Topics in Quantum Physics

Schrodinger's Cat Problem - Time Measurement Accuracies in Quantum Mechanics

by

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Abstract

In this thesis I address two different topics in quantum theory. The first one is the long discussed Schrodinger's cat problem, and the issues related to having a macroscopic superposition state. I show that the quantum theory provides full explanation to the problem. In the second part, I discuss the time measurement related issues in quantum mechanics. Since there does not exist any time operator in quantum mechanics generally, time is not directly measurable. Therefore we should devise other methods to register time. We study different time-energy relations and will find that accurate clocks have high energy uncertainties. If we use accurate clocks in quantum systems to observe their time evolutions, their high energy uncertainties interfere with system's normal evolution and slows it down. I also provide a formal proof to a previously suggested limiting accuracy relation on the measurements of the time-of-arrival experiments.

Table of Contents

Appendices

List of Figures

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Dedication

To my father, my first science teacher, who encouraged my curiosity by giving me gifts like magnets and magnifying glasses to play. His continual support and encouragements during my career has been essential to fulfil my dreams.

"Everything should be made as simple as possible but not simpler."
-Albert Einstein

Part I

Schrodinger's cat, finally dead or alive?

Chapter 1

Stating the Problem

In his 1935 paper [1], Schrodinger has a brief paragraph describing a cat to show a seemingly conceptual problem in quantum mechanics; he discusses and rejects the interpretation that a quantum system is physically some how spread on different parts of a superposition. He emphasizes as follows:

> *"One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following diabolical device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small that perhaps in the course of one hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The first atomic decay would have poisoned it. The Psi function for the entire system would express this by having in it the living and the dead cat (pardon the expression) mixed or smeared out in equal parts.*
>
> *It is typical of these cases that an indeterminacy originally restricted to the atomic domain becomes transformed into macroscopic indeterminacy, which can then be resolved by direct observation. That prevents us from so naively accepting as valid a "blurred model" for representing reality. In itself it would not embody anything unclear or contradictory. There is a difference between a shaky or out-of-focus photograph and a snapshot of*

clouds and fog banks."

This thought experiment has raised many controversial arguments, some times referred as the cat paradox.

The Schrodinger dilemma consists of different part, often mixed carelessly:

- what is the state of the cat in the box?
- Can quantum mechanics provide a full description of the system?
- When the state of the cat change? [when did the cat die?]
- Can we have macroscopic system being in superposition of states? [do we have classical cat-state?]

Each of the questions above should be regarded separately as they refer to different aspects of the problem, not necessarily on equal footing.

We know (at least most of us believe) that the cat in the box is either dead, alive or dying and not in a smeared out state between those alternatives, so is it something missing in order to have a valid quantum mechanical description of the system? Is it that quantum mechanics is incomplete and lacks rules relating the classical and quantal descriptions? Does quantum mechanics have the ability to describe the macroscopic world as good as the microscopic world?

In this Part we first provide the standard quantum mechanical description of the Schrodinger cat's system, and try to answer the above posed questions. In the next chapter, as an example of a macroscopic system being in superposition state, we study a cat in the cat-state[1]. Then we try to elaborate on the obstacles preventing us from observing such a state.

[1] We follow the common conventional term, calling an equal superposition state of all being in two orthogonal states $|0\rangle$ and $|1\rangle$ a cat-state.

Chapter 2

Standard Quantum Mechanical Approach Towards the Schrodinger's Cat

It is instructive to present the standard quantum mechanical approach of the Schrodinger's cat problem and try to find potential problematic parts in it.

First of all let's make some simplifications to the problem, making it *"as simple as possible, however not simpler"*.

In our modified problem, instead of the sophisticated Schrodinger setup to kill the cat, our system is composed of a half reflecting mirror and a photon detector behind it which instantly triggers a laser gun if detects a photon. The laser gun is aimed on an ant, which is held steady under a transparent tape strip; so the laser beam would kill the ant instantly.

The experiment starts by putting an alive ant under the strip onto the specified location into the box with a window which we put the half reflecting mirror there. Sending a single photon toward the window, we investigate the state of the ant. There is a clear analogy between our simple system and Schrodinger's setting. But ours is easier to investigate.

Lets define the states in our subsystems in the box. In the photon subsystem [system I], $|R\rangle$ represents the state of the reflected photon, while $|T\rangle$ is the transmitted one; in the ant subsystem [system II], $|d\rangle$ represents the dead ant and $|l\rangle$ the alive ant[2].

The state of the photon after entering the box and meeting the half-mirror is $\frac{|R\rangle+|T\rangle}{\sqrt{2}}$, and after the time $t_0 = D/c$ (D is the distance of the ant from the initial photon emitter , so t_0 is the time that a photon needs to reach the ant) the state of the ant-photon system is $\frac{|Rl\rangle+|Td\rangle}{\sqrt{2}}$, where we write $|Mn\rangle$ instead of $|M\rangle_I \otimes |n\rangle_{II}$ for brevity. Thus the interaction correlates the ant and photon subsystems and makes the state of the whole system an entangled state. The interaction between subsystems is reversible, as far as we do not separate them.

Thus the state vector of the system changes according to

$$|Photon\rangle_I \oplus |l\rangle_{II} \longmapsto \left(\frac{|R\rangle+|T\rangle}{\sqrt{2}}\right)_I \oplus |l\rangle_{II} \longmapsto \frac{|Rl\rangle+|Td\rangle}{\sqrt{2}},$$

while system's state operator changes as

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}_I \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{II} \longmapsto \begin{bmatrix} \frac{1}{2}|l\rangle\langle l| & \frac{1}{2}|l\rangle\langle d| \\ \frac{1}{2}|d\rangle\langle l| & \frac{1}{2}|d\rangle\langle d| \end{bmatrix}_I = \begin{bmatrix} \frac{1}{2}|R\rangle\langle R| & \frac{1}{2}|R\rangle\langle T| \\ \frac{1}{2}|T\rangle\langle R| & \frac{1}{2}|T\rangle\langle T| \end{bmatrix}_{II}$$
$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}_{I\otimes II}$$

A tricky part is that we must look at the system as a whole when we talk about the entanglement and the coherent states; the individual subsystems of the ant or the photon are not in superposition states as we shall show. There exists no experiment on the ant alone which would be sensitive to the superposition. The common mistake seems to be to think (implicitly) that the system's state vector $\frac{|Rl\rangle+|Td\rangle}{\sqrt{2}}$ is composed of something like $\frac{|R\rangle+|T\rangle}{\sqrt{2}} \otimes \frac{|l\rangle+|d\rangle}{\sqrt{2}}$ or $\frac{|R\rangle+|T\rangle}{\sqrt{2}} \oplus \frac{|l\rangle+|d\rangle}{\sqrt{2}}$. This seems implicit in Schrodinger's description.

A superposition state like $|\phi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ has a coherent density matrix in

[2] As we will discuss in the next chapter, describing an ant by a two dimensional state is not quite true. However that subject does not matter at this part and the basic argument remains intact.

the form of $|\phi\rangle\langle\phi| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, whereas the state operator of the ant or the photon subsystem in the box, which can be constructed by tracing out the irrelevant components of the total state operator, are both of the form of $\rho = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. This state operator represents a non-coherent state. In other words, looking at sole the ant subsystem, the ant state is either alive or dead, but not alive and dead at the same time (the paradoxical issue in the Schrodinger's problem), since the off diagonal components of the state operator are zero. The same is true for the sole photon subsystem. These state operators of the sole ant or photon subsystems do not change any more by coupling to them any extra systems such as a registrar or *Wigner's friend* [2].

The state operators of the subsystem I changes according to

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}_I \longmapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}_I$$

and the subsystem II changes like

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{II} \longmapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}_{II}$$

When these subsystems are looked at individually, the changes to these states are irreversible.

Thus the individual subsystems are not in superposition states; theoretically they appear to be in only one of the two possible states, as we get them experimentally; not in both at the same time. i.e. there exists no experiment which could be performed on the subsystem which could differentiate the ant's state from a classical mixture.

Chapter 3

A Cat in the Cat-State

Now lets take our attention to the question of possibility of having a classical cat in an alive-dead cat-state. Recall that a part of discussion on the Schrodinger's cat problem has been on the existence of a macroscopic superposition state. Here we study a cat in an alive-dead superposition, regardless of the fact that the Schrodinger's cat is not in the cat-state itself, as we discussed before.

Choosing a fairly realistic model for the cat, we show different behaviours of a macroscopic system, for example a cat in the cat-state, as compared to the behaviour of a structure-less quantum particle, like an electron, in a cat-state. In what follows we show the inability to see any positive experimental result from the possible setups wishing to detect the interference between two components of a classical cat-state. As we will show there is no contradictions between the theoretical results and the observations. Thus a quantum mechanical approach can tackle the question in a philosophically correct way.

Looking at a cat or any other living entity, it is clear that when we talk it "living", we mean that its organs are working in a proper way. For example the cat's heart beats, its nerves transmit electrical signals, its respiratory system works, the pupils respond to light, etc... Accordingly when we consider the cat "dead" we mean that some of these features are not present.

Bearing in mind these complexities, we question representing the living cat and the dead cat by states as simple as $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ states. We can expect that this simplification -representing an alive and a dead cat by simple $|1\rangle$ and $|0\rangle$ states- as naive as representing the spin direction of a structure-less particle such as an electron, might finally yield to some counter intuitive results.

As a hint to see aforementioned representation inefficiency, let's consider the operator

$$\hat{O} = |SchrCat\rangle \langle SchrCat| = \left(\frac{|live\rangle + |dead\rangle}{\sqrt{2}}\right)\left(\frac{\langle live| + \langle dead|}{\sqrt{2}}\right) \tag{3.1}$$

$$= \frac{1}{2}\left(|1\rangle + |0\rangle\right)\left(\langle 1| + \langle 0|\right) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Measurements with that operator on a population of dead cats brings 50% of them back to life theoretically. This is something absurd and has never been reported in our daily life. Death has always been a one way road that can not be reversed simply, at least not as successfully as the theory predicts in the above.

This implies that those 2 dimensional states can not ideally represent a cat state. An alive or dead cat is a complex system and we should choose a decent representation to describe it.

There are certain attributes which are specific to aliveness of any living creature, and if any of them changed the being would not be considered alive anymore. These attributes are generally called *vital signs* in biological sciences; so say, the hand position is not of those attributes, while the working of the nerves is an attribute for the live body.

We can define *vital states* as the states which upon change, bring about a change in the living state of the being to dead[3]. All the other states - non vital ones- can be considered as environmental states for life situation of the individual. In this definition we consider the transition between life and death instantaneous, to avoid the possible complexities. Physiologically, however, the death is considered as a process in the body that besides the measurements and observations, should be monitored to be assessed.

If we consider the scale of a living body and the number of different organs and matched *vital signs* we notice there is a large number of different life states all representing an alive cat; furthermore there are numerous other

[3]or in a more conservative notion, aliveness of the being would be questionable.

states all representing a dead cat.

3.1 Equal Number of Possible Dead and Alive States Case

Let's for now suppose that the Hilbert spaces of the life states and the death states are at same dimensions, in other words suppose there is the same number of ways for the cat to be alive or to be dead.

We can define an operator Z with eigenvalues 1 and 0 for live or dead cat and write $|live_i\rangle$ as the basis for the live states and $|dead_i\rangle$ as a basis for the dead state. Consider the unitary and Hermitian operators

$$X\,|live_i\rangle = |dead_i\rangle \qquad\qquad X\,|dead_i\rangle = |live_i\rangle$$

$$Z\,|live_i\rangle = |live_i\rangle \qquad\qquad Z\,|dead_i\rangle = -\,|dead_i\rangle$$

$$Y\,|live_i\rangle = i\,|dead_i\rangle \qquad\qquad Y\,|dead_i\rangle = -i\,|live_i\rangle$$

These X, Y, Z form a "2-level" subsystem of the full Hilbert space, and we can map the full Hilbert space onto the space spanned by the direct product of a two level system $\{|1\rangle, |0\rangle\}$ and a space of "irrelevant" attributes $|\phi_i\rangle$ of dimension half of the dimensionality of the full Hilbert space where $|1\rangle \otimes |\phi_i\rangle = |live_i\rangle$ and $|0\rangle \otimes |\phi_i\rangle = |dead_i\rangle$. Note that this decomposition depends on the arbitrary choice of the basis $|live_i\rangle$ and $|dead_i\rangle$.

Thus the total $2m$ dimensional Hilbert space of the dead plus alive states can be decomposed to two smaller sub-spaces, a 2 dimensional Hilbert space which shows the "life state" of the cat, and a m dimensional Hilbert space representing the details of each dead or alive states $H_{tot} = H_{2\times 2} \otimes H_{m\times m}$.

In this space we can categorize unitary transformations under three types:

1. Type I, which map live states to live states and dead states to dead

states:

$$U_T(t) = \begin{bmatrix} U_l(t) & 0 \\ 0 & U_d(t) \end{bmatrix}$$

with the condition

$$U_l(t)U_l^\dagger(t) = U_d(t)U_d^\dagger(t) = I_m$$

or

$$U_T(t)U_T^\dagger(t) = I_{2m}$$

where I_n is the n-dimensional unit matrix. We call these unitary operators *time evolution operators.*

2. Type II, which map live states to dead states and dead states to live ones:

$$U_L(t) = \begin{bmatrix} 0 & U_r(t) \\ U_k(t) & 0 \end{bmatrix}$$

with the condition

$$U_r(t)U_r^\dagger(t) = U_k(t)U_k^\dagger(t) = I_m$$

or

$$U_L(t)U_L^\dagger(t) = I_{2m}$$

Killing of the cat is caused by the action of such a unitary operator; we call these unitary operators *life-state changing operators.*

3. Type III unitary operators are those which can not be categorized under those previous types. Such operators mix the live and the dead states and make a state which is a mixture of alive and dead states; we call these operators *life-transient operators.*

To do a successful interference experiment on the Schrodinger cat we should choose our Hermitian operator in a way so as to maximize the chance to see the interference between the live and the dead parts of the Schrodinger cat wave function.

The best option would be to have an operator which map the live state part of the Schrodinger cat completely onto its dead state part; however, we do not know the exact configuration of the dead state and alive state parts *per se.*

This is because we do not know the form of time evolution operators which operate on the dead Hilbert space and the alive one. That is the exact form of the time evolution operators $|L_0\rangle \xmapsto{U_L(t)} |L_t\rangle$ and $|D_0\rangle \xmapsto{U_D(t)} |D_t\rangle$.

