LIVING MATHEMATICS EDUCATION

by

Moses Eitan Renert

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate Studies

(Curriculum and Pedagogy)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

August 2011

ABSTRACT

This dissertation searches for possible sources of life in mathematics pedagogy. It is motivated by my observation that much of mathematics education of today is obstructed by inertia. We teach mathematics today using methods and educational philosophies that have changed little in decades of practice, and we generally avoid the harder question of *why do it at all*? I use Wilber's (1995) integral theory, a broad metatheory of psychosocial development, to conceptualize life in general, and aspects of life in mathematics education in particular. Wilber's epistemological framework, called AQAL, describes reality as manifesting in four quadrants – subjective, objective, intersubjective, and interobjective – and in multiple developmental levels. I use AQAL to examine what is revealed about life in mathematics education through these perspectival lenses. The dissertation studies evolutionary dimensions of five related phenomena in mathematics education: purposes of teaching and learning mathematics, human relations in mathematics classes, the subject matter of mathematics, teachers' mathematical knowledge, and ecological sustainability. I connect the diverse evolutions of these phenomena to reveal extant developmental pathologies in mathematics education, such as the Platonic barrier and excessive objectification. Moving beyond critique, the synthesis gestures toward a new emergent pedagogy – *living mathematics education* – that evolves mathematics education past these pathologies. The new pedagogy is elaborated through the examples of an instructional unit on circles and the participatory research methodology of *concept study*. I provide specific suggestions how living mathematics pedagogy may be practiced through dialogical classes, a new purpose of healing the world, a curriculum of sustainability, a skillful blending of Platonic and non-Platonic mathematics, and an improvisatory disposition towards teaching.

PREFACE

Chapter 5 of this dissertation was published in the book *Unpacking pedagogy: New perspectives for mathematics classrooms* (Walshaw, 2010, pp. 177-200). It was written as part of a research project funded by the Social Sciences and Humanities Research Council of Canada with Dr. Brent Davis (PI). Ethics approval for the project was provided by the Beahvioral Research Ethics Board of the University of British Columbia (certificate H09-00972). More details about the design and method of this research program can be found in the paper itself. My authorship of the paper represents 85% of its completion.

Chapter 6 of this dissertation was published in the book *Integral education: New directions for higher learning* (Esbjörn-Hargens, Reams, & Gunnlaugson, 2010, pp. 193-215). It was co-authored with Dr. Brent Davis from the University of Calgary. My authorship of the paper represents 90% of its completion.

Chapter 7 of this dissertation was published in the journal *For the Learning of Mathematics* (Renert, 2011, pp. 20-26). I am its sole author.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGMENTS

While I am listed as the sole author of this dissertation, my research was not carried out in solitude. I am grateful to my family, friends, and students who have contributed to my thinking in so many ways. The conversations, friendships, transformative experiences, and learnings that we shared have extended and enriched my work.

I thank my co-supervisors, Cynthia Nicol and Brent Davis, for allowing me the freedom to explore broad, sometimes unconventional, themes in my research. Through personal example, you have modeled for me the qualities required of excellent academic work: curiosity, courage to question prevailing orthodoxies, willingness to probe deeply, and discernment. Thank you for backing me up every step of the way.

I thank my family: my late grandfather, Mordechai Levit, for instilling in me a love of all things literary from an early age; my father, Jacob Renert, for teaching me the value of analytical thinking and academic rigour; my mother, Galia Renert, for showing me that true learning transcends the confines of formal settings; my brother and business partner, Aaron Renert, for sharing my passion for mathematics pedagogy, while always providing a realistic down-to-earth perspective; and my sister, Miri Renert, for the constant encouragement and inspiration.

I am also thankful to my teachers, colleagues, and friends who have so generously shared their worldviews with me, helping me to complexify my own worldview in the process. They include: Janina and Jim Diodati, Lynne and Ernst Enns, Walt Werner, Gaalen Erickson, Freddie Irani, Edgardo and Katherina Cheb-Terrab, Catherine Hay, Olen Gunnlaugson, Oren Lupo, Kico Gonzalez-Risso and Julia Bonnett, David and Michal Mivasair, Mila Djuras, Faina and Boris Sichon, Mike Braverman, Jason Overy, Noa Ben Mazia, Janice Kamide, Feda Ghaleb, Rachel Moll, Juan Carlos Castro, Ben Shear, Ken Wilber, Sean Esbjörn-Hargens, and Ailsa Campbell.

I thank my students for keeping me academically honest by demonstrating over and over again that no theory of education can capture the complexity of human encounter. My own two

sons, Isaac and Benjamin, have been my best teachers in this respect. Our daily interactions continue to challenge much that I hold to be true about education and breathe new life into the ideas contained in these pages.

My greatest thanks go to my beloved Lisa. This dissertation would not have been possible without your unconditional love, gentleness, and unwavering support. The strength of your spirit is present in every page of this work.

TO MY PARENTS,

GALIA AND JACOB RENERT

CHAPTER 1
LIFE IN MATHEMATICS EDUCATION

From kindergarten to high school, mathematics education spans the whole of our students' educational experience. Over a period of 12 years or more, we teach mathematics to prepare students to be numerate citizens and productive workers. We teach arithmetic to prepare students for algebra; we teach algebra to prepare students for calculus; and then we teach calculus. Incoming mathematics teachers are often successful graduates of this curricular sequence. Once they join the profession, many of them end up perpetuating it through their pedagogy.

It is strangely easy to argue that today's school mathematics is moribund. The limitations of the prevailing modernist-industrial model, which seeks to convey pre-established mathematical truths through transmission pedagogy and which measures learning by means of standardized test scores, are increasingly apparent. International test results (PISA, 2009) show that many students do not succeed with mathematics and continually confront obstacles to engaging with it. Policy decisions of the last decade, which often promoted even more testing and closer scrutiny of test results, have done little to enhance the experience of mathematics teaching and learning.

The emphasis on hyper-instrumentality in mathematics education has likely contributed to increased student disaffection with the subject matter (cf. Ma & Willms, 1999). And yet, postmodern approaches to mathematics education, largely championed by members of the mathematics education research community (e.g., Walshaw, 2004), have gained little traction in actual practice. It is not easy to reconcile the vocabulary of postmodernity with that of formal mathematics and its pedagogy. Indeed, early attempts to do so have resulted in no less than the outbreak of Math Wars (Schoenfeld, 2004). Fluid postmodern notions of identity, discourse, equity, and place often appear incompatible with rigid mathematical formalisms.

Over the past 20 years, I have taught mathematics to students at all levels, from kindergarten to university, and I am particularly experienced in high school mathematics. I have also instructed pre-service and in-service mathematics teachers in my university's teacher-education program. I also own and operate a large for-profit tutoring firm in Western Canada that caters to the ever-growing demand for remedial mathematics education and tutoring support. The high value accorded to mathematics among school subjects has meant that my skills are always in great demand, and I derive direct professional benefit from the ongoing crisis in mathematics education in Western Canada. But despite the success that I have enjoyed in the field of mathematics education, it appears to me that much of the project of mathematics education is obstructed by inertia. We teach mathematics today using methods and educational philosophies that have changed little in decades of practice, and we generally avoid the harder question of: *why do it at all*?

The Oxford English Dictionary defines inertia as "indisposition to motion, exertion, or change." An inert object "lacks vigour." As far as I can tell, when my high school students interact with mathematics, they are rarely invigorated or even moved by their learning experiences. Even though they attend the classes and work on their assignments, it is obvious that most of these teenagers are just going through the motions; they are "doing school". This is evidenced by a question that is all too familiar to us teachers of mathematics: "Will this be on the test?" It is the external inducement provided by testing that drives much of their learning; the subject matter, on the other hand, appears to be inert for them.

Why do so many students experience mathematics in this way? Why don't they enjoy it as I did in school and have continued to do throughout my life? I take genuine pleasure in the analytical demands, orderliness, and beauty of mathematics. Others do too, of course, including some of my own students. But then again, I know of many other intelligent and accomplished people who are practically traumatized by mathematics. In fact, there seem to be far more self-professed victims of mathematics than people who enjoy it, and many more students who want nothing to do with mathematics than students who love it. Why should this be so?

In my early years of teaching I attributed certain students' lack of interest in high-school mathematics to their laziness, to the distraction of teenage hormones, to inadequate teaching in elementary school, to deficiencies in my own instructional methods, to distractions brought on by computers and video gaming, and to a host of other peripheral reasons. I wanted to believe that there had to be an explanation other than the mathematics itself for the students' apathy and discontent. I had to believe that the mathematics I was teaching was intrinsically valuable, for otherwise, there would be no compelling reason to continue teaching it. But, with time, my doubts grew, and I began to realize that I may have been avoiding some hard truths about the enterprise of teaching mathematics. When I looked into the positions of some of the guiding lights in the field, I was surprised to discover some similar conclusions:

> Many teachers today believe their own propaganda – that is, they believe something is wrong with either students or teachers if students do not evince an interest in a given subject. However, after many years of teaching, I have come to believe that this is a great mistake. (Noddings, 1997, p.30)

Like Noddings, I needed many years of teaching to get to a point where I could begin to deconstruct the underlying assumptions of the profession, especially with regards to my basic alignment with its subject matter. My aim in these years was to refine my pedagogy in the hope that better methods of instruction would lead to greater student engagement and improved overall learning. During this time, I became a sought-after mathematics teacher in my city. And still, classroom engagements with mathematics remained inert for many of my students despite my many efforts to change their attitudes and stimulate their creativity. The basic fact remains that students "do" high school mathematics because they are coerced into it. They appreciate my help in making it as painless as possible, and they commit it to short-term memory for their tests, but they mostly forget it just as quickly.

I enrolled in the doctoral program at UBC hoping to study the question, *Where is life in mathematics education?* I initially intended to focus my research on the narrow topic of logarithms, which in my experience of teaching is a particularly traumatizing and lifeless area of the curriculum for many students. If only I could bring life to the teaching of logarithms, I reasoned to myself, then I would surely be able to replicate the experience elsewhere in the curriculum. But to my surprise, the narrow focus on logarithms I had intended for my research could not be sustained. My direction of study compelled me to keep broadening the question. The basic question of *How to teach logarithms?* transformed into *Why teach logarithms?,* then to *Why teach mathematics?*, then to *Why teach?,* and finally to *Why?* Each new question challenged or extended the preceding one in some way, but did not cancel it. All the questions which accumulated as my range of inquiry broadened appeared to be systematically co-implicated. I came to believe that meaningful answers would likely arise simultaneously, and would require a theoretical framework that would be comprehensive enough to include some larger philosophical questions, yet flexible enough to tackle practical questions in mathematics pedagogy.

1.1 Searching for a Theory of Life

What is life? For our purposes, the term *life* stands in opposition to inertia. Life is dynamic in that its energy animates action and expression. To understand this dynamism better, I set out to find what common characteristics are shared by all living systems. Complexity science provided for me a compelling answer.

Capra (2002) offered a systemic analysis of the building blocks of life in diverse phenomena: biological cells, consciousness, and social reality. Capra's synthesis is based on the proposition that there is a fundamental unity to life, and that different living systems exhibit similar patterns of organization. This proposition has been supported by the major findings of complexity science, systems theory, and cognitive science in the past three decades. Accordingly, the pattern of organization of biological systems is the self-generating network. Living systems are cognitive learning systems (cf. Maturana & Varela's Santiago theory, 1972), where cognition is closely related to the process of *autopoiesis*, i.e., the processes of self-organization and self-generation. Living systems are structurally coupled to their environments, and they continuously respond to environmental influences with structural changes. Living systems are also dissipative structures; that is, they are open to the environment and operate far from equilibrium where new forms of order may emerge (Johnson, 2002). Emergence is the creative aspect of life and is the source of development or evolution.

One example of life is a living biological cell. Living cells are membrane-bounded, metabolic networks that are materially and energetically open to their environment. They use a constant flow of matter and energy to repair themselves and self-generate. Because cells operate far from equilibrium, new forms of order may spontaneously emerge within cells through changes in DNA molecules, thereby leading to biological evolution. Another example of life is social culture. A culture is created and sustained by a network of communications that produce meaning (cf. Luhmann, 1990). Each instance of communication gives rise to additional thoughts and communications, and hence the network of communications generates itself autopoietically. Artifacts and texts are the material embodiments of a given culture, and they are used to pass on meaning between generations. Identity is an emergent property of cultural networks that helps to establish their boundaries.

When I first encountered complexity science and its distinct conception of life, I was impressed by the breadth of its application to many different contexts. But I was also put off by its vocabulary. Terms such as *autopoiesis*, *structural coupling*, and *far from equilibrium*, appeared technical, abstract, detached from everyday experience, and far better suited to the analysis of physical phenomena than to the study of human interactions between students and teachers. Indeed, as I came to understand, the discourse of complexity arose out of the exact and natural sciences. The earliest manifestations of complexity thinking were chaos theory in mathematics, nonlinear dynamics in physics, nonequilibrium thermodynamics in chemistry, and systems theory in computer science. Only in the past few decades has complexity thinking been gaining a foothold in the social sciences and humanities: management (Wheatley, 1994), politics (Capra, 2002), economics (Goodwin, 1990), and education (Davis & Sumara, 2006).

Educational research is primarily concerned with interior human phenomena (e.g., knowing and learning). The vocabulary of complexity, on the other hand, derives from empirical observations of exterior phenomena (e.g., vortices in river flows). It follows that any use of complexity thinking in education necessarily should be metaphorical. Unfortunately, the metaphors did sometimes appear to me to be far-fetched or unmanageable when applied to practical teaching. I remained unconvinced that the discourse of complexity would provide comprehensive explanations of social phenomena, even taking into account extended social complexivist theories, such as Juarrero's theory of intentional behaviour (1999). The interior/exterior divide between education and complexity science kept me searching for other theoretical frameworks that might provide a better means of understanding life in mathematics education.

Aside from complexity theory, there was no shortage of theories to choose from in forming my analysis. The prevailing discourse in schools of education nowadays is postmodernism. Postmodernism is an umbrella term for a mass of theories and discourses, all of which share the aspect of rejecting some aspect of modernism. They include: constructivism, social constructionism, structuralism, poststructuralism, critical theory, pragmatism, psychoanalysis, feminism, ecology, existentialism, enactivism, cognitive science, and systems theory, to name but some. During my studies, I attended many presentations that began with the speaker's announcing a given theory to which he or she was committed. The chosen theory, acting as a lens on the world, was touted as the "right one," given a defined set of assumptions and philosophical underpinnings. The theoretical orientation of the day was proposed as the one we would have to employ if we were to understand reality correctly. It was not unusual to encounter thinkers who believed that language and discourse amount to textual reality in their own right, or that power relations govern human behaviour and thought entirely, or that the masculine/feminine duality explains all of the strife in the world. From my perspective, I was willing to acknowledge that each of the theories and discourses that I encountered contained some valid insights. But it was difficult to see how they all fit together.

At the time, I also fostered a favourite theory, that of humanistic dialogue. The works of Martin Buber and Nel Noddings spoke to me directly and deeply. They moved me in ways that other works did not. I often thought that if everyone conceptualized reality as these authors did, the world would be a better place. For a while I considered adopting the lens of intersubjective dialogue as my theoretical framework. But just as I doubted the partiality of other people's commitments to their unitary points of view, I knew that the intersubjective lens was also partial and subject to its own limitations. While human relations are a very important aspect of mathematics classes, the subject matter is also very important. Yet humanistic dialogue had little to offer when it came to the subject matter of mathematics.

At this point it appeared that my choices for a theory of life were either complexity science – an emerging transphenomenal discourse with a convoluted scientifically derived vocabulary – or a whole range of more restrictive, single-cause theories. Neither choice satisfied me. What I needed was a framework that would combine the breadth of complexity science, and its profound understanding of natural and systemic evolution, with the interiority of many postmodern theories. I found this framework in Ken Wilber's integral theory.

1.2 The Need for Integrative Metatheory

— You exposed me to all these theories but gave me no way to sift and winnow among them. How do I know which theory is better or worse than another?

— I'm sorry that you feel that way. That's too bad.

(Van de Ven, 1999, pp. 120-121)

The above fictional conversation between a graduate student and his program director points to a major challenge engendered by theoretical pluralism. Certainly the point of deliberation is not to decide which theories are relatively better or worse. But as theoretical diversity increases, the need to approach the resulting theoretical complexity methodically and to find the links between various theories also increases.

In recent years, mixed-methods research (Johnson and Onwuegbuzie, 2004) has been advanced as a practical response to the ongoing quantitative vs. qualitative disputes in research agendas. Johnson and Turner (2003) recommended that "methods should be mixed in a way that has complementary strengths and non-overlapping weaknesses … It involves the recognition that all methods have their limitations as well as their strengths." (p. 299) But how are the limitations and strengths of different methods to be compared and assessed? Research goals, researched phenomena, and research paradigms are all grounded in theory. So any organized mixing of methods should be carried out with reference to the connections between the underlying theories. It appears that what is needed is integrative metatheoretical research, that is, "the systematic and deliberative study of theories and their constituent conceptual lenses" (Edwards, 2010).

Meta-theorizing is a time-honoured tradition in academic scholarship dating back to Hegel and Marx. In recent decades it has fallen out of favour in academic circles, primarily due to postmodernity's distrust of grand narratives. While some past and present meta-theories may be critiqued justifiably as totalizing, decontextualizing, and marginalizing, these qualities do not inhere in the metatheoretical process. As we shall later see, integrative metatheory can be pluralistic, inclusive, and appreciative of differences. The goal of integrative metatheory is not to develop a *theory of everything* in order to provide a complete description of some pre-established domain. Rather, it is to develop a flexible framework for connecting multiple paradigms and theories within a robust conceptual landscape.

Meta-theorizing is conducted at a higher level of abstraction than ordinary theory. Whereas middle-range theories typically utilize concepts that derive mostly from empirical observations, metatheory utilizes second-order abstractions derived from the analysis of other theories. The abstract nature of metatheory and its objects has led to the modernist critique that metatheory is impractical and not scientific. But social theory, and especially big-picture theory, can have a profound transformative impact on social systems. Giddens pointed to the "double hermeneutic" process (1984, p. xxxii) wherein our theories do not only describe our world but also shape it; people's actions depend on how they interpret their environment.

It is unfortunate that the postmodern and modernist critiques have superseded meta-theorizing in recent decades. Big-picture thinking is more important now than ever before. It is highly doubtful that the current eclecticism of middle-range theories would be sufficient to tackle the immense environmental and social problems that we all face. Fortunately, meta-theorizing is starting to make a comeback in many academic disciplines, whether it is explicitly designated as such or not. Complexity science provides a good example of this resurgence of metatheory. Although complexivists would not necessarily identify themselves as meta-theorists, as there should be no advantage for them in doing so, complexity science is a grand narrative that unites the insights of a host of theories – systems theory, chaos theory, cybernetics, network theory, and enactivism. It describes a common pattern that connects natural phenomena – the network pattern – and tells a compelling story of natural evolution through emergence. As another example, Torbert's *developmental action inquiry* (2004) is an explicit metatheory that draws attention to four different territories of experience in multiple scales. It has been influential in the establishment of organizational transformation as a field of research (see Edwards, 2010).

Probably the most comprehensive integrative metatheory of our time is Wilber's integral theory (1995), also known as the AQAL framework. Integral theory originated from Wilber's early studies (1977) of psychological development and spiritual transformation, and has expanded and matured over a period of nearly 35 years through the successive addition of metatheoretical lenses. Some of AQAL's lenses are: interior-exterior, individual-collective, agency-communion, structural development (stages), multidimensionality (types, lines, states), perspective (first, second third), and methodology (integral methodological pluralism). The integration of these lenses in a unified model makes AQAL a sophisticated multi-perspectival framework for analyzing social reality and psychosocial development.

I came upon integral theory by chance while studying the transformative education literature. Integral theory provided a sophisticated expansive framework in which different paradigms and theories could be situated and connected. The strong analytical bent of the AQAL framework appealed to my mathematical sensibilities. AQAL's emphasis on subjective and intersubjective development made it eminently suitable for the analysis of educational phenomena. I chose AQAL as the theoretical framework for my research because it opened up a productive space for inter-perspectival conversation and synthesis.

1.3 Overview of the Dissertation

Integral theory enabled me to think afresh about five large-scale research questions in mathematics education:

- Why teach mathematics?
- What role do human relations play in the mathematics classroom?
- How has the subject matter of mathematics evolved and in what directions will it evolve next?
- What mathematics do teachers need to know in order to teach mathematics?
- How can mathematics education respond to the central challenge of global society – the problem of ecological sustainability?

I selected these questions because each of them gestures towards a promising source of life in mathematics education – purpose, human relations, subject matter, teacher's knowledge, and environment. My reflections on these questions are the main substance of the present dissertation. They combine to form a response to the overall research question: *What does integral analysis disclose about life in mathematics classrooms?*

Chapter 2 is an overview of integral philosophy and the AQAL framework. Integral theory is sufficiently new to academic discourse that a brief synopsis of its history, elements, methodology, and critiques is warranted. Chapters 3 to 7 are the primary research chapters and they address the five research questions above respectively. Chapter 3 examines the main purposes of mathematics education, past and present, and how they fit together from an integral perspective. It culminates in a proposal of a lively new purpose for mathematics education in our time. Chapter 4 studies the powerful role of intersubjective dialogue in the mathematics classroom. In this chapter, I interrogate the tension between teachers' loyalties to the subject matter of mathematics and the competing need to promote dialogical human encounters. Chapter 5 focuses on the subject matter of mathematics itself. A survey of the history of conceptions of mathematics identifies Platonism as a major barrier to evolution. The second part of the chapter reports on concept study – a participatory research methodology that is making great strides in helping teachers overcome the Platonic barrier. Chapter 6 offers an integral reconceptualization of the popular Mathematics for Teaching (MfT) problem in mathematics education research. It challenges the conventional conception of MfT as a body of knowledge, and argues that MfT is better understood as an open disposition towards mathematics in pedagogical settings. Chapter 7 issues a call to mathematics educators to begin addressing themselves to the problem of ecological sustainability, and offers concrete examples of how we might go about achieving solutions. Integral analysis is used to predict how educators are likely to respond to the call towards sustainable mathematics education. Chapter 8 provides a cross-reading of the preceding chapters and identifies some common themes – overcoming Platonism, emergence vs. stability in mathematics education, the importance of language in mathematics pedagogy, and the expanding boundaries of the mathematical in education. In this chapter, I consider the overall contribution of the dissertation's integral thinking to mathematics education, and whether or not this type of meta-thinking has practical value for mathematics teaching. I conclude with an argument that constant integration of perspectives should be the very essence of the daily work of mathematics teachers.

From a structural perspective, the overall framework of the dissertation is the AQAL matrix. Each of Chapters 3 to 7 was written as a standalone paper intended for publication. Chapters 5 to 6 have been published as book chapters, and Chapter 7 was published as a journal article. I retained the published versions of these chapters intact in the dissertation. I hope that you, the reader, will forgive any repetition that arises as a result. Since Wilber's integral theory is not yet widely accepted by editors of academic journals, some of the chapters employ integral thinking without referring to the theory by name. For the benefit of readers of this dissertation, I precede each chapter with a short synopsis that explains how the chapter ties into the overall

AQAL structure. You will also find some short “teaching interludes” interspersed among the chapters. They contain pedagogical episodes that arose in a Grade 8 instructional unit on circles that I co-taught this year. These are intended to provide brief respites for readers from the theoretical demands of the research chapters.

My research is limned over a large canvas. It addresses big-picture questions and looks for large-scale answers. I ask readers to indulge the necessary use of broad strokes and orientating generalizations in the writing. Rather than ask, *How can you prove that?*, I invite you to ask, *Does it fit? Does it open up new productive paths?* Although research in the social sciences has been plagued by physics-envy for the better part of the 20^{th} century, validation methods of the natural sciences do not often apply to the study of complex human phenomena. A social theory is a story validated by its functional fitness for a given community. If it rings true and produces useful ideas for some readers then it is worthwhile. This dissertation is the phenomenological product of over 20 years of mathematics teaching and 6 years of engagement with the education and integral literatures. It rings true from my perspective, and I hope that it opens up new paths for you too. It is in this spirit that I invite you to enjoy the rest of the work.

CHAPTER 2
AN OVERVIEW OF INTEGRAL THEORY

Developed by the American philosopher Ken Wilber over the past three decades, integral theory is a broad metatheory of psychosocial development. It provides a large-scale framework, called AQAL (Wilber, 1995), for making connections among diverse theories of social reality. AQAL is an analytical space for the systematic integration of many knowledge traditions and paradigms.

In order to place Wilber's integral theory in its historical context, I shall begin this chapter with an overview of integral thinking. I shall then survey some of integral theory's main components: perspectivalism, quadrants and levels, and integral methodological pluralism. The chapter will conclude with some critiques of metatheorizing and personal reflections on integral theory.

2.1 History of Integral Thinking

The roots of integral philosophy can be traced to Georg Wilhelm Hegel, who posited that every domain of reality develops in a dialectical process, wherein synthesis transcends and partially preserves the conflicting dualisms inherent in any given thesis and antithesis. Hegel saw reality as an ongoing process of "becoming," and understood knowledge and consciousness as developmental processes. Dialectic thinking has since been applied in many domains. Karl Marx, for example, interpreted social evolution as a dialectic between techno-economic means and class structure.

In the first half of the 20th century, Alfred Whitehead and Pierre Teilhard de Chardin extended Hegel's understanding of the universe as unfolding, by relating external and internal development. Whitehead conjectured that all natural structures in the cosmos, from atoms to humans, possess some form of consciousness, and hence a subjective interiority. Teilhard de Chardin (1964) later formulated the *law of complexity-consciousness* which holds that consciousness develops in direct proportion to an organism's organizational complexity:

“Complexification due to the growth of consciousness, or consciousness the outcome of complexity: experimentally the two terms are inseparable.” (p. 147)

Developmental psychologists were the first to recognize that consciousness evolves through distinct and universal stages. In 1911, James Mark Baldwin (1906-1911/2007) outlined the first stage model for the dialectical development of human consciousness. It consisted of five distinct stages: *pre-logical, quasi-logical, logical, extra-logical*, and *hyper-logical*. Baldwin’s work had a pivotal influence on Jean Piaget’s subsequent theories of cognitive development in children. Piaget’s work, in turn, inspired numerous other psychologists who formulated stage models for different aspects of consciousness development (e.g., Lawrence Kohlberg, Jane Loevinger, and Abraham Maslow).

Clare Graves (1970) and his followers Don Beck and Christopher Cowan (2006) studied the systemic nature of consciousness development. They proposed a spiralling double-helix topology, which they called *the spiral of development*, to depict the dialectic pattern in which bio-psycho-social systems emerge as humans respond to external life conditions. The spiral of development revolves around the demands of one’s need to adapt to one’s environment and one’s desire to adapt the environment to one’s self. The resulting emergent systems manifest as structures of core values (*value memes*) that serve to organize both individual and collective consciousness.

Spiral Dynamics’ structure-stages bear a close resemblance to the structures of consciousness outlined by Jean Gebser (1984) in his study of human history. In tracing the historical development of nearly every major field of human undertaking (e.g., art, science, language, literature, and philosophy), Gebser discerned an unfolding pattern of transformation that includes five consecutive worldview structures: archaic, magic, mythical, mental-rational, and integral. Each successive structure is characterized by a novel relationship to space and time, while earlier structures continue to operate even as new ones emerge.

Complexity science emerged in the last three decades of the 20th century. It provides a comprehensive new understanding of natural evolutionary systems. Complex systems arise from the co-dependent interactions of autonomous agents, and evolve through the nonlinear dynamic processes of emergence and autopoiesis. Emergence is the process by which the agents cohere into increasingly higher order unities. These new forms of organization manifest transcendent properties, not present preceding forms. Autopoiesis (Maturana & Varela, 1987) is the ongoing process through which complex systems self-organize to regenerate their structures and maintain coherence within their environments.

Complex systems are typically nested and exhibit self-similarity among the qualitatively-different phenomena that are found in their multiple layers of organization. The boundaries of complex systems and their constituent layers are not fixed; they are typically determined by the observer's perspective. Hence, the histories and memories of complex systems are embodied in structure. Indeed, the structure of a complex system is the sum total of its modifications to a given point in time.

Wilber may be seen as the father of integral philosophy of our time. His integral theory is a massive synthesis of results and insights from systems theory, complexity and evolutionary science, postmodern philosophy, and developmental psychology. Wilber's AQAL model (1995) extends current understandings about complexity and evolution in nature to the domains of self and culture. It is a comprehensive map of reality that correlates development in the three realms of nature, self, and culture.

2.2 Integral Perspectivalism

The term *universe* usually refers to the physical realm of nature exclusively. Wilber uses the term *Kosmos* to refer to the expanded mental-physical universe, which also includes interior dimensions of self and culture. Central to AQAL's construction of a map of the Kosmos are the notions of *holon* and *perspective*.

Holons were first proposed by Koestler (1968) as a way to describe complex evolving entities. Koestler noted that biological and social systems were not made up of simple parts, but rather of nested hierarchies of part-wholes, which he called holons. Every component in a system is simultaneously a whole and part of a greater whole. Holarchic systems evolve through a pattern of transcendence and inclusion. So, for example, organisms transcend and include cells, which in turn transcend and include molecules, which in turn transcend and include atoms.

Wilber distinguishes between individual holons that have centered subjectivities and social holons that have distributed subjectivities. A central injunction of integral philosophy is that the capacity for perspective taking is ontologically foundational for all individual holons.

> [T]here are no perceptions anywhere in the real world; there are only perspectives. A subject perceiving an object is *always already* in a relationship of first-person, second-person, and third-person when it comes to the perceived occasions. If the manifest world is indeed panpsychic—or built of sentient beings (all the way up, all the way down)—then the manifest world is built of perspectives, not perceptions … Subjects don't prehend objects anywhere in the universe; rather, first persons prehend second persons or third persons: perceptions are always within actual perspectives. (Wilber, 2006c, p. 4)

Wilber's privileging of perspectives over perception overcomes the modernist misconception that subjects perceive objects, that is, the idea that subjects apprehend an objective pre-given reality. The primacy of perspectives also transcends the postmodern Myth of the Framework (Popper, 1996), which is the belief that all reality is illusory and arbitrarily constructed by the observer. Integral post-metaphysics maintains that while evolution proceeds by creative emergence and there are no ontological pre-givens, existing levels of evolution are rehearsed over time to become ingrained *Kosmic habits*. "…[T]he older the level, the more deeply it has become etched into the Kosmos" (Wilber, 2006c, p. 246). Deep Kosmic habits are concrete and in the course of time have attained an existence that is independent of any particular human individual. Due to the reliability with which they show up in the phenomenological worldspace of humans, Kosmic habits attain an independent existence that all humans must confront. The evolution of consciousness is characterized by the ability to take on an increasing number of perspectives. Although all human perception is filtered through perspectives, the relative degree to which a given perspective has power depends on how much of reality it apprehends.

According to Wilber, all knowing is perspectival. In other words, all experience is interpreted within, and limited by, the conceptual scheme of one's perspective. Some important factors governing human perspectives, for example, are level of development and abstract language. Hence, the integral motto for perspectivalism is "Do not confuse the map with the territory."

2.3 Components of AQAL

AQAL is short for all quadrants, all levels, all lines, all states, and all types. Quadrants, levels, lines, states, and types are five essential metatheoretical perspectival lenses for understanding an evolving Kosmos. They apply at all scales and all contexts. None of them is assigned an ontological or epistemological priority as they all co-arise in the seamless fabric of reality in every moment.

2.3.1 Quadrants

When we investigate any psychosocial phenomenon, we should take two fundamental perspectives into account. The interior-exterior perspective refers to the relationship between subjective experience and objective behaviour. The individual-collective perspective refers to the relationship between the personal and the social. The two perspectives combine to yield the four quadrants – experiential (subjective), behavioural (objective), cultural (intersubjective), and social (interobjective). The quadrants are four interrelated domains of reality and also four perspectives through which we can gain access to these domains.

The four quadrants represent four irreducible domains. A common reductionist mistake, called *quadrant absolutism*, is to privilege one quadrant to the exclusion of the others. For example, when I feel elated after listening to a performance of a violin concerto, the experience of elation can be understood in different ways. From a subjective perspective, I experienced a transcendent feeling of transformation that made me very excited. From an objective perspective, sound waves vibrated in my ear and caused specific neural activity in parts of my brain. From an intersubjective perspective, my culture attaches emotional value to the activity of listening to music. From an interobjective perspective, the piece I listened to belonged to the canon of Western music, which is a specific system for organizing sound. A scientific description of the event that focuses solely on brain activity necessarily misses out on much of the vitality of the experience. To be sure, the feeling of elation in the experience of music has correlates in all quadrants. But the most exciting part of the experience probably resides in the subjective quadrant of intangible experience.

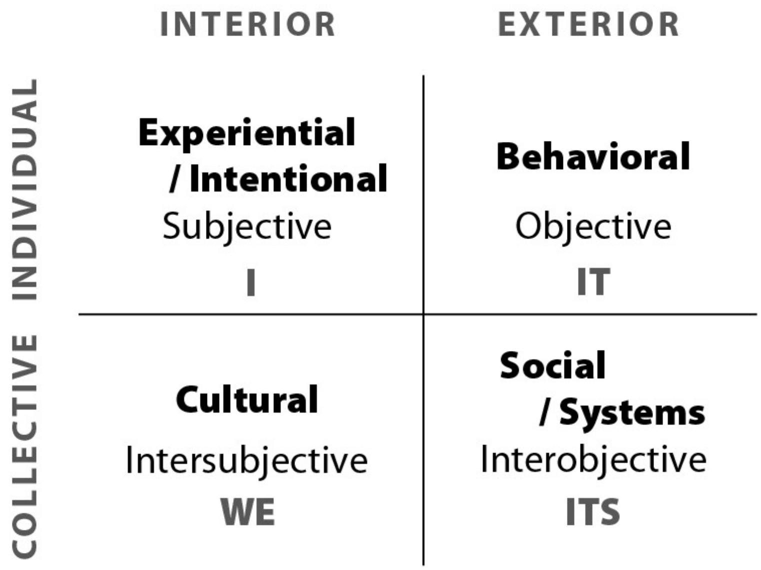


Figure 1. Four quadrants

2.3.2 Levels

The quadrants provide a minimal set of categories for mapping out the development of psychosocial phenomena. The diagonal arrows (see figure 2) represent the spectrum of development in each quadrant, that is, the levels of development through which phenomena in each quadrant have evolved and complexified since the Big Bang. But the linear depiction is somewhat misleading since integral theory does not view development as a rigid, step-by-step linear process.

> Development is not a linear ladder but a fluid and flowing affair, with spirals, swirls, streams, and waves – and what appear to be an almost infinite number of multiple modalities." (Wilber, 2000b, p. 5)

Integral theory uses the terms levels, waves, stages, streams, and structures interchangeably to describe the many facets of development. Development is complex and nonlinear, with moments of progress and regress, stagnation and transcendence. It is characterized by idiosyncratic change within deep patterns of regularity. Recognizing the layers of development within different domains is valuable because it allows practitioners to direct their efforts toward key leverage points in the developmental spectrum. Integral developmental analysis has been applied successfully in such diverse fields as psychotherapy (Marquis, 2007), ecology (Esbjörn-Hargens & Zimmerman, 2009), sustainable development (Hochachka, 2009) and organizational transformation (Edwards, 2010).

