the our procedure for the Lin-Maldacena geometries is clear.

We will consider a line of independent spins interacting with a magnetic field with a strength scaled so that the partition function can be written

$$Z = \sum_{\{s_i\}} e^{-\beta \sum_i s_i}, \tag{5.11}$$

where β is the inverse temperature, and we will take $s_i = \pm 1$.

We will break up our system of N spins into macroscopic subsystems of ℓ spins according to where the size of the subsystems are such that

$$1 \ll \ell \ll N. \tag{5.12}$$

Since the size of each subsystem is large, we may use the canonical ensemble to study them. We find the entropy of each subsystem is given by

$$S = \ell\beta \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}} \ell \log(e^{\beta} + e^{-\beta}), \tag{5.13}$$

and the magnetization (or energy) will be given by

$$M = \ell \frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta}}. \tag{5.14}$$

We can then write an entropy density for the subsystem $s = S/\ell$ in terms of the average magnetization $m = M/\ell$ as

$$s = \frac{1+m}{2}\log\Big(\frac{1+m}{2}\Big) + \frac{1-m}{2}\log\Big(\frac{1-m}{2}\Big) \tag{5.15}$$

This gives the total entropy associated with our coarse-grained spin system as

$$S[m] = \int dx \left[\frac{1+m}{2}\log\Big(\frac{1+m}{2}\Big) + \frac{1-m}{2}\log\Big(\frac{1-m}{2}\Big)\right], \tag{5.16}$$

subject to the constraint that the total magnetization is

$$M = \int dxm. \tag{5.17}$$

Maximizing the entropy functional with respect to m, subject to the constraint, recovers constant m, which is just equivalent to stating that the thermal state has maximal entropy.

Our prescription for finding an entropy function is therefore the following: Break up the system into macroscopic parts (the sets of spins). Determine the entropy for each part and express it as a function of the coarse-grained variables (the average magnetization or energy) of the part. Add up the entropies of each of the parts to find the total entropy, subject to the constraint that the coarse-grained variables are consistent with globally defined quantities (the total magnetization or energy).

5.4.2 Entropy for Coarse-Grained Matrix Model Vacua

Now we move on to the Plane-Wave Matrix Model vacua. In this case, the variable that we use to describe our coarse-grained configurations is the charge density $q(x)$ (recall that we defined $x = z/\sqrt{N}$. Let us now consider the interval $[x, x+dx)$ as a subsystem of our analogue thermodynamic system. The charge in this interval, $q(x)dx$ is given as a sum of independent microscopic variables

$$q(x)dx = Q_n + \cdots + Q_{n+l} \ ,$$

which are also independent of the variables that determine Q outside the interval. Here $n = x\sqrt{N}$ and $l = dx\sqrt{N}$. We assume that the coarse-graining is over macroscopic distances, in other words that the number l of individual degrees of freedom contributing to $Q(x)dx$ is large. Thus, we should have $1 \ll l \ll n$. Now, for the subsystem, we have the partition function

$$Z = \prod_{k=n}^{n+l} \frac{1}{1 - e^{-\beta k}} \ .$$

This gives free energy

$$F \approx lT \ln(1 - e^{-n\beta}) \ ,$$

and energy

$$\begin{aligned} \bar{E} &= \langle nQ_n + \cdots + (n+l)Q_{n+l} \rangle \\ &\approx \frac{nl}{e^{n\beta} - 1} \ . \end{aligned}$$

The entropy is then

$$S = (E - F)/T = l \left[\frac{n\beta}{e^{n\beta} - 1} - \ln(1 - e^{-n\beta}) \right] \ .$$

Note that this is proportional to the size of the interval, so it makes sense to define an entropy density $s(z) = S/l$ or equivalently $s(x) = \sqrt{N}S/l$. We would like to express this in terms of the average charge density $Q(x)$ in the interval, given by

$$\begin{aligned} Q &= \langle Q_n + \cdots + Q_{n+l} \rangle / dx \\ &\approx \sqrt{N}\bar{E}/(nl) \\ &= \sqrt{N} \frac{1}{e^{n\beta} - 1} \ . \end{aligned}$$

Solving for β in terms of Q, and substituting into the formula for s, we find

$$s(x) = \sqrt{N}((q+1)\ln(q+1) - q\ln(q)) \ ,$$

where we have defined $q = Q/\sqrt{N}$.

Thus, we can associate to a coarse-grained configuration described by a charge density $q(x)$ an entropy

$$S[q(x)] = \sqrt{N} \int dx[(q+1)\ln(q+1) - q\ln(q)] \ . \tag{5.18}$$

Allowed vacua of the matrix model are subject to the constraint

$$\int dx x q(x) = 1 \ . \tag{5.19}$$

We can now check that maximizing (5.18) subject to the constraint (5.19) gives the correct result for the charge density. Introducing a Lagrange multiplier for the constraint and varying with respect to q, we find

$$\ln(q+1) - \ln(q) + \Lambda x = 0 \ .$$

This gives

$$q(x) = \frac{1}{e^{\Lambda x} - 1} \ ,$$

and enforcing the constraint yields

$$\Lambda = \pi/\sqrt{6} \ .$$

Thus, we reproduce (5.8).

For more general coarse-grained configurations, it is clear from (5.18) that the entropy will be nonzero if there is any interval (x_1, x_2) for which $q(x)$ is continuous and nonzero. Thus, the only way to have a vanishing entropy functional with a nonzero net charge is to have the charge located at discrete points on the axis such that $q(x)$ is a sum of delta functions, as we have in the microstate configurations.[36] In this case, the entropy vanishes since for large q, we have

$$(q+1)\ln(q+1) - q\ln(q) \sim \ln(q) \qquad \text{(large } q\text{)}$$

and

$$\int \ln(\delta(x-a))dx = 0 \ .$$

Recalling that the D2- and NS5-brane fluxes are quantized properly in the supergravity solutions if and only if the charges are quantized and located at integer values of z, we conclude that the entropy function is zero if and only if $q(x)$ corresponds to a microstate geometry.[37] Consequently, all coarse-grained configurations with non-zero entropy correspond to singular supergravity solutions.

Our formula (5.18) gives the entropy as a simple expression in terms of $q(x)$, which in turn directly determines the geometry. In this sense, it is a

[36] Technically, such a $q(x)$ can only appear as a coarse-grained configuration in the limit where we take the coarse-graining scale to zero. Thus, for any non-zero coarse-graining scale, the entropy will be non-zero for all configurations.

[37] This is analogous to the statement in the spin system we have considered above that the entropy vanishes if we know the precise microstate of the spin system, $m = \pm 1$ everywhere. If we think of the spins as binary digits, this, of course, has to do with the usual notions of information entropy.

geometrical formula for the entropy. We might also ask whether there is any direct relation to a horizon area (or Wald's generalization [84]) in this case. However, as is typical in examples with a large amount of supersymmetry, the singular coarse-grained geometries that we obtain have no horizons.[38] On the other hand, both the curvature and the dilaton diverge at the singularities, so the supergravity solution should receive both α' and string loop corrections. It is possible that the fully corrected solutions have horizons.

Following [52], we might hope that an appropriate definition of a stretched horizon around the singularity[39] would have area that reproduces the entropy (perhaps up to numerical factors). In fact, our setup should provide a very stringent test of any proposed definition of a stretched horizon, if we demand that it correctly reproduces the functional dependence of the entropy on $q(x)$. Unfortunately, as we show in appendix D.2, the necessary location of a stretched horizon whose area would reproduce our entropy is parametrically closer to the singularity than either the radius where the curvature becomes large or the radius where the dilaton becomes large. At this scale, it is probably naive to expect that a simple area would reproduce the entropy.

5.5 Other Ensembles

The $T = 0$ thermal solution we have found is analogous to the 'hyperstar' geometry of [77], dual to the coarse-grained typical state of $\mathcal{N} = 4$ SUSY Yang-Mills theory on S^3 with a $U(1) \in SO(6)$ R-charge equal to energy. For that theory, there is a related geometry known as the 'superstar' that has been understood as the geometry dual to the equilibrium state in a more restricted ensemble for which the number of D-branes in the spacetime is fixed. There are similar restricted ensembles that are natural to consider in our case.

To understand these, we recall that the microstate geometries contain various non-contractible S^3 cycles carrying NS5-brane flux and non-contractible S^6 cycles carrying D2-brane flux. For a given microstate, there will be some

[38] It was shown in [40] that the metric components in a general LM geometry B.1 will be continuous and nonzero (except for points on the conducting discs) for all potentials V satisfying the three dimensional Laplace equation. From this it is straightforward to see that the region outside of the coarse-grained conducting discs is causally connected.

[39] Possible definitions considered in the literature include the locus of points where the curvature becomes strong, where the dilaton becomes strong, where the local temperature equals the Hagedorn temperature, or where microstates begin to differ significantly from each other.

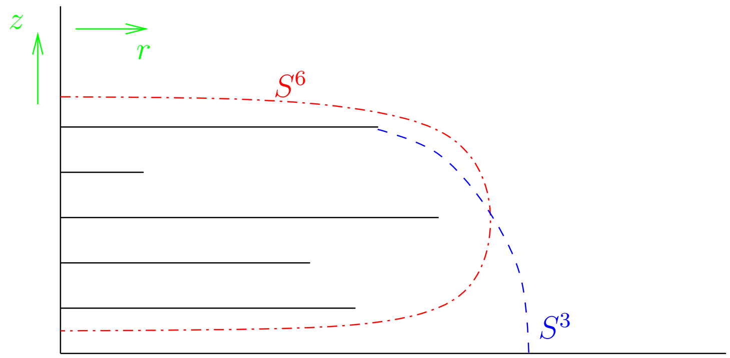


Figure 5.2: Example electrostatics configuration showing the non-contractible cycles S^3 and S^6 carrying the largest amount of NS5-brane and D2-brane flux respectively.

3-cycle in the geometry carrying a maximal number of units N_5 of NS5-brane flux and some 6-cycle carrying a maximal number of units N_2 of D2-brane flux, as shown in figure 5.2. We loosely refer to N_5 and N_2 as the number of NS5-branes and D2-branes in the geometry. Just as we understood the typical states in general, we can also ask about the form of the typical states in ensembles where either N_2 or N_5 or both are fixed.