Bearing in mind the above mentioned problem in choosing operator to do the interference experiment, lets try this handy operator which projects the dead states to the alive ones and vice versa, without changing their distributions:

$$\hat{\Omega}\,|live_i\rangle = \hat{\Omega}\,[|1\rangle \otimes |\phi_i\rangle] = |0\rangle \otimes |\phi_i\rangle = |dead_i\rangle$$

and

$$\hat{\Omega}\,|dead_i\rangle = \hat{\Omega}\,[|0\rangle \otimes |\phi_i\rangle] = |1\rangle \otimes |\phi_i\rangle = |live_i\rangle$$

or in the other representation $\hat{\Omega} = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$, where I_m is the $m \times m$ unit matrix; we can always choose a representation to have our operator look like that. We do the interference experiment on our Schrodinger cat (in the Schrodinger picture of quantum mechanics):

$$\begin{aligned} |SchrCat(t)\rangle &= \frac{|live\rangle + |dead\rangle}{\sqrt{2}} = \frac{(|1\rangle \otimes |L(t)\rangle) + (|0\rangle \otimes |D(t)\rangle)}{\sqrt{2}} \\ &= \frac{\left(\sum_i a_i(t)\,|1\rangle \otimes |\phi_i\rangle\right) + \left(\sum_j b_j(t)\,|0\rangle \otimes |\phi_j\rangle\right)}{\sqrt{2}} \end{aligned}$$

with the normalization

$$\sum_i |a_i(t)|^2 = \sum_j |b_j(t)|^2 = 1$$

We get

$$\langle Schrcat(t)| \hat{\Omega} |Schrcat(t)\rangle = \frac{1}{2}\left(\sum_i \overline{a_i(t)}b_i(t) + a_i(t)\overline{b_i(t)}\right) = Re \langle L(t)|D(t)\rangle .$$

Since at an arbitrary time $|L(t)\rangle$ and $|D(t)\rangle$ are not known for us and can be any of m orthogonal states in the alive and dead Hilbert spaces, we evaluate the mean value of above quantity statistically. This value is zero in all complex vector spaces. This yields to the expectation value of

$$\overline{\langle Schrcat(t)| \hat{\Omega} |Schrcat(t)\rangle} = \overline{Re \langle L(t)|D(t)\rangle} = 0$$

with the standard deviation[4] of $\sigma \propto \frac{1}{\sqrt{m}}$. Thus statistically, the expectation value of our interference experiment would be zero, and the chance to see the interference falls to zero in practise.

However, if we could have conducted an experiment right after the time when the cat changes its state from the alive state to the Schrodinger cat-state[5], using $U_L(0)$ operator -the *life-state changing operator* which acts at $t = 0$ and kills the cat- then (in the Heisenberg picture of quantum mechanics) we get:

$$\begin{aligned}
&\langle SchrCat(0)| U_L(0) |SchrCat(0)\rangle = \\
&\quad = \tfrac{(\langle L(0)|+\langle U_k(0)L(0)|)U_L(0)(|L(0)\rangle+|U_k(0)L(0)\rangle)}{2} \\
&\quad = \tfrac{\langle L(0)|U_r(0)U_k(0)|L(0)\rangle+\langle U_k(0)L(0)|U_k(0)L(0)\rangle}{2} \\
&\quad = \tfrac{1+\langle L(0)|U_r(0)U_k(0)|L(0)\rangle}{2}
\end{aligned}$$

So it would be possible to observe the interference if we knew the exact time of transition.

We should underline that the assumption which the live and the dead Hilbert spaces are at the same dimensions causes a weird situation though; if we kill a cat, killing it again by the same operation, will make it living! This is because we have $U_L(t_n)U_T(t_n - t_0)U_L(t_0) = U_T(\acute{t})$, operating twice

[4]Details of calculation in Appendix A

[5]This is equivalent of knowing the exact configuration of the dead state and alive state parts of the cat at a point in time, in contrast to the previous choice of operator to do the interference experiment.

with any of type II unitary operators, *life-state changing operators*, is equal to action of a type I unitary operator, *time evolution operators*.

3.2 Unequal Number of Possible Dead and Alive States Case

Let us now consider unequal dimensional dead and live Hilbert spaces. In particular let us assume that the dead states span a much bigger Hilbert space than the life states do. We defined *vital states* as the states which upon change, bring about a change in the living state of the being to dead. So for the total number of q *vital states*, we have at least $2^q - 1$ dead states versus any live state[6]. Therefore the dead state Hilbert spaces is much bigger than the live state Hilbert space. Recall also that in the real life we might amputate any part of a dead body creating another dead states, not changing its life situation, while in doing so to an alive body we do not have such freedom.

In the world where the dead states span much greater Hilbert space than the live states, observing the interference is even harder. In this case, since the dimensions of the live and the dead states are not equal, it is not possible to construct a well behaviour Hermitian operator like $\hat{\Omega}$ to do the interference experiment. Also since we do not know the exact configuration of the live and dead parts of the Schrodinger cat state and their time development here as well, a randomly chosen operator like $\hat{A} = |\psi\rangle \langle\phi| + |\phi\rangle \langle\psi|$ again yield to the expectation values of zero with the standard deviation of the order of $\sigma \propto \frac{1}{Q}$, $(Q = 2^q)$.

Lets in this case try the first operator $\hat{O}$ (3.1) and see what its success rate is to bring the dead cats back to life.

$$\hat{O} = |SchrCat\rangle \langle SchrCat| = \frac{1}{\sqrt{2}} (|live\rangle + |dead\rangle) . \frac{1}{\sqrt{2}} (\langle live| + \langle dead|)$$

[6]Having q different *vital states* only one configuration out of 2^q possible configurations have all the *vital states* available.

$$= \frac{1}{2}\begin{bmatrix} |L(t)\rangle\langle L(t)| & |L(t)\rangle\langle D(t)| \\ |D(t)\rangle\langle L(t)| & |D(t)\rangle\langle D(t)| \end{bmatrix}$$

with the states[7] $|L(t)\rangle = \sum_i^n a_i(t)\,|l_i\rangle$ and $|D(t)\rangle = \sum_j^m b(t)\,|d_j\rangle$ and the normalization

$$\sum_i^n |a_i(t)|^2 = \sum_j^m |b_j(t)|^2 = 1$$

This operator brings a dead state, $\left|\acute{D}(t)\right\rangle$, back to life, the live state, $|L(t)\rangle$, with the success chance of $\frac{1}{2}\left\langle D(t)|\acute{D}(t)\right\rangle$ which is of the order of[8] $\frac{1}{\sqrt{m}}$ in average, while bringing a live state, $\left|\acute{L}(t)\right\rangle$, to death with the success chance of $\frac{1}{2}\left\langle L(t)|\acute{L}(t)\right\rangle$ which goes as $\cong \frac{1}{\sqrt{n}}$ in average[9].

Considering that $n \ll m$ [according to our assumptions $\frac{m}{n} \geq 2^q - 1$, where we have q number of *vital states*] these results agree with our daily observation that killing is much easier than the resurrecting a dead body. Also we should emphasize that if we knew the state of a dead body, $\left|\acute{D}(t)\right\rangle$, *per se*, we would be able to construct an operator to bring it back to life with a high success rate. Similar operations happen commonly in hospitals; for example, when physicians find the cause of death is just non-beating heart, they try to bring the patient back to life by executing electric shocks, before the passage of time changes that dead state to another unknown state for

[7] Here note that $|L(t)\rangle$ and $|D(t)\rangle$ are n and m dimensional states respectively, while$|live\rangle$ and $|dead\rangle$ are of $m+n$ dimensions both.

[8] Details of calculation in Appendix A

[9] If someone insists on representing the projection operator $\hat{O}$ in 2×2 basis [dead-alive components], ignoring the internal structure of the cat [or generally any macroscopic measuring device], s/he would end up with something like $\hat{O} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{m}} \\ \frac{1}{2\sqrt{m}} & \frac{1}{2} \end{bmatrix}$ for our first assumption (equal the dead and the live Hilbert spaces) and $\hat{O} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{m}} \\ \frac{1}{2\sqrt{n}} & \frac{1}{2} \end{bmatrix}$ for the second case, which both read $\hat{O} = \begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}$ in the classical realm where $m, n \gg 1$. Note that this decoherence is caused solely because of our intrinsic ignorance about the internal state of the cat [classic measuring device in general] and is not dynamical.

them. So a critical fact which limits our ability to resurrect a deceased body is the **evolution** after the death.

In the previous case of equal number of dead and alive states we encountered the weird situation that killing the cat twice by the same operator resuscitates it. What if we shoot a cat twice in the head in this case? can we bring it back to life?

Clearly now we are not able to make any type II unitary operators, which switches the dead states parts and the life states parts, as we did in the previous case, since the number of the live states and the dead states are not equal and we do not have the life-dead Hilbert space symmetry.

In our present case any life-state changing operator that maps live states to dead states, can not map all the dead states to the live states. This is because the number of the dead states exceeds the number of the live ones. So, inevitably the majority of the dead states will map again on the dead states just with a new configuration. In the limit of $n \ll m$ the action of life-changing operators would look like mapping the live states to the dead states, while redistributing the dead states over the dead Hilbert space. Such behaviour is in accordance with our expectation for a killing operator. Thus by shooting twice a cat, we not only can not resuscitate it, but also we make its dead state configuration more unknown and messy.

Chapter 4

When did the cat die?

So then what changes happen by opening the box and looking into it? This question and another question have been of most challenging questions of quantum mechanics interpretation since the introduction of the Schrodinger cat problem. The other question is *"When did the cat die?"* or in our version of the Schrodinger cat problem "when does the ant die?".

In our simplified version of the notorious cat problem, it is easy to infer that if ever the ant dies that happens at the time $t_0 = D/c$ after we launch the photon towards the box. After this time nothing special happens to the subsystem, neither our looking nor the registrar can kill the ant then! The time of death of the ant never depends on when some one opens the box. Some may bring the question of the possibility of resuscitating the ant, or observing the interference after the time t_0. According to our discussion in the previous chapter, it is not feasible in practise due to the macroscopic nature of the total system.

But what then happens by opening the box and making an observation on the system? The point is that before observing the ant subsystem separately, practically we can not have absolute information about it. The best knowledge that we can acquire theoretically is that its state operator is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and that means it is dead by 50% chance and alive by the same chance[10]. This information remains unchanged until we observe the ant subsystem in the box, or we talk to *Wigner's friend*. At that time our information may change. The configuration of the system does not change then though, the

[10]Having no definite information about the subsystems is clearly due to the setup; we have set the system to behave so. When we send a photon to interact with the half reflecting mirror and be reflected randomly by 50% chance, we should expect then to have no definite information about the subsystems after such interaction.

thing that changes is our ignorance about the subsystems.

This is in harmony with the pragmatic view that no real identity is attributed to the wave function, and its significance is only due to the information that it makes extractable about the system through the mathematical machinery.

Chapter 5

Conclusion

As we discussed in chapter 2, entangling a cat with a photon in the cat-state does not provide us with a cat in the cat-state. Thus the cat would not be in the simultaneous state of life and death.

We see that by entangling the ant's life state with the photon's state in the cat-state, we lose some information about the ant's life state in the box. This is due to the fact that we do not have a proper defined rotated life-dead coordinate to describe the properties of the state of the ant subsystem; however, if for example the entanglement was between an electron in $+X$ spin direction- instead of the initial incoming photon- and another electron initially in $+Z$ direction- instead of an alive ant- interacting according to the Z component of the initial electron's spin, we were able to talk certainly about its final state properties of the spin of the "ant" electron, which would resemble a spin in $+X$ direction properties when we setup experiments to observe interference of the total state.

Thus it seems we should look for obstacles which prevent us from having a large scale system[11] well defined in a cat-state. In other words, why we are not able to simply redefine our coordinates to encompass the state of a classical cat-state as another conceptually well defined pure state.

The arguments presented in section 3, suggest that for the Schrodinger's problem on life status of the cat, the complexities arise from our symmetric attribution to the asymmetric concepts of life and death.

For a general classical system, these kinds of asymmetries are quite basic. In almost all the cases, among lots of different pure states that the macroscopic system can go through, we commonly select few of them, specifically

[11] By the large scale [or interchangeably classical/macroscopic] system, we actually mean a system with a large Hilbert space state, not a system with a large quantum number.

those with lower entropy, and name them differently from the rest of other possible configurations. For example, consider a watch. There is a state [we call it the working clock state] which is distinguished in that some small changes to its components can prevent the clock from working. However, there are many rearrangement of the same components, which are not included in our definition of a working clock. All are called under a single name non-working clock state. Moreover, it is a falsity to count all of them as a single state. Making a cat-state out of clock's working and non-working states, arises similar results as we got for a cat.

In this case, our careless definitions towards asymmetric concepts like life and death, can be extended to any classical objects. As shown in section 3.2, whereby such conditions seeking the entanglement is impossible[12].

In summary we showed that the Schrodinger's cat, in his experimental setup, is not in the cat-state itself, which is therefore not living and dead simultaneously. Also we showed that even if we could have a cat in the cat-state, practically it is impossible to get a positive result in an experimental setup whose goal is to see the interference between the components of the superposition of states of the cat. The result is general and is true for any other macroscopic object with unknown internal structure states. However, there is nothing preventing us from having a macroscopic system being in superposition of states, though not distinguishable from pure states for us in practise.

[12]Here we may introduce another type of decoherence, induced decoherence due to asymmetric definitions, see the footnote (9)

Part II

Time Measurement Accuracies in Quantum Mechanics

Chapter 6

Introduction

One of the first debates in quantum physics was the legality of attributing an element of reality to an unmeasurable quantity. The common pragmatic view is that a property can be assigned to a system only if it can be measured in principle. Physicists are generally interested to find whether there are fundamental limitations on their measurements in general, and whether those limitations have implications in some more fundamental theory.

Time measurements in quantum theory, however, are special ones; there is no time operator. All the physical measurements are happening in time and we can not freeze an event in time to measure its time component.

In classical physics there are no limitations on measurements of space or time in principle. While in quantum physics the structure of the theory, through the commutation relations between operators, suggests fundamental limitations on simultaneous measurement of positions and momentum of the system.

This can suggest existence of a fundamental limitation on time measurements and its possible relation to energy measurements. There are some "proofs" of existence of such limitations, however, the notions they refer to are generally loose and not well defined. We address some of these time-energy relations' derivations later.

Time measurement methods are also a well discussed matter in the subject of time-of-arrival measurements. Ahanarov and Bohm [3] were the first to write such time-of-arrival (TOA) operator. Since then, much literature has been produced on such operators and their interpretations. An excellent review can be found here [4], and also a book-length treatment [5]. Constructing physical models for time measurement was first examined by Allcock [6], [7]; his constructions lead him to the result that time-of-arrival is not a quantum observable. Different physical models for time measurement and possible sources of difficulties arise in measuring the time-of-arrival have been discussed since then and physicists have debated whether this deficiency of quantum mechanics in measuring time-of-arrival is a model dependent defect or some deep concept in the theory.

The way people have approached this problem is to try to find or construct modified TOA operators (as we show there is no TOA operators in general) and get the probability distribution of the time of arrival. In another approach, by constructing several theoretical models for time measurements and solving the Schrodinger equation for them, people tried to find if there are limitations on the TOA measurements.