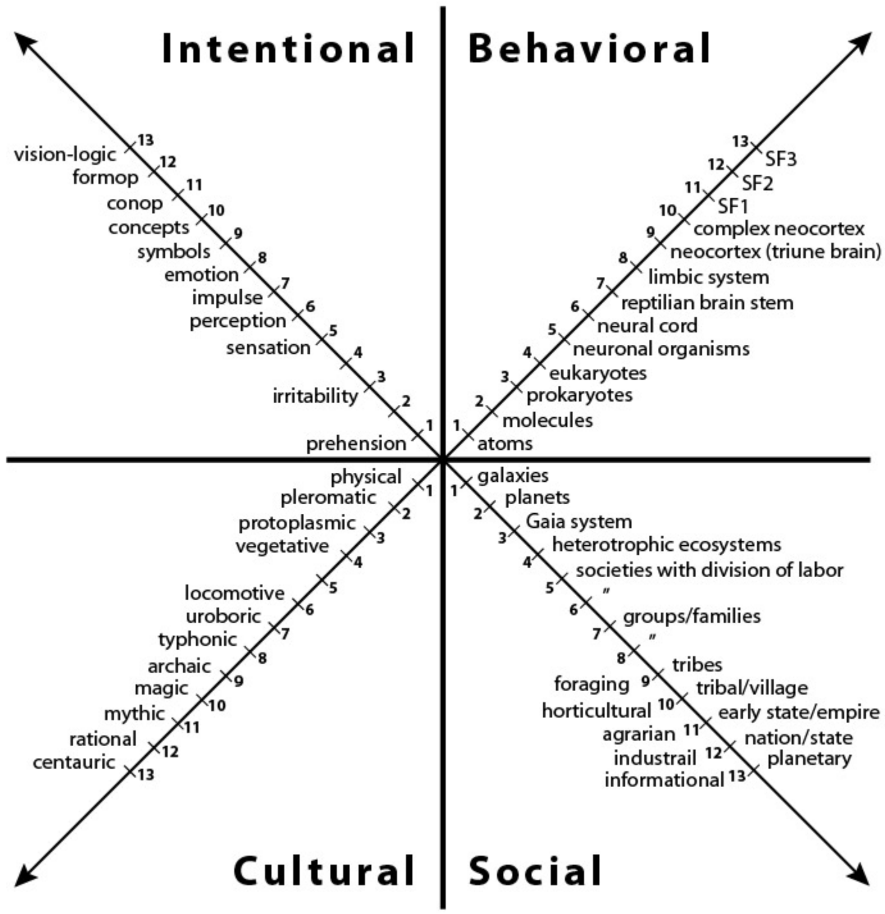


Figure 2. Quadrants and levels

A central proposition of integral theory is that the quadrants tetra-evolve, i.e., development happens simultaneously in all four quadrants. Exterior evolution of nature and society in the right-hand quadrants is paralleled by interior development of self and culture in the left-hand quadrants (figure 2). Wilber offered the notion of *altitude* as a content-independent way of comparing and contrasting development across different domains. He also used colours of the spectrum to denote altitudes. These altitudes are degrees of awareness. Each new altitude opens up an aperture in which new phenomena can arise that are not visible from preceding altitudes. Wilber also described altitudes as levels of consciousness: “Consciousness itself is not a phenomenon, but the space in which phenomena arise" (Wilber, 2006a, p. 68). "A 'level of consciousness'' is simply a measure of the types of things and events that can arise in the first place; a measure of the spaciousness in which a world can appear; a degree of openness to the possibilities of the Kosmos …" (Wilber, 2006c, p. 95).

Kegan's (1994) *subject-object theory* offers a clear way of differentiating between levels of human development: the subject of one level becomes the object of the subject of the succeeding level. Each new level is a different order of consciousness because the former order "is transformed from whole to part, from the very system of knowing to an element in a new system, from subject to object" (p. 128). The new level does not simply negate or replace the preceding one. "Rather, the relationship is transformative, qualitative, and incorporative. Each successive principle subsumes or encompasses the prior principle" (p. 33).

Integral theory's motto of "Transcend and include" captures the manner in which new levels of development incorporate preceding levels in their structures. Wilber asserts that all levels are valuable for the wellbeing of the evolutionary system, and cautions that "transcend without include" is a developmental pathology. Each level of consciousness arises in response to certain life conditions and has its proper application under these conditions. While lower levels are more fundamental and provide possibility, higher levels are more significant and offer new probabilities. Each new level is characterized by increased capacity for perspective taking, and hence thus enables greater inclusivity.

2.3.3 Lines, Types, and States

While my research in this dissertation makes extensive use of the perspectival lenses of quadrants and levels, it does not concentrate on the lenses of lines, types, and states. I shall include short descriptions of these only for the sake of completeness.

Lines refer to specific aspects of human consciousness that develop. Wilber (2000b) mapped more than 20 lines of development in the human psyche, including cognition, morality, role taking, psychosexuality, creativity, altruism, spirituality, values, needs, and worldviews. These lines can and do develop semi-independently. The cognitive line leads development in other lines because cognition determines the breadth of one's awareness. Even though different lines develop at different rates it is still possible to speak broadly of an individual's developmental centre of gravity.

Types refer to different personality categories such as gender and Myers-Briggs type indicators. States refer to temporary states of consciousness and other temporary aspects of reality. They include natural states, such as waking, dreaming, or deep sleep. They also include altered states experienced, such as the Witness and non-duality, which are experienced through meditation.

2.4 Integral Methodological Pluralism

AQAL is a comprehensive map of perspectives on reality. Its descriptive usefulness would be limited however if it were not accompanied by a set of practices to enact and research the territory. Wilber (2006a) proposed *Integral Methodological Pluralism* (IMP) as a companion meta-methodological approach to AQAL. IMP is governed by the recognition that since reality consists of multiple perspectives, truth is disclosed by a plurality of methods and practices. Valid truth claims are those that pass validity tests for their own paradigms in their own fields. The paradigm of one field cannot be used to assert or deny the validity of truth claims brought forth by other paradigms in another field.

Integral theory's corresponding mottos are "everybody's right" and "true, but partial". Every paradigm discloses some aspect of reality, since no one is 100% wrong. And yet no paradigm discloses all of reality, and so every perspective is necessarily partial. In order to access more of reality, we must constantly integrate partial perspectives into grander partialities.

The four quadrants give rise to different categories of validity claims (figure 3). For example, objective claims are assessed for their truth or correspondence, while subjective claims are assessed for their truthfulness, sincerity, or authenticity.

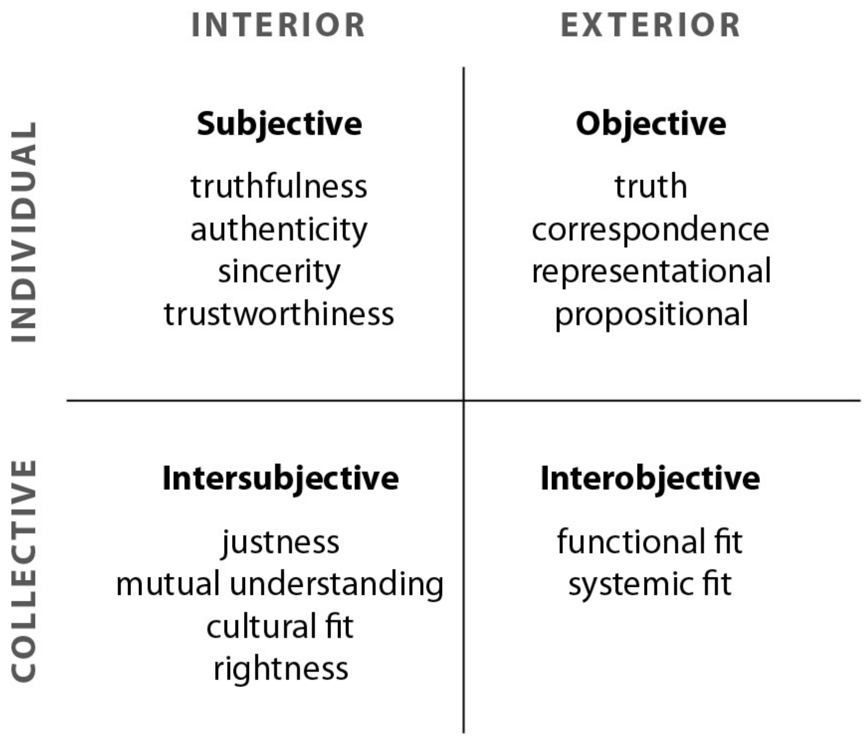


Figure 3. Validity tests in four quadrants

Integral methodological pluralism calls on researchers to use appropriate methodologies for the phenomena under study. As figure 4 shows, each quadrant of AQAL is divided into two methodological zones. Each zone, in turn, represents a family of methods and practices that enact or study phenomena in the quadrant, either from the outside or from the inside. For example, to study or enact my subjective interiority from the inside I may use phenomenology, journaling, or meditation. If someone wanted to study my subjective interiority from the outside, she might use a personality test or interview my friends.

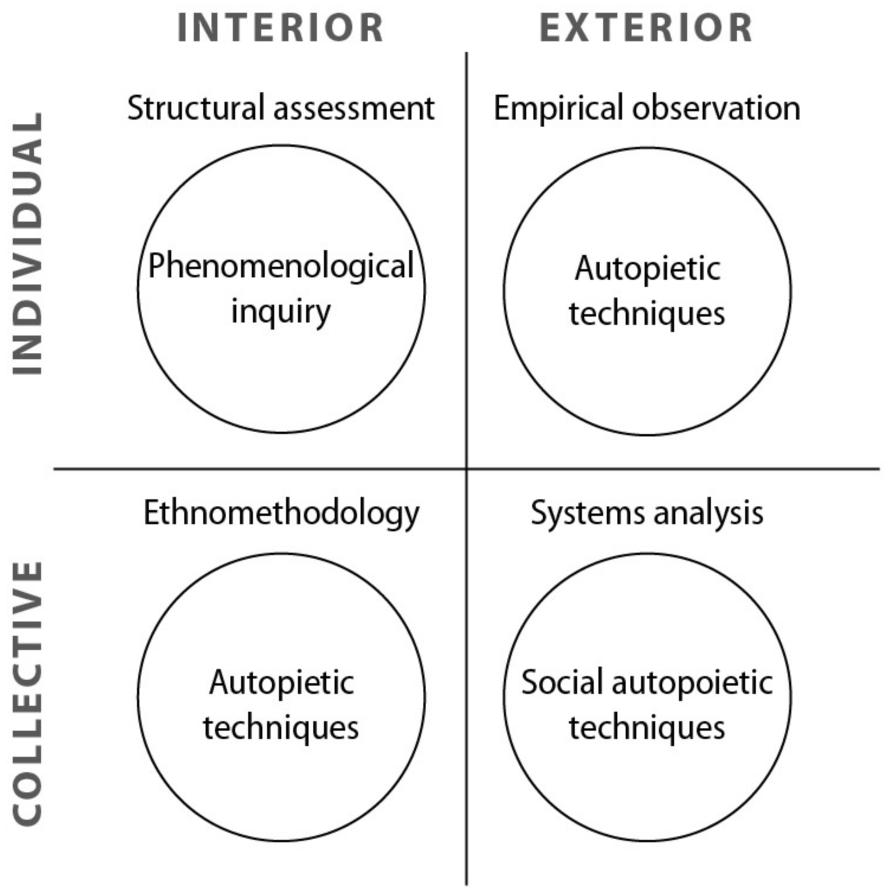


Figure 4. Eight methodological zones

Integral writers (Esbjörn-Hargens, 2006; Martin, 2008) have recently begun operationalizing IMP as a disciplined mixed-mode research methodology. *Integral research* examines phenomena using 1st, 2nd and 3rd person methodologies concurrently. Any data generated is presented in terms of its respective methodology in order to avoid reductionism. AQAL is then used to correlate the different data into a coherent presentation.

2.5 Critiques of Metatheorizing

The breadth and ambition of the metatheoretical project have provoked considerable reaction and criticism among philosophers. Modernist critiques have characterized metatheorizing as removed from practical application and impossible to validate. Postmodern critiques have characterized metatheorizing as totalizing, uncritical, decontextualizing, neglecting the local, and as representing “the view from nowhere.”

In terms of relevance, Giddens (1984) pointed to the double hermeneutic between social theory and society, that is, the "mutual interpretive play between social science and those whose activities compose its subject matter" (p. xxxii). Social theory, and in particular so-called big theory, can play a profound, often unseen, role in shaping social practices and human experience. From a historical perspective, the profound impact that big social theories such as Hegelian dialectics have had on human life are apparent in movements such as Marxism.

The critique that metatheorizing lacks method, and therefore cannot be verified, is largely justified. Until now, most metatheorizing has been conducted through private scholarship. A researcher will read across many disciplines and use personal insights to suggest an overarching framework for integration. One problem of traditional scholarship is that it may lack a solid methodological foundation. Research methods, by their nature, are self-evaluating and include phases that limit the scope and interpretation of their studies. Fortunately, the revival of metatheorizing has lately been accompanied by new interest in methods for large-scale theory building and integrative conceptual research. Edwards' (2010) general method for metatheory building and Sirgy's (1988) method for developing general systems theories are examples of the new methodological turn the field is now taking.

Pluralistic metatheory building, of which Wilber's integral theory is an example, does not aim to totalize or subsume the diversity of all experience into a unified model. Instead, it acknowledges the multiplicity and irreducibility of approaches to social reality and calls for their integration. Integration is the process of building connections among theories rather than unifying or deconstructing them.

Metatheorists frequently survey all extant theories in a given field. As such, they are positioned to raise critical awareness about the relationship between dominant and marginal discourses in the field. Even though metatheorizing uses theory data instead of empirical data, it does enable social researchers to situate their perspectives within a grander framework of competing perspectives. Doing so contextualizes researchers' fieldwork.

Integral theory's deep level of abstraction and broad scope of conceptualization can in some cases give it the appearance of objectifying essentialism. The theory's developmental orientations also make it susceptible to claims that it promotes oppressive hierarchical hegemonies. As Wilber's writings demonstrate, integrative pluralistic metatheory is very much concerned with inclusion, contextualization, and critique in social research. Integral metatheorists would doubtless benefit from further public clarification of their goals, methods, and the limitations of their approaches.

2.6 Personal Reflections on AQAL

A steep learning curve has to be overcome in order to apprehend the ideas that make up integral theory. Wilber is a prolific writer and many of his integral arguments are made through synthesis of modern and postmodern thinking. Familiarity with both is therefore a prerequisite for gaining admission to integral discourse. AQAL then adds layers of metatheoretical terminology. Aside from the epistemological framework discussed so far, Wilber has written about the role of spirit in evolution. Since I do not engage in regular spiritual practice, much of this writing has passed "over my head" as it were. Trusted colleagues whom I have met over the years in the integral community have assured me that my understanding of reality will remain limited until I choose to engage in such internal practice, and they may well be right.

As I studied AQAL's epistemological framework it became increasingly apparent that it offered many useful tools for examining life in mathematics education. The model's scope and ambition were vast, and Wilber's method of orientating generalizations appeared to be a reasonable and fruitful way to tackle big-picture ideas. AQAL helped identify the appropriate domains of application of divergent discourses and practices. Rigid dichotomies and binary categories – e.g., the modern vs. the postmodern, child-centered vs. curriculum-centered pedagogy – began to dissolve for me as I proceeded through Wilber's works.

Through experience, I have come to realize that AQAL is not only a map but also an enactive paradigm of inclusion. I have grown more cognizant of the partiality and fluidity of different points of view in general, and also more compassionate about those who hold views that diverge from my own. I constantly compare and consider perspectives in order to identify how they fit together, and I become concerned when a major perspective is left out of consideration. I even listen to the news of the day differently after having studied in this field. Rather than choose sides on the issues, I take pleasure in tracing the developmental patterns of world events when I am able to do so. Mottos such as "True, but partial" and "Everybody's right" have changed my teaching and altered my personal interactions profoundly.

My engagement with integral thinking over the past four years has complexified my worldview and made me more optimistic. Integral theory maintains that evolution proceeds towards greater capacities for perspective taking, that is, towards greater inclusion and love. It is a hopeful theoretical orientation that endorses the human potentials inherent in cultural evolution. My personal rewards in studying integral theory have thus grown beyond providing me with a theoretical framework for my research.

In the remainder of the dissertation, I will use the terms AQAL, integral philosophy, integral thinking, and integral theory interchangeably to refer to Wilber's integral framework.

Teaching Interlude 1: What's Interesting about Circles?

Our unit on circles began with a video that we found by searching for the keyword "circles" on Youtube. The theme was circles found in nature. Images of circular galaxies changed into images of volcanic smoke rings, which in turn gave changed into images of dolphins playing with ring bubbles. The students and I were fascinated by the beauty of the images. We all wanted to watch the video again. At the end of the viewing, I sensed a feeling of great expansiveness in the room.

We then proceeded to study and derive some of the more standard mathematics of circles: the number π, circumference and area of a circle. I then told the students that they would decide what we should study about circles for the rest of the unit. I asked them to get together into groups and respond to the question, *What's interesting about circles?*

After about 20 minutes of group consideration, the students came up with many questions. Through discussion, the class settled on two of them:

1) How many sides does a circle have? One or infinitely many?
2) Why is the number pi so mysterious?

We took a vote and over four-fifths of the class opted for the first question.

- *Why do you think that a circle has one side?*
- *It just does.*
- *(The student traced an imaginary circumference with her finger.)*
- *So why does a circle have infinitely many sides?*

Two students came up to the blackboard and drew a sequence of geometric shapes.

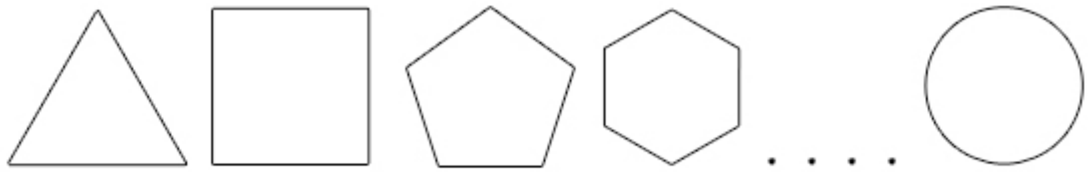


Figure 5. A sequence of polygons with increasing number of sides

- *As the number of sides increases, the shape looks more and more like a circle. So when there are infinitely many sides, it must be a circle.*
- *Looking at your drawing, I see that we have figures with three or more sides, starting with the triangle. And we said that a circle might have one side. So what about figures with two sides? Can you think of some?*

The students paused to think. Two suggestions were offered: a semi-circle and a two-sided figure drawn on a sphere.

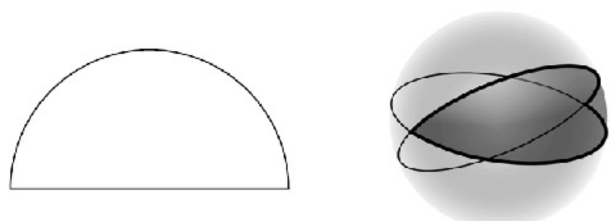


Figure 6. Polygons with two sides

- *Are we allowing our figures to be drawn on spheres now, or are we limiting them to the two-dimensional Euclidean plane?*

A lively debate then began, but the bell rang and we had to stop for the day.

We were off to explore the question, *How many sides does a circle have?* This was not a question I had ever contemplated. So we were all in it together. This question would occupy us for the next three classes.

CHAPTER 3
"WHY LEARN THIS STUFF?": RETHINKING THE PURPOSES OF MATHEMATICS EDUCATION

> My opening research chapter is a critical essay about the purposes of today's modernist mathematics education. It deconstructs the espoused purposes of mathematics education – utility, mental training, and cultural significance – and the unstated purposes of the hidden curriculum – social efficiency and social mobility. The essay is intended for one of the teaching journals in mathematics education (e.g., *Mathematics Teacher*) and is not a standalone piece; it is to be completed by a companion essay on mathematics education and ecological sustainability (see Chapter 7).
>
> The chapter does not employ explicit integral language, but is clearly integral in outlook. The educational purposes it surveys draw from both the individual and collective quadrants. The interplay between the social purpose of social efficiency and the individual purpose of social mobility is a particularly illuminating instance of related co-arising phenomena in different quadrants. From a developmental perspective, all of the purposes under consideration are clearly aligned with the modernist wave, with the exception perhaps of the more traditional cultural significance purpose. My analysis reveals that the development of purpose in mathematics education is currently arrested at the modernist wave.
>
> The chapter ends with a call for practitioners in our discipline to evolve the purposes of mathematics education towards more encompassing and world-centric values. I borrowed my purpose of "healing the world" from the Jewish principle of *Tikkun Olam* ("repair of the world"), a vaguely-defined yet moving notion that invites endless hermeneutic elaboration and active participation.

In celebration of the centenary of *Mathematics Teacher*, the National Council of Teachers of Mathematics' (NCTM) high-school journal, several former presidents of the NCTM were asked to select their favourite past articles for republication. The first article chosen for the August 2006 number was F. L. Wren's "Why Study Mathematics?", originally published in

December 1931. In explaining her selection, Glenda Lappan (2006), the NCTM president in the years 1998 to 2000, remarked on the historical recurrence of articles in the journal that examined persistent problems in mathematics teaching and learning. Lappan was drawn to a class of articles which questioned the place and value of mathematics in the education of young people. Her choice of "Why Study Mathematics?" is certainly one that is topical for me as well. After 20 years of teaching advanced mathematics to high school and university students, I am also more interested in questions of "why" mathematics education matters than in re-hashing "how-to" questions that aim to fix its problems.

After teaching nearly every subject in the high school mathematics curriculum 20 times or more, I have, like all old hands, refined my problem selections, tightened up my sequencing and honed my presentation. When I think of my teaching as a performing art (Sarason, 1999), I feel like an actor who has performed the role of Hamlet for the 200th time. The actor knows the lines, the inflections, what works for the audience and what does not. He has had plenty of time to reflect on his role, and his performance has become predictably nuanced. He is secure in his ability to deliver a satisfactory performance, so he is willing to take more chances. Most importantly, when he performs, the seasoned actor has time to analyze, and to listen to the spaces between the words. Likewise, when I teach, I have time to look into my students' eyes in an attempt to decipher what meaning the experience holds for them.

When it comes to instructing a mathematical topic nowadays—say, logarithms—I have the luxury of time to reflect on the subject's overall value for my students. The problem of *How to teach logarithms* is no longer central to my practice; I have such confidence in the tools I have built up over the years that I mostly take them for granted. But I still have to confront the thorny question of *Why teach logarithms at all?* And on this point, I have not put together any sort of satisfactory answer.

What is the intrinsic value of the subject matter of logarithms? How can the privileged position of school mathematics within the curriculum be justified? I should be in a position to answer these questions clearly. After all, I chose my profession partly because of my love of mathematics, and I readily view the world through the bias of a quantitative lens. Yet, when I step out of my subjective position as a classroom teacher, I find it difficult to spot the cognitive, emotional, and spiritual benefits that mathematics education, as it is practiced today, might offer my students. I question whether, at this time in human history, the compulsory study of logarithms is a wise application of teenagers' resources of time and energy. So unsure I have become about the value of my practice that, in order to teach logarithms, I must sometimes suspend my questioning and just perform the lesson. Fortunately, teaching logarithms, even with a fuzzy sense of purpose, is still a satisfying experience for me. So it goes, I imagine, for many teachers out there.

As an experienced mathematics teacher, I have reached a point in which I am compelled to confront the biggest existential question of my profession: *Why Teach Mathematics?* This chpater is a personal exploration of this question in which I strive to examine the most commonly stated purposes for the teaching and learning of mathematics. I will also examine some purposes that, although not stated explicitly, clearly govern the current practice of mathematics education. I will discuss my own personal response, as an educator, to each one of these purposes, and argue that there is an urgent need to rethink contemporary purposes of math education. I will conclude by proposing a new purpose for mathematics education of today—one of healing the world—which calls for a rethinking of mathematics education as a transformative discipline.

I should note that my question *Why teach mathematics*? focuses on secondary mathematics, the mathematics of later middle school, high school, and early university. Some example topics are: algebra, trigonometry, circle geometry, combinatorics, probability and statistics, and calculus. I distinguish these from topics of elementary mathematics: arithmetical operations, the decimal system, fractions, percents, and measurement. There is no doubt in my mind that all students should acquire skills of basic numeracy. It is hard to imagine how a person can function in today's world without understanding multiplication or percents. On the other hand, it is not hard to imagine how a person functions without understanding logarithms. Indeed, most people do just fine without the benefit of logarithms.

3.1 The Politics and Subjectivity of Educational Purposes

Any discussion of the purposes of mathematics education should begin with the recognition that educational purposes are, by their nature, both political and subjective. The aims of education reflect the wishes, interests, hopes, and dreams of different communities. Consequently, setting and promoting goals in education is fundamentally a political process. It is a way for various groups in society to advance their visions of society's future. It is inevitable that, in a diverse and pluralistic society, the purposes of public schooling will be contentious. The constant struggle for dominance among political interests in free societies is reflected in the ongoing battle to direct the purposes of education.

Educational purposes are also subjective because public education, in mandating compulsory education for all, must remain sensitive to a multiplicity of personal and community contexts, or face backlash. In examining a given purpose for education, it is important to ask both these questions: *Whose purpose is it?* and *To whom does it apply?* Students from low-income families may choose, or be obliged, to study mathematics for quite different reasons than do students from upper-middle-class families. The mathematical needs of citizens in an economically developing society can differ greatly from those of citizens in a post-industrial society. Of course, generational factors also play a part in determining educational purposes, since values and attitudes are liable to change over time in societies, and education is expected to respond to such changes in a coherent and responsible fashion. Keeping in mind that educational purposes are inherently political in nature, and that they depend on social and historical contexts, we may now proceed to examine some of the goals of mathematics education.

3.2 The Utilitarian Purpose

One of the most influential developments in American mathematics education over the past 30 years has been the Standards reform of the NCTM. The document *Principles and Standards of School Mathematics* (2000) made the case for school mathematics as follows:

> We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding. The World Wide Web, CD-ROMs, and other media disseminate vast quantities of

> quantitative information. The level of mathematical thinking and problem solving needed in the workplace has increased dramatically.
>
> In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors.
>
> Students have different abilities, needs, and interests. Yet everyone needs to be able to use mathematics in his or her personal life, in the workplace, and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (Overview section, ¶ 1)

In explaining the need for high school mathematics, *Principles and Standards of School Mathematics* (2000) asserts:

> In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.
>
> Because students' interests and aspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of career and educational options. (Standards by Grade Band section, ¶ 20)

The justification presented by the NCTM for teaching mathematics is exceedingly utilitarian. The argument is that we teach mathematics because students will find need for it at some point in their personal lives, in future education, and in the workplace. The repeated mention of the value of mathematics in the workplace establishes a causal link between workers' mathematical competence and the well-being of the nation's economy. Moreover, mathematics is cast as a means toward social mobility, as it opens the doors of opportunity and leads to economically productive futures for students.

But how convincing is the utilitarian argument really? Ernest (2000) contended that the actual usefulness of school mathematics is greatly overestimated. Admittedly, many of the interconnected systems of commerce and power in modern societies, such as finance, management, and information technology rely heavily on complex mathematics. However, once these systems are set in motion, and refined over decades by technical experts, they require relatively little individual mathematical know-how to sustain. In other words, most people are affected and regulated in a myriad of ways by highly mathematical systems without having to understand these systems' mathematical, logical, and technical underpinnings in any meaningful way. As much as one can happily drive a car without understanding the mechanical intricacies of its transmission, one can operate a computer without knowing binary arithmetic or discrete mathematics. The mathematics of everyday life used in making a purchase at a store, in choosing an insurance or health plan, or even in using a spreadsheet—these being the three illustrative applications chosen by the NCTM—does not go much beyond basic numeracy acquired in elementary school.

Modern society requires a small group of workers to design and control critical information systems, and technicians to program and service them. But even these workers typically do not rely on academic mathematics by and large, but rather they employ specific technical skills that are often learned on the job, outside academic institutions. For example, computer programmers learn new programming languages by referring to programming manuals. While computer programming is in principle a highly mathematical activity, it actually requires a very specific set of technical skills that do not draw directly on school mathematics.

Let's return to logarithms, which are clearly of little use to a consumer who makes a purchase, a salesman who finalizes a sale, and a computer technician who programs the checkout application. Why then do we teach logarithms as part of the common foundation of mathematical ideas recommended by the NCTM for all students? The NCTM (2000) would reason that all options should be kept open for students, as logarithms may well be required for some future studies in mathematics. As it turns out, first-year university calculus courses typically do include instruction in differentiation and integration of logarithmic functions. But this line of reasoning sets up school mathematics as a self-justifying system. One needs arithmetic in order to study algebra, and one needs algebra in order to study calculus. But why does one need calculus? This is where the utilitarian argument breaks down.

Students, in my experience, are rarely convinced by the utility argument and so they hardly ever miss the chance to grumble when I resort to it. This, in turn, leads to a loss of trust between teacher and student. Nowadays, when I face students who ask in exasperation, "When will I ever use this stuff?", I answer, "You may not use it in real life, but you will probably come across it again on tests in future courses." This answer, while truthful, still evades the deeper question being asked, and is therefore mutually unsatisfactory. Nevertheless, I prefer it to the shaky utilitarian response, "You will need it someday. Trust me."

3.3 The Social Efficiency Purpose

Why has the NCTM relied so heavily on the utilitarian argument for justification of its agenda? To answer this question, we should examine the historical situation of the Standards. In the early 1980s, the United States economy was in the grip of a recession, while Asian economies were thriving. In response to the deepening economic crisis, the National Commission on Excellence in Education (1983) published the influential report *A Nation at Risk*. It placed the blame for America's poor economic performance squarely on the shoulders of American education:

> Our Nation is at risk. Our once unchallenged pre-eminence in commerce, industry, science, and technological innovation is being overtaken by competitors throughout the world. ... The educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people. (p. 1)

As Cuban (2003) explained, the growing discontent of corporate leaders with public schooling led to the formation of a coalition between big business and public officials, union leaders, educators, and community activists. Their aim was to enlist public schooling to the cause of training students for skilled jobs. Corporate leaders redefined the notion of vocational training by proposing that academic disciplines, such as English, mathematics, and science, which were once considered liberal arts, provide the training, skills, and attitudes needed to build and sustain a competitive technological economy. A college degree became the accreditation of choice for the corporate workplace.

Since the reformers viewed public schooling as an extension of the national economy, they established new criteria borrowed from the discourse of business management to measure the effectiveness of schools. An effective school, according to this mindset, was one that set high academic standards for students, tested students often, and achieved high test scores. The emphasis on terms such as *effectiveness*, *excellence*, *success*, and *achievement* in educational discourse was meant to advance the narrow goal of higher scores on standardized tests.

The NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics* emerged into a political climate that was conditioned by the admonitions of *A Nation at Risk*. It reflected the NCTM's response to growing political pressures on mathematics educators to bring their practice into line with the perceived requirements of the national economy. It is not surprising therefore that the utility of mathematics in the workplace figured prominently as a justification for the Standards. Even though the NCTM also promoted democratic equality by its call for mathematics education for all students, we may see that even 30 years later the purpose of producing skilled workers persists in the public consciousness as the foremost purpose of mathematics education.

As Cuban (2003) has observed with regret, there is at present one dominant ideological test of a "good" school: it prepares all of its students for college. Similarly, there is currently also one prevailing version of "good" mathematics education: it prepares students for college mathematics and trains them to score high on the mathematics sections of standardized tests. Other critical observers (Purpel & McLaurin 2004; Shapiro, 2006) have likewise issued scathing condemnations of schooling that is mainly dedicated to the goal of social efficiency. They pointed out that the manufacture of workers to fit a pyramid-shaped economic model—with a tiny percentage of high-net-worth individuals above, and larger ranks of lower wage earners at each stage below—necessarily leads to a ruthless process of social ranking of students in schools. Since the economy needs relatively few CEOs but requires many tiers of lower-wage workers down the line, educators and administrators are bound to engage in perpetual assessment, rating, sorting, and ranking of children. The use of standardized tests and the bell curve ensures that only a few students will come out on top.

Moreover, research shows that these top-ranking students are likely to come from privileged households. Studies (National Science Foundation, 2000; Kozol, 1992) have consistently revealed that socio-economic status and race are the most reliable predictors of academic performance and dropout rates. To my mind, these findings may be explained by recognizing that an educational system that molds students to fit the existing economic system necessarily replicates the social injustices inherent to this system.

Its privileged position among school disciplines makes mathematics a convenient tool for social sorting. Most standardized tests required for college admission include substantial mathematics sections. There are many skills to be tested, and mastery of one subject area leads neatly into subsequent ones. A student who misses a link in the chain may find it difficult to catch up, in which case, there are at least as many opportunities for failure. Furthermore, people often associate good performance in school mathematics with intelligence. Students who are not performing well in mathematics may be labelled, and consequently might well see themselves, as being inadequate.

As an educator, I see it as my duty to recognize and nurture the potential that is present in every student. But to be honest, I've tested and ranked students myself for many years. Once I became aware of the hidden curriculum that social efficiency imposes on schooling, I could not continue to do so in good faith. I have altered my entire approach to assessment to reflect this change in attitude. Even though I recognize that a sound economy is important to the welfare of society, I refuse to place the needs of the economy above the needs of my students for understanding, compassion, and support. I trust that human beings who are permitted to flourish in school will grow up to improve their society, and their country's economy, in ways that are far removed from the crude logic of social efficiency.

3.4. The Social Mobility Purpose

Business-driven reforms in education could not have succeeded as well as they have in the past 20 years without broad-based support from the public, and from parents. Why did parents embrace these reforms? Surely parents have more immediate concerns than the ability of American corporations to compete in global markets. In fact, where parents are concerned, we see the goal of social efficiency transformed on an individual level to the educational purpose of social mobility.

Parents and students increasingly view education today as a consumer good whose chief purpose is to provide individuals with social advantage. The social mobility agenda promotes a *meritocratic* system of education. In this system everyone has equal opportunity to compete, the rules of competition are laid out clearly, and the competitors with the greatest merit emerge as winners. Both low-income and upper-middle class parents have bought into this competitive model for different reasons. Low-income parents would like their children to have the opportunity, remote as it may be, for social advancement through academic achievement. Upper-middle class parents want their children to retain their privilege by competing in a system that has always favoured their class.

The *Standards* (National Council of Teachers of Mathematics, 1989) has dovetailed neatly with this meritocratic view of education. It called for equal access for all children to an ambitious mathematical curriculum. It specified standards for curriculum and evaluation that clearly laid out the scope of the competition and how quality was to be judged. The winners would be those who scored highest on standardized tests, and who were admitted to college. From the social mobility perspective, there could hardly be a fairer competition than a standardized test with questions on logarithms.

Both social mobility and social efficiency are driven by economic concerns, but they differ in some important respects. Social mobility views education as a *private good*, benefiting one individual at the expense of another. Social efficiency views education as a *public good*, whose benefits are enjoyed by all members of the community. Social mobility treats education as a form of *exchange value*, to be transacted by the exchange of one's credentials for a job or a comfortable lifestyle. Social efficiency treats education as a form of *use value*, and considers the content and skills learned to be intrinsically useful (Labaree, 1997).

Based on my own experience, I believe that the ongoing ascendancy of the goal of social mobility is transforming present-day education in profound ways. When social mobility is the prime motive for learning, students are apt to be less interested in the subject matter and more interested in the formalisms of the educational process—tests, marks, and credits. Students become proficient in the mechanics of scoring high grades, and in maximizing their results with the least amount of effort. They also engage in an ongoing "bargaining" process with the teacher, in an attempt to optimize work-for-credit ratios (Sedlak, Wheller, Pullin & Cusick, 1986).

It hardly matters whether the subject matter is logarithms or Sanskrit poetry. The all-pervading question is *Will it be on the test?* Hence the test becomes the ultimate authority dictating the approach to any given subject matter. But once the test is marked, returned, and incorporated into a cumulative average, its subject matter loses what little relevance it had, and is likely forgotten before the ink is dry on high-school diplomas.

Students become producers of credentials that will later be exchanged for tangible value in the job marketplace. Some of the students also become consumers of supplementary education in the process, from tutoring to preview courses. The need to gain advantage over other students in a zero-sum game results in demands for more stratification, more specialized programs, more specialized streams, and more honours to go on one's transcript. Education becomes a series of routines and rituals whose purpose is the quantification of merit. It carries no meaning, and yet students still go through the motions of "getting through" the necessary steps to achievement, and teachers still take skills inventories to measure results.

As a mathematics educator who chose his profession because of his love of mathematics, and his love of human beings, I refuse to allow my practice to be dragged down to the level of mechanical routine. I view social mobility as an anti-educational goal that threatens the entire project of education. When learning is replaced by the acquisition of credentials, when the meaning of the subject matter is of little consequence, when students vie for individual advantage and take no notice of their community, the spirit of schooling as laboratory for constructive social development withers away.

3.5 The Mental Training Purpose

The notion that engagement with mathematics trains people to think clearly and logically is another common rationale for teaching mathematics. For instance, Wren (1931/2006) argued that, "the abstractions of algebra, the formal logic of a geometrical demonstration, the induction and deduction, the synthesis and analysis that characterizes mathematical thought give mental training that can be found in no other field of endeavour" (p. 7).

In my own experience, I have noticed that while some parents may question the utility argument because they have found little downstream use for advanced mathematics in their lives, nearly all parents seem to firmly believe that advanced mathematics should be taught because it trains children in logical thinking and problem solving. This mental training purpose is closely related to the popular belief that knowledge of mathematics is a sign of superior intelligence, a belief that can be traced to proponents in Hellenic antiquity, such as Plato, who regarded mathematics as the best training for the mind.