To do this, we note that the total number of units of NS5-brane flux is given by the largest j for which $Q_j \neq 0$, while the number of units of D2-brane flux is given by the total charge $\sum_j Q_j$. If we consider a Young diagram with Q_j rows of length j, then N_2 and N_5 are the total number of rows and columns in the Young diagram respectively. The problem of studying typical Young diagrams with a fixed number of rows (or equivalently a fixed number of columns) is precisely the one studied in [77] to understand typical states in the hyperstar ensemble of LLM geometries, while the problem of studying typical Young diagrams with a fixed number of rows and columns is precisely the one studied in [77] to determine the typical configurations in the (generalized) superstar ensemble. Thus, we can directly carry over those results to find the $q(x)$.

5.5.1 Fixed N_5

For fixed N_5, we simply restrict the partition function (5.4) to $n \leq N_5$. The expected value of Q_n is given by the same formula,

$$\langle Q_n \rangle = \frac{1}{e^{\beta n} - 1} \,, \tag{5.20}$$

but now the expected value of N is

$$\begin{aligned}\langle N\rangle &= \sum_{n=1}^{N_5} \frac{n}{e^{\beta n}-1} \\ &\approx N_5^2 f(\beta N_5)\ , \qquad\qquad f(x) \equiv \frac{1}{x^2}\mathrm{Li}_2(1-e^{-x})\ .\end{aligned}$$

Thus, we obtain a charge density

$$Q(z) = \frac{1}{e^{\frac{z}{N_5} f^{-1}(N/N_5^2)}-1}\ , \qquad\qquad z \le N_5\ .$$

Note that in the unrestricted ensemble, the typical extent of the charge distribution was of order $\sqrt{N}$, so we only have a significant difference from the unrestricted ensemble when N_5 is of order $\sqrt{N}$ or smaller. One interesting case is that where we fix N_5 to some large but finite value in the large N limit. In this case, we find

$$\beta = \frac{N_5}{N}\ ,$$

and

$$Q(z) \approx \frac{N}{N_5 z}\ .$$

In this case, our estimate (5.10) for the size of the discs gives $r \sim \sqrt{\lambda/N_5^3}$, so the discs are large compared to their separations for $\lambda \gg N_5^3$. In this case, we need to use the methods of appendix D.1 to determine the appropriate coarse-grained geometry.

5.5.2 Fixed N_2 or Fixed N_2 and N_5

For fixed N_2 (with either fixed or unrestricted N_5), it is simplest to work in a grand canonical ensemble where we introduce a chemical potential for N_2 and tune it to get the correct value. We will therefore consider the partition function

$$Z(\beta,\mu) = \sum_{Q_j} e^{-\sum(\beta j+\mu)Q_j}. \tag{5.21}$$

From this, we obtain a charge distribution

$$\langle Q(z)\rangle = \frac{1}{e^{\beta z+\mu}-1}\ , \tag{5.22}$$

where β and μ are fixed by demanding

$$N = \left\langle \sum_j jQ_j \right\rangle = \sum_{j=1}^{N_5} \frac{j}{e^{\beta j+\mu}-1}\ , \tag{5.23}$$

as before, and

$$\left\langle \sum_j Q_j \right\rangle = \sum_{j=1}^{N_5} \frac{1}{e^{\beta j+\mu} - 1} \ . \tag{5.24}$$

In general, β and μ are complicated functions of N_2 and N_5, but as pointed out in [77], there is a simple special case where we take $\beta \to 0$ with fixed μ. This gives the solution in the case where we restrict

$$N_2 N_5 = 2N \ .$$

In this case, the charge density is constant

$$Q(z) = \frac{N_2}{N_5} \ , \qquad\qquad 0 \le z \le N_5,$$

and the supergravity solution may be written explicitly in terms of ordinary functions. This case corresponds to a triangular Young diagram, which in the LLM case gives rise to the original superstar geometry.

We also get a simple expression for the charge distribution in the case where N_2 is large but fixed in the large N limit with N_5 unrestricted. In this case, a straightforward calculation gives

$$Q(z) = \frac{N_2^2}{N_5} e^{-\frac{N_2}{N_5} z} \ .$$

5.6 Higher Dimensional $SU(2|4)$ Symmetric Theories

In this chapter thus far, we have focused on the Plane-Wave Matrix Model. However, as we have described in the preceding chapters, Lin and Maldacena [40] also identified supergravity duals to the vacua of other, higher dimensional, field theories with $SU(2|4)$ supersymmetry. These are the aforementioned maximally supersymmetric YM on $R \times S^2$, $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, and type IIA Little String Theory on S^5. [1, 39, 40]. Aspects of the relations among these theories and the Plane-Wave Matrix Model have been discussed in [68, 69], and in chapter 1.

In this section we will analyze these theories in the same way as we have the Plane-Wave Matrix Model. For the higher-dimensional theories, the construction of dual supergravity solutions differs only in the boundary conditions for the electrostatics problem. The individual microstates are still distinguished by the locations and charges of finite-sized conducting discs, so the coarse-graining procedure and the entropy functional are exactly the same as in the Plane-Wave Matrix Model.

5.6.1 Maximally Supersymmetric Yang-Mills Theory on $R \times S^2$

Field theory

We will first consider maximally supersymmetric YM on $R \times S^2$. We will briefly recall the relevant aspects of the theory, we have described above. This theory can be derived as a limit of the Plane-Wave Matrix Model [39], or of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ in the limit $k \to \infty$ [40].

The field content of this theory is the same as the usual low-energy D2-brane gauge theory, with an $SU(N)$ gauge field together with fermions and seven scalar fields. Six of the scalar fields are associated with the $SO(6)$ R-symmetry of the theory. The remaining one comes from the dimensional reduction when the $k \to \infty$ limit is taken in $\mathcal{N} = 4$ SYM on S^3/Z_k. We will refer to this scalar as Φ. The vacua of this field theory are given by $\Phi = -\mathrm{diag}(n_1, n_2, \ldots, n_N)$, and $F = dA = \Phi \sin\theta d\theta d\phi$, where the n_i are integers, and θ and ϕ are the usual coordinates on S^2.

The different vacua of the theory are labelled by the multiplicities of the integers in the vacuum configurations of Φ and F.

Supergravity

We will now recall the features of the dual gravity solutions, which we have described in the preceding chapters. The supergravity duals to vacua of the maximally supersymmetric YM on $R \times S^2$ have isolated sets of finite sized charged conducting discs and the background potential

$$V_\infty = W_0(r^2 - 2z^2) \ . \tag{5.25}$$

A diagram of these solutions was given in figure 3.1. As before, non-singular solutions will have discs with radii R_i chosen so that the charge density vanishes at the edge of each disc.

Corresponding to a vacuum with N_i copies of the integer n_i will be an electrostatics configuration with discs at positions $d_i = \pi n_i/2$ carrying charge $Q_i = \pi^2 N_i/8$. The gauge theory parameters are related to the electrostatics ones as $g^2_{\mathrm{YM}} \propto 1/W_0$, and $N = \sum N_i$.

In similar fashion to the Plane-Wave Matrix Model case, we can find the potential for the system with coarse-grained charge density Q to be

$$V(r,z) = W_0(r^2 - 2z^2) + \int_{-\infty}^{\infty} dz' \frac{Q(z')}{\sqrt{r^2 + (z-z')^2}}. \tag{5.26}$$

Typical states

As we have described above, the vacua of this theory are labelled by a set of integers and their multiplicities. Since the integers specifying the vacuum can be arbitrarily large (the only restriction is that the sum of multiplicities is N), we have an infinite number of vacua in this case. In the electrostatics picture, this corresponds to the fact that the discs are allowed to sit anywhere on the z-axis, with the only restriction that the total charge is N. As a result, quantities such as the charge at any location will average to zero, and we cannot see any natural way to define a typical configuration in this case for the unrestricted ensemble.

On the other hand, we do get a well defined thermal configuration in an ensemble where we fix the number of NS5-branes, as in section 5.5. This corresponds to fixing the separation between the highest and lowest disc. For the $SU(N)$ theory, we should demand also that the sum of integers times their multiplicities is zero, so we end up with a finite set of vacuum states. For coarse-grained typical states, the total charge N will be evenly distributed between the N_5 plates, so the coarse-grained charge density will be

$$Q(z) = \frac{N}{N_5} , \qquad\qquad -\frac{N_5}{2} \leq z \leq \frac{N_5}{2} .$$

Another way to obtain a non-trivial electrostatics configuration is to recall the definition of this theory as a $k \to \infty$ limit of $\mathcal{N} = 4$ SYM on S^3/Z_k. If we instead take a limit in which $N \to \infty$ and $k \to \infty$ with $N/k = \xi$ fixed then the resulting theory will have a $T = 0$ thermal state arising from the electrostatics potential $V(r,z) = W_0(r^2 - 2z^2) - (\pi\xi)/(2)\ln(r)$. The corresponding geometry will have a string like singularity with entropy density

$$s = (1+\xi)\ln(1+\xi) - \xi\ln\xi. \tag{5.27}$$

5.6.2 $\mathcal{N} = 4$ Yang-Mills Theory on $R \times S^3/Z_k$

Field theory

Let us now turn to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, and recall some of its relevant properties. This theory and its vacua can be obtained from $\mathcal{N} = 4$ SYM on S^3 in the following manner, as outlined in [2]. We can coordinatize the S^3 using the metric

$$ds^2_{S^3} = \frac{1}{4}[(2d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2] \tag{5.28}$$

where θ and ϕ are the usual coordinates on S^2, and ψ is an angular variable with period 2π. The orbifold is obtained by identifying $\psi \sim \psi + 2\pi/k$. The vacua of the field theory are given by the space of flat connections, modulo gauge transformations, on S^3/Z_k. The orbifold allows for vacua of the form $A = -\text{diag}(n_1, n_2, \ldots, n_N)d\psi$, so that $e^{2\pi n_i/k}$ are k^{th} roots of unity. This ensures that A has unit holonomy around the full angular direction ψ, which is topologically trivial. To label the vacua uniquely, we will restrict the integers n_i to be on the interval $[0, k)$.

Supergravity

In the supergravity picture, the background potential for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ is the same as in (5.25), but the electrostatics configuration is required to be periodic in z with period $\pi k/2$. Even though the background potential is not periodic in z, the part of the potential that determines the charge densities on the discs is. So the electrostatics solution will have a periodic part that arises from the charged discs in addition to the background piece.

The periodic arrays of conducting discs are, in turn, related to the vacua of the field theory. For a vacuum that has N_i repetitions of the integer n_i, the corresponding electrostatics configuration will have a set of charged conducting discs at positions $z = \pi n_i/2, \pi(n_i \pm k)/2, \pi(n_i \pm 2k)/2, \ldots$, each carrying charge $\pi^2 N_i/8$. The gauge theory parameters are given in terms of the electrostatics parameters by $g^2_{\text{YM}} k \propto 1/W_0$ and $N = \sum N_i$.