Recent works of Oppenheim et al. [18], [19], [22], [23], addressing several models for time-of-arrival measurements, suggest a fundamental limitations in measuring time-of-arrival; In their works they verified the suggested relation, however they left the proof as an interesting open question.

Here our approach is as previous. We present the arguments that reject the existence of the time operators as well as time-of-arrival operators in quantum mechanics in general. Then we discuss different time-energy relations, their derivations and interpretations. Equipped with those time-energy relations, we use some of them in the study of quantum systems which we use to measure the time, the *quantum clocks*. As we show the dynamical bounds we get relating the systems energy with its evolution rate, put a bound on the quantum clocks minimum energy, relating it to the clocks time resolutions.

Energy considerations on the clock's energy and the system's energy interacting with the clock to be measured, lead us to derive an accuracy limiting relation on the time-of-arrival measurements. The proof is general and for any measurement of the quantum time of arrival. Thus we provide a general quantum mechanical limitation on the measurement of the time of arrival. Finally we mention some application of the accuracy limiting relation in the context of quantum physics.

We will be working in natural units where $\hbar = c = 1$.

Chapter 7

Time Operators

The parameter t in the Schrodinger equation enters as an external continuous parameter, and in quantum physics, the time appears not to correspond to any physically measurable operator.

It was Pauli [8] who first demonstrated that generally there does not exist any operator associated with time. As we repeat his proof in below, such a *time operator* should be conjugated to the Hamiltonian and for a stable system with a Hamiltonian which is bound from below[13], it is not possible to construct such an operator.

In this chapter we consider properties of time related operators and the constraints that their existence exert on physical systems Hamiltonian.

7.1 Time Operators

Pauli [8] has proved that having an operator corresponding to physical time for general systems with arbitrary Hamiltonian is not possible. His proof is simple and instructive and we represent it here.

Suppose that there exist a self-adjoint time operator $\hat{T}$ conjugate to the Hamiltonian, i.e.

$$[\hat{H}, \hat{T}] = i \tag{7.1}$$

We show that such a time operator appears as an energy shift operator to energy states. If (7.1) is true then by induction we can have

$$[\hat{H}, \hat{T}^n] = in\hat{T}^{n-1}, n \geq 0 \tag{7.2}$$

($\hat{T}^0 \equiv 1$), supposing a well behaved operator to guaranty higher powers.

[13] Or mathematically equivalent bounded from above.

Together with $[\hat{H}, \hat{T}^0] = 0$, multiplying (7.2) by $(i\varepsilon)^n/n!$,where ε is an arbitrary parameter with dimension of energy, summing over n from $n = o$ to $n = \infty$, one finds

$$[\hat{H}, e^{i\hat{T}\varepsilon}] = -\varepsilon e^{i\hat{T}\varepsilon}, n \geq 0 \tag{7.3}$$

Therefore, according to equation (7.3) any energy eigenstate $|E\rangle$ satisfies

$$\hat{H}e^{i\hat{T}\varepsilon}|E\rangle = (E - \varepsilon)e^{i\hat{T}\varepsilon}|E\rangle$$

That means that the operator $\hat{T}$ generates unitary energy translations.

Since in the physical world Hamiltonians are generally bounded from below, such an operator can not exist. However it is clear that for unbounded Hamiltonians with continuous and homogeneous energy spectrum we can associate an operator whose eigenvalues represent the time.

7.2 Time-of-Arrival Operators:

It has been suggested that one can construct a time-of-arrival operator that registers the time when an event first occurs. These types of operators seemed not to have the same problems as do the time operators, since these operators do not evolve with time[14] and do not need to be conjugate to the Hamiltonian. However it is easy to show that, in general, these operators do not exist either [9].

Suppose we have a time-of-arrival operator $\hat{\tau}$ with eigenstates $|\tau\rangle$. Then according to the standard quantum theory, the probability amplitude of arrival time of the state $|\psi\rangle$ at instant $t = \tau$ would be given by $\langle\tau|\psi\rangle$. In other words this is the probability amplitude that a particle with the state $|\psi\rangle$ arrives to a certain point at the time $t = \tau$.

Now transform the state $|\psi\rangle$ forward through time by an amount t_0, i.e.

$$|\psi\rangle \rightarrow \left|\acute{\psi}\right\rangle = e^{-i\hat{H}t_0}|\psi\rangle$$

[14] By definition time of arrival is independent of the flow of time and does not change as the time goes on, i.e. if I come to my office at 8am this statement is independent of time.

This transformation is general for any state $|\psi\rangle$; it seems plausible to expect that the probability amplitude transforms accordingly to

$$\langle\tau|\psi\rangle \to \langle\tau + t_0|\psi(\tau + t_0)\rangle = \left\langle \tau + t_0|e^{-i\hat{H}t_0}\psi \right\rangle$$

This transformation should not change the probability amplitude of arrival time of the state. So it follows that the time-of-arrival eigenstates, $|\tau\rangle$, should satisfy

$$|\tau + t_0\rangle = e^{i\hat{H}t_0}|\tau\rangle \tag{7.4}$$

This is the same as saying that by backward translation time through by arbitrary amount t_0 any time-of-arrival eigenstate corresponding to a time of arrival τ transforms to another eigenstate, corresponding to a time of arrival $\tau + t_0$ [15].

However the above property of time-of-arrival eigenstates, as we will see implies the existence of an energy shift operator $\hat{T}$ satisfying $[\hat{H}, \hat{T}] = i$, as equation (7.1). Let us start with the equation (7.1) and show that property (7.4) is a general property of energy shift operators with continuous Hamiltonians.

Again by induction we can write

$$[\hat{H}^n, \hat{T}] = in\hat{H}^{n-1}, n \geq 0$$

($\hat{H}^0 \equiv 1$), supposing a well behaved Hamiltonian. Introducing an arbitrary parameter t_0 with dimension of time and following similar procedure as in the previous section, one finds

$$[e^{i\hat{H}t_0}, \hat{T}] = -t_0 e^{i\hat{H}t_0}, n \geq 0$$

If $|\tau\rangle$ represents a complete set of orthonormal eigenstates of $\hat{T}$, then we have

$$\hat{T}e^{i\hat{H}t_0}|\tau\rangle = (\tau + t_0)e^{i\hat{H}t_0}|\tau\rangle$$

[15] All of us are familiar with this fact, especially during the first days after daylight saving begins.

which is the property (7.4).

Therefore, in general, we can not have time-of-arrival operators. The exception here also is for systems with continuous homogeneous unbounded energy spectrum.

Chapter 8

Time-Energy Relations

We argued that generally we can not have a quantum time operator and the time is not a quantum observable. There are several time-energy relations proposed in physics which generally relate the uncertainty in time $\triangle t$ (various possible interpretations) to the energy uncertainty $\triangle E$. The interpretation of those time-energy relations is a part of quantum measurement theory, and need extra attention apart from its derivation.

Unlike the position-momentum uncertainty relation, there has not been a unique quantitative expression of what really is referred to by $\triangle t$ in time-energy relations. Needless to say, incorrect application of the time-energy relation can lead to a great deal of confusion. In this chapter we will try to improve on deriving these relations and their interpretations.

8.1 Inaccuracies Versus Uncertainties

In the theory of quantum mechanics, the measurement of a self-adjoint operator is theoretically possible to any desired accuracy. One can measure an observable A corresponding to an operator $\hat{A}$ as accurately as desired. One may also perform the measurement on another self-adjoint operator $\hat{B}$ which does not commute with operator $\hat{A}$. However, quantum mechanics forbids us from performing the measurements on both observables simultaneously while getting meaningful results.

Consider two non-commuting operators $\hat{A}$ and $\hat{B}$ which do not evolve with time. We can, for example, do measurements on two ensembles of identically prepared particles. On the first group we can measure observable A as accurately as we want, while measuring B on the other group with full accuracy. This is allowed by the theory, i.e. we would get zero inaccuracy for each individual measurement.

Plotting the results of our measurements, we will find the distribution of the measurement results for both operators, $\hat{A}$ and $\hat{B}$, which have normal width of $\triangle A$ and $\triangle B$ respectively. One will find that regardless of the individual results, we always find $\triangle A \triangle B \geq \frac{1}{2}$. This is the uncertainty relation that quantum theory refers to, and it holds between the distribution of measurement results of two non-commuting operators A and B even when there is no theoretical limitation on the accuracy of individual independent measurements.

However, if the experimenter uses devices with a deficiencies which prevent them from registering the results accurately, then the results of each measurement would be inaccurate; so, the results of each observable A and B are inaccurate with inaccuracies depending on devices. This is the familiar inaccuracy which appears here as well, however this type of inaccuracy seems to be avoidable in classical physics.

In other words inaccuracies happen on individual measurements and are related to the measuring devices and the measuring procedures, where uncertainties are related to the system under study itself and are evident practically on an ensemble of identically prepared systems.

For measurements of usual quantum observables there are no such limitations on inaccuracies. However we will find that for quantum time measurement, one has to conduct the experiment inaccurately in order to have a successful registration, and attempting to improve the time accuracies above a certain amount would result in no registration. In quantum time measurements, we will see that the theory excludes such accurate time measurements and yield to an aborted experiment. Such limitation is inherently different from the Heisenberg uncertainty type limitations that we normally encounter in quantum mechanics.

8.2 Different Time-Energy Relation Derivations in Quantum Mechanics

Since the advent of quantum mechanics, the time-energy relation has had a different basis than the standard position-momentum[16] uncertainty relation

$$\triangle q \triangle p \geq \frac{1}{2}$$

In quantum mechanics time is a parameter, not an operator as the position q, therefore the usual quantum mechanical approach of deriving the uncertainties can not be used to derive uncertainties in time, if such a thing ever exists (some authors have argued that such uncertainty relations do not exist for a parameter like time, and any such derivation is erroneous and not general enough [10], [3]).

Time-energy relations, derivations and interpretations, can be categorized mainly in the several classes that follow[17]. We address different derivations of such time-energy relations and try to elaborate on their various interpretations for the quantum measurement theory purposes.

8.2.1 Spread of the Wave Packet

From the Fourier transform properties between conjugate variables we know that a wave packet with the frequency width of $\triangle\omega$ has a spread $\triangle t$ in the conjugate coordinate which obeys the relation $\triangle\omega \triangle t \geq \frac{1}{2}$. Using this alongside with Planck relation $E = \hbar\omega$, there is a time-energy relation on wave packet spread that reads

$$\triangle E \triangle t \geq \frac{1}{2} \tag{8.1}$$

In this derivation, however we should note that the wave function is not

[16] This relation is between the eigenvalues of position and the eigenvalues of momentum, not between the sole position and momentum coordinates; in the same token if we measure the time by reading some operator's eigenvalue, it is not strange to expect similar relation on time readings.

[17] This list was partially done before in [17].

a physical identity in real space, but rather a mathematical representation for information associated to the particle; so we can not say for example that $\triangle t$ is the time that the wave function sweeps a point in space, or any similar justifying notion.

Writing the wave function as

$$\psi(q,t)=\sum_E c_E\psi_E(q)e^{-iEt}$$

where $\psi_E(q)$ is the energy eigenvectors of the Hamiltonian of the system, and c_E's are coefficients of those eigenvectors in the wave function, we find that given a wave packet of width $\triangle E$ in energy space, it follows that $\triangle t$ can be interpreted as the time within which the wave packet does not change "significantly" [21].

The above interpretation for $\triangle t$ is loose; we should also bear in mind that for a single particle the wave function does not have any ontological significance other than its usual epistemological role [11]. The above relation at best can only be interpreted as a Heisenberg uncertainty type limitation on an ensemble of similarly prepared systems.

8.2.2 Reducing to Position-Momentum Relation

Having Heisenberg uncertainty relation on momentum and position, using group velocity definition $v=\frac{\partial E}{\partial p}$, if we take $\triangle E=\frac{\partial E}{\partial p}\triangle p=v\triangle p$ and $\triangle t=\frac{\triangle q}{v}$ we can write

$$\triangle E\triangle t=\triangle q\triangle p\geq\frac{1}{2} \tag{8.2}$$

The general implication of this derivation of the relation is not clear at all, and the derivation can only be justified on a few measurement models in quantum mechanics [15] where the time parameter is associated linearly with the position eigenvalue q [since it is assumed $\triangle t=\frac{\triangle q}{v}$]; and clearly this is not the case in general.

8.2.3 Variability Time

We can also consider time evolution of an arbitrary quantum operator $\hat{A}$ under Heisenberg equation of motion, $\frac{d\hat{A}}{dt} = -i[H, A]$; using standard operator algebra for two non-commuting operators $\hat{A}$ and $\hat{H}$ we have $\triangle H \triangle A \geq \frac{1}{2} \left| \left\langle \frac{dA}{dt} \right\rangle_{av} \right|$. Defining $\triangle\tau_{\hat{A}} = \triangle A \left\{ \left\langle \frac{d\hat{A}}{dt} \right\rangle_{av} \right\}^{-1}$ we can have a "time"-energy relation [12] that reads

$$\triangle E \triangle \tau_{\hat{A}} \geq \frac{1}{2} \tag{8.3}$$

In the above relation although $\triangle E$ is the standard measure of the statistical spread of energy, $\triangle\tau_{\hat{A}}$ is not any statistical spread or inaccuracy of our observable, but rather a characteristic time for variability of the observable $\hat{A}$. This variability time, $\triangle\tau_{\hat{A}}$, is not a time parameter in any sense, only having time dimension. From its definition, it is clear that $\triangle\tau_{\hat{A}}$ is an ensemble averaged quantity, i.e an uncertainty relation, and the above relation is not an accuracy-type energy-time relation.

8.2.4 Wigner Derivation

Wigner [13] has considered two weighted probability values [variances] of time and energy with respect to some fixed values t_0 and E_0 as

$$\tau^2(q) = \frac{\int (t - t_0)^2 \left|\psi(q,t)\right|^2 dt}{\int \left|\psi(q,t)\right|^2 dt}$$

which is weighted by the wave function at a fixed q, $\psi(q,t) = \langle q|\Psi(t)\rangle$; and

$$\epsilon^2(q) = \frac{\int (E - E_0)^2 \left|\eta(q,E)\right|^2 dt}{\int \left|\eta(q,E)\right|^2 dt}$$

weighed with respect to the Fourier transform of the same wave function, $\eta(q,E) = \int \psi(q,t) e^{iEt} dt$.

He showed that the relation

$$\tau(q)\epsilon(q) > \frac{1}{2} \tag{8.4}$$

holds at constant q, since $\psi(q,t)$ and $\eta(q,E)$ are Fourier transforms of each other.

In general t_0 and E_0 are not the average values of the time and energy. They are some fixed reference points to evaluate the moments with respect to. Thus ϵ^2 and τ^2 should not in general be identified with $\triangle E$ and $\triangle t$.