Current popular endorsement of the mental training purpose in mathematics education is based on a transfer theory of learning. This theory maintains that skills and knowledge learned in one context can be applied to others. How does this theory apply to the current practice of mathematics instruction? What thinking skills do we teach in our mathematics classrooms and how do they transfer to other areas of students' lives? These questions are rarely, if ever, explored in the mathematics education literature.

When we examine the process of problem solving as it is commonly practiced in today's schools, we find that much of it is linear and convergent, as opposed to creative and divergent. Students are told that every mathematics problem they encounter in class has a solution, and they are typically challenged to find it in a few minutes or less. Because problems are often posed immediately after new mathematical concepts are taught, students are trained to consider each problem in light of the latest bit of conceptual mathematics they have learned. The teacher then verifies whether or not the students' solutions follow the approved procedure for producing the correct answer.

What messages about thinking and problem solving are communicated by this process? Students learn that every problem has a defined solution, and that only one route to the solution is the right one. Problem solving is rendered not as an exploratory process of discovering new truths, but rather as a process of confirming extant truths. The ultimate judge for the means to this end is an authoritative adult. Students frequently come to believe that any problem whose solution takes more than a few minutes to unravel cannot be solved, or at least that they cannot solve it, and thus the whole undertaking should be abandoned as pointless (Schoenfeld, 1994). Unless we wish our students to grow up to be mentally complacent adults, who submit readily to authority when seeking answers, we should be quite concerned about the messages that are communicated by our instructional practices.

I do not doubt that engagement with mathematics can offer our students many beneficial habits of mind. But we, as math educators, must first understand and explicate for ourselves which habits of mind we wish to instill in our students. As long as we keep asking our students to find the roots of 4th-degree polynomials in the vague hope that they will become better thinkers, we will never be able to unlock the real potential of educational transfer.

Papert (2005) said that "You can't think about thinking without thinking about thinking about something" (p. 366). I believe that educators who aspire to teach clear thinking and good habits of mind must be concerned with the quality of their subject matter, especially in terms of the intellectual interest that it generates. Students will not engage meaningfully with lifeless, irrelevant, and programmatic subject matter. On the other hand, good habits of mind can be fostered through a variety of logico-mathematical domains of application that extend beyond the traditional curriculum of arithmetic, algebra, geometry, and calculus. Formal and conversational logic, games of strategy and chance, recreational mathematics, and the mathematics of finance and money, are all fertile grounds for problem solving. These approaches can easily yield meaningful, inspiring, and practical problems for study. Perhaps if the discipline of mathematics education were renamed *formal thinking*, educators would not be as limited by traditional perceptions of what ought to comprise the approved curriculum.

3.5 The Cultural Significance Purpose

Jerome Bruner (1996) described education as an agent of culture making. Since education has the power to shape future culture, the public has traditionally called on educators to teach by means of the finest examples of past and present human culture. Indeed, mathematics stands as one of the most important intellectual achievements of humanity, and it is appropriate to argue that it is worthy of study as an important cultural text in its own right. This argument, however, raises several questions for educators.

First, mathematics is only one of many fundamental intellectual accomplishments in human history. What criteria should be used to judge which cultural texts are more or less worthy of study in school? Consider philosophy, for example. It is another important subject, but there is hardly any mention of it in today's schools. My Grade 12 students typically cannot name three philosophers, let alone outline key ideas from the canon of Western philosophy.

Second, we may ask whether our current practice of mathematics education, with its strong focus on technical competence, really provides students with the tools to appreciate the cultural significance and importance of mathematics. Ernest (2000) listed various aspects of awareness that may be construed as appreciation of mathematics. These include: qualitative understanding of some of the big ideas of mathematics, such as infinity, symmetry, recursion, and chaos; understanding of the main branches of mathematics, their interconnections, and the unity of mathematics; and awareness of mathematics' foundational role in culture, art, science, and life. These aspects play very minor roles in today's mathematics education, and are downplayed in our current curriculum and pedagogy. Now and then textbook writers do include some historical or contextual information about mathematical concepts, but such information is almost always treated as an informational "extra." Since it is on the periphery of what material must be studied for the almighty test, it is usually ignored.

Teaching mathematics as a cultural text would require a profound reassessment of current teaching practices. Mathematics classes may come to resemble social studies classes, as the focus would shift away from technical competence to mathematical appreciation. Teachers would have to be trained in an entirely new curriculum. Yet I doubt that most mathematics educators see a pressing need for such a change at all; so used we have become to seeing mathematics education as a process of skill acquisition. But as long as our students keep performing synthetic division of polynomials in the absence of any context, we cannot justify the practice of mathematics education on cultural grounds.

3.6 The Need for Relevant Purposes

Teaching is an intentional activity and ideally there should be a strong relation between the expressed aims and the realized practices of mathematics education. Where this link fails to obtain there is an area of disequilibrium and inconsistency which creates stresses for teachers and students. (Ernest, 2000, p. 10)

As my review has shown, the actual practices of today's mathematics education generally fail to meet the espoused goals of the discipline. Secondary mathematics is not very useful in the daily lives of most adults; prevailing pedagogical methods are unlikely to turn students into better thinkers in other areas of their lives; and the instructional focus on technical competence comes at the expense of mathematics appreciation. I have also discussed the two unstated economic purposes that drive much of today's schooling: social mobility and social efficiency. Both these purposes are inherently conservative, in that they seek to perpetuate existing modernist society and its structures. I explained how these two aims have shaped the present-day educational system into a meritocracy, and the educational process into a competition for credentials. The dominant educational experience of the past 30 years has focused on standardized testing and grades rather than on the human quality of the educational encounter. The discrepancy between stated goals and actual practice has contributed to a pervasive loss of meaning, both among students and teachers, as epitomized by the question we hear so often: "Why are we learning this stuff?"

In my own practice, I have sensed that my inability to provide a convincing answer to students has led to a loss of trust on their part. As trust is vital to authentic communication, the discourse of mathematics education today has become largely divorced from real-world lived experience. If there are meaningful connections between $log_2 8$ and my students' out-of-class lives, I find it hard to tell what they would be. I deeply regret that my students rarely experience mathematics as useful, significant, meaningful, or relevant. My classroom sometimes resembles a work camp, in which students perform tedious labour in order to receive a payoff in the form of marks and credits. It is therefore no wonder that students rarely experience joy, curiosity, fun, surprise, intellectual struggle, discovery, and achievement in connection with mathematics.

I believe that the absence of active debate on the goals of mathematics education has contributed to the stagnation of the discipline. Indeed, secondary mathematics education of today is very similar to that of a century ago. We still teach logarithms, and we still use "chalk and talk" methods as the dominant mode of instruction. The most notable difference between now and then is that the number of students who are taught logarithms in this way has increased dramatically. Sometimes it seems that the entire project runs on past precedent alone. This impasse will surely hold back the evolution of mathematics education as a discipline in the 21st century.

Many of today's educational arguments are won on economic grounds. An argument could be made that we should continue our current educational practices without reflection, because we are satisfying the needs of the market for knowledge workers. But by focusing exclusively on the economy, this type of argument ignores the even bigger challenges that confront humanity in our time. Given the problems that our society faces, an attitude of complacency is simply not one that we can afford.

We live in times of hyper-accelerated change, in which the proliferation of problems seems to outpace society's ability to solve them. Some examples are: unsustainable growth, plundering of the earth's resources, pollution and global warming, poverty, famine and hunger, disease, war, clashes of cultures, and disastrous combinations of these ills. Sociologist John Berger (1999) compared modern life to a painting by Hieronymous Bosch:

> There is no continuity between actions; there are no pauses, no paths, no pattern, no past and no future. There is only the clamor of the disparate, fragmentary present. Everywhere there are surprises and sensations, yet nowhere is there any outcome. Nothing flows through; everything interrupts. (p 4)

Even North Americans and Western Europeans, who are surely the most affluent people in human history, experience chronic social and economic troubles—crime, racism, depression, anxiety, isolation, alienation, and a widening gap between rich and poor—that seem to resist every new scheme for improvement that society devises. As Wheatley (2002) observed:

> Almost everyone is experiencing life as more stressful, more disconnected, and less meaningful than just a few years ago. It's not only that there's more change, or that change is now continuous. It's the nature of the change that is upsetting. (p. 14)

Erich Fromm (1976) pointed out that in our culture, "having" has become much more important than "being." In other words, consumption of material goods has replaced our concern with lived experiences and relationships. The primary drive of a free-market economy to mold people into compliant, diligent consumers can often override individual emotional and spiritual needs for human contact, relationship, and meaning. This ongoing conflict between the needs of the economy and the actual needs of human beings has led to a pervasive crisis of meaning in society (Shapiro, 2006).

There is, I think, a direct counterpart to this crisis of meaning in the warnings about the continued viability of life on earth that we hear from scientific communities. In November 1992, 1700 of the world's leading scientists, including the majority of Nobel laureates in the sciences, issued a Warning to Humanity, which cautioned:

> If not checked, many of our current practices put at serious risk the future that we wish for human society and the plant and animal kingdoms, and may so alter the living world that it will be unable to sustain life in the manner that we know. (Union of Concerned Scientists, 1992, p 1)

A recent UN-sponsored study (Millennium Ecosystem Assessment, 2005) of ecosystems by 1360 scientists in 95 countries pointed out that ecosystem degradation will worsen significantly over the next 50 years. It included the warning that, “Any progress achieved in addressing the goals of poverty and hunger eradication, improved health, and environmental protection is unlikely to be sustained if most of the ecosystem 'services' on which humanity relies continue to be degraded” (p. 2). In his book, *Our Final Century*, Martin Rees (2003), former president of the British Association for the Advancement of Science, concluded that “The odds are no better than fifty-fifty that our present civilization . . . will survive to the end of the present century . . . unless all nations adopt low-risk and sustainable policies based on present technology” (p. 8).

Given the uncertainty of the future to which we might well condemn our children, and the enormity of the problems that they may have to face, we as educators can do much better than engage in systematic social ranking of our students. We can seek purposes that honour the original reasons for which many of us chose the profession—our love of children, and our love of mathematics. We can adopt purposes that make mathematics relevant in the lives of our students, and that begin to address the urgent problems facing the world.

3.7 Mathematics Education for Healing the World

I propose that mathematics education of our time should adopt a new moral purpose—mathematics education for healing the world. Education for healing the world is essentially about setting an emotionally compelling goal that will frame personal and communal meaning-making around the urgent questions of our time. The goal is explicit and broad. It opens up a wide spectrum for individual and collective hermeneutic elaboration and subsequent action. As healing the world occurs within one's self, one's immediate community, one's nation, and the world—every step in this direction, at any scale, is worthwhile and potentially rewarding.

The goal of healing the world calls on mathematics educators to deliberately connect their discipline with the environment and living systems in which it is embedded. Some mathematics education researchers have already begun to make these connections. The Critical Mathematics Education movement (e.g., Skovsmose, 1994; Gutstein, 2006) has been relating mathematics to issues of social justice. Complexivist researchers (e.g., Davis and Simmt, 2003) have been examining the transformative potential of systems thinking and collectivity in mathematical settings. And humanistic researchers (e.g., Davis and Hersh, 1986) have studied the ways in which mathematical assumptions both enable and constrain humans' perception and construction of their world. The notion of education for healing the world can serve to bind these different approaches to a common purpose, and to engender a transformative movement in education.

I believe that the educational purpose of healing the world can capture teachers' imaginations and build commitment, if only because it elevates teachers to the position of leaders in social transformation. In the best of all worlds, the answer to the question *Why teach mathematics?* will become self-evident: We teach mathematics to bring forth the best world possible for ourselves, and for our children. What this world might look like remains to be continually negotiated through living educational encounters.

In introducing her selected article for the anniversary volume of the *Mathematics Teacher*, Glenda Lappan (2006) concluded:

> The thought provoking articles from the early years of the *Mathematics Teacher* seem to have given way to a primary focus on classroom-ready activities. Perhaps we should

> strive over the next decade to hit a balance between classroom activities and substantive debate on what we teach and why. (p. 4)

By concentrating on classroom procedures to the detriment of debating and revitalizing the goals of education, we educators have allowed our discipline to lose sight of its social purpose. We should not continue along this path of complacency, for otherwise, history will certainly condemn us for burying our heads in the sand, and for distracting ourselves with worn-out ideas rather than facing the challenges of our time.

Teaching Interlude 2: A Song about π

On that day, I told my students that I wished to start the class with a song called "π," written and performed by one of my favourite artists, Kate Bush (2005). I distributed the lyrics and we began to listen.

> Sweet and gentle sensitive man
> With an obsessive nature and deep fascination
> For numbers
> And a complete infatuation with the calculation
> Of π
> Oh he love, he love, he love
> He does love his numbers
> And they run, they run, they run him
> In a great big circle
> In a circle of infinity
>
> 3.1415926535 897932
> 3846 264 338 3279
> 50288419 716939937510
> 582319749 44 59230781
> 6406286208 821 4808651 32
> 82306647 0938446095 505 8223…

The melody of the song is very repetitive, especially when the singer recites the digits of π. After a while some students became bored and started to giggle. At the end of the song, I asked how many people did not like the song. About half of the students raised their hands.

- *Why not?*
- *Because it's not very interesting.*
- *Would you rather listen to a song of Lady Gaga?*
- *Yes, definitely.*
- *Do you think that Kate intended for the song to be boring?*
- *Yes. Even the idea of saying the digits of π is boring.*

– *So is it a deliberate choice? Why would an artist choose to make her song boring?*

The class was silent for a while. Then some students offered their observations:

– *The music fits with the lyrics. The words talk about numbers that run in a big circle of infinity.*

– *Hey guys, I bet that Kate thinks that a circle has infinitely many sides.*

– *Yes. And this is her way to create infinity in music … very repetitive and dreamy.*

The conversation then turned to the lyrics.

– *What kind of a song is this? Is it a song about math?*

– *No. It's a love song.*

– *Who is it written for?*

– *A mathematician.*

– *How many love songs do you know that are written for mathematicians?*

– *Not many.*

– *Who does Lady Gaga write love songs for?*

– *Hot guys.*

The class began to discuss the differences between popular dance music and art music. They considered the aims and challenges of each musical style. We then continued our conversation.

– *When you grow up, do you think that you'll be the type of person to whom Lady Gaga writes a love song?*

– *No way.*

– *Why not?*

– *We're not hot. We don't look anything like the guys in her videos.*

– *What do you think Kate Bush loves about the man to whom she wrote her song?*

– *His complex mind.*

– *What else?*

– *He is obsessed with the number* π.

– *How many of you tend to obsess over things?*

(Many students raised their hands.)

– *What are you obsessed about?*

– *Science fiction! Music! Quantum physics! Shakespeare!*

The students were proud to put forward their so-called “obsessions.” It was obvious that they identified with them strongly.

We continued to explore ordinary love, and what it means to love an ordinary person. The students were very interested in the topic because they could imagine themselves growing into ordinary adults, and were happy to think about the love relationships in which they would be involved. They wanted to be loved for who they were, for their obsessions and eccentricities.

CHAPTER 4
SPEAKING VOLUMES: HUMAN RELATIONS IN THE MATHEMATICS CLASS

Mathematics education for healing the world is a four-quadrant project. This chapter examines the importance of human relations, which reside in the intersubjective (LL) quadrant. It is the most personal chapter in my dissertation because it employs a humanistic perspective with which I identify closely; it also speaks directly to my own instructional style.

I frame the discussion of human relations in this chapter in terms of a pedagogical paradox. I then proceed to approach the paradox from two very different theoretical lenses – relational ontology and complexity theory. While AQAL is not mentioned explicitly, I use integrative framework to examine the phenomenon of human relation simultaneously on the individual and systemic scales. The resulting synthesis provides a comprehensive explanation of the pivotal role of human encounters in mathematics classes. From this perspective, the chapter raises some fundamental questions about the inter-quadrant interplay between the subject matter of mathematics and human relations.

While the chapter is likely to appeal to journals on humanistic and transformative education, I do hope to publish it in a mainstream mathematics education journal at some point, where it is likely to be read by more mathematics educators. This might not be achievable as the chapter's strong advocacy of human relations is provocative to some extent in its critique of the transcendent role accorded to mathematics in the context of mathematics pedagogy and research.

Should mathematics teaching be concerned with the goal of curriculum coverage? This is a question I have contemplated often over my 20 years of teaching.

When I began teaching mathematics, I believed that the whole of the assigned curriculum should be covered during the course of my classes. My thinking at that time was that mathematics teaching could not properly succeed otherwise. I presumed that my job was to teach to the curriculum so that all students would meet the prescribed learning outcomes. My hope was that if I could teach well enough, and explain the required concepts in a coherent and logical fashion, most of my students would have no trouble performing up to standard in mathematics.

But, as I learned through experience, this is not how things work in practice. Although I did not realize it then, my view of teaching was coloured by the *myth of coverage* (Battista, 1999), which is the belief if the curriculum and its constituent topics are covered by instruction then students learn. Yet despite my best efforts to present the mathematics curriculum in clear and understandable terms, a substantial number of my students year in and year out just "did not get it."

To overcome this problem, I kept refining my lessons and explanations over the early years of teaching. But even though I became increasingly confident in my ability to teach some of my students effectively, I still could not see why I was regularly failing to reach the other students. When I asked myself why this was so, I noticed that the less-able students typically did not show much interest in mathematics; and their interest was not increased by my teaching. So I concluded that their lack of interest stemmed from a lack of proper motivation. If only I could make high school mathematics more interesting, I thought, these students would learn in step with their better-achieving peers. But once again, I was proven wrong.

In my efforts to build up the motivation of the less interested students, I researched practical applications and historical contexts of the mathematics we were studying. However, many topics of school mathematics did not lend themselves to applied examples and contexts. And when I did find mathematical applications, they often seemed artificial and far removed from my students' frames of experience. Similarly, the history of mathematics was not particularly appealing for my students, and seemed to do nothing for their motivations to learn. So I was resigned to the fact that trying to build up intrinsic motivation for learning in some of my students was pretty much a lost cause. Like many teachers, I found myself fostering learning in those students who were more interested and motivated to perform in line with the objectives set out in the curriculum.

Noddings (1997) pointed out that different students have preferences for different topics, and so it is a mistake for teachers to think that something is wrong when some students do not take interest in a given subject. Since I wished that all of my students, even those who weren't particularly interested in mathematics, should have meaningful experiences in my mathematics classes. I began to stray from strict coverage of the topics in the curriculum. Some of the conversations in my classes began to revolve around matters raised by my students instead of mathematics. In short order, I was discussing with my teenage students issues that appeared to concern them a lot. These included: their uncertainty about how to plan their futures, their social lives, difficulties that they were experiencing with adults, their dreams of earning and spending a lot of money, their critiques of society, the music and movies that they were enjoying, and, above all, their hopes and fears about love relationships.

Straying from the prescribed curriculum was not easy at first; I felt that I was failing at my job as a teacher by using valuable class time for purposes that could not be readily connected to mathematics. However, the benefits of my decision to wander off the mandated course of topics were evident almost immediately. The classes grew more lively and inclusive, as more students started to participate in discussions. For a while, I was concerned that discussion of extra-curricular topics would not leave me enough time to cover the program of studies. To my surprise, this was not the case. Students were more attentive, and their increased receptivity enabled me to get through my lessons more quickly, and to explore the mathematics in greater depth. Many of the weaker students were showing considerably more interest in mathematics, and the test scores of most of them improved.

Easing off of my former commitment to coverage of mathematical topics led to improved learning of mathematics by my students. This was by standard expectations a paradox, especially where the less able students were concerned. Since then, free engagement with students about non-mathematical topics has become a hallmark of my teaching. And I have developed techniques for doing it, just as I would for teaching formal prescribed topics. I have often seen the learning of mathematics enhanced as a result. Therefore, I now believe that mathematics teaching need not be directed toward the goal of curriculum coverage, and that "staying on task" need not be an overriding concern for mathematics teachers. I have developed a different approach to teaching mathematics, which draws on what I call *the paradox of reduced coverage*.

What happens when classroom discussion wanders away from mathematics? The students and I are still relating to one another, but not for the strict purpose of learning mathematics. Indeed, our interactions often do not follow any specific purpose or objective. We are able to relax our specific social roles of "math teacher" and "math students," and to interact as one adult talking with a group of teenagers. And yet, this simple human inter-relationality leads us to an enhanced engagement with mathematics, seemingly through increased interest and motivation.

In this chapter, I will approach the paradox of reduced coverage from the theoretical perspectives of relational ontology and complexity science. I will examine how mathematics and inter-human relationality converge in the mathematics classroom.

4.1 Relational Ontology and the Pedagogy of Relation

The philosophical importance of inter-human relations has been long acknowledged in the Western tradition. The philosopher Martin Buber and literary theorist Mikhail Bakhtin were exponents of the primacy of inter-human relations. Buber and Bakhtin each developed an ontological theory of human relations. “In the beginning is the relation” (Buber, 1923/1970, p. 69). What exist first and foremost are relations and not objects or persons. “To be means to communicate dialogically” (Bakhtin, 1929/1984, p. 252). A human being is defined in his or her human quality only by being in dialogical relations with other humans.

At the heart of both theories is the notion of dialogue. The term *dialogue* refers here to more than a conversation between two people; it signifies many different types of relationality. Jenlink and Banathy (2005) surveyed the meanings of dialogue. Dialogue in its broadest sense refers to the entire web of human relationality. It is also a specific type of relation, which Buber called *I—Thou*, characterized by mutuality, directness, presentness, intensity, and ineffability. Dialogue is also a mode of social discourse in which human beings encounter each other with inclusivity, and without objectification. It is also an attitude of profound openness and receptivity that requires a temporary transcendence of preoccupations with the self. Sidorkin (1999) outlined characteristics of dialogue: it is beyond time and space; it knows neither genesis nor causality; there are no objects, only subjects, in dialogue; it does not include social or psychological structures; it precedes language; it establishes intersubjectivity, and enables participants to create understanding across differences.

If the notion of dialogue appears elusive and difficult to achieve in practice, Bakhtin (1929/1984) explained that this is due to the predominance of monological discourse in the modern use of language. Dialogue cannot be properly described by terms that refer only to monological, unidirectional communication. This is because monologue is characterized by radical objectification, and by separation of ideas expressed in language from individual speakers. Monological discourse is fully alienated and presents objects as being “in the world” and not in the discourse itself (Sfard, 2008). Monological narratives take the form of a single, transcendent, and disembodied voice, which Bakhtin referred to as “the voice of the life itself,” “the voice of nature,” and “the voice of God.” Monological truth consists of unitary meanings, whereas dialogical truth is always shared through communication, and manifests in a multitude of voices that are mutually addressed and in constant discursive tension. Sidorkin (1999) argued that even some postmodern discourses, in replacing a grand monologue with multiple isolated micro-monologues that still aim at specific meanings, are also monological. Be it in modern or postmodern discourse, the language of subject-object relations, or *I—it* relations as Buber called them, makes it almost impossible to theorize the realm of the dialogical using the historically established philosophical terms of ontology and metaphysics.

And yet, both Buber and Bakhtin insisted that dialogue is a universal phenomenon that is available to all of human discourse. “All actual life is encounter” (Buber, 1923/1970, p. 62); “All else is the means; dialogue is the end” (Bakhtin, 1929/1984, p. 252). For them, dialogue is both an essential aspect of communication and the highest point of inter-human engagement through discourse.

Buber and Bakhtin described dialogue as pure, unmediated relation. In contrast, the educative relation between teacher and students is typically conditioned by a power imbalance between them, and is mediated by a presumed commitment to the instrumental purpose of learning the subject matter at hand. We may ask then whether or not dialogue is even possible in educational settings. Sidorkin (2002) maintained that even though educative relations are asymmetrical, because they are motivated not only by mutuality but also by instrumentality, it is possible for a qualitatively different form of dialogue to emerge in educative environments. He called on teachers to become adept at directing the relations in their classrooms toward moments of emergent dialogue.

In educational theory, feminist theorists were the first to recognize and analyze the primacy of human relations. Some theoretical constructs founded on inter-human relationality are Noddings’ (1986) ethics of care, Martin’s (1992) notion of the schoolhome, and Gilligan’s (1982) feminist ethics. Noddings, who has perhaps been the most successful in bringing relational thinking to the mainstream of educational theory, suggested that educators should be "taking relations as ontologically basic" (p. 4).

Since the early 2000s, a diverse group of educational researchers has addressed the central philosophical notion of *pedagogy of relation*. Together they issued the *Manifesto of Relational Pedagogy* (Noddings et al., 2004), which called for a reorganization of schooling around human relations. Relational pedagogy starts with the assumption that learning motivation is mainly a function of relations. If we accept that the Self is the intersection of multiple relations that include the individual, then it may be that the formation and development of human identity also depends on relations. Human relations exist in and through shared practices of which the most complex is language. So the goal of schooling should be to create dialogical communities within discourses.

4.2 Monologues of the Mathematics Class

Mathematics as it is taught today may be the most monological of all mandated school subjects. Current pedagogical practices present mathematics to students as a collection of irrefutable propositions to be mastered and memorized. Each mathematical result, be it $2 + 2 = 4$ or Pythagoras' theorem, represents an absolute truth that exists "in the world," apart and distinct from those who come to know it. Mathematical truths are communicated, more often than not, by way of an alienated "voice of the textbook." Once students are told that $2 + 2$ cannot equal anything but *4*, there is no opportunity for them to question or contemplate alternatives to this absolute mathematical result. Even though students may employ various methods to arrive at their answers to a given mathematical problem, these answers are ultimately verified by comparison with the predetermined and correct results of formal mathematics. English pedagogy, on the other hand, is very different. English students are frequently invited to offer personal interpretations of literary texts. Their interpretations are assessed not on the basis of their being right or wrong, true or false, but rather on their coherence, sophistication, and fit in context.

In addition to the subject matter itself, another factor that contributes to the monological nature of mathematics education is the prevalent focus on performative instrumentality. Sidorkin (1999) described a person of dialogical integrity as one whose notion of self organizes around the need to remain unfinalized and open to the polyphony of human voices. Since any choice inevitably reduces plurality and impoverishes life, Sidorkin advised teachers not to hasten students to draw conclusions when faced with a problem. "School tries to complete human growth, while it should uncomplete it" (p. 67). Mathematics pedagogy appears to be failing on all these counts. The ubiquitous quizzes, tests, exercises, and drills in mathematics thoroughly train students in monological decision-making. Not only are students required to make decisions about methods for solutions quickly and efficiently, but they also must always arrive at the correct answers in order to be right. Failure on any of these competencies – imitation, speed, or accuracy – supposedly shows that a student is deficient in mathematics. Such behaviouristic expectations promote the opposite of learning as plurality and polyphony.

In my experience, extra-curricular conversations in my classes serve to alleviate the stranglehold of monologism. Students have opportunities to express themselves in varied and novel ways. Unlike formalized "math talk," which is typically directed at a simple binary, "is this answer true or false?," our extra-curricular conversations tend to promote greater complexity of viewpoints. Unlike mathematical discourse, which sometimes alienates and excludes students, the extra-curricular topics of engagement often enable students to position themselves within a conversation and mutual discourse. In most cases the extra-curricular conversations do not lead to a finalized outcome, and no one feels that such an outcome is necessary for the fulfilment of pedagogical purpose. For instance, in a conversation about music, students may express many different preferences and tastes. It is perfectly alright to end such a conversation without a definite resolution, as the real outcome is an enhanced appreciation of human plurality.

4.3 The Problem of Weak Motivation

Much of contemporary mathematics education practice and research is founded on the assumption that children want to learn mathematics. Many mathematics educators believe that children are driven by natural curiosity to inquire about arithmetic, algebra, and geometry. When children do not display the curiosity assumed by educators, the failing is often attributed to a lack motivation on the part of the students. More broadly, when student interest flags, something must be wrong, either with the child, with the teacher, or with the pedagogy, but certainly not with the subject matter. Noddings (1997) advised teachers not to "believe their own propaganda" (p. 30).

Clearly, there is a large gap between the perceptions of students and of teachers when it comes to interest in school disciplines. Heath (1994) reported that 57 percent of students in suburban schools viewed their classes as boring, but only 11 percent of their faculty members did so. Neill (1961) wrote, “most of the school work that adolescents do is simply a waste of time, of energy, of patience” (p. 25). As a mathematics teacher, I have a large personal stake in arguing that Neill is off the mark, and that school mathematics is useful, interesting, and not a waste of my students’ time. But even if I am interested in mathematics, it is misguided of me to assume that my students would be as interested in it as I am. And if they are not, then Neill’s argument is compelling; mathematics education is a form of oppression for students who are bored by mathematics.

On the whole, my experience as a teacher has shown me that mathematics itself is often a weak motivator for the learning of mathematics, especially for students in their teens. Even educators who disagree with me would likely agree that mathematics does not motivate all students at all times. Educators must face the question of what might make uninspired students want to learn mathematics. The pedagogy of relation offers one possible answer – quality human relations are likely the strongest attractor for students to participate in school life of every sort, and therefore, in mathematics as well.

The extra-curricular conversations in my classes enable me to build relations with students who are not particularly interested in mathematics. These relations almost always start out with my taking interest in whatever interests my students. This practice conforms to Noddings’ (1997) observation that teachers who work with the present motives and expressed needs of their students show respect for their students as Others, and consequently these teachers enable dialogue. As I gain the students’ trust and respect over time, our relations transform into personal ones, in which the students want to be around me because they enjoy my company. Some of these students then begin to show more interest in learning mathematics. It is not that mathematics has suddenly become intrinsically motivating. The students take interest in mathematics because they like their interactions with me, and mathematics happens to be one of my areas of interest.

Sidorkin (2002) suggested that this sequence of transformations of relations – from a teleological relation aimed at the students’ motives, to a personal relation, to a teleological relation aimed at learning – is a valuable one for teachers who consciously manage the economy of relations in their classes. Students are willing to learn because this is a price they pay for the relations they enjoy at school. Teachers pay back with the currency that students need – appreciation, compassion, love, and a lot of attention. Teleological relations, that is, ones mediated by predetermined purposes, are necessarily monological. But the formation of a teacher-student personal relation opens up the possibility of dialogue. This possibility is multiplied when the collective aspects of classroom relations are considered.

4.4 Complexity Science and Network Theory

The educative relation between students and their teacher is one of many overlapping relations that are to be found in any mathematics class. Researchers who examined the systemic nature of classroom collectives (Davis & Sumara, 2006; Davis & Simmt, 2003) proposed that these collectives can be understood and analyzed as complex learning systems. Due to their emergent nature, complex systems are typically nested and exhibit self-similarity among the qualitatively-different phenomena found in their multiple levels of organization. A fundamental assertion of complexity science is that nested phenomena must be studied at the level of their emergence. Davis and Sumara (2006) listed three key conditions that are necessary for emergence in dynamic systems: a tension between diversity and redundancy of agents; sufficient density of neighbour interactions under decentralized control; and enabling constraints that mediate the tension between systemic coherence and change.

The structures of complex systems can be represented and analyzed using graphical network models. In these networks, nodes represent agents, and links represent agent interactions. Research in network theory since the late 1990s (Barabási, 2003; Watts, 2003) has revealed that dynamic networks exhibit a scale-free connectivity pattern in which a few nodes, called hubs, are densely connected, and most nodes are weakly connected. In this topology, nodes tend to cluster into small networks, which in turn cluster into grander networks. These structural patterns have been used to explain the behaviours of complex systems operating in diverse fields, such as sociology and biology. Drawing on the results of complexity science and network theory, we will proceed to investigate two important networks which operate in any given math class: the human relations network and the mathematics network.

4.5 The Human Relations Network

The system of human relations in the mathematics class can be represented by a network in which the students and their teacher are nodes, and the relations among them are links. Students may relate to each other in a variety of ways. For example, John and Mary take mathematics together, but also play woodwinds in the school band. Both of them might have run for the school's social committee last year, but Mary won; John was upset by his loss. Mary and John's sisters are old friends. John might be thinking of asking Mary out on a date, but Mary is interested in Tim. Due to the diversity of their social positionings, Mary and John would relate to each other in multiple ways, all of which could be represented by links that connect Mary to John in the human relations network.

Social roles and positionings divide the students in a given mathematics class into many subgroups: female students, students who are good in mathematics, students who play football, and students who listen to Lady Gaga, for instance. Each of these subgroups is represented by a cluster in the network diagram. Some students are very sociable, and they act as relational hubs in the diagram; others keep to themselves, and they are more sparsely linked. The teacher is also a hub because, at minimum, she is linked to each student in the class through the educative teacher-student relation.

In his study of dialogical schools, Sidorkin (1999) identified complexity as a necessary condition for emergence of dialogue in schools. Complexity, according to Sidorkin, is "a feature of school culture that allows a multitude of human voices to coexist without ever merging or coming to an agreement on anything at all." (p. 120) Sidorkin's condition of complexity correlates with Davis and Sumara's (2006) conditions of agent diversity and density of neighbour interactions. In network terms, dialogical communities are represented by densely linked networks, as each pair of co-existing voices creates a new relation, that is, a new link.

Conducting a mathematics class within the narrow constraints afforded by a prescribed curriculum may diminish the complexity of interactions that arise in the class. The numerous human relations and divergent voices that are present in the collective of the class may not find their proper expression when communication is limited to mandated topics. In such an environment, people may be able to build only partial relations amongst each other, as such relations may occur only with reference to the mathematics curriculum. For instance, a student who is identified as being “bad at math” because of his low scores on quizzes and tests might only be able to interact with his peers in the rigid negative terms of under-achievement. In network terms, mathematics classes that are narrowly focused on the objectives set out in the curriculum are represented by skeletal, sparsely linked networks of relations. The typical organizational strategy of mathematics classrooms is one of a centralized network with the teacher as the only hub. Fuite (2005) hypothesized that teachers’ commitments to covering large amounts of curriculum material as efficiently as possible may bring about this centralized organizational strategy.

When my class discussions move away from the defined topics of mathematics, students are given opportunities to be heard and identified in multiple and novel ways. They get to take on differing roles because they have the chance to express affiliation with different human groups. New relations are formed, as clusters in the human network are activated. Most importantly, with the addition of new links, the whole network grows. Growth is a necessary condition for the viability of dynamic networks (Barabási, 2003). It appears then that extra-curricular conversations serve to advance the complexity of the human community and its network of relations.

4.6 The Network of Mathematics

Our discussion so far has focused on the importance of human relations and dialogue. We have seen that strict adherence to curriculum coverage in mathematics classes runs counter to dialogical engagement. At this point, one may ask, *why should we bother with teaching mathematics at all?* If, as relational ontologists advocate, human meeting in dialogue is the true aim of education, then what has mathematics got to do with achieving this aim? Before we investigate the relation between subject matter and dialogical relationality, it is important to realize that mathematics and mathematical understanding are complex systems in their own right.

Mowat (2008) argued that mathematics is a dynamic evolving complex form with emergent properties. She used Lakoff and Núñez’s (2000) theory of embodied mathematics to describe a network model of mathematical understanding, in which the nodes are conceptual domains, and the links are conceptual metaphors. Subjective mathematical knowledge is understood to be a collection of concepts whose meanings are determined by metaphorical relations. Sfard (2008), who reached similar conclusions in her study of mathematical discourse, also pointed to the developmental dimension of the mathematics network:

> ….mathematical discourse, especially when frozen in the form of a written text, can be seen as a multi-level structure, any layer of which may give rise to, and become the object of another discursive stratum. From this description, mathematics emerges as an autopoietic system – a system that contains the objects of talk along with the talk itself and that grows incessantly "from inside" when new objects are added one after another. (p. 129)

4.7 Relating the Two Networks

Both the human relations network and the mathematics network are present in every mathematics class. We may ask: what is the interplay between subject matter and human relations in a given math class? In other words, how are they positioned relative to one another? In this section, I will argue that the two networks are co-implicated within the context of the mathematics class.

The Russian educational theorist Liudmila Novikova, who studied collective education in the Soviet Union, posited that schools are living holistic social systems, with their own cultures and ecologies. Since Novikova's work remains untranslated to English in its entirety, I will draw on the interpretation of the relevant sections of her writings cited in Sidorkin (2002). When schools are viewed as holistic systems, all aspects of school life, including formal curriculum and informal peer interactions, are interrelated and equally significant. Novikova maintained that any school can be analyzed in terms of its relational field and core activity.