Here the potential for the system with coarse-grained charge density Q is

$$V(r, z) = W_0(r^2 - 2z^2) + \int_{-\infty}^{\infty} dz' \frac{Q(z')}{\sqrt{r^2 + (z - z')^2}}, \tag{5.29}$$

where Q has a of period $\pi k/2$.

Typical states

Having described the field theory vacua and the corresponding auxiliary electrostatics configurations, we would like to consider the typical state.

To find the typical configuration in this case, we can use the partition function (5.21) with $\beta = 0$. We can fix μ by imposing

$$N = k\frac{1}{e^{\mu} - 1}, \tag{5.30}$$

which means

$$e^{-\mu} = \frac{N}{N + k}, \tag{5.31}$$

and the typical vacuum will have $q = N/k$.

Up to an overall constant, the electrostatic potential can be found outside the charge distribution to be $V(r,z) = W_0(r^2 - 2z^2) - (\pi N)/(2k)\ln(r)$. It is singular, and has an entropy of

$$S = k\left(\left(1+\frac{N}{k}\right)\ln\left(1+\frac{N}{k}\right) - \frac{N}{k}\ln\left(\frac{N}{k}\right)\right). \tag{5.32}$$

5.6.3 Type IIA Little String Theory on S^5

Field theory

Type IIA Little String Theory on S^5, has figured prominently in chapters 2 and 3 of this thesis. We recall that it was defined originally by its supergravity dual, found in [40] and described below. Using this supergravity dual, it has been argued in this thesis can be defined by particular double-scaling limits of either the Plane-Wave Matrix Model [1], the maximally supersymmetric YM on $R \times S^2$ or $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ [2].

Supergravity

Let us recall briefly its supergravity solution. In this case, for the theory associated with k fivebranes we have two infinite conducting plates separated by a distance k. As shown by Lin and Maldacena [40], we can have a non-trivial potential

$$V(r,z) = \frac{1}{g_0} I_0\left(\frac{r}{k}\right)\sin\left(\frac{z}{k}\right) \tag{5.33}$$

between the plates for which the corresponding geometry has an infinitely long throat carrying NS5-brane flux. The parameter g_0 is related to the size of the sphere on which the NS5-branes sit, as measured in units of α' (the dimensionful coupling of the Little String Theory).

We can consider adding additional charged conducting discs to this system while keeping the number of units of NS5-brane flux fixed. In the electrostatics picture, this corresponds to adding some number of finite charged conducting discs in the region between the two infinite discs. The discs can sit at positions $d_i = \pi n_i/2$, where the integers n_i are in the interval $[1, k)$, and carry finite charges N_i.

Typical states

As for the 2+1 dimensional case, the number of vacua here is infinite if we allow arbitrary configurations finite discs in between the infinite conducting plates. However, it is interesting to consider some restricted ensembles.

First, we add some fixed number N of units of D0-brane flux. This requires that

$$\sum_i iN_i = N \ .$$

In this case, the counting problem is identical to that is section 5.5.1, so we obtain the same typical charge distribution. Of course, the supergravity solution will be different here, since the background potential is now (5.33).

Alternatively, we could consider an ensemble of geometries in which the number of units of D2-brane charge is fixed. In that case it is again convenient to use (5.21) with $\beta = 0$. Fixing the asymptotic charge we find that

$$N_2 = (N_5 - 1)\frac{1}{e^{\mu} - 1}, \tag{5.34}$$

which can be inverted to give

$$e^{-\mu} = \frac{N_2}{N_2 + N_5 - 1}. \tag{5.35}$$

The typical state will have

$$\langle Q_j \rangle = \frac{1}{e^{\mu} - 1} = \frac{N_2}{N_5 - 1}, \tag{5.36}$$

and the entropy of this configuration is, for $N_5 \gg 1$,

$$S = N_5 \left(\left(1 + \frac{N_2}{N_5}\right) \ln \left(1 + \frac{N_2}{N_5}\right) - \frac{N_2}{N_5} \ln \left(\frac{N_2}{N_5}\right) \right). \tag{5.37}$$

Chapter 6

Conclusion and Future Directions

The present thesis has examined a variety of topics in the context of gauge/gravity duality for field theories with $SU(2|4)$ supersymmetry. We have argued that Little String Theory arises as a double-scaling limit of each of the $SU(2|4)$ symmetric theories. We found that in each case the theory arises as a large N limit in which the coupling constant is scaled to infinity in a controlled way, thereby providing a Lagrangian definition of Little String Theory on S^5. The double scaling limits we found in each case were such that $\lambda/N \to 0$, in contrast to the M-theory limit in which $\lambda/N \to \infty$. This contrast is interesting because Little String Theory is a five-dimensional theory without gravity, and M-theory is an eleven-dimensional gravitational theory. We have also given a solution to the supergravity equations for any general configuration that is dual to a vacuum of the Plane-Wave Matrix Model or maximally supersymmetric YM on $R \times S^2$. Having these solutions will be very useful for strong coupling calculations in these gauge theories that require studying embeddings of strings or branes in the background geometry. There is also the possibility that these solutions may be of use in fluid mechanics, where problem of the radiation of waves from a single submerged disc can be reduced to the electrostatics problem for a pair of charged conducting discs. Our more general solutions may have applications to more general "dock problems" in fluid mechanics. An application of these supergravity solutions to the calculation of new instanton amplitudes was also given. Finally, we studied coarse-graining of the supergravity solutions, and proposed an entropy functional for singular geometries. We showed that this entropy functional vanishes for any microstate geometry, but is non-zero for any singular geometry.

The results in this thesis concerning double-scaling limits and Little String Theory clearly indicate that the strong-coupling dynamics of the $SU(2|4)$ symmetric field theories is very rich. One question that would be interesting to understand is the way in which the S^5 on which the Little String Theory lives, or indeed the full background geometry, emerges at strong coupling. It would seem that the application of the ideas set out by Berenstein and collaborators [85, 86], may provide some insight into this. In that work, in the context of $\mathcal{N} = 4$ SYM, a simplified matrix model has been proposed to study supersymmetric sectors of the theory at strong coupling. In the vacuum state, the eigenvalues of the matrices spread out to form an S^5 which is taken to be the S^5 in the dual geometry, and the spectrum of excitations about the vacuum configuration were shown to correspond to the spectrum of near BPS states with large $U(1)$ R-charge. In the present context, for certain vacua of the Plane-Wave Matrix Model, the same matrix model used by Berenstein and collaborators should describe the strong coupling dynamics. The S^5 of eigenvalues ought to correspond to the S^5 in the geometry that sits at the tip of a conducting disc, and the spectrum of excitations should correspond to near BPS states on a background that comes from taking a plane-wave limit near the tip of the disc. We noted in chapter 2 that information from the supergravity solution for a single disc above an infinite conducting plane can be used to extract the spectrum of near BPS excitations on the plane-wave geometry near the tip of the conducting disc. This is true more generally, however, and our general solutions to the electrostatics problems dual to general vacua of to the Plane-Wave Matrix Model and maximally supersymmetric YM on $R \times S^2$ can be used to extract near BPS spectra for the field theories in an arbitrary vacuum. It would be an interesting and non-trivial check of the gauge/gravity correspondence to determine if the spectrum agrees with a perturbative calculation in the field theory. This setting might also provide other laboratories for studying the integrability issues that are currently of great interest in $\mathcal{N} = 4$ SYM.

Another interesting and related issue stems from the various large N limits that can be taken in these theories. As we found in chapters 2 and 3, the double-scaling limits we have giving Little String Theory have $\lambda/N \to 0$, whereas the M-theory limit of the Plane-Wave Matrix Model has $\lambda/N \to \infty$. The vastly different nature of the emergent geometries and theories in these limits suggests that coupling constant dependence is very important when discussing emergent geometries. It would be interesting to understand the nature of an intermediate limit. How fast must λ scale with N to produce a gravitational theory? How does the dimensionality of the theory change as we vary the ratio λ/N?

To further allow the application of this duality to study the $SU(2|4)$ theories at strong coupling, more investigation is needed to fill out the dictionary relating gauge theory observables to geometric calculations. A basic gauge theory observable is the Wilson loop. Preliminary investigations of supersymmetric versions of this operator suggest that the relative coupling to the adjoint scalar fields in the field theories has a nice geometric interpretation as an angle in the (r, z)-plane in the electrostatics picture. This angle also seems to determine whether the Wilson loop operator is able to distinguish between the different gauge theory vacua.

A possible future direction stemming from the work on coarse-graining is to further the understanding of the role of representation theory in AdS/CFT. One of the reasons why Lin and Maldacena were able to identify supergravity solutions with vacua of the Plane-Wave Matrix Model is that these vacua are protected by supersymmetry, and so would be expected to correspond to some supergravity solutions. The representation theory of $SU(2|4)$, in turn, determines that these states are protected by supersymmetry. It seems, therefore, that one could look for examples of bubbling geometries, starting by considering the representation theory of Lie superalgebras. This is not the path that is usually taken. One future direction would be to better understand the role that representation theory plays in the other known examples of bubbling geometries, for example the case of bubbling Wilson loops [87, 88]. A speculative benefit of such a study is that the extension of the coarse-graining ideas developed in this thesis are more challenging in that context. However, they might be facilitated by a better understanding of the representation theory underlying that case. Further study of the representation theory of other Lie superalgebras might also lead to the identification of other gauge/gravity duality candidates. See also [89] for related discussion.

In addtion to this represntation theoretic question, another future direction is to get a better understanding of the Plane-Wave Matrix Model at strong coupling, and there now exist numerical techniques [90, 91] that should enable this to be done. The gauge/gravity correspondence is most well understood when comparisons are made of quantities that are supersymmetric. However, it would be extremely useful to be able to push this understanding to non supersymmetric quantities as well. Numerical simulation provides an avenue for doing this, and we are currently engaged in such an investigation. The Plane-Wave Matrix Model and its gravitational duals are a particularly good setting for this. The Plane-Wave Matrix Model is superficially quite simple, being just quantum mechanics. However, as we described, the model has many interesting limits that include theories with or without gravity, and theories in 3, 6 or 11 dimensions, as well as dual geometries with very non-trivial topology. The results in this thesis will provide key information guiding the understanding of the gravity side dual to the numerical investigation.