Following the derivation, Wigner mentions that "the uncertainty relation does not apply to time and energy *in abstracto* but to the life-time of a definite state of a system". His interpretation of $\tau(q)$ is "spread in time of presence at a definite quantum mechanical state".

From the derivation of this "spread-time"-energy relation, it is implied that the above relation is an uncertainty type limitation over an ensemble of identically prepared systems.

8.2.5 Geometric Method

Anandan and Aharonov [16] have derived another time-energy relation by applying geometrical concepts to the quantum mechanics.

Using Schrodinger equation, $i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$, we can Taylor expand $|\psi(t+dt)\rangle$ to second order in dt as

$$|\psi(t+dt)\rangle = |\psi(t)\rangle - \frac{idt}{\hbar}H|\psi(t)\rangle - \frac{dt^2}{2\hbar}\left(i\frac{dH}{\hbar}|\psi(t)\rangle + \frac{1}{\hbar}H^2|\psi(t)\rangle\right) + O(dt^3)$$

to get

$$|\langle\psi(t)|\psi(t+dt)\rangle|^2 = 1 - \triangle E^2 dt^2 + O(dt^3) \tag{8.5}$$

with $(\triangle E)^2 = \langle H^2\rangle - \langle H\rangle^2$ as usual definition.

They define a metric on a general n-dimensional Bloch type sphere to measure the distance between states:

$$ds^2 = g_{\mu\bar{\nu}}dZ^\mu d\bar{Z}^\nu \equiv 1 - \left|\left\langle\psi|\acute{\psi}\right\rangle\right|^2$$

where Z^μ's are state coordinates on that state space. It is a well defined metric, recall that the distance of any state from itself is zero, while it has maximum distance with any orthogonal state which is 1.

To calculate the general form of the metric, we consider two state vectors $|\psi\rangle$ and $\left|\acute{\psi}\right\rangle = |\psi\rangle + |d\psi\rangle + \frac{1}{2}\left|d^2\psi\right\rangle + O(\left|d^3\psi\right\rangle)$. Differentiating the normalization relation $\langle\psi|\psi\rangle = 1$ gives $\langle\psi|d\psi\rangle + \langle d\psi|\psi\rangle = 0$ and differentiating once more gives $2\langle d\psi|d\psi\rangle + \left\langle\psi|d^2\psi\right\rangle + \left\langle d^2\psi|\psi\right\rangle = 0$. Therefore

$$1 - \left|\left\langle\psi|\acute{\psi}\right\rangle\right|^2 = \langle d\psi|d\psi\rangle - \langle\psi|d\psi\rangle\langle d\psi|\psi\rangle$$

Thus

$$g_{\mu\bar{\nu}} = \langle\partial_\mu\psi|\partial_{\bar{\nu}}\psi\rangle - \langle\psi|\partial_\mu\psi\rangle\langle\partial_{\bar{\nu}}\psi|\psi\rangle$$

In the two-dimensional case the metric reduces to

$$ds^2 = \frac{dZd\bar{Z}}{Z\bar{Z}} = \frac{dx^2 + dy^2}{r^2} = d\theta^2 + \sin^2\theta d\phi^2$$

which is the metric on the Bloch sphere. Representing a two dimensional state on the Bloch sphere is by assigning to any state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ a point on the sphere with the coordinate (θ, ϕ).

As we sketch the proof in Appendix B they show that the minimum distance in time-geodesic on the Bloch type sphere that should pass for an arbitrary state in order to get to an orthogonal state satisfies the relation $t_\perp \bigtriangleup E = \frac{\pi}{2}$.

This relation means that in general an arbitrary state can only evolve to an orthogonal state in times $t_\perp$ which satisfy the relation:

$$t_\perp \bigtriangleup E \geq \frac{\pi}{2} \tag{8.6}$$

The above relation holds for any individual state and is not an uncertainty type time-energy relation. It puts a lower bound on the time that any individual state can evolve to its orthogonal state *dynamically.*

8.2.6 Orthogonality Time

Margolus and Levitin [14] have calculated another bound on the minimum time needed for any state of a given system to evolve into an orthogonal state. This "orthogonality time" bound looks similar to the the former time-energy relations that we discussed.

We follow the original derivation here: Consider an arbitrary quantum state which can be written as a superposition of energy eigenstates ,

$$|\psi(0)\rangle = \sum_n c_n |E_n\rangle$$

For simplicity assume system has discrete energy spectrum, and choose lowest energy level $E_0 = 0$ therefore

$$E_n > 0 \quad \forall n > 0.$$

After some arbitrary time t, the state evolves to

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |E_n\rangle$$

Let

$$S(t) = \langle\psi(0)|\psi(t)\rangle = \sum_n |c_n|^2 e^{-iE_n t}$$

and look for limits on smallest time which makes $S(t) = 0$. Looking at the real part of $S(t)$, we get

$$\begin{aligned} Re(S) &= \sum_n |c_n|^2 \cos(E_n t) \geq \sum_n |c_n|^2 \left(1 - \frac{2}{\pi}(E_n t + \sin(E_n t))\right) \\ &= 1 - \frac{2\langle E\rangle_{av}}{\pi} t + \frac{2}{\pi} Im(S) \end{aligned}$$

where we have used the relation

$$\cos\alpha \geq 1 - \frac{2}{\pi}(\alpha + \sin\alpha) \qquad \forall \alpha > 0$$

For the desired time t_{orth} we should have both $Re(S) = 0$ and $Im(S) = 0$, so the above equation yields to

$$\langle E\rangle_{av} t_{orth} \geq \frac{\pi}{2} \tag{8.7}$$

This shows there exists a bound on orthogonality time of any quantum state with mean energy $\langle E\rangle_{av}$ (setting lowest energy level to zero[18]). Any state with mean energy $\langle E\rangle_{av}$ can not evolve dynamically to an orthogonal state in times less than $\tau \sim \frac{1}{\langle E\rangle_{av}}$. Note that we have always $\langle E\rangle_{av} > 0$ since $\langle E\rangle_{av}$ is an indicator of the energy spread of the quantum state; if a quantum state has zero energy spread it is in an energy eigenstate and thus does not change its state under the action of a time independent Hamiltonian.

This bound holds for each individual state but is not an uncertainty type time-energy relation. Clearly in cases of limited state space dimension (like a spin with angular momentum l) we would have cycles of orthogonal states as time passes (different cycles are possible for different subsets of states).

8.3 Time-Energy Relations Comparison

As we see there are a number of different time-energy relations derivations, whose results may look similar in the form but are not necessarily equivalent. It is common to see in books that they claim to prove an uncertainty relation while the proof actually demonstrates one of the time relations (8.7) or (8.6) (see for example [21]).

There is a clear distinction among these time-energy relations. The relations (8.1), (8.2), (8.3) and (8.4) are at best statistical relations on an ensemble of identically prepared quantum systems, resembling the position-momentum relation, and have no implication on individual measurements; however the last two relations (the geometric method (8.6) and the orthogonality time (8.7) derivations) are special time-energy relations which are fundamentally different from the previous ones.

These two relations are universal and can stand on equal footing with the position-momentum uncertainty relation; and in contrast to the other time-energy relations, interpretations of these two relations are clear and

[18] We can repeat the derivation by choosing the highest energy level of the system equal to zero if the energy is bounded from above. Thus depending on the energy spectrum of the system, we can choose to set $\langle E\rangle_{av} = E_{max} - E_{mean}$ or $\langle E\rangle_{av} = E_{mean} - E_{min}$. Using the smaller one in the orthogonality time relation gives the better estimate of the orthogonality time.

exact.

Equations (8.6), (8.7) refer to the minimum time needed for evolving of any system from a quantum state to the orthogonal one. They both put a limit on the dynamical time[19] needed for such a change. This time is not directly measurable though, due to the collapse of the wave function; any attempt to observe the system, collapse its wave function in a non-dynamical way and destroys the natural time evolution.

Those two equations may look inconsistent though, since it is possible to have an energy spectrum with quite different $\langle E \rangle_{av}$[20] and $\triangle E$ values. For a simple two level system in which the orthogonality time is directly calculable both equations produce the exact result and are equivalent since $\triangle E$ and $\langle E \rangle_{av}$ are equal. For systems with energy spectrum with different $\langle E \rangle_{av}$ and $\triangle E$ the possible inconsistency fades if we recall that both of these equations put only **lower limits** on the orthogonality time, and in all the cases we have

$$Max\left(\frac{\pi}{2 \triangle E}, \frac{\pi}{2 \langle E \rangle_{av}}\right) \leq t_{orth} < \infty \tag{8.8}$$

Choosing the non-trivial bound (which gives higher result) should not be hard if we have the system's Hamiltonian and so its energy spectrum. We only need to take the smallest among $\triangle E$ and $\langle E \rangle_{av} = E_{max} - E_{mean}$ or $\left\langle \acute{E} \right\rangle_{av} = E_{mean} - E_{min}$ (see footnote 18).

[19] Clearly these relations are not related to the strange collapse of the wave function; in that yet "non-dynamical" phenomenon any state can be projected to an orthogonal state at anytime with some probability that can be calculated by Born formula in quantum mechanics.

[20] Average energy is only meaningful if we set an origin for the zero energy; as follows from the derivation, we set the zero energy as the lowest energy of the system in the derivation of the orthogonality time relation.

Chapter 9

Quantum Time Measuring Devices

In the previous chapter we considered different time-energy relations and possible limitations they may bring about, such as the limitations on the time that a state can evolve to an orthogonal state. In this chapter equipped to those relations we apply relevant ones to find possible limitations they impose on the quantum measuring devices.

In the pragmatic view, without having a measuring device to measure a system parameter, that parameter has no "real" meaning. Therefore, in order to discuss quantum time measurements it is necessary to construct realistic models of time measuring devices, which we shall call by the general name *clock*, and study their behaviour in relation to the quantum systems.

As we saw, time is not an observable in quantum theory. Therefore we should make models of devices that have some observable states changing linearly with the passage of time; we measure the time then by making measurements on those observable states of the clock, which we generally refer to as "pointer states".

9.1 Quantum Models for Clocks

A physical clock is a system whose pointer state changes with time in a linear way. Physical clocks which we use in everyday life are periodic systems; the way we measure the time with them is by enumerating the number of full periods besides the clock pointer's reading, namely 7:29 AM means 7 full rotation of minute pointer and 29/60 of a full rotation after the midnight. By having clocks which count different time periods- years, months, days, hours, minutes, seconds, etc...- we determine time. This makes it possible to have a convenient clock with limited dimension.

There are two class of clocks, linear clocks which continuously measure the time and periodic clocks which their reading jumps and resets after they rich the maximum time that they can hold.

9.1.1 Linear Clocks

The simplest model for a clock can be a massive free particle which is moving with constant speed. The position of the particle changes linearly with time, so if we measure its position we can deduce the time.

Ideal linear clock has the Hamiltonian

$$H_{clock} = P_y$$

with y the pointer.

The above Hamiltonian can be constructed from the free particle Hamiltonian $H = \frac{P^2}{2m}$. If m is very large taking $\tilde{P} = P_y - P_0$, the Hamiltonian can be written as $H = \frac{P_0^2}{2m} + \tilde{P}\frac{P_0}{m} + \frac{\tilde{P}^2}{2m}$. If we only look at States with $\left\langle \tilde{P}^2 \left(\frac{P_0}{m}\right)^2 \right\rangle \gg \left\langle \tilde{P}^4 \right\rangle$ for the period of measurement, then we can neglect the $\tilde{P}^2$ term and we have linear clock [19].

In order to read the time we measure the coordinate y conjugate to P_y. The clocks linearity is evident by considering the Heisenberg equation of motion for variable y,

$$y(t) - y(t_0) = -i \int_{t_0}^{t} [H_{clock}, y]\, dt = t - t_0.$$

For the initial state of the clock's pointer of the form of $f(y)$ we can calculate the time evolution of the pointer as

$$\begin{aligned} f(y,t) &= e^{-iHt} f(y,0) \\ &= e^{-iHt} \int F(p)\, e^{ip_y y} dp_y = \int e^{-ip_y t} F(p)\, e^{ip_y y} dp_y \\ &= \int F(p)\, e^{ip_y (y-t)} dp_y = f(y-t,0)\,. \end{aligned}$$

Therefore due to this Hamiltonian, any pointer state keeps its form exactly whilst changes position; that means the clock accuracy does not change by time.

The Hamiltonian for this ideal clock is unbounded and therefore it is possible to find a time operator for it. The inaccuracy of time measurements by this clock is given by $\delta t = \delta y$. As we discussed in section 8.1 there is no limitation on any *individual* measurements of the position coordinate and therefore the readings of the clock.

However in order to read the time accurately we need to make the initial state of the clock's pointer close to an eigenstate of $y(t_0)$; the spread of this initial value will be an inaccuracy in the clock. Keeping in mind that the Hamiltonian does not commute with position, we see that demanding an accurate reading cause a disturbed Hamiltonian and a great uncertainty in H_{clock}.

In the case of an *ideal* linear clock model we are able to have a "time operator" whose readings represent the time for us; however, in general systems with bounded Hamiltonian we can not find such operators or such operators do not evolve linearly with time.

9.1.2 Periodic Clocks

Asher Peres [20] considered another model for the quantum clock, using the Larmor precession of spin states to register the time. These clocks are periodic and their resolution can be made arbitrarily fine by using bigger spin state systems.

Consider a state of this clock given by this normalized state vector

$$|\psi(t)\rangle = (2j+1)^{-1/2} \sum_{n=-j}^{j} e^{-in\omega t} |n\rangle$$

which $|n\rangle$'s are orthogonal eigenstates of J_z , ω is a positive constant and j is a large positive integer or half integer. The clock Hamiltonian $H = \omega J_z$ satisfies $H_c |n\rangle = n\omega |n\rangle$ where $n = -j, -j+1, \cdots, j$. The above state vector evolves with time by going through the sequences of the orthogonal basis vectors, $|\phi_m\rangle$, which are defined by:

$$|\phi_m\rangle \quad = \quad (2j+1)^{-1/2} \sum_{n=-j}^{j} e^{-i\frac{2\pi nm}{2j+1}} |n\rangle \qquad m = -j, \cdots, j$$

Writing the state functions in the periodic angular space bases $\langle\theta|n\rangle = (2\pi)^{-1/2} e^{in\theta}$ and $\phi(m,\theta) = \langle\theta|\phi_m\rangle$, the above can be written as:

$$\phi(m,\theta) = \frac{1}{\sqrt{2\pi N}} \frac{\sin\left(\frac{N}{2}\left(\theta - \frac{2\pi m}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\theta - \frac{2\pi m}{N}\right)\right)} \qquad N = 2j+1$$

which can be regarded as pointer states of the clock pointing towards $\theta = \frac{2\pi m}{N}$ direction.