The notion of *relational field* should already be familiar from our earlier discussion of networks. The term refers to the pattern of relations within a complex system; that is, the topology described by the system's network diagram. The notion of *core activity* is less familiar. It is related to Vygotsky's and Leontev's activity theory, which seeks to understand human activities as complex socially situated phenomena. Activity theorists view learning as a process that grows out of collective action. The engagement of learners in different kinds of human activities enables the transformation of objective, extra-personal knowledge into the subjective knowledge of lived experience.

The concept of *core activity* can also be analyzed in terms of Davis and Sumara's (2006) conditions of emergence. Since human beings possess intentionality, human collectives gather around shared purposes and manifest in shared practices. The similarity of purpose shared by the members of the collective contributes to agent redundancy within the system. Moreover, the initial reason for the formation of any collective, which Buber (1923/1970) called the *original relational incident*, is a shared memory embedded in the structure of the system. This formative event constrains the system's range of neighbour interactions and adaptations to the environment. For example, the members of an orchestra are involved in a social collective that was formed for the purpose of creating music. They are likely to engage with each other in all regards that relate to making music, but they don't usually solve mathematical problems together. To the extent that the core activity of a collective social system promotes the conditions of agent redundancy and enabling constraints, it provides the system with internal coherence.

Next, we will examine the core activities, relational fields, and emergent phenomena of the human relations and mathematics networks. The results of our analysis are summarized in Table 1.

	Mathematics	Human Relations
Core activity	Mathematizing	Meeting / Encounter
Relational field		
Nodes	Mathematical concepts	Human beings
Links	Mathematical metaphors	Inter-human relations
Emergent phenomena	Subjective mathematical understanding Mathematical knowledge production Novel mathematics	Knowledge production about inter-human relations Intersubjectivity Discursive realms (e.g., dialogue)

Table 1. Characteristics of the mathematics and human relations networks

The core activity of the mathematics network is *mathematizing*, that is, human engagement with objects and processes of mathematics. The relational field consists of mathematical concepts, which are connected by embodied metaphors that inform their meanings. The emergent phenomena include subjective mathematical understanding, mathematical knowledge production, and novel mathematics.

The core activity of the human relations network is meeting, or human encounter. The relational field consists of persons who are connected by inter-human relations. The emergent phenomena include knowledge production about inter-human relationality, intersubjectivity, and discursive realms, such as dialogue.

People usually have a clear sense of what it means "to do math," and so the core activity of the mathematics network is strongly defined. Mathematizing also often produces recognizable mathematical artifacts, such as solutions and proofs. On the other hand, the core activity of the human relations network – encounter among individuals – is more difficult to define than mathematical activity. The products of human encounters, such as intersubjectivity, are abstract and often defy absolute definitions. Some people may not even consider encountering other human beings in dialogical relations to be an activity at all. Therefore, mathematizing, in its various manifestations – learning mathematics, teaching mathematics, and solving mathematical problems – is more easily conceived of as a shared practice than are human encounters.

On the other hand, the relational field of the human relations network is more open for individuals to inhabit than the relational field of the mathematics network. Humans have developed rich vocabularies to describe and negotiate the immense variety of relations found in social situations. There are family relations, friendship relations, intimate relations, educative relations, and power relations, to name but a few. People are constantly engaged in interpreting, positioning, and transacting human relations. The dense topology of the human relations network offers a myriad of possibilities for rich interactions and emergent novelty. Of course, this is not to suggest that the relational field of the mathematics network does not offer rich prospects for relations as well. Clearly, the variety of mathematical metaphors used to make sense of mathematical concepts can serve as a basis for many meaningful interactions. But, given the many drives and instincts that make up human psychology, mathematical metaphors are usually subordinate to the dynamics of inter-human relations in social settings.

We may now see that both the human relations and mathematical networks play important, but different, roles in a given mathematics class. The mathematics network provides a well-defined shared practice around which the collective gathers. It contributes to the collective's sense of stability and coherence. The human relations network provides a complex relational field for rich interactions. It contributes to the collective's capacity for change and novelty. Both networks are essential for conditioning evolution of the math class collective.

4.8 Mathematics Class as a Grand Network

The symbiotic relation between the networks of mathematics and human relations suggests that the mathematics class is a grander network than either network considered individually. I call this total network *the mathematics class network*. Table 2 summarizes the core activity, relational field, and emergent phenomena of this network.

	Mathematics Class
Core activity	Mathematizing and meeting
Relational field	
Nodes	Human beings and mathematical concepts
Links	Inter-human relations and mathematical metaphors
Emergent phenomena	Subjective mathematical understanding
	Knowledge production about mathematics and inter-human relations
	Novel mathematics
	Intersubjectivity
	Discursive realms (e.g., dialogue)

Table 2. Characteristics of the math class network

The mathematics class network is greater than the sum of its components. Its hybrid objects and relations bring to light the interdependence of subject matter and human relations in mathematics education. Once educators realize the important role that intersubjective relationality plays in the emergence of mathematical understanding, they are faced with a number of important questions: *How can human relations in the classroom be managed to bring about meaningful engagement with mathematics? Conversely, in what ways does the subject matter stimulate or inhibit dialogue and the formation of dialogical communities?*

Mathematics educators tend to frame the goals of mathematics education in terms of learners' proficiencies in mathematics. Mathematics instruction is therefore directed toward promoting subjective mathematical understandings in learners. The communal dimensions of the class collective are typically regarded as incidental to the mathematics being taught, and sometimes, they are seen as extraneous factors that interfere with the learning process. Relational ontologists, on the other hand, frame the goals of education in terms of the learners' exposure to dialogical encounters. They emphasize the primacy of human encounter over disciplinary knowledge production. Which of the two viewpoints is the correct one with reference to the communal relations in a given mathematics class? *Do humans meet in order to learn mathematics, or do they learn math in order to meet?* Our mathematics class network model indicates that all these actions and motives co-arise simultaneously. It is counterproductive to differentiate the foreground from the background in the integral classroom network, as all of its elements are interrelated. Discounting either mathematics or human relations in the teaching and learning of mathematics negates the mathematics class' primary core activity and relational field.

4.9 Towards Dialogical Mathematics Classes

From a complexity perspective, one broad purpose of mathematics education is to expand the space of learners' interpretive possibilities with respect to the project of being a human, co-existing with other humans, as part of a more-than-human world, by way of mathematization. Achieving this purpose requires that learners become full participants in mathematical discourse, which in turn necessitates that mathematics classes become dialogical communities.

The prospect for dialogue is already alive in any given mathematics class, since dialogue can emerge from the class's intricate network of human relations at any point. Stimulating dialogical emergence requires teachers to be attentive to the various discourses in their classrooms, and to the roles that mathematics and the mathematics curriculum play in shaping these discourses. The challenge for teachers is to invite and sustain dialogue that includes their students and mathematics in the same discourse. Treating mathematics monologically often leads to monological discourses that exclude students from participation.

Sidorkin (1999) identified three conditions for dialogical emergence in educative settings: complexity, civility, and carnival. As discussed earlier, complexity requires that a plurality of voices be allowed to co-exist in the class. The second condition, civility, ensures that these voices be mutually addressed, and that they attend to each other. For example, Davis and Simmt (2006) proposed that explicit awareness of the many metaphors that underlie a mathematical concept would help learners appreciate the importance of the concept. Mathematics teachers can inspire a plurality of mutually-addressed voices in different ways: engaging a plurality of mathematical metaphors, taking plurality of approaches to solving a given problem, introducing a plurality of curricular and non-curricular topics, eliciting a plurality of student interests, and recognizing and valuing a plurality of human relations. By bringing all these pluralities into classroom discourse, and by explicitly supporting such pluralities, teachers can open up opportunities for students to become participants in the full discursive space created by open dialogue. It is hoped that students who are exposed to plurality in different aspects of classroom discourse would be more likely to understand that mathematical truths are open to interpretation. If students do come to this understanding, they might also come to view mathematics a discourse, rather than a set of disembodied, ever-lasting, logical certainties that are impervious to change and discussion.

The third condition, carnival, refers to a cultural phenomenon described by Bakhtin (1941/1965). The spirit of the carnivalesque is often expressed through laughter. The carnival is a short-lived suspension of the restraints of social conduct and authority. It is often manifested through ludic celebrations and parodies of accepted social hierarchies. It can promote dialogue by granting permission for saying and doing things that would otherwise seem frivolous or offensive under normal power relations. The times in which mathematics is not discussed in my classes can provide the third space needed for carnivalesque discussions that break down the seriousness of mathematics as a subject. These playful digressions often include laughter and jokes about school and about the inflexibility of the curriculum.

Sidorkin (1999) suggested that laughter is an integral part of learning, since humans cannot make sense of ideas unless they have the freedom to challenge, deconstruct, and ridicule them. I have found that one of the most effective ways to engage my students' interests with the subject of mathematics is to politicize it. By opening up each topic in the curriculum to interrogation, and by openly questioning whether it is useful or interesting, I invite my students to mediate the subject matter through their own viewpoints. Over time, students realize that the mathematics curriculum is not a sacred text, but rather an arbitrary set of discursive choices made by educational authorities. Questioning the mathematics curriculum in this way leads my students to interact with mathematics more freely, since they are able to see the subject matter in a new and different light. Dialogical mathematics classes enable students to see themselves, their viewpoints and modes of understanding, within the same order of discourse as the learning that they are expected to gain.

4.10 Mathematics as Dialogue

This chapter began with a paradoxical outcome which I noticed during the course of my teaching: reduced emphasis on curriculum coverage, and free engagement with non-mathematical topics in my classes, often leads to improved learning of mathematics. Two theoretical frameworks, relational ontology and complexity science, were used to examine this counter-intuitive outcome. The relational ontology perspective considers dialogical relations to be of primary value in classroom interactions. From this perspective it appears that, when the monological grip of the mathematics curriculum is loosened in my classes, a space for genuine dialogue is created. This opportunity, in turn, contributes to improved interactions with my students. The complexity science perspective discloses two networks that are present in every mathematics class: the human relations network and the mathematics network. The mathematics network has a clearly identifiable core activity, while the human relations network is defined by a rich and complex relational field. Since the networks are co-implicated, the extra-curricular discussions in my classes condition the emergence of mathematical understanding by activating the relational field of the human relations network.

Given vital interplay between inter-human relations and mathematics in the classroom, why do so many mathematics teachers organize their instruction strictly around curriculum coverage? Why are some teachers reluctant to depart from mathematics, and why do they view moments of spontaneous dialogue in their classes as counter-productive distractions or “noise?” Why is it that so much of mathematics education research is focused on mathematics and human cognition, and not on inter-human relations? In my view, these tendencies in teaching practice occur because of the prevalent monological discourses of mathematics. When mathematics is viewed as a collection of disembodied and objective truths, which stand separate and apart from the human beings who enact them, it may well make sense to organize mathematics instruction around predetermined curricular objectives. However, if mathematics educators were to take a dialogical view of their subject matter, their approach to instruction would likely shift away from strict adherence to curriculum coverage. From the dialogical perspective, the act of involving the learner dialogically in mathematical discourse is the most important part of teaching. Mathematics instruction should be organized with careful attention to inter-human relationality.

Some educators might argue that bringing non-mathematical interactions into class is a waste of time, and that mathematics is sufficiently engaging and diverse as subject to foster dialogue. Indeed, I admire teachers who enable dialogue in their classes while adhering strictly to mathematics. However, given the problem of weak motivation discussed earlier, I believe that it would be far easier for many teachers to initiate dialogues around their students' interests, and then bring mathematics into the discourse once dialogues are established.

In my experience, conversing freely with my students without restricting myself to the mandated curriculum has been deeply rewarding, and ultimately very productive. The freedom to share in my students' interests and to bring up interesting topics enables me to become a full participant in the classroom discourse. As a teacher, I am not just a limited being, an agent of the curriculum or of mathematics, but a complete person who happens to be passionate about mathematics. Even when the conversations turn to music, or money, or love, or anything else that teenagers enjoy talking about, I often make sense of these themes through my mathematical orientation. I cannot really do otherwise, since mathematics is an integral part of my being. Mathematics is a mode of discourse that I use to interpret my reality. Numbers, probabilities, and formal thinking are never completely absent from my conversations with my students, no matter what the topic at hand may be.

Mathematics speaks volumes in my classes when school math is silent. Far from relinquishing the task of teaching mathematics, I use these occasions to engage with mathematics dialogically. To put it differently, I use my position as a relational hub in the mathematics class, which is to say a teacher and interlocutor, to show my students what it means to be a mathematizing human, co-existing with other humans, in a more-than-human world.

Teaching Interlude 3: Rope around the Equator

Suppose that the earth is a perfect sphere and that I tie a rope tightly around the equator. How long will that rope be? What information do you need to know in order to answer this question?

The students asked me how long the radius of the earth was. I told them that it was 6391 kilometers long. Individually and in small groups, they then quickly calculated that the rope would be about 40,000 kilometers long.

Now, suppose that I don't wish the rope to be so tight. I add 10 more meters of rope to the existing 40,000 kilometers, and then loosen the new longer rope evenly around the equator. Do you all agree that a gap will form between the earth and the rope?

The students nodded in agreement.

How tall will this gap be? What could pass through it? An amoeba, an ant, a child, or a mountain? Please give me your best estimate.

We took a vote. Almost all of the students thought that an ant could pass through the gap. A few thought that the gap would be too small for an ant to pass through, and voted for the amoeba.

OK. Now please calculate the height of the gap.

The students worked out some answers. After some discussion, everyone agreed that the correct answer was about 1.6 meters long. In other words, the gap is tall enough to let a child pass through. The students gave a collective gasp of surprise. I then told them that this result had surprised me too when I first came across it, and that it still does. Intuitively, it seems that when we lengthen a rope of 40,000 kilometers by 10 meters, the loosening effect should be negligible.

How can it be that the gap is large enough to allow a child through? How would you explain this result to a person who cannot calculate it?

I posed this question to the students, even though I did not know the answer myself. The students took about 15 minutes to come up with answers. A number of explanations were offered but they were mostly verbal reiterations of the calculations that were just performed. Then one of the students raised his hand and provided this answer:

> *For us, a gap of 1.6 meters looks big. But this gap 1.6 meters is added to the radius of the earth. If you compare 1.6 meters to the radius of the earth, which is 6391 kilometers, you can see that it is not large at all. In fact, it's tiny.*

There it was – a perfectly clear and sensible explanation that had eluded me for years. The proportional increase of 1.6 meters is negligible, even though a child can pass through the gap. The student was able to explain the result even though common intuition was unreliable in thinking through the question. His explanation was not one that I would expect to find in a mathematics textbook. New mathematics was created right there, in the moment, and it was brilliant. The incident made me smile for days afterwards. I was thankful to have trusted the class' collective intelligence enough to pose the question.

CHAPTER 5

LIFE IN MATHEMATICS: EVOLUTIONARY PERSPECTIVES ON SUBJECT MATTER

> If the last chapter veered off the subject matter of mathematics to some extent, this present chapter focuses squarely on mathematics as it manifests in the objective (UR) quadrant. I use developmental analysis to construct an evolutionary stage model for conceptions of mathematics. My reading of the model reveals that, largely due to the legacy of Platonism, the development of mathematics is stunted at the formalist/modernist stage. I argue that the reluctance to evolve mathematics past this stage robs the subject matter of much of its vitality.
>
> The explicit use of integral theory in the chapter permitted me to examine what the next stage in the evolution of mathematics, the integral stage, might look like. The chapter considers different approaches to transcend Platonism and to promote evolution of mathematics. One approach, detailed in considerable length, is *concept study*, a research methodology for interrogating teachers' mathematical knowledge that consciously activates multiple quadrants and levels. I use the integral lens to interpret the results of an extended concept study on multiplication.

Edgardo Cheb-Terrab is an applied mathematician who specializes in developing algebraic algorithms for Maple, a mathematical software package for symbolic computation. His algorithms solve classes of problems, including differential equations and special functions.

In 1999, Cheb-Terrab used his software to investigate solutions of Abel equations—a class of first-order non-linear differential equations that was first described by the Norwegian mathematician Niels Abel in the 1820s. By the end of the 20th century, the mathematics research literature contained solutions to nearly 40 types of Abel equations; each of these types was thought to require its own method of solution. Cheb-Terrab showed that all these types are special cases of an 8-parameter hyper-class. By devising a computer algorithm for solving all equations of this hyper-class, he expanded the range of solvable Abel equations far beyond what was thought possible. These results could not have been derived without a computer.

Even though one may have expected such innovation to be greeted enthusiastically, Cheb-Terrab's work has met with suspicion, and even dismissal by mainstream algebraists. "For many of them, it was heresy; surely a computer cannot solve problems that were impenetrable to such great mathematicians as Abel and Liouville" (E.S. Cheb-Terrab, personal communication, November 2, 2008). Ironically, Cheb-Terrab's censure by the research community came at a time when thousands of mathematicians, engineers, and physicists were already using his algorithms, and verifying the solutions obtained, in a variety of applications. It took four years, and a lengthy review process, to publish the findings (Cheb-Terrab & Roche, 2003).

Cheb-Terrab has since developed innovative computer-based solutions to other long-standing problems in algebra. Yet he regularly confronts obstacles to acceptance. As Cheb-Terrab (personal communication, December 13, 2008) noted, "[m]athematicians have received with discomfort almost every algorithm that I developed which seemed to challenge 'established truths'; but, in fact, these established truths were never anything more than incomplete truths holding back progress in their fields."

The issue of what constitutes acceptable mathematics innovation points to a prevalent orthodoxy among mathematicians around the question *"What is mathematics?"* Even though today's digital technologies enable new mathematical understandings, many mathematicians are unwilling to accept computerized solutions as "real mathematics." For them, the only true mathematics is that which manifests in the time-honoured mechanisms of formal proof. As Cheb-Terrab's example illustrates, these mathematicians, in their strict conformity to traditional modes of mathematical knowledge production, may be stunting the evolution of mathematics and contributing to stagnation of mathematical research.

Likewise, I believe that part of the blame for the current stagnation of mathematics education and pedagogy can be attributed to a shared orthodoxy among educators around the question, "*What is mathematics?"* From an evolutionary perspective, I understand the term "orthodoxy" as referring to rigid adherence to a particular worldview, and refusal to acknowledge and participate in the evolution of consciousness. Mathematics pedagogues often perceive mathematics as a treasured, monolithic, even sacred, body of knowledge, which must be preserved and passed on to future generations. This perception of mathematics often leads to a model of instruction that centers on transmission of stable knowledge.

This chapter explores the range of worldviews that respond to the question, "*What is mathematics?"* I will use integral theory to analyze the stages through which conceptions of mathematics have evolved to date, and where they are likely to evolve next. I examine barriers to the evolution of teachers' views on mathematics, and some approaches to overcome these barriers. I conclude with a discussion of the implications that an evolutionary view of mathematics holds for pedagogy, and in particular the need for educators to balance stable and emergent dimensions of mathematics in their instruction.

5.1 Stages of Mathematics

Davis (1996) traced the history of mathematics, and proposed five major eras, or mentalities into which it may be divided: *oral, pre-formalist, formalist, hyper-formalist,* and *post-formalist*. Integral philosophy suggests that these mentalities may form a dialectic sequence of increasingly complex human conceptions of mathematics, that is, a stage model for mathematics. I will show in this section that the five mentalities are developmental stages, by correlating them with the broader structure-stages, or worldviews, used by integral writers (e.g., Wilber, 2006a; McIntosh, 2007) to describe the development of consciousness. The worldviews are: *archaic, tribal, traditional, modernist, post-modern,* and *integral*. I will also show that the mentalities satisfy the subject-object mutuality that characterizes developmental processes, as indicated by Kegan (1994).

At each mentality, or stage, I explore these questions:

1. How does this stage respond to the question, "What is mathematics?"
2. What is the connection between mathematics and the natural world?
3. How does this stage position the relationship between knowledge and knower?
4. What mathematical technologies are used at this stage?
5. How are mathematical truths validated?

5.1.1 The Oral Stage

This stage refers to societies that existed before the invention of writing in different parts of the world. In oral cultures, mathematics and mathematical meanings are found only in immediate experience and practical action. Mathematics is tightly bound to the knower's immediate environment in the natural world. Mathematical objects are classified by practical situation; numbers, for instance, are used as adjectives, rather than as nouns. Mathematical knowledge manifests in human processes, such as counting. The main technology of mathematics at this stage is oral narrative. The oral stage corresponds to the tribal stage of consciousness, in which tribal myths and immediate experience establish validity and truth.

5.1.2 The Pre-formalist Stage

The invention of writing brought mathematics into the visual-representational realm. In the process, mathematical knowledge was preserved in symbol systems that enabled it to be understood over time and distance, and so it attained a similar separation from its knowers as the textual inscriptions of natural languages. At this stage, mathematics is understood as a mode of reasoning about unchanging forms, or essences, in the natural world. Mathematical knowledge resides outside of the knower, and is discovered by empirical observation. The technology of writing accords mathematics an independent existence through abstract objects, such as numbers, abstract categories, and geometric forms. This stage corresponds to traditional consciousness, in which scriptures that express the universal mythic order are the standards of validity and truth.

5.1.3 The Formalist Stage

The formalist stage of mathematical history originally emerged briefly in Ancient Greece, but it came into full fruition only in the early modern European era of Newton and Descartes. Mathematics is a distinct discipline in the formalist stage, with a separate body of knowledge and knowledge-producing methodology. The methodology of formal logic applies strict derivation rules to fundamental propositions, or axioms, in order to produce new mathematical results. At this stage, mathematical axioms, such as Euclid's Postulates, are framed in terms of observation from the natural world, though the description of the natural world is only one of several concerns of mathematics. The technologies of the formalist stage include the mechanism of formal proof, calculating devices, and formal mathematical representations, such as the Cartesian plane. This stage corresponds to modernist consciousness, in which reason is the standard of validity and truth.

5.1.4 The Hyper-formalist Stage

The hyper-formalist stage arrived at the beginning of the 20th century, when the emergence of non-Euclidean geometries enabled mathematicians to manipulate the axioms of geometry. Mathematicians of the time, such as Hilbert and Russell, set out to reconstruct mathematics as a purely formal system with little or no correspondence to the natural world. Mathematical knowledge exists at this stage only to the extent that it can be derived within the logical parameters of a given formal system. The technologies of the hyper-formalist stage include non-standard logics and abstract grammars. The hyper-formalist stage is in a sense an extreme extension of formalist consciousness, as truth and validity are established solely by syntactic adherence to the rules of invented formal systems.

5.1.5..The Post-formalist Stage

In the 1930s, Gödel's proof of the incompleteness of formal systems challenged the hyper-formalist project, and cast mathematical certainty into doubt. The post-formalist stage regards mathematics as a socially-constructed interpretive discourse, rooted in our need to make sense of our environments and to construct our reality. Far from being separate from knowers, mathematical knowledge at this stage is embodied and enacted by both individual and collective knowers. The principal interpretive technology of this stage is deconstruction. This stage corresponds to the post-modern stage of consciousness, in which notions of validity and truth are themselves taken to be social and discursive constructions.

In Table 3, I list the subject and object at each of the five stages described thus far. The subjects respond to the question, "*what is mathematics?*", and the objects address the question, "*what tools does mathematics use?*"

Stage	Subject *(The nature of mathematics)*	Object *(The tools of mathematics)*
Oral	Human processes and interactions about immediate phenomena	Objects in the environment
Pre-formalist	A mode of reasoning about essences	Abstract mathematical objects
Formalist	A separate discipline with a formal mode of reasoning	Rationality
Hyper-formalist	A formal system	Formalism
Post-formalist	A socially-constructed interpretive discourse	Discourse

Table 3. Subject-object mutuality in Davis' stages of mathematics

As can be seen in Table 3, the stages follow a pattern of subject-object mutuality, in which the subject of one stage of development becomes the object of the subject of the next stage of development. At the oral stage, mathematics consists of immediate experiences and human processes, such as counting. At the pre-formalist stage, the results of these processes are reified into abstract mathematical objects, such as numbers. Mathematics at this stage is a mode of reasoning that uses abstract objects to make sense of observed phenomena in the universe. At the formalist stage, the mode of reasoning itself is formalized into the mental construct that we know as “rationality.” Mathematics at this stage is the body of knowledge derivable by formal proof from initial axioms drawn from the natural world. At the hyper-formalist stage, the process of formalism itself is objectified, and formal proof is seen as one of many possible formal logical systems. Mathematics at this stage consists of all results that can be derived by applying constructed logical rules to arbitrary initial axioms. Finally, at the post-formalist stage, both rationality and formalism are treated as discursive forms within a social matrix. Mathematics at this stage becomes a socially-constructed interpretive discourse.

The subject-object mutuality observed over the history of mathematics appears to confirm that the different mentalities represent a coherent evolution of conceptions of mathematics. This evolution follows a dialectic pattern, and proceeds through distinct developmental stages. I will now proceed to use this evolutionary understanding of mathematics to examine the current prevailing conceptions of mathematics, and their impact on mathematics pedagogy.

5.2 Current Conceptions of Mathematics

Much of mathematics education practice of today seems to reside in the formalist and pre-formalist conceptions of mathematics. Mathematics educators often regard mathematics as a static body of knowledge that represents extra-human reality. As Ernest (1985) noted, this Platonic view of mathematics greatly constrains mathematics pedagogy.

> By locating the source of mathematics in a pre-existing static structure, Platonism results in a static body-of-knowledge view of mathematics. Platonism discounts both man as a creator of mathematics and the importance of dynamic processes in mathematics. In educational terms this corresponds with the view of mathematics as an inert body of knowledge which instruction transmits to the student. (p. 607)

Ernest refers to an important connection between conceptions of mathematics and pedagogy. When mathematics is viewed as transcendent and essential, it follows that the teacher's role is to be a faithful conduit and gatekeeper for the established knowledge that informs the subject. This view supports a transmissive pedagogical model, which measures success according to conformity with pre-determined results, and employs systems of evaluation and discipline that aim for methodological conformity. Ernest argues against this model from a post-formalist perspective that values the enacted, creative, and dynamic human dimensions of mathematics.

Ernest is not alone among mathematics education researchers in criticizing transmission pedagogy that is founded on Platonic conceptions of mathematics. In the past three decades, a significant number of researchers have promoted what we are calling a post-formalist perspective of mathematics in education. They include mathematicians (e.g., Davis & Hersh, 1981) who reframed mathematics as one of the humanities, philosophers of mathematics (e.g., Lakatos, 1976) who analyzed the evolutionary dynamics implicit in mathematics production, and mathematics educators (e.g., Applebaum, 1995) who drew on cultural studies to resituate mathematics in social, ethical, and ideological terms. While these researchers have been instrumental in unearthing and elucidating some of the problematics of traditional and modernist education, postmodern thinking has gained little traction in the actual practice of mathematics education. Among all subjects in the school curriculum, mathematics seems most resistant to postmodern discourses of diversity and intersubjectivity.

From an evolutionary perspective, the problem is one of stagnation, or arrested development. Even though there is no shortage of ideas about the directions in which mathematical pedagogy might evolve, most mathematics educators remain largely indifferent to innovation. Notions of pragmatics, diversity, subjectivity, and intersubjectivity inevitably come up against the limiting Platonic conceptions of mathematics, and must contend with the certitude of "*2+2=4*." I now proceed to explore the question, "*What are some of the barriers to transcending Platonism?"*

5.3 Barriers to Evolution

Modernist consciousness is characterized by rationality, and relies on the scientific method and objective reasoning for truth validation. For these reasons, mathematics has become the paramount technology of modernity. Without the tools of mathematics, neither physics, nor computers, nor stock markets would exist. The rapid pace of technological development in modern times would not have been possible without the time-honoured truths of arithmetic, algebra, and calculus. Nowadays, mathematics plays an increasingly vital role in areas such as genetics, neuroscience, and ecology. Its significance is so ascendant in the information age that Baker (2008) has proposed that the *numerati* have now overtaken the *literati* in the role of defining cultural possibility.

It is important to note that the advances of mathematics in modern times were achieved with a Platonic perspective in mind. In other words, Platonic mathematics has served humanity well for many centuries. Given this complex interplay of intellectual inertia, economic investment, and emotional commitment, it is not surprising that there is massive resistance to questioning, much less transforming the Platonic narrative. I often encounter individuals who argue passionately that mathematical truths, such as "*2+2=4*," are as real in our world as material objects, such as trees. Indeed, the Platonist position holds that mathematical truths are "more real" than trees, since they represent ideal forms that transcend material existence. Modernist consciousness tends to regard mathematics as the guidebook to the phenomena of the physical world. This prevailing view of mathematics is not likely to change given the spectacular success of the present utilitarian conjunction between mathematics and scientific progress.

Educators are generally better positioned than the public at large to appreciate the shortcomings of the Platonic view of mathematics, as the defects of present-day school mathematics can often be traced to educational purposes that are supported philosophically by this view. Yet, since educators operate within cultural and social structures that give rise to modern mathematics curricula, they are more likely to seek pragmatic instructional solutions than to deconstruct and revise their views on the subject matter. In my work, I have encountered many teachers who consider the question *"what is mathematics?"* as having little or no relevance to their practice.

Educators who wish to consider alternatives to Platonic assumptions about mathematics have to contend with certain cognitive barriers. For example, post-formalist perspectives are concerned with issues of intersubjectivity, justice, equity, class, race, and gender in mathematics. As Table 3 shows, the objects of post-formalist mathematics are discursive formations; they are very different from the mathematical objects of previous stages—numbers, formal proofs, and abstract grammars. This qualitative difference manifests in the language of mathematics education practice and is very difficult to overcome. I regularly hear complaints from my pre-service teacher education students that my classroom elaborations of post-formalist perspectives are "not mathematics," and that they belong instead to the disciplines of the social sciences and the humanities. When I speak with practicing teachers, I often sense that my post-formalist stance is removed from their daily experiences of prescribed learning outcomes, tests, marks, and correct answers.

Transcending Platonic assumptions about the nature of mathematical truth and ontology requires not only a profound change in language, but also a negotiation of some very difficult philosophical questions: *Is mathematics discovered or invented? What is the role of formal mathematics? What is truth? Can 2+2 equal anything other than 4? If so, what are the implications for our understanding of the natural world?* The absence of clear answers to these questions makes the certitude of Platonic mathematics all the more appealing to the average teacher.

Two accounts from the history of mathematics illustrate the staying power of the assumptions of Platonism in education. The New Math reforms of the 1960s sought to introduce a hyper-formalist perspective through a curriculum that included set theory, number bases other than 10, and Boolean algebras. New Math challenged the long-standing correspondence between school mathematics and phenomena in the natural world. In brief, these reforms failed within a few years, largely in response to complaints that this approach to mathematics was removed from the students' everyday experiences, and that many teachers and parents did not understand it fully. In 1989, the publication of NCTM's Standards heralded the introduction of an equity agenda that called for "mathematics for all." From an epistemological standpoint, the Standards challenged the traditional view of mathematics as a fixed body of knowledge, by focusing on four strands: mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections. The implementation of reforms in California in the mid-1990s which were based on the Standards led to full-scale math wars (cf. Schoenfeld, 2004) that divided the educational community across the United States for the better part of a decade.

Integral philosophers (e.g., Wilber, 2006a) assert that new worldview systems emerge only when previous ones become insufficient for dealing with the problematics created by changing life conditions. As Dewey (1910) similarly said of the transition from one worldview to another, "We do not solve them, we get over them" (p. 18). A critical mass of disorienting events or cognitive dissonances is required to move thinking to a new vantage point from which new aspects of reality can be seen. From my earlier discussion, it appears that the critical mass required to transcend Platonism is still building toward transition. Yet it strikes me that, given the immense benefits that Platonic mathematics has brought to humanity, opposing the material achievements that are commonly linked to it would be counter-productive to the project of evolving mathematics. An approach that values and integrates the different stages is likely needed. I will now proceed to explore what such an approach might entail.

5.4 The Integral Perspective

Academic turf battles in mathematics education are the result of the attempts of the traditional, modernist, and postmodern consciousnesses to assert their perspectives in the field. The New Math reforms and the math wars of the 1990s are two examples of such battles. As we have seen, when it comes to perspectives on subject matter, the two main camps consist of those who view mathematics as transcendent and stable, and those who view it as embodied and emergent. While educators (Ball et al., 2005) agree that the disputes within the discipline do not serve the needs of students, there is little agreement on how to reconcile the seemingly opposed claims of the different camps within mathematics education. My preceding analysis of the evolution of mathematics, combined with insights of integral philosophy, may offer one starting point for reconciliation.

Integral thinking highlights the importance of the evolutions of consciousness and culture to global well-being. Integral writers (e.g., Gebser, 1984; Wilber, 2006a) pointed to the emergence of a new stage of consciousness, or epistemological interpretive framework, which they called “integral consciousness.” In contrast to mental-rational consciousness that precedes it, integral consciousness is multi-perspectival, that is, characterized by lack of attachment to monological perspectives. Since all preceding structures of consciousness in the spiral are transparent to integral awareness, it is able to integrate them and “live through” them, rather than be controlled by any one of them. While the integral structure does not come with a ready-made set of values, it seeks to solve problems by bringing as many perspectives as possible to bear on given situations in order to arrive at appropriate, yet never permanent, solutions.

Integral consciousness seeks to promote the health of the entire evolutionary spiral by acknowledging the dignities and contributions of every significant historical worldview—including the traditional, modernist, and post-modern. Likewise, it seeks to overcome constructively the limitations of these worldviews. Doing so requires the emergent capacity of *vision-logic*, which Kegan (1994) described as “the capacity to see conflict as a signal of our overidentification with a single system” (p.351). Vision-logic is network logic. It is the ability to see how different elements of an evolutionary system work together dialectically, to “live through” their different perspectives, and to harmonize them by valuing each element for its contribution to the entire system.

From an integral perspective, the question of whether mathematics is Platonic or embodied, stable or emergent, is founded on false dichotomies. Mathematics is all of these things and more. Every conception of mathematics has developed in response to a different set of life conditions, and its values and practices have their appropriate applications under these life conditions. The evolution of new conceptions, or stages, does not negate earlier ones, but rather exhibits a dialectic pattern of transcendence and inclusion. In this pattern, new stages retain the robust structures of earlier stages, while complexifying them into higher-order unities. In other words, mathematics is both Platonic and embodied, stable and emergent, objective and socially-constructed, fixed and enacted, a science and a branch of the humanities. The observer’s vantage point and context determine which aspect of mathematics is revealed at any given moment.

With this understanding in mind, we can see that formal mathematics is a narrative that captures and codifies the more stable dimensions of mathematics. It does so by turning human processes and thought patterns into abstract discursive objects which, over time, take on transcendent, universal qualities. In her analysis of the development of mathematical discourse, Sfard (2008) explained that the process of objectification is indispensible, as it provides mathematical discourse with its principal advantage—the ability to represent complex ideas with concision and compactness. By eliminating the temporal dimension of phenomena, objectification also helps users of mathematics cope with the fluidity of human experience.

From a post-formalist perspective, Platonic conceptions of mathematics often appear to be wrongheaded and even dangerous. Postmodern critiques (e.g., Appelbaum, 1995; Walshaw, 2004) have deconstructed Platonism, and identified various problems that arise when formal mathematics is treated as the only “real” mathematics. Among these problems are: marginalization of other mathematics, hegemony of Western thought, inequitable access to mathematics, and alienation of knowers. However, from an integral perspective, Platonism should be valued for its appropriate contributions in certain contexts, just as post-formalist perspectives ought to be honoured for their appropriate contributions in other contexts. One should not seek to defeat Platonism, but rather to harmonize the stability that it has conferred on mathematics with emergent and subjective potentialities.

The difference between integral and post-formalist perspectives views on Platonism signals that the integral perspective may be a new stage in the evolution of mathematics that will succeed the post-formalist stage. As discussed earlier, at the post-formalist stage, mathematics is a socially-constructed interpretive discourse. The integral stage organizes interpretive discourses into evolving bio-psycho-social systems. At this stage, mathematics becomes an evolving system of interpretive discourses, or perspectives. As Table 4 shows, Kegan's subject-object mutuality is again confirmed.

Stage	Subject *(The nature of mathematics)*	Object *(The tools of mathematics)*
Post-formalist	A socially-constructed interpretive discourse	Discourse
Integral	An evolving system of interpretive discourses (perspectives)	Interpretive discourses

Table 4. Subject-object mutuality in the latest two stages in the evolution of mathematics

5.5 Technologies of Mathematics

The integral perspective invites us to harmonize the Platonic conception of mathematics with emergent and embodied mathematics. As we embark on this project of integration, we accept that the practice of mathematics pedagogy has so far favoured Platonism almost exclusively over emergence-embodiment. So any intervention that would ease pedagogy's pervasive bias toward Platonism is welcome.