The results of this thesis, therefore, are well placed to help answer the more general questions: What are NS5-branes? How far can we push gauge/gravity duality for non-supersymmetric quantities? and How does geometry emerge from gauge theory?

Bibliography

[1] H. Ling, A. R. Mohazab, H.-H. Shieh, G. van Anders, and M. Van Raamsdonk, *Little string theory from a double-scaled matrix model*, *JHEP* **10** (2006) 018, [hep-th/0606014].

[2] H. Ling, H.-H. Shieh, and G. van Anders, *Little string theory from double-scaling limits of field theories*, *JHEP* **02** (2007) 031, [hep-th/0611019].

[3] G. van Anders, *General Lin-Maldacena solutions and PWMM instantons from supergravity*, *JHEP* **03** (2007) 028, [hep-th/0701277].

[4] H.-H. Shieh, G. van Anders, and M. Van Raamsdonk, *Coarse-Graining the Lin-Maldacena Geometries*, *JHEP* **09** (2007) 059, [0705.4308].

[5] M. Le Bellac, *Thermal field theory.* Cambridge University Press, Cambridge, UK, 1996.

[6] Y. Nambu and G. Jona-Lasinio, *Dynamical model of elementary particles based on an analogy with superconductivity. II*, *Phys. Rev.* **124** (1961) 246–254.

[7] V. A. Miransky, *Dynamical symmetry breaking in quantum field theories.* World Scientific, Singapore, 1993.

[8] H. Liu, K. Rajagopal, and U. A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, *Phys. Rev. Lett.* **97** (2006) 182301, [hep-ph/0605178].

[9] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[10] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B72** (1974) 461.

[11] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111].

[12] I. R. Klebanov, *TASI lectures: Introduction to the AdS/CFT correspondence*, hep-th/0009139.

[13] J. M. Maldacena, *Lectures on AdS/CFT*, hep-th/0309246.

[14] H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, Boulder, Colorado, 2nd ed., 1999.

[15] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[16] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[17] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, *Phys. Rev.* **D58** (1998) 046004, [hep-th/9802042].

[18] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131].

[19] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843–882, [hep-th/0412141].

[20] P. Kovtun, D. T. Son, and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601, [hep-th/0405231].

[21] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[22] J. Polchinski, *String Theory*, vol. 1. Cambridge University Press, Cambridge, UK, 1998.

[23] J. Polchinski, *String Theory*, vol. 2. Cambridge University Press, Cambridge, UK, 1998.

[24] R. A. Adams, *Calculus of several variables.* Addison-Wesley, Don-Mills, Ontario, 4th ed., 1999.

[25] G. T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B360** (1991) 197–209.

[26] J. Polchinski, *Lectures on D-branes*, `hep-th/9611050`.

[27] C. V. Johnson, *D-brane primer*, `hep-th/0007170`.

[28] C. V. Johnson, *D-branes*. Cambridge University Press, Cambridge, UK, 2003.

[29] R. Haberman, *Elementary applied partial differential equations: with Fourier series and boundary value problems*. Prentice-Hall, Upper Saddle River, New Jersey, 3rd ed., 1998.

[30] D. Berenstein, J. M. Maldacena, and H. Nastase, *Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills*, *JHEP* **04** (2002) 013, [`hep-th/0202021`].

[31] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, *M theory as a matrix model: A conjecture*, *Phys. Rev.* **D55** (1997) 5112–5128, [`hep-th/9610043`].

[32] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, *Matrix perturbation theory for M-theory on a PP-wave*, *JHEP* **05** (2002) 056, [`hep-th/0205185`].

[33] B. de Wit, J. Hoppe, and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys.* **B305** (1988) 545.

[34] N.-w. Kim, T. Klose, and J. Plefka, *Plane-wave matrix theory from $\mathcal{N} = 4$ super Yang-Mills on $R \times S^3$*, *Nucl. Phys.* **B671** (2003) 359–382, [`hep-th/0306054`].

[35] M. Nakahara, *Geometry, Topology and Physics*. Institute of Physics, Bristol, 2nd ed., 2003.

[36] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, *Protected multiplets of M-theory on a plane wave*, *JHEP* **09** (2002) 021, [`hep-th/0207050`].

[37] A. Yong, *WHAT IS... a Young tableau?*, *Notices of the American Mathematical Society* **54** (2007) 240–241.

[38] J. J. Sakurai, *Modern Quantum Mechanics*. Addison-Wesley, Reading, Massachusets, revised ed., 1994.

[39] J. M. Maldacena, M. M. Sheikh-Jabbari, and M. Van Raamsdonk, *Transverse fivebranes in matrix theory*, *JHEP* **01** (2003) 038, [hep-th/0211139].

[40] H. Lin and J. M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev.* **D74** (2006) 084014, [hep-th/0509235].

[41] G. Grignani, L. Griguolo, N. Mori, and D. Seminara, *Thermodynamics of theories with sixteen supercharges in non-trivial vacua*, *JHEP* **10** (2007) 068, [arXiv:0707.0052 [hep-th]].

[42] H. Lin, O. Lunin, and J. M. Maldacena, *Bubbling ads space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025, [hep-th/0409174].

[43] T. Ortín, *Gravity and Strings.* Cambridge University Press, Cambridge, UK, 2004.

[44] J. D. Jackson, *Classical Electrodynamics.* Wiley, New York, 3rd ed., 1998.

[45] O. Aharony, *A brief review of 'little string theories'*, *Class. Quant. Grav.* **17** (2000) 929–938, [hep-th/9911147].

[46] N. Seiberg, *New theories in six dimensions and matrix description of M- theory on T^5 and T^5/Z_2*, *Phys. Lett.* **B408** (1997) 98–104, [hep-th/9705221].

[47] R. M. Wald, *General Relativity.* University of Chicago Press, Chicago, 1984.

[48] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B379** (1996) 99–104, [hep-th/9601029].

[49] C. G. Callan and J. M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys.* **B472** (1996) 591–610, [hep-th/9602043].

[50] O. Lunin and S. D. Mathur, *Statistical interpretation of Bekenstein entropy for systems with a stretched horizon*, *Phys. Rev. Lett.* **88** (2002) 211303, [hep-th/0202072].

[51] S. D. Mathur, *The quantum structure of black holes*, *Class. Quant. Grav.* **23** (2006) R115, [hep-th/0510180].

[52] A. Sen, *Extremal black holes and elementary string states*, *Mod. Phys. Lett.* **A10** (1995) 2081–2094, [hep-th/9504147].

[53] M. Berkooz, M. Rozali, and N. Seiberg, *Matrix description of M theory on T^4 and T^5*, *Phys. Lett.* **B408** (1997) 105–110, [hep-th/9704089].

[54] O. Aharony and M. Berkooz, *IR dynamics of $d = 2$, $N = (4,4)$ gauge theories and DLCQ of 'little string theories'*, *JHEP* **10** (1999) 030, [hep-th/9909101].

[55] N. Arkani-Hamed, A. G. Cohen, D. B. Kaplan, A. Karch, and L. Motl, *Deconstructing (2,0) and little string theories*, *JHEP* **01** (2003) 083, [hep-th/0110146].

[56] A. Giveon and D. Kutasov, *Comments on double scaled little string theory*, *JHEP* **01** (2000) 023, [hep-th/9911039].

[57] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034, [hep-th/9909110].

[58] J. Polchinski and M. J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[59] N.-w. Kim and J. Plefka, *On the spectrum of pp-wave matrix theory*, *Nucl. Phys.* **B643** (2002) 31–48, [hep-th/0207034].

[60] N. Kim and J.-H. Park, *Superalgebra for M-theory on a pp-wave*, *Phys. Rev.* **D66** (2002) 106007, [hep-th/0207061].

[61] J. Nicholson, *The electrification of two parallel circular discs*, *Philos. Trans. R. Soc. Lond. A* **224** (1924) 303.

[62] R. Love, *The electrostatic field of two equal circular co-axial conducting disks*, *Quarterly Journal of Mechanics & App. Maths.* **2** (1949) 428.

[63] I. Sneddon, *Mixed boundary value problems in potential theory.* North-Holland, Amsterdam, 1966.

[64] D. Porter and D. Stirling, *Integral equations: a practical treatment, from spectral theory to applications.* Cambridge University Press, Cambridge, UK, 1990.

[65] P. H. Ginsparg and G. W. Moore, *Lectures on 2-d gravity and 2-d string theory*, hep-th/9304011.

[66] J. K. Erickson, G. W. Semenoff, and K. Zarembo, *Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B582** (2000) 155–175, [hep-th/0003055].

[67] H. Ebrahim, *Semiclassical strings probing NS5 brane wrapped on S^5*, *JHEP* **01** (2006) 019, [hep-th/0511228].

[68] G. Ishiki, Y. Takayama, and A. Tsuchiya, *$\mathcal{N} = 4$ SYM on $R \times S^3$ and theories with 16 supercharges*, *JHEP* **10** (2006) 007, [hep-th/0605163].

[69] G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, *Embedding of theories with $SU(2|4)$ symmetry into the plane wave matrix model*, *JHEP* **11** (2006) 089, [hep-th/0610038].

[70] L. Delves and J. Mohamed, *Computational methods for integral equations*. Cambridge University Press, Cambridge, UK, 1985.

[71] N. Drukker and D. J. Gross, *An exact prediction of $\mathcal{N} = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896–2914, [hep-th/0010274].

[72] H. Lin, *Instantons, supersymmetric vacua, and emergent geometries*, *Phys. Rev.* **D74** (2006) 125013, [hep-th/0609186].

[73] J.-T. Yee and P. Yi, *Instantons of M(atrix) theory in pp-wave background*, *JHEP* **02** (2003) 040, [hep-th/0301120].

[74] S. D. Mathur, *The fuzzball proposal for black holes: An elementary review*, *Fortsch. Phys.* **53** (2005) 793–827, [hep-th/0502050].

[75] V. Balasubramanian, B. Czech, K. Larjo, D. Marolf, and J. Simon, *Quantum geometry and gravitational entropy*, arXiv:0705.4431 [hep-th].

[76] K. Larjo, *On the existence of supergravity duals to D1–D5 CFT states*, *JHEP* **07** (2007) 041, [arXiv:0705.4433 [hep-th]].

[77] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, *The library of Babel: On the origin of gravitational thermodynamics*, *JHEP* **12** (2005) 006, [hep-th/0508023].

[78] A. Buchel, *Coarse-graining 1/2 BPS geometries of type IIB supergravity*, *Int. J. Mod. Phys.* **A21** (2006) 3495–3502, [hep-th/0409271].