Time reading can be done by reading the "clock time" operator

$$\mathbb{T}_c = \tau \sum_m m |\phi_m\rangle \langle\phi_m|$$

where τ is the time resolution of the clock.

The time resolution of this clock can be related to other parameters considering $\omega = 2\pi/(2j+1)\tau$, that gives us the time resolution of $\tau = \frac{2\pi}{(2j+1)\omega}$. This amount of time is also the time that a pointer state ϕ_m goes to the next orthogonal state, i.e. it is the orthogonality time.

The clock shows the true time if the time measurement happens to be at times $t = m\tau$. In other times with a high probability the clock pointer points to one of the adjacent pointer times. The probability of finding value

$m\tau$ at time t falls as

$$P(m,t) = |\langle\phi_m|\phi_t\rangle|^2 = \left(\frac{1}{N}\sum_{n=-j}^{j} e^{-in\frac{2\pi}{N}\left(m-\frac{t}{\tau}\right)}\right)^2$$
$$= \frac{1}{N^2}\frac{\sin^2\left(\frac{N}{2}\left(\frac{2\pi}{N}\left(m-\frac{t}{\tau}\right)\right)\right)}{\sin^2\left(\frac{1}{2}\left(\frac{2\pi}{N}\left(m-\frac{t}{\tau}\right)\right)\right)} \tag{9.1}$$

[Fig.9.1] which has a spread that is proportional to $\frac{1}{N}$.

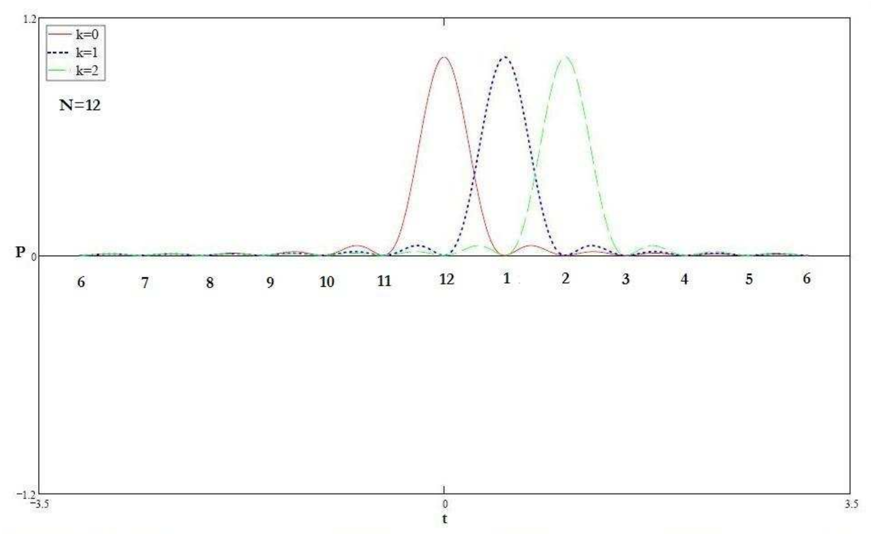

Figure 9.1: Periodic clock's time reading probability for the case of N=12. The curves show the probability distributions of getting the first 3 pointers with time.

To calculate the time uncertainty in our measurements we need to calculate $\langle\mathbb{T}_c\rangle_t = \sum_m m\tau P(m,t)$ and $\langle\mathbb{T}_c^2\rangle_t = \sum_m m^2\tau^2 P(m,t)$ at arbitrary times and get $\triangle\mathbb{T}_c = \sqrt{\langle\mathbb{T}_c^2\rangle_t - \langle\mathbb{T}_c\rangle_t^2}$ which is plotted for the case $j = 6$ in Fig.9.2.

It appears in the plot that the maximum uncertainty in the time measurement depends on when we measure the time. This rather unexpected result is due to the clocks time discontinuity which jumps from time $t = j$ to $t = -j$. We can however, always redefine our nearest clock pointer as $t = 0$

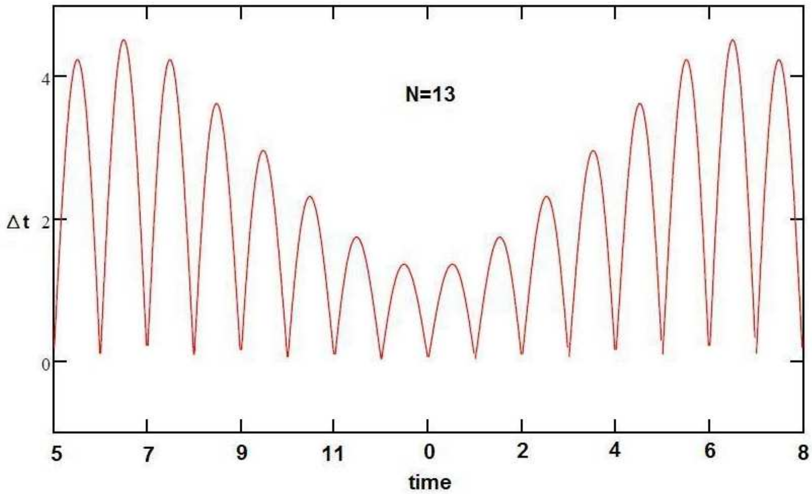

Figure 9.2: Uncertainty in time measurement $\triangle t$ for the case $N = 2j+1 = 13$

pointer. Thus the true amount of the worst time uncertainty is the lowest peak in the plot . The uncertainty of time measurements clearly is zero for the times which $t = m\tau$.

A straightforward calculation of the commutator $[\mathbb{T}_c, H_c]$ shows that it does not satisfy $[\mathbb{T}_c, H_c] = i\hbar$ as it might be expected. The calculation gives

$$\langle\phi_n| [\mathbb{T}_c, H_c] |\phi_m\rangle = i\hbar \frac{2\pi i(n-m)/N}{1 - exp\,(2\pi i(n-m)/N)} \tag{9.2}$$

Having $\triangle E = (\omega/\sqrt{3})(j^2 + j)^{1/2}$ and $\langle E\rangle_{av} = j\omega$ we can verify time-energy relations (8.6), (8.7) . Equation (8.6), in the large j limit, gives us $dt_\perp \geq \frac{\pi\sqrt{3}}{2\omega j}$ for the minimum time that a pointer state can evolve to another orthogonal state, while equation (8.7) gives us $t_{orth} \geq \frac{\pi}{2\omega j}$ for that amount. So in this model clock we verify that $\tau > max(dt_\perp, t_{orth})$ as is expected from our constructed lower bounds on the orthogonality time in the previous chapter.

9.2 General Properties of Quantum Clocks

As we mentioned earlier quantum time is being measured by means of observing the dynamical behaviour of a "clock" system. Quantum clocks are quantum systems with definite observable pointer states being able to register the time of an event. Pointer states by definition pass through a succession of states at constant time intervals under dynamics of the clock. We discussed the main two models of quantum clocks in the previous section. Nevertheless there are certain properties that are general for all physical clocks regardless of their detail.

9.2.1 Clock's Resolution

A quantum clock is characterized as a system that passes through a sequence of distinguishable pointer states at equal time intervals. Pointer states of clocks should, by definition, evolve by the clock resolution time [clock's accuracy] to the next immediate orthogonal state. The clock can not be used to register times shorter than its orthogonality time in a reliable way. This implies that the best time resolution of the clock, τ, is equal to the orthogonality time of its pointer states.

In the intermediate times between the clocks pointer times, clock's reading is inaccurate. This is due to the fact that the quantum clock is a discrete time registrar. The clock reading however, most probably will be one of the neighbouring pointer times, since the contributions from the far pointer states are negligible and the probability of getting them falls off fast (cf. Eq (9.1)).

9.2.2 Clock's Energy

Ideal clock systems have no bound on energy, and therefore, as we saw, it is in principle possible to construct a time operator for them. On the other hand, physical clocks should have lower bounds in their energy in order to be stable, and assigning an operator to measure the time is not possible. However, it is possible to use physical clocks to approximate ideal ones, having both the duration that the clock should operate and the accuracy needed [30].

As we mentioned, the clock's best resolution τ is the same as its orthogonality time by definition, and this consideration along with the relations we found on the orthogonality time, gives us a bound on the minimum mean energy of the clock system and its accuracy. Thus using equation (8.7) we can write

$$\tau \geq \frac{\pi}{2\left\langle E_{clock}\right\rangle_{av}} \tag{9.3}$$

where $\left\langle E_{clock}\right\rangle_{av}$ is clock's mean energy provided setting its lowest populated energy level zero. This shows that there is a direct proportionality between the accuracy of the clock and its energy; the more accurate the time measurement we want, the more energetic the clock we need to use. This general rule has not been explicitly proved previously in the literature; although has been verified studying clock models on a case by case basis.

There is another relation derived from equation (8.6), which puts a bound on the clock's normal energy width and its accuracy

$$\tau \geq \frac{\pi}{2 \triangle E_{clock}} \tag{9.4}$$

Therefore the more accurate the clocks the more uncertain their energy.

If we consider the clock energy spread around its mean-energy, it spans at least by

$$\triangle\mathcal{E}_{clock} = Max\left(\left\langle E_{clock}\right\rangle_{av}, \triangle E_{clock}\right). \tag{9.5}$$

Thus highly accurate clocks have large energy spread around their mean energy.

9.2.3 A "Time"-Energy Relation

As mentioned before in order to read the quantum time we use the clock pointer states. Quantum clocks can be used to read the time for just a single reading; from this aspect quantum clocks work like stopwatches. Obviously the time readings can have statistical spread due to quantum nature of the pointer state and that has nothing to do to the nature of the time.

Here is a good place to make the distinction between the TIME that we read in clocks and the TIME which is a coordinate in our four dimensional space-time. A similar distinction has been made clearly for space, as we distinguish the result of location measurements by *position* "q" contrasting with the coordinate component of location which we refer to as "x". Also it is important to emphasize that there is no relation between the space-time coordinates and the properties of a system. Though it looks trivial, it is widely ignored that there is a fundamental difference between the position "q" and the coordinate "x": there exists a limiting relation between eigenvalues of the position and eigenvalues of the momentum for a certain system, not for the space-time coordinates.

Unfortunately, unlike the case of the position "q" and the coordinate "x", there is not a clear distinction when physicists refer to the time as a coordinate and when they refer to it as a clock reading; they both have been shown by the symbol t which is confusing when they are not distinguished clearly in discussions.

Let us represent the coordinate component of time by t and call it time, while the result of time measurement shall be called *tempo,* $\mathbb{T}$, which is the result of the reading of the quantum pointer state vector, ϕ. We show there exists an uncertainty type relation between readings of the clock pointer state, tempo, and its energy, similar to the position-momentum uncertainty relation.

In the most general form of the clock's Hamiltonian, in order to have pointer states which evolve linearly with time, we should have a Hamiltonian that reads as

$$H_{clock} = -i\frac{\partial}{\partial\phi} \tag{9.6}$$

Having such Hamiltonian, our quantum pointer state vectors ϕ, evolve linearly with time and can be used for the time measurements.

Now a relation between normal width of pointer state readings, tempo, and the clock's energy follows straightforward from the general uncertainty principle:

$$\triangle H_{clock} \triangle \phi \geq \frac{1}{2} |[H_{clock}, \phi]|$$

$$\triangle E_{clock} \triangle \mathbb{T} \geq \frac{1}{2}$$

Therefore, there exists an uncertainty relation between tempo and clock's energy, quite similar to Heisenberg position-momentum uncertainty relation; calling this relation a time-energy relation is a misinterpretation, though.

This tempo-energy uncertainty relation can be used in experimental results on ensembles of both identically prepared clocks and systems; using it over readings of a single clock is meaningless.

We can see that the normal width of the pointer state readings is of the same order of magnitude as the clock's accuracy, as it should be expected [cf. Eq(9.4)].

In the above derivation of the tempo-energy uncertainty relation, we should note that the relation $[\phi, H_{clock}] = i\hbar$ holds only if we define the range of ϕ from $-\infty$ to ∞. In the subspace of periodic functions with the period 2π, like in our periodic clock setup the amount of the commutator depends on states as we already mentioned in Eq(9.2).

9.2.4 Other Fundamental Limitations on Quantum Clocks

Salecker and Wigner [31] showed, by applying quantum limitations and relativity principles, that there are limits on the clock's mass and duration of the measurement. They showed the mass of the clock should exceed a certain value which depends on the accuracy of the clock and the time interval which clock is used, as well as the size of the clock.

The ingredients to drive those physical limits are general principles of quantum mechanics and general relativity which should be satisfied in the low-energy regime of space time. Since we like to confine our discussion in this thesis only to pure quantum mechanical aspects of the problem, we mention those derivations in the Appendix C for the sake of completeness.

Chapter 10

Quantum Time Measurements

Until now we have considered the dynamics of clocks on their own; however in measuring the time we use the clocks which are connected to the systems under study. So we need to consider the system-clock compound dynamics and study the effects that the coupling would have on the quantum system. Here as we will see, the main concern is energy considerations for the clock-system complex. We study how introducing a clock to a quantum system may affect the dynamical evolution of that system and its time variables.

10.1 Measuring With a Clock

As we saw, the energy spread of the clock is directly related to its maximum time resolution (equation (9.3)): $\langle H_{clock} \rangle \geq \frac{1}{\tau}$, so for any desired accuracy there is a minimum energy spread and uncertainty in energy (equation (9.4)) required for the clock, regardless of its internal structure.

We take $g_{interaction}$ to represent certain conditions where the clock start working. For example $g_{int} = \theta(x - x_0)$ represent the case where the clock works only if $x \geq x_0$. Our total Hamiltonian then can be written as

$$H_{tot} = H_{sys} + \alpha g_{int} H_{clock}.$$

To consider the rate of the clock pointer's evolution with time, we may use Heisenberg equation of motion for the clock Hamiltonian (9.6). Thus, we will have

$$\left|\frac{\partial \phi}{\partial t}\right| = |[H_{tot}, \phi]| = |[H_{clock}, \phi]| = \left|\alpha\left[\frac{\partial}{\partial \phi}, \phi\right]\right| = |\alpha|\,,$$

for the evolution rate of the clock's pointer. We already know that we can adjust the rate that the clock registers the time by setting the clock's energy span, equations (9.3) and (9.6). The above relation shows that we can also adjust the rate that the clock registers the time by setting α in the coupling term of H_{tot} . This property is interesting since it implies that in physical systems, a time measurement with a highly accurate clock loosely coupled to the system is equivalent to a time measurement with an almost inaccurate clock; however this clock has a big coupling within the system.

So the overall accuracy of time measurement experiments relates to both the energy spread of the clock and the coupling constant as $\alpha \langle E_{clock} \rangle \geq \frac{1}{\tau}$. Without loss of generality, we will keep $\alpha = 1$ and attribute the accuracy of the time measurements only to an *equivalent* clock with $\langle E_{clock} \rangle$ equal to $\alpha \langle E_{clock} \rangle$.