I have searched for such interventions in my work with pre-service and practicing teachers, work which I conducted together with Brent Davis. At first, we opted for a philosophical approach, and asked our students to consider the question, "*Is mathematics discovered or created?"* Most of them appeared to be uninterested in this highly theoretical question, and even those who attempted to respond were soon tangled in abstractions. We concluded that concrete examples were needed to encourage meaningful debate around the topic.

As we tried out different mathematical examples, we discovered that most of them led back to Platonic identifications held by the pre-service teachers; only a few examples prompted more emergent reactions. For instance, no teacher in the class was willing to believe that the equation $2+2=4$ is a human construction. The students argued, with considerable conviction and passion, that this equation represents a universal truth that holds true for everyone, everywhere, and for all time. In fact, they argued, it would still hold true if there were no knowers in the universe to know it. On the other hand, when we introduced the abstract operator '↑' as $a \uparrow b = 2a - b$, the students were unanimous in their agreement that the equation $3 \uparrow 2 = 4$ is a human construction. They explained that the abstract operator '↑' had just been defined by us, and so it must be a human creation. When we suggested that the operator '+' was also defined by humans some time in history, some students responded that '+' is a "real" operator, while '↑' is not.

In general, we found that our students showed an emotional commitment to the transcendence and timelessness of the results of formal mathematics. The more elementary the result, the more committed students were to its permanent status. Wilber (2006a) referred to significations that have such transcendent status as *Kosmic habits*; in Foucauldian language they are called *technologies*. The older the habit or technology, the more self-evident and secure it is to those who participate in it. For example, while the rudimentary equation $2+2=4$ was created by social agreement at some early point in human history, once it was formalized into language and conventional signs, it acquired an independent and objective existence beyond a series of historically determined symbols. With every new generation, the reality of $2+2=4$ became more entrenched and transcendent. Some may go so far as to say that it has an ontological status apart from actual objects and significations within language. Likewise, many concepts of school mathematics have been in use for so long that they have attained fixed meanings that conceal the circumstances that gave rise to them over time. Yet teaching and learning of mathematics are in fact heavily dependent on multiple meanings that emerge through interpretation. As fixed as *2+2=4* may seem to us, as determiners of its metaphysical status, every pedagogical encounter enhances the collective meaning of *2+2=4* by occasioning unique subjective meanings.

In order to reveal the constructed nature of mathematical technologies, we continued searching for examples of interpreted objects from school mathematics that are not as contrived as the abstract operator '↑'. In our search, we identified coordinate geometry as a useful example. Many of our students readily agreed that the invention of the coordinate plane by Descartes in 1637 allowed mathematicians to conceive of geometric figures, such as circles, in a way that is radically different from that used by Greek geometers. We then probed further by asking, "*What are circles?"* Some of the teachers began to appreciate that rather than being pre-existing objects, circles are created through human interpretive frames.

Logarithms turned out to be another good example, as the students realized that they were invented by Napier in 1614 as a technology for multiplying large numbers. We found that our pre-service teachers preferred the language of "technologies of mathematics" to that of "emergent mathematics." We suppose that their preference is attributable to their familiarity with the metaphors of technological development, and to persistent Platonic commitments to original meanings.

Coordinate geometry and logarithms proved to be productive examples for our pre-service teachers to consider because, from a historical perspective, the emergence of these concepts has provided new metaphors for pre-existing mathematical objects. As our experiences indicate, the emergent-embodied nature of mathematics can be disclosed by engaging concrete and familiar mathematical examples that lend themselves to interpretation through multiple images and metaphors. We will now proceed to describe an ongoing study that we have been conducting with a group of experienced teachers around the meanings of multiplication.

5.6 Emergent-Embodied Mathematics in Action

Our study has unfolded over a period of two years. It involved a group of 11 experienced middle-school teachers who gathered in monthly concept-study meetings to discuss and deconstruct different curriculum topics. These ongoing meetings have been conceived as collective knowledge-producing occasions, through which mathematics educators identify, interpret, interrogate, invent, and elaborate images, metaphors, analogies, examples, exemplars, exercises, gestures, and applications that are invoked in efforts to support the development of students' mathematical understandings. The concept of multiplication has received the most attention from the group. Indeed, discussions of other concepts have regularly gravitated to the subject of multiplication.

We began with the direct question, *What is multiplication?* After the two most obvious answers— 'repeated addition' and 'grouping'—were given, we asked, *And what else?* The rest of the morning was organized around discussions of pedagogical difficulties that crop up in teaching multiplication, investigations of when and how elaborations are introduced, and analyses of teaching resources for multiplication. The end result was a listing of metaphors, images, analogies, and applications, as shown in Figure 7.

Multiplication involves

- repeated grouping
- repeated addition
- sequential folding
- layering
- the basis of proportional reasoning
- grid-generating
- dimension-changing
- intermediary of adding and exponentiation
- opposite/inverse of division
- stretching or compressing of number-line
- magnification
- branching
- rotating a number line
- linear function
- scaling
- and so on …

Figure 7. A teacher-generated list of interpretations of multiplication

The teachers seemed surprised at the lengthy list of realizations of the signifier "multiplication" that they were able to generate. Their surprise was summarized by the comment, "Apparently we don't have a good handle on what we know yet." The teachers also recognized that the point of the list was not to provide an exhaustive summary of interpretations of multiplication, but rather to indicate the range of associations that were accessible to the group on this day.

The next major development on the subject of multiplication took place several months later, when a few of the teachers urged the group to organize all of the realizations in the list according to grade levels and thematic categories. The resulting chart is shown in Figure 8

.

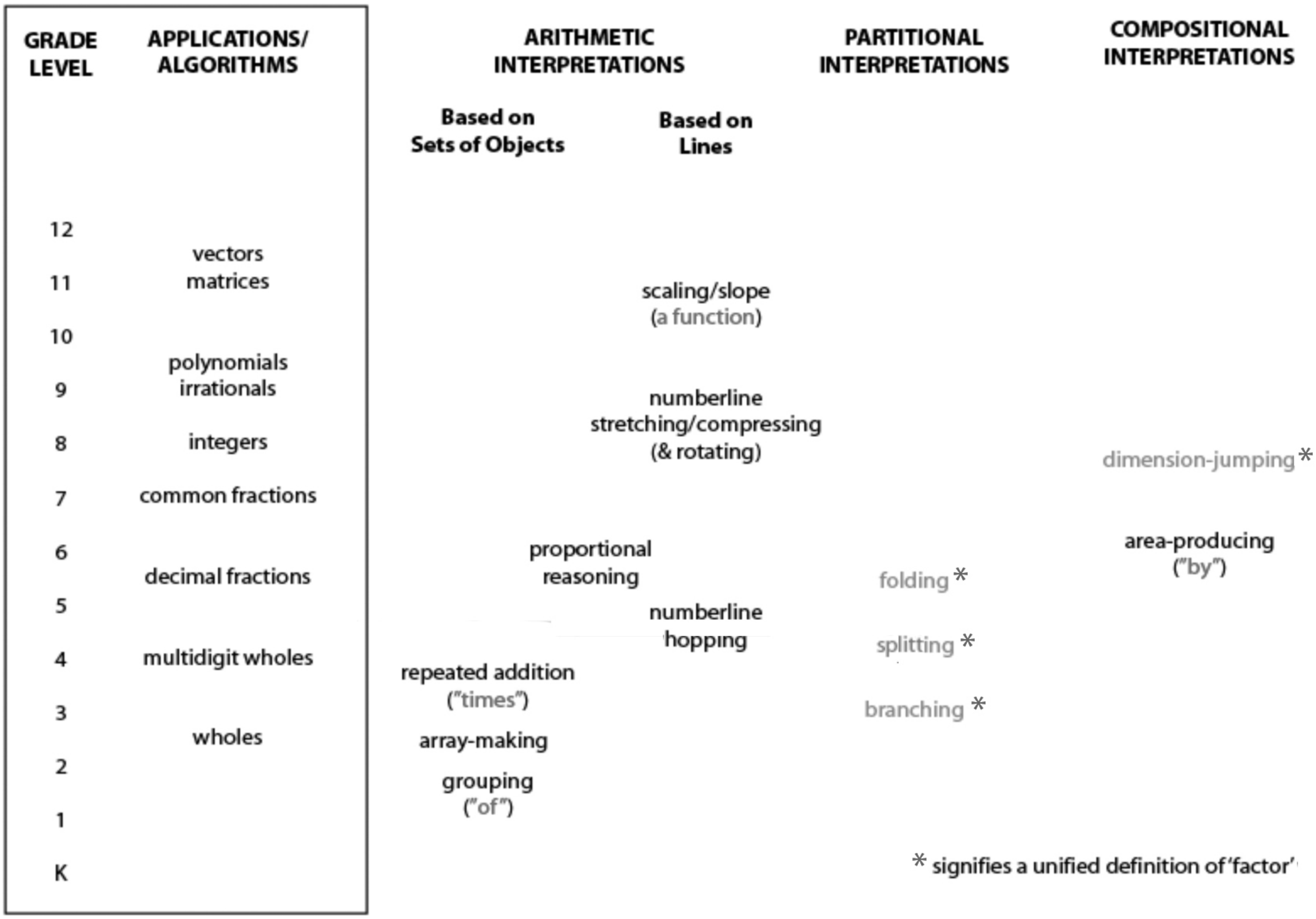


Figure 8. An evolving landscape of the concept of multiplication

Upon examining their mapping, the teachers were surprised to realize that distinct and coherent strands of interpretation are systematically developed over the K–12 experience. The different realizations of multiplication, far from being random or isolated, were organized into grander interpretive structures. The teachers acknowledged that they had in fact participated in the systematic development of these structures prior to the study, without having reflected on their participation.

The next development took place when the group tackled the question, *Is 1 prime?* The teachers began to explore the relevance and implications of different realizations of multiplication to the question. In the discussion, the teachers often framed their remarks as "If …, then …" statements, locating their comments within specific metaphorical domains. In other words, the teachers were consciously engaging in analogical, as opposed to logical, reasoning. Some of the results are presented in Figure 9.

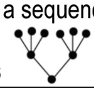
If multiplication is …	… then a product is:	… a factor is:	… a prime is:	Is 1 prime?
REPEATED ADDITION	sum (e.g., $2 \times 3 = 2 + 2 + 2 = 3 + 3$)	either an addend or a count of addends	a product that is either a sum of 1's or itself.	**NO**: 1 cannot be produced by *repeatedly* adding any whole number to itself.
GROUPING	a set of sets (e.g., 2×3 means either 2 sets of three items or 3 sets of two)	either the number of items in a set, or the number of sets	a product that can only be made when one of the factors is 1	**YES**: 1 is one set of one.
BRANCHING	the number of end tips on a 'tree' produced by a sequence of branchings (e.g., 2×3 means )	a branching (i.e., to multiple by n, each tip is branched n times)	a tree that can only be produced directly (i.e., not as a combination of branchings)	**NO**: 1 is a starting place/point … a pre-product as it were.
FOLDING	number of discrete regions produced by a series of folds (e.g., 2×3 means do a 2-fold, then a 3-fold, giving 6 regions)	a fold (i.e., to multiply by n, the object is folded in n equal-sized regions using $n-1$ creases)	a number of regions that can only be folded directly	**NO**: no folds are involved in generating 1 region
ARRAY-MAKING	cells in an m by n array	a dimension	a product that can only be constructed with a unit dimension	**YES**: an array with one cell must have a unit dimension

Figure 9. Some analogical implications of different realizations of multiplication

The teachers came to appreciate through first-hand experience that humans are not merely logical creatures, but association-making beings whose capacity for formal reason operates alongside their predisposition for making connections (cf. Lakoff & Johnson, 1999). This appreciation became the central point of the engagement, so much so that one of teachers remarked, "No wonder the kids find that so difficult."

A recent layer in the group's ongoing concept study of multiplication is that of shared reconciliation of seemingly different realizations of multiplication into unified blends. The first blend was a grid-based representation of multiplication, shown in Figure 10, which pulls together several realizations—including repeated addition, array-making, and area-making—as it highlights procedural similarities in handling additive multiplicands across diverse number systems and algebraic applications.

x	30	+ 5	
20	600	100	700
+ 5	150	25	175
	750	125	875

x	3	+ 0.5	
2	6	1	7
+ 0.5	1.5	0.25	1.75
	7.5	1.25	8.75

x	3	+ $\frac{1}{2}$	
2	6	1	7
+ $\frac{1}{2}$	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$
	$7\frac{1}{2}$	$1\frac{1}{4}$	$8\frac{3}{4}$

x	$3n$	+ 5	
$2n$	$6n^2$	$10n$	$6n^2 + 10n$
+ 5	$15n$	25	$15n + 25$
	$6n^2 + 15n$	$10n + 25$	$6n^2 + 25n + 25$

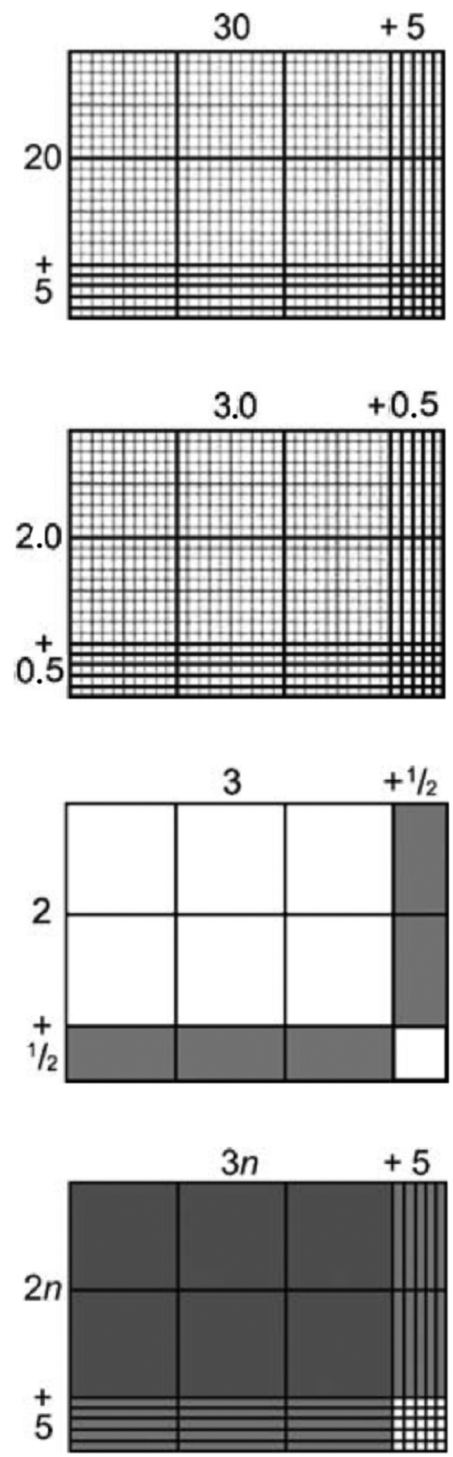


Figure 10. A grid-based blend that highlights the similarities of multiplicative processes involving additive multiplicands

Another blend was created when a teacher noted that the number-line-stretching interpretation could be combined with the mapping-function interpretation, as shown in Figure 11. This blend led directly to an intuitive graphical "proof" of the result that the product of two negative numbers must be positive. The teachers were very satisfied in having created new mathematics, which had not been previously encountered by anyone in the room.

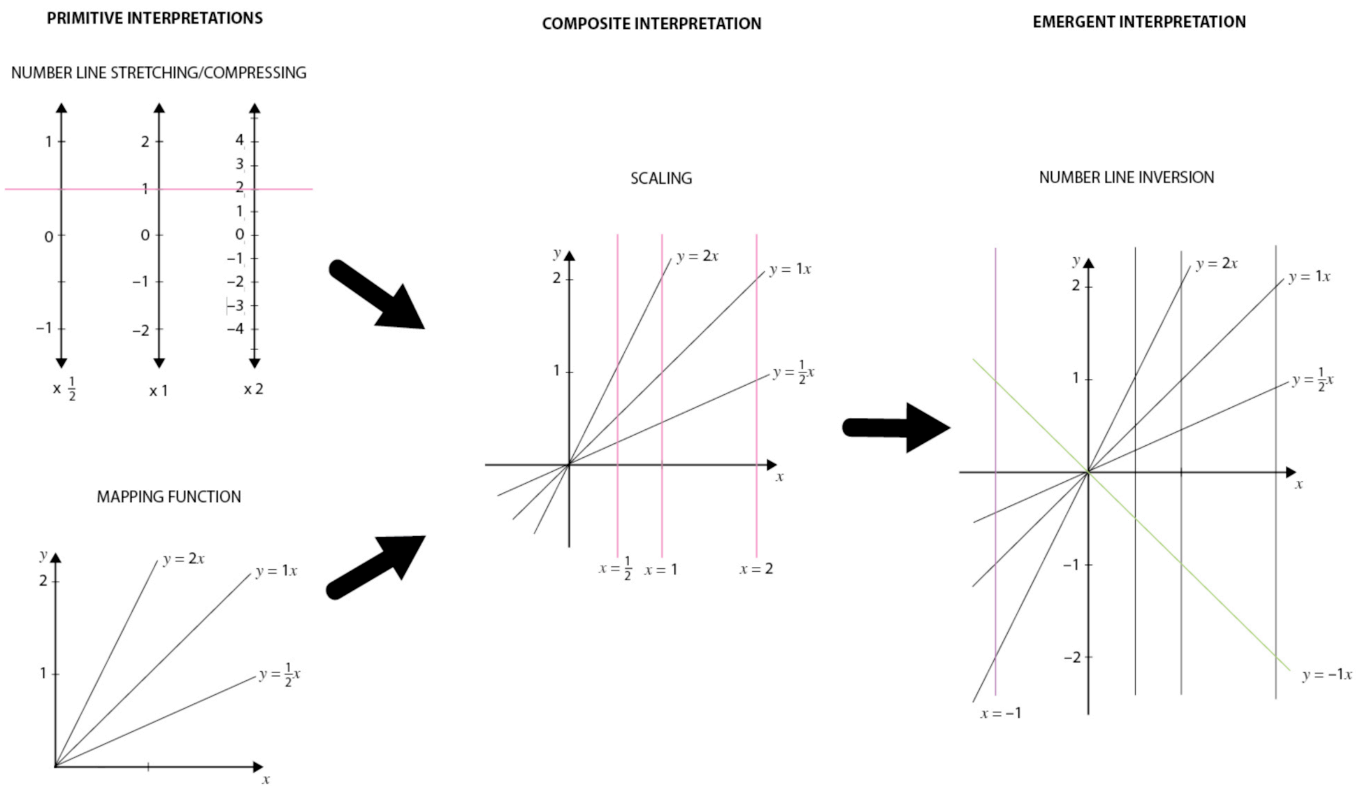


Figure 11. A graph-based blend that combines linear models of multiplication

By opening up a familiar mathematical concept for hermeneutic questioning and elaboration, our concept study of multiplication underscored the dynamic, embodied, and enacted dimensions of mathematical knowledge. As the teachers explored tacit layers of mathematical knowledge, and as they constructed emergent knowledge through collaboration, their Platonic assumptions about the nature of mathematical knowledge were challenged. Reflecting on the group's extended engagement with multiplication, some of the teachers said that they "have really been able to get inside the idea," and to "feel as though [they're] really contributing to how multiplication is understood." Their understandings of multiplication stood in sharp contrast with their general conception of mathematics. As two of the teachers put it, "It feels like it's outside of us …", and "…we feel like we're outside of it." The teachers felt alienated from (Platonic) mathematics, yet inhabiting, and even responsible for, (participatory) multiplication. The contrast reflects the need for mathematics educators to transcend Platonism, and to develop a more participatory and generative pedagogy.

5.7 Living Mathematics Pedagogy

The integral perspective, which recognizes the systemic evolutionary dimensions of mathematics, calls on teachers to embody and enact a living pedagogy that promotes the evolution of culture and consciousness. To achieve this goal, educators should be aware of the structures of evolution in their field, pay attention to the evolutionary tensions among different perspectives on mathematics, and harmonize these tensions as they arise in pedagogical moments.

As discussed previously, the evolution of mathematics is governed by a pronounced tension between stability and novelty. Stability manifests in conceptions of mathematics as Platonic, fixed, explicit, and formal. Novelty manifests in conceptions of mathematics as emergent, embodied, tacit, enacted, and participatory. The stable, transcendent conception of mathematics dominates mathematics teaching and learning at the moment. Therefore, educators who wish to promote cultural evolution in mathematics need to find and employ skillful means for transcending Platonism. In my work, I have identified several principles that may assist educators in this task. They are: elaborate the specific, encourage multiplicity, and use active language. I will examine each of these principles in turn.

As the concept study of multiplication has shown, elaborative engagement with specific mathematical concepts can be an effective way to uncover emergent-embodied features of mathematics. My experience indicates that the more basic the mathematical concept under scrutiny, the more confidently participants engage with it .The process of elaboration reduces expectations for performative instrumentality, which are prevalent in current mathematics pedagogy. The focus of mathematical inquiry then shifts from the question "*What is the correct answer?"* to the question "*What else is there to say about the mathematical situation?"* The shift takes us from exclusive and fixed mathematical meanings to shared and subjective meanings. The activity of hermeneutic elaboration enables participants to enact, reflect on, and explicate tacit and embodied knowings. In the process, participatory learning environments that can accommodate and encourage multiple opinions are created.

Multiplicity is a crucial element of living pedagogy, since the “bumping together” of different ideas is essential for the emergence of novelty (cf. Davis and Sumara, 2006). Mathematics educators can promote multiplicity in teaching and learning in a variety of ways. Some of them are: employing a multiplicity of mathematical metaphors and images to understand a mathematical concept, assessing a multiplicity of approaches to solving a given problem, and engaging a multiplicity of curricular and non-curricular topics in instruction. It is hoped that students who are exposed to multiplicity and plurality in classroom discourse will be open to the multiple meanings and different interpretation of mathematical concepts.

The language that educators choose to use may either enable or constrain students’ interpretive possibilities. For example, the instructions “*find the correct answer*” and “*think of different ways in which you may approach this mathematical situation*” may initiate different learning processes. Platonic mathematics is characterized by a comprehensive, fully-alienated discourse about mathematical objects. Educators who would like to approach mathematics differently should strive to alter their discursive practices by replacing some references to objects with descriptions of human actions. For example, a teacher may define a triangle by saying, “*A triangle is a 3-sided polygon,*” or alternatively by saying, “*We shall call any 3-sided polygon a triangle.*” The latter utterance reminds learners that the human act of naming is co-implicated in every mathematical definition. Admittedly, altering the well-rehearsed alienated discourse of Platonic mathematics is no small task. However, engaging in this project can sharpen pedagogues’ sensitivities to the subtle ways in which discourse shapes conceptions of mathematics in their classes.

Educators who adopt the three principles—elaborate the specific, encourage multiplicity, and use active language—will be providing some of the necessary conditions for emergence of novelty (cf. Davis & Sumara, 2006) and dialogue (cf. Sidorkin, 1999) in their classes. These educators should be ready to receive the new mathematical meanings that are sure to appear. Students today are exposed to vast amounts of mathematical information through television, video games, and the Internet. This information provides numerous living contexts for novel constructions of mathematical meanings. When these contexts come up in classroom interactions, teachers ought to take interest in them, since they are the learners' true lived experience. Teachers should be willing to investigate these contexts together with their students, explicate the mathematics that emerges, and connect this new mathematics to pre-existing mathematical results.

An integral living pedagogy offers educators the opportunity to teach an ever-evolving and innovative subject matter. Teachers who are aware of the dialectic processes of evolution may negotiate the rich stability of formal mathematics with the refreshing novelty of their students' emergent interpretations. Teachers and students co-create mathematics by elaborating, deconstructing, and infusing mathematical concepts with new layers of meaning. The integral perspective views both pre-established and emergent contexts, the past and the present, as valuable sources of inspiration for hermeneutic elaboration.

Living mathematics pedagogy situates teachers as vital participants in the creation of mathematical possibilities. Far from being peripheral agents who passively transmit established results of mathematics, teachers give shape and substance to cultural mathematics. An integral pedagogy of life stands to breathe new life into school mathematics and cultural mathematics, just as Edgardo Cheb-Terrab's computer tools breathe new life into long-standing problems of research mathematics.

Teaching Interlude 4: Do Circles Exist?

Our investigations into the question of how many sides a circle has took many unexpected turns. Quite early on, we discovered the need to be very precise about our terminology and the ways in which we use terms. *What is meant by "side"? Must a side be straight? If not, in what sense does a triangle have three sides?*

When the students researched these questions on the Internet, they came across some different, and occasionally conflicting, answers. Since the terms "polygon" and "vertices" were used in some of the answers, the class had to incorporate them into the discourse. They also came upon new and unusual mathematical figures: the *digon*, a two-sided polygon on a spherical space, and the *apeirogon*, an open polygon with infinitely many sides. All these discoveries brought up even more questions. *Is a circle a polygon? What makes an octagon an octagon– its eight vertices, its eight sides, or both? Can a polygon cross itself? Does a polygon need to be closed?*

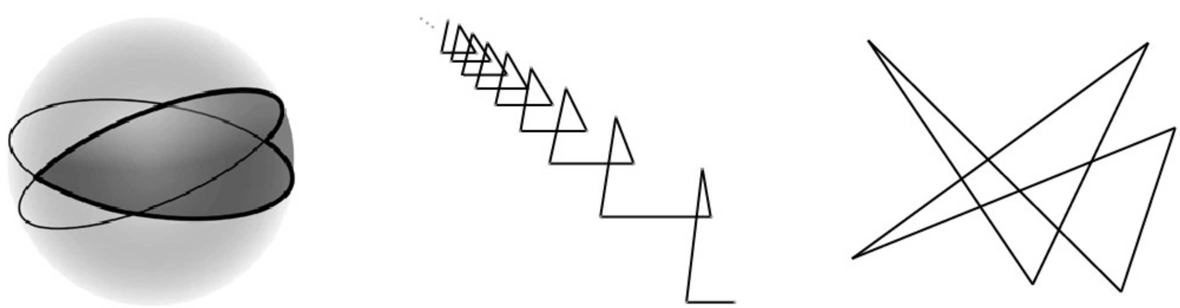


Figure 12. Unusual polygons: a digon, an apeirogon, and a polygon that crosses itself

The students became aware that mathematical terms have contested meanings and that language plays an important role in humans' construction of mathematics. In this context, one student brought up an interesting perspective to bear.

- *There is no such thing as a circle.*
- *Why are you saying that?*

- *When you draw a circle with a pencil, the tip of the pencil has a certain thickness. Every time you move the tip, you create a very short side. So even though it looks like a circle, it is really a polygon with lots of sides.*

Another student responded:

- *But we know what a circle is. A circle is a collection of points, and points have no thickness. So, if we can think about a circle, it must exist, even though we cannot draw it accurately*.

The students looked puzzled.

- *What does it mean when we say that something "exists?" Does it need to exist physically or only in our minds?*

The students were looking at me, hoping that I would sort this out for them – one of the biggest questions of philosophy no less. What aspect of teacher preparation, I wondered, should have prepared me for this moment?

CHAPTER 6
AN OPEN WAY OF BEING: INTEGRAL RECONCEPTUALIZATION OF MATHEMATICS FOR TEACHING

Building on last chapter's developmental analysis of mathematics, this chapter studies teachers' mathematical knowledge as a source of life in the mathematics classroom. If mathematics manifests differently at different levels of consciousness, then the mathematics that teachers need to know in order to teach mathematics is necessarily multi-layered as well. I use a comprehensive AQAL analysis to correlate evolution of teachers' knowledge in all quadrants in order to reveal emergent dimensions of mathematics for teaching.

The chapter was published in *Integral education: New directions for higher learning* (Esbjörn-Hargens, Reams, & Gunnlaugson, 2010), the first academic book to contemplate integral education. Unlike the other chapters in the dissertation, which were written mainly for mathematics educators, this chapter is meant to introduce issues of mathematics education to a largely integral audience. Its strong analytical orientation would not therefore be unfamiliar to its intended readership.

Mathematics for Teaching (MfT) is a burgeoning branch of mathematics education research framed by the question, *What mathematics do teachers need to know in order to teach mathematics?* Here I offer a genealogy of the field by correlating the evolutions of the objective, subjective, interobjective, and intersubjective strands of MfT. In the process, I point to multiple evolutionary tensions, including stability versus novelty of mathematical knowledge, school math versus grander mathematics, mathematics as a science versus mathematics as a humanity. I argue that teachers' mathematical knowledge should be understood as an open disposition – that is, as a pedagogical readiness to recognize evolutionary tensions as they arise and to harmonize them dialectically. This open disposition infuses the teaching of mathematics with meaning and life, and promotes cultural evolution.

6.1 What is "Mathematics for Teaching"? – Four Answers

Mathematics for teaching is an area of mathematics education research that studies connections between teaching and the subject matter of mathematics. Research in this area is organized around the primary question, "What *mathematics* do teachers need to know in order to teach mathematics?"

For much of the 20th century, answers were taken to be self-evident. It was assumed that knowledge of advanced mathematics was required in order to teach grade school mathematics. This assumption is still enacted in the vast majority of teacher education programs today. Pre-service teachers are typically required to obtain college credits in post-secondary mathematics, and prospective secondary teachers are often required to complete university degrees in mathematics prior to enrolling in education. However, contrary to this line of reasoning, as Begle (1979) showed, there is little or no correlation between teachers' college credits in mathematics and the performance of their students.

Interest in MfT arose in the late 1990s in response to this worrisome finding. At that time, most researchers (e.g., Ball & Bass, 2000) framed their work in terms of Shulman's (1986) notion of Pedagogical Content Knowledge (PCK). PCK is a specialized type of teachers' knowledge that links content and pedagogy. It includes, for example, familiarity with certain forms of abstract representation that a teacher might use to help students better comprehend complex mathematical ideas. Developing this notion in the context of mathematics, Ma's (1999) study of contrasts between Chinese and American teachers provided evidence of highly specialized knowledge in elementary mathematics teaching. Ma used the term *Profound Understanding of Fundamental Mathematics* (PUFM) to refer to this knowledge, and described it as a broad awareness of the horizontal and longitudinal connections among the concepts that comprise grade-school mathematics curricula. This contribution triggered a revision of the orienting question of MfT among researchers, as they began to ask, "What *specialized mathematics* (i.e., PCK) do teachers need to know in order to teach mathematics?"

But the answers remained elusive. It was not clear what set of specialized concepts and results could constitute a body of mathematics useful for teaching. Oriented by this problematic, Ball and Bass (2003) suggested a new vantage point by pointing out that the mathematical knowledge of teachers is not static, and that it should be thought of as knowledge-in-action. In their view, mathematics teaching is a form of mathematical practice that includes interpretation and evaluation of students' work, correlation of students' mathematical results with the processes of their production, construction of meaningful explanations, and assessment of curriculum materials. Ball and Bass called for a practice-based theory of MfT that focuses on the specific knowledge that teachers use in their daily work. As might be expected, they also proposed an updated framing question for the study of MfT: "What mathematical knowledge *is entailed by the work* of teaching mathematics?"

Ball and Bass's preliminary research focused attention on a key process of teachers' mathematical practice that they called *unpacking.* Unpacking is the prying apart and explicating of mathematical ideas to make sense of their constituent images, analogies, and metaphors. Whereas mathematicians often convert their ideas into highly condensed representations to facilitate mathematical manipulation, teachers employ the reverse process of unpacking ideas to reveal and explain the meanings of mathematical constructs. Adler and Davis (2006) have also studied unpacking, along with other aspects of teachers' mathematical work. Their somewhat worrisome findings indicated that mathematical ideas addressed in teacher education courses in South Africa are predominantly compressed, not unpacked.

The three framings of MfT noted above share one key quality: they are constrained in scope to the immediate worlds of teachers and their students. Davis and Simmt (2006) critiqued this limitation. Drawing on a complexity framework, they argued that teachers' knowledge of established mathematics is inseparable from knowledge of how mathematics is established. Any distinction between the two is inherently problematic since it ignores the similar non-linear dynamics that underlie categories of both knowledge and knowing. Davis and Simmt broadened the notion of MfT to include multiple nested systems:

> Mathematical knowing is rooted in our biological structure, framed by bodily experiences, elaborated within social interactions, enabled by cultural tools, and part of an ever-unfolding conversation of humans and the biosphere. (p. 315)

Using this expanded view of mathematics, Davis and Simmt explored various complex phenomena to reveal some essential aspects of MfT. According to their analysis, teachers must have access to the interconnected images and metaphors that underlie mathematical concepts, and must also be skilled at translating among different mathematical representations. Acquiring and enacting such skills requires a strong disciplinary background, including familiarity with the interrelationships of mathematical ideas, and the histories of their emergence. Teachers should be aware of the recursive, non-linear processes by which mathematical concepts are elaborated, especially through a curriculum. Teachers should also recognize the crucial importance of collectivity for knowledge production, and be adept at engaging and mobilizing social groupings.

As this brief overview of MfT shows, four key answers have been offered to the question of what constitutes mathematical knowledge for teaching. They are:

1) teachers need to know more advanced math than the math they are teaching;
2) teachers need to know specialized mathematics (i.e. PCK);
3) teachers' mathematical knowledge is enacted in their daily work and must be unpacked; and
4) teachers' mathematical knowledge is embodied in multiple, nested, co-implicated systems of cultural mathematics, institutionalized education, and personal learning.

Each of the answers suggests a different research question for the field. The four research questions, respectively, are:

1) How much advanced mathematics do teachers need to know?
2) What specialized mathematics do teachers need to know?
3) What mathematics is entailed in teachers' daily work? and
4) How do complex dynamics shape teachers' mathematical knowledge?

Examination of the four answers reveals a clear pattern in which successive answers offer increasingly expansive interpretations of mathematical knowledge. The trends are from knowledge as static to knowledge as dynamic, from knowledge as Platonic to knowledge as embodied, and from knowledge as pre-established to knowledge as emergent. These trends suggest that the answers are not random, and that the field of MfT is on an evolutionary path.

What may this evolution signal for the future MfT research? Do the four answers contradict one another, or can they perhaps be integrated into a coherent whole? In the remainder of this chapter, I will use the AQAL framework to investigate these questions.

6.2. Quadrants of MfT

The four quadrants of the AQAL map are four major dimensions of being in the world, and represent the interior and exterior of individual and collective experience. They provide four fundamental perspectives on every complex evolving phenomenon. If we accept that MfT is evolving, then it is open to elucidation through the four-fold lens of the quadrants (see fig. 13).

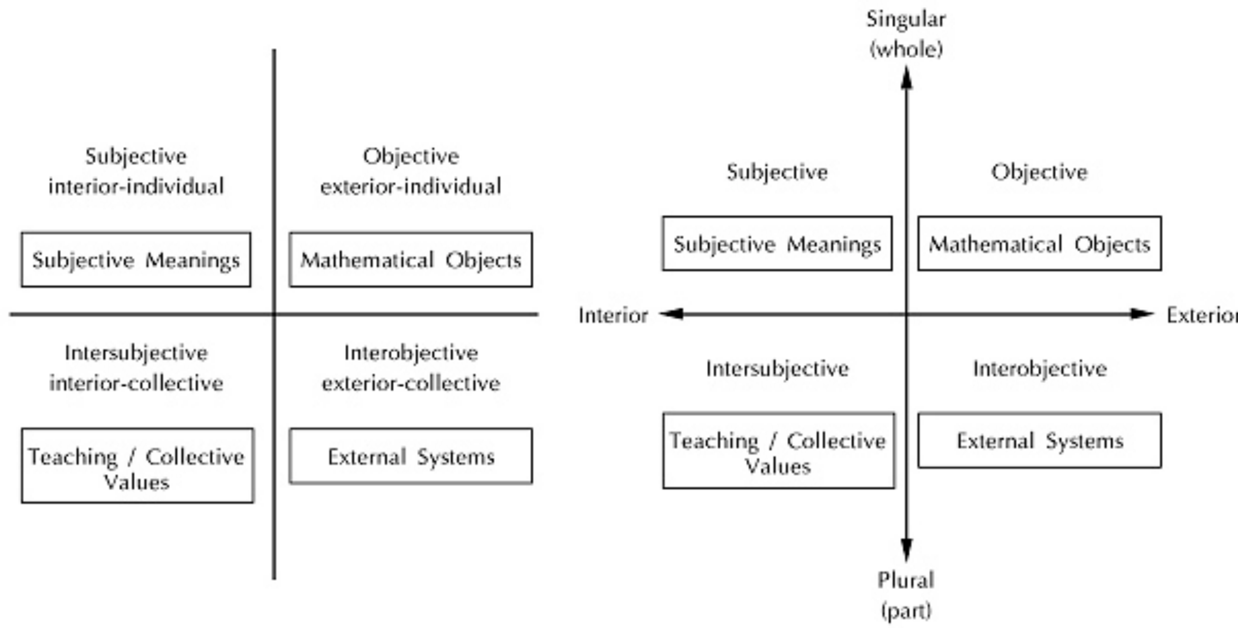


Figure 13. Quadrants of MfT: AQAL and process views

The common reference point that is perhaps easiest to identify is the objective dimension of mathematics as a representational system. There is little question that MfT includes the objects of mathematics – fractions, Pythagoras' theorem, logarithms, formal proofs, and the like. These objects populate the long lists of prescribed learning outcomes. Indeed, mathematics education as it is practiced today is focused to a large extent on training students' proficiencies in manipulating mathematical objects.