[79] P. G. Shepard, *Black hole statistics from holography*, *JHEP* **10** (2005) 072, [hep-th/0507260].

[80] P. J. Silva, *Rational foundation of GR in terms of statistical mechanic in the AdS/CFT framework*, *JHEP* **11** (2005) 012, [hep-th/0508081].

[81] R. C. Myers and O. Tafjord, *Superstars and giant gravitons*, *JHEP* **11** (2001) 009, [hep-th/0109127].

[82] D. Bak, S. Siwach, and H.-U. Yee, *1/2 BPS geometries of M2 giant gravitons*, *Phys. Rev.* **D72** (2005) 086010, [hep-th/0504098].

[83] L. D'Errico, W. Mueck, and R. Pettorino, *Stretched horizon and entropy of superstars*, *JHEP* **05** (2007) 063, [hep-th/0703223].

[84] R. M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038].

[85] D. Berenstein, *Large N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125, [hep-th/0507203].

[86] D. Berenstein and R. Cotta, *A Monte-Carlo study of the AdS/CFT correspondence: an exploration of quantum gravity effects*, hep-th/0702090.

[87] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys.* **A22** (2007) 1353–1374, [hep-th/0601089].

[88] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026, [hep-th/0604133].

[89] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An index for 4 dimensional super conformal theories*, hep-th/0510251.

[90] M. Hanada, J. Nishimura, and S. Takeuchi, *Non-lattice simulation for supersymmetric gauge theories in one dimension*, arXiv:0706.1647 [hep-lat].

[91] K. N. Anagnostopoulos, M. Hanada, J. Nishimura, and S. Takeuchi, *Monte carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature*, arXiv:0707.4454 [hep-th].

[92] S. Coleman, *Aspects of Symmetry*. Cambridge University Press, New York, 1985.

[93] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*. Oxford University Press, Oxford, 4th ed., 2002.

[94] V. Hutson, *The circular plate condenser at small separations*, *Proc. Camb. Phil. Soc.* **59** (1963) 211.

[95] N. V. Suryanarayana, *Half-BPS giants, free fermions and microstates of superstars*, *JHEP* **01** (2006) 082, [hep-th/0411145].

[96] S. Giombi, M. Kulaxizi, R. Ricci, and D. Trancanelli, *Half-BPS geometries and thermodynamics of free fermions*, *JHEP* **01** (2007) 067, [hep-th/0512101].

[97] P. Martin and L. Farina, *Radiation of water waves by a heaving submerged horizontal disc*, *Journal of Fluid Mechanics* **337** (1997) 365–379.

Appendix A

Quantum Mechanical Instantons

This appendix reviews some basic aspects of instantons in quantum mechanics. Fuller treatments are given in, e.g. [92, 93]. Instantons are solutions to the euclidean equations of motion for a quantum mechanical system. They are particularly important in situations where a system has minima that are classically degenerate, and determine the rate of tunnelling between the minima.

To be concrete, we will consider a non-relativistic particle in one dimension with a potential of the form

$$V = \frac{1}{a^2}(1 - \cos ax) = \frac{2}{a^2}\sin^2\frac{ax}{2} \tag{A.1}$$

where a is some parameter with dimensions of length^{-1}. This is a periodic potential with minima at $x = 2n\pi/a$. The action for the system has the form

$$S = \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{2}{a^2}\sin^2\frac{ax}{2}\right). \tag{A.2}$$

The corresponding euclidean action is

$$S_E = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{2}{a^2}\sin^2\frac{ax}{2}\right). \tag{A.3}$$

To maintain consistency with the discussion in chapter 4, we will define a "superpotential" W according to

$$\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 = V(x) = \frac{2}{a^2}\sin^2\frac{ax}{2}, \tag{A.4}$$

which means that

$$W = -\left(\frac{2}{a}\right)^2 \cos\frac{ax}{2} + \text{const.} \tag{A.5}$$

Having introduced this superpotential, we can write the action as

$$S_E = \frac{1}{2} \int dt \left(\dot{x}^2 + \left(\frac{\partial W}{\partial x} \right)^2 \right). \tag{A.6}$$

We will complete the square by adding and subtracting $2\dot{x}\frac{\partial W}{\partial x}$ under the integrand, giving

$$\begin{aligned} S_E =& \frac{1}{2} \int dt \left(\dot{x} - \frac{\partial W}{\partial x} \right)^2 + \int dt x \frac{\partial W}{\partial x} \\ =& \frac{1}{2} \int dt \left(\dot{x} - \frac{\partial W}{\partial x} \right)^2 + W|_{x_i}^{x_f} , \end{aligned} \tag{A.7}$$

where in the second line we have used the chain rule to evaluate the second integral, and x_i and x_f are the initial and final points of the motion of the particle. Note that the first term in this expression is a complete square, and is therefore positive. The minimum action will, therefore, be given by a configuration for which this term is identically zero, and the corresponding minimum value of the action will be $W|_{x_i}^{x_f}$. In the language of chapter 4 the superpotential provides a bound for the Euclidean action, and instanton configurations that satisfy $\dot{x} - \frac{\partial W}{\partial x} = 0$ saturate the bound, or are "BPS". Explicit solutions for a particle that tunnels between the minima at 0 and $2\pi/a$ are given by

$$x(t) = \frac{4}{a} \tan^{-1} \left(e^{t-t_0} \right), \tag{A.8}$$

where t_0 is a constant that determines the time the particle is at the point π/a.

Let us verify that the instanton configurations $\dot{x} - \frac{\partial W}{\partial x} = 0$ solve the Euclidean equation of motion, which is

$$\ddot{x} = \frac{\partial V}{\partial x} = \frac{\partial W}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial x} \right). \tag{A.9}$$

Considering the left hand side we have

$$\begin{aligned} \text{LHS} &= \ddot{x} \\ &= \frac{d}{dt} \dot{x} \\ &= \frac{dx}{dt} \frac{\partial}{\partial x} \dot{x} \\ &= \frac{\partial W}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial x} \right) \\ &= \text{RHS}. \end{aligned} \tag{A.10}$$

Therefore these configurations solve the Euclidean equation of motion.

To make the interpretation of this clearer, we can compare this with the transmission coefficient for a potential barrier that can be calculated using the WKB approximation in quantum mechanics (e.g. [92]). The transmission coefficient is given by

$$T = e^{2\int_{x_i}^{x_f} dx\sqrt{2(E-V)}}. \tag{A.11}$$

Consider a particle in with the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{2}{a^2}\sin^2\frac{ax}{2}. \tag{A.12}$$

A particle with zero energy will have a transmission coefficient given by

$$T = e^{-2\int dx\sqrt{2V}} \tag{A.13}$$

with the potential we have defined in (A.1). Using the superpotential we gave in (A.5), this becomes

$$T = e^{-2W|_{x_i}^{x_f}}, \tag{A.14}$$

so we see that the superpotential determines the rate of tunnelling between vacua in this system.

Appendix B

Lin-Maldacena Supergravity Solutions

The general Lin-Maldacena $SU(2|4)$-symmetric supergravity ansatz (suppressing an overall factor of α' in the metric) is given by [40]

$$
\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V} - 2\dot{V}}{-V''}\right)^{\frac{1}{2}} \left\{ \frac{-4\ddot{V}}{\ddot{V} - 2\dot{V}} dt^2 - \frac{2V''}{\dot{V}}(d\rho^2 + d\eta^2) + 4d\Omega_5^2 + 2\frac{V''\dot{V}}{\Delta} d\Omega_2^2 \right\} \\
e^{4\Phi} &= \frac{4(\ddot{V} - 2\dot{V})^3}{-V''\dot{V}^2\Delta^2} \\
C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V} - 2\dot{V}} dt \\
F_4 &= dC_3, \qquad\qquad C_3 = -4\frac{\dot{V}^2 V''}{\Delta} dt \wedge d^2\Omega, \\
H_3 &= dB_2, \qquad\qquad B_2 = 2\left(\frac{\dot{V}\dot{V}'}{\Delta} + \eta\right) d^2\Omega, \\
\Delta &\equiv (\ddot{V} - 2\dot{V})V'' - (\dot{V}')^2 \,.
\end{aligned}
\tag{B.1}
$$

Their explicit solution corresponding to Little String Theory on S^5 is

$$\begin{aligned}
ds_{10}^2 &= N\Big(-2r\sqrt{\frac{I_0}{I_2}}dt^2 + 2r\sqrt{\frac{I_2}{I_0}}d\Omega_5^2 + \sqrt{\frac{I_2}{I_0}}\frac{I_0}{I_1}(dr^2 + d\theta^2) \\
&\qquad + \sqrt{\frac{I_2}{I_0}}\frac{I_0 I_1 s^2}{I_0 I_2 s^2 + I_1^2 c^2}d\Omega_2^2\Big) \\
B_2 &= N\Big(\frac{-I_1^2 cs}{I_0 I_2 s^2 + I_1^2 c^2} + \theta\Big)d^2\Omega \\
e^\Phi &= g_0 N^{3/2} 2^{-1}\left(\frac{I_2}{I_0}\right)^{\frac{3}{4}}\left(\frac{I_0}{I_1}\right)^{\frac{1}{2}}\left(I_0 I_2 s^2 + I_1^2 c^2\right)^{-\frac{1}{2}} \\
C_1 &= -g_0^{-1}\frac{1}{N}4r\frac{I_1}{I_2}(I_0^2 s^2 + I_1^2 c^2)dt \\
C_3 &= -g_0^{-1}\frac{4 I_0 I_1^2 s^3}{I_0 I_2 s^2 + I_1^2 c^2}dt \wedge d^2\Omega
\end{aligned} \tag{B.2}$$

where $I_n(r)$ are the usual Bessel functions, $s \equiv \sin(\theta)$, and $c \equiv \cos(\theta)$.

Appendix C

More Details of the Electrostatics Solutions in Simple Cases

C.1 Solution of the Electrostatics Problem with Two Discs

In this appendix, we describe the solution of the electrostatics problem described in section 2.3, namely to find a solution of Laplace's equation

$$\nabla^2 V = 0$$

such that $V = 0$ at $z = 0$, $V = \Delta$ for $\{z = \kappa, 0 < r < 1\}$ and V behaves as $zr^2 - \frac{2}{3}z^3$ for large $z^2 + r^2$.