10.2 A Toy Model: Spin System Coupled to a Linear Clock

To see the effect of coupling a linear clock to a quantum system, we study a simple exactly solvable model.

For a simple 2-level quantum spin system with the Hamiltonian $H = \sigma_x$, we know that for the initial state of $|\psi_0\rangle = |\downarrow_z\rangle$, we get the spin precession around the x axis and the spin points towards the same initial direction half the times in average.

If we couple a linear clock to such a system, we can exactly solve the compound clock-spin system, and see the effect of the accuracy of the coupled clock in the afore mentioned observation that is the time that the spin on average points to the initial direction. As we will find coupling an accurate clock with high energy uncertainty to the system, will stop the time evolution of the spin systems.

We can write our 2-level spin system coupled to a clock Hamiltonian according to:

$$H = \frac{\alpha}{2}\sigma_x + \left(\frac{1+\sigma_z}{2}\right)p.$$

Therefore the clock works only if the system is in the upper spin state, $|\uparrow_z\rangle$, and stops otherwise. The Hamiltonian can be written as

$$\begin{aligned} H &= \frac{p}{2}I + \frac{\alpha}{2}\sigma_x + \frac{p}{2}\sigma_z = \\ & \frac{p}{2}I + \frac{\Omega}{2}\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right) \end{aligned}$$

where

$$\Omega = \sqrt{p^2+\alpha^2}$$

and

$$\cos\theta = \frac{p}{\Omega}$$

and

$$\sin\theta = \frac{\alpha}{\Omega}.$$

The time evolution follows

$$\begin{aligned} |0_p(t)\rangle &= e^{-iHt}\,|0_p(0)\rangle \\ &= e^{-i\left(\frac{p}{2}I+\frac{\Omega}{2}(\cos\theta\sigma_z+\sin\theta\sigma_x)\right)t}\,|0\rangle\,|p\rangle \\ &= e^{-i\frac{p}{2}t}\left(\cos\frac{\Omega t}{2}I - i\sin\frac{\Omega t}{2}\left(\cos\theta\sigma_z+\sin\theta\sigma_x\right)\right)|0\rangle\,|p\rangle \end{aligned}$$

which result in

$$|0_p(t)\rangle = e^{-i\frac{p}{2}t}\left(\left(\cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta\right)|0\rangle - i\sin\frac{\Omega t}{2}\sin\theta\,|1\rangle\right)|p\rangle\,.$$

For the full clock system we can write

$$|0_{clock}(t)\rangle = \int dpF(p)e^{-i\frac{p}{2}t}\,|p\rangle\left(\left(\cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta\right)|0\rangle - i\sin\frac{\Omega t}{2}\sin\theta\,|1\rangle\right)$$

We need to calculate

$$\langle 0_{clock}(t)|\, q \,|0_{clock}(t)\rangle = \langle 0_{clock}(t)|\, i\frac{d}{dp}\,|0_{clock}(t)\rangle$$

to find the average time the system spends in upper spin state. We have $\Omega = \sqrt{p^2+\alpha^2}$ and so $\frac{d}{dp} = \frac{d\Omega}{dp}\frac{d}{d\Omega} = \frac{p}{\Omega}\frac{d}{d\Omega}$.

We get for

$$\begin{aligned} i\frac{d}{dp}\,|0_{clock}(t)\rangle \;\; &= \;\; \frac{t}{2}\,|0_{clock}(t)\rangle - \int dpF(p)\,|p\rangle\, ie^{-i\frac{p}{2}t}\frac{t}{2}\frac{p}{\Omega}\sin\frac{\Omega t}{2}\,|0\rangle \\ &- \int dpF(p)\,|p\rangle\, e^{-i\frac{p}{2}t}\frac{t}{2}\frac{p}{\Omega}\cos\frac{\Omega t}{2}\cos\theta\,|0\rangle \\ &- \int dpF(p)\,|p\rangle\, e^{-i\frac{p}{2}t}\frac{1}{\Omega}\sin\frac{\Omega t}{2}\,|0\rangle \\ &- \int dpF(p)\,|p\rangle\, e^{-i\frac{p}{2}t}\frac{p^2}{\Omega}\left(\frac{-1}{\Omega^2}\right)\sin\frac{\Omega t}{2}\,|0\rangle \\ &+ \int dpF(p)\,|p\rangle\, e^{-i\frac{p}{2}t}\frac{t}{2}\frac{p}{\Omega}\cos\frac{\Omega t}{2}\sin\theta\,|1\rangle \\ &+ \int dpF(p)\,|p\rangle\, e^{-i\frac{p}{2}t}\frac{p}{\Omega}\left(\frac{-\alpha}{\Omega^2}\right)\sin\frac{\Omega t}{2}\,|1\rangle \end{aligned}$$

The non-vanishing terms which involve time -assuming a symmetric clock wave function in the momentum space- are

$$\begin{aligned} \langle q\rangle \;\; &= \;\; \frac{t}{2} - \frac{t}{2}\left\langle\left(\frac{p}{\Omega}\right)^2\sin^2\frac{\Omega t}{2}\right\rangle - \frac{t}{2}\left\langle\left(\frac{p}{\Omega}\right)^2\cos^2\frac{\Omega t}{2}\right\rangle \\ &- \left\langle\frac{1}{2\Omega}\sin\Omega t\right\rangle + \left\langle\frac{p^2}{2\Omega^3}\sin\Omega t\right\rangle \\ &= \frac{t}{2} - \frac{t}{2}\left\langle\left(\frac{p}{\Omega}\right)^2\right\rangle - \left\langle\frac{1}{2\Omega}\left(\frac{\alpha}{\Omega}\right)^2\sin\Omega t\right\rangle \end{aligned}$$

Considering a Gaussian distribution for the clock wave function

$$F(p) = \sqrt[4]{\frac{2}{\pi p_0^2}}\,e^{-\frac{p^2}{p_0^2}},$$

where p_0 is the clock energy uncertainty. The second term $\left\langle\left(\frac{p}{\Omega}\right)^2\right\rangle =$

$\int dpF^2(p)\left(\frac{p}{\Omega}\right)^2$ then can be integrated analytically as

$$\left\langle \left(\frac{p}{\Omega}\right)^2 \right\rangle = 1 + \sqrt{\pi}\frac{\sqrt{2}\alpha}{p_0} e^{\frac{\sqrt{2}\alpha}{p_0}^{\ 2}} \left(erf\left(\frac{\sqrt{2}\alpha}{p_0}\right) - 1 \right)$$

where $erf(x)$ is the error function and is defined as $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$.

Using the error function series expansion for the limiting cases of $p_0 \gg \alpha$ and $\alpha \gg p_0$, we get

$$\left\langle \left(\frac{p}{\Omega}\right)^2 \right\rangle == \begin{cases} 1 - \sqrt{\frac{\pi}{2}}\frac{2\alpha}{p_0} \longrightarrow 1 & p_0 \gg \alpha \\ \left(\frac{p_0}{2\alpha}\right)^2 \longrightarrow 0 & \alpha \gg p_0 \end{cases}$$

To Evaluate the third term we notice that for times $t \gg 1$ we can ignore this term due to fast oscillations of the sinusoidal term we can ignore this term safely. For short times this term adds a small oscillating value that can be estimated by recalling that $-1 \le \sin \Omega t \le 1$ and so $\left| \left\langle \frac{1}{2\Omega}\left(\frac{\alpha}{\Omega}\right)^2 \sin \Omega t \right\rangle \right| \le \left\langle \frac{1}{2\Omega}\left(\frac{\alpha}{\Omega}\right)^2 \right\rangle$. However we have

$$\left\langle \frac{1}{2\Omega}\left(\frac{\alpha}{\Omega}\right)^2 \right\rangle = \begin{cases} \left\langle \frac{1}{2\alpha} \right\rangle & p_0 \ll \alpha \\ \left\langle \frac{\alpha^2}{2p^3} \right\rangle & p_0 \gg \alpha \end{cases} \longrightarrow 0$$

and thus this term is negligible in both limiting cases.

Adding all terms together we get for the expectation time that our spin system initially in the down state, spends on the upper state to be

$$\langle q_t \rangle = \frac{t}{2}\sqrt{\pi}\frac{\sqrt{2}\alpha}{p_0} e^{\frac{\sqrt{2}\alpha}{p_0}^{\ 2}} \left(1 - erf\left(\frac{\sqrt{2}\alpha}{p_0}\right) \right)$$

as is plotted in Fig.10.1. In the two limiting cases, high accuracy clock, $p_0 \gg \alpha$ and low accuracy clock, $\alpha \gg p_0$, it simplifies to

$$\langle q_t \rangle \quad = \begin{cases} \frac{t}{2}\sqrt{\frac{\pi}{2}}\frac{2\alpha}{p_0} \longrightarrow 0 & p_0 \gg \alpha \ (\textit{high accuracy clock}) \\ \frac{t}{2}\left(1 - \left(\frac{p_0}{2\alpha}\right)^2\right) \longrightarrow \frac{t}{2} & \alpha \gg p_0 \ (\textit{low accuracy clock}) \end{cases} .$$

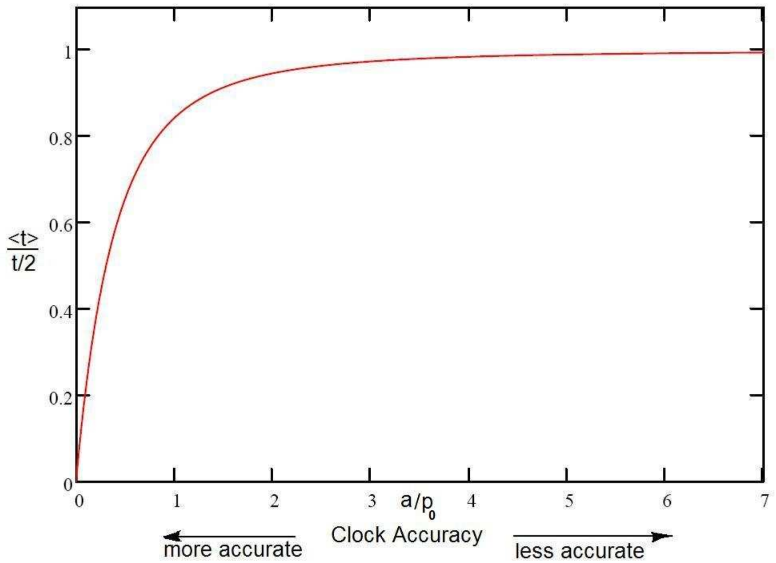

Figure 10.1: Exact form of the "normalized" expectation value of the measured time by a linear clock.
The clock with energy uncertainty p_0 is coupled to a two level spin system with energy a.

As it is visible from the plot in Fig.10.1, when the clock's energy uncertainty is getting higher than the systems energy, systems evolution slows down very dramatically. Therefore coupling a clock with energy uncertainty higher than the initial systems energy to the spin system, considerably slows down the spin system evolution.

We can also plot the time that the clock registers -the amount of time that the spin system spends in the upper state- versus the clocks energy uncertainty (see Figs.10.2, 10.3, 10.4 and 10.5 in the following pages). As we mentioned the system time evolution -with fast damping small oscillations-slows down more and more as we use more and more accurate clocks to observe its time evolution.

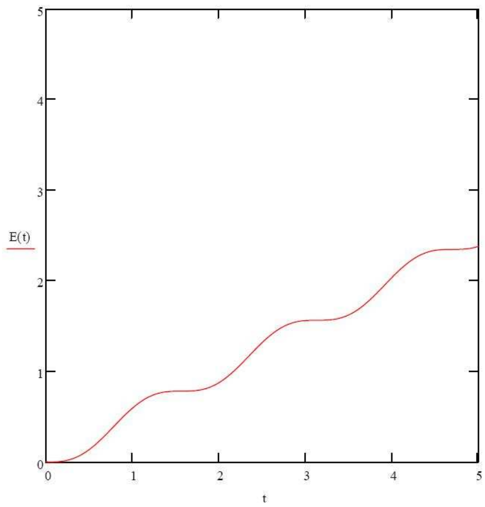

Figure 10.2: Case $p_0 = 0.1a$

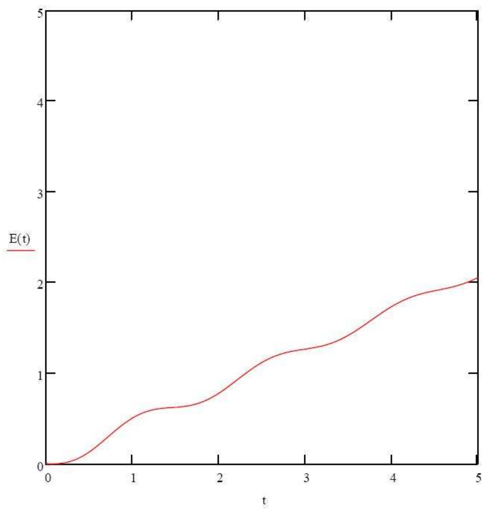

Figure 10.3: Case $p_0 = a$

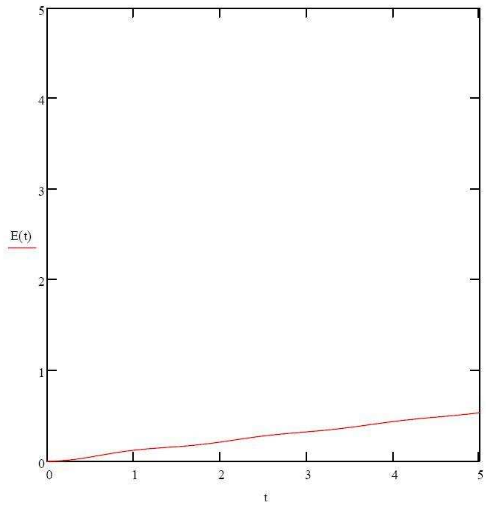

Figure 10.4: Case $p_0 = 10a$

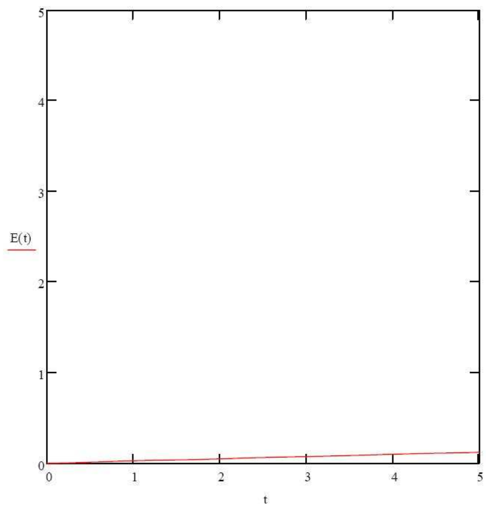

Figure 10.5: Case $p_0 = 50a$

Chapter 11

Measurement of the Time-of-Arrival

The measurement of the time-of-arrival has been an interesting subject both for the experimental physicists and the theoretical ones. Different approaches have been conducted to discuss the problem and explore different aspects of the problem.