Since teachers are entrusted with promoting and assessing their students' mathematical proficiencies, it seems reasonable to require them to be proficient in manipulating mathematical objects. In other words, a teacher should be able to do well on a trigonometry test as a qualification for teaching trigonometry to his or her students. It may well follow from this that the more adept mathematics teachers are in manipulating mathematical objects, the better equipped they will be to teach mathematics in their classrooms. But Shulman's (1986) notion of pedagogical content knowledge revealed that effective teaching is dependent not only on *how much mathematics* teachers know, but also on *how they know mathematics*. The subjective interpretations that teachers attach to mathematical objects play an important role in their teaching.

An examination of the subjective dimension of MfT highlights the importance of the subjective meanings that both teachers and learners access while engaging with mathematics. Learners of mathematics often appear to be guided by the goal of sufficiency; that is, they access only as many representations as are needed to make sense of a mathematical object in order to meet the desired learning outcomes. Skilled teachers, on the other hand, are guided by the goals of depth and inclusivity. They employ multiple representations, images, and metaphors to maneuver among the numerous diverse meanings that arise in pedagogical encounters. The difference between teachers' and learners' modes of access to networks of mathematical meanings may provide a clue as to why, as Begle (1979) discovered, proficiency in advanced mathematics does not necessarily contribute to better teaching. When future teachers study advanced mathematical topics, such as calculus and linear algebra, they approach them as first-time learners, and are likely guided by sufficiency considerations (i.e., the upcoming college test). Perhaps incoming teachers stand to benefit more from re-examining elementary mathematics from an advanced standpoint. Such a study would allow these teachers to re-examine familiar mathematical topics, such as arithmetic and algebra, and to become acquainted with the multiplicities of meanings that inhere in even the most rudimentary mathematical object.

The subjective dimension of MfT includes not only mathematical meanings, but also attitudes and affect in both teachers and learners. So we may ask: *What emotions does mathematics evoke in teachers and learners? What aspects of mathematics enliven the learning process and pique learners' interests, and what aspects lead to boredom and disengagement? Why do so many children, and later many adults, fear mathematics as they seem to do?* Lastly, *what might teachers do to mitigate the distress experienced by some students when they try to learn mathematics?* These questions are commonly asked by mathematics teachers at all levels of instruction, and are therefore rightly considered part of MfT.

Shifting focus to the intersubjective dimension of MfT, we note that most teaching and learning of mathematics take place in communal settings. That is, the notion of collectivity is central to the practice of mathematical education – a detail that has been recognized in a large literature developed around socio-cultural theories (e.g., Bruner, 1996).

Intersubjectivity refers to the shared meanings and values of the learning collective. It is the cultural substrate that underlies the teaching and learning of mathematics. Intersubjectivity is realized when a group of students comes to a shared agreement that degrees are easier to manipulate than radians, and when students believe that "doing math is good for you because it makes you think," and when someone is considered "smart" because he can solve mathematics problems. Easy to overlook, intersubjectivity is the often-transparent "the way things are."

Sidorkin (2002) suggested that community and fellowship are the strongest attractors that schools offer to children. If that is the case, then teachers of mathematics should take great interest in the ways that the subject matter of mathematics promotes the formation of communities. A teacher who can deconstruct collective meanings and effect changes in shared values has the potential to have a profound influence on students. In what ways can mathematics contribute to intersubjective relationality? Conversely, how can human relationality in the classroom be channelled to bring about lively engagement with mathematics? These questions belong to the epistemological field of MfT.

If values and shared meanings represent the interiority of collectives, then systems and institutions represent their exteriority. Every culture finds external expression in the social systems it creates. We shift our focus to the interobjective dimension of MfT.

When students ask, "but when will I ever use this?" they are referring to a mathematical object's external utility. The value of the object is seen to derive from its usefulness for larger systems. Textbooks often use applications of mathematics to broaden the scope of mathematical concepts. Indeed, external utility can act as a powerful motivator when students take interest in the external applications in which the mathematics is embedded. As a teacher, I have yet to meet a student who wasn't at least a bit interested in the mathematics of personal wealth creation. This monetary motive for learning mathematics illustrates how interwoven mathematics and its teaching are with a host of different social systems, including those of science, technology, and economics.

Of all such external systems, schooling systems appear to exert the most profound influence on MfT. One can hardly think about arithmetic, algebra, and geometry without simultaneously conjuring images of drills, textbooks, tests, and marks. A teacher who navigates the interobjective dimension of MfT skillfully, and who possesses a healthy critical awareness of the institutional dimensions of school mathematics would be able to utilize external systems to stimulate student interest in mathematics.

Our four-quadrant survey revealed four fundamental dimensions of MfT. The objective (exterior-singular) dimension deals primarily with the objects of mathematics. The subjective (interior-singular) dimension deals with personal meanings, emotions, and attitudes associated with the teaching of mathematics. The intersubjective (interior-plural) dimension deals with shared meanings and values. And the interobjective (exterior-plural) dimension deals with external systems that enfold and are enfolded in mathematics and teaching.

Awareness of the four quadrants and their underlying dynamics can greatly broaden a teacher's field of vision. When a class is struggling with quadratic equations, for example, the teacher might study the underlying algebra to identify different types of equations and suitable methods for solving each type. She might also explore individual and collective meanings that students attach to the equations in an effort to better anticipate potential sources of difficulty. When she inquires about how students feel about the subject matter, she may well find that they are not much interested in it. Moreover, she may realize that the quiz scheduled for the following day is causing the students a great deal of anxiety. She may then consider ways in which to engage the network of intersubjective relationships in the classroom to create a collective ethos that is more conducive to the study of algebra. She may also choose to bring in an interesting optimization problem from the area of personal finance.

The important point in this example is that the mathematical knowledge required by the teacher is not limited to understanding quadratic equations. Educators who choose to privilege one quadrant to the exclusion of all others may be thought of as quadrant absolutists. For instance, a teacher who believes that good learning hinges on proficiency in algebra absolutizes the exterior-singular quadrant. This teacher's view may be too narrow to notice or include personal meanings and shared values. When a researcher asserts that all knowledge is socially constructed, he likewise absolutizes the interior-plural quadrant and may fail to take notice of objective realities or personal constructions of mathematical meanings.

This is not to say that educators should feel compelled to examine every occasion of MfT painstakingly from all four perspectives. Clearly each event requires its own balance of emphases, and each teacher brings her own strengths and preferences. Still, a quick check of all four bases is likely to promote more creative paths to engaging with the event.

6.3 Waves of MfT: Correlating the Evolutionary Strands

A process view of the four quadrants (Roy, 2006) reveals them to be non-dualistic. As shown in figure 13, every occasion is governed by a tension along the interior/exterior axis, and by a tension along the singular/plural (whole/part) axis. These dynamic tensions combine to produce the primary structures that are the quadrants of MfT.

The quadrants themselves are not static but rather evolving complex structures. In the context of MfT we discern four interrelated strands of evolution:

1) the evolution of mathematics;
2) the evolution of subjective cognition;
3) the evolution of collective values and conceptions of teaching; and
4) the evolution of systems in which mathematics and teaching are embedded.

Integral philosophy adds that these evolutions move through increasingly complex structure-stages. Each successive structure-stage both transcends and includes the patterns of its predecessors. My goal in this section is thus to identify a series of structure-stages that reflects the evolution of MfT, by examining some of the evolutionary strands of MfT – mathematics, teaching, and cognition – in greater detail.

In his examination of the evolution of mathematics, Davis (1996) outlined five stages: oral, pre-formalist, formalist, hyper-formalist, and post-formalist. In the pre-formalist stage, mathematics attains independent existence through abstract objects (e.g., number) and a distinct mode of reasoning. At this stage mathematics is seen as describing essential, unchanging forms in the natural world. In the formalist stage, mathematics becomes a distinct discipline with a rigorous methodology for knowledge production. This methodology, embodied in the "formal proof," begins with axioms that are believed to represent unshakeable truths (e.g., 2 parallel lines will never meet), and derives new truths through deductive reasoning. The hyper-formalist stage does away with the need for truths to correspond to the material world, seeking only internal coherence among propositions. Finally, the post-formalist stage conceives of mathematics as a socio-cultural interpretive system that is rooted in human construction of reality.

In another study, Davis (2004) traced a genealogy of teaching. He identified eight principal stages in the history of teaching: mystical, religious, rationalist, empiricist, structuralist, poststructuralist, complex, and ecological. Each of these has a distinct conception of knowledge, and hence assigns different meanings and purposes to the activity of teaching. For instance, the religious stage conceives of the universe as complete and unchanging, and of divine knowledge as being revealed by a higher authority. It follows that teaching at the religious stage seeks to induct the learner into revealed truths. The structuralist stage, on the other hand, regards the universe as emergent and continuously changing. Personal learning and collective knowledge are framed in terms of embodiment and social agreement. It follows that structuralist teaching seeks to enculturate the learner by facilitating personal interpretation and construction of meaning.

When it comes to cognitive development, the early stages are very familiar. They are Piaget's stages of cognitive development in children: sensorimotor, preoperational, concrete operational, and formal operational. The formal operational mind's capacity to take third-person perspectives enables the emergence of perspectival rationality, scientific objectivity, and world-centric judgments of fairness and care. Subsequent research in developmental psychology (e.g., Commons, Richards, & Armon, 1984; Cook-Greuter, 2005; Kegan 1982, 1994) has pointed to the existence of postformal stages of development in adults. Since the postformal mind is able to take even more perspectives (fourth- and fifth-person perspectives) than the formal operational mind, it is open to what Gebser called integral-aperspectival awareness – the bringing together of multiple perspectives and contexts without unduly privileging the monological perspective of the subject. Wilber (2000b) used the term *vision-logic* to refer to postformal cognition, and distinguished at least two stages of postformal development. In the early vision-logic stage, the learner moves into a cognition of dynamic relativism and plurality. In the mid- and late vision-logic stages, the learner enacts a cognition of dynamic dialecticism and holism.

I next proceed to elucidate stages in the evolution of MfT by correlating the stages of MfT's various evolutionary strands. My synthesis is summarized in figure 14.

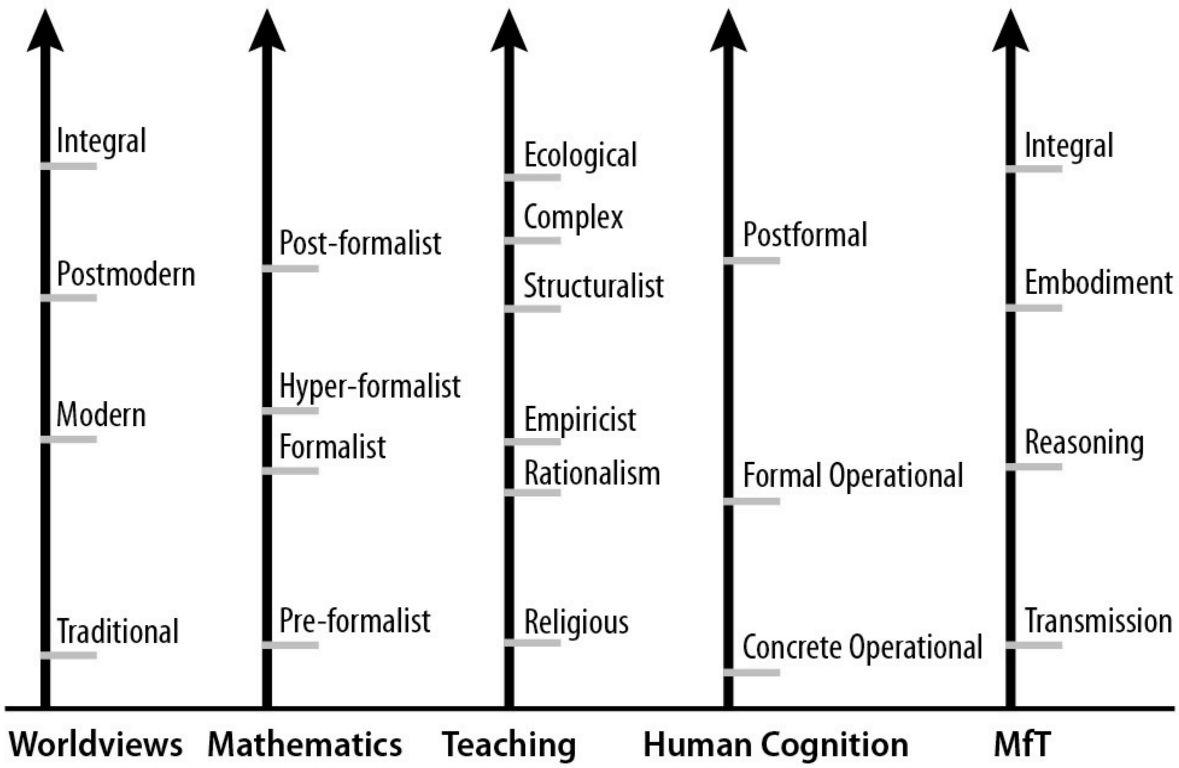


Figure 14. Correlation of the evolutionary strands of MfT

Integral philosophy offers an instructive perspective on possible interrelations of these disparate evolutions. Gebser (1949/1984) posited that the stages of development in human consciousness mirror epochal stages in human history. Each stage of consciousness is a coherent system for sense-making, a natural epistemology that arises from the need of societies and individuals to respond to external life conditions in a specific period. Each stage of consciousness brings forth a value-based worldview that organizes individuals' perceptions and interpretations of reality. Graves (1970) referred to the dialectic pattern of the emergence of worldviews as the spiral of development. McIntosh (2007) developed this notion through a list of eight stages of consciousness development: *archaic, tribal, warrior, traditional, modernist, postmodern, integral,* and *postintegral*. Some of the stages are historical, some are current, and some can only be anticipated. The three stages most commonly found in the developed world are *traditional, modernist,* and *postmodern*. I examine some of their characteristics, and consider how the stages of MfT may be conceptualized through them.

It bears mention that my discussion of the stages of consciousness is necessarily broad and framed in general terms. The stages of consciousness mentioned thus far do not correspond to types of people or to particular individuals. The reference is to categories of consciousness within people that manifest through groups of people, entire societies, or phases of history. Since consciousness is comprised of numerous developmental lines (e.g., cognition, ego, needs, morals, values), its stages are never entirely fixed or rigid in their scope and application. Wilber (2000b) used the term *waves of consciousness* to point out that the stages of consciousness are fluid and overlap one another, much as bands of colour coincide in the visible spectrum. Keeping in mind this understanding of the terms used in this discussion, we may proceed to look at MfT as seen through the traditional, modern, and postmodern waves of consciousness.

6.3.1 The Traditional Wave

The *traditional* wave of consciousness is ethnocentric and conformist. Individuals operating within its pattern seek belonging in a reference group and are willing to submit to the group's centralized authority. This wave usually draws on foundational scriptures for truth validation. In the traditional wave, mathematics is seen as an ideal form within creation. Its definite structures and unquestionable truths appeal to traditional consciousness. Teaching in this frame seeks to induct students into a set of authorized texts, be they religious scriptures or texts of mathematics. Success is measured according to conformity with pre-determined results. Traditional teaching correlates primarily with Davis's religious stage of teaching, and its control structures persist into Davis's rationalist and empiricist stages. Mathematics lends itself to traditional teaching practices more readily than do some other school subjects, as evidenced, for example, in blind acceptance of recondite rules (e.g. "invert and multiply").

MfT, as viewed by traditional consciousness, is primarily concerned with teachers' proficiency in transmitting incontrovertible truths. Guardians of the discipline, teachers are to be faithful conduits of established knowledge. They must be masters of the subject matter. The more mathematics teachers know, the more ready they are to indoctrinate others.

How teachers know mathematics is of little interest in the traditional wave because there is only one mathematics to know in the first place. According to this way of thinking, individuals who have the university credentials to prove their proficiency in mathematics should make fine teachers. Indeed, the best mathematics teachers should be expert mathematicians. For many years, this view of MfT afforded mathematicians the status of ultimate authorities on matters of mathematics education. Different versions of this view still prevail in higher education, where a PhD in mathematics is often the sole credential for instructing university-level courses.

6.3.2 The Modernist Wave

The *modernist* wave of consciousness is characterized by individuality and rationality. Individuals operating within its pattern seek to achieve personal autonomy, status, and material wealth, usually through competition with peers. Since this wave appeals to the scientific method and to objective reasoning for truth validation, mathematics has become its quintessential technology. Its tools are critical for physics, computers, and stock markets. Correspondingly, its methods of formal proof and rigorous logic are prevalent in the discourses of the exact and social sciences. Correlating with formalist and hyper-formalist stages of mathematics evolution, the modernist view of mathematics promotes the discipline to top-tier status among school subjects. Every child is required to study mathematics in each grade of K-12.

Teaching in the modernist wave seeks to help learners construct logically coherent understandings. It offers systematic progress through the subject matter by employing linear curricular structures. In fact, modernist schooling practices draw explicitly on efficiency-oriented industrial processes, treating students and curricula as inputs and final grades as outputs. Hence frequent assessments (i.e., quality controls) play a key role in the process of learning. Examinations and grade promotion create a meritocratic environment in which learners often compete for preferred standings along a bell-curve of achievement. Mathematics fits well in the project of assessing and ranking students for placement in a stratified society, serving as both milieu and means of sorting. Its apparent neutrality appeals to the modernist sense of objective fairness. Modernist teaching thus correlates with rationalist and empiricist stages of teaching.

MfT, as viewed by modernist consciousness, is primarily concerned with teachers' proficiency in providing clear instruction. Since learners are regarded as rational beings, subjective sense-making is a central focus of the modernist wave of MfT. Not only must teachers be well versed in results of arithmetic, algebra, and geometry, they must also be aware of the multiple ways in which learners make, connect, and apply multiple meanings of mathematics. Moreover, since teachers are expected to train students to "think mathematically" and to "problem solve," teachers also need to be able to analyze learners' thinking patterns, distinguish valid thought processes from erroneous ones, and understand the sources of students' errors.

6.3.3 The Postmodern Wave

The *postmodern* wave of consciousness is pluralistic and inclusive. It is characterized by sensitivity to others, especially those marginalized by dominant discourses. Appealing to local and subjective determinations of truth, postmodernity deconstructs Western mathematics to reveal the ways in which it has been used to dominate other ways of knowing. It also deconstructs school mathematics to reveal that, far from being an objective medium of pure rationality, mathematical discourse has a part in producing the disparities of power that prevail in modern societies. This view correlates with the post-formalist stage of mathematics development.

The postmodern wave of teaching regards learning as an ongoing construction of meanings that give shape to learners' subjective worlds. Here teachers provide suitable contexts for students' constructions, facilitating the complex, non-linear, recursively elaborative process of learning. Postmodern educators prefer formative assessments that help guide the learning process to summative assessments that are used to compare and rank.

Of all school subjects, mathematics is the least open to the advances of postmodern ideas. Of all the school subjects, mathematics often serves as the exemplar of stable and universal truths: *"2+2 equals 4. It always has been and always will be."* The postmodern view of mathematical results is that they are dependent on human subjects who construct their meanings, and thus mathematical results are regarded as collective constructions. Mathematical symbols are afforded meaning by people who interpret and enact them. So, in the postmodern view, what matters most is not whether *2+2 equals 4*, but rather the constellation of contexts, uses, and discursive conventions that bring forth and perpetuate such "truths." As non-Euclidean geometries and abstract grammars show, mathematical boundaries are subject to the rules of the discourses in which they are situated.

MfT, as viewed by postmodern consciousness, is primarily concerned with the multiple co-implicated networks that embody mathematics. This wave sees mathematical knowledge as enabled by biology, conditioned by culture, and situated in social experience. Thus it extends

MfT's reach to include psychology, sociology, cultural studies, history, philosophy, and other co-evolving spheres of human activity. The postmodern wave of MfT also highlights the collective dimension of mathematical knowledge production. Postmodernist viewpoints tend to promote an egalitarian attitude that seeks to include all learners in the production of diverse interpretations and multiple meanings. When a learner claims that "*2+2 equals 5,*" traditional and modernist consciousness would call this an error. Postmodern consciousness would frame it as an issue of relative fit that provides an opportunity for deepening of meaning and for the emergence of mathematical novelty – for all participants in the mathematical conversation.

My examination of MfT through traditional, modernist, and postmodern lenses reveals three distinct waves of MfT: *transmission, reasoning,* and e*mbodiment.* The first wave, transmission, is concerned with perpetuation of established mathematical truths. It focuses on the extent and depth of teachers' formal disciplinary knowledge. The second wave, reasoning, is concerned with learners' rational engagement with mathematics. It focuses on the ways in which teachers know the mathematics entailed in their work. And the third wave, embodiment, is concerned with the collective networks that embody and enact mathematics. Its focus on the network dynamics that enable the emergence of mathematical knowledge in the classroom reflects a recognition of the emergent nature of mathematical knowledge. Recall that these are the same foci that we encountered in the survey of the history of MfT. We can now see that this history represents movement along the spiral of development: from the traditional, to the modernist, and on to the postmodern wave of consciousness.

It may seem a little surprising that the field of MfT has experienced such rapid evolution in its short history. But since it is situated at the intersection of mathematics and teaching, two fields which already possess rich evolutionary histories of their own, it is not surprising that researchers drew on existing insights from these fields to conceptualize MfT. In the process, MfT's evolutionary timeline was abbreviated.

The history of MfT also represents a step-wise movement among the quadrants. The first stage, transmission, focuses primarily on the exterior-singular quadrant of mathematical objects. The second stage, reasoning, focuses primarily on the interior-singular quadrant of subjective meanings. The third stage, embodiment, focuses primarily on the interior-plural quadrant of intersubjective networks and the exterior-plural quadrant of interobjective systems. Sfard (2005) detected a similar pattern in her survey of mathematics education research. She noted that the 1960s and 1970s were the era of the curriculum, the 1980s and 1990s were almost exclusively the era of the learner, and the first years of the 2000s were the era of the teacher. It appears that each new wave of consciousness dialectically opens up and draws attention to a new quadrant.

6.4 Integral MfT

So far I have referred to the spiral of development in tracing the past and present of MfT. We can also use it to anticipate what might lie ahead – and so I turn now to the next wave of consciousness in the spiral, the integral wave.

Before proceeding, a qualification is in order. I am not attempting to predict the future of MfT with any degree of certainty. The waves of consciousness are dynamically evolving patterns of organization that arise from human activity in response to lived conditions. And so there is no telling what emergent forms MfT may take during the integral wave. However, the contours of the integral wave that are manifest in other areas of human activity might help to anticipate what eventual directions MfT might take.

The integral wave of consciousness is an emergent wave that succeeds the postmodern. As such it responds dialectically to the apparent failings of postmodern thinking: its inability to solve current global crises, its value relativism, and its uncompromising rejection of the traditional and modernist worldviews. The integral wave also responds to the culture wars that divide the traditional, modernist, and postmodern worldviews, as each tends to discount the contributions and values of the others. The integral worldview recognizes the importance of the evolution of consciousness and culture to global wellbeing. It is therefore committed to the health of the entire evolutionary spiral. Integral consciousness acknowledges the contributions of every significant historical worldview. At the same time, it recognizes and rejects pathological aspects of these worldviews.

Just as the modernist wave gave rise to the epistemological capacity of reason, the integral wave employs a new emergent capacity, called *vision-logic*. Vision-logic is the ability to see how different elements of an evolutionary system work together dialectically, to experience their different perspectives from within, and to harmonize them by valuing each element for its contribution to the entire system. Wilber (2000a) described it as follows:

> The point is to place each proposition alongside numerous others, so as to be able to see, or "to vision," how the truth or falsity of any one proposition would affect the truth or falsity of all the others. Such panoramic or vision-logic apprehends a mass network of ideas, how they influence each other, what their relationships are. (p. 288)

Vision-logic is network logic. It is aperspectival and stands in contrast to the commitment of earlier waves to monological perspectives. Mathematics education is a field marked by academic turf battles that result from the attempts of traditional, modernist, and postmodern consciousnesses to impose their monological perspectives on education. The math wars (Schoenfeld, 2004) that raged in the United States in the 1990s have left the field divided and polarized. While educators (e.g., Ball et al., 2005) agree that these disputes within the discipline do not serve the needs of students, there is little agreement on how to reconcile the seemingly opposed claims of different camps.

The three waves of MfT – transmission, reasoning, and embodiment – also represent seemingly irreconcilable perspectives on the relationship between mathematics and teaching. Is school mathematics pre-established or emergent? Should teachers be committed to the curriculum, to rationality, or to the creativity of students? Should students be assessed on the basis of the final answer, the logic of its derivation, or their committed engagement? These questions divide math education practice and research. This is where integral awareness may offer a starting point for reconciliation. It can promote cultural evolution by valuing the three waves of MfT for the contributions that they make in various contexts, and by disposing of their respective weaknesses. Doing so requires the open attitude enabled by vision-logic.

6.5 Between Agency and Communion: Evolutionary Tensions of MfT

What are some of the main points of disagreement among the transmission, reasoning, and embodiment waves? The three perspectives diverge over these questions:

1) Is mathematical knowledge stable or emergent?
2) Is mathematical knowledge Platonic or embodied? In other words, does mathematics manifest in mathematical objects or in enacted subjective meanings?

3) Who is the learner – an individual or the collective?
4) What is the subject matter – school math or accumulated cultural mathematics?
5) Should mathematics be taught as a science or as humanity?

These questions seem to represent a set of irreconcilable dualities. However, the integral wave casts them as necessary, inescapable, and dynamic tensions, which are necessary for the evolution of MfT. Theorists have argued that evolution proceeds through the complementary processes of differentiation and integration (e.g., Laszlo, 1996; Wilber, 1995). Differentiation is the tendency of every complex system to push outwards and participate in a level of organization more complex than its own. Integration points to the competing tendency of every organizational level to pull inwards toward self-unity and wholeness. Each of the five questions cited above is a particular instance of the interplay between agency and communion within the field of MfT.

Mathematical knowledge for teaching, as viewed by the integral wave, entails an awareness by teachers of the tensions that shape cultural evolution in their field, and the capability to harmonize these tensions dialogically in pedagogical contexts. I will now examine the five types of tensions more closely, in order to see how teachers might translate them into practice.

Two of the tensions are already familiar: the interior-exterior tension between mathematics as Platonic and mathematics as embodied, and. the singular-plural tension between mathematics as individual and mathematics as collective. As discussed, these tensions combine to form the axes of the epistemological field that we call the quadrants of MfT. We have already seen how awareness of the four quadrants can enhance teachers' ability to respond to pedagogical situations skillfully. Next, I will examine the remaining tensions: stability versus novelty, math versus mathematics, and mathematics as science versus mathematics as a branch of the humanities.

6.5.1 Stability versus Novelty of Mathematical Knowledge

One of the main tensions among the transmission, reasoning, and embodiment stages of MfT revolves around conceptualizations of mathematics. The first sees mathematics as inherent in nature and transcendent; the second sees it as manifest in rational structures; the third sees it as emergent and socially constructed. In an integral perspective, all are partially correct.

Mathematical truths are emergent, as shown by the fact that there was a time in human history when "2" carried no meaning. Of course, pairs of objects existed, but encounters with quantities were not interpreted numerically. At some point, humans began to notice the "twoness" of different sets of objects, and found ways to signal, retain, and generalize their experiences of "two." Lodged in collective situations, the initial meaning of "two" was constituted in social action. But once 2 was formalized through shared signs, it began an inexorable evolution from a situation-dependent adjective (two *somethings*) to an objective and independent noun (a *two*). With every new generation that signified the number two, the reality of "two" became more entrenched and transcendent.

Wilber (2006a) refers to significations that have such transcendent status as Kosmic habits. The older the habit, the more self-evident and secure its rehearsed meanings come to be for those who participate in it. Many concepts of school mathematics have been in use for so long that school mathematics often appears to embody unitary meanings. Yet teaching and learning of mathematics are in fact heavily dependent on multiple meanings that emerge through interpretation. As fixed as "two" may seem to us, as determiners of its metaphysical status, every pedagogical encounter enhances the collective meaning of "two" by occasioning unique subjective meanings.

This brings us to the question of radical novelty. How realistic is it to expect brand new results of formal mathematics to emerge in the classroom? It is doubtful that something as momentous as a new conceptual representation of the number two might come from any given elementary school classroom. But provided the right environment, students can take unorthodox approaches to mathematical problems, and do produce unexpected results. From my own experiences as a teacher, I have noticed that when I open up the environment to discussion that allows novelty, my students often surprise me with new mathematical insights. They arrive at insights by intuition, or through pictures, or by thinking about video games. They rarely arrive at them by strict application of logical rules to initial axioms. It frequently seems that I am learning new mathematics from my students as they build their understandings.

The integral wave of MfT recognizes that school mathematics is both stable and emergent. Once teachers become conscious of the dynamic interplay between stability and novelty in the mathematics classroom, they can derive much interest and satisfaction from it.

Sensing and responding to the dynamism of each mathematical context is part of a teacher's daily work, and is essential when selecting appropriate pedagogical responses.

6.5.2 "math" versus Mathematics

Not many people use the phrase "school mathematics." For most, it is simply "math" (with a small "m"). From the standpoint of most children, math is likely to be identical with invert-and-multiply, quizzes, sohcahtoa, and the myriad of other situation-specific fragments of mathematics education. When a child protests that she "hates math," her aversion often stems from some central aspect of mathematics education practice, such as rote memorization, repetitive exercises, or high-stakes testing. Yet the same child might use different forms of mathematics outside of the classroom without being particularly aware that she is doing mathematics, nor resenting it in the least. She may enjoy solving Sudoku puzzles, playing card games, or following the standings in a tennis tournament, for example. It is likely that she would not resent these mathematical engagements because she would not identify them with small-m math.

Many teachers choose mathematics as their discipline because they see themselves as mathematical. Yet once they enter their practice in the classroom, many switch perforce to teaching the orthodoxies of school math. They direct their instruction to prescribed outcomes, teaching to tests, meeting grade-level expectations, and insisting that students "show all their work." The integral wave recognizes that the practice of teaching mathematics is provided with coherence as a discipline, and attains a uniform professional identity, by enforcing these orthodoxies. However, the profession's general adherence to such practices robs mathematics education of life and dynamism, and disconnects school mathematics from children's lived experience. What is more, this tendency can dilute the mathematical interests of teachers to the point that they forget what it was that made them choose teaching mathematics as their vocation.

Integral teachers are conscious of the interplay between "math" and mathematics. They continuously strive to find ways to enliven curriculum by connecting it to what we term "grander mathematics." This may be as simple as starting the lesson with an intriguing puzzle, or making a reference to an interesting statistical item from the news, or talking to the students about what the exchange rate of the dollar actually means to their buying power. Such little sidetracks are part of teachers' tools of the trade, and they serve to broaden mathematics and keep it meaningful for both the students and themselves.

6.5.3 Mathematics as a Science versus Mathematics as a Humanity

The practices of mathematics education not only distance school math from informal mathematics used outside of the classroom, they also determine which human engagements are considered to be properly mathematical. These early determinations carry through to the perceptions of adults, setting up rather rigid distinctions between mathematical and non-mathematical engagements. For example, many people would agree that a Sudoku puzzle is a mathematical activity, since it involves numbers. But fewer people are liable to view a cryptic crossword as a mathematical activity. Yet the thinking processes of solving cryptic crosswords – the decomposition of words, methodical analysis of clues – are akin to mathematical thinking. While it does not much matter if people see cryptic crosswords as mathematical, the common inability to perceive mathematical aspects of serious issues of public concern can be very consequential.

Consider the question of what mathematics might have to do with the debate on global warming. Some people may see little connection between the melting of the polar ice caps and small-m math. Issues of global warming may be discussed in social studies or science classes, but typically not in mathematics classes, which focus on procedures that isolate mathematics as a formal, abstract and value-neutral form of knowledge. As a result, mathematical meanings are often absent from discussions of pressing issues such as global warming, except perhaps when the citation of statistics is called upon to support a given interpretation of the problem.

Yet our collective understandings of global warming have everything to do with mathematics. Comprehending the magnitude of the problem requires that vast orders of magnitude be reduced to the scale of individual experience. Logarithms, for example, provide a mathematical way to understand very large and very small numbers. Unfortunately, formal instruction of logarithms in schools focuses on manipulations of abstract expressions, and essentially achieves the opposite of providing students with the tools for active sense-making when they are confronted with the possible causes and consequences of global warming.

Davis and Hersh (1986) cautioned against the loss of meaning that results from this manner of mathematical abstraction: “The spirit of abstraction and the spirit of compassion are often antithetical” (p. 290). MfT, as seen by the integral wave, includes an awareness of the unseen ways in which mathematics is implicated in the human sphere, and is mindful of the consequences of excessive abstraction.

6.6 Conclusion: MfT as an Open Way of Being

In this chapter, I used complexity science and integral philosophy to trace the evolution of MfT. In the process, I identified three waves of MfT: transmission, reasoning, and embodiment. I then sought to anticipate what the future of MfT may hold at the integral wave.

I find that mathematical knowledge for teaching is not a fixed set of mathematical results and processes, but rather an open way of being with mathematics in different educational contexts. What is called for is a broad awareness of the dynamic evolutionary tensions that are at play during each pedagogical encounter with mathematics. MfT at the integral wave must include a willingness to “live in” these tensions dialogically, not privileging either one of their dual ends. Living in dynamic evolutionary tensions also requires teachers to be open to the many perspectives through which pedagogical occasions may be interpreted and engaged. The best pedagogical responses, according to an evolutionary understanding of MfT, are those that promote cultural evolution and life in the classroom.

How can this open attitude be cultivated among teachers? The short answer is to increase awareness. Awareness of the manifold evolutions that underlie mathematics education can empower teachers to participate in them in a thoughtful way. Recognition that seemingly irreconcilable dualities in mathematics education are in fact productive evolutionary tensions can encourage teachers to become less committed to monological perspectives. For example, once a teacher recognizes that mathematics is simultaneously stable and emergent, she no longer needs to commit to a single perspective. By taking this step, she would be freer to explore the lively interplay between stability and novelty in her mathematics classroom.

In order to live in the evolutionary tensions, teachers of mathematics should know about and live through the opposing perspectives in their discipline. Each of the four historical answers to the question “*What mathematics do teachers need to know in order to teach mathematics?*” addresses certain perspectives in this opposition, but not others. Studies in advanced mathematics, for example, deepen teachers' perspectives on established mathematics, while pedagogical content knowledge enhances teachers' awareness of how subjective meaning-making takes place. Since there will always be new perspectives to know and to harmonize, no closed, static body of knowledge can ever be the whole of a teacher's mathematical knowledge.

MfT, when understood as an open way of being, asks teachers to always remain curious about mathematics and the ways in which it connects to human experience. The career of a mathematics teacher offers a path of growth and deepening through encounters with new perspectives and the ongoing process of harmonizing evolutionary tensions. Being skilled at negotiating the tensions of MfT in the moment is the teacher's true wisdom of practice.

Teaching Interlude 5: Are Circles Efficient?

How are circles different from squares? Let's think about adjectives that describe circles and others that describe squares.

There was a big show of hands. The students described circles as "circular, curvy, perfect, and infinite" and squares as "jaggedy, edgy, and straight."

These are rather predictable descriptions. Let's go deeper. For example, which is blue and which is red?

A quick survey showed that over two thirds of the students thought that circles were blue and squares were red.

OK. Try this then. Which of them is sad?

The students were far less certain about this one, although generally they thought that circles were sadder than squares. I then asked them to get into groups and generate lists of adjectives under the headings "circles" and "squares." We then summarized the results.

Circles:	happy, thoughtful, wise, soft, realistic, feel better, warmer, life affirming, transparent, dynamic, sleepy.
Squares:	heavy, masculine, nerdy, deadly.