To start, we write

$$V = zr^2 - \frac{2}{3}z^3 + \phi$$

so that ϕ vanishes at infinity and at $z = 0$ and is given by

$$\phi(r) = \Delta + \frac{2}{3}\kappa^3 - \kappa r^2 \equiv \beta(1 - \alpha r^2)$$

on the upper plate. Axially symmetric solutions to the Laplace equation that are regular at $r = 0$ and vanish as $r \to \infty$ are linear combinations of functions

$$e^{\pm zu} J_0(ru),$$

where u is a continuous parameter. If we denote by V_+ and V_0 the function ϕ in the regions $z \geq d$ and $0 \leq z \leq d$ respectively, then we must have

$$\begin{aligned} V_0 &= \int_0^\infty B(u) \sinh(zu) J_0(u) \\ V_+ &= \int_0^\infty C(u) e^{-zu} J_0(u) \end{aligned} \tag{C.1}$$

taking into account the boundary conditions at $z = 0$ and $z = \infty$.

We now take into account the boundary conditions at $z = d$. First, we must have $V_0 = V_+$ at $z = d$, so it must be that

$$\begin{aligned} B(u) &= 2\beta u^{-1} e^{-du} A(u) \\ C(u) &= 2\beta u^{-1} \sinh(du) A(u) \end{aligned}$$

for some function $A(u)$ (the prefactors have been chosen for later convenience). Also, we have $\partial_z V_0 = \partial_z V_+$ at $z = \kappa$ for $r > 1$ and $V_0 = V_+ = \beta(1 - \alpha r^2)$ for $0 < r < 1$. These will be satisfied if

$$\int_0^\infty A(u) J_0(ru) du = 0 \qquad r > 1 \qquad \text{(C.2)}$$

$$\int_0^\infty du u^{-1}(1 - e^{-2\kappa u}) A(u) J_0(ru) = 1 - \alpha r^2 \qquad 0 < r < 1 \qquad \text{(C.3)}$$

This set of equations is of the type considered in [63], chapter 4.6, "Dual integral equations with Hankel kernel and arbitrary weight function". For completeness, we review the solution of this type of equation in appendix C.3. For our specific case, the solution is given by

$$A(u) = \frac{2u}{\pi} \int_0^1 f(t) \cos(ut) dt \ ,$$

where $f(t)$ is an even function of t satisfying the integral equation

$$f(x) - \int_{-1}^1 K(x,t) f(t) dt = g(x) \ , \qquad \text{(C.4)}$$

where

$$g(x) = 1 - 2\alpha x^2 \qquad \text{(C.5)}$$

and

$$K(x,t) = \frac{1}{\pi} \frac{2\kappa}{4\kappa^2 + (x-t)^2} \ .$$

The solution f of the integral equation may be given as a series (2.10). Substituting our solution for A in (C.1) gives the simpler result (2.7), valid for all z.

C.1.1 Limiting Forms of the Solution

In our discussion of scaling limits of the theory, it will be useful to have some explicit results for the behaviour of the electrostatics solution for large and small values of κ.

Large κ

For large κ, the kernel K has a small norm, so the series solution (2.10) is well approximated by its leading term. Thus,

$$f(x) \to g(x) = 1 - 2\alpha x^2 \ .$$

In order that the charge density vanishes at the edge of the plate, equation (2.11) implies that $f(1)$ must vanish. Thus, we must have

$$\alpha = \frac{1}{2} \ ,$$

which (for large κ) implies

$$\Delta \to -\frac{2}{3}\kappa^3 + 2\kappa \ .$$

This implies that $\beta = 2\kappa$, so that using (2.12), we have

$$q \to \frac{8}{3\pi}\kappa \ .$$

As a check, we can match these results with those in appendix C.2, where we have solved the electrostatics problem after first taking the limit of large d with fixed R.

Small κ

For small κ, the problem is more difficult to study, since the norm of the kernel K approaches one (the kernel approaches a delta function), so the series solution (2.10) converges very slowly. The work [94] studied in detail the case of charged conducting discs at small separation without a background potential, but some of the general discussion there is helpful in our case also.

Up to corrections of (fractional) order $\sqrt{\kappa}$, the leading small κ behaviour of the solution of the integral equation (C.4) is [94]

$$f(t) \to \kappa^{-1} \int_{-1}^{1} k(s,t) g(s) ds \ ,$$

where

$$k(s,t) = \frac{1}{2\pi} \log \left\{ \frac{1 - st + (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}}{1 - st - (1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}} \right\} \ .$$

This corresponds to the approximation that $\sigma(r)$ varies as $\kappa^{-1}\phi(r)$.

In our case, g is given by (C.5), so we find that

$$f(t) \to \frac{1}{2\kappa}\left\{(1-t^2)^{\frac{1}{2}} - \alpha(1-t^2)^{\frac{1}{2}} + \frac{2}{3}\alpha(1-t^2)^{\frac{3}{2}}\right\} . \qquad \text{(C.6)}$$

Now, in order that σ vanishes at the edge of the disc, (2.11) implies that $f(1)$ vanishes and that the $f'(1)$ is bounded. We see that the latter condition is satisfied only if $\alpha \to 1$ in this limit.[40] Thus, in the limit of small κ, we have

$$f^{\kappa}(t) \to \frac{1}{3\kappa}(1-t^2)^{\frac{3}{2}} . \qquad \text{(C.7)}$$

From $\alpha \to 1$, we infer that

$$\Delta \to \kappa \qquad \text{(C.8)}$$

in the limit, and using (2.12) we have

$$q(\kappa) \to \frac{1}{8} .$$

We are particularly interested in the small κ behaviour of the integral (2.25), which after a change of variables becomes

$$I = \int_0^1 dy \sin(\pi y)\left[\phi_\kappa(1,\kappa y) - \left\{\Delta_\kappa y - \kappa y + \frac{2}{3}\kappa^3 y^3\right\}\right] . \qquad \text{(C.9)}$$

The largest terms are the first two terms in curly brackets, each of which give a contribution of order κ, but (C.8) implies that these cancel. Using numerical methods described in appendix E, we have estimated the remaining contributions. We find that there is a further cancellation between the ϕ term and the term in curly brackets, both of which give contributions of order $\kappa^2 \ln(\kappa)$. The net result is that the integral behaves for small κ as[41]

$$I \approx 0.04\kappa^2 .$$

[40] For finite values of κ, the series solution (2.10) may be used to show that $f'(1)$ is actually bounded for all t. On the other hand, corrections to the formula (C.6) are generally non-zero at $t = \pm 1$. Thus, the vanishing charge density constraint for the exact solution at finite κ comes from the condition $f(1) = 0$. We have verified numerically in appendix E that this condition gives the same result $\alpha \to 1$ in the limit $\kappa \to 0$ as the condition that $f'(1)$ is bounded applied to the leading approximation.

[41] Note that here we cannot simply use the leading approximation (C.7) for f. The reason is that the integral over t receives most of its contribution close to the boundaries of the interval $[-1, 1]$ where the leading approximation goes to zero like a $3/2$ power. Corrections to (C.7), which are fractionally small in the bulk of the interval, become more important in the region near the boundaries, and result in a modified behaviour for the integral.

While this result is numerical, we have performed an analytic check on the method. If we normalize the function

$$\tilde{\phi}(x,y) = C\left[\phi_\kappa(1+x,\kappa y) - \left\{\Delta_\kappa y - \kappa y + \frac{2}{3}\kappa^3 y^3\right\}\right]$$

by choosing C such that (for example) $\tilde{\phi}(0, 0.5) = 1$, then $\tilde{\phi}$ should have a well defined limit as $\kappa \to 0$. This must be a nontrivial solution to the Laplace equation in the case where we have one infinite conducting plate (at $y = 0$), and one semi-infinite conducting plate (at $y = 1$, $x < 0$), with vanishing potential on both plates (recall that the term involving Δ ensures that $\tilde{\phi}$ vanishes on both plates). This reduces to a two-dimensional problem that may be solved using conformal mapping techniques [40]. As explained in [40], if we define a complex coordinate $z = x + iy$ and another complex variable

$$w = 2\partial_z V = \partial_x V - i\partial_y V \ .$$

Then the Laplace equation ensures that the mapping between z and w is analytic. Further, $\partial_x V$ is everywhere non-negative and vanishes at the plates, so the region outside the plates must map to the right half plane, with the plates mapping to the imaginary axis. The explicit transformation that achieves this (unique up to transformations generated by translations, scalings, and inversions that fix the imaginary axis) is [40]

$$z = iw + \frac{1}{\pi}\ln(w) + \frac{i}{2} + \frac{1}{\pi} + \frac{1}{\pi}\ln(\pi) \ . \tag{C.10}$$

This implies

$$V = Re(\int^z w(z')dz') = u(\frac{1}{\pi} - v) \ ,$$

where $w = u + iv$. From (C.10), we find that u and v are determined in terms of x and y by

$$\begin{aligned} x &= -v + \frac{1}{2\pi}\ln(u^2+v^2) + \frac{1}{\pi} + \frac{1}{\pi}\ln(\pi) \\ y &= u\frac{1}{\pi}\tan^{-1}\left(\frac{v}{u}\right) + \frac{1}{2} \ , \end{aligned}$$

so we have a relatively explicit result for the potential in the limit. After normalizing the result in the same way that we normalized $\tilde{\phi}$, we find excellent agreement with our numerical results for $\tilde{\phi}$ in the limit of small κ.

C.2 Single disc solution

In this appendix, we solve the electrostatics problem obtained in the limit of large d with $V_0 d = W_0$ fixed and R fixed, to give a more direct derivation of the asymptotically near-horizon D2-brane solution derived by Lin and Maldacena.

To define this solution, we want to find the potential for a charged conducting disc of radius R and charge Q in a background potential

$$V = W_0(r^2 - 2\eta^2) \ . \tag{C.11}$$

In this case, the potential W_0 will eventually be related to the charge, Q, and the radius, R, by the condition that the charge density vanishes at the edge of the disc.

By the superposition and scale-invariance properties of electrostatics, the solution must take the form

$$V = W_0 R^2 \phi(r/R, \eta/R),$$

where ϕ is the solution to the problem with $W_0 = 1$ and $R = 1$. Further if q is the total charge on the disc required in this simplified problem so that the charge density vanishes at the edge of the disc, we must have

$$Q = W_0 R^3 q.$$

To start, we would like to find a solution to Laplace's equation with boundary conditions that the potential is fixed to Δ on a disc of radius 1 at $z = 0$ and becomes $r^2 - 2z^2$ at infinity. The potential, Δ, and the total charge on the disc, q, will be fixed by demanding that the charge density vanishes at the edges of the disc.