It is important to emphasis that the determination of the time-of-arrival involves only a single act of measurement that corresponds to the registration of the particle by the detector. In each run of experiment, at certain times we can check if the particle has entered the detector, and by then we can construct the probability distribution of the time-of-arrival. This approach has been followed by several works [28], [29].

From another prospect we can discuss limitations that might arise during accurate time-of-arrival measurements. A limitation on the maximum accuracy has been discussed for several models in recent works of Oppenheim et al. [18], [19], [22], [23] . They have suggested a general limiting relation on the maximum accuracy achievable on time-of-arrival measurements. The proposed limitation was verified on several models but the proof is left as "an interesting open question".

We present their approach in this chapter, and follow this aspect of the time-of-arrival registration

11.1 Oppenheim et al. Approach to the Time-of-Arrival Problem

In their simplest analysis of the problem Oppenheim et al. have applied a toy model clock to measure the time-of-arrival of a free particle in one dimension at a certain point x_0 [19]. They have shown that time-of-arrival can not be precisely defined and measured quantum mechanically in their model measurement.

Their toy model is based on the Hamiltonian $H_{tot} = \frac{P^2}{2m} + \theta(x - x_0)P_c$, where P_c is the clock's Hamiltonian and P is the momentum of the particle moving in the x direction. $\theta(z)$ is the step function which is equal to one for positive z's and zero otherwise. The clock runs only if the particle is to the left of $x = x_0$. The equations of motion for the particle and the clock's pointer X_c are $\dot{x} = \frac{P}{m}$, $\dot{P} = P_c\delta(x - x_0)$ and $\dot{X}_c = \theta(x - x_0)$, $\dot{P}_c = 0$.

The time-of-arrival that the clock registers would be $X_c(\infty) = X_c(t_0) + \int_{t_0}^{\infty} \theta(x(t) - x_0)dt$. In the classical case, back-reaction on the particle can be made negligible choosing small P_c. So the undisturbed solution for the particle would be $x(t) = x(t_0) + \frac{P}{m}(t - t_0)$ and the clock finally reads $t_A = X_c(\infty) = -m\frac{x(t_0)}{P}$, which is the classical result for the time-of-arrival measured from t_0.

However, in the quantum mechanics we can expect a strong back reaction, since in the limit $\triangle X_c = \triangle t_A \rightarrow 0$, P_c has a large uncertainty. That is the measurement affects the particle strongly. By solving for the wave function for the problem, they showed that the probability to stop the clock has the form of $\left(\frac{E+E_c}{E}\right)^{\frac{1}{2}} \left(\frac{2\sqrt{E}}{\sqrt{E}+\sqrt{E+E_c}}\right)^2$ where E and E_c are the energies of the incoming particle and the clock respectively.

This probability is almost one only if $\frac{E}{E_c} > 1$; on the other hand the uncertainty principle for the clock implies that possible value of E_c is of the order of $\frac{1}{\tau}$ if the average clock energy goes to zero. So in order to trigger the clock we should have $\langle E \rangle \tau > 1$. If the aforementioned constraint be violated, i.e. if $\frac{E}{E_c} \not> 1$ then the incoming particle reflects back because of the clocks induced energy barrier at the detection point x_0 and therefore no detection is done.

They also have considered different models for the particle detectors, and arrived at the same kind of difficulties found in this model. From their analysis of the effect of the back-reaction of the clock on the system on different models, they have suggested a relation for the maximum accuracy possible in measurements of the time-of-arrival, τ, which is $\tau > \frac{1}{E}$, where E is the incoming particle's typical energy.

This relation has some ambiguity on what really is referred to by E. In their works the E is being interpreted sometimes as the typical initial kinetic energy of the incoming particle[21] ([18] page 40) and some other times, the expectation value of the particle's energy([18] page 83).

11.2 General Limitation on Time-of-Arrival Measurements

In our approach which is similar to Oppenheim et al. approach, we derive a relation for the same concept. In this approach, we apply the relation we found for the clock's energy and accuracy, on a quite simple and general model. This will lead us to a formal proof of the relation for the accuracy of the time-of-arrival measurements.

Consider the following general setup for the time-of-arrival experiment: A clock is set to measure the time-of-arrival of a free particle at the point $x = 0$

$$H_{tot} = \frac{P_x^2}{2m} + \theta(x)H_{clock}$$

where P_x is the incoming particle's momentum. In this setup, the moving particle in the x direction, interacts with the clock for locations after the origin. The clock only starts working if the particle passes the location $x = 0$. The time-of-arrival then can be known by reading the clock after some long time[22].

We can set the mean energy of the clock equal to zero and deal with a

[21] What is the zero of energy?

[22] By this setting we avoid possible complexities related to the future collapse of the clock's wave function.

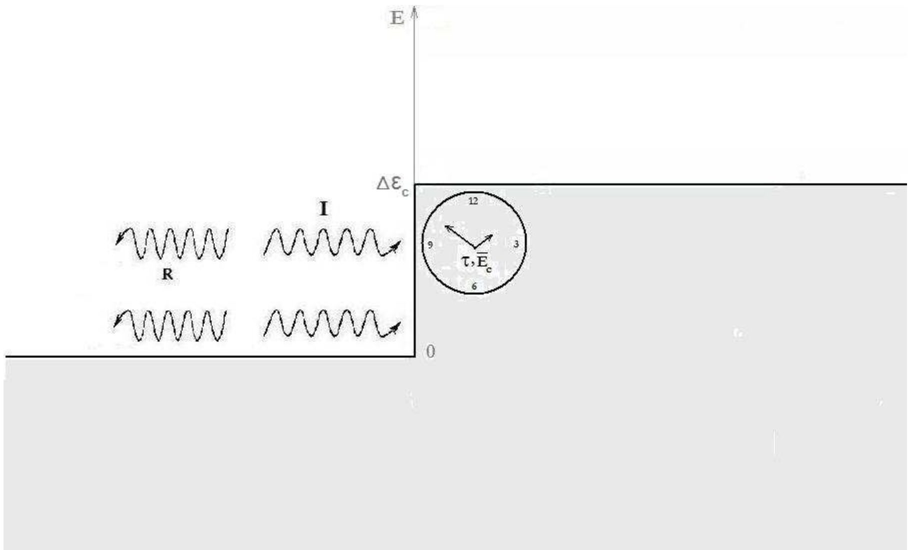

Figure 11.1: Clock's Induced Step Potential

clock with zero mean energy and an energy spread of $\triangle\mathcal{E}_{clock}$ around the mean energy as defined in equation (9.5).

A simple analysis for such a model can be done by looking for the least clock energy that would allow the particle to pass the origin and turn the clock on[23], and be registered. If the particle can not pass the origin and trigger the clock the experiment is unsuccessful in registering the arrival of the particle.

From the particle's point of view, introducing the clock into the problem is equivalent to having an energy step with the height of $\pm\triangle\mathcal{E}_{clock}$. From basic quantum mechanics, we know that if the energy of the particle is less than the energy step height, it will be reflected back and there is no chance to find the particle in the other side of the origin. This fact gives us a necessary condition on the incoming particle's energy in order for the particle to pass the origin and switch the clock

$$E_p \not< \triangle\mathcal{E}_{clock} \tag{11.1}$$

[23] Or off, depending on the direction of the incoming particle and the settings.

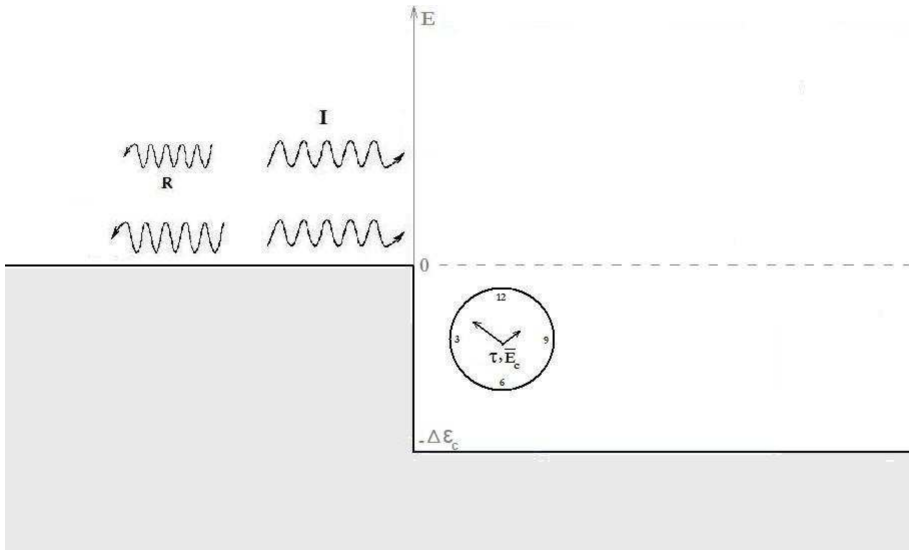

Figure 11.2: Clock Coupled to the System with Negative Coupling Coefficient

Considering this illustration for the problem, we can write the condition for a successful time-of-arrival measurement as $E_p \geq \triangle\mathcal{E}_{clock}$, combining with equation (9.3) we get

$$\tau \nless \frac{1}{E_p} \tag{11.2}$$

There is however also another situation in measure the time-of-arrival, which seems to provide us with more successful results. There is times that the clocks energy is in its lowest, and the particle interacts with a negative energy step of the depth of $\triangle\mathcal{E}_{clock}$ (Fig. 3). In that case the equation (11.1) is satisfied anyway; however, we should also consider the back reaction arises by introducing the sudden energy change at origin $x = 0$. This will give us another sufficient condition in order to have a successful measurement.

Solving Schrodinger equation for the energy step with the depth of $\triangle\mathcal{E}_{clock}$ we get $T = \frac{4kq}{(k+q)^2}$ for transmission coefficient, with $k = \sqrt{2mE_p}$ and

$q = \sqrt{2m\left(E_p + \triangle\mathcal{E}_{clock}\right)}$, which result in the transmission coefficient

$$T = \frac{4\sqrt{1+\frac{\triangle\mathcal{E}_{clock}}{E_p}}}{\left(1+\sqrt{1+\frac{\triangle\mathcal{E}_{clock}}{E_p}}\right)^2}$$

To have a successful measurement, i.e. almost certain transmission of particle from the origin, we need to have $E_p > \triangle\mathcal{E}_{clock}$. Combining this with equation (9.3) we have

$$\tau > \frac{1}{E_p} \tag{11.3}$$

Equation (11.3) is basically what that had been verified in the works of Oppenheim et al. [18], [19], [22], [23] for several well constructed models.

In order to avoid the back reaction causes by the energy step (clock), they also have suggested a model for gradual triggering the clock [19]. As they have shown, while it is possible, in principle, to introduce a correcting potential to smooth the step, the introduction of any additive potential changes the problem and the registered time does not reflect the arrival time of the free particle any more.

11.3 Stating the Time-of-Arrival Inaccuracy Relation

We derived the limitation for the accuracy of the time-of-arrival measurements. The relation we derived is basically the relation proposed by Oppenheim et al in [18], [19], [22], [23]. However as mentioned earlier, in their relation it is not clear to which energy they refer exactly in the relation $\tau > \frac{1}{E}$.

From our derivation and illustrations, we can establish the correct interpretation of E in the time-of-arrival accuracy relation: *there exists a fundamental limitation on the accuracy which we can use to register the time-of-arrival. The accuracy of the measurements can not get better than* $\tau > \frac{1}{\overline{E_p}}$ *where* $\overline{E_p}$ *is the mean energy of the incoming particle, to be measured in* *compare to the clock's zero energy.* This is clear from the illustrations lead to the deriving the equations (11.3).

11.4 Inaccuracy in the Time-of-Arrival Measurement, a General Feature of Quantum Physics?

The main issue that we faced in this chapter which prevents us from successful registration of the time-of-arrival for a free particle was the reflection of the particle off the step potential induced by the clock. This feature is a dynamical property of the non-relativistic free particle's Hamiltonian $H = \frac{P^2}{2m}$.

We can avoid this effect, however, in some other setups which particles would follow other Hamiltonians such as $H = P$ (in case of relativistic particles) or $H = P^3$.

For the Hamiltonian $H = P$, the Schrodinger equation can be written as $H\psi = -i\frac{d}{dx}\psi$. Therefore for an incoming particle from left we have

$$\psi_L(x,t) = A_0 e^{ip(x-t)} + A_R e^{ip(-x-t)}$$

and

$$\psi_R(x,t) = A_T e^{i(p-\triangle\mathcal{E}_{clock})(x-t)}.$$

Imposing the boundary conditions at origin $x = 0$, that are $\psi_L(0,t) = \psi_R(0,t)$ and $i\frac{d}{dx}(\psi_R - \psi_L) = \triangle\mathcal{E}_{clock}$, gives us $A_T = A_0$ and $A_R = 0$. Thus we do not get any reflection off the step potential in this Hamiltonian case. Similar analysis can be done for the case $H = P^3$; one may alternatively consider the phase velocity of the incoming wave for this case $v_p = p^2$ which is always positive. Therefore we do not get any reflection in this case as well. In such cases we have one way particles that switch the clock anyway.

Therefore the limitation we found on the accuracy of time-of-arrival registration is a general feature only in the case of measurements on the non-relativistic free particles.

A key feature of the construction we presented in this chapter is that our derivation for the limiting relation on the accuracy of time-of-arrival measurements is not tied to specific choice of the Hamiltonian that describes the non-relativistic particle's dynamics prior to the moment of registration. We may measure the time-of-arrival of a non-relativistic particle successfully, provided we have the mean energy of the particle just before entering the detector. The procedure and accuracy setup that one follows to measure the time-of-arrival should not depend in principle on particles past dynamics.

Chapter 12

Applications and Examples

12.1 Quantum Zeno Effect

As we illustrated by an example in section 10.2, attempting to observe a system's evolution with accuracies higher than system's orthogonality time $t \sim \frac{1}{\triangle\varepsilon}$ can dramatically slow down the system's evolution.

In formal quantum literature such effect, named *"quantum Zeno effect"* or the "*watched pot paradox*" has been discussed widely [24], [25] and is experimentally verified [26]. It says that continuous observation of an unstable system cause the system to not changes its state.