- *It appears that you associated many positive adjectives with circles and negative ones with squares. Why?*
- *It's because circles are perfect. They cannot be criticized.*
- *What makes circles perfect?*
- *They have no beginning and no end.*
- *Is it good to be perfect? Are you perfect?*

The students thought about this question for a while. After some discussion, they agreed that imperfection is one of the essential traits of being human, and that humans are beautiful in their imperfection.

- *What shape is this classroom?*
- *A rectangle.*
- *How about the school?*
- *It's rectangular too.*
- *If we feel so good about circles why do we build our school in the shape of a rectangle? Can you imagine what this class would look like if it were in the shape of a circle?*

The students clearly disapproved of this suggestion.

- *It would feel weird.*
- *Could you build the entire school in the shape of a circle? Would it be good to do so?*

Again, the students clearly disliked the idea.

- *It would be very inefficient to build a circular school.*
- *In what sense?*
- *It would not use space well. There would be gaps between the classes. There will be unused space.*
- *According to this logic, flowers should also be square, in order to maximize efficiency. How many of you would like flowers to be square?*

The students did not think this would be a good idea.

- *Who decides what is efficient and what is not? Isn't it a value judgment that we humans impose?*

The class then launched into a conversation about individual and collective values, and how they constrain our choices. We explored how adjectives, such as *beautiful* or *efficient*, mediate the ways in which we interact with our environment. Some students who were so inclined then drew renditions of circular schools.

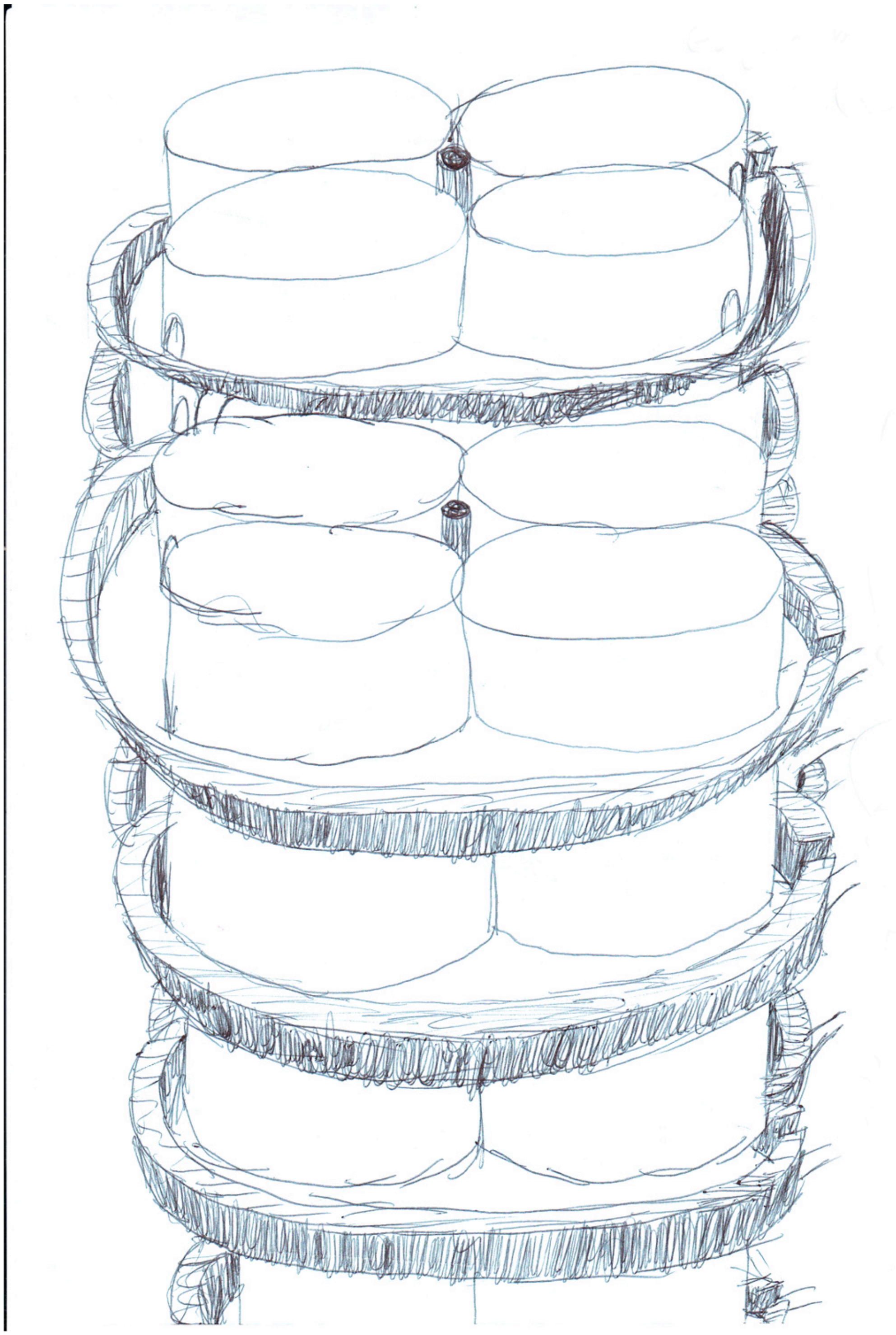


Figure 15. A student rendition of a circular school

CHAPTER 7

MATHEMATICS FOR LIFE: SUSTAINABLE MATHEMATICS EDUCATION

The last chapter described the integral wave as an emergent wave that responds dialectically to the failures of the preceding waves to solve current global problems. Without a doubt, achieving ecological sustainability in technologically advanced societies of the present and future is the greatest challenge facing humankind and the planet itself. In this chapter, I explore connections between mathematics education and ecological sustainability.

In AQAL terms, this chapter is located primarily in the interobjective (LR) quadrant of systems. It considers how mathematics education is implicated in natural systems, and how it can become part of society's transformation to sustainable living. The chapter goes beyond life in mathematics classes to study the co-creative relationship between mathematics and the natural world. It calls for sustainable mathematics education, which is a tangible and urgent expression of the more general purpose of healing the world (Chapter 3). I use a developmental model to contemplate how mathematics educators might respond to this call.

The chapter was recently published as an article in the journal *For the Learning of Mathematics* (Renert, 2011). I hope that its publication initiates a productive conversation among mathematics educators about sustainability.

We live in troubling times. The ecological systems that sustain life on earth are stressed gravely and degrading rapidly. History tells of several societies – the Mayans, Easter Islanders, and Sumers – whose inability to shift course, even when they recognized the harm caused by their unsustainable practices, led to their demise (Wright, 2004). Humans today are confronted by the challenge of managing multiple large-scale ecological problems simultaneously, including climate change, loss of biodiversity, and depletion of natural resources (IPCC, 2008; Millennium Ecosystems Assessment, 2005).

As a mathematics teacher and researcher, I am experiencing a growing disconnect between the preoccupations of my professional life and the increasingly loud calls around me to attend to the problems of ecological sustainability. For years it was easier to busy myself with metaphors of multiplication than to contemplate imminent environmental catastrophes. But over time, I have realized that I cannot be a genuine educator and also avoid the greater challenges that will confront my students in the future. Yet, it has been quite difficult to conceive of how my practice should change in order to respond appropriately to the challenges that we all face. I believe that many other mathematics educators are also facing these critical questions. And so I wonder: how should we reconcile the urgent need to act for the future with the practices of mathematics education of today?

This chapter is an initial attempt to answer this question in my own practice. I write it in the hope that it contributes to a much-needed sustaining conversation among mathematics educators about reorienting our shared practice. I begin by examining the role that sustainability has played in education in general and in mathematics education in particular. I present a model of possible responses to sustainability in mathematics education and apply the model to two extended examples: *large numbers* and *chaos*. Finally, I reflect on the two examples to outline some possible features of *sustainable mathematics education*.

7.1 Ecological Sustainability and Mathematics Education

Ecological sustainability is a broad term with multiple contested meanings. As Esbjörn-Hargens and Zimmerman's (2009) survey shows, numerous approaches to *ecology* exist, many of which are at odds with each other. These approaches are enacted simultaneously at various scales – the biological, the personal, the cultural, the social, the economic, and the biospheric. The term *sustainability* generally refers to the ability of living systems to endure over time; since the 1980s, it has been widely used to describe humans' long-term survival and wellbeing (*cf.*, WCED, 1987).

Although environmental issues have attracted considerable attention in education since the 1960s, ecological sustainability became a major focus of education only in the 1990s (Palmer, 1998). Following the 1992 UN Conference on Environment and Development, environmental education assumed a more activist and future-oriented stance, as evidenced by the rise of *education for sustainable development* (Hopkins & McKeown, 1999), *transformative education* (O'Sullivan, 1999), *futures education* (Hicks & Slaughter, 1998), and *sustainable education* (Sterling, 2004). These trends share a common critique of the ways in which current educational systems perpetuate an unsustainable industrial/modernist model of growth (see Orr, 2004).

School mathematics has traditionally organized some of its applications around the needs of the moment. This is why examples drawn from commerce, such as giving change and buying carpet, are so common in classroom teaching. Occasional references to the environment can also be found in past and present curriculum documents (*e.g.,* Ontario Department of Education, 1951, p. 144). But by and large, ecology has played only a negligible role in mathematics pedagogy. Sustainability has likewise attracted little attention in mathematics education research. Why is this so? I believe that it is the legacy of Platonism. Mathematics is popularly conceived of as a pure body of knowledge, independent of its environment, and value-free (*e.g.*, Hardy, 1940). From the Platonist perspective, connections between global warming and the topics found in mathematics textbooks, such as fractions or quadratic equations, are not readily apparent.

In the past two decades, social constructivist readings (e.g., Ernest, 1998) and critical mathematics education (Skovsmose, 1994) have challenged Platonic assumptions about mathematics by underscoring the political and sociological dimensions of its teaching and learning. The Rethinking Schools movement (Gutstein and Peterson, 2006), for instance, has been instrumental in raising awareness about the ways in which mathematics pedagogy is implicated, both culturally and ethically, in issues of social justice, such as racism, equity, gender, and democracy. A critical stance could also be constructive for mathematics educators who wish to approach issues of the environment, such as climate change (Barwell, 2010). To date, however, these issues have not been a major focus of critical mathematics education.

And so ecological sustainability and mathematics education remain largely unconnected in the research literature. Yet, many connections can be made, as the following statement about food production illustrates:

> The efficiency with which various animals convert grain into protein varies widely. With cattle in feedlots, it takes roughly 7 kilograms of grain to produce a 1-kilogram gain in live weight. For pork, the figure is over 3 kilograms of grain per kilogram of weight gain. For poultry it is just over 2, and for herbivorous species of farmed fish (such as carp,

tilapia, and catfish), it is less than 2. As the market shifts production to the more grain-efficient products, it raises the productivity of both land and water. (Brown, 2009, p. 226)

This statement is qualitatively different from examples about giving change or buying carpet in that it could implicate learners in responsibility for the earth and compel them toward an ethic of conservation. It discloses the reality that a bite of beef stresses the earth's limited agricultural resources 3.5 times more than does a bite of chicken. It suggests that a diet of vegetables and herbivorous fish may provide a ready solution for eliminating over 65% of the pollution caused by protein production.

Like most other information communicated about the environment, the statement relies on mathematical reasoning and numbers. Making sense of it requires some sophistication in proportional reasoning—a major strand of school mathematics.

The mathematics is not simple and any conclusions need to be worked out in larger systemic contexts that include science and society. Issues of sustainability call for an interdisciplinary conversation. Mathematics educators can bring important perspectives to bear on this conversation, due to their familiarity with the vast network of metaphors, exemplars, applications, and algorithms that underlie proportional reasoning (see Confrey, Maloney, Nguyen, Mojica, & Myers, 2009).

Integrating the environment into the discourse of the mathematics classroom signals the possibility of a more genuine mathematics education – one that is not so much about acquiring certain competencies but about noticing the world differently, seeing proportional reasoning in multiple contexts, making connections, and moving to ethical action as a result of increased awareness.

7.2 Educational Approaches to Sustainability

Given the imperatives of sustainability, how might mathematics educators react to this call for change? Judging from the track record of environmental education outside mathematics – which runs the gamut from avoidance to transformation – a developmental stage model may be useful for anticipating the responses to sustainability in mathematics education. The model that I propose adapts two existing stage models of approaches to sustainability to settings of mathematics education.

The first stage model is Sterling's (2004) model of educational responses to sustainability. It derives from Bateson's (1972/2000, p. 279) logical categories of learning: *first-order learning,* which proceeds within agreed boundaries and does not challenge basic values; *second-order learning,* which reflects critically on the assumptions that govern first-order learning; and *third-order learning,* which involves a creative shift of consciousness made possible by deep awareness of alternative worldviews. Sterling's model consists of three broad stages: *accommodation, reformation*, and *transformation*.

The second of the stage models is Edwards' (2010) model of organizational approaches to sustainability. It applies to contexts of organizational transformation in general, rather than to educational contexts in particular. It employs a developmental lens to identify seven narrower stages: *subsistence, avoidance, compliance, efficiency, commitment, local sustaining,* and *global sustaining*.

Table 5 shows my combined reading of both models as applied to contexts of mathematics education. In comparing the two models, I found that their stages correlate quite easily, and that the two models are complementary. Sterling's focus on education and educational biases clarifies how knowledge about sustainability is interpreted by educators at different stages. Edwards' focus on organizational change reveals stakeholders' power positions and worldviews on sustainability at different stages.

Type of Educational Response (Sterling)	Stages of Organizational Sustainability (Edwards)	Description
Accommodation = Education *about* sustainability • Has a content/knowledge bias. • Can be assimilated easily within existing educational paradigms. • Assumes that knowledge about sustainability is uncontested and can be codified and transmitted	Subsistence	Sustainability is considered only as it relates to survival of current educational practices. Stakeholders are concerned solely with perpetuating current practices (*e.g.*, *"Don't bother me. I have to prepare these students for the SAT."*)
	Avoidance	Sustainability is seen as an attack by opposition groups on the status quo. Stakeholders exhibit ignorance and apathy towards the negative impact of current educational activities on the environment (*e.g.*, *"Math has nothing to do with sustainability. It's the science teacher who should be thinking about it.")*
	Compliance	Sustainability is seen as an imposition. Stakeholders conform to traditional ethics, and comply grudgingly with top-down regulation as a way of circumventing more demanding regulation (*e.g.*, *"If I solve a couple of mathematical problems about population explosion, then perhaps no one will bother me."*)
Reformation = Education *for* sustainability • Includes a values and capability bias. • Involves some reformation of the existing paradigm, but essentially leaves it intact. • Still assumes that we know the values/knowledge/skills needed for sustainability, but includes critical thinking.	Efficiency	Sustainability is considered to be a source of potential profit/benefit (*e.g.*, *"I can use examples drawn from sustainability to motivate my students to learn about logarithms"*.)
	Commitment	Sustainability is valued for balancing educational, social, economic, and environmental concerns. Schooling is seen as connected with the outside community in a societal network. Stakeholders are committed in principle and go beyond regulatory compliance. (*e.g.*, *"Sustainability is the most important issue that our society faces. Topics of sustainability should be a large part of the curriculum in my math class."*)
Transformation = Education *as* sustainability • Knowing is seen as approximate, relational, and provisional. • Involves a transformative epistemic learning response by the educational paradigm. • The process of sustainable development is essentially one of learning, while the context of learning is essentially that of sustainability.	Local Sustaining	Sustainability is valued as a way of developing education into the future. Stakeholders devise and implement transformational strategies for moving towards goals that support host communities (*e.g.*, "*Mathematics education itself is a living complex system. We should promote maximum vitality in the system for the benefit of students and their communities.*")
	Global Sustaining	Sustainability is embodied within all aspects of the educational process and is seen in global and intergenerational terms. Stakeholders make connections between multiple layers of purpose that include: physical, economic, environmental, emotional, social, and spiritual. (*e.g.*, "*Based on my understanding at this moment, I would like to reshape mathematics education as an integral project which addresses every student's body, mind, and spirit, for the benefit of society and the planet at large. However, I realize that my actions might actually exacerbate problems of sustainability in ways I cannot see or understand.* ")

Table 5. A stage model of approaches to sustainability in mathematics education

One practical difficulty that follows from Table 5 is that is not easy to imagine how mathematics education might be enacted beyond one's present stage. In my case, for example, I could not quite see how the Local Sustaining level would be enacted. What would it mean for a mathematics educator to "*value sustainability as a way of developing education into the future*" and to "*devise and implement transformation strategies for moving towards goals that support host communities*"? These words were just abstractions until I shifted my thinking to specific examples. I will next discuss how mathematics teachers might approach two examples – large numbers and chaos – through the model's interpretive lenses of *accommodation, reformation,* and *transformation*.

7.3 Large Numbers

Humans emit 29 trillion (2.9×10^{13}) kilograms of carbon to the atmosphere each year. Like most other numbers that describe ecological quantities, it is a large number. But how much carbon is this? We cannot readily imagine this amount, let alone have a felt bodily sensation of it. This quantity is an abstraction that we put into the category of *large numbers*.

Barrow (1992) distinguished between the *notion of counting* and the *notion of quantity*. Whereas *number sense* refers to humans' ability to transact numbers appropriately, *quantity sense* refers to humans' ability to comprehend magnitude and size. When it comes to large numbers, our number sense is almost entirely divorced from any quantity sense. Humans' ability to count – that is, to use numbers as symbolic representations of quantities – provides us with a powerful mechanism for storing, recalling, and manipulating cultural information. But humans' inability to feel large numbers is very problematic in our dealings with ecology and the environment.

Emotions play a crucial role in decision-making and human action (Damasio, 1994). If we do not *feel* numbers, then our emotional access to the physical phenomena they represent is much diminished. The emphasis on number sense in mathematics education has led Wagner and Davis (2010) to caution that current curricular and pedagogical methods may exacerbate students' deficit in comprehending quantities. They called for "mathematics classroom experiences that can help students feel the weight of number" (p. 48). Their call becomes all the more urgent in the context of sustainability. Mathematics educators can respond to it through accommodation, reformation, or transformation.

An *accommodating* response might be to include a new unit of study in the curriculum, under the heading *Orders of Magnitude*. Middle-school students would be taught and then tested on the use of small integers to describe the sizes of large numbers. For example, the number 1,250,000,000 is of order 9. Note that this educational response does not overcome the separation of number sense from quantity sense. The students are still engaging in activities that develop their number sense only.

A *reforming* response might recognize that much of our appreciation of scale is processed through our visual system and devise suitable experiential classroom activities. The film *Powers of Ten* (Eames and Eames, 1977), for instance, tries to impart a sense of the scale of the universe through a series of images. Wagner and Davis (2010) described how they used grains of rice in various containers to represent numbers of various magnitudes. Chemist Nate Lewis offered a particularly effective analogical account of carbon pollution:

> Imagine you are driving in your car and every mile you drive you throw a pound of trash out your window. And everyone else on the freeway in their cars and trucks is doing the exact same thing, and people driving Hummers are throwing two bags out at a time – one out the driver-side window and one out the passenger-side window. How would you feel? Not so good. Well, that is exactly what we are doing; you just can't see it. Only what we are throwing out is a pound of CO_2 – that's what goes into the atmosphere on average, every mile we drive. (Friedman, 2008, p. 34)

Lewis's analogy is powerful because it compares one type of pollution with another, one pound of carbon with one pound of trash. The image of freeways piled up with garbage arouses a physical sense of disgust, and is likely to open up an opportunity for critical discussion of carbon pollution among students.

A *transforming* response subsumes the accommodating and reforming responses and goes beyond them. It will see the value in learning about orders of magnitude, while discounting the strong focus on computational accuracy. It will embrace teachers' ingenuity at devising meaningful experiential activities that open up a space for critique. But it will also recognize that teachers are not the only source of ingenuity in mathematics classrooms, and that deconstruction and critique ought to be followed by innovation and transformation.

One transformative approach to the problem of feeling large numbers is to pose it directly to students as an intractable problem in mathematics. A generative prompt might be: "*Many adults are having a hard time comprehending large numbers and as a result find it difficult to relate to issues of the environment. How would you explain the meaning of some large numbers (for example, the number of kilograms of carbon emitted daily into the atmosphere) to adults in your life in order to move them to action?*"

This prompt suggests a new kind of school mathematical problem solving. Most mathematical problem solving in today's classrooms relies on the unchallenged assumptions that each problem has one correct answer and that the teacher knows this answer. Students' creativity is therefore limited to replicating solutions that are already known by an adult. In contrast, the solutions to many problems of sustainability are not known *a priori*, and in some cases there is no certainty that solutions can be found at all. A different order of ingenuity is required to approach these problems, one that we may call *radical creativity*. The prompt also shifts the responsibility of knowledge production from the teacher to the entire classroom collective. It connects knowledge with political action and can empower students to act locally to bring about change in their own communities.

7.4 Chaos

Chaos theory and fractal geometry provide another ready way to connect mathematics education to the environment. Chaos theory is the mathematics of complex dynamic systems, and fractal geometry is often described as the geometry of the natural world. Between them they provide formal and visual metaphors for understanding the nonlinear dynamic patterns of living systems.

Since the enlightenment, the way humans conceive of the world has been guided by the reductionist scientific paradigm, which maintains that complicated systems can be disassembled and reassembled at will. The mathematical equations of classical physics provide a fully dissociated description of nature and suggest that natural phenomena are predictable and can be controlled. Newton's second law, $F = ma$, for example, predicts with complete certainty that if the mass, m, is increased by a factor of 3, then the force, F, will also increase by a factor of 3. This linear mode of reasoning is at the basis of every mathematical equation that we teach at school.

But complexity science, whose early roots can be traced to Poincaré and the invention of chaos theory (Waldrop, 1992) shows that complex systems are holistic, indivisible, and do not lend themselves to piecemeal analysis. They are open, evolving systems that maintain their identity in the face of constant environmental flux through the iterative processes of self-organization (autopoiesis) and emergence. Autopoiesis employs two types of feedback: negative feedback regulates activity and keeps it within a set range, while positive feedback amplifies and can drive the system towards instability. Unstable systems far from equilibrium may reach bifurcation points at which new forms of organization emerge. We can think of the emergence of increasingly more complex novel structures as the creative dimension of living systems.

Self-organization and emergence are nonlinear dynamic processes. While linear systems change smoothly in response to small influences, nonlinear systems can be very sensitive to initial conditions and tiny perturbations because of the amplifying effects of feedback. Nonlinearity places complex systems beyond human capacity to predict and control. As Meadows (2005) observed, the most we can do is try to encourage the structures that help complex systems run themselves.

Complexity science sees nature as whole: interconnected, seamless, and organic. Current structures of school mathematics generally do not reflect or support this vision. The binary *right-or-wrong* logic enacted repeatedly in school mathematical discourse presupposes absolute certainty. The overriding emphasis on quantification and measurement reinforces the belief that aspects of our world that can be quantified are more important than those that cannot (*cf*., Baker, 2008). The systemic connections between the measurable and non-measurable – e.g., between rapid economic expansion and ecological values – are rarely made explicit. And since values themselves cannot be measured, mathematics comes to be regarded as value-free.

Again, chaos can be taught through an accommodating, a reforming, or a transforming approach.

Chaos theory derives from the study of nonlinear differential equations that is far beyond the level of high school mathematics. So any *accommodating* curriculum of chaos that focuses primarily on the mathematics will have to treat the theme of chaos broadly, rather than emphasize mathematical detail. A good example of such treatment is provided by Burger and Starbird (2005), who present non-linearity through difference equations and Julia sets.

The theme of chaos very much lends itself to a *reforming* approach that connects mathematics with the environment. Reforming teachers could critique linear notions about sustainability by explicating and elaborating the meanings of complex dynamics. For example, scientists warn that the earth will warm up by 1.1–6.4 degrees Celsius by the end of the century (IPCC, 2008). A common response to this warning is to dismiss it with the thought, "*That's not too bad. I actually prefer a slightly warmer winter*." This line of thinking is strictly linear in presuming that a small rise in the earth's average temperature would lead to a small fluctuation in daily climate. Unfortunately it does not apply to the world's nonlinear weather systems, since it does not take into account the greater extremes that increases in standard deviations bring. It also does not take into account the vast impact of wider climate fluctuations on ice sheets, oceans, storms, and crops. A surprising, yet telling, statistic is that the difference between the earth's temperature today and in the last ice age is only 5–6 degrees Celsius.

This example suggests that a descriptive modelling approach may be an effective means with which to explore chaos with our students. This approach would rely on technology to facilitate simulations of phenomena in multiple variables, such as weather patterns. [1] The models could accept probability distributions or even difference equations as inputs, and set in motion iterative simulations that employ a mix of stochastic and chaotic processes. Since multivariate models overcome the common restriction in school algebra of using only single-variable functions, their descriptive power far exceeds that of algebraic equations.

Descriptive modelling is a powerful problem-solving tool in cases where a single approximating formula does not suffice. It would support a problem-based pedagogy in which teachers and students search for the right mathematics required to make sense of real-life problems. The focus of teaching and learning would shift from prescribed lists of mathematical topics to identifying, selecting, using, and evaluating appropriate mathematical processes. The students' toolbox would be extended beyond algebra and allow far greater versatility. It would include stochastic processes, which are arguably more suited to understanding natural phenomena than deterministic ones (Eigen and Winkler, 1993). Information within the models would be represented both numerically and visually, and students would employ both quantitative and qualitative reasoning techniques to draw conclusions from it.

A shift in mathematics education from algebraic equations to qualitative visual reasoning and descriptive analysis will clearly be *transformative*. At the same time, it is easy to think of other transforming opportunities that teaching chaos affords. Concepts such as nonlinearity, emergence, and wholeness carry deep metaphorical meanings that can reshape our understanding of causality, creativity and spirituality (Briggs and Peat, 2000; Juarrero, 1999). It is up to us, educators, to choose to what degree we are prepared to engage analogical reasoning to enable new understandings of humanity's place within the environment.

The metaphors of chaos show that humans, far from being separate life forms in a controllable universe, are systemically implicated at every level of life on earth (Briggs and Peat, 2000). As an example of how these metaphors may transform human action, consider the common belief that ecological problems are just too great for any one person to do anything about. This belief is founded on a linear argument that proceeds by quantitative comparison: "*The problem is very big. I am small. Big is greater than small. So there's nothing I can do.*" This line of reasoning may lead to resignation and inaction on the part of individuals. Chaos can help change the way we think about power and influence. It teaches us that complex systems cannot be controlled, but can be accessed and perhaps influenced through the myriad of feedback loops they contain. This notion has been illustrated by Lorenz's (1972) metaphor of the butterfly whose flapping wings in Brazil could set off a tornado in Texas.

The metaphor of butterfly power is very empowering. It suggests that each of us individually can make a difference, and that the consequences of our actions may be far more profound than we expect. Since the implications of our current actions cannot be predicted, butterfly power also calls on us to act with humility. And so, the mathematical notion of chaos gives rise to a new ethic: attentiveness to the present as a way to act right for an uncertain future (*cf.*, Varela, 1999).

7.5 Sustainable Mathematics Education

The extended examples of large numbers and chaos allow us to discern some likely defining characteristics of *sustainable mathematics education*.

Sustainable mathematics education is the project of reorienting mathematics education towards environmentally-conscious thinking and sustainable practices. It is a change effort that we cannot afford to ignore. Even though sustainable mathematics education is motivated by urgent issues of survival, it need not adopt the pessimistic tone of many writings on ecology (*e.g.*, Deffeyes, 2001). Dire projections are typically founded on the idea that humanity has already passed, or will soon pass, an ecological point of no return – a climate tipping point, peak oil, and the like. While pessimism may be an appropriate response to current conditions, it is neither helpful nor constructive for educators. Since school is a social institution situated at the intersection between present society and the promise of what society may become, educators are more likely to succeed in their work with messages of hope and possibility. Some writers (e.g., Hawken, 2007; Edwards, 2005) have suggested that humanity is at the threshold of a sustainability revolution no less significant than the Industrial Revolution. It will transform our unsustainable industrial practices and set humanity on a new course of ecological harmony for the future.

The notion of *transformation for sustainability* provides mathematics education with a clear generative purpose. What are some pathways for mathematics educators into the sustainability revolution? Brown's (2009) comprehensive survey of practical solutions to problems of sustainability serves as one example of a possible launching point for a sustainable mathematics curriculum of action and hope. Mathematics plays an integral role in many of these solutions: renewable power, smart energy grids, reforestation and carbon sequestering, changes in food production and consumption, and a cradle-to-cradle, zero-waste, new materials industry. The current lack of political will or urgency to implement sustainable solutions on a large scale can be understood, according to Sterling's and Edwards's stage models, as arrested development in the way we see nature and our role in it. I believe that this is where mathematics teachers can make a real difference. Sustainable mathematics education can help evolve the ways in which we see the world by evolving the ways in which we understand and use mathematics.

Sustainable mathematics education is about seeing the world anew through renewed mathematics. It is concerned not only with feeling large numbers, but also with feeling the global situation. It trades linear metaphors of certainty and separation for complexity metaphors of possibility and connection. It helps us relinquish our desire for deterministic predictability and embrace the contingency and stochastic probabilities of each living moment. It turns mathematics from a collection of objects, or a series of competencies, into an open-ended state of observing the world. It aims towards a more complete and appropriate mathematics, and from this position it calls on us to engage in ethical action for healing the world.

Having been shielded by the perceived neutrality of our discipline, mathematics educators are latecomers to environmental education. One benefit is that we may sidestep mistakes made by those who preceded us. We can recognize from the start that what we are aiming for is a paradigm shift and that accommodating responses alone will not be enough:

> [T]he crisis/opportunity of sustainability requires second – and where possible – third order learning responses by cultural and educational systems. There is a double learning process at issue here: cultural and educational systems need to engage in deep change in order to facilitate deep change – that is, need to transform in order to be transformative. (Sterling, 2004, p. 15)

Some mathematics educators are likely to adopt accommodating approaches at the start and they are to be welcomed for taking steps in the right direction. But in setting the more ambitious end goal of transformation early on, and in promoting awareness around it, we may create the conditions necessary for the emergence of second- and third-order learning responses.

Sterling's (2004) quote points to the interdependency of multiple co-implicated systems in sustainable mathematics education: learning systems, ecological systems, cultural systems, and systems of mathematics and science. The nested, self-similar nature of these systems suggests that we should promote maximal vitality and co-enact sustainable practices in all of them simultaneously. A paradigm shift of mathematics education, founded on metaphors of chaos and complexity, would recognize that the mathematics class itself is a living complex system, integrally embedded and open to exchanges with its environment (*cf*., Davis and Sumara, 2006).

Just as the borders between class, school, community, society, and ecology are likely to be continually challenged and blurred, so will the disciplinary boundaries between mathematics and other fields. The descriptive modelling approach described earlier, for instance, demands interdisciplinarity if its models are to be useful for the analysis of real-life phenomena.

Admittedly, many of today's teacher education programs tend to favour disciplinary specialization and thus may leave classroom teachers poorly equipped to act as interdisciplinary authorities. Complexity thinking suggests that a new, and perhaps more effective, kind of interdisciplinarity is needed, one that does not depend on one individual to be knowledgeable in every field. The new interdisciplinarity, which we may call *transdisciplinarity*, consists of decentralized networks of specialists who work in concert towards a common goal (Davis and Sumara, 2006). A joint collaboration of mathematics, science, and social studies teachers on a common modelling project would be an example of transdisciplinarity in a school environment. Transdisciplinarity is further enabled by network technologies, such as the Internet, which allow ready access to diverse communities of disciplinary experts.

If mathematics education is to undergo transformation, we would be wise to start by transforming the way reform itself is done. One of the lessons of chaos is that creative emergence cannot be controlled top-down. It is a bottom-up project that involves the diverse contributions of many interacting participants. We are these participants – educators, researchers, and students who are passionate about mathematics and the role it can play in the world.

A new ethic has presented itself to energize our practice with purpose and meaning – the ethic of mathematics for life.

Teaching Interlude 6: Hermeneutic Circularity

At one point, I wanted to consolidate the class' research on the question of how many sides a circle has. I asked the students to write answers that they had found on one board and any remaining questions on another board. There were two answers and over two dozen new questions.

- *Do you think that there is a definitive answer to the question of how many sides a circle has?*
- *No.*
- *Why can we not find a definitive answer?*
- *It all depends on what you mean by "side." You could be restricted to the Euclidean plane, or you could have it on a sphere. It is possible that the question doesn't make sense. For example, if we say that a circle is not a polygon then it doesn't make sense to count sides.*
- *Suppose I invited a Geometry specialist from the university to visit our class, and tell us the answer to this question. Would you accept it as the definitive answer?*
- *No. It's just his opinion.*
- *How many of you would like me to call the university and find such a person?*

  All hands rose enthusiastically.

The students were clearly comfortable with their constructed mathematics. They were also willing to defend it in the presence of an expert. I suggested that we keep revisiting the question in the future, as we continue to gain new perspectives from other studies in mathematics.

There were no quizzes or tests, and no one felt the poorer for their absence. Instead, I asked students to work individually on a project called, "What is interesting *for me* about circles?" The projects are to be presented to the class at any time before the end of the school year. One of the students is currently studying circles in Shakespearean drama. Another is studying vortices in black holes. Several students have been experimenting with subtractive fashion design, in which a one-piece garment is created by cutting circular holes into a large piece of cloth.

CHAPTER 8
CONCLUDING THOUGHTS

Let us return now to the overall research question posed in Chapter 1: *What does integral thinking disclose about life in mathematics classrooms?*

In this last chapter of the dissertation, I will draw from across the research work previously presented in order to identify common themes and conclusions elucidated by the integral perspective. I will begin by considering how the research chapters are unified by AQAL's quadrants and levels views. The AQAL synthesis offers a new pedagogy of *living mathematics education.* I will describe characteristics of this of this new pedagogy and consider its feasibility. I will then close by contemplating the contributions of the dissertation and directions for future research.

8.1 Reflections on the Dissertation as a Whole

A new paradigm is powerful when it not only critiques prevailing paradigms but also makes us reconsider and reinterpret familiar phenomena in new ways. The AQAL framework, in providing an analytical space for integrating multiple perspectives, enables such a reconsideration of the phenomenon of life in mathematics education.

In reflecting on the dissertation as a whole, I have come to appreciate that the different research chapters represent a step-wise traversal of the four quadrants of the AQAL matrix – objective (UL), intersubjective (LL), objective (UR), and interobjective (LR). Each chapter studies evolving phenomena in at least one of the quadrants. When the different evolutions are correlated, four levels of mathematics education become apparent – traditional, modernist, postmodern, and integral. By surveying all quadrants and all levels, the dissertation advances an integral grand narrative of evolution towards living mathematics education.

8.1.1 Quadrant View of the Dissertation

As stated previously, the quadrants are four essential and irreducible perspectives on reality. As figure 16 shows, different quadrants are home to different likely sources of life in mathematics education.

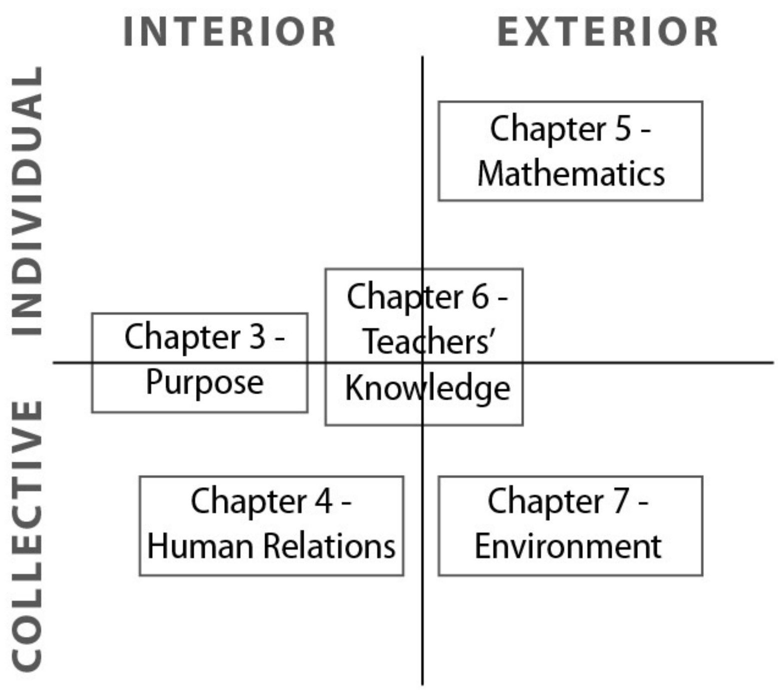


Figure 16. Quadratic view of the research chapters

In Chapter 3, I revisited the continuing discussion of the purposes of mathematics education. These purposes are inner stories and rationalizations, both personal (UL) and communal (LL), which provide the teaching and learning of mathematics with their *raison-d'être*. Given the complexity of challenges that global society faces, I proposed that evolving the purpose of mathematics education from individual utility to collective necessity is likely to fill the field with new life. In Chapter 7, I identified ecological sustainability as a ready domain in which mathematics for healing the world can make a difference. Since mathematics educators have been largely absent from the greater conversation about ecology, the chapter's call for *sustainable mathematics education* is an attempt to connect mathematics pedagogy explicitly with natural systems and the environment (LR). A mutually co-creative symbiosis between mathematics education and life is thus established.