We begin by writing

$$V = r^2 - 2z^2 + \phi \ ,$$

such that ϕ vanishes at infinity and

$$\phi(r, z = 0) = \Delta - r^2; \ .$$

If we denote by V_+ the potential ϕ for $z > 0$, then separating variables gives

$$V_+(r, z) = \int_0^\infty u^{-1} A(u) e^{-uz} J_0(ur) du \ . \tag{C.12}$$

By symmetry, the potential V_- for $z < 0$ must be

$$V_-(r,z) = V_+(r,-z) \ .$$

Finally, we require that

$$\begin{aligned} V_+ = V_- &= \Delta - r^2 && 0 \le r \le 1 \ , \\ \partial_z V_+ - \partial_z V_- &= 0 && r > 1. \end{aligned}$$

These imply the dual integral equations

$$\int_0^\infty A(u) J_0(ru) du = 0 \qquad r > 1 \qquad \text{(C.13)}$$

$$\int_0^\infty duu^{-1} A(u) J_0(\rho u) = \Delta - 2r^2 \qquad 0 < r < 1, \qquad \text{(C.14)}$$

which are of the type considered in appendix C.3, with $k(u) = 0$ and $h(r) = \Delta - r^2$. In this case, the integral equation is trivial, so the result is

$$A(u) = \frac{2u}{\pi} \int_0^1 f(t) \cos(ut) dt$$

with

$$f(t) = g(t) = \Delta - 2r^2 \ .$$

Substituting for A in (C.12) gives

$$V_+ = \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{r^2 + (z+it)^2}}$$

The charge density on the disc is given by

$$\begin{aligned} \sigma(r) &= \frac{1}{4\pi} (\partial_z V_-(r,-\epsilon) - \partial_z V_+(r,\epsilon)) \\ &= \frac{1}{2\pi} \int_0^\infty A(u) J_0(ur) \\ &= \frac{1}{2\pi} \left[\frac{f(1)}{(1-r^2)^{\frac{1}{2}}} - \int_r^1 ds \frac{f'(s)}{(s^2 - r^2)^{\frac{1}{2}}} \right] \\ &= \frac{\Delta - 2}{\pi^2} \frac{1}{\sqrt{1-r^2}} + \frac{4}{\pi^2} \sqrt{1 - r^2} \ . \end{aligned}$$

In order that the charge density vanishes on the tip, we need

$$\Delta = 2 \ ,$$

so finally

$$\sigma(r) = \frac{4}{\pi^2}\sqrt{1-r^2}$$

and the total charge is

$$q = \frac{8}{3\pi} \ .$$

If we like, we can write an explicit solution for the potential using oblate spherical coordinates (see [63], section 3.3), but we will not need it here.

C.3 Dual integral equations

In this appendix, we review the solution of dual integral equations of the form

$$\int_0^\infty A(u)J_0(ru)du \quad = \quad 0 \qquad\qquad r > 1 \quad \text{(C.15)}$$

$$\int_0^\infty duu^{-1}(1+k(u))A(u)J_0(\rho u) \quad = \quad h(r) \qquad\qquad 0 < r < 1 \ , \text{(C.16)}$$

following chapter 4.6 of [63]. In general, the solution is given as

$$A(u) = \frac{2u}{\pi}\int_0^1 f(t)\cos(ut)dt \ ,$$

where $f(t)$ is the solution to a Fredholm integral equation of the second kind,

$$f(x) + \int_0^1 \tilde{K}(x,t)f(t) = g(x) \ ,$$

with

$$g(x) = \frac{d}{dx}\int_0^x \frac{uh(u)du}{\sqrt{x^2-u^2}}.$$

The kernel K is given in terms of $k(u)$ by

$$\tilde{K}(x,u) = \frac{x}{u}\frac{1}{\sqrt{2\pi}}\{K_c(|x-u|) - K_c(x+u)\} \ ,$$

with

$$K_c(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty k(t)\cos(\xi t)dt \ .$$

Appendix D

More Details on Coarse-Graining

D.1 Coarse-Graining for Large Discs

For certain parameter values, or in restricted ensembles, the typical states are such that the radii of the discs are large compared to their separations. As we noted above, in this case, the macroscopic description will replace the closely spaced discs with a solid material that conducts only in the horizontal directions.

Such a conductor has the following properties. Since the charges are free to rearrange themselves in the directions perpendicular to z, they will do so in such a way that the final potential inside the conductor is a function only of z, ensuring that the electric field in the r and θ directions is zero. There will generally be some charge distribution inside the conductor, given by

$$\rho(z) = -\frac{1}{4\pi} V''(z) \; , \tag{D.1}$$

so ρ is also a function only of z. The remaining charge will build up at the surface of the conductor. In general, the shape $R(z)$ for the conductor, and the linear charge distribution $Q(z)$ on the conductor, together with some fixed background potential will determine the charge density $\rho(z)$ inside the conductor and the surface charge density $\sigma(z)$, determined from $\rho(z)$ via

$$Q(z) = \pi R^2(z)\rho(z) + 2\pi R(z)\sigma(z)\sqrt{1 + (R'(z))^2} \; . \tag{D.2}$$

On the other hand, for some special choice of $R(z)$, the surface charge density will vanish. This is the coarse-grained analogue of the constraint that the charge density should vanish at the tip of the discs.

D.1.1 The Variational Problem

We will now set up the mathematical problem that determines $R(z)$ and $\rho(z)$ from $Q(z)$. We start by assuming some fixed $R(z)$ and $Q(z)$.

Outside the conductor, the potential will be given by

$$V_+(r,z) = V_0(r,z) + \tilde{V}(r,z) ,$$

where $\tilde{V}$ is the potential due to the charges in the conductor, which should vanish at large r and z. Since $\tilde{V}$ is an axially-symmetric solution of Laplace's equation, we can expand it in terms of Bessel functions,

$$\tilde{V}(z) = \int_0^\infty \frac{du}{u} A(u) e^{-zu} J_0(ru) .$$

Inside the conductor, the potential will be some function $V_-(z)$. The unknown functions $A(u)$ and $V_-(z)$, together with the charge density $\rho(z)$ inside the conductor and the charge density $\sigma(z)$ on the surface of the conductor will be determined by the two equations (D.1) and (D.2), and the boundary condition

$$\vec{E}_+(R(z),z) - \vec{E}_-(z) = 4\pi\sigma(z)\hat{n} . \tag{D.3}$$

In our case, we wish to fix $R(z)$ by the constraint that the surface charge density vanishes. Then the electric field must be continuous across the boundary of the conductor, and since the electric field is vertical inside, we must have $\partial_r V(R(z),z) = 0$. Explicitly, we have

$$\partial_r V_0(R(z),z) - \int_0^\infty du e^{-zu} A(u) J_1(R(z)u) = 0 . \tag{D.4}$$

This determines $R(z)$ in terms of $A(u)$. Given this, the potential inside the conductor is determined by the z component of the boundary condition (D.3), or simply by continuity of the potential across the boundary, so

$$V_-(z) = V_0(R(z),z) + \int_0^\infty \frac{du}{u} A(u) e^{-zu} J_0(R(z)u) .$$

Finally, we can use (D.1) and (D.2) to write an equation relating $A(u)$ and $Q(z)$,

$$Q(z) = -\frac{1}{4} R^2(z)(\partial_z^2 V_0(z) + \int_0^\infty du u A(u) e^{-zu} J_0(R(z)u)) . \tag{D.5}$$

To summarize, $A(u)$ is determined by the integral equation (D.5) where $R(z)$ is determined in terms of A via (D.4).

In practise, it is far simpler to determine $R(z)$ and $Q(z)$ given some $A(u)$, or more generally some solution to the Laplace equation that arises from any set of axially symmetric localized charges. We could also parametrize our solution to the Laplace equation via the multipole data rather than the function $A(u)$. As an example of this approach, we can come up with an explicit coarse-grained supergravity solution starting with the simplest non-trivial solution $\tilde{V}$, namely the potential from a dipole localized at the origin (the infinite conducting plane at $z = 0$ forces the potential to be an odd function of z.). In this case, we have

$$\tilde{V}(r,z) = p\frac{z}{(r^2+z^2)^{\frac{3}{2}}} .$$

The radial electric field for the full potential is then

$$E_r(r,z) = -\partial_r V_+(r,z) = -2V_0 rz + 3p\frac{rz}{(r^2+z^2)^{\frac{5}{2}}} .$$

Requiring that this is zero gives $r = 0$ or $z = 0$ or

$$z^2 + r^2 = x^2 ,$$

where we define

$$x = \left(\frac{3p}{2V_0}\right)^{\frac{1}{5}} .$$

Thus, in this case, the profile of the conductor is spherical. From (D.5), we can now determine the corresponding charge density $Q(z)$. We find

$$Q(z) = \frac{5}{2}V_0 z(x^2 - z^2) .$$

As a check, we find that the total dipole moment for this configuration is

$$\int_0^\infty dz 2zQ(z) = p .$$

So we have at least one example where we know both the geometry and the Young diagram explicitly. Note that for this case, the typical height for the plates and the typical size are the same, of order x. In terms of the field theory parameters, we have $V_0 \sim 1/g^2$ and $p = 2N$, so $x \sim \lambda^{\frac{1}{5}}$. Thus, our coarse-grained description should be valid as long as λ is large. The typical charge on one of the plates in the corresponding microstate geometries is $Q \sim V_0 x^3 \sim N/\lambda^{\frac{2}{5}}$. In section 5.3, we saw that this charge is of order one for typical distributions, so it is only for $\lambda \sim N^{\frac{5}{2}}$ that the geometry we have constructed has an entropy of the same order of magnitude as the thermal state. (It is important to note that for a fixed configuration of discs (i.e. fixed $p/V_0 \sim \lambda$), the corresponding charge distribution changes as a function of N.)

D.2 Stretched Horizons

In this appendix we investigate the possibility that the area (or some generalization of area) of a suitably defined stretched horizon might reproduce the entropy formula (5.18).[42] We focus on a particularly simple specific example of a coarse-grained geometry, and find that a stretched horizon whose area would reproduce the entropy would necessarily be parametrically closer to the singularity than both the scale x_s where the string coupling becomes of order one, and the scale x_c where the curvature becomes string scale.