Briefly the argument goes as follows: Suppose an unstable system with initial state $|\psi\rangle$, evolving after a time t to the state $e^{-iHt}|\psi\rangle$. Thus the probability of survival of the initial state is $\left|\left\langle\psi, e^{-iHt}\psi\right\rangle\right|^2 \simeq 1 - (\triangle H)^2 t^2$ for small times[24] and where $(\triangle H)^2 = \langle H\psi, H\psi\rangle - \langle\psi, H\psi\rangle^2$. It follows if the projector operator on the initial state $|\psi\rangle$ is measured after a short time t, the probability to find the positive result is $1 - (\triangle H)^2 t^2$. However, if the measurement is performed n times at intervals t/n, the probability that all the results will be positive, is $\left(1 - (\triangle H)^2 \left(\frac{t}{n}\right)^2\right)^n > 1 - (\triangle H)^2 t^2$. The left hand side then tends to 1 as $n \to \infty$. That means the initial state under continual observation never evolves with time.

This effect is similar to an effect in classical mechanics where coupling an oscillating system to another high frequency oscillating system halts the original system's oscillation [25].

[24] We consider here only Hamiltonians with finite moments so Taylor series expansion exists.

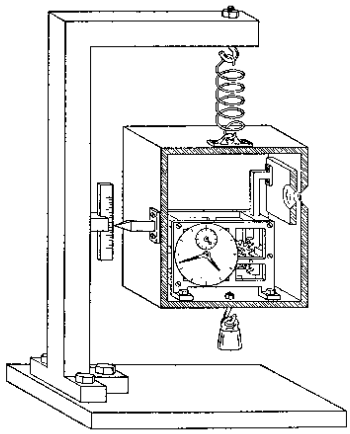

Figure 12.1: Einstein box thought experiment of 1930 as designed by Bohr. Einstein box was supposed to prove the violation of the “spread-time”-energy uncertainty relation. (reproduced by courtesy of the Niels Bohr archive, Copenhagen)

12.2 Bohr-Einstein Debate, the Einstein Box

In one of the epistemological challenges that Albert Einstein presented against Niels Bohr on the Copenhagen interpretation of quantum mechanics, he proposed a weighting gedanken experiment contradicting “spread-time”-energy uncertainty relation, Eq(8.1).

In this experiment a photon from among those inside the box is allowed to escape through the hole which is temporarily opened by a shutter. The opening time is controlled by a clock which is part of the box system. Einstein argued that it is possible to calculate the energy emitted by the outgoing photon through weighing the box before and after the opening period. Thus it seems that one can obtain an arbitrarily precise value for the energy of the photon, while at the same time the time period of emission of the outgoing photon can be as short as one desires, by setting the clock appropriately. This conclusion is in contrary to the “spread-time”-energy uncertainty relation (8.1).

Bohr’s rebuttal [34] was based on the observation that the accuracy of the weighing process is limited by the inaccuracy of the box momentum, which in turn limits the inaccuracy of the position by virtue of the position-momentum uncertainty relation. The uncertainty in the box position entails an uncertainty in the rate of the clock, as a consequence of the fact that energy has weight [33]. Thus, the accuracy of calculating the photon energy and the uncertainty of the opening time of the shutter do satisfy the aforementioned uncertainty relation.

We can present a more direct way of rejecting Einstein's challenge rather than using Bohr's sophisticated way of reasoning. The argument makes no assumptions concerning the method of measurement and is simply based on a property of quantum clocks.

This argument goes as follows: If the photon energy is to be calculated with the inaccuracy δE from the difference of box energies before and after the opening period, then those energies should be well defined within δE. So the Einstein's box energy uncertainty, $\triangle E$, must satisfy $\triangle E \leq \delta E$.

However, the time-energy relation (8.6) between the orthogonality time and energy uncertainty, gives us the conclusion that the Einstein box system needs at least the time $\tau_0 \simeq \frac{\hbar}{\triangle E}$ in order to evolve from the initial "closed-shutter" state to the orthogonal "open-shutter" state. During this time it is intrinsically indeterminate whether the shutter is open or closed. Accordingly, the time interval that the photon can pass the shutter is indeterminate by an amount of at least τ_0.

The argument, in contrast to Bohr's argument, does not use the position-momentum relation for the box components; instead it refers directly to the dynamical quantum features of the box. Close analogy can be made for this Einstein box and the time-of-arrival problem; the above limitations admits the interpretation that it is impossible to determine the energy and time of passage of a particle with accuracies better than that $\tau_0 \simeq \frac{\hbar}{\triangle E}$.

Chapter 13

Conclusion

We have argued different time-energy relations, their derivations and interpretations. Like other discrete quantities in quantum mechanics we see that quantum system's evolution is also discrete in time, and a system goes to next state in times not less than $\frac{1}{\triangle\varepsilon}$ where $\triangle\varepsilon$ is the amount of the energy span of the system around its mean energy value .

Also we showed that in quantum physics we do not have time operator and using the quantum clocks leads to the notion of discrete time registration. The quantum nature of the clock pointers result in an energy-time uncertainty for an ensemble of clock-systems. This uncertainty is not over the time coordinate. It arises since we have to correspond the "time" to some other observable eigenvalue for which that uncertainty relation holds- it is a secondary uncertainty relation, not a fundamental one.

There exist another fundamental relation between the accuracy of a clock (or pragmatically speaking "time" as it measures) and its energy, due to quantum discreteness of its evolution..

In the time-of-arrival problem, we showed that energy considerations place a limit on the time accuracy of time-of-arrival registration for a non-relativistic particle. This time accuracy limit has been suggested before in the works of Oppenheim et al. while its proof had been left as an open question. We provided the formal prove to that suggested limitation.

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Appendix A

Expectation Value and the Standard Deviation of Two Random Vectors Dot Product $< u|v >$

For two randomly chosen unit vectors in a m dimensional complex vector space the mean value of their dot product is zero due to symmetry; since as much as it can be positive, it can be negative. Thus in average it is zero: $\overline{\langle u|v\rangle} = 0$.

To evaluate the standard deviation we need to calculate

$$\sigma^2 = \left(\overline{\langle u|v\rangle^2} - \overline{\langle u|v\rangle}^2\right).$$

We can choose the z direction to correspond with the direction of the first unit vector and the internal product simplifies to $\langle u|v\rangle = \cos\theta$ where $\theta\epsilon[0,\pi]$ in a m dimensional unit sphere. Therefore $\sigma^2 = \overline{(\cos\theta)^2} = \int_0^\pi \cos^2\theta d\Omega_m$, where $\Omega_m$is the solid angle in m dimensional sphere.

To evaluate the above mentioned integral, we notice that the relevant part of the solid angle which works as weight function in relation to the $\cos^2\theta$ and involves θ, is the “surface” on the “sphere” above the direction which the second vector points and that is proportional to $\sin^{m-2}\theta d\theta$. Therefore

$$\begin{aligned}
\sigma^2 \quad = \quad & \overline{(\cos\theta)^2} = \int_0^\pi \cos^2\theta d\Omega_m \propto \int_0^\pi \cos^2\theta \sin^{m-2}\theta d\theta = \\
& \int_0^\pi \left(\sin^{m-2}\theta - \sin^m\theta\right) d\theta = \int_0^\pi \sin^{m-2}\theta d\theta - \int_0^\pi \sin^m\theta d\theta = \\
& \left(-\frac{\sin^{m-3}\theta\cos\theta}{m-2}\bigg|_0^\pi + \frac{m-3}{m-2}\int_0^\pi \sin^{m-4}\theta d\theta\right) - \\
& \left(-\frac{\sin^{m-1}\theta\cos\theta}{m}\bigg|_0^\pi + \frac{m-1}{m}\int_0^\pi \sin^{m-2}\theta d\theta\right) \\
& = \left(\frac{m-3}{m-2}\int_0^\pi \sin^{m-4}\theta d\theta\right) - \left(\frac{m-1}{m}\frac{m-3}{m-2}\int_0^\pi \sin^{m-4}\theta d\theta\right) \\
& = \left(1 - \frac{m-1}{m}\right)\left(\frac{m-3}{m-2}\int_0^\pi \sin^{m-4}\theta d\theta\right) \\
& = \frac{1}{m}\left(\frac{m-3}{m-2}\int_0^\pi \sin^{m-4}\theta d\theta\right) = \\
& \frac{1}{m}\frac{m-3}{m-2}\frac{m-5}{m-4}\cdots\begin{cases} \frac{1}{2}\pi & even\, m \\ \frac{1}{2} & odd\, m \end{cases} \xrightarrow{m\gg 1} \frac{1}{m}
\end{aligned}$$

Therefore in a m dimensional vector space we get $\overline{\langle u|v\rangle^2}$ goes to zero as $\frac{1}{m}$ and so the standard deviation of the aforementioned quantity is $\sigma \propto \sqrt{\frac{1}{m}}$

Appendix B

Geometrical Method to Derive Orthogonality Time

We will now sketch the outline of Anandan and Aharonov [16] method to derive the orthogonality time. By using Schrodinger equation, $i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$, we can Taylor expand $|\psi(t+dt)\rangle$ to second order in dt as

$$|\psi(t+dt)\rangle = |\psi(t)\rangle - \frac{idt}{\hbar}H|\psi(t)\rangle - \frac{dt^2}{2\hbar}\left(i\frac{dH}{\hbar}|\psi(t)\rangle + \frac{1}{\hbar}H^2|\psi(t)\rangle\right) + O(dt^3)$$

to get

$$|\langle\psi(t)|\psi(t+dt)\rangle|^2 = 1 - \triangle E^2 dt^2 + O(dt^3) \tag{B.1}$$

with $(\triangle E)^2 = \langle H^2\rangle - \langle H\rangle^2$ as usual definition. Thus redefining the origin of time we have

$$|\langle\psi(0)|\psi(dt)\rangle|^2 = 1 - \triangle E^2 dt^2 + O(dt^3) \tag{B.2}$$

We can think of a sphere, similar to Bloch sphere, as a geometrical representation of the pure state space of a certain dimensional quantum system. For a two dimensional quantum system, it is easy to visualize the Bloch sphere construction; the mapping of two components of a pure state to two dimensional surface of the Bloch sphere is made by assigning to any state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ a point on the sphere with the coordinate (θ, ϕ). We can extend this notion for any higher dimensional state space in principle.

On the surface of such sphere they defined the metric

$$ds^2 = g_{\mu\bar{\nu}}dZ^{\mu}d\bar{Z}^{\nu} \equiv 1 - |\langle\psi|\psi + d\psi\rangle|^2 \tag{B.3}$$

where Z^{μ}'s are coordinates in that state space. Using this metric we can measure the distance between states; it is a well defined metric, distance of any state from itself is zero, while it has maximum distance with its orthogonal state which is 1.

The general form of the metric can be derived by considering the normalization and the freedom of phase of a component. Lets choose Z_1 to be real and with $\sum_i |Z_i|^2 = 1$ we get $Z_1 = \sqrt{1 - \sum_{i=2} Z_i Z_i^*}$. Thus the form of the metric is

$$g_{ij^*} dZ_i dZ_j^* = \frac{\frac{1}{4}\left(\sum_{i=2} dZ_i Z_i^* + Z_i dZ_i^*\right)^2}{1 - \sum_{i=2} Z_i Z_i^*} + \sum_{i=2} dZ_i dZ_i^*$$

If we consider the time evolution of any state to its orthogonal state, at any time we can write

$$|\psi(t)\rangle = \cos\zeta\,|\psi_0\rangle + \sin\zeta\,|\psi_\perp\rangle$$

with $\zeta = 0$ corresponds to no time evolution and $\zeta = \frac{\pi}{2}$ happens when the state evolves to the orthogonal state for the first time. If we expand the quantum state around $\zeta = 0$, we get

$$|\psi(dt)\rangle = \left(1 - \frac{d\zeta^2}{2}\right)|\psi_0\rangle + d\zeta\,|\psi_\perp\rangle + O\left(d\zeta^3\right)$$

So the metric reads

$$ds^2 = 1 - |\langle\psi(0)|\psi(dt)\rangle|^2 = 1 - \left(\left(1 - \frac{d\zeta^2}{2}\right)\right)^2 = d\zeta^2 \qquad \text{(B.4)}$$

Now if we demand that the evolution of the system to its orthogonal state happens only due to the change of the time coordinate, substituting Eq(B.2) into Eq(B.4) gives us the relation $\triangle E dt = d\zeta$. So for the minimum time -the distance on the time-geodesic on the surface of the Bloch type sphere- that should pass for an arbitrary state in order to get to an orthogonal state, we get the relation $t_\perp \triangle E = \frac{\pi}{2}$.

Thus in general, where the evolution of the state is not confined only to the time coordinate, an arbitrary state can evolve to an orthogonal state only in times $t_\perp$ which hold the relation:

$$t_\perp \triangle E \geq \frac{\pi}{2} \tag{B.5}$$

Appendix C

Fundamental Limitations on Quantum Clocks

Using the fundamental principles of quantum mechanics and general relativity on the short space-time distances, we can extract some limitations on the accuracy, mass, size and maximum duration of the usability of a clock.

Following Salecker and Wigner [31] proof, the argument briefly goes as this: If the clock pointer has a linear spread of δr, then its momentum uncertainty is $\hbar/\delta r$. After a time τ, its position spread grows to $\delta r(t) = \delta r + \hbar\tau m^{-1}(\delta r)^{-1}$ with the minimum at $\delta R = (\hbar\tau/m)^{1/2}$. After total running time T for the clock, the linear spread can grow to

$$\delta R > \left(\frac{\hbar T}{m}\right)^{\frac{1}{2}}$$

However for the clock to measure time with the accuracy of τ, its pointer should have a small enough spread in position that a quantum of light which strikes it to read the time can be determined within the required accuracy τ, thus $\delta R \lesssim c\tau$. That means we require that clock pointer wave function to be confined to a region of the size $c\tau$ throughout the running time T. Accordingly we will have the lower bound on m

$$m \gtrsim \frac{\hbar}{c^2\tau}\left(\frac{T}{\tau}\right) \tag{C.1}$$

for a given T and τ. This limit is more restrictive than the "time"-energy uncertainty relation since it requires repeated measurement of time to not to cause significant disturbance over accuracy over the total running time T.

Further limits has been found by Ng and van Dam [32] on the mass of a clock pointer and its accuracy. Their argument is quite simple: Let the clock be a simple system consisting of two parallel mirrors, each of mass $m/2$ between which a beam of light bounces. For the clock to be able to resolve time intervals as small as τ, the mirrors must be separated by a distance of $d \lesssim c\tau$. Besides that d is necessarily larger than Schwartzchild radius of Gm/c^2 for the mirrors, in order that the time registered by the clock can be read off at all. From these requirements an upper bound on mass of the clock follows:

$$\tau \gtrsim \frac{Gm}{c^3} \tag{C.2}$$

Substituting Eq.(C.1) into Eq.(C.2), we can relate accuracy of a clock to its maximum interval which it can keep the time accurate

$$T \lesssim \tau \left(\frac{\tau}{t_P}\right)^2$$

where $t_P = \sqrt{\frac{\hbar G}{c^5}}$ is the *Planck* time. Thus the better the accuracy of the clock, the shorter it can keep the time accurate. With the *Planck* time being about 10^{-44}sec, this bound on T does not seem to have any practical consequences yet.