Mathematics education's insularity from the natural environment is largely due to the pervasive legacy of Platonism. In Chapter 5, I studied the evolution of mathematics (UR) in order to identify approaches to overcoming the Platonic barrier. One successful approach described in the chapter is *concept study*, a novel integrative research methodology used to interrogate teachers' mathematical knowledge. In Chapter 6, I kept exploring teachers' knowledge of the subject matter as a possible source of life. The chapter offers an extensive integral reconceptualization of the familiar mathematics-for-teaching problem by correlating evolution in all four quadrants (UL, LL, UR, LR). The resulting synthesis reveals that at higher evolutionary stages, the practice of teaching becomes an open and lively improvisation.

In Chapter 4, I surveyed the literature on intersubjective dialogue (LL) and contemplate some integral connections with the subject matter of mathematics (UR). *What sorts of human relations bring life to mathematical engagements, and what mathematics brings life to human interactions?* My study of intersubjectivity in Chapter 4 illustrates an important advantage of AQAL's quadrant view. It illumines overlooked quadrants and signals the need to make more connections.

8.1.2 Levels View of the Dissertation

As instructive as the quadrant view has been for this study, the evolutionary perspective of the levels view has proven even more so. I used the metatheoretical lens of development extensively in the dissertation to study the evolutions of phenomena in all four quadrants of mathematics education. However, my stage models should not be interpreted as rigid categorical imperatives. They are valuable inasmuch as they provide general orientating guidelines about the patterns and potentials of evolution.

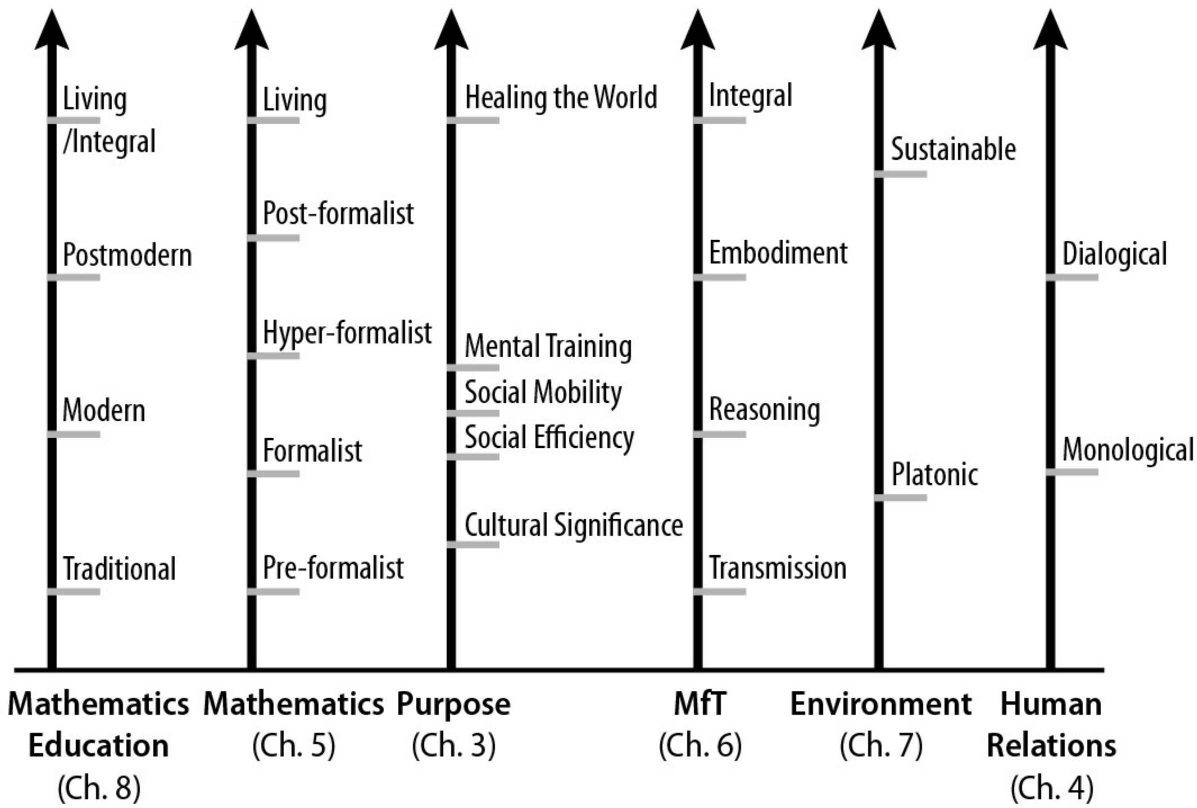


Figure 17. Correlation of the evolutionary strands of mathematics education

Figure 17 correlates the different stage models to reveal three broad extant worldviews on mathematics education – the traditional, the modernist, and the postmodern – and a fourth emergent perspective – the integral. These worldviews are natural epistemologies that arose dialectically in the course of history and that organize individuals' interpretations of reality (Gebser 1949/1984, Kegan 1994). The *traditional* worldview is ethnocentric and conformist. The *modernist* worldview is individualistic and rational. The *postmodern* worldview is pluralistic and inclusive. The emergent integral worldview is aperspectival, and the first to recognize the systemic pattern that unifies the other worldviews.

Different dimensions of the integral stage of mathematics education – dialogical classes, non-Platonic mathematics, an open way of teaching, the purpose of healing the world, and sustainable mathematics education – are worked out in the research chapters of this dissertation. As viewed by the integral wave, mathematics is a multifaceted system whose evolution co-manifests in four quadrants. From this perspective, the purpose of teaching mathematics is to promote the wellbeing of the entire evolutionary system depicted in figure 17. Integral teachers employ an expansive pedagogy that strives to continuously integrate further aspects of reality. As more perspectives are included, the boundaries of that which is deemed mathematical and that which is not recognized as such are blurred. Within an integral pedagogy, mathematics is regarded as inseparable from the environment.

From a developmental perspective, the ongoing process of integration is the primary source of life in mathematics classrooms. Whether teachers integrate diverse purposes, conceptions of mathematics, or conceptions of teaching and learning, it is their responsiveness in the moment to living situations that infuses their practice with life.

8.2 Living Mathematics Education

8.2.1 From Platonism to Living Mathematics

Persistent evolutionary challenges confront mathematics teachers who wish to enact the integral imperative in their pedagogy. Some of these challenges arise from the commitments of the traditional, modernist, and even postmodern worldviews. Many suffocating false dichotomies in mathematics education can be traced back to the Platonic barrier: stability vs. emergence in mathematical discourse, formal mathematics vs. cultural mathematics, the mathematical vs. the non-mathematical, and disciplinarity vs. transdisciplinarity. From an integral perspective, the Platonic barrier is a developmental pathology that stunts potential growth; mathematics education appears to be moribund because it suffers from a case of arrested development.

Integral philosophy prescribes the remedy of *skillful means.* The term borrows from Buddhist teachings and refers to the adept use of discourses and practices to satisfy needs of stakeholders at different levels of the evolutionary spiral while advancing the movement of the spiral as a whole. In the case of mathematics education, overcoming the Platonic barrier and evolving mathematics to the embodied-enacted stage would require a strategic mix of old and new classroom vocabulary and practices. The application of skillful means should not be mistaken for clever rhetoric designed to convert unsuspecting Platonist practitioners into adherents of embodied mathematics. Rather, it is an integrative practice that sincerely values the contributions of all levels of the evolutionary spiral, including those of Platonism.

An evolution that transcends formal mathematics will necessarily also include it. Without a doubt, formal mathematics stands as a model of originality, sophistication, and beauty among human intellectual creations. It is also to be credited with many of the technological improvements in life conditions made possible by modernist progress. At the same time, formal mathematics does not hold exclusive claim to all that is mathematical. Many other and different mathematics reside outside of formal texts; they manifest in personal and collective interpretations, and in social and cultural practices.

Integral thinking gestures towards the possibility of synthesizing Platonic and non-Platonic mathematics to produce a grander mathematics that includes and transcends both paradigms. I call the resulting synthesis *living mathematics*. This term connotes both the life that abounds in mathematics when viewed through integral eyes, and the never-ending evolutionary process in which mathematics keeps reaching out to connect to even further aspects of reality. The pedagogy that responds to living mathematics and enacts its richness in mathematics classrooms is *living mathematics education*.

8.2.2 Living Mathematics Education in Action

Is the integral transformation possible within the context of today's mathematics education? I am convinced that the answer is yes. Living mathematics education is already practiced, even if its practitioners do not necessarily call it by that name. Admittedly, I have met only a handful of educators who enact their pedagogy on such an extensive scale, but in each case the expansiveness of their vision was unmistakable. I report on two of these instances in the dissertation.

The first instance is the account of the participatory research methodology of concept study (Chapter 5), pioneered by Brent Davis. At first glance, concept study appears similar to other techniques for surveying teachers' mathematical knowledge. However, closer inspection reveals that something quite extraordinary happens in concept study groups. The process starts with the question *"What is X?",* where X represents the mathematical concept under consideration. The first few answers are usually predictable and routine. They tend to be the Platonic textbook answers. Then comes the follow-up question *"And what else is there?"* As more and more answers are generated from the group members' past experiences and elaborated hermeneutically, the participants become aware that some of these answers cannot be found in any textbook. So what kind of mathematics are they dealing with? The shift to living mathematics begins from the moment that this question comes into play.

As the teachers proceed to blend metaphorical meanings into new structures, even more new mathematics emerges. Over time, as teachers become comfortable with the process of generating new mathematics, their Platonic commitments are relaxed. A new openness emerges. How far a concept study might proceed and how much of reality is allowed to come into the conversation are determined by collective agreement.

The second instance of living mathematics education is the account of the circles unit in a Grade 8 class. I was fortunate to co-teach this unit with Freddie Irani, an enlightened teacher who has taught me much about the boundless possibilities of integral teaching. The study of circles began with rather standard instruction: properties of circles, area, and circumference. We then asked the students to get into groups and discuss the question *"What's interesting about circles?"* (teaching interlude 1). We preferred this question to its concept study counterpart, *"What are circles?",* because it is less objectifying and invites personal preferences and value judgments. The class voted to study the question *"Do circles have one side or infinitely many sides?"*. It was a completely unexpected question, one to which neither of us co-teachers had a ready answer.

The class then explored this question over four meetings. We used all resources at our disposal, from personal reflections to Google searches. Many of the explorations required calculations of areas of circles and polygons. The students were eager to use results of formal mathematics as a means to getting to the larger answers they were seeking. Emergent answers gave rise to even more questions (teaching interludes 4 and 6). We constantly came up against issues of language and definition and, at one time, we even grappled with some philosophical questions of existence (teaching interlude 4).

Our progress was non-linear and we sometimes found ourselves stuck. I used these opportunities to direct the students' attention to other interesting issues that involve circles. For example, a song about π led to explorations of self-identity (teaching interlude 2). The famous "rope around the equator" problem resulted in a brilliant and unexpected solution by one of the students (teaching interlude 3). Here was emergent mathematics at its finest.

At the end of the four meetings, general consensus was reached that no definitive answer to the question *"Do circles have one side or infinitely many sides?"* could be reached (teaching interlude 6). Any given answer would necessarily represent just one perspective and would be constrained by its author's assumptions and use of language. The greater question of "What is interesting about circles?" was left open for further exploration.

The examples of concept study and the circles unit demonstrate that living mathematics education is not only possible, but is not difficult to condition in ordinary educative situations. In both cases, all four quadrants and different levels were accessed, and integrative connections were made in many instances.

8.2.3 Practical Considerations

It is not easy to imagine how the project of integral mathematics education could be scaled up for wide use in the context of today's school systems. The imperative to "include everything" represents a radical new pedagogy for which no instruction manual exists or could exist. It is also not clear to what extent mathematics educators would be willing or able to enact living mathematics education in their classrooms. At this point, it is useful to consider whether or not integral mathematics education is practical at all.

My answer is that it is and it is not. Living mathematics pedagogy is not practical because we teach mathematics to every student at every grade level. Armies of teachers are needed to carry out instruction on this scale. Since it is nearly impossible to find so many people with a keen interest in the profession, mathematics is sometimes taught by teachers who do not have a particularly strong feel for it as a subject matter. It is one thing to ask teachers to teach Pythagoras' Theorem by reading the textbook's explanation out loud and following it up with a drill and a test. It is quite another to ask them to have an involved, lively interaction with their students around the question "What is interesting about right triangles?". Teachers' responsiveness and improvisational capacity depend to a large extent on their own degree of comfort with mathematics. The systemic shortage of human resources in mathematics education makes it unlikely that living mathematics education can be implemented everywhere.

Living mathematics education is practical, on the other hand, because I have witnessed firsthand the transformative impact that it can have on teachers. Some of the teachers participating in our concept study groups have already begun to broaden their instruction as a result of their participation. I predict that the pace of experimentation will accelerate as teachers gain more confidence in their improvisational skills. The AQAL directive "include everything" does not prescribe what should or should not be included at any particular moment; and so, teachers can enjoy considerable freedom as they try to figure out what works in their own situations. Even though not every teacher will embrace this approach, teachers who are deliberate about changing their pedagogical practices are likely to experience success and to reap the rewards of transformation.

I believe that living mathematics education is practical also because I witness the daily struggles of my students, as they try to reconcile the incredible amount of information to which they are exposed through the Internet with the narrow foci of school instruction. The widespread networking of human knowledge, made possible by digital technologies in the last decade, is profoundly changing the ways in which children absorb and react to information. On the whole, my students are far more sophisticated, globally aware, and knowledgeable than those of 20 years ago. As the trend towards greater complexity continues, I believe that many mathematics teachers would welcome living mathematics pedagogy as a way to keep growing professionally by staying relevant to their students.

Finally, I answer that living mathematics education is practical because I believe that our society will face dramatic changes in the coming decades. The biggest of these is likely to be the transformation to sustainable modes of living. Sustainable mathematics education, with its metaphors of chaos and complexity, can become a catalyst for living mathematics pedagogy, as it reshapes our understanding of reality.

8.3 Reflections on Contributions to the Field and Future Research Directions

The main contribution of this dissertation is the new transcendent perspective on the field of mathematics education that it offers, and the new map that emerges to help navigate the field's past histories and evolutionary potentials. As I argued in the Introduction, mathematics education is at a crossroads at which it would greatly benefit from a big-picture organizing map. My introduction of integral discourse to mathematics education has facilitated new readings of some longstanding quandaries: questions of purpose, ontological status of mathematics, teachers' knowledge of mathematics, classroom relations, and environment. In each case, integral theory has enabled a structured analysis of past epistemologies and suggested directions for conscious evolution.

Since integral thinking is both connected and connecting, some of the research chapters have contributed to the field by building bridges between mathematics education and other discourses not typically associated with it. Specifically, Chapter 4 made new connections with intersubjective dialogue, especially as articulated by Buber, Bakthin, Sidorkin, and Noddings; Chapter 7 made new connections with the greater conversation on ecology and sustainability.

As useful as the integral map might be in providing an organizing perspective, the territory it describes can only be enacted with lived experiences. The present dissertation has contributed to integral studies by providing a lived instance of integral principles in a hitherto unconnected field. I hope that educators in other disciplines would be inspired by this work to explore the evolutions of pedagogy in their respective areas of scholarship.

In contemplating future research directions and paradigms, I am struck by the manner in which almost every chapter in this work opens up a new area of research in mathematics education. AQAL's high-level metatheoretical gaze leaves a lot of room for future midlevel theory building and applications. Some chapters pose direct questions for future research: *What subject matter promotes better human relations in the classroom? How can mathematics pedagogy engage with ecological sustainability?* Others outline new research possibilities by reframing existing questions. For example, *what is the impact of our new understanding of mathematics-for-teaching as an open disposition on the training and professional development of mathematics teachers?*

Integral philosophy and metatheory building have just begun to enter academic discourse in the humanities and social sciences. As this philosophical orientation develops, more lenses will be available through which mathematics education might be analyzed. The systemic relationships among these lenses should also be further clarified. This dissertation made use of only two foundational epistemological lenses within Ken Wilber's AQAL matrix: quadrants and levels. The use of just these two lenses resulted in an explosion of new ideas for evolving the field. I predict that future integral research in mathematics education will also activate AQAL's foundational lenses of lines, types, and states. Such research will investigate connections between mathematics education and feelings, aesthetics, beauty, and spirit.

In my own work, I am excited to have identified research foci to keep me busy for at least a decade or more. I plan to continue researching the methodology of concept study with Brent Davis and to broaden its scope to include more aspects of the four quadrants, especially the subjective (UL) dimension. I am excited to continue working with Freddie Irani on projects of living pedagogy, similar to the circles unit, and to report on the mathematics and life that emerge. Our next project is to teach chaos to high school students. It will enable us to observe firsthand how students interpret and react to issues of sustainability. Above all, I am looking forward to developing a curriculum of living mathematics education for young children, grades K-3. Experience has taught me that the greatest impact in mathematics education is made in the earliest grades.

8.4 Concluding Remarks

When reflecting on the breadth and flexibility of living mathematics education, I am reminded of an integral science educator who once told me that, when it comes to science, he finds it difficult to know what to exclude. "Science is everything," he said, as he stretched his arms as wide as possible to illustrate his point. It's funny, because this is exactly how I feel now about mathematics. Mathematics is everything too. And if language is everything as well, and so are social studies and art, why do we need school disciplines at all?

Integral thinking in education clearly indicates the need for interdisciplinarity and transdisciplinarity. And yet, I believe that each discipline still has its own perspective, its own special insights, and its own unique flavours to impart. I would not feel authentic in writing this dissertation from any other perspective but that of mathematics education. If disciplines are understood as lenses through which life is viewed and experienced, then mathematics is my lens. Life tastes different through it.

As I bring my reflections on the research in this dissertation to a close, I am grateful to have found answers to so many of the questions, big and small, that arose at the start of my studies.

Why? Because life is interesting when I open up to it. Every moment carries the promise of development.

Why teach (mathematics)? Teach in order to evolve. Teach because each moment of living (mathematics) education gives rise to new learning for everyone, including myself. Teach because it is interesting.

How to teach logarithms better? Start by asking students, "*What is interesting about logarithms?*" Then listen carefully and respond in the moment to whatever comes back. Allow as many points of view to enter the dialogue and then integrate them. Take pleasure in improvising.

My search for life in mathematics education has taken a circular path; it began with logarithms and ends with them too. In the process, I discovered that life manifests in four quadrants and multiple developmental scales. Logarithms are a primary mathematical lens through which humans can access and conceptualize the notions of scalability and nestedness. As such, they provide an excellent example of why mathematics will always remain interesting – it affords humans the opportunity to become more intimately acquainted with life.

Living mathematics presents teachers with a generative global question for endless elaboration in localized educative settings: *"What is interesting about mathematics?"* Are we courageous enough to adopt it as our living pedagogy? Are we ready for the life that will erupt?

BIBLIOGRAPHY

Adler, J., & Davis, Z. (2006). Opening another black box: Research mathematics for teaching in mathematics teacher education. *Journal for Research in Mathematics Education*, *37*(4), 270-296.

Appelbaum, P. (1995). *Popular culture, educational discourse, and mathematics*. Albany, NY: State University of New York Press.

Baker, S. (2008). *The numerati*. New York: Houghton Mifflin.

Bakthin, M. M. (1965). *Rabelais and his world* (H. Iswolsky, Trans.). Cambridge: MIT Press. (Original work published 1941)

Bakthin, M. M. (1984). *Problems of Dostoevsky's poetics* (C. Emerson, Trans.). Minneapolis: University of Minnesota Press. (Original work published 1929)

Baldwin, J.M. (2007). *Thought and things: A study of the development and meaning of thought, or genetic logic* (Vols. 1- 6). Whitefish, MT: Kessinger Publishing. (Original work published 1906-1911)

Ball, D.L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics . In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Ball, D.L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

Ball, D.L., Ferrini-Mundy, J., Kilpatrick, J., Milgram, J., Schmid, W., & Schaar, R. (2005). Reaching for common ground in K-12 mathematics education [Electronic version]. *Notices of the American Mathematical Society, 52*(9), 1055-1058.

Barabási, A.-L. (2003). *Linked: How everything is connected to everything else and what it means for business, science, and everyday life*. London: Penguin Books.

Barrow, J. (1992). *Pi in the sky: Counting, thinking, and being*. Oxford, UK: Clarendon Press.

Barwell, R. (2010). Climate change and mathematics education: A critical mathematics education perspective. In Pinto, M. F. & Kawaski, T. F. (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*. Vol. 2, pp. 193-200. Belo Horizonte, Brazil: PME.

Bateson, G. (2000). The logical categories of learning and communications. In *Steps to an ecology of mind: Collected essays in anthropology, psychiatry, evolution and epistemology.* Chicago: University of Chicago Press. (Original paper published 1972)

Battista, M. T. (1999). The mathematical miseducation of America's youth: Ignoring research and scientific study in education. *Phi Delta Kappan, 80(*6)*,* 424-433.

Beck, D.E., & Cowan, C.C. (2006). *Spiral dynamics: Mastering values, leadership, and change*. Malden, MA: Blackwell Publishing.

Begle, E.G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature.* Washington, DC: Mathematical Association of American and National Council of Teachers of Mathematics.

Briggs, J., & Peat, F. D. (2000). *Seven life lessons of chaos: Spiritual wisdom from the science of change.* New York: Harper Collins.

Brown, L. R. (2009). *Plan B 4.0: Mobilizing to save civilization.* London: Norton.

Bruner, J. (1996). *The culture of education.* Cambridge, MA: Harvard University Press.

Buber, M. (1970). *I and thou* (W. Kaufmann, Trans.). New York: Charles Scribner's Sons. (Original work published 1923)

Burger, E. B., & Starbird, M. (2005). *The heart of mathematics: An invitation to effective thinking*. Emeryville, CA: Key College.

Bush, K. (2005). π. On *Aerial Album* CD. UK: EMI; US:Columbia

Capra, F. (2002). *The hidden connections.* London: Harper Collins.

Cheb-Terrab, E.S., & Roche, A.D. (2003). An Abel ordinary differential equation class generalizing known integrable classes, *European Journal of Applied Mathematics, 14*(2), 217-229.

Commons, M.L., Richards, F.A., & Armon, C. (Eds.) (1984). *Beyond formal operations: late adolescent and adult cognitive development*. New York: Praeger.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 345-352). Thessaloniki, Greece: PME.

Cook-Greuter, S. (2005). Ego development: Nine levels of increasing embrace. Unpublished manuscript. Retrieved June 25, 2008, from http://www.cook-greuter.com

Damasio, A. (1994). *Descartes' error: Emotion, reason and the human brain.* New York: Putnam.

Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.

Davis, B. (2004). *Inventions of teaching: A genealogy*. Mahwah, NJ: Erlbaum.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics teaching and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137-167.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61,* 293-319.

Davis, B., & Sumara, D. (2006). *Complexity and education: inquiries into learning, teaching, and research*. Mahwah, NJ: Erlbaum.

Davis, P.J., & Hersh, R. (1981). *The mathematical experience*. Boston: Houghton Mifflin.

Davis, P.J., & Hersh, R. (1986). *Descartes' dream: The world according to mathematics*. Mineola, NY: Dover.

Deffeyes, K. S. (2001). *Hubbert's peak: The impending world oil shortage.* Princeton, MA: Princeton University Press.

Dewey, J. (1910). *The influence of Darwin on philosophy (and other essays).* New York: Henry Holt.

Eames, R., & Eames, C. (Writers and Directors) (1977). *Powers of ten*. [Motion picture] Santa Monica, CA: Pyramid Media.

Edwards, A. R. (2005). *The sustainability revolution: Portrait of a paradigm shift.* Gabriola Island, Canada: New Society.

Edwards, M. G. (2010). *Organisational transformation for sustainability: An integral metatheory.* New York: Routledge.

Eigen, M., & Winkler, R. (1993). *Laws of the game: How the principles of nature govern chance.* Princeton, MA: Princeton University Press.

Ernest, P. (1985). The philosophy of mathematics and mathematics education. *International Journal for Mathematics Education in Science and Technology, 16*(5), 603-612.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics.* New York: SUNY Press.

Esbjörn-Hargens, S. (2006). Integral research: A multi-method approach to investigating phenomena. *Constructivism and the Human Sciences, 11(*1*)*, 79-107.

Esbjörn-Hargens, S., Reams, J. & Gunnlaugson, O. (Eds.). (2010). *Integral education: New directions for higher learning*. Albany, NY: SUNY Press.

Esbjörn-Hargens, S., & Zimmerman, M. E. (2009). *Integral ecology: Uniting multiple perspectives on the natural world.* Boston, MA: Integral Books.

Friedman, T. L. (2008). *Hot, flat, and crowded: Why we need a green revolution – and how it can renew America.* New York: Farrar, Strauss, and Giroux.

Fuite, J. (2005, September). *Network education: Understanding the functional organization of a class.* Paper presented at the Complexity, Science & Society Conference, The University of Liverpool, UK.

Gebser, J. (1984). *The ever-present origin* (N. Barstad & A. Mikunas, Trans.). Athens, OH: Ohio University Press. (Original work published 1949)

Giddens, A. (1984). *The constitution of society: Outline of the theory of structuration*. Cambridge, UK: Polity

Gilligan, C. (1982). In a different voice: Psychological theory and women's development. Cambridge, MA: Harvard University Press.

Goodwin, R.M. (1990). *Chaotic economic dynamics.* Oxford, UK: Clarendon Press

Graves, C.W. (1970) Levels of existence: An open system theory of values. *Journal of Humanistic Psychology, 10*(2), 131-155.

Gutstein, E., & Peterson, B. (Eds.). (2006). *Rethinking mathematics: Teaching social justice by the numbers*. Milwaukee, WI: Rethinking schools.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge: Cambridge University Press.

Hawken, P. (2007). *Blessed unrest: How the largest movement in the world came into being and why no one saw it coming*. New York: Penguin.

Heath, D. H. (1994). *Schools of hope: Developing mind and character in today's youth*. San Francisco: Jossey-Bass.

Hicks, D., & Slaughter, R. (Eds.). (1998). *Futures Education: The World Yearbook of Education*. London: Kogan Page.

Hochachka, G. (2009). *Developing sustainability, developing the self: An integral approach to international and community development.* Victoria, BC: Trafford Publishing.

Hopkins, C., & McKeown, R. (1999). "Education for Sustainable Development." *Forum for Applied Research and Public Policy*, *14*(4), 25-28.

Intergovernmental Panel of Climate Change (IPCC). (2008). *Climate change 2007 synthesis report*. Geneva: IPCC.

Jenlink, P. M., & Banathy, B. H. (2005). Dialogue: Conversation as culture creating and consciousness evolving. In Banathy, B. H. & Jenlink, P. M. (Eds.), *Dialogue as a means of collective communication* (pp. 3-16). New York: Kluwer Academic / Plenum Publishers.

Johnson, B., & Ongwuegbuzie, A.J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher*, *33*(7), 14-26.

Johnson, B., & Turner, L. (2003). Data collection strategies in mixed methods research. In Tashakkori, A. & Teddlie, C. (Eds.) *Handbook of mixed methods in social and behavioural research* (pp. 297-321). Thousand Oaks, CA: Sage Publications.

Johnson, S. (2002). *Emergence: The connected lives of ants, brains, cities and software.* New York: Simon and Schuster.

Juarrero, A. (1999) *Dynamics in action: Intentional behaviour as a complex system*. Cambridge, MA: MIT Press.

Kegan, R. (1982). *The evolving self: Problem and process in human development*. Cambridge, MA: Harvard University Press.

Kegan, R. (1994). *In over our heads: The mental demands of modern life*. Cambridge, MA: Harvard University Press.

Koestler, A. (1968). *The ghost in the machine: The urge to self-destruction – a psychological and evolutionary study of modern man's predicament.* London: Macmillan.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to Western thought.* New York: Basic Books.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Lappan, G. (2006). President's choice: 'Why study mathematics" F. L. Wren. *Mathematics Teacher, 100*(1)*,* 4-13.

Laszlo, E. (1996). *The systems view of the world*. Crestkill, NJ: Hampton Press.

Lorenz, E. (1972, December). *Predictability: Does the flap of a butterfly's wings in Brazil set off a Tornado in Texas?* Paper presented at the meeting of the American Association for the Advancement of Science, Washington, DC.

Luhmann, N. (1990).The autopoiesis of social systems. In Luhmann, N. *Essays on self-reference*. New York: Columbia University Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Ma, X., & Willms, J.D. (1999). Dropping out of advanced mathematics: How much do students and schools contribute to the problem? *Educational Evaluation and Policy Analysis, 21*(4), 365-383.

Marquis, A. (2007). *The integral intake: A guide to comprehensive idiographic assessment in integral psychotherapy*. New York: Routledge.

Martin, J. (2008). Integral research as a practical mixed-method framework: Clarifying the role of integral methodological pluralism. *Journal of Integral Theory and Practice, 3*(2), 155-164.

Martin, J. R. (1992). *The schoolhome: Rethinking schools for changing families*. Cambridge, MA: Harvard University Press.

Maturana, H.T., & Varela, H. (1972). *The tree of knowledge: The biological roots of human understanding*. Boston: Shambhala.

Maturana, H.T., & Varela, H. (1987). *Autopoiesis and cognition: The realization of the living.* New York: Springer.

McIntosh, S. (2007). *Integral consciousness and the future of evolution: How the integral worldview is transforming politics, culture and spirituality*. St. Paul, MN: Paragon House.

Meadows, D. H. (2005). Dancing with systems. In Stone, M. K., & Barlow, Z. (Eds.), *Ecological literacy: Educating our children for a sustainable world.* San Francisco: Sierra Club Books.

Millennium Ecosystems Assessment. (2005). *Ecosystems and human well-being: Biodiversity synthesis.* Washington, DC: World Resources Institute.

Mowat, E. (2008). Making connections: Mathematical understanding and network theory. *For the Learning of Mathematics, 28*(3), 20-27.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Neill, A. S. (1961). *Summerhill: A radical approach to child rearing*. New York: Hart Publishing.

Noddings, N. (1986). *Caring: A feminine approach to ethics and moral education*. Berkeley, CA: University of California Press.

Noddings, N. (1997). Must we motivate? In N. J. Burbules & D. T. Hansen (Eds.), *Teaching and its predicaments* (pp. 29-44). Boulder, CO: Westview Press.

Noddings, N., Bingham, C., Sidorkin, A. M., Biesta, G., Margonis, F., Stengel, B. S., et al. (2004). Manifesto of relational pedagogy: Meeting to learn, learning to meet. In Bingham, C. & Sidorkin, A. M. (Eds.), *No education without relation* (pp. 5-7). New York: Peter Lang.

O'Sullivan, E. (1999). *Transformative learning: Education vision for the 21st century.* Toronto, Canada: Toronto University Press.

Ontario Department of Education. (1951). *Intermediate division: Outline of courses for experimental use.* (Curriculum I:I Revised).

Orr, D. (2004). *Earth in mind: On education, environment, and the human prospect (10th anniversary ed.).* Washington, DC: Island Press.

Palmer, J. A. (1998). *Environmental education in the 21st century: Theory, practice, progress and promise.* London: Routledge.

Popper, K., & Notturno, M.A.(Ed). (1996). *The myth of the framework: In defence of science and rationality.* London: Routledge.

Programme for International Student Assessment (PISA) (2009). Retrieved May 14, 2009, from http://www.oecd.org/documents/61/0,3343,en_2649_

Rees, M. (2003). *Our final hour: A scientist's warning: How terror, error, and environmental disaster threaten humankind's future in this century – on earth and beyond.* New York: Basic Books.

Renert, M. (2011). Mathematics for life: Sustainable mathematics education. *For the Learning of Mathematics. 31*(1), 20-26.

Roy, B. (2006). A process model of integral theory. *Integral Review, 3,* 118-152.

Schoenfeld, A.H. (1994). Reflections on doing and teaching mathematics. In Schoenfeld, A.H. (Ed.). *Mathematical thinking and problem solving* (pp. 53-70). Hillsdale, NJ: Erlbaum.

Schoenfeld, A.H. (2004). The math wars. *Educational Policy, 18*(1), 253-286.

Sfard, A. (2005). What could be more practical than good research? On mutual relations between research and practice of mathematics education. *Educational Studies in Mathematics, 58*(3), 393-413.

Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Sidorkin, A.M. (1999). *Beyond discourse: Education, the self, and dialogue*. Albany, NY: SUNY Press.

Sidorkin, A.M. (2002). *Learning relations: Impure education, deschooled schools, & dialogue with evil*. New York: Peter Lang.

Sirgy, J.M. (1988). Strategies for developing general systems theories. *Behavioral Science, 33*(1), 25-37.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education.* Dordrecht, The Netherlands: Kluwer Academic.

Sterling, S. (2004). *Sustainable education: Re-visioning learning and change.* Foxhole, UK: Green Books.

Teilhard de Chardin, P., (1964). *The future of man*. New York: Collin

Torbert, W.R. (2004). *Action inquiry: The secret of timely and transforming leadership.* San Francisco: Berrett Koehler

Union of Concerned Scientists. (1992). *1992 world scientists' warning to humanity*. Retrieved October 3, 2010, from http://www.ucsusa.org/about/1992-world-scientists.html

Van de Ven, A.H. (1999). The buzzing, blooming, confusing world or organization and management theory: A view from Lake Wobegon University. *Journal of Management Inquiry*. *8*(2), 118-125.

Varela, F. J. (1999). *Ethical know-how: Action, wisdom, and cognition.* Stanford, CA: Stanford University Press.

Wagner, D., & Davis, B. (2010). Feeling number: Grounding number sense in a sense of quantity. *Educational Studies in Mathematics*, *74* (1), 39-51.

Waldrop, M. M. (1992). *Complexity: The emerging science at the edge of order and chaos.* New York: Simon & Schuster.

Walshaw, M. (Ed.). (2004). *Mathematics education within the postmodern*. Greenwich, CT: Information Age.

Walshaw, M. (Ed.). (2010) *Unpacking pedagogy: New perspectives for mathematics classrooms*. Charlotte, NC: Information Age.

Watts, D. J. (2003). *Six degrees: The science of a connected age*. New York: W. W. Norton.

Wheatley, M. (1994). *Leadership and the new science: Discovering order in a chaotic world.* San Francisco: Berrett-Koehler.

Wheatley, M. (2002). *Turning to one another: Simple conversation to restore hope ot the future.* San Francisco: Berrett-Koehler.

Wilber, K. (1977). *The spectrum of consciousness*. Wheaton, IL: Theosophical Publishing House.

Wilber, K. (1995). *Sex, ecology, spirituality: The spirit of evolution*. Boston: Shambhala

Wilber, K. (2000a). *Eye to eye: The quest for the new paradigm* (3rd ed.). Boston: Shambhala.

Wilber, K. (2000b). *Integral psychology: Consciousness, spirit, psychology, therapy*. Boston: Shambhala.

Wilber, K (2000c). *A theory of everything: An integral vision for business, politics, science and spirituality.* Boston: Shambhala

Wilber, K. (2006a). *Integral spirituality: A startling new role for religion in the modern and postmodern world*. Boston: Integral Books.

Wilber, K (2006b). *The Kosmos trilogy vol. II, excerpt C: The ways we are in this together.* Retrieved January 12, 2010, from http://www.kenwilber.com/writings/read_pdf/85

Wilber, K (2006c). *The Kosmos trilogy vol. II, excerpt D: The look of a feeling – the importance of post-structuralism.* Retrieved January 12, 2010, from http://www.kenwilber.com/writings/read_pdf/86

World Commission on Environment and Development (WCED). (1987). *Our common future.* Oxford, UK: Oxford University Press.

Wright, R. (2004). *A short history of progress.* Toronto, Canada: House of Anansi Press.