The geometry we focus on is the thermal state geometry of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. In this case the potential is simply

$$-\frac{N\pi}{2k}\log\rho + V_0(\rho^2 - 2\eta^2) \tag{D.6}$$

where $V_0 \sim \frac{1}{g_{YM}^2 k}$ as identified in chapter 3. The potential is singular at $\rho = 0$, which violates the regularity condition on the LM geometry. The boundary of the coarse-grained conducting discs is at $\rho = r_0\sqrt{\frac{\pi N}{4kV_0}}$, and the supergravity solution is

$$\begin{aligned}
ds_{10}^2 =& \left(\frac{N}{4V_0k\pi}\right)^{\frac{1}{2}} \left\{-4\frac{4V_0k\rho^2}{N\pi}dt^2 + \frac{16kV_0}{4kV_0\rho^2 - N\pi}(d\rho^2 + d\eta^2)\right. \\
&\left. + 4d\Omega_5^2 + \frac{4kV_0\rho^2 - N\pi}{N\pi}d\Omega_2^2\right\}, \\
e^{4\Phi} =& \frac{N\pi k}{8V_0^3(4kV_0\rho^2 - N\pi)^2}, \\
C_3 =& -\frac{(4kV_0\rho^2 - N\pi)^2}{N\pi k}dt \wedge d^2\Omega, \\
B_2 =& 2\eta d^2\Omega.
\end{aligned} \tag{D.7}$$

We see explicitly that the geometry is singular at $\rho = r_0 \sim \sqrt{g^2N}$, which is exactly the edge of the discs, but there is no horizon in this geometry. This solution has been considered in [42], where it was pointed out that the singularity is related to the Z_k orbifold singularity in the IIB language.

We will assume the stretched horizon to be a constant ρ surface respecting the translational symmetry along the η direction. Using $\rho = r_0 + x$, we

[42] These ideas have been explored in the context of coarse-grained LLM microstates [83, 95, 96], though a prescription for defining a stretched horizon that generally reproduces the entropy of coarse-grained states has not emerged.

find the string coupling becomes of order one at

$$x_s = \frac{1}{8\sqrt{\pi}V_0^2} \sim (g_{YM}^2 k)^2. \tag{D.8}$$

The Ricci scalar can be calculated noticing the fibred structure of the metric,

$$R^{string} = 3\sqrt{\frac{V_0 k}{N\pi}} \frac{8V_0 k\rho^2 - N\pi}{4V_0 k\rho^2 - N\pi}. \tag{D.9}$$

We see that it diverges at exactly the boundary of the coarse-grained conducting discs. The curvature becomes of string scale at

$$x_c \sim 1\ . \tag{D.10}$$

In the above we have assumed $g_{YM}^2 N \gg 1$ in order for the supergravity approximation to be valid. As a result, we will be interested in the scale where $g_{YM}^2 N \gg 1 \gg g_{YM}^2 k$, and in particular $N/k \gg 1$. The area of an 8-surface at constant t and $\rho = R(\eta)$ can be calculated to be (in the Einstein frame)

$$A = 2^{11/2}\omega_2\omega_5\sqrt{1+R'^2(z)}\sqrt{(\ddot{V}-2\dot{V})}\dot{V}^{3/2}, \tag{D.11}$$

where ω_2, ω_5 are the volume elements of the two-sphere and five-sphere respectively. Specializing to $R(z) = r_0 + x$ and to the metric (D.7), we get

$$A = 16\omega_2\omega_5\sqrt{N\pi}\frac{(4V_0 k\rho^2 - N\pi)^{3/2}}{k^2}. \tag{D.12}$$

We note that it is a monotonically increasing function with the distance from r_0. Using this and evaluating at x_c, x_s we find

$$\begin{aligned} A_c &\sim \frac{N^{5/4}}{g_{YM}^{3/2} k^2}\ , \\ A_s &\sim (g_{YM}^2 k)^3 A_c\ . \end{aligned} \tag{D.13}$$

The Bekenstein-Hawking entropy formula $S = \frac{A}{G_N}$ gives $G_N = g_s^2 = (g_{YM}^2 k)^2)$

$$\begin{aligned} S_c &\sim \frac{1}{g_{YM}^{11/2} k^{11/4}}\left(\frac{N}{k}\right)^{5/4}, \\ S_s &= (g_{YM}^2 k)^3 S_c. \end{aligned} \tag{D.14}$$

As expected $S_c \gg S_s$. According to the entropy functional (5.18), the entropy associated with the geometry (D.7) is

$$S = -k\ln(N/k) + (N+k)\ln(N/k+1). \tag{D.15}$$

In the large N/k limit it becomes

$$S \sim k(\ln(N/k) + 1), \tag{D.16}$$

which is much smaller than both S_s, and S_c. Here, both α' and string loop corrections are very important. Further, if a horizon (or some stringy analogue) does exist in the fully corrected solution, we may require a highly stringy generalization of area to compare with the entropy. While we have studied only one particular example, we expect that the qualitative features will apply in the general case.

Appendix E

Numerical Methods

In this appendix we will provide more detail on the numerical methods used in this thesis.

An important problem that has occurred throughout this thesis is to find $SU(2|4)$ symmetric solutions of type IIA supergravity. Because the of the symmetries in this ansatz, as Lin and Maldacena showed, the supergravity equations are reduced to a Laplace equation. Due to the boundary conditions for regular solutions, however, solving the supergravity equations is non-trivial, but in many cases possible, as we have discussed above. Although we have found a solution of the supergravity equations under general circumstances, it turns out that to extract detailed information about the solution it is necessary to apply some approximations, and for the types of information we have required, numerical methods are the most appropriate.

We have been able to cast the supergravity equations into a form so that the solution is given by the solution of a Fredholm integral equation of the second kind. Equations of this type are encountered frequently in physics, for example the Lippman-Schwinger equation in quantum mechanics is of this form. In some cases, when the kernel is of a special form, the equations can be solved exactly. Unfortunately, analytic solutions are not known for the present case.

One approach to solving integral equations of this type is to expand the kernel in terms of some basis of known functions, for example the Legendre polynomials would be suitable in the present case, and then to truncate the expansion after a suitable number of terms to give a solution of the desired accuracy (see, e.g. [64]). This technique is not particularly helpful in the present case because an analytic form for integral of the kernel against standard sets of complete functions is not known. These integrations must then be performed numerically, and to obtain reasonably accurate solutions it is computationally expensive.

Another numerical approach, called the Nyström method, turns out to be more appropriate for the present problem. This method is described many places and a nice discussion can be found in [70]. We will summarize the main details here. The crux of this method is to notice that for integral equations of the type

$$\psi(x) + \int_{\alpha}^{\beta} dt K(x,t)\psi(t) = f(x) \tag{E.1}$$

the operator $\mathfrak{K} = \int_{\alpha}^{\beta} dt K(x,t)$ is a linear operator, and so we can write the integral equation as

$$(1 + \mathfrak{K})\psi = f \,. \tag{E.2}$$

The difficulty then arises in inverting the operator $(1 + \mathfrak{K})$. To obtain a tractable problem, one can restrict the interval (α, β) to a finite number of points, and thereby reduce the problem to solving a finite sized linear system, which is straightforward to do numerically. The remaining challenge is then to ensure that the interval is discretized in a suitable way so that there are a sufficient number of points, suitably distributed in the interval to provide a sufficiently accurate numerical solution. The number and distribution depend on the form of the kernel.

Fortunately for us we are provided with a starting point, because the integral equation that we have to solve is a special case of a similar integral equation that is found in the context of fluid mechanics [97]. In that case, the problem under consideration was the radiation of water waves from an oscillating submerged disc. The kernel in question is

$$K(x,t) = \frac{1}{\pi} \frac{2\kappa}{4\kappa^2 + (x-t)^2} \tag{E.3}$$

where κ is a parameter that gives the separation between the discs. In [97] the Nyström method was applied to an integral equation with this kernel, and it was found that for $\kappa \sim 10^{-1}$ a discretization using 60 point Gauss-Legendre quadrature gave good numerical results. In our application, to ensure the constraint that the charge density vanished at the edge of the conducting disc, it was necessary to compare solutions of the integral equation for different right hand sides. The behaviour near the edge of the interval was of particular importance for ensuring the vanishing charge density condition. We found that using the midpoint quadrature rule, instead of Gauss-Legendre, gave better numerical results for the region near the edge of the interval. To solve the numerical system we used the GNU Scientific Library LU-decomposition routine, implemented in the C programming language. Numerical calculations were run on a Linux based workstation with dual 2.80 GHz Intel Pentium 4 CPUs.

We solved the integral equation over a range of values for the parameter κ, and checked that the number of points in the discretization was sufficient so that the results did not depend on the number of points. We also compared the solution of the integral equation in the bulk of the interval to ensure that it gave the same behaviour with both midpoint and Gauss-Legendre quadrature. We checked that the eigenvalues of the matrix in the linear system were sufficiently different from zero to allow the system to have a unique solution. A further check came from considering the solutions of the integral equation for large κ, in which case an analytic form of the solution can be found. We ensured that the numerical results were consistent with this form when we took $\kappa \gg 1$, and that the relevant physical parameters changed continuously as we scaled κ away from this limit. A final check on the numerical results was done for the form of the potential in chapter 2, and was described above in C.1. That was to compare the numerical result for the electrostatics potential with a solution that can be found by conformal mapping in the case where $\kappa \ll 1$. Although it was not possible to fix the overall normalization of the potential from the conformal mapping, excellent agreement was found between the form of the potential obtained by conformal mapping and the result from the Nyström method. We were therefore able to see that the numerical methods appeared to interpolate smoothly between the two extremes in the parameter space in which we had some analytic control over the form of the potential.

Appendix F

Note on Publication

Chapter 2 is based on the paper [1] describing work done in collaboration with H. Ling, A.R. Mohazab, H.-H. Shieh and M. Van Raamsdonk, and contains sections from that paper. I contributed to this project at all levels including the formulation of the method we used, the calculations we performed, and preparing the manuscript. In particular, our main results on the form of the double scaling limits come from a numerical code that I wrote. Chapter 3 is based on the paper [2] describing work done in collaboration with H. Ling and H.-H. Shieh, and contains sections from that paper. The main results of that paper come from results of numerical codes that I wrote, and I wrote the majority of the original manuscript. Chapter 4 is based on the paper [3] describing work I carried out independently, and contains sections from that paper. Chapter 5 is based on [4] describing work done in collaboration with H.-H. Shieh and M. Van Raamsdonk, and contains sections from that paper. I was involved at all stages of the project. Particular contributions include the generalizations to ensembles with different combinations of fixed charges, and a significant amount of writing to the original manuscript.